Narrative Approaches to the International Mathematical Problems.

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a.k.a. Vo Duc Dien

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1663 Liberty Drive
Bloomington, IN 47403
www.authorhouse.com
Phone: 1-800-839-8640

Published by AuthorHouse 4/12/2012

ISBN: 978-1-4685-6850-9 (e)
ISBN: 978-1-4685-6851-6 (sc)

This book is dedicated to my wife Trần Thị Quỳnh-Châu, my children Catherine Diễm Đinh, Alan Huy Đinh, my mother Thùy Đinh and my father in heaven.

It is also dedicated to my beloved sister Nguyễn Thị Hạnh and brother-in-law Nguyễn Thanh Quang, two of the most important persons in my life.

Last but not least, this book is respectfully dedicated to my former professor Lê Chí Đệ of Lidcombe, Australia who had spent countless days taking many of us to mathematical competitions when I was a youngster in school.

I also would like to take this opportunity to thank all my former teachers who had battled the elements to come teach us at our remote coastal village of Thuận An in central Vietnam during the 1970's. Their teaching and encouragement had shaped my education greatly.

Preface

At first, it was not my intention to write a mathematical book let alone one of this magnitude. One day I stumbled upon a web page that has many difficult mathematical problems used in the international and national competitions that had not been solved in decades. I offered to help out and solve them. However, as time has gone by, I have accumulated as huge quantity of these solutions and there are almost no place organized enough for me to post them, and I thought the best way to bring these solutions to the students in the world is to compile them into a book.

I also included the solutions to the problems used for admission to many of the most prestigious colleges that are equally difficult to help the prospective college students with their entrance exams. Many of the problems in this book can be found in the web page *www.mathlinks.ro*. I have donated some of my solutions to the Mathematical Association of America *http://www.maa.org/* for them to sell and raise funds.

Many of my previous work have been published at *www.cut-the-knot.org*. It's the world's largest and most complete mathematical website that has won more than twenty awards from scientific and educational publications.

My books show the global readers how to solve the problems by examples and provide the narrative and analysis to accompany and explain the solutions in details wherever possible. My previous books are now at many technical college and city libraries around the world. See the back pages for the list of some of these libraries.

Steve Dinh
a.k.a. *Vo Duc Dien*

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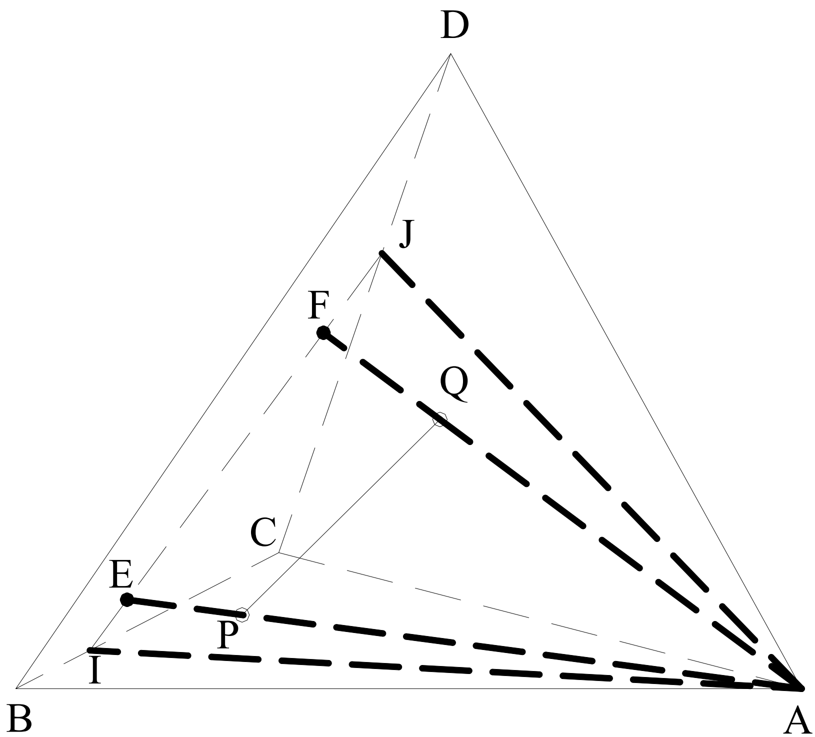
Problem 1 of the United States Mathematical Olympiad 1973

Two points, P and Q, lie in the interior of a regular tetrahedron ABCD. Prove that angle PAQ < 60°.

Solution

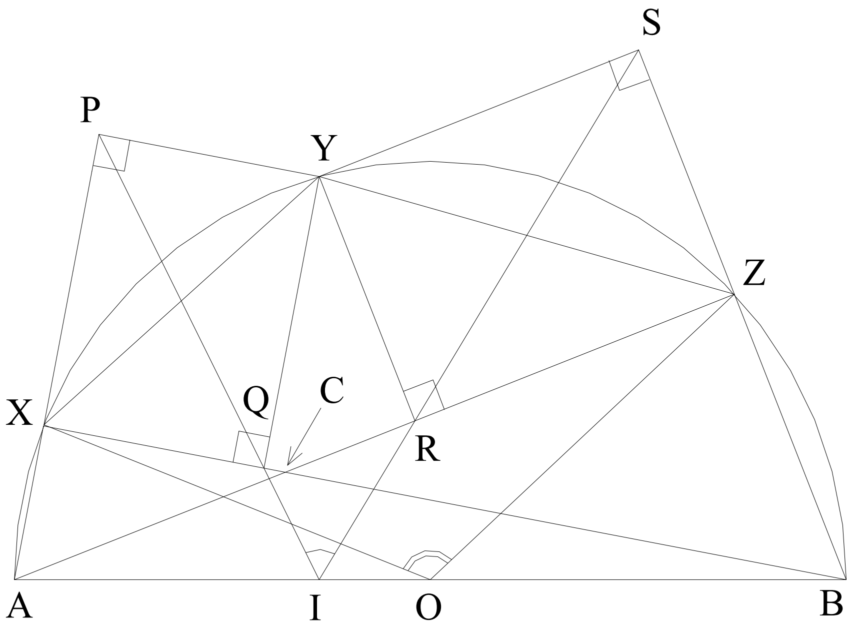


Let the side length of the regular tetrahedron be *a*. Link and extend AP to meet the plane containing triangle BCD at E; link AQ and extend it to meet the same plane at F. We know that E and F are inside triangle BCD and that $\angle$PAQ = $\angle$EAF.
Now let's look at the plane containing triangle BCD with points E and F inside the triangle. Link and extend EF on both sides to meet the sides of the triangle BCD at I and J, I on BC and J on DC. We have $\angle$EAF < $\angle$IAJ.
But since E and F are interior of the tetrahedron, points I and J cannot be both at the vertices and IJ < *a*, $\angle$IAJ < $\angle$BAD = 60°.
Therefore, $\angle$PAQ < 60°.

Problem 1 of the United States Mathematical Olympiad 2010

Let AXYZB be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle$XOZ, where O is the midpoint of segment AB.

Solution



Let AZ intercept BX at C, PQ and RS intercept at I. The acute angle formed by lines PQ and RS is $\angle$PIS = $\angle$PQY + $\angle$SRY − $\angle$QYR = $\angle$PQY + $\angle$SRY – (180° − $\angle$QCR) = $\angle$PQY + $\angle$SRY − $\angle$RCB.

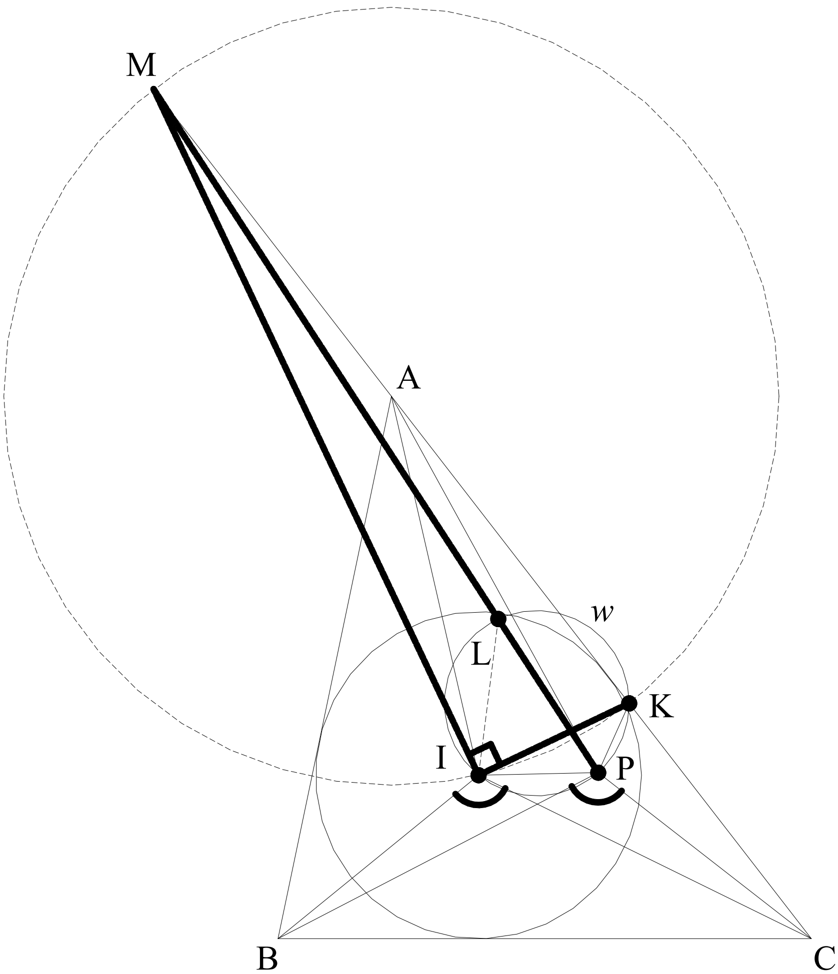
But $\angle$RCB subtends arcs AX and BZ; $\angle$PQY = $\angle$PXY subtends arc AY; $\angle$SRY = $\angle$SZY subtends arc BY.

Therefore, $\angle$PIS subtends arc AY + BY – AX – BZ = arc XZ = ½$\angle$XOZ.

Problem 1 of the International Mathematical Olympiad 2006

Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle$PBA + $\angle$PCA = $\angle$PBC + $\angle$PCB. Show that AP > AI, and that equality holds if and only if P = I.

Solution



We have $\angle$BPC = $\angle$A + $\angle$PBA + $\angle$PCA and $\angle$BIC = $\angle$A + $\angle$IBA + $\angle$ICA. The problem gives us $\angle$PBA +$\angle$PCA = $\angle$PBC

$+ \angle PCB = \frac{1}{2}(\angle ABC + \angle ACB) = \angle IBA + \angle ICA$, and $\angle BPC = \angle BIC$.

Draw a circle with center A that passes through I and intersects AC at K. We have $\angle MPK = 360° - \angle BPC - \angle MPB - \angle KPC$ (i)

But $\angle BPC = \angle BIC$, $\angle MPB = \angle MIB - \angle IMP - \angle IBP$, $\angle KPC = \angle KIC + \angle IKP + \angle ICP$, $\angle IBP = \frac{1}{2}\angle ABC - \angle PBC$, $\angle ICP = \angle PCB - \frac{1}{2}\angle ACB$.

Now equation (i) becomes $\angle MPK = 360° - \angle BIC - \angle MIB + \angle IMP + \angle IBP - \angle KIC - \angle IKP - \angle ICP = 360° - \angle BIC - \angle MIB + \angle IMP + \frac{1}{2}\angle ABC - \angle PBC - \angle KIC - \angle IKP + \frac{1}{2}\angle ACB - \angle PCB$.

But since $\frac{1}{2}(\angle ABC + \angle ACB) = \angle PBC + \angle PCB$, $\angle MPK = (360° - \angle BIC - \angle MIB - \angle KIC) + \angle IMP - \angle IKP$, or
$\angle MIK = 360° - \angle BIC - \angle MIB - \angle KIC = 90°$, or
$\angle MPK = 90° + \angle IMP - \angle IKP = 90° + \angle IMP - \angle ILP$.

Also since $\angle ILP > \angle IMP$, we have $\angle IMP - \angle ILP < 0$, or $\angle MPK < 90°$. Therefore, point P is outside the circle with center A and AP > AI.

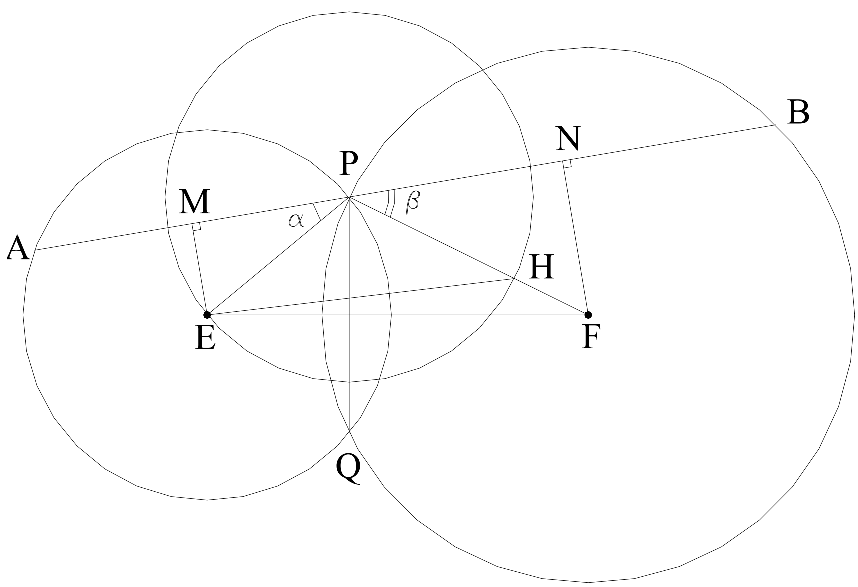
It’s easily seen that equality holds when $P \equiv I$.

The reverse direction is fairly straightforward.

<u>Problem 4 of the United States Mathematical Olympiad 1975</u>

Two given circles intersect in two points P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that AP×PB is a maximum.

<u>Solution</u>



Let E and F be the centers of the small and large circles, respectively, and r and R be their respective radii. Also let M and N be the feet of E and F on AB, respectively, $\alpha = \angle APE$ and $\beta = \angle BPF$.

We have $AP \times PB = 2r\cos\alpha \times 2r\cos\beta = 4rR \times \cos\alpha \times \cos\beta$; $AP \times PB$ is a maximum when the product $\cos\alpha \times \cos\beta$ is a maximum, and we obtain $\cos\alpha \times \cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$.

But $\alpha + \beta = 180° - \angle EPF$ and is fixed, so is its $\cos(\alpha + \beta)$.

So its maximum depends on $\cos(\alpha - \beta)$ which occurs when $\alpha = \beta$.

To draw the line AB:

Draw a circle with center P and radius PE to cut the radius PF at H.

Next, draw a line to parallel EH that passes through P. This line meets the small and large circles at A and B, respectively.

<u>Further observation</u>

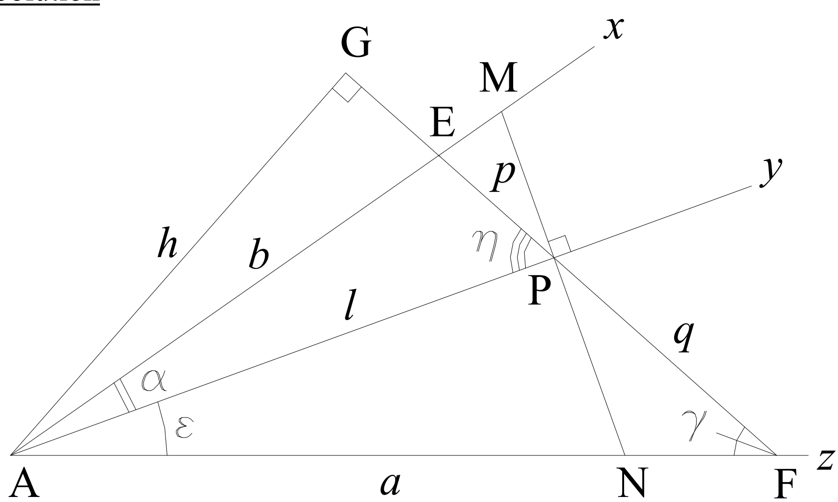
The problem below is derived from the above problem:

Two given circles intersect in two points P and Q. Show how to construct a segment AB passing through P and terminating on the two circles such that the ratio $\frac{AP}{PB}$ equals the ratio of the two radii.

Problem 4 of the United States Mathematical Olympiad 1979

Show how to construct a chord FPE of a given angle A through a fixed point P within the angle A such that $\frac{1}{\mathrm{FP}} + \frac{1}{\mathrm{PE}}$ is a maximum.

Solution



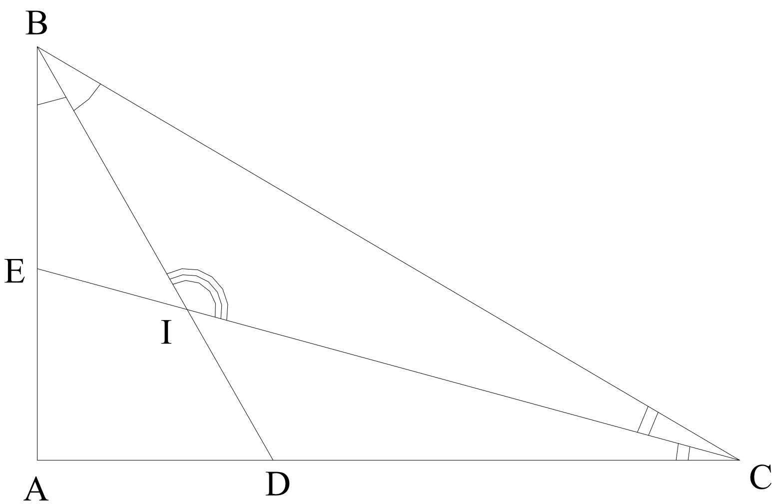
Let EP = p, FP = q, AF = a, AP = l, AE = b, $\angle$EAP = α, $\angle$FAP = ε, $\angle$EPA = η, $\angle$EFA = γ. Extend FE and from A draw a perpendicular line to intercept this extension at G. Now let AG = h.

$\frac{1}{\mathrm{PE}} + \frac{1}{\mathrm{FP}} = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$. The law of the sines gives us $\frac{p+q}{\sin(\alpha+\varepsilon)} = \frac{b}{\sin\gamma}$ and $\frac{b}{p} = \frac{\sin\eta}{\sin\alpha}$, or $\frac{p+q}{pq} = \frac{b\sin(\alpha+\varepsilon)}{pq\times\sin\gamma} = \frac{\sin(\alpha+\varepsilon)\times\sin\eta}{q\sin\gamma\times\sin\alpha}$. But α and ε are constants, so $\frac{1}{p} + \frac{1}{q}$ is a maximum when $\frac{\sin\eta}{q\sin\gamma}$ is a maximum. We also have $\sin\eta = \frac{h}{l}$ and $\sin\gamma = \frac{h}{a}$, and now $\frac{\sin\eta}{q\sin\gamma} = \frac{ha}{qlh} = \frac{a}{ql}$, but l is constant, so $\frac{a}{ql}$ is a maximum when the ratio $\frac{a}{q}$ is a maximum, but $\frac{a}{q} = \frac{\sin(180°-\eta)}{\sin\varepsilon} = \frac{\sin\eta}{\sin\varepsilon}$ and with angle ε fixed, $\frac{a}{q}$ is a maximum when sinη is a maximum or equal to 1 when η = 90° as line MN represents.

Problem 4 of the United States Mathematical Olympiad 2010

Let ABC be a triangle with $\angle A = 90°$. Points D and E lie on sides AC and AB, respectively, such that $\angle ABD = \angle DBC$ and $\angle ACE = \angle ECB$. Segments BD and CE meet at I. Determine whether or not it is possible for segments AB, AC, BI, ID, CI, IE to all have integer lengths.

Solution



Applying the law of the cosine function, we have

$BC^2 = BI^2 + CI^2 - 2\ BI \times CI \times \cos\angle BIC$. But $\angle BIC = 180° - \frac{1}{2} \times (180° - \angle A) = 135°$, and $\cos\angle BIC = -\frac{1}{2}\sqrt{2}$.

The above equation becomes $BC^2 = BI^2 + CI^2 + \sqrt{2}\ BI \times CI$, or

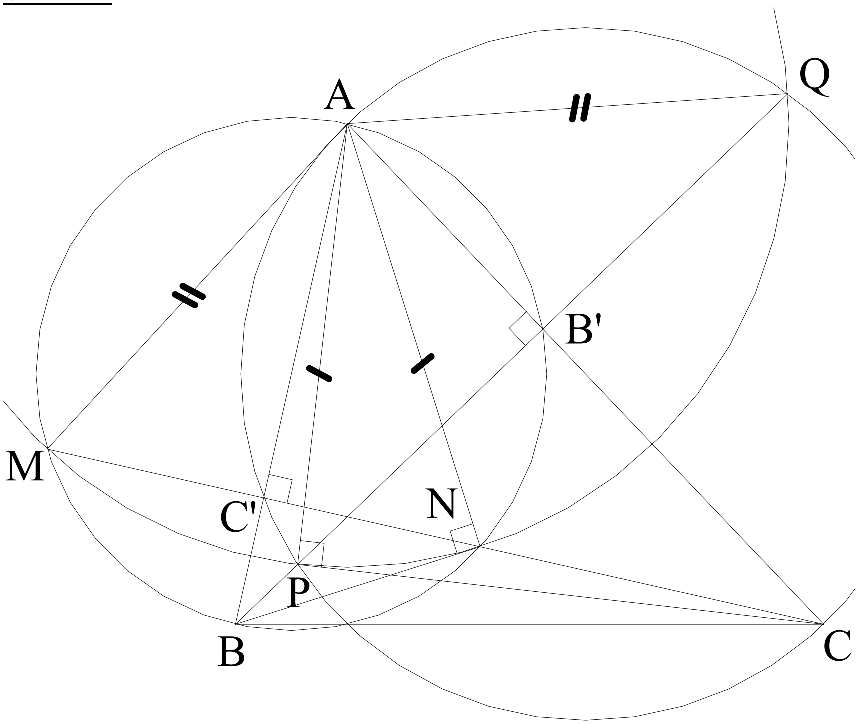
$\sqrt{2}\ BI \times CI = BC^2 - BI^2 - CI^2$.

Now assume that it is possible for segments AB, AC, BI, ID, CI and IE to all have integer lengths. BC^2 is then also an integer because $BC^2 = AB^2 + AC^2$ which, in turn, requires $\sqrt{2}$ BI×CI to be an integer. Since $\sqrt{2}$ is an irrational number, the product of $\sqrt{2}$ with an integer is not an integer. Therefore, our assumption was not possible, and it's not possible for segments AB, AC, BI, ID, CI and IE to all have integer lengths.

<u>Problem 5 of the United States Mathematical Olympiad 1990</u>

An acute-angled triangle ABC is given in the plane. The circle with diameter AB intersects altitude CC' and its extension at points M and N, and the circle with diameter AC intersects altitude BB' and its extensions at P and Q. Prove that the points M, N, P, and Q lie on a common circle.

<u>Solution</u>



We already have AP = AQ and AN = AM. To prove the four points M, N, P and Q to lie on a common circle, it suffices to prove AM = AP. Because AC is the diameter, $\angle APC = 90°$, and we have

$AP^2 = AC^2 - PC^2$, or $AP^2 = AC'^2 + CC'^2 - PC^2$ (i)

and $AM^2 = AC'^2 + C'M^2$ (ii)

Substituting AC^2 from (i) to (ii) to get

$AM^2 = AP^2 - CC'^2 + PC^2 + C'M^2$.

So to prove AM = AP, it suffices to show $CC'^2 = C'M^2 + PC^2$ (iii)

We also have $PC^2 = B'C^2 + PB'^2 = B'C^2 + PB' \times B'Q = B'C^2 + AB' \times B'C = B'C\,(B'C + AB') = B'C \times AC = CM \times CN$. (iv)

Substituting PC^2 from (iv) to (iii), we then need to prove
$CC'^2 = C'M^2 + CM \times CN$ (v)

But $CC' = CM + MC'$, and (v) becomes
$CM^2 + 2\,CM \times MC' + C'M^2 = C'M^2 + CM \times CN$ (vi)

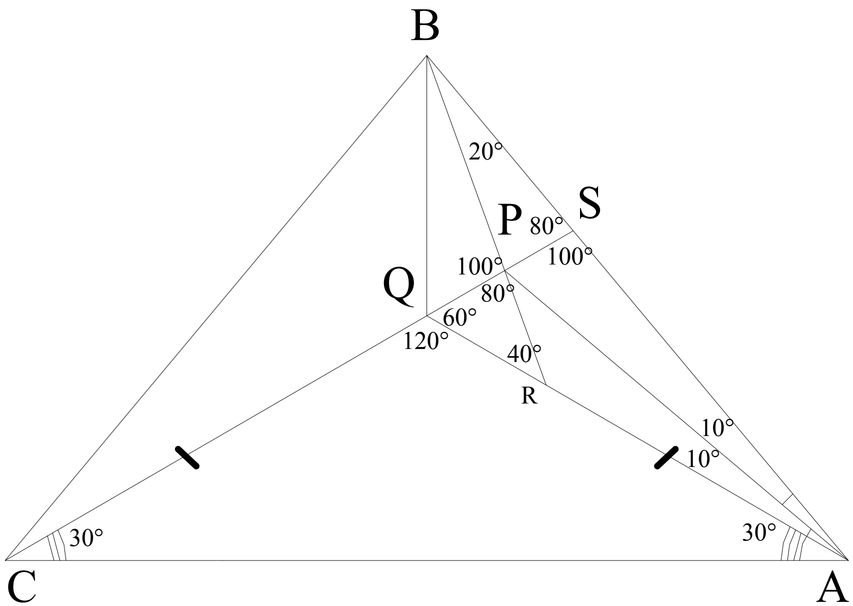
Or we now need to prove
$CM^2 + 2\,CM \times MC' = CM \times CN$ (vii)
or $CM(CM + 2MC') = CM \times CN$.

Because $C'M = C'N$, $CM + 2MC' = CN$, the problem is solved. Therefore, the points M, N, P and Q lie on a common circle with center A and radius AP.

Problem 5 of the United States Mathematical Olympiad 1996

Triangle ABC has the following property: there is an interior point P such that ∠PAB = 10°, ∠PBA = 20°, ∠PCA = 30°, and ∠PAC = 40°. Prove that triangle ABC is isosceles.

Solution



Extend CP to meet AB at S. From A draw a line to meet the extension of BP at R and CP at Q such that □QAP = 10°. We have BR = AR, and
∠BRQ = 40°, ∠QAC = 30°, ∠AQC = 120°, ∠PQR = 60°,
∠QPR = 80°, ∠PSA = 100°, ∠QPB = 100° and ∠BSP = 80°.

It suffices to prove the two triangles QSA and QPB are similar since if they are similar we have ∠QBP = ∠QAS = 20° and ∠QBA = 40°, ∠BQA = 120° = ∠BQC and the two triangles BQC and BQA are congruent and thus BC = BA and the triangle ABC is isosceles.

To prove those two triangles that already have the two equal angles $\angle$QSA = $\angle$QPB = 100° similar, it suffices to prove that

$$\frac{QP}{QS} = \frac{PB}{SA} \qquad \text{(i)}$$

Since AP is bisector of $\angle$QAS, we have $\frac{QP}{QA} = \frac{PS}{SA}$ (ii)

$\frac{QP}{QA} = \frac{PS}{SA} = \frac{QS}{QA + SA}$, or $\frac{QP}{QS} = \frac{QS}{QA + SA}$.

Combining with (i), we now need to prove $\frac{QA}{QA + SA} = \frac{PB}{SA} =$ $\frac{QA - PB}{QA} = \frac{QR + PR}{QA}$ (since BR = AR), or it suffices to prove

$$\frac{QR + PR}{QA} = \frac{PB}{SA} \qquad \text{(iii)}$$

Using the law of the sines, we have

$\frac{QP}{\sin 40^\circ} = \frac{QR}{\sin 80^\circ} = \frac{PR}{\sin 60^\circ} = \frac{QR + PR}{\sin 60^\circ + \sin 80^\circ}$, and $\frac{PS}{\sin 20^\circ} = \frac{PB}{\sin 80^\circ}$, or

$QP = \frac{(QR + PR)\sin 40^\circ}{\sin 60^\circ + \sin 80^\circ}$, and $PS = \frac{PB\sin 20^\circ}{\sin 80^\circ}$.

Substituting QP and PS to (ii), it becomes

$\frac{QR + PR}{QA} \times \frac{\sin 40^\circ}{\sin 60^\circ + \sin 80^\circ} = \frac{PB}{SA} \times \frac{\sin 20^\circ}{\sin 80^\circ}$.

So now we have to prove $\frac{\sin 40^\circ}{\sin 60^\circ + \sin 80^\circ} = \frac{\sin 20^\circ}{\sin 80^\circ}$ (iv)

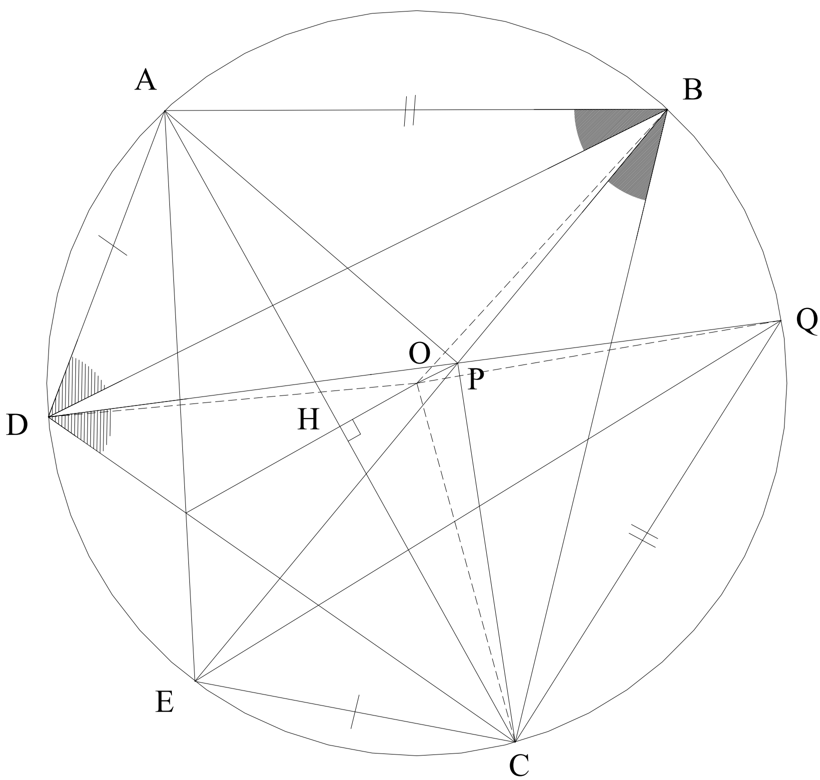
or $\frac{\sin 20^\circ}{\sin 80^\circ} = \frac{\sin 40^\circ - \sin 20^\circ}{\sin 60^\circ}$, or $\frac{\sin 10^\circ}{\sin 30^\circ} = \frac{\sin 20^\circ}{\sin 80^\circ}$, or $\sin 10^\circ \sin 80^\circ =$ $\sin 30^\circ \sin 20^\circ$, or $\frac{1}{2}(\cos 70^\circ - \cos 90^\circ) = \cos 60^\circ \cos 70^\circ$, or $\frac{1}{2} = \cos 60^\circ$, and this is obvious!

<u>Problem 5 of the International Mathematical Olympiad 2004</u>

In a convex quadrilateral ABCD the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside ABCD and satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that ABCD is a cyclic quadrilateral if and only if AP = CP.

<u>Solution</u>

<u>a) Assume point B is already on the circle</u>



Extend DP to intercept the circle at Q and BP to intercept the circle at E.

Let's consider two quadrilaterals ABPD and CQPE.
Since $\angle ABD = \angle EBC \Rightarrow AD = EC$,

$\angle$ABE = $\angle$DBC = $\angle$DQC,
$\angle$DPB = $\angle$EPQ, and
$\angle$ADQ = $\angle$BDC = $\angle$BEC.

Therefore, $\angle$DAB = $\angle$ECQ since the sum of the angles of a quadrilateral is 360°.
Two triangles DAB and ECQ are congruent since $\angle$DAB = $\angle$ECQ, AD = EC and AB = CQ implies DB = EQ

Therefore, triangles DPB and EPQ are also congruent (two angles on each side of DB and EQ are equal which gives us PB = PQ.

Therefore, triangles ABP and CQP are congruent since AB = CQ, PB = PQ and the two angles $\angle$ABP and $\angle$CQP are equal which implies AP = PC.

b) Assume AP = PC and prove ABCD is a cyclic quadrilateral

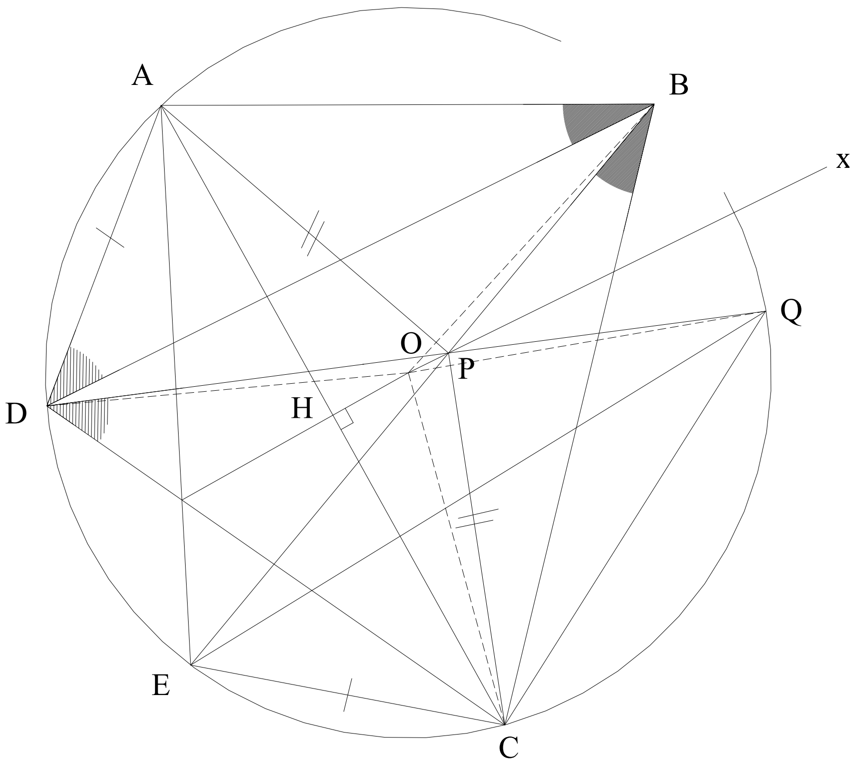
Proof using contradiction

Assume we already have triangle ADC with A, B and C on the circle. From C draw a line that intercepts the circle at E and that CE = AD.

Point P can be chosen anywhere, and draw a line to connect E and P and extend it to cut the circle at B to satisfy the first condition $\angle$ABD = $\angle$PBC since CE = AD.

Now let point O be the center of the circle. We note that OP is the center line of symmetry of AD and CE. Extend DP to intercept the circle at Q.

Assume AP ≠ PC (contradict with fact) then CQ ≠ AB since O*x* (extension of OP) is the line of symmetry. Since Q is on the circle and AB ≠ CQ, $\angle$ADB ≠ $\angle$PDC which implies that B is not symmetrical of point Q with respect to Ox. Therefore, B is not on the circle.

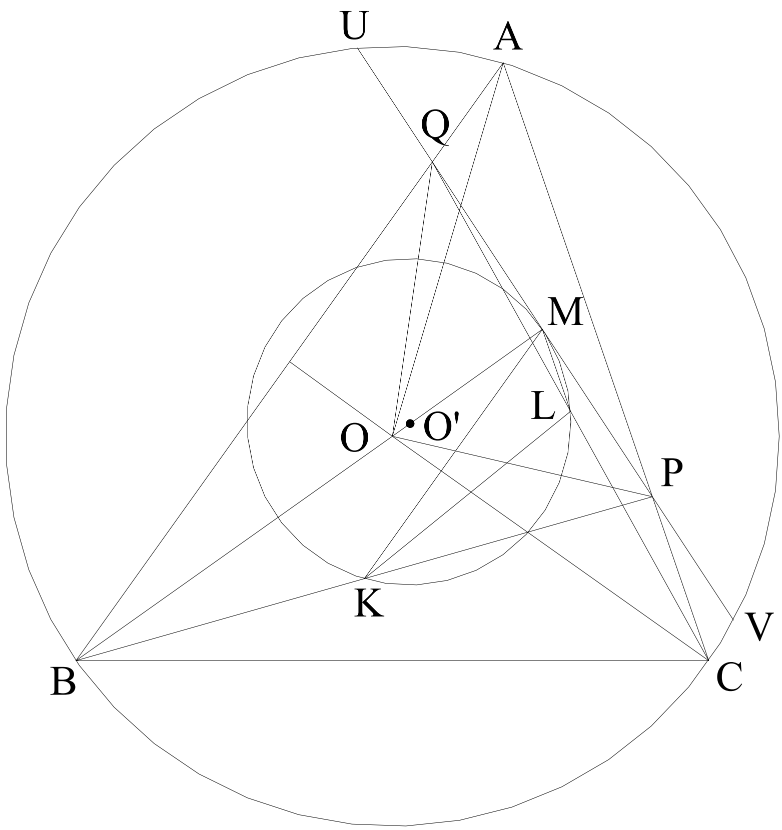


So P has to be on Ox for B to be on the circle or AP = PC, and then ABCD is a cyclic quadrilateral.

Problem 2 of the International Mathematical Olympiad 2009

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that OP = OQ.

Solution



QP tangents with the small circle at M, we have ∠QMK = ∠MLK. M, K and L are midpoints of PQ, BP and QC, respectively; therefore, KM || QB, KM = ½QB (i)
ML || PC, ML = ½PC (ii)
and ∠QMK = ∠MQA, or ∠MLK = ∠MQA.

ML || PC and KM || QB; therefore, $\angle$QAP = $\angle$KML.

The two triangles QAP and KML are similar since their respective angles are equal, and $\frac{ML}{QA} = \frac{KM}{AP}$.

From (i) and (ii), AP×PC = QA×QB (iii)
Extend PQ and QP to meet the larger circle at U and V, respectively.

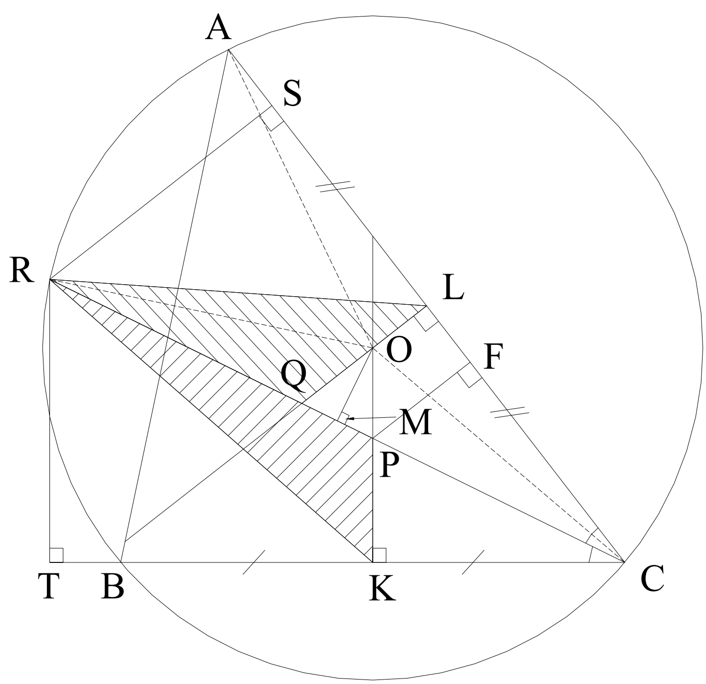
In the larger circle UV intercepts AB at Q, we have
QU×QV = QA×QB, or
QU×(QP + PV) = QA×QB (iv)
UV intercepts AC at P, we have
UP×PV = AP×PC, or
(QU + QP)×PV = AP×PC (v)

From (iv) and (iii), QU×(QP + PV) = AP×PC
Therefore, from (v), QU×(QP + PV) = (QU + QP)×PV.
Or PV = QU and M is also the midpoint of UV and OM $\perp$ UV.
Hence, OP = OQ.

Problem 4 of the International Mathematical Olympiad 2007

In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

Solution:



From R draw the two lines perpendicular to BC and AC and intersect them at T and S, respectively. From P also draw the perpendicular line and intersect AC at F.

To prove the two triangles RPK and RQL to have the same area, it suffices to prove TK×PK = SL×QL (i)

We know KPC and FPC are two congruent triangles, so are the two triangles TRC and SRC. As a result TK = SF and PK = PF, and equation (i) becomes SF×PF = SL×QL which is what is required to be proven or $\frac{PF}{QL} = \frac{SL}{SF}$ (ii)

but $\frac{SL}{SF} = \frac{RQ}{RP}$ since all three lines RS, QL and PF are parallel, and equation (ii) becomes $\frac{PF}{QL} = \frac{RQ}{RP}$, or $\frac{QL}{PF} = \frac{RP}{RQ}$ (iii)

which we still need to prove. Also note that $\frac{QL}{PF} = \frac{QC}{PC}$; equation (iii) can now be written as $\frac{QC}{PC} = \frac{RP}{RQ}$ (iv)

QC = QP + PC, and RP = RQ + QP.

Equation (iv) is equivalent to $1 + \frac{QP}{PC} = 1 + \frac{QP}{RQ}$ we still need to prove, or $\frac{QP}{PC} = \frac{QP}{RQ}$, or PC = RQ (v)

is what needs to be proven.

<u>*Now let's prove it*</u>

Note that O is the center of the circle. From O draw a line to perpendicular to RC and intersect it at M.

$\angle$MOP =$\angle$PCK (because their sides are perpendicular) and $\angle$MOQ =$\angle$PCF for the same reason.

Since $\angle$PCK =$\angle$PCF, $\angle$MOP = $\angle$MOQ are equal, and triangles MOP and MOQ are congruent; therefore, OQ = OP (v)

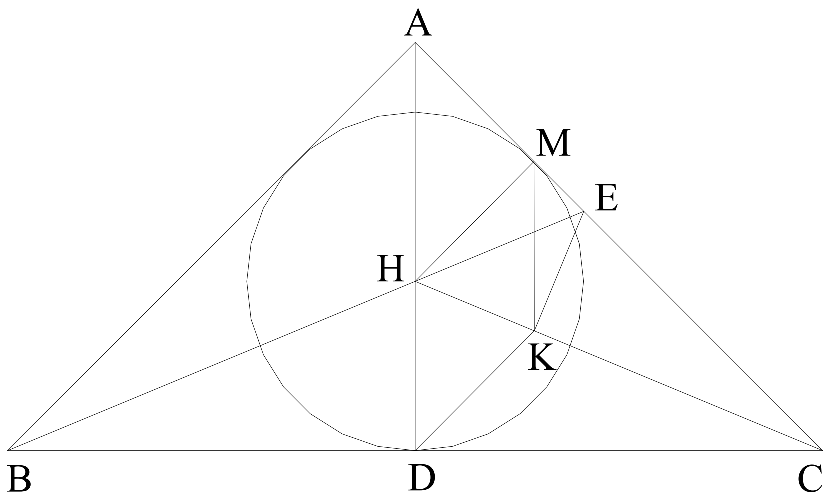
Note that OR = OC, and since O is the center of the circle and $\angle$ROM = $\angle$COM, then $\angle$ROQ =$\angle$COP (vi)

The three conditions (i), (ii) and (iii) make triangles ROQ and COP congruent; therefore, PC = RQ which is the equation (v) required to be proven.

Problem 4 of the International Mathematical Olympiad 2009

Let ABC be a triangle with AB = AC. The angle bisectors of ∠CAB and ∠ABC meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that ∠BEK = 45°. Find all possible values of ∠CAB.

Solution



Extend CK to meet BE at H. Let H be the center of the incircle of triangle ABC. HM = HD. It's easily seen that the two right triangles HDC and HMC are congruent. Therefore, ∠HMK = ∠HDK = 45°. There are two possibilities for ∠MHE. It's either 0° or positive.

a) *Case 1* ∠MHE = 0°

The foot of B on AC is also the midpoint of AC, thus triangle ABC is equilateral and ∠CAB = 60°.

b) *Case 2* ∠MHE > 0°

Problem also requires ∠HEK = 45°; thus the circumcircle of triangle HMK will have point E on it. Draw the circumcircle and we note that HE is the diameter since ∠HME = 90°. Point K is seen as the midpoint of the bottom arc HE. Therefore, $\angle KHE = 45°$, $\angle BHC = 180° - \angle KHE = 135°$, or $\angle B = 22.5°$.

In triangle ABC we have $\angle A + 2\angle B = 90°$.
Therefore, $\angle A = 45°$ or $\angle CAB = 90°$.

Problem 7 of the Canadian Mathematical Olympiad 1971

Let n be a five digit number (whose first digit is non-zero) and let m be the four digit number formed from n by deleting its middle digit. Determine all n such that $\frac{n}{m}$ is an integer.

Solution

Let $n = abcde$ where a, b, c, d and e are positive integers from 0 to 9 and $a \neq 0$.

We then have $m = abde$, and
$n = 10000a + 1000b + 100c + 10d + e$,
$m = 1000a + 100b + 10d + e$.

If $\frac{n}{m} = k$ is an integer, we have
$10000a + 1000b + 100c + 10d + e =$
$1000ak + 100bk + 10dk + ek$ (i)

Now assume that $k > 10$ or $k = 10 + p$ where p is a positive integer; equation (i) becomes

$10000a + 1000b + 100c + 10d + e = 10000a + 1000b + 100d + 10e$
$+ 1000ap + 100bp + 10dp + ep$ (ii)

Now let's find the possible value for p. We have
$p = \frac{100c - 90d - 9e}{1000a + 100b + 10d + e}$, but since $a \neq 0$ and b, c, d and e are all non-negative integers, the denominator is then greater than or equal to 1000 and the numerator is less than 1000, so $p < 1$, and $k > 10$ is not possible.
Similarly, if $k < 10$, $p = \frac{90d + 9e - 100c}{1000a + 100b + 10d + e}$. With the same argument $k < 10$ is not a possibility. Therefore, $k = 10$.

Substituting $k = 10$ into (i), we have $100c = 90d + 9e$ which requires product $9e$ to be a multiple of 10 which is not possible. This equation has the only solution $c = d = e = 0$. So $n = ab000$ where a and b are positive integers where $a = 1 \longrightarrow 9$ and $b = 0 \longrightarrow 9$. The numbers n are

10000, 11000, 12000, 13000, 14000, 15000, 16000, 17000, 18000, 19000,
20000, 21000, 22000, 23000, 24000, 25000, 26000, 27000, 28000, 29000,
30000, 31000, 32000, 33000, 34000, 35000, 36000, 37000, 38000, 39000,
40000, 41000, 42000, 43000, 44000, 45000, 46000, 47000, 48000, 49000,
50000, 51000, 52000, 53000, 54000, 55000, 56000, 57000, 58000, 59000,
60000, 61000, 62000, 63000, 64000, 65000, 66000, 67000, 68000, 69000,
70000, 71000, 72000, 73000, 74000, 75000, 76000, 77000, 78000, 79000,
80000, 81000, 82000, 83000, 84000, 85000, 86000, 87000, 88000, 89000,
90000, 91000, 92000, 93000, 94000, 95000, 96000, 97000, 98000, 99000.

It’s a total of 90 numbers.

Problem 9 of the Irish Mathematical Olympiad 1994

Let w, a, b, c be distinct real numbers with the property that there exist real numbers x, y, z for which the following equations hold:

$x + y + z = 1$ (i)
$xa^2 + yb^2 + zc^2 = w^2$ (ii)
$xa^3 + yb^3 + zc^3 = w^3$ (iii)
$xa^4 + yb^4 + zc^4 = w^4$ (iv)

Express w in terms of a, b, c.

Solution

Multiplying both sides of (i) by a^2, a^3 and a^4, we have
$xa^2 + ya^2 + za^2 = a^2$ (v)
$xa^3 + ya^3 + za^3 = a^3$ (vi)
$xa^4 + ya^4 + za^4 = a^4$ (vii)

Subtracting (ii) from (v), (iii) from (vi) and (iv) from (vii),
$y(a^2 - b^2) + z(a^2 - c^2) = a^2 - w^2$ (viii)
$y(a^3 - b^3) + z(a^3 - c^3) = a^3 - w^3$ (ix)
$y(a^4 - b^4) + z(a^4 - c^4) = a^4 - w^4$ (x)

Now multiplying both sides of (viii) by $a^2 + b^2$,
$y(a^4 - b^4) + z(a^2 - c^2)(a^2 + b^2) = (a^2 - w^2)(a^2 + b^2)$ (xi)

Subtracting (x) from (xi), we have
$z(a^2 - c^2)(b^2 - c^2) = (a^2 - w^2)(b^2 - w^2)$ (xii)

Multiplying both sides of (viii) by $\dfrac{a^2 + ab + b^2}{a + b}$,

$$y(a^3 - b^3) + z(a^2 - c^2)\frac{a^2 + ab + b^2}{a + b} = (a^2 - w^2)\frac{a^2 + ab + b^2}{a + b}$$
(xiii)

Subtracting (xiii) from (ix), we have

$$z(a - c)\left[(a + c)\frac{a^2 + ab + b^2}{a + b} - a^2 - ac - c^2\right] =$$

$$(a - w)\left[(a + w)\frac{a^2 + ab + b^2}{a + b} - a^2 - aw - w^2\right]$$ (xiv)

Now dividing (xiv) by (xii), we have

$$\frac{(a+c)\dfrac{a^2+ab+b^2}{a+b}-a^2-ac-c^2}{(a+c)(b^2-c^2)}=$$
$$\frac{(a+w)\dfrac{a^2+ab+b^2}{a+b}-a^2-aw-w^2}{(a+w)(b^2-w^2)} \qquad \text{(xv)}$$

Expanding (xv) and canceling the same terms to get

$\dfrac{ab^2+b^2c-ac^2-bc^2}{(a+c)(b^2-c^2)}=\dfrac{ab^2+b^2w-aw^2-bw^2}{(a+w)(b^2-w^2)}$, or

$(ab+ac+bc)w^2-c^2(a+b)w-abc^2=0.$

Solving for w,

$$w=\frac{c}{2(ab+ac+bc)}[c(a+b)\pm\sqrt{c^2(a+b)^2+4ab(ab+ac+bc)}].$$

But $c^2(a+b)^2+4ab(ab+ac+bc)=(ac+bc+2ab)^2$; therefore,

$w=\dfrac{c}{2(ab+ac+bc)}[c(a+b)\pm(ac+bc+2ab)]$, or

$w=c$, or $w=-\dfrac{abc}{ab+ac+bc}$ which requires $ab+ac+bc\neq 0$.

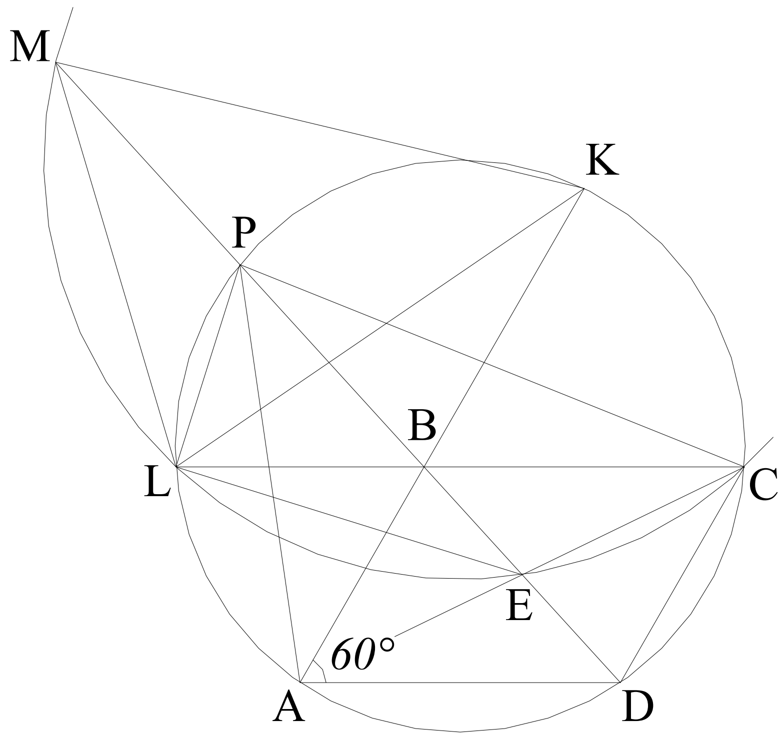
Substitute $w=c$ into (ii) and reverse the above processes; multiply both sides of (ii) by c and subtract it from (iii), etc… We found that when $w=c$, $a=b=c$ which is not allowed by the problem.

Therefore, the only possible solution is $w=-\dfrac{abc}{ab+ac+bc}$.

Problem 9 of the Middle European Mathematical Olympiad 2009

Let ABCD be a parallelogram with $\angle BAD = 60°$ and denote by E the intersection of its diagonals. The circumcircle of triangle ACD meets the line BA at K≠A, the line BD at P≠D and the line BC at L≠C. The line EP intersects the circumcircle of triangle CEL at points E and M. Prove that triangles KLM and CAP are congruent.

Solution



Since K, L, A, C and M, L, E, C are concyclic, we have $\frac{KL}{AC} = \frac{BL}{AB}$, $\frac{AP}{CD} = \frac{AE}{DE}$ and $\frac{ML}{EC} = \frac{BL}{BE}$ or $\frac{ML}{AP} = \frac{EC \times BL \times DE}{BE \times AE \times CD}$.

But also because E is the intersection of the diagonals of the parallelogram ABCD, BE = DE, AE = EC.

It follows that $\dfrac{ML}{AP} = \dfrac{BL}{CD} = \dfrac{BL}{AB} = \dfrac{KL}{AC}$ (i)

We also have $\angle LPD = \angle LCD = \angle BAD = 60°$.

Now chase the angle $\angle KLM = \angle KLP + \angle MLP = \angle KLP + 60° - \angle LMP = \angle KLP + 60° - \angle LME = \angle KLP + 60° - \angle LCE = \angle KLP + \angle ACD = \angle KLP + \angle KLC$ (since AB || CD and KC = AD) $= \angle CLP = \angle CAP$ (subtending arc CP).

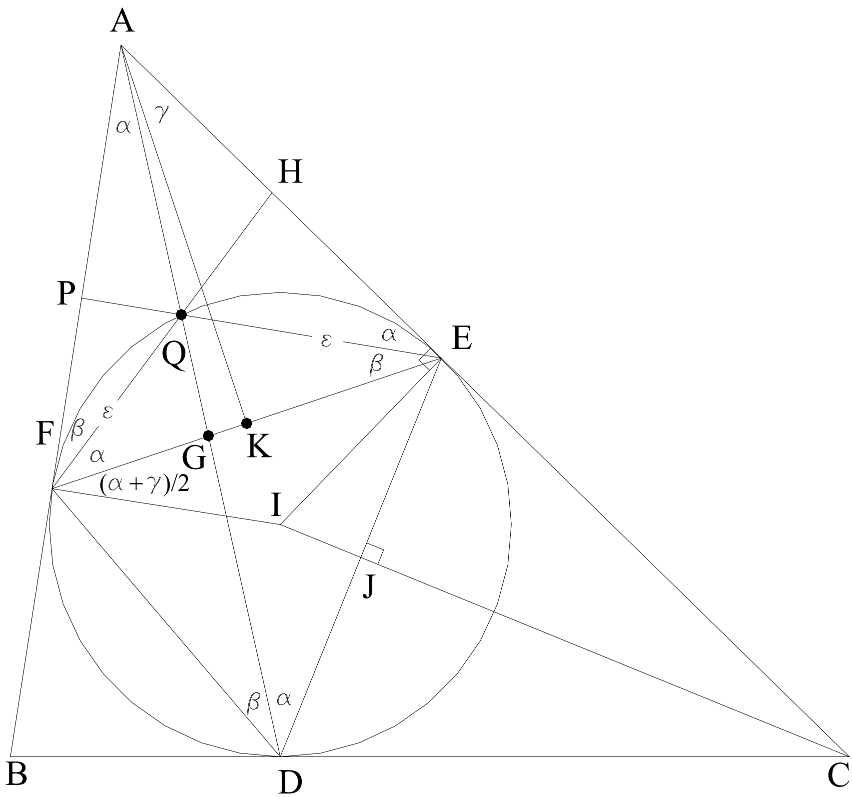
Combining with (i), the two triangles KLM and CAP are similar. Furthermore, since DL = AC (diagonals of isosceles trapezoid ADCL) = DK (diagonals of isosceles trapezoid ADCK).

But since $\angle BAD = 60°$subtends arc DK, it follows that KDL is an equilateral triangle, and KL = DK = AC. This makes the two already similar triangles KLM and CAP congruent.

<u>Problem 2 of the Ibero-American Mathematical Olympiad 1998</u>

The circumference inscribed on the triangle ABC is tangent to the sides BC, CA and AB on the points D, E and F, respectively. AD intersect the circumference on the point Q. Show that the line EQ intersect the segment AF on its midpoint if and only if AC = BC.

<u>Solution</u>



<u>a) *The case of AC = BC*</u>

Extend FQ to meet AC at H and EQ to meet AB at P. Let AD intercept EF at G. Applying Ceva's theorem for the three lines AG, FH and EP, we have $\frac{AP}{PF}\times\frac{FG}{GE}\times\frac{EH}{AH} = 1$.

So, to prove point P, the intersection of EQ and AF, to be the midpoint of AF, it suffices to prove that $\frac{FG}{GE}\times\frac{EH}{AH} = 1$ (i)

Let $\angle FAG = \alpha$, $\angle GAE = \gamma$, $\angle AFQ = \beta$, $\angle AFE = \angle AEF = \varepsilon$ and $\angle EFC = \frac{\alpha+\gamma}{2}$.

We then also have $\angle QDE = \alpha$ (since AB || ED) $= \angle QFE = \angle QEA$ (they subtend the same arc QE) and $\angle AFQ = \angle QEF = \angle QDF = \beta$ (subtending the same arc QF).

Applying the law of the sines to get $\frac{FG}{\sin\alpha} = \frac{AG}{\sin\varepsilon} = \frac{GE}{\sin\gamma}$, or $\frac{FG}{GE} = \frac{\sin\alpha}{\sin\gamma}$, and $EH = FH\times\frac{\sin\alpha}{\sin\varepsilon}$, and $AH = FH\times\frac{\sin\beta}{\sin(\alpha+\gamma)}$, but $\alpha + \gamma = 180° - 2\varepsilon$, and $\sin(\alpha+\gamma) = \sin(180° - 2\varepsilon) = \sin2\varepsilon = 2\sin\varepsilon\cos\varepsilon = 2\sin\varepsilon\cos(90° - \tfrac{1}{2}\angle A) = 2\sin\varepsilon\sin\frac{\angle A}{2} = 2\sin\varepsilon\sin\frac{\alpha+\gamma}{2}$, or $AH = FH\times\frac{\sin\beta}{2\sin\varepsilon\sin\frac{\alpha+\gamma}{2}}$. The equation (i) required to be proven becomes

$$\frac{\sin^2\alpha\sin(\alpha+\gamma)}{\sin\gamma\sin\beta} = 1 \qquad \text{(ii)}$$

But in triangle AFD, $\frac{AF}{\sin\beta} = \frac{FD}{\sin\alpha}$, or $\frac{\sin\alpha}{\sin\beta} = \frac{FD}{AF}$, and in triangle AED, $\frac{\sin\alpha}{\sin\gamma} = \frac{AE}{DE}$. Also in triangle AFK, $\sin\frac{\alpha+\gamma}{2} = \frac{FK}{AF}$ with AE = AF. Equation (ii) then becomes $FD\times EF = DE\times AF$, but FD = EF, or it suffices to prove that $EF^2 = DE\times AF$.

<u>Let's prove it</u>

Again using the law of the sines, in triangle AFK to get $\frac{FK}{\sin\frac{\angle A}{2}} = \frac{EF}{2\sin\frac{\angle A}{2}} = AF$, and in triangle FEJ, $\frac{EJ}{\sin\angle EFJ} = \frac{EJ}{\sin\frac{\angle A}{2}} = EF$, or $EF^2 = \frac{EJ}{\sin\frac{\angle A}{2}}\times AF\times2\sin\frac{\angle A}{2} = 2EJ\times AF = DE\times AF$.

b) *The case of AP = FP*

Since AC is not yet equal to BC, we let $\angle QDE = \psi = \angle QFE = \angle QEA$ (they subtend the same arc QE). We have $\frac{FG}{GE} = \frac{\sin\alpha}{\sin\gamma}$, $EH = FH\times\frac{\sin\psi}{\sin\varepsilon}$, $AH = FH\times\frac{\sin\beta}{2\sin\varepsilon\sin\frac{\alpha+\gamma}{2}}$, and $\frac{\sin\alpha\sin\psi\sin(\alpha+\gamma)}{\sin\gamma\sin\beta} = 1$ which leads to 2EK×FD = DE×AF, or FD×EF = DE×AF, or

$FD = \frac{EJ}{\sin\frac{\angle A}{2}}$ which only occurs when AC = BC.

<u>Problem 2 of the Ibero-American Mathematical Olympiad 2001</u>

The inscribed circumference of the triangle ABC has center at O and it is tangent to the sides BC, AC and AB at the points X, Y and Z, respectively. The lines BO and CO intersect the line Y Z at the points P and Q, respectively. Show that if the segments XP and XQ have the same length, then the triangle ABC is isosceles.

<u>Solution</u>

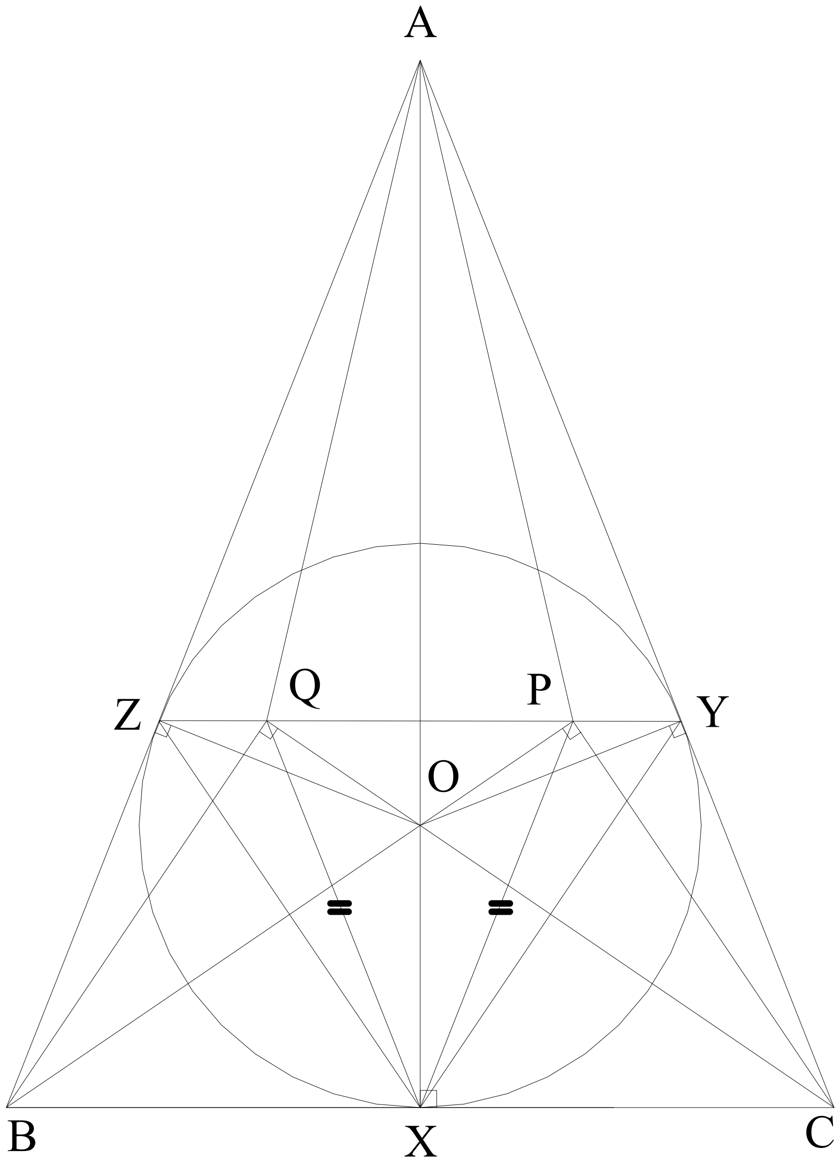
We have $\angle BOQ = \angle BCO + \angle OBC = ½(180° - \angle A) = \angle YZA = \angle ZOA$, and $\angle OZQ = \angle ZAO$ (2 sides perpendicular to each other), or $\angle BOQ + \angle BZQ = \angle YZA + 90° + \angle ZAO = 180°$

Hence, BZQO is cyclic and $\angle BQO = \angle BZO = 90°$. Similarly, $\angle CPO = 90°$ and since $\angle BXO = \angle CXO = 90°$, BZQOX and CYPOX are both cyclic. We also note that BQPC is also cyclic.

Therefore, $\angle QXO = \angle QBO = \angle PCO = \angle PXO$ and triangles QXO and PXO are congruent which leads to OQ = OP and $\angle QOX = \angle POX$, or $\angle QBX = 180° - \angle QOX = 180° - \angle POX = \angle PCX$ (i)

Since $\angle OQP = \angle OPQ$ (OQ = OP) and $\angle OZY = \angle OYZ$ (OZ = OY = radius of the circle), triangles OZQ = triangle OYP or ZQ = YP. Furthermore, $\angle ZQX = 180° - \angle XQP = 180° - \angle XPQ = \angle YPX$ and $\Delta ZQX = \Delta YPX$ which leads to $\angle ZXQ = \angle YXP$.

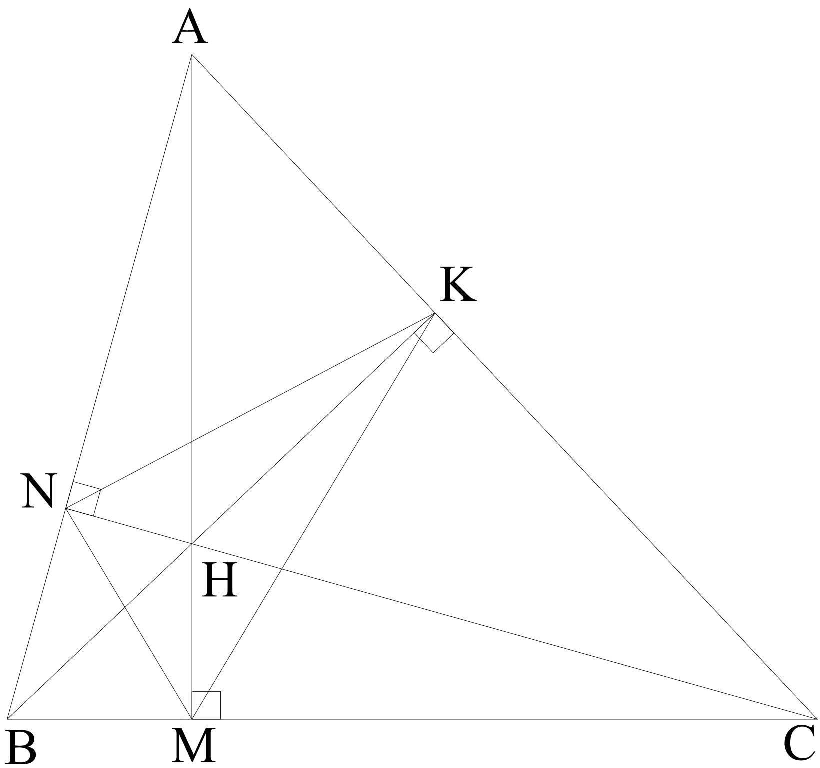
Adding $\angle ZBQ$ to both sides of (i) $\angle QBX = \angle PCX$ to get $\angle ZBX = \angle ZBQ + \angle QBX = \angle ZXQ + \angle QBX = \angle YXP + \angle PCX = \angle YCP + \angle PCX = \angle YCX$, or AB = AC and the triangle ABC is isosceles.


A
Q
P
Z
Y
O
B
X
C

<u>Problem 1 of International Mathematical Talent Search Round 8</u>

Prove that there is no triangle whose altitudes are of lengths 4, 7 and 10 units.

<u>Solution</u>



Let H be the orthocenter of triangle ABC, M, K, N be the feet of H onto BC, AC and AB, respectively. Assuming such a triangle in the problem exists, we have BK = 4, AM = 7 and CN = 10.

The area of the triangle is ½AM×BC = ½BK×AC = ½CN×AB, or 7BC = 4AC = 10AB.

Applying the law of sine, we get
$\frac{BC}{AC} = \frac{\sin A}{\sin B} = \frac{4}{7}$ and $\frac{AB}{AC} = \frac{\sin C}{\sin B} = \frac{4}{10}$, or $\sin A = \frac{4}{7}\sin B = \frac{10}{7}\sin C$.

However, $\angle A = 180° - \angle(B + C)$, $\sin A = \sin[180° - \angle(B + C)] = \sin(B + C)$ and $\sin(B + C) = \sin B\cos C + \cos B\sin C$; the above equation becomes

$\sin B\cos C + \cos B\sin C = \frac{4}{7}\sin B$, or $\frac{1}{4}\cos C + \frac{1}{4}\times\frac{\cos B}{\sin B}\times\sin C = \frac{1}{7}$, or

$\frac{1}{4}\cos C + \frac{1}{10}\cos B = \frac{1}{7}$.

But $\cos C = \sin\angle CAM = \frac{HK}{AH}$, $\cos B = \sin\angle BAM = \frac{HN}{AH}$, and the above equation is equivalent to

$\frac{1}{4}\times\frac{HK}{AH} + \frac{1}{10}\times\frac{HN}{AH} = \frac{1}{7}$, or $\frac{1}{4}\times HK + \frac{1}{10}\times HN = \frac{1}{7}\times AH$.

Now, applying the Ptolemy's theorem to the cyclic quadrilateral ANHK, we get $AN\times HK + AK\times HN = NK\times AH$.

The two previous equations yield
$AN = \frac{1}{4}$, $AK = \frac{1}{10}$ and $NK = \frac{1}{7}$.

Applying the law of cosine, we obtain

$NK^2 = AN^2 + AK^2 - 2AN\times AK\cos A$. Substituting the values for AN, AK and NK into this latest equation, we get

$\frac{1}{49} = \frac{1}{16} + \frac{1}{100} - 2\times\frac{1}{4}\times\frac{1}{10}\cos A$. From here, we solve for cosA which is $\cos A = \frac{1021}{980} = 1.04$, and there's no such angle A to satisfy $\cos A = 1.04$.

Therefore, there is no triangle whose altitudes are of lengths 4, 7 and 10 units.

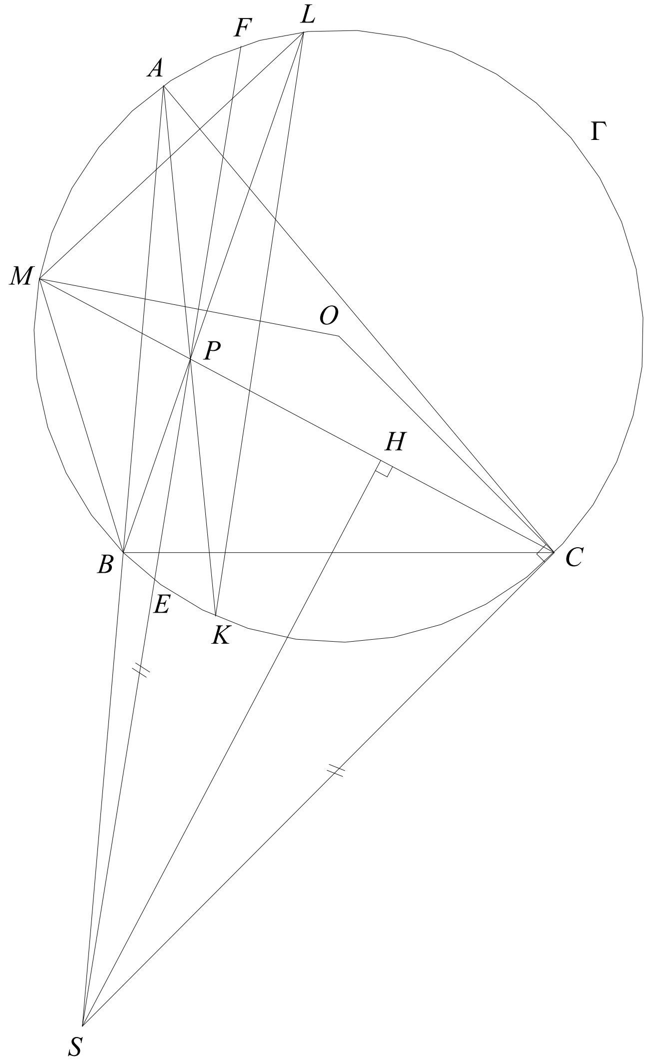
<u>Further observation</u>

This method can be used to verify the altitudes of other triangles.

Problem 4 of the International Mathematical Olympiad 2010

Let P be a point inside the triangle ABC. The lines AP, BP and CP intersect the circumcircle Γ of triangle ABC again at the points K, L and M, respectively. The tangent to Γ at C intersects the line AB at S. Suppose that SC = SP. Prove that MK = ML.

Solution



Let E be the intersection of Γ and SP. Extend SP to meet Γ again at

F. Since SC = SP, $SP^2 = SC^2 = SB \times SA$, or $\frac{SB}{SP} = \frac{SP}{SA}$, and ΔSBP is similar to ΔSPA which causes $\angle SPB = \angle SAP$, or arc BK = arc BE + arc FL, or arc EK = arc FL.

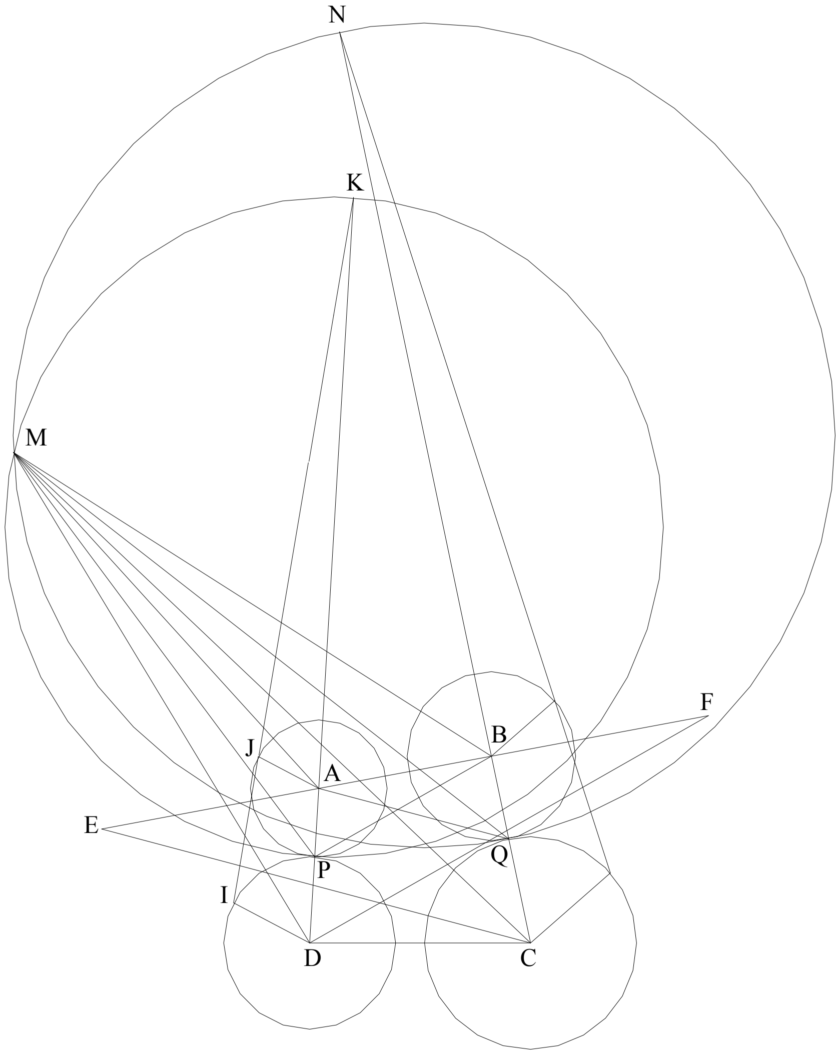
From S draw the perpendicular line to meet PC at H. Since SP = SC, SH is also the bisector of $\angle PSC$. Since $OC \perp SC$ and $SH \perp PC$, $\angle HSC = \angle OCM = \angle OMC$.
But $\angle HSC = \angle HSP$. Therefore, $\angle HSP = \angle OMC$, or $SF \perp OM$, or MF = ME.

Combining with FL = EK, we have MK = ML.

Problem 2 of the Korean Mathematical Olympiad 2007

ABCD is a convex quadrilateral, and AB ≠ CD. Show that there exists a point M such that $\dfrac{AB}{CD} = \dfrac{MA}{MD} = \dfrac{MB}{MC}$.

Solution



Extend BA a segment of AE and AB a segment of BF such that

AE = BF = CD. Link E with C and D with F. From A draw a line to parallel EC and meet BC at Q. From B draw a line to parallel DF and meet AD at P.

We have

$$\frac{AB}{CD}=\frac{AB}{EA}=\frac{BQ}{QC}=\frac{AB}{BF}=\frac{AP}{PD} \qquad \text{(i)}$$

Construct the harmonic subdivision for segment AD by drawing the two circles with incenters D and A and with their radii being DP and AP, respectively.

Draw an arbitrary line from circumcenter D to cut its circle at I, and from A draw AJ (J on the circle with center A) so that AJ || DI. Link and extend IJ to meet the extension of DA at K. The four points D, P, A and K are said to be in harmonic order, and we have

$\frac{AP}{PD}=\frac{AK}{DK}$.

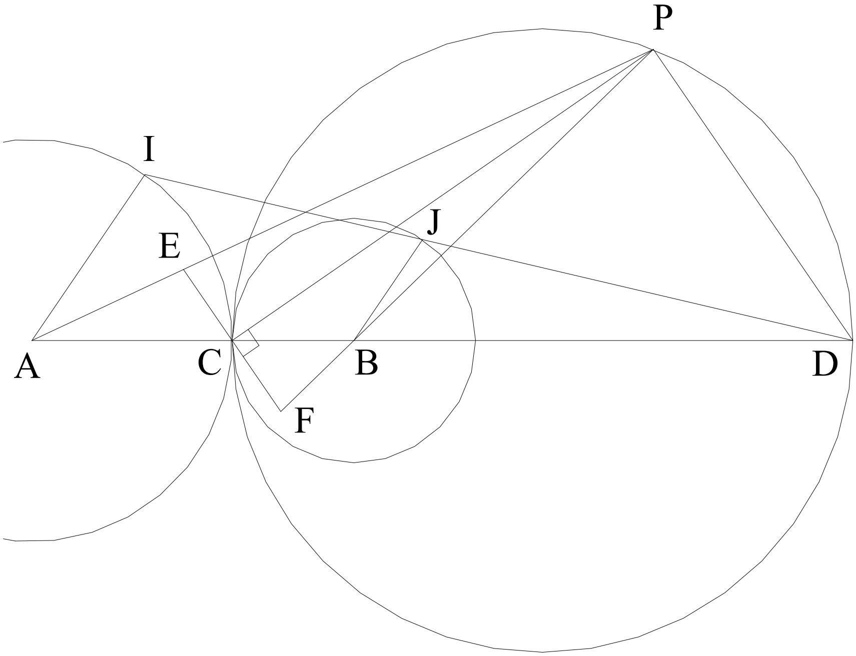
Similarly, construct the harmonic subdivision for segment BC, we get the point N such that $\frac{BQ}{QC}=\frac{BN}{CN}$.

Draw the circles of *Apollonius* with diameters KP and NQ. These two new circles will intercept at two points (in our graph) M whereas, by Ray's theorem (see proof on the next page), we have $\angle AMP = \angle DMP$, or $\frac{MA}{MD}=\frac{AP}{PD}=\frac{AB}{CD}$ (according to (i)), and

$\angle BMQ = \angle CMQ$, or $\frac{MB}{MC}=\frac{BQ}{QC}=\frac{AB}{CD}$,

and at long last $\frac{AB}{CD}=\frac{MA}{MD}=\frac{MB}{MC}$.

Proof



First construct the harmonic subdivision for segment AB using the method on the previous page. Connect and extend IJ to meet the extension of AB at D. Next, draw a so-called *Apollonian circle* with diameter DC. The locus of the points on the plane, for which the ratio of the distances to two fixed points A and B is a constant, in this case equals to $\frac{AC}{CB}$, is the *Apollonian circle*.

Indeed, pick an arbitrary point P on the Apollonius circle. Draw a line through point C that is also parallel to PD; this line meets AP at E. Link and extend PB to meet the extension of EC at F. By Ray's theorem, EC/PD = AC/AD and CF/PD = BC/BD. We have r/R (the ratio of the two radii) = BC/AC = IB/IA = BD/AD (since IA || IB), or AC/AD = BC/BD, and thus EC = CF. Also because EF || PD, and $\angle$CPD = 90°, $\angle$ECP = 90°, and ΔPCE is congruent to ΔPCF, implying that PC is bisecting $\angle$EPF.

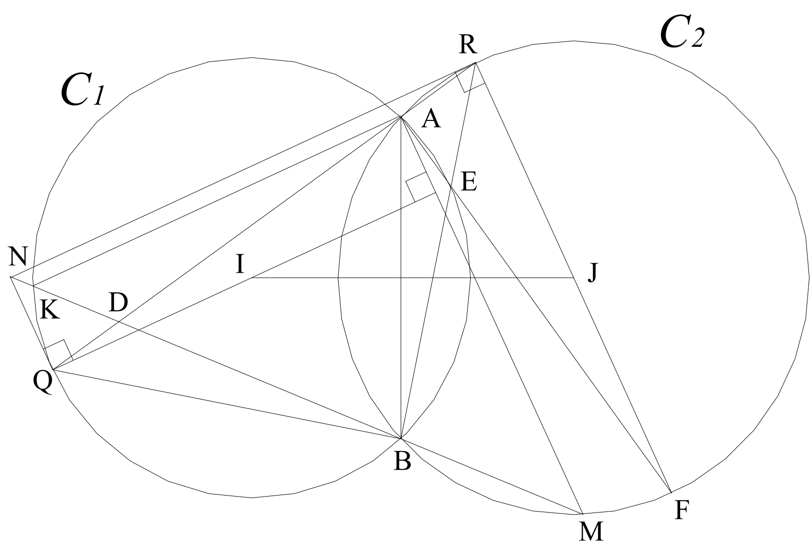
Therefore, $\frac{PA}{PB} = \frac{AC}{CB}$.

<u>Problem 3 of Hong Kong Mathematical Olympiad 2002</u>

Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

a) Prove that the tangent to the second circle at R is parallel to AK.
b) Prove that these two tangents are concurrent with KM.

<u>Solution</u>



a) Let the first circle on the left and the second circle on the right be C_1 and C_2, respectively. Also let their centers be I and J, in that order. Now let QI meet C_1 at E and RJ meet C_2 at F.

Since I and J are the centers, we have $\angle QAE = 90°$ and the three points A, E and F are collinear.
Therefore, $\angle BQE = \angle BAE = \angle BRF$ (i)

Since NQ || AM $\Rightarrow$ $\angle QNB = \angle AMB$, and since $\angle AMB$ and $\angle ARB$ subtend the same small arc AB of C_2,
$\angle NMA = \angle ARB$ (ii)

Therefore, $\angle$QNB = $\angle$QRB (ii),
or BQNR is cyclic, and since BQKA is also cyclic, we have KD×BD = QD×AD, and ND×BD = QD×RD.

From those two equations, we have $\frac{ND}{RD} = \frac{KD}{AD}$, or AK || NR.

b) As a result of BQNR being cyclic, we have
$\angle$QBN = $\angle$QRN (iii)
Also in triangle BQN, $\angle$QNB + $\angle$QBN + $\angle$BQE = 90° (iv)
Substituting $\angle$QNB from (ii), $\angle$QNB from (iii) and $\angle$BQE from (i) into (iv), we have
$\angle$QRB + $\angle$QRN + $\angle$BRF = 90°, or RN $\perp$ RJ, or RN is tangent to C_2.
Therefore, the three segments QN, KM and RN are concurrent.

<u>Further observation</u>

Let RB intercept C_2 at P. Since BQNR is cyclic, $\angle$QNB = $\angle$QRB. $\angle$QNB subtends arc QB – arc QK = $\angle$QRB subtends arc QB – arc AP. From there we conclude that QK = AP, or QP || AK.

Problem 1 of Hong Kong Mathematical Olympiad 2002

Find the value of $\sin^2 1° + \sin^2 2° + \ldots + \sin^2 89°$.

Solution

Let $S = \sin^2 1° + \sin^2 2° + \ldots + \sin^2 89°$.

We can group the sum of the squares as follows:
$S = (\sin^2 1° + \sin^2 89°) + (\sin^2 2° + \sin^2 88°) + (\sin^2 3° + \sin^2 87°) + \ldots + (\sin^2 44° + \sin^2 46°) + \sin^2 45°$.

Every group inside brackets (total of 44 groups) has the form of $\sin^2 a + \sin^2 b$ where $a + b = 90°$.

We have $N = \sin^2 a + \sin^2 b = (\sin a + \sin b)^2 - 2\sin a \sin b = 4\times\sin^2\frac{a+b}{2}\cos^2\frac{a-b}{2} + \cos(a+b) - \cos(a-b)$.

With $a + b = 90°$, $4\times\sin^2\frac{a+b}{2} = 2$, and $\cos(a + b) = 0$.

We now have $N = 2\times\cos^2\frac{a-b}{2} - \cos(a - b)$.

But $\cos(a - b) = 2\times\cos^2\frac{a-b}{2} - 1$; hence $N = 1$.

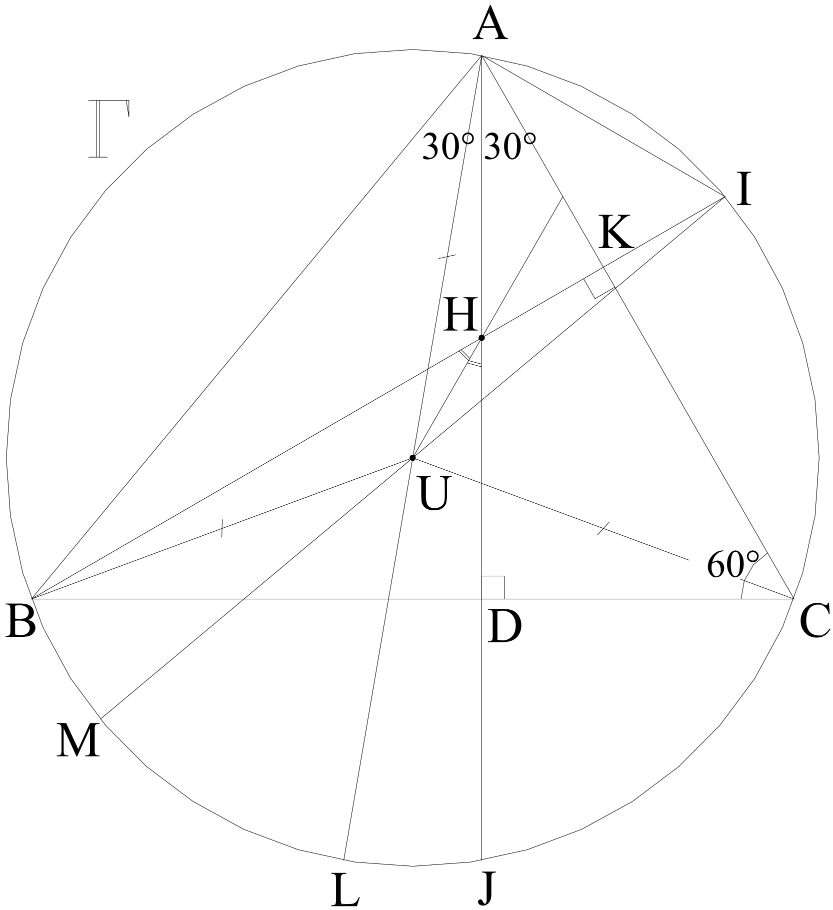
There are 44 groups plus $\sin^2 45°$; and $S = 44 + \frac{1}{2} = 44.5$.

Problem 4 of Austria Mathematical Olympiad 2000

In the acute, non-isosceles triangle ABC with angle C = 60° let U be the circumcenter, H be the orthocenter and D the intersection of the lines AH and BC (that is, the orthogonal projection of A onto BC).

Show that the Euler line HU is the bisector of $\angle$BHD.

Solution



Let the circumcircle of triangle ABC be called Γ. Let K be the foot of B onto AC. Extend BK to meet Γ at I. Extend IU to meet Γ at

M, AU to meet Γ at L, AD to meet Γ at J.

Since U is the center of Γ, triangles AUC and BUC are isosceles with ∠UAC = ∠UCA, ∠UBC = ∠UCB, and with ∠ACB = 60°, we have ∠BAL = 30°.

Also because I is the image of the orthocenter H across AC, KI = KH and with ∠HAK = 30°, triangle AHI is equilateral and ∠AHI = 60°.

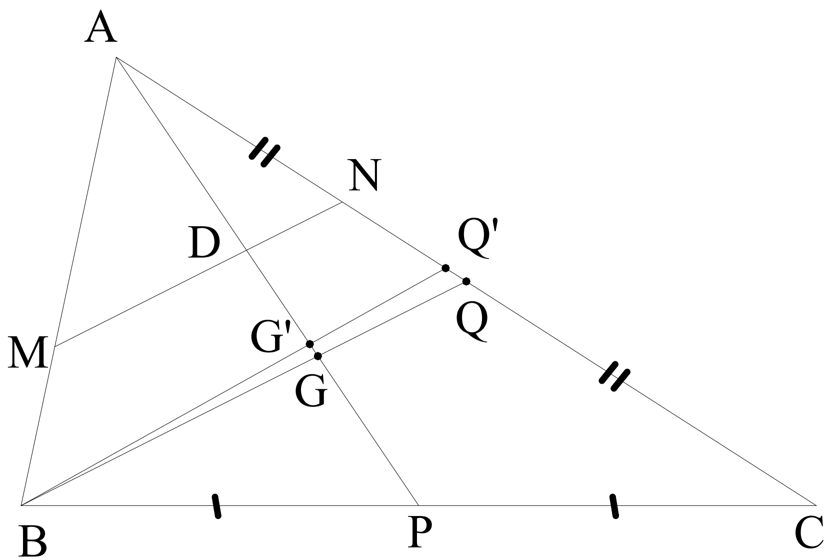
To prove that the Euler line HU is the bisector of the ∠BHD is equivalent to proving HU being the exterior bisector of ∠AHI.

Since ∠AHI = 60°, the arc (BJ + AI) = arc (BL + JC), or arc BJ + arc AI = arc BJ – arc LJ + arc JC, or arc AI + arc LJ = arc JC, but arc JC = arc BL, and we then have arc AI + arc LJ = arc BM + arc ML, but arc ML = arc AI. We now get arc LJ = arc BM, or ∠LAJ = ∠BIM which, in turn, causes triangles AUH and IUH to be congruent and ∠AUH = ∠IUH, or HU is the exterior bisector of ∠AHI, and the proof is complete.

<u>Problem 2 of the Irish Mathematical Olympiad 2010</u>

Let ABC be a triangle and let P denote the midpoint of the side BC. Suppose that there exist two points M and N interior to the sides AB and AC, respectively such that |AD| = |DM| = 2|DN|, where D is the intersection point of the lines MN and AP. Show that |AC| = |BC|.

<u>Solution</u>



From B draw a line to parallel MN and to intercept AC at Q' and AP at G'. Also draw the median BQ to intercept AP at G.

Since BQ' || MN, $\frac{BG'}{G'Q'} = \frac{DM}{DN} = \frac{2}{1}$, and G the centroid of triangle ABC, $\frac{BG}{GQ} = \frac{2}{1}$, or $\frac{BG'}{G'Q'} = \frac{BG}{GQ}$, or GG' || QQ'.

Therefore, G' ≡ G and Q ≡ Q'.

Now since AD = DM, AG = BG, and AP = $\frac{3}{2}$ AG = $\frac{3}{2}$ BG = BQ, and ΔPAB = ΔQBA implying AQ = BP, or |AC| = |BC|.

Problem 2 of Australia Mathematical Olympiad 2008

Let ABC be an acute triangle, and let D be the point on AB (extended if necessary) such that AB and CD are perpendicular. Further, let *tA* and *tB* be the tangents to the circumcircle of ABC through A and B, respectively, and let E and F be the points on *tA* and *tB*, respectively, such that CE is perpendicular to *tA* and CF is perpendicular to *tB*.

Prove that $\frac{\text{CD}}{\text{CE}} = \frac{\text{CF}}{\text{CD}}$.

Solution

We see that ADCE and BDCF are both cyclic which cause
$\angle\text{DFC} = \angle\text{ABC}$ (i)
and $\angle\text{DEC} = \angle\text{BAC}$ (ii)

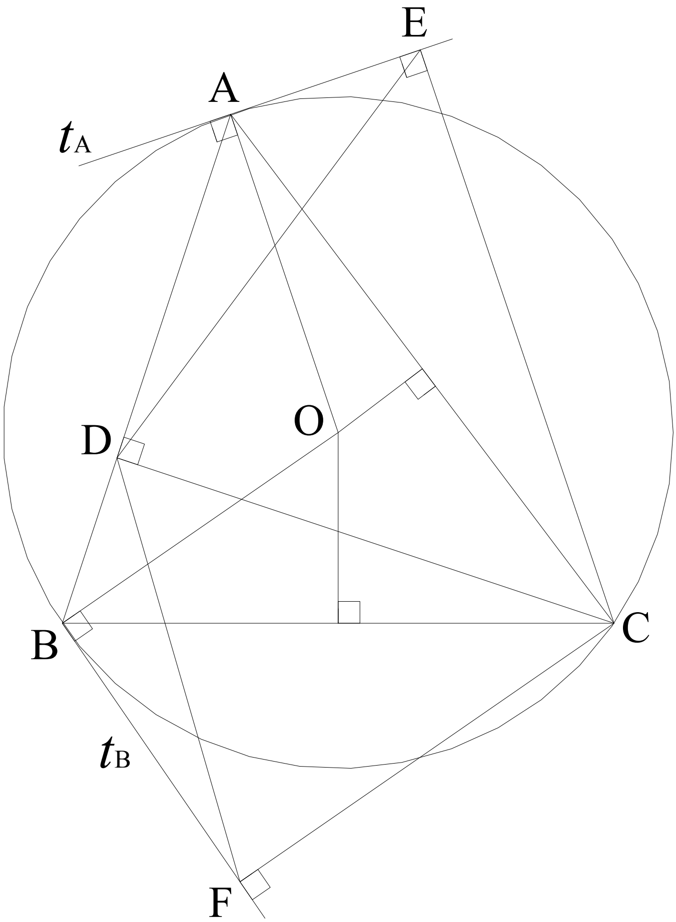
$\angle\text{BAE} + \angle\text{DCE} = \angle\text{ABF} + \angle\text{DCF} = 180°$, but $\angle\text{BAE} = \angle\text{ABF}$ (both subtend larger arc AB); therefore, $\angle\text{DCE} = \angle\text{DCF}$.

However, $\angle\text{ACB}$ subtends smaller arc AB; hence, $\angle\text{ACB} = \angle\text{DCE} = \angle\text{DCF}$ since $\angle\text{ACB}$ also combines with $\angle\text{BAE}$ to be 180°.

Now with the addition of (i), the triangles ABC and DFC are similar because their respective angles are equal.

Similarly, combined with (ii), the two triangles ABC and EDC are also similar for the same reason.

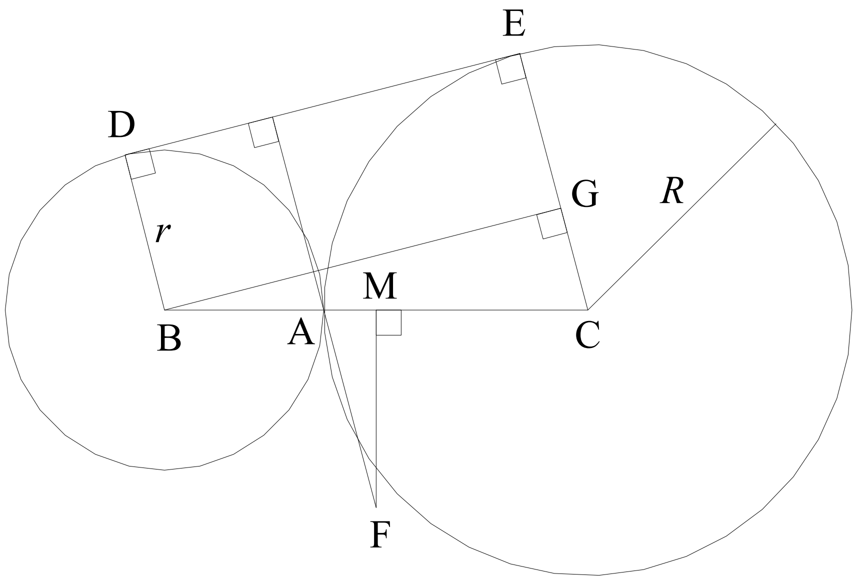
Therefore, triangles DFC and EDC are similar, and $\frac{\text{CD}}{\text{CE}} = \frac{\text{CF}}{\text{CD}}$.


E
A
t_A
O
D
B
C
t_B
F

<u>Problem 6 of the British Mathematical Olympiad 2009</u>

Two circles, of different radius, with centers at B and C, touch externally at A. A common tangent, not through A, touches the first circle at D and the second at E. The line through A which is perpendicular to DE and the perpendicular bisector of BC meet at F. Prove that BC = 2AF.

<u>Solution</u>



Let R and r be the radii of the large and small circles, respectively. Also let M be the midpoint of BC. From B draw the altitude to EC to meet it at G.
Consider two right triangles GBC and MFA, $\angle$GBC = $\angle$MFA (their sides are perpendicular to each other).

Therefore, ΔGBC is similar to ΔMFA, and we have

$$\frac{\mathrm{BC}}{\mathrm{AF}} = \frac{\mathrm{BG}}{\mathrm{MF}} = \frac{\mathrm{GC}}{\mathrm{AM}} = \frac{R-r}{\mathrm{AM}} \qquad \text{(i)}$$

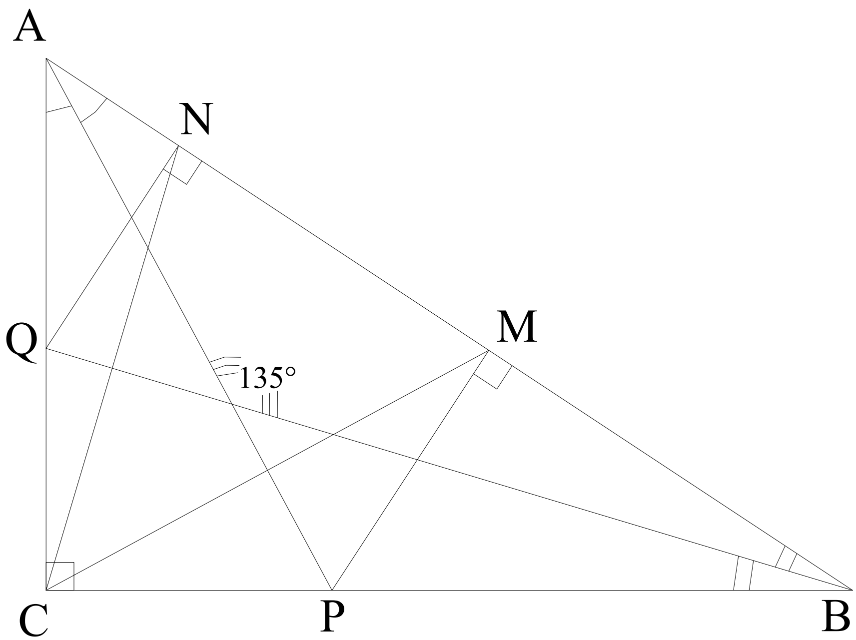
But AM + r = R – AM, or $\mathrm{AM} = \dfrac{R-r}{2}$.

Equation (i) becomes $\dfrac{\mathrm{BC}}{\mathrm{AF}} = 2\times\dfrac{R-r}{R-r} = 2$, or BC = 2AF.

Problem 4 of the British Mathematical Olympiad 1995

ABC is a triangle, right-angled at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

Solution



Extend NC to meet MP at I (not shown on graph). Since QN || PM (because both $\perp$ AB), $\angle$CNQ = $\angle$CIM.

Besides, $\angle$MCN = $\angle$CIM + $\angle$CMP, we then have

$\angle$MCN = $\angle$CNQ + $\angle$CMP (i)

Observe that ΔAPM ≡ (congruent to) ΔAPC, and
ΔBQC ≡ (congruent to) ΔBQN.

We then have AP $\perp$ CM and BQ $\perp$ CN.

$AP \perp CM$ results in $\angle CMP = \angle MCP$, and $BQ \perp CN$ results in $\angle CNQ = \angle NCQ$.
Equation (i) becomes $\angle MCN = \angle NCQ + \angle MCP$.
However, $\angle MCN + \angle NCQ + \angle MCP = 90°$.
Or, $\angle MCN = 45°$.

Further observation

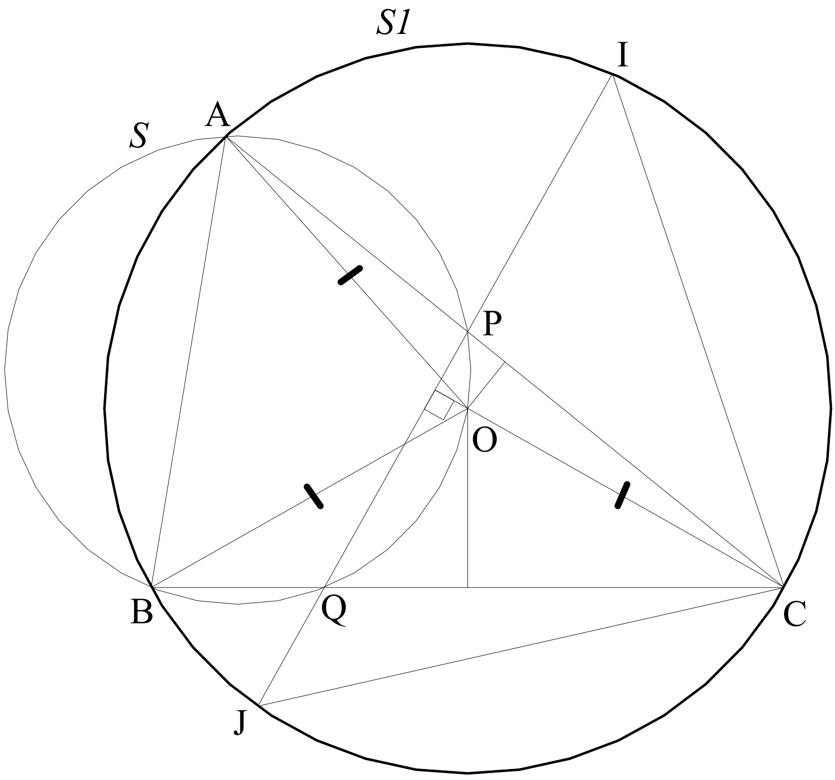
We can prove CP = MP which results in $\angle CMP = \angle MCP$ by using a different method using the angle bisector AP.

Since AP is the angle bisector of $\angle BAC$, we have
$\frac{CP}{PB} = \frac{AC}{AB}$.
Furthermore, the two triangles ABC and PBM are similar making
$\frac{MP}{PB} = \frac{AC}{AB}$.
Those two previous equations give us CP = MP.
Similarly, CQ = NQ resulting in $\angle CNQ = \angle NCQ$.

Problem 3 of the British Mathematical Olympiad 1996

Let ABC be an acute triangle, and let O be its circumcenter. The circle through A, O and B is called S. The lines CA and CB meet the circle S again at P and Q, respectively. Prove that the lines CO and PQ are perpendicular.

Solution



The intersecting secant theorem gives us CP×CA = CQ×CB, or $\frac{CP}{CQ} = \frac{CB}{CA}$ which implies that triangle CPQ is similar to triangle CBA, and $\angle$ABC = $\angle$QPC.

Extend QP to meet the circumcircle of triangle ABC at I on top and J on bottom. Since $\angle$ABC subtends smaller arc AC, and

$\angle$QPC with P inside the circumcircle of triangle ABC subtending arc AI plus arc JC, we have arc AC = arc AI + arc JC, or arc AI + arc IC = arc AI + arc JC, or arc IC = arc JC, or IC = JC.

Furthermore, O is the circumcenter; therefore, CO $\perp$ JI, or CO $\perp$ PQ.

Problem 5 of the British Mathematical Olympiad 1996

Let a, b and c be positive real numbers,

a) Prove that $4(a^3 + b^3) \geq (a + b)^3$
b) Prove that $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$

Solution

a) To prove that $4(a^3 + b^3) \geq (a + b)^3$, it suffices to prove
$4(a^3 + b^3) \geq a^3 + 3a^2b + 3ab^2 + b^3$, or
$3(a^3 + b^3) \geq 3a^2b + 3ab^2$, or
$a^3 + b^3 \geq a^2b + ab^2$, or
$a^3 - a^2b + b^3 - ab^2 \geq 0$, or
$a(a^2 - b^2) + b(b^2 - a^2) \geq 0$, or

$(a - b)(a^2 - b^2) \geq 0$, or
$(a - b)(a - b)(a + b) \geq 0$, or
$(a - b)^2(a + b) \geq 0$.

Since $(a - b)^2 \geq 0$ and $a + b > 0$,
$(a - b)^2(a + b) \geq 0$ is always valid.

b) To prove that $9(a^3 + b^3 + c^3) \geq (a + b + c)^3$, it suffices to prove $9(a^3 + b^3 + c^3) - (a + b + c)^3 \geq 0$. Expanding this latest inequality, we have $9(a^3 + b^3 + c^3) - (a + b + c)^3 \geq 0$.

$9(a^3 + b^3 + c^3) - (a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3a^2c + 3ac^2 + 3b^2c + 3bc^2 + 6abc) \geq 0$.

Rearranging the terms as follows
$3a^3 - 3ab^2 - 3a^2b + 3b^3 + 3a^3 - 3ac^2 - 3a^2c + 3c^3 + 3b^3 - 3bc^2 - 3b^2c + 3c^3 + 2a^3 + 2b^3 + 2c^3 \geq 6abc$, or

$3(a - b)(a^2 - b^2) + 3(a - c)(a^2 - c^2) + 3(b - c)(b^2 - c^2) + 2a^3 + 2b^3 + 2c^3 \geq 6abc$, or

$3(a-b)^2(a+b) + 3(a-c)^2(a+c) + 3(b-c)^2(b+c) + 2(a^3 + b^3 + c^3) \geq 6abc$ which is the inequality we need to prove.

But $(a-b)^2 \geq 0$, and $a + b > 0$, or $(a-b)^2(a+b) \geq 0$.
Similarly, $(a-c)^2 \geq 0$, and $a + c > 0$, or $(a-c)^2(a+c) \geq 0$.
$(b-c)^2 \geq 0$, and $b + c > 0$, or $(b-c)^2(b+c) \geq 0$.

Adding these three inequalities to get
$(a-b)^2(a+b) + (a-c)^2(a+c) + (b-c)^2(b+c) \geq 0$, or
$$3(a-b)^2(a+b) + 3(a-c)^2(a+c) + 3(b-c)^2(b+c) \geq 0 \qquad \text{(i)}$$

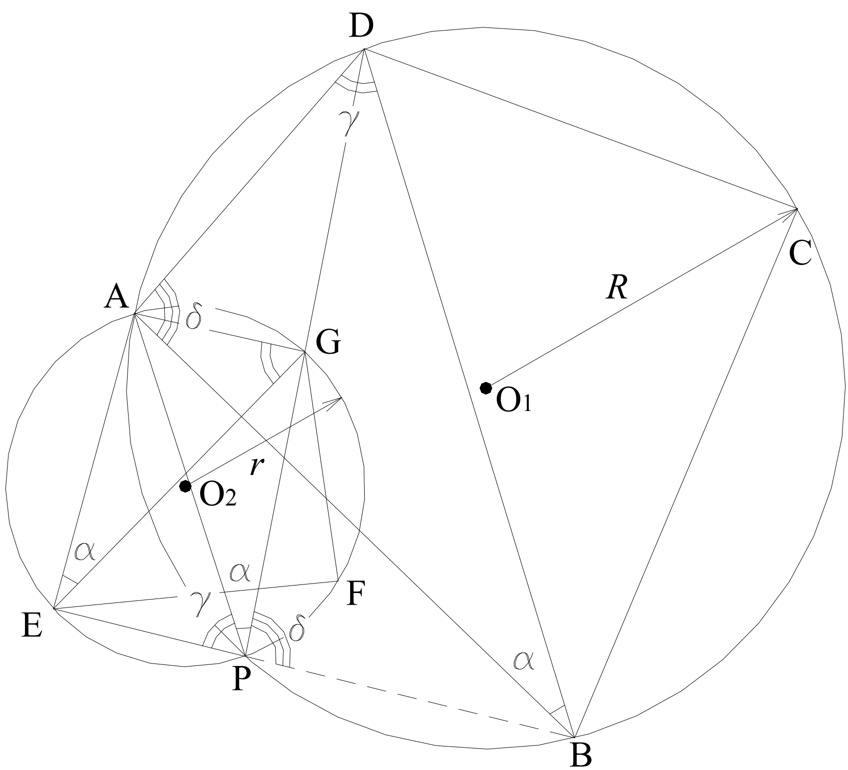
Now applying the AM-GM inequality to get
$a^3 + b^3 + c^3 \geq 3\sqrt[3]{a^3b^3c^3} = 3abc$, or
$$2(a^3 + b^3 + c^3) \geq 6abc \qquad \text{(ii)}$$

Adding (i) to (ii) to get
$3(a-b)^2(a+b) + 3(a-c)^2(a+c) + 3(b-c)^2(b+c) + 2(a^3 + b^3 + c^3) \geq 6abc$.

Problem 3 of Austria Mathematical Olympiad 2002

Let ABCD and AEFG be two similar cyclic quadrilaterals (labeled counter-clockwise). Let P be the second point of intersection of the circumcircles of the quadrilaterals. Show that P lies on the line BE.

Solution



Let R, O_1 and r, O_2 be the radii, circumcenters of the circumcircles of quadrilateral ABCD and AEFG, respectively. Also let $\angle ABD = \alpha$, $\angle ADB = \gamma$ and $\angle BAD = \delta$.

The similarities between the two quadrilaterals give us

$$\frac{AG}{AD} = \frac{r}{R}, \quad \angle AEG = \angle ABD = \alpha, \quad \angle AGE = \angle ADB = \gamma.$$

But both $\angle APG$ and $\angle AEG$ subtend small arc AG, and both

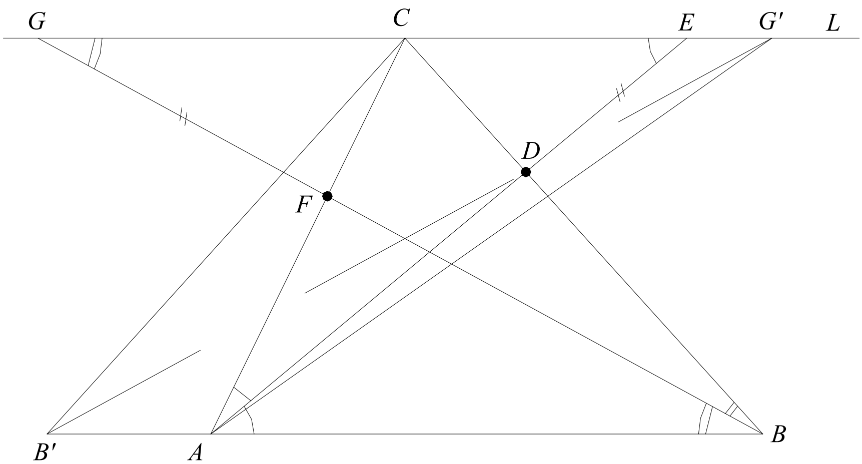
$\angle$AGE and $\angle$APE subtend small arc AE, and we have $\angle$APG = $\angle$AEG = α = $\angle$ABD, $\angle$APE = $\angle$AGE = γ = $\angle$ADB.

Furthermore, since point P being on both circles and $\frac{\mathrm{AG}}{\mathrm{AD}} = \frac{r}{R}$, the three points P, G and D are collinear.
Therefore, $\angle$BAD = δ = $\angle$BPG.
Now, let's add the three angles $\angle$APE + $\angle$APG + $\angle$BPG = $\gamma + \alpha + \delta$ which are the three angles of triangle ABD, and $\angle$APE + $\angle$APG + $\angle$BPG = 180°, or the three points E, P and B are on the same straight line, or point P lies on the line BE.

Problem 8 of the Irish Mathematical Olympiad 1991

Let ABC be a triangle and L the line through C parallel to the side AB. Let the internal bisector of the angle at A meet the side BC at D and the line L at E, and let the internal bisector of the angle at B meet the side AC at F and the line L at G. If |GF| = |DE|, prove that |AC| = |BC|.

Solution



From the law of sines, $\dfrac{BC}{\sin\angle CAB} = \dfrac{AC}{\sin\angle CBA}$. Without loss of generality (WLOG) assuming that BC > AC as shown on the graph, we then have $\sin\angle CAB > \sin\angle CBA$, or $\angle CAB > \angle CBA$.

Since BG and AE are internal bisectors, we have the following equalities $\dfrac{FC}{FA} = \dfrac{BC}{AB}$, and $\dfrac{DC}{DB} = \dfrac{AC}{AB}$.

As assumed earlier BC > AC, $\dfrac{BC}{AB} > \dfrac{AC}{AB}$, and $\dfrac{FC}{FA} > \dfrac{DC}{DB}$ (i)

But also because line L || AB, $\dfrac{FC}{FA} = \dfrac{FG}{FB}$, and $\dfrac{DC}{DB} = \dfrac{DE}{DA}$.

Inequality (i) becomes $\dfrac{FG}{FB} > \dfrac{DE}{DA}$ (ii)

Given FG = DE by the problem, inequality (ii) now becomes FB < DA. Adding FG to the left and DE = FG to the right of it, we have FB + FG < DA + DE, or BG < AE (iii)
Bur since $\angle CAB > \angle CBA$, $\frac{1}{2}\angle CAB > \frac{1}{2}\angle CBA$,
or $\angle CAE > \angle CBG$,
or $180° - 2\angle CAE < 180° - 2\angle CBG$,
or $\angle BCG > \angle ACE$.

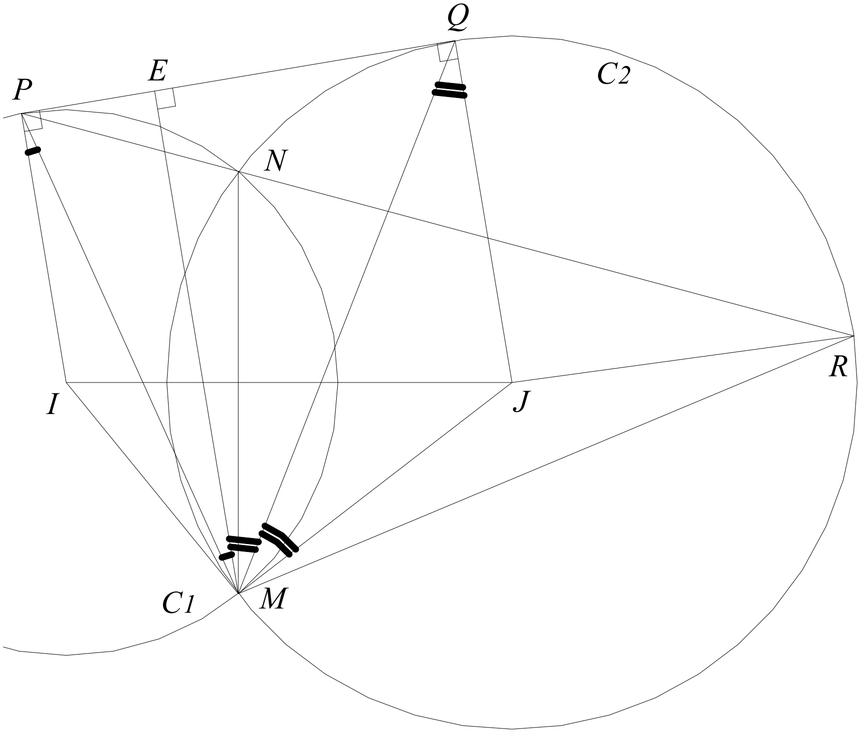
Picking point G' as mirror image of G across C on line L, and B' as the mirror image of B across the vertical line perpendicular to AB through C, we have BG = B'G'. It's evidenced that from BC > AC and GC = BC (triangle GCB is isosceles since $\angle CBG = \angle CGB$ due to line L || AB) > CE = AC, or G'C > CE and B'C > AC.
Combining with $\angle BCG = \angle B'CG' > \angle ACE$, we obtain B'G' = BG > AE which contradicts with the result in (iii).

Therefore, the assumption that BC > AC is false; likewise, the assumption in the opposite direction AC > BC will also be false by following the same argument, and the only possible scenario is that AC = BC.

<u>Problem 1 of the British Mathematical Olympiad 2000</u>

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects angle PMR.

<u>Solution</u>



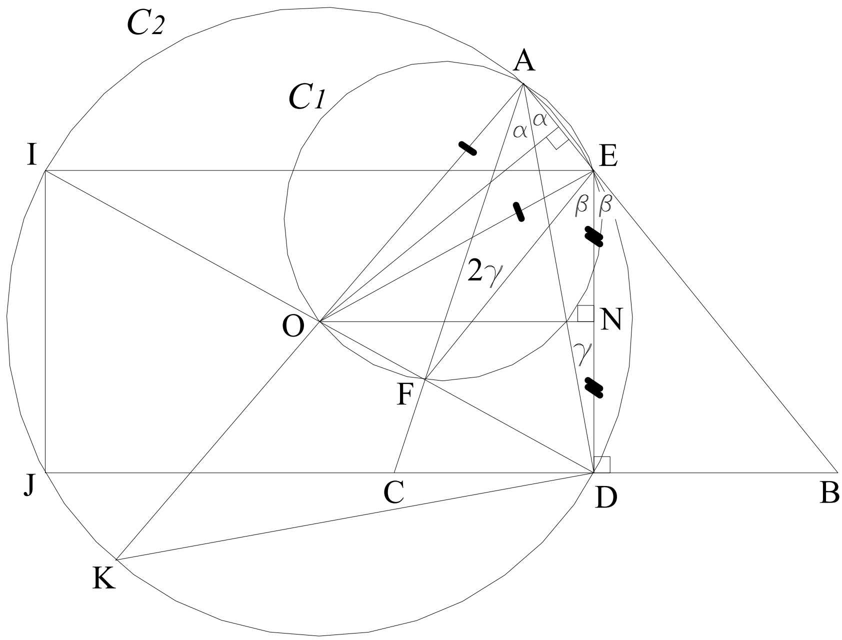
Let I and J be the centers of C_1 and C_2, respectively. From M draw a line to parallel IP and to meet PQ at E. We have ∠PME = ∠MPI, and ∠QME = ∠MQJ. Observe that ∠PNM + ∠RNM = 180°, or the arc PM on C_1 is equivalent to arc RM on C_2, and angles subtending small arc PM on C_1 equals the one subtending small arc RM on C_2, or ∠MIP = ∠MJR, or ½(180°–∠MIP) = ½(180°–∠MJR), or ∠MPI = ∠RMJ. Finally, ∠PMQ = ∠PME + ∠QME = ∠MPI + ∠MQJ = ∠RMJ + ∠QMJ = ∠RMQ, and MQ bisects angle PMR.

Problem 8 of the British Mathematical Olympiad 2001

A triangle ABC has $\angle ACB > \angle ABC$.
The internal bisector of $\angle BAC$ meets BC at D.
The point E on AB is such that $\angle EDB = 90°$.
The point F on AC is such that $\angle BED = \angle DEF$.

Show that $\angle BAD = \angle FDC$.

Solution



Let $\alpha = \angle BAD = \angle DAC$, $\beta = \angle BED = \angle DEF$, $\gamma = \beta - \alpha = \angle ADE$.

We have $\angle AFE = 180° - 2\alpha - \angle AEF = 180° - 2\alpha - (180° - 2\beta) = 2(\beta - \alpha) = 2\gamma$.
Let's draw the circumcircle C_1 of ΔAEF, and let point O be the intersection of the perpendicular bisector of AE and C_1. It's easy to understand that $\angle AOE = \angle AFE = 2\gamma$ as O lies on C_1. Now use O as the center, draw the circle C_2 with the radius of OA = OE.

Since $\angle ADE = \gamma = \frac{1}{2}\angle AOE$, point D is also on C_2. Extend AO to meet C_2 at K.

Observe that AOFE is cyclic in C_1, AKDE is cyclic in C_2, and we have

$\angle AOF + \angle AEF = 180°$, or $\angle AOF = 180° - \angle AEF = 2\beta$.
$\angle AKD + \angle AED = 180°$, or $\frac{1}{2}\angle AOD + \angle AED = 180°$, or $\frac{1}{2}\angle AOD = 180° - \angle AED = \beta$, or $\angle AOD = 2\beta$.

Thus $\angle AOF = \angle AOD$, and the three points O, F and D are collinear. Now extend DO and BC to meet C_2 at I and J, respectively. It's easily recognized that EDJI is a rectangle, and ED = IJ. $\angle BAD$ and $\angle FDC$ subtend arc ED and arc IJ, respectively on C_2; therefore, $\angle BAD = \angle FDC$.

<u>Further observation</u>

The problem can also be solved by

a) Proving that $\Delta ACD \sim$ (is similar to) ΔDCF to imply $\angle BAD = \angle DAC = \angle FDC$. This can be done by proving that $CD^2 = CF \times CA$, or proving the perpendicular bisectors of FD and AF meet on ED.

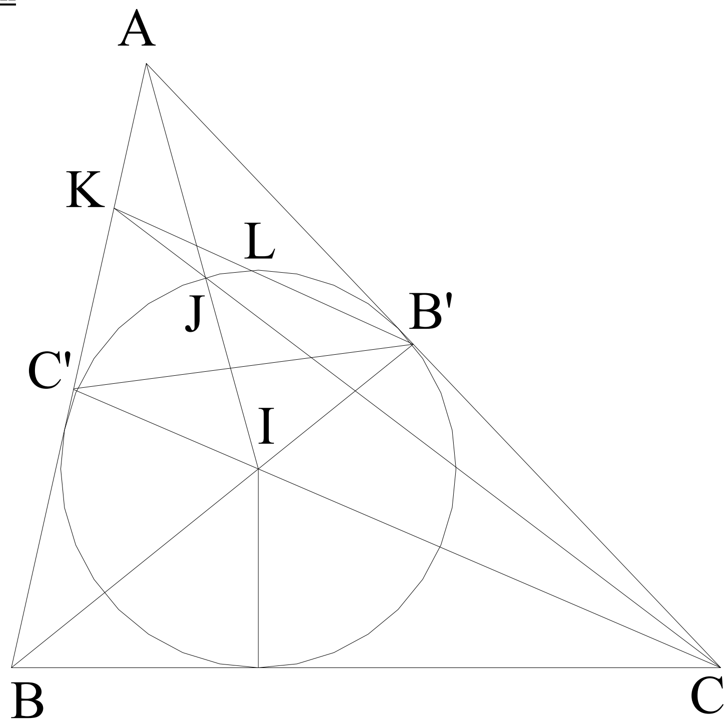
b) Proving that $\Delta AFD \sim \Delta ADB$ to imply $\angle AFD = \angle ADB$ which, in turn, causes $\angle DFC$ to equal $\angle ADC$ making the two triangles in method a) similar. This can be done by proving that $AD^2 = AF \times AB$.

These tasks are left for the reader as exercises.

<u>Problem 5 of Austria Mathematical Olympiad 1988</u>

The bisectors of angles B and C of triangle ABC intersect the opposite sides at points B′ and C′, respectively. Show that the line B′C′ intersects the incircle of the triangle.

<u>Solution</u>



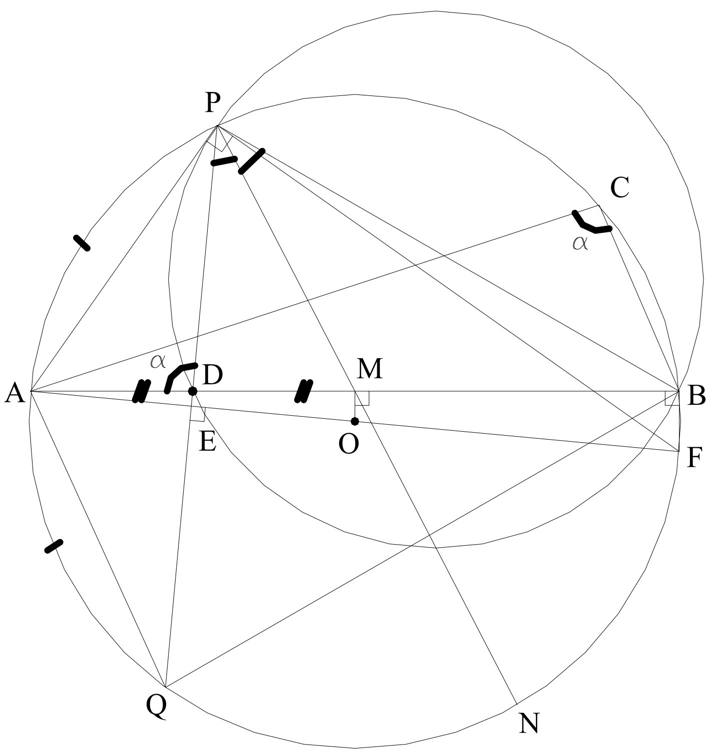
Let I be the incenter, Link AI to meet the incircle at J as shown. It's easily recognized that point J is in the interior of triangle AB'C' and on the bisector of angle BAC. Now link CJ and extend it to meet AB at K. K must be between A and C' because J is in the interior of triangle AB'C'. Furthermore, since AC is tangent to the incircle; therefore, with K on the other side of AC, B'K must intersect the incircle.

However, since both C' and K are on AB and K is between C' and A, C' is at a lower altitude compared to K; i.e., distance from C' to BC is shorter than that from K to BC. Therefore, because B'K already intersects the incircle, B'C' also intersects the incircle.

<u>Problem 7 of the British Mathematical Olympiad 2003</u>

Let ABC be a triangle and let D be a point on AB such that 4AD = AB. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of α with DA where α = ∠ACB. If the circumcircle of ABC meets the half-line ℓ at P, show that PB = 2PD.

<u>Solution</u>



Extend PD to meet the circle at Q. Since α = ∠ACB subtending arc AQB = ∠ADP subtending arcs AP + arc QB, AP = AQ.

From A draw the diameter AF for the circle. Let AF and PQ meet at E. We do have PE = QE, ∠ABF = ∠DEF = 90°, and BDEF is cyclic which implies AD×AB = AE×AF = AE(AE + EF) = AE² + AE×EF (i)

But $AE \times EF = PE \times QE = PE^2$, and equation (i) becomes
$AD \times AB = AE^2 + PE^2 = AP^2$

Given $AB = 4AD$ by the problem, we then have $AP^2 = 4AD^2$, or $\dfrac{AP}{AD} = 2$.

The similarity of ΔADQ and ΔPDB gives us $\dfrac{PB}{PD} = \dfrac{AQ}{AD} = \dfrac{AP}{AD} = 2$, or $PB = 2PD$.

<u>Further observation</u>

Let M be the midpoint of AB; we have to prove that $\dfrac{PB}{PD} = \dfrac{AQ}{AD} = \dfrac{AP}{AD} = 2$*, or* $AP = 2AD = AM$ *which makes PAM an isosceles triangle with* $AP = AM$*. Extend PM to meet the circle again at N. We will need to prove arc AP + BN = arc AQN, or BN = QN, or PN the bisector of* $\angle BPQ$ *which, interestingly, would cause* $\dfrac{BM}{DM} = \dfrac{PB}{PD} = 2$*, and this is a fact.*

One can also solve this problem by proving that $BQ \perp ON$*, or the circumcircle of triangle BDP tangents with AP at P to achieve the statement* $AP^2 = AD \times AB$*.*

<u>Problem 1 of the British Mathematical Olympiad 1997</u>

N is a four-digit integer, not ending in zero, and R(N) is the four-digit integer obtained by reversing the digits of N; for example, R(3275) = 5723.
Determine all such integers N for which R(N) = 4N + 3.

<u>Solution</u>

Let N = $abcd$ where a, b, c and d are positive integers from 0 to 9, and $d \neq 0$ as required by the problem. R(N) = $dcba$.
R(N) = 4N + 3 is now written in terms of a, b, c and d as
$1000d + 100c + 10b + a = 4000a + 400b + 40c + 4d + 3$, or

$$3999a = 996d + 60c - 390b - 3 \qquad \text{(i)}$$

Observe that a is maximum when d and c are maximum and d is minimum, or the maximal possible value for a is the integer value not greater than $\dfrac{996\times 9 + 60\times 9 - 3}{3999}$, or $a \leq 2$.

Also observe that the right expression of (i) is an odd number; therefore, $3999a$ must be an odd number as well, or a must be an odd number smaller than 2, and in that case $a = 1$.

Substituting $a = 1$ into (i) to get

$$4002 = 996d + 60c - 390b \qquad \text{(ii)}$$

Now observe that the units digits for $60c$ and $390b$ are both zero; therefore, the units digit of $996d$ must be 2. Hence, $d = 2$ or 7.

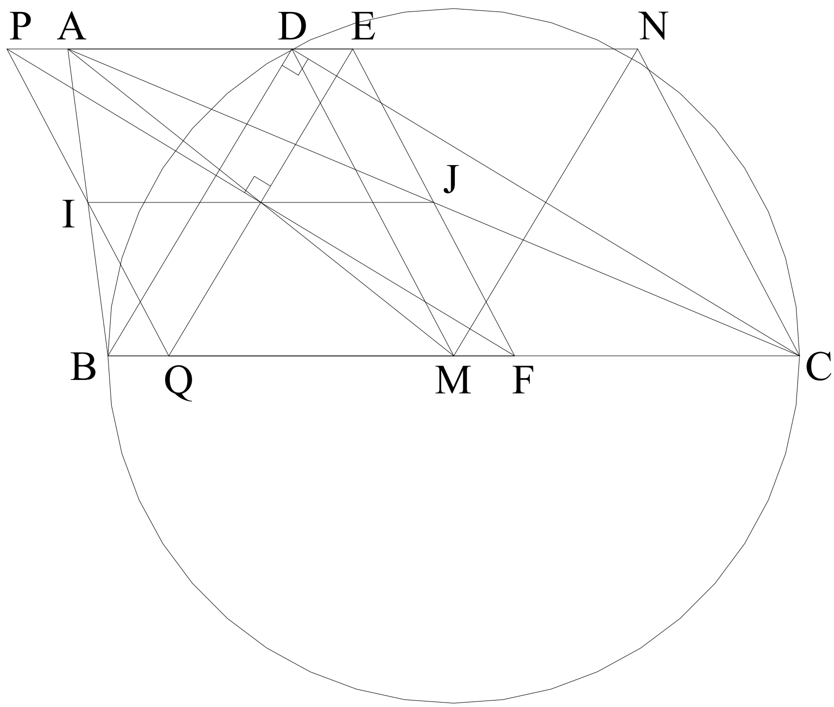
When $d = 2$, we have $2010 = 60c - 390b$ which has no solution in c and b as maximal value for $60c$ is only 540.
When $d = 7$, substituting it into (ii), we have $39b = 297 + 6c$, and the minimum value for b must be 8. Also observe that $297 + 6c$ is an odd number, so $b = 9$ and $c = 9$.

Answer: N = 1997.

<u>Problem 8 of the Russian Mathematical Olympiad 2010</u>

In a acute triangle ABC, the median, AM, is longer than side AB. Prove that you can cut triangle ABC into three parts out of which you can construct a rhombus.

<u>Solution</u>



Geometrically, we know that any right triangle can be cut into three parts to form a rhombus. Let's draw a circle with center M and diameter BC as shown. From A draw a line to parallel BC and meet the circle at D. Now denote (Ω) the area of shape Ω.

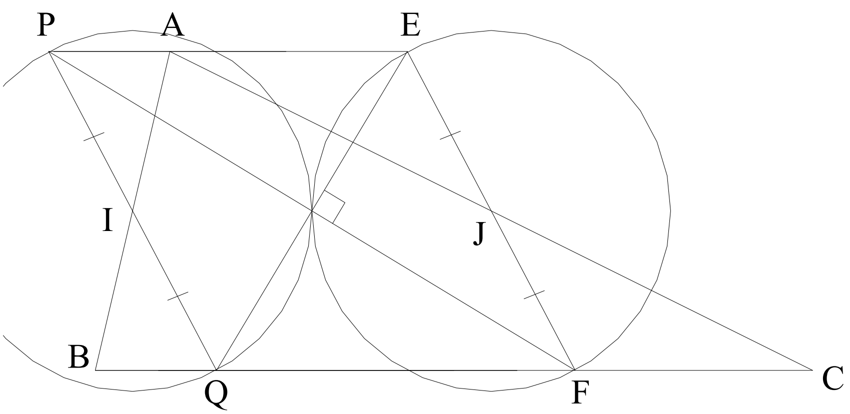
We have (DBC) = (ABC).
Let N be the mirror image of M across the diagonal DC. It's easily seen that DNCM is a rhombus with (DNCM) = (DBC) = (ABC).

Let I and J be the midpoints of AB and AC, respectively. We have

IJ || BC, and IJ = ½BC. Now move the rhombus DNCM to the left so that the midpoint of NC coincides with J. Point D ⟶ P, N ⟶ E, M ⟶ Q and C ⟶ F.

Next, cut the triangle ABC into the three pieces with shape IBQ, AIQFJA and JFC. It's easily seen that ΔIBQ = ΔIAP and ΔJFC = ΔJEA, and the three pieces fit into the rhombus PEFQ.

<u>Further observation</u>



Now that the problem has been solved, it's easier to summarize the way to cut the triangle into three pieces which together form a rhombus. It is as follows:
Pick the midpoints I and J of AB and AC, respectively. Draw two identical circles with centers I and J and with their diameter being half the length of BC. They intercept BC at Q and F, respectively. Draw the line to parallel BC through A. This line should intercept the extensions of QI and FJ at P and E, respectively. The rhombus is PEFQ.

This solution does not cover all the configurations of triangle ABC. There are cases where vertex A is higher than the highest point of the circumcircle of triangle DBC.

Problem 3 of the Middle European Mathematical Olympiad 2010

We are given a cyclic quadrilateral ABCD with a point E on the diagonal AC such that AD = AE and CB = CE. Let M be the center of the circumcircle *k* of the triangle BDE. The circle *k* intersects the line AC at points E and F. Prove that the lines FM, AD and BC meet at one point.

Solution

Extend AD and BC to meet circle *k* at J and I, respectively and to meet each other at P. Since ABCD is cyclic, we have $\angle$DAC = $\angle$DBC, but $\angle$DAC subtends arc = JF minus arc DE on circle *k* while $\angle$DBC subtends arc IE minus arc DE; therefore, IE = JF.

Also since CB = CE, CEB is an isosceles triangle and BF = IE, or JF = BF.

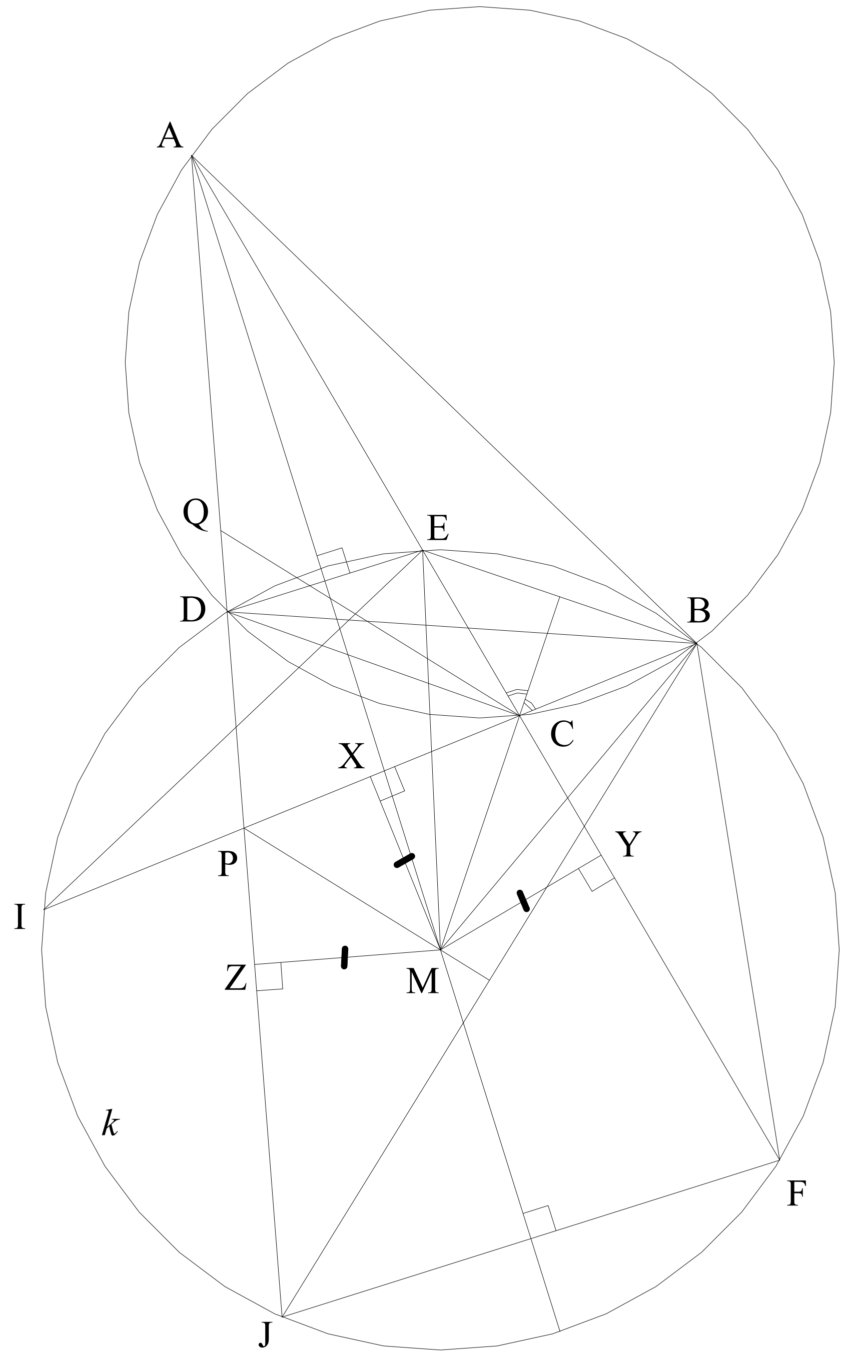
From M drop the three altitudes MX, MY and MZ to BP, AF and AJ, respectively. Because AM and CM are the angle bisectors of $\angle$DAE and $\angle$BCE, respectively, it's easily seen that MX = MY = MZ, or M is on the angle bisector of $\angle$BPJ.

With M being the circumcenter of *k*, MP is perpendicular to BJ as a result, and combining with JF = BF, we conclude that the three points P, M and F are collinear, or that the lines FM, AD and BC meet at the same point P.

Note: It's the author's intention to solve the problem this way and not applying the facts that BFMC and MFDA are cyclic quadrilaterals and that the lines FM, AD and BC meet at the radical center of *k* and two circles.

Further observation

This problem can also be solved by drawing the segment CQ with Q on AD satisfying that PC = PQ, and proving that QC || PM.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Problem 1 of the Ibero-American Mathematical Olympiad 1999

Find all the positive integers less than 1000 such that the cube of the sum of its digits is equal to the square of such integer.

Solution

Let the integer be N = abc where a, b and c are integers from 0 to 9. The problem can be solved by claiming N to be a cube itself as has been done by other authors. Let's solve the problem by not applying this fact.

Observe that the maximum value for a, b and c is 9, and the maximum value of the cube of the sum of its digits is $27^3 = 19683$.

So the maximum value of the square of N is 19683, or

$(100a + 10b + c)^2 \leq 19683$, or $a \leq 1$.

Now assume $a = 1$, $(100 + 10b + c)^2 = (1 + b + c)^3$, or

$10000 + 97b^2 + 1997b + 14bc + 197c = b^3 + c^3 + 3bc^2 + 3b^2c + 2c^2 + 1.$

Observe that the maximum value for the expression on the right is 5995 when $b = c = 9$ is less than 10000 which is the minimum value of the expression on the left; therefore, the above assumption of $a = 1$ is not possible.

Now let $a = 0$, we now have $(10b + c)^2 = (b + c)^3$, or

$$100b^2 - b^3 = c^3 + (3b - 1)c^2 + (3b - 20)bc \qquad \text{(i)}$$

With $b = 9$, substituting it into (i) to get
$7371 = c^3 + 26c^2 + 63c \leq 3402$ which is not allowed.

Continue to substitute the other values for b, we have the solutions of N = 1, 27.

<u>*Problem 3 of Japan's Hitotsubashi University Entrance Exam 2010*</u>

In the *xyz* space with O (0, 0, 0), take points A on the *x*-axis, B on the *xy* plane and C on the *z*-axis such that $\angle OAC = \angle OBC = \theta$, $\angle AOB = 2\theta$, OC = 3. Note that the *x*-coordinate of A, the *y*-coordinate of B and the *z*-coordinate of C are all positive.
Denote H the point that is inside ΔABC and is the nearest to O. Express the *z*-coordinate of H in terms of θ.

<u>Solution</u>

Let [Φ] denote the plane containing shape Φ. Since $B \in [xy]$, $OC \perp OB$. $\Delta BOC \equiv \Delta AOC$ (having 3 respective equal angles and common segment OC). Hence, OA = OB and AC = BC.

Let M be the midpoint of AB. Since point H is nearest to point O, $OH \perp [ABC]$, H must lie on MC ($H \in MC$), and $OH \perp MC$. Let's draw the two-dimensional graph for [MOC] as shown in Figure 2.

Next draw the altitude HI to OC. We then need to find OI, which is the z-coordinate of H, in terms of θ. Since $\Delta OHI \cong$ (is similar to) ΔMCO, $OI = OH \times \frac{OM}{MC}$.
But in figure 2,

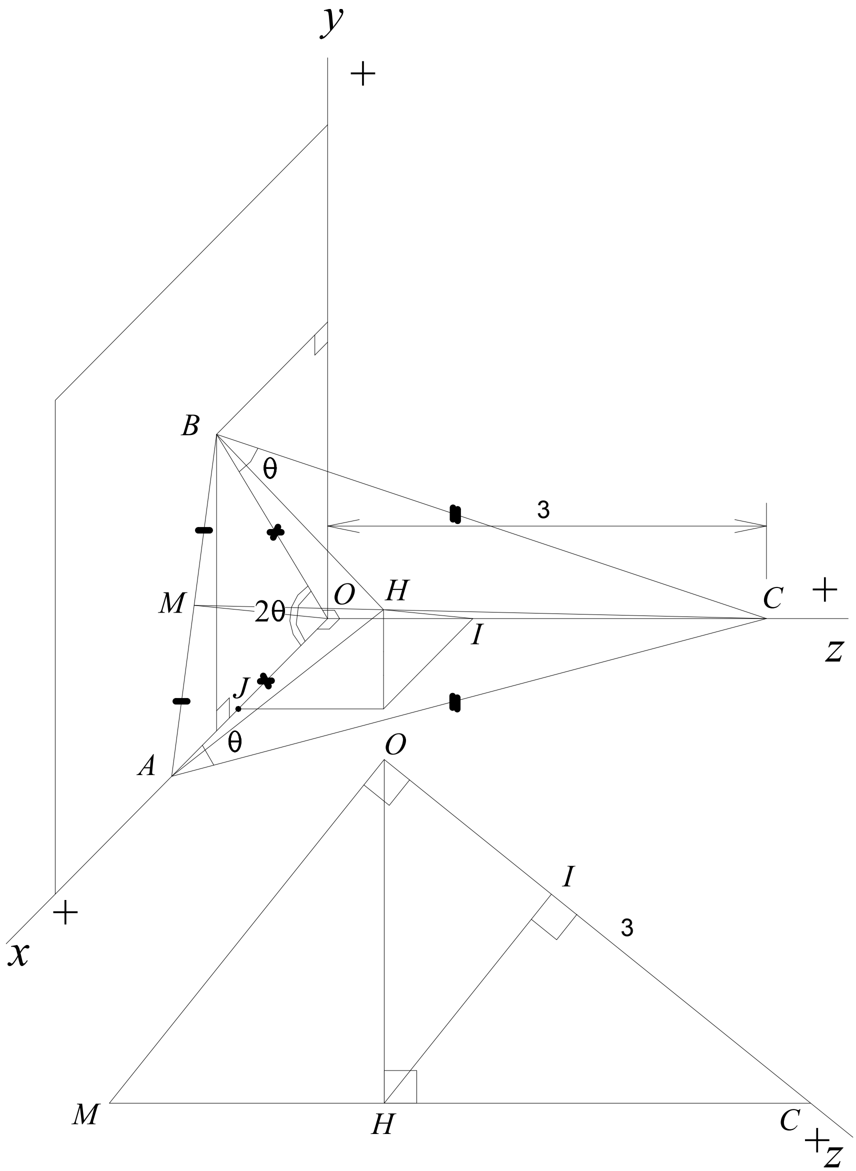
$$OH = OM \times \frac{OC}{MC}, \text{ or } OI = 3 \times \frac{OM^2}{MC^2} \qquad \text{(i)}$$

Also in figure 1,
$OA = OC/\tan\theta = 3/\tan\theta$,
$OM = OA \times \cos\frac{1}{2}\angle AOB = OA \times \cos\theta = 3\cos\theta/\tan\theta = 3\cos^2\theta/\sin\theta$

$$MC = \sqrt{OM^2 + OC^2} = 3\sqrt{\frac{\cos^4\theta}{\sin^2\theta} + 1}$$

Substituting these values to (i) to get

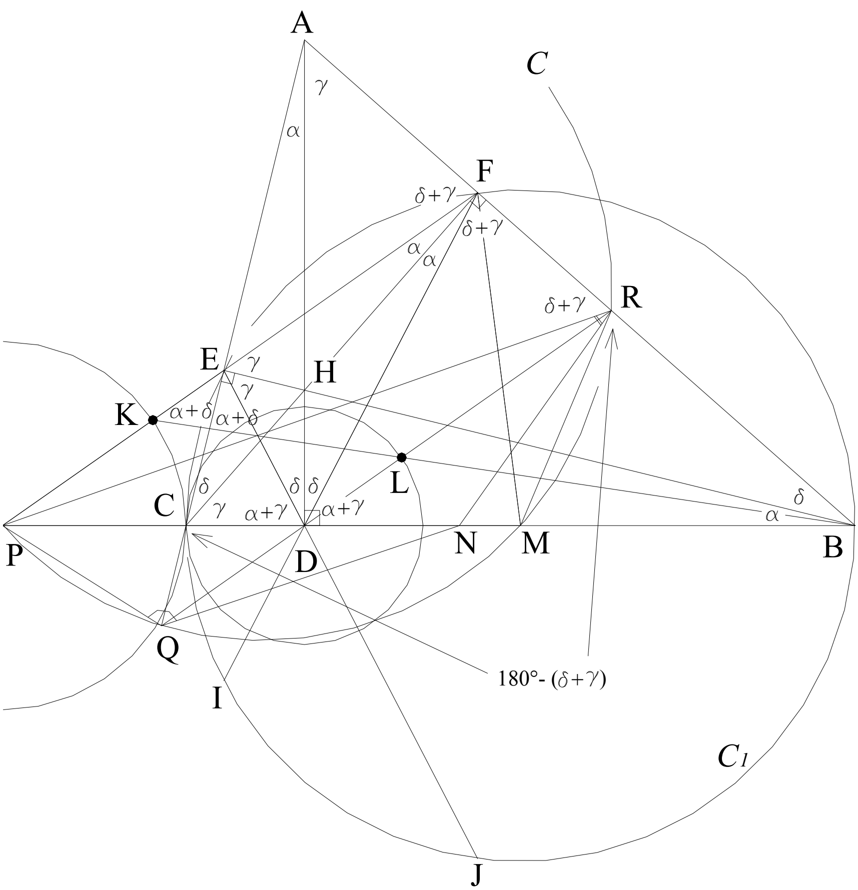
$$\mathrm{OI} = \frac{\dfrac{3\cos^4\theta}{\sin^2\theta}}{1+\dfrac{\cos^4\theta}{\sin^2\theta}} = \frac{3}{1+\dfrac{\sin^2\theta}{\cos^4\theta}}$$, and this completes our analysis.



Problem 24 of the Iranian Mathematical Olympiad 2003

In an acute triangle ABC points D, E, F are the feet of the altitudes from A, B and C, respectively. A line through D parallel to EF meets AC at Q and AB at R. Lines BC and EF intersect at P. Prove that the circumcircle of triangle PQR passes through the midpoint of BC.

Solution



Let H be the orthocenter of triangle ABC. Because ABDE, ACDF, BCEF, AEHF, BDHF and CDHE are the cyclic quadrilaterals, we have the following results when assigning the Greek letters to the angles:

$\alpha = \angle BAD = \angle BCF = \angle FEB = \angle DEB$,
$\delta = \angle ABE = \angle ACF = \angle ADF = \angle ADE$,
$\gamma = \angle CAD = \angle CBE = \angle CFE = \angle CFD$,
and $\delta + \gamma = \angle AEF = \angle ABC = \angle AQR$ (since FE || RQ).
From there, $\angle RBC = \angle RQC = 180° - (\delta + \gamma)$, and BQCR is cyclic which implies $BD \times DC = RD \times DQ = PD \times DM$ (i)
Let the circumcircles of triangle PQR and BEC be C and C_1, respectively, and let M be the intersection of C and BC.

Extend ED and FD to meet the circle C_1 at I and J, respectively. We easily see that ED = DJ and FD = DI, and $BD \times DC = FD \times DJ = FD \times DE$.
Combining with (i) above to get
$PD \times DM = FD \times DE$, or $\dfrac{PD}{FD} = \dfrac{DE}{DM}$.
Now in addition with $\angle FDP = \angle MDE = \alpha + \gamma$, the two triangles FDP and MDE are similar which implies that $\angle DFP = \angle DME$.
But $\angle DFP = \angle DFB + \angle PFB = \angle DFB + \angle AFE = 2(\alpha + \delta)$, or $\angle DME = 2(\alpha + \delta) = 2\angle ACB$.
Because BC is the diameter of C_1 and $\angle DME = 2\angle ACB$, we conclude that M is the midpoint of BC.

Further observation

The four points P, B, D and C form a harmonic sub-division. Draw a circle with center P and radius PB to meet EP at K, and another circle with center D and radius BD to meet QR at L. The reader can attempt to prove the fact that the three points K, L and C are collinear. The circle C_1 is called an Apollonius circle. For more on this, see the previous problem 2 of the Korean Mathematical Olympiad 2007 in this book.

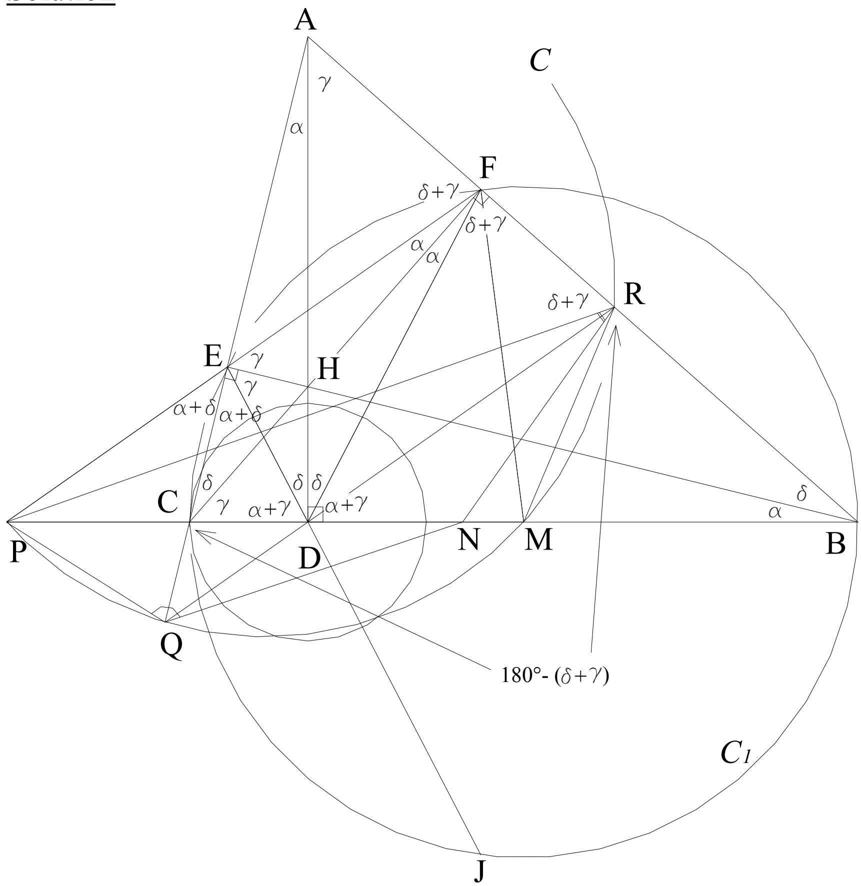
We also have the following equation as a result of the Apollonius circle

$\dfrac{BD}{PB} = \dfrac{DL}{PK} = \dfrac{DC}{PC}$, *and* $\angle PIB = \angle DIP$, $\angle PJB = \angle DJP$.

Problem 5 of Taiwan Mathematical Olympiad 1999

The altitudes through the vertices A, B, C of an acute triangle ABC meet the opposite sides at D, E, F, respectively, and AB > AC. The line EF meets BC at P, and the line through D parallel to EF meets the lines AC and AB at Q and R, respectively. N is a point on the line BC such that $\angle$NQP + $\angle$NRP < 180º. Prove that BN > CN.

Solution

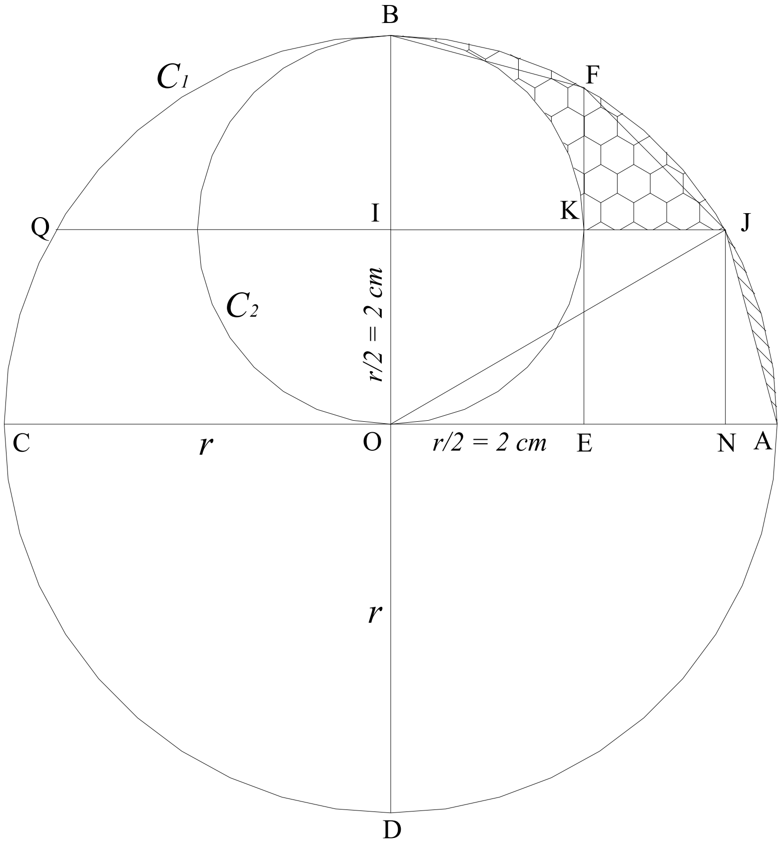


This problem and the previous one are closely related. In the previous problem, we proved BM = CM. And since PQMR is cyclic, $\angle$MQP + $\angle$MRP = 180º. To satisfy the requirement that $\angle$NQP + $\angle$NRP < 180º, point N has to be on the left side of point M on line BC, and we have BN > BM = CM > CN, or BN > CN.

Problem 4 of Hong Kong Mathematical Olympiad 2009

In figure below, the sector OAB has radius 4 cm and $\angle AOB$ is a right angle. Let the semi-circle with diameter OB be centered at I with IJ || OA, and IJ intersects the semi-circle at K. If the area of the shaded region is T cm², find the value of T.

Solution



Draw the full circle with center O and radius r = OA = OB, and assign it circle C_1. Let's also assign C_2 the circle with center I and radius IO. It's easily seen that the radius of C_2 equals $\frac{1}{2}r$. Extend AO and BO to meet circle C_1 at C and D, respectively.

Pick E as the midpoint of OA. From E draw the segment EF (F on C_1) to parallel with OB. It's also easily seen that the two trapezoids OEFB and OIJA are congruent, and AJ = BF.

Extend JI to meet the circle C_1 at Q. The two segments JQ and BD intercept at I inside a circle, and we have IJ×IQ = IB×ID.

But IJ = IQ, and ID = 3IB = $\frac{3r}{2}$, or $IJ^2 = 3IB^2$, and IJ = $r\sqrt{3}/2$.

Now let N be the foot of J onto OA; we have $JA^2 = JN^2 + NA^2 =$

$\frac{1}{4}r^2 + (r - ON)^2 = \frac{1}{4}r^2 + (r - IJ)^2 = \frac{1}{4}r^2 + (r - r\sqrt{3}/2)^2$, or

JA = $r\sqrt{2-\sqrt{3}}$.

In the right triangle JKF with KJ = KF, JF= JK$\sqrt{2}$ = (IJ – IK)$\sqrt{2}$ = $r\sqrt{2-\sqrt{3}}$.

Hence, JA = JF = BF, and the area bounded by arc JA and the straight segments OJ and OA is $\frac{1}{12}(C_1) = \frac{1}{12}\pi r^2$.

The area of the triangle OIJ is $\frac{1}{2}$ IJ×O $= \frac{1}{8}r^2\sqrt{3}$.
The area bounded by arc BK and straight segments BI and IK is

$\frac{1}{4}(C_2) = \frac{1}{16}\pi r^2$.

The area of the shaded region is a quarter of the area of C_1 minus the total of the above three areas which is $\frac{1}{4}\pi r^2 - \frac{1}{12}\pi r^2 - \frac{1}{8}r^2\sqrt{3}$

$-\frac{1}{16}\pi r^2 = \frac{1}{48}r^2\,[5\pi - 6\sqrt{3}]$.

Using the given value r = 4 cm, T = $\frac{1}{3}[5\pi - 6\sqrt{3}]$ cm.

Problem 1 of the Vietnamese Mathematical Olympiad 1992

Let ABCD be a tetrahedron satisfying
a) $\angle ACD + \angle BCD = 180°$, and
b) $\angle BAC + \angle CAD + \angle DAB = \angle ABC + \angle CBD + \angle DBA = 180°$.
Find value of (ABC) + (BCD) + (CDA) + (DAB) if we know AC + CB = k and $\angle ACB = \alpha$. *Note:* (Ω) *denotes the area of shape* Ω.

Solution

Let [Φ] denote the plane containing shape Φ. Lay ΔABD, ΔACD and ΔBCD flat on the plane of ΔABC ([ABC]) in figure 1. The segments are dotted to show that they lie on the same plane [ABC]. Point D of triangle ACD is now at D'; the same point D of triangle BCD is now at D", and that of triangle ABD is now at D"'.

We are already given $\angle ACB = \alpha$. Now let $\angle ACD = \beta$, $\angle BCD = \chi$, $\angle D'''AB = \mu$, $\angle BAC = \varepsilon$, $\angle CAD' = \delta$, $\angle CD'A = \gamma$, $\angle ABD''' = \lambda$, $\angle ABC = \varphi$, $\angle CAD'' = \psi$, AD = a, BD = b and CD = c (as shown in figure 1).

Since $\angle BAC + \angle CAD + \angle DAB = \angle ABC + \angle CBD + \angle DBA = 180°$, which can now be written as $\varepsilon + \delta + \mu = \varphi + \eta + \lambda = 180°$, the three points D', A and D''' form a straight line, so do the three points D", B and D'''.

Also note that AD = AD' = AD''' = a,
BD = BD" = BD''' = b,
CD = CD' = CD" = c,
and since A and B are the midpoints of D'D''' and D"D''', respectively, AB || D'D", and AB = ½D'D''.

Draw the altitude D"K to the extension of D'C. Since $\angle ACD + \angle BCD$ $(\beta + \chi) = 180°$, or $\angle ACD' + \angle ACK = 180°$, we have $\angle D''CK = \angle ACB = \alpha$, and $\angle D'CD'' = 180° - \alpha$, and thus

AB = ½D'D" = c×cos½α (i)

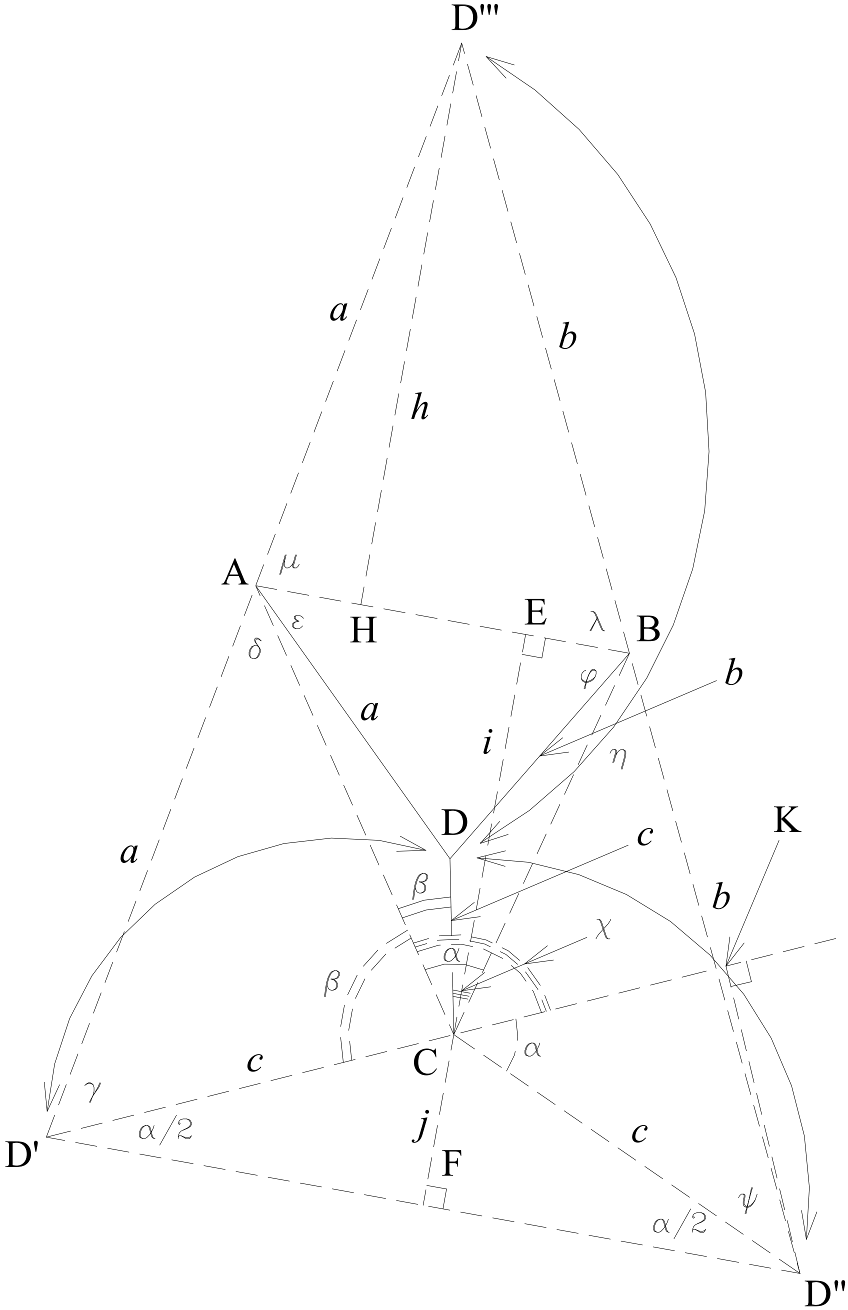


Figure 1. Two dimentional layout.

Now lay flat the two triangles ACD and BCD to share the same side CD as shown in figure 2. Applying the law of sines for ΔABC,

$$\frac{AB}{\sin\alpha} = \frac{AC}{\sin\varphi} = \frac{BC}{\sin\varepsilon} = \frac{AC + BC}{\sin\varphi + \sin\varepsilon}\text{, or } AB = \frac{k\times\sin\alpha}{\sin\varphi + \sin\varepsilon} \qquad \text{(ii)}$$

Equating the two values of AB in (i) and (ii) to get

$$c\times\cos\tfrac{1}{2}\alpha = \frac{k\times\sin\alpha}{\sin\varphi + \sin\varepsilon}\text{, or}$$

$$c\times(\sin\varphi + \sin\varepsilon) = \frac{k\times\sin\alpha}{\cos\tfrac{1}{2}\alpha} = 2k\times\sin\tfrac{1}{2}\alpha\text{, or}$$

$$2c\times\sin\tfrac{1}{2}(\varphi + \varepsilon)\cos\tfrac{1}{2}(\varphi - \varepsilon)= 2k\times\sin\tfrac{1}{2}\alpha\text{, or}$$

$$2c\times\cos\tfrac{1}{2}\alpha\times\cos\tfrac{1}{2}(\varphi - \varepsilon) = 2k\times\sin\tfrac{1}{2}\alpha\text{, or}$$

$$c\times\cos\tfrac{1}{2}(\varphi - \varepsilon) = k\times\tan\tfrac{1}{2}\alpha \qquad \text{(iii)}$$

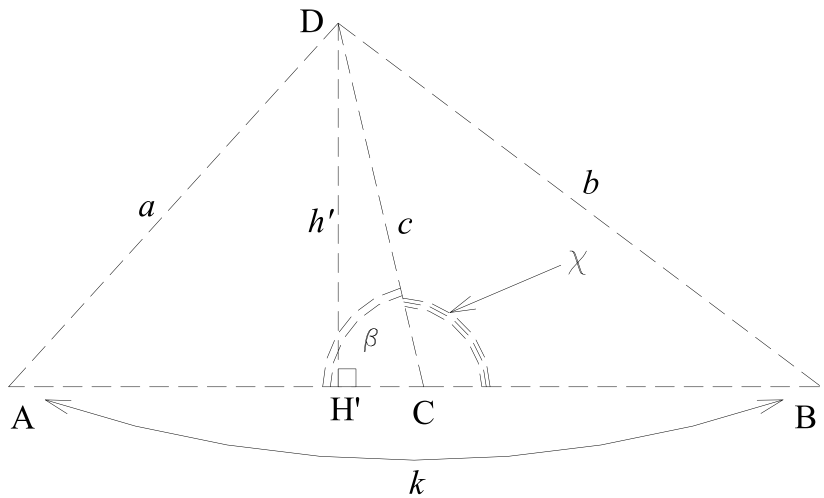


Figure 2

Let S be the value of the area (ABC) + (BCD) + (CDA) + (DAB).
S = (D’D”D’’’) – (D’CD”) = 4(ABD’’’) – (D’CD”).

But (ABD’’’) = ½AB×D’’’H (H is the foot of D’’’ on AB). Now let D’’’H = h, and using AB in (i), we have (ABD’’’) = ½c× cos½α×h.

On the other hand, (D'CD") = ½D"K×c = ½c^2sinα. Therefore, S = 2ch×cos½α – ½c^2sinα = 2ch ×cos½α – c^2sin½αcos½α = c×cos½α (2h – $c$sin½α).

Now from C draw the altitudes CE and CF to AB and D'D", respectively, and let CE = i, CF = j. The above expression for S becomes S = c(2h – j)cos½α = c(2h – j)cos∠CD'D" =

$$c(2h - j)\frac{\text{D'D"}}{2c} = (2h - j)\times\text{AB} = (h + i)\times\text{AB} = 2(\text{ACBD'''}).$$

This result shows that S = 2[(ACD') + (BCD")] = 2(ABD) as shown in figure 2.

Let H' be the foot of D on AB and h'= DH', we obtain S = h'× (AC + CB) = kh' and need to relate h' to k and α. Indeed, because AB || D'D", we have

$\mu = \gamma + \dfrac{\alpha}{2}$, and

$\lambda = \psi + \dfrac{\alpha}{2}$, or

$$\mu - \lambda = \gamma - \psi \qquad \text{(iv)}$$

And since ε + δ + μ = φ + η + λ = 180°, it follows that

$$\varphi - \varepsilon = \delta + \mu - \lambda - \eta. \qquad \text{(v)}$$

Substituting μ – λ from (iv) to (v) and note that ψ – η = β, we have

$$\varphi - \varepsilon = \gamma + \delta - \beta \qquad \text{(vi)}$$

However, in ΔACD' and ΔBCD" with β + χ = 180°, γ + δ = 180° – β, and (vi) becomes φ – ε = 180° – 2β, or ½(φ – ε) = 90° – β.

And equation (iii) becomes c×cos(90°– β) = c×sinβ = $k$tan½α. And now S = h'×(AC + CB) = kh' = k×csinβ = k^2tan½α. This completes our analysis.

Problem 2 of the British Mathematical Olympiad 2005

Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy $x^2 - y^2 = 2k$ for some non-negative integer k.

Solution

Since x and y are positive integers with no prime factors larger than 5, we can express them as follows $x = 2^a \times 3^b \times 5^c$, and $y = 2^d \times 3^e \times 5^f$ where all the values a, b, c, d, e and f take on the values of either 0 or 1, and the possible values for x^2 and y^2 are
$x^2 = 1, 4, 9, 25, 36, 100, 225, 900$.
$y^2 = 1, 4, 9, 25, 36, 100, 225, 900$.

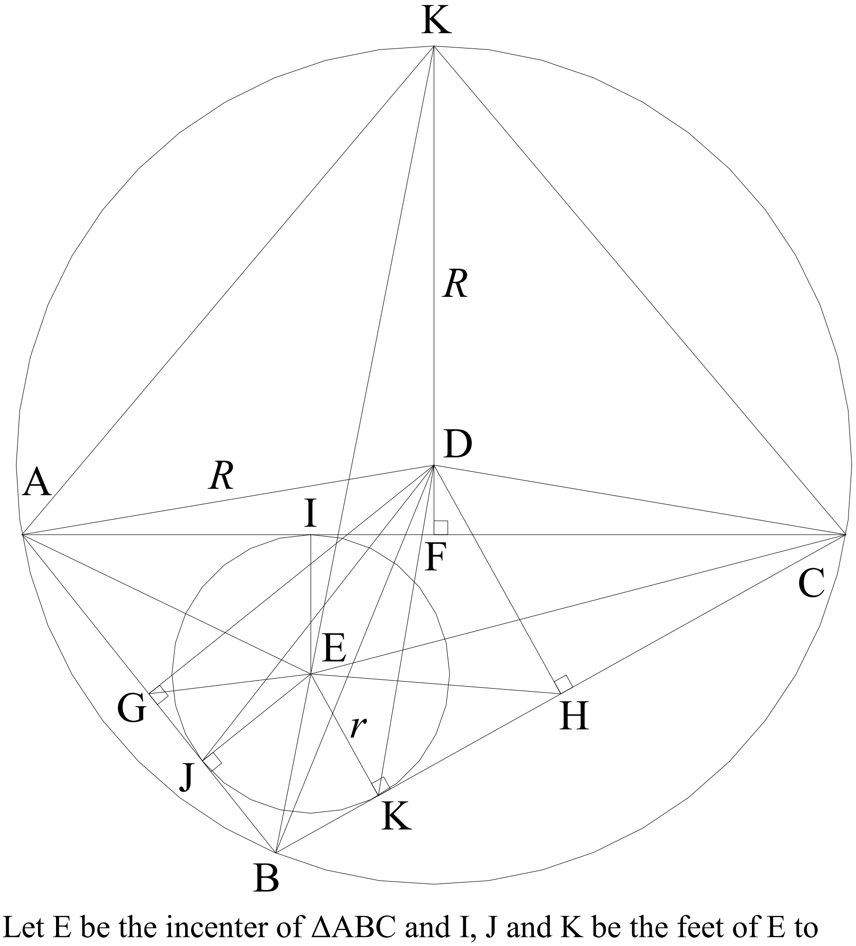
The problem requires $x > y$ and the difference of $x^2 - y^2$ to be an even number. Therefore,
$(x^2, y^2) = (9, 1), (25, 1), (225, 1),$
$(25, 9), (225, 9), (225, 25),$
$(36, 4), (100, 4), (900, 4),$
$(100, 36), (900, 36),$
$(900, 100),$
and finally
$(x, y) =$ (3, 1), (5, 1), (15, 1), (5, 3), (15, 3), (15, 5), (6, 2), (10, 2), (30, 2), (10, 6), (30, 6), (30, 10).

<u>Proof of Carnot's theorem for the obtuse triangle</u>

Let ABC be an arbitrary obtuse triangle. Prove that DG + DH = R + r + DF, where r and R are the inradius and circumradius of triangle ABC, respectively, D the circumcenter of triangle ABC, DF, DG and DH the altitudes to the sides AC, AB and BC, respectively.

(Carnot's theorem is used in a proof of the Japanese theorem for concyclic polygons.)

<u>Solution</u>



Let E be the incenter of ΔABC and I, J and K be the feet of E to

AC, AB and BC, respectively. We have r = EI = EJ = EK, and AI = AJ, CI = CK, BJ = BK. Let a = BJ = BK, and denote (Ω) the area of shape Ω.

Since DG || EJ, EJ ⊥ AB, ΔGEJ and ΔDEJ have the same base r = EJ and the same altitude GJ, hence (GEJ) = (DEJ).
Similarly, (HEK) = (DEK).

Adding those two equations to get (GEJ) + (HEK) = (DEJ) + (DEK).

Now adding (BJEK) to both sides, we obtain (BGEH) = (BJDK). But since G and H are the midpoints of AB and BC, respectively, (BGEH) = ½(AECB), and (BJDK) = (DJB) + (DKB) = ½DG×BJ + ½DH×BK, we then have ½(AECB) = ½DG×BJ + ½DH×BK, or (AECB) = DG×BJ + DH×BK = a(DG + DH).

Moreover, (AECB) = (AEB) + (BEC) = ½r(AB + BC), and now the above equation becomes ½r(AB + BC) = a(DG + DH).

Also note that BJ + BK = AB + BC – AJ – CK = AB + BC – AI – CI, or 2a = AB + BC – AC, and the previous equation can now be written as r(AB + BC) = (AB + BC – AC)×(DG + DH).

Rearranging the above equation to get

$$(AB + BC)\times(DG + DH - r) = AC\times(DG + DH) \qquad \text{(i)}$$

Now note that BGDH is cyclic and by Ptolemy's theorem
GH ×BD = DG×BH + DH×BG, but GH = ½AC and BD = R, and we have $\frac{R\times AC}{2} = \frac{DG\times BC}{2} = \frac{DH\times AB}{2}$, or

$$R\times AC = DG\times BC + DH\times AB \qquad \text{(ii)}$$

But since (ADCB) = (ADC) + (ABC), DF ×AC + r(AB + BC + AC) = DG×AB + DH×BC (iii)

Adding (ii) and (iii), we get
AC(R + r + DF) = DG(AB + BC) + DH(AB + BC) – r(AB + BC),

or AC(R + r + DF) = (AB + BC) ×(DG + DH – r) (iv)

From (i) and (iv), we finally have
AC×(DG + DH) = AC×(R + r + DF), or DG + DH = R + r + DF.

Further observation

This proof was made possible by requests from the readers of my previous book even though the proof of the Carnot's theorem for the acute triangle is already available at the website www.cut-the-knot.org.

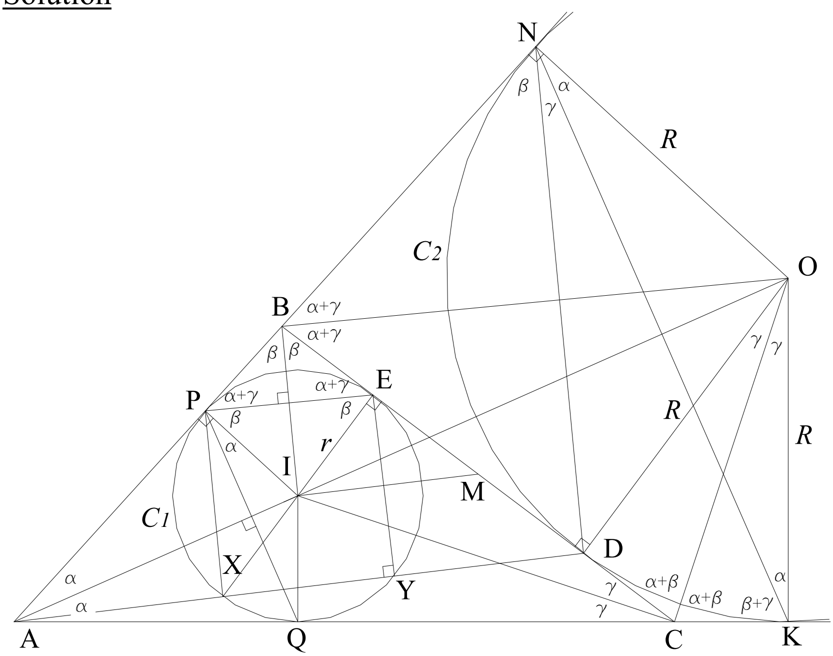
It was brought to the author's attention that for an obtuse triangle the proof is much more complex compared to the existing proof in the website. And for that purpose, the author has intentionally resorted to a more difficult way to prove the theorem instead of basing on the already existing method which used the similarities of the triangles as you have seen it in the above website.

<u>*Problem 1 of Hong Kong Mathematical Olympiad 2007*</u>

Let D be a point on the side BC of triangle ABC such that AB + BD = AC + CD. The line segment AD cut the incircle of triangle ABC at X and Y with X closer to A. Let E be the point of contact of the incircle of triangle ABC on the side BC. Show that

a) EY is perpendicular to AD,
b) XD is 2×IM, where I is the incenter of the triangle ABC and M is the midpoint of BC.

<u>Solution</u>



a) Let the incircle be *C1*. Draw the external circle *C2* with center O for triangle ABC that tangents the extensions of AB and AC at N and K, respectively. It's easily seen that BN = BD, CK = CD and OD⊥BC. Now let α = ∠BAI = ∠CAI = ½∠BAC, β = ∠ABI = ∠CBI = ½∠ABC and γ = ∠ACI = ∠BCI = ½∠ACB. We have α + β + γ = 90°.
Now let P and Q be the tangential points of *C1* with AB and AC,

respectively. We then also have $\alpha = \angle QPI$, $\beta = \angle EPI$.

Note that since OC and IC are the angle bisectors of $\angle DCK$ and $\angle ACB$, respectively, $\angle DCI + \angle DCO = ½\angle ACK = 90°$, or $\angle DOC = \gamma$, $\angle DOK = 2\angle DOC = 2\gamma$.
Therefore, $\angle DNK = \gamma$ (subtending arc DK).

Since both C_1 and C_2 tangent AB and AC with the points of tangent on AC at Q and K, and ray AD cuts C_1 and C_2 at X and D, respectively, we have $\frac{XQ}{DK} = \frac{r}{R}$ where r and R are the radii of C_1 and C_2, respectively.

Therefore, $\angle XPQ = \angle DNK = \gamma$, and $\angle EPX = \angle EPI + \angle QPI + \angle XPQ = \beta + \alpha + \gamma = 90°$, and because PEYX is cyclic, $\angle EPX + \angle EYX = 180°$, or $\angle EYX = 90°$, and EY is perpendicular to AD.

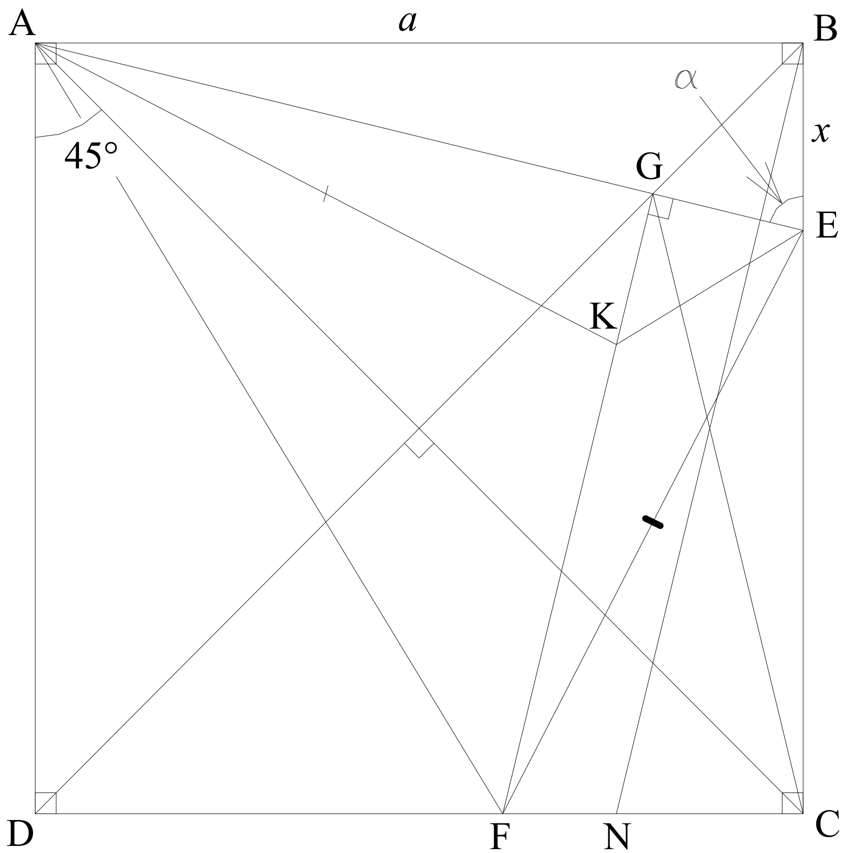
b) Since AP = AQ, BP = BE, CE = CQ and CE = CD + DE, the given equation AB + BD = AC + CD is equivalent to AP + BP + BE + DE = AQ + CQ + CD. Canceling equal terms on both sides, we get BE = CD. Therefore, the midpoint M of BC is also the midpoint of DE, and since $\angle EYX = 90°$, EX is the diameter of C_1, or E, I and X are collinear, and I is the midpoint of EX.

I and M are midpoints of EX and ED, respectively; therefore, $XD = 2\times IM$.

Problem 4 of the Estonian Mathematical Olympiad 2007

In square ABCD, points E and F are chosen in the interior of sides BC and CD, respectively. The line drawn from F perpendicular to AE passes through the intersection point G of AE and diagonal BD. A point K is chosen on FG such that AK = EF. Find ∠EKF.

Solution



Observing that the whole configuration depends on the length of segment BE. Let's calculate the lengths of other segments as functions of variable BE.

Let BE = x, the side length of square ABCD be a, and $\alpha = \angle AEB$.

From B draw a line to parallel with GF and intercept DC at N.

Since GF $\perp$ AE, BN $\perp$ AE, and $\angle$BAE = $\angle$CBN (angles with perpendicular sides), or ΔABE = ΔBCN, and BN = AE.

Per Pythagorean theorem, AE = GA + GE = $\sqrt{a^2 + x^2}$ (i)

and since AD $\parallel$ BC, we have $\frac{GE}{GA} = \frac{GB}{GD} = \frac{BE}{AD} = \frac{x}{a}$ (ii)

or GA = GE$\times\frac{a}{x}$.

Now substituting GA into (i), we get GE = $\frac{x\sqrt{a^2 + x^2}}{a + x}$.

And because GF $\parallel$ BN (=AE), $\frac{GF}{BN} = \frac{GD}{BD}$.

But BD is the diagonal of the square ABCD, and now we have

GF = AE$\times\frac{GD}{BD} = \sqrt{a^2 + x^2}\times\frac{GD}{a\sqrt{2}}$ (iii)

Now combining GB = GD$\times\frac{x}{a}$ from (ii) with

GB + GD = BD = $a\sqrt{2}$, we get GD = $\frac{a^2\sqrt{2}}{a + x}$.

Substituting GD into (iii), we obtain GF = $\frac{a\sqrt{a^2 + x^2}}{a + x}$, and now

EF² = GE² + GF² = $\left(\frac{a^2 + x^2}{a + x}\right)^2$ = AK² = GA² + GK², or

$$GK^2 = \left(\frac{a^2 + x^2}{a + x}\right)^2 - \left(GE\times\frac{a}{x}\right)^2 = \left(\frac{a^2 + x^2}{a + x}\right)^2 - \left(\frac{x\sqrt{a^2 + x^2}}{a + x}\times\frac{a}{x}\right)^2 =$$

$\frac{x^2(a^2 + x^2)}{(a + x)^2}$ = GE², or GK = GE and KGE is a right isosceles triangle, and $\angle$GEK = 45°.

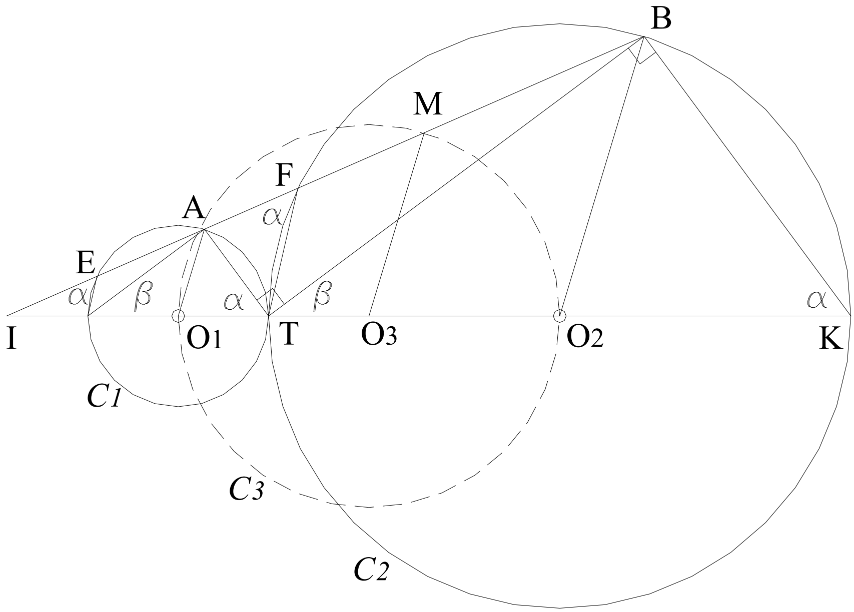
Finally, $\angle$EKF = $\angle$KGE + $\angle$GEK = 135°.

<u>Problem 4 of Hong Kong MO Team Selection Test 2009</u>

Two circles C_1, C_2 with different radii are given in the plane, they touch each other externally at T. Consider any points $A \in C_1$ and $B \in C_2$, both different from T, such that $\angle ATB = 90°$.

a) Show that all such lines AB are concurrent.
b) Find the locus of midpoints of all such segments AB.

<u>Solution</u>



a) Let O_1, r and O_2, R be the circumcenters and radii of C_1 and C_2, respectively. Extend BA to intercept the extension of O_2O_1 at I. Let IB intercept C_1 and C_2 at E (other than A) and F (other than B), respectively, IT intercept C_1 at J (other than T) and C_2 at K (other than T). Also let $\alpha = \angle ATI$, $\beta = \angle BTO_2$.

We have $\alpha + \beta = 90°$. But since JT and TK are the diameters of the two circles, we also have $\angle JAT = \angle TBK = 90°$. Therefore, $\angle AJT = \beta$, and $\angle BKT = \alpha$, and $AJ \parallel BT$, $AT \parallel BK$ all of which, in turn, cause $\frac{\mathrm{IB}}{\mathrm{IA}} = \frac{\mathrm{IT}}{\mathrm{IJ}}$, and $\frac{\mathrm{IB}}{\mathrm{IA}} = \frac{\mathrm{IK}}{\mathrm{IT}}$, or

$\frac{\mathrm{IT}}{\mathrm{IJ}} = \frac{\mathrm{IK}}{\mathrm{IT}}$, or $\frac{\mathrm{IJ} + \mathrm{JT}}{\mathrm{IJ}} = \frac{\mathrm{IT} + \mathrm{TK}}{\mathrm{IT}}$, or $1 + \frac{2r}{\mathrm{IJ}} = 1 + \frac{2R}{\mathrm{IJ} + 2r}$, or

$\frac{r}{\mathrm{IJ}} = \frac{R}{\mathrm{IJ} + 2r}$, or $\mathrm{IJ} = \frac{2r^2}{R - r}$ and is a constant.

We conclude that all such lines AB meet at point I at a distance of $\frac{2r^2}{R - r}$ away from fixed point J on the line that passes through the two circumcenters, or they are concurrent.

b) Pick O_3 and M as the midpoints of O_1O_2 and AB, respectively. $O_3M = \frac{1}{2}(AO_1 + BO_2) = \frac{1}{2}(r + R)$ and is constant no matter where A and B are on the two respective circles C_1 and C_2 as defined by the problem.

Therefore, the locus of midpoints of all such segments AB is a circle with center O_3, which is the midpoint of the two circumcenters, and with radius of $\frac{1}{2}(r + R)$.

Problem 3 of Tokyo University Entrance Exam 2006

Given the point P(0, p) on the y-axis and the line m: $y = (\tan\theta)x$ on the coordinate plane with the origin, where $p > 1$, $0 < \theta < \pi/2$. Now by the symmetric transformation, the line l with slope α as the axis of symmetry, the origin O was mapped the point Q lying on the line $y = 1$ in the first quadrant and the point P on the y-axis was mapped the point R lying on the line m in the first quadrant.

a) Express $\tan\theta$ in terms of α and p.
b) Prove that there exist the point P satisfying the following condition, then find the value of p.
Condition: For any θ ($0 < \theta < \pi/2$) *the line passing through the origin and is perpendicular to the line* l *is* $y = [\tan(\theta/3)]x$.

Solution

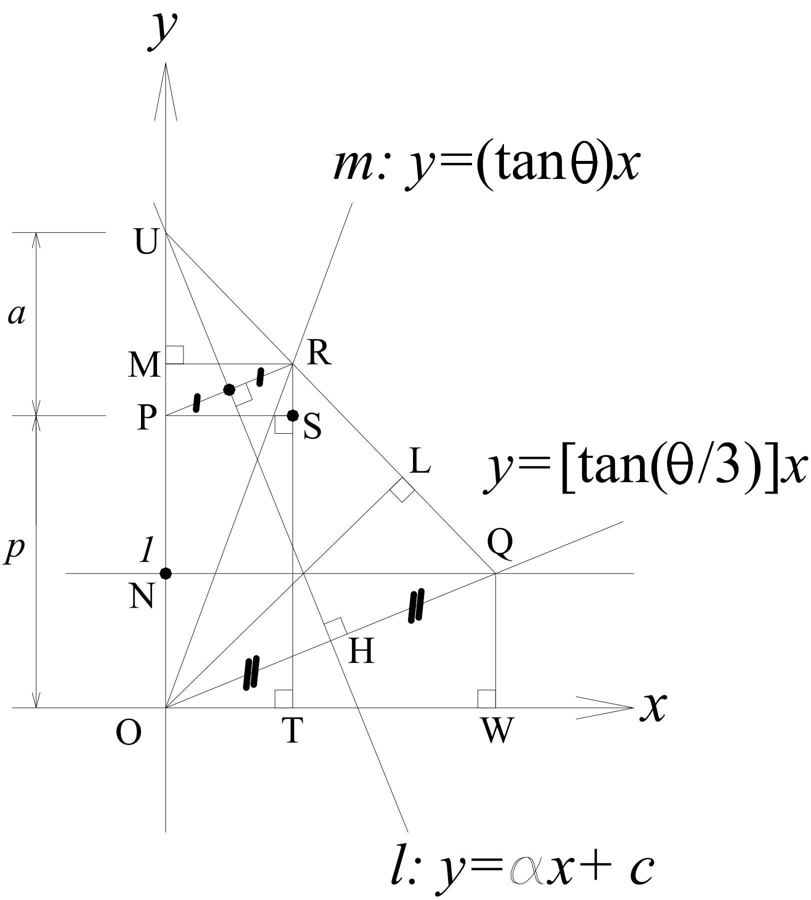
a) Extend QR to meet the y-axis at U; let H be the intersection of the line l with OQ, T and W be the feet of R and Q on the x-axis and S be the foot of P on RT, M and N be the feet of R and Q on the y-axis. Also let PU = a.
Since UV $\perp$ OQ and UO $\perp$ OW, the slope OW/QW = $|\alpha|$, but QW = 1, and OW = $|\alpha|$. Applying the Pythagorean's theorem to get OQ

$= \sqrt{\alpha^2 + 1}$.

The slope OH/UH = $1/|\alpha|$, or OQ $/[2\sqrt{UO^2 - OQ^2/4}\,] =$

$\sqrt{\alpha^2 + 1}\,/[2\sqrt{(a + p)^2 - (\alpha^2 + 1)/4}\,] = 1/|\alpha|$, or $\alpha^2 + 1 = 2(a + p)$, or $a + p = (\alpha^2 + 1)/2$.
Since both PR and OQ are perpendicular with line l, implying PR $\|$ OQ which makes PR/a = OQ/ $(a + p)$, or PR = $a\,\times$OQ/ $(a + p)$.
Furthermore, RS/PR = QW/OQ, or RS = $a/(a + p) = \dfrac{\alpha^2 + 1 - 2p}{\alpha^2 + 1}$.

Therefore, PS $= \sqrt{PR^2 - RS^2} = \sqrt{\dfrac{(\alpha^2 + 1 - 2p)^2}{\alpha^2 + 1} - \dfrac{(\alpha^2 + 1 - 2p)^2}{(\alpha^2 + 1)^2}} =$



$\dfrac{\alpha(\alpha^2+1-2p)}{\alpha^2+1}$. Finally, $\tan\theta$ = RT/ OT = (RS + p)/PS = $\dfrac{1}{|\alpha|}+\dfrac{p}{\mathrm{PS}}$ $= -\dfrac{1}{\alpha} - \dfrac{p(\alpha^2+1)}{\alpha(\alpha^2+1-2p)} = -\dfrac{\alpha^2(p+1)+1-p}{\alpha(\alpha^2+1-2p)}$. The negative sign is taken since α is negative, and the slope for line m $\tan\theta$ is positive.

b) Since $\theta = \angle$ROW = 3$\angle$QOW, $\angle$ROQ = 2$\angle$QOW. Let L be the foot of O on UQ, we must have ΔOLQ = ΔOWQ, and $\angle$LOQ = $\angle$WOQ = $\angle$LOR = θ/3. The point P exists to satisfy the condition when RQ = 2QW. However, RQ = PO = p; therefore, p = 2, and P(0, 2) satisfies the given condition.

Problem 5 of Korean Mathematical Olympiad 2006

In a convex hexagon ABCDEF triangles ABC, CDE, EFA are similar. Find conditions on these triangles under which triangle ACE is equilateral if and only if so is BDF.

Solution

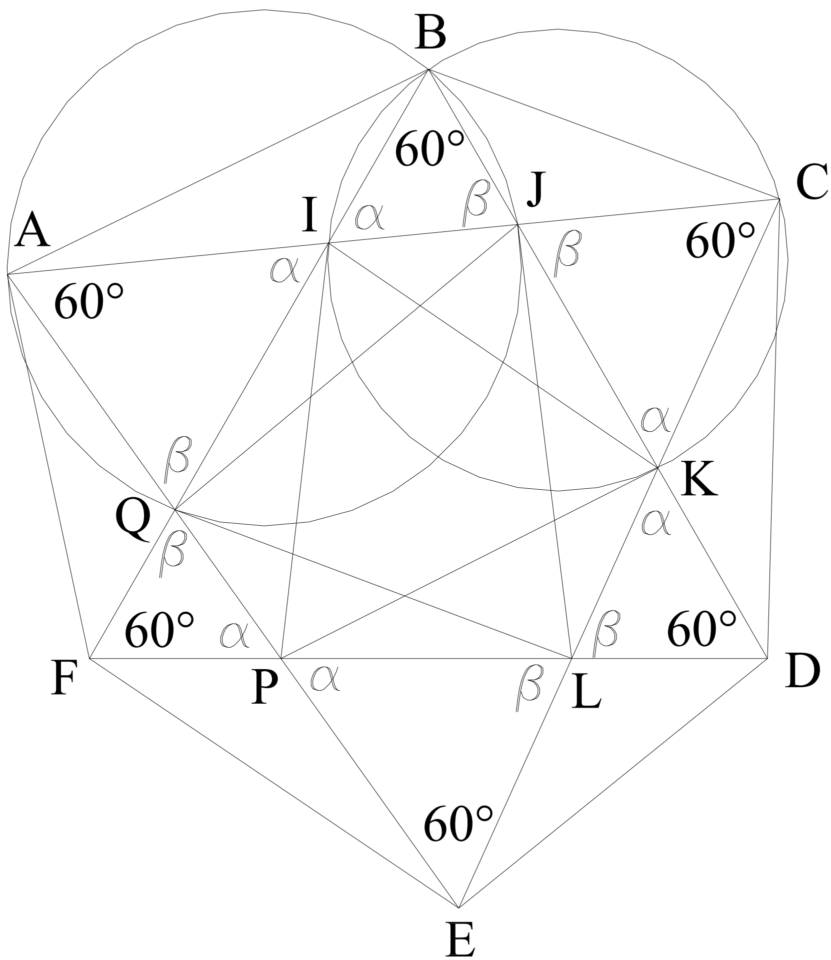


Figure 1

Let I = AC ∩ BF, J = AC ∩ BD, K = BD ∩ CE, L = CE ∩ DF, P = DF ∩ AE, Q = BF ∩ AE. When both triangles ACE and BDF are equilateral, the six 60° angles make the following quadrilaterals cyclic ABJQ, BCKI, CDLJ, DEPK, EFQL and AFPI.

Let $\angle$BIC = α and $\angle$BJA = β; we then also have $\angle$AIF = $\angle$FPA = $\angle$EPD = $\angle$DKE = $\angle$CKB = α, and $\angle$AQB = $\angle$FQE = $\angle$ELF = $\angle$DLC = $\angle$CJD = β.

Therefore, the smallest triangles that have the 60° angles are all similar (≅) ; ΔBIJ ≅ ΔAIQ ≅ ΔFPQ ≅ ΔEPL ≅ ΔDKL ≅ ΔCKJ.

Suppose the two similar ΔABC and ΔCDE are not congruent, super-impose them as seen on figure 2 below. Let C' be the previous C of ΔCDE. Join the two points B and D together (even though B ≡ D both letters are used to differentiate between the two triangles), and make BI ∈DK, BJ ∈DL.

Since AC = C'E, $\angle$BIC = $\angle$DKE = α and $\angle$BJA = $\angle$DLC' = β, we have such a configuration as shown with AC || C'E. Let R = AC ∩ DC' and S = AC ∩ DE.

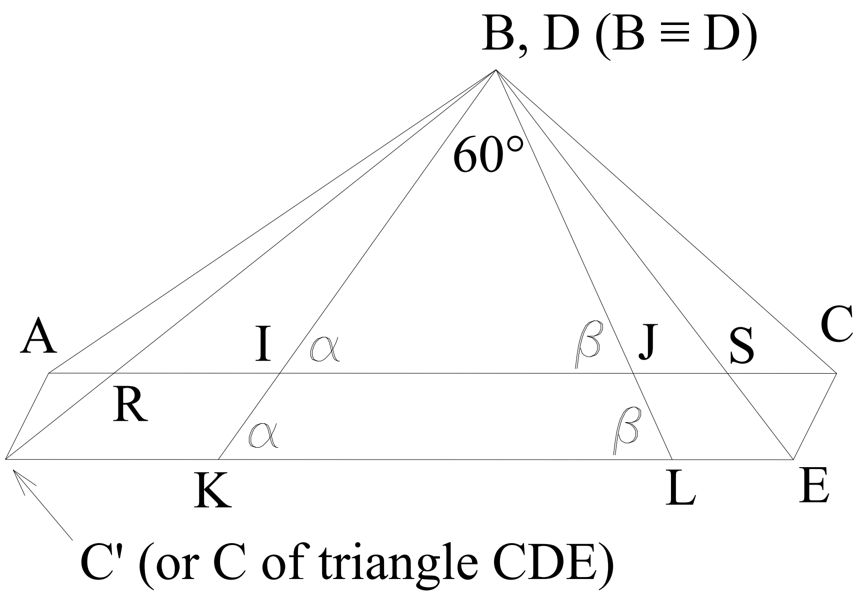


Figure 2

Since ΔABC ≅ ΔC'DE, there are three options for $\angle$ABC:

∠ABC = ∠DC'E (i)
∠ABC = ∠DEC', or (ii)
∠ABC = ∠C'DE (iii)

Option (i): When **∠ABC = ∠DC'E** = ∠BRC, the other angle of ΔABC, ∠BAC has the options of

∠BAC =∠C'DE (i-a)
∠BAC = ∠DEC'. (i-b)

(i-a) But when **∠BAC =∠C'DE**, the remaining angle **∠BCA =∠DEC'** which is not possible since ∠DEC' = ∠BSA = ∠BCA + ∠CDE, or **∠BCA < ∠DEC'.**

(i-b) When **∠BAC = ∠DEC' =** ∠BSA**,** the remaining angle **∠BCA =∠C'DE**, and in ΔABC, ∠ABC > 60° (a condition for ABCDEF to be a convex hexagon), ∠BCA = ∠C'DE > 60° (*) (also another condition for ABCDEF to be a convex hexagon), and ∠BAC = 180° –∠ABC – ∠BCA < 60°. Hence, ∠BSA = ∠BAC < 60°. But ∠BSA = ∠BCA + ∠CDE, or ∠BCA + ∠CDE < 60°, or ∠BCA < 60° which contradicts with condition (*) above. Hence, this option is also not feasible.
Therefore, option (i) is not allowed.

Option (ii): It is equivalent to option (i) with the sides switched.

The only other possible option is
Option (iii): When **∠ABC = ∠C'DE,** since ΔABC and ΔC'DE share the same length for AC = C'E, the two triangles must be congruent with either **AB = C'D and BC = DE**, or **AB = DE and BC = C'D**.

Now let's go back to figure 1.
a) When **AB = CD and BC = DE**, we also have AB = CD = EF, and BC = DE = FA. The three triangles ABC, CDE, EFA are then congruent with the above conditions in addition to AC = CE = EA.

b) When **AB = DE and BC = CD**, B and D are symmetrical with respect to the CF axis since F is a vertex of the equilateral triangle BDF; therefore, AF = FE and EFA is an isosceles triangle which causes the other two congruent triangles ABC and CDE to also be isosceles with AB = BC and CD = DE.

The two results in a) and b) are conditions on these triangles under which triangle ACE is equilateral if and only if so is BDF.

Other graphical configurations such as the one in figure 3 below can be proven in a similar manner.

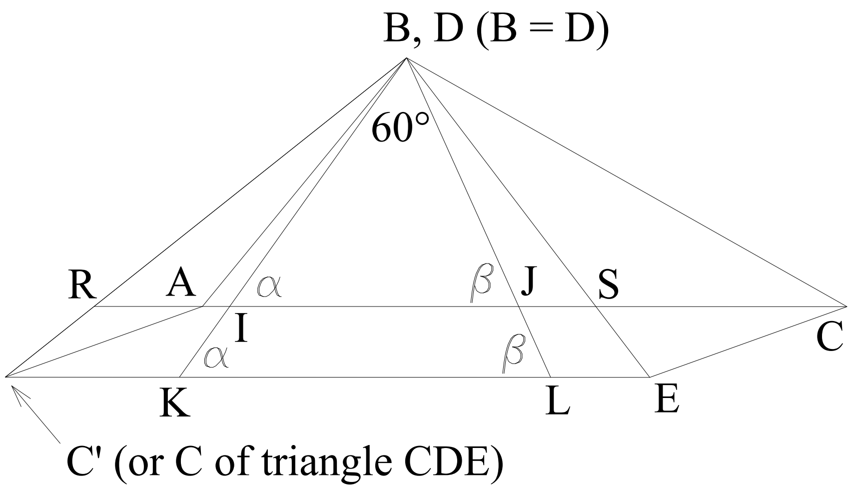
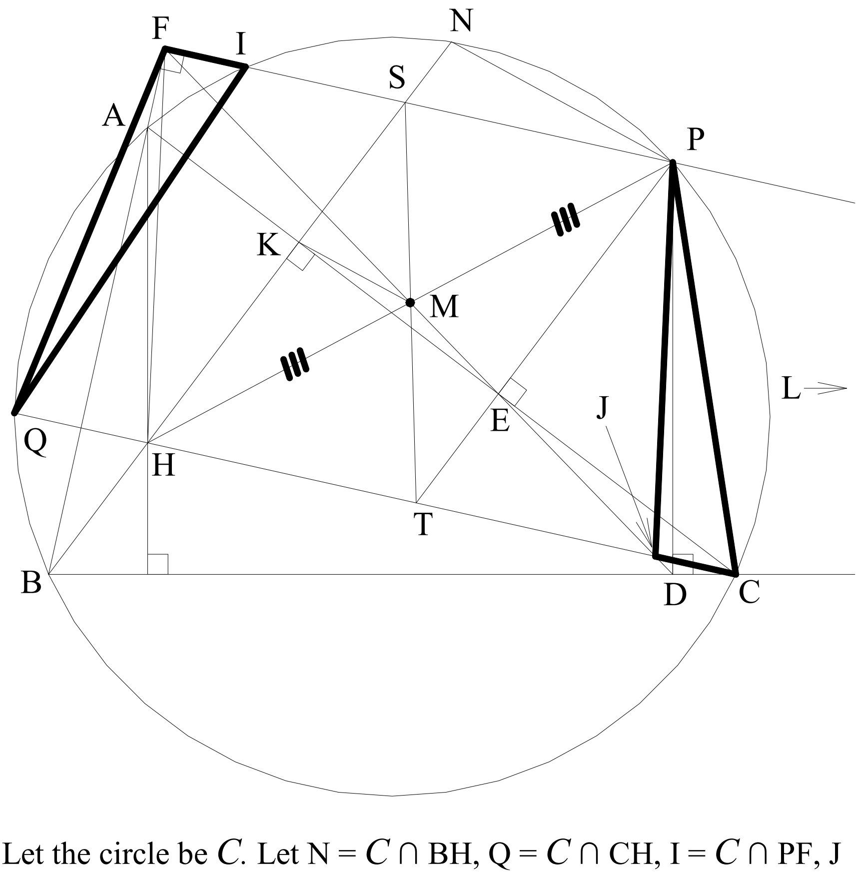


Figure 3

Problem 5 of Taiwan Mathematical Olympiad 1995

Let P be a point on the circumscribed circle of ΔABC and H be the orthocenter of ΔABC. Also let D, E and F be the points of intersection of the perpendicular from P to BC, CA and AB, respectively. It is known that the three points D, E and F are colinear. Prove that the line DEF passes through the midpoint of the line segment PH.

Solution



Let the circle be $\mathcal{C}$. Let N = $\mathcal{C}$ ∩ BH, Q = $\mathcal{C}$ ∩ CH, I = $\mathcal{C}$ ∩ PF, J = QC ∩ DF, S = BN ∩ PF, K = BN ∩ AC, T = PE ∩ QC, M = PH ∩ DF and L = FP ∩ BC (shown with arrow to the right).

Since CH ⊥ AB and FP ⊥ AB, FP || QC and PC = QI and QIPC is

an isosceles trapezoid, and $\angle$QIP = $\angle$IPC, or $\angle$FIQ = $\angle$PCJ. Furthermore, since BIPC and BFPD are cyclic, we have LP×LI = LC×LB and LP×LF = LD×LB, or
$\frac{LI}{LF} = \frac{LC}{LD}$, or IC || FD.

Coupling with FP || QC, FICJ is a parallelogram, and FI = JC. Combining with PC = QI and $\angle$FIQ = $\angle$PCJ proven earlier to get ΔFIQ = ΔJCP, or QF = PJ.

However, point Q on the circumcircle $\mathcal{C}$ of ΔABC is the image of of its orthocenter H across AB by definition, and F is on the extension of BA, QF = FH. Therefore, FH = PJ, and FPJH is a parallelogram.

Hence, M = FJ ∩ PH is the midpoint of PH, and line DEF passes through the midpoint of the line segment PH.

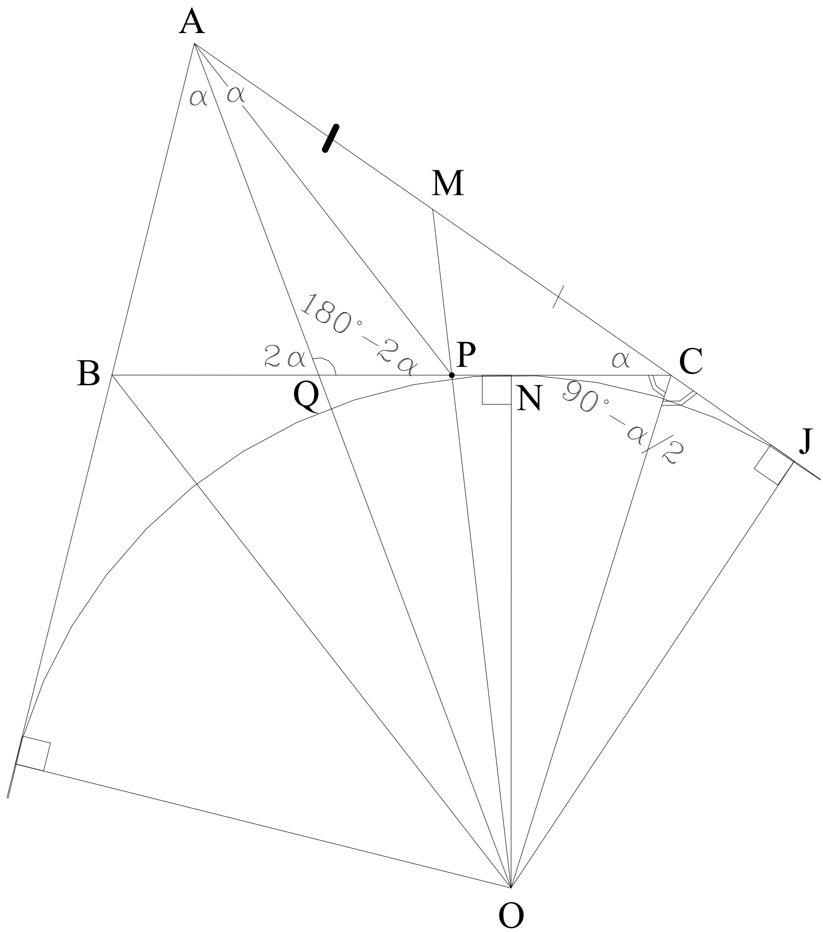
<u>Further observation</u>

Because FP || QC and BN || PT, the problem can be proven by proving that the line DEF passes through the midpoint of the line segment ST, and yet another way to prove the problem is to show that KM || NP since K is the midpoint of HN, thus M is the midpoint of PH.

Problem 4 of Taiwan Winter Camp 2001

Let O be the center of excircle of ΔABC touching the side BC externally. Let M be the midpoint of AC, P the intersection point of MO and BC. Prove that AB = BP, if ∠BAC = 2∠ACB.

Solution



Since O is the circumcenter of the external circle, AO is the angle bisector of ∠BAC. Let Q = AO ∩ BC, J be the foot of O on AC and let ∠BAQ = α. We then also have ∠CAQ = ∠ACB = α, ∠AQB = ∠OQC = 2α. And since OC is also the angle bisector of ∠BCJ, ∠BCO = 90° − α/2.

Applying the law of sines for ΔAQC to get

$$\frac{AQ}{\sin\alpha} = \frac{AC}{\sin(180° - 2\alpha)} = \frac{AC}{\sin 2\alpha}, \text{ or } \frac{AQ}{AC} = \frac{\sin\alpha}{\sin 2\alpha} \qquad \text{(i)}$$

Now for ΔAMO, we have

$$\frac{AM}{\sin\angle QOP} = \frac{MO}{\sin\alpha}, \text{ or } \frac{AM}{MO} = \frac{\sin\angle QOP}{\sin\alpha} \qquad \text{(ii)}$$

For ΔCMO, we have $\dfrac{MC}{\sin\angle COP} = \dfrac{MO}{\sin(\alpha + \angle QCO)} =$

$$\frac{MO}{\sin(90° + \alpha/2)} = \frac{MO}{\cos\alpha/2}, \text{ or } \frac{MC}{MO} = \frac{\sin\angle COP}{\cos\alpha/2} \qquad \text{(iii)}$$

With $AM = MC$, equations (ii) and (iii) give us

$$\frac{\sin\angle QOP}{\sin\alpha} = \frac{\sin\angle COP}{\cos\alpha/2}, \text{ or } \frac{\sin\angle QOP}{\sin\angle COP} = \frac{\sin\alpha}{\cos\alpha/2} \qquad \text{(iv)}$$

For ΔQPO, we have

$$\frac{QP}{\sin\angle QOP} = \frac{PO}{\sin 2\alpha}, \text{ or } \sin\angle QOP = \frac{QP\times\sin 2\alpha}{PO} \qquad \text{(v)}$$

For ΔCPO, we have $\dfrac{CP}{\sin\angle COP} = \dfrac{PO}{\sin(90° - \alpha/2)} = \dfrac{PO}{\cos\alpha/2}$, or

$$\sin\angle COP = \frac{CP\times\cos\alpha/2}{PO} \qquad \text{(vi)}$$

From (v) and (vi), we obtain $\dfrac{\sin\angle QOP}{\sin\angle COP} = \dfrac{QP\times\sin 2\alpha}{CP\times\cos\alpha/2}$ (vii)

From (iv) and (vii), we have $\dfrac{\sin\alpha}{\cos\alpha/2} = \dfrac{QP\times\sin 2\alpha}{CP\times\cos\alpha/2}$, or

$$\frac{QP}{CP} = \frac{\sin\alpha}{\sin 2\alpha} \qquad \text{(viii)}$$

Now from (i) and (viii), $\dfrac{AQ}{AC} = \dfrac{QP}{CP}$ implying that $\angle QAP = \angle CAP$, and $\angle BAP = \alpha + \angle QAP = \alpha + \angle CAP = \angle BPA$, or ABP is an isosceles triangle and $AB = BP$.

Problem 9 of the British Mathematical Olympiad 1999

Consider all numbers of the form $3n^2 + n + 1$, where n is a positive integer.
a) How small can the sum of the digits (in base 10) of such a number be?
b) Can such a number have the sum of its digits (in base 10) equal to 1999?

Solution

a) Let such a number be N. First of all the sum of the digits (let's denote it S) can not be zero because the number then is zero. Observe that $3n^2 + n + 1 = n(3n + 1) + 1$ is an odd number because $n(3n + 1)$ is an even number.

Now assume S = 1, then one scenario is $n = 0$, but this is not allowed because n is required to be positive by the problem.
So if S = 1, N must have the form of N = 10…0 (all zeros after the first digit 1). In such a case $3n^2 + n = n(3n + 1) =$ 9…9 (all digits are 9's) which is not allowed since $n(3n + 1)$ is an even number.

Now assume S = 2; N is now in the form of N = 10…01 (again because N is odd). We then have $N - 1 = 3n^2 + n = n(3n + 1) =$ 10…0. Since n and $3n + 1$ must not be both even or odd, the possible scenarios for N – 1 are N – 1 = 5×2, 25×4, 25×4, 20…0×5, 40…0×25… for which we find no solutions.

Now assume S = 3; N is now in the form of N = 20…01. We then have $N - 1 = 3n^2 + n = n(3n + 1) =$ 20…0. The possible scenarios for N – 1 are N – 1 = 8×25 = 8(3×8 + 1), and $n = 8$. Thus, the sum of the digits is as small as 3.

b) Let $n =$ 3…3 (666 numbers 3's), $3n + 1 =$ 10…0 (666 numbers 0's). $3n^2 + n + 1 =$ 3…30…01 (666 numbers 3's followed by 665 numbers 0's), and such number has the sum of its digits equal to 1999.

Problem 6 of Uruguay Mathematical Olympiad 2009

Is the sum $1^{2009} + 2^{2009} + 3^{2009} + ... + 2008^{2009}$ divisible by 7?

Solution

We can group the expression as
$1^{2009} + 2008^{2009} + 2^{2009} + 2007^{2009} + 3^{2009} + 2006^{2009} + \ldots +$
$1004^{2009} + 1005^{2009} = [(1^7)^7]^{41} + [(2008^7)^7]^{41} + [(2^7)^7]^{41} +$
$[(2007^7)^7]^{41} + [(3^7)^7]^{41} + [(2006^7)^7]^{41} + \ldots + [(1004^7)^7]^{41} +$
$[(1005^7)^7]^{41}$.

Now observe that $x^{41} + y^{41} = (x + y)(x^{40}y^0 - x^{39}y^1 + x^{38}y^2 - x^{37}y^3$
$+ \ldots + x^2y^{38} - x^1y^{39} + x^0y^{40})$.

Hence, $[(a^7)^7]^{41} + [(b^7)^7]^{41}$ has one of its two factor being $(a^7)^7 + (b^7)^7$, and so on $(a^7)^7 + (b^7)^7$ has one of its two factor being $a^7 + b^7$.

We also have $a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 -$
$ab^5 + b^4)$.

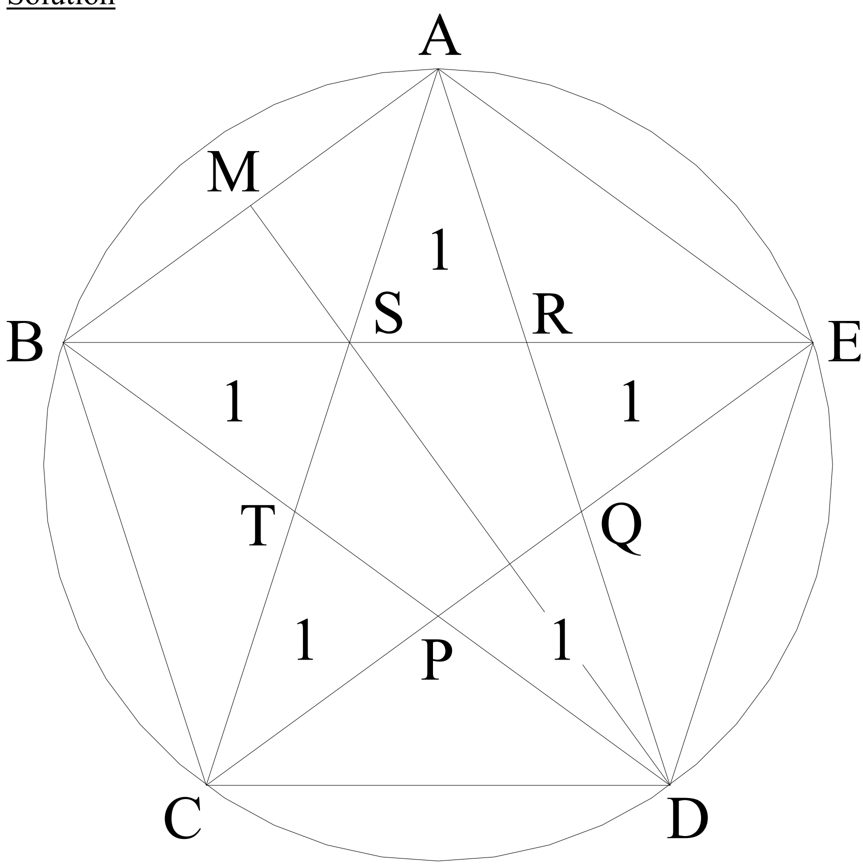
Therefore, the above expression has 1004 pairs of sum of two numbers that have 2009 as their common factor ($a + b = 2009$), and $\frac{2009}{7} = 287$, and thus the whole expression is divisible by 7.

<u>Problem 3 of the Japanese Mathematical Olympiad 1995</u>

In a convex pentagon ABCDE, let S, R, T, P and Q be the intersections of AC and BE, AD and BE, AC and BD, CE and BD, CE and AD, respectively. If all of ΔASR, ΔBTS, ΔCPT, ΔDQP and ΔERQ have the area of 1, then find the area of the following pentagons

a) The pentagon PQRST.
b) The pentagon ABCDE.

<u>Solution</u>



a) Let (Ω) denote the area of shape Ω. We have $(BTA) = 1 + (BSA) = (BRA)$, or the altitudes from T and R to AB have the same length, or RT || AB. Similarly, SQ || AE, RP || DE, TQ ||

CD and SP || BC.

Now let $a = \frac{\text{TD}}{\text{BD}}$, x = (STR), y = (ABS) and z = (PQRT).

It's easily seen that since TR || AB,

$a = \frac{\text{TD}}{\text{BD}} = \frac{\text{TR}}{\text{AB}} = \frac{\text{TS}}{\text{AS}} = \frac{\text{SR}}{\text{BS}} = \frac{\text{DR}}{\text{DA}}$, and

$\frac{x}{y} = a^2$, (i)

$\frac{(\text{ATR})}{(\text{ATB})} = \frac{x + (\text{ASR})}{y + (\text{BST})} = a$, or $\frac{x+1}{y+1} = a$, (ii)

$\frac{(\text{ATD})}{(\text{ABD})} = \frac{x + z + 2}{x + y + z + 3} = a$, (iii)

and $\frac{(\text{DTR})}{(\text{DBA})} = \frac{z + 1}{x + y + z + 3} = a^2$. (iv)

From (i) and (ii), $xy = 1$. Substituting this into (i), $ay = 1$, or $x = a$; therefore, $y = 1/a$.

Also from (iii) and (iv), $x + z + 2 = \frac{z+1}{a}$, or $a + z + 2 = \frac{z+1}{a}$, and $z = \frac{a^2 + 2a - 1}{1 - a}$.

Therefore, (PQRST) = $x + z = \frac{3a - 1}{1 - a}$.

Without loss of generality (WLOG), let $b = \frac{\text{CS}}{\text{CA}}$, if we had started out the process using the ratio $b = \frac{\text{CS}}{\text{CA}}$ instead of ratio $a = \frac{\text{TD}}{\text{BD}}$, (PQRST) = $\frac{3b - 1}{1 - b}$ and (ARE) = $1/b$.

So now $\frac{3a - 1}{1 - a} = \frac{2}{\frac{1}{a} - 1} - 1$, and $\frac{3b - 1}{1 - b} = \frac{2}{\frac{1}{b} - 1} - 1$, or $a = b$ and (ARE) = $1/a$. Writing a and b in ratios again to get $\frac{\text{TD}}{\text{BD}} = \frac{\text{CS}}{\text{CA}}$.

But $\frac{\text{TD}}{\text{BD}} = \frac{\text{DR}}{\text{DA}}$; therefore, $\frac{\text{CS}}{\text{CA}} = \frac{\text{DR}}{\text{DA}}$, or SR || CD.

Similarly, with the same argument, we have CE || AB || TR.

The parallel segments give us $\frac{\text{CE}}{\text{AB}} = \frac{\text{CS}}{\text{AS}}$, or

$\frac{(\text{CAE})}{(\text{ABC})} = \frac{(\text{BCS})}{(\text{ABS})}$, or $\frac{3 + \frac{3a-1}{1-a} + \frac{1}{a}}{1 + \frac{2}{a}} = \frac{1 + \frac{1}{a}}{\frac{1}{a}}$, or

$a^3 + 2a^2 - 1 = 0$. Solving this equation to get

$a_1 = \frac{1}{2\cos 36°}$,

$a_2 = -1$, and

$a_3 = -1 - \frac{1}{2\cos 36°}$ as solutions.

Only positive solution is acceptable, and we take $a = \frac{1}{2\cos 36°}$.

Therefore, the area of the pentagon PQRST is

$$(\text{PQRST}) = x + z = \frac{3a-1}{1-a} = \frac{3 - 2\cos 36°}{2\cos 36° - 1} = 2.24.$$

b) The area of the outer pentagon ABCDE, (ABCDE) is the sum of all the areas, and we have
(ABCDE) = 15.33.

Further observation

This is a difficult problem, and there was no solution for it in the web when this solution is release. Below is my further analysis of the problem.

By finding the ratio $a = \frac{TD}{BD} = \frac{1}{2\cos 36°}$ *which is the inverse of the area of y (area of triangle ABS), we conclude that the pentagon ABCDE is a regular pentagon, meaning that all its angles are equal* $\angle ABC = \angle BCD = \angle CDE = \angle DEA = \angle BAE = 108°$, *and all its sides have the same length; i.e.,* $AB = BC = CD = DE = AE$.

Subsequently, the segments at each vertex of pentagon ABCDE divide its angles equally; i.e., $\angle BAC = \angle CAD = \angle DAE = 108°/3 = 36°$ *which is the angle in solution for a, the ratio* $\frac{TD}{BD}$. *Following is yet more analysis:*

The parallel of the segments $SQ \parallel AE$, $RP \parallel DE$, $TQ \parallel CD$ *and* $SP \parallel BC$ *give us* $\frac{SA}{ST} = \frac{SB}{SR}$, $\frac{SR}{RE} = \frac{RQ}{RA}$, *or*

$$\frac{SA}{RA} = \frac{ST \times SB}{RE \times RQ} \qquad (v)$$

Similarly,

$$\frac{RE}{QE} = \frac{RA \times RS}{QD \times QP} \qquad (vi)$$

$$\frac{QD}{PD} = \frac{QR \times QE}{PT \times PC} \qquad (vii)$$

$$\frac{PC}{TC} = \frac{PD \times PQ}{TB \times TS} \qquad (viii)$$

$$\frac{TB}{SB} = \frac{TC \times TP}{SA \times SR}$$

Since $RT \parallel AB$, $\frac{RQ + QD}{TP + PD} = \frac{RA}{TB}$,

or

$$\frac{RQ}{RA} + \frac{QD}{RA} = \frac{TP}{TB} + \frac{PD}{TB}.$$

Substituting the values from (v), (vi), (vii) and (viii) above for the terms from left to right of the above equation, respectively, it becomes

$$\frac{ST \times SB}{SA \times RE} + \frac{QE \times SR}{RE \times PQ} = \frac{SA \times SR}{SB \times TC} + \frac{PC \times ST}{TC \times PQ}.$$

Or $$\frac{1}{RE}\left[\frac{ST \times SB}{SA} + \frac{QE \times SR}{PQ}\right] = \frac{1}{TC}\left[\frac{SA \times SR}{SB} + \frac{PC \times ST}{PQ}\right],$$

or $$\frac{1}{RE}\left[\frac{ST \times SB \times PQ + SA \times SR \times QE}{SA \times PQ}\right] =$$
$$\frac{1}{TC}\left[\frac{SA \times SR \times PQ + ST \times SB \times PC}{SB \times PQ}\right].$$

But $SA \times SR = ST \times SB$, and the above equation becomes

$$\frac{PQ + QE}{SA \times RE} = \frac{PQ + PC}{SB \times TC}, \text{ or } \frac{PE}{QC} = \frac{(PDE)}{(QDC)} = \frac{SA \times RE}{SB \times TC} = \frac{(RDE)}{(TDC)}.$$

From D draw the altitudes DH and DK to BE and AC, respectively. We have $\frac{(RDE)}{(TDC)} = \frac{DH \times RE}{DK \times TC}$, *and*

$SA \times DK = SB \times DH$, *or* $(SDA) = (SDB)$.

Hence, DS is also the median of triangle ABD. Extending DS to meet AB at M, we have AM = BM.

Obviously, the finding of ABCDE being a regular pentagon makes this point moot.

The following problem is derived from the above problem:

Express the solutions of equation $a^3 + 2a^2 - 1 = 0$ in terms of the trigonometric functions of an angle that has its degree as an integer.

Problem 2 of the Czech and Slovak Mathematical Olympiad 2002

Consider an arbitrary equilateral triangle KLM, whose vertices K, L and M lie on the sides AB, BC and CD, respectively, of a given square ABCD. Find the locus of the midpoints of the sides KL of all such triangles KLM.

Solution

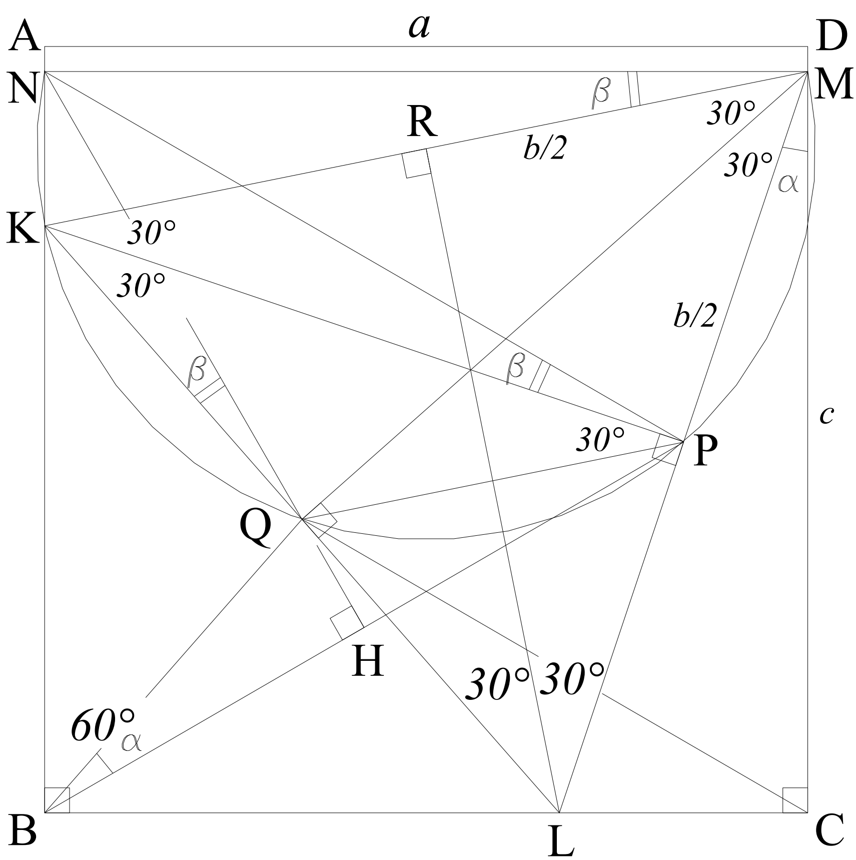


Figure 1

Let the side length of the square ABCD = a, the side length of equilateral triangle KLM = b, N a point on AB such that NM || AD, $\angle$LMC = α, $\angle$NMK = β. Now let P, Q and R be the midpoints of LM, KL and KM, respectively.

Since BKPL, KQPM and NQPM are cyclic, $\angle$KBP = $\angle$KLM = 60°, $\angle$BNP = $\angle$KMP = 60°, $\angle$QNM = $\angle$QLM = 60° and $\angle$QNP = $\angle$QMP = 30°, and thus BNP is an equilateral triangle and NQ is the angle bisector of $\angle$BNP implying QP = QB, $\angle$QPB = $\angle$QBP and NQ $\perp$ BP. Now extend NQ to meet BP at H. We now have BN = NP = BP = MC, and $\angle$QNB = $\angle$QNB = $\angle$MNP = 30°.

Notice that QMCL is also cyclic which causes $\angle$QCM = $\angle$QLM = 60°.

Therefore, point Q lies on the fixed straight line that contains QC. To find the locus we use the worst scenario where point M is at D as in figure 2. Let E, F, N and N' be the midpoints of AD, BC, CD and AB, respectively.

We have $\angle$LMC + $\angle$AMK ($\alpha + \beta$) = 90° – $\angle$KML = 30°, but in triangle AHP in figure 2, we also have $\angle$QAP (30°) + $\angle$APK (β) + $\angle$KPQ (30°) + $\angle$QPB ($\angle$QBP) = 90°, or $\angle$QPB + β = 30°, or $\angle$QPB = α = $\angle$LMC.

Also note that since BQ is the median of triangle KBL, BQ = ½KL = ½b = MP, or the two triangles BQH and MPN are congruent which causes QH = NP.

However, NP = a – N'P (the altitude of the equilateral triangle APB), or NP = $a - \frac{a\sqrt{3}}{2} = \frac{a(2-\sqrt{3})}{2}$, and thus QH = $\frac{a(2-\sqrt{3})}{2}$, and AQ = AH – QH = $\frac{a\sqrt{3}}{2} - \frac{a(2-\sqrt{3})}{2} = a(\sqrt{3} - 1)$.

Therefore, the locus lies on the straight line that contains QC and is from point Q on line AQ such that $\angle$BAQ = 30° and AQ = $a(\sqrt{3} - 1)$ to point P' which is the mirror image of P across the vertical axis EF of the square ABCD, or P' is a distance of $a(2-\sqrt{3})/2$ on the horizontal line away from the midpoint of AB on its left side.

The reason point P', the mirror image of P across EF, is used for the other end of the locus is that when we flip triangle KLM vertically, with respect to EF, we create the other worst case (or best case depending on how we look at it), and point Q will be at P. But since we only consider one side of the configuration, we pick point P' for the other end of the locus.

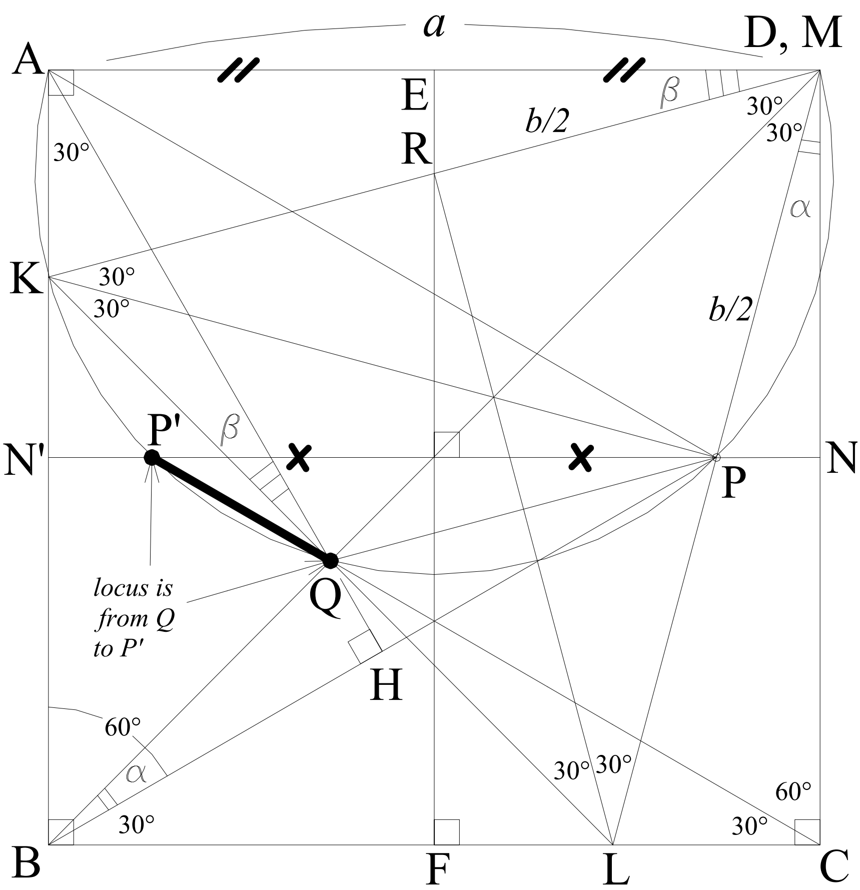
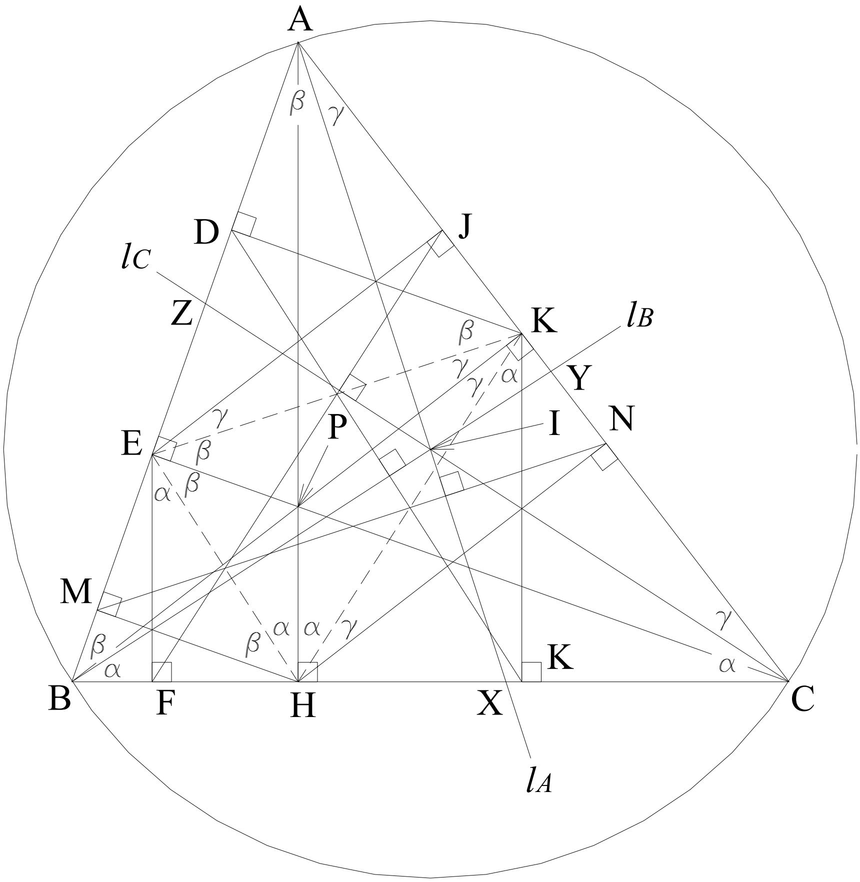


Figure 2

<u>Iceland's problem for International Mathematical Olympiad</u>

For an acute triangle ABC, let H be the foot of the perpendicular from A to BC. Let M, N be the feet of the perpendicular from H to AB, AC, respectively. Define l_A to be the line through A perpendicular to MN and similarly define l_B and l_C. Show that l_A, l_B and l_C pass through a common point O. (*This problem was proposed by Iceland and was never chosen for testing by the IMO organization*.)

<u>Solution</u>



Let K and E be the feet of B and C on AC and AB, respectively, D and L be the feet of K on AB and BC, respectively, F and J be the

feet of E on BC and AC, respectively, M and N be the feet of H on AB and AC, respectively, $X = l_A \cap BC$, $Y = l_B \cap AC$, $Z = l_C \cap AB$, P be the orthocenter of triangle ABC, $\alpha = \angle EHA$, $\beta = \angle KEC$ and $\gamma = \angle EKB$.
By definition of a triangle and because of the parallel segments, we also have
$\alpha = \angle AHK = \angle FEH = \angle HKL$,
$\beta = \angle CEH = \angle MHE = \angle EKD$ and
$\gamma = \angle BKH = \angle KHN = \angle JEK$.
Since MH and CE are perpendicular to AB, MH || EP, and we have $\frac{AE}{AM} = \frac{AP}{AH}$. Also since BK and HN perpendicular to AC, PK || HN, and we have AP/AH = AK/AN. The last two equalities give us AE/AM = AK/AN, or EK || MN, or $l_A \perp EK$, and $\angle BAX = \beta$, $\angle CAH = \gamma$.
Similarly, $\angle CBY = \alpha$, $\angle ABY = \beta$, and $\angle ACZ = \gamma$, $\angle BCZ = \alpha$.

Applying the law of sines to triangles ABX and ACX to get
$\frac{BX}{\sin\beta} = \frac{AX}{\sin\angle ABC}$, $\frac{CX}{\sin\gamma} = \frac{AX}{\sin\angle ACB}$, or $\frac{BX}{CX} = \frac{\sin\beta\times\sin\angle ACB}{\sin\gamma\times\sin\angle ABC}$.

Similarly, for the other triangles
$\frac{CY}{AY} = \frac{\sin\alpha\times\sin\angle BAC}{\sin\beta\times\sin\angle ACB}$ and $\frac{AZ}{BZ} = \frac{\sin\gamma\times\sin\angle ABC}{\sin\alpha\times\sin\angle BAC}$.
Multiply the last three equalities, we get $\frac{BX}{CX}\times\frac{CY}{AY}\times\frac{AZ}{BZ} =$
$\frac{\sin\beta\times\sin\angle ACB}{\sin\gamma\times\sin\angle ABC}\times\frac{\sin\alpha\times\sin\angle BAC}{\sin\beta\times\sin\angle ACB}\times\frac{\sin\gamma\times\sin\angle ABC}{\sin\alpha\times\sin\angle BAC} = 1$.

Therefore, AX, BY and CZ (or l_A, l_B and l_C) are concurrent per Ceva's theorem. Let them meet at a point I.
Note that $\alpha = \angle IBC = \angle ICB$, $\beta = \angle IAB = \angle IBA$ and $\gamma = \angle IAC = \angle ICA$ make the three sides IA = IB = IC, and I is also the circumcenter of triangle ABC, or I = O which is the common designation for the circumcenter of a circle as is done in this problem.

Problem 3 of Hong Kong Mathematical Olympiad 2008

For arbitrary real number x, define $[x]$ to be the largest integer less than or equal to x. For instance, $[2] = 2$ and $[3.4] = 3$. Find the value of $[1.008^8 \times 100]$.

Solution

First get the square of 1.008; we have $1.008^2 = 1.016064$ which is smaller than 1.017 and greater than 1.016.
Or $1.017^4 > 1.008^8 > 1.016^4$.

We now pick these values of 1.017 and 1.016 with the smallest possible numbers after the decimal points to be able to perform manual multiplication.
We then have $1.017^4 = 1.0698$, and $1.016^4 = 1.0656$, or
$1.017^4 \times 100 = 106.98$, and $1.016^4 \times 100 = 106.56$.

Therefore, $106.98 > 1.008^8 \times 100 > 106.56$. In other words, $1.008^8 \times 100$ is in the range of $(106.56, 106.98)$, and $[1.008^8 \times 100] = 106$.

Problem 6 of Hong Kong Mathematical Olympiad 2007

If R is the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7, find the value of R.

Solution

Let $S = 1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$. Denote $R[\frac{S}{7}]$ the remainder of S divided by 7.

But there exists the formula

$a^6 + b^6 = (a + b)(a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5) + 2b^6 = (a + b)F + 2b^6$ where F is the factor equals $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5$.

Therefore,

$6^6 + 1^6 = (6 + 1)F_1 + 2\times1^6 = 7F_1 + 2\times1^6$,

$5^6 + 2^6 = (5 + 2)F_2 + 2\times2^6 = 7F_2 + 2\times2^6$,

$4^6 + 3^6 = (4 + 3)F_3 + 2\times3^6 = 7F_3 + 2\times3^6$,

and $R(S/7) = R[2\times\frac{1^6 + 2^6 + 3^6}{7}]$, and the terms inside the bracket is now manually calculable; it is

$$R[2\times\frac{1^6 + 2^6 + 3^6}{7}] = R[\frac{1588}{7}] = 6.$$

So the remainder of $1^6 + 2^6 + 3^6 + 4^6 + 5^6 + 6^6$ divided by 7 is 6.

<u>Sample problem for the Irish Mathematical Olympiad</u>

Prove that, for every positive integer n which ends in the digit 5, $20^n + 15^n + 8^n + 6^n$ is divisible by 2009. (*This problem was just an example and has never yet been used in any competition.*)

<u>Solution</u>

Let the expression $20^n + 15^n + 8^n + 6^n$ be E.

E is equivalent to $E = 5^n\times4^n + 5^n\times3^n + 2^n\times4^n + 2^n\times3^n =$

$5^n(4^n + 3^n) + 2^n(4^n + 3^n) = (5^n + 2^n)(4^n + 3^n)$.

When n ends in digit 5, we can write $n = 5(2m + 1)$ where m is an integer. E then becomes

$E = [5^{5(2m+1)} + 2^{5(2m+1)}][4^{5(2m+1)} + 3^{5(2m+1)}]$.

But note that

$a^{2m+1} + b^{2m+1} = (a + b)(a^{2m}b^0 - a^{2m-1}b^1 + a^{2m-2}b^2 - \ldots + a^2b^{2m-2}$

$- a^1b^{2m-1} + a^0b^{2m}) = (a + b)F$ where

$F = (a^{2m}b^0 - a^{2m-1}b^1 + a^{2m-2}b^2 - \ldots + a^2b^{2m-2} - a^1b^{2m-1} + a^0b^{2m})$.

Therefore, $[5^{5(2m+1)} + 2^{5(2m+1)}] = (5^5 + 2^5)F_1 = 3157\times F_1$ and $[4^{5(2m+1)} + 3^{5(2m+1)}] = (4^5 + 3^5)F_2 = 1267\times F_2$.

And $E = 3157\times1267\times F_1\times F_2 = 2009\times1991\times F_1\times F_2$.

Therefore, $20^n + 15^n + 8^n + 6^n$ is divisible by 2009 for every positive integer n which ends in the digit 5.

Problem 10 of Hong Kong Mathematical Olympiad 2008

Let $[x]$ be the largest integer not greater than x. If $a = [(\sqrt{3} - \sqrt{2})^{2009}] + 16$, find the value of a.

Solution

Observe that $\sqrt{3} - \sqrt{2} = 1.732 - 1.414 = 0.318$, and the

exponent of a number smaller than 1, no matter to any power of, will always be less than 1, or $(\sqrt{3} - \sqrt{2})^{2009} < 1$, or

$[(\sqrt{3} - \sqrt{2})^{2009}] = 0$, and

$[(\sqrt{3} - \sqrt{2})^{2009}] + 16 = 16$, or $a = 16$.

Problem 3 of Hong Kong Mathematical Olympiad 2007

$208208 = 8^5a + 8^4 b + 8^3 c + 8^2 d + 8e + f$, where a, b, c, d, e and f are integers and $0 \leq a, b, c, d, e, f \leq 7$. Find the value of $a\times b\times c + d\times e\times f$.

Solution

Note that all the terms on the right side are positive, and $8^5\times 7 = 229376 > 208208$, or $a < 7$.

Now assuming that $a = 5$ and all other values b, c, d, e and f are equal to the maximum, or $b = c = d = e = f = 7$, the maximum of $8^5 \times 5 + 8^4 b + 8^3 c + 8^2 d + 8e + f = 163840 + 28672 + 3584 + 448 + 56 + 7 = 196607 < 208208$. Therefore, $a > 5$, and $\boldsymbol{a} \mathbf{= 6}$.

We then have

$8^4 b + 8^3 c + 8^2 d + 8e + f = 208208 – 196608 = 11600$, or

$8(8^3 b + 8^2 c + 8d + e) + f = 11600$, and we know
$8\times 1450 + 0 = 11600$, and $\boldsymbol{f} \mathbf{= 0}$.

From there, we have $8^3 b + 8^2 c + 8d + e = 1450$.

Now $b < 3$; let's pick $\boldsymbol{b} \mathbf{= 2}$, and we get

$8^2 c + 8d + e = 8(8c + d) + e = 1450 – 1024 = 426 = 8\times 53 + 2$,
or $\boldsymbol{e} \mathbf{= 2}$, and $8c + d = 53$, or $\boldsymbol{c} \mathbf{= 6}$ and $\boldsymbol{d} \mathbf{= 5}$.

These values of a, b, c, d, e and f satisfy $0 \leq a, b, c, d, e, f \leq 7$, and

the value of $a\times b\times c + d\times e\times f = 6\times 2\times 6 + 5\times 2\times 0 = 72$.

<u>Problem 8 of Hong Kong Mathematical Olympiad 2007</u>

Amongst the seven numbers 3624, 36024, 360924, 3609924, 36099924, 360999924 and 3609999924, there are *n* of them that are divisible by 38. Find the value of *n*.

<u>Solution</u>

Observe that all the given seven numbers are even, and 38 = 2×19; dividing the seven numbers by 2, they become 1812, 18012, 180462, 1804962, 18049962, 180499962 and 1804999962.

Note that to find if a number is divisible by 19 we multiply 14 by number of hundreds minus two last digit number. We have

1812 → 18×14 – 12 = 240, and this number is not divisible by 19.
18012→ 180×14 – 12 = 2508, ***2508***→ 25×14 – 8 = 342, ***342***→ 3×14 – 42 = 0, and this number is divisible by 19.
180462 → 180462 – 18012×10 (ten times the previous number is divisible by 19) = 342. As in previous case, it is divisible by 19.
1804962 → 1804962 – 1804620 = 342, and it is divisible by 19.
18049962 → 18049962 – 18049620 = 342, and it is divisible by 19.
180499962→180499962 –180499620 = 342, and it is divisible by 19.
1804999962 → 1804999962 –1804999620 = 342, and it is divisible by 19.
There are six of them that are divisible by 19, and *n* = 6.

<u>Further observation</u>

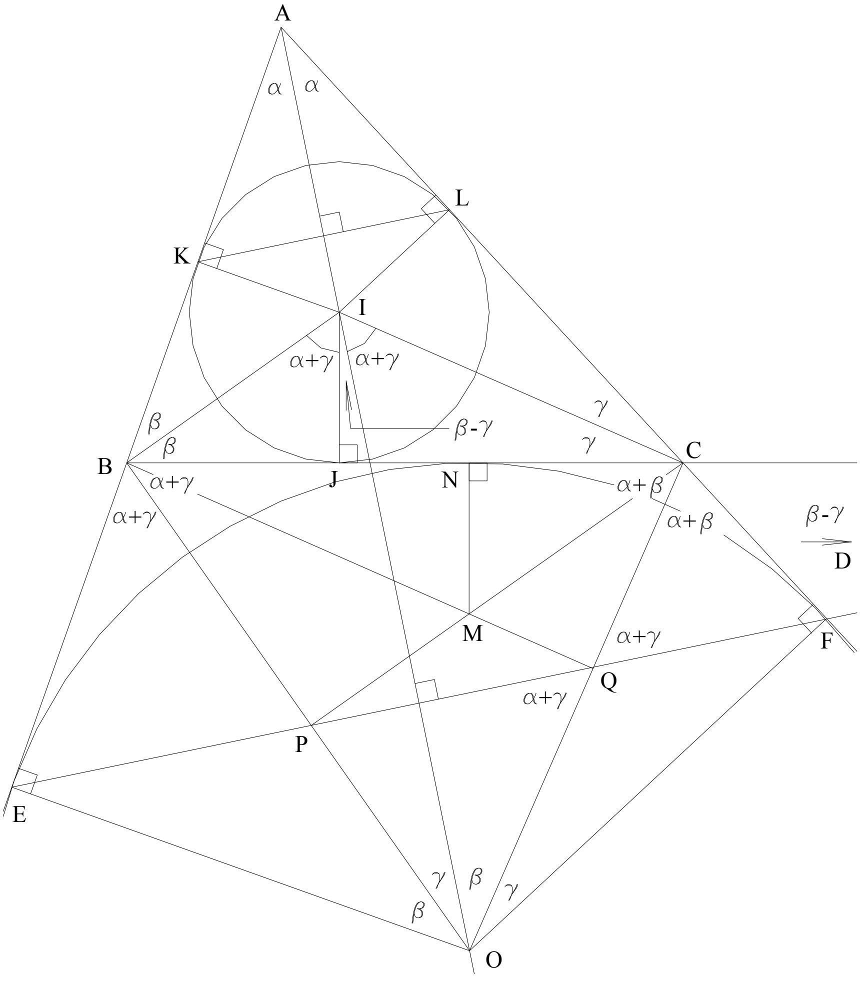
The reader should prove or disprove this statement on another method to verify if a number is divisible by 19:

To find if a number is divisible by 19 we add two times the last digit to the remaining leading truncated number. If the result is divisible by 19, then so is the first number. Apply this rule over and over again until we can verify it without resorting to a calculator.

<u>Problem 2 of the Iranian Mathematical Olympiad 2010</u>

Let O be the center of the excircle *C* of triangle ABC opposite vertex A. Assume *C* touches AB and AC at E and F, respectively. Let OB and OC intersect EF at P and Q, respectively. Let M be the intersection of CP and BQ. Prove that the distance between M and the line BC is equal to the inradius of ΔABC.

<u>Solution</u>



Let I be the incenter of triangle ABC, J and N be the feet of I and M on BC, respectively, D be the intersection of BC and EF (on the right out of the picture with arrow pointing to), α = ∠BAI =

$\angle CAI = \frac{1}{2}\angle BAC$, $\beta = \angle ABI = \angle CBI = \frac{1}{2}\angle ABC$ and $\gamma = \angle ACI = \angle BCI = \frac{1}{2}\angle ACB$.

Since O is the circumcenter of the excircle *C*, the three points A, I and O are on a straight line, and BO, CO are also the angle bisectors of $\angle EBC$ and $\angle FCB$, respectively. We then also have $\angle ICO = \angle BCI + \angle BCO = \frac{1}{2}180° = 90°$ (or $IC \perp CO$).

Similarly, $\angle IBO = 90°$ (or $IB \perp BO$) (i)
$\alpha + \gamma = \angle EBO = \angle OBC$, $\alpha + \beta = \angle BCO = \angle FCO$.

Because $\angle BIO = \angle BAI + \angle ABI = \alpha + \beta$, $\angle BIJ = 90° - \angle CBI = \alpha + \gamma$, $\angle JIO = \angle BIO - \angle BIJ = \alpha + \beta - (\alpha + \gamma) = \beta - \gamma$. Also because E and F are tangent points, we have AE = AF, and $EF \perp AO$. Combining with $IJ \perp BC$, $\angle CDF = \angle JIO = \beta - \gamma$.

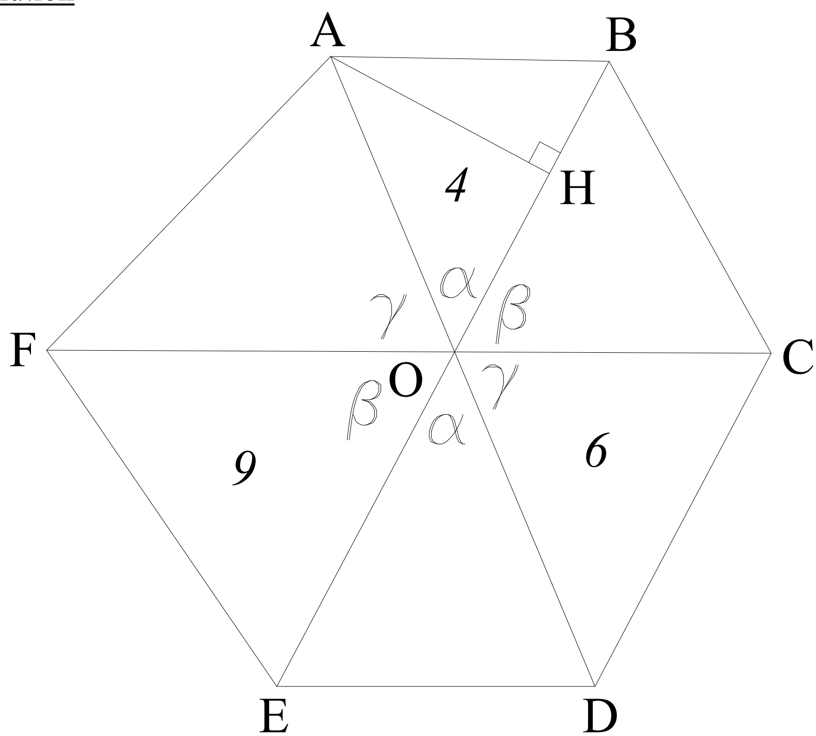
Now $\angle CQF = \angle BCO - \angle CDF = \alpha + \beta - (\beta - \gamma) = \alpha + \gamma$. This makes $\angle CQF = \angle CBP$, $\angle EBO = \angle EQO$ and both BQOE and BCQP cyclic. But since $\angle BEO = 90°$, $\angle BQO = 90°$ (or $BQ \perp CO$), and we have $\angle BPC = 90°$ (or $CP \perp BO$) as a result of BCQP being cyclic and $BQ \perp CO$.

Combining with (i), IC || BQ and IB || CP, or BICM is a parallelogram which makes IJ = MN = *r* which is the inradius of the triangle ABC.

<u>Problem 1 of Belarus Mathematical Olympiad 2004 Category B</u>

The diagonals AD, BE, CF of a convex hexagon ABCDEF meet at point O. Find the smallest possible area of this hexagon if the areas of the triangles AOB, COD, EOF are equal to 4, 6 and 9, respectively.

<u>Solution</u>



Let $\alpha = \angle AOB$, $\beta = \angle BOC$ and $\gamma = \angle COD$. We then also have $\angle DOE = \alpha$, $\angle EOF = \beta$ and $\angle AOF = \gamma$. Denote (Ω) the area of shape Ω. Draw the altitude AH from A onto OB. We have

$(AOB) = 4 = \frac{1}{2}AH \times OB = \frac{1}{2}OA \times OB \times \sin\alpha$, or $OA \times OB \times \sin\alpha = 8$.
Similarly $(COD) = 6$, or $OC \times OD \times \sin\gamma = 12$, and
$(EOF) = 9$, or $OE \times OF \times \sin\beta = 18$.

We also have $2(BOC) = OB \times OC \times \sin\beta$, $2(DOE) = OD \times OE \times \sin\alpha$

and 2(AOF) = OA×OF×sinγ.

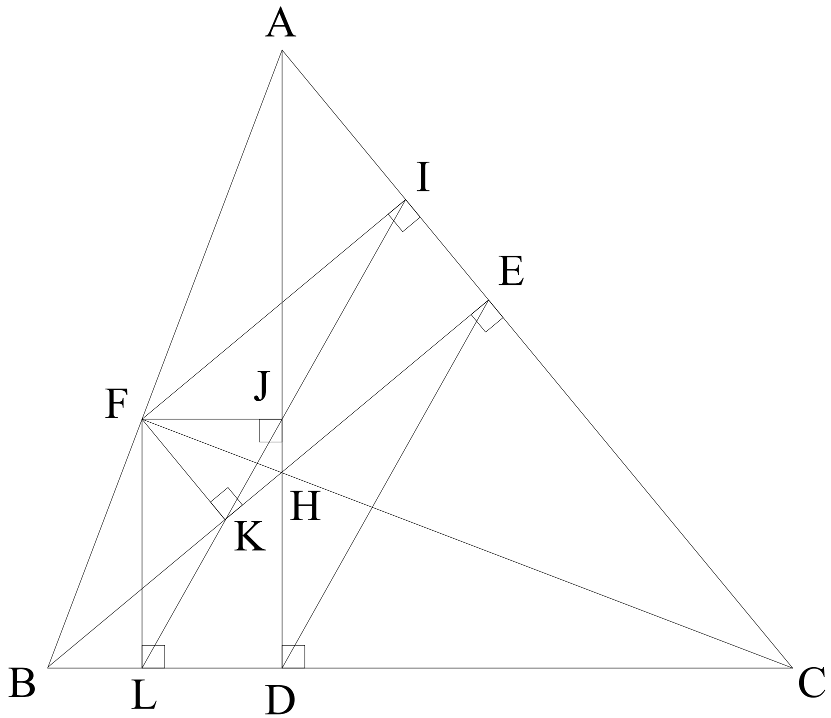
Per AM-GM inequality, 2(BOC) + 2(DOE) + 2(AOF) ≥ $3\sqrt[3]{OB\times OC\times \sin\beta\times OD\times OE\times \sin\alpha\times OA\times OF\times \sin\gamma}$, or
(BOC) + (DOE) + (AOF) ≥ $\frac{3}{2}\times$
$\sqrt[3]{OA\times OB\times \sin\alpha\times OC\times OD\times \sin\gamma\times OE\times OF\times \sin\beta} = \frac{3}{2}\times$
$\sqrt[3]{2(AOB)\times 2(COD)\times 2(EOF)}$ = 18.

The smallest possible area of (BOC) + (DOE) + (AOF) = 18. Therefore, the smallest possible area of this hexagon is the smallest possible area of (BOC) + (DOE) + (AOF) plus the areas of triangles AOB, COD, EOF = 18 + 4 + 6 + 9 = 37.

Problem 5 of Hong Kong Mathematical Olympiad 2007

AD, BE, and CF are the altitudes of an acute triangle ABC. Prove that the feet of the perpendiculars from F onto the segments AC, BC, BE and AD lie on the same straight line.

Solution



Let the feet of the perpendiculars from F onto the segments AC, BC, BE and AD be I, L, K and J, respectively. Also let H be the orthocenter of triangle ABC.

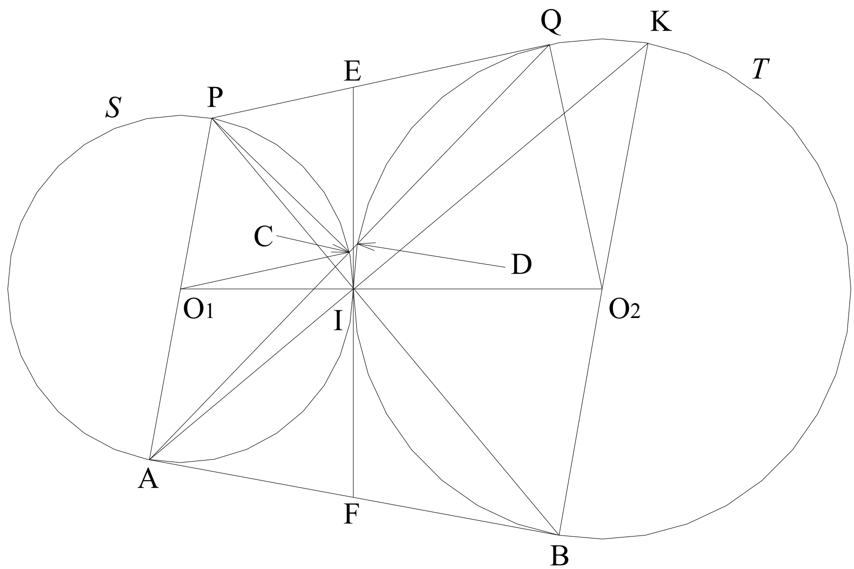
Since FI || HE and FL || HD, we have $\frac{CE}{CI} = \frac{CH}{CF}$ and $\frac{CD}{CL} = \frac{CH}{CF}$, or $\frac{CE}{CI} = \frac{CD}{CL}$, or DE || LI and $\angle$FIL =$\angle$BED.

On the other hand, since both AEDB and AIJF are cyclic and $\angle$BED = $\angle$BAD and $\angle$FIJ =$\angle$BAD, or $\angle$BED = $\angle$FIJ, or $\angle$FIL = $\angle$FIJ, or J is on LI. Similarly, $\angle$FLK = $\angle$FLI, or K is on LI. The four points L, K, J and I are collinear.

Problem 4 of the British Mathematical Olympiad 2006

Two touching circles S and T share a common tangent which meets S at A and T at B. Let AP be a diameter of S and let the tangent from P to T touch it at Q. Show that AP = PQ.

Solution



Let O_1 and O_2 be the circumcenters of the two circles *S* and *T*, respectively, and I the intersection of the vertical tangent of the two circles with O_1O_2 as shown. Let this vertical tangent meet PQ and AB at E and F, respectively.

Since AB is also tangent to both circles, AF = IF = BF and as a result, $\angle AIB = 90°$. On the other hand, AP is the diameter of *S* and $\angle PIA = 90°$, or P, I and B are collinear implying that $\angle O_1IP = \angle O_2IB$, or $\angle O_1PI = \angle O_2BI$ (since both O_1IP and O_2IB are isosceles triangles), or $AP \parallel O_2B$.

Now extend BO_2 to meet circle *T* at K. The three points A, I and K are also collinear since BK is the diameter and $\angle BIK = 90° = \angle AIB$.

Because both PQ and AB tangent *T*, we get $PQ^2 = PI \times PB$ and $AB^2 = AI \times AK$, or $\left(\frac{PQ}{AB}\right)^2 = \frac{PI \times PB}{AI \times AK}$.

But since AP || BK as proven earlier, $\frac{PI}{AI} = \frac{BI}{KI} = \frac{PB}{AK}$.

The previous equation becomes $\left(\frac{PQ}{AB}\right)^2 = \left(\frac{BI}{KI}\right)^2$, or $\frac{PQ}{AB} = \frac{BI}{KI}$.

We also have $\frac{AP}{AB} = \frac{BI}{KI}$ since he two triangles APB and IBK are similar. Therefore, AP = PQ.

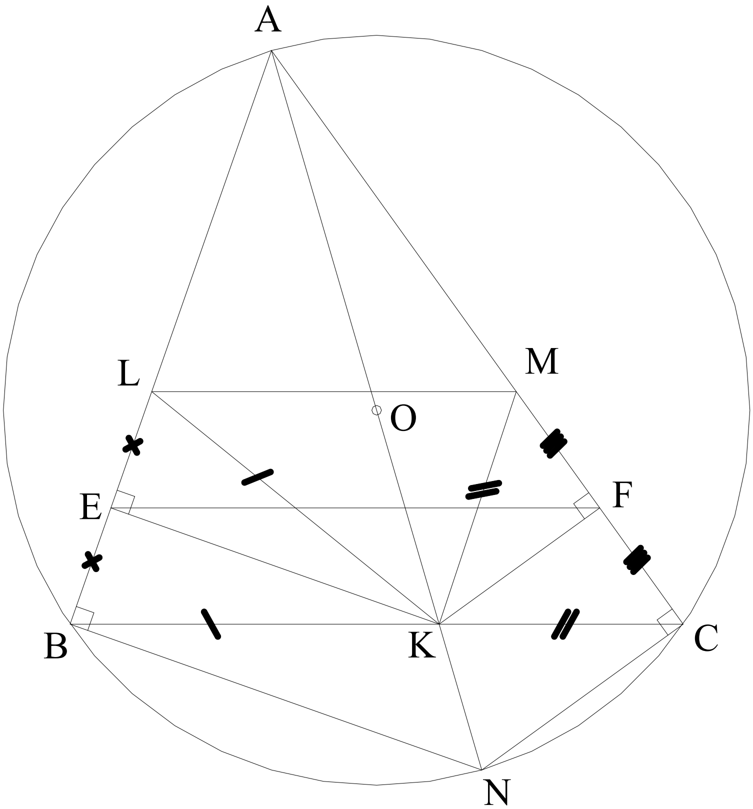
<u>Further observation</u>

Let $C = S \cap AQ$ and $D = T \cap AQ$; since $\angle PAQ = \angle PQA$ and PQ tangents T, we have $\frac{PC}{QD} = \frac{r}{R}$ where r and R are the radii of the circles S and T, respectively.

Problem 2 of the Estonian MO Team Selection Test 2004

Let O be the circumcenter of the acute triangle ABC and let lines AO and BC intersect at point K. On sides AB and AC, points L and M are chosen such that KL = KB and KM = KC. Prove that the segments LM and BC are parallel.

Solution



Extend AO to meet the circle at N. From K draw the altitudes KE and KF to sides AB and AC, respectively. Since AN is the diameter of the circle, NB $\perp$ AB, and NC $\perp$ AC.

Therefore, BN || EK and CN || FK implying that $\frac{AE}{AB} = \frac{AK}{AN}$ and $\frac{AK}{AN}$ $= \frac{AF}{AC}$, or $\frac{AE}{AB} = \frac{AF}{AC}$, or EF || BC and as a result $\frac{AE}{EB} = \frac{AF}{FC}$ (i)

But since KL = KB and KM = KC, the two triangles BKL and CKM are both isosceles, KE and KF are also their medians, respectively. Or EB = EL and FC = FM.

Equation (i) becomes $\frac{AE}{EL} = \frac{AF}{FM}$, *or LM || EF. But EF || BC, as proven earlier, LM || BC.*

<u>Problem 1 of Uruguay Mathematical Olympiad 2009</u>

What is the highest 8-digit number ending in 2009 and is a multiple of 99?

<u>Solution</u>

Let the number be N = $abcd2009$, or

N = $10000000a + 1000000b + 100000c + 10000d + 2009 = 99n$ where n is an integer.

N = $9999990a + 10a + 999999b + b + 99990c + 10c + 9999d + d + 1980 + 29 = 99(101010a + 10101b + 1010c + 101d + 20) + 10a + b + 10c + d + 29$.

Therefore, the remainder R = $10a + b + 10c + d + 29 = 10(a + c + 2) + b + d + 9$ must be divisible by 99.

The maximum value of $10(a + c + 2) + b + d + 9$ is 227, so the three numbers under 227 that are divisible by 99 are 0, 99 and 198, and since the units digit for 198 is 8 is less than and the last digit 9 of $10(a + c + 2) + b + d + 9$, the highest possible value for $a + c + 2$ is

$a + c + 2 = 18$, or $a + c = 16$, and the highest value for $\boldsymbol{a}$ is $\boldsymbol{a} = \mathbf{9}$ when $\boldsymbol{c} = \mathbf{7}$, and $b + d + 9 = 18$.

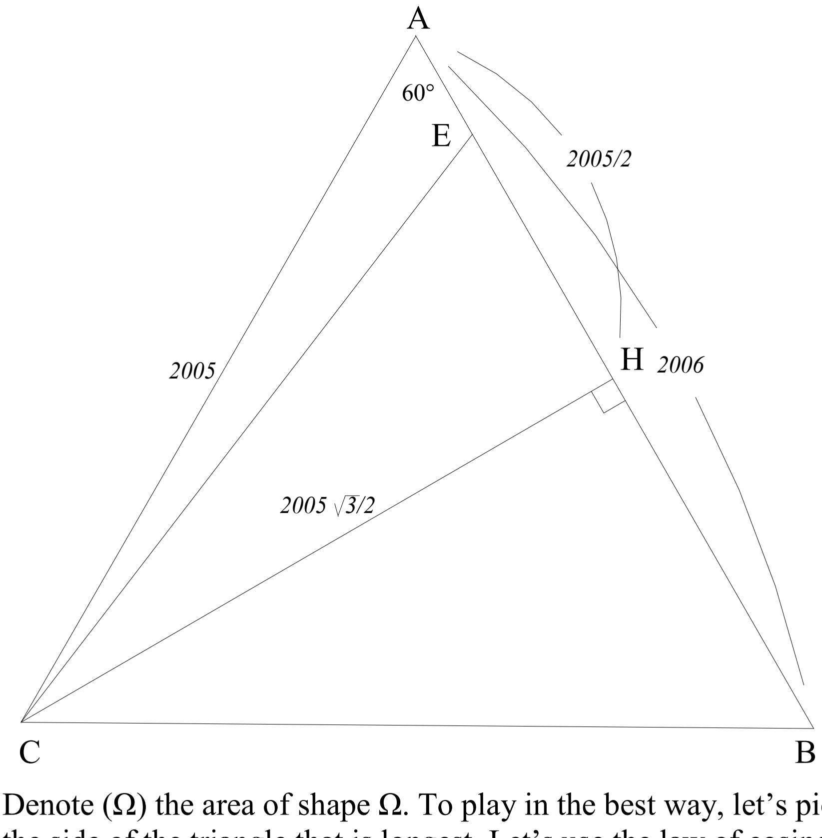
The highest value for b is $\boldsymbol{b} = \mathbf{9}$ and, in turn, $\boldsymbol{d} = \mathbf{0}$**.**

Answer: The highest 8-digit number ending in 2009 and is a multiple of 99 is 99702009.

<u>Problem 4 of Hong Kong Mathematical Olympiad 2007</u>

Given triangle ABC with $\angle A = 60°$, AB = 2005, AC = 2006. Bob and Bill in turn (Bob is the first) cut the triangle along any straight line so that two new triangles with area more than or equal to 1 appear. After that an obtused-angled triangle (or any of two right-angled triangles) is deleted and the procedure is repeated with the remained triangle. The player loses if he cannot do the next cutting. Determine, which player wins if both play in the best way.

<u>Solution</u>



Denote (Ω) the area of shape Ω. To play in the best way, let's pick the side of the triangle that is longest. Let's use the law of cosines to find the distance BC.

$BC^2 = AB^2 + AC^2 - 2AB \times AC \times \cos 60° = 2006^2 + 2005^2 -$

$2\times2006\times2005\times\frac{1}{2} = 2006^2 + 2005^2 - 2006\times2005$. From this, it's easily seen that $2005 < BC < 2006$, or $AC < BC < AB$. So let's pick side AB = 2006 to start the cutting, and to cut ABC into two triangles we have to cut through C. The best way is to cut so that (ACE) = 1.

Draw the altitude CH of triangle ABC. Since $\angle A = 60°$, AH = ½AC = $\frac{2005}{2}$ and CH = $\frac{2005\sqrt{3}}{2}$.

(ACE) = ½CH×AE = 1, or AE = $\frac{4}{2005\sqrt{3}}$.

As long as the cut is above the altitude CH with the distance of $\frac{4}{2005\sqrt{3}}$, the resulting triangle ACE will have the area equal to 1 and it has the obtuse angle AEC that can be discarded away. And we note that as soon as the cut through C goes below line CH, the bottom triangle will have the obtuse angle that must be discarded.

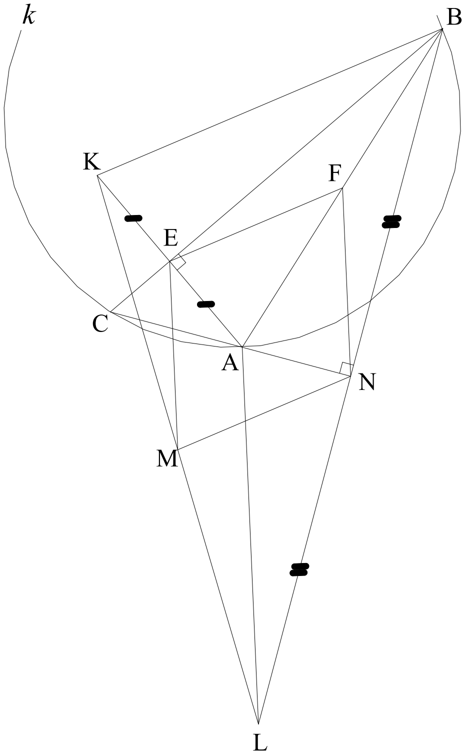
So the distance AH determines the number of cuts n, and $n = \frac{AH}{AE} = 2005\times\frac{2005\sqrt{3}}{8} = 870360.94$

Bob is the first to start, and he cuts the odd numbers of cuts; Bill cuts the even numbers of cuts. When Bob cuts the 870361st time, he encroaches into triangle CHB, discards the lower triangle since it has the obtuse angle and leaves the area of the last remaining triangle exactly equal to 1. He wins.

Problem 4 of the Czech-Polish-Slovak Math Competition 2009

Given a circle *k* and its chord AB which is not a diameter, let C be any point inside the longer arc AB of *k*. We denote by K and L the reflections of A and B with respect to the axes BC and AC. Prove that the distance of the midpoints of the line segments KL and AK is independent of the location of point C.

Solution



Let M, N, F and E be the midpoints of KL, BL, AB and AK, respectively; EF || BK, EF = ½BK, MN || BK, MN = ½BK. Therefore, MN || EF and MN = EF and MNFE is a parallelogram implying that ME || NF and ME = NF.
However, the two right triangles AEB and ANB share the same hypotenuse AB, and thus EF = NF = ½AB which is constant. Hence, MNFE is a rhombus and its side length, ME is one of them, is independent of the location of C.

<u>Problem 1 of the British Mathematical Olympiad 2006</u>

Find four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$.

<u>Solution</u>

Applying the formula $a^2 - b^2 = (a + b)(a - b)$ to get
$3^{32} - 2^{32} = (3^{16} + 2^{16})(3^8 + 2^8)(3^4 + 2^4)(3^2 + 2^2)(3 + 2)(3 - 2) =$
$(3^{16} + 2^{16})\times6817\times97\times13\times5\times1$.

Number 1 is not considered as a prime, whereas 5, 13 and 97 are prime numbers. We need to find another prime number as a factor of the product $(3^{16} + 2^{16})\times6817$.

Let's check the divisibility of the smaller number $3^8 + 2^8 = 6817$ by other prime numbers in the increasing order from 2 to 89 (97 is already found).

This number 6817 is odd and is not divisible by 2.
It's a sum of two numbers and the first number 3^8 is divisible by 3 while the second number 2^8 is not, and 6817 is not divisible by 3.
It does not end with 0 or 5 and is not divisible by 5.
It's not divisible by 7 since 5×number of hundreds (68) – 2 last digit number (17) is not divisible by 7; $68\times5 - 17 = 323$ is not divisible by 7.
It's not divisible by 11 since 10×number of hundreds (68) – 2 last digit number (17) is not divisible by 11; $68\times10 - 17 = 663$ is not divisible by 11.
The prime number 13 is also already found.
It's divisible by 17 since 2×number of hundreds (68) – 2 last digit number (17) is divisible by 17; $68\times2 - 17 = 119$ is divisible by 17.

Thus the four prime numbers less than 100 which are factors of $3^{32} - 2^{32}$ are 5, 13, 17 and 97.

Problem 5 of the British Mathematical Olympiad 2006

For positive real numbers a, b, c, prove that

$(a^2 + b^2)^2 \geq (a + b + c)(a + b - c)(b + c - a)(c + a - b)$.

Solution

$(a + b + c)(a + b - c) = (a^2 + b^2 - c^2 + 2ab)$,
$(b + c - a)(c + a - b) = -(a^2 + b^2 - c^2 - 2ab)$, and
$(a + b + c)(a + b - c)(b + c - a)(c + a - b) = -[(a^2 + b^2 - c^2)^2 - (2ab)^2]$.

So now we have to prove

$(a^2 + b^2)^2 \geq -[(a^2 + b^2 + c^2)^2 - (2ab)^2]$, or

$(a^2 + b^2)^2 \geq -(a^2 + b^2)^2 - c^4 - 2(a^2 + b^2)c^2 + 4a^2b^2$, or

$2(a^2 + b^2)^2 \geq -c^4 - 2(a^2 + b^2)c^2 + 4a^2b^2$, or

$2(a^4 + b^4) \geq -c^4 - 2(a^2 + b^2)c^2$ which is obvious since

$2(a^4 + b^4) \geq 0$, and $-c^4 - 2(a^2 + b^2)c^2 \leq 0$.

Equality occurs when $a = b = c = 0$.

<u>Problem 6 of the British Mathematical Olympiad 2006</u>

Let n be an integer. Show that, if $2 + 2\sqrt{1 + 12n^2}$ is an integer, then it is a perfect square.

<u>Solution</u>

$2 + 2\sqrt{1 + 12n^2} = 2(1 + \sqrt{1 + 12n^2})$, and for it to be a perfect square

$1 + \sqrt{1 + 12n^2} = 2m^2$ where m is an integer, or

$\sqrt{1 + 12n^2} = 2m^2 - 1$. Now squaring both sides, we get

$1 + 12n^2 = 4m^4 - 4m^2 + 1$, or $3n^2 = m^4 - m^2 = m^2(m^2 - 1)$, or

$3 \times n \times n = m \times m \times (m - 1)(m + 1)$.

Now if m is odd, both $m - 1$ and $m + 1$ are even and their difference is 2, and we must have $n^2 = (m - 1)(m + 1) = m^2 - 1$, or $m = 1$ and $\boldsymbol{n = 0}$.

Now if m is even, both $m - 1$ and $m + 1$ are odd and their difference is 2. Therefore, we have the following possibilities:

a) $3 \times 5 \times 5 = (m - 1)(m + 1) \times m \times m$, or $m - 1 = 3$, $m + 1 = 5$, or $m = 4$, and $3 \times 5 \times 5 = 3 \times 5 \times 4 \times 4$ which is not true.

b) $1 \times 3 \times n \times n = (m - 1)(m + 1) \times m \times m$, or $m = 2$ and $\boldsymbol{n = 2}$.

When $\boldsymbol{n = 0}$, $2 + 2\sqrt{1} = 4$ or 0 which are both perfect square.

When $\boldsymbol{n = 2}$, $2 + 2\sqrt{49} = 16$ or -12 of which -12 is not a perfect square.

<u>Further observation</u>

The problem should've stated that $2 + 2\sqrt{1 + 12n^2}$ is a positive integer instead of just being an integer.

Problem 1 of the British Mathematical Olympiad 2007

Find the value of $\dfrac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}$.

Solution

Let N = $\dfrac{1^4 + 2007^4 + 2008^4}{1^2 + 2007^2 + 2008^2}$. We have $1^4 + 2007^4 + 2008^4 =$ $(1^2 + 2007^2 + 2008^2)^2 - 2\times1^2\times2007^2 - 2\times1^2\times2008^2 - 2\times2007^2 \times2008^2 = (1^2 + 2007^2 + 2008^2)^2 - 2 + 2 - 2\times2007^2 - 2\times2008^2 - 2\times2007^2\times2008^2 = (1^2 + 2007^2 + 2008^2)^2 - 2(1^2 + 2007^2 + 2008^2) + 2 - 2\times2007^2\times2008^2 = (1^2 + 2007^2 + 2008^2)^2 - 2(1^2 + 2007^2 + 2008^2) - 2(2007^2\times2008^2 - 1)$.

Therefore,

$N = 1^2 + 2007^2 + 2008^2 - 2 - 2\times\dfrac{2007^2\times2008^2 - 1}{1^2 + 2007^2 + 2008^2} =$

$1^2 + 2007^2 + 2008^2 - 2 - 2\times\dfrac{(2007\times2008 - 1)(2007\times2008 + 1)}{1^2 + 2007^2 + 2008^2}$.

But $1^2 + 2007^2 + 2008^2 = 1^2 + 2007^2 + (2007 + 1)^2 = 1^2 + 2007^2 + 2007^2 + 2\times2007 + 1 = 2(2007^2 + 2007 + 1) = 2(2007\times2008 + 1)$, and N becomes

$N = 2(2007\times2008 + 1) - 2 - (2007\times2008 - 1) = 2007\times2008 + 1 =$ 4030057.

Further observation

This problem can be applied for any two consecutive years after number 1 or any two numbers to replace 2007 and 2008.

<u>Problem 2 of Pan African Mathematical Competition 2004</u>

Is $4\sqrt{4 - 2\sqrt{3}} + \sqrt{97 - 56\sqrt{3}}$ an integer?

<u>Solution</u>

We have $4 - 2\sqrt{3} = (1 - \sqrt{3})^2$ and $97 - 56\sqrt{3} = (7 - 4\sqrt{3})^2$.

Therefore, $\sqrt{4 - 2\sqrt{3}} = \pm(1 - \sqrt{3})$ and $\sqrt{97 - 56\sqrt{3}} = \pm(7 - 4\sqrt{3})$.

We can choose
$4\sqrt{4 - 2\sqrt{3}} = -4 + 4\sqrt{3}$ and $\sqrt{97 - 56\sqrt{3}} = 7 - 4\sqrt{3}$.

Thus $4\sqrt{4 - 2\sqrt{3}} + \sqrt{97 - 56\sqrt{3}} = 3$ which is an integer.

Problem 1 of the British Mathematical Olympiad 1993

Find, showing your method, a six-digit integer n with the following properties: (i) n is a perfect square, (ii) the number formed by the last three digits of n is exactly one greater than the number formed by the first three digits of n. (*Thus n might look like 123124, although this is not a square.*)

Solution

Let $n = abcdef = k^2$ where a, b, c, d, e, f and k are all integers; b, c, d, e, f are from 0 to 9; a is from 1 to 9 ($a \neq 0$, if $a = 0$ then it's only a five-digit integer).

Let's consider the case where $c \leq 8$. We have $n = abcab(c+1)$, and

$$100000a + 10000b + 1000c + 100a + 10b + c + 1 = k^2, \text{ or}$$

$$100100a + 10010b + 1001c = k^2 - 1, \text{ or}$$

$$1001(100a + 10b + c) = k^2 - 1 = (k-1)(k+1) \qquad \text{(i)}$$

$7\times11\times13\times(100a + 10b + c)$ is a product of two consecutive even numbers or consecutive odd numbers $k - 1$ and $k + 1$. Therefore, $100a + 10b + c$ must be a product of two numbers X and Y such that the product $7\times11\times13\times XY$ contains exactly two factors and their difference is equal to 2. Let's find those values for X and Y. One of the possible scenarios is that

$$7\times11X - 13Y = 2 \qquad \text{(ii)}$$
$$XY = 100a + 10b + c \qquad \text{(iii)}$$

From those two equations, X, Y and c must be all even or all odd. We first assume that they're all odd. Note that the units digit of k^2 are 0, 1, 4, 5, 6 or 9, and the units digit of $k^2 - 1$ are then 9, 0, 3, 4, 5 or 8 and with this assumption, $k^2 - 1$ is odd and its units digit are 9, 3 or 5. From (i), we know that this units digit is the same as the value of c; however, $c \leq 8$. Therefore, c is either 5 or 3.

Now let's proceed with $c = 5$. Let $X = 2m + 1$ and $Y = 2n + 1$ where m and n are both integers. Equation (iii) can now be written as

$$(2m + 1)(2n + 1) = 2(2mn + m + n) + 1 = 100a + 10b + c.$$

With $c = 5$, the units digit of $2mn + m + n$ must be 2 or 7. To satisfy this requirement one of the scenarios is for the units digit of m to be 5 and that of n to be 2.

Now rewrite (ii) as $7\times11\times(2m + 1) - 13\times(2n + 1) = 2$, or $26n - 154m = 62$.

Substituting $m = 5$ found above into this latest equation, we get $26n - 154\times5 = 62$, or $n = 32$. From there, $X = 11$ and $Y = 65$, $a = 7$, $b = 1$, $c = 5$, $n = 715716 = 846^2 = k^2$ and $k = 846$.

Answer: $n = 715716$.

Problem 4 of the Czech and Slovak Mathematical Olympiad 2002

Find all pairs of real numbers a, b for which the equation in the domain of the real numbers x

$$\frac{ax^2 - 24x + b}{x^2 - 1} = x$$

has two solutions and the sum of them equals 12.

Solution

Expanding the equation, we get $x^3 - ax^2 + 23x - b = 0$ (i)

Since it has two solutions and the sum of them equals 12, let the solutions be c and d; we have $(x - c)(x - d) = 0$, and $c + d = 12$, or

$x^2 - (c + d)x + cd = 0$. Now multiplying both sides by x,
$x^3 - (c + d)x^2 + cdx = 0$ (ii)

Equating (i) and (ii), we get
$a = c + d = 12$ (iii)
$cd = 23$ (iv)
$b = 0$.

Answer: $a = 12$, $b = 0$.

Now let's confirm! From (iii) and (iv), we get $c^2 - 12c + 23 = 0$, or

$(c, d) = (6 + \sqrt{13}, 6 - \sqrt{13})$, or

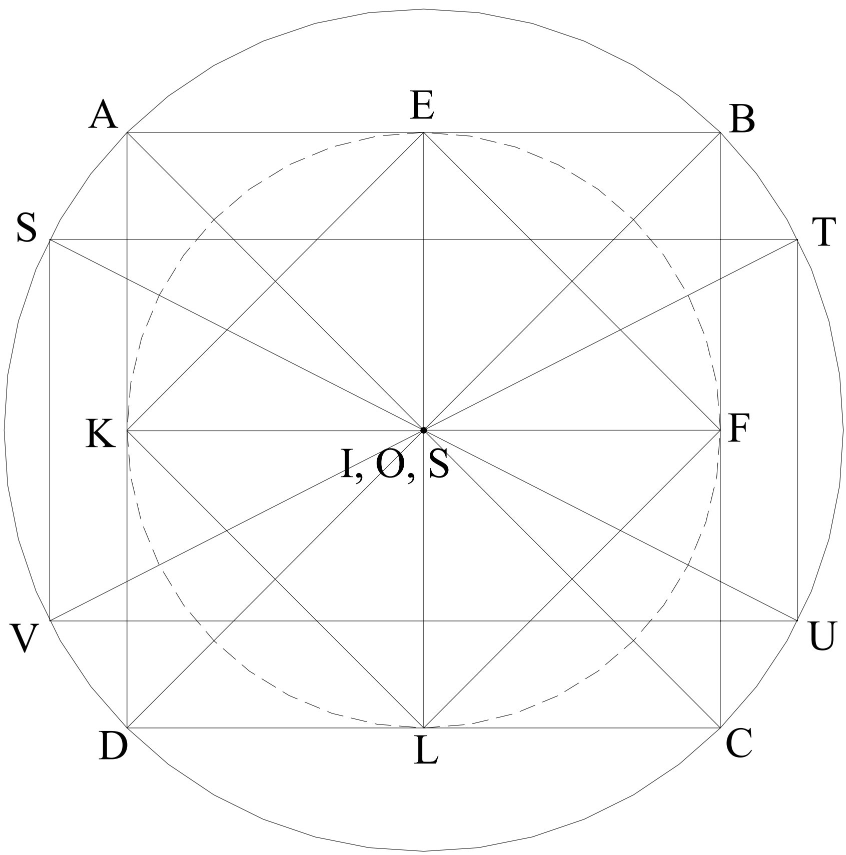
$(c, d) = (6 - \sqrt{13}, 6 + \sqrt{13})$,

and $c + d = 12$.

<u>*Problem 1 of the Brazilian Mathematical Olympiad 1995*</u>

ABCD is a quadrilateral with a circumcircle center O and an inscribed circle center I. The diagonals intersect at S. Show that if two of O, I, S coincide, then it must be a square.

<u>Solution</u>



First, assuming that **O ≡ I** (O coincides I). Let E, F, L and K be the feet of I onto AB, BC, CD and AD, respectively. Since I is the incenter, $\angle$IAE = $\angle$IAK, $\angle$IBE = $\angle$IBF, $\angle$ICF = $\angle$ICL and $\angle$IDL = $\angle$IDK. Moreover, since I is also the circumcenter, IA = IB = IC = ID, and all the triangles AIB, BIC, CID and AID are isosceles making all the eight angles above equal, and each is equal to $\frac{1}{8}$ of 360°, or 45°. This implies that all the angles of ABCD are right angles, and its diagonals also make a right angle, and thus it is a square.

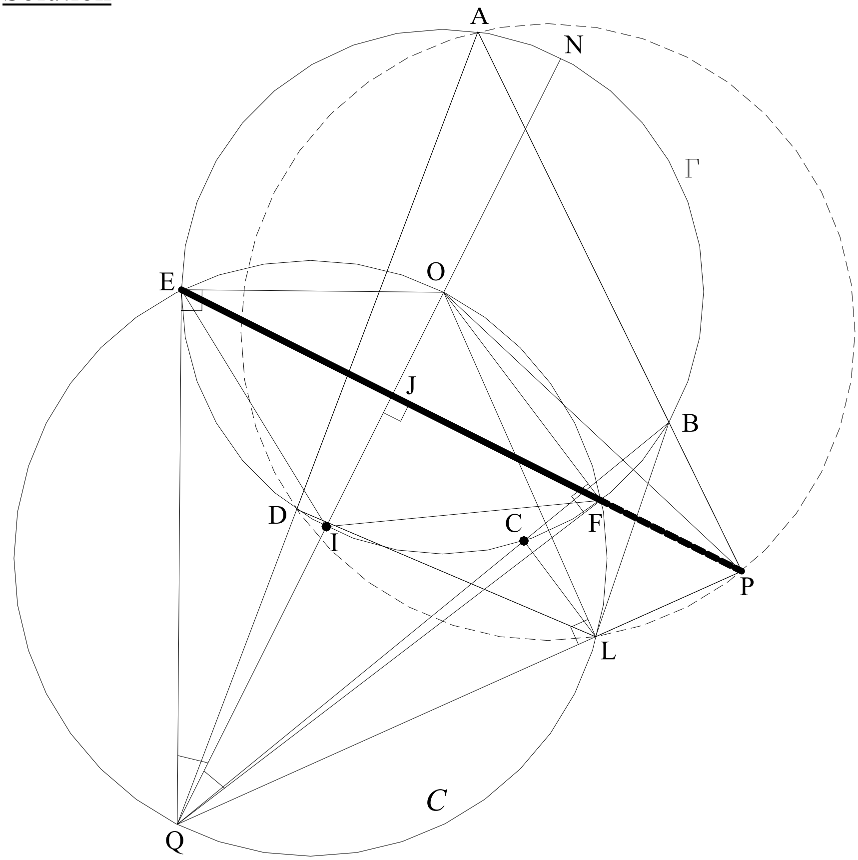
Next, assume that **O ≡ S**. It's easily seen that ABCD is a rectangle and *not necessarily* a square because a rectangle has its diagonals meet each other at a point equidistant to its vertices such as rectangle STUV on the graph.

Lastly, assume that **I ≡ S**. A, I, C and B, I, D are sets of collinear points. Again, since I is the incenter, $\angle IAE = \angle IAK$, $\angle IBE = \angle IBF$, $\angle ICF = \angle ICL$ and $\angle IDL = \angle IDK$. Furthermore, the opposite angles of a cyclic quadrilateral combine to be 180°, and $\angle BAD + \angle BCD = 180°$, or $\angle IAE + \angle IAK + \angle ICF + \angle ICL = 180°$, or $2(\angle IAE + \angle ICF) = 180°$, or $\angle IAE + \angle ICF = 90°$, and $\angle ABC = 180° – (\angle IAE + \angle ICF) = 90°$. Thus all the angles of ABCD are right angles making it a rectangle. Moreover, since the mid-segments EL and KF are also equal, ABCD is a square.

Problem 4 of China Mathematical Olympiad 1997

Let quadrilateral ABCD be inscribed in a circle. Suppose lines AB and DC intersect at P and lines AD and BC intersect at Q. From Q construct the two tangents QE and QF to the circle where E and F are the points of tangency. Prove that the three points P, E, F are collinear.

Solution



Let the circumcircle of ABCD be Γ, O be its center, Let I = OQ ∩ Γ and J = OQ ∩ EF. By definition I is the incenter of triangle QEF since QE and QF are tangents of Γ, and we also have $QO \perp EF$ and EJ = FJ. Since $\angle QEO = \angle QFO = 90°$, EOFQ is cyclic inscribed in a circle which we name *C*. Let L = QP ∩ *C*. Since QO is the diameter of *C*, $\angle QLO = 90°$, and thus OPLJ is cyclic and $QL \times QP = QJ \times QO$ (i)

It suffices to show that $PB \times PA = PL \times PQ$ for us to conclude that the three points E, F and P are collinear since then $PB \times PA = PL \times PQ = PF \times PE$, and E must be on the extension of EF.

But since ABCD is cyclic, $PB \times PA = PC \times PD$, or we need to prove that $PC \times PD = PL \times PQ$, or DCLQ is cyclic. To do this, we need to prove $\angle CLP = \angle ABC$, or BCLP is cyclic or $QL \times QP = QC \times QB$. Moreover, $QC \times QB = QD \times QA$, and combining with (i), we now must prove $QD \times QA = QJ \times QO$. Now extending QO to meet Γ at N, $QD \times QA = QI \times QN$. We need to prove $QI \times QN = QJ \times QO = QJ \times (QJ + JO) = QJ^2 + QJ \times QO$ (ii)

However, the intersecting of two segments EF and QO at J inside *C* gives us $QJ \times QO = EJ \times FJ = FJ^2$. Therefore, equation (ii) becomes $QI \times QN = QJ^2 + FJ^2 = QF^2$ (iii)

Again, since QF is tangent to Γ, and I and N are on Γ and on the straight line containing Q, equation (iii) is obvious, and the proof is complete.

Problem 5 of the Irish Mathematical Olympiad 1988

A person has seven friends and invites a different subset of three friends to dinner every night for one week (7 days). In how many ways can this be done so that all friends are invited at least once?

Solution 1

All possible combinations of seven friends numbered 1 to 7 being invited are as follows with 0 being not invited and 1 being invited

1	2	3	4	5	6	7	Combination Number	
0	0	0	0	0	0	0	1	
0	0	0	0	0	0	1	2	
0	0	0	0	0	1	0	3	
0	0	0	0	0	1	1	4	
0	0	0	0	1	0	0	5	
0	0	0	0	1	0	1	6	
0	0	0	0	1	1	0	7	
0	0	0	0	1	1	1	8	×
0	0	0	1	0	0	0	9	
0	0	0	1	0	0	1	10	
0	0	0	1	0	1	0	11	
0	0	0	1	0	1	1	12	×
0	0	0	1	1	0	0	13	
0	0	0	1	1	0	1	14	×
0	0	0	1	1	1	0	15	×
0	0	0	1	1	1	1	16	
0	0	1	0	0	0	0	17	
0	0	1	0	0	0	1	18	
0	0	1	0	0	1	0	19	
0	0	1	0	0	1	1	20	×
0	0	1	0	1	0	0	21	
0	0	1	0	1	0	1	22	×
0	0	1	0	1	1	0	23	×
0	0	1	0	1	1	1	24	

0	0	1	1	0	0	0	25	
0	0	1	1	0	0	1	26	×
0	0	1	1	0	1	0	27	×
0	0	1	1	0	1	1	28	
0	0	1	1	1	0	0	29	×
0	0	1	1	1	0	1	30	
0	0	1	1	1	1	0	31	
0	0	1	1	1	1	1	32	
0	1	0	0	0	0	0	33	
0	1	0	0	0	0	1	34	
0	1	0	0	0	1	0	35	
0	1	0	0	0	1	1	36	×
0	1	0	0	1	0	0	37	
0	1	0	0	1	0	1	38	×
0	1	0	0	1	1	0	39	×
0	1	0	0	1	1	1	40	
0	1	0	1	0	0	0	41	
0	1	0	1	0	0	1	42	×
0	1	0	1	0	1	0	43	×
0	1	0	1	0	1	1	44	
0	1	0	1	1	0	0	45	×
0	1	0	1	1	0	1	46	
0	1	0	1	1	1	0	47	
0	1	0	1	1	1	1	48	
0	1	1	0	0	0	0	49	
0	1	1	0	0	0	1	50	×
0	1	1	0	0	1	0	51	×
0	1	1	0	0	1	1	52	
0	1	1	0	1	0	0	53	×
0	1	1	0	1	0	1	54	
0	1	1	0	1	1	0	55	
0	1	1	0	1	1	1	56	
0	1	1	1	0	0	0	57	×
0	1	1	1	0	0	1	58	

0	1	1	1	0	1	0	59	
0	1	1	1	0	1	1	60	
0	1	1	1	1	0	0	61	
0	1	1	1	1	0	1	62	
0	1	1	1	1	1	0	63	
0	1	1	1	1	1	1	64	
1	0	0	0	0	0	0	65	
1	0	0	0	0	0	1	66	
1	0	0	0	0	1	0	67	
1	0	0	0	0	1	1	68	×
1	0	0	0	1	0	0	69	
1	0	0	0	1	0	1	70	×
1	0	0	0	1	1	0	71	×
1	0	0	0	1	1	1	72	
1	0	0	1	0	0	0	73	
1	0	0	1	0	0	1	74	×
1	0	0	1	0	1	0	75	×
1	0	0	1	0	1	1	76	
1	0	0	1	1	0	0	77	×
1	0	0	1	1	0	1	78	
1	0	0	1	1	1	0	79	
1	0	0	1	1	1	1	80	
1	0	1	0	0	0	0	81	
1	0	1	0	0	0	1	82	×
1	0	1	0	0	1	0	83	×
1	0	1	0	0	1	1	84	
1	0	1	0	1	0	0	85	×
1	0	1	0	1	0	1	86	
1	0	1	0	1	1	0	87	
1	0	1	0	1	1	1	88	
1	0	1	1	0	0	0	89	×
1	0	1	1	0	0	1	90	
1	0	1	1	0	1	0	91	
1	0	1	1	0	1	1	92	

1	0	1	1	1	0	0	93	
1	0	1	1	1	0	1	94	
1	0	1	1	1	1	0	95	
1	0	1	1	1	1	1	96	
1	1	0	0	0	0	0	97	
1	1	0	0	0	0	1	98	×
1	1	0	0	0	1	0	99	×
1	1	0	0	0	1	1	100	
1	1	0	0	1	0	0	101	×
1	1	0	0	1	0	1	102	
1	1	0	0	1	1	0	103	
1	1	0	0	1	1	1	104	
1	1	0	1	0	0	0	105	×
1	1	0	1	0	0	1	106	
1	1	0	1	0	1	0	107	
1	1	0	1	0	1	1	108	
1	1	0	1	1	0	0	109	
1	1	0	1	1	0	1	110	
1	1	0	1	1	1	0	111	
1	1	0	1	1	1	1	112	
1	1	1	0	0	0	0	113	×
1	1	1	0	0	0	1	114	
1	1	1	0	0	1	0	115	
1	1	1	0	0	1	1	116	
1	1	1	0	1	0	0	117	
1	1	1	0	1	0	1	118	
1	1	1	0	1	1	0	119	
1	1	1	0	1	1	1	120	
1	1	1	1	0	0	0	121	
1	1	1	1	0	0	1	122	
1	1	1	1	0	1	0	123	
1	1	1	1	0	1	1	124	
1	1	1	1	1	0	0	125	
1	1	1	1	1	0	1	126	

1	1	1	1	1	1	0	127
1	1	1	1	1	1	1	128

There are 35 combinations marked with '×' that it could be done.

The simpler way to find the combinations is to group the two friends and then add in the next one, and then the next; then move on by replacing the second friend in the first group with a different one and so on… We will get the same combinations:

1	2	3	2	3	4	4	5	6
1	2	4	2	3	5	4	5	7
1	2	5	2	3	6	4	6	7
1	2	6	2	3	7			
1	2	7	2	4	5	5	6	7
1	3	4	2	4	6			
1	3	5	2	4	7			
1	3	6	2	5	6			
1	3	7	2	5	7			
1	4	5	2	6	7			
1	4	6						
1	4	7	3	4	5			
1	5	6	3	4	6			
1	5	7	3	4	7			
1	6	7	3	5	6			
			3	5	7			
			3	6	7			

Problem 1 of the British Mathematical Olympiad 1996

Consider the pair of four-digit positive integers

$$(M, N) = (3600, 2500).$$

Notice that M and N are both perfect squares, with equal digits in two places, and differing digits in the remaining two places. Moreover, when the digits differ, the digit in M is exactly one greater than the corresponding digit in N.

Find all pairs of four-digit positive integers (M, N) with these properties.

Solution

Let M be represented by four digits *abcd* in that order where a, b, c and d are all integers from 0 to 9. Number N will then be represented by four digits $(a + 1)(b + 1)cd$ in that order.

The problem gives us

$1000a + 100b + 10c + d = m^2$, and
$1000a + 100b + 10c + d + 1100 = n^2$

where m and n are integers.

Subtracting the two equations above to get

$(m + n)(m - n) = 1100$

Now either value $m + n$ or $m - n$ can be a combination of the product of these number(s) where 0 denotes no-selection and 1 denotes selection of the multiplier.

2	2	5	5	11	$m + n$	$m - n$
0	0	0	0	0	1	1100
0	0	0	0	1	11	100
0	0	0	1	0	5	220
0	0	0	1	1	55	20
0	0	1	0	0	same as a combination above	
0	0	1	0	1	same as a combination above	
0	0	1	1	0	25	44
0	0	1	1	1	275	4
0	1	0	0	0	2	550
0	1	0	0	1	22	50
0	1	0	1	0	10	110
0	1	0	1	1	110	10
0	1	1	0	0	same as a combination above	
0	1	1	0	1	same as a combination above	
0	1	1	1	0	50	22
0	1	1	1	1	550	2
1	0	0	0	0	same as a combination above	
1	0	0	0	1	same as a combination above	
1	0	0	1	0	same as a combination above	
1	0	0	1	1	same as a combination above	
1	0	1	0	0	same as a combination above	
1	0	1	0	1	same as a combination above	
1	0	1	1	0	same as a combination above	
1	0	1	1	1	same as a combination above	
1	1	0	0	0	4	275
1	1	0	0	1	44	25
1	1	0	1	0	20	55
1	1	0	1	1	220	5

2	2	5	5	11	$m+n$	$m-n$
1	1	1	0	0	same as a combination above	
1	1	1	0	1	same as a combination above	
1	1	1	1	0	100	11
1	1	1	1	1	1100	1

We found solutions as follows

$m + n = 110$, $m - n = 10$ $m = 60$, $n = 50$
$m + n = 50$, $m - n = 22$ $m = 36$, $n = 14$
$m + n = 550$, $m - n = 2$ $m = 276$, $n = 274$

Therefore, the solutions are

(M, N) = (3600, 2500);
(M, N) = (1296, 0196);

Note that for (M, N) = (76176, 75076) where M and N have more than 4 digits is eliminated.

Problem 1 of Poland Mathematical Olympiad 1997

Let ABCD be a tetrahedron with $\angle$BAD = 60°, $\angle$BAC = 40°, $\angle$ABD = 80°, $\angle$ABC = 70°. Prove that the lines AB and CD are perpendicular.

Solution

Let [Φ] denote the plane containing shape Φ. Lay triangle ADB on [ΔABC] as shown on the next page. Point D is now D'. From D' draw the altitude D'I to AB and extend it to meet AC at C'. Since $\angle$BAD' = 60°, $\angle$BAC = 40° and $\angle$ABC = 70°, we have $\angle$AD'I = 30°, $\angle$AC'I = 50°.

Now let AD = a. It's easily seen that AI = $\frac{a}{2}$ and D'I = $a\frac{\sqrt{3}}{2}$.

Applying the law of sines, we get

D'B = $\frac{a\sqrt{3}}{2\times\sin 80°}$, BI = $\frac{a\sqrt{3}\times\sin 10°}{2\times\sin 80°}$,

AB = AI + BI = $\frac{a}{2} + \frac{a\sqrt{3}\times\sin 10°}{2\times\sin 80°}$, and

AC' = $\frac{a}{2\times\sin\angle AC'I} = \frac{a}{2\times\sin 50°}$.

Now let's prove that

$$\frac{1}{\sin 50°} = 1 + \frac{\sqrt{3}\times\sin 10°}{\sin 80°} \qquad \text{(i)}$$

$$\frac{1}{\sin 50°} = 1 + \frac{\sqrt{3}\times\sin 10°}{\cos 10°}, \text{ or}$$

$$\frac{1}{\sqrt{3}\sin 50°} = \frac{1}{\sqrt{3}} + \tan 10°, \text{ or}$$

$$\frac{1}{\sqrt{3}\sin 50°} = \tan 30° + \tan 10°, \text{ or}$$

$$\frac{1}{\sqrt{3}\sin 50°} = \frac{\sin(30° + 10°)}{\cos 30°\times\cos 10°} = \frac{\sin 40°}{\cos 30°\times\cos 10°}, \text{ or}$$

$$\sqrt{3}\sin 50°\times\sin 40° = \cos 30°\times\cos 10°, \text{ or}$$

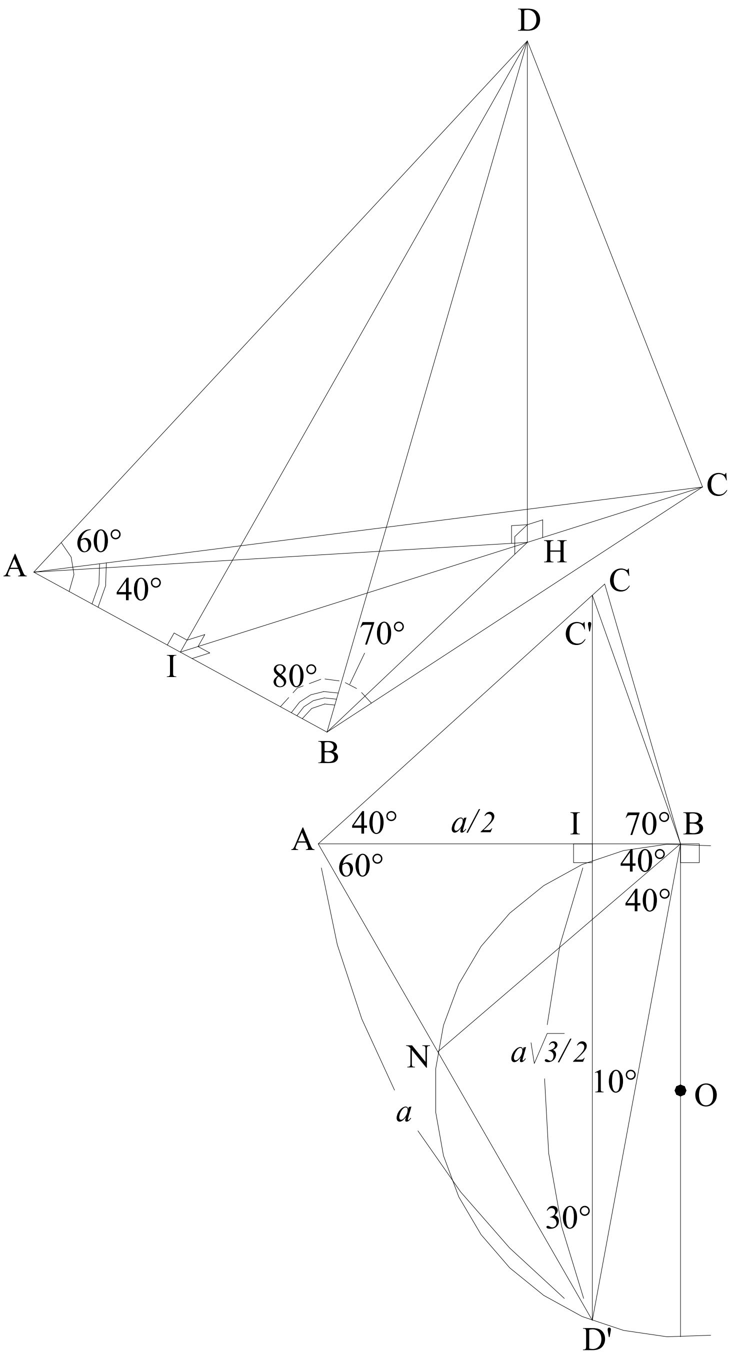


Figure (not to scale)

$\sin50° \times \sin40° = \frac{1}{\sqrt{3}}\cos30° \times \cos10°$, or

$\sin50° \times \sin40° = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\cos10°$, or

$\sin50° \times \sin40° = ½\cos10°$, or

$-½(\cos90° - \cos10°) = ½\cos10°$, and finally this is obvious because $\cos90° = 0$.

Now multiplying both sides of equation (i) by $\frac{a}{2}$, we get

$$\frac{a}{2 \times \sin50°} = \frac{a}{2} + \frac{a\sqrt{3} \times \sin10°}{2 \times \sin80°}.$$

Therefore, AC' = AB and ∠AC'B = ∠ABC = 70°, or C' coincides C, and the three points D, I and C are collinear, and DC ∈[ΔIDC]. This confirms that AB perpendiculars CD.

Further observation

In three-dimensional geometry, when one line falls or lies in a plane that is perpendicular to the other line, we say the two lines perpendicular to each other. A line is said to perpendicular a plane when it is perpendicular to two intersecting lines belonging to the plane.

The reader is encouraged to prove the equation $\frac{1}{\sin50°} = 1 + \frac{\sqrt{3} \times \sin10°}{\sin80°}$ *geometrically based on the fact that* $AN \times AD' = AB^2$ *where N is the intersection of the bisector of ∠ABD' and AD'.*

Problem 1 of British Mathematical Olympiad 1991

Prove that the number $3^n + 2\times17^n$ where n is a non-negative integer, is never a perfect square.

Solution

For $n = 0$, $3^n + 2\times17^n = 1 + 2 = 3$ and is not a square. Now let m be a non-negative integer.

For $n = 1 + 4m$, the units digits of 3^n and 17^n are 3 and 7, respectively, and the units digit of 2×17^n is 4. Therefore, the units digits of $3^n + 2\times17^n$ is $3 + 4 = 7$.

Similarly, for $n = 2 + 4m$, the units digits of 3^n and 2×17^n are 9 and 8, respectively, and the units digit of $3^n + 2\times17^n$ is $9 + 8 = 7$.

And for $n = 3 + 4m$, the units digits of 3^n and 2×17^n are 7 and 6, respectively, and the units digit of $3^n + 2\times17^n$ is $7 + 6 = 3$.

Finally, for $n = 4 + 4m$, the units digits of 3^n and 2×17^n are 1 and 2, respectively, and the units digit of $3^n + 2\times17^n$ is $1 + 2 = 3$.

Thus the units digits of $3^n + 2\times17^n$ are always either 3 or 7, and note that the units digits of perfect squares are either 0, 1, 4, 5, 6 or 9. Therefore, $3^n + 2\times17^n$ where n is a non-negative integer, is never a perfect square.

Further observation

Expanding the idea, we can make a conclusion that for non-negative integers a, b, n, p and q, the number $(10^a p + 3)^n + 2\times(10^b q + 7)^n$ is never a perfect square.

Problem 4 of Poland Mathematical Olympiad 1996

ABCD is a tetrahedron with $\angle$BAC = $\angle$ACD, and $\angle$ABD = $\angle$BDC. Show that AB = CD.

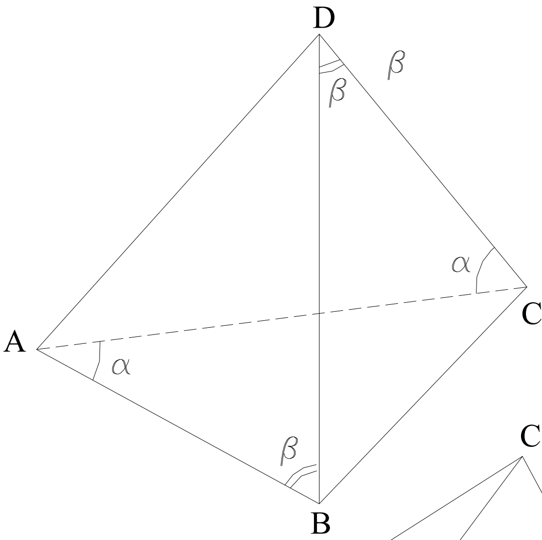
Solution

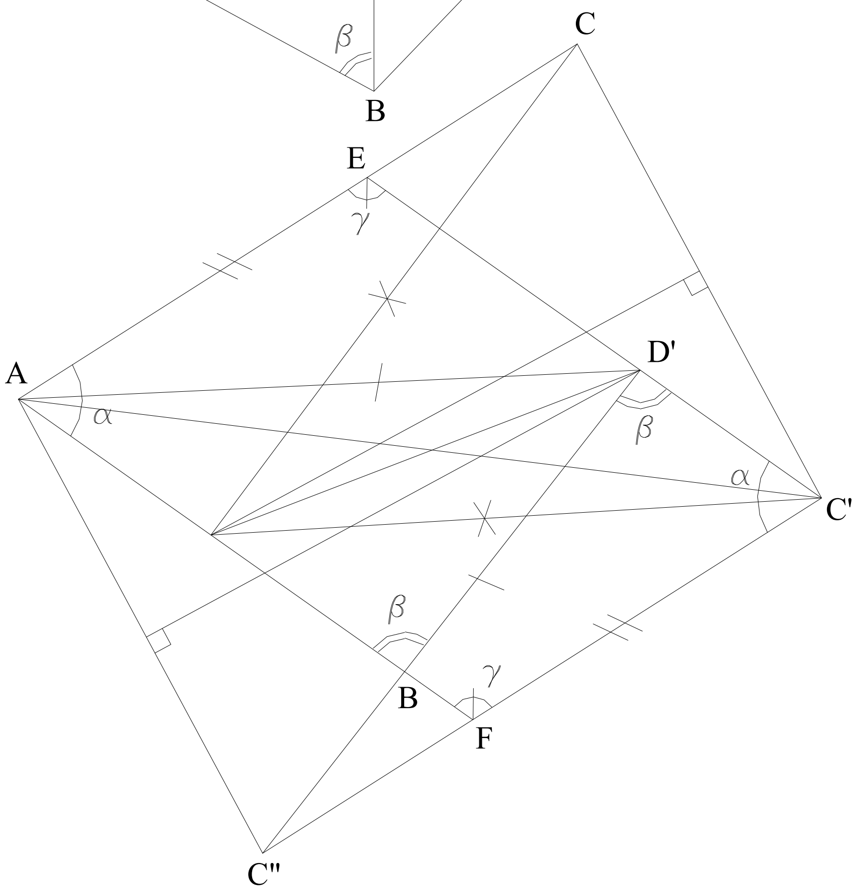
In the figure on the next page that is not drawn to scale, the top part depicts the three-dimensional graph while the bottom one shows the two-dimensional layout. Let α = $\angle$BAC = $\angle$ACD and β = $\angle$ABD = $\angle$BDC and denote [Φ] the plane containing shape Φ.

Lay ΔABD flat on [ΔABC]; its vertex D becomes D' as shown. Then lay ΔBDC flat next to ΔABD' on the same plane. Its vertex C becomes C'. These two triangles share side BD'. Since β = $\angle$ABD' = $\angle$BD'C', AB || C'D'.

Now lay flat ΔACD on the same plane. This triangle shares side C'D' and its vertex A moves to C". Since AB || C'D' and $\angle$BAC = $\angle$ACD = $\angle$D'C'C" = α, AC || C'C". However, AC = C'C" (and equal the original AC on the top graph); therefore, the new quadrilateral ACC'C" is a parallelogram.

Now let M and N be the midpoints of AC" and CC', respectively. Since BC = BC', AD' = D'C", D' is on EC' and on the perpendicular bisector of AC" while B is on AF || EC' and on the perpendicular bisector of CC', and AC" = CC', we thus have MD' = NB, or AD' = BC' (*the reader should try to prove these easy claims*), and AD'C'B is a parallelogram, and thus AB = C'D', or AB = CD on the three-dimensional graph.

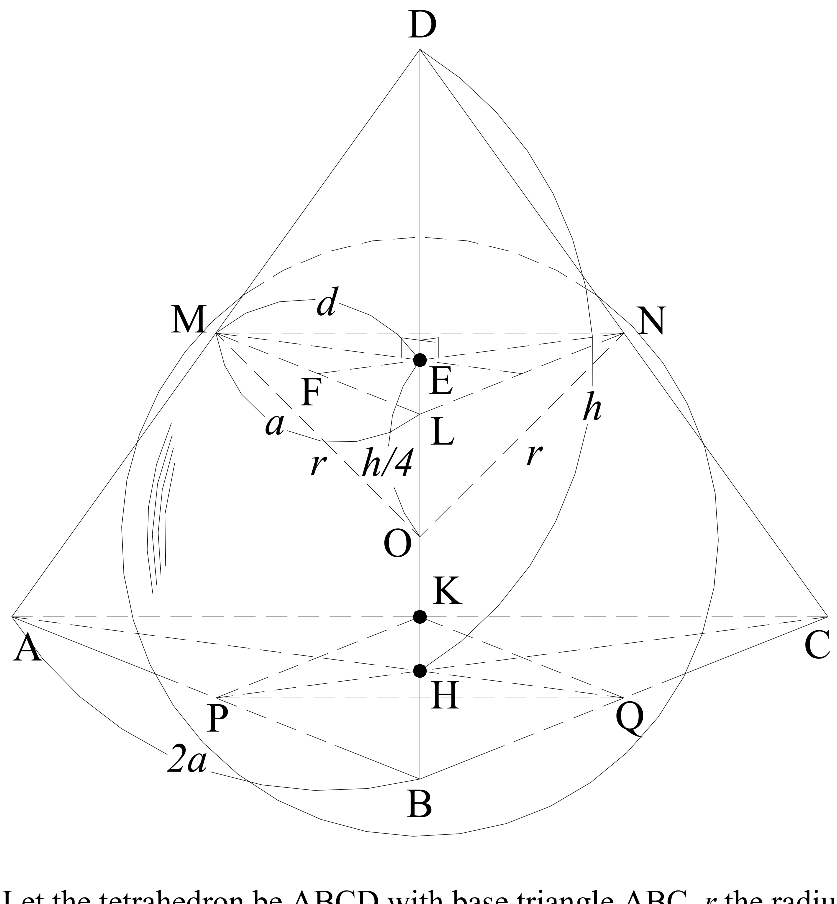

D
β
β
α
C
A
α
β
B


C
E
γ
D'
A
α
β
α
C'
β
γ
B
F
C''

Problem 6 of Hungary Mathematical Olympiad 1999

The midpoints of the edges of a tetrahedron lie on a sphere. What is the maximum volume of the tetrahedron?

Solution



Let the tetrahedron be ABCD with base triangle ABC, *r* the radius of the sphere, (Ω) denote the area of shape Ω, [Φ] denote the plane containing shape Φ, M, N, L, P, Q, K the midpoints of AD, CD, BD, AB, BC and AC, respectively, H the foot of D onto [ABC] and *h* = DH, the height of D above [ABC].
Since M, N, L, P, Q and K are the midpoints, ML = ½AB = KQ, NL = ½BC = PK, MN = ½AC = PQ. The two triangles MNL and

PQK are congruent, and (PQK) = ¼(ABC), or (MNL) = ¼(ABC). But the three points M, N and L are on the sphere or on a circle that lies on a sphere, (MNL) is maximum when it is an equilateral triangle which causes ABC to also be an equilateral triangle. Combining with ΔMNL = ΔPQK and [MNL] || [PQK], [MNL] and [PQK] are equidistant from the center O of the sphere. Therefore, H is the centroid of both triangles ABC and PQK, and OE = ½EH (E is the centroid of triangle MNL).

Because the triangle MNL cuts across the mid-section of the tetrahedron, h = 2DE, or OE = ½EH = $\frac{1}{4}h$.

Now let F the foot of N on ML, a the side length of the equilateral triangle MNL, d = NE. Per Pythagorean's theorem, NF² + MF² = MN², or $(\frac{3d}{2})^2 + (\frac{a}{2})^2 = a^2$, or $\frac{3d}{2} = a\frac{\sqrt{3}}{2}$, or $a = d\sqrt{3}$, and h×(MNL) = $\frac{1}{2}h \times a \times \frac{3d}{2} = \frac{\sqrt{3}}{4}a^2h$.

However, in triangle OEN, $r^2 = EN^2 + OE^2$, or $r^2 = d^2 + (\frac{h}{4})^2$, or $d = \frac{1}{4}\sqrt{16r^2 - h^2}$, or $a = \frac{1}{4}\sqrt{3(16r^2 - h^2)}$.

The volume is $V = \frac{1}{3}\, h \times (ABC) = \frac{4}{3} \times h \times (MNL) = \frac{1}{\sqrt{3}}a^2h = \frac{\sqrt{3}}{16} \times h \times (16r^2 - h^2)$ and is maximum when the product $h(16r^2 - h^2) = 16hr^2 - h^3$ is maximum.

Since r is fixed, taking the derivative of $16hr^2 - h^3$ with respect to h gives us $(16hr^2 - h^3)' = 16r^2 - 3h^2$, and $16r^2 - 3h^2 = 0$ when $h = \frac{4r}{\sqrt{3}}$, and $d = \frac{1}{4}\sqrt{16r^2 - h^2} = r\sqrt{\frac{2}{3}}$, $a = d\sqrt{3} = r\sqrt{2}$.

Finally, the maximum volume is $V_{max} = \frac{\sqrt{3}}{16} \times \frac{4r}{\sqrt{3}}(16r^2 - \frac{16r^2}{3}) = \frac{8}{3}r^3$.

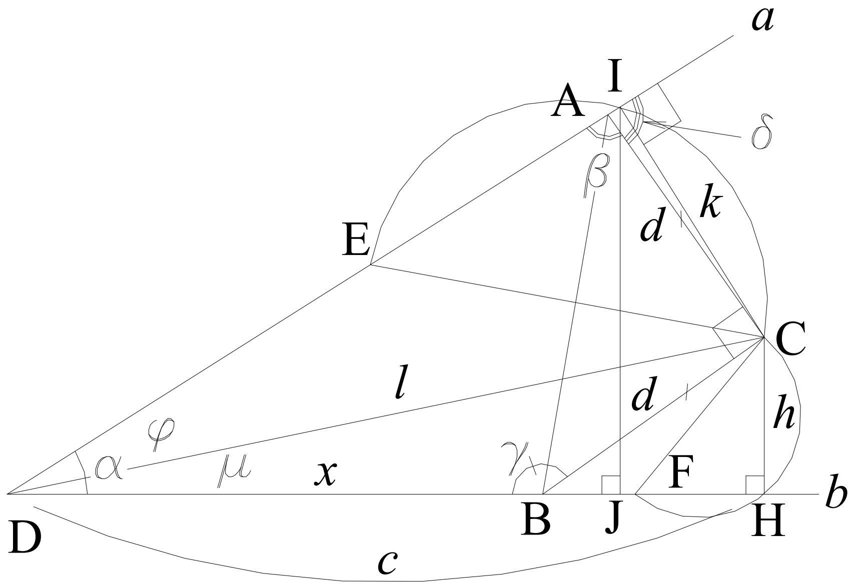
<u>Further observation</u>

The tetrahedron is a regular tetrahedron and $AB \perp DC$, $BC \perp AD$ and $AC \perp BD$.

Problem 5 of International Mathematical Talent Search Round 18

Let *a* and *b* be two lines in the plane, and let C be a point as shown in the figure below. Using only a compass and an unmarked straight edge, construct an isosceles right triangle ABC, so that A is on line *a*, B is on line *b*, and AB is the hypotenuse of triangle ABC.

Solution



Let lines *a* and *b* meet at D, α = ∠ADB, φ = ∠ADC, μ = ∠BDC and *l* = DC. Draw the altitudes CI and CH from C onto the lines *a* and *b*, respectively. Assuming that the isosceles right triangle BCD has been constructed, let *d* = AC = BC, *h* = CH, *k* = CI, *x* = BD, *c* = DH, β = ∠CAD, γ = ∠CBD and δ = 180° – β = ∠CAI.

Applying the law of sines to the triangles ACD and BCD, we get

$$\frac{d}{l} = \frac{\sin\varphi}{\sin\beta} = \frac{\sin\mu}{\sin\gamma} \qquad \text{(i)}$$

However, in the quadrilateral ACBD with ∠C = 90°, β + γ = 270° – α, or γ = 270° – (α + β) and sinγ = sin[270° – (α + β)] =

$\sin[180° – (\alpha + \beta – 90°)] = \sin(\alpha + \beta – 90°) = -\sin[90° – (\alpha + \beta)] = -\cos(\alpha + \beta)$.

Equation (i) becomes $\frac{\sin\varphi}{\sin\beta} = \frac{\sin\mu}{-\cos(\alpha + \beta)}$, or $\frac{\sin\mu}{\sin\varphi} = \frac{-\cos(\alpha + \beta)}{\sin\beta}$.

Furthermore, since $\sin\varphi = \frac{k}{l}$, $\sin\mu = \frac{h}{l}$ and $\cos(\alpha + \beta) = \cos\alpha\cos\beta – \sin\alpha\sin\beta$, the above equation is now equivalent to

$\frac{h}{k} = \frac{\sin\alpha\sin\beta – \cos\alpha\cos\beta}{\sin\beta} = \sin\alpha – \cos\alpha\cot\beta$, or $\cot\beta = \tan\alpha – \frac{h}{k\cos\alpha}$.

Note that $\cot\delta = \cot(180° – \beta) = -\cot\beta = \frac{h}{k\cos\alpha} – \tan\alpha$. But $\cot\delta = \frac{AI}{k}$, or $AI = k\cot\delta = \frac{h}{\cos\alpha} – k\tan\alpha$.

Since h and k are fixed and we can calculate $\cos\alpha$, $\tan\alpha$, we can find the length AI using only a compass and an unmarked straight edge by following this procedure:

1. Pick a point E on line a, draw a circle with diameter EC to meet a at point I.
2. Similarly, pick a point F on line b, draw a circle with diameter FC to meet b at point H.
3. Next, draw a circle with diameter DI to meet b at J.
4. $\tan\alpha = \frac{IJ}{DJ}$ and $\cos\alpha = \frac{DJ}{DI}$ are then determined.
5. From there we can calculate the value of AI.

Problem 2 of Austria Mathematical Olympiad 2004

Solve the equation

$$\sqrt{4 - x\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}}} = \frac{5x - 6 - x^2}{2}.$$

(all the square roots are non-negative)

Solution

The problem makes it simple by allowing us to ignore the negative roots. Since all square roots are non-negative, $5x - 6 - x^2 = (x-2)(-x+3) \geq 0$, or $2 \geq x$ and $x \geq 3$ which is not possible, or $3 \geq x \geq 2$.

We have $1 + (x-5)(x-7) = (x-6)^2$ and $\sqrt{1 + (x-5)(x-7)} = 6 - x$ instead of $x - 6$ since it's negative when $3 \geq x \geq 2$, and

$\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}} = \sqrt{4 - (x-2)(6-x)} = \sqrt{(4-x)^2} = 4 - x$ instead of $x - 4$ for the same reason.

And $\sqrt{4 - x\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}}} = \sqrt{(x-2)^2} = x - 2$.

Equating the two sides, we get

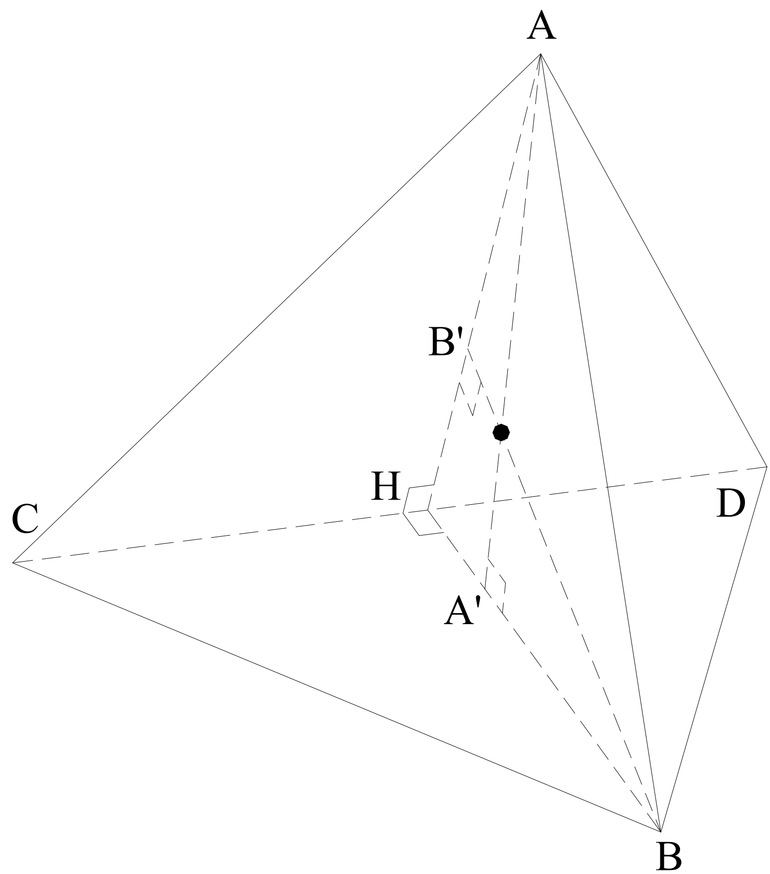
$x - 2 = \dfrac{5x - 6 - x^2}{2}$, or $x^2 - 3x + 2 = 0$, or $(x-1)(x-2) = 0$. Or $x = 1$ or $x = 2$.

But $x = 1$ is outside of $[2, 3]$ and must be rejected. Therefore, $x = 2$ is the only solution.

Problem 3 of the Vietnamese Mathematical Olympiad 1962

Let ABCD be a tetrahedron. Denote by A', B' the feet of the perpendiculars from A and B, respectively to the opposite faces. Show that AA' and BB' intersect if and only if AB is perpendicular to CD. Do they intersect if AC = AD = BC = BD?

Solution



Let's denote [Φ] the plane containing shape Φ.

When AB is perpendicular to CD, draw the altitude AH onto CD with H on CD. Since CD ⊥ AB, CD ⊥ AH and both AB and AH form the plane ABH, CD is perpendicular to plane of triangle ABH (denoted CD⊥[ABH]). Therefore, CD does perpendicular to any line that lies on that plane, and CD⊥BH. Also because CD⊥[ABH] and [BCD] contains CD, [ABH]⊥[BCD].

Combining AA'⊥[BCD] with [ABH]⊥[BCD] to get AA' ∈[ABH].

Similarly, BB' ∈[ABH], and since both AA' and BB' are on the same plane [ABH] and are the altitudes of triangle ABH, AA' and BB' intersect.

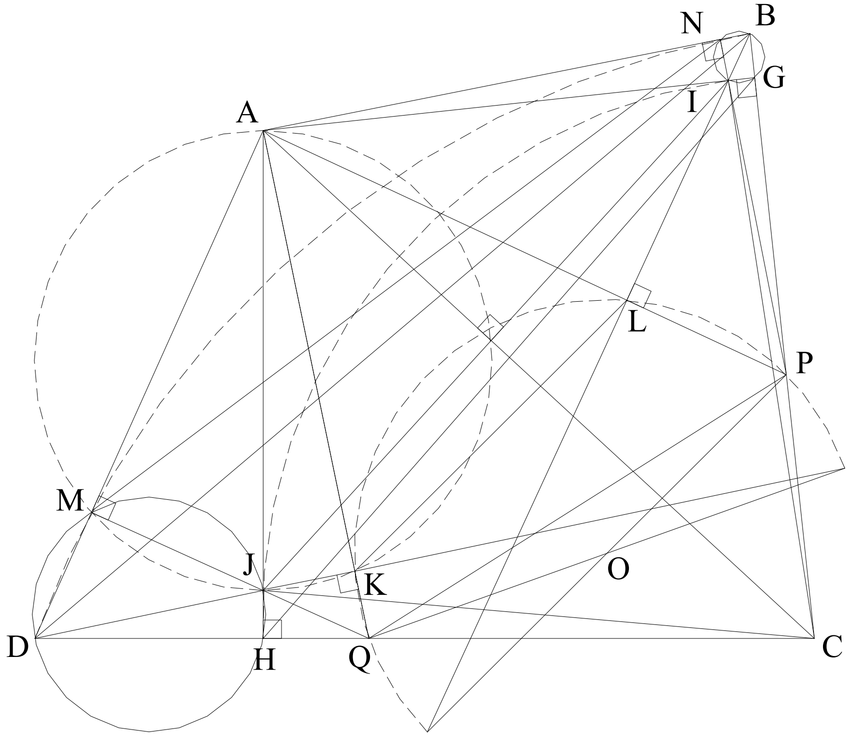
When AA' intersects BB', AA' and BB' are on the same plane containing AB. Since AA'⊥[BCD], AA'⊥CD. Similarly, because BB'⊥[ACD], BB'⊥CD. Together, CD perpendiculars with the plane containing AA' and BB', but this plane contains AB; therefore, AB⊥CD.

They do intersect if AC = AD = BC = BD. In such a situation, the two isosceles triangles ADC and BDC are congruent because of all their respective sides are equal, but A and B are still on the plane [ABH].

<u>Problem 8 of Georgia MO Team Selection Test 2005</u>

In a convex quadrilateral ABCD the points P and Q are chosen on the sides BC and CD, respectively so that $\angle BAP = \angle DAQ$. Prove that the line, passing through the orthocenters of triangles ABP and ADQ, is perpendicular to AC if and only if the triangles ABP and ADQ have the same areas.

<u>Solution</u>



Let I and J be the orthocenters of triangles ABP and ADQ, respectively, and M, H, K, N, G and L be the feet of Q onto AD, A onto CD, D onto AQ, P onto AB, A onto BC and B onto AP, respectively. Our mission is to prove that $IJ \perp AC$. To do that we will prove that

$AJ^2 + CI^2 = AI^2 + CJ^2$, or

$AJ^2 + CG^2 + IG^2 = AI^2 + CH^2 + JH^2$, or

$AJ^2 + AC^2 - AG^2 + IG^2 = AI^2 + AC^2 - AH^2 + JH^2$, or

$AJ^2 - AG^2 + IG^2 = AI^2 - AH^2 + JH^2$, or
$AJ^2 - AG^2 + IG^2 = AI^2 - (AJ^2 + JH^2 + 2AJ \times JH) + JH^2$, or
$= AI^2 - AJ^2 - 2AJ \times JH$, or

$2AJ^2 + IG^2 = AI^2 + AG^2 - 2AJ \times JH$, or
$= AI^2 + (AI^2 + IG^2 + 2AI \times IG) - 2AJ \times JH$, or

$2AJ^2 = 2AI^2 + 2AI \times IG - 2AJ \times JH$, or
$AJ^2 + AJ \times JH = AI^2 + AI \times IG$, or
$AJ(AJ + JH) = AI(AI + IG)$, or
$AJ \times AH = AI \times AG$.

In other words, we need to prove that the quadrilateral JHGI is cyclic.
The problem gives us that the areas of the two triangles ABP and ADQ are equal, or $QM \times AD = PN \times AB$ (i)

Furthermore, we're also given that $\angle BAP = \angle DAQ$. Now it's easily seen that the two triangles AMQ and ANP are similar because all their respective angles are equal which implies that

$$\frac{QM}{AM} = \frac{PN}{AN} \qquad \text{(ii)}$$

From (i) and (ii), we have $AN \times AB = AM \times AD$ (iii)
However, since the two quadrilaterals MDHJ and NBGI have their opposite angles being the right angles, they are cyclic, and we have $AN \times AB = AI \times AG$ and $AM \times AD = AJ \times AH$.

Therefore, equation (iii) becomes $AI \times AG = AJ \times AH$, or JHGI is cyclic and we're done.

The reverse process is fairly straightforward; the reader is encouraged to prove it.

Further observation

We can prove that the quadrilateral JHGI is cyclic by applying the law of cosines.

Since $\angle BAP = \angle DAQ$*,* $\sin\angle BAP = \sin\angle DAQ$*, or* $\frac{DK}{AD} = \frac{BL}{AB}$.

Because the two triangles ABP and ADQ having the same areas, $DK\times AQ = BL\times AP$*. The last two ratios give us* $AB\times AP = AD\times AQ$*, or* $AB\times AP\times\cos\angle BAP = AD\times AQ\times\cos\angle DAQ$ *(iv)*

The law of cosines gives us
$BP^2 = AB^2 + AP^2 - 2AB\times AP\times\cos\angle BAP$*, and*
$DQ^2 = AD^2 + AQ^2 - 2AD\times AQ\times\cos\angle DAQ$.

Combining with (iv), we get
$BP^2 - AB^2 - AP^2 = DQ^2 - AD^2 - AQ^2$.

Now the Pythagorean's theorem yields
$BP^2 - AB^2 - AP^2 = BL^2 + LP^2 - BL^2 - AL^2 - AL^2 - LP^2 - 2\times AL\times LP$
$= -2AL^2 - 2AL\times LP$.

Similarly, $DQ^2 - AD^2 - AQ^2 = -2AK^2 - 2AK\times KQ$*. Therefore,*

$AL^2 + AL\times LP = AK^2 + AK\times KQ$*, or* $AL\times AP = AK\times AQ$*. This implies that KLPQ is cyclic.*

Combining with the two adjacent cyclic quadrilaterals JKQH and LIGP on either side of the cyclic KLPQ, JHGI is cyclic.

Problem 4 of Hong Kong MO Team Selection Test 1994

Suppose that $yz + zx + xy = 1$ and x, y, and $z \geq 0$. Prove that
$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) \leq 4\frac{\sqrt{3}}{9}$.

Solution

Expanding the expression on the left, we get

$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) =$
$x + y + z - z(x^2 + xz + yz) - y(x^2 + xy + \mathrm{yz}) + xyz(xy + xz + yz)$.

Now substituting $yz + zx = 1 - xy$, $yz + xy = 1 - xz$ and $yz + zx + xy = 1$ into the previous expression, we get

$x(1-y^2)(1-z^2) + y(1-z^2)(1-x^2) + z(1-x^2)(1-y^2) =$
$x + y + z - z(x^2 - xy + 1) - y(x^2 - xz + 1) + xyz$.

Next, regrouping this expression to get
$x + y + z - z(x^2 - xy + 1) - y(x^2 - xz + 1) + xyz =$
$x(1 - xy - xz + 3yz) = 4xyz$.
Now it suffices to prove that $4xyz \leq 4\frac{\sqrt{3}}{9}$, or $xyz \leq \frac{\sqrt{3}}{9}$.

Applying the AM-GM inequality for the non-negative values yz, zx and xy, we get $yz + zx + xy \geq 3\sqrt[3]{x^2y^2z^2}$, or
$1 \geq 3\sqrt[3]{x^2y^2z^2}$, or $xyz \leq \frac{1}{\sqrt{27}} = \frac{\sqrt{3}}{9}$, and we're done.

Problem 5 of the Iranian Mathematical Olympiad 2000

In a tetrahedron we know that the sum of angles of all vertices is 180°. (e.g., for vertex A, we have $\angle BAC + \angle CAD + \angle DAB = 180°$.) Prove that the faces of this tetrahedron are four congruent triangles.

Solution

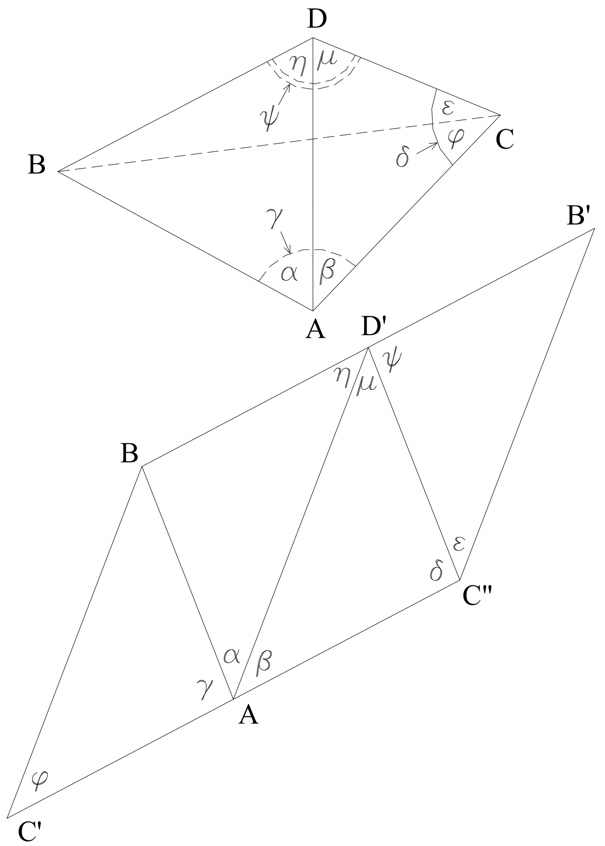


Figure (not to scale)

Let $\alpha = \angle DAB$, $\beta = \angle CAD$, $\gamma = \angle BAC$, $\delta = \angle ACD$, $\varphi = \angle ACB$, $\varepsilon = \angle BCD$, $\eta = \angle ADB$, $\mu = \angle ADC$ and $\psi = \angle BDC$. We have $\alpha + \beta + \gamma = \delta + \varphi + \varepsilon = \eta + \mu + \psi = 180°$.

The top part of the graph on the previous page depicts the three-dimensional tetrahedron while the bottom one the two-dimensional layout of its triangles on the plane that contains the triangle ABC, denoted [ABC], except that the triangle ABC has been flipped 180° counterclockwise around axis AB. Point C of ΔABC moves to C'.

Now lay flat ΔABD but keep side AB at the same position on [ABC], point D of ΔABD ⟶ D'. Continue doing the same for the other two triangles, laying flat ΔADC (side AD is now AD') on [ABC], and then ΔBCD (B ⟶ B', D ⟶ D', C ⟶ C".)

The angles transform to the two-dimensional graph as follows $\alpha = \angle D'AB$, $\beta = \angle C''AD'$, $\gamma = \angle BAC'$, $\delta = \angle AC''D'$, $\varphi = \angle AC'B$, $\varepsilon = \angle B'C''D'$, $\eta = \angle AD'B$, $\mu = \angle AD'C$ and $\psi = \angle B'D'C''$. And these sides are equal BC = BC' = B'C", BD = BD' = B'D', AC = AC' = AC".

Since $\alpha + \beta + \gamma = 180°$ the three points C', A and C" are collinear, so are the three points B, D' and B' since $\eta + \mu + \psi = 180°$. Also because $\delta + \varphi + \varepsilon = 180°$, BC' || B'C". Thus BB'C"C' is a parallelogram and BB' = C'C". With D' and A being the midpoints of BB' and C'C", we have BD' = D'B' = C'A = AC" and AB = C"D'. Therefore, the four triangles in the two-dimensional graph ABC', ABD', ACD' and B'C"D' are congruent because all their respective sides are equal, and the faces of this tetrahedron are four congruent triangles.

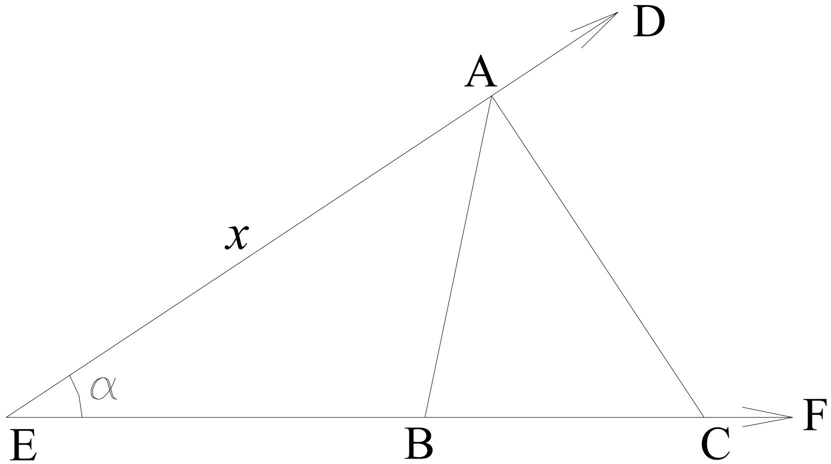
<u>Further observation</u>

As we have seen, we did not touch the vertex B of the tetrahedron. Only three vertices with each having the sum of its angles being 180° is enough for all the faces to be congruent. So we conclude that the sum of angles on the last vertex must also be 180°, and thus we arrive with this problem "Three vertices of a tetrahedron with each having the sum of its angles being 180°, prove that the sum of the angles at the remaining vertex is also 180°."

Problem 3 of Moldova Mathematical Olympiad 2002

Consider an angle $\angle$DEF, and the fixed points B and C on the semi-line EF and the variable point A on ED. Determine the position of A on ED such that the sum AB + AC is minimum.

Solution



Let $\alpha = \angle DEF$, $x = EA$.

Applying the law of cosines to get

$AB = \sqrt{x^2 + EB^2 - 2x\times EB\cos\alpha}$ and

$AC = \sqrt{x^2 + EC^2 - 2x\times EC\cos\alpha}$.
$AB + AC = \sqrt{x^2 + EB^2 - 2x\times EB\cos\alpha} + \sqrt{x^2 + EC^2 - 2x\times EC\cos\alpha}$.

The extreme values of AB + AC is found by first taking its derivative with respect to x. We have

$(AB + AC)' = (\sqrt{x^2 + EB^2 - 2x\times EB\cos\alpha})^{-1/2} \times (x - EB\times\cos\alpha) +$
$(\sqrt{x^2 + EC^2 - 2x\times EC\cos\alpha})^{-1/2} \times (x - EC\times\cos\alpha)$.

Now setting this derivative to zero, we get

$(x - \text{EB}\times\cos\alpha)\sqrt{x^2 + \text{EC}^2 - 2x\times\text{EC}\cos\alpha} + (x - \text{EC}\times\cos\alpha)\times$

$\sqrt{x^2 + \text{EB}^2 - 2x\times\text{EB}\cos\alpha} = 0$, or

$(x - \text{EB}\times\cos\alpha)\sqrt{x^2 + \text{EC}^2 - 2x\times\text{EC}\cos\alpha} = -(x - \text{EC}\times\cos\alpha)$
$\sqrt{x^2 + \text{EB}^2 - 2x\times\text{EB}\cos\alpha}$.

Now squaring both sides, we obtain
$(x - \text{EB}\times\cos\alpha)^2(x^2 + \text{EC}^2 - 2x\times\text{EC}\cos\alpha) = (x - \text{EC}\times\cos\alpha)^2(x^2 + \text{EB}^2 - 2x\times\text{EB}\cos\alpha)$.

Expanding it and we get
$x(\text{EC}^2 - \text{EB}^2) - 2\text{EB}\times\text{EC}\cos\alpha(\text{EC} - \text{EB}) - x\times\cos^2\alpha(\text{EC}^2 - \text{EB}^2) + 2\text{EB}\times\text{EC}\cos^3\alpha(\text{EC} - \text{EB}) = 0$, or

$(\text{EC} - \text{EB})\times\sin^2\alpha[x(\text{EC} + \text{EB}) - 2\text{EB}\times\text{EC}\cos\alpha] = 0$.

However, neither EC – EB nor $\sin\alpha$ is zero since EC > EB and α is not zero, and we must have
$x(\text{EC} + \text{EB}) - 2\text{EB}\times\text{EC}\cos\alpha = 0$, or $x = \dfrac{2\text{EB}\times\text{EC}\cos\alpha}{\text{EB} + \text{EC}}$.

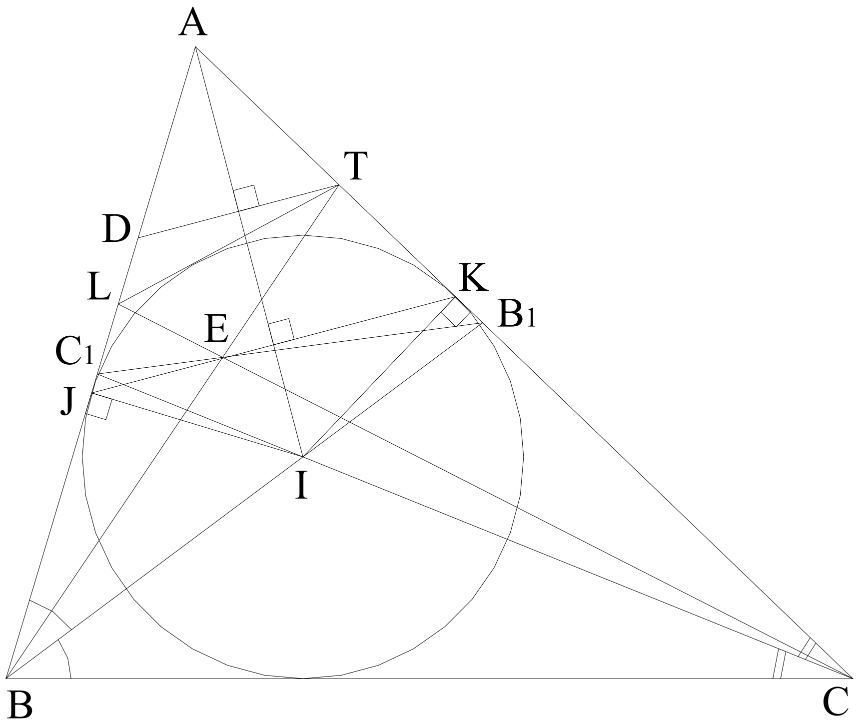
For $x < \dfrac{2\text{EB}\times\text{EC}\cos\alpha}{\text{EB} + \text{EC}}$ and $x > \dfrac{2\text{EB}\times\text{EC}\cos\alpha}{\text{EB} + \text{EC}}$ the value for AB + AC is greater than that of AB + AC when $x = \dfrac{2\text{EB}\times\text{EC}\cos\alpha}{\text{EB} + \text{EC}}$.

We conclude that the position of A on ED such that the sum AB + AC is minimum is when $\text{EA} = \dfrac{2\text{EB}\times\text{EC}\cos\alpha}{\text{EB} + \text{EC}}$.

Problem 15 of Moldova Mathematical Olympiad 2002

In a triangle ABC, the bisectors of the angles at B and C meet the opposite sides B_1 and C_1, respectively. Let T be the midpoint AB_1 Lines BT and B_1C_1 meet at E and lines AB and CE meet at L. Prove that the lines TL and B_1C_1 have a point in common.

Solution



Let I be the incenter of ΔABC, J and K be the feet of I onto AB and AC, respectively, T' and B' the symmetrical points of T and B across AI, respectively, F the intersection of BT and AI.

We have AT = AT', AB = AB', and it's easily seen that the three points T', F and B' are collinear just like the other three points T, F and B which are also collinear.

Without loss of generality, assume $\angle ABC > \angle ACB$, or AC > AB = AB'. Vertically, F is at a higher altitude than E (both on the same segment BT); B' is obviously also at a higher altitude than C; therefore, T' is at a higher altitude than L. And because T' and L are on AB, AT' < AL, or AT < AL.

We also have $AT = TB_1$ (T is the midpoint of AB_1 as given by the problem), but $TB_1 > TK = T'J > LC_1$.

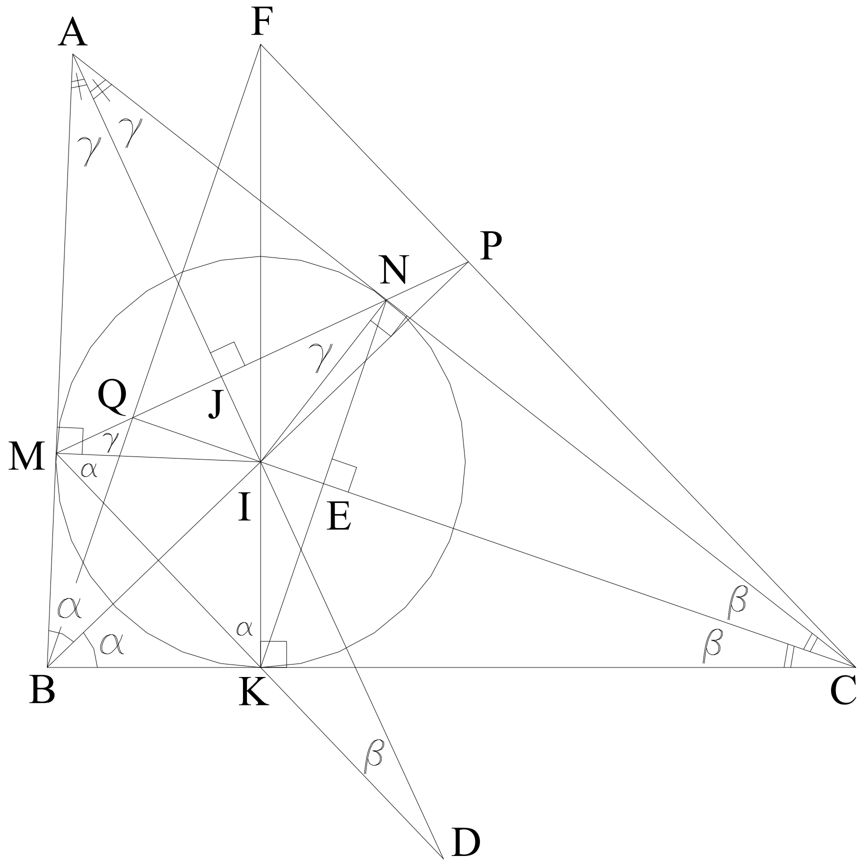
Hence, $\frac{AT}{TB_1} < \frac{AL}{TB_1} < \frac{AL}{LC_1}$.

Since T is the midpoint of AB_1 and L is not the midpoint of AC_1, TL is not parallel to B_1C_1, and they will meet and have a common point.

Problem 7 of Moldova MO Team Selection Test 2003

The sides AB and AC of the triangle ABC are tangent to the incircle with center I of the ΔABC at the points M and N, respectively. The internal bisectors of the ΔABC drawn from B and C intersect the line MN at the points P and Q, respectively. Suppose that F is the intersection point of the lines CP and BQ. Prove that $FI \perp BC$.

Solution



Let $\alpha = \frac{1}{2}\angle ABC$, $\beta = \frac{1}{2}\angle ACB$, $\gamma = \frac{1}{2}\angle BAC$, K be the foot of I onto BC. We have $\alpha + \beta + \gamma = 90°$. Link MK, AI, KN and extend both MK and AI to meet at D, IC to meet KN at E and MN to meet AI at J.

Since AI, BI and CI are the angle bisectors of $\angle BAC$, $\angle ABC$ and $\angle ACB$, respectively, and M, N and K are points of tangencies, we have $AI \perp MN$, $BI \perp MK$ and $CI \perp KN$. These angle equalities result from their sides being perpendicular to each other $\angle KMI = \angle MBI = \alpha$, $\angle ENI = \angle NCI = \beta$ and $\angle JMI = \angle MAI = \angle JNI = \angle NAI = \gamma$.

We have $\angle MDJ = 180° - \angle MJD - \angle DMI - \angle NMI = 180° - 90° - \alpha - \gamma = \beta$.

However, $\angle MDJ = \angle NPI$ because their respective sides perpendicular to each other; therefore, $\beta = \angle NPI = \angle NCI$, and CINP is cyclic which results in $\angle IPC = \angle INC = 90°$, or $BP \perp FC$.

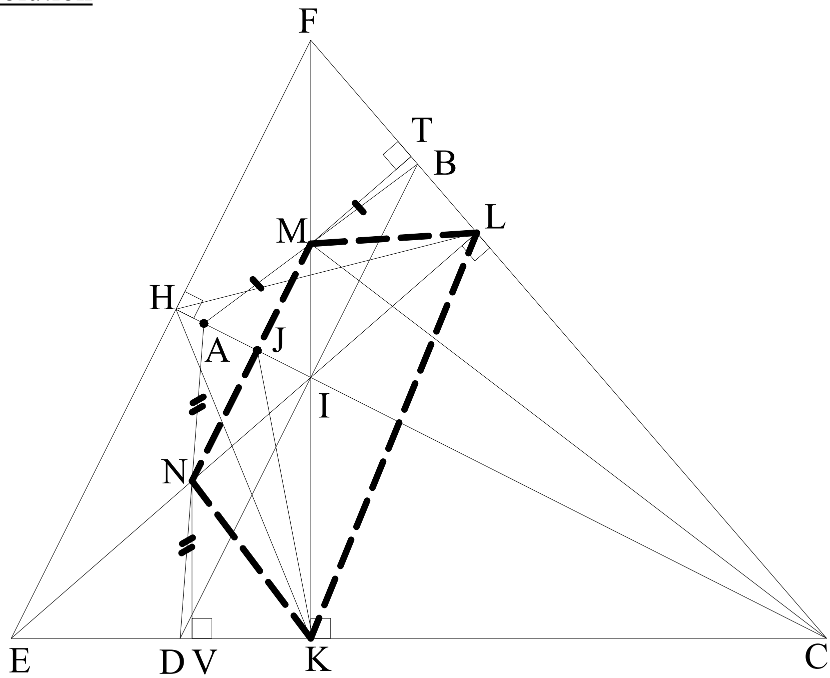
Similarly, $\angle EQN = 180° - \angle QEN - \angle KNI - \angle MNI = 180° - 90° - \beta - \gamma = \alpha$. But $\angle EQN + \angle MQI = 180°$, or $\alpha + \angle MQI = 180°$, or $\angle MBI + \angle MQI = 180°$, and BMQI is cyclic resulting in $\angle BQC = \angle BMI = 90°$, or $CQ \perp FB$.

Combining with $BP \perp FC$ found earlier and with point I being the intersection of BP and CQ, it is the orthocenter of ΔBFC, and thus $FI \perp BC$.

Problem 20 of Indonesia MO Team Selection Test 2009

Let ABCD be a convex quadrilateral. Let M, N be the midpoints of AB, AD, respectively. The foot of perpendicular from M to CD is K, and the foot of perpendicular from N to BC is L. Show that if AC, BD, MK and NL are concurrent, then KLMN is a cyclic quadrilateral.

Solution



Let I be the intersection of MK and NL. Extend CB and KM to meet at F, CD and LN to meet at E. Link EF and extend CI to meet EF at H. Also denote (Ω) the area of shape Ω.

Since $EL \perp FC$ and $FK \perp EC$, I is the orthocenter of ΔEFC, and thus $CH \perp EF$. Now draw the altitudes MT to FC and NV to EC.

We then have $\frac{FM}{FI} = \frac{MT}{IL} = \frac{(MBC)}{(IBC)}$. But since M is the midpoint of AB, (MBC) = (MAC), and the previous equation becomes

$\frac{FM}{FI} = \frac{(MAC)}{(IBC)}$. Now let h_1 and h_2 be the altitudes from M and B onto AC, respectively (h_1 and h_2 not shown on the graph). Again, since M is the midpoint of AB, $h_2 = 2h_1$, and

$$\frac{FM}{FI} = \frac{(MAC)}{(IBC)} = \frac{h_1 \times AC}{h_2 \times IC} = \frac{AC}{2 \times IC} \qquad \text{(i)}$$

Similarly, let h_3 and h_4 be the altitudes from N and D onto AC, respectively (again, they're not shown on the graph). We also have $h_4 = 2h_3$, and

$$\frac{EN}{EI} = \frac{NV}{IK} = \frac{(NDC)}{(IDC)} = \frac{(NAC)}{(IDC)} = \frac{h_3 \times AC}{h_4 \times IC} = \frac{AC}{2 \times IC}.$$

Combining with (i), we get $\frac{FM}{FI} = \frac{EN}{EI}$.

Therefore, MN || EF and $\angle$NMK = $\angle$EFK. But $\angle$EFK = $\angle$HCE because I is the orthocenter and because KILC is cyclic (opposite angles being right angles) $\angle$HCE = $\angle$NLK.

Hence, $\angle$NMK = $\angle$NLK and KLMN is a cyclic quadrilateral.

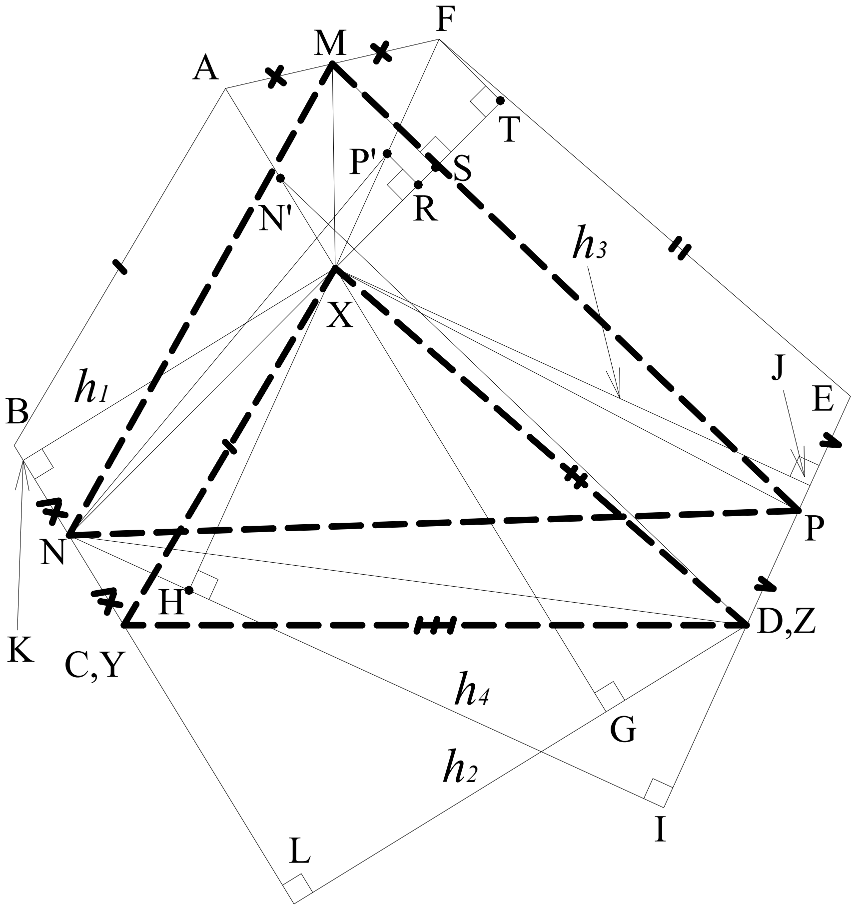
<u>Further observation</u>

Let J be the intersection of MN and H; h_1, h_2, h_3 and h_4 turn out to be MJ, BI, NJ and DI, respectively. Also since $\angle$NMK = $\angle$NLK = $\angle$ICK, KJMC is also a cyclic quadrilateral. And since BD || MN || EF, $\angle$AIB = 90° and with M being the midpoint of AB, MI = MA = MB and $\angle$AMJ = $\angle$IMJ, or $\angle$ABI = $\angle$IMJ = $\angle$EFK = $\angle$HCE, or $\angle$ABD = $\angle$ACD, and thus ABCD is also a cyclic quadrilateral.

Problem A5 Tournament of Towns 2009

Let XYZ be a triangle. The convex hexagon ABCDEF is such that AB, CD and EF are parallel and equal to XY, YZ and ZX, respectively. Prove that the area of triangle with vertices at the midpoints of BC, DE and FA is no less than the area of triangle XYZ.

Solution



Without loss of generality (WLOG), let's assume YZ > XZ > XY. Move the ΔXYZ so that its longest side YZ coincides with side CD of the hexagon ABCDEF; Y ≡ C and Z ≡ D as shown. Let M, N and P be the midpoints of AF, BC and DE, respectively. Denote (Ω) the area of shape Ω.

To prove (MNP) $\geq$ (XYZ), we need to prove that (ABNM) + (MFEP) + (MNCX) + (MPDX) $\geq$ (ABNM) + (MFEP) + (NPDC). In other words, the area of (ABCDEF) minus the area occupied by (XYZ) is greater or equal to the area of (ABCDEF) minus the area occupied by (MNP), or (MNCX) + (MPDX) $\geq$ (NPDC), or

(NXC) + (NXM) + (PXD) + (PXM) $\geq$ (NDC) + (NDP), or
(NXM) + (PXM) $\geq$ (NDC) – (NXC) + (NDP) – (PXD) (i)

Draw the altitudes XK, DL onto BC, the altitudes NI, XJ onto DE. Let h_1 = XK, h_2 = DL, h_3 = XJ and h_4 = NI. Extend AX to meet DL at G and FX to meet NI at H. It's easily seen that $h_2 - h_1$ = DG and $h_4 - h_3$ = NH, and the equation (i) becomes

(NXM) + (PXM) $\geq$ ½CN×DG + ½DP×NH (ii)

Now pick points N' on AX and P' on FX such that XN' = CN and XP' = DP. Equation (ii), which is still required to be proven, is equivalent to (NXM) + (PXM) $\geq$ (DXN') + (NXP').

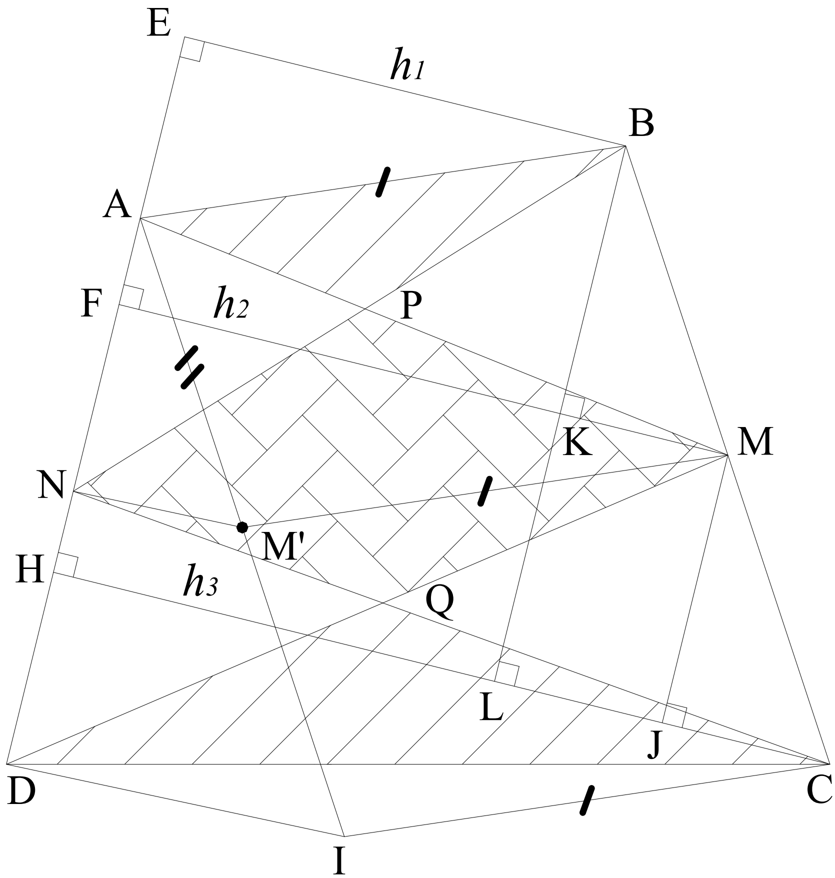
Now let's compare (NXM) with (NXP'). They have the same base NX; since ABCDEF is convex, the altitude MS from M to NX is greater than that from F to NX (FT on the graph), and because XP' = DP < DE = XF (or P' is on the interior of XF), FT is greater than the altitude P'R from P' to NX. Or the altitude MS from M to NX is greater than the altitude P'R from P' to NX, or (NXM) > (NXP').

Whereas ΔPXN' and ΔDXN' have the same base XN', but again since ABCDEF is convex, the altitude from P to N'X is greater than that from D to N'X which is DG, or (PXN') > (DXN'). Now compare (PXM) with (PXN') using the common base PX. The altitude from M to PX is greater than that from N' to PX; therefore, (PXM) > (PXN'), or (PXM) > (DXN'). Combining with (NXM) > (NXP') found above, we get (NXM) + (PXM) > (DXN') + (NXP') which is the equation required to be proven. Equality is achieved when AF = BC = DE = 0, and finally
(NXM) + (PXM) $\geq$ (DXN') + (NXP').

Problem 16 of Moldova Mathematical Olympiad 2002

Let ABCD be a convex quadrilateral and let N on side AD and M on side BC be points such that $\frac{AN}{ND} = \frac{BM}{MC}$. The lines AM and BN intersect at P, while the lines CN and DM intersect at Q. Prove that if $S_{ABP} + S_{CDQ} = S_{MNPQ}$, then either AD || BC or N is the midpoint of DA.

Solution



Draw the altitudes BE, MF and CH to AD and let h_1 = BE, h_2 = MF and h_3 = CH. Denote (Ω) the area of shape Ω. We have S_{ABP} = (ABP), S_{CDQ} = (CDQ) and S_{MNPQ} = (MNPQ), and that (MNPQ) = ½h_2×AD – (APN) – (NQD) and

(ABP) + (CDQ) = $\frac{1}{2}h_1\times$AN + $\frac{1}{2}h_3\times$ND – (APN) – (NQD)
For them to equal (MNPQ) = (ABP) + (CDQ), we must have
$h_2\times$AD = $h_1\times$AN + $h_3\times$ND, or
$h_2\times$AN + $h_2\times$ND = $h_1\times$AN + $h_3\times$ND, or
AN($h_2 - h_1$) = ND($h_3 - h_2$).

The conditions to satisfy the above equation are either

a) AN = ND, or N is the midpoint of AD and $h_2 - h_1 = h_3 - h_2$, or $2h_2 = h_1 + h_3$. These two conditions AN = ND and $2h_2 = h_1 + h_3$ must simultaneously exit. Draw two segments MM' and CI to parallel and equal segment AB (MM' || AB || CI and MM' = AB = CI). It's easily verified that when AN = ND, AM' = M'I (NM' || DI) and BM = MC, $\frac{AN}{ND} = \frac{BM}{MC}$ which is a condition given by the problem that is also satisfied by having AN = ND.

b) $h_2 - h_1 = h_3 - h_2 = 0$, or $h_1 = h_2 = h_3$. This occurs when AD || BC and it does not depend on the lengths of AN and ND since any real values multiplying by zero is zero.

c) AN = $h_3 - h_2$ and ND = $h_2 - h_1$. Let L and K be the feet of B onto HC and FM, respectively, and J the foot of M onto HC. AN = $h_3 - h_2$ = JC, ND = KM, $\frac{AN}{ND} = \frac{JC}{KM} = \frac{MC}{BM}$ (because the two triangles BKM and MJC are similar) = $\frac{BM}{MC}$ (required by the problem), or BM = MC and AN = ND. So this condition can only happen when AN = ND and BM = MC just like conditions in a) above.

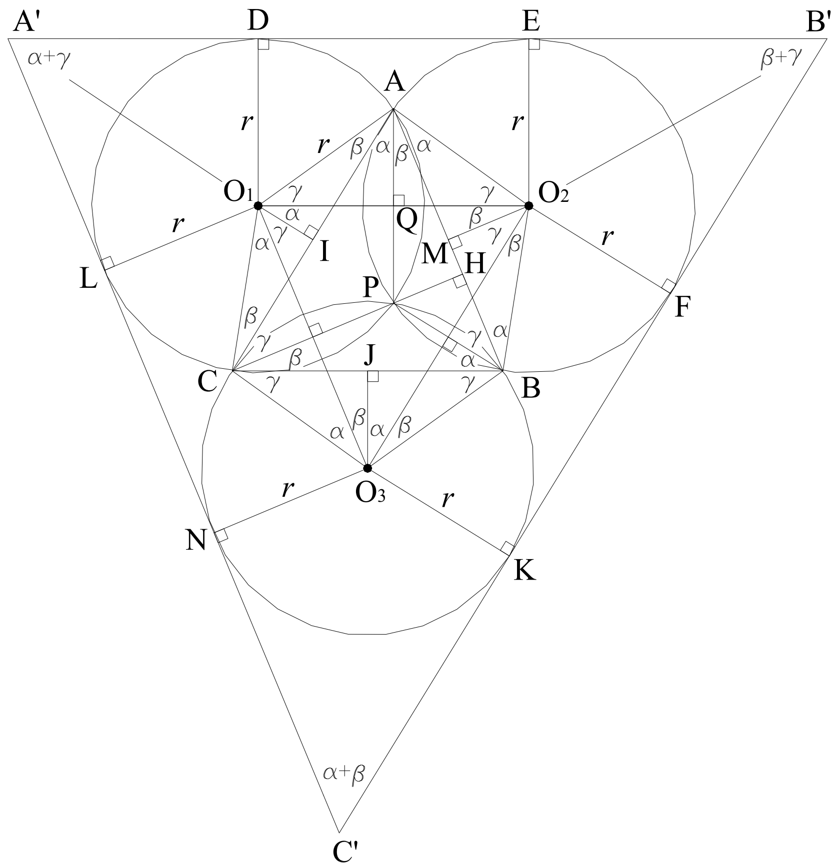
<u>Further observation</u>

If we had not been bound by the requirement that $\frac{AN}{ND} = \frac{BM}{MC}$, *then the third condition in c)* $AN = h_3 - h_2$ *and* $ND = h_2 - h_1$ *is one of the conditions for* $S_{ABP} + S_{CDQ} = S_{MNPQ}$ *to occur. We can also apply the Carpet theorem to solve this problem even though it's very much similar.*

<u>*Problem 3 of Hungary-Israel Binational 1994*</u>

Three given circles have the same radius and pass through a common point P. Their other points of pairwise intersections are A, B, C. We define triangle A'B'C', each of whose sides is tangent to two of the three circles. The three circles are contained in triangle A'B'C'. Prove that the area of triangle A'B'C' is at least nine times the area of triangle ABC.

<u>Solution</u>



Let r be the radius of the circles, O_1, O_2 and O_3 be the centers of the circles where O_1 is nearest to A', O_2 nearest to B' and O_3 nearest to C' as shown. Let (Ω) denote the area of shape Ω. Draw the altitudes O_1D onto A'B', O_2E onto A'B', O_1L onto A'C', O_3N

onto A'C', O_3K onto B'C', O_2F onto B'C', O_1I onto AC and O_3J onto BC. Let $Q = AP \cap O_1O_2$ and $H = CP \cap AB$.

Now let $\alpha = \angle CAP$, $\beta = \angle BAP$, $\gamma = \angle ACP$. We then also have $\alpha = \angle O_2O_1I$ (sides $\perp$ with those of $\angle CAQ$) = $\angle PBC$ (angles subtending same arc PC on identical circles) = $\angle JO_3O_2$ (sides $\perp$ with those of $\angle PBC$);
$\beta = \angle BO_2O_3$ (angle at center of circle subtends one-half arc PB) = $\angle BCP$ (angles subtending same arc PB on identical circles) = $\angle BO_3O_2$ (BO_3O_2 an isosceles triangle with $BO_3 = BO_2 = r$) = $\angle O_1O_3J$ (sides $\perp$ with those of $\angle BCP$), and
$\gamma = \angle O_3O_1I$ (sides $\perp$ with those of $\angle ACP$) = $\angle AO_1O_2$ (angle at center of circle subtends one-half arc AP) = $\angle ABP$ (angles subtending same arc AP on identical circles).

Therefore, in triangle ABC, $\alpha + \beta + \gamma = 90°$, and we also have $\angle AO_1O_3 + \angle BO_3O_1 = 2(\alpha + \beta + \gamma) = 180°$, or $AO_1 \parallel BO_3$, and in triangle ACH, $\angle HAC + \angle ACH = \alpha + \beta + \gamma = 90°$, or $\angle AHC = 90°$, or $CP \perp AB$. But $CP \perp O_1O_2$; therefore, $AB \parallel O_1O_3 \parallel A'C'$.

Similarly, $AC \parallel O_2O_3 \parallel B'C'$ and $BC \parallel O_1O_2 \parallel A'B'$. Hence, ΔABC is similar to $\Delta C'A'B'$, and the ratio of their areas $\dfrac{(\Delta A'B'C')}{(\Delta ABC)}$ equals the square of the ratio of their respective sides. For our purpose, we pick the ratio of their respective sides being $\dfrac{A'B'}{BC}$, and now we have to prove that $\dfrac{(\Delta A'B'C')}{(\Delta ABC)} = \dfrac{A'B'^2}{BC^2} \geq 9$, or $\dfrac{A'B'}{BC} \geq 3$.

However, $A'B' = A'D + DE + EB' = A'D + O_1O_2 + EB' = BC + A'D + EB'$, and $\dfrac{A'B'}{BC} = 1 + \dfrac{A'D + EB'}{BC}$, and it suffices to show

$$\frac{A'D + EB'}{BC} \geq 2 \qquad \text{(i)}$$

Note that because $O_1O_2 \parallel A'B'$, $O_1O_3 \parallel A'C'$ and $O_2O_3 \parallel B'C'$,

$\angle A'B'C' = \angle O_1O_2O_3 = \beta + \gamma$, $\angle B'A'C' = \angle O_2O_1O_3 = \alpha + \gamma$. Furthermore, since $A'O_1$ and $B'O_2$ are the angle bisectors of $\angle B'A'C'$ and $\angle A'B'C'$, respectively, $\angle DA'O_1 = ½(\alpha + \gamma)$ and $\angle EB'O_2 = ½(\beta + \gamma)$, and we have

$A'D = O_1D\times\cot[½(\alpha + \gamma)]$ and $EB' = O_2E\times\cot[½(\beta + \gamma)]$, or
$A'D = r\times\cot[½(\alpha + \gamma)]$ and $EB' = r\times\cot[½(\beta + \gamma)]$.

Also in ΔBJO_3, $BJ = O_3B\times\cos\angle JBO_3$, or $BJ = r\times\cos\gamma$, or $BC = 2BJ = 2r\times\cos\gamma$.

Therefore, the equation still requires to be proven (i) becomes

$$\frac{A'D + EB'}{BC} = \frac{r\times\cot[½(\alpha + \gamma)] + r\times\cot[½(\beta + \gamma)]}{2r\times\cos\gamma}$$

$$\frac{\cot[½(\beta + \gamma)]}{2\cos\gamma} \geq 2, \text{ or}$$

$\cot[½(\alpha + \gamma)] + \cot[½(\beta + \gamma)] \geq 4\cos\gamma$.

Now expanding the lest side, we get
$\cot[½(\alpha + \gamma)] + \cot[½(\beta + \gamma)] = [\cos½(\alpha + \gamma)\times\sin½(\beta + \gamma) + \cos½(\beta + \gamma)\times\sin½(\alpha + \gamma)]/[\sin½(\alpha + \gamma)\times\sin½(\beta + \gamma)] =$
$[½\sin(\frac{\alpha+\beta}{2} + \gamma) - ½\sin\frac{\alpha-\beta}{2} + ½\sin(\frac{\alpha+\beta}{2} + \gamma) + ½\sin\frac{\alpha-\beta}{2}]/$
$[\sin½(\alpha + \gamma)\times\sin½(\beta + \gamma)] = \sin(\frac{\alpha+\beta}{2} + \gamma)/[\sin½(\alpha+\gamma)\times$
$\sin½(\beta + \gamma)]$.

But because $\alpha + \beta + \gamma = 90°$, $\frac{\alpha+\beta}{2} + \gamma = 45° + \gamma/2$, and

$\sin(\frac{\alpha+\beta}{2} + \gamma) = \sin(45° + \gamma/2) = \sin45°\cos(\gamma/2) + \cos45°\sin(\gamma/2) =$
$\sqrt{2}/2[\sin(\gamma/2) + \cos(\gamma/2)]$. We then need to prove
$\sqrt{2}/2[\sin(\gamma/2) + \cos(\gamma/2)]/[\sin½(\alpha + \gamma)\times\sin½(\beta + \gamma)] \geq 4\cos\gamma$, or
$\sin(\gamma/2)+ \cos(\gamma/2) \geq 4\sqrt{2}\cos\gamma\times\sin½(\alpha + \gamma)\times\sin½(\beta + \gamma)$.

However, $\cos\gamma = \cos^2(\gamma/2) - \sin^2(\gamma/2) = [\cos(\gamma/2) + \sin(\gamma/2)]\times[\cos(\gamma/2) - \sin(\gamma/2)]$, and the previous inequality becomes

$\sin(\gamma/2) + \cos(\gamma/2) \geq 4\sqrt{2}[\cos(\gamma/2) + \sin(\gamma/2)] \times [\cos(\gamma/2) - \sin(\gamma/2)]$
$\times \sin\frac{1}{2}(\alpha + \gamma) \times \sin\frac{1}{2}(\beta + \gamma)$, or
$1 \geq 4\sqrt{2}[\cos(\gamma/2) - \sin(\gamma/2)] \times \sin\frac{1}{2}(\alpha + \gamma) \times \sin\frac{1}{2}(\beta + \gamma)$ (ii)

Similarly, $\cos(\gamma/2) - \sin(\gamma/2) = \sin(90° - \gamma/2) - \sin(\gamma/2) =$
$2\cos45°\sin(45° - \gamma/2) = \sqrt{2}\sin(45° - \gamma/2)$, and
$\sin\frac{1}{2}(\alpha + \gamma) = \sin(45° - \beta/2)$ and $\sin\frac{1}{2}(\beta + \gamma) = \sin(45° - \alpha/2)$, and equation (ii) becomes
$1 \geq 4\sqrt{2} \times \sqrt{2}\sin(45° - \gamma/2) \times \sin(45° - \beta/2) \times \sin(45° - \alpha/2)$, or
$\mathbf{1/8 \geq \sin(45° - \gamma/2) \times \sin(45° - \beta/2) \times \sin(45° - \alpha/2)}$, or
$1/4 \geq [\cos(\alpha/2 + \beta - 45°) - \cos(45° + \alpha/2)] \times \sin(45° - \alpha/2)$, or
$1/4 \geq \sin(45° - \alpha/2) \times \cos(45° - \alpha/2 - \beta) - \sin(45° - \alpha/2) \times \cos(45° + \alpha/2)]$, or $1/2 \geq \sin(90° - \alpha - \beta) + \sin\beta - \sin90° + \sin\alpha$, or
$1/2 \geq \cos(\alpha + \beta) + \sin\beta - 1 + \sin\alpha$, or
$3/2 \geq \cos(90° - \gamma) + \sin\beta + \sin\alpha$, or
$\mathbf{\sin\alpha + \sin\beta + \sin\gamma \leq 3/2}$. Now let's prove it.

Let's assign a function $f(x) = \sin(x)$ on [0, 90°]. We have $f'(x) = \cos x$, $f''(x) = -\sin x < 0$, so the curve of the function is concave and we can apply the Jensen's inequality which proclaims that
$[f(\alpha) + f(\beta) + f(\gamma)]/3 \leq f[(\alpha + \beta + \gamma)/3]$.

Given $(\alpha + \beta + \gamma)/3 = 30°$, $f[(\alpha + \beta + \gamma)/3] = f(30°) = \sin30° = 1/2$.
$[f(\alpha) + f(\beta) + f(\gamma)]/3 = (\sin\alpha + \sin\beta + \sin\gamma)/3 \leq 1/2$, or
$\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3}{2}$, and the inequality is proven.

Equality occurs when $\alpha = \beta = \gamma = 30°$, and this completes our analysis.

<u>Further observation</u>

ABO_3O_1, ACO_3O_2 and BCO_1O_2 are all parallelograms, and triangles ABC and $O_1O_2O_3$ are congruent. By proving that $\sin\alpha + \sin\beta + \sin\gamma \leq 3/2$, we indirectly proved that $1/8 \geq \sin(45° - \gamma/2) \times \sin(45° - \beta/2) \times \sin(45° - \alpha/2)$ when $\alpha + \beta + \gamma = 90°$.

Problem 21 of Moldova Mathematical Olympiad 2002

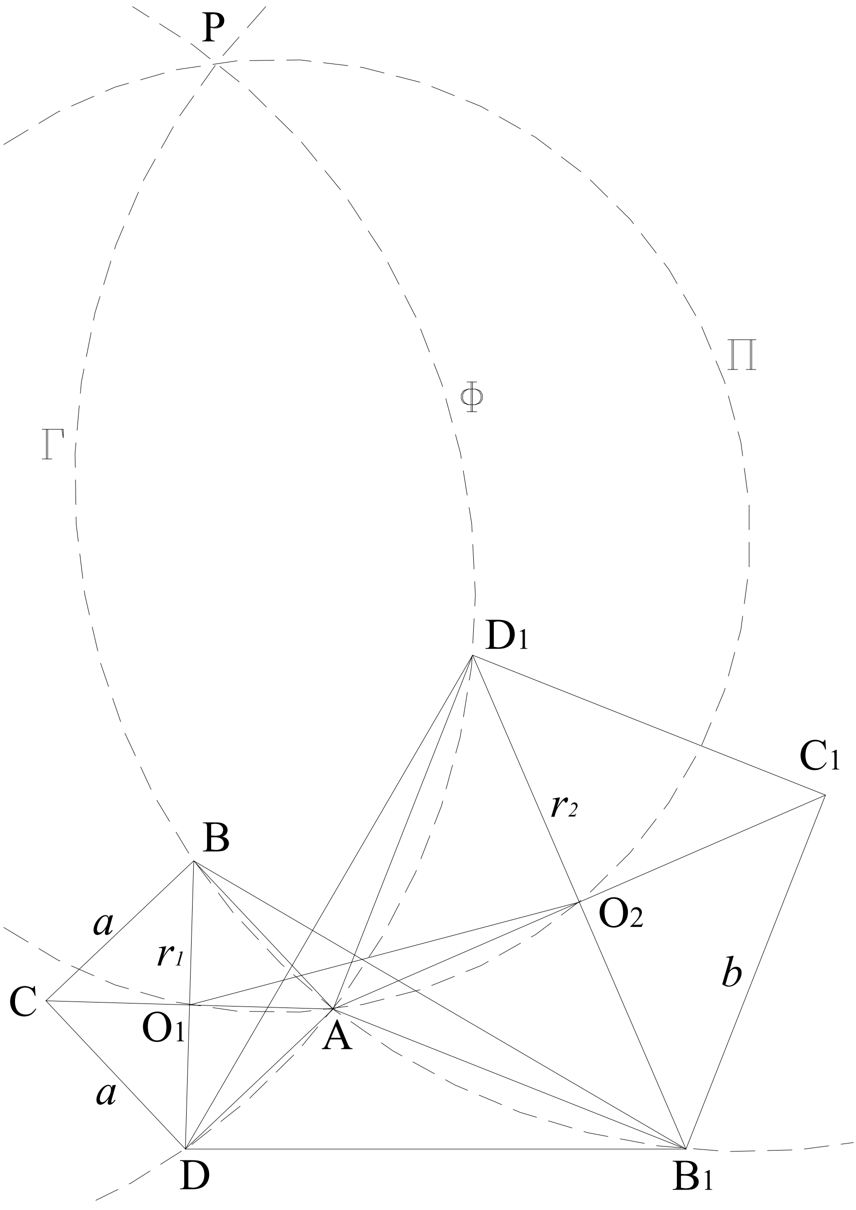
Let the triangle ADB_1 such that $\angle DAB_1 \neq 90°$. On the sides of this triangle externally are constructed the squares ABCD and $AB_1C_1D_1$ with centers O_1 and O_2, respectively. Prove that the circumcircles of the triangles BAB_1, DAD_1 and O_1AO_2 share a common point differs from A.

Solution

Let Γ, Φ and Π be the circumcircles of ΔBAB_1, ΔDAD_1 and ΔO_1AO_2, respectively. Also let a and b be the side lengths of the squares ABCD and $AB_1C_1D_1$, respectively, r_1 and r_2 be the radii of the circumcircles of squares ABCD and $AB_1C_1D_1$, respectively.

It's easily seen that $\angle BAB_1 = \angle DAD_1 = 90° + \angle BAD_1$, and $\Delta BAB_1 = \Delta DAD_1$. Therefore, Γ ≡ (identical to) Φ and $\angle APB = \angle APD$ (subtending arcs with same length a) and $\angle APB_1 = \angle APD_1$ (subtending arcs with same length b). Thus the three points P, B and D are collinear, so are the three points P, D_1 and B_1.

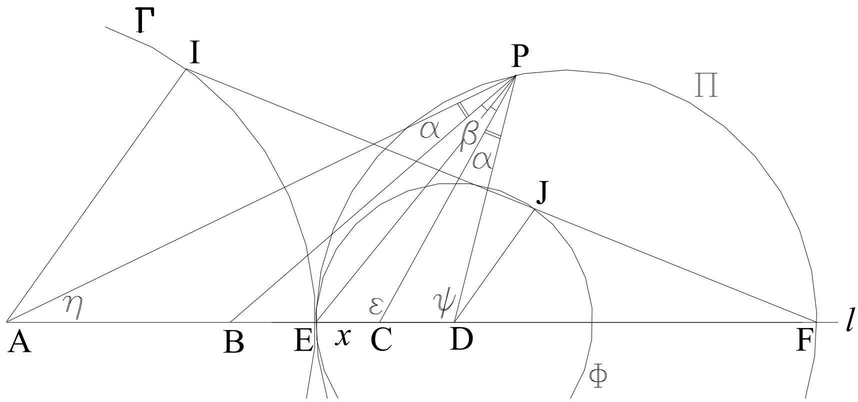
Furthermore, it's easily seen that $\frac{a}{r_1} = \frac{b}{r_2} = \sqrt{2}$, and $\angle O_1AO_2 = \angle O_1AB + \angle BAD_1 + \angle D_1AO_2 = 45° + \angle BAD_1 + 45° = \angle BAB_1$. Therefore, $\Delta O_1AO_2 \cong$ (similar to) ΔBAB_1 and because $\angle APD = \angle APO_1$ and $\angle APB_1 = \angle APO_2$, point P must also be on circle Π, and the circumcircles of the triangles BAB_1, DAD_1 and O_1AO_2 share a common point P differs from A.


P
Γ
Φ
Π
D1
C1
r2
B
O2
a
r1
b
C
O1
A
a
D
B1

Problem 2 of Hungary-Israel Binational 2001

Points A, B, C and D lie on a line *l,* in that order. Find the locus of points P in the plane for which $\angle APB = \angle CPD$.

Solution



Assume that a point P has been established. Let $\alpha = \angle APB = \angle CPD$, $\beta = \angle BPC$, $\eta = \angle BAP$, $\varepsilon = \angle BCP$, $\psi = \angle CDP$. Pick point E on the interior of BC such that $\angle BPE = \angle EPC = \frac{\beta}{2}$ and let $CE = x$.

The problem becomes finding the locus of points P for which $\angle APE = \angle EPD$. Our first task is to find where point E is. Let's find it.

Since PE is the angle bisector of both $\angle APD$ and $\angle BPC$, we have

$$\frac{BE}{CE} = \frac{BP}{CP} \text{ and } \frac{DP}{AP} = \frac{DE}{AE} \qquad \text{(i)}$$

But the law of sines gives us

$\frac{BP}{\sin\eta} = \frac{AB}{\sin\alpha}$, or $BP = \frac{AB\times\sin\eta}{\sin\alpha}$. Similarly, $CP = \frac{CD\times\sin\psi}{\sin\alpha}$, and

$\frac{BP}{CP} = \frac{AB\times\sin\eta}{CD\times\sin\psi}$, or $\frac{BE}{CE} = \frac{AB\times\sin\eta}{CD\times\sin\psi}$.

The law of sines also gives $\frac{\sin\eta}{\sin\psi} = \frac{DP}{AP}$, and now $\frac{BE}{CE} = \frac{AB \times DP}{CD \times AP}$. From (i), we get

$$\frac{BE}{CE} = \frac{AB \times DE}{CD \times AE} = \frac{AB(CD + CE)}{CD(AB + BC - CE)}.$$

Substituting CE = x and BE = BC – x into the above equation, we now have

$$\frac{BC - x}{x} = \frac{AB(CD + x)}{CD(AB + BC - x)}, \text{ or}$$

$$(AB - CD)x^2 + 2AC \times CDx - AC \times BC \times CD = 0.$$

Solving for x, we get

$x = \frac{1}{AB - CD}(-AC \times CD \pm \sqrt{AB \times AC \times BD \times CD})$, but x is positive, and the only valid root is

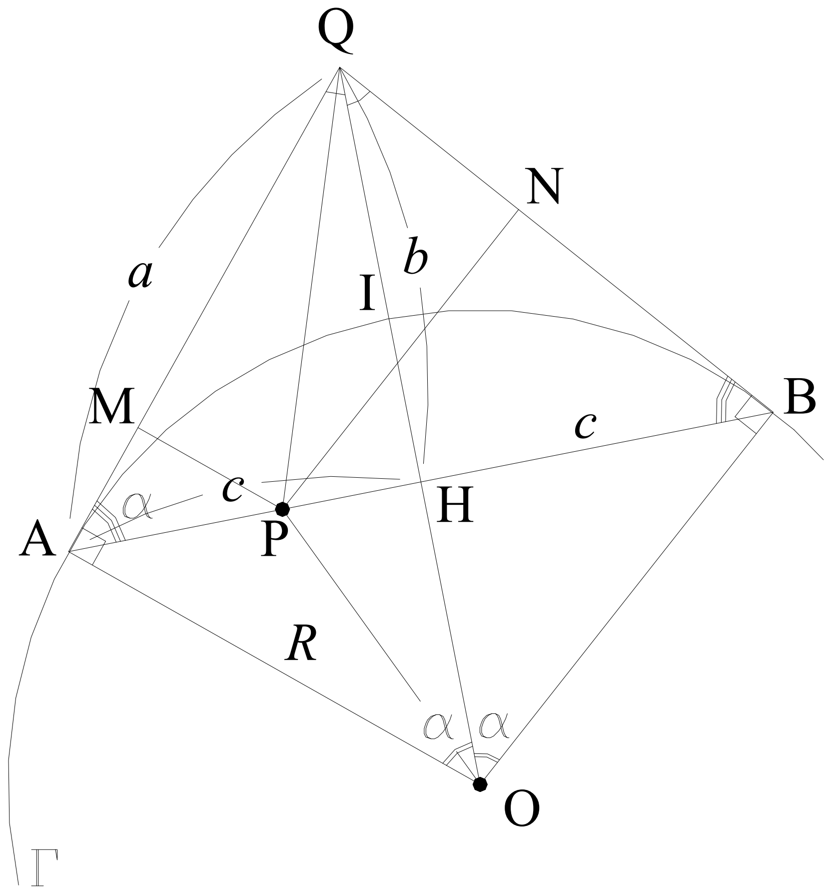
$$x = \frac{1}{AB - CD}(-AC \times CD + \sqrt{AB \times AC \times BD \times CD})$$

So the locus is the circle of *Apollonius* Π with the diameter EF shown on the graph. To draw the circle of *Apollonius* first draw circle Γ with radius AE after we have determined the location of point E from the equation above given the locations of points A, B, C and D; next draw circle Φ with the radius DE. We then draw two arbitrary parallel segments originating from the centers A and D of the two circles Γ and Φ to intercept them at I and J, respectively. Link and extend IJ to meet the extension of AD at F. Then draw the circle of *Apollonius* with the diameter EF. This circle Π is the locus of points P in the plane for which ∠APB = ∠CPD. *See the proof of the Apollonius circle in the second problem of the book.*

Problem 11 of Moldova Mathematical Olympiad 2002

Consider a circle Γ(O, R) and a point P found in the interior of this circle. Consider a chord AB of Γ that passes through P. Suppose that the tangents to Γ at the points A and B intersect at Q. Let M $\in$QA and N $\in$QB such that PM$\perp$QA and PN$\perp$QB. Prove that the value of $\frac{1}{PM}+\frac{1}{PN}$ doesn't depend of choosing the chord AB.

Solution



Let I = OQ ∩ Γ, H = OQ ∩ AB, *a* = QA = QB, *b* = QH, *c* = AH = BH, α = ∠QAB = ∠QBA = ∠QOA = ∠QOB, *R* the radius of the circle Γ and denote (Ω) the area of shape Ω. We have

$$R\left(\frac{1}{\mathrm{PM}}+\frac{1}{\mathrm{PN}}\right)=\frac{\mathrm{OA}}{\mathrm{PM}}+\frac{\mathrm{OB}}{\mathrm{PN}}=\frac{(\mathrm{OAQ})}{(\mathrm{PAQ})}+\frac{(\mathrm{OBQ})}{(\mathrm{PBQ})}.$$

But (OAQ) = (OBQ) = ½$a\times R$, (PAQ) = ½$b\times$AP, (PBQ) = ½$b\times$BP, and we obtain

$$\frac{(\mathrm{OAQ})}{(\mathrm{PAQ})}+\frac{(\mathrm{OBQ})}{(\mathrm{PBQ})}=\frac{a\times R}{b}\times\left(\frac{1}{\mathrm{AP}}+\frac{1}{\mathrm{BP}}\right)=\frac{a\times R}{b}\times\frac{\mathrm{AP}+\mathrm{BP}}{\mathrm{AP}\times\mathrm{BP}}=$$

$$\frac{R}{\sin\alpha}\times\frac{\mathrm{AP}+\mathrm{BP}}{\mathrm{AP}\times\mathrm{BP}}=\frac{R}{\sin\alpha}\times\frac{2c}{\mathrm{AP}\times\mathrm{BP}}=\frac{2R}{\mathrm{AP}\times\mathrm{BP}}\times\frac{cR}{\mathrm{AH}}=\frac{2R^2}{\mathrm{AP}\times\mathrm{BP}}, \text{ or}$$

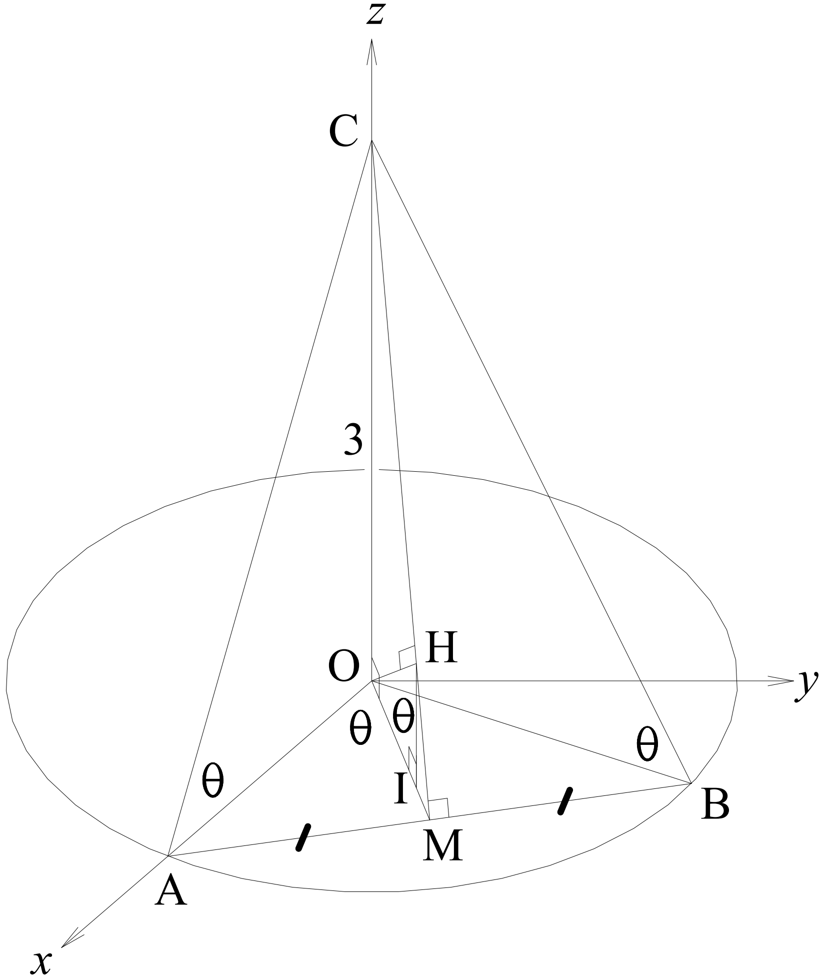
$$\frac{1}{\mathrm{PM}}+\frac{1}{\mathrm{PN}}=\frac{2R}{\mathrm{AP}\times\mathrm{BP}}.$$

Because point P inside the circle Γ, the product AP×BP is always constant no matter where A and B are as long as AB passes through point P, and obviously R is constant. Therefore, $\frac{1}{\mathrm{PM}}+\frac{1}{\mathrm{PN}}=\frac{2R}{\mathrm{AP}\times\mathrm{BP}}$ is constant and does not depend on choosing the chord AB.

Problem 3 of Hitotsubashi University Entrance Exam 2010

In the xyz space with O(0, 0, 0), take points A on the x-axis, B on the xy plane and C on the z-axis such that $\angle$OAC = $\angle$OBC = θ, $\angle$AOB = 2θ, OC = 3. Note that the x coordinate of A, the y coordinate of B and the z coordinate of C are all positive. Denote H the point that is inside ΔABC and is the nearest to O. Express the z coordinate of H in terms of θ.

Solution



Observe that $AC = \frac{OC}{\sin\theta}$, $AO = \frac{OC}{\tan\theta}$, and since the two triangles CAO and AOM are similar, we get $\frac{AM}{AO} = \frac{OC}{AC}$, or $AM = OA\times\frac{OC}{AC}$.

Also the two triangles OCA and OCB are congruent because all their corresponding angles are equal and they share side OC. Therefore, AC = BC and let M be the midpoint of AB. It's easily seen that H is on the segment CM, and $CM \perp AB$. We then have $CM^2 = AC^2 - AM^2$.

The Pythagorean's theorem also gives us $OM^2 = OA^2 - AM^2$.

Draw the altitude HI to the *xy* plane; the *z* coordinate of H is HI, and we have $\frac{HI}{OC} = \frac{HM}{CM}$, or $HI = OC\times\frac{HM}{CM}$.

However, the two triangles OHM and COM are similar, and we get

$$\frac{HM}{OM} = \frac{OM}{CM}, \text{ or } HM = \frac{OM^2}{CM}, \text{ and } HI = OC\times\frac{OM^2}{CM^2} = OC\times\frac{OA^2 - AM^2}{AC^2 - AM^2}$$

$$= OC\times\frac{OA^2 - \dfrac{OA^2\times OC^2}{AC^2}}{AC^2 - \dfrac{OA^2\times OC^2}{AC^2}} = \frac{OA^2\times AC^2 - OA^2\times OC^2}{AC^4 - OA^2\times OC^2} =$$

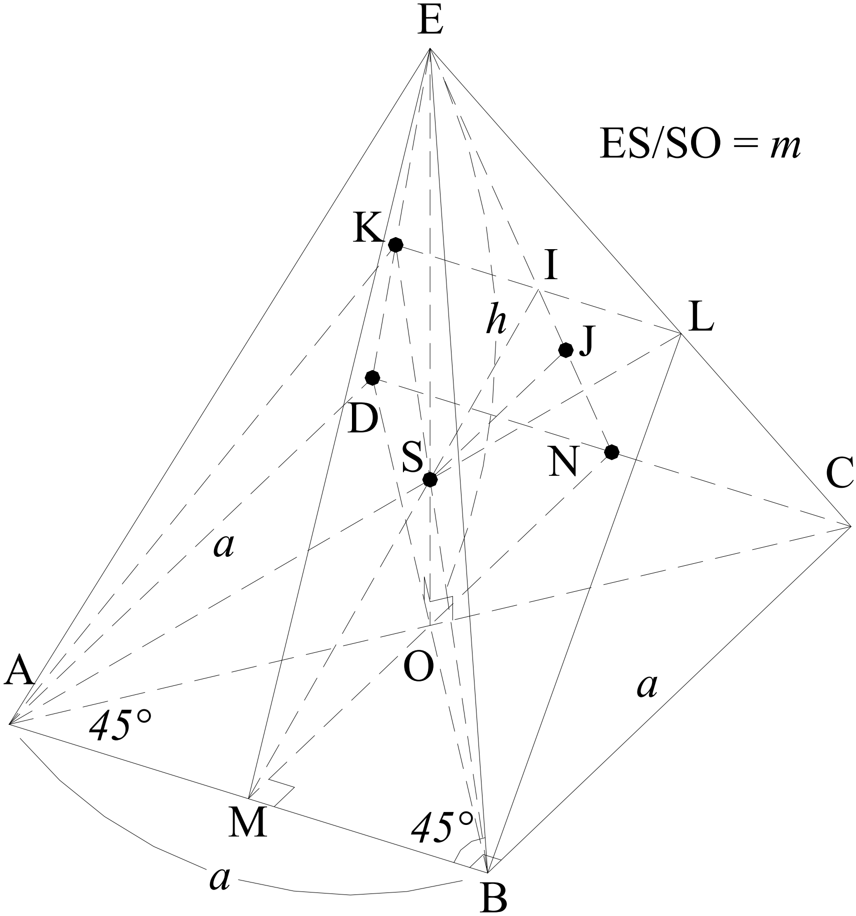
$$3\times\frac{\dfrac{81}{\sin^2\theta\tan^2\theta} - \dfrac{81}{\tan^2\theta}}{\dfrac{81}{\sin^4\theta\tan^2\theta} - \dfrac{81}{\tan^2\theta}} = \frac{3\cos^4\theta}{1 - \sin^2\theta\cos^2\theta}.$$

The *z* coordinate of H in terms of θ is $\frac{3\cos^4\theta}{1 - \sin^2\theta\cos^2\theta}$.

Problem 4 of Moldova Mathematical Olympiad 2006

Let ABCDE be a right quadranular pyramid with vertex E and height EO. Point S divides this height in the ratio ES:SO = *m*. In which ratio does the plane [ABS] divide the lateral area of triangle EDC of the pyramid.

Solution



Let the plane containing A, B and S cut the triangle CDE at K and L with K on ED and L on EC, respectively. Now let M, N and I be the midpoints of AB, CD and KL, respectively. From S draw a segment SJ to parallel ON with J on the plane of triangle CDE.

We have $\dfrac{\text{ES}}{\text{EO}} = \dfrac{1}{\dfrac{\text{EO}}{\text{ES}}} = \dfrac{1}{\dfrac{\text{ES} + \text{SO}}{\text{ES}}} = \dfrac{1}{1 + \dfrac{\text{SO}}{\text{ES}}} = \dfrac{1}{1 + \dfrac{1}{m}}$,

Now let's look at the triangle EMN. We also have

$$\frac{\text{IJ}}{\text{IN}} = \frac{\text{SJ}}{\text{MN}} = \frac{\text{SJ}}{2\text{ON}} = \frac{\text{EJ}}{2\text{EN}} \qquad \text{(i)}$$

But $\dfrac{\text{EJ}}{2\text{EN}} = \dfrac{\text{ES}}{2\text{EO}} = \dfrac{1}{2(1 + \dfrac{1}{m})}$, or $\dfrac{\text{IJ}}{\text{IN}} = \dfrac{1}{2(1 + \dfrac{1}{m})}$.

Also from (i), $\dfrac{\text{IJ}}{\text{IN}} = \dfrac{\text{EJ - IJ}}{2\text{EN} - \text{IN}} = \dfrac{\text{EI}}{\text{EI} + \text{EN}}$, or

$\dfrac{\text{EI} + \text{EN}}{\text{EI}} = \dfrac{\text{IN}}{\text{IJ}} = 2 + \dfrac{2}{m}$, or $1 + \dfrac{\text{EN}}{\text{EI}} = 2 + \dfrac{2}{m}$, or $\dfrac{\text{EN}}{\text{EI}} = 1 + \dfrac{2}{m}$.

Therefore, $\dfrac{\text{EI}}{\text{EN}} = \dfrac{1}{1 + \dfrac{2}{m}}$, and the ratio of the areas of the two triangles EKL and EDC is the square of this ratio $\dfrac{\text{EI}}{\text{EN}}$. Hence, the ratio the plane [ABS] divides the lateral area of triangle EDC of the pyramid is $\dfrac{1}{(1 + \dfrac{2}{m})^2} = (\dfrac{m}{m + 2})^2$.

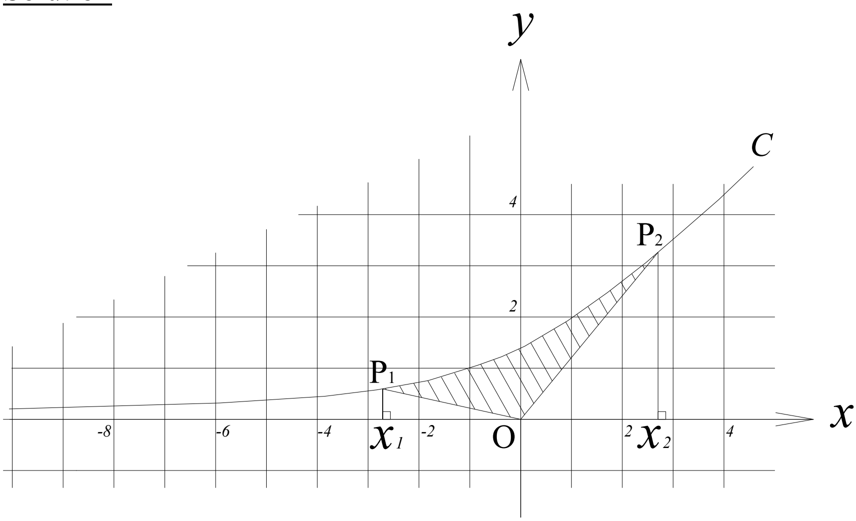
Problem 4 of Tokyo University Entrance Exam 2010

In the coordinate plane with O (0, 0), consider the function C: $y = \frac{1}{2}x + \sqrt{\frac{1}{4}x^2 + 2}$ and two distinct points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ on C.

a) Let H_i (i = 1, 2) be the intersection points of the line passing through P_i (i = 1, 2), parallel to x-axis and the line $y = x$. Show that the area of ΔOP_1H_1 and ΔOP_2H_2 are equal.

b) Let $x_1 < x_2$. Express the area of the figure bounded by the part of $x_1 < x < x_2$ for C and line segments P_1O, P_2O in terms of y_1, y_2.

Solution



a) Denote (Ω) the area of shape Ω. We have

$$(OP_1H_1) = \frac{1}{2}y_1(y_1 - x_1) = \frac{1}{2}\left(\frac{1}{2}x_1 + \sqrt{\frac{1}{4}x_1^2 + 2}\right)\left(-\frac{1}{2}x_1 + \sqrt{\frac{1}{4}x_1^2 + 2}\right)$$
$$= \frac{1}{2}\left(\frac{1}{4}x_1^2 + 2 - \frac{1}{4}x_1^2\right) = 1.$$

Likewise, $(OP_2H_2) = \frac{1}{2}y_2(x_2 - y_2) = \frac{1}{2}(\frac{1}{2}x_2 + \sqrt{\frac{1}{4}x_2^2 + 2})(-\frac{1}{2}x_2 + \sqrt{\frac{1}{4}x_2^2 + 2}) = \frac{1}{2}(\frac{1}{4}x_2^2 + 2 - \frac{1}{4}x_2^2) = 1.$

Therefore, $(OP_1H_1) = (OP_2H_2)$, or the areas of ΔOP_1H_1 and ΔOP_2H_2 are equal.

b) Let the area bounded by the part of $x_1 < x < x_2$ for C be A. We have $A = \int_{x_1}^{x_2}(\frac{1}{2}x + \sqrt{\frac{1}{4}x^2 + 2})\,dx = \frac{1}{2}\int_{x_1}^{x_2}(x + \sqrt{x^2+8})\,dx =$

$[\frac{1}{4}x^2 + \frac{1}{4}x\sqrt{x^2+8} + 2\ln(x + \sqrt{x^2+8})]\Big|_{x_1}^{x_2} = \frac{1}{4}(x_2^2 - x_1^2) + \frac{1}{4}(x_2\sqrt{x_2^2+8} - x_1\sqrt{x_1^2+8}) + 2[\ln(x_2 + \sqrt{x_2^2+8}) - \ln(x_1 + \sqrt{x_1^2+8})]$

$= \frac{1}{2}x_2y_2 - \frac{1}{2}x_1y_1 + 2[\ln(2y_2) - \ln(2y_1)] = \frac{1}{2}x_2y_2 - \frac{1}{2}x_1y_1 + 2\ln(\frac{y_2}{y_1}).$

Whereas,

$(OP_2x_2) = \int_0^{x_2}\frac{y_2}{x_2}x\,dx = \frac{1}{2}x_2y_2$ and $(OP_1x_1) = \int_{x_1}^{0}\frac{y_1}{x_1}x\,dx = -\frac{1}{2}x_1y_1.$

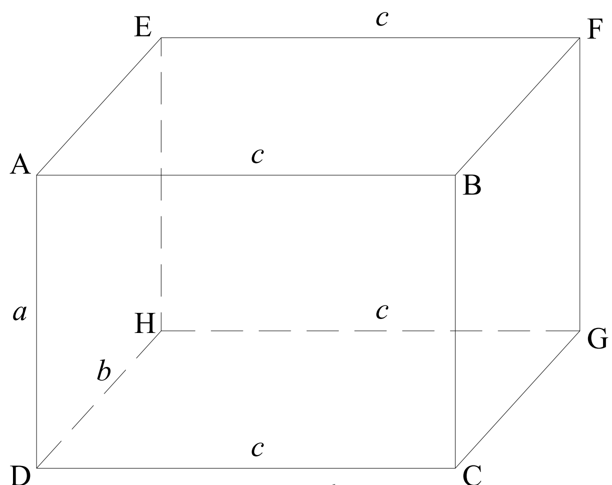
And the area of the figure bounded by the part of $x_1 < x < x_2$ for C and line segments P_1O, P_2O is $A - (OP_1x_1) - (OP_2x_2)$ which is the shaded area on the graph, and it equals $2\ln(\frac{y_2}{y_1})$. *The reader is encouraged to find the areas when both x_1 and x_2 are either on the left or right side of the y-axis.*

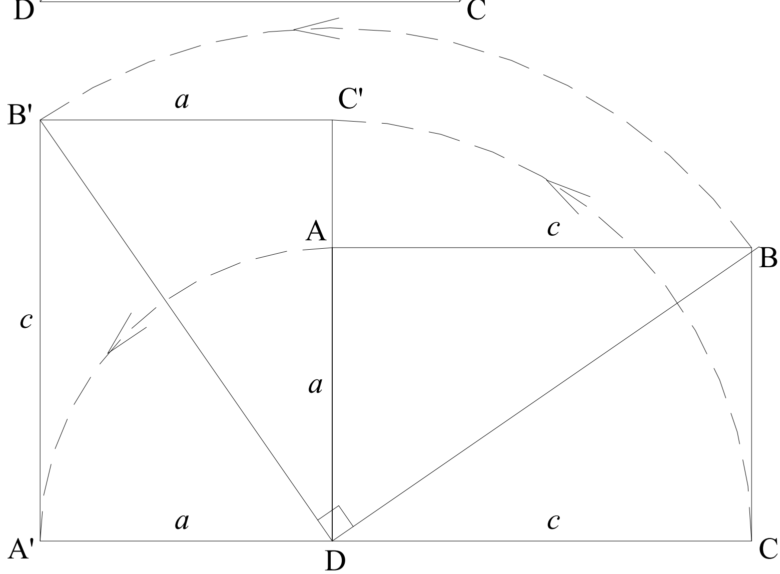
<u>Problem 1 of Tokyo University Entrance Exam 2010</u>

Let the lengths of the sides of a cuboid be denoted a, b and c. Rotate the cuboid in 90° the side with length b as the axis of the cuboid. Denote by V the solid generated by sweeping the cuboid.

a) Express the volume of V in terms of a, b and c.
b) Find the range of the volume of V with $a + b + c = 1$.

<u>Solution</u>





a) After the 90° rotation, point A moves to A', B to B', C to C' as shown. The volume of the solid generated by sweeping the cuboid equals the height b times the combined areas of triangles A'B'D, BCD and area formed by arc BB' and the two segments B'D and BD.

The areas of triangles A'B'D, BCD equal the area of the rectangle ABCD = ac. Now easily note that the angle formed by the two segments B'D and BD is 90°. Therefore, the area formed by arc BB' and the two segments B'D and BD equals a quarter of the area

of the circle with radius of $\sqrt{a^2 + c^2}$ and it is $\frac{1}{4}\pi(a^2 + c^2)$. Therefore, $V = b[ac + \frac{1}{4}\pi(a^2 + c^2)] = abc + \frac{1}{4}\pi b(a^2 + c^2)$.

b) Let's write $V = abc + \frac{1}{4}\pi b(a^2 + c^2) = abc + \frac{1}{4}\pi b[(a + c)^2 - 2ac]$ $= abc(1 - \frac{1}{2}\pi) + \frac{1}{4}\pi b(a + c)^2$.

With $a + b + c = 1$, the maximum of the product abc occurs when $a = b = c = \frac{1}{3}$.

Now let's find the maximum of the other product $P = b(a + c)^2$. Let $x = b$ and $y = a + c$. The task reduces to finding the maximum of $P = xy^2$ given $x + y = 1$.

$P = x(1 - x)^2$, and $P' = 1 - 4x + 3x^2 = 0$ when $x = 1/3$. Carrying out the procedure we found that the maximum of P also occurs when $x = 1/3$, $y = 2/3$, or when $a = b = c = 1/3$, and the range of the volume of V is $0 < V \leq \frac{1 + \frac{1}{2}\pi}{27}$.

<u>Further observation</u>

For your information only: Because $a^2 + c^2 \geq 2ac$, $V \geq abc + \frac{1}{2}\pi abc = abc(1 + \frac{1}{2}\pi)$, but this does not help in determining the minimum value of V since when $b \longrightarrow 0$, $V \longrightarrow 0$.

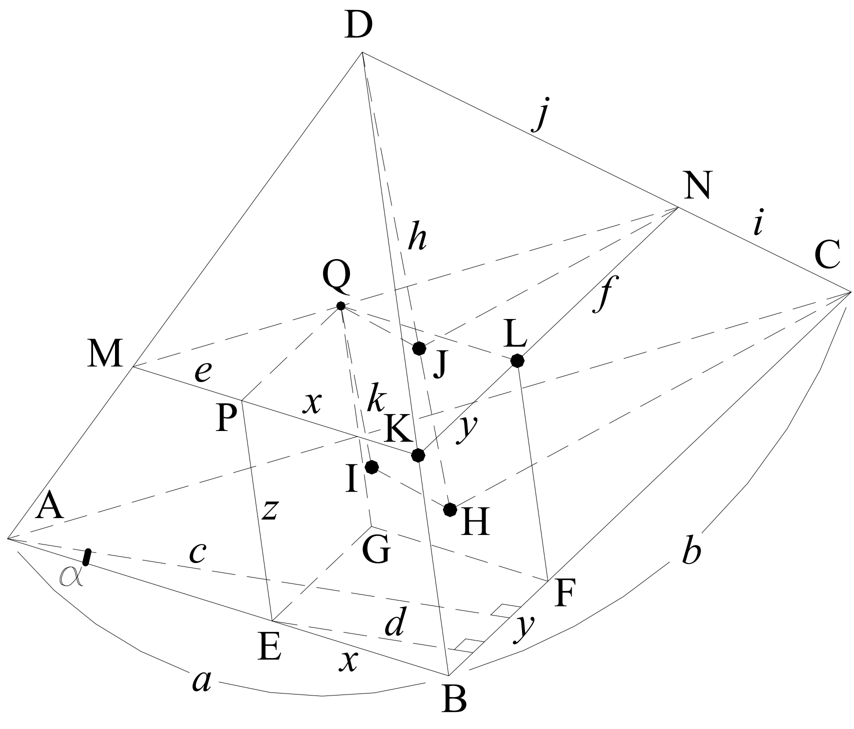
Problem 3 of the Vietnamese Mathematical Olympiad 1990

A tetrahedron is to be cut by three planes which form a parallelepiped whose three faces and all vertices lie on the surface of the tetrahedron.

a) Can this be done so that the volume of the parallelepiped is at least $\frac{9}{40}$ of the volume of the tetrahedron?

b) Determine the common point of the three planes if the volume of the parallelepiped is $\frac{11}{50}$ of the volume of the tetrahedron.

Solution



Denote (Ω) the area of shape Ω and [Φ] the plane containing shape Φ. Let the parallelepiped be BEGF-KPQL where E is on AB, F on BC, G on [ABC] (EG || BC and EG = BF), K on BD, P on [ABD] (KP || AB, KP = BE), L on [BCD] (KL || BC, KL = BF).

Extend KP and KL to meet AD and DC at M and N, respectively. Q, therefore, is on MN and is on [ADC]. Drop the altitudes DH and QI to plane [ABC]. Let h = DH and k = QI, the heights of the tetrahedron and the parallelepiped, respectively. Now also let x = BE, y = BF, z = EP (the three dimensions of the parallelepiped), i = NC, j = DN, c and d the lengths of the altitudes from A and E onto BC, respectively.

a) We are required to prove whether that the volume of the parallelepiped Vp is at least $\frac{9}{40}$ of the volume of the tetrahedron Vt. In other words,

$\mathrm{V}p = kdy \geq \frac{9}{40}\mathrm{V}t = \frac{9}{40}\times\frac{1}{3}\times\frac{hbc}{2} = \frac{3hbc}{80}$, or $2\times\frac{dy}{bc} \geq \frac{3}{40}\times\frac{h}{k}$, or

$\frac{(\mathrm{KPQL})}{(\mathrm{ABC})} \geq \frac{3}{40}\times\frac{h}{k}$ is the condition that needs to be satisfied.

Now note that these triangles are similar to one another ΔABC, ΔMKN, ΔMPQ and ΔQLN because all their corresponding sides parallel to one another, and [MKN] || [ABC] which implies that $\frac{k}{h} = \frac{\mathrm{NC}}{\mathrm{DC}} = \frac{i}{i+j}$, or $\frac{h}{k} = \frac{i+j}{i} = 1 + \frac{j}{i}$.

Also note that dy = (BEGF) = (KPQL) = (MKN) – (MPQ) – (QLN), and the ratios of the areas of similar shapes as

$\frac{(\mathrm{MPQ})}{(\mathrm{MKN})} = \frac{e^2}{(e+x)^2} = \frac{\mathrm{MQ}^2}{\mathrm{MN}^2}$,

$\frac{(\mathrm{QLN})}{(\mathrm{MKN})} = \frac{f^2}{(f+y)^2} = \mathrm{NQ}^2/\mathrm{MN}^2$,

$\frac{(\mathrm{MKN})}{(\mathrm{ABC})} = \frac{(e+x)^2}{a^2} = \frac{\mathrm{MN}^2}{\mathrm{AC}^2} = \frac{j^2}{(i+j)^2}$.

Therefore,

$(\mathrm{MPQ})/(\mathrm{ABC}) = [e/(e+x)]^2\times[(e+x)/a]^2 = e^2/a^2 = \mathrm{MQ}^2/\mathrm{AC}^2$,

$(\mathrm{QLN})/(\mathrm{ABC}) = [f/(f+y)]^2\times[(e+x)/a]^2 = \mathrm{NQ}^2/\mathrm{AC}^2$.

We then have $\frac{(\mathrm{KPQL})}{(\mathrm{ABC})} = \frac{(\mathrm{MKN})}{(\mathrm{ABC})} - \frac{(\mathrm{MPQ})}{(\mathrm{ABC})} - \frac{(\mathrm{QLN})}{(\mathrm{ABC})} = j^2/(i+j)^2 -$

$MQ^2/AC^2 - NQ^2/AC^2 = j^2/(i + j)^2 - (MQ^2 + NQ^2)/AC^2 =$
$j^2/(i + j)^2 - [(MQ + NQ)^2 - 2MQ\times NQ)]/AC^2 =$
$j^2/(i + j)^2 - (MQ + NQ)^2/AC^2 + 2MQ\times NQ/AC^2 =$
$j^2/(i + j)^2 - MN^2/AC^2 + 2MQ\times NQ/AC^2$.

However, $MN^2/AC^2 = j^2/(i + j)^2$, and the above expression becomes $\frac{(KPQL)}{(ABC)} = 2MQ\times NQ/AC^2$. Finally, we need to find the condition to satisfy $2MQ\times NQ/AC^2 \geq \frac{3}{40}\times\frac{i+j}{i}$. Now let $z = \frac{MN}{AC}$ and $w = \frac{MQ}{AC}$. We have $z - w = \frac{NQ}{AC}$, $\frac{i+j}{i} = \frac{1}{1-z}$, and the previous inequality becomes $2w(z - w)(1 - z) \geq \frac{3}{40}$ (i)
or $80wz - 80w^2 - 80wz^2 + 80w^2z - 3 \geq 0$.

We see that the two variables w and z are independent of each other. Let's treat z as a constant and w a variable. Taking the derivative of $f(w) = 80wz - 80w^2 - 80wz^2 + 80w^2z - 3$ with respect to w, we get

$f'(w) = 80z - 160w - 80z^2 + 160wz$. Equating it to zero, we get $f'(w) = (2w - z)(z - 1) = 0$ but $1 > z$ thus $w = \frac{z}{2}$.

We have $f(w = \frac{z}{2}) = 20z^2(1- z) - 3$. Again, since $1 > z, f(\frac{z}{2}) > -3$.

We also find that $f(0) = f(z > \frac{z}{2}) = -3$; therefore, the maximum value of $f(w)$ is $20z^2(1- z) - 3$ occurring at $w = \frac{z}{2}$. Now we need to verify that $20z^2(1 - z) - 3 > 0$ (ii) as required.

Let $g(z) = 20z^2(1- z) - 3$. Again taking the derivative of $g(z)$ we have $g'(z) = 40z - 60z^2$. It equals zero when $z = \frac{2}{3}$. And we have $g(z = \frac{2}{3}) = -\frac{1}{27}$; $g(0) = g(1) = -3$. Therefore, the maximum value of $g(z)$

is $-\frac{1}{27}$ and is negative, and the inequality (ii) can not be satisfied. Thus the volume of the parallelepiped can not be at least $\frac{9}{40}$ of the volume of the tetrahedron under any circumstances.

b) For the volume of the parallelepiped to be $\frac{11}{50}$ of the volume of the tetrahedron, let's set the new value $\frac{11}{50}$ for equation (i), and in this case it's an equality.

$$2w(z-w)(1-z) = \frac{11}{50}\times\frac{1}{3}\times\frac{1}{1-z} \qquad \text{(iii)}$$

The equation is equivalent to

$300(z-1)w^2 - 300z(z-1)w - 11 = 0.$
Solving for w, we get $w = \frac{1}{2}\,[z \pm \sqrt{z^2 + \frac{11}{75(z-1)}}\,]$.

So there are potentially more than one solutions meaning that a tetrahedron can be cut by three planes located at many different locations to form the parallelepiped with the volume being $\frac{11}{50}$ of the volume of the tetrahedron.

Let's pick $z = \frac{2}{3}$ as found in the previous case to find $w = \frac{3}{10}$. So the height of the parallelepiped is one-third ($\frac{1}{3}$) of that of the tetrahedron and vertex Q is located such that MQ = $\frac{3}{10}$ AC.

Problem 3 of Spain Mathematical Olympiad 1994

A tourist office was investigating the numbers of sunny and rainy days in a year in each of six regions. The results are partly shown in the following table:

Region	*Sunny or rainy*	*Unclassified*
A	336	29
B	321	44
C	335	30
D	343	22
E	329	36
F	330	35

Looking at the detailed data, an officer observed that if one region is excluded, then the total number of rainy days in the other regions equals one third of the total number of sunny days in these regions. Determine which region is excluded.

Solution

The total number of sunny and rainy days for all regions is $336 + 321 + 335 + 343 + 329 + 330 = 1994$ days.

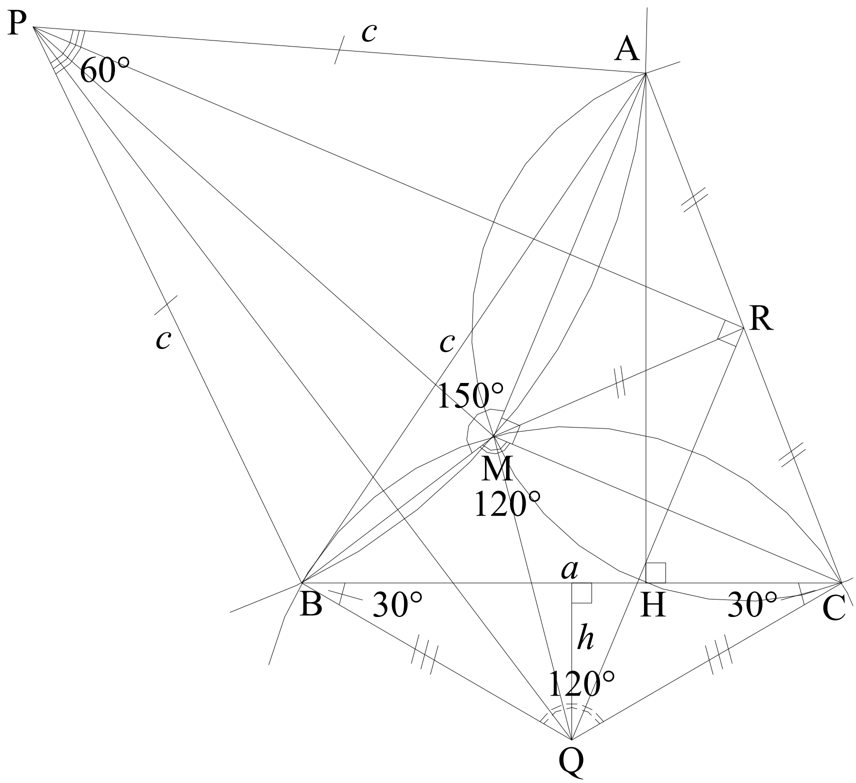
Let the number of sunny or rainy days in the region excluded to be n; the number $\frac{1994 - n}{4}$ must be an integer.
Thus n has to be an even number for $1994 - n$ to be divisible by 4. So n is either 336 or 330, and only $1994 - 330 = 1664$ is divisible by 4.

Therefore, the region excluded is region F.

<u>Problem 26 of India Postal Coaching 2010</u>

Let M be an interior point of a triangle ABC such that $\angle$AMB = 150°, $\angle$BMC = 120°, Let P, Q, R be the circumcenters of the triangles AMB, BMC, CMA, respectively. Prove that (PQR) ≥ (ABC).

<u>Solution</u>



Let BC = *a*, AB = *c*, the length of the altitude from Q to BC be *h*, (Ω) denote the area of shape Ω.

Because PA, PB, PM are the radii of the circumcircle of triangle AMB, QB, QC, QM are the radii of the circumcircle of triangle BMC and RA, RC, RM are the radii of the circumcircle of triangle CMA, these triangles are congruent PMQ and PBQ, PMR and PAR, RMQ and RCQ (because all their respective sides are equal.)

Therefore, (PACQB) = 2(PQR). We can now prove that (PACQB) $\geq$ 2(ABC).

But (PACQB) = (ABC) + (ABP) + (BQC). It suffices to prove that (ABP) + (BQC) $\geq$ (ABC) (i)

Now note that because P is the circumcenter and $\angle$AMB = 150°, $\angle$APB = 2(180° – 150°) = 60°. Combining with the fact that PAB is an isosceles triangle with $\angle$PAB = $\angle$PBA, $\angle$PAB = $\angle$PBA = 60° and PAB is then an equilateral triangle.

Similarly, because Q is the circumcenter and $\angle$BMC = 120°, $\angle$BQC = 2(180° – 120°) = 120° and with BQC being an isosceles triangle with $\angle$QBC = $\angle$QCB, $\angle$QBC = $\angle$QCB = 30°.

The altitude of an equilateral with side length c is known to be $c\frac{\sqrt{3}}{2}$, and (ABP) = $\frac{1}{2}c \times c\frac{\sqrt{3}}{2} = \frac{1}{4}c^2\sqrt{3}$. In the mean time, (BQC) = $\frac{1}{2}a \times h = \frac{1}{4}a^2\tan 30° = \frac{1}{4}\frac{a^2}{\sqrt{3}}$. Now draw the altitude AH from A onto BC, (ABC) = $\frac{1}{2}a \times$AH = $\frac{1}{2}ac \times \sin\angle$ABC, and the inequality (i) required to be proven becomes

$$\frac{1}{4}\left(c^2\sqrt{3} + \frac{a^2}{\sqrt{3}}\right) = \frac{\sqrt{3}}{4}\left(c^2 + \frac{a^2}{3}\right) \geq \frac{1}{2}ac \times \sin\angle ABC$$

Applying the AM-GM inequality, we get

$\frac{\sqrt{3}}{4}\left(c^2 + \frac{a^2}{3}\right) \geq \frac{1}{2}ac$, and (ii) becomes $\frac{1}{2}ac \geq \frac{1}{2}ac \times \sin\angle ABC$.

Because the sine value of any angle is less than or equal to 1, and the previous inequality is true.

Problem 4 of the International Zhautykov Olympiad 2010

Positive integers 1, 2, . . ., n are written on a blackboard ($n > 2$). Every minute two numbers are erased and the least prime divisor of their sum is written. In the end only the number 97 remains. Find the least n for which it is possible.

Solution

Noting that any prime divisor not equal to 2 is an odd number, the sum of two such prime numbers is an even number, and the prime divisor that is also an even number is number 2.

To find the least possible n we should keep a free adder, a number originally on the board in the series 1, 2, . . ., n that is free from any calculations until the end to add to the last prime divisor. Since adding two odd numbers or even numbers creates a new even number that has 2 as its least prime divisor, the problem requires the addition of a prime number to number 2 to become another prime number.

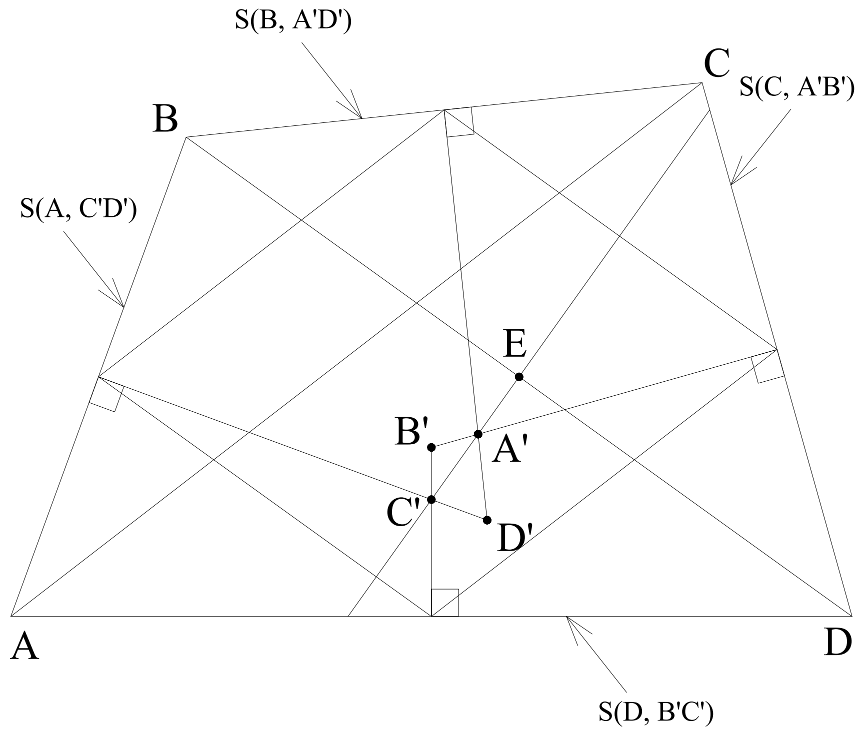
Furthermore, unlike the additions of the even numbers, the total odd numbers in the series 1, 2, . . ., n have to be in pairs plus one more. The pairs of the odd numbers help adding up to exact even numbers in order for us to get 2 as their prime divisors. The two consecutive prime numbers under 97 that satisfy these conditions are 41 and 43, 59 and 61. We pick the pair 41 and 43 to give us the least n and $n = 57$.

Erasing the numbers this way: first erase all the odd numbers in pairs from 1 to 39, their pairings are of no significance, to get 2 at the end, leave number 41 alone. Next erase the rest of the odd numbers from 43 to 57 to get another number 2 on the board (we should have a total of three numbers 2 now.) Then erase all the even numbers except number 54 (which is the free adder.) The remaining numbers on the board are 2, 41, and 54. Now erase 41 and 2 to get 43. Then erase 43 and 54 to get 97.

<u>Problem 6 of the Iranian Mathematical Olympiad 1995</u>

In a quadrilateral ABCD let A’, B’, C’ and D’ be the circumcenters of the triangles BCD, CDA, DAB and ABC, respectively. Denote by S(X, YZ) the plane which passes through the point X and is perpendicular to the line YZ. Prove that if A’, B’, C’ and D’ don't lie in a plane, then four planes S(A, C’D’), S(B, A’D’), S(C, A’B’) and S(D, B’C’) pass through a common point.

<u>Solution</u>



Let [Φ] denote the plane containing shape Φ, *l* be the line that is perpendicular to [ABC] at B and *k* the line perpendicular to [A’DC’] at D, respectively, E = BD ∩ A’C’. It’s easily seen that *l* = S(A, C’D’) ∩ S(B, A’D’) and *k* = S(C, A’B’) ∩ S(D, B’C’).

But since A’ and C’ are the circumcenters of the triangles BCD, and DAB, we have A’B = A’D and C’B = C’D. Therefore, triangles BC'A' and DC'A' are congruent because they also share segment A'C'. It follows that triangles BC'E and DC'E are congruent and we get $\angle$BEC' = 90°, or BD$\perp$C'E and C'E perpendiculars to the plane containing lines *l*, *k* and segment BD.

Now assume point B' does not lie on a plane that contains the other three points A', C' and D'. Since *l*, *k* and BD currently lie on the same plane perpendicular to C'E, folding this plane along C'E would make the two line *l* and *k* to intersect each other, and we conclude that the four planes S(A, C'D'), S(B, A'D'), S(C, A'B') and S(D, B'C') pass through a common point on [BED] after folding.

<u>Further observation</u>

It's also easily seen that B'D'$\perp$AC with the same analysis.

Problem 8 of Hong Kong Mathematical Olympiad 2008

Let $Q = \log_{2+\sqrt{2^2-1}}(2-\sqrt{2^2-1})$. Find the value of Q.

Solution

The equation is equivalent to $(2+\sqrt{2^2-1})^Q = 2-\sqrt{2^2-1}$.

But we have $(2+\sqrt{2^2-1})(2-\sqrt{2^2-1}) = 1$, or

$$\frac{1}{2+\sqrt{2^2-1}} = 2-\sqrt{2^2-1}\text{, or}$$

$$(2+\sqrt{2^2-1})^{-1} = 2-\sqrt{2^2-1}.$$

Therefore, Q = -1.

Problem 9 of Hong Kong Mathematical Olympiad 2008

Let $F = 1 + 2 + 2^2 + 2^3 + \ldots + 2^s$ and $T = \sqrt{\dfrac{\log(1 + F)}{\log 2}}$. Find the value of T.

Solution

We have $1 + 1 + 2 = 2^2$,

$1 + 1 + 2 + 2^2 = 2^3$,

$1 + 1 + 2 + 2^2 + 2^3 = 2^4$, etc...

$1 + F = 2 + 2 + 2^2 + 2^3 + \ldots + 2^s = 2^{s+1}$, and

$$T = \sqrt{\frac{\log(1 + F)}{\log 2}} = \sqrt{\frac{\log(2^{s+1})}{\log 2}} = \sqrt{s + 1}.$$

Problem 2 of Netherlands Dutch Mathematical Olympiad 1998

Let TABCD be a pyramid with top vertex T, such that its base ABCD is a square of side length 4. It is given that, among the triangles TAB, TBC, TCD and TDA one can find an isosceles triangle and a right-angled triangle. Find all possible values for the volume of the pyramid.

Solution

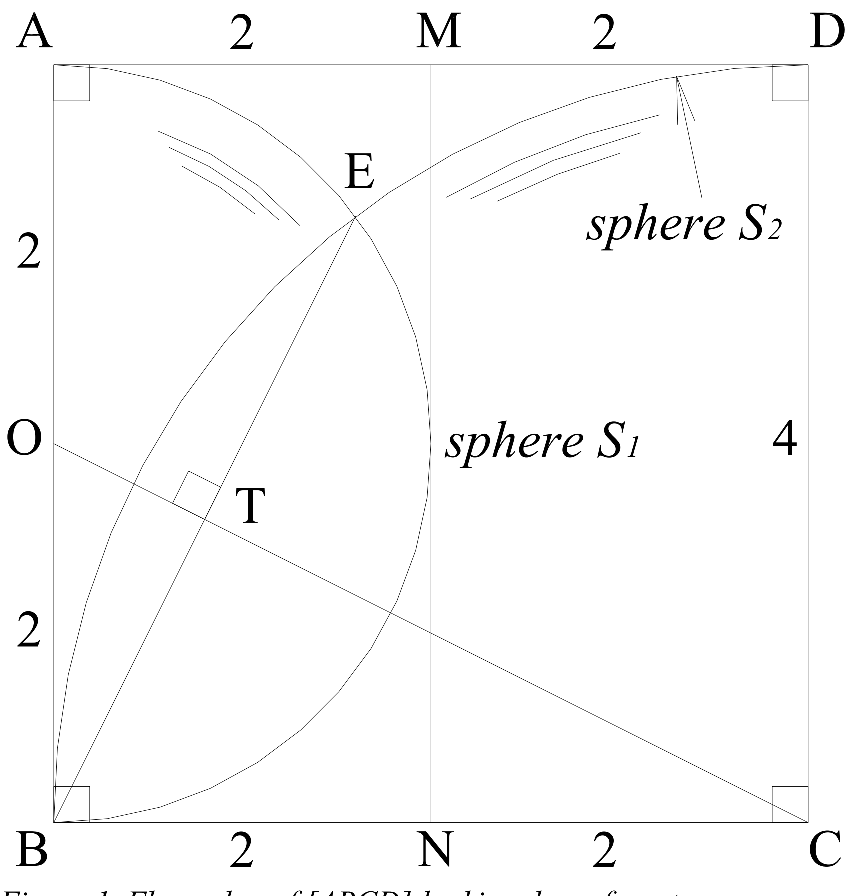


Figure 1. Floor plan of [ABCD] looking down from top

Let [Φ] denote the plane containing shape Φ, (Ω) denote the area of shape Ω, O, M and N be the midpoints of AB, AD and BC, respectively, H the foot of T onto [ABCD], and h = TH, the height of point T above [ABCD], V and V_{max} the volume and the maximum volume of the pyramid, respectively.

Let's ignore the scenarios that cause the pyramid to be degenerate or T to be on [ABCD] or the scenarios that make the volume of the pyramid to be zero except for the special circumstances. Also ignore any scenario where the volume is the same because of symmetry. Note that the volume of the pyramid is given by the equation $V = \frac{1}{3}h \times$(ABCD). The possible scenarios that give rise to the different possible values for the volume of the pyramid among the triangles TAB, TBC, TCD and TDA with one being an isosceles triangle and another the right-angled triangle are:

<u>Scenario 1:</u> ΔTAB with $\angle$TAB = 90° and ΔTAD isosceles with TA = TD. For TA = TD, T has to be on the plane that is perpendicular to [ABCD] and passes through M and N and T must also be on the plane that is perpendicular to [ABCD] and containing segment AD. In other words, T must be on the infinite line that perpendiculars [ABCD] at M. Therefore, in this scenario, h is anywhere from near zero to infinity, and the volume of the pyramid is, therefore, also from near zero to infinity, $\boldsymbol{V \in (0, \infty)}$. This is the same scenario if we replace $\angle$TAB with $\angle$TBA and ΔTAD with ΔTBC because of symmetry, or replacing ΔTAD with an isosceles ΔTBC with TB = TC.

<u>Scenario 2:</u> Same as scenario 1, ΔTAB with $\angle$TAB = 90° and ΔTAD isosceles but with TA = AD, or TD = AD. We find that in either case, the maximum value of h is the side length of ABCD, h = 4, occurring when H ≡ (coincides) A or H ≡ D, and $\boldsymbol{V_{max} = \frac{64}{3}}$, or $\boldsymbol{V \in (0, \frac{64}{3}]}$. This is the same scenario if we replace $\angle$TAB with $\angle$TBA and ΔTAD with ΔTBC because of symmetry.

<u>Scenario 3:</u> Same as scenarios 1 but with $\angle ATB = 90°$ and ΔTBC isosceles with TC = BC = 4 which is depicted in figure 1. We find that T must be on both the sphere S_1 with the center O and the radius of 2 and the sphere S_2 with the center C and radius of 4. It's easily seen that T must be on the plane [OTC] perpendiculars to [ABCD] as shown on figure 2 on the next page, and we have these equations

$OT^2 = h^2 + OH^2$,
$CT^2 = h^2 + CH^2$,
$OC = OH + HC$ and
$OC^2 = OB^2 + BC^2$ (see figure 1). Solving these equations with OB

= OT = 2, BC = TC = 4, we get $OC = 2\sqrt{5}$ making the two circles shown on figure 2 orthogonal, $OH = \frac{2}{\sqrt{5}}$, $CH = \frac{8}{\sqrt{5}}$, $h = \frac{4}{\sqrt{5}}$ and the volume $\boldsymbol{V = \frac{64}{3\sqrt{5}}}$. This is the same scenario if we replace $\angle TBC$ with $\angle TAD$ because of symmetry. Also note that this scenario where TC = BC = CD = 4 also includes the situation for ΔTCD being isosceles with TC = CD.

<u>Scenario 4:</u> Same as scenarios 3 but TB = BC. This is the scenario that causes the volume V to be zero but is worth included here. In this scenario, the sphere S_1 and the sphere with center B and radius of 4, the side length of ABCD do not meet at any point except point A. Therefore, $\boldsymbol{V = 0}$.

<u>Scenario 5:</u> Same as scenarios 3 but in ΔTCD, TC = TD. It's easily seen that the highest point T occurs when $H \equiv O$ and $h = 2$. The maximum volume is $\boldsymbol{V_{max} = \frac{32}{3}}$, or $\boldsymbol{V \in (0, \frac{32}{3}]}$.

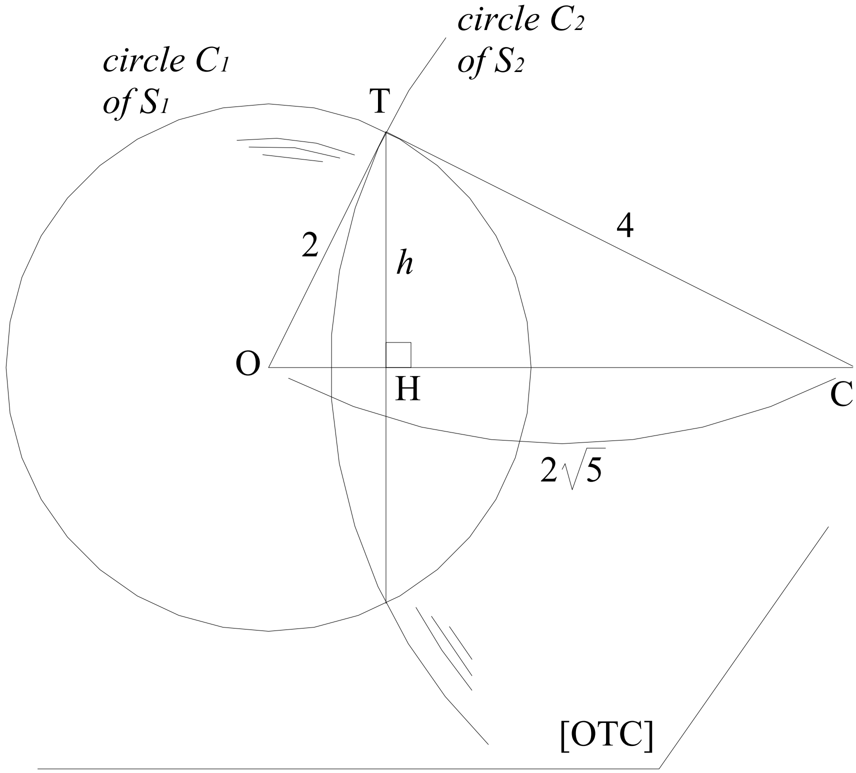
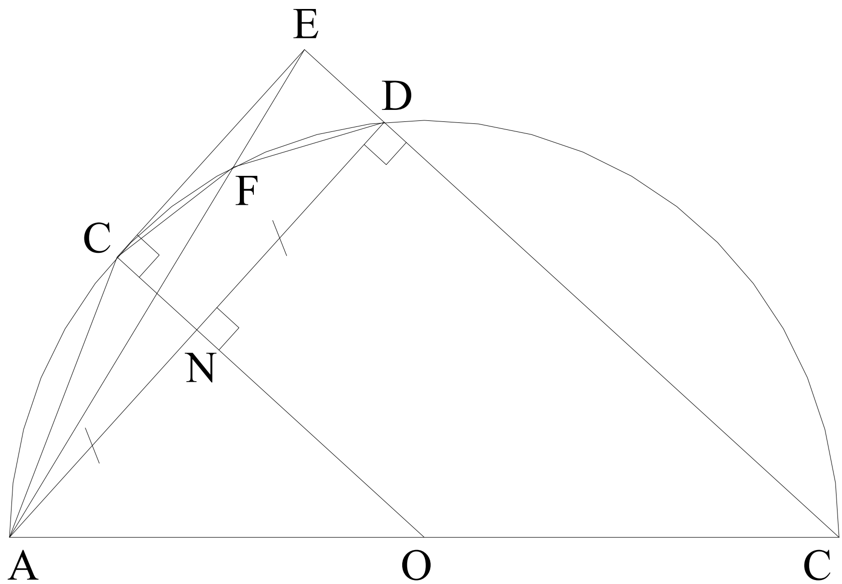


Figure 2. Cross section of plane [OTC]

<u>Problem 6 of Austria Mathematical Olympiad 2001</u>

We are given a semicircle with diameter AB. Points C and D are marked on the semicircle, such that AC = CD holds. The tangent of the semicircle in C and the line joining B and D intersect in a point E, and the line joining A and E intersects the semicircle in a point F. Show that FD > FC must hold.

<u>Solution</u>



Let O be the center of the semi-circle, N = OC ∩ AD. Because AC = CD, AN = ND and OC ⊥ AD. Furthermore, because AD ⊥ DB CEDN is a rectangle, and CE ∥ AD, and it follows that ∠CEA = ∠EAD.

Applying the law of sines for triangle ACE, we get

$$\frac{AC}{\sin\angle CEA} = \frac{CE}{\sin\angle CAE}, \text{ or } \frac{CD}{\sin\angle EAD} = \frac{CE}{\sin\angle CAE}.$$

But CD is the diagonal of rectangle CEDN; therefore, CD > CE, and to satisfy the previous equation, sin∠EAD > sin∠CAE, or ∠EAD > ∠CAE. Since ∠EAD subtends arc FD and ∠CAE subtends arc FC, FD > FC.

<u>Problem 3 of Tokyo University Entrance Exam 2008</u>

A regular octahedron is placed on a horizontal rest. Draw the plan of top-view for the regular octahedron.

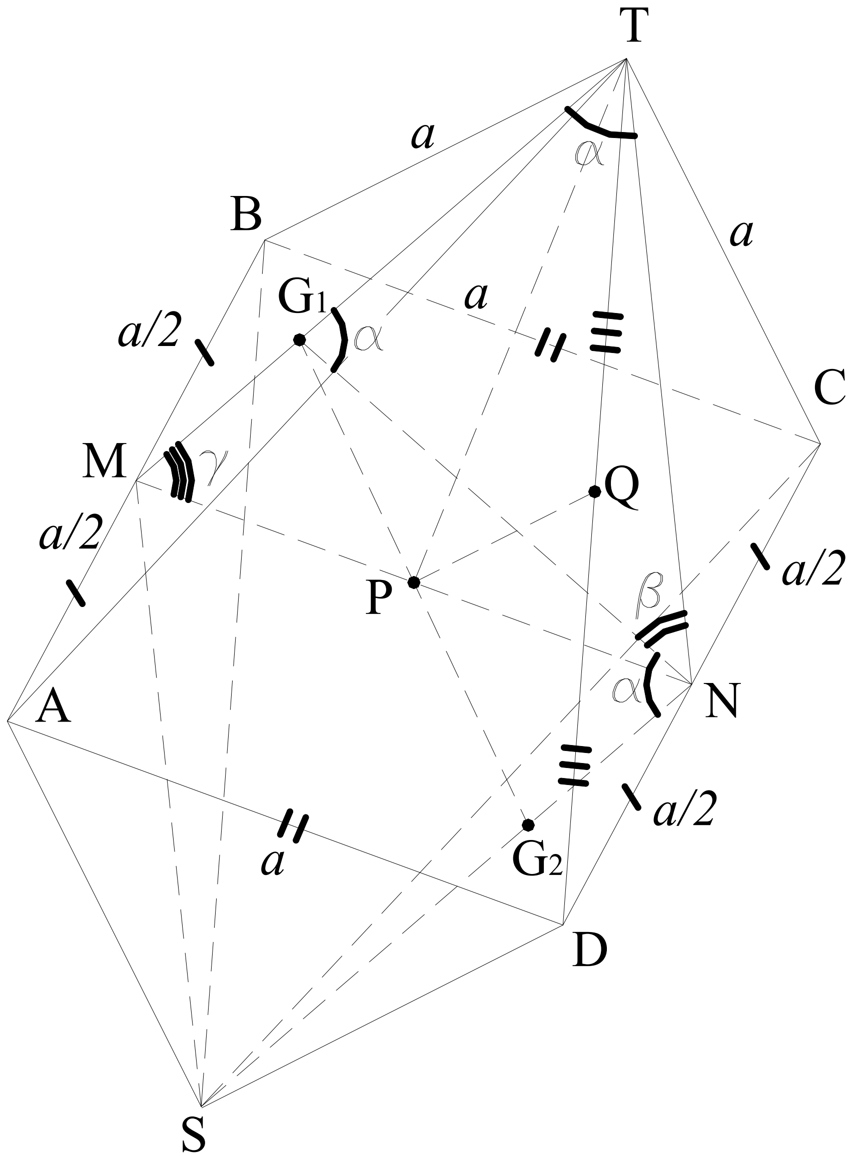
Let G_1, G_2 be the barycenters of the two faces of the regular octahedron parallel to each other. Find the volume of the solid by revolving the regular tetrahedron about the line G_1G_2 as the axis of rotation.

<u>Solution</u>

First, a regular octahedron is a Platonic solid composed of eight equilateral triangles, four of which meet at each vertex. Let the regular octahedron be ABCDTS where ABCD is the square in the middle, T and S are the vertices at top and bottom, respectively. By definition, these triangles are equilateral TAB, TBC, TCD, TAD, SAB, SBC, SCD and TAD.

Let [Φ] denote the plane containing shape Φ, M and N be the midpoints of AB and CD, respectively, G_1 and G_2 be the barycenters located on the parallel triangles TAB and SCD, respectively, P and Q the midpoints of G_1G_2 and TD, repectively. Now let a be the side length of the equilateral triangles, $\alpha = \angle MTN$, $\beta = \angle TNG_1$ and $\gamma = \angle TMN = \angle TNM = \angle MNS = \angle NMS = 90° - \frac{\alpha}{2}$.

Since M and N are the midpoints, we have $TM = TN = \frac{1}{2}a\sqrt{3}$, and with G_1 and G_2 being the barycenters, we get $TG_1 = \frac{2}{3}TM = SG_2 = \frac{a}{\sqrt{3}}$. Now let's find cosα. Applying the law of cosines, we obtain $MN^2 = TM^2 + TN^2 - 2TM\times TN\times\cos\alpha$, or $a^2 = \frac{3}{2}a^2(1 - \cos\alpha)$, or $\cos\alpha = \frac{1}{3}$. We need to verify that G_1G_2 is perpendicular to both [TAB] and [SCD]. Armed with the value of cosα, we can get the



length of segment G_1N which is now $G_1N^2 = TN^2 + TG_1^2 - 2TN \times TG_1 \times \cos\alpha = \frac{3}{4}a^2$, or $G_1N = TN$ and thus $\angle TG_1N = \angle MTN = \alpha$. We also have $\angle G_1NG_2 = \angle TNS - \beta = 2\gamma - \beta = 180° - \alpha - \beta = \alpha$.

Combining with the fact that the points T, G_1, M, S, G_2 and N are coplanar and $\angle TG_1N = \alpha$, TM is parallel to SN. We now find the length G_1G_2.

$G_1G_2^2 = G_1N^2 + G_2N^2 - 2G_1N\times G_2N\times\cos\alpha = \frac{2}{3}a^2$, or $G_1G_2 = \frac{2}{\sqrt{6}}\,a$.

It follows that $G_1N^2 = G_1G_2^2 + G_2N^2 = \frac{3}{4}a^2$, or $G_1G_2 \perp SN$. Because $TM \parallel SN$, we also have $G_1G_2 \perp TM$.

Now since $[TG_1MSG_2N] \perp CD$, we get $G_1D = G_1C = \sqrt{G_1N^2 + DN^2} = a = CD$. The triangle CG_1D is congruent with other triangles of this regular octahedron, and since G_2 is the barycenter of an equilateral triangle, $G_2D = SG_2 = \frac{a}{\sqrt{3}}$. We now have $G_1D^2 = a^2 = G_1G_2^2 + G_2D^2$, or $G_1G_2 \perp G_2D$.

Combining with $G_1G_2 \perp SN$, we conclude that G_1G_2 is perpendicular to both parallel planes [TAB] and [SCD], and combined with the fact that P is the midpoint of G_1G_2, TP = BP = AP = PC = PD = PS.

Using G_1G_2 as the axis of rotation, we now need to find the farthest points away from G_1G_2 of this regular octahedron, and they are T and S which are equidistant from G_1G_2. Meanwhile, Q and the midpoint of SB are the shortest and equidistant from G_1G_2. When revolving the octahedron forms a volume that is symmetrical with respect to the revolving line PQ, and the total volume equals twice the volume obtained by revolving TM and PQ as seen on the graph on the next page.

Now let's find PQ. As previously discovered, TP = PD and

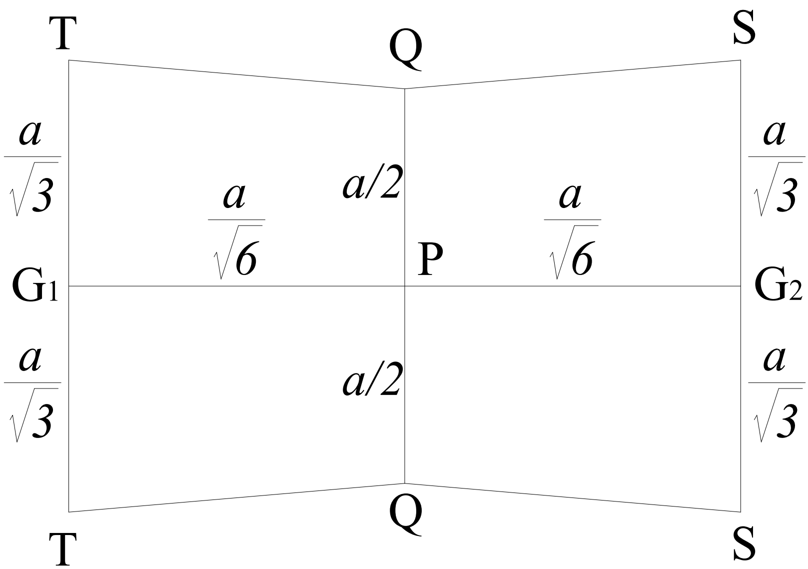
$TP = \sqrt{TG_1^2 + PG_1^2} = \sqrt{TG_1^2 + \frac{1}{4}G_1G_2^2} = \frac{a}{\sqrt{2}}$, $TQ = \frac{1}{2}TD = \frac{a}{2}$.

Therefore, $PQ = \sqrt{TP^2 - TQ^2} = \frac{a}{2}$.

The volume of the regular octahedron when revolving is now $V = 2\times\frac{1}{3}\pi[R^2(d + \frac{a}{\sqrt{6}}) - r^2 d\,]$ where $R = G_1T$, $r = PQ$, d is the distance from the intersection of the extensions of TQ and G_1G_2 to P. Because of the similar triangles, we have

$$\frac{d}{PQ} = \frac{PG_1}{TG_1 - PQ}\text{, or } d = \frac{a(2+\sqrt{3})}{\sqrt{2}}.$$

Finally, $V = \frac{1}{18\sqrt{6}}\pi a^3(7 + 2\sqrt{3})$.



Cross section of the regular octahedron when revolving around G_1G_2.

Problem 2 of the British Mathematical Olympiad 2007

Find all solutions in positive integers x, y, z to the simultaneous equations

$x + y - z = 12$
$x^2 + y^2 - z^2 = 12.$

Solution

Equating the two equations, we get
$x + y - z = x^2 + y^2 - z^2$, or $(y - z)(1 - y - z) = x(x - 1)$.

But since all solutions x, y and z are required to be positive integers, $x(x - 1) \geq 0$ because two consecutive integers must have the same sign. Therefore, $(y - z)(1 - y - z) \geq 0$, or both $y - z$ and $1 - y - z$ must have the same sign. Either both must be smaller than or equal zero, or both greater than or equal zero.

When $y - z \leq 0$, or $y \leq z$, let $y + a = z$ where a is a non-negative integer, we then have
$x - a = 12$, or $x^2 = a^2 + 24a + 144$,
$x^2 + y^2 - z^2 = x^2 - 2ay - a^2 = 12$.
Substituting $x^2 = a^2 + 24a + 144$ into the bottom equation, we get $a(12 - y) = -66$, or $a = -66/(12 - y)$. Since a is an integer, and $66 = 1\times2\times3\times11$, $12 - y = 1, 2, 3, 6, 11, 22, 33, 66$, or $y = 11, 10, 9, 6, 1$ (negative values for y are ignored). The values for a with respect to the values of $y = 11, 10, 9, 6, 1$ are then $a = -66, -33, -22, -11, -6$, and $x = a + 12 = 1, 6$ (negative values for x ignored). Hence, $(x, y, z) = (1, 6, -5)$ and $(6, 1, -5)$ which are rejected because $z < 0$.

When $y - z \geq 0$, $1 - y - z \geq 0$, or $y \leq 1 - z$, but both y and z are positive, and when $z \geq 1$, $y \leq 1 - z \leq 0$, or $y \leq 0$ and not allowed. Thus there are no solutions in positive integers.

If all integer solutions are accepted, we should have
$(x, y, z) = (1, 6, -5), (6, 1, -5), (11, -54, -55), (-54, 11, -55)$.

Problem 6 of the Vietnamese Mathematical Olympiad 1982

Let ABCDA’B’C’D’ be a cube (where ABCD and A’B’C’D’ are faces and AA’, BB’, CC’, DD’ are edges). Consider the four lines AA’, BC, D’C’ and the line joining the midpoints of BB’ and DD’. Show that there is no line which cuts all the four lines.

Solution

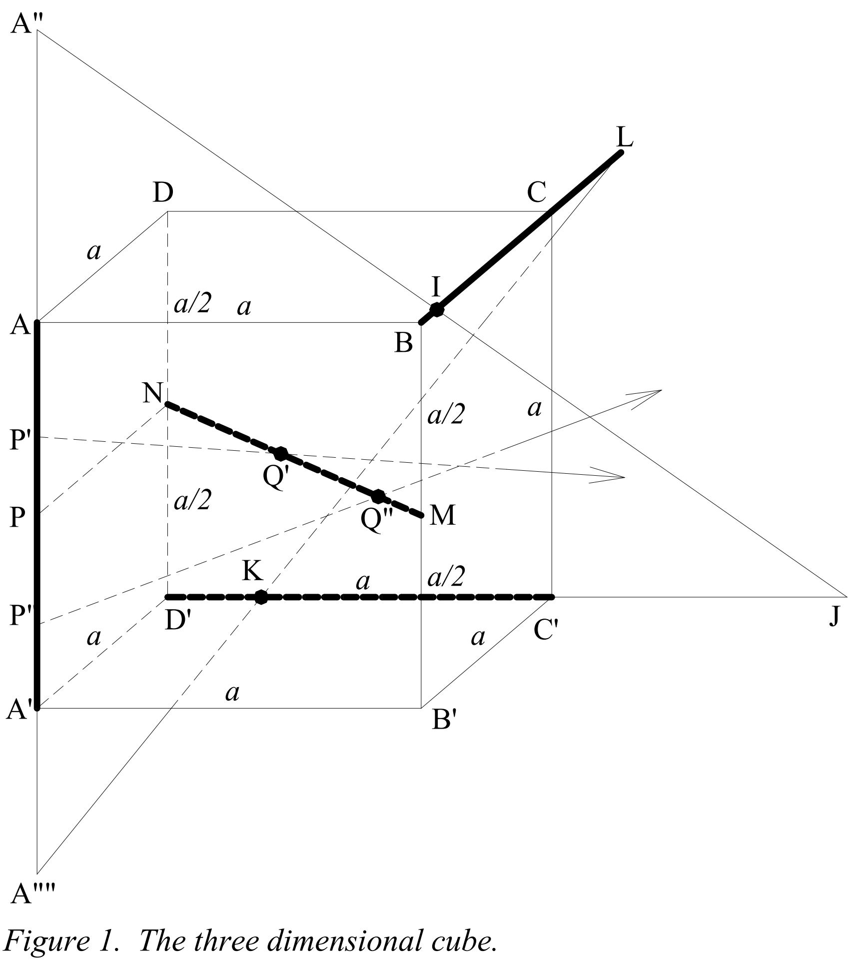


Figure 1. The three dimensional cube.

Let the side length of the cube be *a*, the midpoints of BB’, DD’ and AA’ be M, N and P, respectively, [Φ] denote the plane containing shape Φ. If there is a line, assigned letter *l*, that cuts both lines containing AA' and BC, *l* has to belong to all the planes that perpendicular [ABCD] and sweep from one end point at infinity of the extension of BC on the left to another end point at infinity of extension of CB on the right (see figure below).

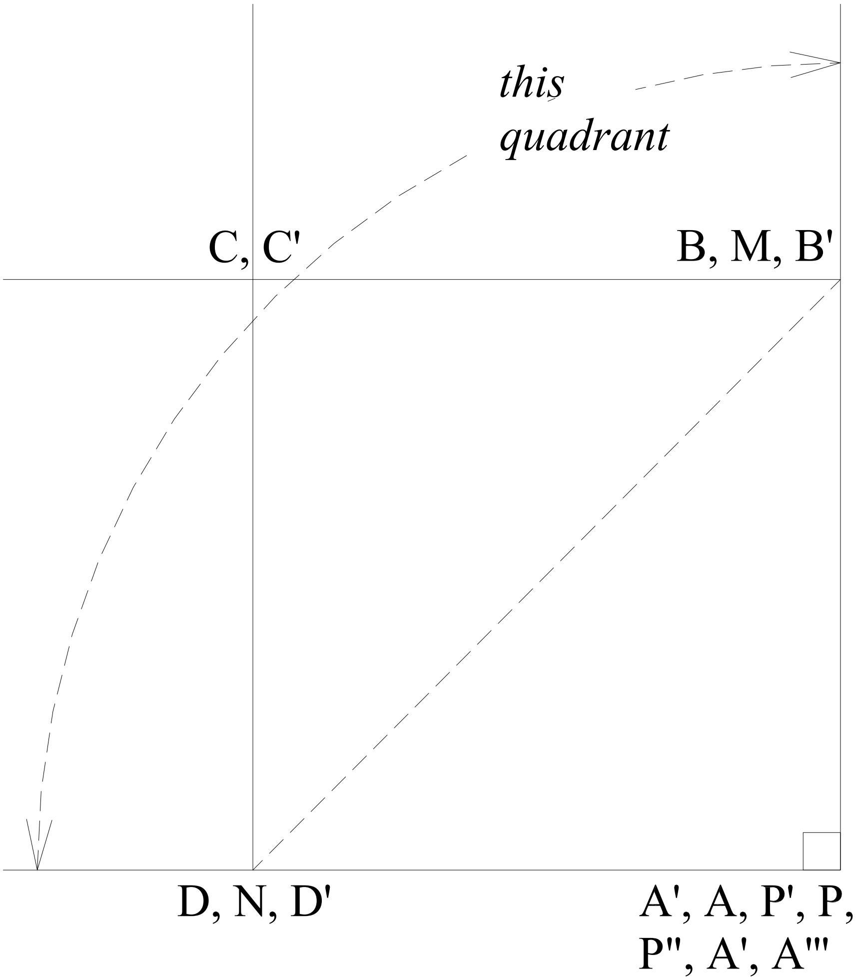


Figure 2. Floor plan of ABCD.

Similarly, if there is a line *l* that cuts both lines containing AA' and D'C', *l* has to belong to all the planes that perpendicular [ABCD] and sweep from one end point at infinity of the extension of C'D' on the bottom to another end point at infinity of extension of D’C’ on the top.

To satisfy those two aforementioned conditions, *l* must belong to the planes that perpendicular [ABCD] and are in the quadrant of space encumbranced by the plane that perpendiculars [ABCD] and on the left side of AB and also the plane that perpendiculars [ABCD] and on top of half-line AD (see figure 2). In other words, *l* must originate or end at the line that perpendiculars [ABCD] at A. Therefore, the opportunity for *l* to also cut the line containing MN only occurs from M to N or segment MN. Now let’s check the scenarios for point A.

If A is at A” which is at or above A, to cut BC and D’C’ line *l* (line A”IJ in figure 1) will miss segment MN (it cannot cut segment MN).

If A is at A’’’ which is at or below A’, to cut D’C’ and BC again line *l* (line A’’’KL in figure 1) will miss segment MN.

If A is at P’ which is at or above P but below A, to cut MN and D’C’ line *l* (line P’Q’ in figure 1) will miss the half-line BC from B to its left.

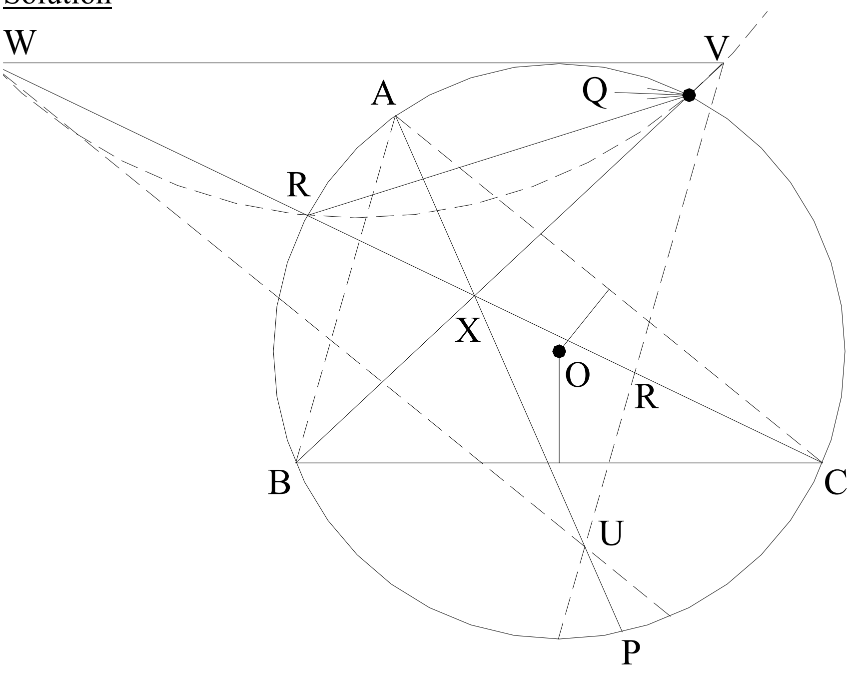
If A is at P” which is at or below P and above A’, to cut MN and BC line *l* (line P”Q” in figure 1) will miss the half-line D’C’ from D’ to the top.

Therefore, there is no line which cuts all the four lines AA’, BC, D’C’ and the line joining the midpoints of BB’ and DD’.

<u>Problem 1 of British Mathematical Olympiad 2011</u>

Let ABC be a triangle and X be a point inside the triangle. The lines AX, BX and CX meet the circumcircle of triangle ABC again at P, Q and R, respectively. Choose a point U on XP which is between X and P. Suppose that the lines through U which are parallel to AB and CA meet XQ and XR at points V and W, respectively. Prove that the points W, R, Q and V lie on a circle.

<u>Solution</u>



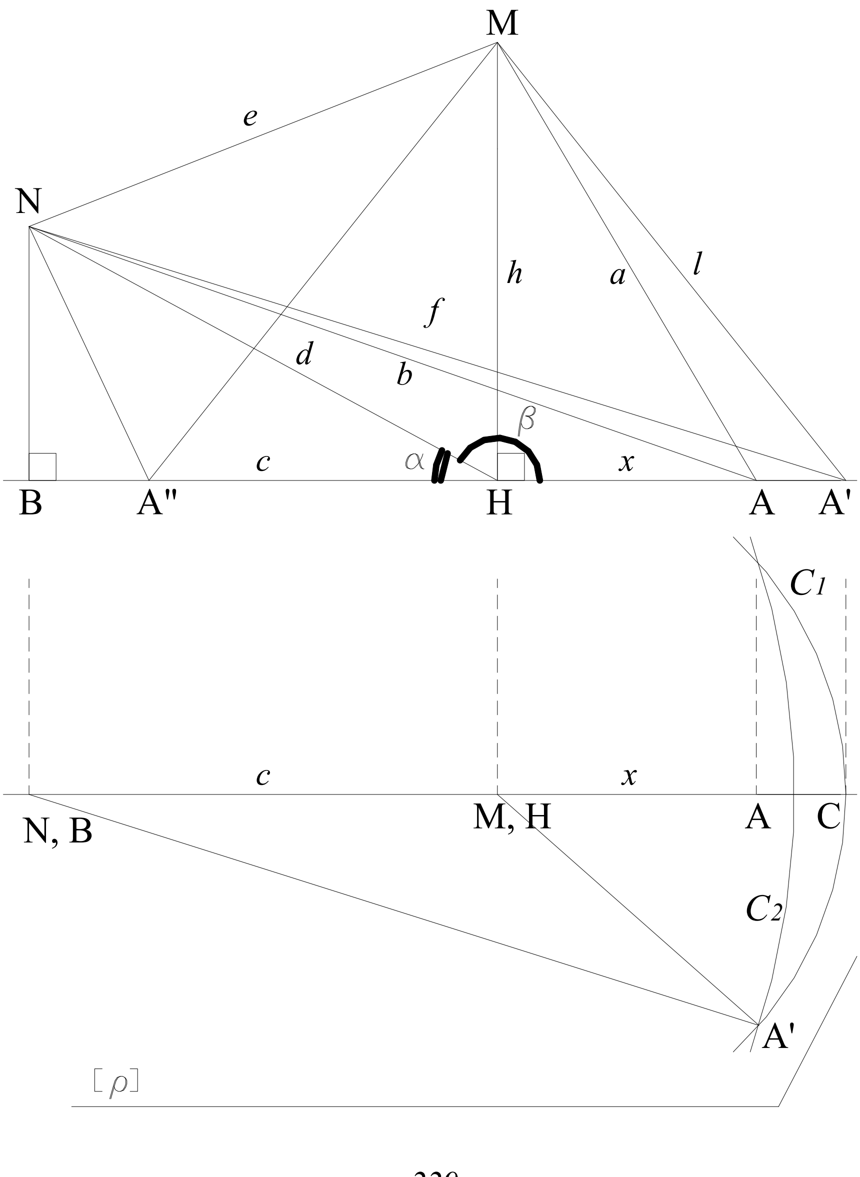
Since AC || UW, we get $\frac{AX}{UX} = \frac{CX}{WX}$ and since AB || UV, we get $\frac{AX}{UX} = \frac{BX}{VX}$. It then follows that $\frac{CX}{WX} = \frac{BX}{VX}$, or CX×VX = BX×WX. On the other hand since X is inside the circle, CX×RX = BX×QX. The two previous equation imply RX×WX = QX×VX, or WRQV is cyclic, or W, R, Q and V lie on a circle.

Note: $\frac{CX}{WX} = \frac{BX}{VX}$ *implies that VW || BC.*

Problem 3 of the Vietnamese Mathematical Olympiad 1981

A plane ρ and two points M, N outside it are given. Determine the point A on ρ for which $\dfrac{AM}{AN}$ is minimal.

Solution



Let $[\Phi]$ denote the plane containing shape Φ, H and B be the feet of M and N onto $[\rho]$, respectively, a = AM, b = AN, c = BH, d = NH, e = MN, h = MH, x = AH, $\alpha = \angle$BHN, $\beta = \angle$AHN. All points given thus far are on the plane $[\Lambda]$ that perpendiculars $[\rho]$ shown on top of the previous graph. The bottom of the graph shows the floor plan looking down from an infinite point on top, away from M, N.

Assume that HM > BN or point M is at a higher altitude above plane $[\rho]$ than point N. We first find the location of point A on $[\Lambda]$ where the ratio $\frac{\text{AM}}{\text{AN}} = \frac{a}{b}$ is minimal. For the case where point A is on H or on its right side ($x \geq 0$), applying the Pythagorean's theorem, we get
$a^2 = x^2 + h^2$, and the law of cosines gives us
$b^2 = x^2 + d^2 - 2xd\cos\beta = x^2 + d^2 + 2xd\cos\alpha = x^2 + d^2 + 2cx$, and
$\frac{a^2}{b^2} = \frac{x^2 + h^2}{x^2 + d^2 + 2cx}$, but note that except for x, all segments h, d and c are constant, and the ratio $\frac{a^2}{b^2}$ is at an extreme value when its derivative $(\frac{a^2}{b^2})'$ is zero, or $(\frac{x^2 + h^2}{x^2 + d^2 + 2cx})' = 0$.
Based on the formula $(\frac{u}{v})' = \frac{vu' - uv'}{v^2}$, we have
$(\frac{x^2 + h^2}{x^2 + d^2 + 2cx})' = \frac{2[cx^2 + (d^2 - h^2)x - h^2c]}{(x^2 + d^2 + 2cx)^2} = 0$ when

$cx^2 + (d^2 - h^2)x - h^2c = 0$, or when

$$x = \frac{1}{2c}[h^2 - d^2 \pm \sqrt{d^4 - 2d^2h^2 + h^4 + 4h^2c^2}] \qquad \text{(i)}$$

We verified and confirmed that at this point A on $[\Lambda]$ where x = AH = $\frac{1}{2c}[h^2 - d^2 \pm \sqrt{d^4 - 2d^2h^2 + h^4 + 4h^2c^2}]$ the ratio $\frac{\text{AM}}{\text{AN}}$ is minimal. This scenario is for $x \geq 0$. Now we note that if point A

happens to be on the left side of H such as point A" on the top part of the graph, we see that A"M > h whereas A"N < NH; therefore, $\frac{A''M}{A''N} > \frac{h}{d} > \frac{a}{b}$ (keep in mind that $\frac{AM}{AN} = \frac{h}{d}$ when $x = 0$).
Another important piece of argument is that for any point A such as this same point A" that positions on the left side of H, we can always find a symmetrical point A' with respect to the vertical segment MH on the right side of H such that A'M = A"M and A'N > A"N which causes $\frac{A''M}{A''N} > \frac{A'M}{A'N} > \frac{a}{b}$ (since A' $\neq$ A).

We conclude that the minimum ratio $\frac{AM}{AN}$ indeed occurs at point A as defined by equation (i) when A is on the plane [Λ].

Now let's expand this idea to three dimensional space and verify whether $\frac{A'M}{A'N} > \frac{a}{b}$ also holds true for any point A' on the plane [ρ] where A' $\neq$ A.
Let *C1* and *C2* be the circles with center H and radius A'H and center B and radius A'B. Indeed, if point A is now at A' on the bottom part of the graph, we can find a point C on the extension of BH such that CM = A'M and A'N < CN (A'N equals the distance from the intersection of C2 and HC to N, see the bottom part of the graph), so that $\frac{A'M}{A'N} > \frac{CM}{CN} > \frac{a}{b}$.

We finally determine that the point A on [ρ] for which $\frac{AM}{AN}$ is minimal happens to be at A on the right of the extension of BH that is a distance AH = x given by the equation (i), and this completes our analysis.

<u>Further observation</u>

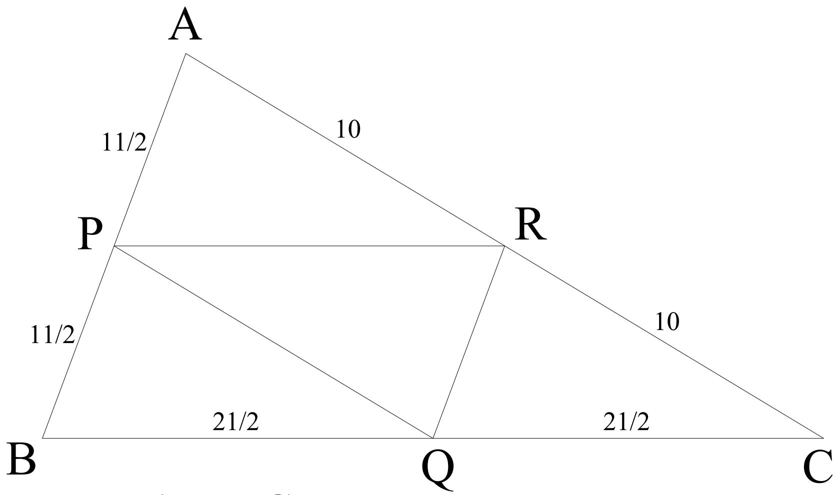
This problem is derived from the above problem:

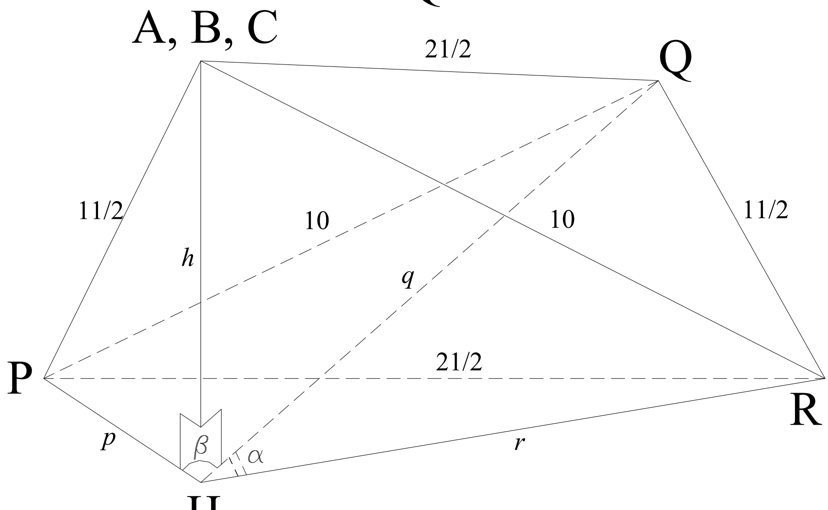
A plane ρ and two points M, N outside it are given. Determine the point A on ρ for which AM + AN is minimal.

<u>*Problem 5 of International Mathematical Talent Search Round 4*</u>

The sides of triangle ABC measure 11, 20, and 21 units. We fold it along PQ, QR, RP where P, Q, R are the midpoints of its sides until A, B, C coincide. What is the volume of the resulting tetrahedron?

<u>Solution</u>





It’s easily seen that after just a single fold on each side points A, B and C coincide. Let H be the foot of the tetrahedron after folding and h = AH = BH = CH, p = PH, q = QH, r = RH, $\alpha = \angle$QHR, $\beta = \angle$QHP.

Since AH is perpendicular to the plane of triangle PQR, applying the Pythagorean's theorem, we get

$h^2 = AP^2 - PH^2 = \frac{121}{4} - p^2$

$h^2 = AR^2 - RH^2 = 100 - r^2$

$h^2 = AQ^2 - QH^2 = \frac{441}{4} - q^2$, or

$$\frac{121}{4} - p^2 = 100 - r^2 = \frac{441}{4} - q^2 \qquad \text{(i)}$$

Now the law of cosines gives us

$RQ^2 = r^2 + q^2 - 2rq\times\cos\alpha$,

$PQ^2 = p^2 + q^2 - 2pq\times\cos\beta$ and

$PR^2 = p^2 + r^2 - 2pr\times\cos(\alpha + \beta)$

Orderly substituting in the real values for RQ, PQ and PR, we get

$$121 = 4r^2 + 4q^2 - 8rq\times\cos\alpha \qquad \text{(ii)}$$
$$100 = p^2 + q^2 - 2pq\times\cos\beta \text{ and} \qquad \text{(iii)}$$
$$441 = 4p^2 + 4r^2 - 8pr\times\cos(\alpha + \beta) \qquad \text{(iv)}$$

Substituting $r = \frac{1}{2}\sqrt{4q^2 - 41}$ from (i) into (ii), we get

$162 = 8q^2 - 4q\sqrt{4q^2 - 41}\times\cos\alpha$, or $\cos\alpha = \frac{4q^2 - 81}{2q\sqrt{4q^2 - 41}}$

and $p = \sqrt{q^2 - 80}$ from (i) into (iii), we get

$90 = q^2 - q\sqrt{q^2 - 80}\times\cos\beta$, or $\cos\beta = \frac{q^2 - 90}{q\sqrt{q^2 - 80}}$

and r and p from above into (iv), we get

$401 = 4q^2 - 2\sqrt{(q^2 - 80)(4q^2 - 41)}\times\cos(\alpha + \beta)$, or

$\cos(\alpha + \beta) = \frac{4q^2 - 401}{2\sqrt{(q^2 - 80)(4q^2 - 41)}}$

Applying the trigonometric formula of the cosine of the sum of two angles $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, we get

$$\frac{4q^2 - 401}{2\sqrt{(q^2-80)(4q^2-41)}} = \frac{4q^2-81}{2q\sqrt{4q^2-41}} \times \frac{q^2-90}{q\sqrt{q^2-80}} -$$

$$\sqrt{1 - \frac{(4q^2-81)^2}{4q^2(4q^2-41)}} \times \sqrt{1 - \frac{(q^2-90)^2}{q^2(q^2-80)}}.$$

Simplifying the above equation, we obtain

$16q^4 - 5832q^2 + 531441 = (q^2 - 81)(484q^2 - 6561)$, or $4q^2 = \dfrac{4437}{13}$,

Therefore, $h = \dfrac{1}{2}\sqrt{441 - 4q^2} = \dfrac{18}{\sqrt{13}}$, and the volume of the

tetrahedron is $V = \dfrac{1}{3}Ah$ where A is the area of the triangle PQR and

it is given by the Heron's formula $A = \sqrt{s(s-a)(s-b)(s-c)}$ where s is semi-perimeter of the triangle, a, b and c are its side lengths, or

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \frac{15}{2}\sqrt{13}.$$

Finally, the volume of the resulting tetrahedron is

$V = \dfrac{1}{3}\times\dfrac{15}{2}\sqrt{13}\times\dfrac{18}{\sqrt{13}} = 45$ cubic units.

<u>Further observation</u>

The trick here is to determine that the foot H of A, B, C after folding falls outside the area of triangle PQR so that $\angle PHR = \alpha + \beta$.

Problem 4 of International Mathematical Talent Search Round 7

In an attempt to copy down from a board a sequence of six positive integers in arithmetic progression, a student wrote down the five numbers

113, 137, 149, 155, 173

accidentally omitting one. He later discovered that he also miscopied one of them. Can you help him and recover the original sequence?

Solution

The differences of the numbers in sequences are 24, 12, 6 and 18. We notice that the difference of the first two numbers is 24 and is twice that of the difference between the next two; 24 also equals the difference of the middle number and the last number; i.e, 24 = 173 – 149.

Therefore, the common difference in the arithmetic progression should be one-half of 24 which is 12, and the original sequence should be

113, 125, 137, 149, 161, 173.

Problem 1 of British Mathematical Olympiad 1990

Find a positive integer whose first digit is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

Solution

Let n, m, p, q, r and s to be non-negative integers. Let the positive integer in question be N = $1\ldots n$.

We start out with the equation $3\times 1\ldots n = \ldots n1$. Therefore, $n = 7$ in order for $3\times 7 = 21$ to have a units digit being 1.

Now we get the next equation to be $3\times 1\ldots m7 = \ldots m71$; the value of m must be $m = 5$ in order for us to get $3\times 5 + 2 = 17$ where 2 is the carry-over from 21 above.

Now we get the next equation to be $3\times 1\ldots p57 = \ldots p571$; the value of p must be $p = 8$ in order for us to get $3\times 8 + 1 = 25$ where 1 is the carry-over from 17 above.

Now we get the next equation to be $3\times 1\ldots q857 = \ldots q8571$; the value of q must be $q = 2$ in order for us to get $3\times 2 + 2 = 8$ where 2 is the carry-over from 25 above.

Now we get the next equation to be $3\times 1\ldots r2857 = \ldots r28571$; the value of r must be $r = 4$ in order for us to get $3\times 4 = 12$ (no carry-over) with the units digit being 2.

Now we get the next equation to be $3\times 1\ldots s42857 = \ldots s428571$; the value of s must be $s = 1$ in order for us to get $3\times 1 + 1 = 4$ where 1 is the carry-over from 12 above; s is also the most significant digit of number N.

Thus the number is 142857.

Problem 2 of the British Mathematical Olympiad 2008

Find all real values of x, y and z such that

$(x + 1)yz = 12$, $(y + 1)zx = 4$ and $(z + 1)xy = 4$.

Solution

Expanding the last two equations, we get
$xz + xyz = xy + xyz = 4$, or $x(y - z) = 0$. This occurs when either $x = 0$ or $y = z$.

If $x = 0$, the last two equations in the problems would not be valid because $0 \neq 4$; therefore, $y = z$.

Let $w = y = z$, and replacing y and z with w in the first two equations of the problems, we then have
$(x + 1)w^2 = 12$
$(w + 1)wx = 4$, or
$w^3 + w^2 - 8w - 12 = 0$, or $(w - 3)(w + 2)^2 = 0$.

From there, we get
$(x, y, z) = (\frac{1}{3}, 3, 3), (2, -2, -2)$.

Problem 1 of International Mathematical Talent Search Round 15

Is it possible to pair off the positive integers 1, 2, 3, . . . , 50 in such a manner that the sum of each pair of numbers is a different prime number?

Solution

The sum of each pair of numbers is required to be a different prime number. There are 25 pairs and thus there must be 25 different prime numbers with the largest prime number just smaller than the sum of the two largest numbers in the series 50 + 49 = 99 or 97.

We know that there are 25 prime numbers smaller or equal to 97, and they are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, and 97.

However, the minimum possible number after pairing is 1 + 2 = 3. So the number 2 in the group of the 25 prime numbers above is excluded, and there are only 24 different prime numbers left for 25 pairs.

Therefore, it is not possible to pair off the positive integers 1, 2, 3, . . . , 50 in such a manner that the sum of each pair of numbers is a different prime number.

Problem 4 of International Mathematical Talent Search Round 15

Suppose that for positive integers a, b, c and x, y, z, the equations $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$ are satisfied. Prove that

$(a + x)^2 + (b + y)^2 \leq (c + z)^2$,

and determine when equality holds.

Solution

Applying the AM-GM inequality for positive integers a, b, c, x, y, z, we get $a^2y^2 + b^2x^2 \geq 2axby$.

Adding $a^2x^2 + b^2y^2$ to both sides, we obtain
$a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \geq a^2x^2 + 2axby + b^2y^2$, or
$(a^2 + b^2)(x^2 + y^2) \geq (ax + by)^2$. Because $a^2 + b^2 = c^2$ and $x^2 + y^2 = z^2$,
$c^2z^2 \geq (ax + by)^2$, and with integers a, b, c, x, y, z positive, we get
$cz \geq ax + by$.

Now multiplying both sides by 2 then adding $c^2 + z^2$ to get
$c^2 + 2cz + z^2 \geq 2ax + 2by + c^2 + z^2$.

Replacing c^2 and z^2 on the right side with $c^2 = a^2 + b^2$ and $z^2 = x^2 + y^2$, it is now equivalent to
$c^2 + 2cz + z^2 \geq a^2 + 2ax + x^2 + b^2 + 2by + y^2$, or
$(a + x)^2 + (b + y)^2 \leq (c + z)^2$. Equality holds when $ay = bx$.

<u>Problem 1 of International Mathematical Talent Search Round 17</u>

The 154-digit number, 19202122 . . . 939495, was obtained by listing the integers from 19 to 95 in succession. We are to remove 95 of its digits, so that the resulting number is as large as possible. What are the first 19 digits of this 59-digit number?

<u>Solution</u>

Let's write the 154-digit number as 19202122232425262728293031323334353637383 9...939495. First remove the first digit 1 which is the most significant digit (MSD), we then have the number 9202122232425262728293031323334353637383 9...939495, and there are 94 more digits to remove.

We note that from number 9 in 19 to another number 9 in 29 there are 19 digits. This is also true from number 9 in 29 to another number 9 in 39, etc...

Therefore, let's remove those 19 digits 2021222324252627282 between the number 9's, and we get 9930313233343536373839404142434445464748 49...939495, and there are 75 more digits to remove.

Repeat the same process three more times and we get 9999960616263646566676869707172737475767 77879...939495.

There are 18 more digits to remove, and it's one digit short between the number 9's. We note that the next highest digit in the series 9606162636465666768 69 is 8; therefore, we remove 17 more digits to get 9999986970717273747576777879...939495.

There is now 1 more digit to remove, and we pick number 6 between numbers 8 and 9. Finally, after all the removals, we have the number 999998970717273747576777879...939495, and the first 19 digits of this number are 9999989707172737475.

Problem 2 of International Mathematical Talent Search Round 17

Find all pairs of positive integers (m, n) for which $m^2 - n^2 = 1995$.

Solution

$m^2 - n^2 = (m - n)(m + n) = 1995 = 1\times3\times5\times7\times19$. Therefore, all the possible values for $m - n$ and $m + n$ are

$m - n$	$m + n$	m	n
1	1995	998	997
3	665	334	331
5	399	202	197
7	285	146	139
15	133	74	59
19	105	62	43
21	95	58	37
35	57	46	11
57	35	46	-11
95	21	58	-37
105	19	62	-43
133	15	74	-59
285	7	146	-139
399	5	202	-197
665	3	334	-331
1995	1	998	-997

All pairs of positive integers (m, n) for which $m^2 - n^2 = 1995$ are

$(m, n) = (998, 997), (334, 331), (202, 197), (146, 139), (74, 59), (62, 43), (58, 37), (46, 11)$.

Problem 4 of International Mathematical Talent Search Round 17

A man is 6 years older than his wife. He noticed 4 years ago that he has been married to her exactly half of his life. How old will he be on their 50th anniversary if in 10 years she will have spent two-thirds of her life married to him?

Solution

Let m and w be the current ages of the man and his wife, YMP, YMC, YMF be the numbers of years the couple had been married 4 years ago, have been married currently and will be married in 10 years, respectively. We get $m = w + 6$ because he's 6 years older.

The number of years the couple had been married four years ago is YMP $= \dfrac{m-4}{2}$.

And presently, they have been married for

YMC = YMP + 4 years, or YMC $= \dfrac{m-4}{2} + 4 = \dfrac{m+4}{2}$ years.

Therefore, in 10 years, they will be married for this number of years YMF $= \dfrac{m+4}{2} + 10$ which is equal to $\dfrac{2}{3}$ of the wife's life at that time, and in 10 years, the wife's age will be $w + 10$, so we get $\dfrac{2}{3}(w+10) = \dfrac{m+4}{2} + 10$.

Combining with the top equation, we solve and obtain $m = 56$. So currently they have been married for YMC $= \dfrac{m+4}{2} = 30$ years.

Their 50th anniversary will occur 20 years later, and he will be $m + 20 = 76$ years old.

Problem 3 of Spain Mathematical Olympiad 1985

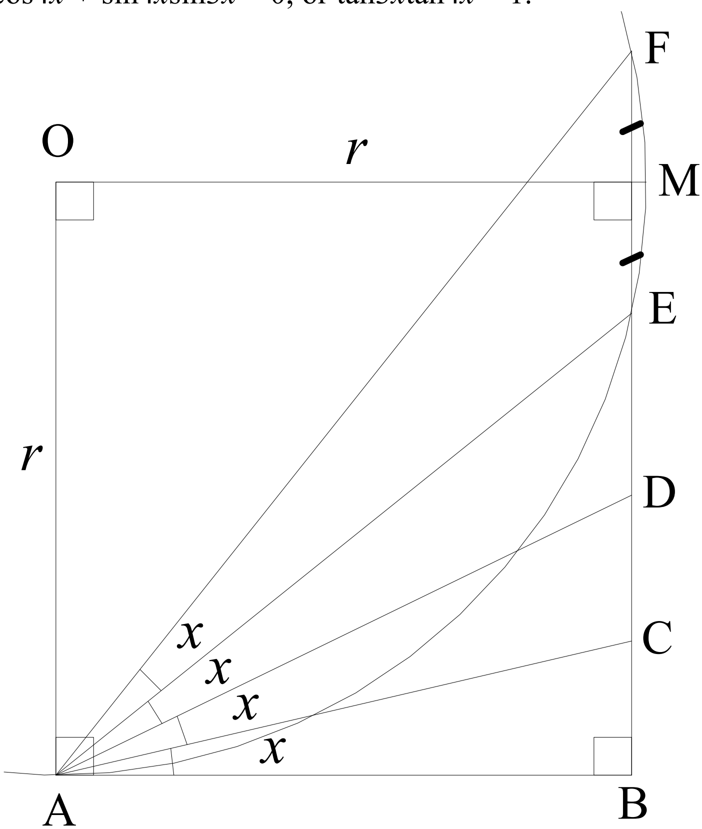
Solve the equation $\tan^2 2x + 2\tan 2x \tan 3x = 1$.

Solution

It is understood that the solutions that make either $\tan 2x$ or $\tan 3x$ to equal ∞ are not accepted. Among them are $x = \pm 30°$ and $x = 45°$.

Method 1

Expanding the equation to get $\frac{\sin^2 2x}{\cos^2 2x} + 2\frac{\sin 2x \sin 3x}{\cos 2x \cos 3x} = 1$, or
$\sin^2 2x \cos 3x + 2\sin 2x \cos 2x \sin 3x = \cos^2 2x \cos 3x$, or
$\cos 3x(\sin^2 2x - \cos^2 2x) + \sin 4x \sin 3x = 0$, or
$-\cos 3x \cos 4x + \sin 4x \sin 3x = 0$, or $\tan 3x \tan 4x = 1$.



It is here that we see the direct relationship between trigonometry and geometry when we can describe the equation $\tan3x\tan4x = 1$ with the geometrical graph as shown where segment AB tangents to a circle with center O and radius r at A and a vertical line starting at B cuts the circle at E and F with $\angle ABF = 90°$, $\angle BAE = 3x$ and $\angle BAF = 4x$. We have $AB^2 = BE \times BF$, $\tan3x = \frac{BE}{AB}$ and $\tan4x = \frac{BF}{AB}$ and thus $\tan3x\tan4x = \frac{BE}{AB} \times \frac{BF}{AB} = 1$.

Let M be the midpoint of arc EF; it's easily seen that since BF || OA, OM || AB and $\angle MAE = \frac{1}{2}x$, or $\angle MAB = 3.5x = 45°$. Therefore $x = \frac{90°}{7}$.

<u>Method 2</u>

Solving the equation $\tan^2 2x + 2\tan2x\tan3x = 1$ for $\tan2x$, we get $\tan2x = -\tan3x \pm \sqrt{\tan^2 3x + 1}$. Now substituting $\tan3x = \frac{\sin3x}{\cos3x}$ into the solution, we have $\tan2x = -\tan3x \pm \sqrt{\frac{\sin^2 3x + \cos^2 3x}{\cos^2 3x}} = -\tan3x \pm \frac{1}{\cos3x} = \frac{-\sin3x \pm 1}{\cos3x}$, or $\frac{\sin2x}{\cos2x} = \frac{-\sin3x \pm 1}{\cos3x}$, or $\sin2x\cos3x =$ $-\sin3x\cos2x \pm \cos2x\cos0°$. Applying the formula $\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$, we get $\sin5x + \sin(-x) = -\sin5x - \sin x \pm (\cos2x + \cos2x)$, or $\sin5x = \pm\cos2x$. We then have either

a) $\cos(90° - 5x) = \cos2x$, or $90° - 5x = 2x$, or $x = \frac{90°}{7}$, or

b) $\cos(90° - 5x) = -\cos2x$, or $90° - 5x = 180° - 2x$ or $x = -30°$,

The only acceptable solutions are $x = \frac{90°}{7}$ and others based on the formulas $\cos[(2n + 1)180° - x] = -\cos x$ and $\cos[2n \times 180° - x] = \cos x$ where n is a non-negative integer.

The reader is urged to find the rest of the solutions.

Problem 5 of International Mathematical Talent Search Round 8

Given that a, b, x and y are real numbers such that
$a + b = 23$,
$ax + by = 79$,
$ax^2 + by^2 = 217$,
$ax^3 + by^3 = 691$.

Determine $ax^4 + by^4$.

Solution

Multiplying both sides of $a + b = 23$ by x, we get
$ax + bx = 23x$. Subtracting $ax + by = 79$ from it,
$b(x - y) = 23x - 79$ (i)

Now multiplying both sides of $a + b = 23$ by x^2, we get
$ax^2 + bx^2 = 23x^2$. Subtracting $ax^2 + by^2 = 217$ from it,
$b(x^2 - y^2) = b(x - y)(x + y) = 23x^2 - 217$ (ii)

Now multiplying both sides of $a + b = 23$ by x^3, we get
$ax^3 + bx^3 = 23x^3$. Subtracting $ax^3 + by^3 = 691$ from it,
$b(x^3 - y^3) = b(x - y)(x^2 + xy + y^2) = 23x^3 - 691$ (iii)

Dividing (ii) by (i), we obtain $x + y = \frac{23x^2 - 217}{23x - 79}$ $(x \neq y)$ (iv)

Dividing (iii) by (i), we obtain $x^2 + xy + y^2 = \frac{23x^3 - 691}{23x - 79}$ (v)

From (iv), $y = \frac{79x - 217}{23x - 79}$; now substituting this into (v) yields
$x^2 + (\frac{79x - 217}{23x - 79})x + (\frac{79x - 217}{23x - 79})^2 = \frac{23x^3 - 691}{23x - 79}$. Simplifying this equation, we come up with $x^2 - x - 6 = 0$, or $x = 3, -2$ which cause $y = -2, 3$ accordingly.

When $(x, y) = (3, -2)$, substituting them into $ax + by = 79$, we come up with the sets of equation $3a - 2b = 79$, along with the existing equation $a + b = 23$ make $a = 25$, $b = -2$.

When $(x, y) = (-2, 3)$, substituting them into $ax + by = 79$, we come up with the equation $-2a + 3b = 79$, along with the existing equation $a + b = 23$ make $a = -2$, $b = 25$.

These set of symmetrical solutions are expected since a, b are interchangable, so are x and y.

Therefore, $ax^4 + by^4 = 25\times3^4 - 2\times(-2)^4 = 1993$, and $ax^4 + by^4 = -2\times(-2)^4 + 25\times3^4 = 1993$ which is the same answer.

<u>Problem 1 of Yugoslav Mathematical Olympiad 2001</u>

Vertices of a square ABCD of side 25/4 lie on a sphere. Parallel lines passing through points A, B, C and D intersect the sphere at points A', B', C' and D', respectively. Given that AA' = 6, BB' = 10, CC' = 8, determine the length of the segment DD'.

<u>Solution</u>

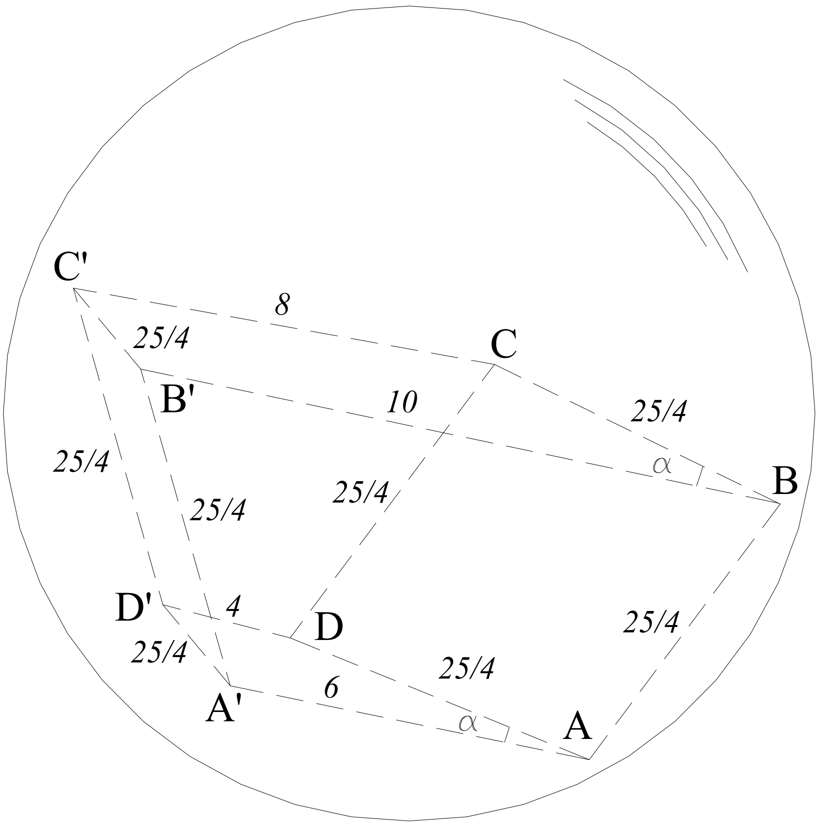


Figure 1. The sphere containing the square (not to scale)

It is easily seen that ABB'A', BCC'B', CDD'C' and ADD'A' are all trapezoids because the parallel lines cut the sphere in equal segments; i.e., AB = A'B', BC = B'C', CD = C'D' and AD = A'D', and they are, therefore, all cyclic. If we cut the sphere across the planes of each of these trapezoids, we would get the surfaces of round circles that circumscribe the trapezoids. Now let $\alpha = \angle CBB'$ and put all the trapezoids on the same plane on a two-dimensional layout as shown in figure 2.

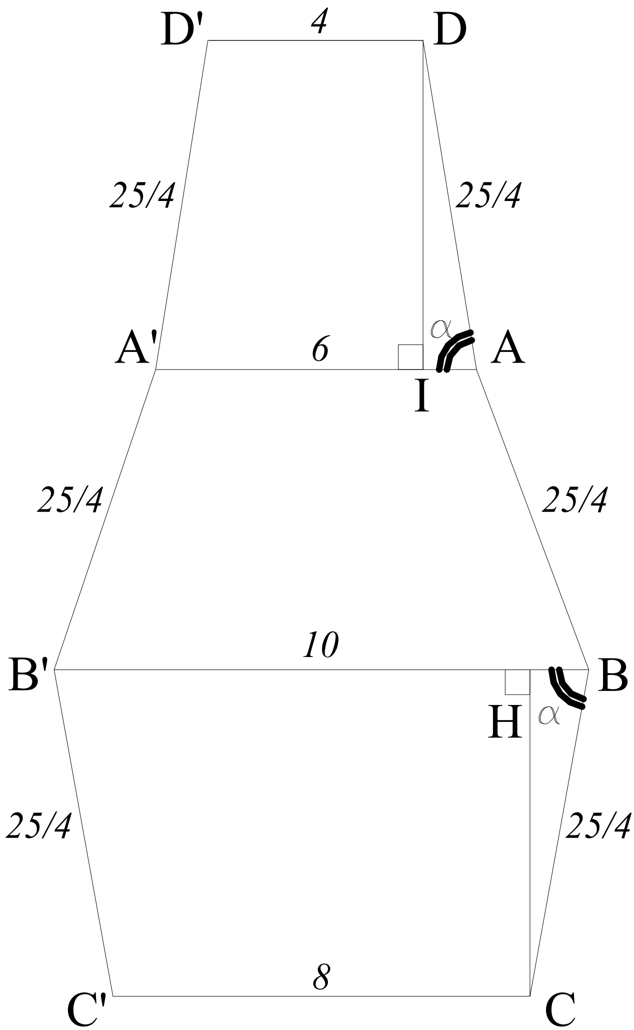


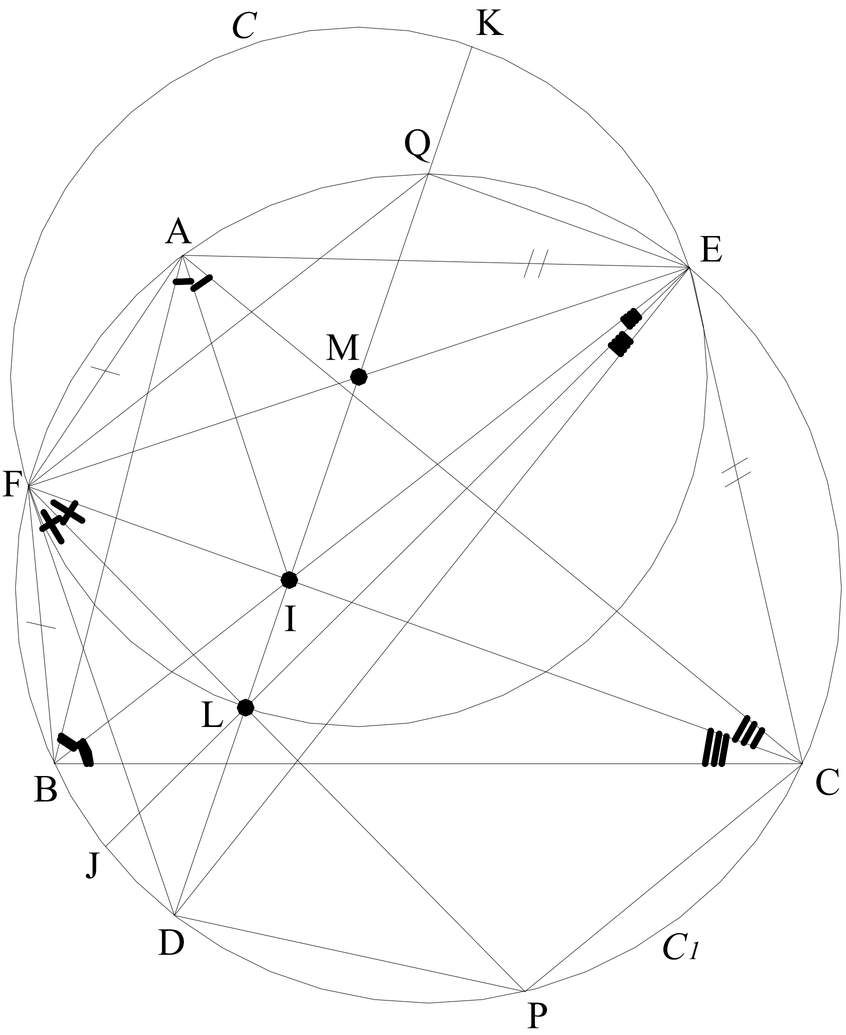
Figure 2. The two-dimensional layout (not to scale)

Let H be the foot of C onto BB' and I of D onto AA'. We have BH = ½(BB' – CC') = 1. Hence, cosα = 4/25. However, because AA' || BB' and AD || BC, $\angle DAA' = \alpha$, $\cos\alpha = \frac{AI}{AD} = \frac{4}{25}$, or AI = 1. Therefore, DD' = AA' – 2AI = 4.

<u>Problem 16 of the Iranian Mathematical Olympiad 2010</u>

In a triangle ABC, I is the incenter, BI and CI cut the circumcircle of ABC at E and F, respectively. M is the midpoint of EF. *C* is a circle with diameter EF. IM cuts *C* at two points L and K and the arc BC of circumcircle of ABC (not containing A) at D. Prove that $\frac{\mathrm{DL}}{\mathrm{IL}}=\frac{\mathrm{DK}}{\mathrm{IK}}$.

<u>Solution</u>



Let C_1 be the circumcircle of triangle ABC. Extend FL to meet C_1 at P. We will first prove that $\angle$DFP = $\angle$CFP, or DP = PC.
Let DK intercept C_1 at Q. Angle $\angle$FQE subtends arc FDE equals arc FB + arc BC + arc EC (i)

But since I is the incenter of ΔABC, BI and CI are angle bisectors of $\angle$ABC and $\angle$ACB, respectively. And we have arc FB = arc FA, and arc EA = arc EC.

Statement (i) becomes: $\angle$FQE subtends arc FDE equals arc FA + arc BC + arc EA = $\angle$FIE.

Also since that M is the midpoint of FE, and with $\angle$FQE = $\angle$FIE, rotating ΔFIE 180° clockwise causes F $\Rightarrow$ E, E $\Rightarrow$F and I $\Rightarrow$ Q.
Or IM = MQ, and EIFQ is a parallelogram which gives us $\angle$FEQ = $\angle$EFI, or EQ || FC making arc FQ = arc EC (ii)
Now notice that EF and LK are diameters of circle *C,* ELFK is a rectangle which makes $\angle$FLM = $\angle$LFM or
arc EC + arc PC = arc FQ + arc DP (iii)

Subtracting (iii) from (ii), we have arc DP = arc PC which implies that $\angle$DFP = $\angle$CFP, or DP = PC.

Similarly, extending EL to meet C_1 at J, we have $\angle$MLE subtending arc JD + arc EQ = $\angle$MEL = arc FB + arc BJ (iv)

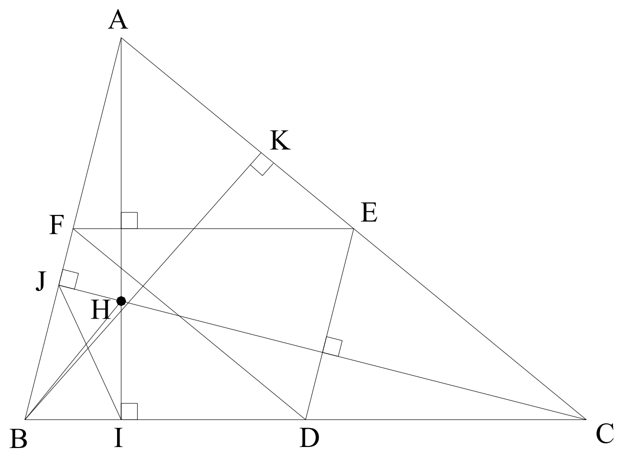
Since EIFQ is a parallelogram, IE || FQ, we have EQ = FB (v)
Subtracting (iv) from (v), we have BJ = JD, or $\angle$IEL = $\angle$DEI.

With E and F on circle *C,* $\angle$DFP = $\angle$CFP (or $\angle$DFI = $\angle$IFL) and $\angle$IEL = $\angle$DEL as have been proven, and LK being the diameter of circle *C*. This circle *C* is known as a circle of Apollonius, and the four points D, L, I and K form a harmonic subdivision from which we can make a conclusion that
$\frac{IL}{DL} = \frac{IK}{DK}$, or $\frac{DL}{IL} = \frac{DK}{IK}$.

Problem 2 of the United States Mathematical Olympiad 1997

ABC is a triangle. Take points D, E and F on the perpendicular bisectors of BC, CA and AB, respectively. Show that the lines through A, B and C perpendicular to EF, FD and DE, respectively are concurrent.

Solution



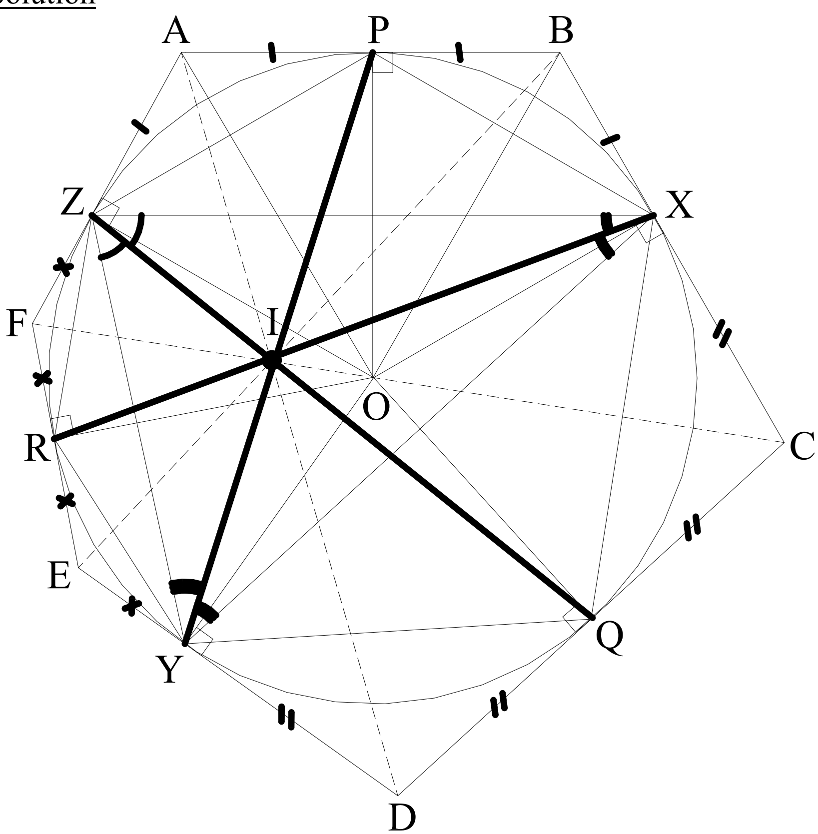
Since D, E and F are the midpoints of BC, AC and AB, respectively, we have EF || BC, DF || AC and DE || AB. The lines through A, B and C perpendicular to EF, FD and DE, respectively, are the altitudes of triangle ABC, and they must be concurrent at a point called the orthocenter.

We can also prove the altitudes of a triangle are concurrent as follows:
Let the altitudes from A, C and B of triangle ABC be AI, CJ and BK, respectively. Let's assume that BK does not pas through H which is the intersection of the other two altitudes.
Since BIHJ and ACIJ are cyclic quadrilaterals, we have
$\angle$JBH = $\angle$JIH, and $\angle$JIH = $\angle$JCA, or $\angle$JBH = $\angle$JCA.
But $\angle$JCA + $\angle$BAC = 90°, or $\angle$JBH + $\angle$BAC = 90°.
On the other hand, $\angle$JBK + $\angle$BAC = 90°, or
$\angle$JBH = $\angle$JBK, or the three points J, H and K are collinear, and the altitudes of a triangle are concurrent.

<u>*Problem 7 of the British Mathematical Olympiad 1999*</u>

Let ABCDEF be a hexagon (which may not be regular), which circumscribes a circle S. (That is, S is tangent to each of the six sides of the hexagon.) The circle S touches AB, CD, EF at their midpoints P, Q, R, respectively. Let X, Y, Z be the points of contact of S with BC, DE, FA, respectively. Prove that PY, QZ and RX are concurrent.

<u>Solution</u>



The distances from a point outside the circle to the points of tangent on either side on the circle are equal. We are also given that the circle S touches AB, CD and EF at their midpoints P, Q and R, respectively; therefore, we have

AP = PB = AZ = BX,
CQ = QD = CX = DY,
ER = RF = EY = FZ.

Since ΔAPO and ΔBPO are right triangles with AP = PB and share PO, they are congruent, and we have OA = OB and ∠AOP = ∠BOP, or ∠ZOP = ∠XOP, and PX = PZ.

Similarly, QX = QY and RY = RZ.

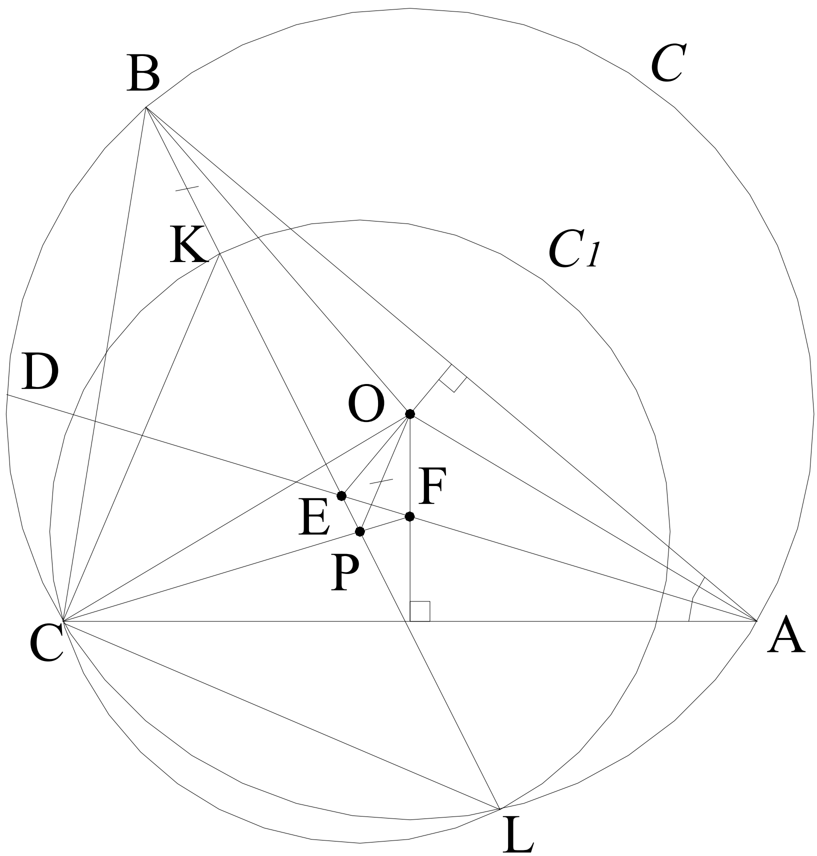
As a result, PY, QZ, RX are the angle bisectors of ΔXYZ and thus are concurrent at its incenter.

<u>Further observation</u>

Prove that AD, PY and QZ are also concurrent.

Problem 1 of Hong Kong Mathematical Olympiad 2000

Let C be the circumcenter of a triangle ABC with AB > AC > BC. Let D be a point on the minor arc BC of the circumcircle, and let E and F be points on AD such that AB $\perp$ OE and AC $\perp$ OF. The lines BE and CF meet at P. Prove that if PB = PC + PO, then $\angle$BAC = 30°.



Solution 1

Draw circle C_1 with center P and radius PC to meet BP at K. We have $\angle$BPC = $\angle$PEA + $\angle$DFC = 2$\angle$EBA + 2$\angle$FAC = 2$\angle$BAC = $\angle$BOC, and the four points B, C, P and O are concyclic.

Extending KP to meet C_1 at L, since BCPO is cyclic, $\angle$BPO = $\angle$BCO = $\angle$CBO (OB = OC = R, the radius of C) = 90°– ½$\angle$BOC = 90° – ½$\angle$BPC = $\angle$CKP, or KC || PO, and $\angle$BPO = $\angle$CKP = $\angle$KCP (PK = PC = r, the radius of C_1) = $\angle$OPF. And since O and P are the circumcenters and OP $\perp$ CL, point L is mirror image of C across OP, and thus L also lies on C. Angle $\angle$BLC now subtends minor arc KC on C_1 and minor arc BC on C. Therefore, $\frac{\text{KC}}{\text{BC}} = \frac{r}{R}$. From $\angle$KCP = $\angle$BCO, $\angle$BCK = $\angle$OCF.

Combining with $\frac{\text{KC}}{\text{BC}} = \frac{r}{R} = \frac{\text{PC}}{\text{OC}}$, ΔBCK and ΔOCP are similar, but given PB = PC + PO by the problem, we then have BK = PO and ΔBCK = ΔOCP implying BC = OC = R, and BOC is an equilateral triangle causing $\angle$BOC = 60° = 2$\angle$BAC, or $\angle$BAC = 30°.

<u>Solution 2</u>

As shown in solution 1, BCPO is cyclic which, by Ptolemy's theorem, gives BP×OC = BC×PO + PC×OB, or BP×R = BC×PO + PC×R, or R×(BP – PC) = BC×PO. Given PB = PC + PO, the previous expression becomes R×PO = BC×PO, or R = BC, and BOC is an equilateral triangle, and we get the same result.

<u>Further observation</u>

We have proven that $\angle$BAC = 30° which implies $\angle$EOF = 30°. We also have $\angle$KPO = $\angle$BCO = 60° = $\angle$BOC = $\angle$BPC. The angle $\angle$OPF is then also equal to 60° (180° – $\angle$BPC – $\angle$BPO). Or PO is the angle bisector of $\angle$EPF. Furthermore, given PB = PC + PO, or BE + EP = CF – PF + PO. But BE = AE, and CF = AF. We then have AF + EF + EP = CF – PF + PO, or EF + EP = – PF + PO, or PO = EF + EP + PF. Thus in the quadrilateral EPFO, PO = EF + EP + PF, $\angle$EOF = 30° and PO is the bisector of $\angle$EPF. This leads us to the next similar problem:

<u>Problem 3 of British Mathematical Olympiad 1990</u>

The angles A, B, C, D of a convex quadrilateral satisfy the relation $\cos A + \cos B + \cos C + \cos D = 0$. Prove that ABCD is a trapezium (British for trapezoid) or is cyclic.

<u>Solution</u>

Applying the formula $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$, we get

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$ and

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$.

a) The equation in the problem is equivalent to

$$\cos\frac{A+B}{2}\cos\frac{A-B}{2} + \cos\frac{C+D}{2}\cos\frac{C-D}{2} = 0 \qquad \text{(i)}$$

However, in an quadrilateral the four angles sum up to equal 360°, or $A + B = 360° - (C + D)$, or $\frac{A+B}{2} = 180° - \frac{C+D}{2}$, and $\cos\frac{A+B}{2} = \cos(180° - \frac{C+D}{2}) = -\cos\frac{C+D}{2}$, and equation (i) becomes

$-\cos\frac{C+D}{2}\cos\frac{A-B}{2} + \cos\frac{C+D}{2}\cos\frac{C-D}{2} = 0$, or $\cos\frac{A-B}{2} = \cos\frac{C-D}{2}$, or $A - B = C - D$, or $A + D = B + C = 180°$ which implies that ABCD is a trapezoid.

b) The equation in the problem is also equivalent to

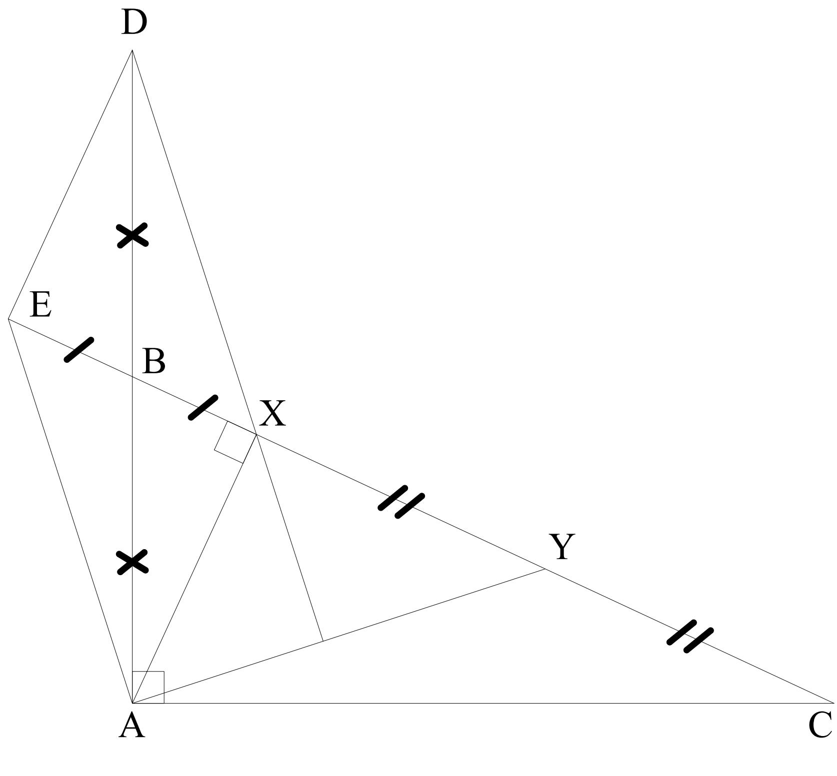
$$\cos\frac{A+D}{2}\cos\frac{A-D}{2} + \cos\frac{B+C}{2}\cos\frac{B-C}{2} = 0$$

Similarly, we get $\cos\frac{A-D}{2} = \cos\frac{B-C}{2}$, or $A + C = B + D = 180°$ which implies that ABCD is cyclic.

<u>Problem 5 of the Irish Mathematical Olympiad 1990</u>

Let ABC be a right-angled triangle with right-angle at A. Let X be the foot of the perpendicular from A to BC, and Y the mid-point of XC. Let AB be extended to D so that |AB| = |BD|. Prove that DX is perpendicular to AY.

<u>Solution</u>



Since $\angle$BAC is a right angle, and AX is perpendicular to BC, we have $AX^2 = BX\times XC = 2BX\times XY$.
Extend CB a segment of BE = BX, the above equation now becomes $AX^2 = EX\times XY$. Therefore, $\angle$EAY is a right angle, and since B is the midpoint of both EX and AD, AEDX is a parallelogram, or EA || DX, and DX is perpendicular to AY.

Problem 7 of the British Mathematical Olympiad 1998

A triangle ABC has $\angle$BAC > $\angle$BCA. A line AP is drawn so that $\angle$PAC = $\angle$BCA where P is inside the triangle. A point Q outside the triangle is constructed so that PQ is parallel to AB, and BQ is parallel to AC. R is the point on BC (separated from Q by the line AP) such that $\angle$PRQ = $\angle$BCA.

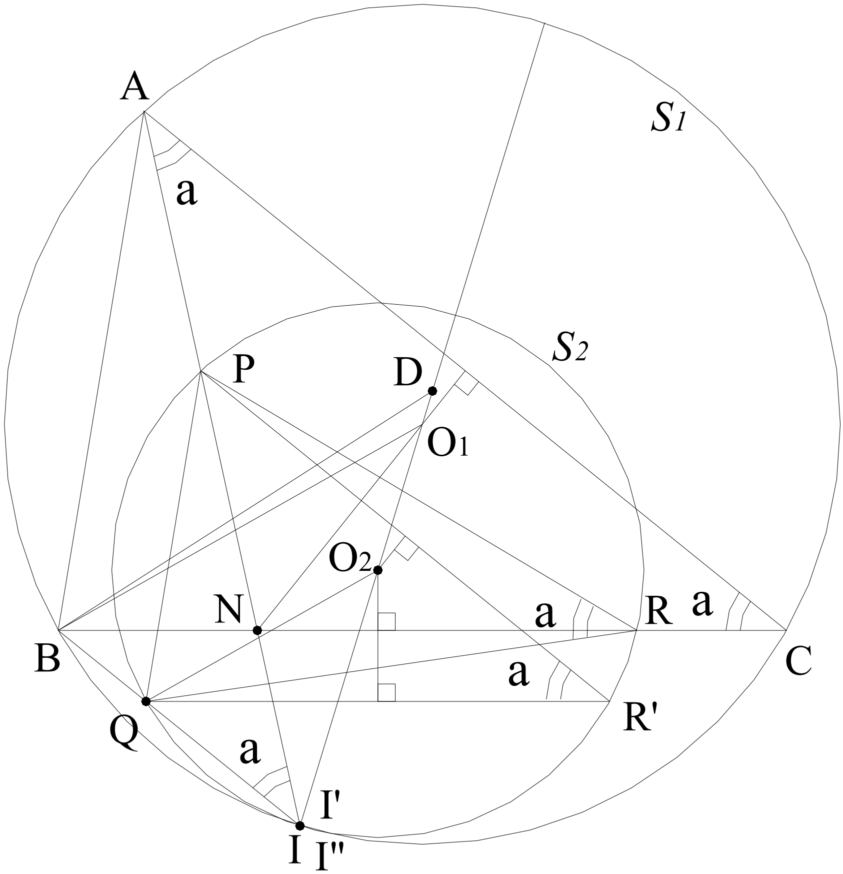
Prove that the circumcircle of ΔABC touches the circumcircle of ΔPQR.

Solution

To pick point R, let's pick point R' to satisfies PR' || AC and QR' || BC. Draw the circumcircle S_2 of triangle QPR' to cut BC at R (nearer to point C). Let S_1 be the circumcircle of triangle ABC, *R* and *r* be the radii of S_1 and S_2, respectively. Let $\angle$PAC = α; we also have $\angle$BCA = $\angle$PRQ = α.
Now let I = AP ∩ BQ, I' = AP ∩ S_1, and I" = AP ∩ S_2.
We're starting out with the understanding that the three points B, Q and I are on a straight line (collinear).
Since BQ ||AC, $\angle$AIB = $\angle$AIQ = α. (i)
Now with I' on S_2, $\angle$PI'Q subtends arc PQ, and $\angle$PI'Q=$\angle$PRQ = α.
Combining with (i), I' ≡ (coincides) I.
Also with I" on S_1, $\angle$AI"B subtends arc AB, and $\angle$AI"B = $\angle$ACB = α.

Again, combining with (i), I" ≡ (coincides) I.
Therefore, the three points I, I' and I" coincide, and let's refer to them as I only from this point on.

Also since $\angle$PRQ = $\angle$BCA and $\angle$PRQ subtends PQ on S_2, and $\angle$BCA subtends AB on S_1, we have



$\frac{QP}{BA} = \frac{r}{R} = \frac{QI}{BI}$. Let O_1 and O_2 be the circumcenters of S_1 and S_2, respectively. Link QO_2. From B draw a line to parallel to QO_2 to meet IO_2 at D. We have $\frac{QI}{BI} = \frac{IO_2}{ID} = \frac{r}{R}$, or R = ID. From there, ID = IO_1, or D ≡ O_1. The three points I, O_2 and O_1 are collinear.

Therefore, we finally conclude that the circumcircle of ΔABC touches the circumcircle of ΔPQR.

Problem 3 of Austria Mathematical Olympiad 2000

Determine all real solutions of the equation

$$|||||| x^2 - x - 1 | - 3 | - 5 | - 7 | - 9 | - 11 | - 13 | = x^2 - 2x - 48$$

Solution

Since the absolute value on the left must be non-negative, $x^2 - 2x - 48 = (x-1)^2 - 7^2$ must be non-negative, or

$x - 1 \geq 7 \Rightarrow (x \geq 8)$, or $x - 1 \leq -7 \Rightarrow (x \leq -6)$.

When $x \geq 8$

$x^2 - x - 1 \geq 55$, and all the other terms inside the absolute value signs are positive. In other words,

$|| x^2 - x - 1 | - 3 | \geq 52 > 0$,
$||| x^2 - x - 1 | - 3 | - 5 | \geq 47 > 0$,
$|||| x^2 - x - 1 | - 3 | - 5 | - 7 | \geq 40 > 0$,
$||||| x^2 - x - 1 | - 3 | - 5 | - 7 | - 9 | \geq 31 > 0$,
$|||||| x^2 - x - 1 | - 3 | - 5 | - 7 | - 9 | - 11 | \geq 20 > 0$, and
$||||||| x^2 - x - 1 | - 3 | - 5 | - 7 | - 9 | - 11 | - 13 | \geq 7 > 0$,

and we can then write
$|||||| x^2 - x - 1 | - 3 | - 5 | - 7 | - 9 | - 11 | - 13 | = x^2 - x - 49$. Now the original equation becomes

$x^2 - x - 49 = x^2 - 2x - 48$, or $x = 1$ which is not greater than or equal to 8. Therefore, there's no solution when $x \geq 8$.

When $x \leq -6$

$x^2 - x - 1 \geq 41$, and all these other terms inside the absolute value signs are positive. In other words,

$|| x^2 - x - 1 | - 3 | \geq 38 > 0$;
$||| x^2 - x - 1 | - 3 | - 5 | \geq 33 > 0$;

$|||| x^2 - x - 1| - 3| - 5| - 7| \geq 26 > 0$;
$||||| x^2 - x - 1| - 3| - 5| - 7| - 9| \geq 17 > 0$;
$|||||| x^2 - x - 1| - 3| - 5| - 7| - 9| - 11| \geq 6 > 0$, and

We can then write
$|||||| x^2 - x - 1| - 3| - 5| - 7| - 9| - 11|$ as $x^2 - x - 36$
and $||||||| x^2 - x - 1| - 3| - 5| - 7| - 9| - 11| - 13|$ as
$|| x^2 - x - 36| - 13|$.

Observe that $x^2 - x - 36 \leq -13$ when $-4.32 \leq x \leq 5.32$.
This range is outside of $x \geq 8$ and $x \leq -6$.

And that $x^2 - x - 36 \geq 13$ when $x \geq 7.52$ and $x \leq -6.52$.
In this range, we did find $x = 1$, and it was not an acceptable solution.

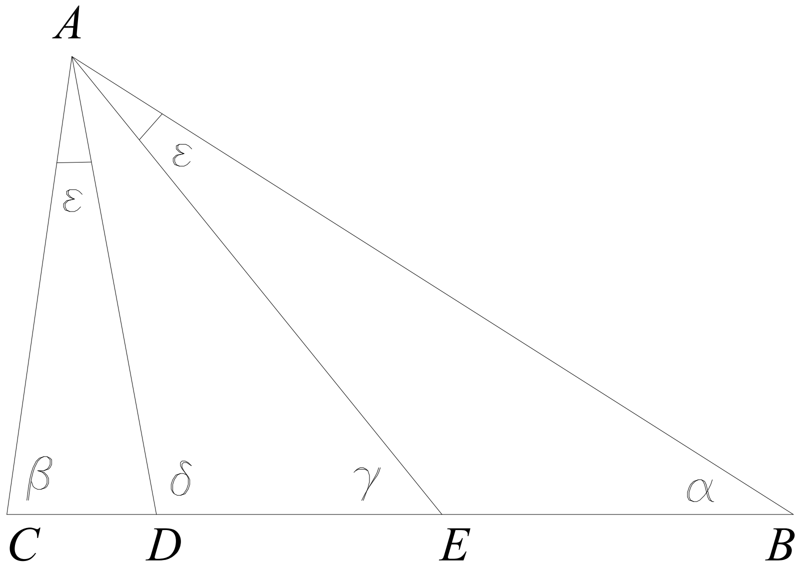
And $x^2 - x - 36 < 13$ when $-6.52 \leq x \leq 7.52$.
Let's only consider the range [-6.52, -6] since $-6 \leq (-6, 7.52] \leq 8$.
When $-6.52 \leq x \leq -6$, $|| x^2 - x - 36| - 13| = -x^2 + x + 49 =$
$x^2 - 2x - 48$, or $2x^2 - 3x - 97 = 0$.
Solving for x, we have $x = 7.75$ and -6.25.

Answer: $x = -6.25$.

<u>*Problem 2 of Belarus Mathematical Olympiad 1997 Category D*</u>

Points D and E are taken on side CB of triangle ABC, with D between C and E, such that $\angle BAE = \angle CAD$. If AC < AB, prove that AC×AE < AB×AD.

<u>Solution</u>



Let $\alpha = \angle ABC$, $\beta = \angle ACB$, $\gamma = \angle AEC$, $\delta = \angle ADB$ and $\varepsilon = \angle BAE = \angle CAD$. To prove AC×AE < AB×AD, it suffices to prove $\frac{AC}{AB} < \frac{AD}{AE}$ (i)

Applying the law of sine function, (i) becomes $\frac{\sin\alpha}{\sin\beta} < \frac{\sin\gamma}{\sin\delta}$, or

$\sin\alpha\times\sin\delta < \sin\beta\times\sin\gamma$, or

$-\frac{1}{2}[\cos(\alpha + \delta) - \cos(\alpha - \delta)] < -\frac{1}{2}[\cos(\beta + \gamma) - \cos(\beta - \gamma)]$, or

$\cos(\alpha + \delta) - \cos(\alpha - \delta) > \cos(\beta + \gamma) - \cos(\beta - \gamma)$ (ii)

but $\alpha + \delta = 180° - \angle BAD = 180° - \angle DAE - \varepsilon = \beta + \gamma$ and (ii) becomes $-\cos(\alpha - \delta) > -\cos(\beta - \gamma)$, or

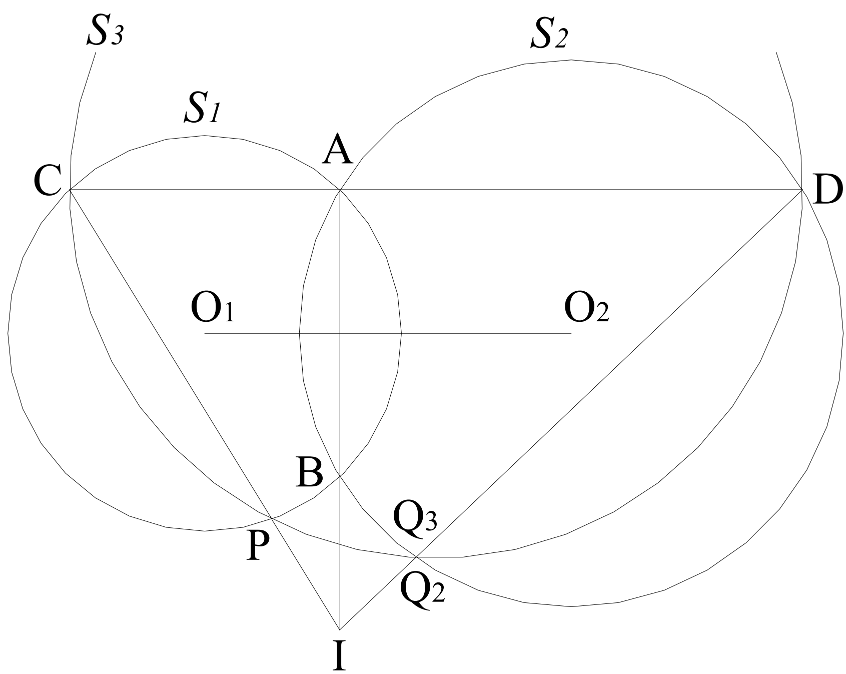
$\cos(\alpha - \delta) < \cos(\beta - \gamma)$, or $\cos(\delta - \alpha) < \cos(\gamma - \beta)$ (iii)

But we're given AC < AB which makes $\beta > \alpha$ and $\delta = \beta + \varepsilon > \alpha + \varepsilon = \gamma$, or $\delta - \alpha > \varepsilon$, and $\gamma - \beta < \varepsilon$, or $\delta - \alpha > \gamma - \beta$, and (iii) is a reality.

Problem 6 of Belarus Mathematical Olympiad 2004

Circles S_1 and S_2 meet at points A and B. A line through A is parallel to the line through the centers of S_1 and S_2 and meets S_1 again at C and S_2 again at D. The circle S_3 with diameter CD meets S_1 and S_2 again at P and Q, respectively. Prove that lines CP, DQ, and AB are concurrent.

Solution



Extend CP to meet the extension of AB at I. Link ID to meet S_3 at Q_3 and S_2 at Q_2.

We have $IP \times IC = IQ_3 \times ID$ since C, P, Q_3 and D are on S_3 (i)

$IP \times IC = IB \times IA$ since C, P, B and A are on S_1 (ii)

$IQ_2 \times ID = IB \times IA$ since A, B, Q_2 and D are on S_2 (iii)

From (ii) and (iii), $IP \times IC = IQ_2 \times ID$ (iv)

From (i) and (iv), $IQ_3 \times ID = IQ_2 \times ID$, or

$IQ_3 = IQ_2$.

But S_3 and S_2 only meet at a single point Q; therefore, $Q_3 \equiv Q_2 \equiv Q$, or the three CP, DQ, and AB are concurrent.

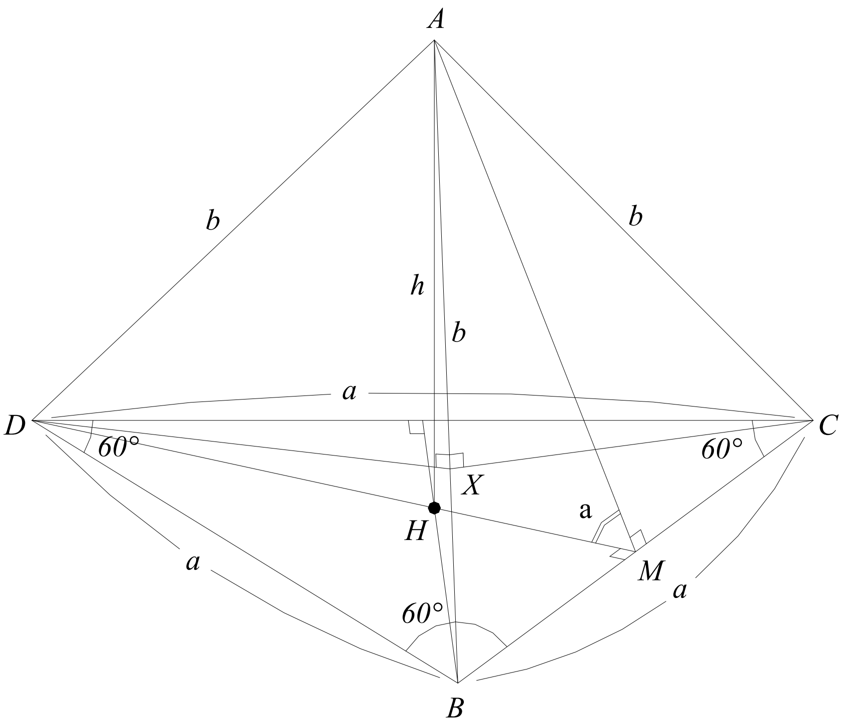
Further observation

Line CD does not have to parallel to the line through the centers of S_1 and S_2. The result is still the same regardless.

Problem 4 of the Vietnamese Mathematical Olympiad 1962

Let be given a tetrahedron ABCD such that triangle BCD equilateral and AB = AC = AD. The height is h and the angle between two planes ABC and BCD is α. The point X is taken on AB such that the plane XCD is perpendicular to AB. Find the volume of the tetrahedron XBCD.

Solution



Let's find the area of triangle XDC denoted (XDC) and the length segment XB. Let a be the side length of triangle BDC and b = AB = AC = AD. Drawing the altitude AM to BC and applying Pythagorean's theorem, we have

$AM^2 = b^2 - \frac{a^2}{4}$, or $AM = \frac{1}{2}\sqrt{4b^2 - a^2}$, and $BC^2 = CX^2 + BX^2$, or $a^2 = CX^2 + BX^2$.

Since M is the midpoint of BC and ABC is an isosceles triangle with AB = AC and triangle BCD is equilateral, $\alpha = \angle AMD$.

We also have $\tan\angle ABC = \frac{AM}{BM} = \frac{CX}{BX} = \frac{\sqrt{4b^2 - a^2}}{a}$.

Now solving the two equations, we have $BX = \frac{a^2}{2b}$, $CX = \frac{a}{2b}\times$

$\sqrt{4b^2 - a^2}$. Applying Heron's formula for (XDC), taking into account that CX = DX, we then have

$(XDC) = \sqrt{s(s-a)(s-CX)^2}$ where $s = \frac{a}{2} + CX = \frac{a}{2} + \frac{a}{2b}\sqrt{4b^2 - a^2}$,

$s - CX = \frac{a}{2}$, $s - a = -\frac{a}{2} + \frac{a}{2b}\sqrt{4b^2 - a^2}$.

After a few computations, $(XDC) = \frac{a^2}{4b}\sqrt{3b^2 - a^2}$.

The volume of the tetrahedron XBCD, by definition, is

$V = \frac{1}{3}(XDC)\times BX = \frac{a^4}{24b^2}\sqrt{3b^2 - a^2}$.

Furthermore, $h^2 = AM^2 - HM^2 = AM^2 - (\frac{DM}{3})^2 = \frac{3b^2 - a^2}{3}$, or

$h = \sqrt{\frac{3b^2 - a^2}{3}}$. The volume is now $V = \frac{a^4 h\sqrt{3}}{24b^2}$. (i)

But $\tan\alpha = \frac{h}{HM} = \frac{6h}{a\sqrt{3}}$, or $a = \frac{6h}{\tan\alpha\sqrt{3}}$, and $b^2 = AM^2 + BM^2 = h^2 +$

$HM^2 + \frac{a^2}{4} = h^2 + \frac{a^2}{12} + \frac{a^2}{4} = h^2 + \frac{a^2}{3}$, or $b^2 = h^2 + \frac{4h^2}{\tan^2\alpha}$.

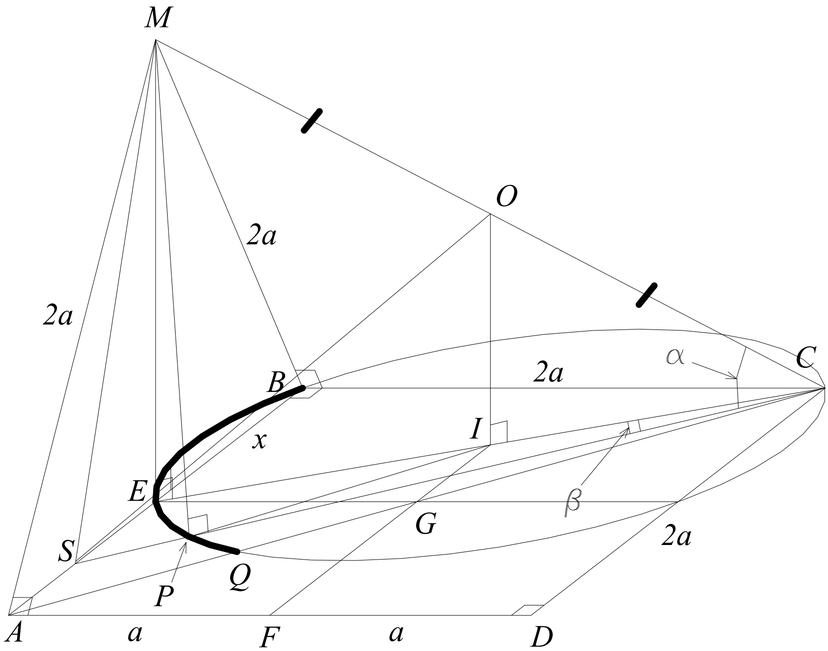
Substituting the values of a and b^2 into (i), the volume of the tetrahedron XBCD $= \frac{6h^3\sqrt{3}}{\tan^2\alpha(\tan^2\alpha + 4)}$.

Problem 4 of the Vietnamese Mathematical Olympiad 1986

Let ABCD be a square of side $2a$. An equilateral triangle AMB is constructed in the plane through AB perpendicular to the plane of the square. A point S moves on AB such that SB = x. Let P be the projection of M on SC and E, O be the midpoints of AB and CM, respectively.

a) Find the locus of P as S moves on AB.
b) Find the maximum and minimum lengths of SO.

Solution



Let $\alpha = \angle MCP$, $\beta = \angle ECP$ and I be the midpoint of CE. The lengths of the other segments are calculated to be CA = CM = $2\sqrt{2}a$, CE = $a\sqrt{5}$, IC = $\frac{1}{2}$CE = $\frac{1}{2}a\sqrt{5}$, ME = $a\sqrt{3}$, SE = $x - a$, MS = $\sqrt{(x-a)^2 + 3a^2}$, SC = $\sqrt{x^2 + 4a^2}$.

a) Applying the law of the cosine function, we have
$MS^2 = CM^2 + SC^2 - 2CM \times SC \times \cos\alpha$, or

$(x-a)^2 + 3a^2 = 8a^2 + x^2 + 4a^2 - 4\sqrt{2}a\times\sqrt{x^2+4a^2}\times\cos\alpha$, or

$\cos\alpha = \dfrac{x+4a}{2\sqrt{2(x^2+4a^2)}}$, but $\cos\alpha = \dfrac{CP}{CM}$; therefore, $CP = 2\sqrt{2}a\times$
$\dfrac{x+4a}{2\sqrt{2(x^2+4a^2)}} = \dfrac{a(x+4a)}{\sqrt{x^2+4a^2}}$.

Again, the law of the cosine function gives us
$SE^2 = CE^2 + SC^2 - 2\,CE\times SC\times\cos\beta$, or $(x-a)^2 = 5a^2 + x^2 + 4a^2 -$
$2a\sqrt{5(x^2+4a^2)}\cos\beta$, or $\cos\beta = \dfrac{x+4a}{\sqrt{5(x^2+4a^2)}}$.

$IP^2 = IC^2 + CP^2 - 2\,IC\times CP\times\cos\beta$, or

$IP^2 = \dfrac{5a^2}{4} + \dfrac{a^2(x+4a)^2}{x^2+4a^2} - a\sqrt{5}\,\dfrac{a(x+4a)}{\sqrt{x^2+4a^2}}\times\dfrac{x+4a}{\sqrt{5(x^2+4a^2)}} = \dfrac{5a^2}{4}$, or

$IP = \frac{1}{2}a\sqrt{5}$ which is a constant, and the locus of P is part of the circle that has its center at I and radius of $\frac{1}{2}a\sqrt{5}$ that passes through point E and is from B to Q where Q is the intersection of the circle and CA.

b) Since I and O are the midpoints of CE and CM, respectively, IO || ME, and the plane containing the three points M, C and E is perpendicular with the plane of the square, IO is then perpendicular with CE and $SO^2 = IO^2 + SI^2$.

But $IO = \frac{1}{2}ME = \frac{1}{2}a\sqrt{5}$ is fixed; the extreme values of SO depend on SI. As S moves on AB, SI is a minimum when S is at the midpoint of EB ($SI = a$), and is a maximum when S is at A when $SI^2 = AI^2 = AF^2 + FI^2$ where F is the midpoint of AD.
Now let G be the midpoint of AC.

$SI^2 = AF^2 + (FG + GI)^2 = a^2 + (a + \frac{a}{2})^2 = \dfrac{13a^2}{4}$, and

$SO^2_{max} = \dfrac{5a^2}{4} + \dfrac{13a^2}{4} = \dfrac{9a^2}{2}$, or $SO_{max} = \dfrac{3a}{\sqrt{2}}$, and

$SO^2_{min} = \dfrac{5a^2}{4} + a^2 = \dfrac{9a^2}{4}$, or $SO_{min} = \dfrac{3a}{2}$.

<u>Problem 4 of the Irish Mathematical Olympiad 2006</u>

Given a positive integer n, let $b(n)$ denote the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n. For example, $b(13) = 6$ because $13 = 1101_2$, which contains as consecutive blocks the binary representations of $13= 1101_2$, $6 = 110_2$, $5 = 101_2$, $3 = 11_2$, $2 = 10_2$ and $1 = 1_2$.

Show that if $n \le 2500$, then $b(n) \le 39$, and determine the values of n for which equality holds.

<u>Solution</u>

Note the first row of number right below. Starting from number 1 on the right, the next number equals the previous one multiplied by 2 from right to left.

2048	1024	512	256	128	64	32	16	8	4	2	1	n
1	0	0	1	1	1	0	0	0	1	0	0	2500
0	1	1	1	1	1	0	1	1	0	0	0	2008

The sum 2500 equals 2048 + 256 + 128 + 64 + 4. On the middle row, the number 1's are placed below the numbers on the top row that sum up to be 2500 and the numbers 0's are under the other numbers as we have seen. But let's pick number 2008 (bottom row) which is smaller than 2500 and 2008 = 1024 + 512 + 256 + 128 + 64 + 16 + 8, and put the number 1's below these numbers that sum up to be 2008.

Therefore, if n = 2008, the number of positive integers whose binary representations occur as blocks of consecutive integers in the binary expansion of n are

1, (1024)
11, (1024, 512)
111, (1024, 512, 256)
1111, (1024, 512, 256, 128
11111, (1024, 512, 256, 128, 64)
111110, (1024, 512, 256, 128, 64, 32

1111101,	(1024, 512, 256, 128, 64, 32, 16
11111011,	(1024, 512, 256, 128, 64, 32, 16, 8)
111110110,	(1024, 512, 256, 128, 64, 32, 16, 8, 4)
1111101100,	(1024, 512, 256, 128, 64, 32, 16, 8, 4, 2)
11111011000,	(1024, 512, 256, 128, 64, 32, 16, 8, 4, 2, 1)

total of eleven numbers so far in addition to the next six numbers

11110,	(512, 256, 128, 64, 32
111101,	(512, 256, 128, 64, 32, 16
1111011,	(512, 256, 128, 64, 32, 16, 8)
11110110,	(512, 256, 128, 64, 32, 16, 8, 4)
111101100,	(512, 256, 128, 64, 32, 16, 8, 4, 2)
1111011000,	(512, 256, 128, 64, 32, 16, 8, 4, 2, 1)

plus the next six numbers

1110,	(256, 128, 64, 32
11101,	(256, 128, 64, 32, 16
111011,	(256, 128, 64, 32, 16, 8)
1110110,	(256, 128, 64, 32, 16, 8, 4)
11101100,	(256, 128, 64, 32, 16, 8, 4, 2)
111011000,	(256, 128, 64, 32, 16, 8, 4, 2, 1)

plus the next six numbers

110,	(128, 64, 32
1101,	(128, 64, 32, 16
11011,	(128, 64, 32, 16, 8)
110110,	(128, 64, 32, 16, 8, 4)
1101100,	(128, 64, 32, 16, 8, 4, 2)
11011000,	(128, 64, 32, 16, 8, 4, 2, 1)

plus the next six numbers

10,	(128, 64, 32
101,	(128, 64, 32, 16
1011,	(128, 64, 32, 16, 8)
10110,	(128, 64, 32, 16, 8, 4)
101100,	(128, 64, 32, 16, 8, 4, 2)
1011000,	(128, 64, 32, 16, 8, 4, 2, 1),

and add in four more numbers 1100, 11000, 100 and 1000.

The total of all the numbers is $11 + 6 + 6 + 6 + 6 + 4 = 39$, and when $n = 2008 \leq 2500$, $b(n) = 39$ as required. The value of n for which equality occurs is $n = 2008$.

Problem 1 of the Canadian Mathematical Olympiad 1992

Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n + 1$ is not an odd prime.

Solution

The product of the first n natural numbers is
$1.2.3.4\ldots\ldots\ldots(n-1)n$.
The sum of the first n natural numbers is
$1 + 2 + 3 + 4 + \ldots + (n-1) + n = 0.5n(n+1)$.
Let k be the resultant of the product divided by the sum, we have

$$k = \frac{1.2.3.4\ldots(n-1)n}{0.5n(n+1)} = \frac{2.2.3.4\ldots(n-1)}{n+1}$$

The following are possibilities for $n + 1$.
1) $n + 1$ is an even number
Let $n + 1 = 2k$, or $n - 1 = 2k - 2 = 2(k - 1)$, and

$$k = \frac{2.2.3.4\ldots(n-1)}{n+1} = \frac{2.2.3.4\ldots(n-2)2(k-1)}{2k} =$$

$$= \frac{2.2.3.4\ldots(k-1)k(k+1)\ldots(n-3)(n-2)(k-1)}{k}$$

$$= 2.2.3.4\ldots(k-2)(k-1)(k+1)(k+2)\ldots(n-3)(n-2)(k-1)$$

So the first case of n not being an odd prime satisfies the problem.

2) $n + 1$ is an odd number
a) It's a prime number
When $n + 1$ is a prime number it can not be factored out to smaller numbers, and thus the product is then not divisible by the sum.

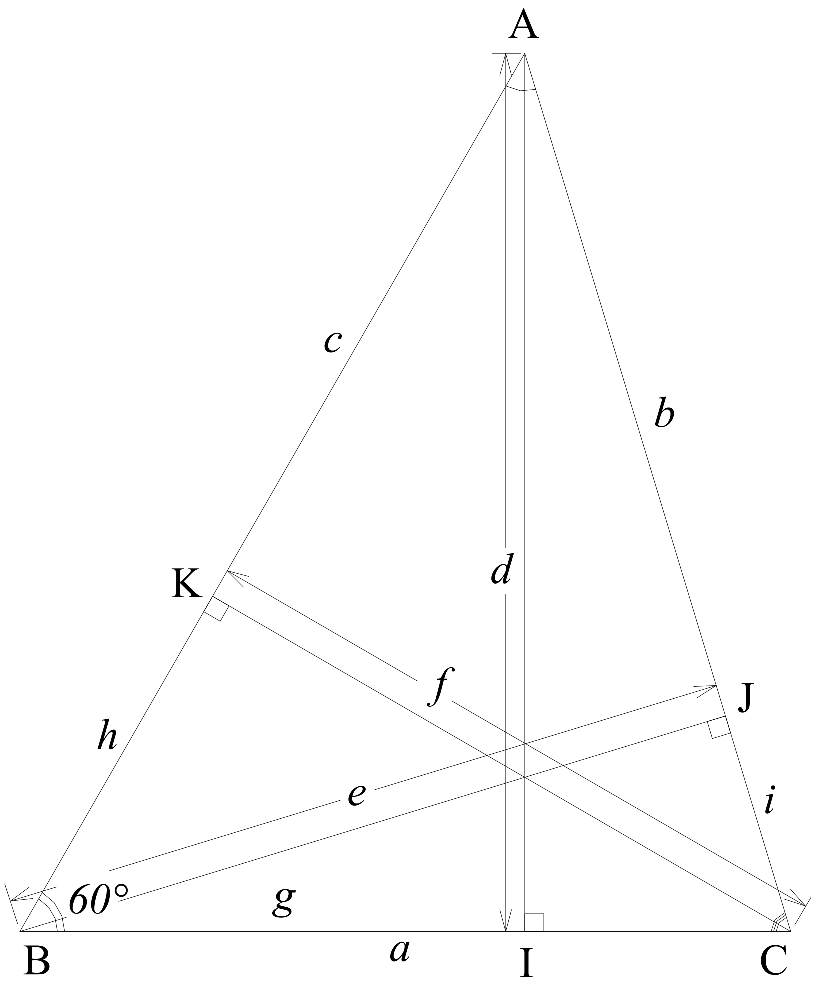
$$k = \frac{2.2.3.4\ldots(n-1)}{n+1}.$$

b) It's not a prime number
When $n + 1$ is not a prime number it can be factored out to two or more numbers that are smaller than itself $n + 1 = m(m + 1)(m + p)\ldots$ and $(m + p) < n - 1$, and thus the product is then divisible by the sum.
Note: n in the problem must be > 2.

Problem 1 of the Ibero-American Mathematical Olympiad 1988

The measures of the angles of a triangle is an arithmetic progression and its altitudes is also another arithmetic progression. Prove that the triangle is equilateral.

Solution



Let I, J, and K be the feet of A, B and C on BC, AC and AB, respectively. Now let BC = a, AC = b, AB = c, AI = d, BJ = e, CK = f, BI = g, CJ = i, BK = h, $\angle$BAC = α, $\angle$ABC = β and ACB = γ.

Assume α is the smallest angle of the triangle and ε is the angle of common difference. We have $\beta = \alpha + \varepsilon$, $\gamma = \alpha + 2\varepsilon$, but the sum of the angles is 180°, we then have $3(\alpha + \varepsilon) = 180°$, or $\beta = \alpha + \varepsilon = 60°$, and $\alpha = 120° - \gamma$.

Now it suffices to prove $a = c$ for the triangle ABC to be equilateral.
Since $\beta = 60°$, we have $a = 2h$, $c = 2g$ and $f^2 = a^2 - h^2 = 3h^2$, or

$f = h\sqrt{3} = a\dfrac{\sqrt{3}}{2}$.

Similarly, $d = c\dfrac{\sqrt{3}}{2}$ and since d, e, and f form another arithmetic

progression, we have $e = \dfrac{f + d}{2} = (a + c)\dfrac{\sqrt{3}}{4}$ (i)

We also have $\sin\alpha = \dfrac{e}{c}$, and $\sin\gamma = \dfrac{e}{a}$, or

$\sin\alpha = \sin(120° - \gamma) = \dfrac{\sqrt{3}}{2}\cos\gamma + \dfrac{e}{2a} = \dfrac{e}{c}$ (ii)

but $\cos\gamma = \dfrac{i}{a}$, (ii) becomes $\dfrac{\sqrt{3}i}{2a} + \dfrac{e}{2a} = \dfrac{e}{c}$ (iii)

Applying the Pythagorean's theorem to right triangle BJC, we have

$i = \sqrt{a^2 - e^2}$.

Now substituting i and e from (i) to (iii), we have

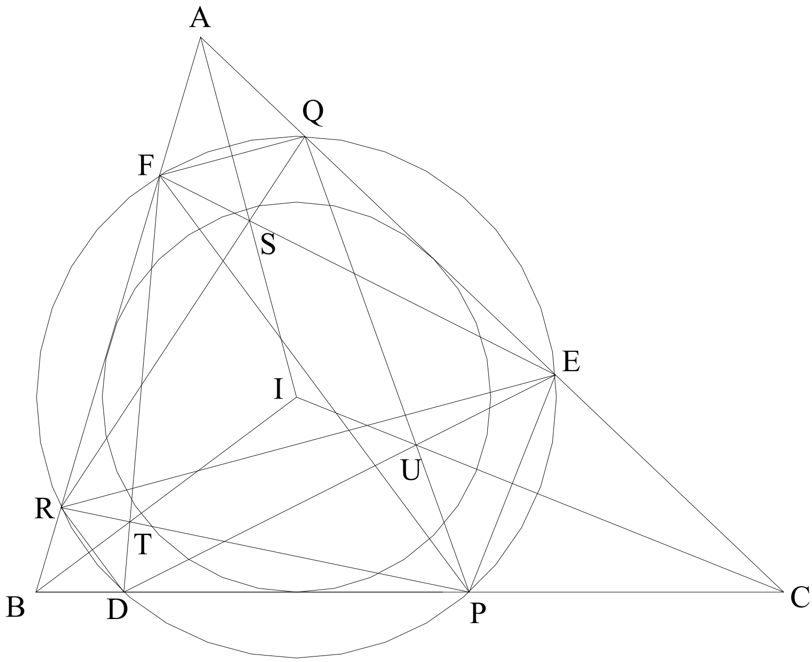
$a^4 + c^4 + a^3c + ac^3 - 4a^2c^2 = 0$, or

$(a - c)^2(a^2 + 3ac + c^2) = 0$, or $a = c$.

Problem 2 of the Ibero-American Mathematical Olympiad 1997

In a triangle ABC draw a circumcircle with its center I being the incicle of the triangle to intersect twice each of the sides of the triangle: the segment BC on D and P (where D is nearer two B), the segment CA on E and Q (where E is nearer to C) and the segment AB on F and R (where F is nearer to A). Let S be the intersection of the diagonals of the quadrilateral EQFR, T be the intersection of the diagonals of the quadrilateral FRDP and U be the intersection of the diagonals of the quadrilateral DPEQ. Show that the circumcircles of the triangles FRT, DPU and EQS have a unique point in common.

Solution



It is easily seen that BR = BD, and BI the angle bisector of $\angle$B cuts RD in two equal segments.
Since we also have BF = BP, D and P are symmetrical images of R and F, respectively with respect to BI, and FRDP is a isosceles trapezoid, and T is on BI.

Similarly, EQFR and DPEQ are also isosceles trapezoids, and S and U are on segments AI and CI, respectively.

With I being the center of the larger circle, $\angle$RID = 2$\angle$RFD, or $\angle$RFT = $\angle$RIT and FRTI is cyclic.

The same arguments apply to DPUI and EQSI. Therefore, the circumcircles of the triangles FRT, DPU and EQS have unique point I in common.

<u>Problem 1 of Tournament of Towns 1987</u>

A machine gives out five pennies for each nickel inserted into it. The machine also gives out five nickels for each penny. Can Peter, who starts out with one penny, use the machine in such a way as to end up with an equal number of nickels and pennies?

<u>Solution</u>

Peter inserts the penny and gets 5 nickels. The process is now for Peter to insert m number of nickels into the machines and still keep n remaining nickels where both m and n are integers from 0 to 5 and $m + n = 5$.

He now has $5m$ pennies and n nickels. Next he would insert a p number of pennies into the machine where p is an integer. He now has $5m - p$ pennies and $5p + n$ nickels. To end up with an equal number of nickels and pennies, he must have $\mathbf{5m - p = 5p + n}$, or $5m = 6p + n$.

However, $m = 5 - n$, and the previous equation $5m = 6p + n$ becomes $\mathbf{25 = 6(p + n)}$ which is not possible because $6(p + n)$ is an even number.

Peter now continues with the process; he inserts a q number of nickels and gets $5m - p + 5q$ pennies and still has $5p + n - q$ nickels. Again to end up with an equal number of nickels and pennies, he must have $5m - p + 5q = 5p + n - q$, or $\mathbf{5m - p = 5p + n - 6q}$, or $\mathbf{25 = 6(q - p + n)}$ which is again not possible.

The process continues to give us the equation that is the same as the previous one with the addition to the right hand side of a product of 6 and the number of pennies or nickels Peter inserts previously. Therefore, he always ends up with an equation that has an odd number 25 on the left side and an even number on the right side which is never possible, and Peter will never be able to end up with an equal number of nickels and pennies.

Problem 1 of the Canadian Mathematical Olympiad 1981

For any real number t, denote by $[t]$ the greatest integer which is less than or equal to t. For example: $[8] = 8$, $[\pi] = 3$ and $[-\frac{5}{2}] = -3$. Show that the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345$$

has no real solution.

Solution

Let $x = i + f$ where i is the integer part or integral part and f the fractional part of x. We have $f < 1$, and $[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 63i + [f] + [2f] + [4f] + [8f] + [16f] + [32f]$.

Since $f < 1$, $[f] = 0$, and we have
$63i + [f] + [2f] + [4f] + [8f] + [16f] + [32f] = 63i + [2f] + [4f] + [8f] + [16f] + [32f] = 12345 = 63\times195 + 60$.

Therefore, $i = 195$, and $[2f] + [4f] + [8f] + [16f] + [32f] = 60$ (i)

Since maximum value of $[nf] = n - 1$, the maximum value of $[2f] + [4f] + [8f] + [16f] + [32f] = 1 + 3 + 7 + 15 + 31 = 57$.

Therefore, equation (i) is not possible, and there is no f that satisfies the equation in the problem, and thus there is no x.

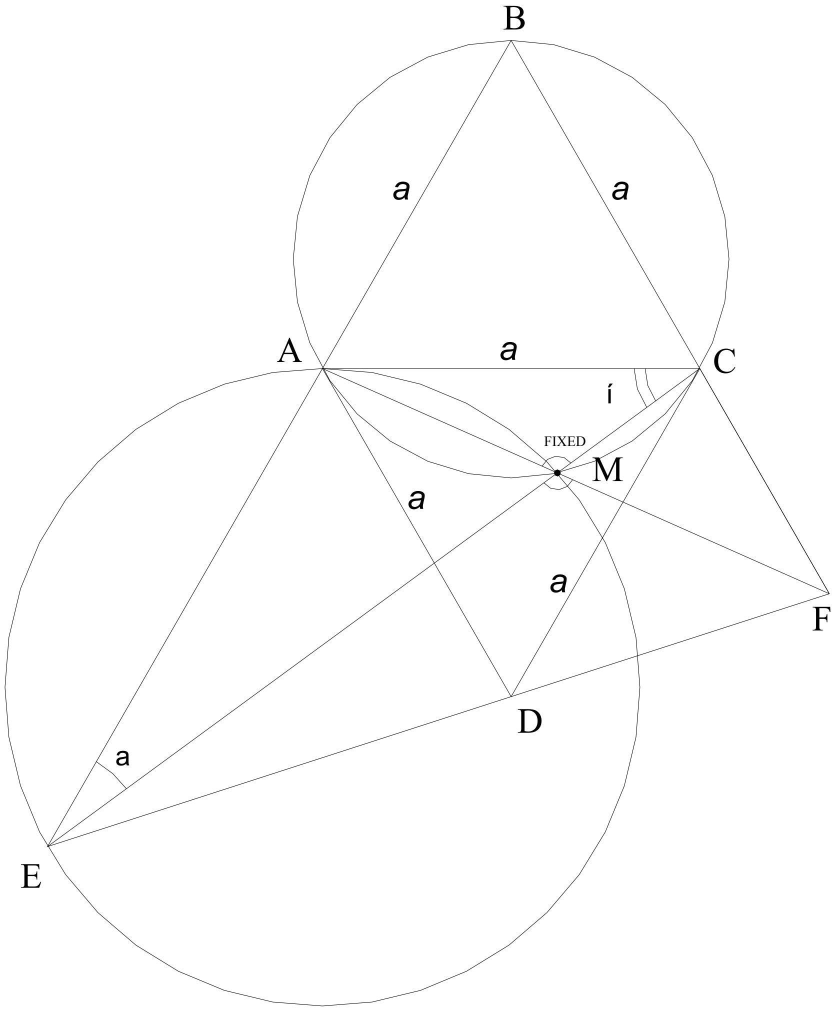
Further observation

We can change the number 12345 to 12344 or 12343 and the problem is still valid.

Problem 1 of Asian Pacific Mathematical Olympiad 1993

Let ABCD be a quadrilateral such that all sides have equal length and angle ABC is 60°. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF. Prove that $CA^2 = CM \times CE$.

Solution



Let a be the length of the equilateral triangle ABC and ACD as

shown and $\angle AEC = \alpha$ and $\angle ACE = \beta$.
We have $\alpha + \beta = 180° - \angle EAC = 180° - 120° = 60°$ (i)

We also have AE || CD and AD || CF; therefore, the two triangles EAD and DCF are similar which causes $\frac{EA}{a} = \frac{a}{CF}$. This makes the two triangles EBC and BCF to also be similar, and as a result $\alpha = \angle CAF$.

Therefore from (i), $\angle CAM + \beta = 60°$, and $\angle AMC = 120°$.

From there the two triangles EAC and AMC are similar because their respective angles are equal. Hence,

$\frac{CE}{CA} = \frac{CA}{CM}$, or $CA^2 = CM \times CE$.

<u>Further observation</u>

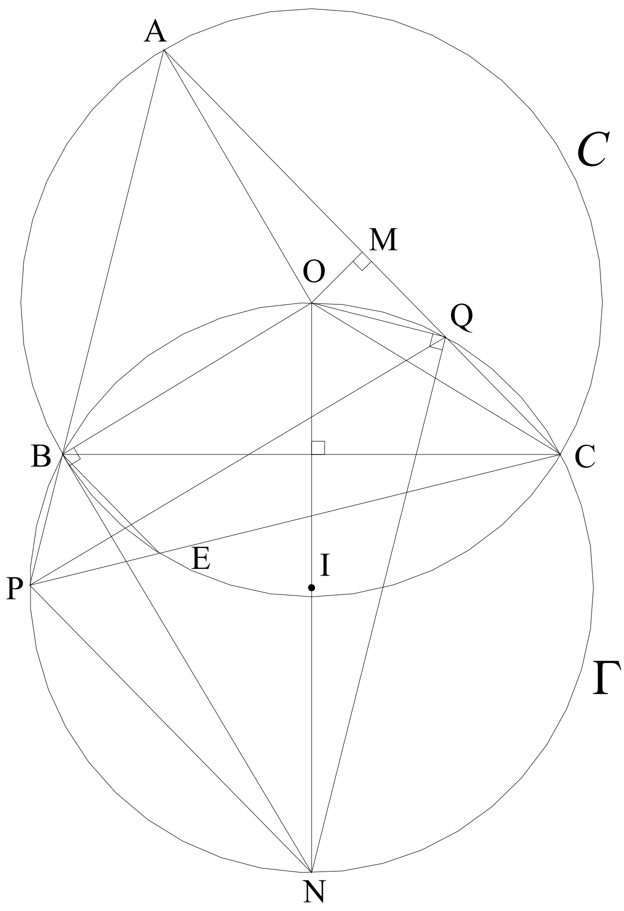
The problem below is derived from the above problem:

Let ABCD be a quadrilateral such that all sides have equal length and angle ABC is 60°. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF. Find the locus of point M.

Problem 1 of the Asian Pacific Mathematical Olympiad 2010

Let ABC be a triangle with $\angle BAC \neq 90°$. Let O be the circumcenter of the triangle ABC and let Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be a diameter of the circle Γ. Prove that the quadrilateral APNQ is a parallelogram.

Solution



Let the circumcircle of triangle ABC be *C,* M be the midpoint of

AC and E the intersection of C with PC. Also let r and R be the radii of C and Γ, respectively.

Consider two right triangles MOC and QON with $\angle$ONQ = $\angle$OCQ (subtends arc OQ). They are thus similar; therefore,

$\frac{\text{MC}}{\text{OC}} = \frac{\text{QN}}{\text{ON}}$, or $\frac{\text{MC}}{r} = \frac{\text{QN}}{2R}$, or $\frac{\text{AC}}{\text{QN}} = \frac{r}{R}$, or $\angle$ABC = $\angle$NPQ.

We also have $\angle$PQN = $\angle$PBN (subtends PN) = 180° − $\angle$OBN − $\angle$ABO = 90° − $\angle$ABO, but $\angle$ABO = $\angle$BAO, $\angle$OBC = $\angle$OCB and $\angle$OAC = $\angle$OCA, or $\angle$OCB + $\angle$OCA = $\angle$ACB = 90° − $\angle$ABO = $\angle$PQN.

Now $\angle$BAC = 180° − $\angle$ABC − $\angle$ACB = 180° − $\angle$NPQ − $\angle$PQN = $\angle$PNQ.

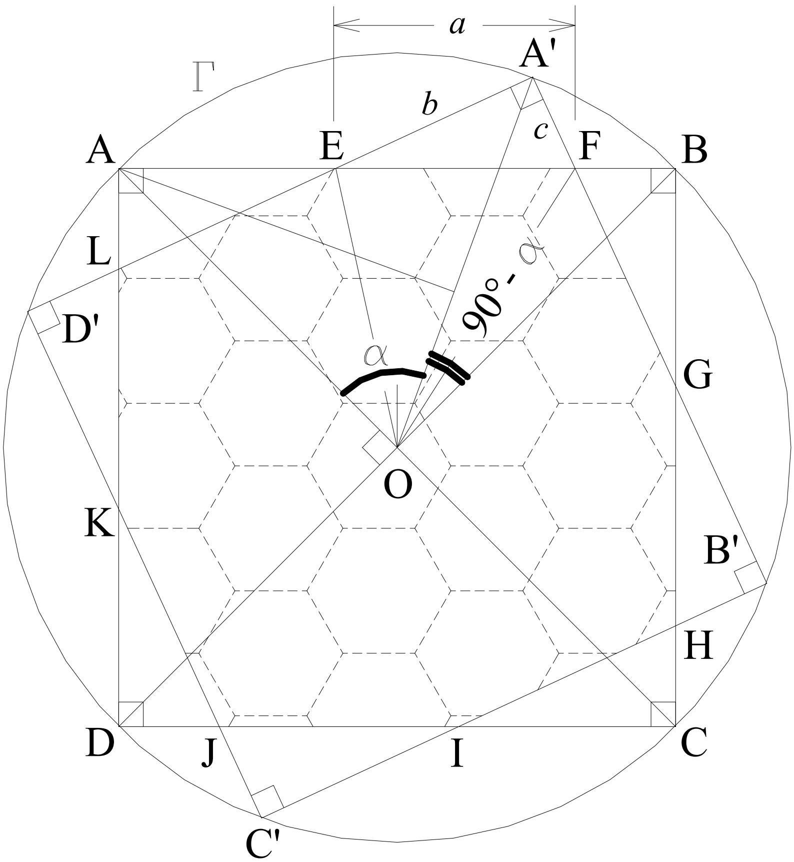
Moreover, $\angle$PNQ = $\angle$PCQ = $\angle$BAC, or AP = PC and BE || AC, and since BN is tangent to C, $\angle$NBE = $\angle$BCE = $\angle$BNP (subtends arc BP), or BE || PN.

Along with BE || AC, we have PN || AC. Combining with $\angle$BAC = $\angle$PNQ, we conclude that APNQ is a parallelogram.

<u>*Problem 1 of Spain Mathematical Olympiad 1998*</u>

A unit square ABCD with center O is rotated about O by an angle α. Compute the common area of the two squares.

<u>Solution</u>



Let the side length of the square be *l*, the rotated square be A'B'C'D' and its sides to meet the sides of ABCD at E, F, G, H, I, J, K and L as shown. We are asked to compute the common area EFGHIJKL.

Let's draw the circumcircle Γ of the squares. Since A'B'C'D' rotates about O, AA' = BB' = CC' = DD' and A'B = AD'. Combining with the fact that the triangles A'BE and AD'E are similar (since AA'BD' is cyclic), we conclude that those two triangles are congruent which gives us AE = A'E. This causes the two already similar triangles ΔAEL and ΔA'EF to be congruent (similar triangles with the two equal side lengths) and ΔAOE = ΔA'OE (all respective sides are equal) which implies that $\angle$AOE $= \frac{1}{2}\alpha$. Similarly, OF is the bisector of $\angle$BOA', and $\angle$BOF $= 45° - \frac{1}{2}\alpha$.

With the same argument, all these triangles are congruent to one another ΔAEL, ΔA'EF, ΔBGF, ΔB'GH, ΔCIH, ΔC'IJ, ΔDKJ and ΔD'KL.

We now need to find the area of one of these triangles. Let a = EF, b = EA' and c = FA'. It suffices to find the product bc.

Since ΔAEL = ΔA'EF = ΔBGF, b = AE and c = BF, or the sum of the perimeter of ΔA'EF equals the side length of the square and equals l, or $a + b + c = l$ (i)

Also because $\angle$AFA' = $\angle$BFB' subtends the equal arcs AA' and BB', $\angle$AFA' = α. We now have $\tan\alpha = \frac{b}{c}$ (ii)

And the right triangle A'EF gives us $a^2 = b^2 + c^2$ (iii)

From (iii), we have $a^2 = (b + c)^2 - 2bc$, or $bc = \frac{1}{2}[(b + c)^2 - a^2] = \frac{1}{2}[(b + c)^2 - (l - b - c)^2] = l(b + c - \frac{l}{2}) = l(\frac{l}{2} - a)$.

The area of EFGHIJKL is equal the area of the square minus four times the area of triangle A'EF $= l^2 - 4\times\frac{bc}{2} = l^2 - 2bc = 2al$.

Now let's find a. Applying the law of sines to triangle EOF, we obtain $\frac{\text{EF}}{\sin\angle\text{EOF}} = \frac{a}{\sin 45°} = \frac{\text{OE}}{\sin\angle\text{EFO}} = \frac{\text{OE}}{\sin(\angle\text{EBO} + \angle\text{FOB})} = \frac{\text{OE}}{\sin[45° + \frac{1}{2}(90° - \alpha)]} = \frac{\text{OE}}{\cos\frac{\alpha}{2}} = \text{OE}\times\sec\frac{\alpha}{2}$.

Similarly, in triangle AOE the law of sines gives us $\frac{\text{OE}}{\sin 45°} =$

$$\frac{\text{OA}}{\sin\angle\text{AEO}} = \frac{\text{OA}}{\sin(\angle\text{EBO} + \angle\text{EOB})} = \frac{\text{OA}}{\sin[45° + \frac{\alpha}{2} + 90° - \alpha]} =$$

$$\frac{\text{OA}}{\sin[90° - (\frac{\alpha}{2} - 45°)]} = \frac{\text{OA}}{\cos(\frac{\alpha}{2} - 45°)}.$$

From the previous two equations, we come up with

$a = \dfrac{\text{OA}\times\sin^2 45°}{\cos\frac{\alpha}{2}\cos(\frac{\alpha}{2} - 45°)}$. However, OA is half length of the diagonal of the square and $\text{OA} = \frac{l}{\sqrt{2}}$, $\cos(\frac{\alpha}{2} - 45°) = \frac{\sqrt{2}}{2}(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})$ and $\sin^2 45° = \frac{1}{2}$, we then have $a = \dfrac{l}{2\cos^2\frac{\alpha}{2} + \sin\alpha}$, and

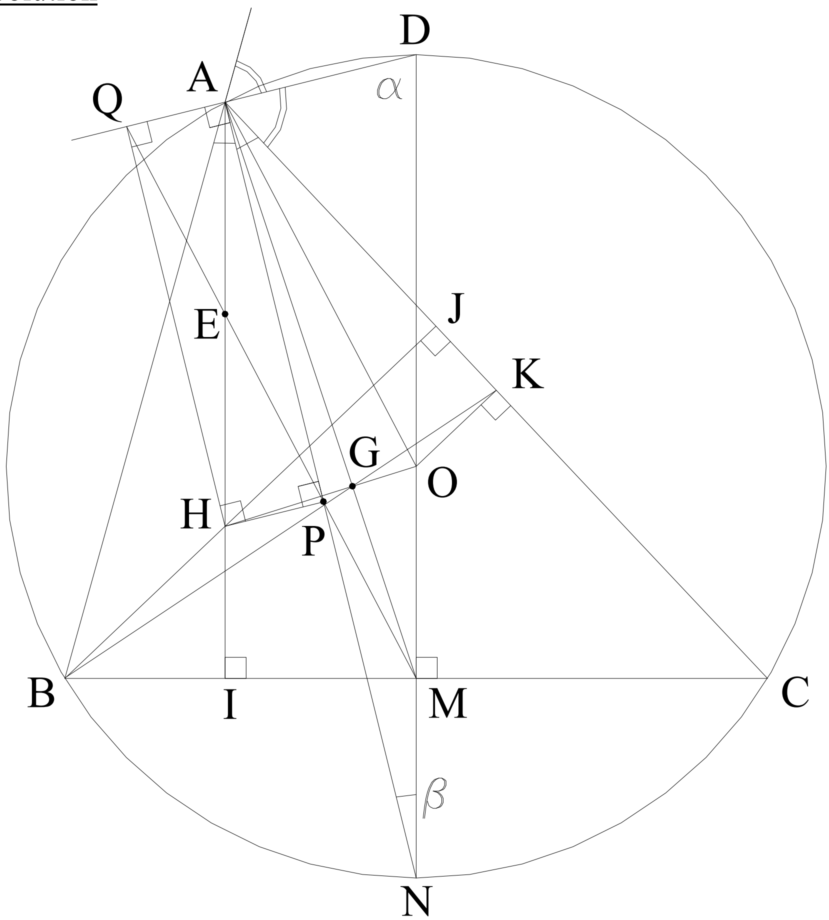
$2al = \dfrac{l^2}{\cos^2\frac{\alpha}{2} + \frac{1}{2}\sin\alpha} = \dfrac{2l^2}{1 + \sin\alpha + \cos\alpha}$ which is the common area of the two squares.

When $l = 1$, the common area is $\dfrac{2}{1 + \sin\alpha + \cos\alpha}$.

Problem 4 of the British Mathematical Olympiad 1987

The triangle ABC has orthocenter H. The feet of the perpendiculars from H to the internal and external bisectors of angle BAC (which is not a right angle) are P and Q. Prove that PQ passes through the middle point of BC.

Solution



Let O, G be the circumcenter and centroid of triangle ABC, respectively, M the midpoint of BC. Extend OM to meet the circumcircle at D and N, D on top and N on bottom as shown. The Euler line contains the orthocenter, centroid and circumcenter of a triangle and H, G and O are collinear and that GM = ½AG, or OM = ½AH. But since AQ and AP are segments belonging to the external and internal bisectors of angle BAC, $\angle QAP = 90°$ and APHQ is a rectangle. Now let E be the intersection of the diagonals of the rectangle APHQ; we have AE = ½AH = OM.

However, since $AI \perp BC$ and $OM \perp BC$, AE || OM and AOME is then a parallelogram which implies that OA || EM.

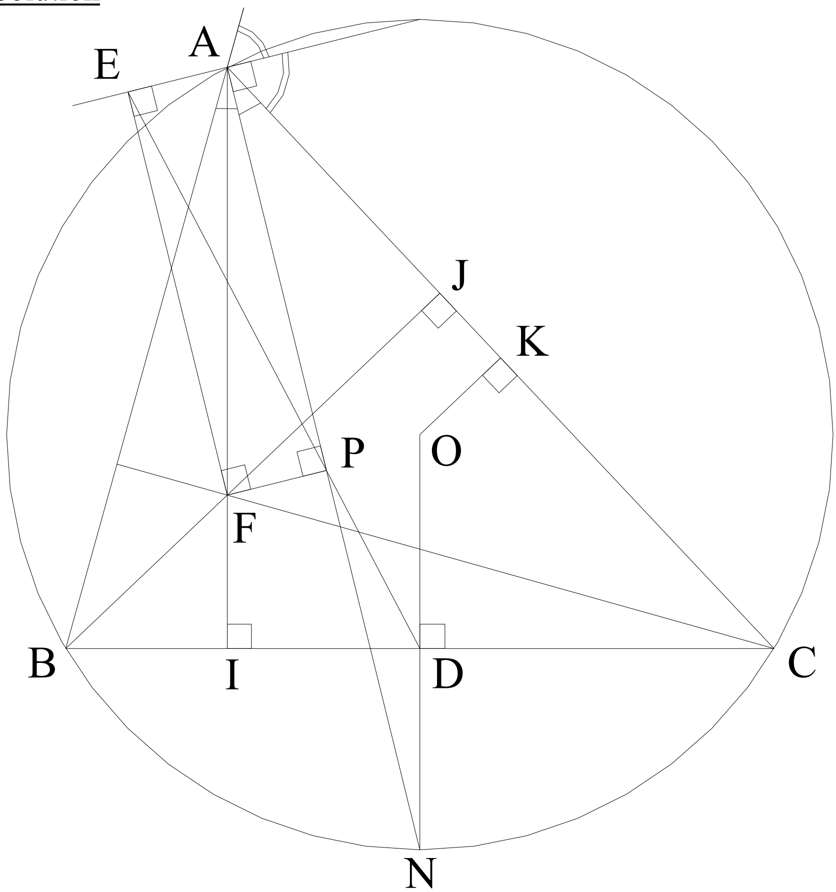
Now let $\alpha = \angle ADN$ and $\beta = \angle AND$, $\alpha + \beta = 90°$ (because DN is the diameter of the circle). Since AI || DN, $\beta = \angle IAN = \angle HQP$, or $\alpha = \angle AQP = \angle DAO$ which implies that OA || QP.

Combining with OA || EM, we conclude that the three points Q, P, M are collinear, or PQ passes through the midpoint of BC.

Problem 5 of India postal Coaching 2010

A point P lies on the internal angle bisector of ∠BAC of a triangle ABC. Point D is the midpoint of BC and PD meets the external angle bisector of ∠BAC at point E. If F is the point such that PAEF is a rectangle then prove that PF bisects ∠BFC internally or externally.

Solution

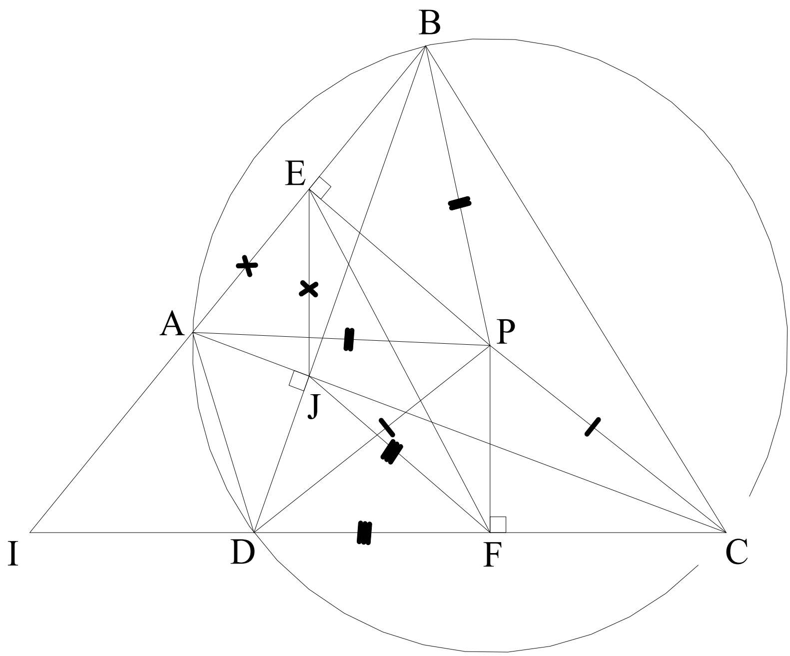


The result of the previous problem indicates that F is the ortho-center of triangle ABC. Therefore, ∠PFC = ∠PAB and ∠PFJ = ∠PAC and, but AP bisects ∠BAC, or ∠PAB = ∠PAC. Thus ∠PFC = ∠PFJ, or PF bisects ∠BFC externally.

Problem 1 of the International Mathematical Olympiad 1998

In the convex quadrilateral ABCD, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside ABCD. Prove that ABCD is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

Solution



Let AC intercept BD at J, AB intercept CD at I, E and F be the midpoints of AB and CD, respectively.

We have $\angle$EAJ = $\angle$EJA, $\angle$EBJ = $\angle$EJB, $\angle$FDJ = $\angle$FJD, $\angle$FJC = $\angle$FCJ and JE = ½AB, JF = ½CD.

Since triangles ABP and CDP have equal areas, we get PE×AB =

PF×CD, or $\frac{PE}{PF} = \frac{CD}{AB} = \frac{JF}{JE}$ (i)

But $\angle EJA + \angle EJB + \angle FJD + \angle FJC = 180°$,

Or $\angle EJB + \angle FJC = 180° - \angle EJA - \angle FJD$.

Therefore, $\angle EJF = \angle EJB + 90° + \angle FJC = \angle EBJ + 90° + \angle FCJ = 180° - \angle EIF = \angle EPF$.

Combining with (i) and the fact that they share segment EF, the triangles JEF and PFE are congruent, and EPFJ is a parallelogram.

It follows that PE = JF = DF and PF = JE = AE, and the two triangles AEP and PFD are congruent which causes PA = PD, or P is the center of the circumcircle passing through A, B, C and D. ABCD is then a cyclic quadrilateral.

Conversely, if ABCD is a cyclic quadrilateral, P is the center of the circumcircle. Since AC is perpendicular to BD, the sum of the angles subtending arcs AB plus CD equal 90°.

Therefore, $\angle APB + \angle CPD = 180°$, or $\angle APE + \angle FPD = 90°$, or $\angle APE = \angle DPF$ and the two triangles APE and DPF are congruent (similar triangles with PA = PD). Hence, triangles ABP and CDP with each having twice the areas of the triangles APE and DPF, respectively, have equal areas.

Problem 2 of Austria Mathematical Olympiad 2005

For how many integer values a with $|a| \leq 2005$ does the system of equations

$$x^2 = y + a$$
$$y^2 = x + a$$

have integer solutions?

Solution

Subtracting the two equations, we have $x^2 - y^2 = y - x$.
a) When $y \neq x$, we can write $(x + y)(x - y) = y - x$, or $x = -y - 1$, so now we know that if a solution of x is an integer, y will also be an integer.

Now substituting $x = -y - 1$ into $y^2 = x + a$, we have

$y^2 + y + 1 - a = 0$ which has roots as $y = ½(-1 \pm \sqrt{4a - 3}\,)$ (i)

y has real solutions when $4a - 3 \geq 0$, and it has integer solution when $4a - 3 = m^2$ where m is an integer.

Since $|a| \leq 2005$, $-2005 \leq a \leq 2005$ and $0 \leq 4a - 3 \leq 8017$ (ii)

Values of integers m to satisfy (ii) are $0 \leq m \leq 89$, or the values for $4a - 3$ are $1^2, 3^2, 5^2, 7^2, \ldots, 89^2$. Among these values we have to find the squares that makes a an integer. Let $m = pq$ where q is the units digit. We have $a = \dfrac{100p^2 + q^2 + 20pq + 3}{4}$.

Note that both $100p^2$ and $20pq$ are divisible by 4; therefore, $q^2 + 3$ has to be divisible by 4, or when units digit $q = 1, 3, 5, 7$ or 9. So all the squares of the odd numbers from 1 to 89 will make a an integer and $\sqrt{4a - 3}$ an odd number which, in turn, makes y in (i) an integer. That's a total of 45 numbers for a.
b) When $y = x$**,** substituting it into the second equation, we have
$y^2 - y - a = 0$

which has roots as $y = ½(1 \pm \sqrt{4a + 1}\,)$
y has real solutions when $4a + 1 \geq 0$, and it has integer solution when $4a + 1 = n^2$ where n is an integer.

Since $|a| \leq 2005$, $-2005 \leq a \leq 2005$, and $0 \leq 4a + 1 \leq 8021$ (iii)

Similarly, values of integers n to satisfy (iii) are $0 \leq n \leq 89$, or the values for $4a + 1$ are $1^2, 3^2, 5^2, 7^2, \ldots, 89^2$. Among these values we have to find the squares that makes a an integer. Let $n = pq$ where q is the units digit. We have $a = \dfrac{100p^2 + q^2 + 20pq - 1}{4} =$ $\dfrac{100p^2 + q^2 + 20pq - 4 + 3}{4}$.

Note that $100p^2$, $20pq$ and -4 are divisible by 4; therefore, $q^2 + 3$ has to be divisible by 4 which ends up with the number of integer a being the same as above, 45 of them.

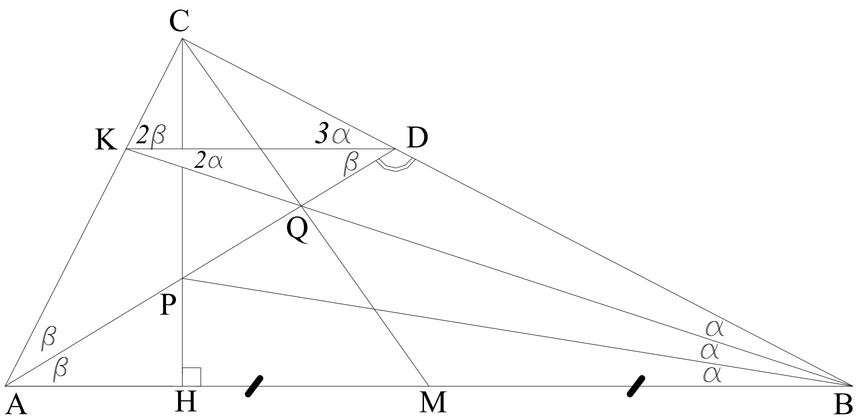
Problem 4 of Indonesia MO Team Selection Test 2010

Let ABC be a non-obtuse triangle with CH and CM are the altitude and median, respectively. The angle bisector of $\angle$BAC intersects CH and CM at P and Q, respectively. Assume that $\angle$ABP = $\angle$PBQ = $\angle$QBC.

a) Prove that ABC is a right-angled triangle, and

b) Calculate $\frac{BP}{CH}$.

Solution



a) Extend BQ to meet AC at K. Let α = $\angle$ABP = $\angle$PBQ = $\angle$QBC, β = $\angle$BAD = $\angle$DAC. Applying the Ceva theorem, we get $\frac{CK}{AK}\times\frac{AM}{BM}\times\frac{BD}{CD} = 1$, but AM = BM, and $\frac{CK}{AK}\times\frac{BD}{CD} = 1$, or $\frac{CK}{AK} = \frac{CD}{BD}$ implying that KD || AB which, in turn, makes $\angle$KDA = $\angle$DAB = β and AKD an isosceles triangle meaning that AK = DK, and $\frac{CK}{AK} = \frac{CD}{BD}$ becomes $\frac{CK}{DK} = \frac{CD}{BD}$, or $\frac{CK}{CD} = \frac{DK}{BD}$.

Now by applying the law of sines, we obtain $\frac{CK}{CD} = \frac{\sin 3\alpha}{\sin 2\beta}$ and $\frac{DK}{BD} = \frac{\sin\alpha}{\sin 2\alpha}$, or $\frac{\sin 3\alpha}{\sin 2\beta} = \frac{\sin\alpha}{\sin 2\alpha}$, or $2\sin 3\alpha\cos\alpha = \sin 2\beta$.

Squaring both sides, we get $4\sin^2 3\alpha\cos^2\alpha = \sin^2 2\beta$ (i)
Assuming that ABC is a right triangle with $\angle C = 90°$ and $3\alpha + 2\beta = 90°$. Per Pythagorean's theorem $\sin^2 3\alpha + \sin^2 2\beta = 1$, or $\sin^2 2\beta = 1 - \sin^2 3\alpha$.

Substituting $\sin^2 2\beta = 1 - \sin^2 3\alpha$ into (i), we have
$1 - \sin^2 3\alpha = 4\sin^2 3\alpha\cos^2\alpha$, or $\sin^2 3\alpha(4\cos^2\alpha + 1) - 1 = 0$, or
$\sin^2 3\alpha[4(1 - \sin^2\alpha) + 1] - 1 = 0$, or $\sin^2 3\alpha(5 - 4\sin^2\alpha) - 1 = 0$ (ii)

Furthermore, $\sin 3\alpha = \sin(2\alpha + \alpha) = \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha = 2\sin\alpha\cos^2\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = \sin\alpha(3\cos^2\alpha - \sin^2\alpha) = \sin\alpha[3(1 - \sin^2\alpha) - \sin^2\alpha] = \sin\alpha(3 - 4\sin^2\alpha)$.
Equation (ii) becomes $\sin^2\alpha(3 - 4\sin^2\alpha)^2(5 - 4\sin^2\alpha) - 1 = 0$ (iii)
Now let $x = \sin^2\alpha$; equation (iii) is equivalent to
$x(3 - 4x)^2(5 - 4x) - 1 = 0$, or $64x^4 - 176x^3 + 156x^2 - 45x + 1 = 0$,
or $(x - 1)(64x^3 - 112x^2 + 44x - 1) = 0$.
Since $x = \sin^2\alpha \neq 1$ ($\alpha \neq 90°$), $64x^3 - 112x^2 + 44x - 1 = 0$.
Solving this cubic equation for x using the Cardano's method, we get $x_1 = 0.0241970181$, $x_2 = 1.177318838$, and $x_3 = 0.548484143$.
When $x_1 = 0.0241970181$, $\sin\alpha = 0.155553907$, $\alpha = 8.948922449°$, or $3\alpha = 26.84676735°$, and $2\beta = 63.15323265°$ and $\angle ACB = 90°$.
When $x_2 = 1.177318838$, $\sin\alpha = 1.085043242$ and it's impossible.
When $x_3 = 0.548484143$, $\sin\alpha = 0.740597153$, $\alpha = 47.78230871°$, or $3\alpha = 143.3469261°$ and the sum of the other two angles is not $90°$.
Hence, the first part of the problem is proven when $\angle C = 90°$, $\angle A = 63.15323265°$ and $\angle B = 26.84676735°$.

Note: The angle measurements are for your information only. The contestant is only required to prove that there exist angles A and B such that their sum is 90°, or in this case $\sin^2 A + \sin^2 B = 1$.

b) $\dfrac{BP}{CH} = \dfrac{\cot 3\alpha}{\cos\alpha} = 2.$

Problem 5 of Spain Mathematical Olympiad 1987

In a triangle ABC, D lies on AB, E lies on AC and $\angle$ABE = 30°, $\angle$EBC = 50°, $\angle$ACD = 20°, $\angle$DCB = 60°. Find $\angle$EDC.

Solution 1

Since the sum of all angles in a triangle is 180°, $\angle$ABC = $\angle$ACB = 80°, $\angle$A = $\angle$ACD = 20°, $\angle$EBC = $\angle$BEC = 50°, and all three triangles ABC, EBC and ADC are isosceles triangles. Now let *a* = AB = AC, *b* = AD = CD, *c* = BC = EC, *d* = BE and *e* = BD.

Applying the law of sines:

In triangle BDC, $\frac{e}{\sin 60°} = \frac{c}{\sin 40°}$; in triangle BEC, $\frac{d}{\sin 80°} = \frac{c}{\sin 50°}$, and $\frac{e}{d} = \frac{\sin 50° \sin 60°}{\sin 40° \sin 80°} = \frac{\sqrt{3}\sin 50°}{2\sin 40° \sin 80°}$. Also in triangle BDE, $\frac{e}{d} = \frac{\sin\angle DEB}{\sin\angle EDB}$; therefore, $\frac{\sin\angle DEB}{\sin\angle EDB} = \frac{\sqrt{3}\sin 50°}{2\sin 40° \sin 80°}$.

Replacing $\angle$DEB = 150° – $\angle$EDB into the equation above to get

$$\frac{\sin(150° - \angle EDB)}{\sin\angle EDB} = \frac{\sqrt{3}\sin 50°}{2\sin 40° \sin 80°}, \text{ or}$$

$$\frac{\sin 150° \cos\angle EDB - \cos 150° \sin\angle EDB}{\sin\angle EDB} = \frac{\sqrt{3}\sin 50°}{2\sin 40° \sin 80°}, \text{ or}$$

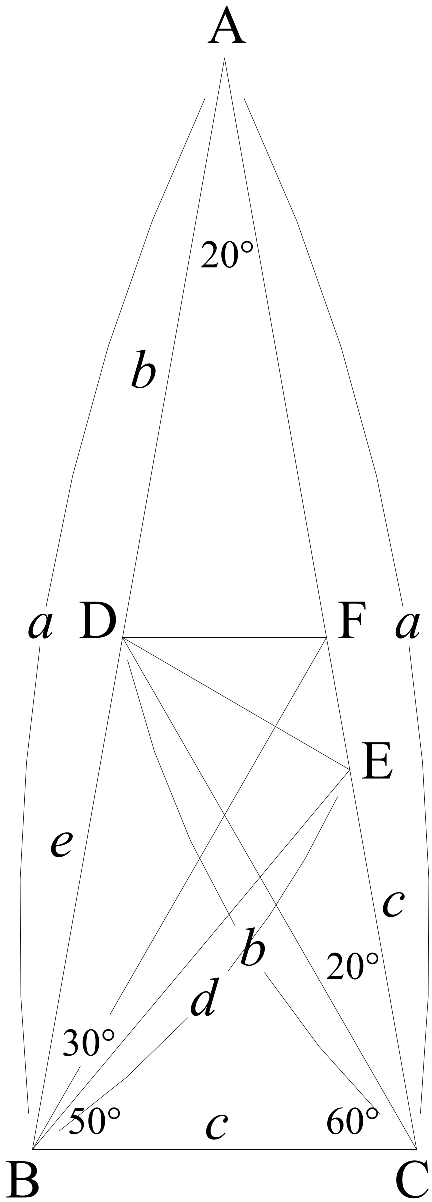
$$\sin 150° \cot\angle EDB - \cos 150° = \frac{\sqrt{3}\sin 50°}{2\sin 40° \sin 80°}.$$

Now note that $\sin 150° = \sin 30° = \frac{1}{2}$ and $\cos 150° = -\cos 30° = -\frac{\sqrt{3}}{2}$.

We then have $\cot\angle EDB + \sqrt{3} = \frac{\sqrt{3}\sin 50°}{\sin 40° \sin 80°}$, or

$$\cot\angle EDB = \sqrt{3}\left(\frac{\sin 50°}{\sin 40° \sin 80°} - 1\right).$$

However, sin50° = cos40° and sin80° = 2sin40°cos40°.



The previous equation is equivalent to

$$\cot\angle \mathrm{EDB} = \sqrt{3}\left(\frac{\cos 40°}{2\sin^2 40° \cos 40°} - 1\right) = \sqrt{3}\left(\frac{1}{2\sin^2 40°} - 1\right) =$$

$\sqrt{3}(\frac{1 - 2\sin^2 40°}{2\sin^2 40°}) = \sqrt{3}\frac{\cos 80°}{2\sin^2 40°}$, or

$\tan\angle EDB = \frac{2\sin^2 40°}{\sqrt{3}\cos 80°} = \frac{2\sin^2 40°}{\sqrt{3}\sin 10°} = \frac{4\sin^2 40°\cos 10°}{2\sqrt{3}\sin 10°\cos 10°} =$
$\frac{4\sin^2 40°\cos 10°}{\sqrt{3}\sin 20°} = \frac{4\sin^2 40°\cos 10°}{\sqrt{3}\cos 70°}$.

But $\frac{4}{\sqrt{3}}\sin^2 40°\cos 10° = \frac{4}{\sqrt{3}}\cos^2 50°\cos 10° = \frac{4}{\sqrt{3}}$
$\cos 50°\cos 50°\cos 10° = \frac{2}{\sqrt{3}}\cos 50°(\cos 60° + \cos 40°) = \frac{2}{\sqrt{3}}\cos 50°(\frac{1}{2}$
$+ \cos 40°) =$
$\frac{1}{\sqrt{3}}\cos 50° + \frac{2}{\sqrt{3}}\cos 50°\cos 40° = \frac{1}{\sqrt{3}}\cos 50° + \frac{1}{\sqrt{3}}(\cos 90° + \cos 10°)$
$= \frac{1}{\sqrt{3}}(\cos 50° + \cos 10°) = \frac{2}{\sqrt{3}}\cos 30°\cos 20° = \cos 20° = \sin 70°$.

Therefore, $\tan\angle EDB = \frac{\sin 70°}{\cos 70°} = \tan 70°$, and $\angle EDB = 70°$, or
$\angle EDC = 70° - \angle BDC = 30°$.

Solution 2

Draw the segment DF with F on AC such that DF || BC. We have $\angle CDF = \angle BCD = 60°$. We can solve the problem by proving that DE is the bisector of $\angle CDF$ to imply that $\angle EDC = ½\angle CDF = 30°$. To do that we need to prove $\frac{DF}{b} = \frac{EF}{c}$. Now let's do it.

In triangle BEF, $\frac{EF}{\sin 10°} = \frac{b}{\sin 130°} = \frac{b}{\sin 50°}$, or $EF = \frac{b\sin 10°}{\sin 50°}$

It suffices to show that $\frac{EF}{c} = \frac{b\sin 10°}{c\sin 50°} = \frac{DF}{b}$.

However, in triangle BCD, $\frac{b}{c} = \frac{\sin 80°}{\sin 40°}$, and $\frac{EF}{c} = \frac{b\sin 10°}{c\sin 50°} =$
$\frac{\sin 80°\sin 10°}{\sin 40°\sin 50°} = \frac{2\sin 40°\cos 40°\sin 10°}{\sin 40°\sin 50°} = \frac{2\cos 40°\sin 10°}{\sin 50°} =$

$\frac{2\cos 40°\sin 10°}{\cos 40°} = 2\sin 10°.$

On the other hand, in triangle CDF,

$\frac{\mathrm{DF}}{b} = \frac{\sin 20°}{\sin\angle\mathrm{CFD}} = \frac{\sin 20°}{\sin(180° - \angle\mathrm{AFD})} = \frac{\sin 20°}{\sin\angle\mathrm{AFD}} = \frac{\sin 20°}{\sin\angle\mathrm{ACB}} = \frac{\sin 20°}{\sin 80°}.$

We now need to prove that $\frac{\sin 20°}{\sin 80°} = 2\sin 10°$, or

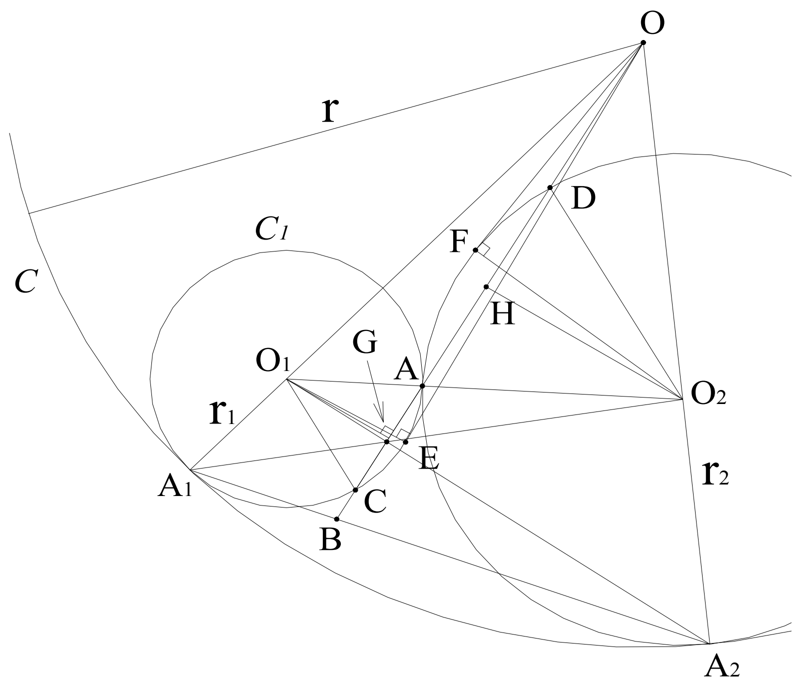
$\sin 20° = 2\sin 80°\sin 10°.$

But $2\sin 80°\sin 10° = \cos 70° - \cos 90° = \cos 70° = \sin 20°$ and our objective is achieved.

Problem 2 Asian Pacific Mathematical Olympiad 1992

In a circle *C* with center O and radius *r*, let C_1, C_2 be two circles with centers O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangent to *C* at A_i and so that C_1, C_2 are externally tangent to each other at A. Prove that the three lines OA, O_1A_2, and O_2A_1 are concurrent.

Solution



Extend OA to meet A_1A_2 at B. From O_1 and O_2 draw the altitudes O_1G and O_2H to OB, respectively. From O draw tangential lines to C_1 and C_2 and to meet them at E and F, respectively.

Use the law of the sines, we have

$$\frac{A_1B}{\sin\angle A_1OB} = \frac{OB}{\sin\angle OA_1B} = \frac{OB}{\sin\angle OA_2B} = \frac{A_2B}{\sin\angle A_2OB},$$

$$\text{or } \frac{A_1B}{A_2B} = \frac{\sin\angle A_1OB}{\sin\angle A_2OB} = \frac{\dfrac{O_1G}{O_1O}}{\dfrac{O_2H}{O_2O}} = \frac{O_1G}{O_1O} \times \frac{O_2O}{O_2H} \tag{i}$$

Because the two triangles GO_1A and HO_2A are similar, we have $\frac{O_1G}{O_1A} = \frac{O_2H}{O_2A}$, or $\frac{O_1G}{O_2H} = \frac{r_1}{r_2}$, and (i) becomes

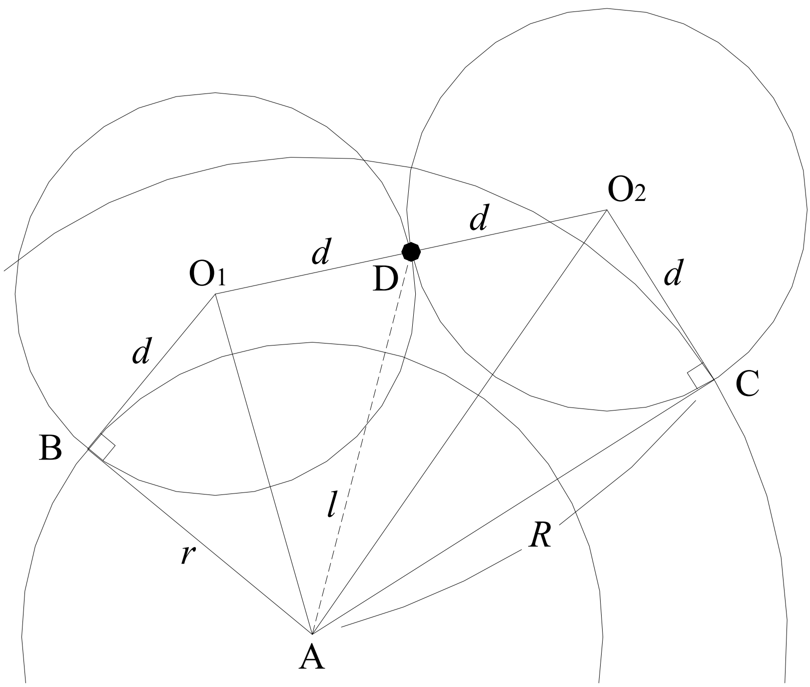
$$\frac{A_1B}{A_2B} = \frac{r_1}{r_2} \times \frac{O_2O}{O_1O} = \frac{r_1}{r_2} \times \frac{r - r_2}{r - r_1}, \text{ or } \frac{A_1B}{A_2B} \times \frac{r_2}{r - r_2} \times \frac{r - r_1}{r_1} = 1.$$

Therefore, per Ceva's theorem, the three lines OA, O_1A_2, and O_2A_1 are concurrent.

Problem 6 of Russia Sharygin Geometry Olympiad 2008

In a plane, given two concentric circles with the center A. Let B be an arbitrary point on some of these circles, and C on the other one. For every triangle ABC, consider two equal circles mutually tangent at the point D, such that one of these circles is tangent to the line AB at point B and the other one is tangent to the line AC at point C. Determine the locus of points D.

Solution



Let O_1 and O_2 be the circumcenters of the equal circles on the left and on the right as shown, respectively, d be their radius. Also let R, r be the radii of the larger and smaller concentric circles, respectively and l = AD.

Per Pythagorean theorem, we have
$d^2 = AO_1^2 - r^2 = AO_2^2 - R^2$, or $AO_1^2 + R^2 = AO_2^2 + r^2$ (i)
and per Stewart's theorem, we have $AO_1^2 \times DO_2 + AO_2^2 \times DO_1 =$

$O_1O_2(AD^2 + DO_1 \times DO_2)$, or $d(AO_1^2 + AO_2^2) = 2d(l^2 + d^2)$, or $AO_1^2 + AO_2^2 = 2(l^2 + d^2)$.

Substituting $d^2 = AO_2^2 - R^2$ into the above equation, we obtain
$AO_1^2 + AO_2^2 = 2l^2 + 2AO_2^2 - 2R^2$, or
$AO_1^2 + R^2 = AO_2^2 + 2l^2 - R^2$ (ii)

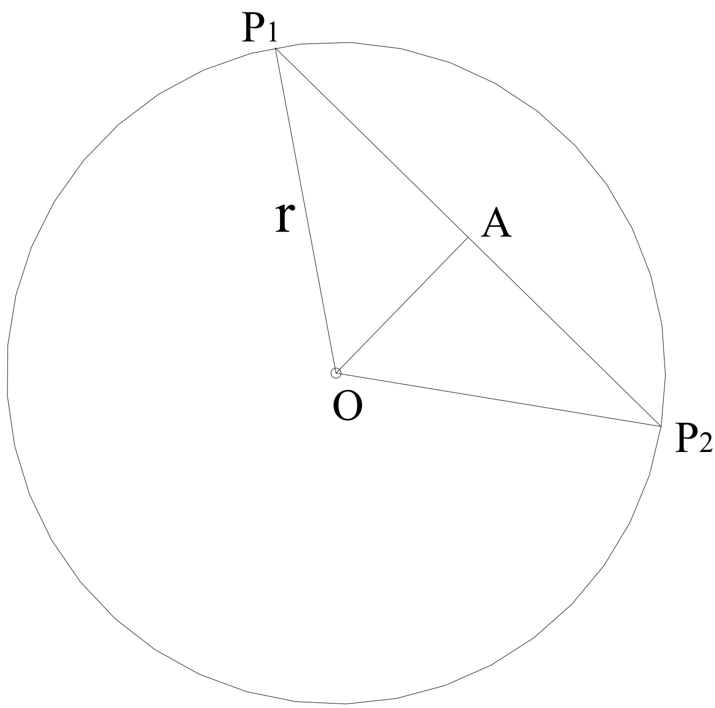
From (i) and (ii), we get $2l^2 = R^2 + r^2$, or $l = \sqrt{\frac{1}{2}(R^2 + r^2)}$.

Thus the locus of points D is a circle with center A and radius of $\sqrt{\frac{1}{2}(R^2 + r^2)}$.

Problem 2 of the Canadian Mathematical Olympiad 1977

Let O be the center of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.

Solution



Let point P ≡ point P_1. Applying the law of the sines, we obtain $OA/\sin\angle OP_1A = r/\sin\angle OAP_1$, or $\sin\angle OP_1A = OA\times \sin\angle OAP_1/r$. Since angle OP_1A has a side passing through the center of the circle, it's an acute angle, and therefore, the angle $\angle OP_1A$ is a maximum when $\sin\angle OP_1A$ is a maximum. Furthermore, since r and OA are constants, $\sin\angle OP1A$ is a maximum when $\sin\angle OAP_1$ is a maximum, and the maximum of a sine of an angle is 1 which will happen when $\angle OAP_1 = 90°$.

Another point $P = P_2$ which is the mirror image of P_1 across A also satisfies this requirement.

Problem 2 of the Canadian Mathematical Olympiad 1978

Find all pairs a, b of positive integers satisfying the equation
$2a^2 = 3b^3$.

Solution

The product on the left side $2a^2$ is an even number, so $3b^3$ has to be an even number, and b^3, therefore, has to be an even number, or b to be an even number. Let $b = 2n$ where n is a positive integer.
We then have $b^2 = 4n^2$; now rewrite $2a^2 = 3b^3$ as $\frac{2a^2}{b^2} = 3$, or
$\frac{2a^2}{4n^2} = 3b$, or $\frac{a^2}{2n^2} = 3b$, or $a^2 = 6bn^2$.

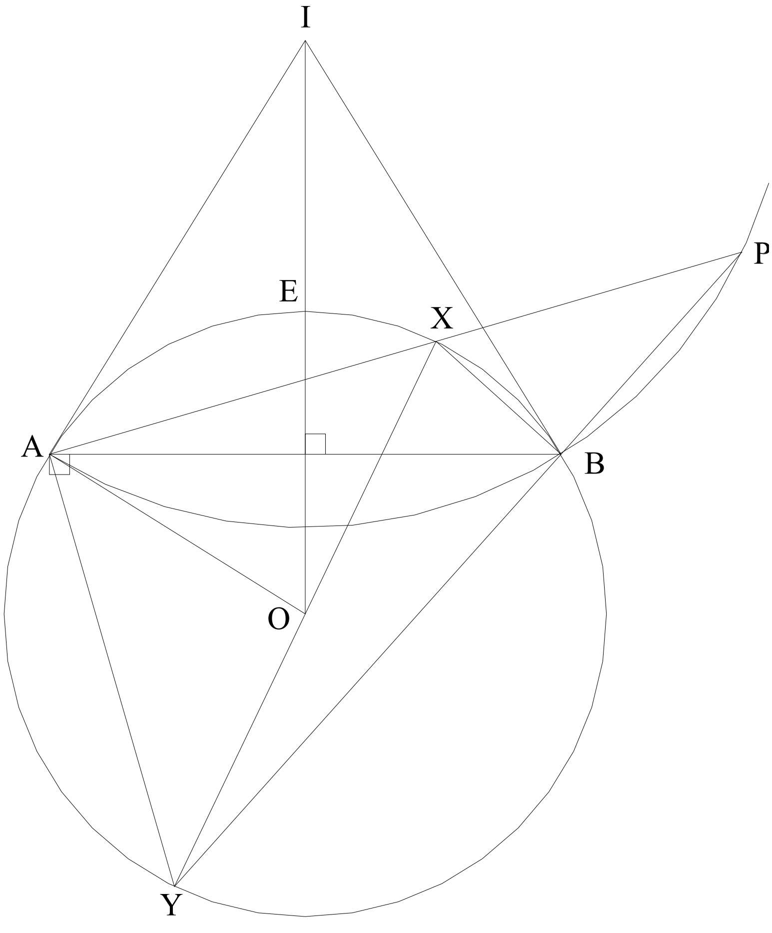
Since a^2 and n^2 are already squares of two numbers, $6b$ must be a square of another number. Let it be $6b = m^2$, or $b = 6k^2$ where k is a positive integer. Now substituting it to the original equation to get
$a^2 = \frac{3b^3}{2} = 3^2 \times 6^2 \times (k^3)^2$, or $a = 18k^3$.

And the solutions are $(a, b) = (18k^3, 6k^2)$ where k is a positive integer. For example, for $k = 23$, $a = 18 \times 23^3 = 219006$ and $b = 6 \times 23^2 = 3174$ is a set of solution when $2 \times 219006^2 = 3 \times 3174^3 = 95{,}927{,}256{,}072$.

Problem 2 of the United States Mathematical Olympiad 1976

If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter.

Solution



Draw the tangents of the circle at A and B to meet at I. Let IO intercept the circle at E between I and O.

We have $\angle AIO + \angle AOI = 90°$, but $\angle AOI = \angle AOE = \frac{1}{2}\angle AOB = \angle AYB$ (O is center of circle and both $\angle AOB$ and $\angle AYB$ subtends arc AB).

It follows that $\angle AIO + \angle AYB = 90°$.
On the other hand, since XY is the diameter of the circle

$$\angle XAY = 90° = \angle PAY, \text{ or}$$
$$\angle APY + \angle AYP = 90°.$$

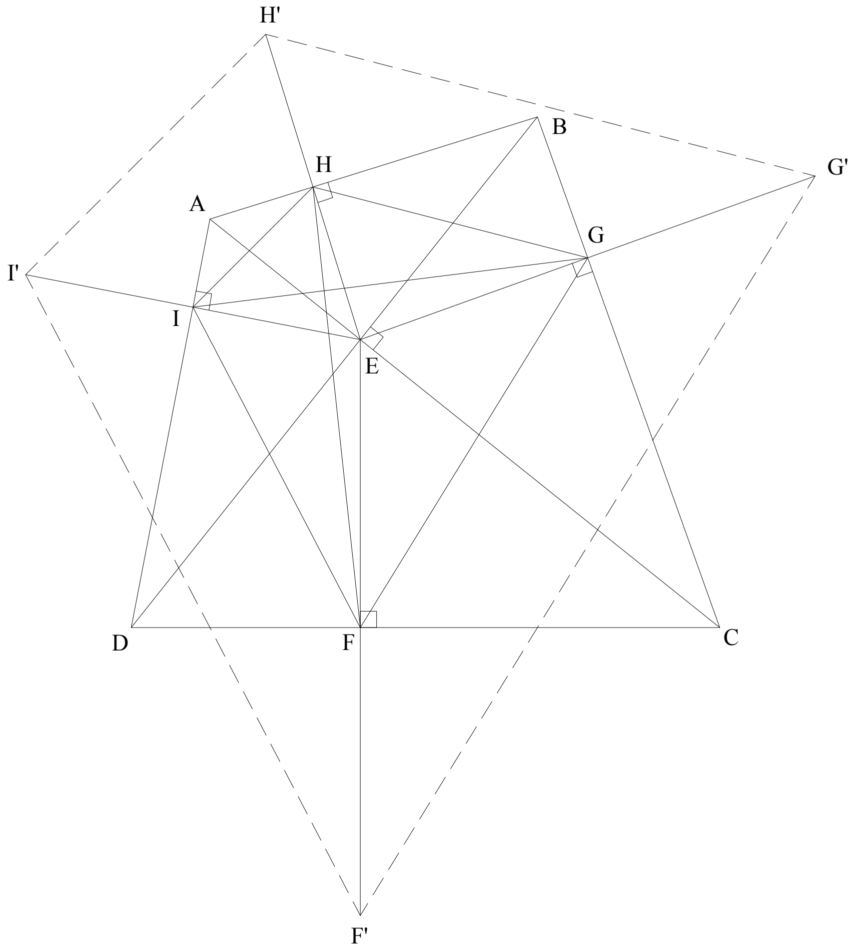
Therefore, $\angle APY = \angle APB = \angle AIO = \frac{1}{2}\angle AIB$, or P is on the circle with fixed center I and fixed radius IA.

The locus is then a circle with center I and radius IA.

Problem 2 of the United States Mathematical Olympiad 1993

Let ABCD be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection. Prove that the reflections of E across AB, BC, CD, DA are concyclic.

Solution



Let the feet from E to the four sides AB, BC, CD and DA be H, G, F and I as shown. Instead of proving the reflections of E across AB, BC, CD, DA (H', G', F', and I') are concyclic we can prove H, G, F and I are concyclic because each foot is the midpoint of the distance from E to its reflection, and the two quadrilaterals are similar.

The four quadrilaterals EHBG, EGCF, EFDI and EIAH are cyclic since they have opposite right angles; we have

$\angle$EHG = $\angle$EBG, $\angle$EFG = $\angle$ECG, $\angle$EFI = $\angle$EDI, and $\angle$EHI = $\angle$EAI.

But since AC $\perp$ BD, we have $\angle$EBG + $\angle$ECG = 90° and $\angle$EDI + $\angle$EAI = 90°, or

$\angle$EHG + $\angle$EFG + $\angle$EFI + $\angle$EHI = 180°, or
$\angle$IHG + $\angle$IFG = 180° and H, G, F and I are concyclic.

Further observation

The problem below is derived from the above problem:

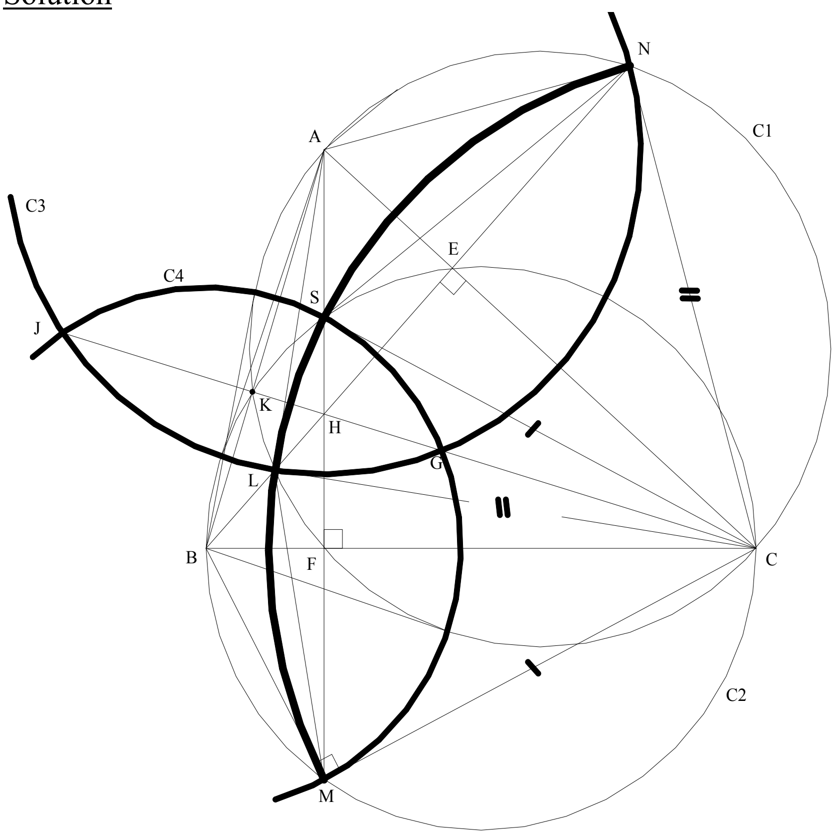
Let ABCD be a convex quadrilateral such that diagonals AC and BD intersect at right angles, and let E be their intersection, and C be the circumcircle of triangle FGH where F, G and H are reflections of E across AB, BC and CD, respectively. Let I and J are points where DC intercepts the circle C and K is where the altitude line of triangle EDC from E intercept the circle C. Prove that E is the orthocenter of triangle KIJ.

Problem 3 of Austria Mathematical Olympiad 2005

In an acute-angled triangle ABC two circles C1 and C2 are drawn whose diameters are the sides AC and BC. Let E be the foot of the altitude h*b* on AC and let F be the foot of the altitude h*a* on BC. Let L and N be the intersections of the line BE with the circle C1 (L on the line BE) and let K and M be the intersections of the line AF with the circle C2 (K on the line AF).

Show that KLMN is a cyclic quadrilateral.

Solution



Let D be the foot of the altitude from C to AB. Since E is on the circle C2, BE $\perp$ AC and because AC is also the diameter of C1, AC is then the perpendicular bisector of LN. Therefore, CN = CL.

Similarly, since BC is the diameter of C2 and F is on circle C1, BC is perpendicular bisector of KM and CK = CM.

Now it suffices to prove CM = CN so that the four points K, L, M, and N will lie on a circle with center at C and radius CN = CM = CK = CL.

Since the two triangles AFC and BEC are similar, we have $\frac{CF}{CE} = \frac{CA}{CB}$, or $CF \times CB = CE \times CA$, or $CF(CF + BF) = CE(CE + AE)$, or $CF^2 + CF \times BF = CE^2 + CE \times AE$ (i)

But BMC is a right triangle at M and F its foot on BC, we have $CF \times BF = MF^2$.

Similarly $CE \times AE = NE^2$.

Now, rewrite (i) as $CF^2 + MF^2 = CE^2 + NE^2$, or

$CM^2 = CN^2$, and we're done.

<u>Further observation</u>

Draw circle C3 with center A and radius AN = AL and circle C4 with center B and radius BM = BK. Let them intercept each other at J and G with G inside the circles. We can conclude that the four points D, H, G and C are collinear since $CM^2 = CN^2 = CG \times CJ$.

Problem 3 of the Canadian Mathematical Olympiad 1973

Prove that if p and $p + 2$ are both prime integers greater than 3, then 6 is a factor of $p + 1$.

Solution

Since p and $p + 1$ are prime integers, they are not divisible by 2 and we can express
$p = 2k + 1$ (k is an integer),
$p + 2 = 2k + 3$, or
$p + 1 = 2(k + 1)$, or
2 is a factor of $p + 1$,

and since they are not divisible by 3,
$p = 3n + 1$, or
$p = 3n + 2$ (n is an integer), but if
$p = 3n + 1$, then
$p + 2 = 3(n + 1)$ which is divisible by 3, so the only option is
$p = 3n + 2$, and $p + 2 = 3n + 4$, or
$p + 1 = 3(n + 1)$, or 3 is also a factor of $p + 1$.

Both 2 and 3 are factors of $p + 1$ then $2 \times 3 = 6$ is a factor of $p + 1$.

Problem 3 of the Canadian Mathematical Olympiad 1978

Determine the largest real number z such that
$x + y + z = 5$
$xy + yz + xz = 3$

and x, y are also real.

Solution

From the top equation, $z = 5 - (x + y)$. To find the largest real number z we will find the numbers x and y such that $x + y$ is smallest.

From the bottom equation $z = \dfrac{3 - xy}{x + y}$, or $5 - (x + y) = \dfrac{3 - xy}{x + y}$.

Rearranging this equation, we have $y^2 + (x - 5)y + x^2 - 5x + 3 = 0$

which has two roots as $y = ½(5 - x \pm \sqrt{-3x^2 + 10x + 13})$.

Therefore, $x + y = x + ½(5 - x \pm \sqrt{-3x^2 + 10x + 13}) = ½(5 + x \pm \sqrt{-3x^2 + 10x + 13})$.

And $x + y$ is at extreme when its derivative is equal to zero

$(5 + x \pm \sqrt{-3x^2 + 10x + 13})' = 0$, or $1 \pm ½\dfrac{1}{\sqrt{-3x^2 + 10x + 13}}\times(-3x^2 + 10x + 13)' = 1 \pm ½\dfrac{1}{\sqrt{-3x^2 + 10x + 13}}(-6x + 10) = 0$.

Rearranging this equation and square both sides, we have

$3x^2 - 10x + 3 = 0$. This equation has solutions $x = 3$, and $x = \dfrac{1}{3}$.

Substitute these x values to $x + y = ½(5 + x \pm \sqrt{-3x^2 + 10x + 13})$

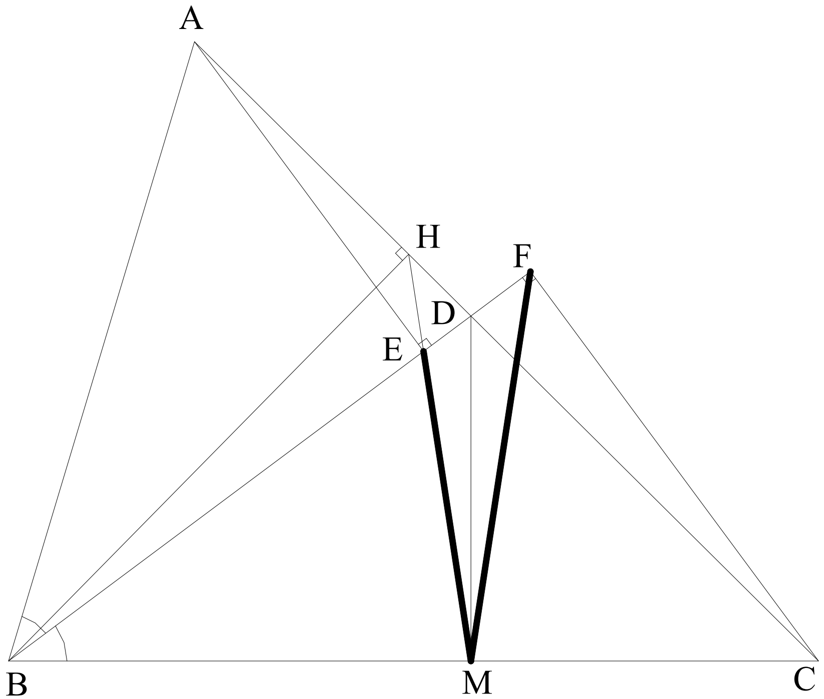
When $x = 3$, $x + y = 6$ and 2.

When $x = \dfrac{1}{3}$, $x + y = \dfrac{2}{3}$ and $\dfrac{14}{3}$. Therefore, $x + y$ is a minimum when $x + y = \dfrac{2}{3}$, and the largest value of z is $\dfrac{13}{3}$.

<u>Problem 4 of the Ibero-American Mathematical Olympiad 2002</u>

In a triangle ABC with all its sides of different length, D is on the side AC, such that BD is the angle bisector of ∠ABC. Let E and F, respectively, be the feet of the perpendicular drawn from A and C onto the line BD and let M be the point on BC such that DM is perpendicular to BC. Show that ∠EMD = ∠DMF.

<u>Solution</u>



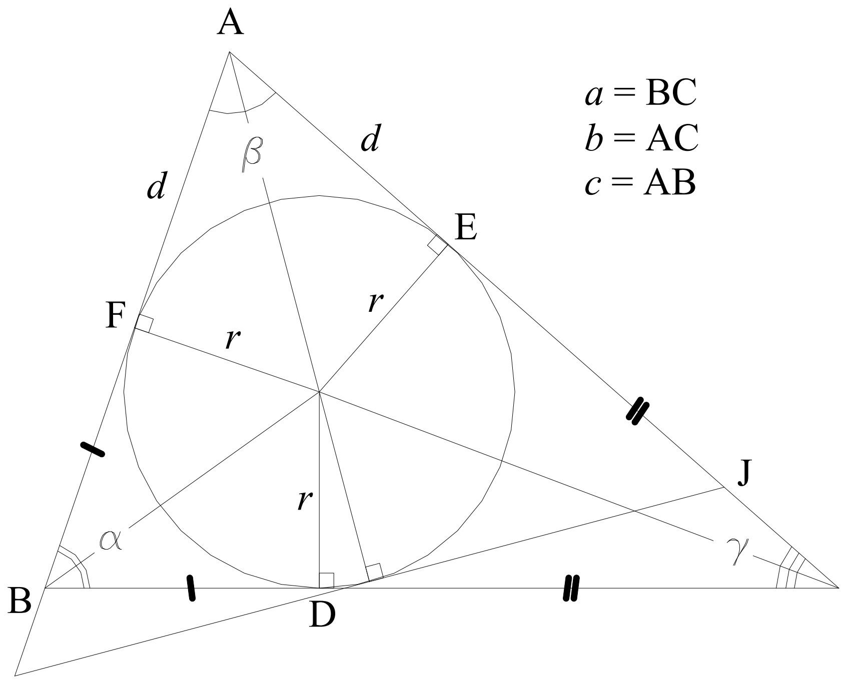
From B draw altitude to AC and meet it at H. We have the following cyclic quadrilaterals AHEB, BHDM and MDFC.

Hence, ∠BHE = ∠BAE = 90° − ½∠B = ∠BDM = ∠BHM
∠BHE = ∠BHM; therefore, H, E and M are collinear,
and we have ∠EMD = ∠HMD = ∠HBD = ∠HBE = EAH = ∠EAD = ∠DCF (AE, CF both perpendicular EF) = ∠DME.

Problem 3 of the Canadian Mathematical Olympiad 1980

Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

Solution



Let β = ∠A, α = ∠B, γ = ∠C, r be the radius of the incircle, D, E and F are the points the incircle tangents with BC, CA and AB, respectively. Now let a = BC, b = AC, c = AB and d = AF = AE.

Note that BF = BD, CE = CD and we have
$a + b + c = 2d + 2\text{BD} + 2\text{CD} = 2d + 2(\text{BD} + \text{CD}) = 2d + 2a$ (i)
Since ∠A and r are fixed, d is also fixed, and ***the minimum value of a + b + c is obtained when a is a minimum.***
From (i), $a = b + c - 2d$ (ii)

Now applying the law of the sines, we obtain

$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha} = \frac{c}{\sin\gamma}$, or $c = \frac{b\sin\gamma}{\sin\alpha}$.

Substituting them into (ii), we have

$a = b + \frac{b\sin\gamma}{\sin\alpha} - 2d = b\times\frac{\sin\alpha + \sin\gamma}{\sin\alpha} - 2d = a\times\frac{\sin\alpha + \sin\gamma}{\sin\beta} - 2d$, or

$a = \frac{2a}{\sin\beta}[\cos\frac{1}{2}(\alpha - \gamma)\sin\frac{1}{2}(\alpha + \gamma)] - 2d$, or

$a = \frac{2d\sin\beta}{2\cos\frac{1}{2}(\alpha - \gamma)\sin\frac{1}{2}(\alpha + \gamma) - \sin\beta}$.

Since d, $\sin\beta$ and $\sin\frac{1}{2}(\alpha + \gamma)$ are all constants, a is minimum when $\cos\frac{1}{2}(\alpha - \gamma)$ is a maximum or when it's equal to 1, or when $\alpha - \gamma = 0$ or $\alpha = \gamma$.

The triangle has the least perimeter when $\angle$B = $\angle$C as in triangle AIJ shown on the graph.

Problem 3 of Canadian Mathematical Olympiad 1983

The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?

Solution

There are two methods to prove this problem. One using a mathematical volume calculation, the other is easily proven using visual effect beyond doubt.

The first method

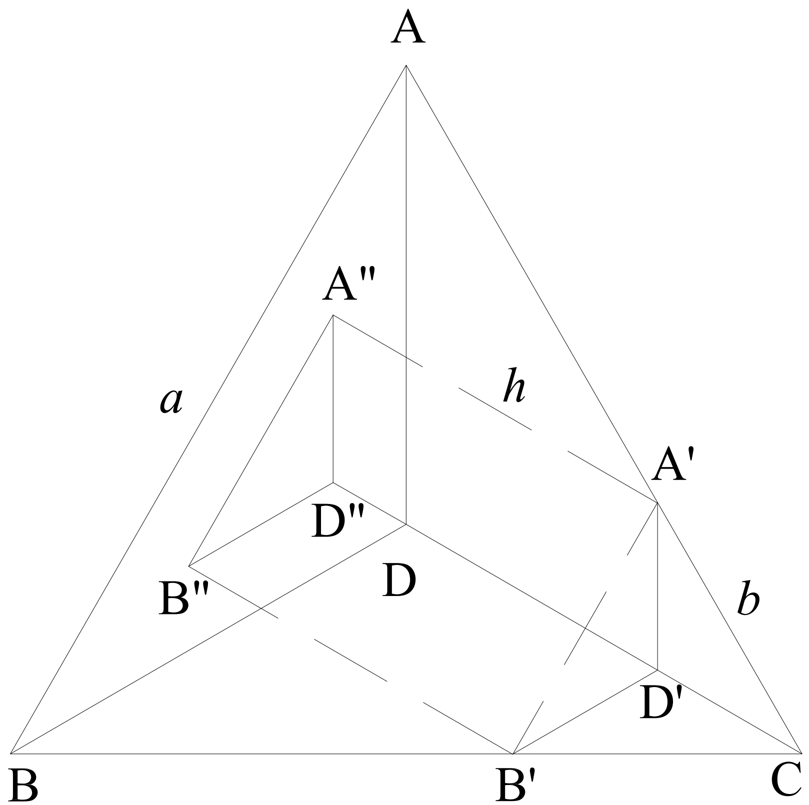
The volume of a regular tetrahedron is given generally as

V = Area of base×height of altitude/3 = $\frac{a^3\sqrt{2}}{12}$ where a is the side of the tetrahedron. The area of an equilateral triangle which is the base of a tetrahedron depends on its side. So the volume of a tetrahedron depends on its area.

The second method

Assume having two tetrahedra ABCD and A'B'CD' with different side lengths a and b and $a > b$ as shown in the graph where they both are laying flat and being looked straight down. It's called the floor plan.

We can always make one of their vertices to coincide (vertex C in this case) and the sides A'D'C to lie completely on the plane of ADC and the same for B'D'C to lie on the plane of BDC. The volume of tetrahedron ABCD, therefore, completely covers that of tetrahedron A'B'CD'. So the area of the faces ABC and A'B'C of the tetrahedra determine their volumes.

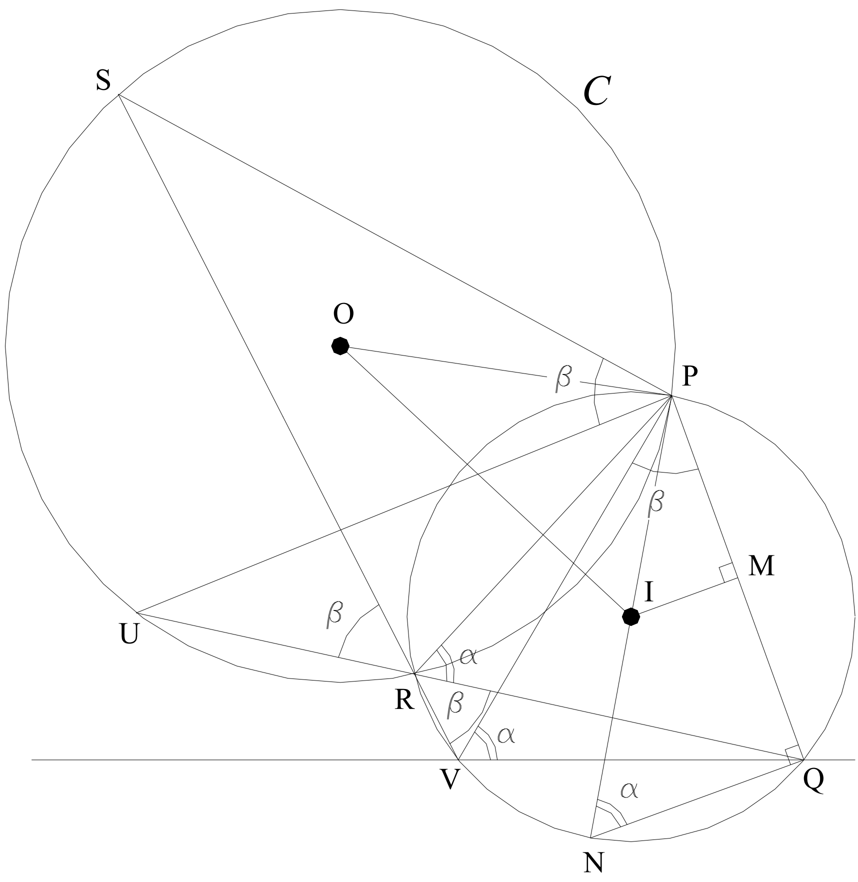


Or from A', B' and D' draw projections A", B", and D" to plane ABD, the volume of the tetrahedron ABCD is greater than that of A'B'CD' at least the volume of A"B"D"A'B'D' which is the area we can calculate without resorting to the formula above.

Problem 3 of the Irish Mathematical Olympiad 2007

The point P is a fixed point on a circle and Q is a fixed point on a line. The point R is a variable point on the circle such that P, Q and R are not collinear. The circle through P, Q and R meets the line again at V. Show that the line VR passes through a fixed point.

Solution



Let *C* be the circle where the fixed point P is on. Link QR and VR and extend them to intercept *C* at U and S, respectively. Let I and IN be the center and the diameter of the circumcircle of triangle PQR, respectively. Now let β = ∠SPU. We also have β = ∠SRU = ∠VRQ = ∠VPQ, and let α = ∠PRQ = ∠PVQ = ∠PNQ.

We have $\angle SPQ = \angle UPQ + \angle SPU$ (i)

But O and I are centers of the two circles and P and R are their intersections, we then have $\angle IOP = \frac{1}{2}\angle ROP = \angle PUQ$, and similarly $\angle OIP = \angle PQU$. The two triangles OPI and UPQ are then similar, and $\angle OPI = \angle UPQ$.

Equation (i) is now equivalent to

$\angle SPQ = \angle OPI + \beta = \angle OPI + \angle VPQ = \angle OPI + 180° - \angle PQV - \angle PVQ = \angle OPI + 180° - \angle PQV - \angle PNQ = \angle OPI + 180° - \angle PQV - (90° - \angle NPQ) = \angle OPI + 180° - \angle PQV - 90° + \angle NPQ = \angle OPQ + 90° - \angle PQV$.

Since both angles $\angle OPQ$ and $\angle PQV$ are constants, $\angle SPQ$ is then constant and VR passes through a fixed point.

Problem 3 of the British Mathematical Olympiad 2005

Let ABC be a triangle with AC > AB. The point X lies on the side BA extended through A, and the point Y lies on the side CA in such a way that BX = CA and CY = BA. The line XY meets the perpendicular bisector of side BC at P. Show that $\angle$BPC + $\angle$BAC = 180°.

Solution

Since BX = AC and AB = YC, we have AX = AY. Let $\angle$BXY = α, we also have $\angle$AXY = $\angle$AYX = $\angle$PYC = α. Now let β = $\angle$BAY; β = $\angle$AXY + $\angle$AYX = 2α.

From B draw a segment BJ such that BJ ||AY and BJ = AY. ABJY is a parallelogram, and AB = YJ = YC and $\angle$BAC = $\angle$JYC = 2α, or 2α = $\angle$PYC + $\angle$JYP = α + $\angle$JYP, or $\angle$JYP = α.

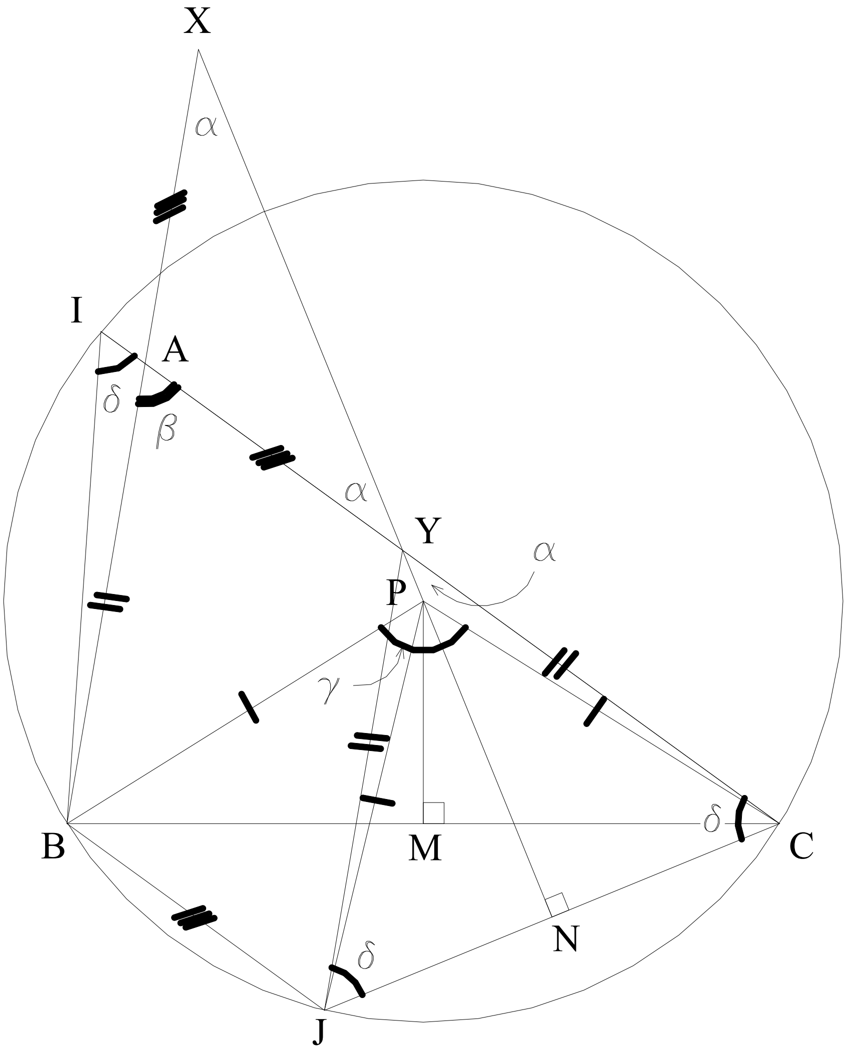
Now link JC and extend XY all the way to meet JC at N. Triangle JYN = triangle CYN.

Since they share YN, YJ = YC and $\angle$JYN = $\angle$CYN. Therefore, YN $\perp$ JC and PB = PC = PJ or P is the circumcenter of triangle BCJ.

Now draw the circumcircle of triangle BCJ, extend CA to meet the circle at I. Since P is the circumcenter, $\angle$BPC = 2$\angle$BIC because both angles subtend the same arc BC. And since BJ || IC, BI = JC, we have JI = BC and $\angle$BIC = $\angle$JCI = δ as shown.

These two equations $\angle$BPC = 2$\angle$BIC and $\angle$BIC = $\angle$JCI = δ give us $\angle$BPC = 2$\angle$JCI = 2δ.

But 2δ = $\angle$JCI + $\angle$YJC = $\angle$AYJ.
Since ABJY is a parallelogram, $\angle$AYJ + $\angle$BAC = 180°, or
2δ + $\angle$BAC = 180°, or $\angle$BPC + $\angle$BAC = 180°.


X
α
I
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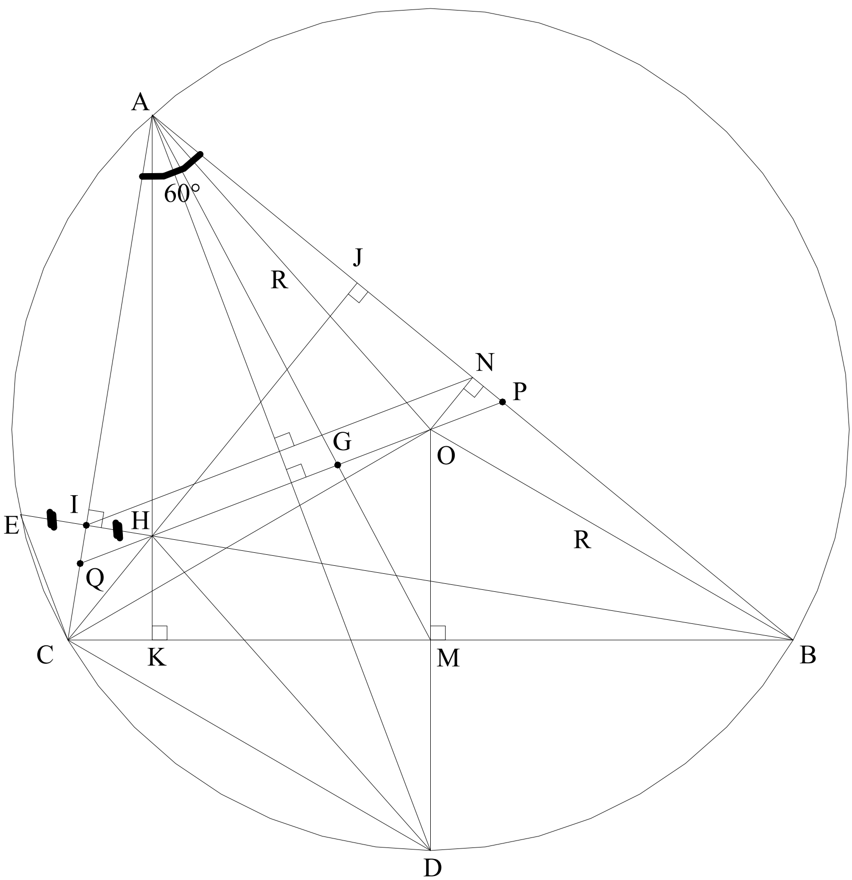
<u>*Problem 3 of the British Mathematical Olympiad 2006*</u>

Let ABC be an acute-angled triangle with AB > AC and ∠BAC = 60°. Denote the circumcenter by O and the orthocenter by H and let OH meet AB at P and AC at Q. Prove that PO = HQ.

<u>Solution</u>



Let *R* be the radius of the circle, M and N the midpoints of BC and AB, respectively. Extend OM to meet the circle at D, CH to meet AB at J, BH to meet AC at I and the circle at E.

Since ∠BAC = 60°, ∠ABE = ACJ = ∠ACE (subtends arc AE) = ∠CAD (arc CD = ½arc CB) = 30°, or CE || AD.

We also have $\angle$COD = 2$\angle$CAD = 60° and OC = OD = R make OCD an equilateral triangle and since CM $\perp$ OD, we have $\angle$OCB = $\angle$DCM = $\angle$OBC = 30°

$\angle$ABE = $\angle$ABO + $\angle$OBE = $\angle$OBC = $\angle$CBE + $\angle$OBE = 30°, or $\angle$CBE = $\angle$ABO = $\angle$OAB.

Combining with $\angle$CBE = $\angle$CAK = $\angle$IAH, we have $\angle$OAB = $\angle$IAH.

Since AIB is a right triangle and N is the midpoint of AB, we have AN = NB = NI, and with $\angle$BAC = 60°, triangle ANI is equilateral and AI = AN.

Now two right triangles AIH and ANO are congruent since all their corresponding angles are equal and AI = AN.

Therefore, AH = AO = R and since AH || OD it makes AODH a rhombus and the diagonal lines AD $\perp$ HO.

Also since $\angle$AIN = 60°, and $\angle$IAD = 30° makes AD $\perp$ IN; we now have IN || HO, or $\angle$ANI = $\angle$APQ = $\angle$AIN = $\angle$AQP = 60°.

The two triangles AHQ and AOP have all their corresponding angles equal to one another and its sides AH = AO and are congruent.

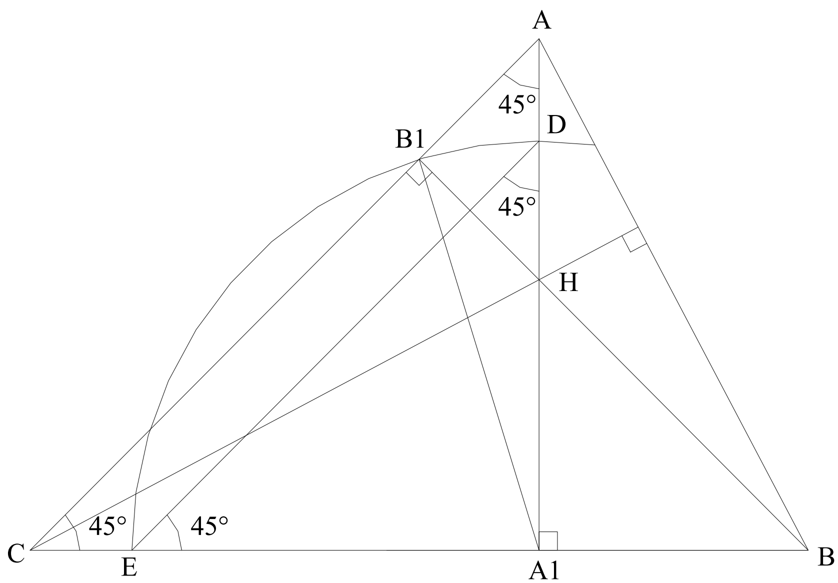
Therefore, PO = HQ.

Problem 3 of Romanian Mathematical Olympiad 2006

In the acute-angle triangle ABC we have $\angle ACB = 45°$. The points A_1 and B_1 are the feet of the altitudes from A and B, respectively. H is the orthocenter of the triangle. We consider the points D and E on the segments AA_1 and BC such that $A_1D = A_1E = A_1B_1$. Prove that

a) $A_1B_1 = \sqrt{(A_1B^2 + A_1C^2)/2}$.
b) CH = DE.

Solution



a) Since $\angle ACB = 45°$, $\angle CAA_1 = 45°$, $A_1C = A_1A$ and $A_1D = A_1E$ causes $\angle A_1ED = \angle A_1DE = 45°$.

$A_1B^2 + A_1C^2 = AB^2 - AA_1^2 + AC^2 - AA_1^2 =$

$AB^2 + AC^2 - 2AA_1^2$ (i)

But $2AA_1^2 = AA_1^2 + A_1C^2 = AC^2$, and (i) becomes

$A_1B^2 + A_1C^2 = AB^2$ (ii)

Since $\angle AB_1B = \angle AA_1B = 90°$, AB_1A_1B is cyclic, and the two triangles HB_1A_1 and HAB are similar, we have $\dfrac{A_1B_1}{AB} = \dfrac{HB_1}{HA}$.

Furthermore, the three triangles A_1ED, B_1AH and A_1CA are also similar, we have $\dfrac{A_1B_1}{DE} = \dfrac{A_1D}{DE} = \dfrac{AA_1}{AC} = \dfrac{HB_1}{HA}$.

Hence, $\dfrac{A_1B_1}{AB} = \dfrac{A_1B_1}{DE}$, or AB = DE.

Equation (ii) now becomes $A_1B^2 + A_1C^2 = DE^2$.

We also have $2A_1E^2 = A_1E^2 + A_1D^2 = DE^2$, or

$A_1B^2 + A_1C^2 = 2A_1E^2 = 2A_1B_1{}^2$.

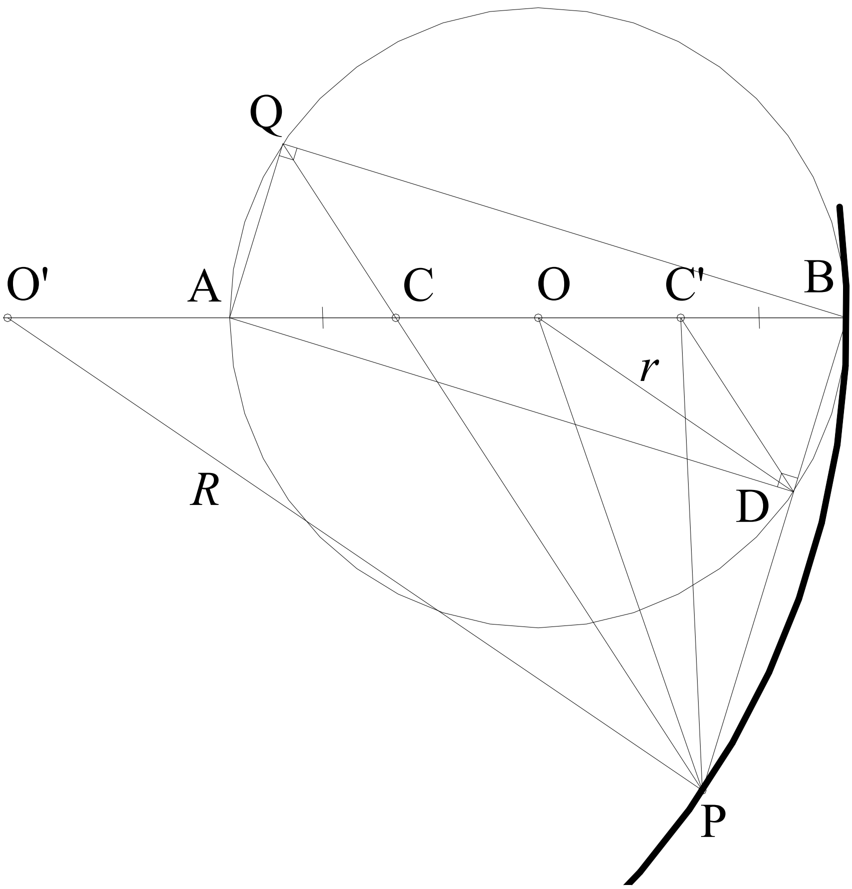
Therefore, $A_1B_1 = \sqrt{(A1B^2 + A1C^2)/2}$.

b) We have $A_1H = A_1B$ (right isoceles triangle A_1HB), and $\angle HCA_1 = \angle BAA_1$ (sides perpendicular), $\angle CA_1H = \angle AA_1B = 90°$; the two triangles CHA_1 and ABA_1 are congruent and CH = AB = DE (AB = DE from part a).

Problem 4 of the Canadian Mathematical Olympiad 1976

Let AB be a diameter of a circle, C be any fixed point between A and B on this diameter, and Q be a variable point on the circumference of the circle. Let P be the point on the line determined by Q and C for which $\frac{AC}{CB} = \frac{QC}{CP}$. Describe, with proof, the locus of the point P.

Solution



Let D be the intersection of the circle and BP. From $\frac{AC}{CB} = \frac{QC}{CP}$, we have $\frac{AC}{QC} = \frac{CB}{CP}$, and triangles ACQ and BCP are similar since we also have $\angle ACQ = \angle BCP$.

The similarity of the triangles gives us $\frac{AC}{CB} = \frac{QC}{CP} = \frac{AQ}{BP}$, and AQBD is a rectangle and AQ = BD, $\angle AQP = \angle BPQ$, $\angle QAB = \angle ABP$, AQ || BP and QB || AD.

Now pick the point C' as the image of point C across center O of the circle. We have $\frac{AC}{CB} = \frac{BC'}{CB} = \frac{AQ}{BP} = \frac{BD}{BP}$.

Let r and O be the diameter and center of the circle, respectively. Link OD and from P draw a line to parallel with OD to meet AB extension at O'.

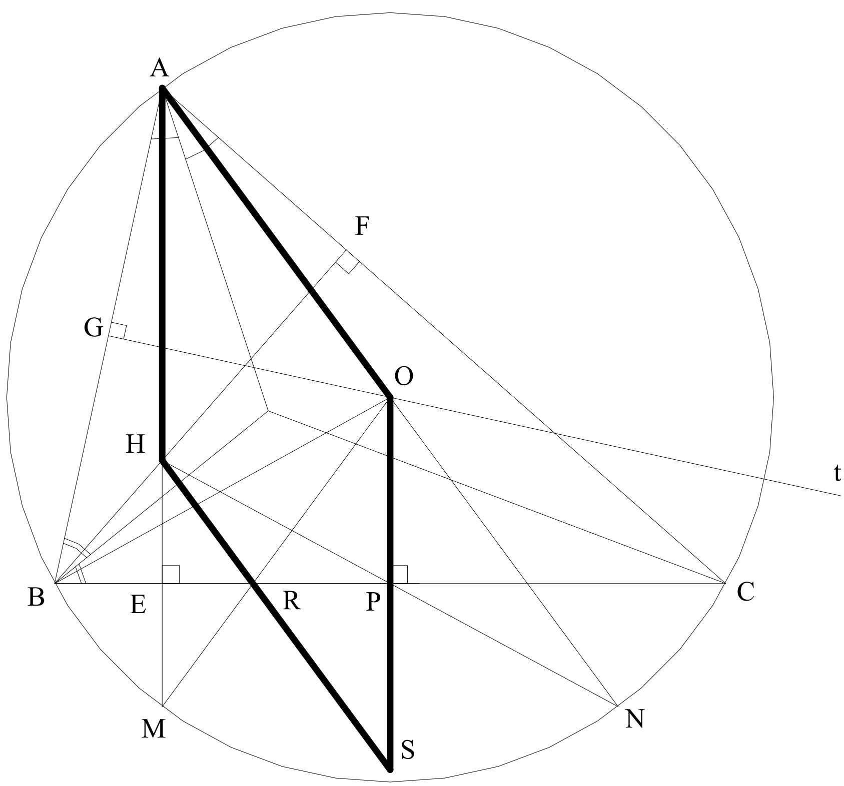
We have $\frac{OD}{O'P} = \frac{BD}{BP} = \frac{AC}{CB}$, or $O'P = \frac{OD \times CB}{AC} = \frac{r \times CB}{AC}$, and $\frac{OB}{O'B} = \frac{OD}{O'P} = \frac{AC}{CB}$, or O'B = O'P.

We conclude that the locus of the point P is a circle (only portion bold arc shown) with center at O' and radius $R = O'P = \frac{r \times CB}{AC}$.

Problem 4 of the Ibero-American Mathematical Olympiad 1997

In an acute triangle ABC, let AE and BF be its altitudes, and H the orthocenter. The symmetric line of AE with respect to the angle bisector of angle A and the symmetric line of BF with respect to the angle bisector of angle B intersect each other on the point O. The lines AE and AO intersect again the circumscribed circumference to ABC on the points M and N respectively. Let P be the intersection of BC with HN; R the intersection of BC with OM; and S the intersection of HR with OP. Show that AHSO is a parallelogram.

Solution



Let G be the midpoint of AB. Draw line *t* linking G and O. The problem gives us ∠BAO = ∠EAF = ∠FBE (sides perpendicular)

= ∠ABO, or OA = OB. So the center of the circumcircle is on line *t*. The problem also gives us ∠BAM = ∠CAN and the arcs BM = CN, or BC || MN and ∠AMN = 90° and AN is the diameter of the circumcircle. AN intercepts line *t* at O, and O is thus the center of the circumcircle.

Since H is the orthocenter, point M on the circle is, therefore, its image across BC.

We also have HE = EM, and since BC || MN, HP = PN. P and O are midpoints of HN and AN, respectively, we have OP || AE which is one of the requirements for AHSO to be a parallelogram. The second requirement is for AO to parallel HS.

Since HM || OS and R is on symmetric segment BC, HS = OM = radius of the circle = OA.

Therefore, AO || HS and it is the second requirement.

Problem 3 of the Canadian Mathematical Olympiad 1977

N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Solution

Let's write $N = 7b^2 + 7b + 7 = 7(b^2 + b + 1) = n^4$, or $b^2 + b + 1 = 7^3 \times m^4$ where m is a positive integer.

The smallest positive integer b for which N is the fourth power of an integer is when $m = 1$, or $b^2 + b + 1 = 7^3$, or $b(b + 1) = 342$.

We have $18 \times 19 = 342$, or $b = 18$, and then
$N = 7(18^2 + 18 + 1) = 7^4$.

Problem 3 of Belarus Mathematical Olympiad 2004

Find all pairs of integers (x, y) satisfying the equation
$y^2(x^2+ y^2 - 2xy - x - y) = (x + y)^2(x - y)$.

Solution

Expanding, eliminating and combining terms, we have
$y^2(y - x)^2 = x^2(y + x)$.

Therefore, $y + x$ must be a square of an integer. Let $y + x = n^2$ where n is an integer.

The above equation can be written as $y(y - x) = \pm nx$.

Let's look at the case where $y(y - x) = nx$.
Substituting $y = n^2 - x$ into the above equation, we have
$2x^2 - n(3n + 1)x + n^4 = 0$. Now solving for x, we have

$x = \frac{n}{4}[3n + 1 \pm\sqrt{n^2 + 6n + 1}]$ which requires $n^2 + 6n + 1$ to be a square of another integer. Let $n^2 + 6n + 1 = m^2$.

Solving for n, we have $n = -3 \pm \sqrt{m^2 + 8}$.
Now $m^2 + 8$ must be a square or $m = \pm$ 1which makes $n = 0$ or $n = -6$.

When $n = 0$, $x = y = 0$.
When $n = -6$, $x = 27$, $y = 9$
$x = 24$, $y = 12$.
And the other case $y(y - x) = -nx$.

Similarly, the same procedure gives us $n = 3 \pm \sqrt{m^2 + 8}$, and we end up having the same pairs of (x, y) as above.

Therefore, the three pairs of integers to satisfy the equation are $(x, y) = (0, 0)$, $(27, 9)$ and $(24, 12)$.

<u>Problem 2 of the Vietnamese Regional Competition 1977</u>

Compare $\frac{2^3+1}{2^3-1}\cdot\frac{3^3+1}{3^3-1}\cdot\ .\ .\ .\ \frac{100^3+1}{100^3-1}$ with $\frac{3}{2}$.

<u>Solution</u>

Rewrite the given expression $\frac{2^3+1}{2^3-1}\cdot\frac{3^3+1}{3^3-1}\cdot\ .\ .\ .\ \frac{100^3+1}{100^3-1}$ as

$$(2^3+1).\ \frac{3^3+1}{2^3-1}\cdot\frac{4^3+1}{3^3-1}\cdot\ .\ .\ .\ \frac{100^3+1}{99^3-1}\cdot\frac{1}{100^3-1}=$$

$$\frac{3^3+1}{2^3-1}\cdot\frac{4^3+1}{3^3-1}\cdot\ .\ .\ .\ .\ \frac{100^3+1}{99^3-1}\cdot\frac{2^3+1}{100^3-1} \qquad \text{(i)}$$

Note that $a^3+b^3=(a+b)(a^2-ab+b^2)$ and $a^3-b^3=(a-b)(a^2+ab+b^2)$.

With $b=1$, we have $a^3+1=(a+1)(a^2-a+1)$ and $(a-1)^3-1=(a-2)(a^2-a+1)$, and $\frac{a^3+1}{(a-1)^3-1}=\frac{a+1}{a-2}$.

We then write (i) as

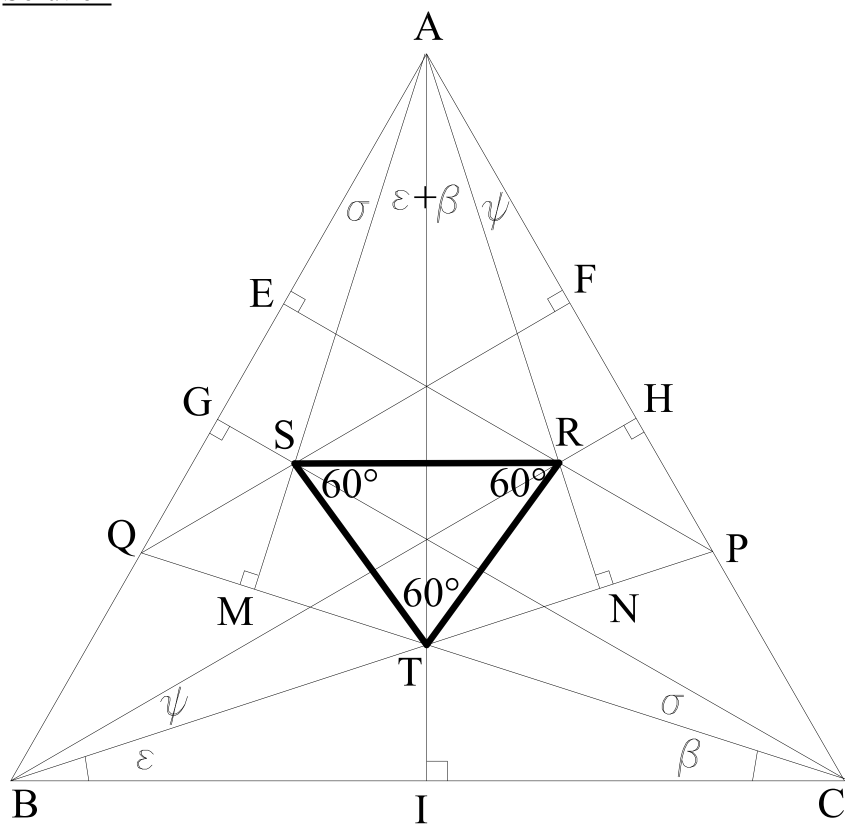
$$\frac{3+1}{3-2}\cdot\frac{4+1}{4-2}\cdot\ .\ .\ \frac{100+1}{100-2}\cdot\frac{2^3+1}{100^3-1}=\frac{99\times101}{2\times3}\cdot\frac{2^3+1}{100^3-1}=\frac{3}{2}\times\frac{9999}{999999}.$$

We conclude that $\frac{2^3+1}{2^3-1}\cdot\frac{3^3+1}{3^3-1}\cdot\ .\ .\ .\ \frac{100^3+1}{100^3-1}<\frac{3}{2}$.

Problem 3 of Asian Pacific Mathematical Olympiad 2002

Let ABC be an equilateral triangle. Let P be a point on the side AC and Q be a point on the side AB so that both triangles ABP and ACQ are acute. Let R be the orthocenter of triangle ABP and S be the orthocenter of triangle ACQ. Let T be the point common to the segments BP and CQ. Find all possible values of $\angle$CBP and $\angle$BCQ such that triangle TRS is equilateral.

Solution



Let a be the side length of equilateral triangle ABC and $\angle$CBP = α, $\angle$BCQ = β, $\angle$SBR = δ, $\angle$ABS = σ, $\angle$TBR = ψ.

We have $\dfrac{\mathrm{BT}}{\sin\beta} = \dfrac{a}{\sin(\alpha + \beta)}$, or $\mathrm{BT} = \dfrac{a\sin\beta}{\sin(\alpha + \beta)}$.

$\frac{\mathrm{BP}}{\sin 60°} = \frac{\mathrm{PC}}{\sin\alpha} = \frac{a}{\sin(\alpha + 60°)}$. But $\alpha + 60° = 90° - \psi$; therefore, $\sin(\alpha + 60°) = \cos\psi$.

$\mathrm{BP} = \frac{a\sin 60°}{\cos\psi}$ and $\mathrm{PC} = \frac{a\sin\alpha}{\cos\psi}$; $\mathrm{AP} = \frac{\mathrm{NP}}{\sin\psi}$, or

$\mathrm{NP} = \mathrm{AP}\times\sin\psi = (a - \mathrm{PC})\sin\psi = a\sin\psi(1 - \frac{\sin\alpha}{\cos\psi})$.

$\mathrm{TN} = \mathrm{BP} - \mathrm{BT} - \mathrm{NP}$.

$\mathrm{TN} = \frac{a\sin 60°}{\cos\psi} - \frac{a\sin\beta}{\sin(\alpha + \beta)} - a\sin\psi(1 - \frac{\sin\alpha}{\cos\psi})$

Now for RN: $\frac{\mathrm{RN}}{\sin\psi} = \frac{\mathrm{BN}}{\cos\psi}$ or $\mathrm{RN} = \mathrm{BN}\times\tan\psi = (\mathrm{BP} - \mathrm{NP})\tan\psi$.

$\mathrm{RN} = [\frac{a\sin 60°}{\cos\psi} - a\sin\psi(1 - \frac{\sin\alpha}{\cos\psi})] \tan\psi$.

$\mathrm{TR}^2 = \mathrm{TN}^2 + \mathrm{RN}^2 = [\frac{a\sin 60°}{\cos\psi} - \frac{a\sin\beta}{\sin(\alpha + \beta)} - a\sin\psi(1 - \frac{\sin\alpha}{\cos\psi})]^2 + [\frac{a\sin 60°}{\cos\psi} - a\sin\psi(1 - \frac{\sin\alpha}{\cos\psi})]^2\tan^2\psi$.

Using the same process to find TS, we have

$\mathrm{TS}^2 = \mathrm{TM}^2 + \mathrm{SM}^2 = [\frac{a\sin 60°}{\cos\sigma} - \frac{a\sin\alpha}{\sin(\alpha + \beta)} - a\sin\sigma(1 - \frac{\sin\beta}{\cos\sigma})]^2 + [\frac{a\sin 60°}{\cos\sigma} - a\sin\sigma(1 - \frac{\sin\beta}{\cos\sigma})]^2\tan^2\sigma$.

So for TR = TS one obvious solution is that $\alpha = \beta$, $\psi = \sigma$ to make the corresponding terms of TR^2 and TR^2 above equal, and when $\alpha = \beta$ the points P and Q are symmetrical across AI where I is the foot of A to BC.

Since SA = SB and CG $\perp$ AB and CQ $\perp$ AM, we then also have $\angle$ABS = $\angle$SAB = $\angle$TCS = σ.
Also because SA = SB and CG is perpendicular to AB and CQ perpendicular to AM, we then also have $\angle$ABS = $\angle$SBA = $\angle$TCS = σ.

Assume a solution has been attained and that $\angle$CBP = α_1 and $\angle$BCQ = β_1 are the angles required for triangle TRS to be equilateral. We will prove that for every unique value of angle α_1 there is one and only one corresponding angle β_1 to satisfy the problem.

Indeed, let's keep angle α_1 and increase $\angle$BCQ. As we do so point T moves to T' closer to N and RT' < RT, or RT decreases.

We also know that $\angle$MAN = $\alpha + \beta$. So $\angle$MAN increases by the same amount of the increase of $\angle$BCQ, and simultaneously $\angle$GAS also decreases by the same amount. Therefore, as we increase $\angle$BCQ, point S moves to S' closer to point G and RS' > RS, or RS increases.

The same but opposite effect occurs if we decrease $\angle$BCQ. Therefore, TR will no longer equal SR if $\angle$BCQ $\neq \beta_1$. So for every angle α there is only one unique angle β to satisfy the condition for triangle TRS to be equilateral.

We also know that $\angle$CBP = $\angle$BCQ is a condition for ST = RT. So point T has to always be on AI, or $\alpha = \beta$. Now let's find $\angle\alpha$.

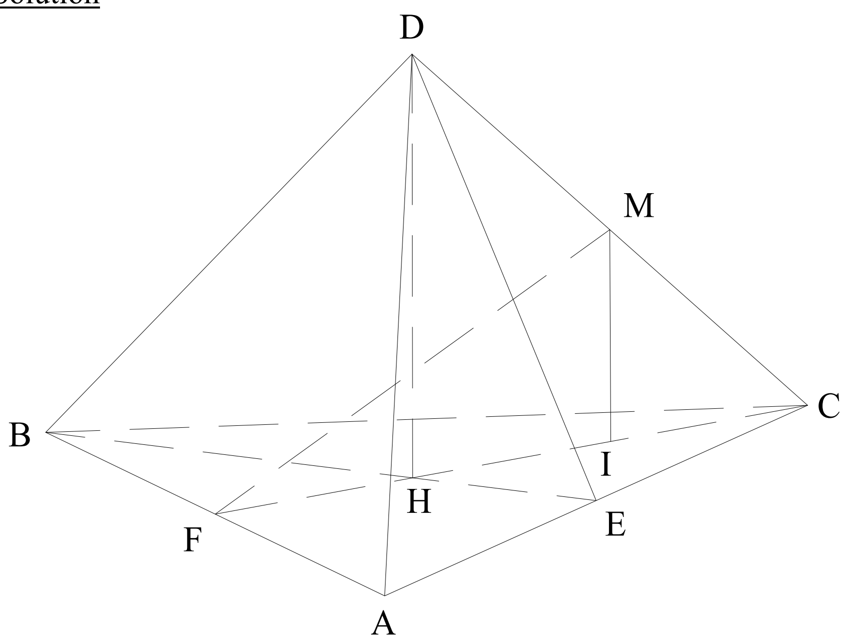
Since triangle TRS is equilateral, and R is on the bisector BH of $\angle$ABC, we have SR $\parallel$ BC, ST $\parallel$ AC and RT $\parallel$ AB, or BH is the bisector of $\angle$SBT or $\delta = \psi$. We now have $\sigma = 30° - \delta = 30° - \psi = \alpha$. We also have $\angle$BCG = $\sigma + \beta = 30°$.

Therefore, $\alpha = \beta = \delta = \sigma = \psi = 15°$, or $\angle$CBP = $\angle$BCQ = 15°.

Problem 3 of the Balkan Mathematical Olympiad 1988

Let ABCD be a tetrahedron and let d be the sum of squares of its edges' lengths. Prove that the tetrahedron can be included in a region bounded by two parallel planes, the distances between the planes being at most $\frac{1}{2}\sqrt{\frac{d}{3}}$.

Solution



Let E, F and M be the midpoints of AC, AB and DC, respectively; also let the edge's length of the tetrahedron be l.

From D and M draw the altitudes to the plane containing triangle ABC and to meet it at H and I, respectively. We will prove that the tetrahedron fits into the parallel planes with DC and AB on either plane.

The sum of squares of six lengths is $6l^2 = d$, or $l = \sqrt{d/6}$.

Consider the equilateral triangle DAC, $DE^2 = l^2 - l^2/4$, or

$DE = \frac{l\sqrt{3}}{2}$. We also have BE = DE and since H is also the centroid of triangle ABC, $HE = \frac{BE}{3} = \frac{DE}{3} = \frac{l\sqrt{3}}{6}$.

Consider right triangle DHE where $DH^2 = DE^2 - HE^2 = \frac{3l^2}{4} - \frac{3l^2}{36} = \frac{2l^2}{3}$, or $DH = l\sqrt{\frac{2}{3}} = \sqrt{\frac{d}{6}} \times \sqrt{\frac{2}{3}} = \frac{\sqrt{d}}{3}$.

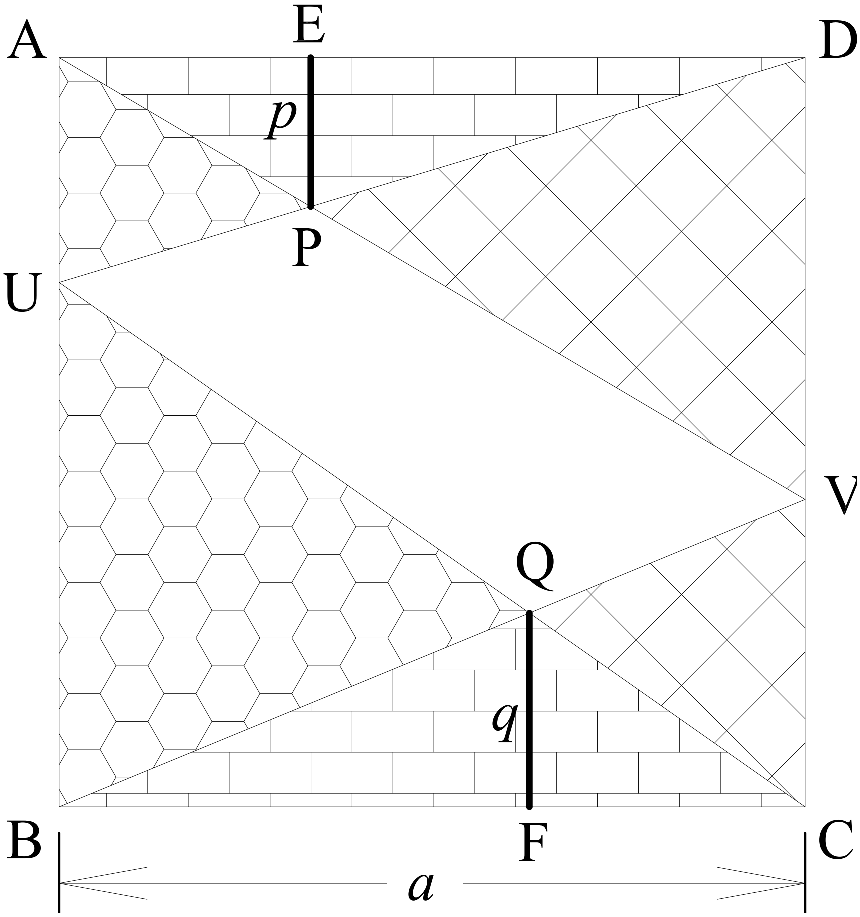
Now $FM^2 = MI^2 + FI^2 = (DH/2)^2 + (2HE)^2 = \frac{1}{2}\sqrt{\frac{d}{3}}$, but as we can see FM is orthogonal to AB (in triangle BMA) and it's also orthogonal to DC (in triangle DFC).

Therefore, the plane containing AB and the plane containing DC that are both orthogonal to FM are parallel to each other. The tetrahedron, therefore, fits into the two planes being at most $\frac{1}{2}\sqrt{\frac{d}{3}}$ apart.

Problem 3 of the Canadian Mathematical Olympiad 1992

In the diagram, ABCD is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral PUQV.

Solution



Let the side of the square be a. From P and Q draw perpendiculars to AD and BC, respectively, and let PE = p and QF = q. Let's also denote (Ω) the area of shape Ω.

Note that the area of the quadrilateral PUQV is maximum when the total of the shaded areas is minimum.

It's easily seen that the total areas shaded with honey and bricks (AUD) + (BUC) = $\frac{1}{2}a$ (AU + UB) = $\frac{1}{2}a^2$ and is constant. So now the total areas shaded with squares (PDV) + (QVC) must be minimal.

But also note that (PDV) + (QVC) = (ADV) + (BCV) – (APD) – (BQC) = $\frac{1}{2}a^2$ – (APD) – (BQC)
so (PDV) + (QVC) is minimal when (APD) + (BQC) is maximal.

(APD) + (BQC) = $\frac{1}{2}a(p + q)$ so the requirement now is for $p + q$ to be a maximum.
Since both EP and QF || with the vertical sides of the square, we have

$$\frac{p}{\mathrm{AU}} = \frac{\mathrm{DE}}{a} = \frac{a - \mathrm{AE}}{a} = 1 - \frac{\mathrm{AE}}{a} = 1 - \frac{p}{\mathrm{DV}}, \text{ or } \quad p\times\frac{\mathrm{AU} + \mathrm{DV}}{\mathrm{AU} \times \mathrm{DV}} = 1,$$

or $$p = \frac{\mathrm{AU} \times \mathrm{DV}}{\mathrm{AU} + \mathrm{DV}}$$

Similarly, $$q = \frac{\mathrm{BU} \times \mathrm{VC}}{\mathrm{BU} + \mathrm{VC}}$$

$$p + q = \frac{\mathrm{AU} \times \mathrm{DV}}{\mathrm{AU} + \mathrm{DV}} + \frac{\mathrm{BU} \times \mathrm{VC}}{\mathrm{BU} + \mathrm{VC}} =$$

$$\frac{\mathrm{AU} \times \mathrm{DV} \times \mathrm{BU} + \mathrm{AU} \times \mathrm{DV} \times \mathrm{VC} + \mathrm{AU} \times \mathrm{BU} \times \mathrm{VC} + \mathrm{BU} \times \mathrm{VC} \times \mathrm{DV}}{\mathrm{AU} \times \mathrm{BU} + \mathrm{AU} \times \mathrm{VC} + \mathrm{DV} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}}$$

$$= \frac{\mathrm{AU} \times \mathrm{BU}\,(\mathrm{DV} + \mathrm{VC}) + \mathrm{DV} \times \mathrm{VC}\,(\mathrm{AU} + \mathrm{BU})}{\mathrm{AU} \times \mathrm{BU} + \mathrm{AU} \times \mathrm{VC} + \mathrm{DV} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}} =$$

$$\mathrm{a}\times\frac{\mathrm{AU} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}}{\mathrm{AU} \times \mathrm{BU} + \mathrm{AU} \times \mathrm{VC} + \mathrm{DV} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}}$$

Now divide both numerator and denominator by sum of products AU×BU + DV×VC, we have

$$p + q = a/(1 + \frac{\mathrm{AU} \times \mathrm{VC} + \mathrm{DV} \times \mathrm{BU}}{\mathrm{AU} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}})$$

so now for $p + q$ to be a maximum, $\dfrac{\mathrm{AU} \times \mathrm{VC} + \mathrm{DV} \times \mathrm{BU}}{\mathrm{AU} \times \mathrm{BU} + \mathrm{DV} \times \mathrm{VC}}$ has to be a minimum. Let it be k.

But AU = a – BU and DV = a – VC, and now

$$k = \frac{\text{AU} \times \text{VC} + \text{DV} \times \text{BU}}{\text{AU} \times \text{BU} + \text{DV} \times \text{VC}} \text{ becomes}$$

$$k = \frac{(a - \text{BU})\ \text{VC} + (a - \text{VC})\ \text{BU}}{(a - \text{BU})\ \text{BU} + (a - \text{VC})\ \text{VC}}$$

$$= \frac{a\ (\text{VC} + \text{BU}) - 2\ \text{VC} \times \text{BU}}{a\ (\text{VC} + \text{BU}) - (\text{VC}^2 + \text{BU}^2)}$$

$$= \frac{a\ (\text{VC} + \text{BU}) - 2\ \text{VC} \times \text{BU}}{a\ (\text{VC} + \text{BU}) - 2\ \text{VC}\times\text{BU} - (\text{VC} - \text{BU})^2}$$

$$= 1/[1 - \frac{(\text{VC}-\text{BU})^2}{a\ (\text{VC}+\text{BU}) - 2\ \text{VC}\times\text{BU}}]$$

for k to be minimum the denominator of

$$\frac{(\text{VC}-\text{BU})^2}{a\ (\text{VC}+\text{BU}) - 2\ \text{VC}\times\text{BU}} \text{ has to be a maximum and}$$

$$\frac{(\text{VC}-\text{BU})^2}{a\ (\text{VC}+\text{BU}) - 2\ \text{VC}\times\text{BU}} \text{ to be a minimum. Note that the}$$

denominator is not zero, and the square $(\text{VC} - \text{BU})^2$ is always greater than or equal to zero, and it's a minimum when it's zero or when VC = BU.

So to maximize the area of the quadrilateral PUQV, U and V has to be on a horizontal line between the top and bottom sides of the square ABCD. The maximal area of PUQV is then equal

$$a^2 - \tfrac{1}{2}a^2 - \tfrac{1}{2}(a/2)\times a = \tfrac{1}{4}a^2.$$

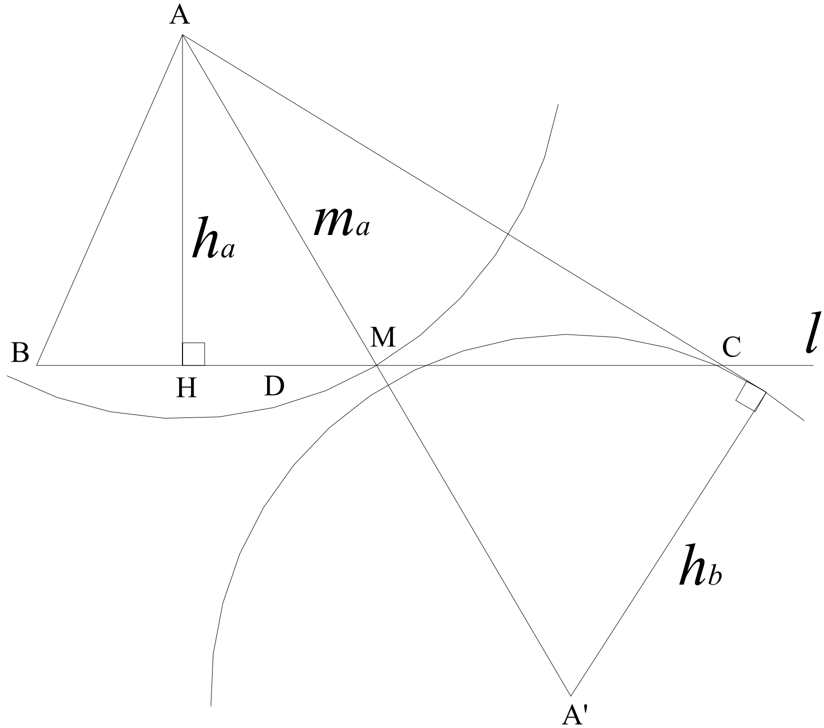
<u>Further observation</u>

We can also resort to the Carpet theorem for a simpler analysis.

Problem 4 of the International Mathematical Olympia 1960

Construct triangle ABC given h_a, h_b (the altitudes from A and B) and m_a, the median from vertex A.

Solution

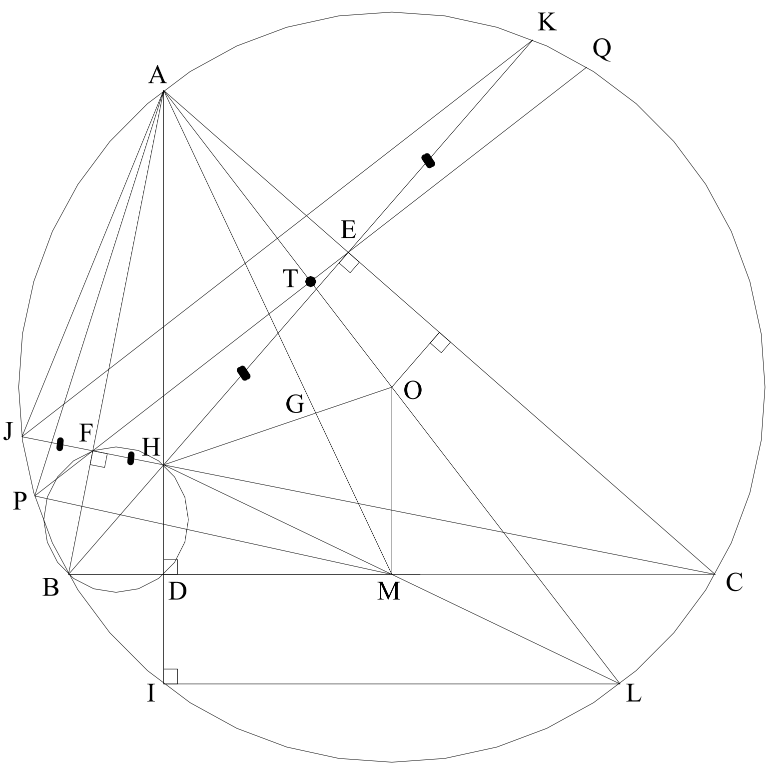


Draw line l. Both points B and C will be on this line. Pick an arbitrary point H on l. From H draw a segment HA perpendicular to l and with a length equal h_a. Draw a circle with center A and radius m_a to intercept line l at M. Extend AM and pick point A' at the extension so that MA' = MA. (A' is point symmetry of A across M).
Draw a circle *C* with center A' and radius h_b. Then draw the tangential line from A to circle *C*. This tangential line will intercept line l at C. Point B is the image of C across point M.

Problem 5 of the Ibero-American Mathematical Olympiad 1999

An acute triangle 4ABC is inscribed in a circumference of center O. The highs of the triangle are AD; BE and CF. The line EF cut the circumference on P and Q.
a) Show that OA is perpendicular to PQ.
b) If M is the midpoint of BC, show that $AP^2 = 2AD \times OM$.

Solution



a) Extend HE, HF and HD to meet the circle at K, J and I, respectively. Since H is the orthocenter of triangle ABC, the three points K, J and I are images of H across the three sides AC, AB and BC of the triangle ABC, respectively.
Therefore,

HE = EK and HF = FJ, or FE || JK and arc KQ = arc JP;

$\angle$PEB (subtends arcs KQ and PB) = $\angle$JKB (subtends arc JB) = $\angle$JCB = $\angle$BAI.

Extend AO to meet the circle at L. Since AL is the diameter of the circle, AI $\perp$ IL and IL || BC, BI = CL, and $\angle$BAI = $\angle$CAL = $\angle$PEB. In addition to AC $\perp$ BE, OA $\perp$ PQ.

b) Let PQ intersect AO at T. Since OA $\perp$ PQ, we have $AP^2 = AT^2 + PT^2 = AT^2 + (PF + FT)^2 = AT^2 + PF^2 + FT^2 + 2PF \times FT = AT^2 + PF(PF + FT) + FT^2 + PF \times FT = AT^2 + PF \times PT + FT^2 + PF \times FT = AT^2 + PF(PT + FT) + FT^2 = AT^2 + PF \times FQ + FT^2 = AT^2 + AF \times FB + FT^2 = AF^2 + AF \times FB = AF \times (AF + FB) = AF \times AB$.

But FHDB is cyclic and we have $AF \times AB = AH \times AD = 2\, AD \times OM$. Also since triangles AHG and MOG are similar and G is the centroid of triangle ABC, $MG = \frac{1}{2}AG$.

Problem 6 of the Canadian Mathematical Olympiad 1971

Show that, for all integers n, $n^2 + 2n + 12$ is not a multiple of 121.

Solution

Assuming $n^2 + 2n + 12$ is a multiple of 121, we have
$(n + 1)^2 + 11 = 121k$ where k is an integer, or
$(n + 1)^2 = 121k - 11 = 11(11k - 1)$.

Since 11 is a prime integer, for $(n + 1)^2 = 11(11k - 1)$ to occur we must have $11k - 1 = 11m^2$ where m is also an integer, or $11(m^2 - k) = -1$, or $m^2 - k = -1/11$ which is a fraction and is not possible.

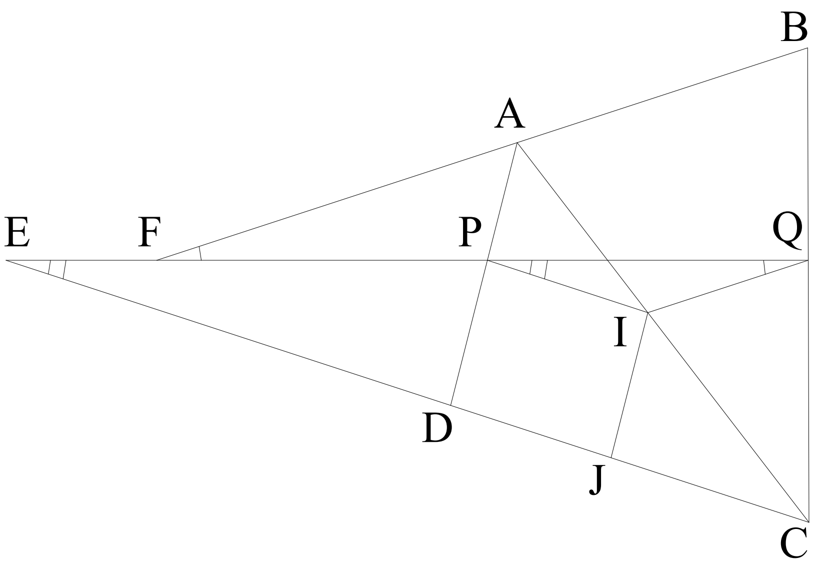
So the assumption that $n^2 + 2n + 12$ is a multiple of 121 is not possible.

Problem 6 of the Ibero-American Mathematical Olympiad 1987

Let ABCD be a plain convex quadrilateral. P, Q are points of AD and BC respectively such that $\frac{AP}{PD} = \frac{AB}{DC} = \frac{BQ}{QC}$.

Show that the angles that are formed by the lines PQ with AB and CD are equal.

Solution



From P draw a line || to DC and intercept AC at I. Link IQ. We have IQ || AB. We then have $\angle QEC = \angle QPI$ and $\angle QFB = \angle PQI$
To prove that the angles that are formed by the lines PQ with AB and CD are equal,
we then need to prove $\angle QPI = \angle PQI$ or $IP = IQ$.

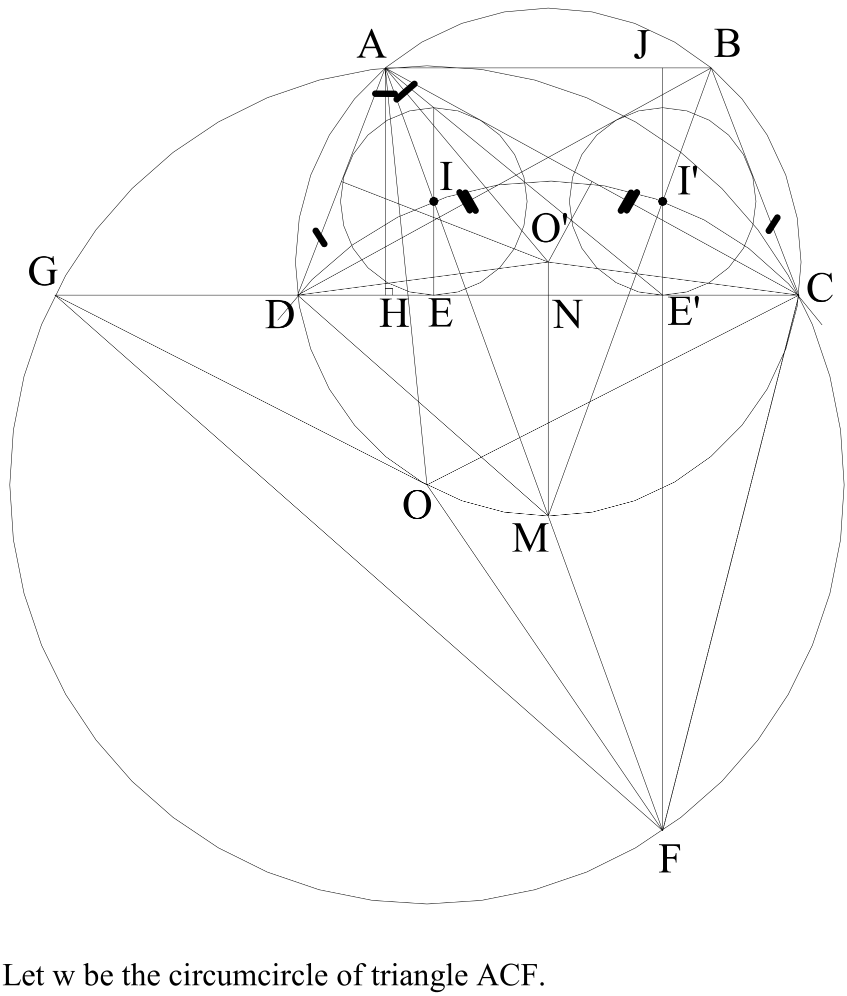
From I draw a line || to AD and intercept DC at J. We have
$\frac{IQ}{AB} = \frac{IC}{AC} = \frac{JC}{DC}$, or $\frac{AB}{DC} = \frac{IQ}{JC}$.

We also have $\frac{AB}{DC} = \frac{AP}{PD} = \frac{IP}{JC}$; therefore, $IP = IQ$.

Problem 6 of the United States Mathematical Olympiad 1999

Let ABCD be an isosceles trapezoid with AB$\parallel$CD. The inscribed circle w of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle$DAC such that EF$\perp$CD. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

Solution



Let w be the circumcircle of triangle ACF.
Draw the incircle of triangle ADC with center at I'; this circle is symmetrical of the incircle of triangle BDC with respect to the axis

passing through centers of AB and DC. Draw the circumcircle w1 of triangle ADC to intercept AF at M.

Since AF is the bisector of $\angle$DAC, we have MD = MI' = MI, and M is the center of circle w2 as shown.
From M draw line perpendicular to DC and meets it at N. Since N is the midpoint of EE', MI' = MF and therefore, F is on circle w2.

For circle w1, we have	$AP\times PM = DP\times PC$	(i)
For circle w, we have	$AP\times PF = GP\times PC$	(ii)
From (i) and (ii),	$PM/PF = DP/GP$	(iii)
or	$MD \parallel GF$ and	
	$MD/GF = PM/PF$	
or	$GF = MD\times PF/PM$	(iv)

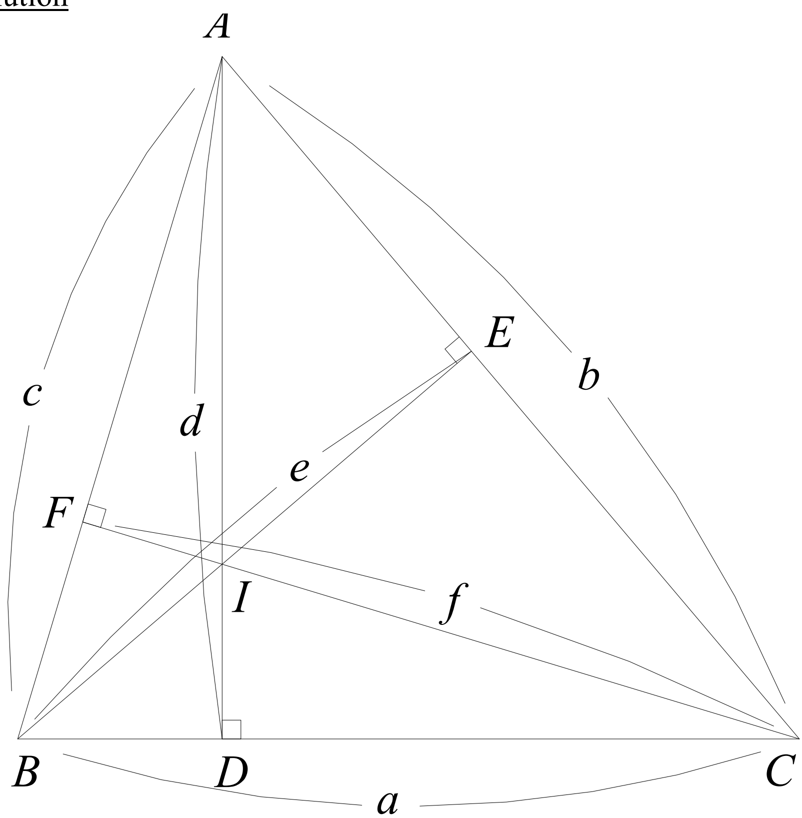
For circle w2, we have	$IP\times PF = DP\times PC$	(v)
From (v) and (ii), we have	$IP/AP = DP/GP$	(vi)
From (vi) and (iii), we have	$IP/AP = PM/PF$	(vii)
From (vii),		
$IP/AP = PM/PF = (IP+PM)/(AP+PF) = MI/AF$		(viii)
From (iv) and (viii),	$GF = MD\times AF/MI = AF$	

Therefore, triangle AFG is isosceles.

Problem 7 of Belarus Mathematical Olympiad 2004

Let be given two similar triangles such that the altitudes of the first triangle are equal to the sides of the other. Find the largest possible value of the similarity ratio of the triangles.

Solution



Let the first triangle be ABC and the feet from A, B and C to the opposite sides be D, E and F, respectively. Now let BC = a, AC = b, AB = c and AD = d, BE = e and CF = f.
Without los of generality, assume $a \geq b \geq c$. Because twice the area of triangle ABC = $ad = be = cf$, our assumption makes $f \geq e \geq d$.
To find the largest possible value of the similarity ratio of the triangles we need to find the largest possible ratio $\frac{f}{a}$ or largest possible $\cos\angle$FCB.

The similarity of the triangles ADB and CFB gives us

$$\frac{d}{c} = \frac{f}{a} \qquad \text{(i)}$$

and similarity of triangles AEB and AFC,

$$\frac{e}{c} = \frac{f}{b} \qquad \text{(ii)}$$

And because the altitudes of the first triangle are equal to the sides of the second, we also have

$$\frac{f}{a} = \frac{e}{b} \qquad \text{(iii)}$$

From (ii), $e = \frac{cf}{b}$; substituting it into (iii), we then have $b^2 = ac$.

Now the law of cosines gives us $b^2 = a^2 + c^2 - 2ac\times\cos\angle ABC$,

or $ac = a^2 + c^2 - 2ac\times\cos\angle ABC$, or $\cos\angle ABC = \frac{a^2 + c^2 - ac}{2ac}$ = $\frac{a^2 + c^2}{2ac} - \frac{1}{2}$. But $\angle ABC + \angle FCB = 90°$; therefore, $\cos\angle FCB$ =

$\sin\angle ABC = \sqrt{1 - \cos^2\angle ABC}$. Hence, $\cos\angle FCB$ is largest when $\cos^2\angle ABC$ is smallest or when $\frac{a^2 + c^2}{2ac} - \frac{1}{2}$ is smallest, or when $\frac{a^2 + c^2}{2ac}$ is smallest which happens when $a = c$ (per AM-GM inequality) which makes $\frac{a^2 + c^2}{2ac} = 1$.

Thus, the largest possible $\cos\angle FCB = \sqrt{1 - ¼} = ½\sqrt{3}$ when $a = b = c$ and the triangle ABC and its similar triangle are both equilateral.

<u>Further observation</u>

It depends on how one defines the similarity ratio; the similarity ratio could be the ratio of the side of the larger triangle to the corresponding side of the smaller one. In such a case, the

similarity ratio is the inverse of the above result which is $2\sqrt{3}/3$. This ratio is the largest and could be the solution required.

Problem 7 of the Canadian Mathematical Olympiad 1969

Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

Solution

Adding $2ab$ to both sides, we have $(a + b)^2 = 2(ab + 4c + 3)$, or $ab + 4c + 3 = 2d^2$ where d is an integer. Since $2d^2$ is even, the product ab must be an odd number and both ***a and b must be odd numbers***.

Now let $a = 2m + 1$ and $b = 2n + 1$ where m and n are integers. Substituting them into the original equation, we have
$(2m + 1)^2 + (2n + 1)^2 = 2(4c + 3)$, or
$4m^2 + 4m + 4n^2 + 4n + 2 = 2(4c + 3)$, or
$m^2 + m + n^2 + n = 2c + 1$, or
$$m(m + 1) + n(n + 1) = 2c + 1 \qquad \text{(i)}$$

Now note that the product of two consecutive numbers is always an even number since one of them is an even number.

Therefore, the sum on the left of (i) is an even number whereas the one on the right is an odd number. So the original requirements of both a and b being odd numbers are also not possible.

Therefore, we can not find integers for a, b and c to satisfy the problem.

Problem 1 of Austria Mathematical Olympiad 2004

Determine all integers a and b such that $(a^3 + b)(a + b^3) = (a + b)^4$.

Solution

Expanding the equation and canceling the same terms, we have $(b^2 - 4)a^2 - 6ab - 4b^2 + 1 = 0$.

Solving for a, we have $a_1 = \dfrac{2b^2 + 3b - 2}{b^2 - 4}$, and $a_2 = \dfrac{-2b^2 + 3b + 2}{b^2 - 4}$.

If $a_1 = \dfrac{2b^2 + 3b - 2}{b^2 - 4} = \dfrac{2b - 1}{b - 2}$ $(b \neq -2) = 2 + \dfrac{3}{b-2}$ $(b \neq 2)$
which is an integer when $b = 1$, $b = 3$ and $b = 5$.

If $a_2 = \dfrac{-2b^2 + 3b + 2}{b^2 - 4} = -\dfrac{2b + 1}{b + 2}$ $(b \neq 2) = -2 + \dfrac{3}{b+2}$ $(b \neq -2)$
which is an integer when $b = -1$, $b = -3$ and $b = -5$.

Answers: (a, b) = (-3, -5), (-5, -3), (-1, -1), (1, 1), (5, 3) and (3, 5).

Problem 2 of the Irish Mathematical Olympiad 1994

Let A, B, C be three collinear points with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the opposite side. Prove that the centroids of the triangles are the vertices of an equilateral triangle. Prove that the centroid of this triangle lies on the line AC.

Solution

a) Let a = AB, b = BC and $c = a + b$, and let J, K, R, H and G be the feet from A to BD, C to BE, E to BC, A to CF and C to AF, respectively. Also let X, Y and Z be the centroids of equilateral triangles ABD, BCE and ACF, respectively.

From X and Z draw perpendicular lines to meet CG and CK at P and Q, respectively.

Observe that AX = GP, HZ = CQ, $XP = AG = HC = ZQ = \frac{c}{2}$,

and $GZ = \frac{1}{3}CG = \frac{c}{2\sqrt{3}}$, and $AX = \frac{a}{\sqrt{3}}$.

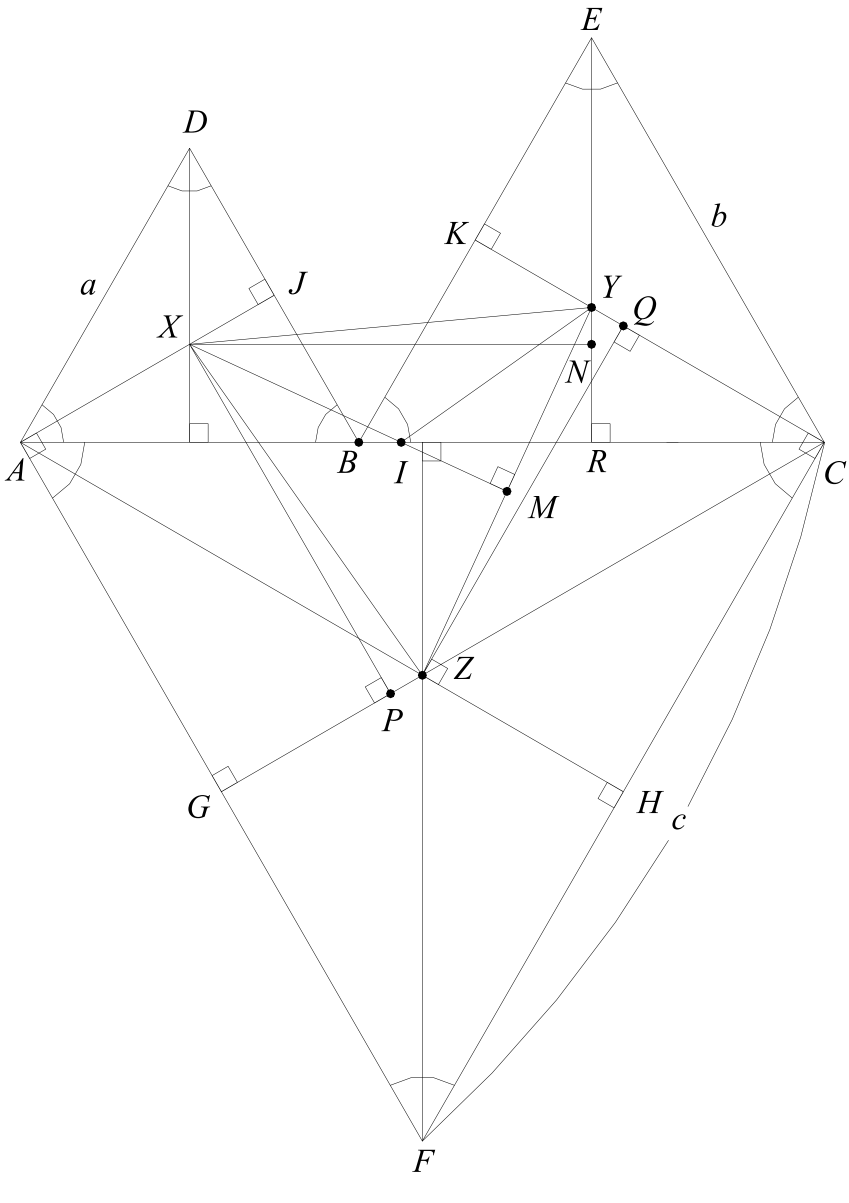
Now let $\angle PXZ = \alpha$, $\tan\alpha = \frac{PZ}{XP} = \frac{GZ - GP}{XP} = \frac{GZ - AX}{AG} = \frac{c-2a}{\sqrt{3}c}$.

Similarly, $\tan\angle QZY = \frac{c-2a}{\sqrt{3}c}$.

But $c = a + b$, or $c - 2a = 2b - c$, and
$\tan\alpha = \tan\angle QZY$, or $\alpha = \angle QZY$.

Since XP || AF and ZQ || FC, XP and ZQ will intercept each other at an angle of 60°, and XZ and YZ also intercept each other at the same angle.

With the addition of $PZ = \frac{c-2a}{2\sqrt{3}} = \frac{2b-c}{2\sqrt{3}} = QY$ and XP = ZQ as mentioned earlier, the two triangles XPZ and ZQY are congruent which makes XZ = ZY and thus XYZ is an equilateral triangle.



b) Now let S and M be the feet of X on AB and YZ and XM intercepts AC at I.

We have to prove that IX = IY or $IS^2 + XS^2 = IR^2 + YR^2$ (i)

But $XS^2 = \frac{a^2}{12}$, $YR^2 = \frac{b^2}{12}$, $IR = \frac{c}{2} - IS$, and (i) becomes

$\mathrm{IS}^2 + \dfrac{a^2}{12} = (\dfrac{c}{2} - \mathrm{IS})^2 + \dfrac{b^2}{12}$, or $\mathrm{IS} = \dfrac{b+c}{6}$ is what we have to prove.

But we also have $\tan\angle\mathrm{SXI} = \tan(60° + \alpha) = \dfrac{\mathrm{IS}}{\mathrm{XS}}$ (ii)

$$\tan\angle\mathrm{SXI} = \tan(60° + \alpha) = \frac{\sin 60°\cos\alpha + \cos 60°\sin\alpha}{\cos 60°\cos\alpha - \sin 60°\sin\alpha}.$$

However, $\mathrm{PZ} = \dfrac{c-2a}{2\sqrt{3}}$, $\mathrm{XP} = \dfrac{c}{2}$, and

$$\frac{\sin 60°\cos\alpha + \cos 60°\sin\alpha}{\cos 60°\cos\alpha - \sin 60°\sin\alpha} = \frac{\sqrt{3}\,\mathrm{XP} + \mathrm{PZ}}{\mathrm{XP} - \sqrt{3}\mathrm{PZ}} = \frac{2c-a}{\sqrt{3}a}.$$

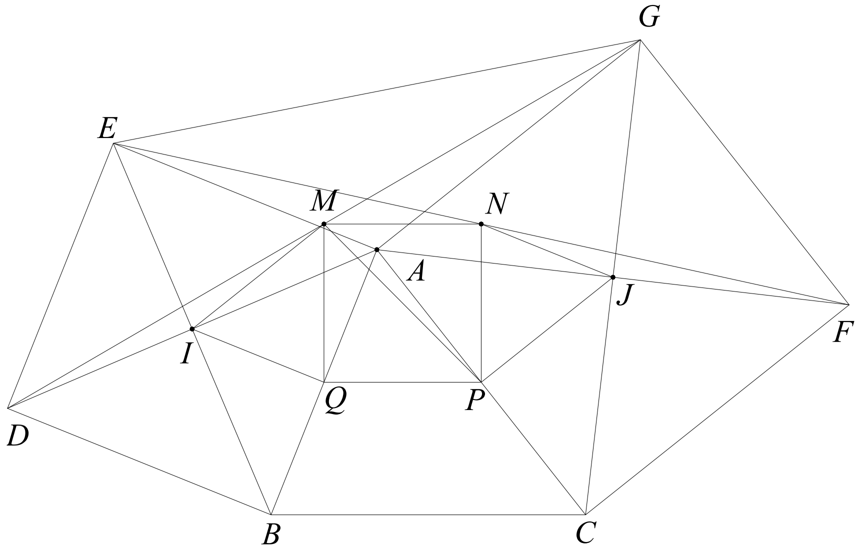
From (ii), $\mathrm{IS} = \mathrm{XS}\tan(60° + \alpha) = \dfrac{a}{2\sqrt{3}} \cdot \dfrac{2c-a}{\sqrt{3}a} = \dfrac{2c-a}{6}$.

So now it suffices to prove $2c - a = b + c$, or $c = a + b$, and this is obvious.

<u>Problem 2 of Poland Mathematical Olympiad 2001</u>

ABC is a given triangle. ABDE and ACFG are the squares drawn outside of the triangle. The points M and N are the midpoints of DG and EF, respectively. Find all the values of the ratio MN : BC.

<u>Solution</u>



Let P, Q, I and J be the midpoints of AC, AB, AD and AF, respectively. We observe the following QP || BC, QP = ½BC, IQ || AE, IQ = ½AE, NJ || AE, NJ = ½AE, IM || AG, IM = ½AG, JP || AG, JP = ½AG.
From there, IQ || NJ and IQ = NJ, IM || PJ and IM = PJ, ΔMIQ = ΔPJN, and we have MQ = NP, and ∠IMQ = ∠JPN (i)

On the other hand, since IM || PJ, ∠IMP = ∠JPM.
Combining with (i), we have ∠QMP = ∠NPM, and with MQ = NP as proved earlier, MNPQ is a parallelogram. Therefore, MN || QP and MN = QP, or MN : BC = 1 : 2.
For a triangle ABC with obtuse angle BAC, the proof is similar.

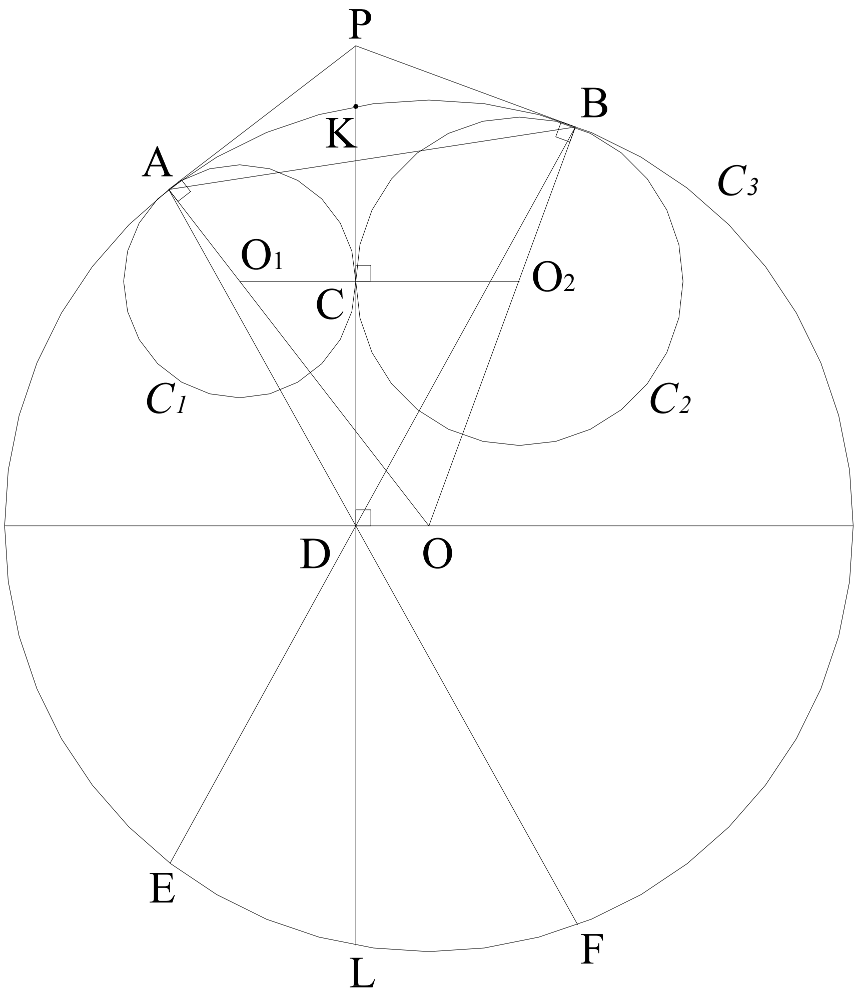
<u>Further observation</u>

Prove that MNPQ is a square.

Problem 3 of Balkan Mathematical Olympiad 1993

Circles C_1 and C_2 with centers O_1 and O_2, respectively, are externally tangent at point C. A circle C_3 with center O touches C_1 at A and C_2 at B so that the centers O_1, O_2 lie inside C_3. The common tangent to C_1 and C_2 at C intersects the circle C_3 at K and L. If D is the midpoint of the segment KL, show that $\angle ADB = \angle O_1OO_2$.

Solution



Observe that $\angle O1OO2 = \angle AOB$.

Let the tangents of C_1 and C_2 at A and B, respectively, meet at P. P is on the extension of LK.

Since OA $\perp$ PA, OB $\perp$ PB and because D is midpoint of KL and O is the circumcenter, OD $\perp$ KL.

The two quadrilaterals APBO and PBOD are cyclic implying that ABOD to also be cyclic on the same circle. Therefore, $\angle$AOB = $\angle O_1OO_2$ = $\angle$ADB.

<u>Further observation</u>

From the result we have arc AB×2 = arc AB + arc EF, or arc AB = arc EF, and since OD is on the diameter of the circumcircle, E and F are images of A and B across the diameter, respectively, which makes KL to be the angle bisector of ∠ADB.

Problem 5 of the Canadian Mathematical Olympiad 1972

Prove that the equation $x^3 + 11^3 = y^3$ has no solution in positive integers x and y.

Solution

Rearrange the equation to $(x - y)^3 + 3x^2y - 3xy^2 = -11^3$ or $(y - x)[(y - x)^2 + 3xy] = 11^3$.
Since 11 is a prime integer, $y - x$ will take on the possible values of 1, 11, 11^2 or 11^3.
If $y - x = 1$, then $3xy = 11^3 - 1 = 1330$, or $xy = \frac{1330}{3}$ which is not an integer, and this is not a possible scenario.

If $y - x = 11$, then $3xy = 0$ and either x or y must be 0 and not positive as required.

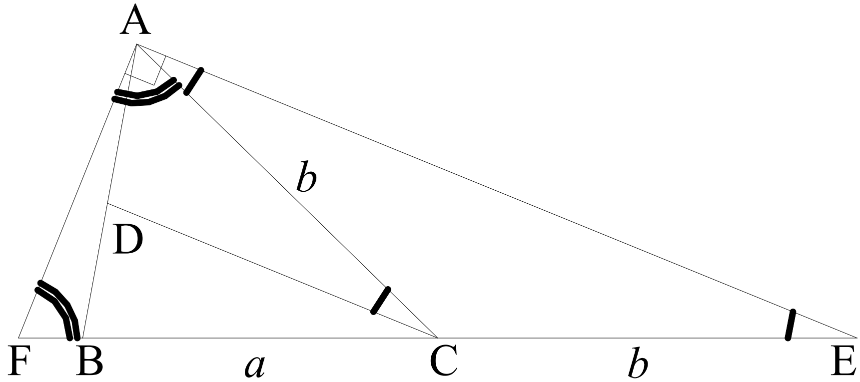
If $y - x = 11^2$, then $3xy = 11 - 11^2$, or $xy < 0$ and either x or y must be negative and not both being positive as required.

If $y - x = 11^3$, then $3xy = 1 - 11^3$, or $xy < 0$ which is the same as in the previous case.

Problem 5 of the Canadian Mathematical Olympiad 1969

Let ABC be a triangle with sides of lengths *a*, *b* and *c*. Let the bisector of the angle C cut AB in D. Prove that the length of CD is
$$\frac{2ab\times\cos\frac{C}{2}}{a+b}.$$

Solution



Extend BC to the right a length of CE = AC = *b*. From A draw the perpendicular to AE to meet the extension of CB to the left at F.

Since AC = CE, $\angle$AEB = ½$\angle$ACB = $\angle$DCB and CD || AE and we have $\frac{CD}{AE} = \frac{a}{a+b}$, or CD = $\frac{a\times AE}{a+b}$ (i)

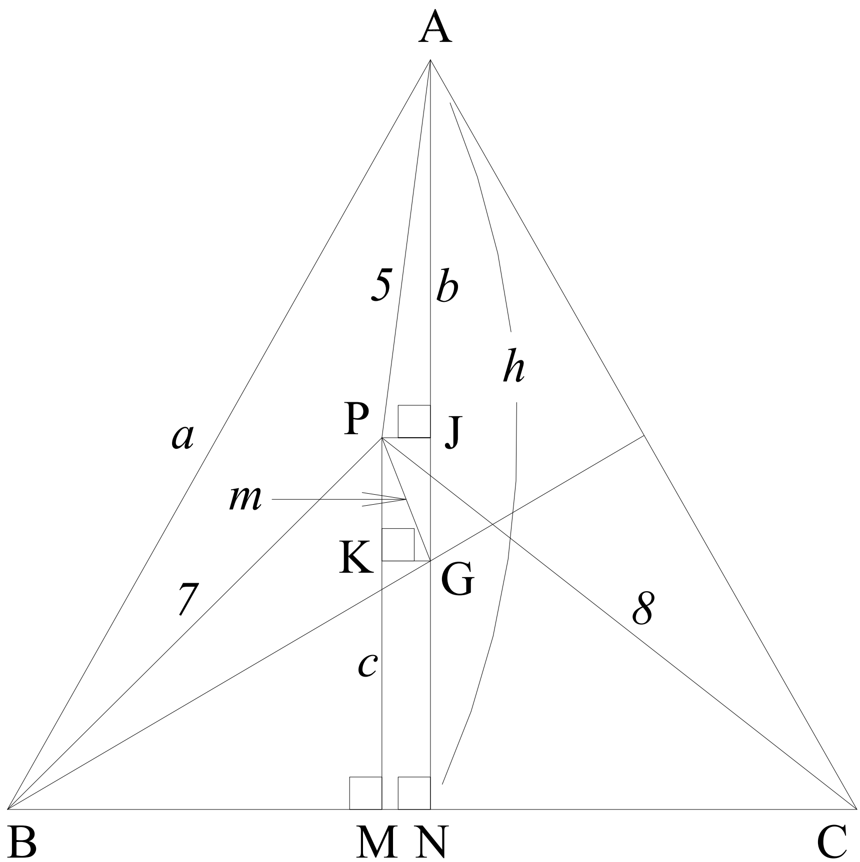
We also have $\angle$AFE = 90°− $\angle$AEF = 90°− $\angle$CAE = $\angle$FAC or CAF is isosceles with CA = CF = *b* and $\cos\frac{C}{2} = \cos\angle AEF = \frac{AE}{2b}$,

or AE = $2b\times\cos\frac{C}{2}$, and (i) now becomes CD = $\frac{2ab\times\cos\frac{C}{2}}{a+b}$.

Problem 2 of the Ibero-American Mathematical Olympiad 1985

Let P be a point in the interior of the equilateral triangle ABC such that PA = 5, PB = 7, PC = 8. Find the length of the side of the triangle ABC.

Solution



There is an existing formula relating the distances from a point inside a triangle to its vertices expressed as follows $a^2 + b^2 + c^2 = 3(d^2 + e^2 + f^2 - 3m^2)$ where a, b and c are the lengths of the sides of the triangle, d, e and f the distances from that point to the vertices, and m the distance from that point to the triangle's centroid.

Let a be the side length of the equilateral triangle; applying the above formula, we have $a^2 = d^2 + e^2 + f^2 - 3m^2$.

In our case let $d = 5$, $b = 7$ and $f = 8$ as given by the problem, we now get $a^2 = 138 - 3m^2$ (i)

Now let's find m = PG as shown on the graph. Let AJ = b, we have

$m^2 = PJ^2 + PK^2 = 25 - b^2 + (\frac{2}{3}h - b)^2 = 25 + \frac{4}{9}h^2 - \frac{4}{3}hb$ where $h = \frac{a\sqrt{3}}{2}$ is the equilateral triangle's altitude.

Substituting m^2 into (i), we have $a^2 = 63 - \frac{4}{3}h^2 + 4hb$ (ii)

Now substituting h into (ii), we have

$$2a^2 = 63 + 2ab\sqrt{3}\text{, or } b = \frac{2a^2 - 63}{2a\sqrt{3}} \qquad \text{(iii)}$$

Now let s be the semi-perimeter of triangle PBC, $s = \frac{a + 15}{2}$, and the area of triangle PBC using Heron's formula is

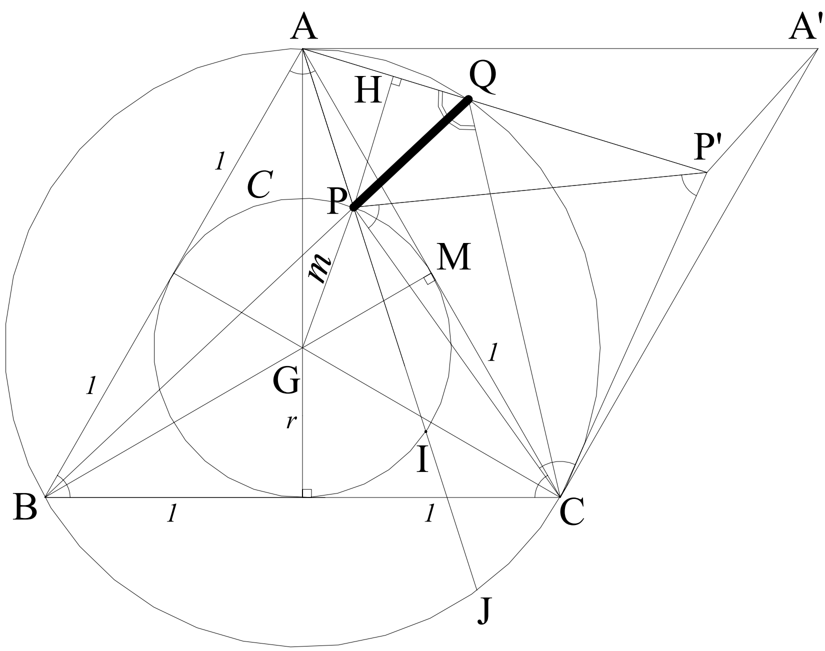
area of triangle PBC $= \sqrt{(a + 15)(a + 1)(a - 1)(15 - a)} = 2a(h - b)$,
or $(a^2 - 225)(a^2 - 1) = -4a^2(\frac{a\sqrt{3}}{2} - b)^2$ (iv)

Substituting b from (iii) into (iv), we have
$(a^2 - 225)(a^2 - 1) = -4a^2(\frac{a\sqrt{3}}{2} - \frac{63}{2a\sqrt{3}})^2$.
Let $x = a^2$; the above equation reduces to a quadratic equation $x^2 - 138x + 1161 = 0$, or $x^2 = 9$ and $x^2 = 129$. Therefore, $a = 3$ and $a = 11.36$. Length a can not be less than 5; we then pick 11.36 as the answer.

<u>*Problem 3 of the Ibero-American Mathematical Olympiad 1992*</u>

In a equilateral triangle of length 2, it is inscribed a circumference *C*.
a) Show that for all point P of *C* the sum of the squares of the distance of the vertices A, B and C is 5.
b) Show that for all point P of *C* it is possible to construct a triangle such that its sides has the length of the segments AP, BP and CP, and its area is $\frac{1}{4}\sqrt{3}$.

<u>Solution</u>



a) Let G be the centroid of triangle ABC, m = GP. The existing formula relating the distances from a point to the vertices of a triangle is $a^2 + b^2 + c^2 = 3(d^2 + e^2 + f^2 - 3m^2)$ where a, b and c are the three sides of the triangle and d, e and f are distances from the point to the triangle's vertices. Applying the formula to this case with $m = \mathrm{GP} = r = 1/\sqrt{3}$, $4 = \mathrm{AP}^2 + \mathrm{BP}^2 + \mathrm{CP}^2 - 3/3$, or $\mathrm{AP}^2 + \mathrm{BP}^2 + \mathrm{CP}^2 = 5$.

b) Rotate triangle ABC 60° clockwise around point C. We have A $\rightarrow$ A', B $\rightarrow$ A, P $\rightarrow$ P' and AP' = BP, PC = PP'= P'C and the triangle AP'P has its side lengths of the segments AP, BP and CP.

Now draw triangle ABC's circumcircle, and let Q be the intersection of AP' with the circumcircle. Since after the rotation, triangle ABP = triangle A'AP', $\angle$ABP = $\angle$A'AP', the three points B, P and Q are collinear. Let H be the foot of P to AP', the area of the triangle APP' = ½PH×AP'. Now extend AP to intercept the two circles at I and J, respectively. We have PQ×PB = AP×PJ, but since the two circles share the same centers, AP = IJ and PQ×PB = AP×AI = AM² = 1.

However, PB = AP', and $\angle$AQB = $\angle$BQC = 60°, and

PH = ½PQ$\sqrt{3}$, the area of the triangle APP' = ½PH×AP' =

(¼PQ$\sqrt{3}$)×PB = ¼$\sqrt{3}$ PQ×BP = ¼$\sqrt{3}$.

<u>Further observation</u>

The following are drawn from this problem:

1. The sum of the distances from point Q to vertices of triangle ABC is $AQ^2 + BQ^2 + CQ^2 = 8$ *(i)*

since $GQ = 2r = 2/\sqrt{3}$ *or* $BQ^2 = 8 - AQ^2 - CQ^2$.

2. Using the law of the cosine function $AC^2 = AQ^2 + CQ^2 - 2\,AQ\times CQ\cos 120°$ *or* $AQ^2 + CQ^2 + AQ\times CQ = 4$ *(ii)*
or $(AQ + CQ)^2 = AQ\times CQ + 4$ *(iii)*

Now subtract (ii) from (i), we have $BQ^2 = AQ\times CQ + 4$
Combining with (iii), we have $BQ = AQ + CQ$.

3. The area of triangle with length segments AP, BP and CP is always constant as long as P is on the inner circle. One can derive another problem to find the locus of the points P in the *plane of an equilateral triangle ABC for which the triangle formed with PA, PB and PC has constant area.*

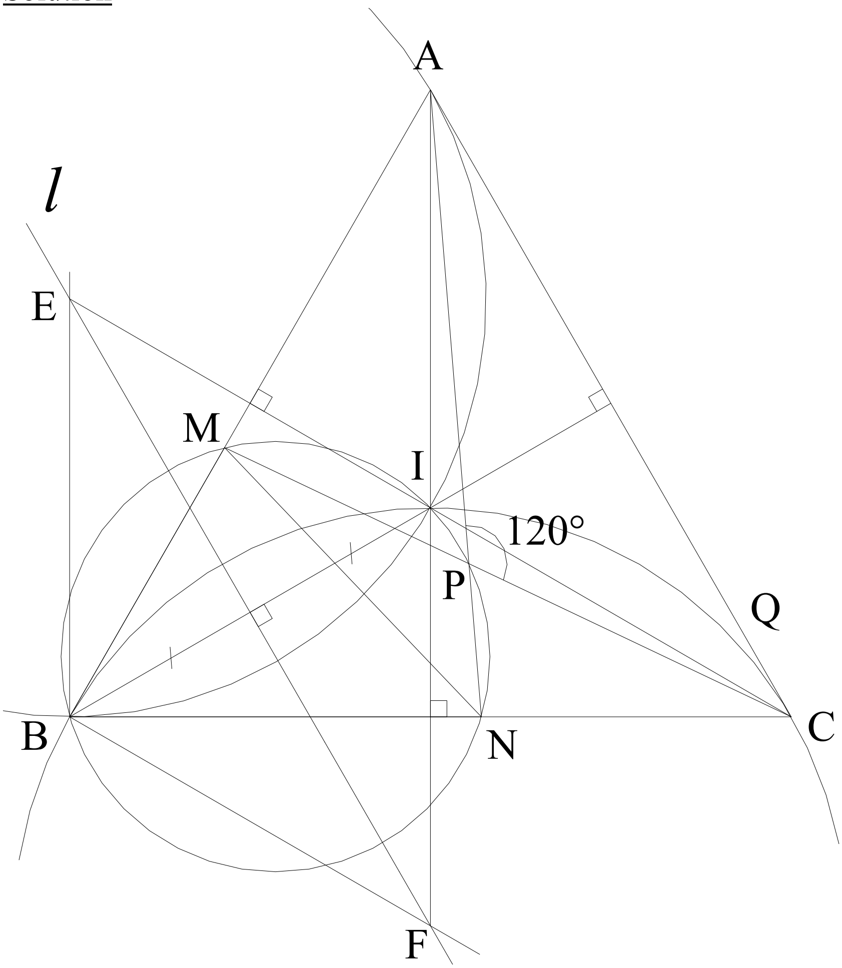
4. The problem can be reversed: If for every point P in the interior of a triangle, one can construct a triangle having sides equal to PA, PB and PC then the triangle is equilateral.

5. In triangle ABC, AB is the longest side. Prove that for any point P in the interior of the triangle, $PA + PB > PC$.

<u>Problem 3 of the Ibero-American Mathematical Olympiad 2002</u>

Let P be a point in the interior of the equilateral triangle ABC such that $\angle APC = 120°$. Let M be the intersection of CP with AB, and N the intersection of AP and BC. Find the locus of the circumcenter of the triangle MBN when P varies.

<u>Solution</u>



Since $\angle MPN = 120°$ and ABC is an equilateral triangle and $\angle ABC = 60°$, BMPN is cyclic.

We also noted that the circumcircle of triangle MBN has to pass through I, the incenter/circumcenter/centroid/orthocenter of triangle ABC.

So, the circumcenter of triangle MBN passes through two fixed points B and I.

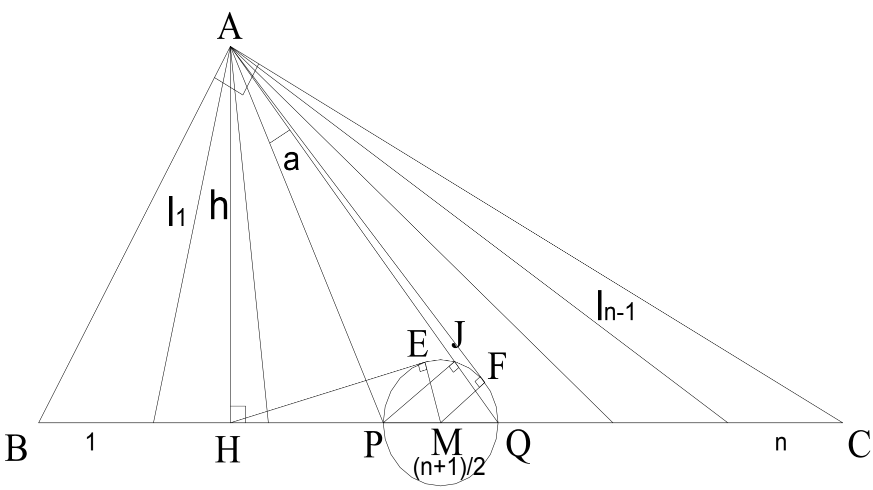
Thus the locus is on line *l*, the bisector of BI and *l* || AC, and is from E to F excluding points E and F where EF = AC, the length of the triangle ABC, since beyond those two points the circles do not cut the side of triangle ABC.

Problem 3 of the International Mathematical Olympiad 1960

In a given right triangle ABC; the hypotenuse BC, of length a, is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A, the segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove that

$$\tan\alpha = \frac{4nh}{(n^2 - 1)a}.$$

Solution



Let the two left and right segments meeting at A to make up angle α meet BC at P and Q, respectively. We have $PQ = \frac{a}{n}$. Let H be the foot of A to BC, and let AH be h.

From P draw a line to perpendicular to and meet AQ at J. Draw a circle with radius PQ and let its radius be r. From A draw AF to tangent the circle at F; from H draw AE to tangent the circle at E.

We have $\tan\alpha = \frac{PJ}{AJ} = \frac{PJ\times AQ}{AJ\times AQ} = \frac{PJ\times AQ}{AF^2} = \frac{PJ\times AQ}{AM^2 - r^2} =$

$$\frac{\mathrm{PJ}\times\mathrm{AQ}}{\mathrm{AH}^2+\mathrm{HM}^2-r^2}=\frac{\mathrm{PJ}\times\mathrm{AQ}}{\mathrm{AH}^2+\mathrm{HE}^2}=\frac{\mathrm{PJ}\times\mathrm{AQ}}{\mathrm{AH}^2+\mathrm{HP}\times\mathrm{HQ}}.$$

But PJ×AQ is twice the area of APQ, and it is also equal AH×PQ; we then have

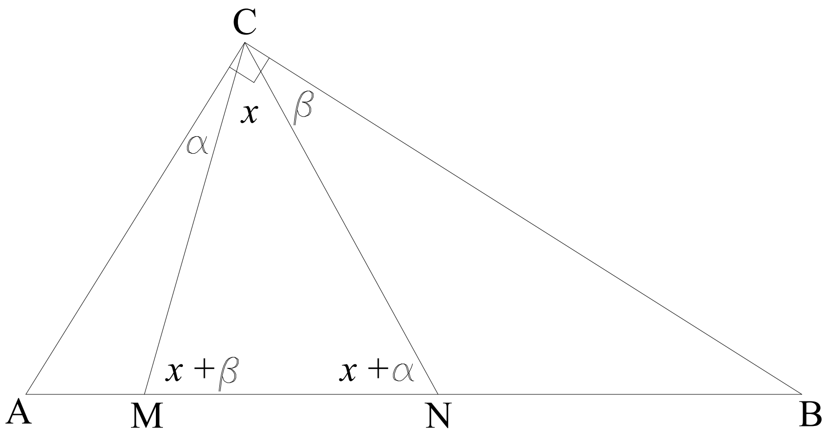
$$\tan\alpha=\frac{\mathrm{AH}\times\mathrm{PQ}}{\mathrm{AH}^2+\mathrm{HP}\times\mathrm{HQ}}=\frac{\mathrm{AH}\times\mathrm{PQ}}{\mathrm{AH}^2+(\mathrm{MH}-\frac{a}{2n})(\mathrm{MH}+\frac{a}{2n})}=$$

$$\frac{ha}{n(\mathrm{AH}^2+\mathrm{MH}^2-\frac{a^2}{4n^2})}=\frac{ha}{n(\mathrm{AM}^2-\frac{a^2}{4n^2})}=\frac{ha}{n(\frac{a^2}{4}-\frac{a^2}{4n^2})}=\frac{4nh}{(n^2-1)a}.$$

Problem 1 of Tournament of Towns 1993

Point M and N are taken on the hypotenuse of a right triangle ABC so that BC = BM and AC = AN. Prove that the angle MCN is equal to 45 degrees.

Solution



Let $\alpha = \angle ACM$, $\beta = \angle BCN$ and $x = \angle MCN$. Since BC = BM and AC = AN, both triangles ACN and BCM are isosceles with $\angle ACN = \angle ANC$ and $\angle BCM = \angle BMC$, or

$\angle ANC = x + \alpha$ and $\angle BMC = x + \beta$, and we have

$$\angle ANC = x + \alpha = \angle B + \angle BCN = \angle B + \beta \qquad \text{(i)}$$
$$\angle BMC = x + \beta = \angle A + \angle ACM = \angle A + \alpha \qquad \text{(ii)}$$

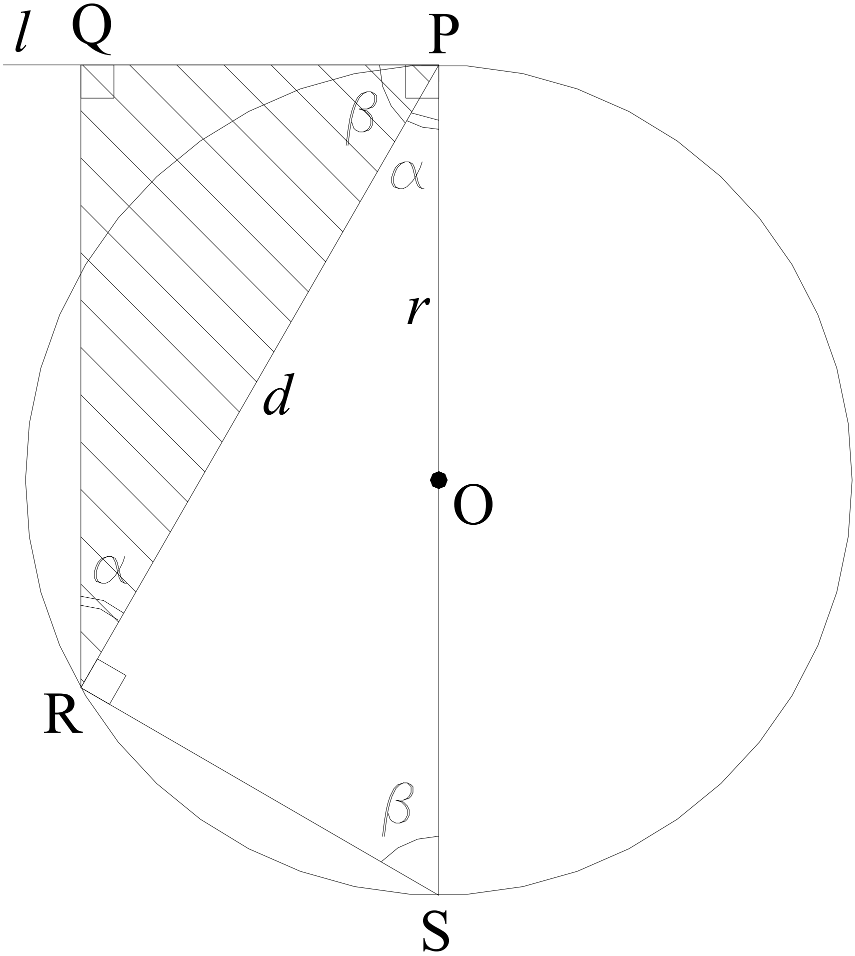
Substituting α from (i) into (ii), we get

$x + \beta = \angle A + \angle B + \beta - x$, or $2x = \angle A + \angle B = 180° - \angle ACB = 90°$, or $x = 45°$, and $\angle MCN = 45°$.

Problem 2 of the Canadian Mathematical Olympiad 1981

Given a circle of radius r and a tangent line l to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to l with Q on l. Determine the maximum of the area of triangle PQR.

Solution



Let O be the center of the circle. Link and extend PO to meet the circle at S. Now let $\beta = \angle QPR$ and $\alpha = \angle QRP$. We then also have $\beta = \angle$PSR and $\alpha = \angle$RPS. Let d = PR and denote (Ω) the area of shape Ω. We have (PQR) = $\frac{1}{2}$PQ×RQ.

But PQ = $d\sin\alpha$, RQ = $d\sin\beta$, and (PQR) = $\frac{1}{2}d^2\sin\alpha\sin\beta$ (i)

But in triangle PRS, $\sin\beta = \frac{d}{2r}$, and $\sin\alpha = \frac{RS}{2r}$.

Substituting them into (i), we have

$$(PQR) = \frac{d^3\times RS}{8r^2} \quad \text{(ii)}$$

With radius r being a constant, to find the maximum value of (PQR) we now need to find the maximum value of function $f(d) = d^3\times$RS, and we're stuck. But there's a trick. Recall that a function reaches its extreme (maximum or minimum) points when its derivative is zero?

However, the function $f(d)$ above has two variables d and RS. Now let's try to reduce it to a single variable d by relating RS to variable d and to eliminate RS. We have RS = $\sqrt{4r^2 - d^2}$ so now $f(d) = d^3\times \sqrt{4r^2 - d^2}$ and $f'(d)$ is the derivative of $f(d)$ with respect to the changing variable d, and it is the derivative of the product of two differentiable functions. We have the formula for derivative

$Dx(u\,.\,v) = u.\,Dxv + v.\,Dxu.$

Therefore, $f'(d) = [d^3\sqrt{4r^2 - d^2}\,]' = d^3[\sqrt{4r^2 - d^2}\,]' + [\sqrt{4r^2 - d^2}\,]\times (d^3)'$, but we also have $Dx(x^n) = nx^{n-1}$, and now

$$f'(d) = [\frac{1}{2}\frac{d^3}{\sqrt{4r^2 - d^2}}]\,Dd(4r^2 - d^2) + 3d^2\sqrt{4r^2 - d^2} =$$

$$[\frac{1}{2}\frac{d^3}{\sqrt{4r^2 - d^2}}](0 - 2d) + 3d^2\sqrt{4r^2 - d^2} = -\frac{d^4}{\sqrt{4r^2 - d^2}} + 3d^2\sqrt{4r^2 - d^2}\,.$$

The derivative $f'(d) = 0$ when $\frac{d^4}{\sqrt{4r^2 - d^2}} = 3d^2\sqrt{4r^2 - d^2}$, or when

$d^2 = 3(4r^2 - d^2)$, or when $d^2 = 3r^2$ or $d = r\sqrt{3}$.

We know the minimum of (PQR) occurs when it's a degenerate triangle either by having R at P (R ≡ P and $d = 0$) or R at S (R ≡ S and $d = 2r$) and (PQR) = 0. Neither is the case when $d = r\sqrt{3}$ when (PQR) is a maximum.

When $d = r\sqrt{3}$, $\sin\beta = \frac{d}{2r} = \frac{\sqrt{3}}{2}$, or $\beta = 60°$ as seen on the graph, and the maximum area of triangle PQR is

$$(PQR)_{max} = \frac{d^3\sqrt{4r^2 - d^2}}{8r^2} = \frac{(r\sqrt{3})^3\sqrt{4r^2 - d^2}}{\sqrt{8r^2}} = \frac{3}{2}\sqrt{\frac{3r^2}{2}}.$$

<u>Problem 2 of Canadian Mathematical Olympiad 1985</u>

Prove or disprove that there exists an integer which is doubled when the initial digit is transferred to the end.

<u>Solution</u>

Assume that there is such an integer N.

Let N = $n_0 n_1 n_2 \ldots n_{n-1} n_n$ ($n_0 \neq 0$) and $2n_0 n_1 n_2 \ldots n_{n-1} n_n =$

$n_1 n_2 \ldots n_{n-1} n_n n_0$. Since the number on the left is even, the units digit of the number on the right must also be even, or n_0 is an even digit. Expanding the above equation, we have

$2n_0 \times 10^n + 2n_1 \times 10^{n-1} + 2n_2 \times 10^{n-2} + \ldots + 2n_{n-1} \times 10 + 2n_n =$

$n_1 \times 10^n + n_2 \times 10^{n-1} + \ldots + n_{n-1} \times 102 + n_n \times 10 + n_0$.

Now regroup them all to get $n_0 (2\times10^n - 1) - 8n_1\times10^{n-1} - 8n_2 \times$

$10^{n-2} - \ldots - 8n_{n-1}\times10 - 8n_n = 0$ (i)

Since n_0 is even, $n_0 = 2, 4, 6$ or 8.

When $n_0 = 2$, divide the left side of (i) by 2, we have

$2\times10^n - 1 - 4n_1 \times 10^{n-1} - 4n_2 \times10^{n-2} - \ldots - 4n_{n-1} \times10 - 4n_n = 0$.

We see that the left side is now an odd number which is not zero, so $n_0 \neq 2$.

When $n_0 = 4$, dividing the left side of (i) by 4, we have

$2\times10^n - 1 - 2n_1 \times 10^{n-1} - 2n_2 \times 10^{n-2} - \ldots - 2n_{n-1} \times 10 - 2n_n = 0$.

Again the left side is an odd number and not zero, so $n_0 \neq 4$

When $n_0 = 6$ or 8, the second number $n_1 n_2 \ldots n_{n-1} n_n n_0$ has one more digit than the first number $n_0 n_1 n_2 \ldots n_{n-1} n_n$, so $n_0 \neq 6$ and $n_0 \neq 8$.

We conclude that there exists no integer which is doubled when the initial digit is transferred to the end.

Problem 2 of Canadian Mathematical Olympiad 1987

The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.

Solution

Converting the number xyz in base b to base 10, we have $xb^2 + yb + z = 1987$, but $x + y + z = 1 + 9 + 8 + 7 = 25$.

Subtracting the two equations, we have $(b - 1)[x(b + 1) + y] = 1987 - 25 = 1962 = 2\times3\times3\times109$, and all these numbers are prime.

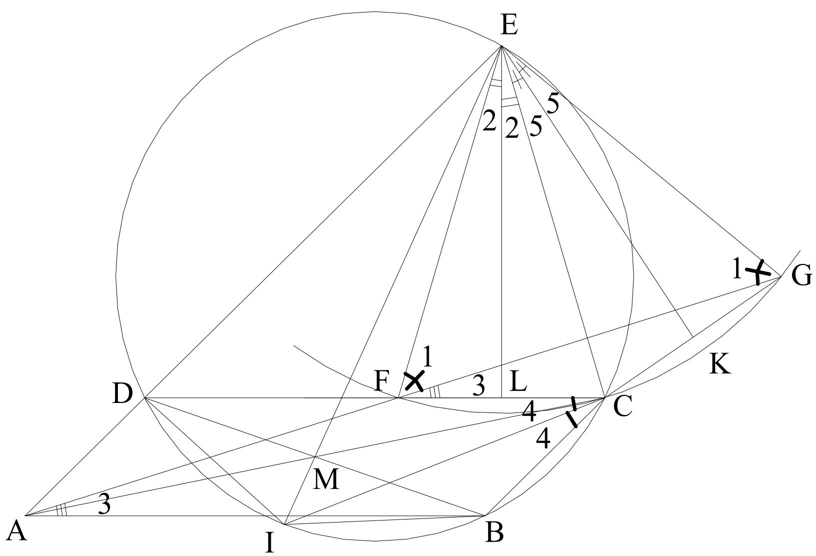
We can only find solution of $b - 1 = 19 - 1$, and $x(b + 1) + y = 5\times(19 + 1) + 9$.
$z = 25 - 5 - 9 = 11$ (or $z = \text{B}$).

Answer: $x = 5$, $y = 9$, $z = \text{B}$ and $b = 19$.

Problem 2 of the International Mathematical Olympiad 2007

Consider five points A, B, C, D and E such that ABCD is a parallelogram and BCED is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that EF = EG = EC. Prove that l is the bisector of angle DAB.

Solution



Based on Simson-Wallace's theorem the feet of projections of E down onto the three sides of triangle BCD are M, L and K are collinear as seen on the graph.

Since L and K are midpoints of FC and CG, respectively, therefore, LK || FG, and triangle ACG has LK intersect AC at its midpoint M. Therefore, M is also midpoint of DB since ABCD is a parallelogram.

Thus there is only one unique point M to satisfy conditions that M is on DB and also collinear with K and L, and M is also the foot of

E to DB. Extend EM to cut the circle at I. Since EI is the perpendicular bisector of DB because DM = BM it is understood that I is midpoint of arc DB. Therefore, $\angle DCI = \angle BCI = \angle 4$ as denoted on the graph. EI is also the diameter of the circle.

Therefore, $\angle ECI = 90°$.
$\angle ICB = \angle CEK$ (they both have sides perpendicular to each other), or $\angle 4 = \angle 5$ (i)

In triangle EFL: $\angle 1 + \angle 2 + \angle 3 = 90°$
In triangle EFG: $2\times(\angle 1 + \angle 2 + \angle 5) = 180°$, or
$\angle 1 + \angle 2 + \angle 5 = 180°$. Therefore, $\angle 3 = \angle 5$.

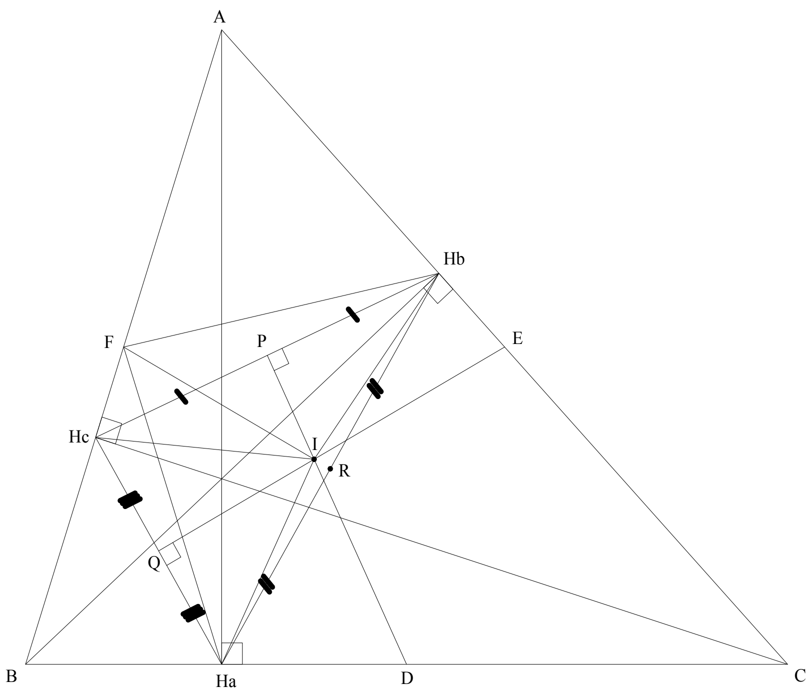
From (i), $\angle 3 = \angle 4$, or $\angle 3 = ½\angle DCB = ½\angle DAB$ or AG is bisector of $\angle DAB$ which is the answer.

Problem 4 of Austria Mathematical Olympiad 2009

Let D, E and F be the midpoints of the sides of the triangle ABC (D on BC, E on CA and F on AB). Further let HaHbHc be the triangle formed by the base points of the altitudes of the triangle ABC. Let P, Q and R be the midpoints of the sides of the triangle HaHbHc (P on HbHc, Q on HcHa and R on HaHb).

Show that the lines PD, QE and RF share a common point.

Solution



Let I be the intersection of DP and EQ. Since BHcC and BHbC are right triangles, BHcHbC is cyclic, and with D being the midpoint of diameter BC, DHc = DHb.

Combining with P being the midpoint of HbHc, we have DP $\perp$ HbHc.

Similarly, AHcHaC is cyclic, EHc = EHa and EQ $\perp$ HaHc. Also FR $\perp$ HaHb.

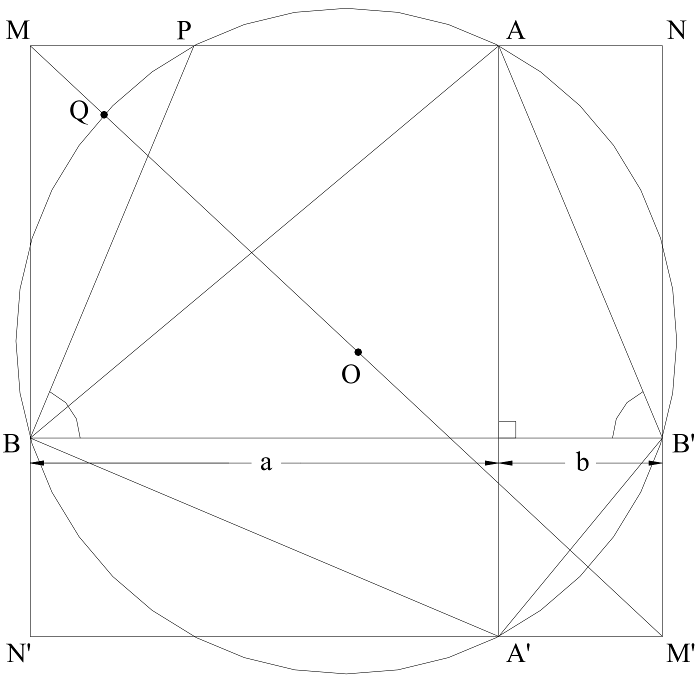
Since I is on DP and EQ and DP $\perp$ HbHc, EQ $\perp$ HaHc, we have IHb = IHc and IHa = IHc or IHa = IHb or IR $\perp$ HaHb since R is also the midpoint of HaHb.

Combining with FR $\perp$ HaHb, the three points F, I and R are collinear, or the lines PD, QE and RF share a common point I.

Problem 4 of Asian Pacific Mathematical Olympiad 1995

Let *C* be a circle with radius *R* and center O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles SAMB, SBN'A', SA'M'B', and SB'NA. Find the set of all points M, N', M', and N when A moves around the whole circle.

Solution



Let *r* be the radius of the circle, *a* = SB and *b* = SB'. Also let MA and MO intercept the circle at P and Q, respectively.

Now let MQ = *c*. Since MA || BA, we have BP = B'A, ∠PBB' = ∠AB'B, or ∠PBM = ∠AB'N, and triangle PBM = triangle AB'N. Therefore, MP = NA = SB' = *b*.

From point M outside the circle, we have MP×MA = MQ×(MQ +

$2r$), or $a \times b = c\,(c + 2r)$, or $c^2 + 2rc - ab = 0$, and we have $c = -\,r \pm \sqrt{R^2 + ab}$. Therefore, OM $= c + r = \sqrt{R^2 + ab}$.

The same proof can be used for other points N', M' and N; we have

OM = ON' = OM' = ON $= \sqrt{R^2 + ab}$.

Since S is a fixed point inside circle C, the product ab is fixed. From there we conclude that the set of all points M, N’, M’, and N when A moves around the whole circle is a circle with the radius

R = OM $= \sqrt{R^2 + ab}$.

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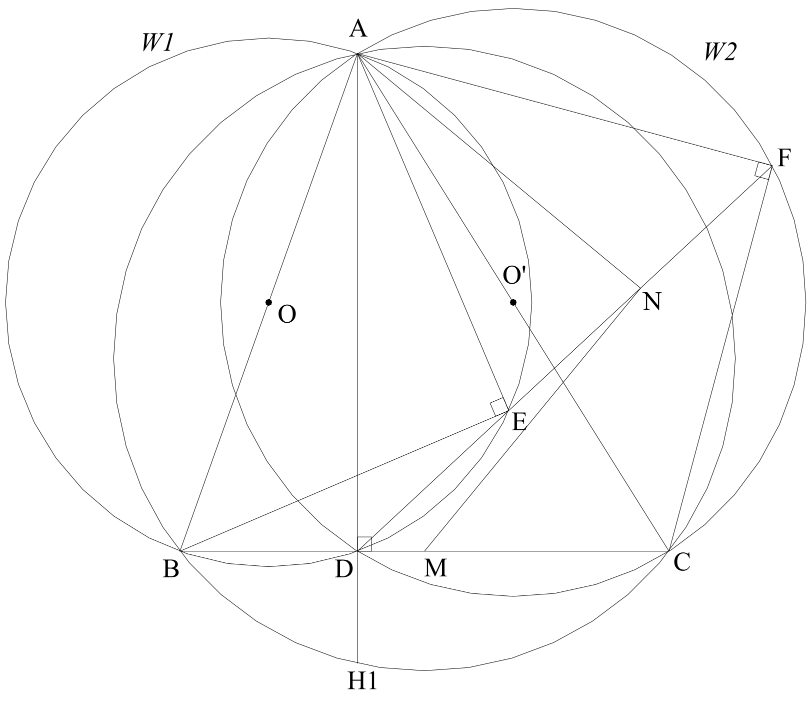
Problem 4 of Asian Pacific Mathematical Olympiad 1998

Let ABC be a triangle and D the foot of the altitude from A. Let E and F be on a line through D such that AE is perpendicular to BE, AF is perpendicular to CF, and E and F are different from D. Let M and N be the midpoints of the line segments BC and EF, respectively. Prove that AN is perpendicular to NM.

Solution



Let $w1$ and $w2$ be the circles with diameter AB and AC, respectively. E is on $w1$ because AE is perpendicular to BE and F is on $w2$ because AF is perpendicular to CF. Let AN intercept $w1$ at Q and $w2$ at P. Point N is outside $w1$, and we have NE×ND = NQ×NA since N is midpoint of EF, NF×ND = NQ×NA (i)
For $w2$, we have NF×ND = NP×NA (ii)
From (i) and (ii), NQ×NA = NP×NA, or NQ = NP.
Combining with MB = MC, we have MN || PC. Since AP is perpendicular to PC and MN || PC, AN is perpendicular to NM.

<u>Problem 4 of the Canadian Mathematical Olympiad 1970</u>

a) Find all positive integers with initial digit 6 such that the integer formed by deleting this 6 is $\frac{1}{25}$ of the original integer.
b) Show that there is no integer such that deletion of the first digit produces a result which is $\frac{1}{35}$ of the original integer.

<u>Solution</u>

a) Let $N = N_0N_1N_2 \ldots\ldots N_n$ ($n \longrightarrow$ infinity) be such integers.
$N_0 = 6$, and $N = 6N_1N_2 \ldots\ldots N_n$.
By deleting the initial digit 6, we have $M = N_1N_2 \ldots\ldots N_n$ and $\frac{N}{M} = 25$, and $N - M = 60 \ldots\ldots 0$ (n number of 0's) $= 24M$, or
$\frac{N-M}{24} = M = \frac{60\ldots0}{24} = \frac{6\times10\ldots0}{6\times4} = 10\ldots0/4 = 250\ldots0$ ($n - 2$ numbers 0's).

So N = 625, 6250, 62500, 625000, 6250000, 62500000, etc…

b) With the similar approach, assuming there are such integers $N = N_0N_1N_2 \ldots\ldots N_n$ ($n \longrightarrow$ infinity) where $N_0 = 1 \longrightarrow 9$ (integers), and $N = N_0N_1N_2 \ldots\ldots N_n$.
By deleting the initial digit, we have $M = N_1N_2 \ldots\ldots N_n$, $\frac{N}{M} = 35$, and $N - M = N0\ldots0$ (n numbers 0's) $= 34M$, or
$\frac{N-M}{34} = M = \frac{N0\ldots0}{34} = \frac{N_0\times10\ldots0}{34}$, and since N_0 takes on the integer values of $1 \longrightarrow 9$, $N_0\times10\ldots\ldots0$ is not divisible by 34.

Therefore, there are no such integers N as we assumed there were.

Problem 4 of Canadian Mathematical Olympiad 1971

Determine all real numbers a such that the two polynomials $x^2 + ax + 1$ and $x^2 + x + a$ have at least one root in common.

Solution

The roots for $x^2 + ax + 1 = 0$ are $x = \dfrac{-a \pm \sqrt{a^2 - 4}}{2}$, and the roots for $x^2 + x + a = 0$ are $x = \dfrac{-1 \pm \sqrt{1 - 4a}}{2}$.

Equating the roots of the two equations, we have

$-a \pm \sqrt{a^2 - 4} = -1 \pm \sqrt{1 - 4a}$, or $a^2 - 2a + 1 = a^2 - 4 + 1 - 4a \pm 2\sqrt{-4a^3 + a^2 + 16a - 4}$, or

$(2a + 4)^2 = 4(-4a^3 + a^2 + 16a - 4)$, or $a^3 - 3a + 2 = 0$ (i)

or $a^3 - 2a^2 + a + 2a^2 - 4a + 2 = (a + 2)(a^2 - 2a + 1) = (a + 2)(a - 1)^2 = 0$, or the solution for (i) are $a = -2$ and $a = 1$.

When $a = -2$

The roots for $x^2 + ax + 1 = x^2 - 2x + 1 = (x - 1)^2 = 0$ is $x = 1$.
The roots for $x^2 + x + a = x^2 + x - 2 = (x - 1)(x + 2) = 0$ are $x = 1$ and $x = -2$, so their common root is $x = 1$.

When $a = 1$

The roots for $x^2 + ax + 1 = x^2 + x + a = x^2 + x + 1$, so they are the same equation and their roots are $x = \dfrac{-1 \pm \sqrt{-3}}{2}$, and -3 is negative so they have no real roots.

Problem 6 of Tokyo University Entrance Exam 2010

Given a tetrahedron with four congruent faces such that OA = 3, OB = $\sqrt{7}$, AB = 2. Denote by *L* a plane which contains three points O, A and B.

a) Let H be the foot of the perpendicular drawn from the point C to the plane *L*. Express vector OH in terms of vectors OA and OB.
b) For a real number *t* with 0 < *t* < 1, let Pt, Qt be the points which divide internally the line segments OA, OB into *t* : 1 – *t*, respectively. Denote by *M* a plane which is perpendicular to the plane *L*. Find the sectional area S(*t*) of the tetrahedron OABC cut by the plane *M*.
c) When *t* moves in the range of 0 < *t* < 1, find the maximum value of S(*t*).

Solution

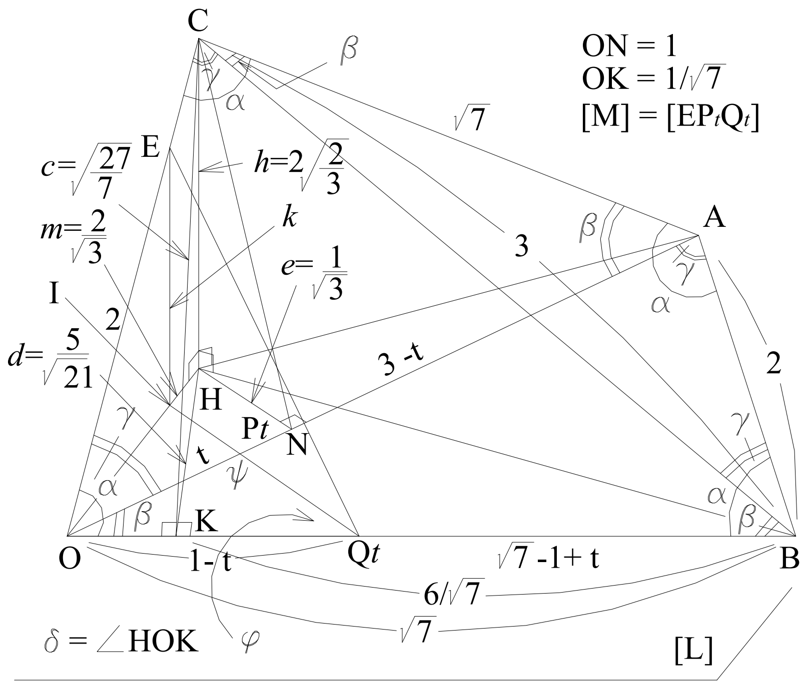


Figure 1 (not to scale)

Denote (Ω) the area of shape Ω and [Φ] the plane containing shape Φ. Let the angles facing sides with length 3 be α, angles facing sides with length 2 be β and angles facing sides with length $\sqrt{7}$ be γ. In other words,
$\alpha = \angle ABO = \angle ACO = \angle BAC = \angle BOC$,
$\beta = \angle ACB = \angle AOB = \angle CAO = \angle CBO$,
$\gamma = \angle ABC = \angle AOC = \angle BAO = \angle BCO$.

Now let's compute the angles α, β, and γ β based on the law of cosines on triangle ABO,
$\cos\alpha = (AB^2 + BO^2 - AO^2)/(2\times AB\times BO) = 1/(2\sqrt{7})$, or $\alpha = 79.11°$,
$\cos\beta = (AO^2 + BO^2 - AB^2)/(2\times AO\times BO) = 2/\sqrt{7}$, or $\beta = 40.89°$,
$\cos\gamma = (AB^2 + AO^2 - BO^2)/(2\times AB\times AO) = 1/2$, or $\gamma = 60°$.

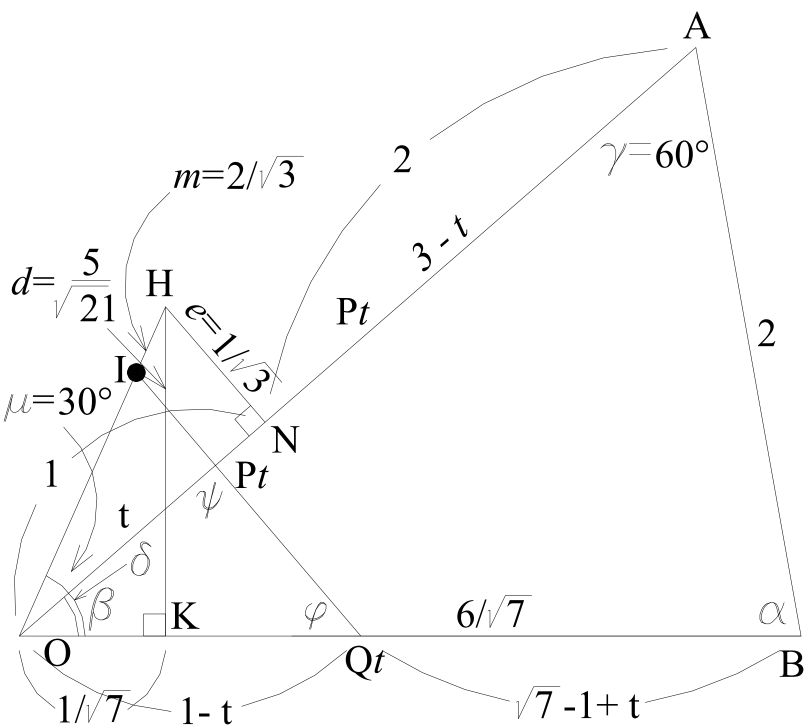


Figure 2 (not to scale)

Now let's look at the triangle ABO as shown on figure 2 above. Let P_tQ_t intercept OH at I, HK and HN be the altitudes from H onto OB and H onto OA, respectively. Let h = CH be the height of the tetrahedron, k = EI where E is the intersection of plane M with CO or E be the highest point of plane M cutting the tetrahedron, c = CK, d = HK, e = HN and m = OH.

We have $BK^2 + c^2 = BC^2$, or $BK^2 + c^2 = 9$ and

$$(OB - BK)^2 + c^2 = OC^2, \text{ or } (\sqrt{7} - BK)^2 + c^2 = 4.$$

These two equations give $BK = 6/\sqrt{7}$, $c = \sqrt{27/7}$, and $OK = 1/\sqrt{7}$. Similarly, $ON^2+CN^2 = 4$ and $(3-ON)^2+CN^2 = 7$, or $ON = 1$, $CN = \sqrt{3}$.

Now let $\delta = \angle HOK$, $\mu = \angle HON$ (now $\beta = \delta - \mu$), $\varphi = \angle P_tQ_tO$ and $\psi = \angle Q_tP_tO$. We have

$$\cos\beta = \cos(\delta + \mu) = \cos\delta\cos\mu - \sin\delta\sin\mu = \frac{1}{\sqrt{7m^2}} - \frac{de}{m^2} = \frac{2}{\sqrt{7}}.$$

However, in ΔKHO and ΔNHO, $m^2 = e^2 + 1 = d^2 + 1/7$, and the previous equation becomes $m^2(3m^2 - 4) = 0$, or $m = \dfrac{2}{\sqrt{3}}$. From this we have $e = 1/\sqrt{3}$, $d = 5/\sqrt{21}$, $h^2 = OC^2 - m^2$, or $h = 2\sqrt{2/3}$, $\cos\delta = \frac{1}{2}\sqrt{3/7}$ and $\cos\mu = \sqrt{3}/2$, or $\delta = 70.89°$ and $\mu = 30°$.

a) So the relationship between vector OH and OA, OB is that OH $= m = \dfrac{2}{\sqrt{3}}$ where $\mu = \angle AOH = 30°$ and $\delta = \angle BOH = 70.89°$.

b) In figure 1, [M] cuts the tetrahedron at E on OC, P_t on OA and Q_t on OB as shown. And on figure 2, as the author understands, $OP_t = t$ and $OQ_t = 1 - t$ (*if not, the reader can change these numbers accordingly*.)

Let's also assign $\varphi = \angle P_tQ_tO$ and $\psi = \angle Q_tP_tO$. We then obtain $P_tQ_t^2 = t^2 + (1 - t)^2 - 2t(1 - t)\cos\beta = 2t^2 - 2t + 1 - 4t(1 - t)/\sqrt{7} = 2t(t - 1)(1 + 2/\sqrt{7}) + 1$, or $P_tQ_t = \sqrt{2t(t - 1)(1 + 2/\sqrt{7}) + 1}$.

Applying the law of cosines to triangles P_tIO and Q_tIO, we get
$IP_t^2 = IO^2 + t^2 - 2IO\times t\times\cos\mu = IO^2 + t^2 - IO\times t\sqrt{3}$.
$IQ_t^2 = IO^2 + (1-t)^2 - 2IO(1-t)\cos\delta = IO^2 + (1-t)^2 - IO(1-t)\sqrt{3/7}$,
Solving these equations noting that $IP_t + P_tQ_t = IQ_t$, we get

$$\sqrt{IO^2 + t^2 - IO\times t\sqrt{3}} + \sqrt{2t(t-1)(1 + 2/\sqrt{7}) + 1} =$$
$$\sqrt{IO^2 + (1-t)^2 - IO(1-t)\sqrt{3/7}}.$$

Solving this equation for IO and letting the value of IO be v[t]. We have $\frac{k}{h} = \frac{OI}{OH}$, or $k = OI\sqrt{2} = \sqrt{2}v[t]$.

The area of triangle P_tEQ_t is
$(P_tEQ_t) = ½k\times P_tQ_t = ½\sqrt{2}v[t]\times\sqrt{2t(t-1)(1 + 2/\sqrt{7}) + 1}$, or
$S(t) = ½\sqrt{2}v[t]\times\sqrt{2t(t-1)(1 + 2/\sqrt{7}) + 1}$.

c) To find the maximum value of $S(t)$ when t moves in the range of $0 < t < 1$, we will need to take the derivative of $S(t)$ with respect to t and then find the local maximum.

$$S'(t)|_{0<t<1} = \{½\sqrt{2}v[t]\times\sqrt{2t(t-1)(1 + 2/\sqrt{7}) + 1}\}'|_{0<t<1}.$$

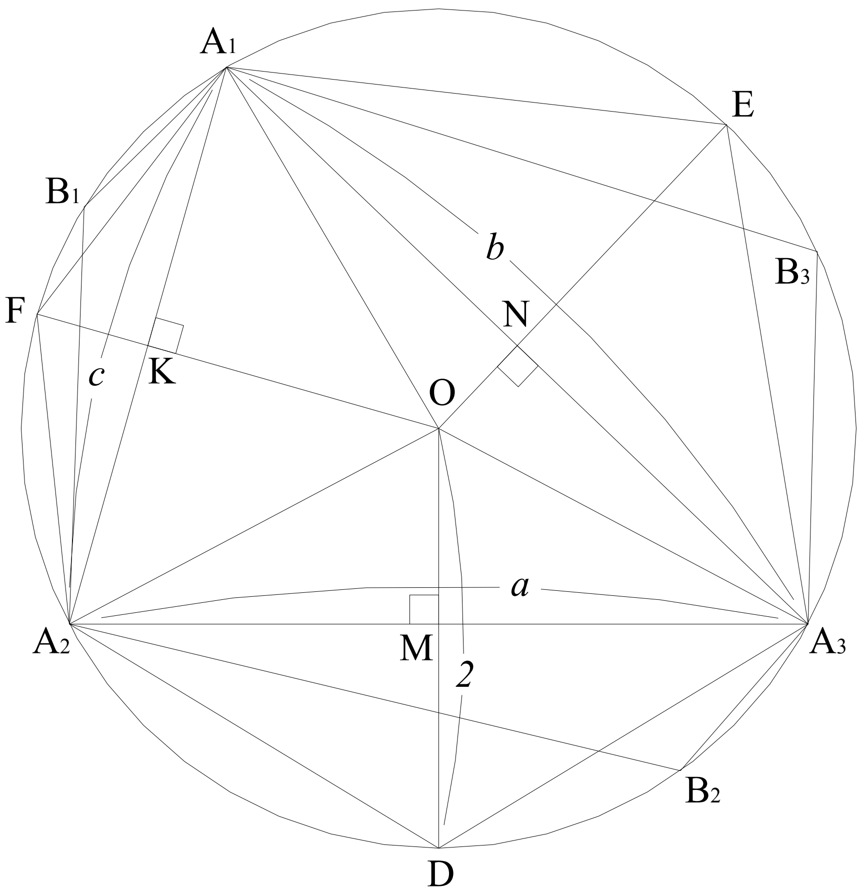
<u>Further observation</u>

The measurements in degrees of angles that are not apparently easily recognized are not really necessary, not required and are for reference only as contestants were not allowed calculator access. This solution shows the details to solve the problem. The reader is urged to carry out the calculations which prove to be lengthy. With the proper understanding of the problem, however, the equations could have been less complex than they really are.

Problem 22 of Tournament of Towns 2008

A 30-gon $A_1A_2 \ldots A_{30}$ is inscribed in a circle of radius of 2. Prove that one can choose a point B_k on the arc A_kA_{k+1} for $1 \leq k \leq 29$ and a point B_{30} on the arc $A_{30}A_1$, such that the numerical value of the area of the 60-gon $A_1B_1A_2B_2 \ldots A_{30}B_{30}$ is equal to the numerical value of the perimeter of the original 30-gon.

Solution



Let's work on a 3-gon (a triangle) then expand the idea to a 30-gon. Let O and r be the circumcenter and circumradius of ΔABC, a, b and c the side lengths of A_2A_3, A_1A_3 and A_1A_2, respectively, M, N and K the midpoints of A_2A_3, A_1A_3 and A_1A_2, respectively.

Denote (Ω) the area of shape Ω, and num[n] the numerical value of n.

Extend OM, ON and OK to intercept the circle at D, E and F, respectively.

We have num[$(A_1B_1A_2B_2A_3B_3)$] = $(A_1A_2A_3) + (A_1B_1A_2) + (A_2B_2A_3) + (A_1B_3A_3)$, or
$a + b + c = (A_1A_2A_3) + (A_1B_1A_2) + (A_2B_2A_3) + (A_1B_3A_3)$ (i),

but OD = OE = OF = r = 2, and we have

num[a] = $a\times 1 = a\times OD/2 = (A_2OA_3D)$, num[$b$] = $b\times 1 = (A_1OA_3E)$ and num[c] = (A_1OA_2F), and

$a + b + c = (A_2OA_3D) + (A_1OA_3E) + (A_1OA_2F) = (A_1A_2A_3) + (A_1FA_2) + (A_2DA_3) + (A_1EA_3)$, and equation (i) becomes

$(A_1A_2A_3) + (A_1FA_2) + (A_2DA_3) + (A_1EA_3) = (A_1A_2A_3) + (A_1B_1A_2) + (A_2B_2A_3) + (A_1B_3A_3)$, or

$(A_1FA_2) + (A_2DA_3) + (A_1EA_3) = (A_1B_1A_2) + (A_2B_2A_3) + (A_1B_3A_3)$

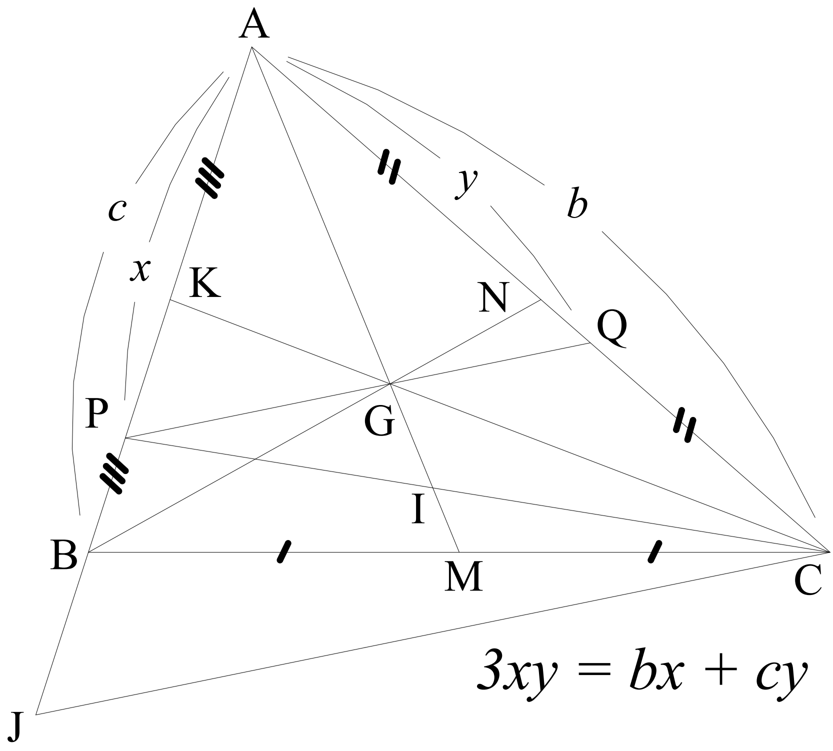
To satisfy the above equation, we need to choose point B_1 at F, B_2 at D and B_3 at E, the midpoints of arcs A_1A_2, A_2A_3 and A_1A_3, respectively. In such a case, the numerical value of the area of the hexagon $A_1B_1A_2B_2A_3B_3$ is equal to the numerical value of the perimeter of the triangle $A_1A_2A_3$.

This solution also applies to a 30-gon. Just choose points B_k at the midpoints of arcs A_kA_{k+1} for $1 \le k \le 29$ and point B_{30} at the midpoint of arc $A_{30}A_1$; the numerical value of the area of the 60-gon $A_1B_1A_2B_2...A_{30}B_{30}$ is then equal to the numerical value of the perimeter of the original 30-gon.

Problem 2 of British Mathematical Olympiad 1988

Points P, Q lie on the sides AB, AC respectively of triangle ABC and are distinct from A. The lengths AP, AQ are denoted by x, y respectively, with the convention that $x > 0$ if P is on the same side of A as B, and $x < 0$ on the opposite side; similarly for y. Show that PQ passes through the centroid of the triangle if and only if $3xy = bx + cy$ where $b = \text{AC}$, $c = \text{AB}$.

Solution



Let G be the centroid of triangle ABC, M, N and K the midpoints of BC, AC and AB, respectively. Let I be the intersection of AM and CP.

We now have PB = $c - x$, PK = BK – PB = $\frac{c}{2} - (c - x) = \frac{2x - c}{2}$.

Here, we will show that when PQ passes through the centroid of the triangle ABC then $3xy = bx + cy$.

Indeed, per Ceva's theorem when PQ passes through G, we have

$\frac{\text{AK}}{\text{PK}}\times\frac{\text{PI}}{\text{CI}}\times\frac{\text{CQ}}{\text{AQ}} = 1$, or $\frac{c}{2x-c}\times\frac{\text{PI}}{\text{CI}}\times\frac{\text{CQ}}{y} = 1$, or $c\times\text{CQ}\times\frac{\text{PI}}{\text{CI}} = 2xy - cy$, or

$$c\times\text{CQ}\times\frac{\text{PI}}{\text{CI}} = 2xy - cy \qquad \text{(i)}$$

or $c\times\text{CQ}\times\frac{\text{PI}}{\text{CI}} + xy = 3xy - cy$.

We're required to prove that $3xy = bx + cy$, or $3xy - cy = bx$. The assignment now is for us to prove that $c\times\text{CQ}\times\frac{\text{PI}}{\text{CI}} + xy = bx$, or

$x(b-y) = x\times\text{CQ} = c\times\text{CQ}\times\frac{\text{PI}}{\text{CI}}$, or $\frac{x}{c} = \frac{\text{PI}}{\text{CI}}$.

However, from (i), $\frac{\text{PI}}{\text{CI}} = \frac{2xy-cy}{c\times\text{CQ}}$, and it suffices to prove that

$$x = \frac{2xy-cy}{\text{CQ}}, \text{ or } \frac{x}{y} = \frac{2x-c}{\text{CQ}} \qquad \text{(ii)}$$

From C draw a segment to parallel PQ and intercept the extension of AB at J. We get $\frac{x}{y} = \frac{\text{PJ}}{\text{CQ}}$. (iii)

But since G is the centroid of triangle ABC, GC = 2GK, and because GP || CJ, $\frac{\text{PJ}}{\text{PK}} = \frac{\text{GC}}{\text{GK}} = 2$ and PJ = 2PK = $2x - c$, and (iii) is equivalent to (ii) and we're done.

The reverse direction is fairly straight-forward, and the reader is encouraged to prove it.

Problem 2 of Austria Mathematical Olympiad 1989

Find all triples (a, b, c) of integers with $abc = 1989$ and $a + b - c = 89$.

Solution

Note that $abc = 3\times3\times13\times17$ and we have all these possible combinations of (a, bc) which are
$(a, bc) = (-1, -1989), (-3, -663), (-9, -221), (-13, -153), (-17, -117), (-39, -51), (-51, -39), (-117, -17), (-153, -13), (-221, -9), (-663, -3), (-1981, -1)$, and on the positive side
$(1, 1989), (3, 663), (9, 221), (13, 153), (17, 117), (39, 51), (51, 39), (117, 17), (153, 13), (221, 9), (663, 3), (1981, 1)$.

When $a = 1$ and $bc = 1989$, $b - c = 88$, we get $b^2 - 88b - 1989 = 0$, and $b = 44 \pm \sqrt{3925}$, and b is not an integer because 3925 is not a perfect square.

Similarly, when $a = 3$ and $bc = 663$, $b - c = 86$, we get $b^2 - 86b - 663 = 0$. Following the same process we have $b = 43 \pm \sqrt{43^2 + 663}$, and b is not an integer since the units digit of $43^2 + 663$ is 2 which is different from the units digit of a perfect square of an integer which is either 0, 1, 4, 5, 6 or 9 (this is a quick way to verify if a number is a perfect square without having to carry out the calculation).

When $a = 9$ and $bc = 221$, $b - c = 80$, $b^2 - 80b - 221 = 0$;
when $a = 13$ and $bc = 153$, $b - c = 76$, $b^2 - 76b - 153 = 0$;
when $a = 17$ and $bc = 117$, $b - c = 72$, $b^2 - 72b - 117 = 0$, and there are no solutions in integers for the above three quadratic equations.

When $a = 39$ and $bc = 51$, $b - c = 50$, $b^2 - 50b - 51 = 0$, and $b = 25 \pm \sqrt{676} = 51$ or -1, and the solutions are $(a, b, c) = (39, 51, 1), (39, -1, -51)$.

When $a = 51$ and $bc = 39$, $b - c = 38$, $b^2 - 38b - 39 = 0$, and $b = 19 \pm \sqrt{400} = 39$ or -1, and the solutions are $(a, b, c) = (51, 39, 1), (51,$

-1, -39).

When $a = 117$ and $bc = 17$, $b - c = -28$, $b^2 + 28b - 17 = 0$;
when $a = 153$ and $bc = 13$, $b - c = -64$, $b^2 + 64b - 13 = 0$;
when $a = 221$ and $bc = 9$, $b - c = -132$, $b^2 + 132b - 9 = 0$;
when $a = 663$ and $bc = 3$, $b - c = -574$, $b^2 + 574b - 3 = 0$;
when $a = 1981$ and $bc = 1$, $b - c = -1892$, $b^2 + 1892b - 1 = 0$, and there are no solutions in integers for the above five quadratic equations.

When $a = -1$ and $bc = -1989$, $b - c = 90$, $b^2 - 90b + 1989 = 0$, and $b = 45 \pm 6 = 51$ or 39, and the solutions are $(a, b, c) = (-1, 51, -39)$, $(-1, 39, -51)$.

When $a = -3$ and $bc = -663$, $b - c = 92$, $b^2 - 92b + 663 = 0$;
when $a = -9$ and $bc = -221$, $b - c = 98$, $b^2 - 98b + 221 = 0$;
when $a = -13$ and $bc = -153$, $b - c = 102$, $b^2 - 102b + 153 = 0$;
when $a = -17$ and $bc = -117$, $b - c = 106$, $b^2 - 106b + 117 = 0$, and there are no solutions in integers for the above four equations.

When $a = -39$ and $bc = -51$, $b - c = 128$, $b^2 - 128b + 51 = 0$., and $b = 25 \pm \sqrt{676} = 51$ or -1, and the solutions are $(a, b, c) = (39, 51, 1)$, $(39, -1, -51)$.

When $a = -51$ and $bc = -39$, $b - c = 140$, $b^2 - 140b + 39 = 0$, and $b = 19 \pm \sqrt{400} = 39$ or -1, and the solutions are $(a, b, c) = (51, 39, 1)$, $(51, -1, -39)$.

When $a = -117$ and $bc = -17$, $b - c = 206$, $b^2 - 206b + 17 = 0$;
when $a = -153$ and $bc = -13$, $b - c = 242$, $b^2 - 242b + 13 = 0$;
when $a = -221$ and $bc = -9$, $b - c = 310$, $b^2 - 310b + 9 = 0$;
when $a = -663$ and $bc = -3$, $b - c = 752$, $b^2 - 752b + 3 = 0$;
when $a = -1981$ and $bc = -1$, $b - c = 2070$, $b^2 - 2070b + 1 = 0$, and there are no solutions in integers for the above five equations.

The solutions are $(a, b, c) = (-1, 51, -39)$, $(-1, 39, -51)$, $(39, 51, 1)$, $(39, -1, -51)$, $(51, 39, 1)$, $(51, -1, -39)$.

Problem 3 of Canadian Mathematical Olympiad 1975

For a positive number such as 3.27, 3 is referred to as the integral part of the number and .27 as the decimal part. Find a positive number such that its decimal part, its integral part, and the number itself form a geometric progression.

Solution

Let x and y be the integral and decimal part of the number to be found, respectively, and r the common ratio of the geometric progression. We are given $yr = x$ and $xr = x + y$.

Now divide the latter equation by the former one, side by side, to get $\frac{x}{y} = 1 + \frac{y}{x}$, but $\frac{x}{y} = r$ and we now have $r^2 - r - 1 = 0$, or $r = \frac{1 \pm \sqrt{5}}{2}$.

When $r = \frac{1 + \sqrt{5}}{2}$, note that since y is the decimal part and is always smaller than 1, $y < 1$ and $yr < r$, but $yr = x < r = \frac{1 + \sqrt{5}}{2} =$ 1.618. However, x is the integral part and is less than 1.62, it must be $x = 1$ ($x = 0$ is not acceptable), and now $y = \frac{x}{r} = \frac{2}{1 + \sqrt{5}} = .618$, and the whole number to be found is 1.618.

When $r = \frac{1 - \sqrt{5}}{2} < 0$, since y is the decimal part and is always non-negative, $yr = x$ causes x to be non-positive which is outside the scope of the problem.

Problem 1 of International Mathematical Talent Search Round 16

Prove that if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Solution

There exists a formula

$$a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc) + 3abc.$$

So if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

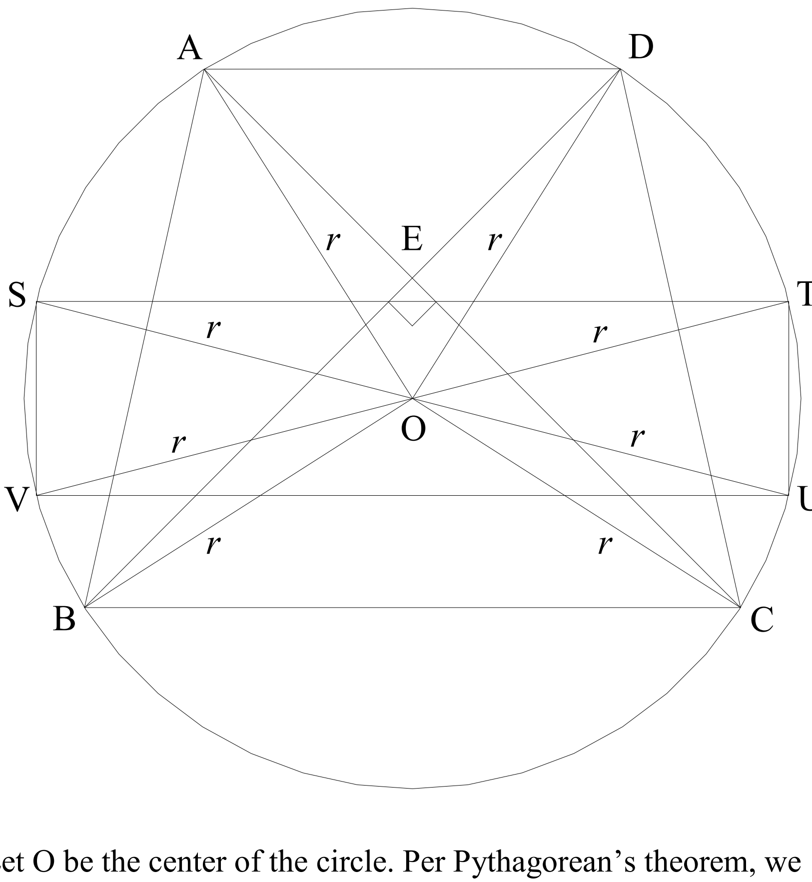
Problem 3 of British Mathematical Olympiad 1991

ABCD is a quadrilateral inscribed in a circle of radius r. The diagonal AC, BD meet at E. Prove that if AC is perpendicular to BD then

$$EA^2 + EB^2 + EC^2 + ED^2 = 4r^2 \qquad (*)$$

Is it true that if (*) holds then AC is perpendicular to BD? Give a reason for your answer.

Solution



Let O be the center of the circle. Per Pythagorean's theorem, we get $EA^2 + ED^2 = AD^2$ and $EB^2 + EC^2 = BC^2$. Now applying the law of cosines, we have

$AD^2 = OA^2 + OD^2 - 2OA\times OD\cos\angle AOD = 2r^2(1 - \cos\angle AOD)$.

Likewise, $BC^2 = 2r^2(1 - \cos\angle BOC)$.

But since $AC \perp BD$, together smaller arcs AD and BC subtend half the circle, or $\angle AOD + \angle BOC = 180°$, and $\cos\angle BOC = \cos(180° - \angle AOD) = -\cos\angle AOD$.

Therefore, $BC^2 = 2r^2(1 + \cos\angle AOD)$, and $EA^2 + EB^2 + EC^2 + ED^2 = 2r^2(1 - \cos\angle AOD) + 2r^2(1 + \cos\angle AOD) = 4r^2$.

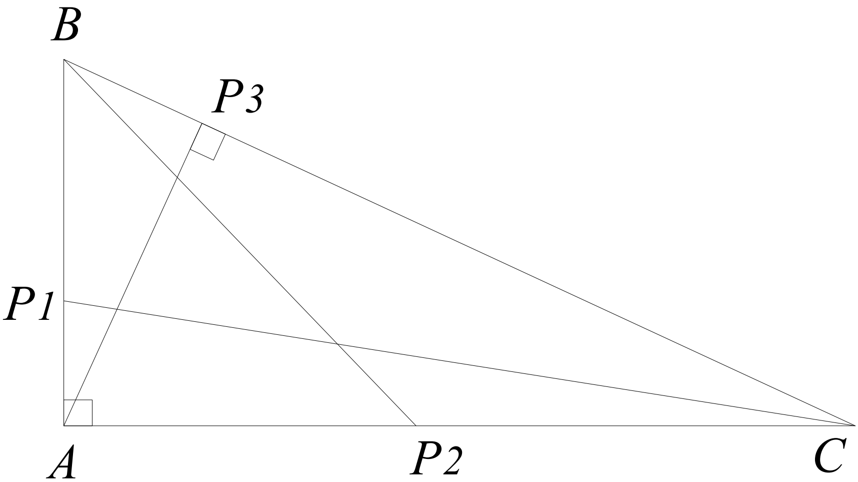
It is *not true* that if (*) holds then AC is perpendicular to BD. Let a rectangle STUV with $ST > TU$ being inscribed in the same circle. In this case $E \equiv$ (coincides with) O, AC is SU and BD is VT, $EA^2 + EB^2 + EC^2 + ED^2 = OS^2 + OT^2 + OU^2 + OV^2 = 4r^2$, but SU is not perpendicular to VT. If SU is perpendicular to VT, STUV would become a square and $ST = TU$ which is not the condition we assumed earlier.

Problem 3 of the British Mathematical Olympiad 2000

Triangle ABC has a right angle at A. Among all points P on the perimeter of the triangle, find the position of P such that

AP + BP + CP is minimized.

Solution



There are six possible positions to cover all the scenarios for point P: P is at point A, point B, point C, between AB, between AC, and between BC.

Case 1 When P is at point A (P coincides with A),
AP + BP + CP = AB + AC.

Case 2 When P is at point B (P coincides with B),
AP + BP + CP = AB + BC.

Case 3 When P is at point C (P coincides with B),
AP + BP + CP = AC + BC.
Since the shortest distance from C to AB is AC, for these three cases, AB + AC < AB + BC, and AB + AC < AC + BC, and cases 2 and 3 are eliminated.

Case 4 When P is between A and B, $AP + BP + CP = AB + CP_1 > AB + AC$. This case 4 is also eliminated.

Case 5 When P is between A and C, $AP + BP + CP = AC + BP_2 > AC + AB$. This case 5 is also eliminated.

Case 6 When P is between B and C, $AP + BP + CP =$ BC plus the distance from A to BC, but the minimum distance is the perpendicular AP_3, and $AP + BP + CP = BC + AP_3$. Now we're proving that $BC + AP_3 > AB + AC$ (i)

Indeed, assume (i) is a true statement. Let's square both sides; we have $BC^2 + AP_3^2 + 2\times BC\times AP_3 > AB^2 + AC^2 + 2\times AB\times AC$ (ii)

The Pythagorean's theorem gives us $BC^2 = AB^2 + AC^2$, (ii) becomes $AP_3^2 + 2\times BC\times AP_3 > 2\times AB\times AC$ (iii)

Observe that $AB\times AC$ is twice the area of triangle $ABC = BC\times AP_3$; the inequality (iii) then becomes $AP_3^2 > 0$, and this is a fact.

Therefore, this final case 6 is also eliminated, and the position of P such that $AP + BP + CP$ is minimized is when P is at point A.

Problem 9 of Canadian Mathematical Olympiad 1970

Let $f(n)$ be the sum of the first n terms of the sequence

0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, . . .

a) Give a formula for $f(n)$.
b) Prove that $f(s + t) - f(s - t) = st$ where s and t are positive integers and $s > t$.

Solution

a) Since $f(i) = 0$ does not belong to the sequence, $f(n + 1)$ is really $f(n)$ in a sense, so we have to subtract 1 from the n sequence, and have $f(n) = \frac{n-1}{2}(\frac{n-1}{2} + 1)$.
b) Therefore, $f(s + t) = \frac{s+t-1}{2} \times \frac{s+t-1}{2}$, and $f(s - t) = \frac{s-t-1}{2} \times \frac{s-t+1}{2} = st$.

Problem 1 of British Mathematical Olympiad 1988

Find all integers a, b, c for which

$(x - a)(x - 10) + 1 = (x + b)(x + c)$ for all x.

Solution

Expanding both sides and group them to get
$x^2 - (a + 10)x + 10a + 1 = x^2 + (b + c)x + bc$. Now by equating the same terms, we get $b + c = - a - 10$ and $bc = 10a + 1$, or
$b^2 + (a + 10)b + 10a + 1 = 0$.
Solving for b, we obtain $b = \frac{1}{2}(-a - 10 \pm \sqrt{a^2 - 20a + 96})$. So now for b to be an integer, $a^2 - 20a + 96 = (a - 10)^2 - 4$ must be a square of an integer. Let that integer be n, we must have
$(a - 10)^2 - 4 = n^2$, or $(a - 10)^2 = n^2 + 4$.

This only occurs when $(a - 10)^2 = 4$ and $n^2 = 0$, or $a - 10 = \pm 2$, or $(a, b, c) = (12, -11, -11), (8, -9, -9)$.

<u>Problem 1 of Canadian Mathematical Olympiad 1973</u>

a) Solve the simultaneous inequalities, $x < \frac{1}{4x}$ and $x < 0$; i.e, find a single inequality equivalent to the two given simultaneous inequalities.
b) What is the greatest integer which satisfies both inequalities $4x + 13 < 0$ and $x^2 + 3x > 16$?
c) Give a rational number between 11/24 and 6/13.
d) Express 100000 as a product of two integers neither of which is an integral multiple of 10.
e) Without the use of logarithm tables evaluate $1/\log_2 36 + 1/\log_3 36$.

<u>Solution</u>

a) Because $x < 0$ multiplying both sides of $x < \frac{1}{4x}$ by x, we get $x^2 > \frac{1}{4}$. Hence, $x > \frac{1}{2}$ or $x < -\frac{1}{2}$. However, $x < 0$; therefore, $x < -\frac{1}{2}$.

b) $4x + 13 < 0$ gives $x < -3.25$ and the greatest integer to satisfy this inequalities is -4. Now the derivative of $x^2 + 3x - 16$, or $(x^2 + 3x - 16)' = 0$ when $x = -\frac{3}{2}$ and the minimum value of $x^2 + 3x - 16$ occurs at $x = -\frac{3}{2}$, and $x^2 + 3x - 16 = 0$ when $x = \frac{1}{2}(-3 \pm \sqrt{73})$ or $x^2 + 3x > 16$ when $x > \frac{1}{2}(-3 + \sqrt{73})$ or when $x < -\frac{1}{2}(3 + \sqrt{73})$, and the greatest integer that is smaller than $-\frac{1}{2}(3 + \sqrt{73})$ is -6. Therefore, -6 is the greatest integer to satisfy both inequalities $4x + 13 < 0$ and $x^2 + 3x > 16$.

c) Let's convert $\frac{11}{24}$ and $\frac{6}{13}$ into ratios with the same denominators, we have $\frac{11}{24}=\frac{286}{624}$ and $\frac{6}{13}=\frac{288}{624}$. Therefore, the rational number is $\frac{287}{624}$ because $\frac{11}{24}=\frac{286}{624}<\frac{287}{624}<\frac{288}{624}=\frac{6}{13}$.

Another easy way to do this is by adding the numerators together to get the new numerator 11 + 6 = 17 and adding the denominators together to get the new denominator 24 + 13 = 37, and the rational number is $\frac{17}{37}$. We do have $\frac{11}{24}<\frac{17}{37}<\frac{6}{13}$.

d) 100000 = 5×2×5×2×5×2×5×2×5×2 = 32×3125.

e) $\frac{1}{\log_2 36}+\frac{1}{\log_3 36}=\frac{1}{\log_2(3\times3\times2\times2)}+\frac{1}{\log_3(3\times3\times4)}=\frac{1}{2+2\log_2 3}$ $+\frac{1}{2+2\log_3 2}=(2+\log_2 3+\log_3 2)\,/\,[2(1+\log_2 3)(1+\log_3 2)]=$

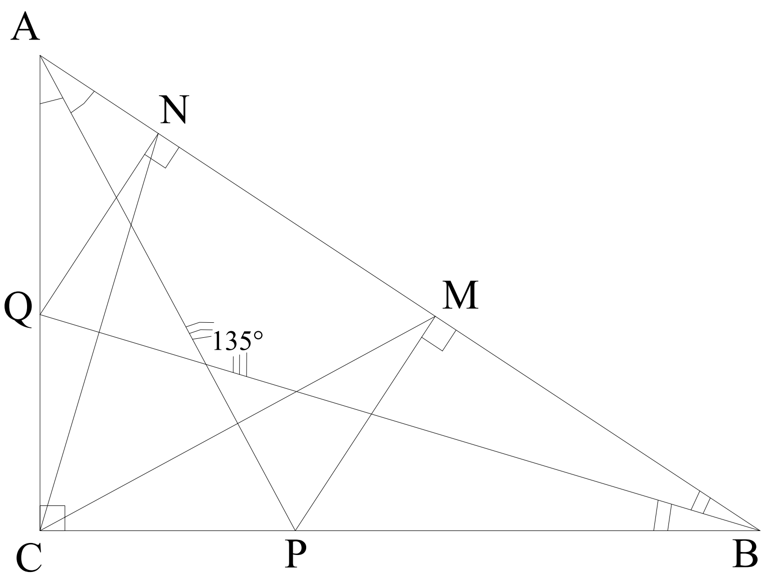
$(2+\log_2 3+\log_3 2)\,/\,[2(1+\log_2 3+\log_3 2+\log_2 3\times\log_3 2)]=$

$(2+\log_2 3+\log_3 2)\,/\,[2(2+\log_2 3+\log_3 2)]=\frac{1}{2}$.

Problem 4 of the British Mathematical Olympiad 1995

ABC is a triangle, right-angled at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q, respectively. M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

Solution



Extend NC to meet MP at I (not shown on graph). Since QN || PM (because both $\perp$ AB), $\angle$CNQ = $\angle$CIM.

Besides, $\angle$MCN = $\angle$CIM + $\angle$CMP, we then have

$$\angle MCN = \angle CNQ + \angle CMP \qquad \text{(i)}$$

Observe that ΔAPM ≡ (congruent to) ΔAPC, and
ΔBQC ≡ (congruent to) ΔBQN.

We then have AP $\perp$ CM and BQ $\perp$ CN.

AP $\perp$ CM results in $\angle$CMP = $\angle$MCP, and BQ $\perp$ CN results in $\angle$CNQ = $\angle$NCQ.

Equation (i) becomes $\angle$MCN = $\angle$NCQ + $\angle$MCP.

However, $\angle$MCN + $\angle$NCQ + $\angle$MCP = 90°.
Or, $\angle$MCN = 45°.

Further observation

We can prove CP = MP which results in $\angle CMP = \angle MCP$ by using a different method using the angle bisector AP.
Since AP is the angle bisector of $\angle BAC$, we have
$$\frac{CP}{PB} = \frac{AC}{AB}.$$

Furthermore, the two triangles ABC and PBM are similar making
$$\frac{MP}{PB} = \frac{AC}{AB}.$$

Those two previous equations give us CP = MP.
Similarly, CQ = NQ resulting in $\angle CNQ = \angle NCQ$.

Problem 1 of Hong Kong Mathematical Olympiad 2009 (Event 3)

Find the smallest prime factor of $101^{303} + 303^{101}$.

Solution

$101^{303} + 303^{101} = (101^{101})^3 + 101^{101}\times 3^{101} = 101^{101}[(101^{101})^2 + 3^{101}]$
$= 101\times 101^{100}[(101^{101})^2 + 3^{101}] = 3\times 37\times 101^{100}[(101^{101})^2 + 3^{101}]$.

Therefore, the smallest prime factor is 3.

Problem 1 of Hong Kong Mathematical Olympiad 2009 (Event 2)

$p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{10} + 2^{11}$, find the value of p.

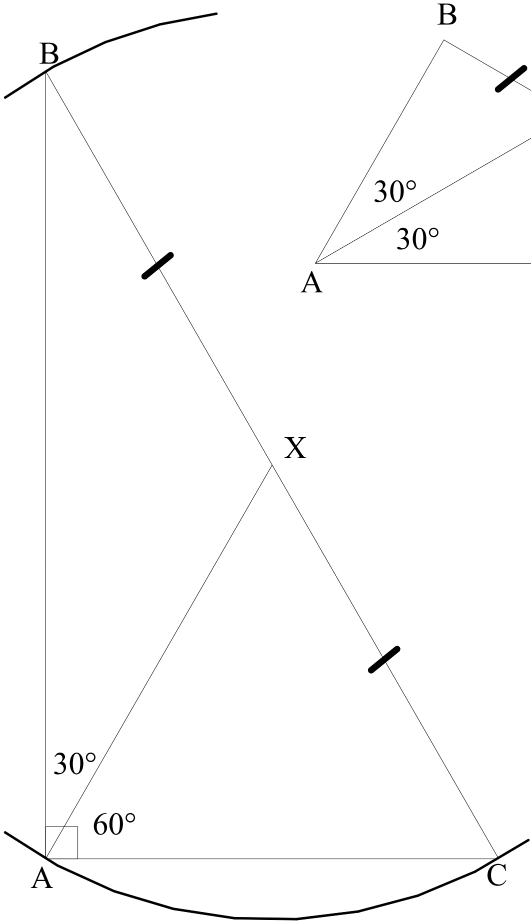
Solution

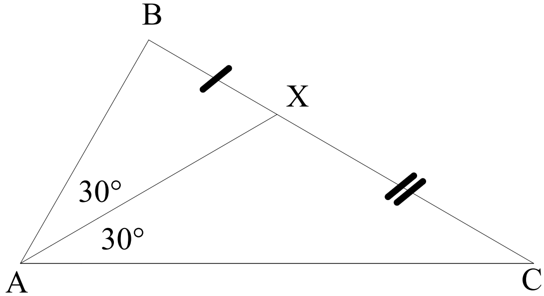
We have $-2^{10} = -2\times2^{10} + 2^{10} = -2^{11} + 2^{10}$, and $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 - 2^{11} + 2^{10} + 2^{11} = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 + 2^{10}$.
Similarly, $-2^9 = -2\times2^9 + 2^9 = -2^{10} + 2^9$, and we now get $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^9 + 2^{10} = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^8 - 2^{10} + 2^9 + 2^{10} = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^8 + 2^9$.
Continue with $-2^8 = -2\times2^8 + 2^8 = -2^9 + 2^8$, and p becomes $p = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^8 + 2^9 = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^7 - 2^9 + 2^8 + 2^9 = 2 - 2^2 - 2^3 - 2^4 - \ldots - 2^7 + 2^8$.
Now $-2^7 = -2\times2^7 + 2^7 = -2^8 + 2^7$, and $p = 2 - 2^2 - 2^3 - 2^4 - 2^5 - 2^6 + 2^7$.
Next, $-2^6 = -2\times2^6 + 2^6 = -2^7 + 2^6$, and $p = 2 - 2^2 - 2^3 - 2^4 - 2^5 - 2^6 + 2^7 = 2 - 2^2 - 2^3 - 2^4 - 2^5 - 2^7 + 2^6 + 2^7 = 2 - 2^2 - 2^3 - 2^4 - 2^5 + 2^6$. We now proceed with $-2^5 = -2\times2^5 + 2^5 = -2^6 + 2^5$, and $p = 2 - 2^2 - 2^3 - 2^4 + 2^5$.
Finally, with $-2^4 = -2\times2^4 + 2^4 = -2^5 + 2^4$, $p = 2 - 2^2 - 2^3 - 2^4 + 2^5 = 2 - 2^2 - 2^3 - 2^5 + 2^4 + 2^5 = 2 - 2^2 - 2^3 + 2^4 = 2 - 4 - 8 + 16 = 6$.

<u>Problem 2 of the British Mathematical Olympiad 1994</u>

In triangle ABC the point X lies on BC.
a) Suppose that $\angle$BAC = 90°, that X is the midpoint of BC, and that $\angle$BAX is one third of $\angle$BAC. What can you say and prove about triangle ACX?
b) Suppose that $\angle$BAC = 60°, that X lies one third of the way from B to C, and that AX bisects $\angle$BAC. What can you say and prove about triangle ACX?

<u>Solution</u>





(ii)

a) Since $\angle$BAC = 90°, ΔABC can be circumcircled by a circle that has BC as its diameter and X its circumcenter.

Therefore, XA = XB = XC and ΔACX is an isoceles triangle. Combining with $\angle BAX = \frac{1}{3}\angle BAX = 30°$, $\angle CAX = 60°$, and ΔACX becomes an equilateral triangle.

b) Applying the law of sines for ΔABX and ΔACX, we have

$\frac{BX}{\sin 30^\circ} = \frac{AX}{\sin\angle B}$, and $\frac{CX}{\sin 30^\circ} = \frac{AX}{\sin\angle C}$, respectively.

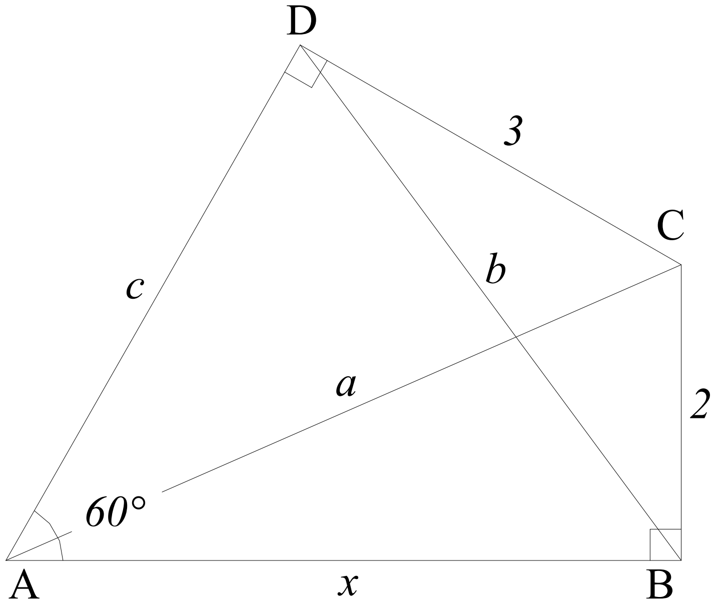
But CX = 2BX; we then have $2\sin\angle C = \sin\angle B = \sin(120° - \angle C)$, or $2\sin\angle C = \sin 120°\cos\angle C - \cos 120°\sin\angle C = \frac{\sqrt{3}}{2}\cos\angle C + \frac{1}{2}\sin\angle C$, or $\sqrt{3}\sin\angle C = \cos\angle C$, or $\tan\angle C = \frac{\sqrt{3}}{3}$, or $\angle C = 30°$.

And ΔACX is an isosceles triangle with $\angle C = \angle CAX = 30°$.

Problem 3 of Hong Kong Mathematics Olympiad 2009

In the figure below, if $\angle A = 60°$, $\angle B = \angle D = 90°$, BC = 2, CD = 3 and AB = x. Find the value of x.

Solution



Let a = AC, b = BD and c = AD. Since the sum of two angles $\angle B$ and $\angle D$ is 180°, ABCD is cyclic and $\angle C = 180° - \angle A = 120°$.

Applying the Ptolemy's theorem to a cyclic quadrilateral ABCD, we get $3x + 2c = ab$. But $a^2 = x^2 + 4 = c^2 + 9$, or $a = \sqrt{x^2 + 4}$, $c^2 = x^2 - 5$ and $c = \sqrt{x^2 - 5}$.
Now applying the law of cosines, we get $b^2 = BC^2 + CD^2 - 2BC\times CD\cos\angle C = 19$, or $b = \sqrt{19}$.

Substituting the values of c, a, and b into $3x + 2c = ab$ to get $3x^4 - 52x^2 - 256 = 0$. Solve for x^2 and we have $x^2 = 64/3$, or $x = 8/\sqrt{3}$.

Problem 7 of Canadian Mathematical Olympiad 1975

A function $f(x)$ is periodic if there is a positive number p such that $f(x + p) = f(x)$ for all x. For example, $\sin x$ is periodic with period 2π. Is the function $\sin(x^2)$ periodic? Prove your assertion.

Solution

Assuming the function $\sin(x^2)$ is periodic, we then have $\sin(x^2 + \theta) = \sin(x^2)$ where θ is a fixed angle. Applying the formula for sine of a sum of two angles, we get

$\sin(x^2 + \theta) = \sin(x^2)\cos\theta + \cos(x^2)\sin\theta$, and now
$\sin(x^2)\cos\theta + \cos(x^2)\sin\theta = \sin(x^2)$, or
$\sin(x^2)(\cos\theta - 1) = -\cos(x^2)\sin\theta$, or
$\sin(x^2)(\cos\theta - \cos 0°) = -\cos(x^2)\sin\theta$.

But $\cos\theta - \cos 0° = -2\sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$. The above equation then becomes

$\sin(x^2)\sin^2\frac{\theta}{2} = \cos(x^2)\sin\frac{\theta}{2}\cos\frac{\theta}{2}$, or

$\sin(x^2)\sin\frac{\theta}{2} = \cos(x^2)\cos\frac{\theta}{2}$, or

$\tan\frac{\theta}{2} = \cot(x^2)$.

From there, $\frac{\theta}{2} = \tan^{-1}[\cot(x^2)]$, or

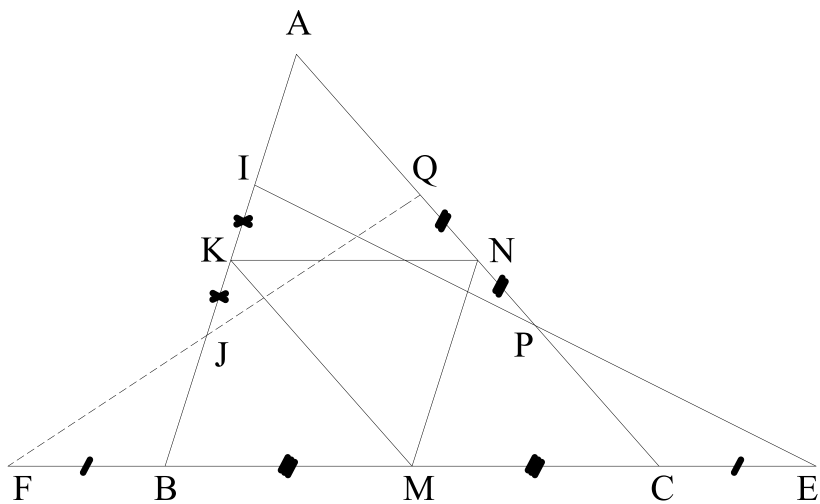
$\theta = 2\tan^{-1}[\cot(x^2)]$ which is a function and not a fixed angle.

Therefore, the function $\sin(x^2)$ is not periodic.

Problem 3 of Austria Mathematical Olympiad 1985

A line meets the lines containing sides BC, CA, AB of a triangle ABC at E, P, I, respectively. The points F, Q, J are symmetric to E, P, I with respect to the midpoints of BC, CA, AB, respectively. Prove that F, Q and J are collinear.

Solution



According to Menelaus' theorem, the three given collinear points E, P and I give us $\frac{EB}{EC}\times\frac{CP}{AP}\times\frac{AI}{BI}=1$.

But because the points F, Q, J are symmetric to E, P, I with respect to the midpoints of BC, CA, AB, respectively, we have EB = FC, EC = FB, CP = AQ, AP = CQ, AI = BJ, BI = AJ. Now replace all the segments on the above equation with their counterparts to get

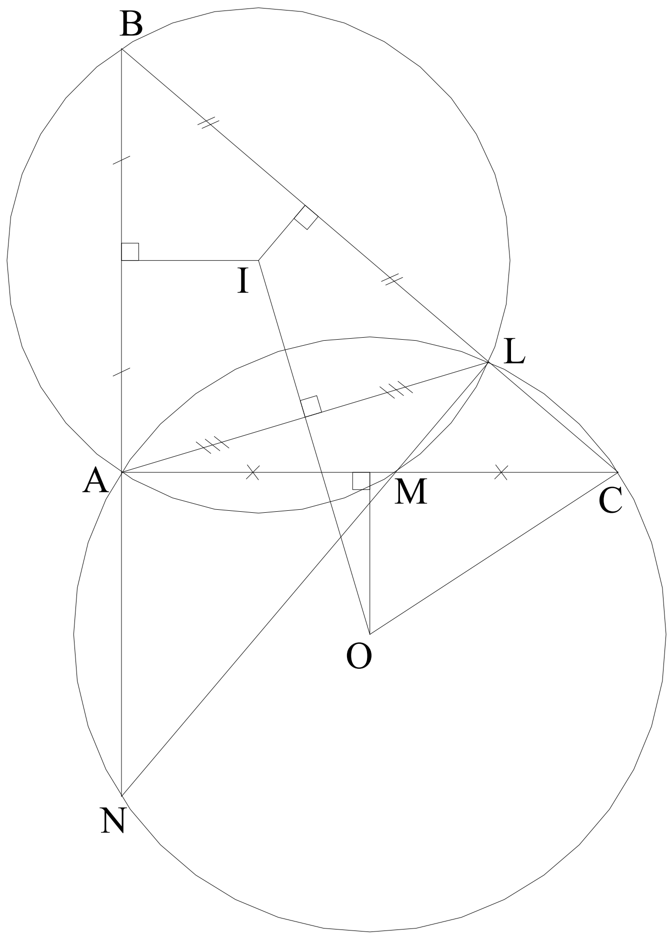
$$\frac{FC}{FB}\times\frac{AQ}{CQ}\times\frac{BJ}{AJ}=1.$$

And again, per Menelaus' theorem, the previous equation implies that the three points F, Q and J are collinear.

Problem 3 of British Mathematical Olympiad 2010

Let ABC be a triangle with ∠CAB a right-angle. The point L lies on the side BC between B and C. The circumcircle of triangle ABL meets the line AC again at M and the circle of triangle CAL meets the line AB again at N. Prove that L, M and N lie on a straight line.

Solution



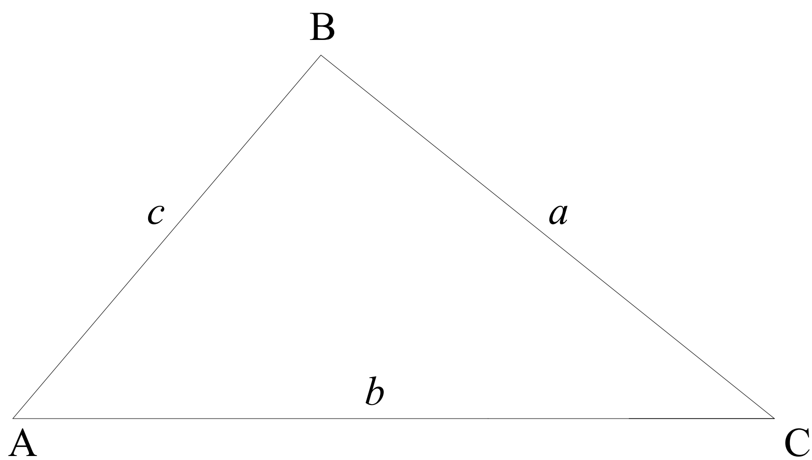
Since ABLM and ALCN are cyclic, ∠BLM = 180° – ∠CAB = 90° and ∠CLN = ∠CAN = 90° = ∠BLM, but B, L and C are on a straight line; therefore, L, M and N are also on a straight line.

Problem 1 of the Uzbekistan Mathematical Olympiad 2008

Let triangle ABC with AB = c, AC = b, BC = a, R the circum-radius, p the half perimeter of triangle ABC.

If $\frac{a\cos A + b\cos B + c\cos C}{a\sin A + b\sin B + c\sin C} = \frac{p}{9R}$ then find the value of cosA.

Solution



The problem does not specify whether we need to find the value of cosA as a function of the other parameters of the triangle or the value of cosA as a specific number. Let's find the value of cosA as a number.

Assuming $\angle B = 90°$, $\cos\angle B = 0$ and $\sin\angle B = 1$, and the left side of the equation in the problem becomes $\frac{a\cos A + b\cos B + c\cos C}{a\sin A + b\sin B + c\sin C} = \frac{a\cos A + c\cos C}{a\sin A + b + c\sin C}$. Note that because $\angle A + \angle C = 90°$, $\cos C = \sin A$ and $\sin C = \cos A$, and the equation of the problem is now equivalent to $\frac{a\cos A + c\sin A}{a\sin A + b + c\cos A} = \frac{p}{9R}$.

Now substituting $\sin A = \frac{a}{b}$, $\cos A = \frac{c}{b}$, $p = \frac{a+b+c}{2}$ and $R = \frac{a}{2\sin A}$

$= \frac{b}{2}$ into the previous equation to get $\frac{2ac}{a^2 + b^2 + c^2} = \frac{a + b + c}{9b}$, or

$18abc = (a + b + c)(a^2 + b^2 + c^2)$.

The Pythagorean theorem gives us $a^2 = b^2 - c^2$ or $a = \sqrt{b^2 - c^2}$, and the above equation is now equivalent to

$9c\sqrt{b^2 - c^2} = b(b + c + \sqrt{b^2 - c^2})$. Now dividing both sides by $\sqrt{b^2 - c^2}$ to get $9c = b(\frac{b + c}{\sqrt{b^2 - c^2}} + 1) = b(\frac{b + c}{\sqrt{(b + c)(b - c)}} + 1) = b(\sqrt{\frac{b + c}{b - c}} + 1)$. Again, divide both sides by b, and we have

$\frac{9c}{b} = 1 + \sqrt{\frac{b + c}{b - c}} = 1 + 1\sqrt{\frac{1 + \frac{c}{b}}{1 - \frac{c}{b}}}$, or $9\cos A - 1 = \sqrt{\frac{1 + \cos A}{1 - \cos A}}$, or $(9\cos A - 1)^2 = \frac{1 + \cos A}{1 - \cos A}$, or

$81\cos^2 A - 99\cos A + 20 = 0$. Solving for cosA and we obtain $\cos A = \frac{11 \pm \sqrt{41}}{18}$. We conclude that there are at least two values of cosA if $\angle B = 90°$ to satisfy the requirement by the problem that $\frac{a\cos A + b\cos B + c\cos C}{a\sin A + b\sin B + c\sin C} = \frac{p}{9R}$.

Problem 3 of the Irish Mathematical Olympiad 2001

Prove that if an odd prime number p can be expressed in the form $x^5 - y^5$, for some integers x, y, then $\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2}$ for some odd integer v.

Solution

Note that $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$. Because p is an odd prime number, there are two possibilities when either $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$ and $p = x - y$ is the odd prime number, or $x - y = 1$ and $p = x^4 + x^3y + x^2y^2 + xy^3 + y^4$ is the odd prime number itself.

Let's consider the case when $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1$ and $p = x - y$ is the odd prime number. But $x - y > 1$, or $x > y$, and $(x, y) = (1, 0)$ satisfies the equation $\sqrt{\frac{4p+1}{5}} = \frac{v^2+1}{2}$. However, $p = x - y = 1$ is not a prime number as defined.

If $y \neq 0$, expand the expression $x^4 + x^3y + x^2y^2 + xy^3 + y^4$ to get $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = (x + y)^4 - xy[3(x + y)^2 - xy]$.

Now there two scenarios for x and y: both having the same sign (positive or negative) or opposite signs (one positive and the other negative or vice-versa).

When both x and y are having the same sign, every term on the left side of the previous equation is positive and its left side is greater than 1.

On the other hand, when both x and y are having the opposite signs, $-xy$ is positive and the right side of the previous equation is also positive and is greater than 1. Therefore, the case when $x^4 + x^3 y + x^2y^2 + xy^3 + y^4 = 1$ is not possible.

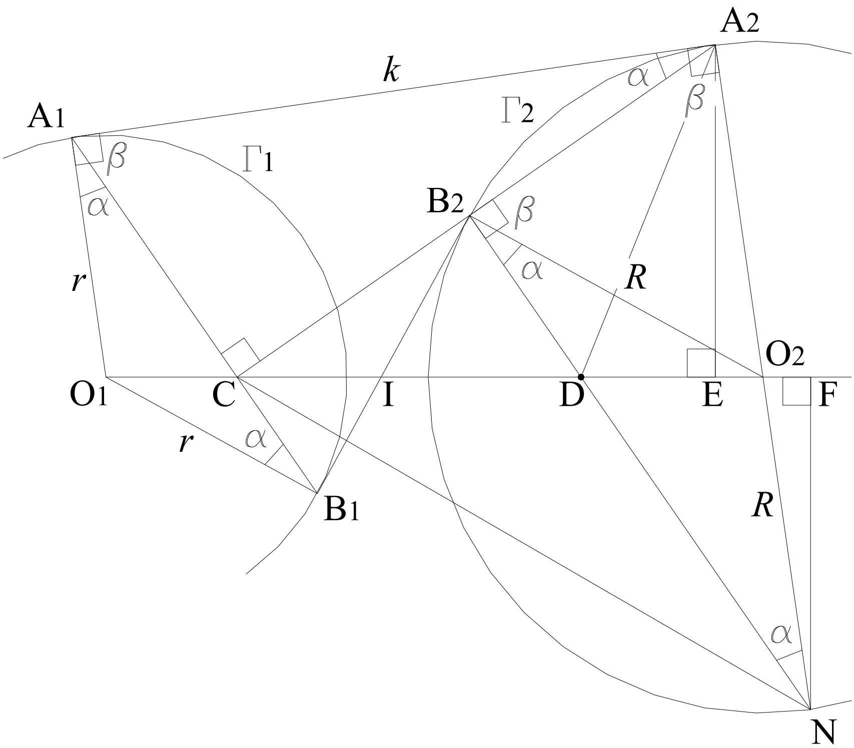
Now we consider the other case when $x - y = 1$, or $x = y + 1$, and $p = x^4 + x^3y + x^2y^2 + xy^3 + y^4 = (y + 1)^4 + (y + 1)^3y + (y + 1)^2y^2 + (y + 1)y^3 + y^4 = 5y^4 + 10y^3 + 10y^2 + 5y + 1$, and $\frac{4p + 1}{5} = 4(y^4 + 2y^3 + 2y^2 + y) + 1$, or $\sqrt{\frac{4p + 1}{5}} = \sqrt{4(y^4 + 2y^3 + 2y^2 + y) + 1} = \frac{v^2 + 1}{2}$, or $16(y^4 + 2y^3 + 2y^2 + y) + 4 = (v^2 + 1)^2$.

But v is some odd integer; let $v = 2n + 1$ where n is an integer. We then have
$16(y^4 + 2y^3 + 2y^2 + y) + 4 = [(2n + 1)^2 + 1]^2 = 16(n^4 + 2n^3 + 2n^2 + n) + 4$, or $n = y$, and $v = 2y + 1$ which is an odd integer.

Problem 3 of Poland Mathematical Olympiad 2008

Disjoint circles Γ_1 and Γ_2, with O_1, O_2 as their respective centers, are tangent to the line k at A_1, A_2. The point C on the segment O_1O_2 satisfies $\angle A_1CA_2 = 90°$. Let B_1 be the second intersection point of A_1C with Γ_1 and B_2 be the second intersection point of A_2C with Γ_2. Prove that B_1B_2 is tangent to the circles Γ_1, Γ_2.

Solution



Extend A_2O_2 to meet Γ_2 at N. Link B_2N to meet O_1O_2 at D. Draw the altitudes A_2E and NF to O_1O_2. Let $\alpha = \angle O_1A_1B_1 = \angle O_1B_1A_1$, $\beta = \angle CA_1A_2$ ($\alpha + \beta = 90°$), r and R be the radii of Γ_1 and Γ_2, respectively. Also denote (Ω) the area of shape Ω.

Since $O_1A_1 \perp A_1A_2$ and $A_1C \perp A_2C$, $\alpha = \angle A_1A_2C$, and also since

$NA_2 \perp A_1A_2$ and NA_2 is the diameter of Γ_2, $NB_2 \perp A_2B_2$ which causes $A_1B_1 \parallel NB_2$ and $\alpha = \angle O_2NB_2 = \angle O_2B_2N$. Because $\angle A_1O_1B_1 = 180° - 2\alpha = \angle NO_2B_2$ and $O_1A_1 \parallel O_2N$, $O_1B_1 \parallel O_2B_2$.

The parallel segments cause these triangles to be similar $\Delta A_1O_1B_1 \cong$ (similar to) ΔNO_2B_2 and $\Delta CO_1B_1 \cong \Delta DO_2B_2$ which give rise to the ratios

$$\frac{A_1B_1}{NB_2} = \frac{O_1B_1}{O_2B_2} = \frac{r}{R} \text{ and } \frac{CB_1}{DB_2} = \frac{O_1B_1}{O_2B_2} = \frac{r}{R}, \text{ or } \frac{A_1B_1}{NB_2} = \frac{CB_1}{DB_2} = \frac{A_1B_1 - CB_1}{NB_2 - DB_2} = \frac{CA_1}{ND}, \text{ or } \frac{DB_2}{ND} = \frac{CB_1}{CA_1}.$$

Also note that $\Delta A_2EO_2 = \Delta NFO_2$ (similar triangles with $A_2O_2 = NO_2 = R$). Therefore, $A_2E = NF$, and $A_2E \times CD = NF \times CD$, or $(A_2CD) = (NCD)$.

However, $(A_2CD) = ½DB_2 \times CA_2$ and $(NCD) = ½CB_2 \times ND$, and we have $DB_2 \times CA_2 = CB_2 \times ND$.
Rewrite the previous equation as $\frac{DB_2}{ND} = \frac{CB_2}{CA_2}$. But $\frac{DB_2}{ND} = \frac{CB_1}{CA_1}$ and $\frac{CB_2}{CA_2} = \frac{CB_1}{CA_1}$ implying that $\Delta CB_1B_2 \cong \Delta CA_1A_2$, or $\angle CB_1B_2 = \angle CA_1A_2 = \beta$, or $\angle O_1B_1B_2 = \alpha + \beta = 90°$, or B_1B_2 is tangent to the circle Γ_1.

We had proven earlier that $O_1B_1 \parallel O_2B_2$, or B_1B_2 is also tangent to the circle Γ_2.

Problem 1 of British Mathematical Olympiad 2009

Find all integers x, y and z such that

$x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$.

Solution

Rewrite $x^2 + y^2 + z^2 = 2(yz + 1)$ as $x^2 + y^2 - 2yz + z^2 = 2$, or $x^2 + (y - z)^2 = 2$.

Since all x, y and z are integers, we must have $x^2 = (y - z)^2 = 1$ and the only four possible combinations of x and $y - z$ are

$(x, y - z) = (-1, -1), (-1, 1), (1, -1), (1, 1)$.

When $x = -1$ and $y - z = -1$, $y + z = 4018 - x = 4019$, $2y = 4018$, or $y = 2009$, and $(x, y, z) = (-1, 2009, 2010)$.

When $x = -1$ and $y - z = 1$, $2y = 4020$, or $y = 2010$, and $(x, y, z) = (-1, 2010, 2009)$.

When $x = 1$ and $y - z = -1$, $2y = 4016$, or $y = 2008$, and $(x, y, z) = (1, 2008, 2009)$.

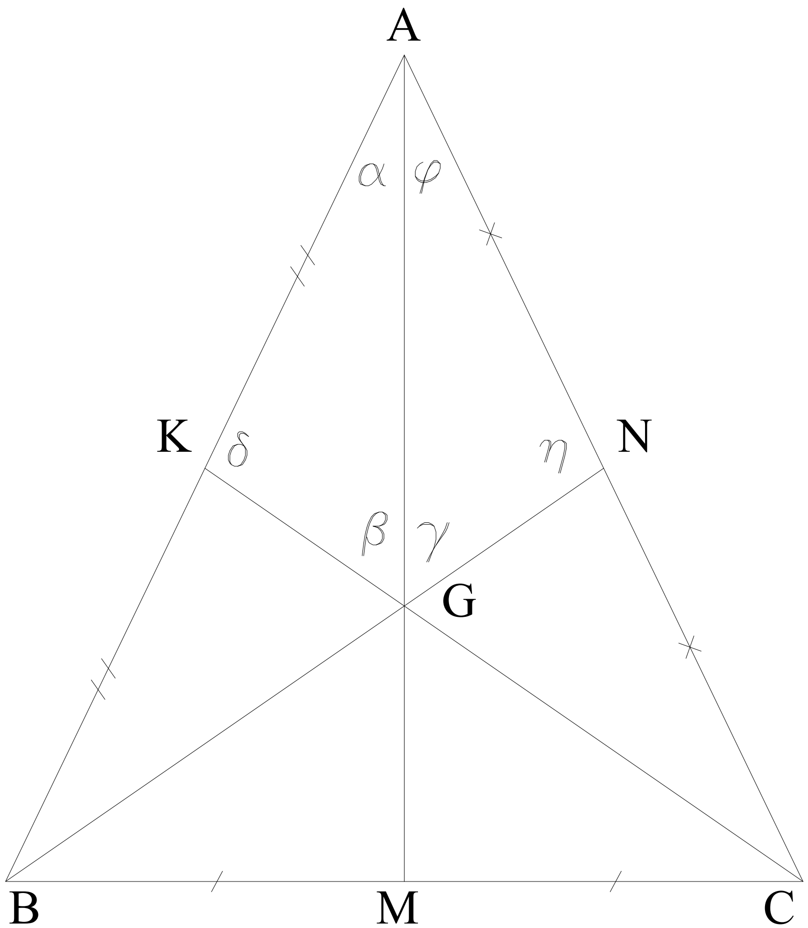
When $x = 1$ and $y - z = 1$, $2y = 4018$, or $y = 2009$, and $(x, y, z) = (1, 2009, 2008)$.

Al integers x, y and z are $(x, y, z) = (-1, 2009, 2010), (-1, 2010, 2009), (1, 2008, 2009)$ and $(1, 2009, 2008)$.

Problem 2 of Spain Mathematical Olympiad 1996

Let G be the centroid of a triangle ABC. Prove that if AB + GC = AC + GB, then the triangle is isosceles.

Solution



Let M, N and K be the midpoints of BC, AC and AB, respectively, $\alpha = \angle KAG$, $\beta = \angle AGK$, $\delta = \angle AKG$, $\varphi = \angle NAG$, $\gamma = \angle AGN$ and $\eta = \angle ANG$.
Since G is the centroid of the triangle ABC, we have GC = 2GK, GB = 2GN, and the given equation AB + GC = AC + GB becomes

AK + GK = AN + GN. Now applying the law of sines, we get $\frac{\sin\delta}{AG} = \frac{\sin\alpha}{GK} = \frac{\sin\beta}{AK} = \frac{\sin\alpha + \sin\beta}{GK + AK}$.

Similarly, $\frac{\sin\eta}{AG} = \frac{\sin\varphi}{GN} = \frac{\sin\gamma}{AN} = \frac{\sin\varphi + \sin\gamma}{GN + AN}$, or $\frac{\sin\delta}{\sin\alpha + \sin\beta} = \frac{AG}{GK + AK} = \frac{AG}{GN + AN} = \frac{\sin\eta}{\sin\varphi + \sin\gamma}$.

However, $\delta = 180° - (\alpha + \beta)$, $\sin\delta = \sin(\alpha + \beta)$, and $\sin\eta = \sin(\varphi + \gamma)$, the equation $\frac{\sin\delta}{\sin\alpha + \sin\beta} = \frac{\sin\eta}{\sin\varphi + \sin\gamma}$ is now equivalent to

$$\frac{\sin(\alpha + \beta)}{\sin\alpha + \sin\beta} = \frac{\sin(\varphi + \gamma)}{\sin\varphi + \sin\gamma} \tag{i}$$

Furthermore, $\sin\alpha + \sin\beta = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha - \beta}{2}$, $\sin\varphi + \sin\gamma = 2\sin\frac{\varphi + \gamma}{2}\cos\frac{\varphi - \gamma}{2}$ and $\sin(\alpha + \beta) = 2\sin\frac{\alpha + \beta}{2}\cos\frac{\alpha + \beta}{2}$, $\sin(\varphi + \gamma) = 2\sin\frac{\varphi + \gamma}{2}\cos\frac{\varphi + \gamma}{2}$. The equation (i) can now be written as $\frac{\cos\frac{\alpha + \beta}{2}}{\cos\frac{\alpha - \beta}{2}} = \frac{\cos\frac{\varphi + \gamma}{2}}{\cos\frac{\varphi - \gamma}{2}}$. It's easily seen that $\alpha + \beta = \varphi + \gamma$ and $\alpha - \beta = \varphi - \gamma$ is a solution of the above equation, or $\alpha = \varphi$.

Again, applying the law of sines, we obtain $\frac{\sin\angle ABC}{\sin\alpha} = \frac{AM}{BM} = \frac{AM}{CM} = \frac{\sin\angle ACB}{\sin\varphi}$, or $\sin\angle ABC = \sin\angle ACB$.

But because these two angles are both less than 180°, therefore, $\angle ABC = \angle ACB$ and ABC is an isosceles triangle.

Problem 6 of Spain Mathematical Olympiad 1996

A regular pentagon is constructed externally on each side of a regular pentagon of side 1. The figure is then folded and the two edges of the external pentagons meeting at each vertex of the original pentagon are glued together. Find the volume of water that can be poured into the obtained container.

Solution

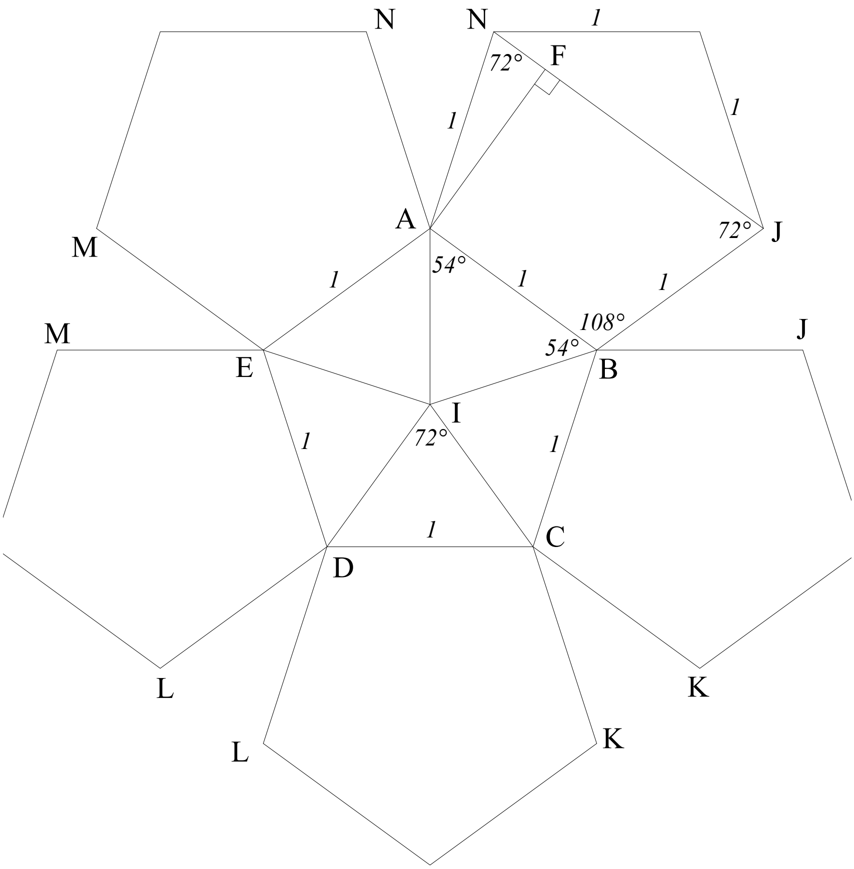


Figure 1: Two-dimensional graph (not to scale).

Let the regular pentagon be ABCDE and I be its center. By definition, all their angles are equal; i.e., $\angle A = \angle B = \angle C = \angle D = \angle E = 180° \times 3/5 = 108°$, and $\angle IAB = \angle IBA = 108°/2 = 54°$. Let

N, J, K, L and M be the vertices of the external pentagons as shown in figure 1. It's easily seen that NJ || AB and $\angle$ANJ = 180° – $\angle$NAB = 72° (before folding in figure 1). After folding their vertices coincide to make another pentagon NJKLM with center O as shown in figure 2 where point O is directly overhead of point I, and OI is the shortest distance between the two pentagons.

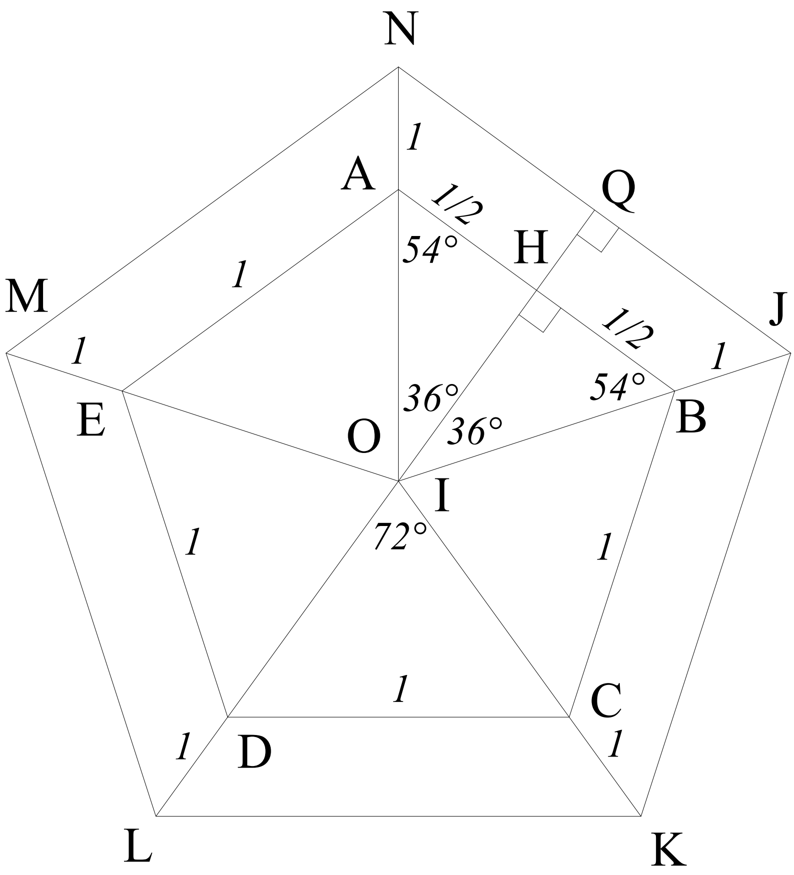


Figure 2: Three-dimensional top view after folding the external pentagons, ABCDE on the bottom plane and NJKLM on the top plane that parallels to the bottom one.

Now let l = AB = AN and draw the altitude IH onto AB and the altitude OQ onto NJ, we get $\sin 36° = \dfrac{\text{AH}}{\text{IA}} = \dfrac{\text{AB}}{2\text{IA}}$, or $\text{IA} = \dfrac{l}{2\sin 36°}$.

Similarly, in triangle ONQ, ON = $\frac{\text{NJ}}{2\sin 36°}$.

In figure 1, drawing the altitude AF to NJ, we get NF = AN× cos∠ANF, NF = $l\cos 72°$ and NJ = AB + 2NF = $l(1 + 2\cos 72°)$. ON becomes ON = $\frac{l(1+2\cos 72°)}{2\sin 36°}$; ON – IA = $\frac{l(1+2\cos 72°)}{2\sin 36°} - \frac{l}{2\sin 36°} = \frac{l\cos 72°}{\sin 36°}$ and $\frac{\text{IA}}{\text{ON} - \text{IA}} = \frac{1}{2\cos 72°}$.

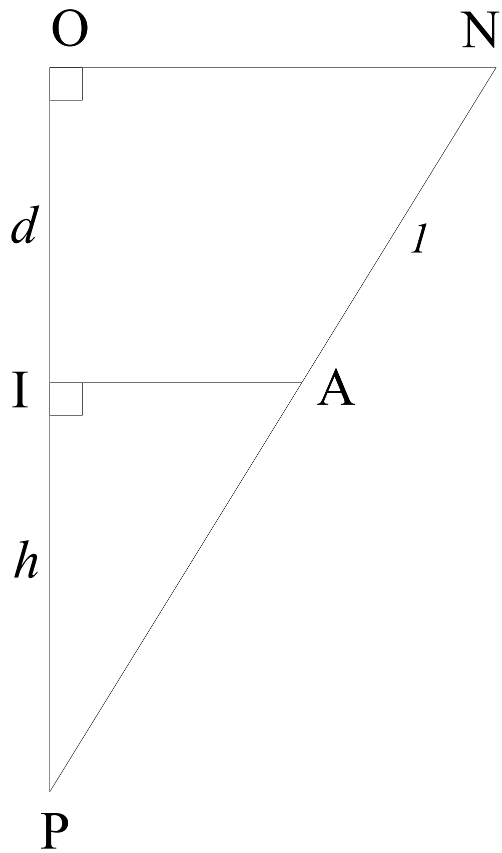


Figure 3: Two-dimensional cross section of O, N, A, I (not to scale).

On the other hand, figure 3 depicts the two-dimensional cross section of O, N, A, I where ON || IA and the two segments NA and OI meet at P. Let d = OI and h = PI.

We have $d = \sqrt{\mathrm{AN}^2 - (\mathrm{ON} - \mathrm{IA})^2}$ (per Pythagorean's theorem) =

$$\sqrt{l^2 - \frac{l^2\cos^2 72°}{\sin^2 36°}} = \frac{l\sqrt{\sin^2 36° - \cos^2 72°}}{\sin 36°} \text{ and } \frac{h}{d} = \frac{\mathrm{IA}}{\mathrm{ON} - \mathrm{IA}} = \frac{1}{2\cos 72°}$$

or $h = \dfrac{l\sqrt{\sin^2 36° - \cos^2 72°}}{2\sin 36°\cos 72°}$.

Furthermore, in triangles IAH and ONQ, $\tan 36° = \dfrac{l}{2\mathrm{IH}} = \dfrac{\mathrm{NJ}}{2\mathrm{OQ}}$, or

$\mathrm{IH} = \dfrac{l}{2\tan 36°}$ and $\mathrm{OQ} = \dfrac{\mathrm{NJ}}{2\tan 36°} = \dfrac{l(1 + 2\cos 72°)}{2\tan 36°}$.

Since we've been dealing with ΔONJ whose area equals one-fifth that of pentagon NJKLM and with ΔIAB whose area also equals one-fifth that of pentagon ABCDE. The volume of one-fifth of the water that can be poured into the obtained container is the volume of tetrahedron PONJ minus that of tetrahedron PIAB, and it is

$$\frac{1}{5}\mathrm{V} = \frac{1}{3}\,[(h + d)\times\text{Area of }\Delta\mathrm{ONJ} - h\times\text{Area of }\Delta\mathrm{IAB}], \text{ or}$$

$$\mathrm{V} = \frac{5}{3}\,[(h + d)\times\text{Area of }\Delta\mathrm{ONJ} - h\times\text{Area of }\Delta\mathrm{IAB}] =$$

$$\frac{5}{6}\,[(h + d)\times\mathrm{OQ}\times\mathrm{NJ} - h\times\mathrm{IH}\times\mathrm{AB}].$$

Substituting in the values, we obtain

$$\mathrm{V} = \frac{5}{6}\,[(\frac{l\sqrt{\sin^2 36° - \cos^2 72°}}{2\sin 36°\cos 72°} + \frac{l\sqrt{\sin^2 36° - \cos^2 72°}}{\sin 36°})\frac{l^2(1 + 2\cos 72°)^2}{2\tan 36°}$$

$$-\frac{l\sqrt{\sin^2 36° - \cos^2 72°}}{2\sin 36°\cos 72°}\times\frac{l}{2\tan 36°}\times l] = \frac{5l^3\sqrt{\sin^2 36° - \cos^2 72°}}{24\sin 36°\tan 36°\cos 72°}\,[(1+$$

$2\cos72°)^3 - 1] = \dfrac{5l^3\sqrt{\sin^2 36° - \cos^2 72°}}{12\sin36°\tan36°}(4\cos^2 72° + 6\cos72° + 3)$,

or $V = \dfrac{5l^3}{12\sin^2 36°}\sqrt{4\sin^6 36° - 9\sin^4 36° + 6\sin^2 36° - 1}\times(16\sin^4 36° - 28\sin^2 36° + 13) =$

$\dfrac{5l^3}{48\sin^2 36°}\sqrt{(4\sin^2 36° - 3)^3 - 12\sin^2 36° + 11}\times[(4\sin^2 36° - \dfrac{7}{2})^2 + \dfrac{3}{4}]$.

According to the result of the previous problem, an isosceles triangle with each equal base angle of 36°, its base length of b, the length of each equal side of a, we have $a = \frac{1}{2}b(\sqrt{5} - 1)$, or

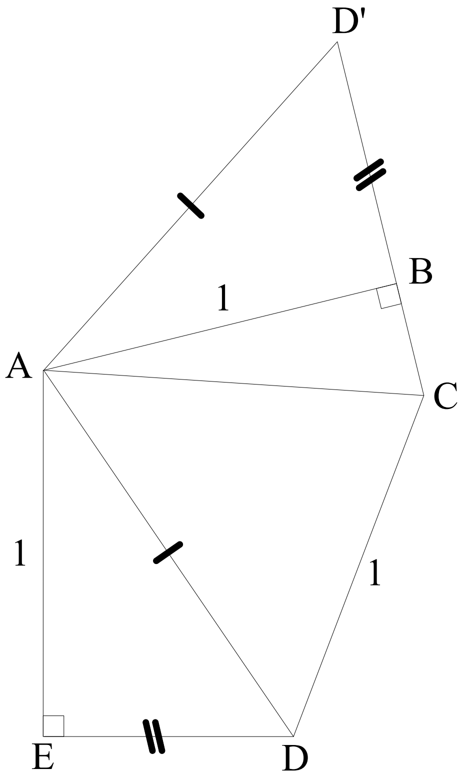
$b = \dfrac{2a}{\sqrt{5} - 1}$. From there, $\sin36° = \dfrac{1}{a}\sqrt{a^2 - \dfrac{1}{4}b^2} = \dfrac{\sqrt{25 - 11\sqrt{5}}}{(3 - \sqrt{5})\sqrt{2}}$.

Substituting this value of sin36° into the latest equation of V to get V = 2.55 cubic units.

Problem 2 of Junior Balkan Mathematical Olympiad 1998

Let ABCDE be a convex pentagon such that AB = AE = CD = 1, $\angle$ABC = $\angle$DEA = 90° and BC + DE = 1. Compute the area of the pentagon.

Solution



Pick a point D′ on the extension of CB such that BD′ = DE. Now we have CD′ = BC + DE = CD = 1.
Furthermore, the two right triangles AED and ABD′ are congruent because their corresponding sides are equal AE = AB and BD′ = DE. Therefore, AD = AD′.
Now the two triangles ACD and ACD′ are congruent because their corresponding sides are equal AD = AD′, CD = CD′ = 1 and the common side AC. Therefore, the area of triangle ACD equals that of triangle ACD′, or the area of the pentagon is twice the area of triangle ACD′ which is AB×CD′ = 1.

Problem 4 of International Mathematical Talent Search Round 19

Suppose that f satisfies the functional equation

$$2f(x) + 3f\left(\frac{2x+29}{x-2}\right) = 100x + 80.$$

Find $f(3)$.

Solution

Trying $x = 3$ into the equation, we get $2f(3) + 3f(35) = 380$.

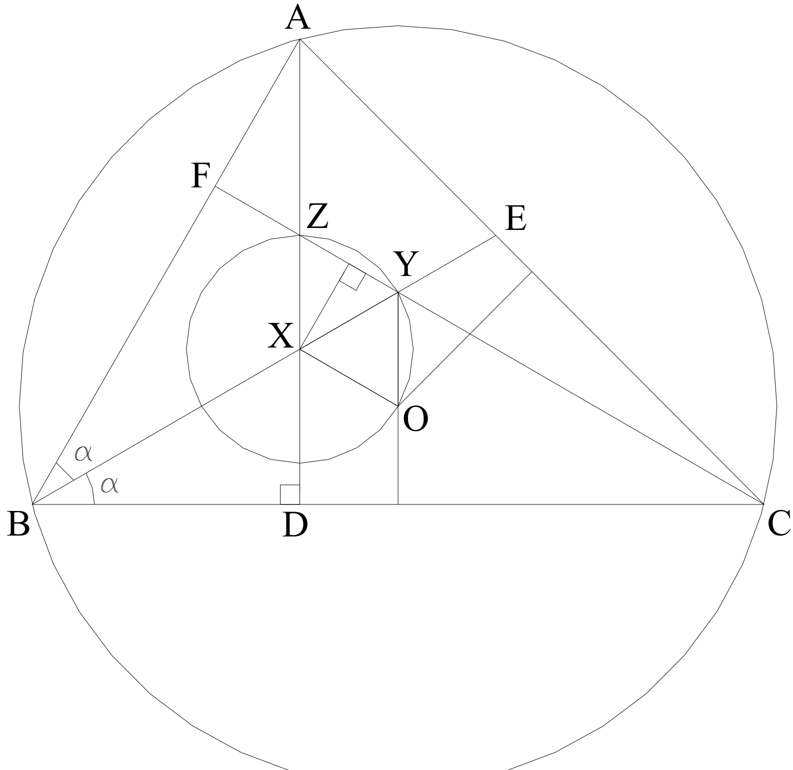
Now trying $x = 35$, we get $2f(35) + 3f(3) = 3580$.

The two above equations give us $f(3) = 1996$.

<u>Problem 2 of the Central America Mathematical Olympiad 2011</u>

In a scalene triangle ABC, D is the foot of the altitude through A, E is the intersection of AC with the bisector of $\angle ABC$ and F is a point on AB. Let O be the circumcenter of ABC and $X = AD \cap BE$, $Y = BE \cap CF$, $Z = CF \cap AD$. If XYZ is an equilateral triangle, prove that one of the triangles OXY, OYZ, OZX must be equilateral.

<u>Solution</u>



Let $\alpha = ½\angle B$ and r be the radius of the circumcenter of ΔABC. If ΔXYZ is equilateral $\angle XYZ = \angle YXZ = \angle XZY = 60° = \angle BXD = \angle AZF$, $\angle BCF = 90° - \angle CZD = 30°$ and $CF \perp AB$. Therefore, $\alpha = 30°$, $\angle B = 60°$, $\angle BAX = 30°$. Since $\alpha = \angle ABX = \angle BAX = 30°$, ΔXAB is isosceles with $AX = BX$. Because ΔOAB is also isosceles with OA = OB = r, $\angle$OAB = $\angle$OBA, we then have $\angle$OAX = $\angle$OBX and ΔOAX is congruent with ΔOBX or $\angle$AOX = $\angle$BOX and OX is the bisector of $\angle$AOB and OX$\perp$AB. Combining with CF$\perp$AB, OX || CF, or $\angle$OXY = $\angle$ZYX = 60°.

Similarly, because $\angle$YCB = 30° = $\angle$YBC, ΔYBC is isosceles and YB = YC; ΔOBY is congruent with ΔOCY or $\angle$BYO = $\angle$CYO and OY is the bisector of $\angle$BYC and OY$\perp$BC. Combining with AD$\perp$BC, OY || AD, or $\angle$XYO = $\angle$ZXY = 60°.

Hence, $\angle$XOY is also 60° and OXY is an equilateral triangle.

Problem 1 of the Irish Mathematical Olympiad 2001

Find, with proof, all solutions of the equation $2^n = a! + b! + c!$ in positive integers a, b, c and n. (Here, ! means "factorial".)

Solution

Since 2^n is an even number, the possible scenarios for $a!$, $b!$ and $c!$ are that one of them an even number while the other two odd numbers or all of them are even numbers.

Let's examine the former scenario where *one of the factorial is an even number while the other two odd numbers*. Without loss of generality, let $a!$ be an even number while both $b!$ and $c!$ be odd numbers. A factorial of a number is an odd number when the number itself equals to 0 or 1; i.e., $b! = c! = 1$ when $b = c = 0$ (by definition $0! = 1$) or $b = c = 1$, and $2^n = a! + b! + c!$ becomes $2^n = 2(1 + \frac{a!}{2})$, and $\frac{a!}{2}$ must be an odd number that makes $1 + \frac{a!}{2}$ a power of 2. We find $a = 3$ to satisfy this requirement.

Also because $a!$, $b!$ and $c!$ are interchangeable, we have drawn the following conclusion:
$(a, b, c, n) = (0, 0, 3, 3), (0, 3, 0, 3), (3, 0, 0, 3), (1, 1, 3, 3), (1, 3, 1, 3), (3, 1, 1, 3)$.

Now let's look at the scenario where *all the factorials a!, b! and c! are even numbers*. Let $a! = 2p$, $b! = 2q$ and $c! = 2s$ where p, q and s are all integers. We now have
$2^n = 2(p + q + s)$, or $p + q + s = 2^{n-1}$. Note that $p + q + s \geq 3$, and $n > 2$. Once again, the possible scenarios for p, q and s are that one of them an even number while the other two odd numbers or all of them are even numbers.

Assume *p is even with $p = 3\times4m$ (3 and 4 are the next two factors of the factorial) where m is an integer and both q and s are odd*. In this scenario, both q and s must be equal to 3, and we have $2^n = 2(3 \times 4m + 3 + 3) = 2 \times 3(4m + 2)$, and this is not possible since there is no factor of 3 on 2^n on the left side.

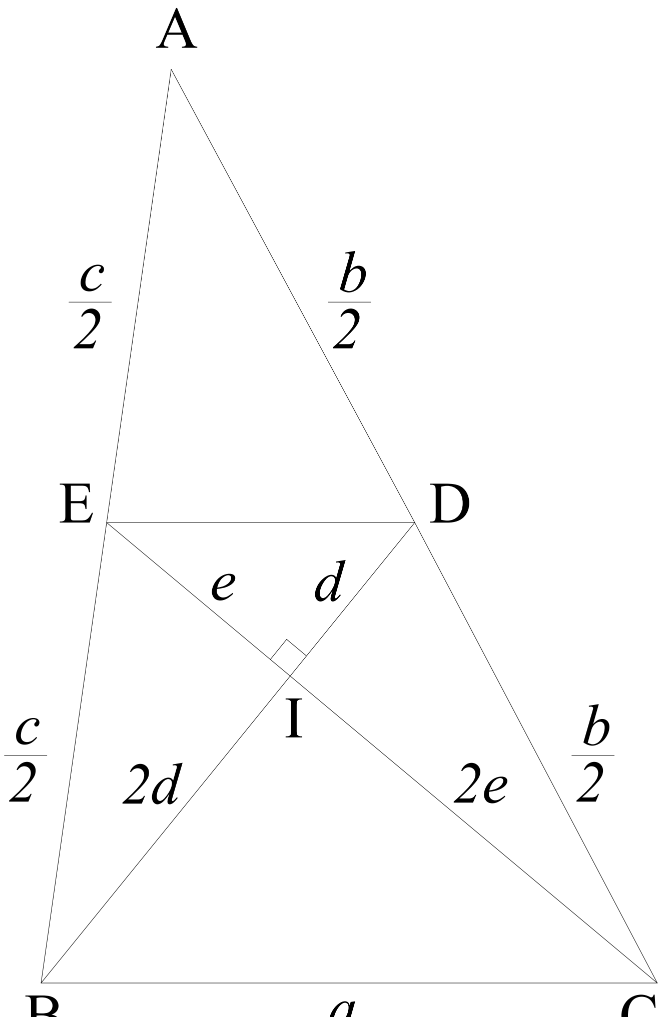
Now in the scenario of *all p, q and s being even numbers*, they all must have 3 and 4 as their factors. This is the same as the previous scenario right above where there is no factor of 3 on 2^n.

Therefore, the only solutions are $(a, b, c, n) = (0, 0, 3, 3), (0, 3, 0, 3), (3, 0, 0, 3), (1, 1, 3, 3), (1, 3, 1, 3), (3, 1, 1, 3)$.

Problem 2 of the Irish Mathematical Olympiad 2001

Let ABC be a triangle with sides BC, CA, AB of lengths a, b, c, respectively. Let D, E be the midpoints of the sides AC, AB, respectively. Prove that BD is perpendicular to CE if, and only if, $b^2 + c^2 = 5a^2$.

Solution



Let I be the intersection of BD and CE, e = IE, d = ID and α =

$\angle$BIC. Since D and E are the midpoints of AC and AB, respectively, DE || BC and DE = BC/2 = $a/2$, and triangles IDE and IBC are similar, and we have $\frac{\text{ID}}{\text{IB}} = \frac{\text{IE}}{\text{IC}} = \frac{\text{DE}}{\text{BC}} = \frac{1}{2}$. Therefore, IB = $2d$ and IC = $2e$.

Now let BD perpendicular to CE. Applying the Pythagorean theorem to get $BC^2 = IB^2 + IC^2$, or

$a^2 = 4d^2 + 4e^2$, or $5a^2 = 20d^2 + 20e^2 = 16d^2 + 4e^2 + 16e^2 + 4d^2 = 4BE^2 + 4CD^2 = c^2 + b^2$.

Conversely, let $b^2 + c^2 = 5a^2$. Applying the law of cosine, we have

$BC^2 = IB^2 + IC^2 - 2IB \times IC\cos\alpha$, or $a^2 = 4d^2 + 4e^2 - 8de\cos\alpha$ and

$$5a^2 = 20d^2 + 20e^2 - 40de\cos\alpha \qquad \text{(i)}$$

However, $BE^2 = IB^2 + IE^2 - 2IB \times IE\cos(180° - \alpha)$, or
$c^2/4 = 4d^2 + e^2 - 4de\cos(180° - \alpha) = 4d^2 + e^2 + 4de\cos\alpha$, or
$c^2 = 16d^2 + 4e^2 + 16de\cos\alpha$. Similarly, $b^2 = 16e^2 + 4d^2 + 16de\cos\alpha$.

Adding the two previous equations, we obtain

$$b^2 + c^2 = 20d^2 + 20e^2 + 32de\cos\alpha \qquad \text{(ii)}$$

Now equate (i) and (ii) to get $72de\cos\alpha = 0$. This only occurs when $\cos\alpha = 0$, or $\alpha = 90°$, or BD is perpendicular to CE.

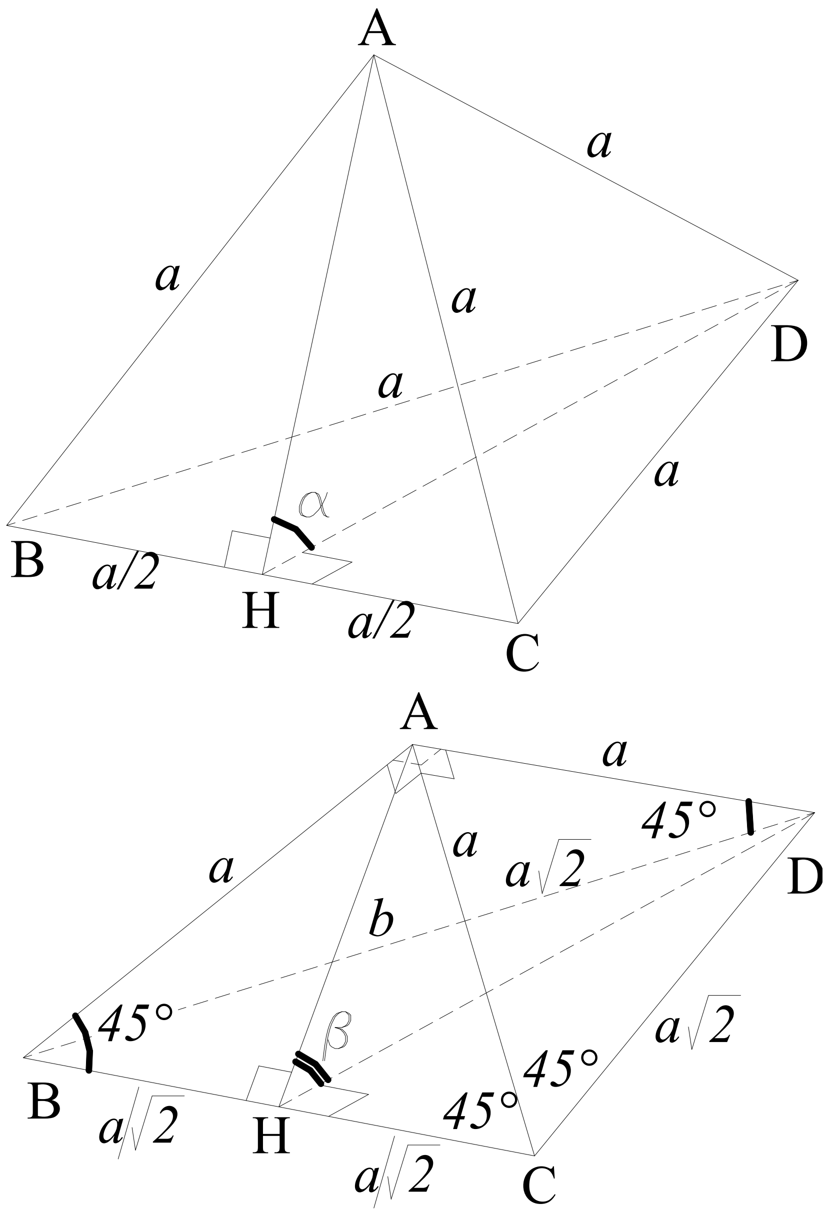
Problem 2 of the Canadian Mathematical Olympiad 1979

It is known in Euclidean geometry that the sum of the angles of a triangle is constant. Prove, however, that the sum of the dihedral angles of a tetrahedron is not constant.
Note. *A tetrahedron is a triangular pyramid, and a dihedral angle is the interior angle between a pair of faces.*

Solution

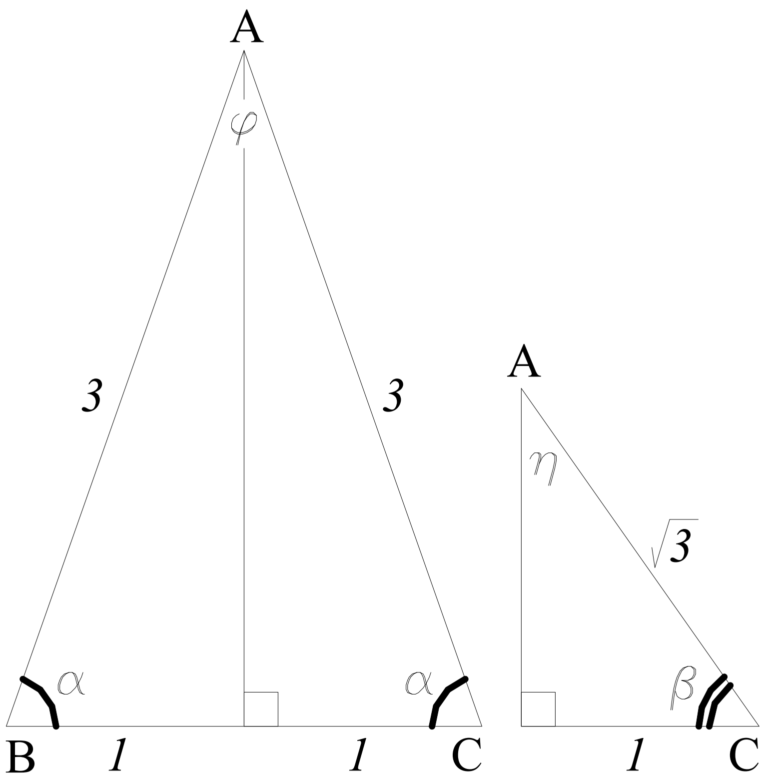
Let's prove this by adding all the dihedral angles of a regular tetrahedron with all its faces being equilateral triangles, and of a right tetrahedron with three of its faces being right isosceles triangles and with their right angles sharing the same vertex and the remaining face being an equilateral triangle as shown on the graph on the next page.

For the regular tetrahedron, let a be its side length and H be the midpoint of BC. The measures of all six dihedral angles are the same and equal $\alpha = \angle AHD$. Applying the law of cosines to triangle AHD, we get $AD^2 = AH^2 + DH^2 - 2AH\times DH\times\cos\alpha$, or $a^2 = \frac{3}{4}a^2 + \frac{3}{4}a^2 - 2\times\frac{3}{4}a^2\times\cos\alpha$, or $\cos\alpha = \frac{1}{3}$. The sum of all the angles is $6\alpha = 6\cos^{-1}\frac{1}{3} = 423.17°$.
Whereas, in the case of the other tetrahedron already defined, the sum of the angles equals three times the right angle plus three times the angle β where $\beta = \angle AHD$, but $AH = ½BC = \frac{a}{\sqrt{2}}$ and $DH = \sqrt{2a^2 - a^2/2} = a\sqrt{\frac{3}{2}}$, and the law of cosines gives us $AD^2 = AH^2 + DH^2 - 2AH\times DH\times\cos\beta$, or $\beta = \cos^{-1}\frac{1}{\sqrt{3}}$, and the sum is equal to $3(90° + \beta) = 270° + 3\cos^{-1}\frac{1}{\sqrt{3}} = 434.21°$. Clearly, the two sums are different.



A regular tetrahedron with four congruent equilateral-triangled faces (above) and a tetrahedron with three congruent right isosceles triangles having their right-angles joining at A (below).

The measures of the sums in degrees are for your information only. To prove the difference between the sums, let's compare one-third of 6α or 2α to 90° + β.



As seen on the graph above, we can compare the angles φ and η instead because φ = 180° – 2α and η = 180° – (90° + β). Again, applying the law of cosines, $\cos\varphi = \frac{7}{9} \neq \cos\eta = \sqrt{\frac{2}{3}}$.

Problem 5 of Malaysia National Olympiad 2010 Muda Category

Find the number of triples of nonnegative integers (x, y, z) such that $x^2 + 2xy + y^2 - z^2 = 9$.

Solution

$x^2 + 2xy + y^2 - z^2 = (x + y)^2 - z^2 = 9$ can only happen with non-negative integers (x, y, z) when

$(x + y)^2 = 9$ and $z^2 = 0$, or $(x + y)^2 = 25$ and $z^2 = 16$, or

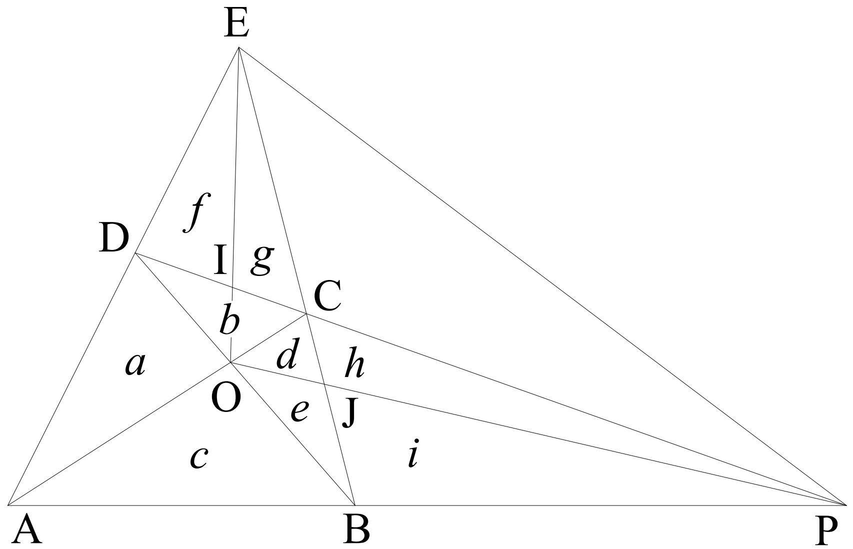
$x + y = \pm 3$ and $z = 0$, or $x + y = \pm 5$ and $z = \pm 4$.

The answers are $(x, y, z) = (a, -3 - a, 0)$, $(a, 3 - a, 0)$, $(a, -5 - a, -4)$, $(a, -5 - a, 4)$, $(a, 5 - a, -4)$, $(a, 5 - a, 4)$, where a is an integer.

Problem 2 of the Iranian Mathematical Olympiad 1993

In the figure below, areas of triangles AOD, DOC, and AOB are given. Find the area of triangle OEF in terms of areas of these three triangles.

Solution



Let (Ω) denote the area of shape Ω, I = OE $\cap$ DC, J = OF $\cap$ BC, a = (AOD), b = (DOC), c = (AOB) which are the three areas given by the problem, d = (COJ), e = (BOJ), f = (IDE), g = (ICE), h = (CJF), i = (BJF).

We have $\dfrac{\text{OA}}{\text{OC}} = \dfrac{a}{b} = \dfrac{c}{(d+e)}$, or $d + e = \dfrac{bc}{a}$ (i)

Now note that $\dfrac{(\text{EBO})}{(\text{EDO})} = \dfrac{g + (\text{ICO}) + \dfrac{bc}{a}}{f + (\text{IDO})} = \dfrac{c}{a}$, or

$ag + bc + (\text{ICO})a = c(f + (\text{IDO}))$ (ii)

$\dfrac{(\text{EDC})}{(\text{BDC})} = \dfrac{(\text{EAC})}{(\text{BAC})}$, or $\dfrac{f+g}{b + \dfrac{bc}{a}} = \dfrac{f+g+a+b}{c + \dfrac{bc}{a}}$, or

$$f + g = \frac{b(a+b)(a+c)}{a(c-b)} \qquad \text{(iii)}$$

$$\frac{g}{f} = \frac{\text{(ICO)}}{\text{(IDO)}} \qquad \text{(iv)}$$

and (IDO) + (ICO) = b (v)

Solve four equations (ii), (iii), (iv) and (v) with four unknowns f, g, (ICO) and (IDO). By substituting (ICO) = b – (IDO) into (iv) and (ii) we get

$\frac{g}{f} = \frac{b}{\text{(IDO)}} - 1$ or $\text{(IDO)} = \frac{bf}{f+g}$ and

$ag + ab + bc - \text{(IDO)}a = c(f + \text{(IDO)})$, or

$$(f+g)(ab + ag + bc - cf) = bf(a+c) \qquad \text{(vi)}$$

Now substitute $f + g$ from (iii) into equation (vi) to get

$$(a+b)(ab + ag + bc) = f(bc + 2ac - ab) \qquad \text{(vii)}$$

and by substituting $g = \frac{b(a+b)(a+c)}{a(c-b)} - f$ into equation (vii)

$f = \frac{b(a+b)(a+c)^2}{(c-b)(a^2+bc+2ac)}$ and $g = \frac{bc(a+b)^2(a+c)}{a(c-b)(a^2+bc+2ac)}$.

$\frac{f}{g} = \frac{a(a+c)}{c(a+b)}$. Therefore,

$$\text{(ICO)} = b - \frac{bf}{f+g} = \frac{b}{\frac{f}{g}+1} = \frac{bc(a+b)}{a^2+2ac+bc}.$$

$$\frac{g}{\text{(ICO)}} = \frac{(a+b)(a+c)}{a(c-b)}.$$

We also have $\frac{d+h+b}{a} = \frac{i+e}{c} = \frac{d+e+h+i+b}{a+c}$ (viii)

$$\frac{h+i}{b+d+e} = \frac{h+i+d+e+c}{a+b} \qquad \text{(ix)}$$

Substitute $d + e = \frac{bc}{a}$ from (i) into (viii) and simplify to get

$\frac{a(h+i)}{ab+bc} = \frac{a(h+i\ +c)+bc}{a(a+b)}$, or $a^2(h+i)(a+b) = ab(h+i)(a+c) +$ $bc(a+b)(a+c)$, or $h+i = \frac{bc(a+b)(a+c)}{a(a^2-bc)}$.

On the other hand, substituting the values of $d+e$ and $h+i$ into equation (viii) gives $d+h = \frac{bc(a+b)}{a^2-bc}$.

Lastly, we also have $\frac{\text{(IFO)}}{\text{(ICO)}} = \frac{\text{(IFE)}}{g} = \frac{\text{(IFO)}+\text{(IFE)}}{\text{(ICO)}+g} = \frac{\text{(OEF)}}{\text{(ICO)}+g}$'

but (IFO) = (ICO) + $d+h$, and the area of triangle OEF in terms of areas of these three triangles is

(OEF) = $[\text{(ICO)}+g]\times[\text{(ICO)}+d+h]/\text{(ICO)} = [1+\frac{g}{\text{(ICO)}}]\times[\text{(ICO)}$ $+\,d+h] = [1+\frac{(a+b)(a+c)}{a(c-b)}]\times bc(a+b)[\frac{1}{a^2+2ac+bc}+\frac{1}{a^2-bc}]$.

Finally (OEF) = $\frac{2bc(a+b)(a+c)}{(c-b)(a^2-bc)}$.

Problem 6 of the Irish Mathematical Olympiad 1993

The real numbers x, y satisfy the equations

$$x^3 - 3x^2 + 5x - 17 = 0 \qquad \text{(i)}$$
$$y^3 - 3y^2 + 5y + 11 = 0 \qquad \text{(ii)}$$

Find $x + y$.

Solution

Let's try the monic formula for roots for the cubic equation that has the form of $x^3 + ax^2 + bx + c = 0$. The solutions for this equation are

$$x_1 = -\frac{a}{3}$$

$$-\frac{1}{3}\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c + \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

$$-\frac{1}{3}\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c - \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

$$x_2 = -\frac{a}{3} + \frac{1 + i\sqrt{3}}{6} \times$$

$$\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c + \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

$$+\frac{1 - i\sqrt{3}}{6} \times$$

$$\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c - \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

$$x_3 = -\frac{a}{3} + \frac{1 - i\sqrt{3}}{6} \times$$

$$\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c + \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

$$+\frac{1 + i\sqrt{3}}{6} \times$$

$$\sqrt[3]{\frac{1}{2}[2a^3 - 9ab + 27c - \sqrt{(2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3}\,]}$$

where only x_1 is the solution in real number.

In equation (i), a = -3, b = 5, c = -17 and $-\frac{a}{3} = 1$, $2a^3 - 9ab + 27c$ = -378, $4(a^2 - 3b)^3$ = -864, and the solution x in real number is $x = 1 - \frac{1}{3}\sqrt[3]{-189 + 33\sqrt{33}} - \frac{1}{3}\sqrt[3]{-189 - 33\sqrt{33}}$.

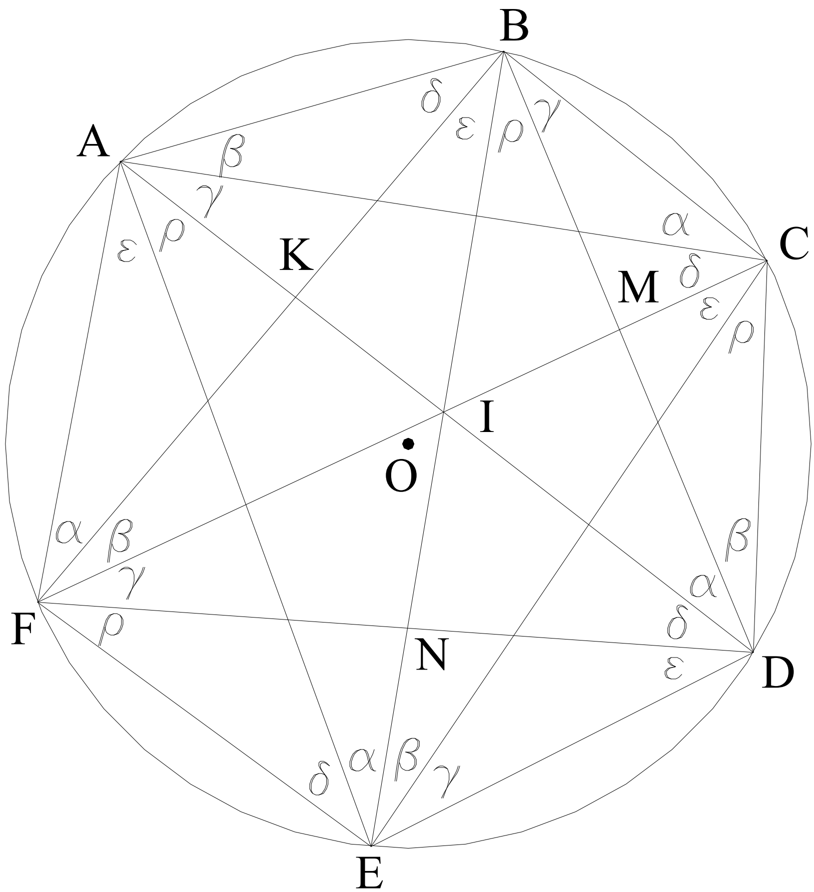
Similarly, in equation (ii), a = -3, b = 5, c = 11 and $-\frac{a}{3} = 1$, $2a^3 - 9ab + 27c$ = 378, $4(a^2 - 3b)^3$ = -864, and the solution y in real number is $y = 1 - \frac{1}{3}\sqrt[3]{189 + 33\sqrt{33}} - \frac{1}{3}\sqrt[3]{189 - 33\sqrt{33}}$.

But $\sqrt[3]{-189 + 33\sqrt{33}} = \sqrt[3]{189 - 33\sqrt{33}}$ and $\sqrt[3]{-189 - 33\sqrt{33}} = \sqrt[3]{189 + 33\sqrt{33}}$; therefore, $x + y = 2$.

Problem 1 of Mediterranean Mathematics Olympiad 2008

Let ABCDEF be a convex hexagon such that all of its vertices are on a circle. Prove that AD, BE, and CF are concurrent if and only if $\frac{AB}{BC} \times \frac{CD}{DE} \times \frac{EF}{FA} = 1$.

Solution



Let $\alpha = \angle ACB = \angle ADB = \angle AEB = \angle AFB$, $\beta = \angle BAC = \angle BFC = \angle BEC = \angle BDC$, $\gamma = \angle CBD = \angle CAD = \angle CFD = \angle CED$, $\delta = \angle ABF = \angle ACF = \angle ADF = \angle AEF$, $\varepsilon = \angle EAF = \angle EBF = \angle ECF = \angle EDF$, $\rho = \angle DCE = \angle DBE = \angle DAE = \angle DFE$, $K = AD \cap BF$, $M = BD \cap CF$, and $N = BE \cap DF$ as shown on the graph.

Given the equation $\frac{AB}{BC}\times\frac{CD}{DE}\times\frac{EF}{FA} = 1$, let's prove that AD, BE, and CF are concurrent. Per Ceva's theorem, AD, BE, and CF (or DK, BN, and FM) are concurrent if and only if $\frac{BK}{FK}\times\frac{FN}{DN}\times\frac{DM}{BM} = 1$.

But according to the law of sines, in triangle ABK, $\frac{BK}{AK} = \frac{\sin(\beta + \gamma)}{\sin\delta}$ and in triangle AFK, $\frac{FK}{AK} = \frac{\sin(\varepsilon + \rho)}{\sin\alpha}$, or $\frac{BK}{FK} = \frac{\sin\alpha\sin(\beta + \gamma)}{\sin\delta\sin(\varepsilon + \rho)}$.

Similarly, $\frac{FN}{DN} = \frac{\sin\varepsilon\sin(\alpha + \delta)}{\sin\rho\sin(\beta + \gamma)}$ and $\frac{DM}{BM} = \frac{\sin\gamma\sin(\varepsilon + \rho)}{\sin\beta\sin(\alpha + \delta)}$.

Now multiply the three terms to get $\frac{BK}{FK}\times\frac{FN}{DN}\times\frac{DM}{BM} = \frac{\sin\alpha\sin(\beta + \gamma)}{\sin\delta\sin(\varepsilon + \rho)}$ $\times\frac{\sin\varepsilon\sin(\alpha + \delta)}{\sin\rho\sin(\beta + \gamma)}\times\frac{\sin\gamma\sin(\varepsilon + \rho)}{\sin\beta\sin(\alpha + \delta)} = \frac{\sin\alpha\sin\gamma\sin\varepsilon}{\sin\beta\sin\rho\sin\delta}$.

It suffices to prove that $\frac{\sin\alpha\sin\gamma\sin\varepsilon}{\sin\beta\sin\rho\sin\delta} = 1$.

Again applying the law of sines to triangles ABC, CDE and AEF, we get $\frac{\sin\alpha}{\sin\beta} = \frac{AB}{BC}$, $\frac{\sin\gamma}{\sin\rho} = \frac{CD}{DE}$ and $\frac{\sin\varepsilon}{\sin\delta} = \frac{EF}{FA}$.

The problem gives us $\frac{AB}{BC}\times\frac{CD}{DE}\times\frac{EF}{FA} = 1$, or $\frac{\sin\alpha\sin\gamma\sin\varepsilon}{\sin\beta\sin\rho\sin\delta} = 1$, or $\frac{BK}{FK}\times\frac{FN}{DN}\times\frac{DM}{BM} = 1$, and we're done.

The reverse process is fairly straight-forward. If the three segments are concurrent, apply Ceva's theorem to get $\frac{BK}{FK}\times\frac{FN}{DN}\times\frac{DM}{BM} = 1$. From there we follow the same path as we have done above to come up with $\frac{AB}{BC}\times\frac{CD}{DE}\times\frac{EF}{FA} = 1$.

Problem 1 of International Mathematical Talent Search Round 2

What is the smallest integer multiple of 9997, other than 9997 itself, which contains only odd digits?

Solution

Let U(m) denote the units digit of integer m, n be the smallest integer multiple of 9997 that contains only odd digits; n must be a product of 9997 and an odd number that has an odd units digit a, and a = 1, 3, 5, 7 or 9.

When a = 1	f = U(7b) must be an even number for 9 + f to be
. . . . 9997	an odd number, so are g and h. Hence, b = 0, c = 0
× $edcb$1	and d = 0. The next digit e must be odd and
. . . . 9997	smallest for the integer multiple to be odd and
9997hgf_	smallest, and e = 1. The multiplier $edcb$1 is now
99979997	$edcb$1 = 10001 and n = 99979997.

When a = 3	Similarly, f = U(7b) must be an even number, so
. . . . 9997	are g and h, or b = 0, c = 0 and d = 0. The next
× $edcb$3	digit e = 1, and $edcb$3 = 10003 > 10001. Number
. . . 29991	99999991 is greater than 99979997 when a = 1,
9997hgf_	and this result is rejected because it's not the
99999991	smallest integer multiple.

When a = 7	
. . . . 9997	
× $edcb$7	Similarly, f = U(7b) must be an even number, so
. . . 69979	are g and h, or b = 0, c = 0 and d = 0 and $edcb$7 =
xxxxhgf_	e0007. Whatever the next digit for e will make
xxxx9979	$edcb$7 to be greater than 10001 when a = 1.

When a = 9	
. . . . 9997	Similarly, f = U(7b) must be an even number, so
× $edcb$9	are g and h, or b = 0, c = 0 and d = 0, and $edcb$9 =
. . . 89973	e0009. Whatever the next digit for e will make
xxxxhgf_	$edcb$9 to be greater than 10001 when a = 1.

When $a = 5$

. . . . 9997
$\times \quad edcb5$
. . . 49985
xxxxhgf_

Similarly, $f = \mathrm{U}(7b)$ must be an odd number, and $b = 1, 3, 5, 7$ or 9.

When $a = 5$, $b = 1$

. . . . 9997
$\times \quad edc15$
. . . 49985
9997_
149955
xxxgf__

Now $c = 0$, $d = 0$ in order for f and g to be even numbers and $edc15 = e0015 > 10001$ when $a = 1$.

When $a = 5$, $b = 3$

. . . . 9997
$\times \quad edc35$
. . . 49985
29991_
349895
xxxgf__

c must be odd for f to be odd and $c = 1, 3, 5, 7, 9$.

When $a = 5$, $b = 3$ and $c = 1$

. . . . 9997
$\times \quad ed135$
. . . 49985
29991_
349895
9997__
1349595
xxhgf__

Now $d = 0$, and whatever digit for e will make $e0135 > 10001$ when $a = 1$.

When $a = 5$, $b = 3$ and $c = 3$

. . . . 9997
$\times \quad ed335$
. . . 49985
29991_
349895

349895	(same number from the bottom row of last page)
29991__	
3348995	Now d is odd and d = 1, 3, 5, 7, 9.

When $a = 5$, $b = 3$, $c = 3$ and $d = 1$

. . . . 9997
× e1335
. . . 49985
29991_
349895
29991__
3348995
x9997___
13345995 Now e is odd and e1335 > 10001.

When $a = 5$, $b = 3$, $c = 3$ and $d = 3$

. . . . 9997	
× e3335	
. . . 49985	
29991_	
349895	**When the multiplier *dcba* = 3335, the integer**
29991__	**multiple is 33339995, and it contains only**
3348995	**the odd digits** and is smaller than 99979997.
29991___	This is the new number we use to compare with
33339995	the rest of the results encountered, if there is any.

When $a = 5$, $b = 3$, $c = 3$ and $d = 5$, 7 or 9 even if we find an integer multiple that contains only the odd digits, it is still greater than the previous result because 5, 7 or 9 is greater than 3, and these cases are ignored.

When $a = 5$, $b = 3$ and $c = 5$

. . . . 9997
× ed535
. . . 49985
29991_
349895
49985__

49985__ (last line copied to here)
5348395 Now *d* is odd, and let's try *d* = 1.

When *a* = 5, *b* = 3 and *c* = 5 and *d* = 1
. . . . 9997
× *e*1535
. . . 49985
29991_
349895
49985__
5348395
x9997___
15345395 Now *e* is odd, and *e*1535 > 3335.

When *a* = 5, *b* = 3, *c* = 5 and *d* = 3, 5, 7 or 9 even if we find an integer multiple that contains only the odd digits, it is still greater than integer multiple 33339995, and these cases are ignored.

When *a* = 5, *b* = 3 and *c* = 7
. . . . 9997
× *ed*735
. . . 49985
29991_
349895
69979__
7347795 Now *d* = 0, and *e*0735 > 3335.

When *a* = 5, *b* = 3 and *c* = 9
. . . . 9997
× *ed*935
. . . 49985
29991_
349895
89973__
9347195 Now *d* = 0, and *e*0935 > 3335.

When *a* = 5, *b* = 5
. . . . 9997

```
. . . . 9997        (last line copied to here)
×    edc55
. . . 49985
   49985_
   549835           Now c must be an odd number; c = 1, 3, 5, 7, 9.
```

When $a = 5$, $b = 5$ and $c = 1$

```
. . . . 9997
×    ed155
. . . 49985
   49985_
   549835
   9997__
  1549535           Now d = 0 and e0155 > 3335.
```

When $a = 5$, $b = 5$ and $c = 3$

```
. . . . 9997
×    ed355
. . . 49985
   49985_
   549835
  29991__
  3548935           Now d must be an odd number; d = 1, 3, 5, 7, 9.
```

When $a = 5$, $b = 5$, $c = 3$ and $d = 1$

```
. . . . 9997
×    e1355
. . . 49985
   49985_
   549835
  29991__
  3548935
x9997___
13545935            Now e must be odd and e1355 >3335.
```

When $a = 5$, $b = 5$, $c = 3$ and $d = 3, 5, 7$ or 9 even if we find an integer multiple that contains only the odd digits, it is still greater than integer multiple 33339995, and these cases are ignored.

When $a = 5$, $b = 5$ and $c = 5$
. . . . 9997
× *ed*555
. . . 49985
49985
549835
49985
5548335 Now d must be an odd number; let's try $d = 1$.

When $a = 5$, $b = 5$, $c = 5$ and $d = 1$
. . . . 9997
× *e*1555
. . . 49985
49985
549835
49985
5548335
x9997
15545335 Now e is odd, and $e1555 > 3335$.

When $a = 5$, $b = 5$, $c = 5$ and $d = 3, 5, 7$ or 9 even if we find an integer multiple that contains only the odd digits, it is still greater than integer multiple 33339995 because all 3555, 5555, 7555, 9555 or higher multipliers are greater than 3335, and these cases are ignored.

When $a = 5$, $b = 5$ and $c = 7$
. . . . 9997
× *ed*755
. . . 49985
49985
549835
69979
7547735 Now $d = 0$ and $e0755 > 3335$.

When $a = 5$, $b = 5$ and $c = 9$
. . . . 9997
× *ed*955

```
×    ed955        (last line copied to here)
. . . 49985
   89973_
 9547135          Now d = 0 and e0955 > 3335.
```

When $a = 5$, $b = 7$

```
. . . . 9997
×    edc75
. . . 49985
   69979_
   749775         Now c = 0, d = 0 and edc15 = e0015 > 3335.
```

When $a = 5$, $b = 9$

```
. . . . 9997
×    edc95
. . . 49985
   89973_
   949715         Now c = 0, d = 0 and edc15 = e0015 > 3335.
```

Finally, we conclude that the smallest integer multiple of 9997, other than 9997 itself, which contains only odd digits is $9997 \times 3335 = 33339995$.

<u>*Problem 6 of Canadian MO Qualification Repechage 2011*</u>

In the diagram, ABDF is a trapezoid with AF parallel to BD and AB perpendicular to BD. The circle with center B and radius AB meets BD at C and is tangent to DF at E. Suppose that x is equal to the area of the region inside quadrilateral ABEF but outside the circle, that y is equal to the area of the region inside ΔEBD but outside the circle, and that $\alpha = \angle EBC$. Prove that there is exactly one measure α, with $0° \le \alpha \le 90°$, for which $x = y$ and that this value of α satisfies $\frac{1}{2} < \sin\alpha < \frac{1}{\sqrt{2}}$.

<u>Solution</u>

Let's shade the areas x and y as shown on the graph and denote (Ω) the area of shape Ω. Also let the circle and its radius be Γ and r, respectively, $a = AF = EF$, $b = BD$ and $c = ED$.

The area bounded by Γ and segments AB, BE is $\frac{\pi r^2}{360°}(90° - \alpha)$,

and $x = (ABEF) - \frac{\pi r^2}{360°}(90° - \alpha)$.

Similarly, $y = (BED) - \frac{\pi r^2\alpha}{360°}$. When $x = y$, we have

$$(ABEF) - \frac{\pi r^2}{360°}(90° - \alpha) = (BED) - \frac{\pi r^2\alpha}{360°}, \text{ or}$$

$$r(a - \frac{1}{2}c) = \frac{\pi r^2}{360°}(90° - 2\alpha), \text{ or } a - \frac{1}{2}c = \frac{\pi r}{360°}(90° - 2\alpha), \text{ or}$$

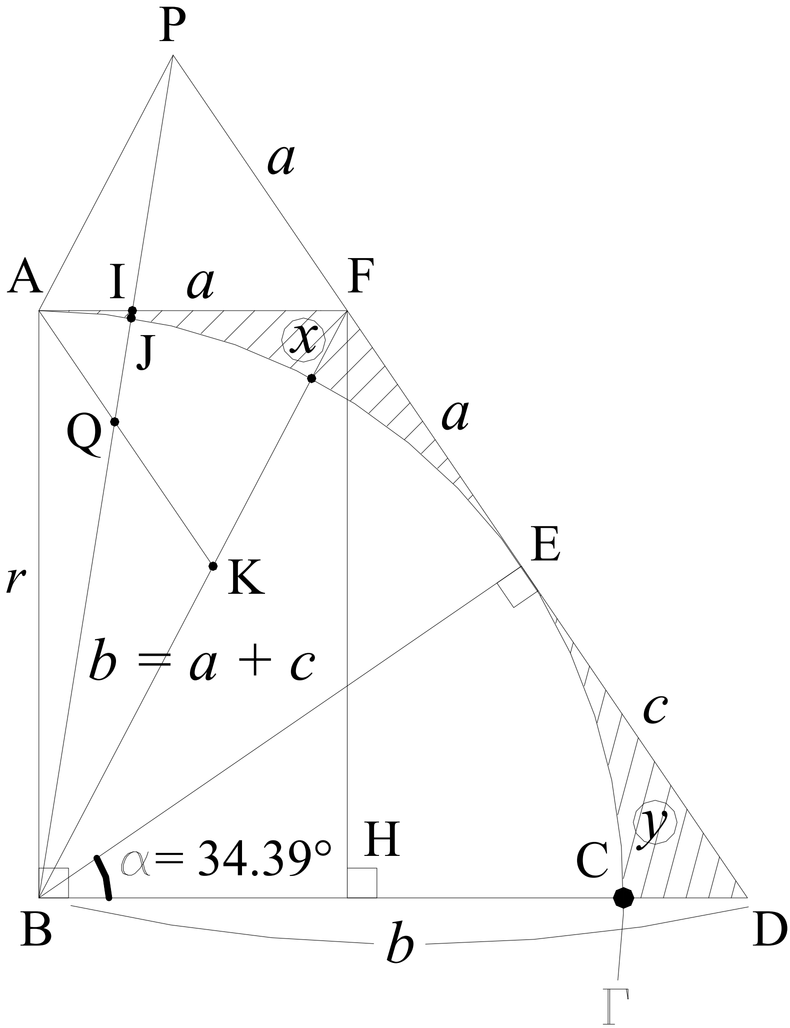
$$\alpha = 45° + \frac{180°}{\pi r}(\frac{1}{2}c - a) \qquad \text{(i)}$$

Now draw the altitude FH onto BD. Applying the Pythagorean theorem to get $(a + c)^2 = r^2 + (b - a)^2$, and $b^2 = r^2 + c^2$.

From there, we have $b = \frac{r^2 + a^2}{2a}$, $c = \frac{r^2 - a^2}{2a}$ and $b = a + c$.

Substitute c into equation (i) to get $\alpha = 45°(1 + \frac{r^2 - 5a^2}{\pi ra})$ (ii)

$$\alpha = 45°(1 + \frac{1}{\pi}\times\frac{r}{a} - \frac{5}{\pi}\times\frac{a}{r}) = 45°[1 + \frac{1}{\pi}\cot(45° - \frac{\alpha}{2}) - \frac{5}{\pi}\tan(45° - \frac{\alpha}{2})].$$



However, $\cot(45° - \frac{\alpha}{2}) = \cos(45° - \frac{\alpha}{2})/\sin(45° - \frac{\alpha}{2})$ and $\cos 45° = \sin 45°$, and we now obtain $\cot(45° - \frac{\alpha}{2}) = (\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})/(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2})$ and $\tan(45° - \frac{\alpha}{2}) = (\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2})/(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2})$. Equation (ii) is equivalent to $\alpha = 45°[1 - \frac{2}{\pi\cos\alpha}(2 - 3\sin\alpha)]$ (iii)

The angle α in this expression is unique in the first quadrant (0° to 90°), and there is exactly one measure α to satisfy this condition. (*For your information α = 34.38129675° even though we're not asked to find its measure.*)

To prove that this value of α satisfies $\frac{1}{2} < \sin\alpha < \frac{1}{\sqrt{2}}$, extend EF a segment to equal itself, FP = a; draw segment AK || FP with K on BF. Now let I = AF ∩ BP, Q = AK ∩ BP, the area made up by segments EF, FI, IJ and Γ be w and the area made up by segments AI, IJ and Γ be z ($x = w + z$)..

Note that $\sin\alpha = \frac{c}{b} = \frac{c}{a+c}$; proving $\frac{1}{2} < \sin\alpha$ is equivalent to proving $\frac{1}{2} < \frac{c}{a+c}$ or $a < c$. Assuming that $a = c$, then the area inside triangle BEF but outside Γ equals y (or $\frac{1}{2}x = y$). This is not true because $x = y$; therefore, $a < c$.

Next proving $\sin\alpha < \frac{1}{\sqrt{2}}$ is equivalent to proving α < 45° or 45° + $\frac{180°}{\pi r}(\frac{1}{2}c - a) < 45°$ (from (i)), or $2a > c$. Indeed, since ∠AFP = ∠D, the two isosceles triangles AFP and BDF are similar and ∠APF = ∠BFD, or AP || KF, PF = AK and PF > AQ because AK > AQ. Since the two triangles PIF and QIA are similar with PF > AQ, we conclude that (PIF) > (QIA) > (JIA) > z. Therefore, w + (PIF) > $w + z = x = y$, or $2a > c$.

<u>Further observation</u>

It's easily seen that the two triangles BHF and FEB are congruent which causes ∠HBF = ∠EFB and the triangle BDF is isosceles with b = a + c. The area of the quadrilateral ABDF equals the sum of a quarter of the area of the circle ($\frac{\pi r^2}{4}$), x and y, or $x + y + \frac{\pi r^2}{4} = 2x + \frac{\pi r^2}{4} = 2[ar - \frac{\pi r^2}{360°}(90° - \alpha)] + \frac{\pi r^2}{4} = (ABDF) = \frac{1}{2}r(a + b)$*, or b*

$= 3a - \frac{\pi r}{2}(1 - \frac{\alpha}{45^\circ}) = \frac{r^2 + a^2}{2a}$, or $5a^2 - \pi r(1 - \frac{\alpha}{45^\circ})a - r^2 = 0$, or $r^2 - 5a^2 = \pi r(\frac{\alpha}{45^\circ} - 1)a$, and we come up with the same equation (ii).

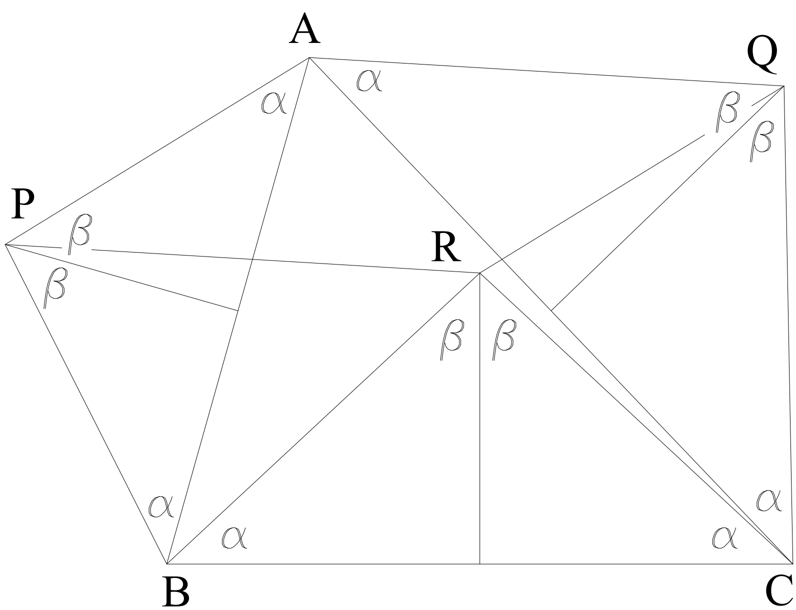
By proving that $\alpha < 45^\circ$, from (iii) we obtain $2 - 3\sin\alpha > 0$, or $\sin\alpha < \frac{2}{3}$ which is even smaller than $\frac{1}{\sqrt{2}}$.

As an exercise, we should try to solve the equation $\alpha = 45^\circ[1 - \frac{2}{\pi\cos\alpha}(2 - 3\sin\alpha)]$.

Problem 7 of Australia Mathematical Olympiad 2010

On the edges of a triangle ABC are drawn three similar isosceles triangles APB (with AP = PB), AQC (with AQ = QC) and BRC (with BR = RC). The triangles APB and AQC lie outside the triangle ABC and the triangle BRC is lying on the same side of the line BC as the triangle ABC. Prove that the quadrilateral PAQR is a parallelogram.

Solution



The way the three triangles APB, AQC and BRC are similar, $\angle PAB = \angle PBA = \angle QAC = \angle QCA = \angle RBC = \angle RCB$, and let them equal α. Now let $\beta = 90° - \alpha = \frac{1}{2}\angle APB = \frac{1}{2}\angle AQC = \frac{1}{2}$ $\angle BRC$.

The similarity of the mentioned triangles gives us $\frac{BP}{BR} = \frac{AB}{BC}$.

Combining with $\angle PBR = \alpha + \angle ABR = \angle ABC$, the two triangles

PBR and ABC are similar. Applying the exact same argument, the two triangles ABC and QRC are also similar. Therefore, the two triangles PBR and QRC are similar to each other which implies that $\angle BPR = \angle RQC$, and $\angle APR = 2\beta - \angle BPR = 2\beta - \angle RQC = \angle AQR$.

Furthermore, the similarity of the triangles also gives $\angle BRP = \angle ACB$ and $\angle CRQ = \angle ABC$.

Successively, $\angle PRQ = 360° - 2\beta - \angle BRP - \angle CRQ = 180° - 2\beta + 180° - (\angle ACB + \angle ABC) = 2\alpha + \angle BAC = \angle PAQ$.

Combining with the earlier result $\angle APR = \angle AQR$, we conclude that the quadrilateral PAQR is a parallelogram.

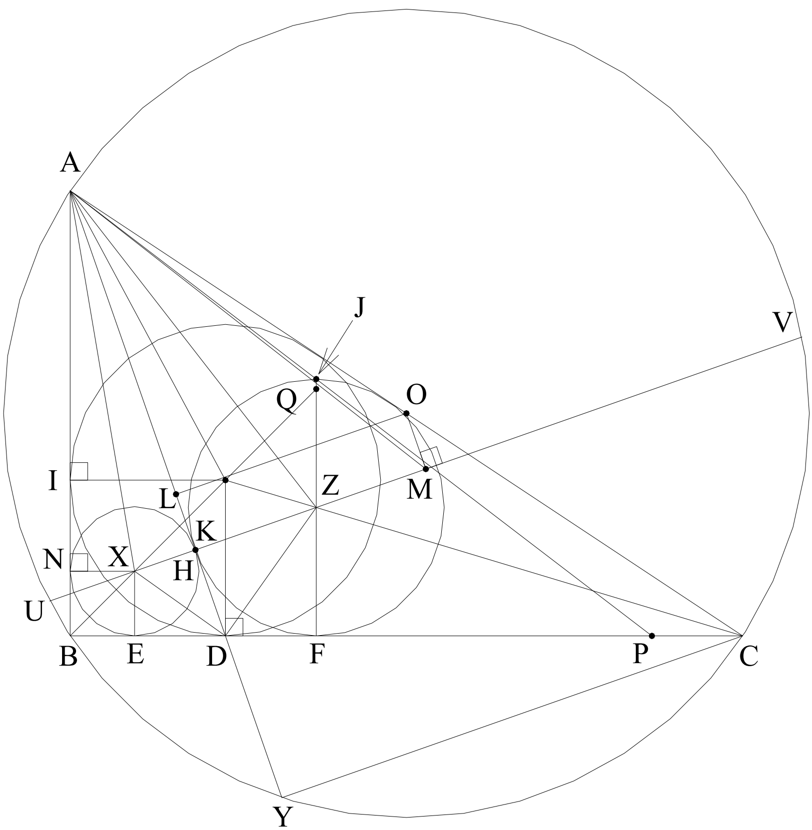
<u>Further observation</u>

Now that PAQR is a parallelogram, AP = QR; the two triangles BPR and RQC are congruent.

__Problem 5 of Turkey Mathematical Olympiad 2007__

Let ABC be a triangle with $\angle B = 90°$. The incircle of ABC touches the side BC at D. The incenters of triangles ABD and ADC are X and Z, respectively. The lines XZ and AD are intersecting at the point K. XZ and circumcircle of ABC are intersecting at U and V. Let M be the midpoint of line segment [UV]. AD intersects the circumcircle of ABC at Y other than A. Prove that $|CY| = 2|MK|$.

__Solution__



Let the incircle of triangle ABC touch the side AB at I, H and F be the feet of Z onto AD and CD, respectively, N and E the feet of X onto AB and BD, respectively, L the midpoint of AY and O the

circumcenter of triangle ABC. We need to show that the two incircles of triangles ABD and ACD are tangent to each other at H or K. To do this we need to prove that AH = AN.
It's easily seen that AC + AD + CD = 2(AH + DH + CF), or

AH = ½(AC + AD + CD) – (DH + CF) = ½(AC + AD – CD).

Similarly, AN = ½(AB + AD – BD), and AH = AN when AB + CD = AC + BD, but AC = AI + CD, and the previous equation becomes AB = AI + BD (i)

Since BD = BI is the inradius of triangle ABC, the equation (i) is true. Therefore, the two incircles of triangles ABD and ACD are tangent to each other at H. The tangential point is also on the line connecting the two centers, and H coincides with K, and AK⊥UV.

Because M is the midpoint of UV and O is the circumcenter, OM ⊥UV. Combining with AK⊥UV, we have LK || OM.

It's also because ∠B = 90°, AC is the diameter of the circumcircle and ∠AYC = 90°. Since O and L are the midpoints of AC and AY, respectively, OL || CY and OL = ½CY, and thus we also have OL⊥AY, or OL || MK and OMKL now becomes a rectangle which implies that OL = MK.

Therefore, we finally have |CY| = 2|MK|.

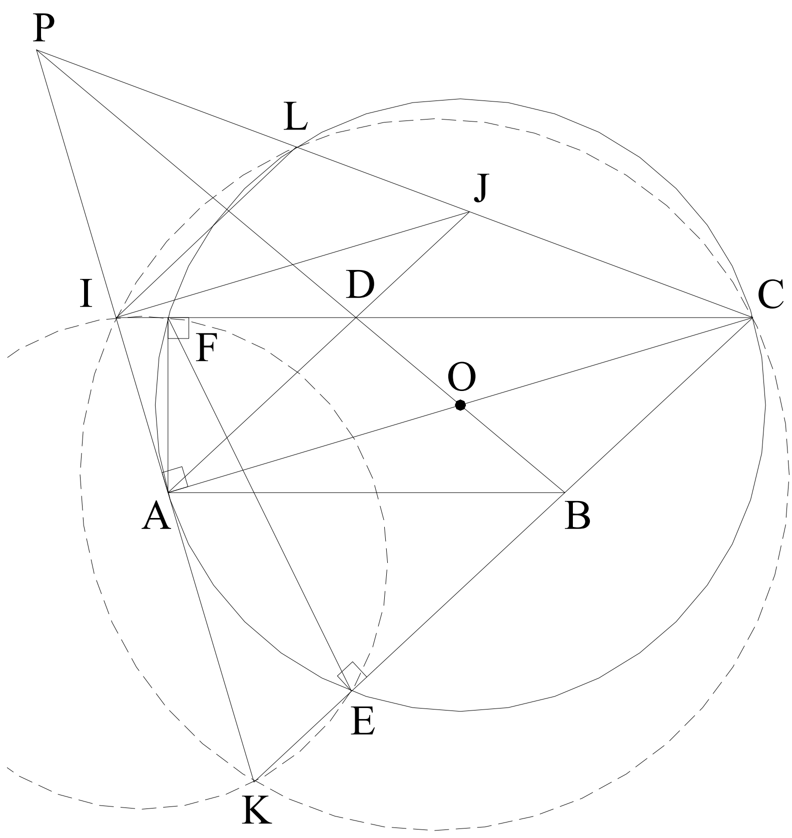
<u>Further observation</u>

Extend FZ to meet the incircle of triangle ACD at J as shown. Now link and extend AJ to meet BC at P. By definition, we should now have DF = CP.
Since the problem is already proven, we can also draw conclusion that DE = DF = CP. Now extend BX to meet FJ at Q. Since DE = DF, the incenter of triangle ABC is also the midpoint of XQ.
We also see that if S is the incenter of triangle ABC, these triangles are similar to each other: triangles AXD and ASZ, triangles AXS and ADZ.

Problem 2 of Turkey MO Team Selection Test 1996

In a parallelogram ABCD with $\angle A < 90°$, the circle with diameter AC intersects the lines CB and CD again at E and F, and the tangent to this circle at A meets the line BD at P. Prove that the points P, E, F are collinear.

Solution



Extend PA and CE to meet at K, CF to meet PA at I, AD to meet PC at J, PC to meet the circle with diameter AC at L. Since O is the midpoint of AC, per Ceva's theorem, IJ || AC. Also note that AJ || CK because ABCD is a parallelogram.

That brings us to $\dfrac{PI}{PA} = \dfrac{PJ}{PC} = \dfrac{PA}{PK}$, or $PA^2 = PI \times PK$.

However, because AECL is cyclic and A is the tangential point, $PA^2 = PL \times PC$. Therefore, per the intersecting secant theorem $PI \times PK = PL \times PC$, or IKCL is also cyclic as shown.

We also have $CF \times CI = CF(CF + FI) = CF^2 + CF \times FI = CF^2 + AF^2 = AC^2 = CE^2 + AE^2 = CE^2 + CE \times EK = CE(CE + EK) = CE \times CK$. This makes IKEF a cyclic quadrilateral.

Combining with LCEF being cyclic and both KI, CL intersecting at P, we conclude that the three points P, E, F are collinear.

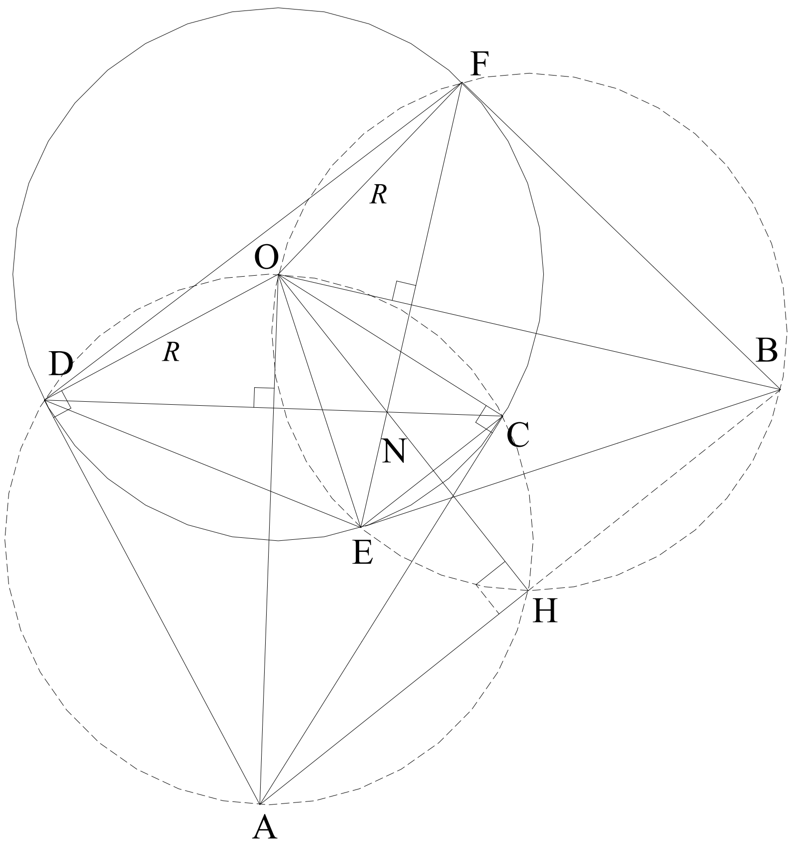
<u>Further observation</u>

The proof of IKCL being a cyclic quadrilateral is just a bonus; it's not required to prove the problem.

Problem 6 of Pan African 2009

Points C, E, D and F lie on a circle with center O. Two chords CD and EF intersect at a point N. The tangents at C and D intersect at A, and the tangents at E and F intersect at B. Prove that ON $\perp$ AB.

Solution



Draw the circle with radius OA; extend ON to meet this circle at point H. We do have $\angle$OHA = 90° and per the intersecting chord theorem (when two chords intersect each other inside a circle, the products of their segments are equal), ON×NH = DN×NC because both D and C are also on this circle. We are also given the fact that the four points C, E, D and F lie on a circle, and thus EN×NF =

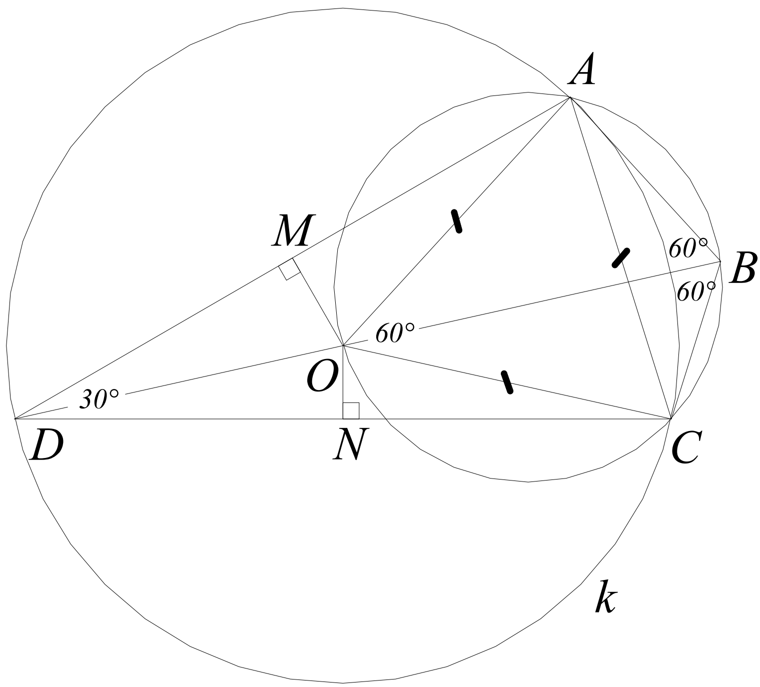
$DN \times NC$, or $ON \times NH = EN \times NF$. This makes H to be on the circle that has OB as its diameter as shown, or $\angle OHB = 90°$. We then have $\angle OHA + \angle OHA = 180°$, or the three points A, H and B are collinear.

In other words, ON is perpendicular to AB.

Problem 7 of Belarus Mathematical Olympiad 1997

If ABCD is a convex quadrilateral with $\angle$ADC = 30° and BD = AB + BC + CA, prove that BD bisects $\angle$ABC.

Solution



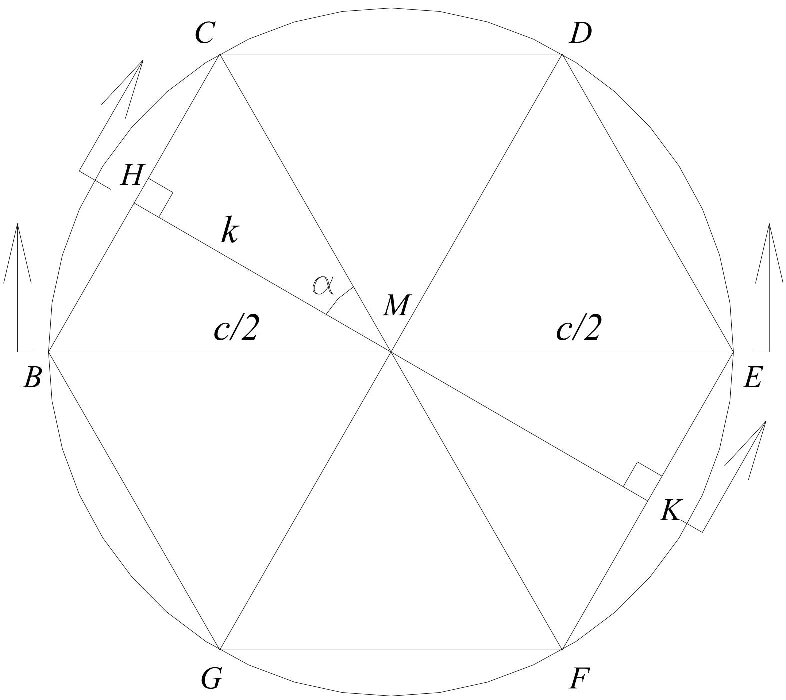
Draw the circumcircle *k* of triangle ACD with center O and let *R* be its radius. Because $\angle$AOC = 2$\angle$ADC = 60° and OA = OC = *R*, AOC is an equilateral triangle with side length *R*.

Extending DO to meet the circumcircle of the equilateral triangle AOC at B; ABCO is cyclic, and by Ptolemy's theorem, we get OB×AC = AB×OC + BC×OA, or OB×*R* = AB×*R* + BC×*R*, or OB = AB + BC. Now add DO = *R* to both sides to get BD = *R* + AB + BC = AB + BC + CA, and we have found point B to form the convex quadrilateral ABCD described in the problem. Also since both angles $\angle$OBA and $\angle$OBC subtend equal arcs OA = OC, $\angle$OBA = $\angle$OBC = 60°, and BD bisects $\angle$ABC.

Problem 2 of the Vietnamese Mathematical Olympiad 1986

Let R and r be the respective circumradius and inradius of a regular 1986-gonal pyramid. Prove that $\frac{R}{r} \geq 1 + \frac{1}{\cos\frac{\pi}{1986}}$ and find the total area of the surface of the pyramid when equality occurs.

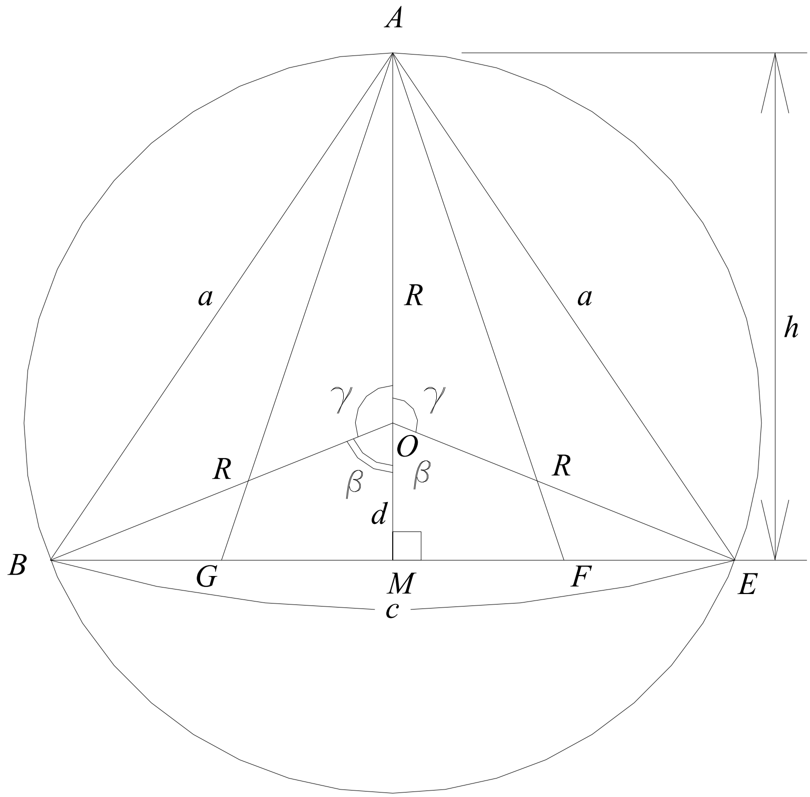
Solution



FLOOR PLAN

For easy visualization, the graphs depict a hexagon BCDEFG instead of a 1986-gon BCDEFG….

Let A be the peak of the pyramid and M its foot onto the base which is a 1986-gon, a the side length of the 1986-gon and a = AB = AE = …, b the altitude from A to the side and b = AH = AK = … (see I-J cross section), $c/2$ = BM = ME = … is the radius of the circumcircle of the base, h = AM the altitude from A to its base (or the height of the pyramid), k = MH (MH$\perp$BC) the shortest distance from the center of the base to its side. Also let O and I be the center of the circumsphere of the pyramid and center of the sphere inscribing it, respectively, $\alpha = \angle$BMH $= \angle$CMH = …, $\beta = \angle$BOM $= \angle$EOM, $\gamma = \angle$AOB $= \angle$AOE, $2\eta = \angle$AHM, or $\eta = \angle$IHA $= \angle$IHM.

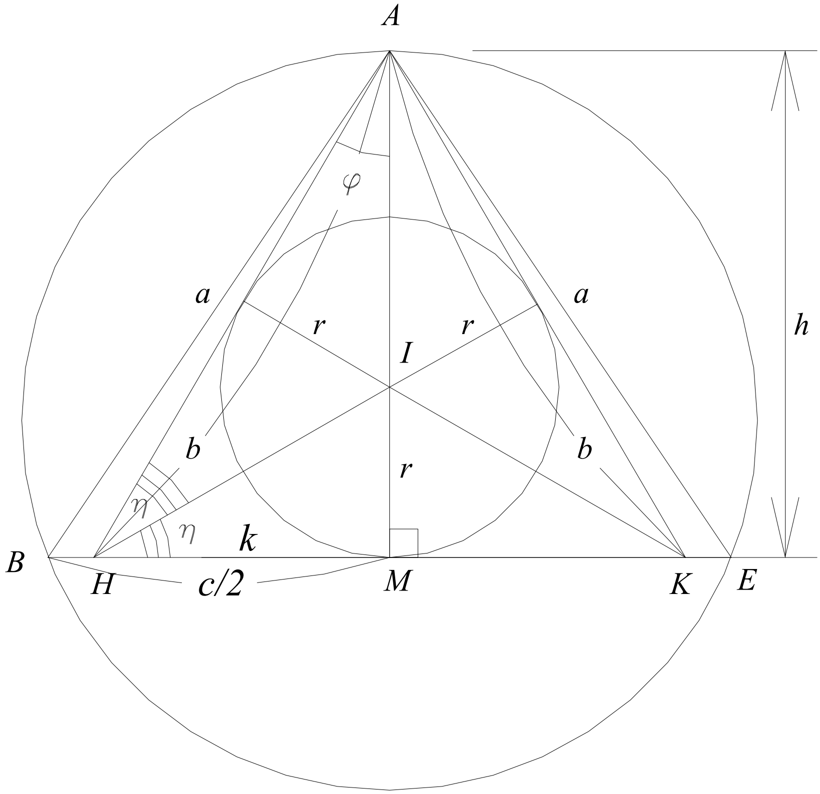


B-E CROSS SECTION

Now notice that in the inequality of the problem $\frac{R}{r} \geq 1 + \frac{1}{\cos\alpha} = \frac{1 + \cos\alpha}{\cos\alpha}$ or $r \leq \frac{R\cos\alpha}{1 + \cos\alpha}$ the only variable is r; everything else is fixed. We, therefore, are required to find that the maximum value of r must be equal to $\dfrac{R\cos\alpha}{1+\cos\alpha}$. The trick here is that we must find a relationship between r and another single variable of the configuration, if there is any, so that we can take the derivative of r with respect to this variable, then set the numerator of the derivative to zero in order to find the extreme value of r. And now let's attempt to find this variable.

In the I-J cross section below, since IH is the bisector of $\angle$AHM, we have $\tan\eta = \dfrac{r}{k} = \dfrac{\mathrm{AI}}{b} = \dfrac{r+\mathrm{AI}}{k+b} = \dfrac{h}{k+b}$.

But $h = R + d$ and now $r = k\times\dfrac{R+d}{k+b} = \dfrac{R+d}{1+b/k}$ (i)



I-J CROSS SECTION

From the floor plan $\alpha = \frac{360°}{2\times1986} = \frac{\pi}{1986}$, $k = \frac{c}{2}\times\cos\alpha$, $c = 2R\sin\beta$, or $k = R\cos\alpha\sin\beta$, $d = R\cos\beta$, $\sin^2\beta = 1 - \cos^2\beta = \frac{R^2 - d^2}{R^2}$, $b^2 = (R + d)^2 + k^2$, or $b = \sqrt{(R + d)^2 + k^2}$, and $\frac{b}{k} = \sqrt{1 + \frac{R + d}{\cos^2\alpha(R - d)}}$.

Substitute these values into (i) to get $r = \dfrac{R + d}{1 + \sqrt{1 + \frac{R + d}{\cos^2\alpha(R - d)}}}$.

Now notice that r is a function of a single variable d, the distance from the center of the circumsphere of the pyramid to its base M.

We're only interested in finding the value of d at which the derivative of r, denoted r', is zero and are ignoring the denominator of r'. The numerator of the derivative of r with respect to d is

$$1 + \sqrt{1 + \frac{R + d}{\cos^2\alpha(R - d)}} - \frac{R(R + d)}{\cos^2\alpha(R - d)^2}\times\frac{1}{\sqrt{1 + \frac{R + d}{\cos^2\alpha(R - d)}}},$$

and is equal to zero when

$$\cos^2\alpha(R - d)^2\sqrt{1 + \frac{R + d}{\cos^2\alpha(R - d)}} + \cos^2\alpha(R - d)^2\left[1 + \frac{R + d}{\cos^2\alpha(R - d)}\right] - R(R + d) = 0.$$

Replace d with $R\cos\beta$ to get

$$\cos^2\alpha(1 - \cos\beta)^2\sqrt{1 + \frac{1 + \cos\beta}{\cos^2\alpha\sin^2\beta(1 - \cos\beta)}} + \cos^2\alpha(1 - \cos\beta)^2\left[1 + \frac{1 + \cos\beta}{\cos^2\alpha(1 - \cos\beta)}\right] - (1 + \cos\beta) = 0.$$

Now multiplying both the numerators and denominators of the ratios with $1 + \cos\beta$, we have

$$\cos^2\alpha(1 - \cos\beta)^2\sqrt{1 + \frac{(1 + \cos\beta)^2}{\cos^2\alpha\sin^2\beta}} + \cos^2\alpha(1 - \cos\beta)^2\left[1 + \frac{(1 + \cos\beta)^2}{\cos^2\alpha\sin^2\beta}\right] - (1 + \cos\beta) = 0, \text{ or}$$

$$\cos^2\alpha\times\frac{(1-\cos\beta)^2}{\cos\alpha\sin\beta}\sqrt{\cos^2\alpha\sin^2\beta+(1+\cos\beta)^2}+\cos^2\alpha(1-\cos\beta)^2$$
$$+(1-\cos\beta)^2\frac{(1+\cos\beta)^2}{\sin^2\beta}-(1+\cos\beta)=$$
$$\cos\alpha\times\frac{(1-\cos\beta)^2}{\sin\beta}\times\sqrt{\cos^2\alpha\sin^2\beta+(1+\cos\beta)^2}+\cos^2\alpha(1-\cos\beta)^2+$$
$$\sin^2\beta-(1+\cos\beta)=0.$$

Next, multiply both sides by $\sin\beta$ and rearrange the terms to obtain $\cos\alpha\times(1-\cos\beta)^2\sqrt{\cos^2\alpha\sin^2\beta+(1+\cos\beta)^2}=\sin\beta(1+\cos\beta)-$ $\cos^2\alpha\sin\beta(1-\cos\beta)^2-\sin^3\beta=0.$

Now divide both sides by $\cos\alpha\times(1-\cos\beta)^2$; the result is

$$\sqrt{\cos^2\alpha\sin^2\beta+(1+\cos\beta)^2}=\frac{\sin\beta(1+\cos\beta)}{\cos\alpha(1-\cos\beta)^2}-\cos\alpha\sin\beta-$$
$$\frac{\sin^3\beta}{\cos\alpha(1-\cos\beta)^2}.$$

Continue by squaring both sides and deleting the equal terms

$$(1+\cos\beta)^2=\frac{\sin^2\beta(1+\cos\beta)^2}{\cos^2\alpha(1-\cos\beta)^4}+\frac{\sin^6\beta}{\cos^2\alpha(1-\cos\beta)^4}-$$
$$2\times\frac{\sin^2\beta(1+\cos\beta)}{(1-\cos\beta)^2}-2\times\frac{\sin^4\beta(1+\cos\beta)}{\cos^2\alpha(1-\cos\beta)^4}+2\times\frac{\sin^4\beta}{(1-\cos\beta)^2}.$$

Now divide both sides by $(1+\cos\beta)^2$, knowing that $\sin^2\beta=(1+\cos\beta)(1-\cos\beta)$, $\sin^4\beta=(1+\cos\beta)^2(1-\cos\beta)^2$ and $\sin^6\beta=(1+\cos\beta)^3(1-\cos\beta)^3$, to transform the equation into, terms by terms

$$1=\frac{\sin^2\beta}{\cos^2\alpha(1-\cos\beta)^4}+\frac{1+\cos\beta}{\cos^2\alpha(1-\cos\beta)}-2\times\frac{1}{1-\cos\beta}-2\times$$
$$\frac{1+\cos\beta}{\cos^2\alpha(1-\cos\beta)^2}+2,\text{ or } 0=1+\frac{1+\cos\beta}{\cos^2\alpha(1-\cos\beta)^3}+$$
$$\frac{1+\cos\beta}{\cos^2\alpha(1-\cos\beta)}-2\times\frac{1}{1-\cos\beta}-2\times\frac{1+\cos\beta}{\cos^2\alpha(1-\cos\beta)^2}.$$

We proceed by multiplying both sides by $\cos^2\alpha(1-\cos\beta)^3$ to get

$0 = \cos^2\alpha(1 - \cos\beta)^3 + 1 + \cos\beta + (1 + \cos\beta)(1 - \cos\beta)^2 - 2\cos^2\alpha\times$ $(1 - \cos\beta)^2 - 2(1 + \cos\beta)(1 - \cos\beta)$, or $0 = (1 + \cos\beta)[\cos^2\beta -$ $\cos^2\alpha(1 - \cos\beta)^2]$.

But $1 + \cos\beta \neq 0$; hence, $\cos^2\beta - \cos^2\alpha(1 - \cos\beta)^2 = 0$, or $(1 - \cos^2\alpha)\cos^2\beta + 2\cos^2\alpha\cos\beta - \cos^2\alpha = 0$.

Solving for $\cos\beta$, we get $\cos\beta = \dfrac{\cos\alpha}{1 + \cos\alpha}$ or $\cos\beta = \dfrac{-\cos\alpha}{1 - \cos\alpha}$.

From there we verify and confirm that with $\cos\beta = \dfrac{\cos\alpha}{1 + \cos\alpha}$ the value of the radius r attains its maximum (*refer to a calculus book on how to do this*).

Replacing $\cos\beta = \dfrac{\cos\alpha}{1 + \cos\alpha}$ into $r = \dfrac{R + d}{1 + \sqrt{1 + \dfrac{R + d}{\cos^2\alpha(R - d)}}}$ with

$d = R\cos\beta$, we get $r = \dfrac{R\cos\alpha}{1 + \cos\alpha}$ which is the result we seek, and the first part of the problem is proven.

When equality occurs or when $r = \dfrac{R\cos\alpha}{1 + \cos\alpha}$, the area of the base which is the 1986-gon is 1986 times the area of triangle BCM, and it equals $1986\times\frac{1}{2}\times\mathrm{BC}\times k = 1986\times\frac{1}{2}\times c\sin\alpha\times\frac{1}{2}c\times\cos\alpha$ (where $c =$

$2R\sin\beta) = 1986R^2\times\sin\alpha\cos\alpha\times\dfrac{1 + 2\cos\alpha}{(1 + \cos\alpha)^2}$. Whereas the total area of the isosceles, equal and slanted triangles that share the common vertice A is 1986 times the area of triangle ABC.

This area equals $1986\times\frac{1}{2}\times\mathrm{BC}\times b = 1986\times\frac{1}{2}c\times\sin\alpha\times\sqrt{(R + d)^2 + k^2}$ $=$

$1986R^2\sin\alpha\times\sin\beta\times\sqrt{\cos^2\alpha[1 - \dfrac{\cos^2\alpha}{(1 + \cos\alpha)^2}] + [1 + \dfrac{\cos\alpha}{1 + \cos\alpha}]^2} =$

$1986R^2\sin\alpha\times\dfrac{1 + 2\cos\alpha}{1 + \cos\alpha} = 1986R^2\sin\alpha(1 + \dfrac{\cos\alpha}{1 + \cos\alpha})$.

The total area of the surface of the pyramid when equality occurs is

$1986R^2 \times \sin\alpha\cos\alpha \times \dfrac{1 + 2\cos\alpha}{(1 + \cos\alpha)^2} + 1986R^2 \sin\alpha \times \dfrac{1 + 2\cos\alpha}{1 + \cos\alpha} =$

$1986R^2 \times \sin\alpha(1 + \dfrac{\cos\alpha}{1 + \cos\alpha})^2 = 1986R^2 \times \sin\alpha(1 + \dfrac{r}{R})^2 =$

$1986\sin\alpha(R + r)^2 =$ where $\alpha = \dfrac{\pi}{1986}$.

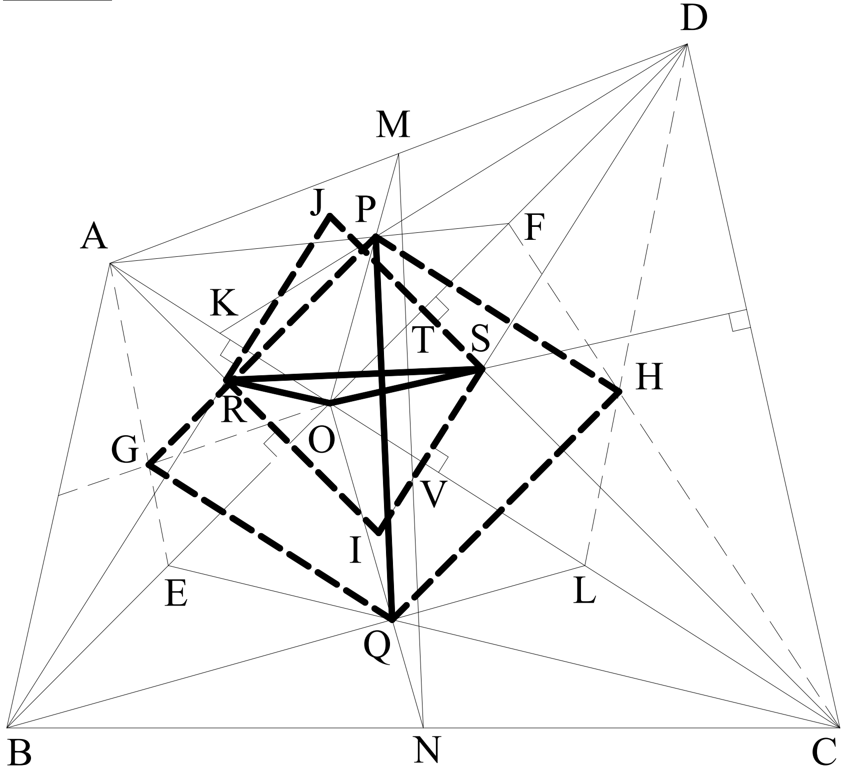
<u>Further observation</u>

When the inner sphere is largest or $r = \dfrac{R\cos\alpha}{1 + \cos\alpha}$, $d = R\cos\beta = \dfrac{R\cos\alpha}{1 + \cos\alpha} = r$. *We conclude that the centers of the two spheres coincide with each other, or the spheres are concentric.*

<u>*Problem 5 of British Mathematical Olympiad 1990*</u>

The diagonal of a convex quadrilateral ABCD intersect at O. The centroids of triangles AOD and BOC are P and Q; the orthocenters of triangles AOB and COD are R and S, respectively. Prove that PQ is perpendicular to RS.

<u>Solution</u>



Extend AR and DS to meet at I, BR and CS to meet at J. Let G and H be the centroids of the other two triangles AOB and COD, respectively, E and F the midpoints of BO and DO, respectively, T the intersection of CS and BD, V the intersection of DS and AC.

Since both BR and DS are perpendicular to AC, RJ || SI. The same can be said about RI and SJ, or RI || SJ, and RISJ is a parallelogram.

On the other hand, since P, G, Q and H are the centroids, we obtain PG $\parallel$ BD and $\frac{PG}{EF} = \frac{AG}{AE} = \frac{2}{3}$, or PG $= \frac{2}{3}\times EF = \frac{1}{3}\times BD$. The same reasoning applies to QH and BD. We then have QH $\parallel$ BD $\parallel$ PG and QH $= \frac{1}{3}\times BD = PG$. Therefore, PGQH is also a parallelogram, and we have the ratio $\frac{PG}{GQ} = \frac{BD}{AC}$.

Now because AR $\perp$ BD and PG $\parallel$ BD, AR $\perp$ PG, or RI $\perp$ PG. With the similar reasoning RJ $\perp$ GQ.

The similarity of the four triangles BJT, DST, CSV and AIV gives us $\frac{ST}{DT} = \frac{JT}{BT} = \frac{ST + JT}{BT + DT} = \frac{SJ}{BD} = \frac{SV}{CV} = \frac{VI}{AV} = \frac{SV + VI}{CV + AV} = \frac{SI}{AC}$, or $\frac{SJ}{SI} = \frac{BD}{AC} = \frac{PG}{GQ} = \frac{PG}{PH}$.

Therefore, the two parallelograms RISJ and PGQH are similar with their respective sides perpendicular to one another RI $\perp$ PG, RJ $\perp$ GQ, and thus their respective diagonals must also be perpendicular to each other and PQ $\perp$ RS.

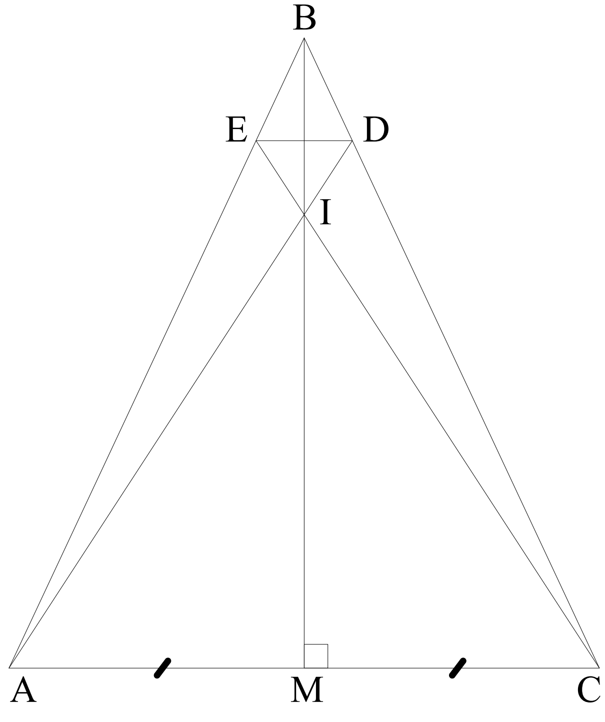
<u>Further observation</u>

Because PQ $\perp$ RS, we also have MN $\perp$ RS since PQ $\parallel$ MN.

Problem 9 of Russia Sharygin Geometry Olympiad 2010

A point inside a triangle is called "*good*" if three cevians passing through it are equal. Assume for an isosceles triangle ABC with AB = BC the total number of "*good*" points is odd. Find all possible values of this number.

Solution



Let the cevians from A, B and C be AD, BM and CE, respectively. The problem gives us AD = BM = CE. Apply the law of sines to get AD/AB = sin∠ABC/sin∠ADB = CE/BC = sin∠ABC/sin∠CEB, or ∠ADB = ∠CEB and ∠BAD = ∠BCE which makes the two triangles ABD and CEB to be congruent. Subsequently, AD meets CE at a point I on the cevian BM which is also the bisector of ∠ABC. So there is only one unique segment length AD = CE = BM, and only one "*good*" point.

<u>Sample Mathematical Olympiad Problem</u>

Given triangle ABC, its orthocenter H and its altitude AD, BE and CF such that the perimeters of the triangles AHB, AHC and BHC are the same. Prove that triangle ABC is equilateral. (*This problem was proposed but has never been selected for any competition.*)

<u>Solution</u>

Superimpose the two triangles ABH and CBH where the vertex C of triangle BHC coincides with vertex A of triangle AHB, and vertices B and H of triangle BHC are renamed to B′ and H′, respectively as shown. Assuming that AB ≠ BC and AH ≠ CH, the problem gives us

AB + AH = AB′ + AH′.

Therefore, BB′ = HH′, and BH must intercept B′H′ at a point. Let's call it I, inside triangle ABH' and certainly inside angle BAH.

Assign the Greek letters to the angles as shown. From I draw the altitudes IP and IQ to BB′ and HH′, respectively. It's easily seen that IB > IB′ and IH > IH′, or BH > B′H′ which causes the perimeters of triangles ABH and CBH to be different.

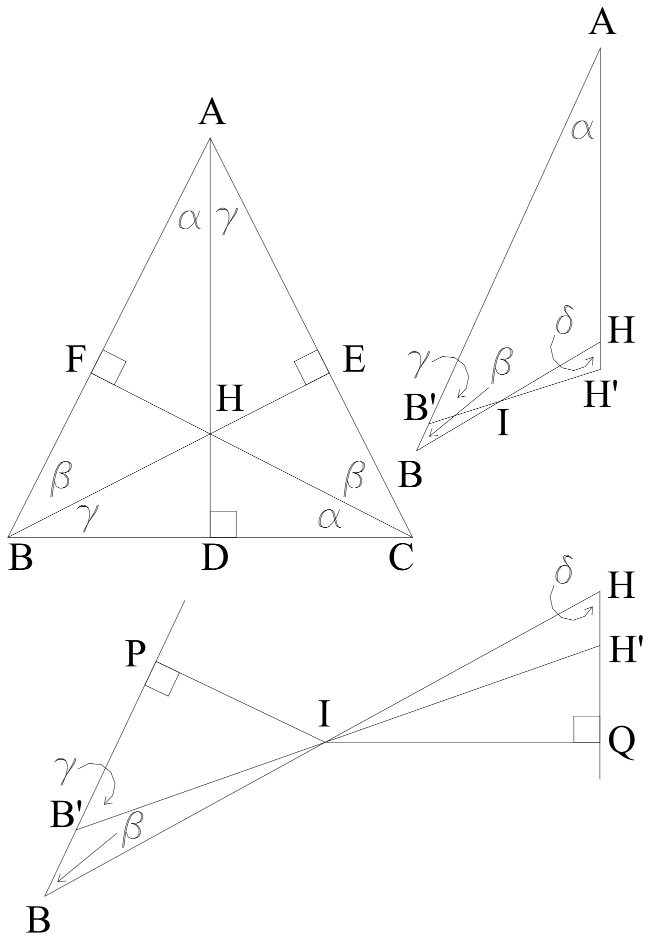
Therefore, our assumption that AB ≠ BC and AH ≠ CH is not possible, and thus AB = BC.

The same argument can be used for one of these triangles and triangle ACH making AB = AC.

ABC is then an equilateral triangle.

<u>Further observation</u>

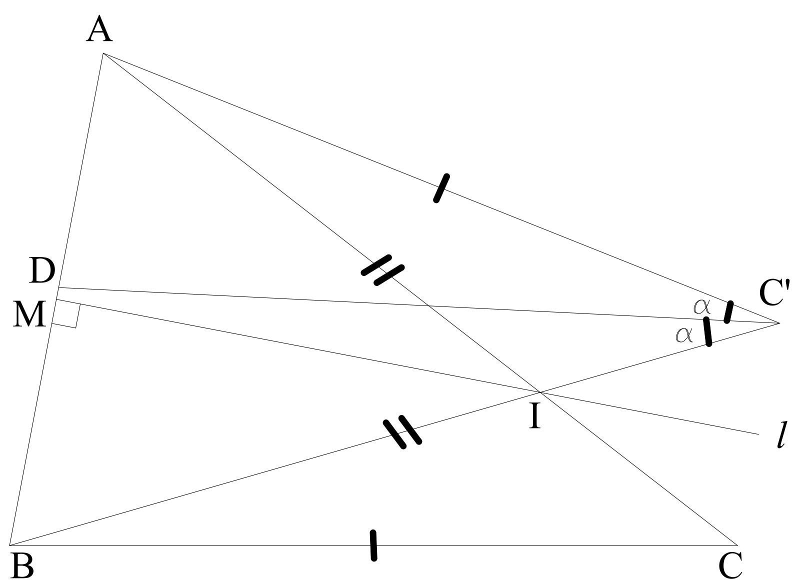
There are possibly many other methods that can be utilized to solve this problem.


A
α γ
F
E
H
β β
γ α
B D C
A
α
δ H
γ β
B' H'
I
B
δ H
P H'
I
Q
γ
B' β
B

Problem 10 of Russia Sharygin Geometry Olympiad 2010

Let three lines forming a triangle ABC be given. Using a two-sided ruler and drawing at most eight lines construct a point D on the side AB such that $\frac{AD}{BD} = \frac{BC}{AC}$.

Solution



Draw the perpendicular bisector of segment AB and name it *l* as shown. Locate point C’ which is the symmetrical point of C across *l*. Next, draw the bisector of angle AC’B to meet segment AB at D. Since C’D is the bisector, we get $\frac{AD}{BD} = \frac{AC'}{BC'} = \frac{BC}{AC}$, and D is the point that needs to be constructed.

Further observation

Finding a point is probably a more appropriate term than constructing a point. One should construct a line and not a point as is the term used in the problem.

Problem 1 of the Russian Mathematical Olympiad 2008

Do there exist 14 positive integers such that, upon increasing each of them by 1, their product increases exactly 2008 times?

Solution

Let the 14 positive integers be a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}, a_{13} and a_{14}.

We are given $[(a_1+1)(a_2+1)(a_3+1)(a_4+1)(a_5+1)(a_6+1)(a_7+1)(a_8+1)(a_9+1)(a_{10}+1)(a_{11}+1)(a_{12}+1)(a_{13}+1)(a_{14}+1)]/[a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}] = 2008$, or

$$(1+\frac{1}{a_1})(1+\frac{1}{a_2})(1+\frac{1}{a_3})(1+\frac{1}{a_4})(1+\frac{1}{a_5})(1+\frac{1}{a_6})(1+\frac{1}{a_7})(1+\frac{1}{a_8})(1+\frac{1}{a_9})(1+\frac{1}{a_{10}})(1+\frac{1}{a_{11}})(1+\frac{1}{a_{12}})(1+\frac{1}{a_{13}})(1+\frac{1}{a_{14}}) = 2008 \qquad \text{(i)}$$

It's easily seen that we must have at least seven integers a's with all their values equal to 1's. Because if only six a's or less with all their values equal to 1's, without loss of generality, let $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$, we then have

$$2^6(1+\frac{1}{a_7})(1+\frac{1}{a_8})(1+\frac{1}{a_9})(1+\frac{1}{a_{10}})(1+\frac{1}{a_{11}})(1+\frac{1}{a_{12}})(1+\frac{1}{a_{13}})(1+\frac{1}{a_{14}})$$

$\leq 2^6\times(1+\frac{1}{2})^8 = 1640.25$ (when the rest of the remaining a's are minimum and equal to 2's) < 2008.

Now since seven values of a's are 1's, the equation (i) reduces to

$$2^7(1+\frac{1}{a_8})(1+\frac{1}{a_9})(1+\frac{1}{a_{10}})(1+\frac{1}{a_{11}})(1+\frac{1}{a_{12}})(1+\frac{1}{a_{13}})(1+\frac{1}{a_{14}}) = 2008,$$

or $16(a_8+1)(a_9+1)(a_{10}+1)(a_{11}+1)(a_{12}+1)(a_{13}+1)(a_{14}+1) = 251a_8a_9a_{10}a_{11}a_{12}a_{13}a_{14}$.

Because 251 is a prime number, it's only natural to let one of the terms on the left side equal 251. Again, without loss of generality, let $a_{14} + 1 = 251$, or $a_{14} = 250$, and the above equation becomes

$8(a_8 + 1)(a_9 +1)(a_{10} + 1)(a_{11} + 1)(a_{12} + 1)(a_{13} + 1) = 125a_8 a_9 a_{10} a_{11}$
$\times a_{12} a_{13} = 5\times5\times5a_8 a_9 a_{10} a_{11} a_{12} a_{13}$.

Next let $a_{11} = a_{12} = a_{13} = 4$, and the previous equation is equivalent to $(a_8 + 1)(a_9 +1)(a_{10} + 1) = 8a_8 a_9 a_{10}$. And now we can see that with $a_8 = a_9 = a_{10} = 1$ this latest equation is also satisfied.

Therefore, there exist 14 positive integers such that, upon increasing each of them by 1, their product increases exactly 2008 times and they are $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 1$, $a_{11} = a_{12} = a_{13} = 4$ and $a_{14} = 250$.

Problem 6 Tournament of Towns 2008

Let ABC be a non-isosceles triangle. Two isosceles triangles AB'C with base AC and CA'B with base BC are constructed outside of triangle ABC. Both triangles have the same base angle φ. Let C_1 be a point of intersection of the perpendicular from C to A'B' and the perpendicular bisector of the segment AB. Determine the value of $\angle AC_1B$.

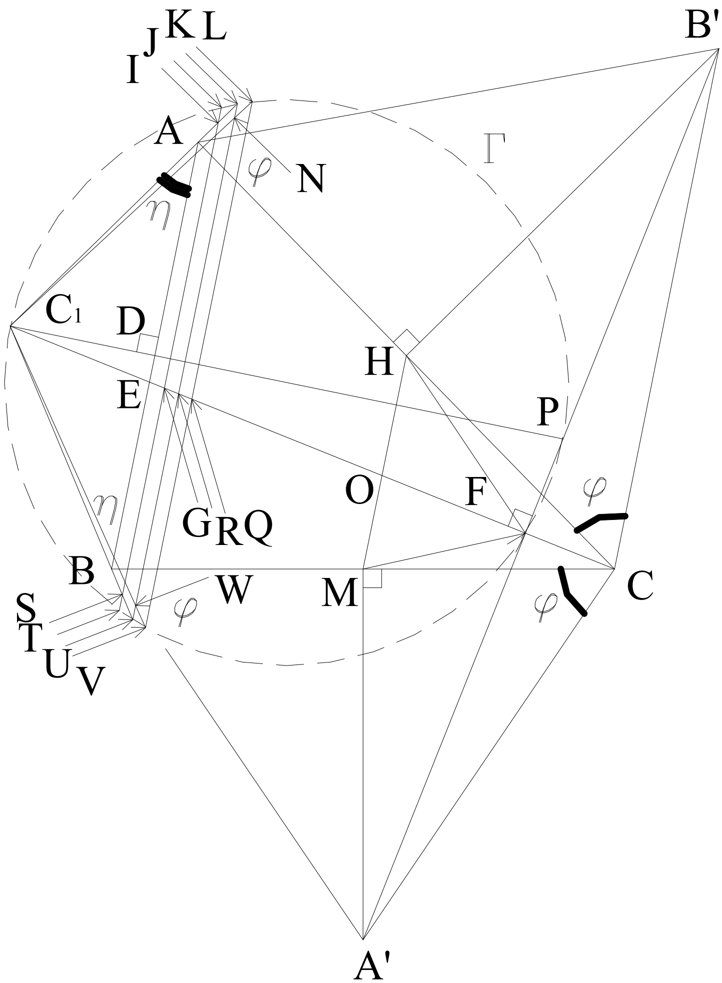
Solution

Let M, H and D be the midpoints of BC, AC and AB, respectively, $F = A'B' \cap C_1C$, $O = MH \cap C_1C$, $P = A'B' \cap C_1D$. Draw the circle Γ with diameter C_1P. Now let $K = C_1A \cap \Gamma$, $U = C_1B \cap \Gamma$, $E = AB \cap C_1C$, $R = KU \cap C_1C$. Since CB'HF and CA'MF are cyclic, $\angle HFC_1 = \angle MFC_1 = ½\angle AB'C = ½\angle BA'C = 90° - \varphi$, and because M and H are the midpoints as defined, MH || AB, OH/OM = EA/EB. For C_1P is the diameter and also the bisector of $\angle AC_1B$, extensions $C_1K = C_1U$ and KU || AB. From there, we get OH/OM = RK/RU.

Now extend FH and FM to meet the circle Γ and assume that these extensions do not meet Γ at K and U, respectively. Instead we assume they meet at points J and T on the left side of K and U, respectively and then L and V, on the right side of K and U, respectively, and then prove that these are not true.

First, assuming that $J = FH \cap \Gamma$ and $T = FM \cap \Gamma$. Let $G = C_1C \cap JT$, $I = C_1K \cap JT$, $S = C_1U \cap JT$. Since $\angle HFC_1 = \angle MFC_1$, we have $C_1J = C_1T$ or JT || AB || KU, and OH/OM = EA/EB = GI/GS = GJ/GT = (GJ – GI)/(GT – GS) = IJ/ST = 1 (because IJ = ST); therefore, GJ = GT which is false because G is not on the diameter C_1P.

Now assume that $L = FH \cap \Gamma$ and $V = FM \cap \Gamma$. Let $Q = CC_1 \cap LV$, $N = C_1L \cap KU$, $W = C_1V \cap KU$.

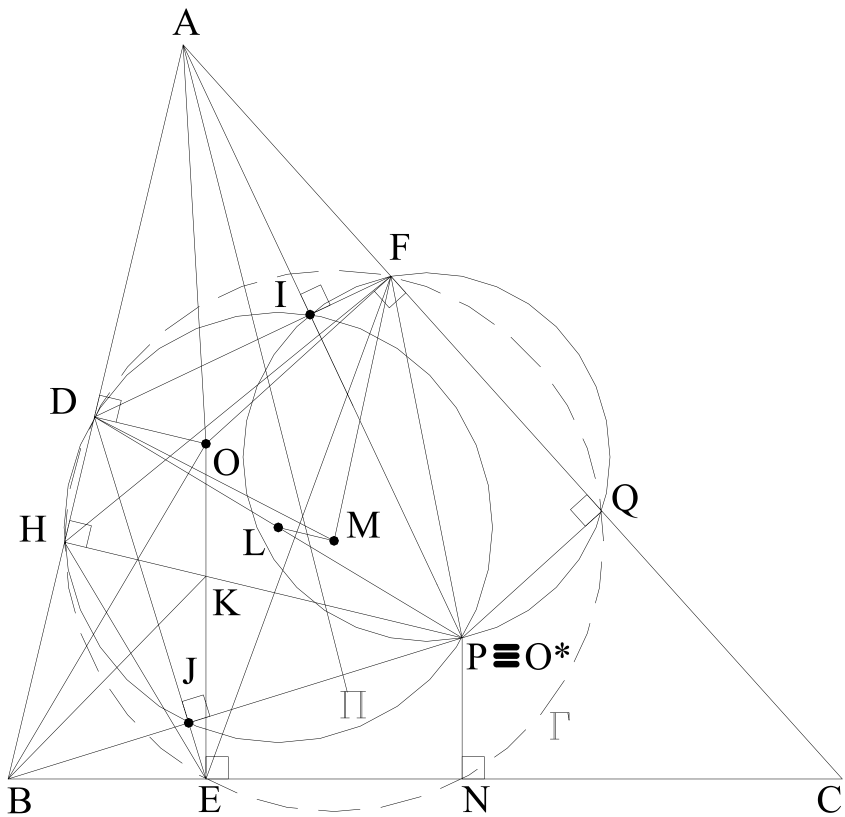


With the similar analysis, we get RN = RW which is again not true because R is not on the diameter C_1P. Therefore, the three points F, H and K must be on a straight line, and so are the three points F, M and U which imply that $\angle AC_1B = \angle KC_1U = 180° - \angle KFU = 180° - \angle HFM = 180° - \angle AB'C = 2\varphi$.

Problem 4 of Bulgaria Mathematical Olympiad 2011

Point O is inside triangle ABC. The feet of perpendicular from O to AB, BC, CA are D, E, F, respectively. Perpendiculars from A and B, respectively to DF and ED meet at P. Let H be the foot of perpendicular from P to AB. Prove that D, E, F, H are concyclic.

Solution



Let the intersection of AP and DF be I, the intersection of BP and DE be J. Now draw the circumcircle Γ of triangle DFH to intersect AC at Q, BC at N (Γ is known as the pedal circle of point O) and apply the intersecting secant theorem (when two secant lines intersect each other outside a circle, the products of their segments

are equal) to get AD×AH = AF×AQ. We will prove that this circle also passes through point E.
Indeed, since ∠DHP = ∠DIP = ∠DJP = 90°, D, I, P, J and H are concyclic and we have AD×AH = AI×AP, and thus AD×AH = AF×AQ; therefore, FQPI is also cyclic and because ∠FIP = 90°, ∠FQP = 180° – ∠FIP = 90°.

Since PH⊥AB, PQ⊥AC and H, Q are on the pedal circle Γ, we conclude that point P is the isogonal conjugate of point O, denoted O*, and Γ is also the pedal circle of point P, or O*. Hence, by definition, PN⊥BC and ∠ENP = 90°, and EJPN is also another cyclic quadrilateral because the sum of two of its opposite angles is 180° which causes BJ×BP = BE×BN.

However, because D, H, J and P are concyclic, the intersecting secant theorem also gives us BJ×BP = BH×BD, and now BH×BD = BE×BN to imply that the four points D, H, E and N are on the same circle Γ and the problem is proven.

Further observation

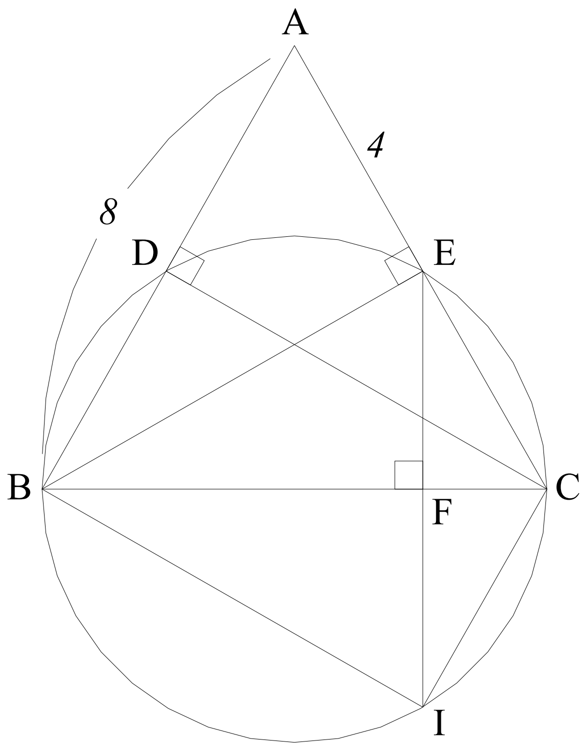
Let's try to solve this modified problem:

Let K, Π, L and M be the intersection of OE and HP, the circumcircle of triangle DEH, the circumcenter of Π, and the circumcenter of Γ, respectively. Prove that ∠OBK = ∠DML.

Problem 4 of Hong Kong Mathematical Olympiad 2004

In the figure below, BEC is a semicircle and F is a point on the diameter BC. Given that BF:FC = 3:1, AB = 8 and AE = 4. Find the length of EC.

Solution



Per the intersecting secant theorem, AD×AB = AE×AC, or 2AD = AC = 4 + EC; $CD^2 = AC^2 - AD^2$, or $CD^2 = 4AD^2 - AD^2 = 3AD^2$, or $2CD = \sqrt{3}(4 + EC)$. Draw the other half of the circle and extend EF to meet it at I. We have BE = BI and EC = IC. We also have BF:FC = (BE:EC)×(BI:IC), or $BE = EC\sqrt{3}$. The area of the area of triangle ABC is ½BE×AC = ½CD×AB, or $EC\sqrt{3}(4 + EC) = 4\sqrt{3}(4 + EC)$, and EC = 4.

<u>Problem 2 of Hong Kong Mathematical Olympiad 2009</u>

Let n be the integral part of $\dfrac{1}{\frac{1}{1980}+\frac{1}{1981}+\ldots+\frac{1}{2009}}$; find the value of n.

<u>Solution</u>

Let D be the denominator and $D = \frac{1}{1980}+\frac{1}{1981}+\ldots+\frac{1}{2009} =$ $(\frac{1}{1980}+\frac{1}{2009})+(\frac{1}{1980+1}+\frac{1}{2009-1})+\ldots+(\frac{1}{1980+14}+\frac{1}{2009-14})$.

Note that D is a sum of 15 pairs of the sums of two numbers inside the brackets, and
$D = (\frac{1}{1980}+\frac{1}{2009})+(\frac{1}{1981}+\frac{1}{2008})+\ldots+(\frac{1}{1994}+\frac{1}{1995}) =$
$\frac{3989}{1980\times2009}+\frac{3989}{1981\times2008}+\ldots+\frac{3989}{1994\times1995}$.

Now note that $\frac{3989}{1980\times2009} > \frac{3989}{1981\times2008} > \ldots > \frac{3989}{1994\times1995}$.
Therefore, $15\times\frac{3989}{1980\times2009} > D > 15\times\frac{3989}{1994\times1995}$, and
$\frac{1994\times1995}{15\times3989} > \frac{1}{D} > \frac{1980\times2009}{15\times3989}$, or $71.232 > \frac{1}{D} > 71.228$.

We conclude that $n = 71$.

<u>Further observation</u>

Keep the number at one end of the denominator constant; find the number at the other end such that the problem is still valid. In other words, *Let n be the integral part of* $\dfrac{1}{\dfrac{1}{1980}+\dfrac{1}{1981}+\ldots+\dfrac{1}{m}}$ *; find the maximum value of integer m that makes n a unique integer, or*

Let n be the integral part of $\dfrac{1}{\dfrac{1}{m}+\dfrac{1}{1981}+\ldots+\dfrac{1}{2009}}$ *; find the minimum value of integer m that makes n a unique integer.*

<u>Problem 3 of Hong Kong Mathematical Olympiad 2009 (Event 2)</u>

Given that x is a positive real number and $x \cdot 3^x = 3^{18}$. If k is a positive integer and $k < x < k + 1$, find the value of k.

<u>Solution</u>

The problem asks us to find the integral part of the value of x that falls in between k and $k + 1$. Basically, find the estimate value of x.

From $x \cdot 3^x = 3^{18}$, we get $x = 3^{18-x}$. Now note that when x increases, 3^{18-x} decreases and vice-versa.

When $x = 15$, we have $3^{18-x} = 3^3 = 27$.
When $x = 16$, we have $3^{18-x} = 3^2 = 9$.

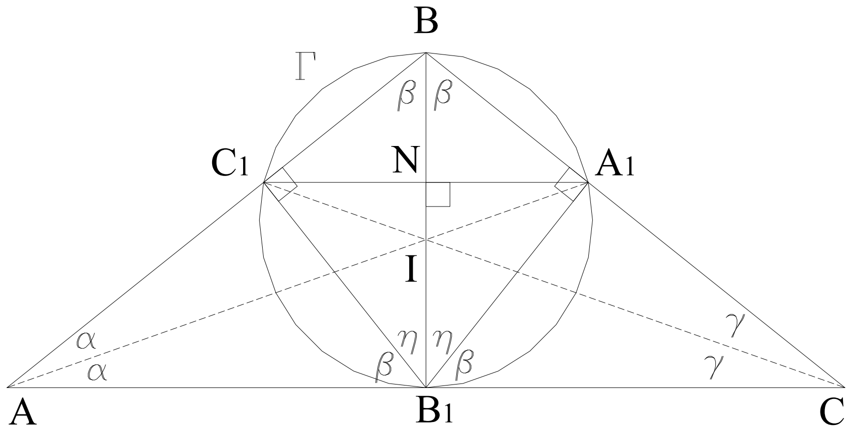
Because $27 > 15.... > 9$, so when $x \in (15, 16)$, $x = 3^{18-x}$.

Therefore, $k = 15$.

<u>*Problem 6 of Mongolian Mathematical Olympiad 2000*</u>

In a triangle ABC, the angle bisectors at A, B, C meet the opposite sides at A_1, B_1, C_1, respectively. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then $\frac{AC}{AB+BC} = \frac{AB}{AC+CB} + \frac{BC}{BA+AC}$.

<u>Solution</u>



Let I be the incenter of triangle ABC, Γ the circumcircle of quadrilateral $BA_1B_1C_1$, $\alpha = ½\angle BAC$, $\beta = ½\angle ABC$, $\gamma = ½\angle ACB$ and $\eta = \angle BB_1C_1$. We have $\alpha + \beta + \gamma = 90°$.

Since $BA_1B_1C_1$ is cyclic and BB_1 is the bisector of $\angle C_1BA_1$, BB_1 must be the diameter of Γ, and $\beta = \angle A_1C_1B_1 = \angle C_1A_1B_1$, $\angle BC_1B_1 = \angle BA_1B_1 = 90°$ to imply that $\angle B_1AC_1 + \angle AB_1C_1 = 2\alpha + \angle AB_1C_1 = 2\alpha + ½\angle C + \angle B_1C_1C = 2\alpha + \gamma + \angle B_1C_1C = 90°$.

Combining with $\alpha + \beta + \gamma = 90°$, we get $\alpha + \angle B_1C_1C = \beta = \angle A_1C_1B_1 = \angle A_1C_1C + \angle B_1C_1C$, or $\alpha = \angle A_1C_1C$, and AC_1A_1C is also cyclic which implies that $\gamma = \angle AA_1C_1$.
However, $\gamma = \angle AA_1C_1 = \angle IA_1C_1 = \angle IC_1A_1 = \angle CC_1A_1 =$

α, and ABC is an isosceles triangle with AB = BC and $BB_1 \perp AC$ which gives us $\angle AB_1C_1 = \beta$. But $\eta + \beta = 90°$, or $\eta = 2\alpha$.

The equation required to be proven $\frac{AC}{AB + BC} = \frac{AB}{AC + CB} + \frac{BC}{BA + AC}$ now reduces to $\frac{AC}{2AB} = \frac{2AB}{AB + AC}$, or $\frac{AB_1}{AB} = \frac{2}{1 + \frac{2AB_1}{AB}}$, or

$\cos 2\alpha = \frac{2}{1 + 2\cos 2\alpha}$, or $2\cos^2 2\alpha + \cos 2\alpha - 2 = 0$.

Now let's prove it. Indeed, let N be the intersection of BB_1 and A_1C_1, $\cos 2\alpha = \frac{B_1C}{BC} = \frac{IB_1}{IB}$ (because IC is the bisector of $\angle BCB_1$) = $\frac{A_1C}{B_1C} = \frac{B_1N}{A_1B_1}$. On the other hand, we also have $\cos 2\alpha = \frac{B_1C}{BC} = \frac{AC}{2BC}$ (because CC_1 is the bisector of $\angle ACB$) = $\frac{A_1C}{2A_1B} = \frac{B_1N}{2BN}$ (because $A_1C_1 \parallel AC$). Those two previous results give us $2BN = A_1B_1$.

We also have $\cos\beta = \frac{BN}{A_1B} = \frac{A_1B}{BB_1}$, or $\frac{A_1B_1}{2A_1B} = \frac{A_1B}{BB_1}$, or

$2A_1B^2 = A_1B_1 \times BB_1 = 2BB_1^2 - 2A_1B_1^2$, or $\frac{A_1B_1}{BB_1} = 2 - 2\frac{A_1B_1}{BB_1}^2$, or $2\cos^2 2\alpha + \cos 2\alpha - 2 = 0$, and we're done.

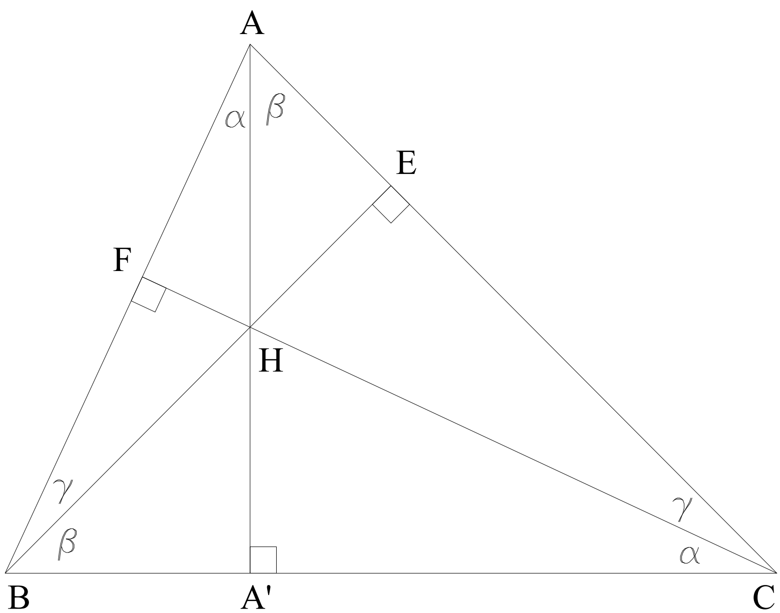
<u>Further observation</u>

Solving for $\cos 2\alpha$, we, get $\cos 2\alpha = \frac{1}{4}(\sqrt{17} - 1)$, or $\alpha = 19.33°$.

Problem 3 of Spain Mathematical Olympiad 2003

The altitudes of triangles ABC meet at H. We know that AB = CH. Determine the angle BCA.

Solution



Let (Ω) denote the area of shape Ω. Note that BDHE is cyclic because the sum of its opposite angles is 180°.

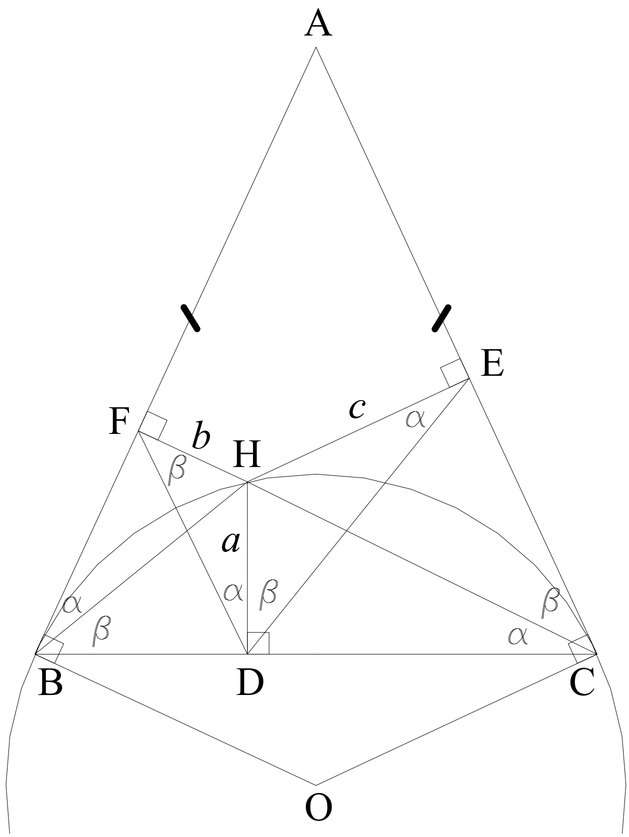
Applying the intersecting secant theorem, we get CD×BC = CH×CF.

But AB = CH; therefore, CD×BC = AB×CF = 2(ABC) = AD×BC, or AD = CD, and ADC is a right isosceles triangle, and angle BCA is 45°.

Problem 3 of Spain Mathematical Olympiad 2006

ABC is an isosceles triangle with AB = AC. Let H be a point on the circle tangent to the sides AB at B and AC at C. We call a, b, and c the distances from H to the sides BC, AB and AC, respectively. Prove that $a^2 = bc$.

Solution



Let the feet of H onto BC, AC and AB be D, E and F, respectively, Let $\alpha = \angle FBH$, $\beta = \angle DBH$. Because BDHF is cyclic (sum of opposite angle is 180°), we also have $\alpha = \angle FDH$ (subtends same arc FH as $\angle FBH$ does) and $\beta = \angle DFH$.

However, $\angle BCH$ and $\angle CBH$ also subtend arcs BH and CH,

respectively, we have $\alpha = \angle BCH$ and $\beta = \angle ECH$.
Also because CDHE is cyclic, $\alpha = \angle DEH$ and $\beta = \angle EDH$.
Now applying the law of sines to triangle DFH, we get

$$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha}, \text{ or } a = \frac{b\sin\beta}{\sin\alpha}.$$

And in triangle DEH, we get

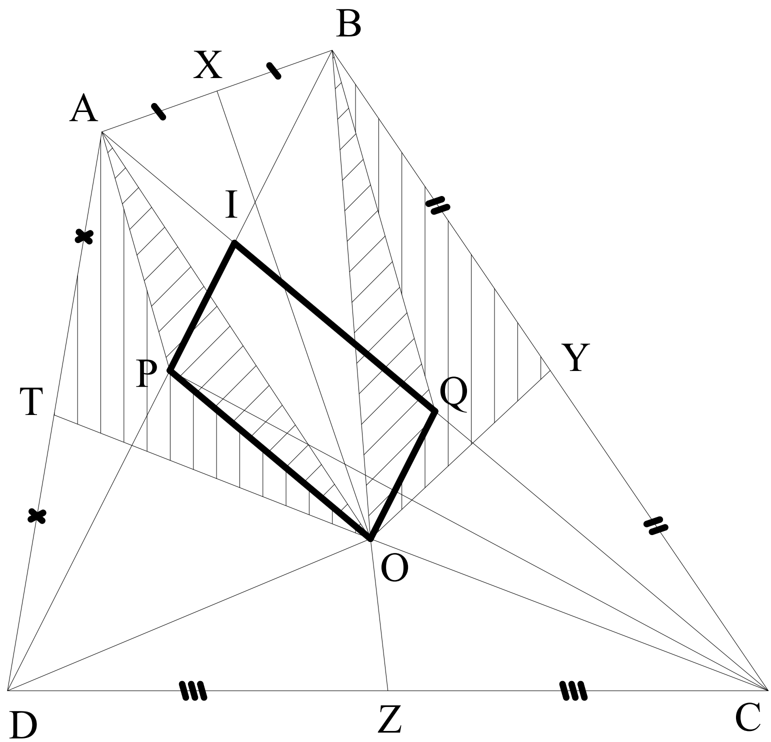
$$\frac{a}{\sin\alpha} = \frac{c}{\sin\beta}, \text{ or } a = \frac{c\sin\alpha}{\sin\beta}.$$

Multiplying those two previous equations gives us $a^2 = bc$.

<u>Problem 3 of Spain Mathematical Olympiad 2004</u>

ABCD is a quadrilateral with P and Q the midpoints of the diagonals BD and AC, respectively. The line through P and is parallel to AC meets the line through Q and is parallel to BD at O; X, Y, Z and T are the midpoints of AB, BC, CD and AD, respectively. Prove that the four quadrilaterals OXBY, OYCZ, OZDT and OTAX have the same area.

<u>Solution</u>



Let (Ω) denote the area of shape Ω and I be the intersection of the diagonals of ABCD.

Since P and Q are the midpoints of BD and AC, respectively, the triangles ABQ and CBQ have the same base length AQ = CQ and same height from B to AC; therefore, (ABQ) = (CBQ). The same situation applies to the triangles AOQ and COQ, and we have (AOQ) = (COQ), or (ABQO) = (CBQO).

Expanding the areas of both sides to get
(AOX) + (BOX) + (BOQ) = (CBQ) + (COQ).
But because X is the midpoint of AB, (AOX) = (BOX), (COQ) + (CBQ) = (BOY) + (COY) – (BOQ) = 2(COY) – (BOQ), the previous equation becomes
2(BOX) + (BOQ) = 2(BOY) – (BOQ), or
½[2(BOX) + (BOQ)] = ½[2(BOY) – (BOQ)], or
(BOX) + ½(BOQ) = (BOY) – ½(BOQ), or
(BOX) = (BOY) – (BOQ) = (BYOQ) (i)

Similarly, on the left side of the configuration, we get
(AOX) = (ATOP) (ii)
And because (AOX) = (BOX), equations (i) and (ii) give us (ATOP) = (BYOQ).

Now note that if we consider OP the base of triangle AOP; OP is also a side of parallelogram POQI, and because IQ || OP, the height of triangle AOP is the same height of this parallelogram from IQ to OP; therefore, (AOP) = ½(POQI).

Similarly, (BOQ) = ½(POQI), and (AOP) = (BOQ).

Finally, (OXBY) = (BOX) + (BOQ) + (BYOQ) = (AOX) + (AOP) + (ATOP) = (OTAX), and the first two of the four quadrilaterals are proven to have the same area.

The above result implies that (AOT) = (BOY), but (DOT) = (AOT) and (COY) = (BOY), and now (DOT) = (COY) (iii)

Now proceed with the same argument; (CDP) = (CBP) because P is the midpoint of BD; expand the areas of both sides of the equation to get (COZ) + (DOZ) + (DOP) + (COP) = (BOY) + (COY) + (BOP) – (COP).

But (COZ) = (DOZ), (BOY) = (COY) and (DOP) = (BOP), and we now have (COZ) + (COP) = (BOY) = (BOQ) + (BYOQ),

But (COP) = (BOQ) = ½(POQI); successively, (COZ) = (BYOQ).
Adding (COY) to both sides, we obtain
(OYCZ) = (BYOQ) + (COY) = (BYOQ) + (BOY) (iv)

From (i), (BYOQ) = (BOX), and equation (iv) is equivalent to (OYCZ) = (BOX) + (BOY) = (OXBY), and the next two of the four quadrilaterals are proven to have the same area.

Finally, (OYCZ) = (COZ) + (COY) = (DOZ) + (DOT) (because (COY) = (DOT) in (iii)) = (OZDT).

We now have (OXBY) = (OTAX) = (OYCZ) = (OZDT).

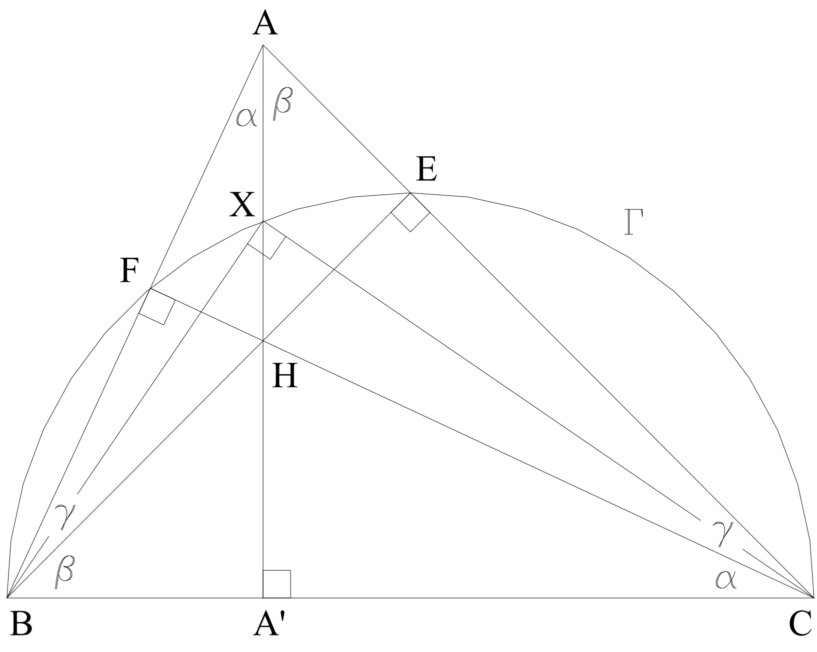
<u>Problem 2 of Spain Mathematical Olympiad 2002</u>

In a triangle ABC, A' is the foot of the vertex A onto BC and H the orthocenter.

a) Given a positive real number k such that $\frac{AA'}{HA'} = k$, find the relationship between the angles B and C as a function of k.

b) If B and C are fixed, find the locus of the vertex A for each value of k.

<u>Solution</u>



a) Let the feet of B and C onto AC and AB be E and F, respectively, $\alpha = \angle BAA' = \angle BCF$, $\beta = \angle CAA' = \angle CBE$, $\gamma = \angle ABE = \angle ACF$.

We have $\tan\angle B = \frac{AA'}{BA'}$, $\tan\beta = \frac{HA'}{BA'}$, and $\frac{AA'}{HA'} = k = \frac{\tan\angle B}{\tan\beta} = \frac{EB\times\tan\angle B}{EC} = \tan\angle B\times\tan\angle C$.

b) $k = \tan\angle B \times \tan\angle C = \frac{AA'}{BA'} \times \frac{AA'}{A'C}$. Now draw a circle Γ with diameter BC to cut AA' at X.

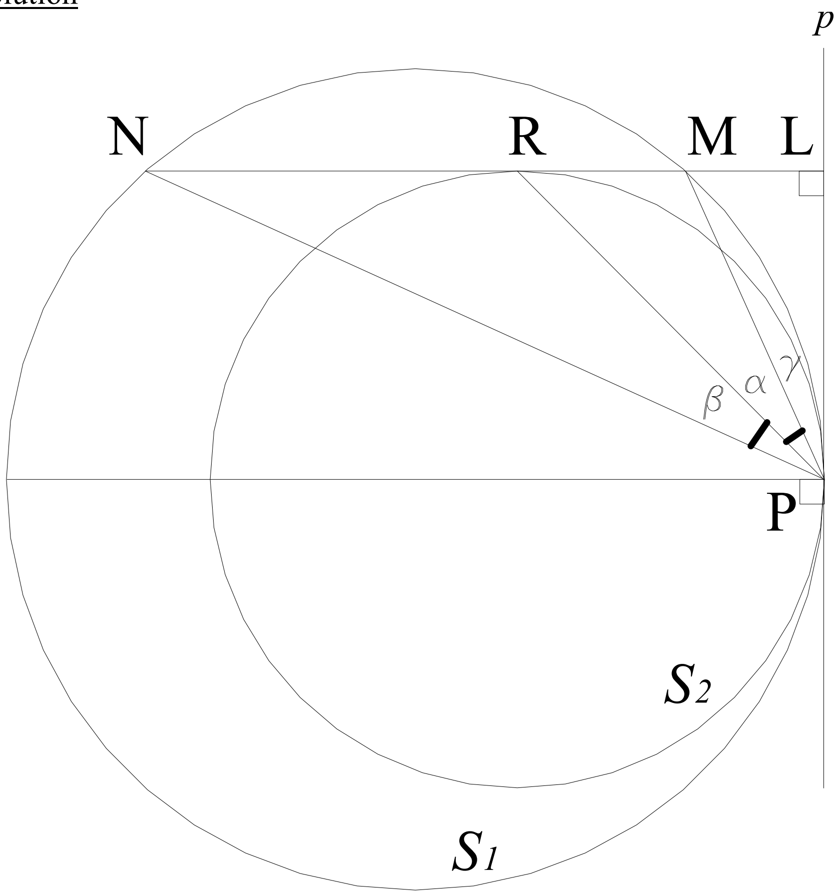
We have BA'×A'C = A'X², and $k = (\frac{AA'}{A'X})^2 = (\frac{AX + A'X}{A'X})^2 = (1 + \frac{AX}{A'X})^2$, or $\frac{AX}{A'X} = \sqrt{k} - 1$.

So when B and C are fixed, the locus of the vertex A for each value of k satisfies the condition $\sqrt{k} = \frac{AX}{A'X} + 1$. Pick any point X on the circle Γ, A is a point above circle Γ such that AX $\perp$ BC and $\frac{AX}{A'X} = \sqrt{k} - 1$.

Problem 1 of British Mathematical Olympiad 1985

Two circles S_1 and S_2 each touch a straight line *p* at the same point P. All points of S_2, except P, are in the interior of S_1. A straight line *q* (i) is perpendicular to *p*; (ii) touches S_2 at R; (iii) cuts *p* at L; and (iv) cuts S_1 at N and M, where M is between L and R.
a) Prove that RP bisects angle MPN.
b) If MP bisects angle RPL, find, with proof, the ratio of the areas of S_1 and S_2.

Solution



a) Let r and R be the radii of S_2 and S_1, respectively, PR meet S_1 at S, PM meet S_2 at I, $\alpha = \angle RPM$, $\beta = \angle RPN$ and $\gamma = \angle MPL$. Since both LR and LP tangent to S_2 at R and P, LR = LP, and because $\angle$RLP = 90°, RLP is a right isosceles triangle; hence, $\angle$LRP = $\angle$LPR = 45°.

We now have $\alpha = \angle RPL - \gamma = 45° - \gamma$, and $\beta = \angle LRP - \angle LNP$ 45° – $\angle$LNP. But $\angle$LNP subtends arc MP of S_1, or $\angle LNP = \gamma$.

Therefore, $\alpha = \beta$, or RP bisects angle MPN.

b) Note that α subtends arc RI on S_2 and SM on S_1. Because of this, $\frac{R}{r} = \frac{\mathrm{SM}}{\mathrm{RI}}$, and the ratio of the areas of S_1, denoted $A(S_1)$, and S_2, denoted $A(S_2)$, is $\frac{A(S_1)}{A(S_2)} = \frac{\pi R^2}{\pi r^2} = \frac{R^2}{r^2} = \frac{\mathrm{SM}^2}{\mathrm{RI}^2}$.

However, $\angle$MSP = $\angle$LNP subtends arc MP on S_1 and $\angle IRP = \gamma = \angle LNP$ subtends arc IP on S_2, $\angle MSP = \angle IRP = \gamma$. And when MP bisects angle RPL, $\alpha = \beta = \gamma = 45°/2 = 22.5°$, MSP, IRP, RNP are all isosceles triangles, and SM = MP, RI = IP, NR = RP, RI || SM || NP.

We now obtain $\frac{A(S_1)}{A(S_2)} = \frac{\mathrm{SM}^2}{\mathrm{RI}^2} = \frac{\mathrm{MP}^2}{\mathrm{IP}^2} = \frac{\mathrm{NM}^2}{\mathrm{NR}^2}$.

Applying the law of sines, $\frac{\mathrm{NM}}{\sin\angle \mathrm{NPM}} = \frac{\mathrm{NM}}{\sin 45°} = \frac{\mathrm{NP}}{\sin\angle \mathrm{NMP}} = \frac{\mathrm{NP}}{\sin(90° + 22.5°)} = \frac{\mathrm{NP}}{\sin 112.5°}$, or $\mathrm{NM} = \frac{\mathrm{NP}\times\sin 45°}{\sin 112.5°}$.

Similarly, $\frac{\mathrm{NR}}{\sin 22.5°} = \frac{\mathrm{NP}}{\sin\angle \mathrm{NRP}} = \frac{\mathrm{NP}}{\sin 135°}$, or $\mathrm{NR} = \frac{\mathrm{NP}\times\sin 22.5°}{\sin 135°}$

Finally, $\frac{A(S_1)}{A(S_2)} = \frac{\mathrm{NM}^2}{\mathrm{NR}^2} = \frac{(\sin 135°\times\sin 45°)^2}{(\sin 112.5°\times\sin 22.5°)^2}$.

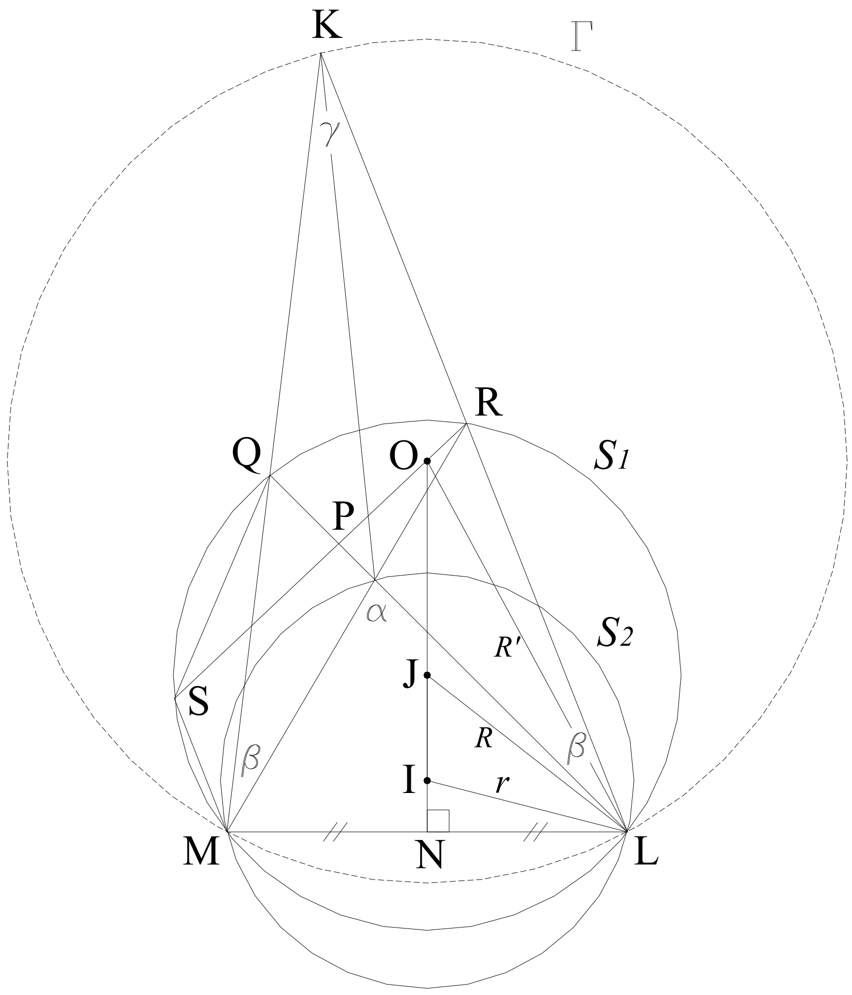
Applying the existing formula $\cos a - \cos b = -2\sin\frac{a+b}{2}\sin\frac{a-b}{2}$,

$$\frac{A(S_1)}{A(S_2)} = \frac{(\cos 180° - \cos 90°)^2}{(\cos 135° - \cos 90°)^2} = 2.$$

<u>*Problem 5 of British Mathematical Olympiad 2010*</u>

Circles S_1 and S_2 meet at L and M. Let P be a point on S_2. Let PL and PM meet S_1 again at Q and R, respectively. The lines QM and RL meet at K. Show that, as P varies on S_2, K lies on a fixed circle.

<u>Solution</u>



Let $\alpha = \angle MPL$, $\beta = \angle QMR = \angle QLR$ (because they subtend the same arc QR on S_1) and $\gamma = \angle MKL$. Since P is on S2, α subtends fixed arc ML on S_2 and is always a fixed angle. Furthermore, it also subtends arcs ML plus arc QR on S_1; thus the length of arc QR is also constant even though their locales vary. Therefore, β which subtends arc QR is also always a constant.

We also have $\alpha = \angle MKP + \angle LKP + \angle KMP + \angle KLP = \gamma + 2\beta$, or $\gamma = \alpha - 2\beta$ is constant. We conclude that K is on a fixed circle with ML as a chore.

Because Γ, S_1 and S_2 share the same chore ML, their centers are collinear. Let O, J and I be the centers of Γ, S_1 and S_2 and R', R and r the radii of these circles in the exact same order. O, J and I lie on a straight line that is also the perpendicular bisector of ML as shown.

From M draw a line to parallel KL to meet S_1 at S; we have arc SR = arc ML, and now γ subtends arc SR – arc QR = arc SQ. Subsequently, $\frac{R'}{R} = \frac{\text{ML}}{\text{SQ}}$, or $R' = \text{OL} = \frac{\text{ML}}{\text{SQ}} \times R$, and thus we have been able to determine the center and radius of circle Γ that is the locus of point K.

Problem 4 of the Vietnamese Mathematical Olympiad 1989

Are there integers x, y, not both divisible by 5, such that $x^2 + 19y^2 = 198\times10^{1989}$?

Solution

There are four possible combinations for x and y: x and y are both odd numbers, x odd y even, x even y odd, or x even y even.
When x and y are both odd, let $x = 2m + 1$ and $y = 2n + 1$ where m and n are both integers. The equation in the problem is written as $(2m + 1)^2 + 19(2n + 1)^2 = 198\times10^{1989}$, or $4[m(m + 1) + 19n(n + 1)] + 20 = 4\times25\times198\times10^{1987}$. Dividing both sides by 4, we get $m(m + 1) + 19n(n + 1) + 5 = 25\times198\times10^{1987}$, and this equation is not allowed because the two products $m(m + 1)$ and $n(n + 1)$ of consecutive numbers are even, and the expression on the left is an odd number while $25\times198\times10^{1987}$ is an even number.
When either one of them is odd and the other even, which are the middle two combinations, the sum of $x^2 + 19y^2$ is an odd number whereas 198×10^{1989} is an even number which again is not allowed.
When x and y are both even, and x is divisible by 5 while y is not; x must have the units digit 0; therefore, the units digit of x^2 must also be 0 which makes the units digit of y^2 to be 0 because the units digit of 198×10^{1989} is 0, or that of y to be 0, and thus y is divisible by 5, and this scenario is not allowed by the problem.
When x and y are both even, and y is divisible by 5 while x is not. Applying the similar argument, because y is even and is divisible by 5, its units digit must be 0 which makes the units digit of $19y^2$ to be 0 which requires that of x^2 to be 0, and thus x is divisible by 5, and this scenario is also not allowed by the problem.

Further observation

Try to solve the problem with neither x nor y divisible by 5.

<u>Problem 2 of Tournament of Towns 2008</u>

Twenty-five of the numbers 1, 2, ... , 50 are chosen. Twenty-five of the numbers 51, 52, … , 100 are also chosen. No two chosen numbers differ by 0 or 50. Find the sum of all 50 chosen numbers.

<u>Solution</u>

For the first set of 50 numbers from 1 to 50 let's choose the numbers from 1 to 25.

For the second set of 50 numbers from 51 to 100 let's choose the numbers from 76 to 100.

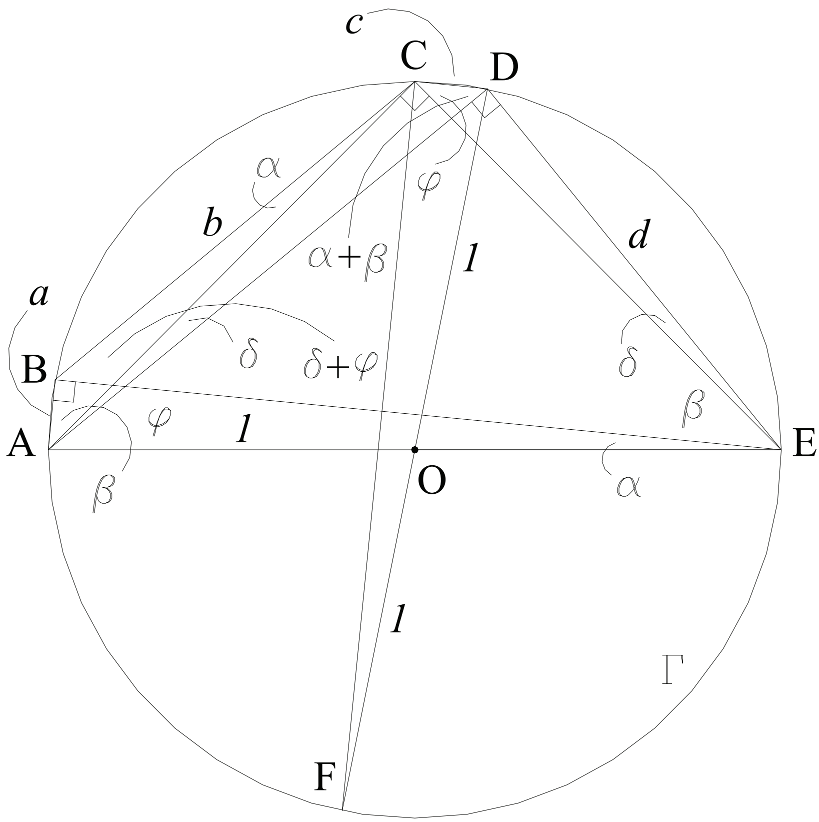
No two chosen numbers differ by 0 or 50. The minimal difference between the two numbers is 1, and the maximal difference between the two numbers is $100 - 1 = 99$.

The sum of all 50 chosen numbers is $(1 + 2 + \ldots + 25) + (76 + 77 + \ldots + 100) = 25[(100 + 1) + (99 + 2) + \ldots + (76 + 25)] = 25\times 101 = 2525$.

Problem 4 of Turkey MO Team Selection Test 1997

A convex ABCDE is inscribed in a unit circle, AE being its diameter. If AB = a, BC = b, CD = c, DE = d and $ab = cd = \frac{1}{4}$, compute AC + CE in terms of a, b, c, d.

Solution



Let α = ∠ACB = ∠AEB, β = ∠BAC = ∠BEC, δ = ∠CAD = ∠CED, φ = ∠DAE = ∠DCE and O be the circumcenter of the unit circle Γ. We then have α + β = ∠ADC and δ + φ = ∠CBE. Applying the law of sines to triangle ABC, we get $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} =$

$\frac{\text{AC}}{\sin\angle\text{ABC}} = \frac{\text{AC}}{\sin(90° + \delta + \varphi)} = \frac{\text{AC}}{\cos(\delta + \varphi)}$. However, in any triangle $\frac{a}{\sin\alpha} = 2R$ where R is the radius of its circumcircle (this is easily obtained by applying the law of sines to triangle ABE).

In our case $R = 1$ and $\frac{\text{AC}}{\cos(\delta + \varphi)} = 2$, or AC $= 2\cos(\delta + \varphi)$.
Similarly, CE $= 2\cos(\alpha + \beta)$. Adding the two terms to get

$$\text{AC} + \text{CE} = 2[\cos(\alpha + \beta) + \cos(\delta + \varphi)] = 2[\cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\delta\cos\varphi - \sin\delta\sin\varphi] \qquad \text{(i)}$$

Now extend DO to meet Γ at F; $\delta = \angle\text{CFD}$, and $\cos\delta = \frac{\text{CF}}{2R} = \frac{\text{CF}}{2} = \frac{1}{2}\sqrt{4R^2 - c^2} = \frac{1}{2}\sqrt{4 - c^2}$.

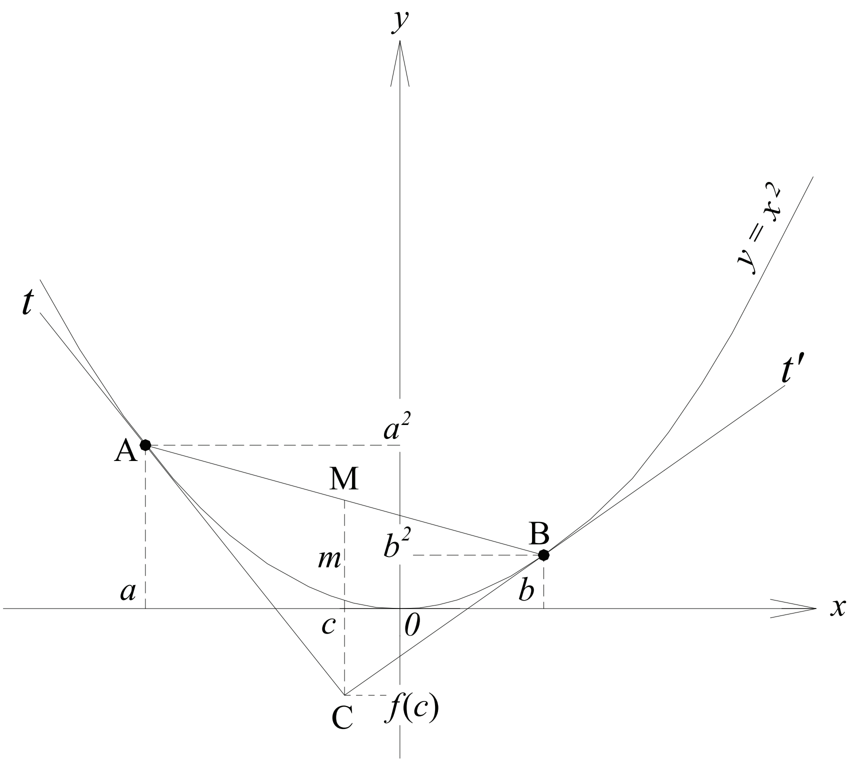
Similarly, $\cos\varphi = \frac{1}{2}\sqrt{4 - d^2}$, $\cos\alpha = \frac{1}{2}\sqrt{4 - a^2}$, $\cos\beta = \frac{1}{2}\sqrt{4 - b^2}$, $\sin\alpha = \frac{a}{2R} = \frac{a}{2}$, $\sin\beta = \frac{b}{2}$, $\sin\delta = \frac{c}{2}$ and $\sin\varphi = \frac{d}{2}$.
Also given $ab = cd = \frac{1}{4}$, equation (i) is now equivalent to

$$\text{AC} + \text{CE} = \frac{1}{2}[\sqrt{(4 - c^2)(4 - d^2)} - cd + \sqrt{(4 - a^2)(4 - b^2)} - ab] = \frac{1}{2}\sqrt{(4 - a^2)(4 - b^2)} + \frac{1}{2}\sqrt{(4 - c^2)(4 - d^2)} - \frac{1}{4} = \sqrt{\frac{257}{64} - a^2 - b^2} + \sqrt{\frac{257}{64} - c^2 - d^2} - \frac{1}{4}.$$

Problem 1 of Spain Mathematical Olympiad 1999

The lines t and t' are tangent to the parabola of equation $y = x^2$ at points A and B intersect at point C. The median of the triangle ABC corresponds to the vertex C has length m. Determine the area of triangle ABC in terms of m.

Solution



Graph not drawn to scale

Let's pick a worst case scenario where point A is on the left side of the y-axis, B on its right side. Let the coordinates of point A be (a, a^2), those of point B be (b, b^2) and of point C be $(c, f(c))$.

The equation of a tangent to a curve through a point with coordinates $(p, f(p))$ having slope $f'(p)$ is, by definition, expressed as $y - f(p) = f'(p)(x - p)$ where $f'(p)$ is the derivative of the curve at point $(p, f(p))$, and in this case $f'(x) = (x^2)' = 2x$.

Applying the equation to the two tangential lines t and t', we get
t: $y - a^2 = 2a(x - a)$ and t': $y - b^2 = 2b(x - b)$, or
t: $y = 2ax - a^2$ and t': $y = 2bx - b^2$.
At point C where the two lines meet, we have $2ac - a^2 = 2bc - b^2$, or $2c(a - b) = a^2 - b^2 = (a - b)(a + b)$, and since $a \neq b$, $c = ½(a + b)$, and the x-coordinate of point C is the midpoint of the segment connecting the x-coordinates of points A and B.

In addition to point M being the midpoint of AB, the segment CM is vertical and is parallel to the y-axis. The y-coordinate of C is $f(c) = 2ac - a^2 = a(a + b) - a^2 = ab$.

The equation for the line that passes through points A and B is $y(\mathrm{AB}) = \dfrac{b^2 - a^2}{b - a}x + d = (a + b)x + d$. At point $\mathrm{A}(a, a^2)$, we have $a^2 = (a + b)a + d$, or $d = -ab$, and $y(\mathrm{AB}) = (a + b)x - ab$.

To find the area of triangle ABC, denoted (ABC), we move the x-axis down and make it pass through point C. Since a is negative in this case, to move it up we add a positive length $-f(c) = -ab$; the equation of the curve become $y = x^2 - f(c) = x^2 - ab$; the equation for t becomes $y(t) = 2ax - a^2 - ab$, for t': $y(t') = 2bx - b^2 - ab$, and $y(\mathrm{AB}) = (a + b)x - 2ab$.

The area is now (ABC) = $\int_a^c [y(\mathrm{AB}) - y(t)]dx + \int_c^b [y(\mathrm{AB}) - y(t')]dx =$

$$\int_a^c [(a+b)x - 2ab - (2ax - a^2 - ab)]dx + \int_c^b [(a+b)x - 2ab - (2bx - b^2 - ab)]dx =$$

$$\int_a^c [(b - a)x + a^2 - ab]dx + \int_c^b [(a - b)x + b^2 - ab]dx = [\frac{1}{2}(b - a)x^2 +$$

$$ax(a - b) + \text{constant}]\Big|_a^c + [\frac{1}{2}(a - b)x^2 + bx(b - a) + \text{constant}]\Big|_c^b =$$

$$\frac{1}{2}(b - a)c^2 + ac(a - b) - \frac{1}{2}(b - a)a^2 - a^2(a - b) + \frac{1}{2}(a - b)b^2 +$$

$b^2(b-a) - \frac{1}{2}(a-b)c^2 - bc(b-a) = (b-a)(\frac{1}{2}c^2 - ac - \frac{1}{2}a^2 + a^2 - \frac{1}{2}b^2$

$+ b^2 + \frac{1}{2}c^2 - bc) = (b-a)\times(\frac{1}{2}a^2 + \frac{1}{2}b^2 + c^2 - ac - bc)$.

However, $c = \frac{1}{2}(a+b)$, and (ABC) $= (b-a)[\frac{1}{2}a^2 + \frac{1}{2}b^2 + \frac{1}{4}(a+b)^2 -$

$\frac{1}{2}(a+b)^2] = (b-a)[\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}(a+b)^2] = (b-a)[\frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{4}(a^2$

$+ 2ab + b^2)] = \frac{1}{4}(b-a)(a^2 - 2ab + b^2) = \frac{1}{4}(b-a)^3$.

Meanwhile, $m = \frac{1}{2}(a^2 - ab + b^2 - ab) = \frac{1}{2}(b-a)^2$.

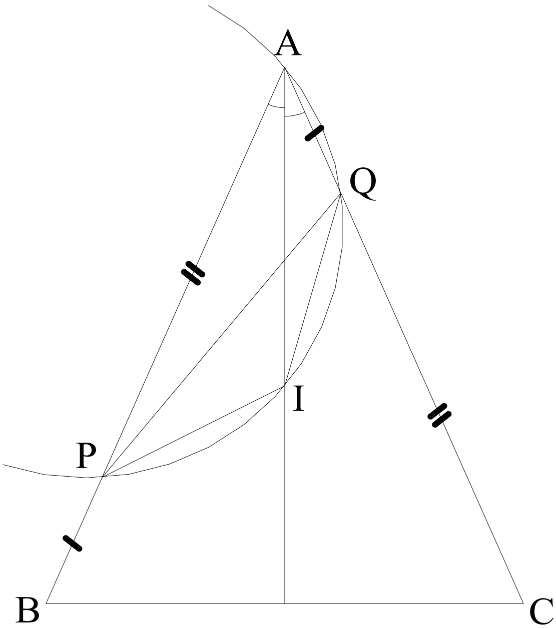
Finally, (ABC) $= m\sqrt{\frac{m}{2}}$.

The reader is encouraged to try this problem with both points A and B on the left or the right sides of the y-axis.

<u>Problem 2 of the Irish Mathematical Olympiad 2006</u>

P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that AP = CQ. Moreover, neither P nor Q is a vertex of ABC. Prove that the circumcircle of the triangle APQ passes through the circumcenter of the triangle ABC.

<u>Solution</u>



Let the circumcircle of triangle APQ intercept the bisector of $\angle$A of triangle ABC at I. We have $\angle$PAI = $\angle$QAI and PI = QI.

Since AQIP is cyclic, we have $\angle$AQI + $\angle$API = 180°, or 180° − $\angle$API = $\angle$AQI. Now consider the two triangles BPI and AQI, we have BP = AQ, PI = QI, and $\angle$BPI = 180° − $\angle$API = $\angle$AQI; thus they are congruent and thus AI = BI.

Since AI is the bisector of $\angle$BAC and ABC is an isosceles triangle with AB = AC, AI is also the altitude to BC, and BI =CI. Hence, BI = CI = AI, or I is the circumcenter of triangle ABC.

Problem 2 of the Irish Mathematical Olympiad 2007

Prove that a triangle ABC is right-angled if and only if $\sin^2A + \sin^2B + \sin^2C = 2$.

Solution

Let the three side lengths of triangle ABC be a, b and c. Applying the law of the sines, we obtain

$$\frac{a^2}{\sin^2A} = \frac{b^2}{\sin^2B} = \frac{c^2}{\sin^2C} = \frac{a^2 + b^2 + c^2}{\sin^2A + \sin^2B + \sin^2C} = \frac{a^2 + b^2 + c^2}{2},$$

and the law of the cosines gives us $a^2 = b^2 + c^2 - 2bc\times\cos A$.
Now substituting a^2 into the above equation

$$\frac{a^2}{\sin^2A} = \frac{a^2 + b^2 + c^2}{2} = \frac{2(b^2 + c^2 - bc\times\cos A)}{2} = b^2 + c^2 - bc\times\cos A,$$

or $\quad a^2 = (b^2 + c^2 - bc\times\cos A)\sin^2A$,
$\quad b^2 + c^2 - 2bc\times\cos A = (b^2 + c^2 - bc\times\cos A)\sin^2A$, or

$(b^2 + c^2)(1 - \sin^2A) = bc\times\cos A(2 - \sin^2A)$,
$(b^2 + c^2)\cos^2A = bc\times\cos A(1 + \cos^2A)$,
$(b^2 + c^2)\cos A = bc\times(1 + \cos^2A)$,
$bc\times\cos^2A - (b^2 + c^2)\cos A + bc = 0$.

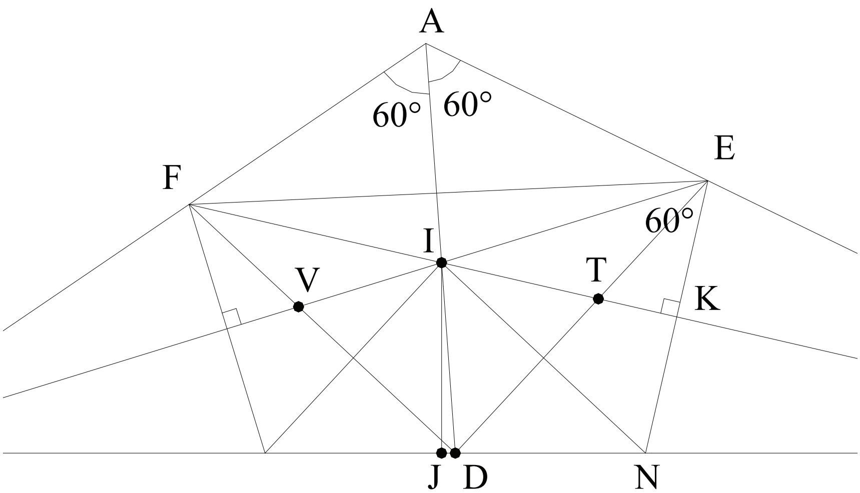
Solving for cosA, we have $\cos A = \frac{b}{c}$ and $\frac{c}{b}$; this implies that either angle B or angle C is a right angle.

Now if the triangle is right-angled, we have one of the angles being 90°. Without loss of generality, assume it's angle A and $\sin^2A = 1$, we have $\sin^2B = \frac{b^2}{a^2}$ and $\sin^2C = \frac{c^2}{a^2}$, and $\sin^2B + \sin^2C = \frac{b^2 + c^2}{a^2} = 1$. Therefore, $\sin^2A + \sin^2B + \sin^2C = 2$.

Problem 2 of the British Mathematical Olympiad 2005

In triangle ABC, $\angle BAC = 120°$. Let the angle bisectors of angles A, B and C meet the opposite sides in D, E and F, respectively. Prove that the circle on diameter EF passes through D.

Solution

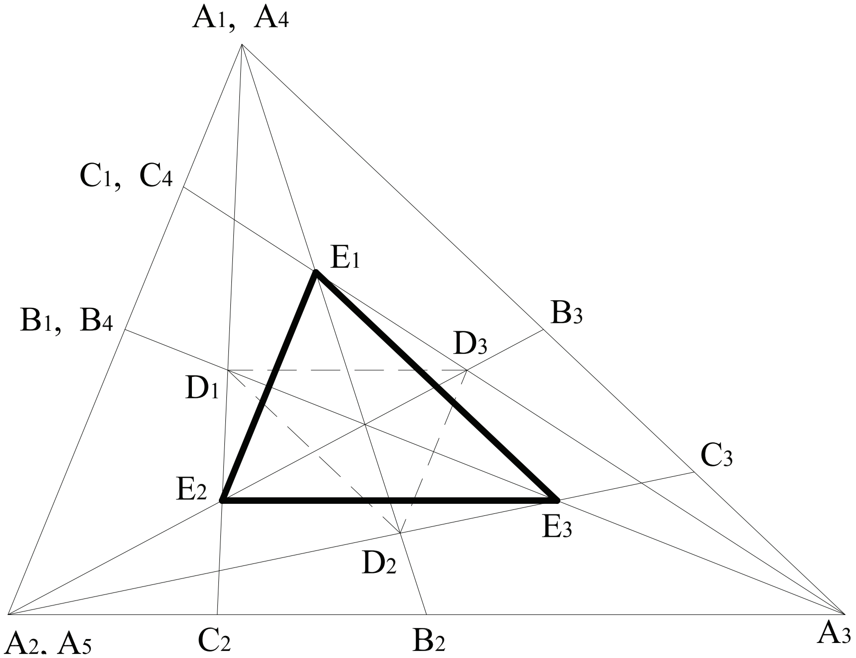


Let BE meet CF at I and J be the foot of I on BC. From E draw the perpendicular to CF to meet CF and BC at K and N, respectively. Also let CF meet ED at T, BE meet FD at V. We have $\angle BID = \angle ABI + \angle BAI = 90° - \frac{1}{2}\angle C = \angle JIC$, or $\angle BIJ = \angle DIC$.
It's easily seen that $\angle EIK = \frac{1}{2}(\angle B + \angle C) = 30°$, or $\angle BIC = 150°$ and $\angle IEK = 90° - \angle EIK = 60°$, and since CI is the perpendicular bisector of EN, IE = IN and $\angle INE = 60°$. It follows that IEN is an equilateral triangle and $\angle NIK = 30°$. We now have $\angle IND = \angle NIK + \frac{1}{2}\angle C = 30° + \frac{1}{2}\angle C = 30° + (30° - \frac{1}{2}\angle B) = 60° - \frac{1}{2}\angle B$, and $\angle DIN = \angle DIC - 30° = \angle BIJ - 30° = 90° - \frac{1}{2}\angle B - 30° = 60° - \frac{1}{2}\angle B$, or $\angle IND = \angle DIN$, and DE is bisector of $\angle IDN$.
Similarly, on the other side, DF is bisector of $\angle IDB$.
Therefore, $\angle FDE = 90°$, and the circle on diameter EF passes through D.

Problem 3 of Asian Pacific Mathematical Olympiad 1989

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Solution



Connect and extend A_1D_3 to meet A_2A_3 at A, A_2D_1 to meet A_1A_3 at B and A_3D_2 to meet A_1A_2 at C.

Applying Ceva's theorem for the three lines A_1C_2, A_3B_1 and A_2B, we have $\dfrac{A_2C_2 \times A_3B \times A_1B_1}{A_3C_2 \times A_1B \times A_2B_1} = 1$. Since $A_3C_2 = 3 \times A_2C_2$ and $A_1B_1 = A_2B_1$, we have $\dfrac{A_1B}{A_3B} = \dfrac{1}{3}$ and $\dfrac{A_1C_1}{A_2C_1} = \dfrac{1}{3}$. Therefore, $BC_1 \parallel A_2A_3$, $\dfrac{C_1B}{A_2A_3} = \dfrac{A_1C_1}{A_1A_2} = \dfrac{1}{4}$ and $\dfrac{A_2B_2 \times A_3B \times A_1C_1}{A_3B_2 \times A_1B \times A_2C_1} = 1 \times 3 \times \dfrac{1}{3} = 1$.

Hence, per Ceva's theorem point E_1 is on A_2B.

With the same argument, E_2 is on A_3C and E_3 is on A_1A.

Now applying Ceva's theorem for the three lines E_1B_2, A_2D_3 and A_3D_1 that meet at G, we get $\dfrac{E_1D_3}{A_3D_3} = \dfrac{E_1D_1}{A_2D_1}$ or $D_1D_3 \parallel A_2A_3$.

Similarly, $D_2D_3 \parallel A_1A_2$ and $D_1D_2 \parallel A_1A_3$. For the three lines GB_2, A_2E_3 and A_3E_2 that meet at D_2, we obtain $\dfrac{GE_3}{A_3E_3} = \dfrac{GE_2}{A_2E_2}$, or $E_2E_3 \parallel A_2A_3$.

Also similarly, $E_1E_2 \parallel A_1A_2$ and $E_1E_3 \parallel A_1A_3$; $\Delta D_1D_2D_3$ and $\Delta E_1E_2E_3$ are similar triangles since their corresponding sides are parallel to each other. The parallel lines give us $\dfrac{E_1D_1}{E_1A_2} = \dfrac{D_1D_3}{A_2A_3} = \dfrac{D_1D_3}{2AC_2} = \dfrac{A_1D_1}{2A_1C_2} = \dfrac{A_2D_1}{2A_2B}$, or $\dfrac{A_2D_1}{2E_1D_1} = \dfrac{A_2B}{E_1A_2} = \dfrac{E_1A_2 + E_1B}{E_1A_2} = 1 + \dfrac{E_1B}{E_1A_2} = 1 + \dfrac{C_1B}{A_2A_3} = 1 + \dfrac{1}{4} = \dfrac{5}{4}$, or $2\dfrac{E_1D_1}{A_2D_1} = 2\dfrac{E_1D_1}{A_3D_3} = \dfrac{4}{5}$, and $\dfrac{E_1D_3}{A_3D_3} = \dfrac{2}{5}$.

Adding 1 to both sides to get $1 + \frac{E_1D_3}{A_3D_3} = \frac{7}{5}$, or $\frac{A_3D_3 + E_1D_3}{A_3D_3} = \frac{7}{5}$,

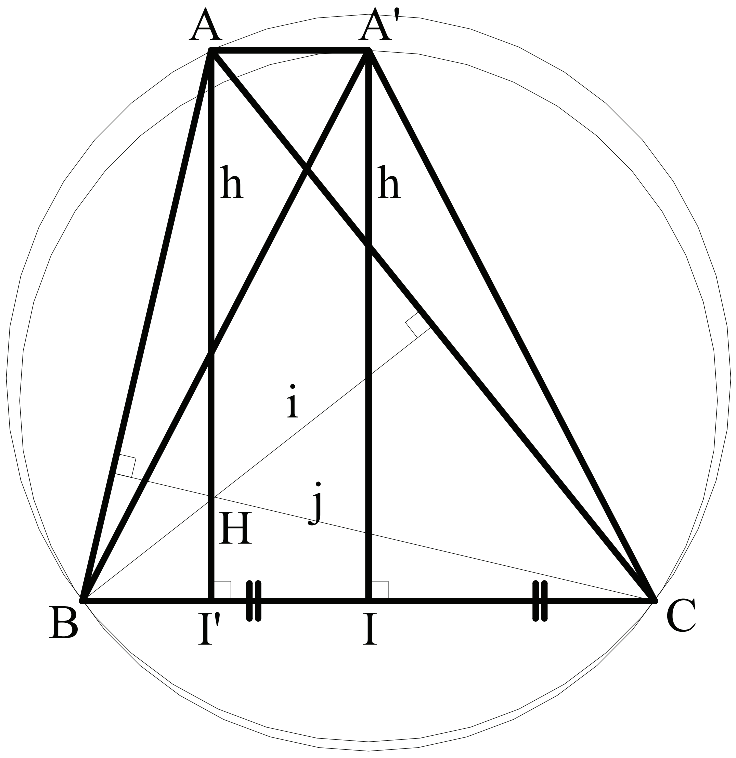
$\frac{E_1A_3}{A_3D_3} = \frac{7}{5}$, $\frac{A_3D_3}{E_1A_3} = \frac{5}{7}$, $\frac{D_2D_3}{E_2E_1} = \frac{5}{7}$.

Thus the ratio of the corresponding sides of the two similar triangles $\Delta D_1D_2D_3$ and $\Delta E_1E_2E_3$ is equal to $\frac{5}{7}$. Therefore, the ratio of the area of $\Delta D_1D_2D_3$ to the area of $\Delta E_1E_2E_3$ is equal to the square of the ratio of their corresponding sides and is $(\frac{5}{7})^2 = \frac{25}{49}$.

Problem 3 of Asian Pacific Mathematical Olympiad 1990

Consider all the triangles ABC which have a fixed base BC and whose altitude from A is a constant *h*. For which of these triangles is the product of its altitudes a maximum?

Solution



From B draw line perpendicular to AC and from C draw line perpendicular to AB. We have the altitudes *i* and *j*, respectively. The problem asks for the product *h* times *i* times *j* to be maximum. But *h* is constant, so $i \times j$ must be a maximum.
The area of the triangle ABC is also constant since base BC is fixed.
But twice the area of triangle ABC = $h \times$AC = $i \times$AC = $j \times$AB. From there, $i \times$AC$\times j \times$AB equals the square of twice the area of triangle ABC and is constant.

The multiplication of two products ($i \times j$) and (AC×AB) is a constant, for one to be maximum ($i \times j$) the other has to me minimum, we must find AC and AB so that AB×AC is a minimum.

Let R be the radius of the circumcircle of triangle ABC, and a, b and c as the lengths of its sides, there exists the formula:

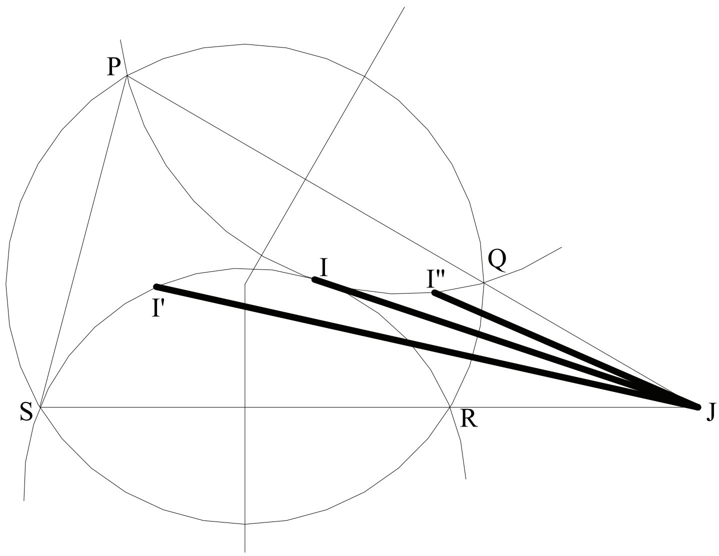
Area of ABC $= \frac{abc}{4R}$.

Area of ABC is fixed as we know, so for the product of the three sides to be a minimum (one side is already fixed, or the product of the two sides to be a minimum) the denominator R has to be minimum, or the circumcircle has to be smallest (A $\longrightarrow$ A') and A'B = A'C. The triangle is isosceles.

<u>Problem 3 of Asian Pacific Mathematical Olympiad 1995</u>

Let PQRS be a cyclic quadrilateral such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set I of points of tangency of circles in these two sets.

<u>Solution</u>



Extend PQ and SR to intercept each other at J. Since PQRS is a cyclic quadrilateral $\angle PSR + \angle PQR = 180°$ or $\angle PSR = \angle RQJ$. Similarly, $\angle SPQ = \angle QRJ$.

And the two triangles JPS and JRQ are similar; therefore, $\frac{JQ}{JR} = \frac{JS}{JP}$, or $JQ \times JP = JS \times JR$.

From J draw the two lines tangential to the bottom and top circles and assume that the two tangential points are different, respectively, are I’ and I” on the bottom and top circles as shown.

We have $JI'^2 = JR \times JS$, and $JI''^2 = JQ \times JP$.

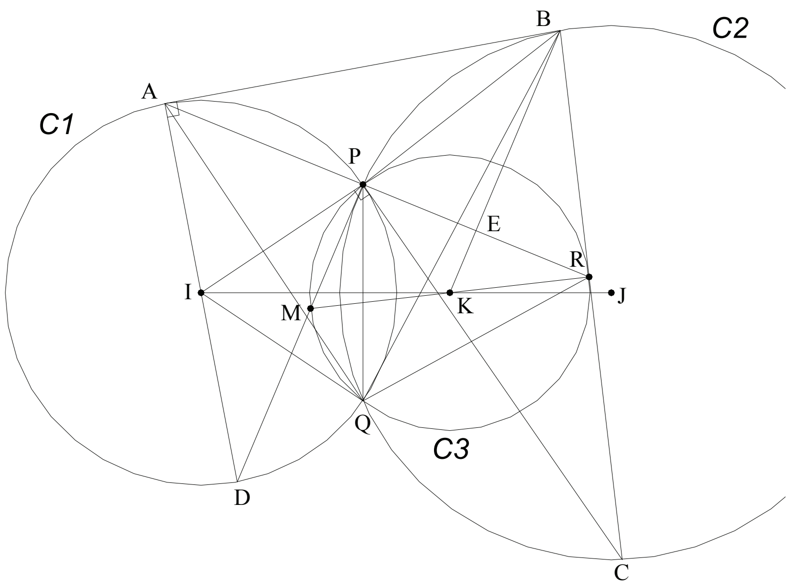
With the assumption that the tangential JI’ and JI” are not coincided, the two circles are either overlap or not touching each other at all which is not true with the given condition of the problem. Therefore, for the two circles to tangent I’ must coincide I” and also coincide with I, or $JI^2 = JR \times JS$ which is a constant.

So the set of points of tangency of the two circles is a circle with center at J and radius $r = \sqrt{JR \times JS}$ or $r = \sqrt{JQ \times JP}$.

<u>*Problem 3 of Asian Pacific Mathematical Olympiad 1999*</u>

Let C_1 and C_2 be two circles intersecting at P and Q. The common tangent, closer to P, of C_1 and C_2 touches C_1 at A and C_2 at B. The tangent of C_1 at P meets C_2 at C, which is different from P, and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

<u>Solution</u>



Let C_3 be the circumcircle of triangle PQR and I be the center of circle C_1.
We have $\angle RAQ + \angle IPQ = 90°$ (they combine to cut half the circle). Therefore, $\angle RAQ = \angle QPC$ (i)
But $\angle QPC = \angle QBC$, and $\angle RAQ = \angle RBQ$, or A, B, R, Q are concyclic.
We have $\angle BPR = \angle BAR + \angle ABP$ (ii)
But $\angle BAR = \angle BQR$ (because of cyclic ABRQ) and $\angle ABP = \angle PQB$, and equation (ii) becomes $\angle BPR = \angle PQB + \angle BQR = \angle PQR$, or BP is tangential to circle C_3.

Now extend AI to intercept circle C_1 at D. Link PD to intercept C_3 at M. We have $\angle DAQ = \angle DPQ = \angle MRQ$ (iii)

and $\angle PAB + \angle API = 90°$ and $\angle API + \angle RPC = 180° - \angle IPC = 90°$, or $\angle PAB = \angle RPC$ (iv)

Combining (i) and (iv) we have $\angle QAB = \angle QPR$
But $\angle QAB + \angle DAQ = 90°$; therefore, $\angle QPR + \angle DPQ = \angle MPR = 90°$, or MR is the diameter of C_3.

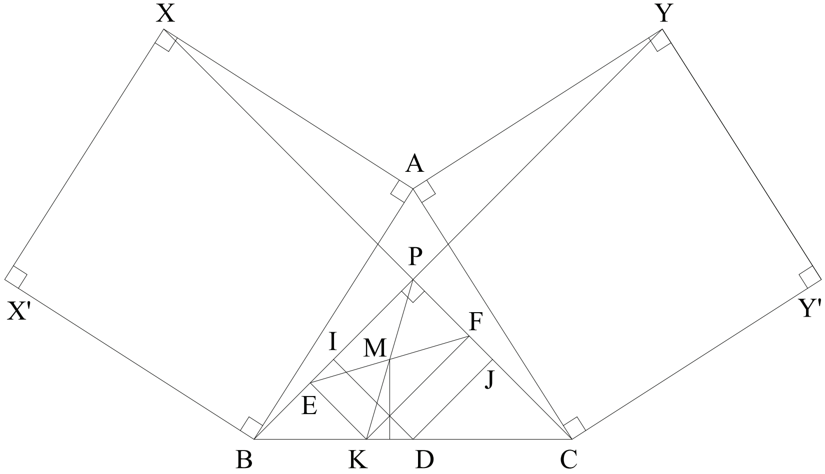
In the cyclic quadrilateral ABRQ $\angle QAB + \angle QRB = 180°$ (v)
Adding $\angle DAQ$ and subtract $\angle MRQ$ from (iii) to the left side of (v), we have $\angle QAB + \angle DAQ + \angle QRB - \angle MRQ = 180°$, or $90° + \angle MRP = 180°$, or $\angle MRP = 90°$.

Since MR is diameter of C_3 as proven earlier, therefore, BR is also tangential to circle C_3.

<u>*Problem 1 of Turkey MO Team Selection Test 1998*</u>

Squares BAXX' and CAYY' are drawn on the exterior of a triangle ABC with AB = AC. Let D be the midpoint of BC, and E and F be the feet of the perpendiculars from an arbitrary point K on the segment BC to BY and CX, respectively.
a) Prove that DE = DF.
b) Find the locus of the midpoint of EF.

<u>Solution</u>



a) Since both squares are congruent because AB = AC, AB = AX = AC = AY and $\angle XAC = 90° + \angle BAC = \angle BAY$. Therefore, the two isosceles triangles XAC and YAB are now congruent and $\angle AXC = \angle ABY$ or $XC \perp BY$. Also since the two squares are symmetrical with respect to the AP axis, BPC is a right isosceles triangle and PFKE is a rectangle. Draw the two perpendiculars from D to meet BP and CP at I and J, respectively; I and J are midpoints of BP and CP, respectively and DI = DJ. The parallel segments give us

$$\frac{EI}{KD} = \frac{BE}{BK} = \frac{EI + BE}{KD + BK} = \frac{BI}{BD} = \frac{CJ}{CD} = \frac{JF}{KD}, \text{ or } EI = JF.$$

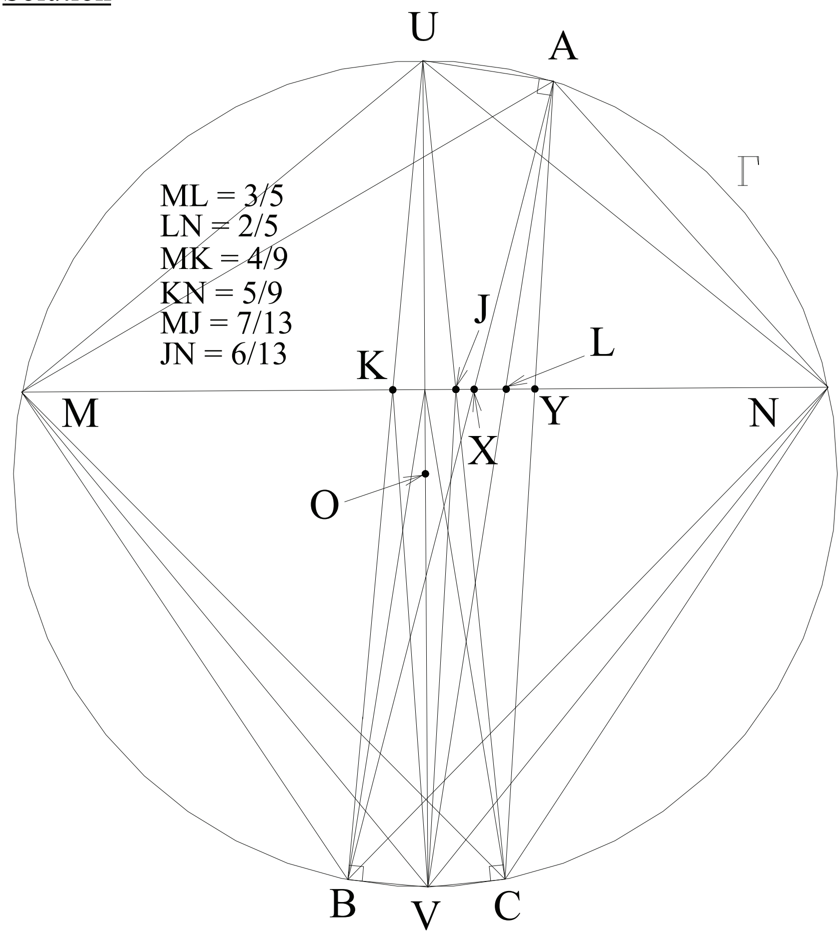
Combining with DI = DJ, the two right triangles DIE and DJF are congruent and we finally have DE = DF.

b) Since PFKE is a rectangle and M is the midpoint of diagonal EF, it is also the midpoint of diagonal PK. Therefore, its distance from BC is always constant and equals one-half the altitude from P to BC. We conclude that the locus of the midpoint of EF is the segment IJ.

Problem 2 of the Argentine MO Team Selection Test 2008

Triangle ABC is inscribed in a circumference Γ. A chord MN = 1 of Γ intersects the sides AB and AC at X and Y, respectively, with M, X, Y, N in that order in MN. Let UV be the diameter of Γ perpendicular to MN with U and A in the same semi-plane respect to MN. Lines AV, BU and CU cut MN in the ratios $\frac{3}{2}$, $\frac{4}{5}$ and $\frac{7}{6}$, respectively (start counting from M). Find XY.

Solution



It's difficult to draw the graph for this problem, but let's start by drawing chord MN across the circle. This chord MN will dictate the rest of the configuration. Draw the diameter UV to perpendicular MN. From V draw AV to cut MN at L such that $\frac{\text{ML}}{\text{LN}} = \frac{3}{2}$. From U draw BU to cut MN at K such that $\frac{\text{MK}}{\text{KN}} = \frac{4}{5}$, and from U draw CU to cut MN at J such that $\frac{\text{MJ}}{\text{JN}} = \frac{7}{6}$. All points M, N, U, V, A, B and C are on the circle. Let I = MN ∩ UV; I is the midpoint of MN and MI = NI = $\frac{1}{2}$.

The ratios give us ML = $\frac{3}{5}$, LN = $\frac{2}{5}$, MK = $\frac{4}{9}$, KN = $\frac{5}{9}$, MJ = $\frac{7}{13}$ and JN = $\frac{6}{13}$.

Let the two chords MN and UB intersect at K inside the circle Γ, and we have $\frac{\text{MK}}{\text{KN}} = \frac{\text{MU}}{\text{UN}} \times \frac{\text{MB}}{\text{BN}}$.

But MU = UN and subsequently $\frac{\text{MK}}{\text{KN}} = \frac{4}{5} = \frac{\text{MB}}{\text{BN}}$.

Similarly, $\frac{\text{ML}}{\text{LN}} = \frac{\text{MA}}{\text{AN}} \times \frac{\text{MV}}{\text{VN}}$, but MV = VN and $\frac{\text{ML}}{\text{LN}} = \frac{3}{2} = \frac{\text{MA}}{\text{AN}}$.

Continue with $\frac{\text{MX}}{\text{XN}} = \frac{\text{MA}}{\text{AN}} \times \frac{\text{MB}}{\text{BN}} = \frac{3}{2} \times \frac{4}{5} = \frac{6}{5}$.

Furthermore, MX + XN = 1, and we then obtain MX = $\frac{6}{11}$.

With the same argument, we have $\frac{\text{MJ}}{\text{JN}} = \frac{\text{MU}}{\text{UN}} \times \frac{\text{MC}}{\text{CN}} = \frac{\text{MC}}{\text{CN}} = \frac{7}{6}$, and $\frac{\text{MY}}{\text{YN}} = \frac{\text{MA}}{\text{AN}} \times \frac{\text{MC}}{\text{CN}} = \frac{3}{2} \times \frac{7}{6} = \frac{7}{4}$.

Once again, with MY + YN = 1, we get MY= $\frac{7}{11}$.

Finally, XY= MY – MX = $\frac{7}{11} - \frac{6}{11} = \frac{1}{11}$.

Problem 4 of International Mathematical Talent Search Round 2

Let a, b, c, and d be the areas of the triangular faces of a tetrahedron, and let h_a, h_b, h_c, and h_d be the corresponding altitudes of the tetrahedron. If V denotes the volume of the tetrahedron, prove that

$(a + b + c + d)(h_a + h_b + h_c + h_d) \geq 48V.$

Solution

Applying Cauchy-Schwarz's inequality, we have

$(\sqrt{a}^2 + \sqrt{b}^2 + \sqrt{c}^2 + \sqrt{d}^2)(\sqrt{h_a}^2 + \sqrt{h_b}^2 + \sqrt{h_c}^2 + \sqrt{h_d}^2) \geq (\sqrt{ah_a} + \sqrt{bh_b} + \sqrt{ch_c} + \sqrt{dh_d})^2.$

But the volume of a tetrahedron is given by $V = \frac{1}{3}ah_a = \frac{1}{3}bh_b = \frac{1}{3}ch_c = \frac{1}{3}dh_d$, or $ah_a = bh_b = ch_c = dh_d$, and the above inequality becomes

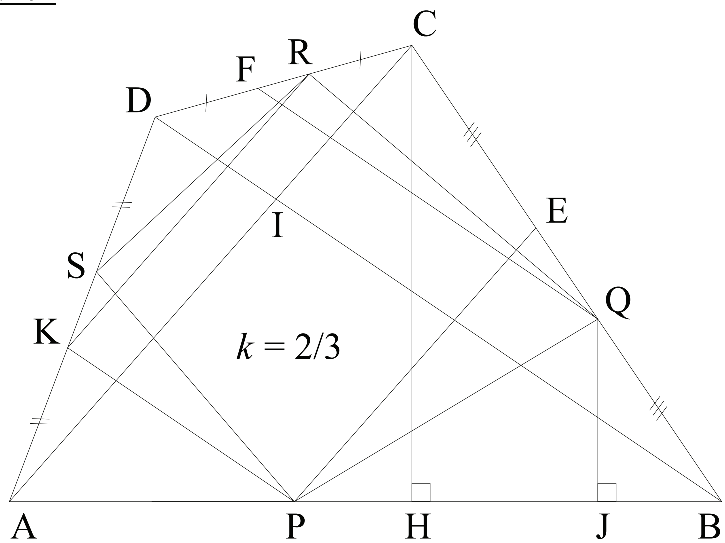
$(\sqrt{a}^2 + \sqrt{b}^2 + \sqrt{c}^2 + \sqrt{d}^2)(\sqrt{h_a}^2 + \sqrt{h_b}^2 + \sqrt{h_c}^2 + \sqrt{h_d}^2) \geq (4\sqrt{ah_a})^2 = 16ah_a = 48V.$

In other words, $(a + b + c + d)(h_a + h_b + h_c + h_d) \geq 48V.$

<u>*Problem 3 of International Mathematical Talent Search Round 3*</u>

Find k if P, Q, R and S are points on the sides of quadrilateral ABCD so that $\frac{AP}{PB} = \frac{BQ}{QC} = \frac{CR}{RD} = \frac{DS}{SA} = k$, and the area of quadrilateral PQRS is exactly 52% of the area of quadrilateral ABCD.

<u>Solution</u>



This is the graph for $k = \frac{2}{3}$*; the next graph is for* $k = \frac{3}{2}$.

Let (Ω) denote the area of shape Ω. From C and Q draw the altitudes CH and QJ to AB. From P draw PE (E on BC) such that PE || AC. Point Q is now on BC such that BQ = CE. From Q draw QF (F on CD) such that QF || BD. Point R is now on CD such that CR = DF. From R draw RK (K on AD) such that RK || AC. Point S is now on AD such that DS = AK.

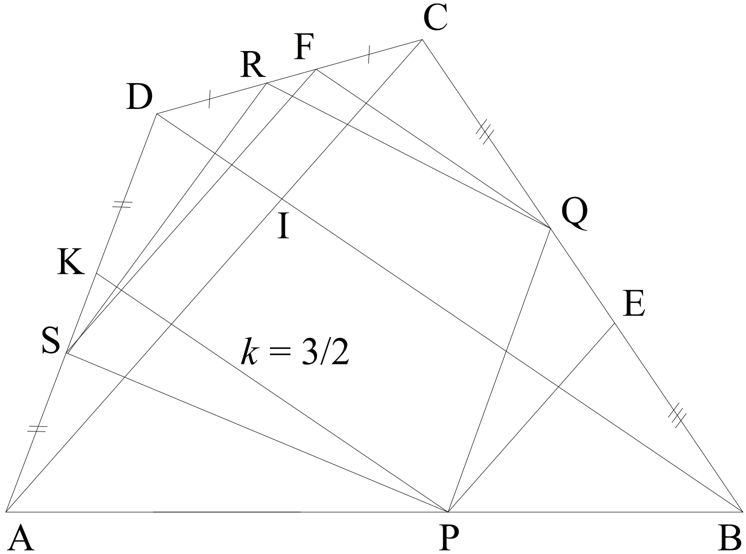
We now have $(BPQ) = \frac{1}{2}\times QJ\times PB = \frac{1}{2}\times QB\sin\angle QBJ\times PB = \frac{1}{2}\times QB\times\frac{CH}{BC}\times PB = \frac{1}{2}\times CH\times\frac{CE}{BC}\times PB = \frac{1}{2}\times CH\times PB\times\frac{AP}{AB} = (BCP)\times\frac{AP}{AB} = (BCP)\times\frac{1}{\frac{AP+PB}{AP}} = (BCP)\times\frac{1}{1+\frac{PB}{AP}} = (BCP)\times\frac{1}{1+\frac{1}{k}} = (BCP)\times\frac{k}{1+k}$, or $(BPQ) = (BCP) \times \frac{k}{1+k}$ (i)

However, $\frac{(BCP)}{(BCP)+(ACP)} = \frac{(BCP)}{(ABC)} = \frac{PB}{AB} = \frac{PB}{PB+AP} = \frac{1}{1+\frac{AP}{PB}} =$ $\frac{1}{1+k}$, and equation (i) is now equivalent to $(BPQ) = (ABC)\times\frac{k}{1+k}$ $\times\frac{1}{1+k} = (ABC)\times\frac{k}{(1+k)^2}$. Similarly, $(CQR) = (BCD)\times\frac{k}{(1+k)^2}$, $(DRS) = (ACD)\times\frac{k}{(1+k)^2}$, and $(APS) = (ABD)\times\frac{k}{(1+k)^2}$.

Adding up all these areas $(BPQ) + (CQR) + (DRS) + (APS) =$ $\frac{k}{(1+k)^2}[(ABC) + (BCD) + (ACD) + (ABD)] = \frac{k}{(1+k)^2}\times 2(ABCD)$.

But $(BPQ) + (CQR) + (DRS) + (APS) = (ABCD) - (PQRS)$, or $(PQRS) = (ABCD) - \frac{k}{(1+k)^2}\times 2(ABCD)$.

From there we obtain $\frac{(PQRS)}{(ABCD)} = 1 - \frac{2k}{(1+k)^2} = \frac{1+k^2}{(1+k)^2} = \frac{52}{100}$, or $6k^2 - 13k + 6 = 0$. Solving this quadratic equation for k, we get $k = \frac{2}{3}$, or $k = \frac{3}{2}$. Both of these solutions are acceptable.



<u>*Problem 13 of the Iranian Mathematical Olympiad 2010*</u>

In a quadrilateral ABCD, E and F are on BC and AD, respectively such that each of the area of triangle AED or triangle BFC is $\frac{4}{7}$ of the area of ABCD. R is the intersection point of diagonals of ABCD. It's also given that $\frac{AR}{RC} = \frac{3}{5}$, and $\frac{BR}{RD} = \frac{5}{6}$.

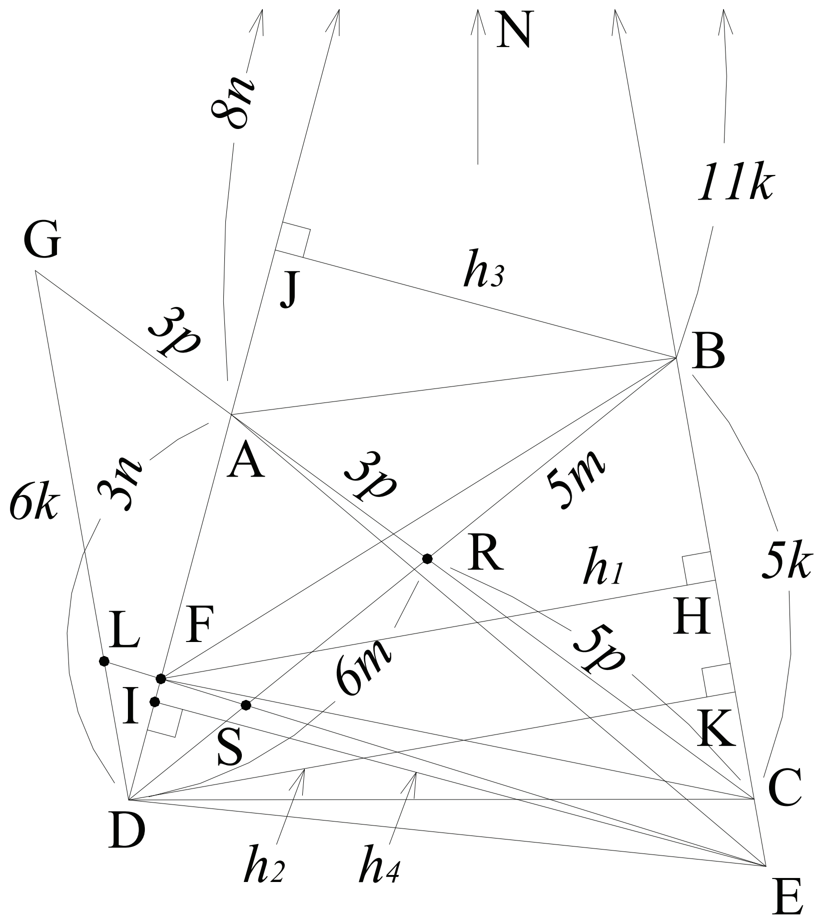
a) In what ratio does EF cut the diagonals?

b) Find $\frac{AF}{FD}$.

<u>Solution</u>

Let (Ω) denote the area of shape Ω, N be the intersection of the extensions of DA and CB, S be the intersection of BD and EF (if there is such an intersection), *n*, *m*, *p* and *k* be the positive real numbers. The different sets of dimensions of the segments can be represented in proportions of these numbers as shown on the graph on the next page.

We have $\frac{(ADR)}{(CDR)} = \frac{AR}{RC} = \frac{3}{5}$, $\frac{(CDR)}{(CBR)} = \frac{RD}{BR} = \frac{6}{5}$; $\frac{(ADR)}{(CBR)} = \frac{3}{5}\times\frac{6}{5} = \frac{18}{25}$,
From there, $\frac{(CDR)+(CBR)}{(ADR)} = \frac{(BDC)}{(ADR)} = \frac{5}{3} + \frac{25}{18} = \frac{55}{18}$.
Similarly, $\frac{(ABR)}{(CBR)} = \frac{AR}{RC} = \frac{3}{5}$, $\frac{(CDR)}{(CBR)} = \frac{6}{5}$, or $\frac{(ABR)}{(CDR)} = \frac{3}{5}\times\frac{5}{6} = \frac{1}{2}$.
From there, $\frac{(CDR)+(CBR)}{(ABR)} = \frac{(BDC)}{(ABR)} = 2 + \frac{5}{3} = \frac{11}{3}$.
Adding the two terms $\frac{(ADR)}{(BDC)} + \frac{(ABR)}{(BDC)} = \frac{(ABD)}{(BDC)} = \frac{18}{55} + \frac{3}{11} = \frac{3}{5}$,
and $1 + \frac{(ABD)}{(BDC)} = \frac{(ABCD)}{(BDC)} = 1 + \frac{3}{5} = \frac{8}{5}$, or $\frac{(BDC)}{(ABCD)} = \frac{5}{8}$ and $\frac{(ABD)}{(ABCD)} = 1 - \frac{5}{8} = \frac{3}{8}$. Since $\frac{(BDC)}{(ABCD)} = \frac{5}{8} > \frac{4}{7} = \frac{(BFC)}{(ABCD)}$, as given by the problem, point F must be closer to point N than point D is.



Graph drawn to scale.

Follow the same procedure, we get $\frac{(CDR)}{(CBR)} = \frac{6}{5}$, $\frac{(CDR)}{(ABR)} = \frac{6}{5} \times \frac{5}{3} = 2$.

From there, $\frac{(ABR) + (CBR)}{(CDR)} = \frac{(ABC)}{(CDR)} = \frac{1}{2} + \frac{5}{6} = \frac{4}{3}$, or $\frac{(CDR)}{(ABC)} = \frac{3}{4}$.

Similarly, the previous results $\frac{(ADR)}{(CBR)} = \frac{18}{25}$ and $\frac{(ADR)}{(ABR)} = \frac{6}{5}$ give us $\frac{(ABR) + (CBR)}{(ADR)} = \frac{(ABC)}{(ADR)} = \frac{5}{6} + \frac{25}{18} = \frac{20}{9}$, or $\frac{(ADR)}{(ABC)} = \frac{9}{20}$.

Adding the two terms to get $\frac{(ADR)}{(ABC)} + \frac{(CDR)}{(ABC)} = \frac{(ACD)}{(ABC)} = \frac{3}{4} + \frac{9}{20} =$

$\frac{6}{5}$, and $1 + \frac{(ACD)}{(ABC)} = \frac{(ABCD)}{(ABC)} = 1 + \frac{6}{5} = \frac{11}{5}$, or $\frac{(ABC)}{(ABCD)} = \frac{5}{11} < \frac{4}{7} = \frac{(BFC)}{(ABCD)}$ as given by the problem.

Therefore, point F must be further away from point N than point A is. Combining with the earlier result that point F must be closer to point N than point D is, we conclude that F must be on the interior of segment AD.

On the other hand, $1 + \frac{(ABC)}{(ACD)} = \frac{(ABCD)}{(ACD)} = 1 + \frac{5}{6} = \frac{11}{6}$, and $\frac{(ACD)}{(ABCD)} = \frac{6}{11} < \frac{4}{7} = \frac{(AED)}{(ABCD)}$.

Because of this, point E must be further away from point N than point C is; in other words, point E is on the extension of BC. Thus EF does not cut the diagonal AC but does cut the diagonal BD. We only need to find the ratio that EF cuts the diagonal BD into.

The problem gives us $\frac{(AED)}{(ABCD)} = \frac{(BFC)}{(ABCD)} = \frac{4}{7}$, or $\frac{(AED)}{(ABD)} = \frac{4}{7}\times\frac{8}{3} = \frac{32}{21}$ and $\frac{(BFC)}{(BDC)} = \frac{4}{7}\times\frac{8}{5} = \frac{32}{35}$.

From F and D drop the two altitudes FH and DK onto BC where we let FH= h_1 and DK = h_2 as shown. We then obtain $\frac{(BFC)}{(BDC)} = \frac{h_1}{h_2} = \frac{32}{35}$. Likewise, from E and B drop the two altitudes EI and BJ onto DA where BJ = h_3 and EI= h_4. We also found earlier that $\frac{(AED)}{(ABD)} = \frac{h_4}{h_3} = \frac{32}{21}$.

Now extend RA a segment AG to equal itself, AG = AR. Since $\frac{RG}{RC} = \frac{RD}{RB}$, DG || BC. If we let DG = $6k$, AR = AG = $3p$, AD = $3n$, RD = $6m$, we will have BC = $5k$.

The parallel segments give us $\frac{NA}{AD} = \frac{AG}{AC} = \frac{3}{8}$, or NA = $8n$.

Similarly, $\frac{DG}{CN} = \frac{3}{8}$ and with DG = $6k$, CN = $16k$, or NB = $11k$.

Because $h_1 \parallel h_2$, $\frac{h_1}{h_2} = \frac{\mathrm{NF}}{\mathrm{ND}} = \frac{32}{35}$, or $\mathrm{NF} = \frac{32}{35} \times \mathrm{ND} = \frac{32}{35} \times 11n$. We then have $\mathrm{AF} = \mathrm{NF} - \mathrm{NA} = \frac{32}{35} \times 11n - 8n = \frac{72n}{35}$, and $\mathrm{FD} = \mathrm{AD} - \mathrm{AF} = 3n - \frac{72n}{35} = \frac{33n}{35}$.

The ratio $\frac{\mathrm{AF}}{\mathrm{FD}}$ becomes $\frac{\mathrm{AF}}{\mathrm{FD}} = \frac{72n}{33n} = \frac{24}{11}$.

Segment $\mathrm{NF} = \mathrm{NA} + \mathrm{AF} = 8n + \frac{72n}{35} = \frac{352n}{35}$, and the ratio $\frac{\mathrm{NF}}{\mathrm{FD}} = \frac{32}{3}$.

Now extend EF to meet DG at L. We have $\frac{\mathrm{NE}}{\mathrm{DL}} = \frac{\mathrm{NF}}{\mathrm{FD}} = \frac{32}{3}$ (i)

As we found earlier $\frac{(\mathrm{ABD})}{(\mathrm{AED})} = \frac{h_3}{h_4} = \frac{21}{32}$. Because $h_3 \parallel h_4$, $\frac{h_3}{h_4} = \frac{\mathrm{NB}}{\mathrm{NE}}$, or $\mathrm{NE} = \frac{32}{21} \times \mathrm{NB} = \frac{32}{21} \times 11k = \frac{352k}{21}$, and now $\mathrm{BE} = \mathrm{NE} - 11k = \frac{121k}{21}$.

From (i), $\mathrm{DL} = \frac{3}{32} \times \mathrm{NE} = \frac{3}{32} \times \frac{352k}{21} = \frac{11k}{7}$.

The ratio that EF cuts the diagonal BD is $\frac{\mathrm{BS}}{\mathrm{DS}} = \frac{\mathrm{BE}}{\mathrm{DL}} = \frac{\frac{121k}{21}}{\frac{11k}{7}} = \frac{11}{3}$.

<u>Further observation</u>

Let's find the ratio of the areas of ABCD and NDC.

$\frac{RG}{RC} = \frac{2AR}{RC} = \frac{6}{5} = \frac{RD}{BR}$; *therefore, DG || BC which directly gives us the ratio* $\frac{NC}{DG} = \frac{AC}{AG} = \frac{AC}{AR} = \frac{AR + RC}{AR} = 1 + \frac{RC}{AR} = 1 + \frac{5}{3} = \frac{8}{3}$, *or* $\frac{DG}{NC} = \frac{3}{8} = \frac{AD}{AN} = \frac{(ADC)}{(ANC)} = \frac{(AED)}{(AEN)}$.

We also have $\frac{(NDC)}{(ADC)} = \frac{(ADC) + (ANC)}{(ADC)} = 1 + \frac{(ANC)}{(ADC)} = 1 + \frac{8}{3} = \frac{11}{3}$, $\frac{BN}{BC} = 1 + \frac{RD}{BR} = 1 + \frac{6}{5} = \frac{11}{5}$, *and* $\frac{(ANC)}{(ABC)} = \frac{(ABC) + (ABN)}{(ABC)} = 1 +$

$\frac{(ABN)}{(ABC)} = 1 + \frac{BN}{BC} = 1 + \frac{11}{5} = \frac{16}{5}$, *or* $\frac{(ABC)}{(ANC)} = \frac{5}{16}$.

Moreover, $\frac{1}{1 + \frac{(ABN)}{(ABC)}} = \frac{1}{1 + \frac{BN}{BC}} = \frac{1}{1 + \frac{11}{5}} = \frac{5}{16}$, *or* $\frac{(ANC)}{(NDC)} = \frac{AN}{DN} =$
$\frac{1}{1 + \frac{AD}{AN}} = \frac{1}{1 + \frac{3}{8}} = \frac{8}{11}$, *or* $\frac{(ABC)}{(NDC)} = \frac{(ABC)}{(ANC)} \times \frac{(ANC)}{(NDC)} = \frac{5}{16} \times \frac{8}{11} = \frac{5}{22}$.

Therefore, $\frac{(ADC) + (ABC)}{(NDC)} = \frac{(ABCD)}{(NDC)} = \frac{3}{11} + \frac{5}{22} = \frac{1}{2}$, *or the area of quadrilateral ABCD equals one-half the area of triangle NDC.*

Problem 1 of International Mathematical Talent Search Round 4

Use each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly twice to form two distinct prime numbers whose sum is as small as possible. What must be this minimal sum be? *(Note: The five smallest primes are 2, 3, 5, 7 and 11.)*

Solution

We have $2 + 5 - 3 - 4 + 7 \times 8 - 9 \times 6 = 2$ and

$$\frac{9 - 4 + 8 - 3 + 7 - 2 + 6 - 1}{5} - 1 = 3$$

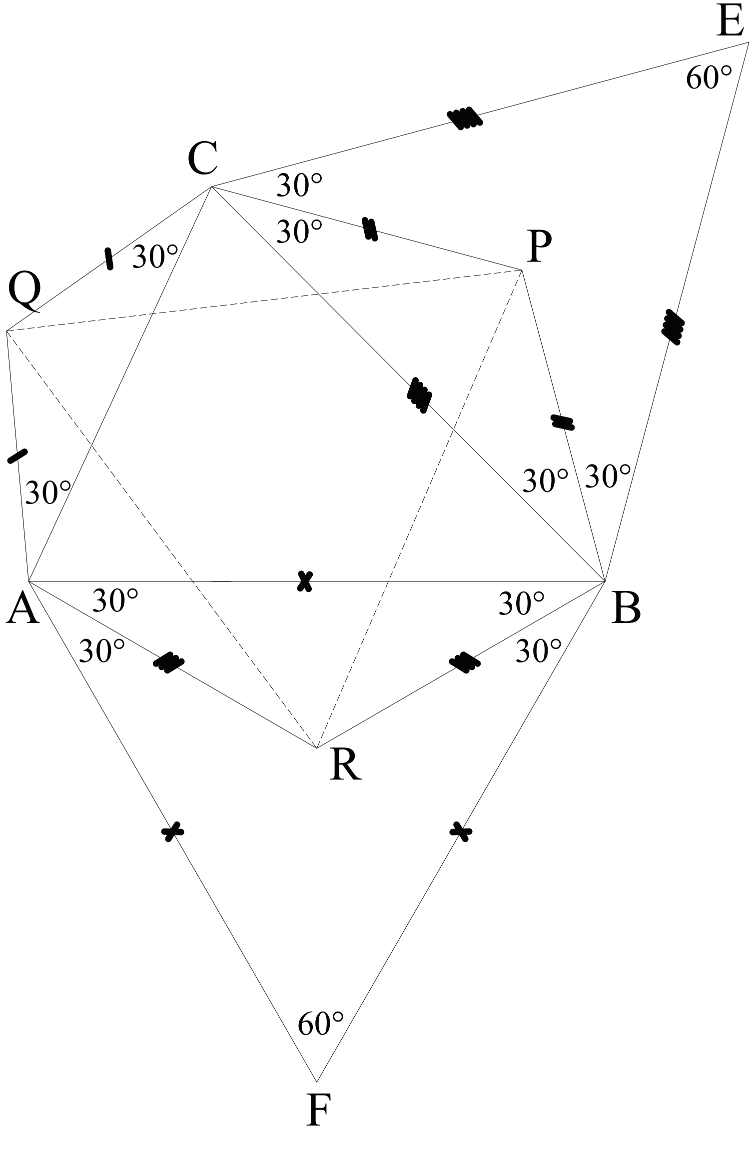
in which each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 was used exactly twice to form two distinct prime numbers 2 and 3 whose sum is 5 which is as small as possible.

The minimal sum is 5.

Problem 4 of International Mathematical Talent Search Round 4

Let ΔABC be an arbitrary triangle, and construct P, Q, and R so that each of the angles marked is 30°. Prove that ΔPQR is an equilateral triangle.

Solution



Draw two equilateral triangles ABE and BCF; C and E are on opposite sides of AB; A and F are on opposite sides of BC. Because the two isosceles triangles ACQ and BCP are similar and with the two new equilateral triangles, we have $\frac{QC}{PC} = \frac{AC}{BC} = \frac{AC}{CF}$ and $\angle QCP = \angle C + 2\times 30° = \angle ACF$, triangles QCP and ACF are similar which implies $\frac{QP}{AF} = \frac{QC}{AC}$.

Also because of the same reason as above, $\frac{QA}{RA} = \frac{AC}{AB} = \frac{AC}{AE}$ and $\angle QAR = \angle A + 2\times 30° = \angle CAE$, triangles QAR and CAE are also similar which gives us $\frac{QR}{CE} = \frac{QA}{AC} = \frac{QC}{AC} = \frac{QP}{AF}$ (i)

Furthermore, again because of the two new equilateral triangles, we have AB = BE, $\angle ABF = \angle B + 2\times 30° = \angle EBC$ and BF = BC which make the two triangles ABF and EBC to be congruent which implies AF = CE.

Equation (i) is now equivalent to $\frac{QR}{CE} = \frac{QP}{CE}$, or QR = QP.

With the similar approach, we also get QR = PR, and ΔPQR is an equilateral triangle.

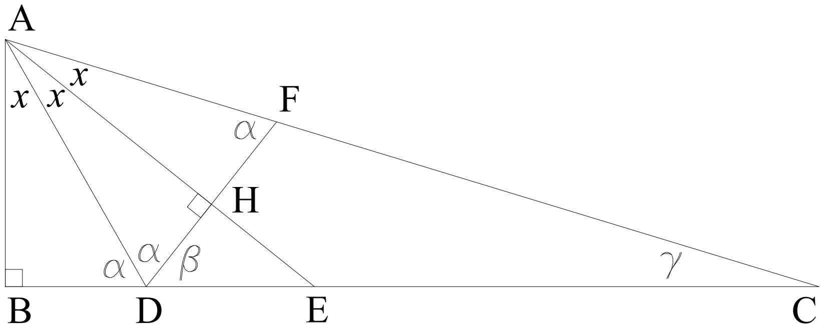
<u>Further observation</u>

This problem is somewhat similar to the Napoleon's theorem. It is also similar to problem 1 of Hong Kong Mathematical Olympiad 2010 described as "ABC is an arbitrary triangle. Draw three regular polygons on the external part of ABC with three edges. Find all possible n, so that the triangle formed by the three centers of polygon is equilateral".

<u>Problem 3 of International Mathematical Talent Search Round 41</u>

Suppose $\dfrac{\cos 3x}{\cos x} = \dfrac{1}{3}$ for some angle x, $0 \le x \le \dfrac{\pi}{2}$. Determine $\dfrac{\sin 3x}{\sin x}$ for the same x.

<u>Solution</u>



Draw the two right triangle ABC and ABD with the right angle at B and AC = 3AD. The bisector of $\angle$CAD meets BC at E. From D draw the perpendicular to meet AE and AC at H and F, respectively. We then have $\cos\angle \text{BAD} = \dfrac{\text{AB}}{\text{AD}} = 3\times\dfrac{\text{AB}}{\text{AC}} =$ $3\cos\angle\text{BAC}$, or $x = \angle\text{BAD}$, $3x = \angle\text{BAC}$ and $2x = \angle\text{CAD}$. Now let $\angle\text{ADB} = \alpha$, $\angle\text{CDF} = \beta$,
and $\angle\text{ACB} = \gamma$, or $\gamma = 90° - 3x$.

It's easily seen that the three right triangles ABD, AHD and AHF are congruent because they each have a right angle, the same angles x and triangles ABD, AHD share side AD while triangles AHD, AHF share side AH. Now let a = AD, $3a$ = AC, $2a$ = FC, b = BD = DH = HF. We now have $\alpha + \beta = 90° + x$.
Applying the law of sines to triangle CDF, we get $\dfrac{2a}{\sin\beta} = \dfrac{2b}{\sin\gamma}$, or

$\dfrac{a}{\sin\beta} = \dfrac{b}{\cos 3x}$, or $\dfrac{b}{a} = \dfrac{\cos 3x}{\sin\beta} = \sin x$.
However, ABDH is a cyclic quadrilateral because $\angle$B = $\angle$AHD = 90°, or $\beta = \angle\text{BAH} = 2x$.

The equation $\frac{\cos 3x}{\sin\beta} = \sin x$ becomes $\frac{\cos 3x}{\sin 2x} = \sin x$, or
$\sin 2x \sin x = \cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x$, or
$2\sin 2x \sin x = \cos 2x \cos x$, or $4\sin x \cos x \sin x = \cos 2x \cos x$, or
$4\sin^2 x = \cos 2x = 1 - 2\sin^2 x$, or $6\sin^2 x = 1$, or $\sin x = \frac{1}{\sqrt{6}} = \frac{b}{a}$.

Again apply the law of sines to triangle ACD, we get $\frac{\text{CD}}{\sin 2x} = \frac{\text{CD}}{2\sin x \cos x} = \frac{3a}{\sin(\alpha + \beta)} = \frac{3a}{\sin(90° + x)} = \frac{3a}{\cos x}$, or $\frac{\text{CD}}{2\sin x} = 3a$, or
$\text{CD} = 6a\sin x$.

Now without loss of generality, let $b = 1$, $a = \sqrt{6}$, $\text{AC} = 3\sqrt{6}$, $\text{CD} = 6\sqrt{6}\times\frac{1}{\sqrt{6}} = 6$. Hence, $\text{BC} = \text{BD} + \text{CD} = 7$, and $\sin 3x = \frac{\text{BC}}{\text{AC}} = \frac{7}{3\sqrt{6}}$.

Finally, $\frac{\sin 3x}{\sin x} = \frac{\frac{7}{3\sqrt{6}}}{\frac{1}{\sqrt{6}}} = \frac{7}{3}$.

Problem 1 of International Mathematical Talent Search Round 41

Determine the unique positive integers m and n for which the approximation $\frac{m}{n} = .2328767$ is accurate to the seven decimals; i.e., $0.2328767 \leq \frac{m}{n} < 0.2328768$.

Solution

$0.2328767 = \frac{2328767}{10000000}$ and $0.2328768 = \frac{2328768}{10000000}$. We are now need to determine the unique positive integers m and n for which $\frac{2328767}{10000000} \leq \frac{m}{n} < \frac{2328768}{10000000}$.

The answer is $\frac{m}{n} = \frac{2328767 + 2328768}{10000000 + 10000000} = \frac{4657535}{20000000} = \frac{931507}{4000000}$ because $0.2328767 \leq \frac{931507}{4000000} = 0.23287675 < 0.2328768$ as required.

Further observation

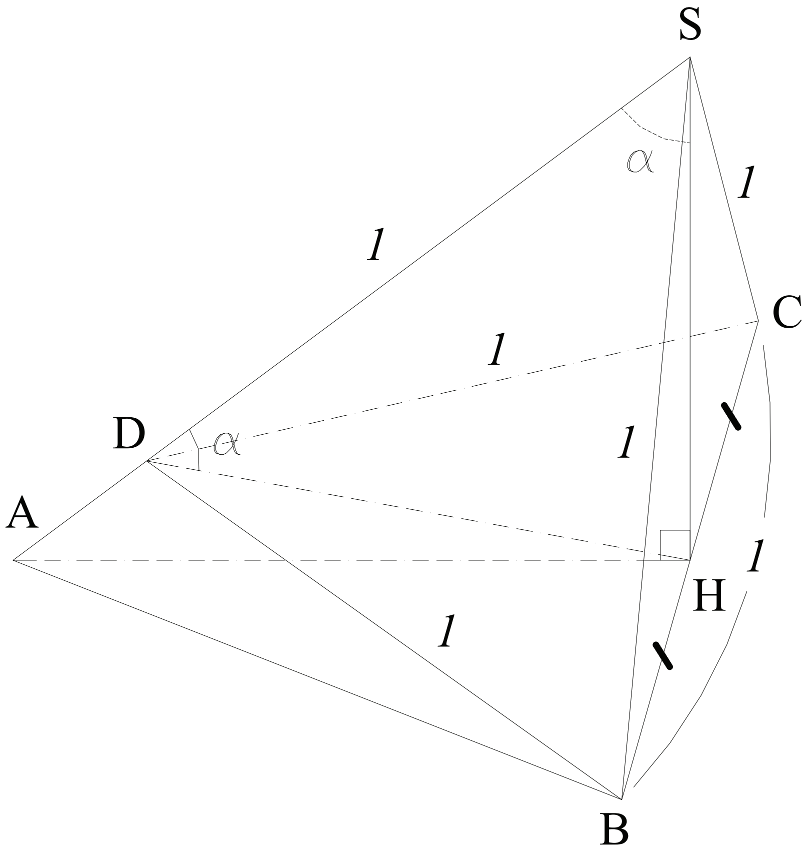
The above method can be used to solve the problem 3 of the British Mathematical Olympiad 1987 where it is asked to find a pair of integers r and s such that $0 < s < 200$ *and* $\frac{45}{61} > \frac{r}{s} > \frac{59}{80}$. *Also prove that there is exactly one such pair.*

The answer is $\frac{45+59}{61+80} = \frac{104}{141}$, *and it satisfies the problem because* $\frac{45}{61} > \frac{104}{141} > \frac{59}{80}$ *where* $141 < 200$ *as required.*

<u>Problem 4 of the Vietnamese Mathematical Olympiad 1964</u>

The tetrahedron SABC has the faces SBC and ABC perpendicular to each other. The three angles at S are all 60° and SB = SC = 1. Find its volume.

<u>Solution</u>



Pick point D on side SA such that SD = SB = SC = 1; SBCD is a regular tetrahedron with all four faces being equilateral triangles and side length of 1. Therefore, if H is the midpoint of BC, SH and DH are the altitudes of congruent equilateral triangles SBC and DBC, respectively, and SH = DH.

Per the Pythagorean theorem, SH = DH = $\sqrt{SB^2 - BH^2} = \frac{\sqrt{3}}{2}$. Now let $\alpha = \angle ASH = \angle SDH$ and $180° - 2\alpha = \angle SHD$. Applying the

law of sines, we get $\frac{SD}{\sin\angle SHD} = \frac{DH}{\sin\angle ASH}$, or $\frac{SD}{\sin(180° - 2\alpha)} = \frac{SD}{\sin 2\alpha} = \frac{SD}{2\sin\alpha\cos\alpha} = \frac{DH}{\sin\alpha}$, or $\frac{SD}{2\cos\alpha} = DH$; $\cos\alpha = \frac{SD}{2DH} = \frac{SD}{2SH}$.

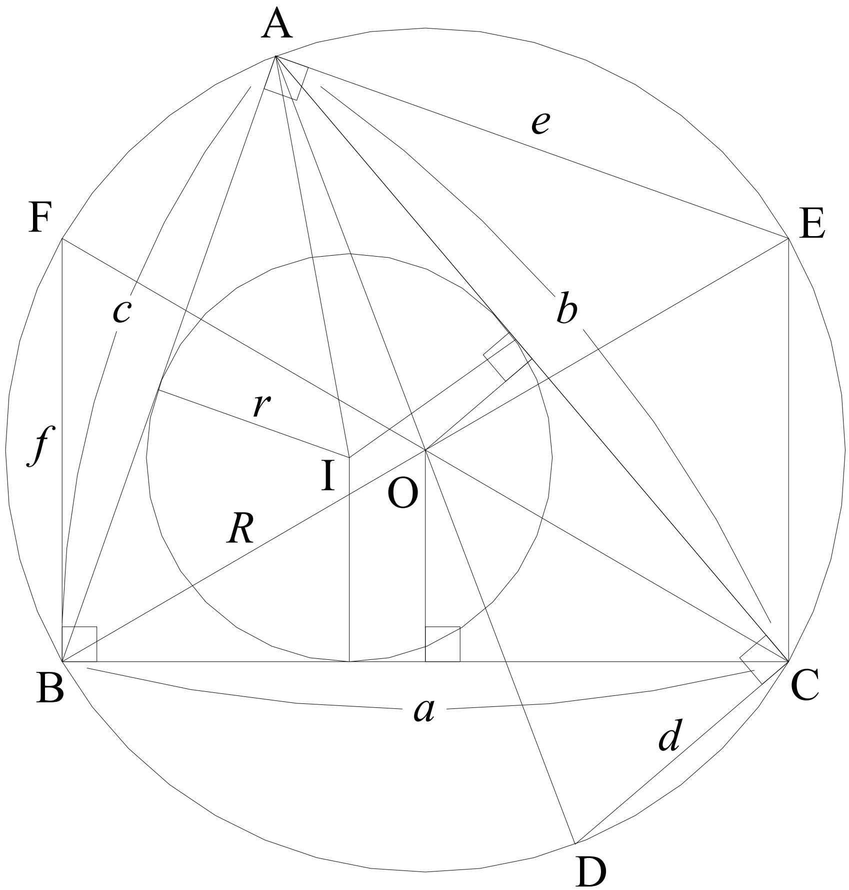
Substituting in the values for SD and SH, we now obtain $\cos\alpha = \frac{1}{\sqrt{3}} = \frac{SH}{SA}$ (because the faces SBC and ABC perpendicular to each other and $SH \perp AH$), or $SA = \sqrt{3} \times SH = \frac{3}{2}$, $AH = \sqrt{SA^2 - SH^2} = \sqrt{\frac{9}{4} - \frac{3}{4}} = \frac{\sqrt{6}}{2}$.

AH is also the height of tetrahedron ASBC with SBC as its base triangle. The volume of tetrahedron SABC, or tetrahedron ASBC is given as $V = \frac{1}{3}AH \times (\text{Area of SBC}) = \frac{1}{3}AH \times \frac{1}{2}BC \times SH = \frac{1}{3} \times \frac{\sqrt{6}}{2} \times \frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{8}$.

Problem 5 of the Vietnamese Mathematical Olympiad 1964

The triangle ABC has perimeter p. Find the side length AB and the area S in terms of $\angle$A, $\angle$B and p.

Solution



Let a = BC, b = AC, c = AB and R be the circumradius of triangle ABC. For simplicity, let's denote A = $\angle$A, B = $\angle$B and C = $\angle$C.

We have $2R = \dfrac{a}{\sin A}$, and according to the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{c}{\sin[180° - (A + B)]} = \frac{c}{\sin(A + B)} =$$

$\frac{a+b+c}{\sin A+\sin B+\sin C}=\frac{a+b+c}{\sin A+\sin B+\sin[180°-(A+B)]}=\frac{a+b+c}{\sin A+\sin B+\sin(A+B)}$, or $c=\frac{p\sin(A+B)}{\sin A+\sin B+\sin(A+B)}$ which is the answer for the first question of the problem.

Similarly,
$a=\frac{p\sin(B+C)}{\sin B+\sin C+\sin(B+C)}$ and $b=\frac{p\sin(A+C)}{\sin A+\sin C+\sin(A+C)}$.

However, the sum of B + C = 180° – A, and sin(B + C) = sin(180° – A) = sin A; therefore, a, b and c become $a=\frac{p\sin A}{\sin A+\sin B+\sin C}$, $b=\frac{p\sin B}{\sin A+\sin B+\sin C}$ and $c=\frac{p\sin C}{\sin A+\sin B+\sin C}$.

Per Heron's formula if $s=\frac{p}{2}$ is the semi-perimeter of triangle ABC, its area is $\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{\frac{p}{2}(\frac{p}{2}-a)(\frac{p}{2}-b)(\frac{p}{2}-c)}$;

$\frac{p}{2}(\frac{p}{2}-a)(\frac{p}{2}-b)(\frac{p}{2}-c)=\frac{p}{16}(p-2a)(p-2b)(p-2c)=$
$\frac{p^4}{16}\times\frac{(\sin B+\sin C-\sin A)(\sin A+\sin C-\sin B)(\sin A+\sin B-\sin C)}{(\sin A+\sin B+\sin C)^3}$

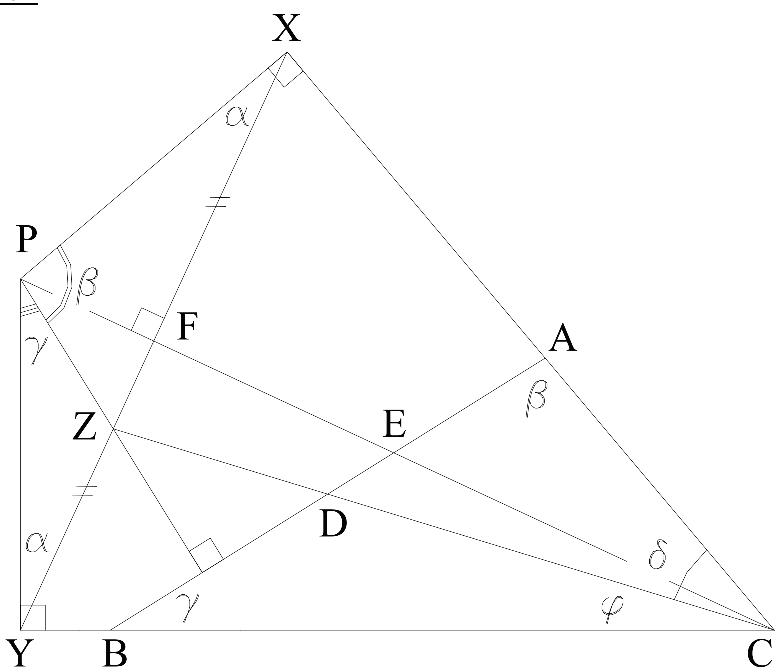
$$S=\sqrt{\frac{p}{2}(\frac{p}{2}-a)(\frac{p}{2}-b)(\frac{p}{2}-c)}=\frac{p^2}{4}\times\frac{1}{\sin A+\sin B+\sin C}\times\sqrt{\frac{(\sin B+\sin C-\sin A)(\sin A+\sin C-\sin B)(\sin A+\sin B-\sin C)}{\sin A+\sin B+\sin C}}$$

Replace sinC with sin(A + B) in order to have S in terms of ∠A, ∠B and p.

__Problem B6 of British Mathematical Olympiad 1974__

X and Y are the feet of the perpendiculars from P to CA and CB respectively, where P is in the plane of triangle ABC and PX = PY. The straight line through P which is perpendicular to AB cuts XY at Z. Prove that CZ bisects AB.

__Solution__



Since the altitudes from P to CA and CB are equal, PX = PY, P is on the bisector CP of ∠ACB, and ∠PXY = ∠PYX. Now let α = ∠PXY = ∠PYX, β = ∠XPZ = ∠BAC (their respective sides perpendicular to each other), γ = ∠YPZ = ∠ABC (for the same reason), δ = ∠XCZ, and φ = ∠YCZ.

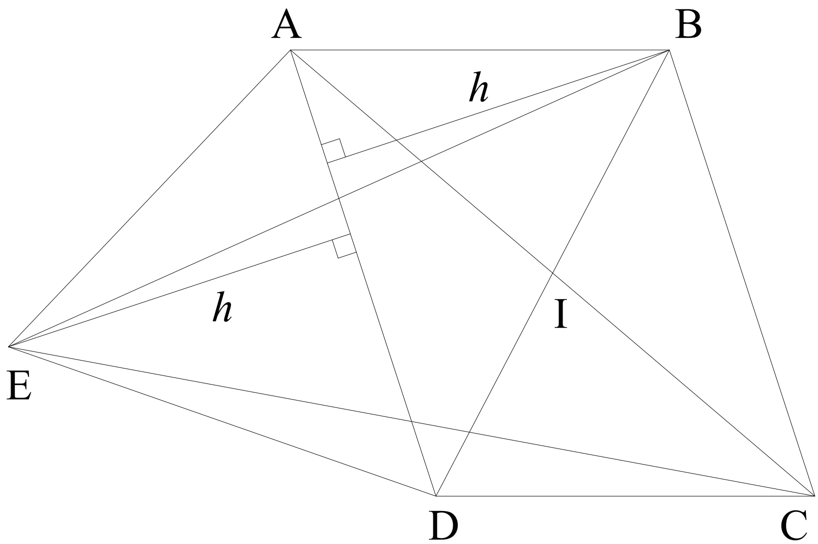
Applying the law of sines to triangles PZX and PZY, we get $\frac{XZ}{PZ} = \frac{\sin\beta}{\sin\alpha}$ and $\frac{YZ}{PZ} = \frac{\sin\gamma}{\sin\alpha}$, respectively, or $\frac{XZ}{YZ} = \frac{\sin\beta}{\sin\gamma}$.

Similarly, in triangles XZC and YZC, we have $\frac{XZ}{CZ} = \frac{\sin\delta}{\sin(90° - \alpha)}$ and $\frac{YZ}{CZ} = \frac{\sin\varphi}{\sin(90° - \alpha)}$, or $\frac{XZ}{YZ} = \frac{\sin\delta}{\sin\varphi} = \frac{\sin\beta}{\sin\gamma}$, or $\frac{\sin\delta\sin\gamma}{\sin\beta\sin\varphi} = 1$.
However, in triangles ACD and BCD, the law of sines gives us $\frac{AD}{CD} = \frac{\sin\delta}{\sin\beta}$ and $\frac{BD}{CD} = \frac{\sin\varphi}{\sin\gamma}$, or $\frac{AD}{BD} = \frac{\sin\delta\ \sin\gamma}{\sin\beta\sin\varphi} = 1$, or AD = BD and CZ bisects AB.

Problem 3 of Austria Mathematical Olympiad 2001

In a convex pentagon, the areas of the triangles ABC, ABD, ACD and ADE are all equal to the same value F. What is the area of the triangle BCE?

Solution



Let (Ω) denote the area of shape Ω and I = AC ∩ BD.

Since (ABC) = (ABD) the altitudes from C and D to AB are equal which means AB || CD and (AID) = (ABD) – (AIB) = (ABC) – (AIB) = (BIC).

Similarly, (ABC) = (ACD) makes the altitudes from D and B to AC to be equal and with (AID) = (BIC), we have AI = IC. Combining with AB || CD, ABCD is a parallelogram.

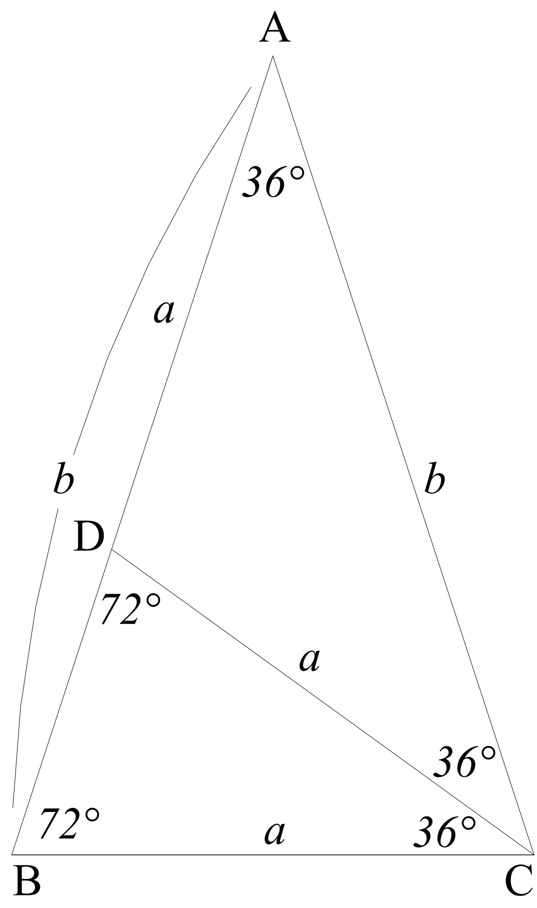
Also since (ADE) = (ABD) the altitudes from E and B to AD are equal. Let it be h.

We have (BCE) = ½2h×BC = 2F.

Problem 4 of Spain Mathematical Olympiad 1994

In a triangle ABC with $\angle$A = 36° and AB = AC, the bisector of the angle at C meets the opposite side at D. Compute the angles of ΔBCD. Express the length *a* of side BC in terms of the length *b* of side AC without using trigonometric functions.

Solution



Since ABC is an isosceles triangle and $\angle$BAC = 36°, $\angle$B = $\angle$C = ½(180° – 36°) = 72°. CD bisects $\angle$C and $\angle$BCD = 36°, $\angle$BDC = 72°. BCD is then an isosceles triangle itself with BC = CD = *a*. Also because $\angle$BAC = $\angle$ACD = 36°, CD = AD = *a*.

Now applying Stewart's theorem, we get $AC^2 \times BD + BC^2 \times AD = AB(CD^2 + AD \times BD)$, or $b^2 \times (b - a) + a^3 = b[a^2 + a(b - a)]$, or $a^3 - 2ab^2 + b^3 = 0$.

However, $a^3 - 2ab^2 + b^3 = a^3 - ab^2 - ab^2 + b^3 = a(a^2 - b^2) - b^2(a - b) = (a - b)(a^2 + ab - b^2) = 0$.
But $a \neq b$ or $a^2 + ab - b^2 = 0$. Solving for a, we get $a = \frac{1}{2}b(\sqrt{5} - 1)$.

<u>Further observation</u>

We can measure sin18° from this result. Draw the altitude from A of triangle ABC.

$$\sin 18° = \frac{a}{2b} = \frac{1}{4}(\sqrt{5} - 1).$$

Problem 7 Baltic Way 1995

Prove that $\sin^3 18° + \sin^2 18° = \frac{1}{8}$.

Solution

Based on the result of the previous problem, $\sin 18° = \frac{1}{4}(\sqrt{5} - 1)$, we have $\sin^3 18° = [\frac{1}{4}(\sqrt{5} - 1)]^3$ and $\sin^2 18° = [\frac{1}{4}(\sqrt{5} - 1)]^2$, and $\sin^3 18° + \sin^2 18° = [\frac{1}{4}(\sqrt{5} - 1)]^2(1 + \frac{1}{4}(\sqrt{5} - 1)] = \frac{1}{32}(3 - \sqrt{5})(3 + \sqrt{5}) = \frac{1}{32}(3^2 - \sqrt{5}^2) = \frac{1}{8}$.

<u>Problem 1 of the Vietnamese Mathematical Olympiad 1982</u>

Determine a quadric polynomial with integral coefficients whose roots are cos72° and cos144°.

<u>Solution</u>

We have $\cos 72° = 2\cos^2 36° - 1 = 2(2\cos^2 18° - 1)^2 - 1 = 2(4\cos^4 18° - 4\cos^2 18° + 1) - 1 = 8\cos^4 18° - 8\cos^2 18° + 1$.

Based on the result of the previous problem, $\sin 18° = \frac{1}{4}(\sqrt{5} - 1)$,

$\sin^2 18° = [\frac{1}{4}(\sqrt{5} - 1)]^2 = \frac{1}{8}(3 - \sqrt{5})$, $\cos^2 18° = 1 - \sin^2 18° = \frac{1}{8}(5 + \sqrt{5})$, $\cos^4 18° = \frac{5}{32}(3 + \sqrt{5})$.

Let's find the value of cos72°, $\cos 72° = 8\cos^4 18° - 8\cos^2 18° + 1 = 8\cos^4 18° - 8\cos^2 18° + 1 = \frac{5}{4}(3 + \sqrt{5}) - 5 - \sqrt{5} + 1 = \frac{1}{4}(-1 + \sqrt{5})$,

and $\cos 144° = 2\cos^2 72° - 1 = 2[\frac{1}{4}(-1 + \sqrt{5})]^2 - 1 = -\frac{1}{4}(1 + \sqrt{5})$.

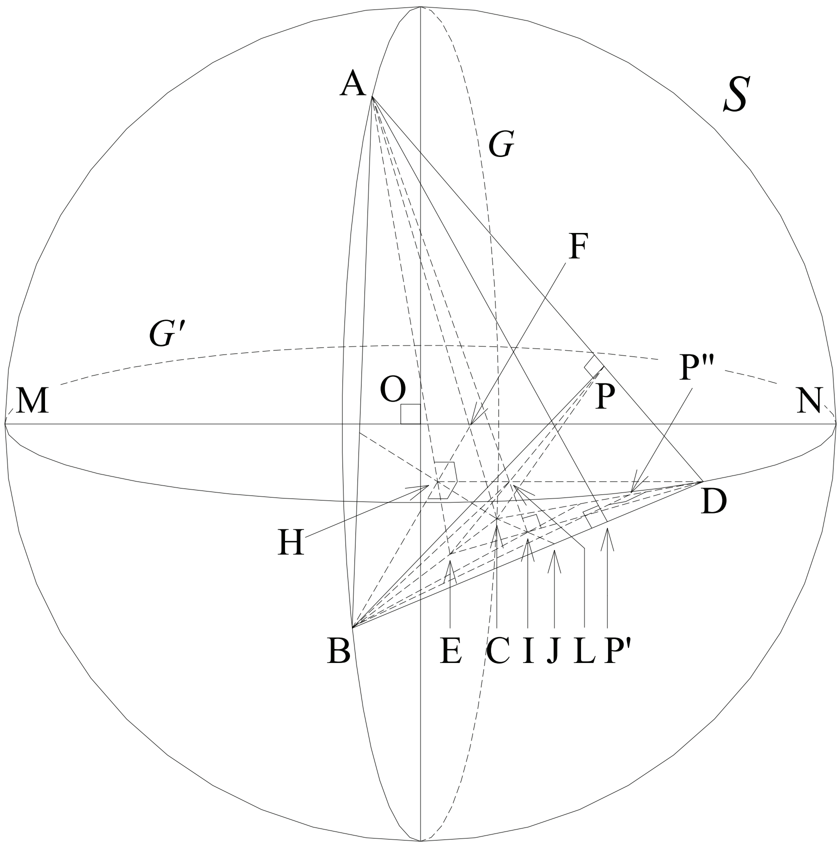
Now write the quadric polynomial as $(x - \cos 72°)(x - \cos 144°) = [x - \frac{1}{4}(-1 + \sqrt{5})][x + \frac{1}{4}(1 + \sqrt{5})] = (4x + 1 - \sqrt{5})(4x + 1 + \sqrt{5}) = (4x + 1)^2 - 5 = 4x^2 + 2x - 1 = 0$.

Answer: The quadric polynomial with integral coefficients whose roots are cos72° and cos144° is $4x^2 + 2x - 1 = 0$.

Problem 5 of the Vietnamese Mathematical Olympiad 1994

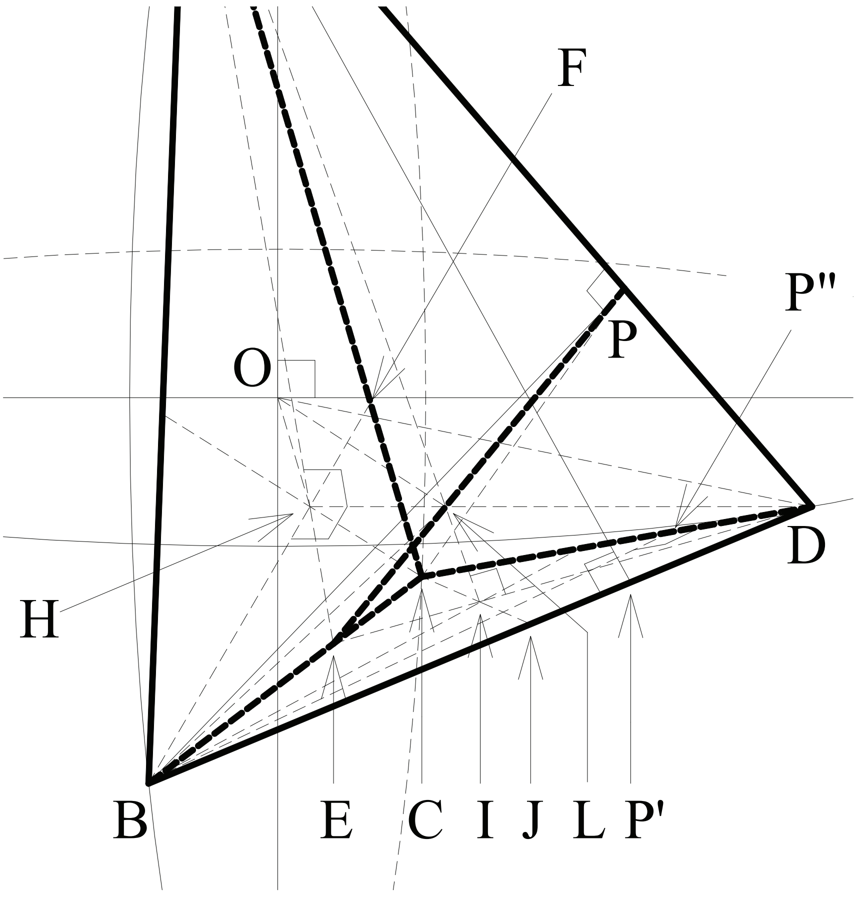
S is a sphere with center O. *G* and *G'* are two perpendicular great circles on *S*. Take A, B, C on *G* and D on *G'* such that the altitudes of the tetrahedron ABCD intersect at a point. Find the locus of the intersection.

Solution



Let [Φ] denote the plane containing shape Φ, *G* and *G'* be the vertical and horizontal great circles with radius *R* = OA = OB = OC = OD, *G'* to touch the westernmost and easternmost points of the sphere at M and N, respectively, the altitudes of the tetrahedron ABCD to intersect at L. Extend AL to meet [BCD] at I, DL to meet [ABC] at H, DI to meet BC at E.

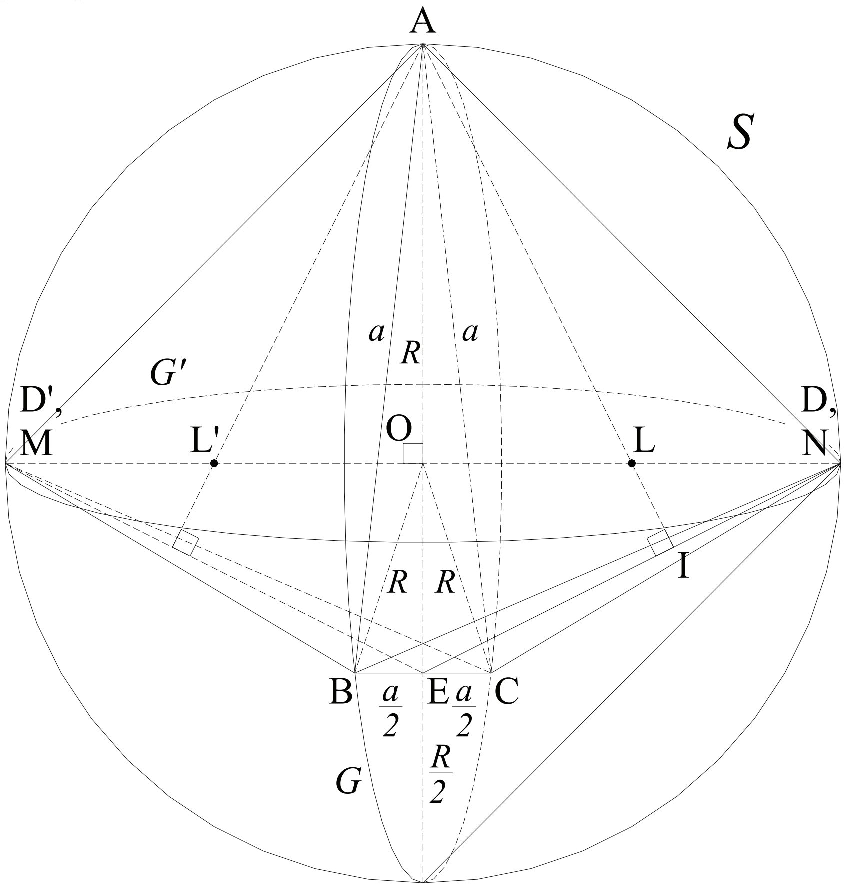
Let's analyze the triangle ADE. According to the Pythagorean theorem, $OL^2 = OH^2 + HL^2$, but $OH^2 = OD^2 - HD^2 = R^2 - HD^2$. We then have $OL^2 = R^2 - HD^2 + HL^2 = R^2 - (DL + HL)^2 + HL^2 = R^2 - DL^2 - HL^2 - 2DL\times HL + HL^2 = R^2 - DL^2 - 2DL\times HL = R^2 - DL(DL + 2HL) = R^2 - DL\times DH - DL\times HL$.



Now extend EL to meet AD at P; quadrilateral AHLP is cyclic because of its two opposite right angles, and this gives us $DL\times DH = DP\times AD$. Hence, $OL^2 = R^2 - DP\times AD - DL\times HL$ (i)

Since AI $\perp$ [BCD], AI $\perp$ BC, and since DH $\perp$ [ABC], DH $\perp$ BC. Now because BC is perpendicular to both AI and DH with both

AI and DH lie on the plane [ADE], BC⊥[ADE]; therefore, when we rotate the plane [BCD] around the BC axis counter-clockwise an angle ∠DEP which equals angle ∠DAI, AD⊥[BCP]. Thus, AD⊥[BCP] or AP⊥BP, and point P is on the circle that lies on [ABD] with radius AB.



With the similar reasoning, if P’ and P” are the counterparts of point P on BD and CD, respectively that are obtained by going through the same process, we will also get $OL^2 = R^2 - DP'\times BD - DL\times HL = R^2 - DP''\times CD - DL\times HL$ (ii)

In addition, we also have BD⊥AP’ and BP”⊥CD, or point P’ is

also on the same circle mentioned above, and both points P' and P" are on the circle that lies on [BCD] with radius BC. With the same argument, both points P and P" are on the circle that lies on [ACD] with radius AC.
The two equations (i) and (ii) together give us R^2 – DP×AD – DL×HL = R^2 – DP'×BD – DL×HL = R^2 – DP"×CD – DL×HL, or DP×AD = DP'×BD = DP"×CD.

With this result, we conclude that the three circles with radius AB, BC and CD are equal, or AB = BC = CD, and ABC must be an equilateral triangle. Therefore, if the two great circles are kept stationary as they are, point D must always be at either M or N, the westernmost and easternmost points defined and depicted earlier.

The side length of the equilateral triangle ABC circumscribed in a circle with radius R is $a = R\sqrt{3}$. In the previous graph, $DE^2 = R^2 + OE^2 = R^2 + (\frac{R}{2})^2 = \frac{5R^2}{4}$, or $DE = \frac{R\sqrt{5}}{2}$, $EI\times DE = EO\times EA = \frac{R^2}{2}$, or $EI = \frac{R}{\sqrt{5}}$, $DI = DE - EI = \frac{R\sqrt{5}}{2} - \frac{R}{\sqrt{5}} = \frac{3R}{2\sqrt{5}}$, $DL\times OD = DI\times DE = \frac{3R}{2\sqrt{5}} \times \frac{R\sqrt{5}}{2}$, or $DL = \frac{3R}{4}$, or $OL = OD - DL = R - \frac{3R}{4} = \frac{R}{4}$.

The locus are two points L and L' that lie on diameter MN, on either sides of center O, are equidistant and are $\frac{R}{4}$ from point O, the center of the sphere and ABC is an equilateral triangle.

If we rotate the great circles to cover the entire sphere, the locus will be a concentric sphere with the same center O and radius $\frac{R}{4}$, a quarter of the radius of the original sphere given in the problem.

Problem 2 of Tournament of Towns 1984

Prove that among 18 consecutive three digit numbers there must be at least one which is divisible by the sum of its digits.

Solution

Note that among 18 consecutive three digit numbers there must be one number which is a multiple of 18 which are 18, 36, 54,..., 990. These numbers can be denoted *abc* where $100a + 10b + c = 18n$ and n is an integer. The above case does not apply for numbers from 0 to 17 which we can pick number 2 that is divisible by the sum of its digits which is also 2.

If we start the 18 consecutive three digit numbers from 19, the 18th number is 36 which is twice the amount of 18, etc… , and the sum of the digits of one of these numbers that are multiples of 18 is always either 9 or 18; i.e., $a + b + c = 9$, or $a + b + c = 18$. This fact can be manually verified because there are only $[\frac{1000}{18}] = 55$ such numbers. ([m] *denotes the largest integer not greater than* m.)

Therefore, $100a + 10b + c = 18n$ is divisible by $a + b + c$ which is either 9 or 18, and among 18 consecutive three digit numbers there must be at least one which is divisible by the sum of its digits.

Further observation

This problem is really a trick with the usage of the language. Instead of asking that any three digit number that is a multiple of 18 is divisible by the sum of its digits, the author resorted to a more tricky way with the usage of the language to make it really more interesting and to divert the attention of the contestant in a way to make the problem harder than it should be.

Problem 4 of Canada Students Math Olympiad 2011

Circles Γ_1 and Γ_2 have centers O_1 and O_2 and intersect at P and Q. A line through P intersects Γ_1 and Γ_2 at A and B, respectively, such that AB is not perpendicular to PQ. Let X be the point on PQ such that XA = XB and let Y be the point within AO_1O_2B such that AYO_1 and BYO_2 are similar. Prove that $2\angle O_1AY = \angle AXB$.

Solution

Extend XA and XB to meet Γ_1 and Γ_2 at D and C, respectively, and let $\angle AXB = 2\alpha$, E and F the midpoints of arcs AD and BC, respectively.

Per the intersecting secant theorem, we obtain $XA\times XD = XP\times XQ = XB\times XC$. Therefore, ABCD is cyclic and since XA = XB, AD = BC and AB || CD. Now respectively let CD cut Γ_1 and Γ_2 at I and J, O be the circumcenter of ABCD. Next, let AI meet BJ at Y, M and N be the midpoints of arcs PI and PJ, respectively. It's easily seen that AD = PI = BC = PJ because AB || CD.

Extend IP to meet XA at K and JP to meet AB at L. Since KI is a line symmetric of KD across KO_1 and $KO_1 \parallel XO$, $KI \parallel XB$, or $\angle AKP = \angle AXB = 2\alpha$. Since E and M are the midpoints of arcs AD and PI, respectively as we defined, $O_1E \perp KA$ and $O_1M \perp KP$, or $\angle O_1EM = \angle O_1ME = \angle O_1KE = \frac{1}{2}\angle AKP = \alpha$.

Now rotate $\angle O_1EM$ clockwise around center O_1 an amount that equals arc EA or half that of arc AD. Point E will move to A and M will move to I because arc EA $= \frac{1}{2}$ arc AD $= \frac{1}{2}$ arc PI = arc MI. In other words, $\angle O_1AI = \angle O_1AY = \angle O_1EM = \alpha$.

Similarly, on circle Γ_2, we have $\angle O_2BJ = \angle O_2BY = \angle O_2FN = \angle O_2LB = \angle OXB = \alpha$, or $\angle O_1AY = \angle O_2BY = \alpha$, and $\angle AO_1I =$

$\angle BO_2J = 180° - 2\alpha$, or $\dfrac{AI}{BJ} = \dfrac{R}{r}$ where R and r are the radii of Γ_1 and Γ_2, respectively.

Also since AB || CD, $\dfrac{AY}{BY} = \dfrac{YI}{YJ} = \dfrac{AY + YI}{BY + YJ} = \dfrac{AI}{BJ} = \dfrac{R}{r} = \dfrac{AO_1}{BO_2}$.

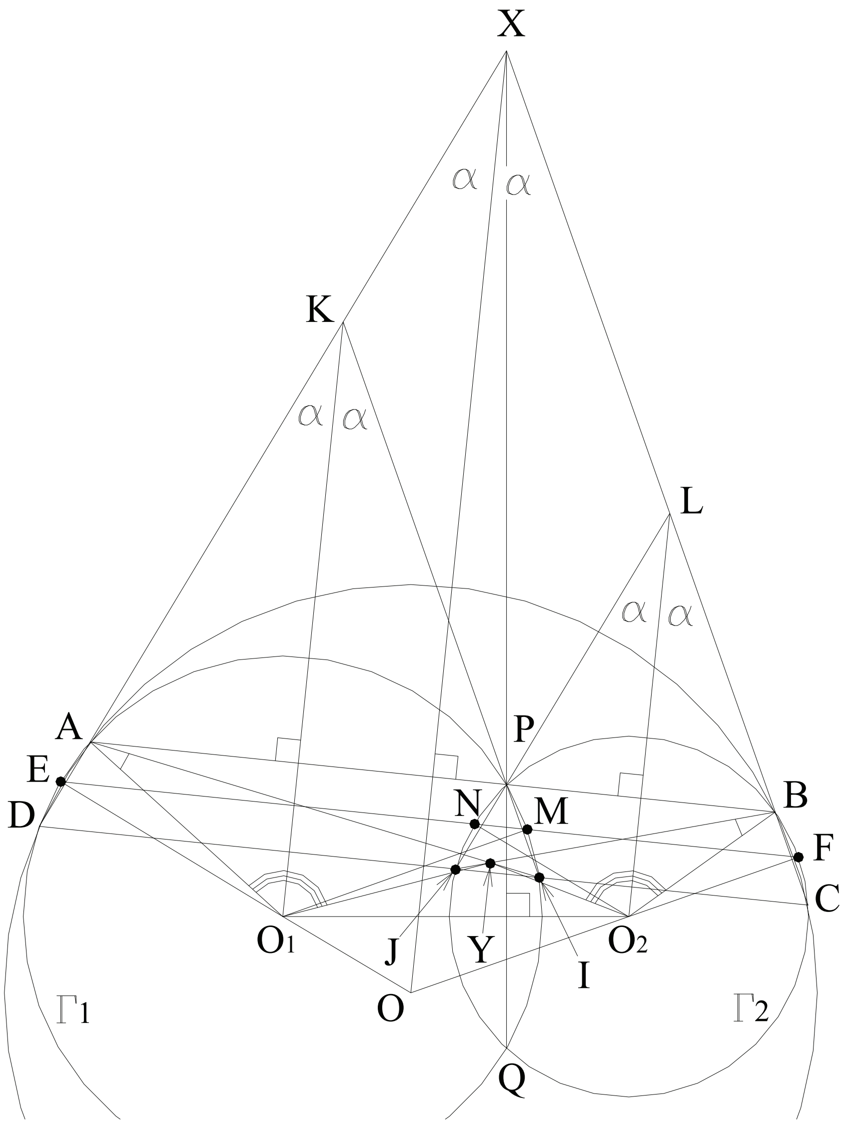


Figure 1

Combining with the earlier result $\angle O_1AY = \angle O_2BY = \alpha$, the two triangles AYO_1 and BYO_2 are similar which gives us the configuration described by the problem.

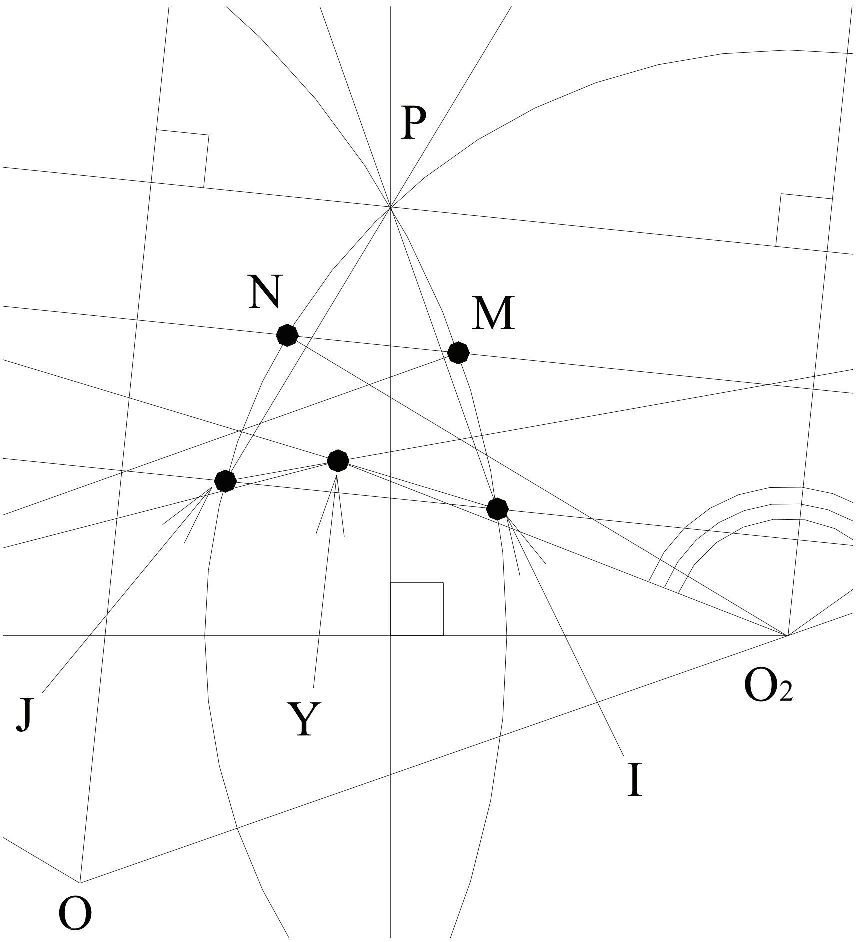


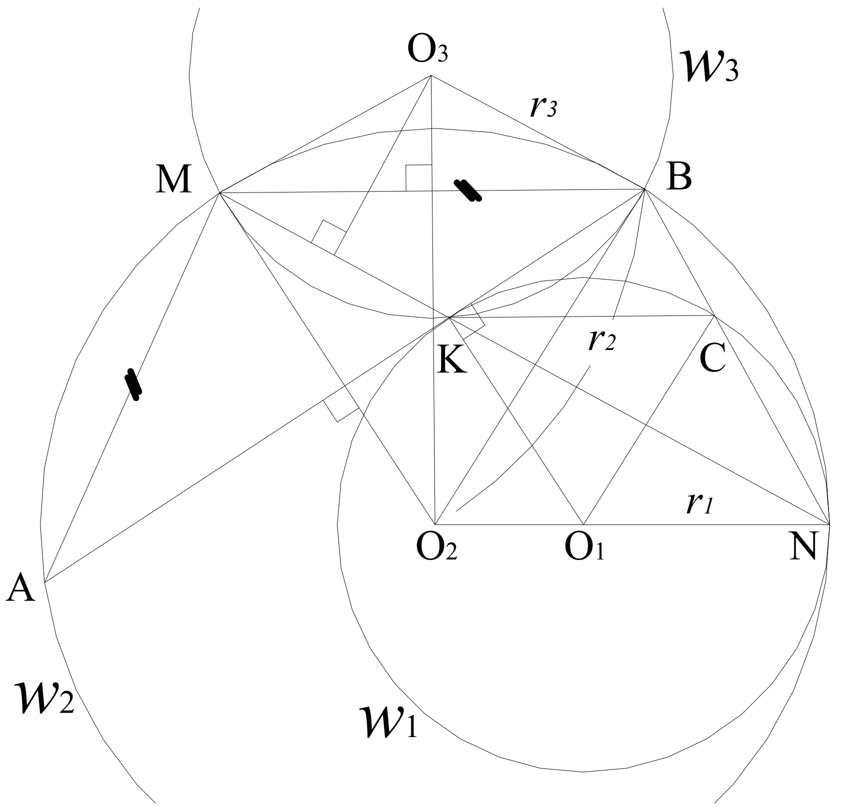
Figure 2 (enlargement of vital portion of figure 1)

And as proven earlier, $\angle O_1AY = \angle O_2BY = \alpha = \frac{1}{2}\angle AXB$, or $2\angle O_1AY = \angle AXB$, and we're done.

Problem 2 of the Russian Mathematical Olympiad 2001

Let the circle w_1 be internally tangent to another circle w_2 at N. Take a point K on w_1 and draw a tangent AB which intersects w_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that the circumradius of the triangle KBM doesn't depend on the choice of K.

Solution



Let w_3 be the circumcircle of triangle KBM, O_1, O_2, O_3 and r_1, r_2 and r_3 be the circumcenters and radii of w_1, w_2 and w_3, respectively. Link BN to meet w_1 at C. Since $\angle$BMN subtends arc BN on w_2 and arc BK on w_3, $\frac{\text{BN}}{\text{BK}} = \frac{r_2}{r_3}$. Also note that because ½$\angle BO_3M$ +

$\angle BKM = 180°$, or $\angle BO_3O_2 + \angle BKM = 180°$, $\angle BO_3O_2 = \angle BKN$.

Moreover, $\angle BNK = \angle BNM = \angle BO_2O_3$ (because $\angle BNM$ subtends arc BM while $\angle BO_2M = 2\angle BO_2O_3$ with $\angle BO_2O_3$ subtending half the same arc BM on w_2), the two triangles BO_2O_3 and BNK are similar which implies that $\frac{O_3B}{O_2B} = \frac{r_3}{r_2} = \frac{BK}{BN}$, or $r_3 = r_2 \times \frac{BK}{BN}$.

Now according to the intersecting secant theorem with BK being the tangent to w_1, $BK^2 = BC \times BN$, or $r_3 = r_2 \times \sqrt{\frac{BC}{BN}}$.
Furthermore, since $O_1K \perp AB$ and $O_2M \perp AB$, $O_1K \parallel O_2M$ and $\angle KO_1N = \angle MO_2N$, but $\angle KCN = 180° - ½\angle KO_1N = 180° - ½\angle MO_2N = \angle MBN$ which implies that $KC \parallel MB$.

That result gives us $\frac{CN}{BN} = \frac{KN}{MN} = \frac{r_1}{r_2} = \frac{O_1N}{O_2N}$, or $O_1C \parallel O_2B$ and we get $\frac{BC}{BN} = \frac{O_2O_1}{O_2N} = 1 - \frac{r_1}{r_2}$.

Therefore, $r_3 = r_2\sqrt{1 - \frac{r_1}{r_2}} = \sqrt{r_2(r_2 - r_1)}$ and is independent of the locations of point K.

<u>Problem 2 of Austria Mathematical Olympiad 2001</u>

Determine all real solutions of the equation

$$(x+1)^{2001} + (x+1)^{2000}(x-2) + (x+1)^{1999}(x-2)^2$$
$$+ \ldots + (x+1)^2(x-2)^{1999} + (x+1)(x-2)^{2000} + (x-2)^{2001} = 0.$$

<u>Solution</u>

Let $S = (x+1)^{2001} + (x+1)^{2000}(x-2) + (x+1)^{1999}(x-2)^2$
$+ \ldots + (x+1)^2(x-2)^{1999} + (x+1)(x-2)^{2000} + (x-2)^{2001}$.

And also let

$(x+1)^{2001} = n_1$,

$(x+1)^{2000}(x-2) = n_2$,

$(x+1)^{1999}(x-2)^2 = n_3$,

…

$(x+1)^2(x-2)^{1999} = n_{2000}$,

$(x+1)(x-2)^{2000} = n_{2001}$,

$(x-2)^{2001} = n_{2002}$.

Let's compare term by term n_1 with n_{2002}, n_2 with n_{2001}, n_3 with n_{2000}, n_4 with n_{1999}, etc…

We have

$n_1/n_{2002} = (\frac{x+1}{x-2})^{2001} = (1 + \frac{3}{x-2})^{2001}$,

$n_2/n_{2001} = (1 + \frac{3}{x-2})^{1999}$,

$n_3/n_{2000} = (1 + \frac{3}{x-2})^{1997}$,

$n_4/n_{1999} = (1 + \frac{3}{x-2})^{1995}$,

….

If $\boldsymbol{x = 0.5}$, $1 + \frac{3}{x-2} = -1$, and

$n_1/n_{2002} = (1 + \frac{3}{x-2})^{2001} = -1$, and $n_1 = -n_{2002}$,

$n_2/n_{2001} = (1 + \frac{3}{x-2})^{1999} = -1$, and $n_2 = -n_{2001}$,

$n_3/n_{2000} = (1 + \frac{3}{x-2})^{1997} = -1$, and $n_3 = -n_{2000}$,

$n_4/n_{1999} = (1 + \frac{3}{x-2})^{1995} = -1$, and $n_4 = -n_{1999}$,

.....

or $n_1 + n_2 + \ldots + n_{1001} = -(n_{1002} + n_{1003} + \ldots + n_{2002})$,
and thus $x = 0.5$ is a solution.

Similarly,
Now if $\boldsymbol{x < 0.5}$, $1 + \frac{3}{x-2} > -1$, and $(1 + \frac{3}{x-2})^{2001} > -1$, or

$n_1 > -n_{2002}$,

$n_2 > -n_{2001}$,

$n_3 > -n_{2000}$,

$n_4 > -n_{1999}$,

…

And $S > 0$.

If $\boldsymbol{0.5 < x < 2}$, $1 + \frac{3}{x-2} < -1$, and $(1 + \frac{3}{x-2})^{2001} < -1$, or

$n_1 < -n_{2002}$,

$n_2 < -n_{2001}$,

$n_3 < -n_{2000}$,

$n_4 < -n_{1999}$,

…

And $S < 0$.

If $\boldsymbol{x = 2}$, $S = 3^{2001} > 0$.

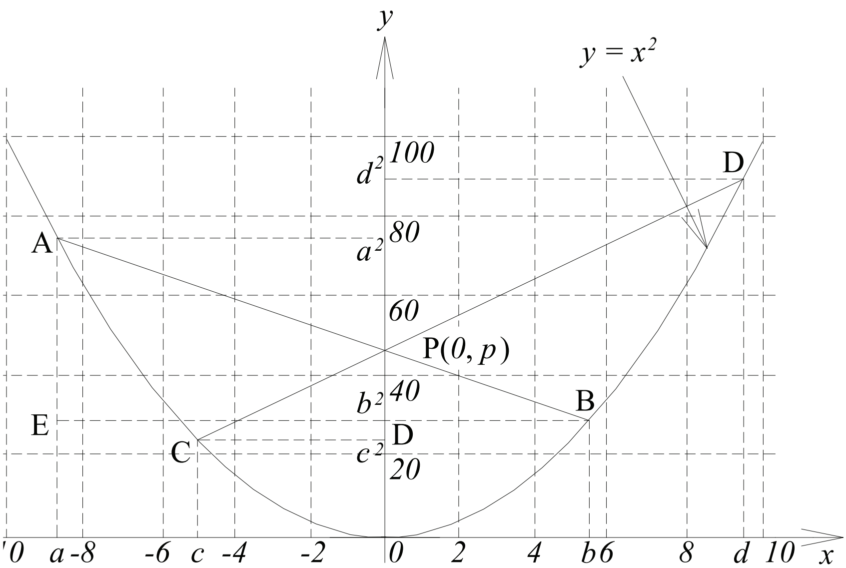
If $\boldsymbol{x > 2}$, all terms of S are positive and $S > 0$.

Hence, $x = 0.5$ is the only solution.

<u>*Problem 1 of Tournament of Towns 2007 Senior Level*</u>

A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c, respectively.

<u>Solution</u>



Assuming that the altitude of point A is higher than that of point B and AB intersects CD at point P(0, p) on the y-axis as shown. Now from B draw the horizontal line to meet the y-axis and the vertical line that passes through A at D and E, respectively.

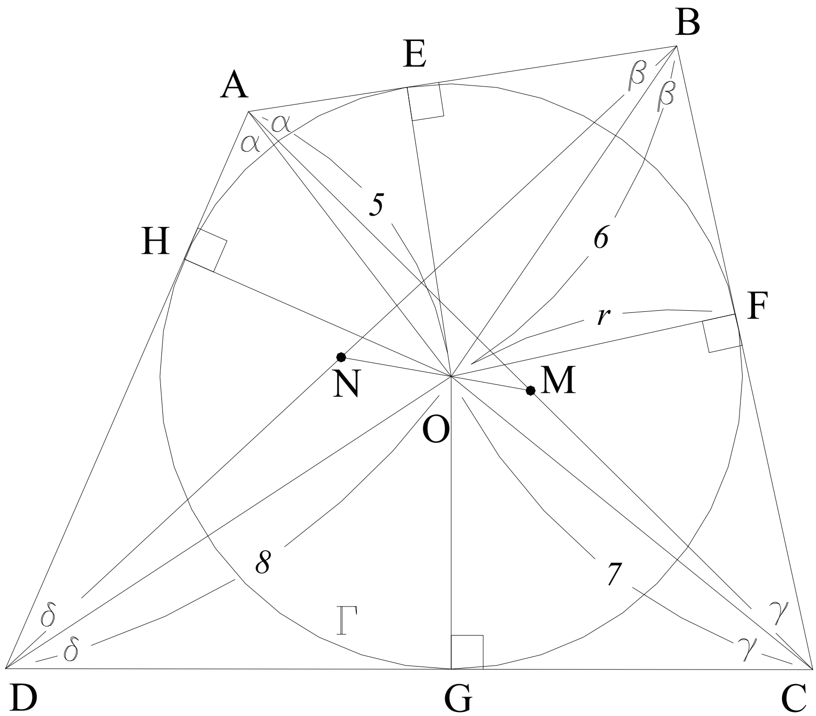
We have $\dfrac{\text{PD}}{\text{AE}} = \dfrac{\text{BD}}{\text{BE}}$, or $\dfrac{p - b^2}{a^2 - b^2} = \dfrac{b}{b - a}$, and $p = -ab$.

The equation of line CD is $y_{\text{CD}} = \dfrac{d^2 - c^2}{d - c}x + p = (c + d)x + p$ where $c + d$ is its slope. However, the slope of this line also equals $\dfrac{\text{PD}}{\text{CD}} = \dfrac{p - c^2}{-c} = -\dfrac{p}{c} + c$. Therefore, $c + d = -\dfrac{p}{c} + c$, or $d = -\dfrac{p}{c} = \dfrac{ab}{c}$.

<u>Problem 1 Set 6 of India Postal Coaching 2011</u>

Let ABCD be a quadrilateral with an inscribed circle, center O. Let AO = 5, BO = 6, CO = 7, DO = 8. If M and N are the midpoints of the diagonals AC and BD, determine $\frac{OM}{ON}$.

<u>Solution</u>



<u>Method 1 by finding the radius of the incircle</u>

Let Γ be the inscribed circle, r its radius, Γ touches AB, BC, CD and AD at E, F, G and H, respectively, $\alpha = \angle OAE = \angle OAH$, $\beta = \angle OBE = \angle OBF$, $\gamma = \angle OCF = \angle OCG$, and $\delta = \angle ODG = \angle ODH$. It's easily seen that $2(\alpha + \beta + \gamma + \delta) = 360°$, the total sum of four angles of ABCD, or $\alpha + \beta + \gamma + \delta = 180°$. We then have $\sin\alpha = \frac{r}{5}$, $\sin\beta = \frac{r}{6}$, $\sin\gamma = \frac{r}{7}$, $\sin\delta = \frac{r}{8}$, and $\cos\alpha = \frac{\sqrt{25 - r^2}}{5}$,

$\cos\beta = \frac{\sqrt{36 - r^2}}{6}$, $\cos\gamma = \frac{\sqrt{49 - r^2}}{7}$, $\cos\delta = \frac{\sqrt{64 - r^2}}{8}$.

Applying Stewart's theorem, we get $OA^2 \times MC + OC^2 \times MA = AC(OM^2 + MA \times MC)$, but $MA = MC = \frac{1}{2}AC$, and we now have

$OA^2 + OC^2 = 2(OM^2 + \frac{1}{4}AC^2)$, or

$$OM^2 = \frac{1}{2}(OA^2 + OC^2) - \frac{1}{4}AC^2 \quad \text{(i)}$$

Substitute the values to get $OM^2 = 37 - \frac{1}{4}AC^2$.

Similarly, $OB^2 + OD^2 = 2(ON^2 + \frac{1}{4}BD^2)$, or $ON^2 = 50 - \frac{1}{4}BD^2$.

The ratio becomes $\frac{OM}{ON} = \sqrt{\frac{148 - AC^2}{200 - BD^2}}$.

Furthermore, the law of cosines gives us $AC^2 = OA^2 + OC^2 - 2OA \times OC \times \cos\angle AOC = 25 + 49 - 70 \times \cos(\angle AOB + \angle BOC) = 74 - 70(\cos\angle AOB \times \cos\angle BOC - \sin\angle AOB \times \sin\angle BOC) = 74 - 70\{\cos[180° - (\alpha + \beta)] \times \cos[180° - (\beta + \gamma)] - \sin[(180° - (\alpha + \beta)] \times \sin[180° - (\beta + \gamma)]\} = 74 - 70 \times [\cos(\alpha + \beta) \times \cos(\beta + \gamma) - \sin(\alpha + \beta) \times \sin(\beta + \gamma)] = 74 - 70\cos(\alpha + \beta + \beta + \gamma) = 74 - 70\cos(180° - \delta + \beta) = 74 + 70\cos(\beta - \delta) = 74 + 70(\cos\beta\cos\delta + \sin\beta\sin\delta) = 74 + 70(\frac{\sqrt{36 - r^2}}{6} \times \frac{\sqrt{64 - r^2}}{8} + \frac{r}{6} \times \frac{r}{8}) =$

$\frac{1}{48}[74 \times 48 + 70(\sqrt{36 - r^2} \times \sqrt{64 - r^2} + r^2)]$, and

$$\mathbf{148 - AC^2 = \frac{1}{48} \times [74 \times 48 - 70(\sqrt{36 - r^2} \times \sqrt{64 - r^2} + r^2)].}$$

Meanwhile, $BD^2 = OB^2 + OD^2 - 2OB \times OD \times \cos\angle BOD = 36 + 64 - 96 \times \cos(\angle AOB + \angle AOD) = 100 - 96(\cos\angle AOB \times \cos\angle AOD - \sin\angle AOB \times \sin\angle AOD) = 100 - 96\{\cos[180° - (\alpha + \beta)] \times \cos[180° - (\alpha + \delta)] - \sin[(180° - (\alpha + \beta)] \times \sin[180° - (\alpha + \delta)]\} = 100 - 96 \times [\cos(\alpha + \beta) \times \cos(\alpha + \delta) - \sin(\alpha + \beta) \times \sin(\alpha + \delta)] = 100 - 96\cos(\alpha + \beta + \alpha + \delta) = 100 - 96\cos(180° - \gamma + \alpha) = 100 + 96\cos(\alpha - \gamma) = 100$

$+ 96(\cos\alpha\cos\gamma + \sin\alpha\sin\gamma) = 100 + 96(\frac{\sqrt{25-r^2}}{5}\times\frac{\sqrt{49-r^2}}{7} + \frac{r}{5}\times\frac{r}{7}) =$

$\frac{1}{35}[100 + 96(\sqrt{25-r^2}\times\sqrt{49-r^2} + r^2)]$, and

$\mathbf{200 - BD^2 = \frac{1}{35}\times[100\times35 - 96(\sqrt{25-r^2}\times\sqrt{49-r^2} + r^2)]}$. Hence,

$$\frac{\mathrm{OM}}{\mathrm{ON}} = \sqrt{\frac{35[74\times48 - 70(\sqrt{36-r^2}\times\sqrt{64-r^2} + r^2)]}{48[100\times35 - 96(\sqrt{25-r^2}\times\sqrt{49-r^2} + r^2)]}} \qquad \text{(ii)}$$

At this point we note that there is an existing theorem that gives us the formula for the inradius that can be found at the this web link http://forumgeom.fau.edu/FG2010volume10/FG201005.pdf where it says that

If u, v, x and y are the distances from the incenter to the vertices of a tangential quadrilateral, then the inradius is given by the formula $r = 2\sqrt{\frac{(M-uvx)(M-vxy)(M-xyu)(M-yuv)}{uvxy(uv+xy)(ux+vy)(uy+vx)}}$ *where* $M = \frac{uvx + vxy + xyu + yuv}{2}$.

Applying this formula to the problem, we get

$M = \frac{5\times6\times7 + 6\times7\times8 + 7\times8\times5 + 8\times5\times6}{2} = 533$, and

$r = 2\sqrt{\frac{(533-210)(533-336)(533-280)(533-240)}{1680(30+56)(35+48)(40+42)}} =$

$2\sqrt{\frac{11\times17\times19\times23\times197\times293}{1680\times86\times83\times82}} = 4.38034787$, or

or $r^2 = \frac{4\times11\times17\times19\times23\times197\times293}{1680\times86\times83\times82}$.

From this value of r^2 we find that

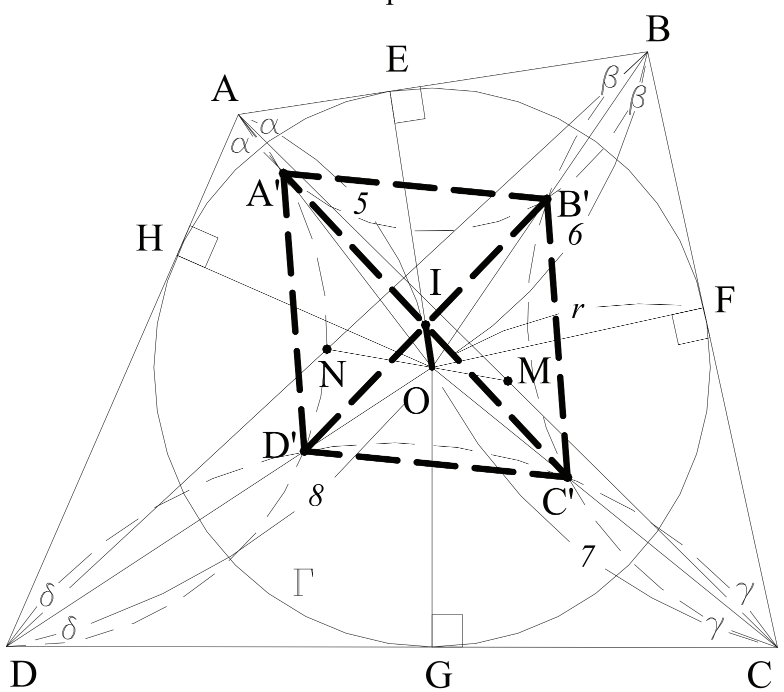
$$\frac{74\times48 + 70(\sqrt{36-r^2}\times\sqrt{64-r^2} + r^2)}{100 + 96(\sqrt{25-r^2}\times\sqrt{49-r^2} + r^2)} = \frac{35}{48}.$$

Therefore, from (ii) we finally have $\frac{\mathrm{OM}}{\mathrm{ON}} = \frac{35}{48}$.

<u>Method 2 by applying inversion and without having to find the radius</u>

Let's inverse the points A, B, C and D with respect to center O to become A', B', C' and D', respectively. By the inversion formula, OA×OA' = OB×OB'= OC×OC' = OD×OD' = r^2. This implies that all AA'B'B, BB'C'C, CC'D'D and DD'A'A are cyclic quadrilaterals which cause ∠OA'B' = ∠OBA = β, ∠OA'D' = ∠ODA = δ, or ∠B'A'D' = ∠OA'B' + ∠OA'D' = β + δ.

Similarly, ∠B'C'D' = ∠OBC + ∠ODC = β + δ, or ∠B'A'D' = ∠B'C'D'. By the same token, ∠A'B'C' = ∠A'D'C' = α + γ, and A'B'C'D' is a parallelogram. Let its diagonals A'C' and B'D' bisect at I which is also their midpoints.



Now note that because OA×OA' = OC×OC' = r^2, AA'C'C is cyclic which implies that the two triangles OA'C' and OCA are similar, and we have $\frac{OA'}{OC'} = \frac{OC}{OA}$, $\frac{A'C'}{AC}$, $= \frac{OC'}{OA}$, or A'C' = AC×$\frac{OC'}{OA}$.

Again, applying the Stewart's theorem to triangle OA'C' with I being the midpoint of A'C", we get $OA'^2 \times IC' + OC'^2 \times IA' = A'C'(OI^2 + IA' \times IC')$, but $IA' = IC' = \frac{1}{2}A'C'$, and we now have $OA'^2 + OC'^2 = 2(OI^2 + \frac{1}{4}A'C'^2)$, or $OI^2 = \frac{1}{2}(OA'^2 + OC'^2) - \frac{1}{4}A'C'^2$, but $OA'^2 \times OA^2 = OC'^2 \times OC^2 = r^4$, and now $OI^2 = \frac{1}{2}(\frac{r^4}{OA^2} + \frac{r^4}{OC^2}) - \frac{1}{4}AC^2 \times (\frac{OC'}{OA})^2 = \frac{r^4}{OA^2 \times OC^2}[\frac{1}{2}(OC^2 + OA^2) - \frac{1}{4}AC^2] = \frac{r^4}{OA^2 \times OC^2} \times OM^2$ (see equation (i)), or $OI = \frac{r^2}{OA \times OC} \times OM$.

Similarly, applying the Stewart's theorem to the other triangle OB'D' and follow the same procedure, we would end up with $OI = \frac{r^2}{OB \times OD} \times ON$.

Equating the two expressions for OI, we get $\frac{r^2}{OA \times OC} \times OM = \frac{r^2}{OB \times OD} \times ON$, or $\frac{OM}{ON} = \frac{OA \times OC}{OB \times OD} = \frac{35}{48}$ which is the same result harvested in the first method.

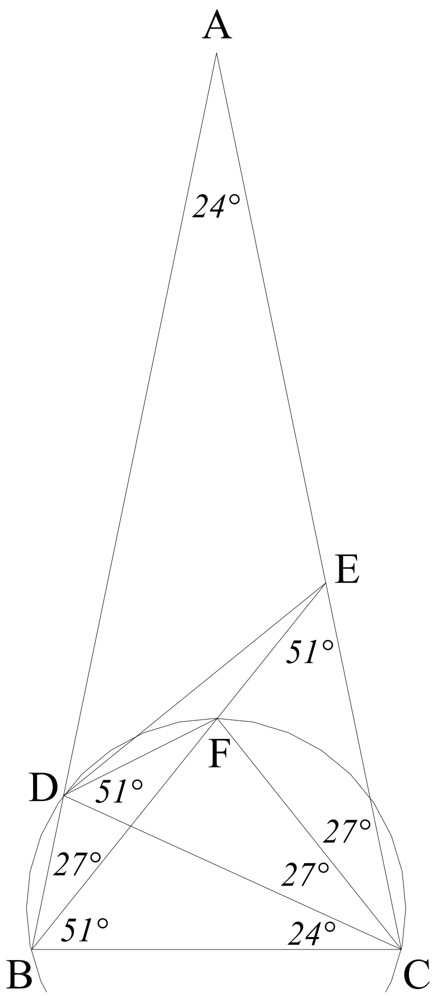
<u>Further observation</u>

The first method involves rigorous calculation. It is included, however, to show the reader that there is actually a different approach to solve the problem besides the inversion method which appears to be less complicated.

<u>Problem 5 of International Mathematical Talent Search Round 21</u>

Assume that triangle ABC, shown below, is isosceles, with ∠ABC = ∠ACB = 78°. Let D and E be points on sides AB and AC, respectively, so that ∠BCD = 24° and ∠CBE = 51°. Determine, with proof, ∠BED.

<u>Solution</u>



Because ∠ABC = ∠ACB = 78°, ∠BCD = 24° and ∠CBE = 51°,

$\angle$DCE = 54°, $\angle$DBE = 24° and $\angle$BEC = 51°.

Draw the bisector for $\angle$DCE to meet BE at F; $\angle$DCF = $\angle$ECF = 27° = $\angle$DBE. Therefore, BCFD is cyclic, and as a result $\angle$CDF = $\angle$CBF = 51°. The two triangles DCF and ECF are now congruent because all their respective angles are equal and they also have common side CF.

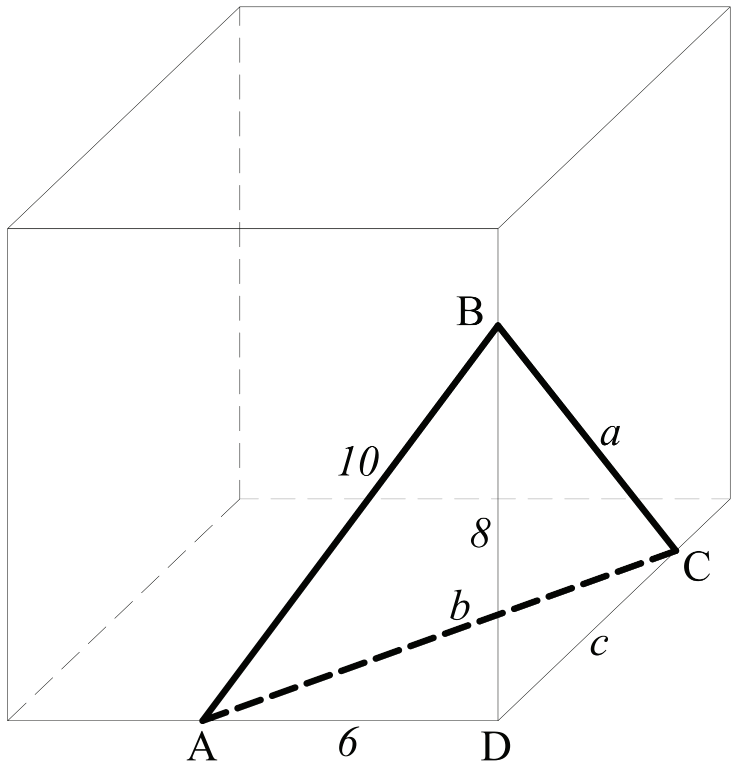
Hence, DF = EF, and DEF is an isosceles triangle with $\angle$DEF = $\angle$FDE. However, $\angle$DFE = $\angle$DCE + $\angle$CDF + $\angle$CEF = 54° + 2×51° = 156°.

Finally, $\angle BED = \angle FED = \frac{1}{2}(180° - \angle DFE) = 12°$.

Problem 4 of International Mathematical Talent Search Round 22

As shown below, a large wooden cube has one corner sawed off forming a tetrahedron ABCD. Determine the length of CD, if AD = 6, BD = 8 and area of triangle ABC = 74.

Solution



Let a = BC, b = AC and c = DC. Per Pythagorean's theorem, $AB^2 = AD^2 + BD^2 = 100$, or AB = 10. Let s be the semi-perimeter of triangle ABC, $s = \frac{1}{2}(10 + a + b)$.

Per Heron's formula, the area of ABC is $\sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{4}\sqrt{(10 + a + b)(10 + a - b)(b + a - 10)(b - a + 10)} = 74$, or

$\sqrt{(10 + a + b)(10 + a - b)(b + a - 10)(b - a + 10)} = 296$, or
$(10 + a + b)(10 + a - b)(b + a - 10)(b - a + 10) = 296^2$, or
$[(10 + a)^2 - b^2][b^2 - (a - 10)^2] = 296^2$, or
$$(100 + 20a + a^2 - b^2)(b^2 - a^2 + 20a - 100) = 296^2 \qquad \text{(i)}$$

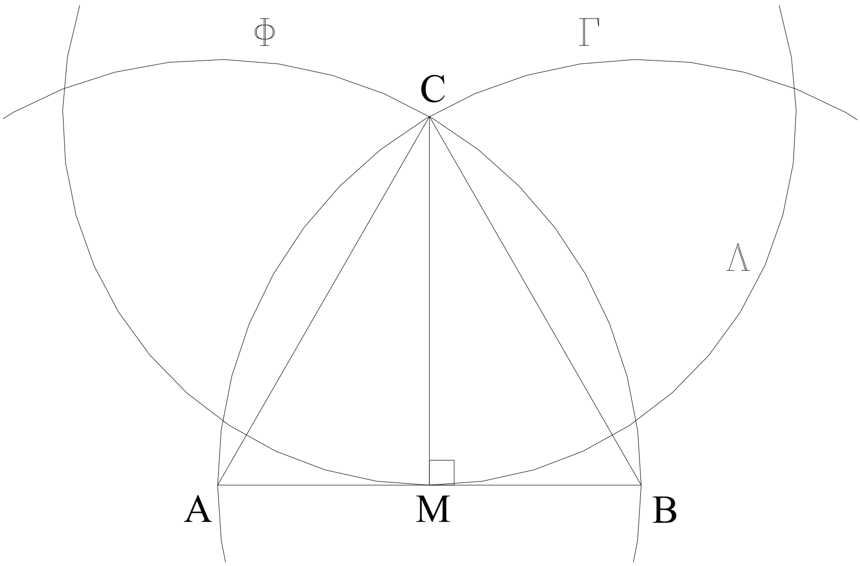
Furthermore, the Pythagorean's theorem also gives us
$a^2 = \text{BD}^2 + c^2 = 64 + c^2$ and $b^2 = \text{AD}^2 + c^2 = 36 + c^2$

Substituting a^2 and b^2 into (i) to get $(100 + 20a + 64 + c^2 - 36 - c^2)(36 + c^2 - 64 - c^2 + 20a - 100) = (20a + 128)(20a - 128) = 296^2$, or $400a^2 - 128^2 = 296^2$, or $a^2 = \dfrac{296^2 + 128^2}{400} = 260$, or $64 + c^2 = 260$, or $c = 14$.

<u>Problem 5 of International Mathematical Talent Search Round 27</u>

Is it possible to construct in the plane the midpoint of a given segment using compasses alone (i.e., without using a straight edge, except for drawing the segment)?

<u>Solution</u>



The answer is yes. Draw two identical circles Φ and Γ with centers A and B and their radius equals the length of segment AB. Let these circles meet at a point C. Triangle ABC is equilateral because AB = BC = CA. Next draw another circle Λ with center at C that touches segment AB at a point, say M. The length of this segment can be calculated as it is the altitude of equilateral triangle ABC.

Point M is the midpoint of the given segment AB.

Problem 1 of International Mathematical Talent Search Round 22

In 1996 nobody could claim that on their birthday their age was the sum of the digits of the year in which they were born. What was the last year prior to 1996 which had the same property?

Solution

Assume that in 1996 someone could claim that on their birthday their age was the sum of the digits of the year in which they were born, and the four digit of the year in which they were born is *abcd* where all a, b, c and d are integers with a from 0 to 1 and the others from 0 to 9. With this assumption, their age on their birthday in 1996 is $1996 - 1000a - 100b - 10c - d = a + b + c + d$, or $1996 - 1001a - 101b - 11c - 2d = 0$.

Judging the above equation, we know that a can not be equal to 0 because in such a case even with $b = c = d = 9$ which make $101b + 11c + 2d$ a maximum, the left side of the equation is still positive. Therefore, $a = 1$, and the previous equation becomes $995 = 101b + 11c + 2d$.

With similar reasoning, we get $b = 9$, $c = 7$ and $2d = 9$. Since $2d$ is an even number and can not equal 9 our assumption is false. In fact, no one in the world was able to make that claim in 1996.

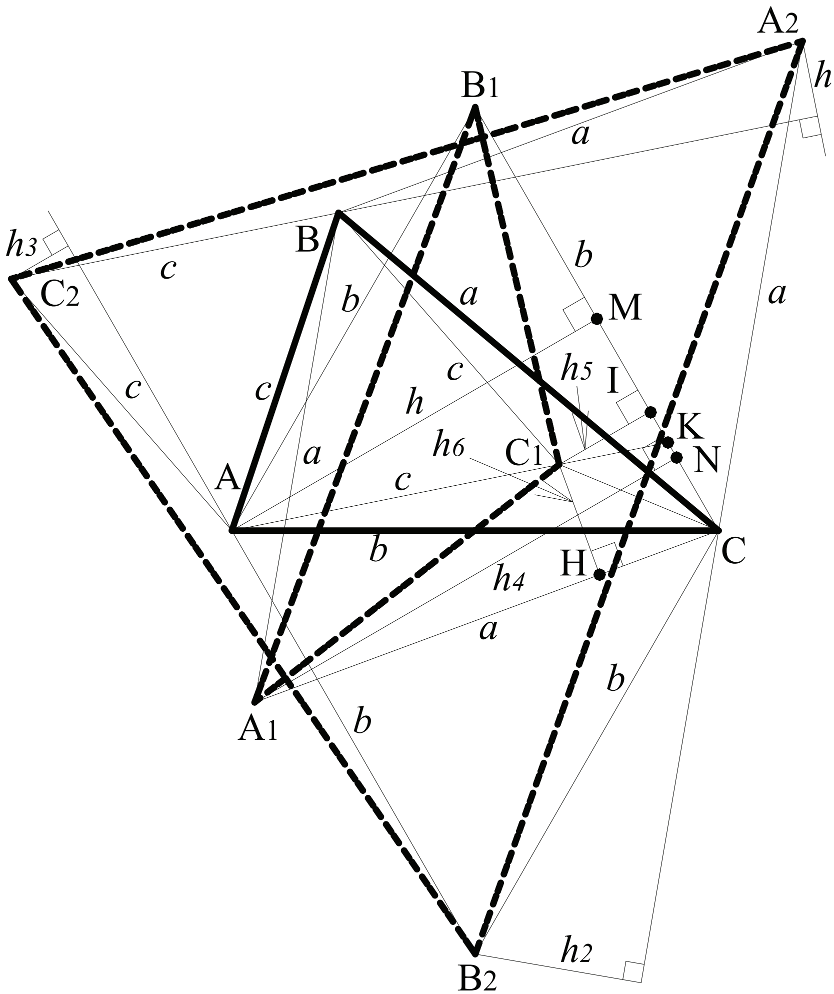
Our goal is to find the year prior to 1996 by following the similar approach to finally get the value of $2d$ to be an odd number. We found 1985 to be that year because in 1985, their age on their birthday is $1985 - 1000a - 100b - 10c - d = a + b + c + d$, or $1985 - 1001a - 101b - 11c - 2d = 0$. Again $a = 1$, and $984 = 101b + 11c + 2d$ which forces $b = 9$ and $75 = 11c + 2d$, or $c = 6$ and $2d = 9$.

The reader is encouraged to find the last year prior to 1985 which had the same property by following this approach.

Problem 3 of the Vietnamese Mathematical Olympiad 1982

Let be given a triangle ABC. Equilateral triangles BCA_1 and BCA_2 are drawn so that A and A_1 are on one side of BC, whereas A_2 is on the other side. Points B_1, B_2, C_1, C_2 are analogously defined. Prove that $S(A_2B_2C_2) = 5S(ABC) - S(A_1B_1C_1)$.

Solution



For simplification, let's denote (Ω) the area of shape Ω instead of

$S(\Omega)$ as seen in the description of the problem. Let M be the mid-point of B_1C, I the foot of C_1 to B_1C, K the intersection of AC_1 and B_1C, N the foot of A_1 to B_1C and H the foot of C_1 to A_1C. Also let $BC = a$, $AC = b$, $AB = c$, h, h_1, h_2, h_3, h_4, h_5, h_6 be the altitudes from A to B_1C, A_2 to BC_2, B_2 to A_2C, C_2 to AB_2, A_1 to B_1C, C_1 to B_1C and C_1 to A_1C, respectively. We also employ the letter A for $\angle BAC$, letter B for $\angle ABC$ and letter C for $\angle ACB$.

From the graph, we have
$\angle A_2BC_2 = 360° - B - 120° = 240° - B$, and $\sin\angle A_2BC_2 = \sin[180° - (B - 60°)] = \sin(B - 60°)$.

Similarly, $\angle B_2AC_2 = 240° - A$ and $\sin\angle B_2AC_2 = \sin[180° - (A - 60°)] = \sin(A - 60°)$,
$\angle A_2CB_2 = 120° + C$ and $\sin\angle A_2CB_2 = \sin(60° - C)$.

Hence, $h_1 = a\sin\angle A_2BC_2 = a\sin(B - 60°) = \frac{1}{2}a(\sin B - \sqrt{3}\cos B)$,

$h_2 = b\sin\angle A_2CB_2 = b\sin(60° - C) = \frac{1}{2}b(\sqrt{3}\cos C - \sin C)$,

$h_3 = c\sin\angle B_2AC_2 = c\sin(A - 60°) = \frac{1}{2}c(\sin A - \sqrt{3}\cos A)$.

The areas of the triangles A_2BC_2, AB_2C_2 and A_2B_2C are

$(A_2BC_2) = \frac{1}{2}h_1\times BC_2 = \frac{1}{2}h_1c = \frac{1}{4}ac(\sin B - \sqrt{3}\cos B)$,

$(A_2B_2C) = \frac{1}{2}h_2\times A_2C = \frac{1}{2}h_2a = \frac{1}{4}ab(\sqrt{3}\cos C - \sin C)$, and

$(AB_2C_2) = \frac{1}{2}h_3\times AB_2 = \frac{1}{2}h_3b = \frac{1}{4}bc(\sin A - \sqrt{3}\cos A)$.
And note that A_2BC, AB_2C and ABC_2 are the equilateral triangles

and their areas are $\frac{a^2\sqrt{3}}{4}$, $\frac{b^2\sqrt{3}}{4}$ and $\frac{c^2\sqrt{3}}{4}$, respectively.

Now the area of triangle $A_2B_2C_2$ is
$(A_2B_2C_2) = (ABC) + (A_2BC) + (AB_2C) + (ABC_2) + (A_2BC_2) +$

$(AB_2C_2) - (A_2B_2C) = (ABC) + \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) + \frac{1}{4}[ac(\sin B - \sqrt{3}\cos B) + bc(\sin A - \sqrt{3}\cos A) - ab(\sqrt{3}\cos C - \sin C)] = (ABC) + \frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) + \frac{1}{4}[ac(\sin B - \sqrt{3}\cos B) + bc(\sin A - \sqrt{3}\cos A) + ab(\sin C - \sqrt{3}\cos C)]$ (i)

However, $ac\sin B = bc\sin A = ab\sin C = 2(ABC)$, and according to the law of cosines, $ac\cos B = \frac{1}{2}(a^2 + c^2 - b^2)$, $bc\cos A = \frac{1}{2}(b^2 + c^2 - a^2)$, $ab\cos C = \frac{1}{2}(a^2 + b^2 - c^2)$.

Equation (i) becomes $(A_2B_2C_2) = \frac{5}{2}(ABC) + \frac{\sqrt{3}}{8}(a^2 + b^2 + c^2) = \frac{5}{2}(ABC) + \frac{1}{2}[(A_2BC) + (AB_2C) + (ABC_2)]$ (ii)

Now let's calculate the area of triangle $A_1B_1C_1$.
$\angle A_1CB_1 = 60° + \angle ACA_1 = 60° + 60° - C = 120° - C$, and
$\sin\angle A_1CB_1 = \sin(60° + C)$.
Therefore, $h_4 = a\sin\angle A_1CB_1 = a\sin(60° + C)$, and

$(A_1B_1C) = \frac{1}{2}h_4b = \frac{1}{2}ab\sin(60° + C) = \frac{1}{4}ab(\sqrt{3}\cos C + \sin C) = \frac{ab\sqrt{3}}{4} \times\cos C + \frac{1}{4}ab\sin C = \frac{ab\sqrt{3}}{4}\cos C + \frac{1}{2}(ABC) = \frac{\sqrt{3}}{8}(a^2 + b^2 - c^2) + \frac{1}{2}\times (ABC) = \frac{1}{2}[(ABC) + (A_2BC) + (AB_2C) - (ABC_2)]$.
$\angle MAK = \angle MAC - \angle CAC_1 = 30° - (A - 60°) = 90° - A$.

Since h is the altitude of an equilateral triangle with side length b,

$h = b\frac{\sqrt{3}}{2}$, and $\frac{h_5}{h} = \frac{C_1K}{AK} = \frac{C_1K}{c + C_1K}$, but $\cos\angle MAK = \cos(90° – A)$

$= \sin A = \frac{h}{c + C_1K} = \frac{b\sqrt{3}}{2(c + C_1K)}$, or $c + C_1K = \frac{b\sqrt{3}}{2\sin A}$, or $C_1K =$

$\frac{b\sqrt{3}}{2\sin A} – c$, and $h_5 = b\frac{\sqrt{3}}{2} – c\sin A$. Similarly, $h_6 = a\frac{\sqrt{3}}{2} – c\sin B$.

$(B_1CC_1) = \frac{1}{2}h_5b = \frac{1}{2}b(b\frac{\sqrt{3}}{2} – c\sin A) = \frac{b^2\sqrt{3}}{4} – \frac{bc\sin A}{2} = \frac{b^2\sqrt{3}}{4} –$
$(ABC) = (AB_2C) – (ABC)$.

$(A_1CC_1) = \frac{1}{2}h_6a = \frac{1}{2}a(a\frac{\sqrt{3}}{2} – c\sin B) = \frac{a^2\sqrt{3}}{4} – \frac{ac\sin B}{2} = \frac{a^2\sqrt{3}}{4} –$
$(ABC) = (A_2BC) – (ABC)$.

$(A_1B_1C_1) = (A_1B_1C) – (A_1CC_1) – (B_1CC_1) = \frac{\sqrt{3}}{8}(a^2 + b^2 – c^2) +$

$\frac{1}{2}(ABC) – \frac{a^2\sqrt{3}}{4} + (ABC) – \frac{b^2\sqrt{3}}{4} + (ABC) = \frac{5}{2}(ABC) – \frac{a^2\sqrt{3}}{8} –$

$\frac{b^2\sqrt{3}}{8} – \frac{c^2\sqrt{3}}{8} = \frac{5}{2}(ABC) – \frac{1}{2}[(A_2BC) + (AB_2C) + (ABC_2)]$ (iii)

Adding equations (ii) and (iii) yields
$(A_2B_2C_2) + (A_1B_1C_1) = 5(ABC)$, or
$(A_2B_2C_2) = 5(ABC) – (A_1B_1C_1)$, or as expressed with the notation used in the problem $S(A_2B_2C_2) = 5S(ABC) – S(A_1B_1C_1)$.

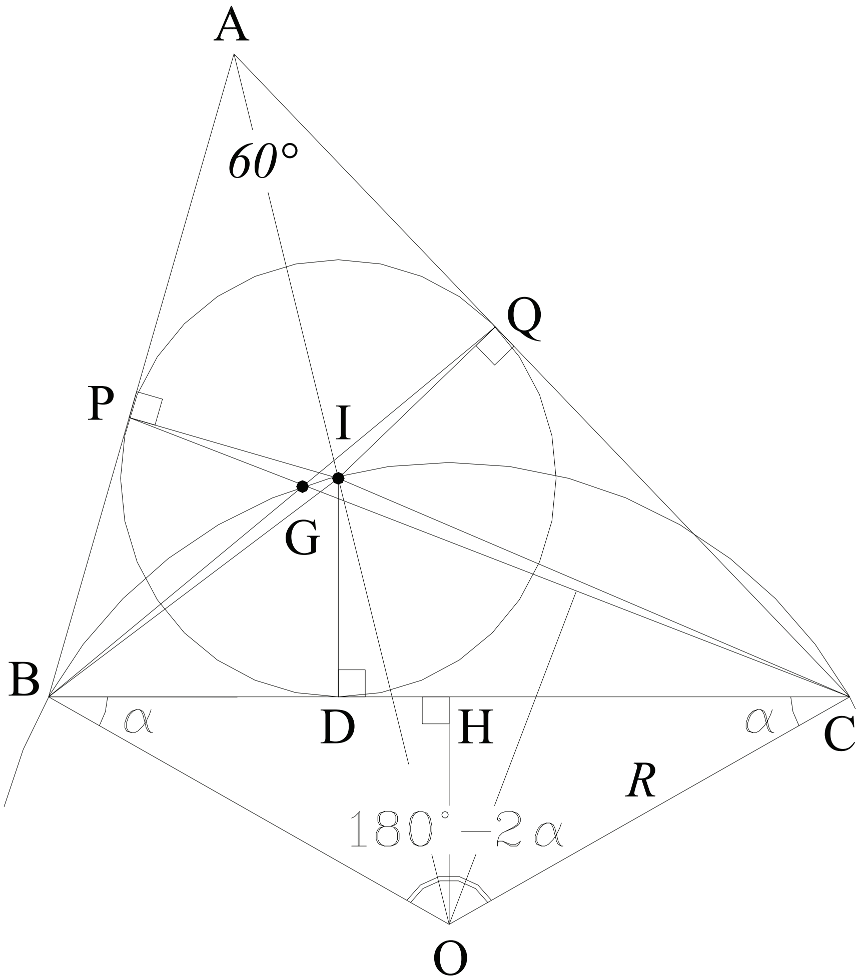
Further observation

The problem description the reader may have found in the web and elsewhere when it says that $S(ABC) + S(A_1B_1C_1) = S(A_2B_2C_2)$ *is not correct as the proof of this problem has attested. The author has not only proven that fact but also come up with the correct equation* $S(A_2B_2C_2) = 5S(ABC) – S(A_1B_1C_1)$.

<u>Problem 1 of Canada Students Math Olympiad 2011</u>

In triangle ABC, $\angle$BAC = 60° and the incircle of ABC touches AB and AC at P and Q, respectively. Lines PC and QB intersect at G. Let *R* be the circumradius of BGC. Find the minimum value of $\frac{R}{\text{BC}}$.

<u>Solution</u>



Let I, O and H be the incenter of ΔABC, circumcenter of ΔBGC and the foot of O onto BC, respectively, $\alpha = \angle OBC = \angle OCB$ and $\angle BOC = 180° - 2\alpha$. Applying the law of sines to triangle BOC, we get $\frac{BC}{\sin\angle BOC} = \frac{R}{\sin\angle OBC}$ or $\frac{BC}{\sin(180° - 2\alpha)} = \frac{R}{\sin\alpha}$.

But $\sin(180° - 2\alpha) = \sin 2\alpha$ and the previous equation becomes $\frac{BC}{\sin 2\alpha} = \frac{R}{\sin\alpha}$, and the ratio $\frac{R}{BC} = \frac{\sin\alpha}{\sin 2\alpha} = \frac{\sin\alpha}{2\sin\alpha\cos\alpha} = \frac{1}{2\cos\alpha}$ which is minimal when $\cos\alpha$ is maximal or $\cos\alpha = 1$, or $\alpha = 0°$, or $O \equiv H$, the midpoint of BC. In such a case, $R = \frac{1}{2}BC$ and $\angle BGC = 90°$ when $A \equiv B$ or $A \equiv C$. But this would cause the triangle ABC to be in a degenerate state. So the lower limit of $\frac{R}{BC}$ when $B \rightarrow A$ or $C \rightarrow A$ is $\frac{1}{2}$.

Problem 1 of British Mathematical Olympiad 2011

One number is removed from the set of integers from 1 to n. The average of the remaining numbers is $40\frac{3}{4}$. Which integer was removed?

Solution

From the equation $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$, we note that n can not be any number because if n is too large, the average of the remaining numbers will also be too large. Therefore, there's a limit on how large n is, and let's find that limit.

The minimum average after removing a number in the series $1 + 2 + \ldots + n$, with the exception of the first number 1 and the last number n, is $\dfrac{\frac{n(n+1)}{2} - (n-1)}{n-1} \le 40\frac{3}{4}$, or $\frac{n(n+1)}{2(n-1)} \le 41\frac{3}{4}$, or $2n^2 - 165n + 167 \le 0$, or $n \le \frac{1}{4}(165 + \sqrt{165^2 - 1336}) = 81.48$, or $n \le 81$.

Now let the integer that is removed be m. The average of the remaining number is $A = [\frac{n(n+1)}{2} - m]/(n-1) = 40\frac{3}{4} = \frac{163}{4}$, or $n(n+1) - 2m = \frac{163}{2}(n-1)$, or $2n^2 - 161n - 4m + 163 = 0$.

Solving this quadratic equation for n, we get

$$n = \frac{1}{4}(161 \pm \sqrt{32m + 24617}).$$

Therefore, $\frac{1}{4}(161 \pm \sqrt{32m + 24617}) \le 81$, or

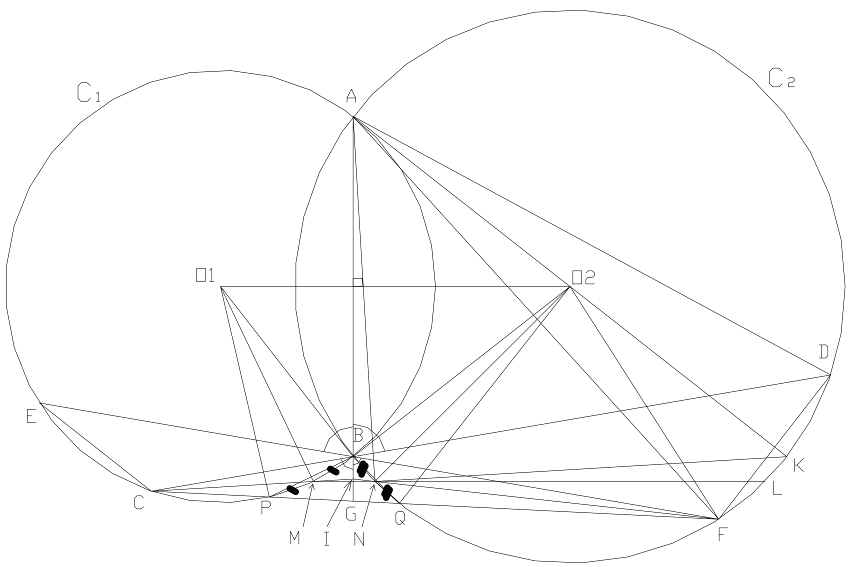
$$\sqrt{32m + 24617} \le 163, \text{ or } 32m + 24617 \le 26569, \text{ or } m \le 61.$$

$32m + 24617$ is a perfect square when $m = 1, 61$, and we conclude that the integer that was removed was 61 and $n = 81$.

Problem 4 of Morocco Mathematical Olympiad 2011 (Day 3)

Two circles C_1 and C_2 intersect at A and B. A line passing through B intersects C_1 at C and C_2 at D. Another line passing through B intersects C_1 at E and C_2 at F; CF intersects C_1 and C_2 at P and Q, respectively. Make sure that in your diagram, B, E, C, A, P $\in C_1$ and B, D, F, A, Q $\in C_2$, in this order. Let M and N be the midpoints of the arcs BP and BQ, respectively. Prove that if CD = EF, then the points C, F, M, N are concylic, in this order.

Solution



Extend AB to meet CF at G. We are going to prove that BG is the bisector of $\angle$CBF. We have CB×CD = CQ×CF, and FB×EF = FP×CF.

Dividing the two above equations, knowing CD = EF, we get $\frac{CB}{FB} = \frac{CQ}{FP}$. We also have PG×CG = GB×GA = QG×FG, or $\frac{QG}{PG} = \frac{CG}{FG} = \frac{QG + CG}{PG + FG} = \frac{CQ}{PF}$.

It follows that $\frac{CG}{FG} = \frac{CB}{FB}$, or $\angle CBG = \angle FBG$ and BG is the bisector of $\angle CBF$.

So now the three bisectors CM, FN and BG coincide. Let them meet at I on BG. We now have IM×IC = IB×IA = IN×IF, or $\frac{IM}{IN} = \frac{IF}{IC}$, or the two triangles IMN and IFC are similar, meaning $\angle IMN = \angle IFC$, but $\angle IMN + \angle CMN = 180°$, or $\angle CMN + \angle NFC = 180°$. Therefore, C, F, M and N are concyclic.

<u>Further observation</u>

Let's prove that MN || O_1O_2 where O_1 and O_2 are centers of C_1 and C_2, respectively. Since $\angle CBG = \angle FBG$, we have $\angle ABD = \angle FBG$, or AD = AF.

Let K be the midpoint of arc FD, AK is then the diameter of C_2.

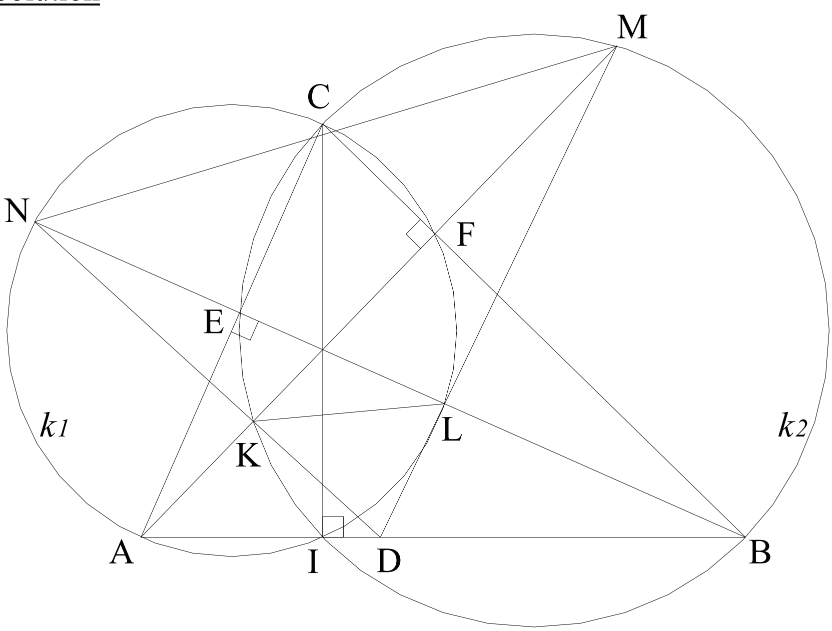
$\angle MCP = ½\angle BCP = ½$arc (DF – BQ) = arc (KF – NQ), or $\angle MCP + \angle NFQ$ = arc FK. Extend MN to meet C_2 at L, $\angle LNF = \angle MCP$.

Therefore, $\angle KNL = \angle NFQ = \angle BAN$ (subtending arc NB = NQ). But $\angle ANK = 90°$, or $AN \perp NK$; hence, $NL \perp AG$ or MN || O_1O_2.

Problem 3 of Austria Mathematical Olympiad 2005

In an acute-angled triangle ABC two circles k_1 and k_2 are drawn whose diameters are the sides AC and BC. Let E be the foot of the altitude h_b on AC and let F be the foot of the altitude h_a on BC. Let L and N be the intersections of the line BE with the circle k_1 (L on the line BE) and let K and M be the intersections of the line AF with the circle k_2 (K on the line AF). Show that KLMN is a cyclic quadrilateral.

Solution



Let H be the orthocenter of ΔABC; i.e., H = AF ∩ BE. Extend CH to meet AB at I. We have CI ⊥ AB, and I lies on both k_1 and k_2. By the intersecting chord theorem: in circle k_1, HC×HI = HN×HL, and in k_2, HC×HI = HK×HM. Or HN×HL = HK×HM. Therefore, according to the intersecting chord theorem, KLMN is a cyclic quadrilateral.

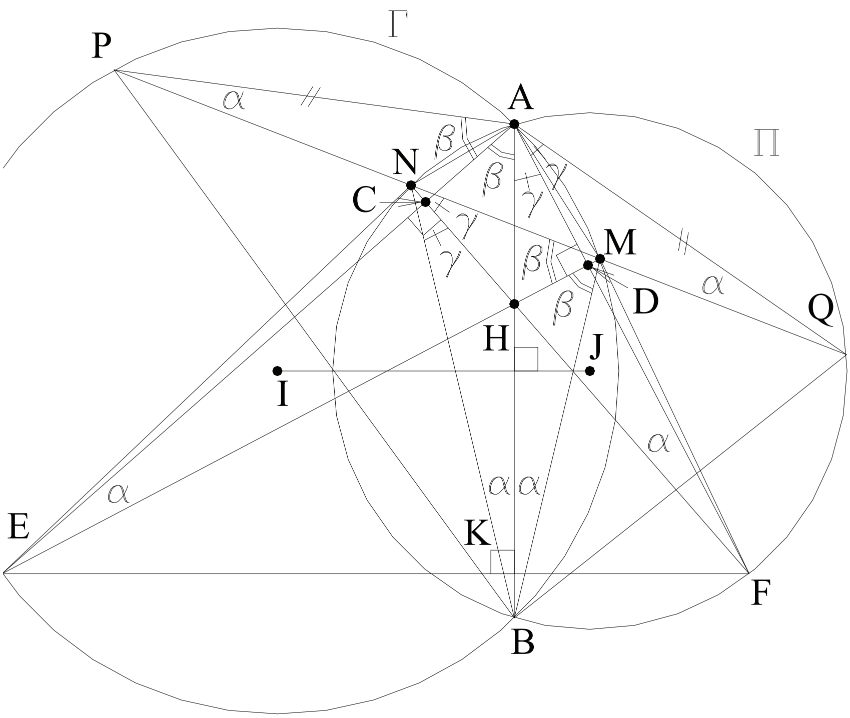
Further observation

Prove that the two segments NK and ML meet at a point on AB.

Problem 3 of pre-Vietnamese Mathematical Olympiad 2011

Two circles Γ and Π intersect at A and B. Take two points P, Q on Γ and Π, respectively, such that AP = AQ. The line PQ intersects Γ and Π, respectively at M and N. Let E, F, respectively be the centers of the two arcs BP and BQ (which do not contain A). Prove that MNEF is a cyclic quadrilateral.

Solution



Since AP = AQ, APQ is an isosceles triangle. Let α = ∠APQ = ∠AQP. Because both ∠AEM and ∠ABM subtend arc AM of Γ we also have ∠AEM = ∠ABM = α. Similarly, because both ∠AFN and ∠ABN subtend arc AN of Π we also have ∠AFN = ∠ABN = α, or ∠ABM = ∠ABN = α, and AB is the bisector of ∠MBN.

Furthermore, since E and F are the midpoints of the two arcs

BP and BQ, we get $\angle$BME = $\angle$NME = $\angle$BAE = $\angle$PAE = β (let them equal β), and $\angle$BNF = $\angle$MNF = $\angle$BAF = $\angle$QAF = γ (let them equal γ), or ME and NF are the bisectors of $\angle$BMN and $\angle$BNM, respectively.

Therefore, the three bisectors AB, ME and NF of triangle BMN are concurrent and let them meet at point H on AB. H is thus the incenter of triangle BMN.

Now applying the intersecting chord theorem, we get MH×EH = AH×BH = NH×FH, or MNEF is a cyclic quadrilateral.

<u>Further observation</u>

Now let C = AE ∩ FN, D = AF ∩ EM and K = AB ∩ EF. We will prove that H is also the orthocenter of triangle AEF.

Indeed, since $\angle$AEM = $\angle$AFN = α, $\angle$CED = $\angle$AEM = $\angle$AFN = $\angle$CFD = α which implies that CEFD is also cyclic and $\angle$ECF = $\angle$EDF.

However, $\angle$ECF = $\angle$EAF + $\angle$AFN = $\beta + \gamma + \alpha$, and $\angle$EDF = EAF + $\angle$AEM = $\beta + \gamma + \alpha$, or $\angle$ECF + $\angle$EDF = 2$\angle$ECF = 2($\beta + \gamma + \alpha$) = $\angle$PAQ + $\angle$APQ + $\angle$AQP (the three angles of triangles APQ) = 180°, or $\angle$ECF = $\angle$EDF = 90°. This implies that ED $\perp$ AF and FC $\perp$ AE, and H is the orthocenter of triangle AEF.
It also makes AK $\perp$ EF and EF || IJ where I and J are the centers of circles Γ and Π, respectively.

This problem is reminiscent of the previous two problems where the three angle bisectors meet outside the circles and the two segments intersecting on the other segment connecting the two points where the circles meet.

Inversion can be used to solve this problem.

Problem 3 of International Mathematical Talent Search Round 4

Prove that a positive integer can be expressed in the form $3x^2 + y^2$ if and only if it can also be expressed in the form $u^2 + uv + v^2$, where x, y, u and v are positive integers.

Solution

Express $u^2 + uv + v^2$ in the form $(ax)^2 + ax(bx + cy) + (bx + cy)^2 = a^2x^2 + abx^2 + acxy + b^2x^2 + 2bcxy + c^2y^2 = 3x^2 + y^2$.

From there, we get $a^2 + ab + b^2 = 3$, $a = -2b$ and $c^2 = 1$, or $(a, b, c) = (2, -1, \pm 1), (-2, 1, \pm 1)$ and

$(2x)^2 + 2x(-x + y) + (-x + y)^2 = (2x)^2 + 2x(-x - y) + (-x - y)^2 = (-2x)^2 + (-2x)(x + y) + (x + y)^2 = (-2x)^2 + (-2x)(x - y) + (x - y)^2 = 3x^2 + y^2$.

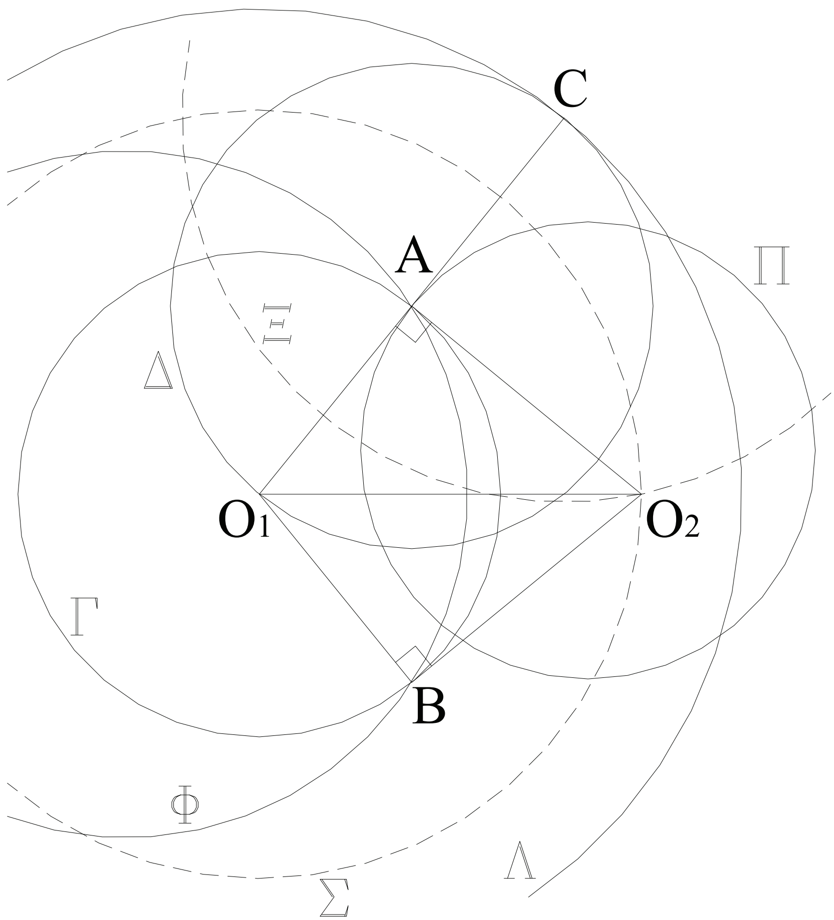
Therefore, $(u, v) = (2x, -x + y), (2x, -x - y), (-2x, x + y), (-2x, x - y)$.

The reader is encouraged to prove the problem in the other direction.

Problem 5 of International Mathematical Talent Search Round 13

Armed with just a compass – no straightedge – draw two circles that intersect at right angles; that is, construct overlapping circles in the same plane, having perpendicular tangents at the two points where they meet.

Solution



Draw arbitrary circles Γ with center O_1 and Φ with Φ larger than Γ and overlapping Γ at two points A and B as shown. Next, draw circle Δ with center A and radius AO_1. Continue by drawing circle

Λ with center O_1 to tangent circle Δ at a point; let this point be C. It's easily seen that the three points O_1, A and C are collinear and that $AO_1 = AC$.

Proceed by drawing two identical circles Σ and Ξ. These circles have the same radius that is less than O_1C but greater than O_1A with center O_1 and C, respectively. They meet at point O_2 with both points O_2 and B on the same side of O_1C.

Finally, draw circle Π with center O_2 and radius O_2A. Because Σ and Ξ are identical, $O_2C = O_2O_1$ and O_2O_1C is an isosceles triangle. Also since A is the midpoint of O_1C, O_2A is perpendicular to O_1A. Similarly, O_2B is perpendicular to O_1B.

The lines O_1C, O_1O_2, O_2A, O_1B and O_2B are only drawn with the straightedge to show that the two circles Γ and Π intersect at right angles.

Problem 10 of Austria Mathematical Olympiad 2006

Let A be a nonzero integer. Solve the following system in integers:

$$x + y^2 + z^3 = A \quad \text{(i)}$$
$$\frac{1}{x} + \frac{1}{y^2} + \frac{1}{z^3} = \frac{1}{A} \quad \text{(ii)}$$
$$xy^2z^3 = A^2 \quad \text{(iii)}$$

Solution

From (ii), with $x \neq 0, y \neq 0, z \neq 0$, we get $\frac{1}{x} + \frac{1}{y^2} = \frac{z^3 - A}{Az^3}$.

Now substitute $A - z^3 = x + y^2$ from (i) into it to obtain $\frac{1}{x} + \frac{1}{y^2} = \frac{-(x + y^2)}{Az^3}$, or $\frac{x + y^2}{xy^2} = \frac{-(x + y^2)}{Az^3}$, or $Az^3(x + y^2) = -xy^2(x + y^2)$, or $(x + y^2)(Az^3 + xy^2) = 0$.

The only relevant scenario is $x + y^2 = 0$, or $x = -y^2$. Substituting $x + y^2 = 0$ into (i) we end up with $A = z^3$. Successively, substituting $A = z^3$ into (iii), we get $xy^2 = z^3$, but $x = -y^2$ and now $A = -x^2 = -y^4 = z^3 = < 0$.

For this condition $A = -y^4 = z^3 < 0$ to prevail, we must have $A = -n^{12}$ where n is a nonzero integer.

From there, we conclude that $(x, y, z) = (-n^6, \pm n^3, -n^4)$ or $(x, y, z) = (-n^{12}, \pm n^3, -n^2)$.

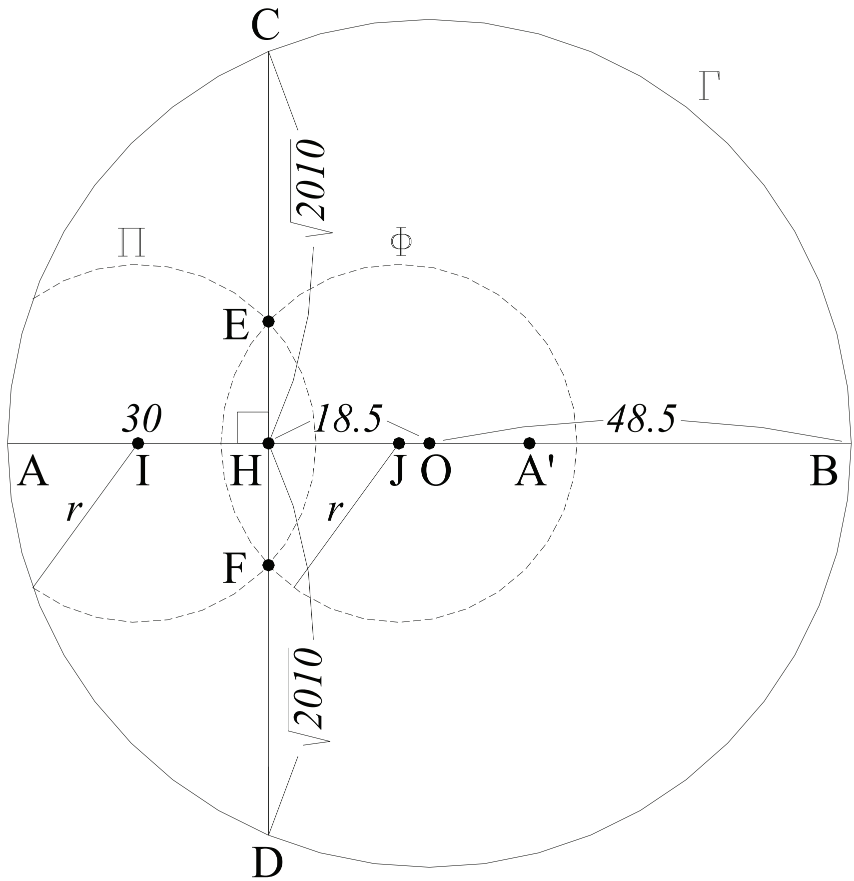
Further observation

Solve the problem when both x and z are positive integers.

Problem 7 of Malaysia National Olympiad 2010 Sulung Category

A line segment of length 1 is given on the plane. Show that a line segment of length $\sqrt{2010}$ can be constructed using only a straight-edge and a compass.

Solution



Given a line segment of length 1, we can draw segments AH and HB with all three points A, H and B on a straight line such that AH = 30, HB = 67. Pick A' as the point symmetry of A with respect to point H. Next, draw circle Π with center I which is the midpoint of

AH and radius r greater than one-half of AH but smaller than AH. Continue by drawing circle Φ with center J which is the midpoint of A'H and the same radius r. Let these circles Π and Φ meet at E and F. It's easily seen that $EF \perp AB$. Now draw the circle Γ with diameter AB. Extend EF to meet Γ at the two points C and D.

According to the intersecting chord theorem, $AH \times HB = HC \times HD$, but since H is on the diameter and $CD \perp AB$, $HC = HD$, and $AH \times HB = HC^2$, or

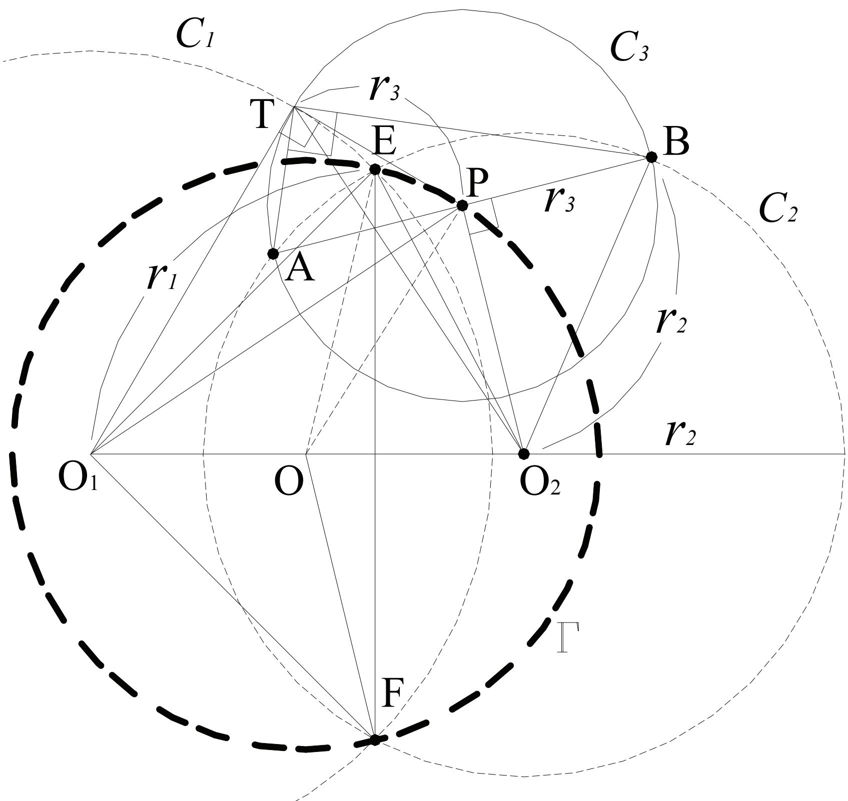
$HC^2 = 30 \times 67 = 2010$, or $HC = \sqrt{2010}$.

We have only used a straightedge and a compass.

Problem 4 Set 4 of India Postal Coaching 2011

Let C_1, C_2 be two circles in the plane intersecting at two distinct points. Let P be the midpoint of a variable chord AB of C_2 with the property that the circle on AB as diameter meets C_1 at a point T such that PT is tangent to C_1. Find the locus of P.

Solution



Let C_1, C_2 intersect at two distinct points E and F. Draw circle C_3, and let r_1, r_2 and r_3 be the radii of C_1, C_2 and C_3, respectively. Also let O_1 and O_2 be the centers of C_1 and C_2, respectively, and O be the midpoint of O_1O_2.

Per Stewart's theorem, in triangle O_1EO_2, we get

$$O_1E^2 \times OO_2 + O_2E^2 \times OO_1 = O_1O_2(OE^2 + OO_1 \times OO_2) \qquad \text{(i)}$$

But $O_1E = r_1$, $O_2E = r_2$, $OO_1 = OO_2 = \frac{1}{2}O_1O_2$, and (i) becomes

$$r_1^2 + r_2^2 = 2(OE^2 + \frac{1}{4}O_1O_2^2) \qquad \text{(ii)}$$

Now in triangle O_1PO_2, the Stewart theorem gives us
$O_1P^2 \times OO_2 + O_2P^2 \times OO_1 = O_1O_2(OP^2 + OO_1 \times OO_2)$.
Similarly, this equation transforms into

$$O_1P^2 + O_2P^2 = 2(OP^2 + \frac{1}{4}O_1O_2^2) \qquad \text{(iii)}$$

However, applying the Pythagorean's theorem to get $O_1P^2 = O_1T^2 + TP^2 = r_1^2 + r_3^2$, and $O_2P^2 = O_2B^2 - BP^2 = r_2^2 - r_3^2$, and equation (iii) is equivalent to $r_1^2 + r_2^2 = 2(OP^2 + \frac{1}{4}O_1O_2^2)$ (iv)

Comparing (ii) and (iv), we obtain $OE^2 + \frac{1}{4}O_1O_2^2 = OP^2 + \frac{1}{4}O_1O_2^2$, or $OP = OE$, and it is fixed.

Therefore, the locus of points P is part of a circle with center at the midpoint of the segment connecting the centers of the two circles C_1 and C_2 and with radius being the distance from this midpoint to one of the two distinct points these two circles intersect at each other.

<u>Further observation</u>

As we know that the locus is not the whole circle Γ as shown, the reader is encouraged to find the limit of this locus on Γ.

Problem 8 of Malaysia National Olympiad 2010 Bongsu category

Find the last digit of $7^1 \times 7^2 \times 7^3 \times \ldots \times 7^{2009} \times 7^{2010}$.

Solution

$7^1 \times 7^2 \times 7^3 \times \ldots \times 7^{2009} \times 7^{2010} = 7^{1+2+3+\ldots+2009+2010} = 7^{2010\times2011/2}$
$= 7^{2021055}$.

Now denote $u(m)$ the units digit of m where m is an integer; we have $u(7^0) = 1$, $u(7^1) = 7$, $u(7^2) = 9$, $u(7^3) = 3$, $u(7^4) = 1$, and the process repeats itself… In other words, $u(7^{4n}) = 1$, $u(7^{4n+1}) = 7$, $u(7^{4n+2}) = 9$, $u(7^{4n+3}) = 3$ where $n = 0, 1, 2, 3, \ldots$.

We have $2021055 \equiv 3 \pmod 4$, or $7^{2021055} = 7^{4\times505263+3}$. Therefore, $u(7^{2021055}) = 3$.

Answer: The last digit of $7^1 \times 7^2 \times 7^3 \times \ldots \times 7^{2009} \times 7^{2010}$ is 3.

<u>Problem 9 of Malaysia National Olympiad 2010 Bongsu category</u>

Show that there exist integers m and n such that $\frac{m}{n} = \sqrt[3]{\sqrt{50}+7} - \sqrt[3]{\sqrt{50}-7}$.

<u>Solution</u>

Note that there exists the formula $(a - b)(a^2 + ab + b^2) = a^3 - b^3$. Let $a = \sqrt[3]{\sqrt{50}+7}$ and $b = \sqrt[3]{\sqrt{50}-7}$. Multiply $\sqrt[3]{\sqrt{50}+7} - \sqrt[3]{\sqrt{50}-7}$ by $\frac{\sqrt[3]{(\sqrt{50}+7)^2} + \sqrt[3]{\sqrt{50}+7}\times\sqrt[3]{\sqrt{50}-7} + \sqrt[3]{(\sqrt{50}-7)^2}}{\sqrt[3]{(\sqrt{50}+7)^2} + \sqrt[3]{\sqrt{50}+7}\times\sqrt[3]{\sqrt{50}-7} + \sqrt[3]{(\sqrt{50}-7)^2}}$ to get $\frac{m}{n} = \frac{\sqrt[3]{(\sqrt{50}+7)^3} - \sqrt[3]{(\sqrt{50}-7)^3}}{\sqrt[3]{(\sqrt{50}+7)^2} + \sqrt[3]{\sqrt{50}+7}\times\sqrt[3]{\sqrt{50}-7} + \sqrt[3]{(\sqrt{50}-7)^2}} = \frac{14}{\sqrt[3]{(\sqrt{50}+7)^2} + \sqrt[3]{\sqrt{50}+7}\times\sqrt[3]{\sqrt{50}-7} + \sqrt[3]{(\sqrt{50}-7)^2}} = \frac{14}{\sqrt[3]{99+70\sqrt{2}} + 1 + \sqrt[3]{99-70\sqrt{2}}}$.

Now by putting $99 \pm 70\sqrt{2}$ in the form of $(c\sqrt{2})^3 \pm 3\times(c\sqrt{2})^2 d + 3c\sqrt{2}d^2 \pm d^3 = (c \pm d)^3$, we get $2c^3 + 3cd^2 = 70$ and $6c^2d + d^3 = 99$, or $d(6c^2 + d^2) = 1\times3\times3\times11$.

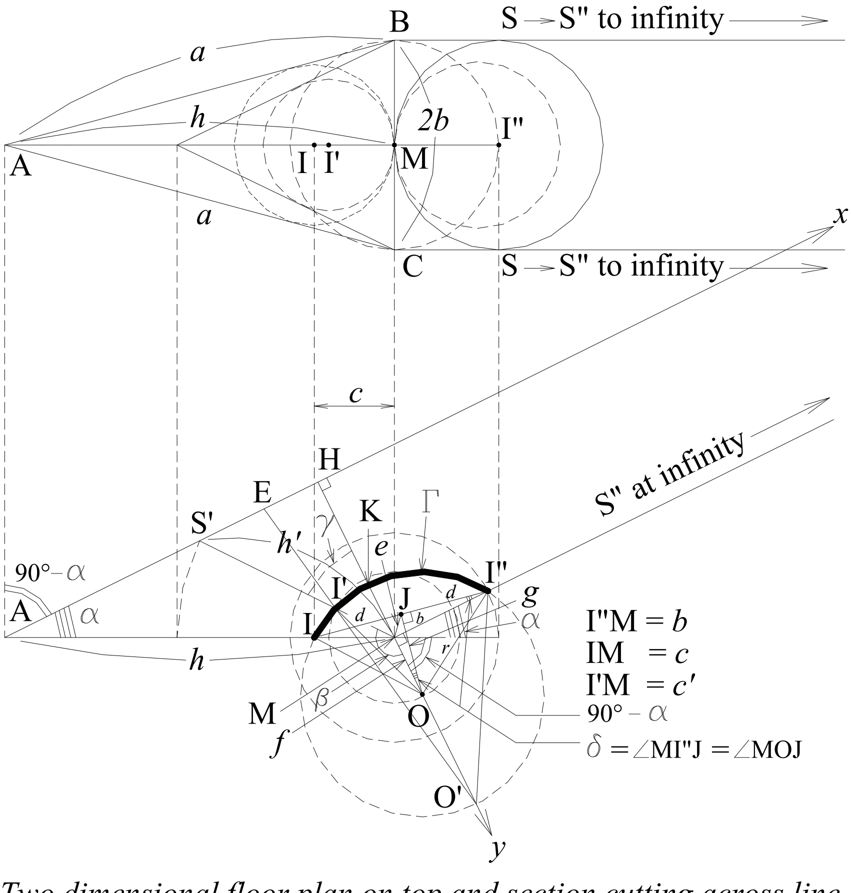
From there, $c = 2$, $d = 3$, or $99 \pm 70\sqrt{2} = (3 \pm 2\sqrt{2})^3$, $\sqrt[3]{99+70\sqrt{2}} = 3 + 2\sqrt{2}$, $\sqrt[3]{99-70\sqrt{2}} = 3 - 2\sqrt{2}$.

Finally, $\frac{m}{n} = \frac{14}{7} = \frac{2p}{p}$ where p is an integer. We conclude that there exist integers m and n such that $\frac{m}{n} = \sqrt[3]{\sqrt{50}+7} - \sqrt[3]{\sqrt{50}-7}$.

<u>*Problem 2 of the Vietnamese MO Team Selection Test 1985*</u>

Let ABC be a triangle with AB = AC. A ray A*x* is constructed in space such that the three planar angles of the trihedral angle ABC*x* at its vertex A are equal. If a point S moves on A*x*, find the locus of the incenter of triangle SBC.

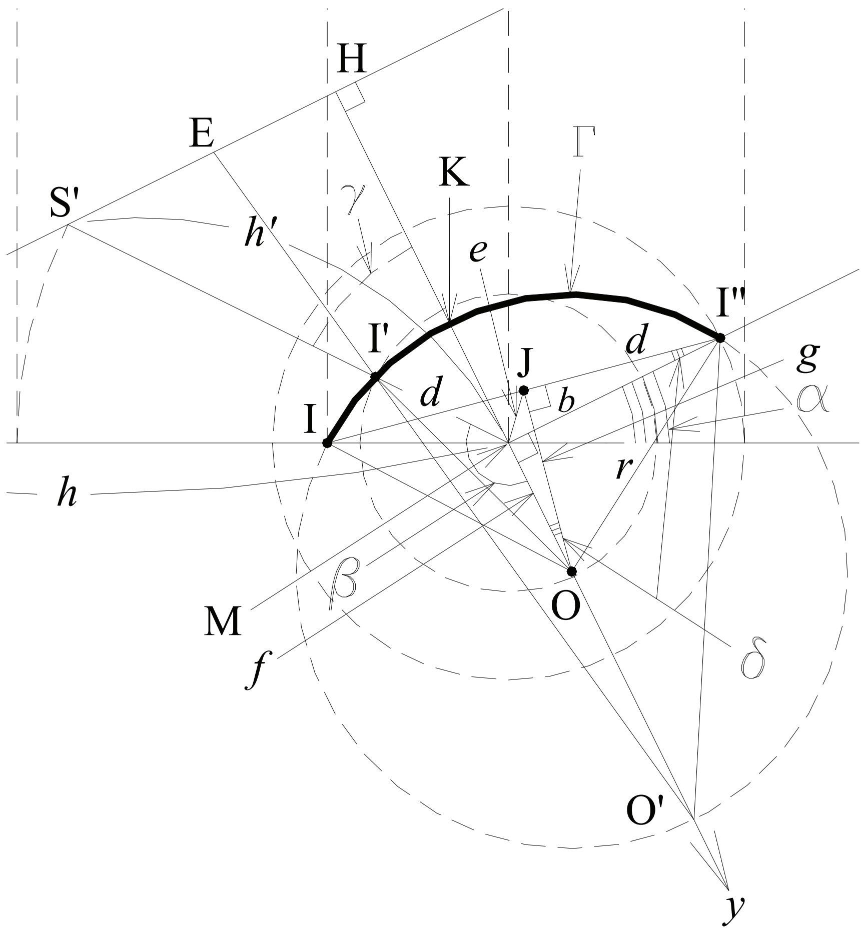
<u>Solution</u>



Two dimensional floor plan on top and section cutting across line AM on the bottom.

Let the moving S on A*x* be S', I and M be the incenter of triangle ABC and midpoint of BC, respectively, I' be the incenter of triangle S'BC, H be the foot of M on A*x*, ∠SAM = α, AB = AC =

a, BC = $2b$, IM = c, AM = h, S'B = S'C = a', I'M = c', S'M = h' and $\angle$I'MH = γ. When S' gets to infinity, let it be S'' and I'' be the incenter of triangle S''BC. At that point we consider I''M = BC/2 = b, S''B || S''C and S''M || S''A. Now draw the perpendicular bisector of II'' to meet My, the extension of HM, at O. Let $\angle$I'MO = β (or $\beta + \gamma = 180°$), J be the midpoint of II'' and d = IJ = JI'', e = MJ, f = OM, g = OJ, r = OI = OI'', $\delta = \angle$MI''J = $\angle$MOJ (because I''JMO is cyclic in circle denoted Π with $\angle$I''JO = $\angle$I''MO = 90°). Let (Ω) denote the area of shape Ω.



We will prove that the locus of the incenter of triangle SBC is the **boldface** arc II'' from I to I'' (when S moves from A to infinity on Ax) of the circle denoted Γ with center at O and radius r.

Indeed, in triangle ABC, (ABC) = $hb = c(a + b)$. Similarly, in triangle S'BC, (S'BC) = $h'b = c'(a' + b)$.

Dividing these two equations to get $\frac{h'}{h} = \frac{c'}{c} \times \frac{a' + b}{a + b}$ (i)

Now applying the law of cosines to triangle I'MO, we have
$OI'^2 = c'^2 + f^2 - 2c'f \times \cos\beta = c'^2 + r^2 - b^2 - 2c'f \times \cos\beta$.

Our goal is to prove that OI' = r. From the above equation, we now need to prove that $0 = c'^2 - b^2 - 2c'f \times \cos\beta = c'^2 - b^2 + 2c'f \times \cos\gamma$, or $c'^2 = b^2 - 2c'f \times \cos\gamma$, or $1 = (\frac{b}{c'})^2 - 2\frac{f}{c'} \times \cos\gamma$, but $\cos\gamma = \frac{MH}{h'} = \frac{h\sin\alpha}{h'}$

and $1 = (\frac{b}{c'})^2 - 2\frac{f}{c'} \times \frac{h\sin\alpha}{h'}$ (ii)

Let's find the value for f.

Applying the law of sines to triangle II''M, $\frac{c}{\sin\delta} = \frac{2d}{\sin(180° - \alpha)} = \frac{2d}{\sin\alpha}$, or $\boldsymbol{\sin\delta = \frac{c\sin\alpha}{2d}}$.

Next, apply the Stewart theorem to the same triangle with median MJ = e; we get $I''M^2 \times IJ + IM^2 \times I''J = II''(MJ^2 + IJ \times I''J)$, or $b^2 + c^2 = 2(e^2 + d^2)$, or $e^2 = \frac{1}{2}(b^2 + c^2) - d^2$ (iii)

Furthermore, according to the law of cosines, in the same triangle, $II''^2 = I''M^2 + IM^2 - 2 \times I''M \times IM \times \cos\angle IMI''$, or $4d^2 = b^2 + c^2 - 2 \times bc\cos(180° - \alpha) = b^2 + c^2 + 2bc\cos\alpha$, or $\boldsymbol{d = \frac{1}{2}\sqrt{b^2 + c^2 + 2bc\cos\alpha}}$.

Equation (iii) becomes $e^2 = \frac{1}{2}(b^2 + c^2) - d^2 = \frac{1}{2}(b^2 + c^2) - \frac{1}{4}(b^2 + c^2 + 2bc\cos\alpha) = \frac{1}{4}(b^2 + c^2 - 2bc\cos\alpha)$, or $\boldsymbol{e = \frac{1}{2}\sqrt{b^2 + c^2 - 2bc\cos\alpha}}$.

Continue by applying the law of sines. In triangles I''MJ and OMJ, we have $\frac{e}{\sin\angle MI''J} = \frac{e}{\sin\delta} = \frac{d}{\sin\angle I''MJ}$ and $\frac{e}{\sin\angle MOJ} = \frac{e}{\sin\delta} = \frac{g}{\sin(90° + \angle I''MJ)} = \frac{g}{\cos\angle I''MJ}$, respectively, or $g = \frac{e}{\sin\delta} \times$

$\cos\angle \mathrm{I''MJ} = \frac{e}{\sin\delta}\sqrt{1 - \sin^2\angle \mathrm{I''MJ}} = \frac{e}{\sin\delta}\sqrt{1 - \frac{d^2\sin^2\delta}{e^2}} = \frac{1}{\sin\delta}\times$

$\sqrt{e^2 - d^2\sin^2\delta}$, and $g^2 = \frac{e^2}{\sin^2\delta} - d^2$. Therefore, $r^2 = d^2 + g^2 = \frac{e^2}{\sin^2\delta}$

and $\boldsymbol{r = \frac{e}{\sin\delta}}$.

We should also open a note here to say that since I"JMO is cyclic in circle Π and triangle I"JM is circumscribed in Π, the diameter of Π equals $\frac{\mathrm{MJ}}{\sin\angle \mathrm{MI''J}}$ or $\boldsymbol{r = \frac{e}{\sin\delta}}$ which is the same result.

Substitute in the values for e and $\sin\delta$ obtained earlier, we get

$r = \frac{d}{c\sin\alpha}\sqrt{b^2 + c^2 - 2bc\cos\alpha} = \frac{1}{2c\sin\alpha}\sqrt{b^2 + c^2 - 2bc\cos\alpha}\times$

$\sqrt{b^2 + c^2 + 2bc\cos\alpha} = \frac{1}{2c\sin\alpha}\sqrt{(b^2 + c^2)^2 - 4b^2c^2\cos^2\alpha}$.

Successively, $f^2 = r^2 - b^2 = \frac{1}{4c^2\sin^2\alpha}[(b^2 + c^2)^2 - 4b^2c^2\cos^2\alpha -$

$4b^2c^2\sin^2\alpha] = \frac{1}{4c^2\sin^2\alpha}[(b^2 + c^2)^2 - 4b^2c^2] = \frac{1}{4c^2\sin^2\alpha}(b^2 - c^2)^2$, or

$\boldsymbol{f = \frac{1}{2c\sin\alpha}(b^2 - c^2)}$.

Now substituting c' from (i) into (ii), equation (ii) that is still required to be proven becomes

$1 = \frac{h^2b^2(a' + b)^2}{h'^2c^2(a + b)^2} - 2\times\frac{h^2(b^2 - c^2)(a' + b)\sin\alpha}{2h'^2c^2(a + b)\sin\alpha}$, or $1 = \frac{h^2b^2(a' + b)^2}{h'^2c^2(a + b)^2} -$

$\frac{h^2(b^2 - c^2)(a' + b)}{h'^2c^2(a + b)}$, or $\frac{c^2h'^2}{h^2} = \frac{b^2(a' + b)^2}{(a + b)^2} - \frac{(b^2 - c^2)(a' + b)}{a + b}$.

But from (i), $\frac{a' + b}{a + b} = \frac{ch'}{c'h}$ and the previous equation is equivalent to

$\frac{c^2h'^2}{h^2} = \frac{b^2c^2h'^2}{c'^2h^2} - (b^2 - c^2)\frac{ch'}{c'h}$. Next, by making the denominators the same and then dividing both sides by ch', we get

$cc'^2h' = ch'b^2 - c'hb^2 + c'c^2h$ or $\dfrac{cc'}{b^2} = \dfrac{ch' - c'h}{c'h' - ch}$ or $\dfrac{c}{b}\times\dfrac{c'}{b} = \dfrac{\dfrac{c}{b}\times\dfrac{h'}{b} - \dfrac{c'}{b}\times\dfrac{h}{b}}{\dfrac{c'}{b}\times\dfrac{h'}{b} - \dfrac{c}{b}\times\dfrac{h}{b}}$.

Now let $2\varepsilon = \angle ABC$, $2\xi = \angle S'BC$. It's easily seen that both 2ε and 2ξ are different from 90°. We then have $\dfrac{c}{b} = \tan\varepsilon$, $\dfrac{h}{b} = \tan 2\varepsilon$, $\dfrac{c'}{b} = \tan\xi$, $\dfrac{h'}{b} = \tan 2\xi$, and the above equation that is still required to be proven can be written as

$\tan\varepsilon\tan\xi = \dfrac{\tan\varepsilon\tan 2\xi - \tan\xi\tan 2\varepsilon}{\tan\xi\tan 2\xi - \tan\varepsilon\tan 2\varepsilon}$, or

$\tan\varepsilon\tan\xi(\tan\xi\tan 2\xi - \tan\varepsilon\tan 2\varepsilon) = \tan\varepsilon\tan 2\xi - \tan\xi\tan 2\varepsilon$, or

$\tan\varepsilon\tan^2\xi\tan 2\xi - \tan\xi\tan^2\varepsilon\tan 2\varepsilon = \tan\varepsilon\tan 2\xi - \tan\xi\tan 2\varepsilon$, or

$\tan\varepsilon\tan 2\xi(\tan^2\xi - 1) = \tan\xi\tan 2\varepsilon(\tan^2\varepsilon - 1)$, or

$\dfrac{\sin\varepsilon}{\cos\varepsilon}\times\dfrac{\sin 2\xi}{\cos 2\xi}(\dfrac{\sin^2\xi}{\cos^2\xi} - 1) = \dfrac{\sin\xi}{\cos\xi}\times\dfrac{\sin 2\varepsilon}{\cos 2\varepsilon}(\dfrac{\sin^2\varepsilon}{\cos^2\varepsilon} - 1)$, or

$\dfrac{\sin\varepsilon}{\cos\varepsilon}\times\dfrac{2\sin\xi\cos\xi}{\cos^2\xi - \sin^2\xi}(\dfrac{\sin^2\xi - \cos^2\xi}{\cos^2\xi}) = \dfrac{\sin\xi}{\cos\xi}\times\dfrac{2\sin\varepsilon\cos\varepsilon}{\cos^2\varepsilon - \sin^2\varepsilon}(\dfrac{\sin^2\varepsilon - \cos^2\varepsilon}{\cos^2\varepsilon})$,

or $\dfrac{\sin\varepsilon}{\cos\varepsilon}\times\dfrac{\sin\xi}{\cos\xi} = \dfrac{\sin\xi}{\cos\xi}\times\dfrac{\sin\varepsilon}{\cos\varepsilon}$, or $\tan\varepsilon\tan\xi = \tan\xi\tan\varepsilon$ which is a true equality, and we're finally done with our analysis.

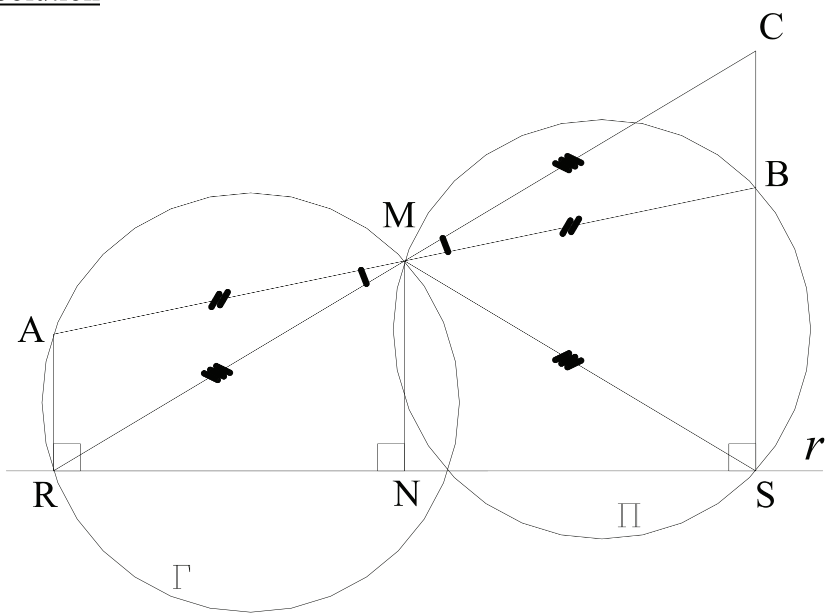
Further observation

Inversion can be used to find the locus of the incenter of triangle SBC by inverting the line Ax with respect to point O', which is on the extension My such that OO' = r, and some radius R. Let K be the incenter of triangle SBC when S is at H and E the intersection of Ax and O'I'. We can show that O'K×O'H = O'I'×O'E. The inversion of the line Ax is part of the circle Γ, and the locus is the smaller arc II" on the circle when S moves from A to infinity as shown.

Problem 3 of Italian Mathematical Olympiad 2002

Let A and B are two points on a plane, M be the midpoint of AB, *r* be a line, R and S be the projections of A and B onto *r*. Assuming that A, M, and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM.

Solution



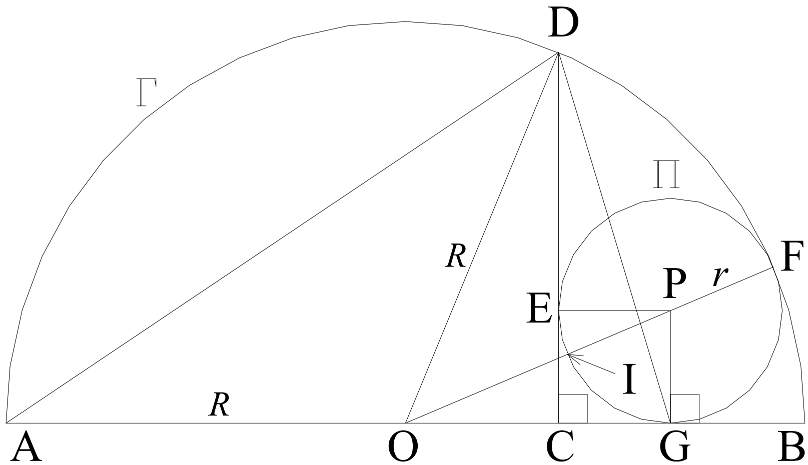
Extend RM and SB to meet at C. Since both AR and BS are perpendicular to *r*, AR || BC. Therefore, $\angle$ARM = $\angle$BCM. Now let N be the foot of M on *r*, because M is the midpoint of AB, N is the midpoint of RS, and the two triangles MRN and MSN are congruent, so are the two triangles MRA and MCB. Together they give us MR = MS = MC and $\angle$ARM = $\angle$BCM = $\angle$BSM.

We know that the diameter of the circumcircle of triangle AMR, denoted Γ, is $\dfrac{\mathrm{AM}}{\sin\angle\mathrm{ARM}} = \dfrac{\mathrm{BM}}{\sin\angle\mathrm{BCM}} = \dfrac{\mathrm{BM}}{\sin\angle\mathrm{BSM}}$ which is the diameter of the circumcircle of triangle BSM denoted Π as shown.

<u>Problem 3 of Italian Mathematical Olympiad 2003</u>

A semicircle is given with diameter AB and center O. Let C be an arbitrary point on the segment OB. Point D on the semi-circle is such that CD is perpendicular to AB. A circle with center P is tangent to the arc BD at F and to the segment CD and AB at E and G, respectively. Prove that the triangle ADG is isosceles.

<u>Solution</u>



Let the semicircle be Γ and the circle with center P be Π, R and r be the radii of Γ and Π, respectively, O be the center of Γ, $\angle$BOD = α, and I be the intersection of Π and OP.

Applying the law of cosines to get $AD^2 = 2R^2(1 + \cos\alpha)$. We need to prove that $AD^2 = AG^2$, or $2R^2(1 + \cos\alpha) = (R + OG)^2 = R^2 + 2R\times OG + OG^2$, or $R^2 + 2R^2\cos\alpha = 2R\times OG + OG^2$ (i)

But $\cos\alpha = \dfrac{OC}{OD} = \dfrac{OC}{R}$, and (i) becomes $R^2 + 2R\times OC = 2R\times OG + OG^2$, or $R^2 - 2R(OG - OC) = OG^2$, or $R^2 - 2R\times CG = OG^2$. However, $CG = EP = r$ and the previous equation is equivalent to $R^2 - 2rR = OG^2$, or $R(R - 2r) = OG^2$, or $R(OF - IF) = OG^2$, or $OI\times OF = OG^2$. Because OG is tangent to the circle Π at G, this statement is true according to the intersecting chord theorem.

Problem 4 of Italian Mathematical Olympiad 2002

Find all values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers.

Solution

Given α, β, γ as roots, we then write $x^3 - 3x + n = 0$ as
$(x - \alpha)(x - \beta)(x - \gamma) = 0$, or
$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$.

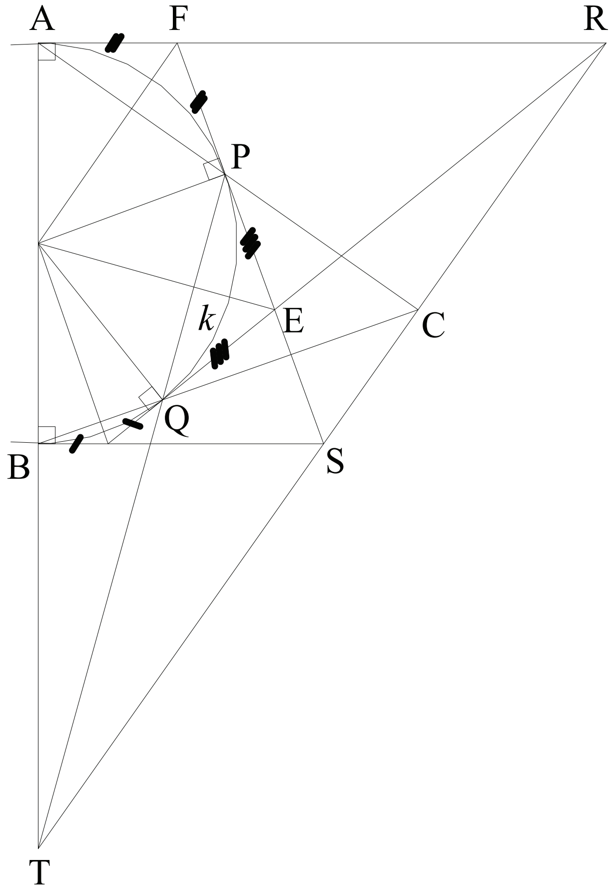
Equating the coefficients to get
$\alpha + \beta + \gamma = 0$,
$\alpha\beta + \beta\gamma + \alpha\gamma = -3$, and
$\alpha\beta\gamma = -n$.

From there, $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \alpha\gamma) = 0$, or
$\alpha^2 + \beta^2 + \gamma^2 = 6$.
Therefore, $(\alpha^2, \beta^2, \gamma^2)$ is a permutation of (4, 1, 1), and the only values for n are $n = 2$ or $n = -2$.

Problem 6 of Austria Mathematical Olympiad 2003

Let ABC be an acute-angled triangle. The circle *k* with diameter AB intersects AC and BC again at P and Q, respectively. The tangents to *k* at A and Q meet at R, and the tangents at B and P meet at S. Show that C lies on the line RS.

Solution



Extend AB and PQ to meet at T. Consider quadrilateral ABQP as a hexagon AABQQP with lengths AA = QQ = 0 and as a hexagon ABBQPP with lengths BB = PP = 0. Both hexagons are inscribed in circle *k*.

According to Pascal's theorem, the extensions of the opposite segments of a hexagon meet at points which lie on a straight line. Therefore, for hexagon AABQQP, the three points R = AA ∩ QQ, T = AB ∩ PQ and C = AP ∩ BQ are on a straight line.

Similarly, for hexagon ABBQPP, the three points S = BB ∩ PP, C = AP ∩ BQ and T = AB ∩ PQ are also on a straight line.

Therefore, the three points R, C and S are also on a straight line. Or C lies on the line RS.

Problem 2 of Australia Mathematical Olympiad 2010

Let the number of different divisors of the integer n be N(n); e.g. 24 has the divisors 1, 2, 3, 4, 6, 8, 12 and 24, so N(24) = 8. Determine whether the sum

N(1) + N(2) + . . . + N(1998)

is odd or even.

Solution

Let's recall the property of a divisor function: *For a non-square integer every divisor d of n is paired with divisor n/d of n and N(n) is then even; for a square integer one divisor (namely $\sqrt{n}$) is not paired with a distinct divisor and N(n) is then odd.*

For example, the non-square integer 24 = 1×2×3×4×6×8×12×24, 1 is paired with 24/1, 2 is paired with 24/2, 3 is paired with 24/3, 4 is paired with 24/4. For a square integer 64 = 1×2×4×8×16×32×64, 1 is paired with 64/1, 2 is paired with 64/2, 4 is paired with 64/4 = 16, 8 is not paired with any other divisor, and 8 = $\sqrt{64}$.

From 1 to 1998 there are 44 square integers because $44^2 = 1936 < 1998$ and $45^2 = 2025 > 1998$. Hence, there are 1998 – 44 = 1954 non-square integers.

Therefore, there are a sum of 1954 of sums of even divisors and another sum of 44 of sums of odd divisors combining to make the sum N(1) + N(2) + . . . + N(1998) an even number.

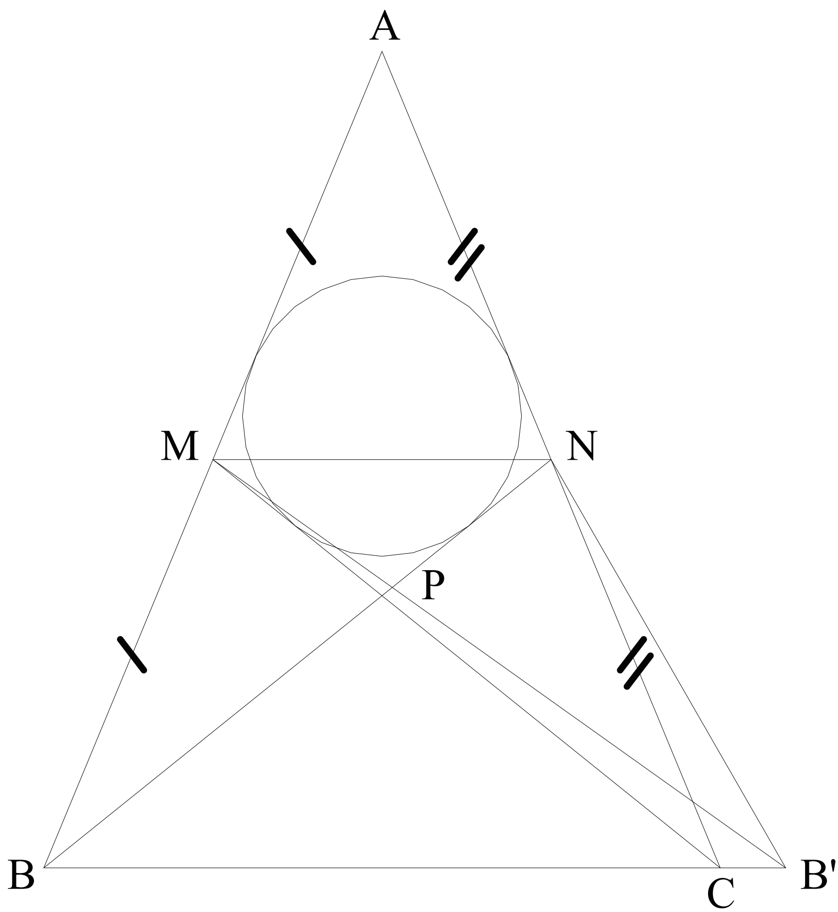
Further observation

Find the sum of divisors of N(1) + N(2) + . . . + N(1998).

Problem 2 of the Ibero-American Mathematical Olympiad 1987

In a triangle ABC, M and N are the midpoints of the sides AC and AB respectively, and P is the point of intersection of BM and CN. Show that if it is possible to inscribe a circumference in the quadrilateral ANPM, then the triangle ABC is isosceles.

Solution



Since M and N are the midpoints of the sides of triangle ABC, area of the triangles ABN = area of triangle ACM = ½ area of triangle ABC. These two triangles also share the same incircle; therefore, their perimeters are also equal since the area of a triangle equals pne-half the product of radius of incircle with the perimeter. We then have AM + BM + BN + AN = AM + MC + NC + AN,

or BM + BN = MC + NC (i)

We know that MN || BC; let's pick point B' as mirror image of B with respect to the perpendicular bisector of MN and is also perpendicular to BC, and assume B' ≠ C.

If B' is on the right of C then BM + BN = B'M + B'N > MC + NC since B'M > CM and B'N > NC.

If B' is on the left of C, then BM + BN < MC + NC. So to satisfy (i) we must have B' ≡ C (B' coincides with C), and therefore, BM = CN, and the triangle ABC is isosceles with AB = AC.

Problem 4 of Mongolian Mathematical Olympiad 1999

Is it possible to place a triangle with area 1999 and perimeter 1999^2 in the interior of a triangle with area 2000 and perimeter 2000^2?

Solution

Let the sides of the second triangle with area 2000 be a, b and c, R be its inradius and r be the inradius of the first triangle with area 1999.

The area of the second triangle is $\frac{1}{2}R(a + b + c) = 2000$, or $\frac{1}{2}R\times$ $2000^2 = 2000$, and $R = \frac{1}{1000}$.

Similarly, the inradius of the first triangle is $r = \frac{2}{1999}$ which is greater than that of the second triangle, $r > R$.

Therefore, it is not possible to place a triangle with area 1999 and perimeter 1999^2 in the interior of a triangle with area 2000 and perimeter 2000^2 because the inradius of the former is greater than that of the latter one.

Problem 4 of International Mathematical Talent Search Round 18

Let a, b, c, d be distinct real numbers such that $a+b+c+d=3$ and $a^2+b^2+c^2+d^2=45$. Find the value of the expression

$$\frac{a^5}{(a-b)(a-c)(a-d)}+\frac{b^5}{(b-a)(b-c)(b-d)}+\frac{c^5}{(c-a)(c-b)(c-d)}+\frac{d^5}{(d-a)(d-b)(d-c)}.$$

Solution

Note that $\frac{b^5}{(b-a)(b-c)(b-d)}=\frac{-b^5}{(a-b)(b-c)(b-d)}$. Now let's add the first two terms $\frac{a^5}{(a-b)(a-c)(a-d)}-\frac{b^5}{(a-b)(b-c)(b-d)}=$
$\frac{a^2b^2(a^3-b^3)+cd(a^5-b^5)-ab(c+d)(a^4-b^4)}{(a-b)(a-c)(a-d)(b-c)(b-d)}=$
$\frac{a^2b^2(a^2+ab+b^2)+cd(a^4+a^3b+a^2b^2+ab^3+b^4)}{(a-c)(a-d)(b-c)(b-d)}-$
$\frac{ab(c+d)(a+b)(a^2+b^2)}{(a-c)(a-d)(b-c)(b-d)}$ (these two terms should be written as a single ratio but the width of the page does not allow it).

With the denominator as $(a-c)(a-d)(b-c)(b-d)$, the numerator of the first two items can be expressed as $a^4b^2+a^3b^3+a^2b^4+a^4cd+a^3bcd+a^2b^2cd+ab^3cd+b^4cd-a^4bc-a^4bd-a^2b^3c-a^2b^3d-a^3b^2c-a^3b^2d-ab^4c-ab^4d=(b-c)(a^4b+a^3b^2+a^2b^3-a^4d-a^3bd-ab^3d-a^2b^2d)-b^4c(a-d)$.

The sum of the first two terms becomes

$$\frac{a^4b+a^3b^2+a^2b^3-a^4d-a^3bd-ab^3d-a^2b^2d}{(a-c)(a-d)(b-d)}-\frac{b^4c}{(a-c)(b-c)(b-d)}$$
$$=\frac{ab(a^2+ab+b^2)}{(a-c)(b-d)}-\frac{a^4d}{(a-c)(a-d)(b-d)}-\frac{b^4c}{(a-c)(b-c)(b-d)}.$$

Now let's add in the other two terms. The whole expression is equivalent to

$$\frac{ab(a^2+ab+b^2)}{(a-c)(b-d)}-\frac{a^4d}{(a-c)(a-d)(b-d)}-\frac{b^4c}{(a-c)(b-c)(b-d)}+\frac{c^5}{(c-a)(c-b)(c-d)}+\frac{d^5}{(d-a)(d-b)(d-c)}.$$

Leave the first term $\frac{ab(a^2+ab+b^2)}{(a-c)(b-d)}$ of this new expression alone and continue by adding the last four terms

$$-\frac{a^4d}{(a-c)(a-d)(b-d)}-\frac{b^4c}{(a-c)(b-c)(b-d)}+\frac{c^5}{(c-a)(c-b)(c-d)}+\frac{d^5}{(d-a)(d-b)(d-c)}=-\frac{a^4d}{(a-c)(a-d)(b-d)}-\frac{b^4c}{(a-c)(b-c)(b-d)}+\frac{c^5}{(a-c)(b-c)(c-d)}-\frac{d^5}{(a-d)(b-d)(c-d)}=$$

$$\frac{cd(b^4-c^4)-bc^2(b^3-c^3)}{(a-c)(b-c)(b-d)(c-d)}-\frac{cd(a^4-d^4)-ad^2(a^3-d^3)}{(a-c)(a-d)(b-d)(c-d)}=$$

$$\frac{cd(b^2+c^2)(b+c)-bc^2(b^2+bc+c^2)}{(a-c)(b-d)(c-d)}-\frac{cd(a^2+d^2)(a+d)-ad^2(a^2+ad+d^2)}{(a-c)(b-d)(c-d)}=$$

$$-\frac{b^3c+b^2c^2+bc^3+a^3d+a^2d^2+ad^3-c^3d-c^2d^2-cd^3}{(a-c)(b-d)}.$$

Now add in the term we left out to get the original sum again, and it equals $\frac{ab(a^2+ab+b^2)}{(a-c)(b-d)}-$

$$\frac{b^3c+b^2c^2+bc^3+a^3d+a^2d^2+ad^3-c^3d-c^2d^2-cd^3}{(a-c)(b-d)}=\frac{1}{(a-c)(b-d)}\times[a^3(b-d)+a^2(b^2-d^2)+a(b^3-d^3)-c^3(b-d)-c^2(b^2-d^2)-c(b^3-d^3)]=\frac{1}{(a-c)}[a^3-c^3+(a^2-c^2)(b+d)+(a-c)(b^2+bd+d^2)]=$$

$$a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd.$$

However, $2(ab + ac + ad + bc + bd + cd) = (a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2) = 3^2 - 45 = -36$, or $ab + ac + ad + bc + bd + cd = -18$.

Finally, the value of the expression is $a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd = 45 - 18 = 27$.

Further observation

It's easily seen that if (a, b, c, d) is a permutation of $(5, -4, 2, 0)$, then $a + b + c + d = 3$ and $a^2 + b^2 + c^2 + d^2 = 45$. We can substitute these values into the expression of the problem to verify our result, and it also equals 27.

Problem 3 of the Korean Mathematical Olympiad 2000

A rectangle ABCD is inscribed in a circle with center O. The exterior bisectors of $\angle$ABD and $\angle$ADB intersect at P; those of $\angle$DAB and $\angle$DBA intersect at Q; those of $\angle$ACD and $\angle$ADC intersect at R, and those of $\angle$DAC and $\angle$DCA intersect at S. Prove that P, Q, R, and S are concyclic.

Solution

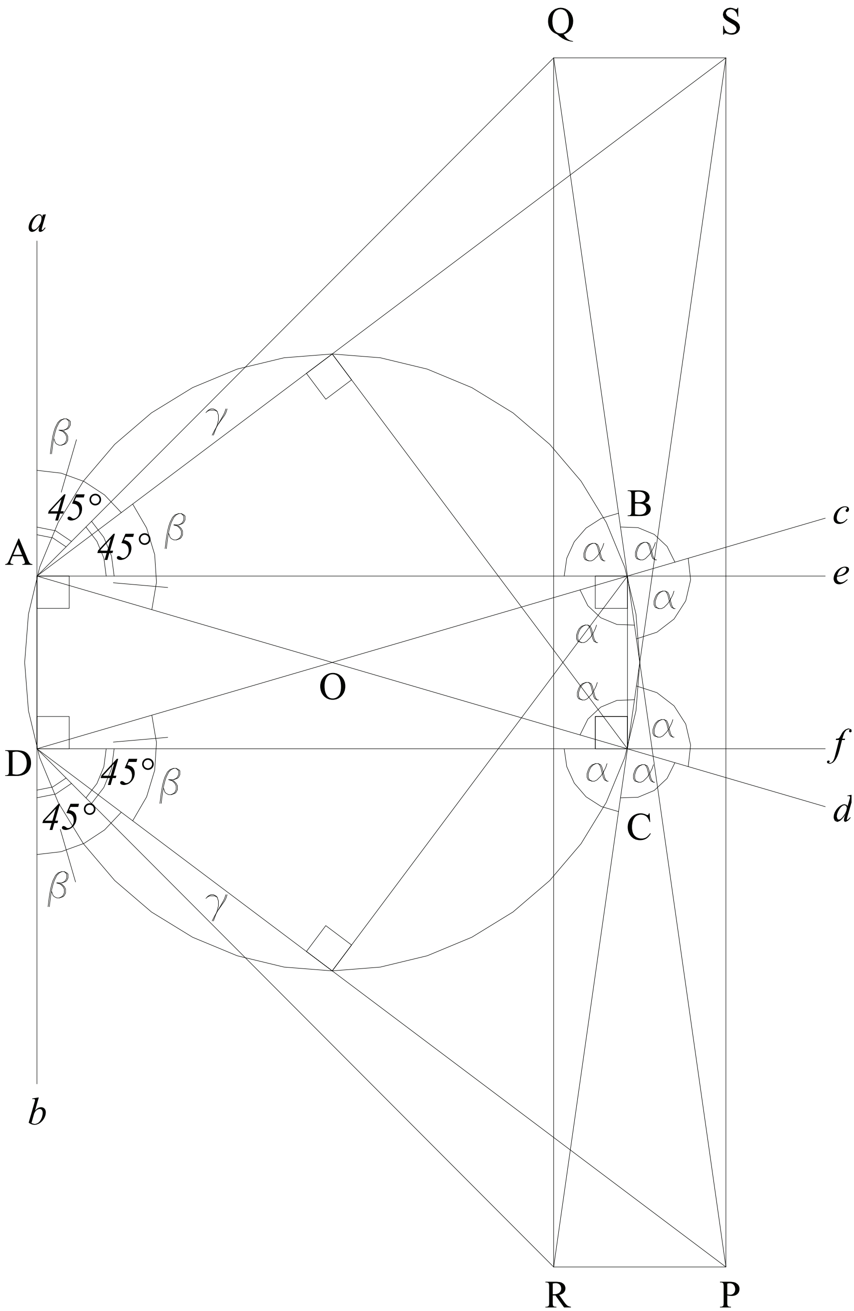
Let *a* be the line that starts from D and passes through A, *b* be the line that starts from A and passes through D, *c* be the line that starts from D and passes through B, *d* be the line that starts from A and passes through C, *e* be the line that starts from A and passes through B, and *f* be the line that starts from D and passes through C. Let $\alpha = \angle ABQ = \angle QBc = \angle DBP = \angle PBe$, $\beta = \angle SAa = \angle SAC$, $\angle QAa = \angle QAB = \angle RDb = \angle RDC = 90°/2 = 45°$. Since ABCD is a rectangle, we also have $\alpha = \angle ACS = \angle SCf = \angle DCR = \angle RCd$, $\beta = \angle PDb = \angle PDB$.

The two triangles ABQ and DCR are congruent because AB = CD (parallel sides of rectangle ABCD), $\angle QAB = \angle RDC = 45°$ and $\angle ABQ = \angle DCR = \alpha$. So are the two triangles ACS and DBP because AC = BD (the diagonals of ABCD), $\angle SCA = \angle PBD = \alpha$ and $\angle SAC = \angle PDB = \beta$. Furthermore, the two triangles ABQ and DCR are symmetrical across axis MN where M is the midpoint of AD and N the midpoint of BC. So are the two triangles ACS and DBP; they are symmetrical across the same axis MN. Hence, QS = RP, QR $\perp$ MN and SP $\perp$ MN, or QR $\|$ SP.

Therefore, P, Q, R, and S are concyclic.

Further observation

It's also easily seen that PQRS is a rectangle.


Q
S
a
γ
β
45°
45°
β
B
c
A
α
α
e
α
α
O
α
f
D
45°
β
α
α
45°
d
C
β
γ
b
R
P

<u>Problem 1 of International Mathematical Talent Search Round 27</u>

Are there integers M, N, K, such that M + N = K and

a) each of them contains each of the seven digits 1, 2, 3, … , 7 exactly once?
b) each of them contains each of the nine digits 1, 2, 3, … , 9 exactly once?

<u>Solution</u>

a) Note that if M, N and K each contains each of the seven digits 1, 2, 3, … , 7 exactly once, the sum of all the individual digits is 1 + 2 + 3 + 4 + 5 + 6 + 7 = 28 which is not evenly divisible by 3.

However, $M \equiv 1 \pmod 3$, and $N \equiv 1 \pmod 3$. This should give us $K = M + N \equiv 2 \pmod 3$. But the sum of all the individual digits of K is also 28 which contradicts with the previous statement $K \equiv 2$ (mod 3). Therefore, the answer is no. There are no integers M, N and K to satisfy the condition required for this part.

b) Since 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45, and $M \equiv 0 \pmod 3$, $N \equiv 0 \pmod 3$ and $K \equiv 0 \pmod 3$. Now let's try to subtract the two integers 987654321 – 123456789 = 864197532 which satisfies the requirement for this part.

Therefore, for this part the answer is yes, and M = 123456789, N = 864197532, K = 987654321.

Problem 1 of Italy Mathematical Olympiad 2003

Find all three digit integers n which are equal to the integer formed by three last digit of n^2.

Solution

Let the three digit integer n be abc where a is the hundreds digit, b the tens digit and c the units digit. We have $n = 100a + 10b + c$ and $n^2 = (100a + 10b + c)^2 = 10{,}000\times a^2 + 1{,}000\times 2ab + 100\times(2ac + b^2) + 10\times 2bc + c^2$ (i)

Denote the units digit of an integer m be $u(m)$, its tens digits be $t(m)$ and its hundreds digit be $h(m)$. It's easily seen that $u(n^2) = u(c^2)$ from equation (i). We also note that the units digit of a integer that equals the units digit of its own square is when the integer equals 0, 1, 5 or 6 because $u(0^2) = u(0) = 0$, $u(1^2) = u(1) = 1$, $u(5^2) = u(25) = u(5) = 5$ and $u(6^2) = u(36) = u(6) = 6$.

Let's try $c = 0$ to see if there is any three digit integer n ending with 0 that satisfies the problem. From (i), $b = t(n) = t(10\times 2bc) = u(2bc)$. However, $c = 0$, and $u(2bc) = 0$. Thus $b = 0$. Also according to (i), $a = h(n) = u(2ac + b^2) = 0$ because $c = b = 0$. Hence, for $c = 0$, ***the three digit integer is n = 000***.

Now try $c = 1$. Again, from (i), $b = t(n) = u(2bc) = u(2b)$. In order for $b = u(2b)$ to hold, $b = 0$. And $a = h(n) = u(2ac + b^2) = u(2a)$, or $a = 0$. Hence, for $c = 1$, ***the three digit integer is n = 001***.

Now try $c = 5$, $c^2 = 25$, $u(n) = u(25) = 5$. The carryover is 2 and $b = t(n) = u(2bc + 2) = u(10b + 2) = 2$ and the carryover is also 2. Now $a = h(n) = u(2ac + b^2 + 2) = u(10a + 4 + 2) = 6$. Hence, for $c = 5$, ***the three digit integer is n = 625***.

For $c = 6$, $c^2 = 36$, $u(n) = u(36) = 6$. The carryover is 3 and $b = t(n) = u(2bc + 3) = u(12b + 3) = u(2b + 3) = 7$. Thus $b = 7$ and the carryover is 8. Now $a = h(n) = u(2ac + b^2 + 8) = u(12a + 49 + 8) = 3$. Hence, for $c = 6$, ***the three digit integer is n = 376***.

<u>Problem 3 of Spain Mathematical Olympiad 1988</u>

Prove that if one of the number $25x + 3y$, $3x + 7y$ (where $x, y \in \mathbb{Z}$) is a multiple of 41, then so is the other.

<u>Solution</u>

$\mathbb{Z}$ means a set of integers, and $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
We have found that for $x = 5$ and $y = 13$, $25\times5 + 3\times13 = 164 = 41\times4$.
However, $3\times5 + 7\times13 = 106 \neq 41\times n$ where n is an integer.
Therefore, the problem is not valid.

<u>Further observation</u>

The problem description was copied from the web at this link *<u>http://www.imomath.com/othercomp/Spa/SpaMO88.pdf</u>*

It could be an error made by the moderator who entered the problem's description into the web.

To test for divisibility by 41: Subtract four times the last digit from the remaining leading truncated number. If the result is divisible by 41, then so was the first number. Apply this rule over and over again as necessary. For example: $30873 \rightarrow 3087 - 4\times3 = 3075 \rightarrow 307 - 4\times5 = 287 \rightarrow 28 - 4\times7 = 0$; *remainder is zero and so 30873 is also divisible by 41.*

Problem 4 of Germany Mathematical Olympiad 1998

Do there exist three consecutive odd integers whose sum of squares is a four-digit number having all its digits equal?

Solution

Let the three consecutive odd integers be $2n + 1$, $2n + 3$ and $2n + 5$ where n is an integer. The sum of their squares is $(2n + 1)^2 + (2n + 3)^2 + (2n + 5)^2 = 12n^2 + 36n + 35$.
Assuming there exist three consecutive odd integers whose sum of squares is a four-digit number having all its digits equal, we then have $12n^2 + 36n + 35 = aaaa = 1000a + 100a + 10a + a$ where a is a digit from 0 to 9.

Denote the units digit of an integer m be $u(m)$. We must have $a = u(12n^2 + 36n + 35) = u(2n^2 + 6n + 5)$. Now let's set up this table to list all the possible units digit values of this term.

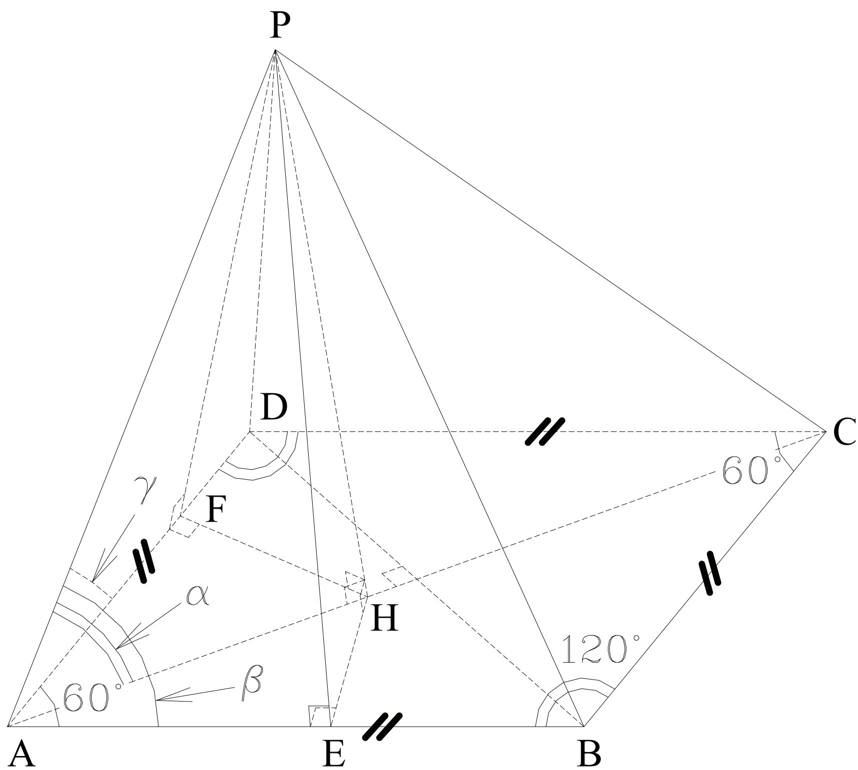
n	$u(6n)$	$u(n^2)$	$u(2n^2)$	$a = u(2n^2 + 6n + 5)$
0	0	0	0	5
1	6	1	2	3
2	2	4	8	5
3	8	9	8	1
4	4	6	2	1
5	0	5	0	5
6	6	6	2	3
7	2	9	8	5
8	8	4	8	1
9	4	1	2	1

So the only possible values for digit a are 1, 3 or 5 as seen, or $aaaa = 1111$, 3333 or 5555. We now need to solve the three equations $12n^2 + 36n + 35 = 1111$, $12n^2 + 36n + 35 = 3333$, and $12n^2 + 36n + 35 = 5555$, or $3n^2 + 9n - 269 = 0$, $6n^2 + 18n - 1649 = 0$ and $n^2 + 3n - 460 = 0$. The only equation that has the integer solutions is the last one with $n = 20, -23$, and the three consecutive odd integers are 41, 43, 45 or -45, -43, -41.

Problem 5 of International Mathematical Talent Search Round 25

As shown in the figure on the right, PABCD is a pyramid, whose base, ABCD, is a rhombus with $\angle DAB = 60°$. Assume that $PC^2 = PB^2 + PD^2$. Prove that $PA = AB$.

Solution



Let $\alpha = \angle PAC$, $\beta = \angle PAB$, $\gamma = \angle PAD$, $a = AB = BC = CD = AD$. Since ABCD is a rhombus with $\angle DAB = 60°$, both ABD and BCD are equilateral triangles and $BD = a$, $AC = a\sqrt{3}$.
According to the law of cosines, we have

$PC^2 = PA^2 + AC^2 - 2PA\times AC\cos\alpha = PA^2 + 3a^2 - 2aPA\sqrt{3}\cos\alpha$,
$PB^2 = PA^2 + AB^2 - 2PA\times AB\cos\beta = PA^2 + a^2 - 2aPA\cos\beta$, and
$PD^2 = PA^2 + AD^2 - 2PA\times AD\cos\gamma = PA^2 + a^2 - 2aPA\cos\gamma$.

We are given $PC^2 = PB^2 + PD^2$, or

$$PA^2 + 3a^2 - 2aPA\sqrt{3}\cos\alpha = 2PA^2 + 2a^2 - 2aPA(\cos\beta + \cos\gamma), \text{ or}$$

$$a^2 - 2aPA\sqrt{3}\cos\alpha = PA^2 - 2aPA(\cos\beta + \cos\gamma), \text{ or}$$

$$a^2 - PA^2 = 2aPA(\sqrt{3}\cos\alpha - \cos\beta - \cos\gamma).$$

Now let's assume that P is on the plane that is perpendicular to the plane of rhombus ABCD and passes through AC. In other words, assuming $\beta = \gamma$, the previous equation becomes

$$a^2 - PA^2 = 2aPA(\sqrt{3}\cos\alpha - 2\cos\beta) \qquad \text{(i)}$$

From P draw the altitude PH to the plane of ABCD, the altitudes HE to AB and HF to AD where E and F are on AB and AD, respectively. We then have $\cos\beta = \dfrac{AE}{PA} = \dfrac{\sqrt{AH^2 - EH^2}}{PA}$ and

$$\cos\alpha = \frac{AH}{PA} = \frac{AH\cos\beta}{\sqrt{AH^2 - EH^2}} = \frac{\cos\beta}{\sqrt{1 - \dfrac{EH^2}{AH^2}}} = \frac{\cos\beta}{\sqrt{1 - \sin^2 30°}} = \frac{\cos\beta}{\cos 30°},$$

or $\sqrt{3}\cos\alpha - 2\cos\beta = 0$. From equation (i) we now get $a^2 = PA^2$, or PA = AB.

Problem 3 of Italy Mathematical Olympiad 2009

A natural number n is called nice if it enjoys the following properties:
• the expression is made up of 4 decimal digits;
• the first and third digits of n are equal;
• the second and fourth digits of n are equal;
• the product of the digits of n divides n^2.
Determine all nice numbers.

Solution

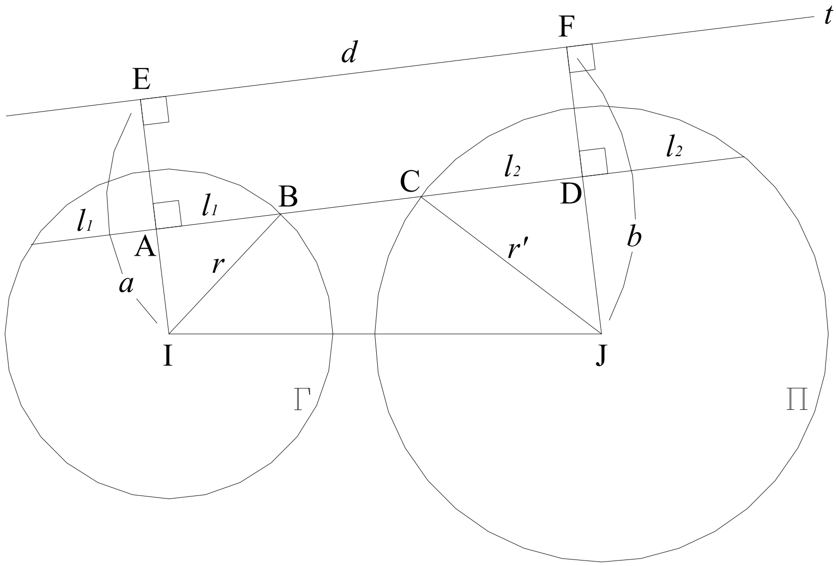
Let a be the first and third digit and b the second and fourth digit of n. In other words, $n = abab$. The value of n is $n = 1000a + 100b + 10a + b = 1010a + 101b$.

The product of the digits of n divides n^2 gives us the equation $(1010a + 101b)^2 = 101^2(10a + b)^2 \equiv 0 \pmod{a^2b^2}$, or
$101^2(\frac{10a + b}{ab})^2 = 101^2(\frac{10}{b} + \frac{1}{a})^2 = m^2$ where m is an integer.
Now substitute the values for b from 1 to 9 to find the corresponding ones for a, if there is any.
When $b = 1$, $\frac{10}{b} + \frac{1}{a} = 10 + \frac{1}{a}$ is an integer when $a = 1$ and $\boldsymbol{n = 1111}$.
When $b = 2$, $\frac{10}{b} + \frac{1}{a} = 5 + \frac{1}{a}$ is an integer when $a = 1$ and $\boldsymbol{n = 1212}$.
When $b = 3$, $\frac{10}{b} + \frac{1}{a} = \frac{10}{3} + \frac{1}{a}$; there is no value for a to make $\frac{10}{3} + \frac{1}{a}$ an integer.
When $b = 4$, $\frac{10}{b} + \frac{1}{a} = \frac{10}{4} + \frac{1}{a}$ is an integer when $a = 2$ and $\boldsymbol{n = 2424}$.
When $b = 5$, $\frac{10}{b} + \frac{1}{a} = 2 + \frac{1}{a}$ is an integer when $a = 1$ and $\boldsymbol{n = 1515}$.
When $b = 6$, $\frac{10}{b} + \frac{1}{a} = \frac{10}{6} + \frac{1}{a}$ is an integer when $a = 3$ and $\boldsymbol{n = 3636}$.
Similarly, when $b = 7$, 8 or 9, there is no value for a to make b an integer.

<u>*Problem 2 of Spain Mathematical Olympiad 1992*</u>

Given two circles of radii r and r' exterior to each other, construct a line parallel to a given line and intersecting the two circles in chords with the sum of lengths l.

<u>Solution</u>



Let the two circles be Γ and Π, their respective radii be r and r', the given line be t as shown, I and J be the centers of Γ and Π, respectively. Draw the altitudes IE and JF to t. The line parallels to t that needs to be constructed cuts IE, Γ, Π and JF at A, B, C and D, respectively with B and C between A and D. Now let a = IE, b = JF, d = EF, and the values of these segments a, b and d are given. Also let l_1 = AB, l_2 = CD. We're also given $2(l_1 + l_2) = l$.

Applying the Pythagorean theorem, DJ = $\sqrt{r'^2 - l_2^2}$; DF = b – DJ = $b - \sqrt{r'^2 - l_2^2}$ = AE. From here, AI = a – AE = $a - b + \sqrt{r'^2 - l_2^2}$.

Similarly, we have AI = $\sqrt{r^2 - l_1^2}$, or $\sqrt{r^2 - l_1^2} = a - b + \sqrt{r'^2 - l_2^2}$.

Now square both sides of the previous equation to get

$r^2 - l_1^2 = (a - b)^2 + 2(a - b)\sqrt{r'^2 - l_2^2} + r'^2 - l_2^2$, or

$r^2 - r'^2 - l_1^2 + l_2^2 - (a - b)^2 = 2(a - b)\sqrt{r'^2 - l_2^2}$, or

$$[r^2 - r'^2 - l_1^2 + l_2^2 - (a - b)^2]^2 = 4(a - b)^2(r'^2 - l_2^2) \qquad \text{(i)}$$

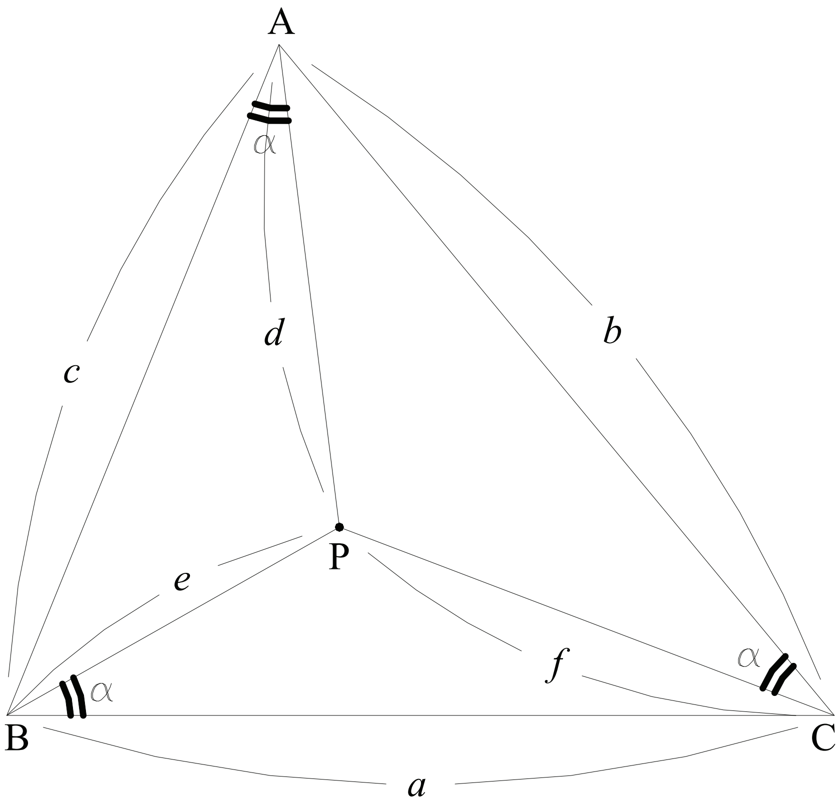
In addition to equation (i), as mentioned earlier, we also have the expression $2(l_1 + l_2) = l$ (ii)

In other words, we have two equations (i) and (ii) with two unknowns l_1 and l_2. Solving those equations to get the individual values for the unknowns. IA then can be found and the line can be constructed.

Problem 5 of Spain Mathematical Olympiad 1992

Given a triangle ABC, show how to construct the point P such that $\angle$PAB = $\angle$PBC = $\angle$PCA. Express this angle in terms of $\angle$A, $\angle$B, $\angle$C using trigonometric functions.

Solution



Let α = $\angle$PAB = $\angle$PBC = $\angle$PCA, a = BC, b = AC, c = AB, d = AP, e = BP and f = CP.

Applying the law of sines, in triangle APB, we get

$$\frac{e}{\sin\alpha} = \frac{c}{\sin[180° - \alpha - (\angle B - \alpha)]} = \frac{c}{\sin\angle B}, \text{ or } e = \frac{c\sin\alpha}{\sin\angle B} \qquad \text{(i)}$$

Similarly, in triangle BPC, $f = \dfrac{a\sin\alpha}{\sin\angle C}$.

In triangle APC, $d = \dfrac{b\sin\alpha}{\sin\angle A}$, and in triangle ABC, $\dfrac{a}{c} = \dfrac{\sin\angle A}{\sin\angle C}$.

Equation (i) becomes $e = \dfrac{a\sin\alpha\sin\angle C}{\sin\angle A\times\sin\angle B}$.

Now applying the law of cosines to triangle BPC to get
$a^2 = e^2 + f^2 - 2ef\cos\angle BPC = e^2 + f^2 + 2ef\cos C$.
Substituting *e* and *f* into the previous equation gives us

$$a^2 = \frac{a^2\sin^2\alpha\sin^2\angle C}{\sin^2\angle A\times\sin^2\angle B} + \frac{a^2\sin^2\alpha}{\sin^2\angle C} + \frac{2a^2\sin^2\alpha\times\sin\angle C\times\cos\angle C}{\sin\angle A\times\sin\angle B\times\sin\angle C}$$

or $1 = \sin^2\alpha\left(\dfrac{\sin^2\angle C}{\sin^2\angle A\times\sin^2\angle B} + \dfrac{1}{\sin^2\angle C} + \dfrac{2\cos\angle C}{\sin\angle A\times\sin\angle B}\right)$ (ii)

However, $\sin^2\angle C = \sin^2[180° - (\angle A + \angle B)] = \sin^2(\angle A + \angle B) = (\sin\angle A\times\cos\angle B + \cos\angle A\times\sin\angle B)^2 = \sin^2\angle A\times\cos^2\angle B + 2\sin\angle A\times\cos\angle B\times\cos\angle A\times\sin\angle B + \cos^2\angle A\times\sin^2\angle B$. Hence,

$$\frac{\sin^2\angle C}{\sin^2\angle A\times\sin^2\angle B} = \cot^2\angle B + 2\cot\angle A\times\cot\angle B + \cot^2\angle A.$$

and $1 = \sin^2\angle C + \cos^2\angle C$, or $\dfrac{1}{\sin^2\angle C} = 1 + \cot^2\angle C$.

Lastly, $\cos\angle C = \cos[180° - (\angle A + \angle B)] = -\cos(\angle A + \angle B) = -(\cos\angle A\times\cos\angle B - \sin\angle A\times\sin\angle B) = \sin\angle A\times\sin\angle B - \cos\angle A\times\cos\angle B$, and $\dfrac{2\cos\angle C}{\sin\angle A\times\sin\angle B} = 2 - 2\cot\angle A\times\cot\angle B$.

Adding all the three terms to get $\dfrac{\sin^2\angle C}{\sin^2\angle A\times\sin^2\angle B} + \dfrac{1}{\sin^2\angle C} + \dfrac{2\cos\angle C}{\sin\angle A\times\sin\angle B} = 3 + \cot^2\angle A + \cot^2\angle B + \cot^2\angle C$.

Therefore, equation (ii) is equivalent to
$1 = \sin^2\alpha(3 + \cot^2\angle A + \cot^2\angle B + \cot^2\angle C)$, or

$$\sin\alpha = \frac{1}{\sqrt{3 + \cot^2\angle A + \cot^2\angle B + \cot^2\angle C}}, \text{ or}$$

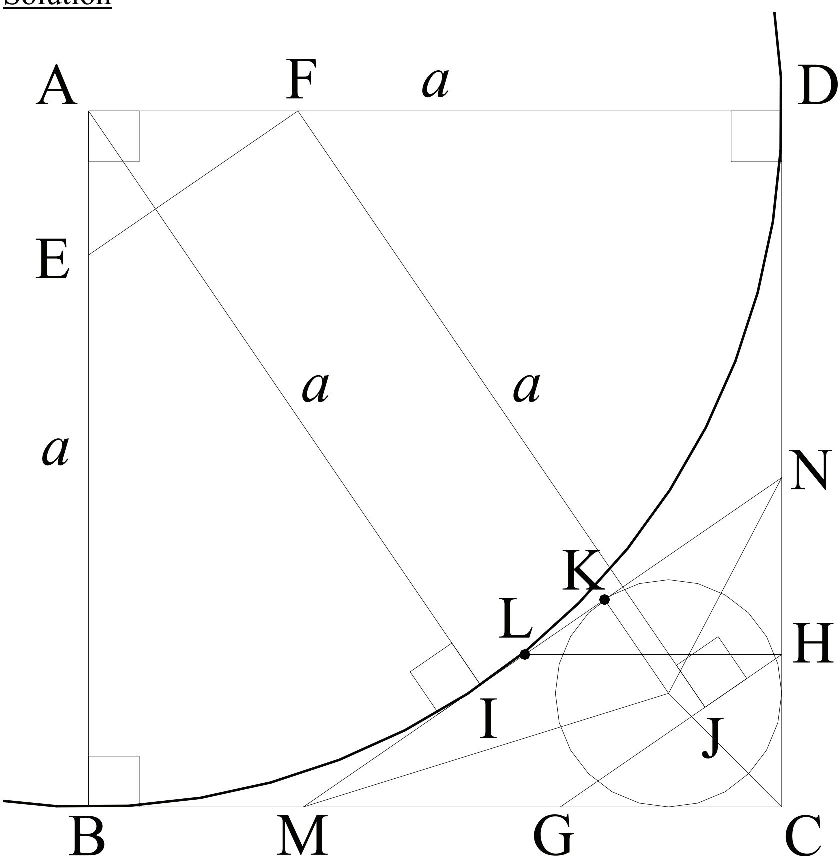
$$\alpha = \sin^{-1}\frac{1}{\sqrt{3 + \cot^2\angle A + \cot^2\angle B + \cot^2\angle C}}.$$

Based on this result we can construct the segments PA, PB, PC and the point P such that $\alpha = \angle PAB = \angle PBC = \angle PCA$.

Problem 2 Asian Pacific Mathematical Olympiad 2003

Suppose ABCD is a square piece of cardboard with side length a. On a plane are two parallel lines l_1 and l_2, which are also a units apart. The square ABCD is placed on the plane so that sides AB and AD intersect l_1 at E and F respectively. Also, sides CB and CD intersect l_2 at G and H respectively. Let the perimeters of triangle AEF and triangle CGH be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Solution



It's easily seen that the two triangles AEF and CHG are similar.

We have $\frac{\mathrm{HC}}{\mathrm{AE}} = \frac{\mathrm{GC}}{\mathrm{AF}}$.

Pick points M and N on BC and DC, respectively such that NH = AE and MG = AF.

We then have $\frac{\mathrm{HC}}{\mathrm{NH}} = \frac{\mathrm{GC}}{\mathrm{MG}}$, or GH || MN.
From H draw a line parallel to AD and intercept MN at L. Triangles AEF and HNL are congruent. Therefore, AE = NH, AF = LH = MG, EF = LN, GH = ML, and

m_1 = AE + AF + EF,

m_2 = HC + GC + GH.

$m_1 + m_2$ = AE + AF + EF + HC + GC + GH = NH + HC + GC + MG + ML + LN = NC + MC + MN, or $m_1 + m_2$ is the perimeter of triangle MCN.

From F draw a line perpendicular to and intercept GH at J, we have FJ = a as given by the problem. Similarly, from A draw a line perpendicular to and intercept MN at I, we have FJ = AI = a.

That proves to us that line MN is tangential to the circle with radius a and center A. Therefore, the parameter of triangle MCN equals BC + DC = 2a, or $m_1 + m_2$ is a constant.

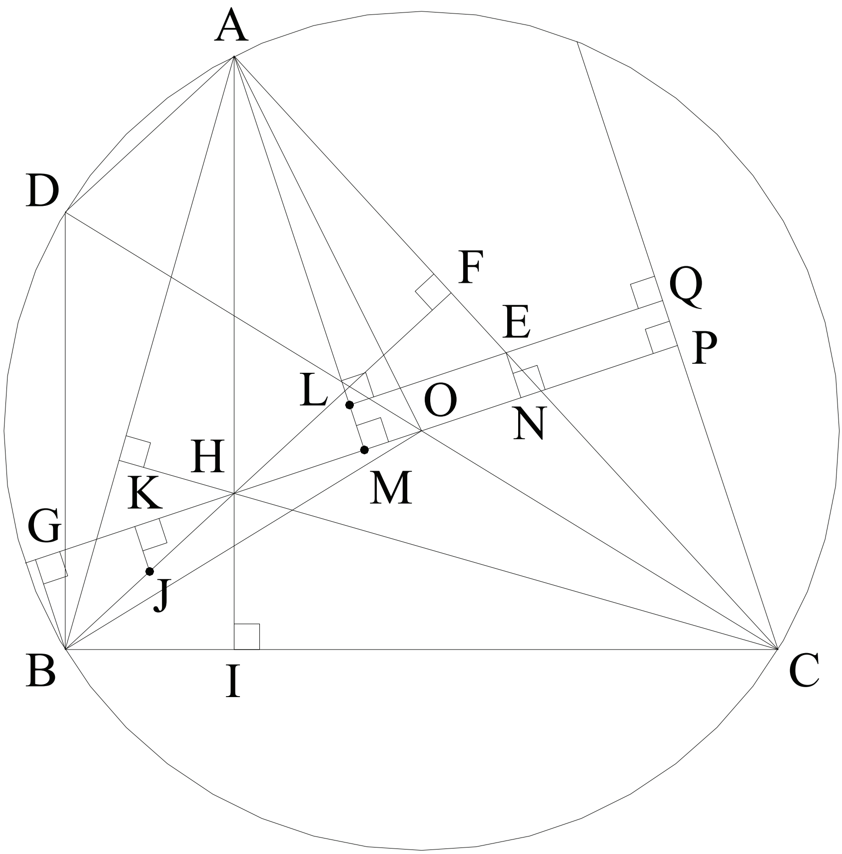
Further observation

Let K be the foot of incenter of incircle of triangle MCN to MN. Prove that IK = MN – 2×KN.

<u>*Problem 2 of Asian Pacific Mathematical Olympia 2004*</u>

Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Prove that the area of one of the triangles AOH, BOH and COH is equal to the sum of the areas of the other two.

<u>Solution 1</u>



From the vertices of triangle ABC A, B and C draw the altitudes to OH and let them intercept the extension of OH at M, G and P, respectively.

Since the three triangles AOH, BOH and COH have the same base OH, to prove the area of AOH is the sum of the areas BOH and COH, it suffices to prove that AM = GB + PC (i)

Extend CO to meet the circle at D. Since CD is the diameter of the circle, we have $\angle DAC = \angle DBC = \angle BFC = \angle AIC = 90°$. Or AD || HB, and DB || AH; therefore, AD = HB.

O and E are also midpoints of DC and AC, respectively, we have $OE = ½AD$, or $OE = ½HB$. Let J be the midpoint of BH; from J draw the altitude to OH and cuts the extension of OH at K. We have $KJ = ½GB$ (ii)
$HJ = ½HB = OE$ and $\angle KHJ = \angle OHF = \angle NOE$.

From E draw the altitude to OH and intercept it at N. The two triangles JKH and ENO are then congruent; we have KJ = EN.

Draw the line parallel to OH through E and intercepts AM and PC at L and Q, respectively; we then have
KJ = EN = LM = QP,
AL = QC (iii)

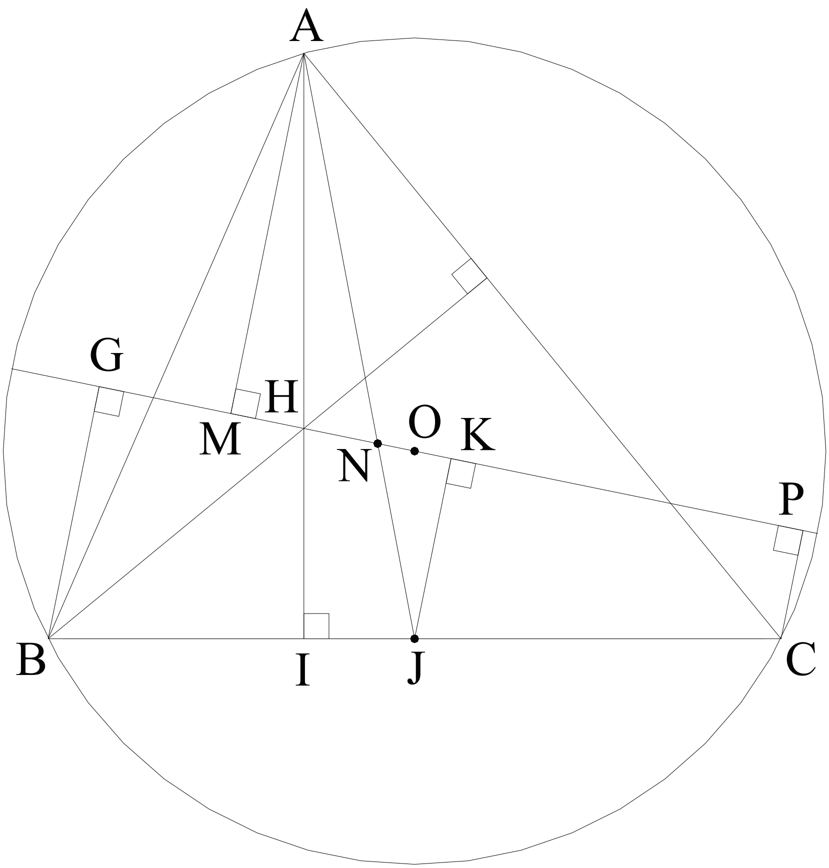
Combining with (ii), we get GB + PC = 2 KJ + QC – QP.
From (iii), GB + PC = LM + QP + AL – QP, or GB + PC = AM which is the condition (i) we set out to prove.

<u>Solution 2</u>

From the three vertices A, B and C of ΔABC draw orthogonal lines to OH and intercept it at M, G and P, respectively. The three triangles AOH, BOH and COH share the same base OH, so to prove the areas of AOH to equal the areas of the other two it suffices to prove AM = GB + PC.

Let J be the midpoint of BC. AJ intercepts OH at N. From J draw the line to perpendicular and intercept OH at K. We see that GB + PC = 2JK. We then need to prove AM = 2JK. Note that in a triangle, the three points centroid, orthocenter and circumcenter collinear; therefore, N is also the centroid of ΔABC

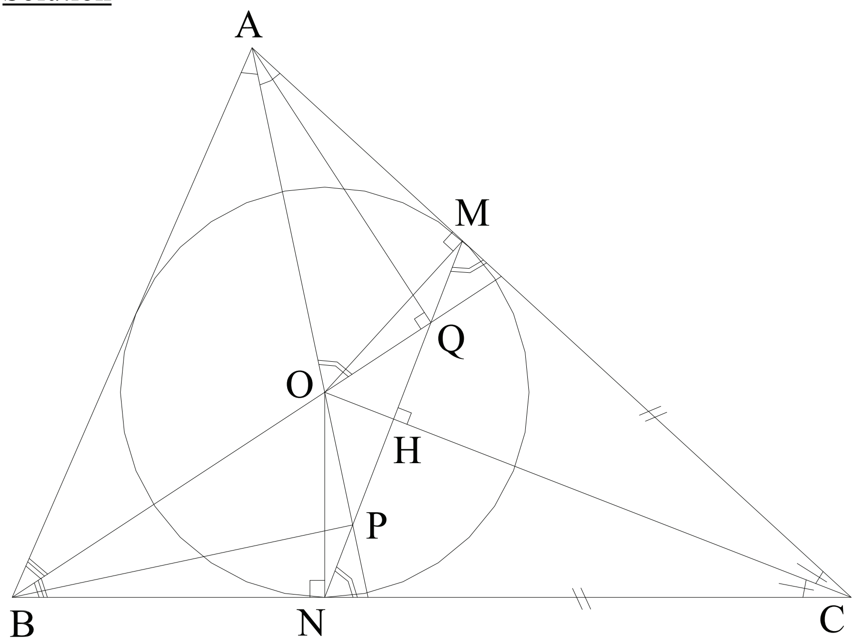
Hence, AN = 2NJ, or AM = 2JK because the two triangles AMN and JKN are similar.


A
G
H
O
K
M
N
P
B
I
J
C

Problem 4 of the Ibero-American Mathematical Olympiad 1989

The incircle of the triangle ABC, is tangential to both sides AC and BC at M and N, respectively. The angle bisectors of the angles A and B intersect MN at points P and Q, respectively. Let O be the incenter of the triangle ABC. Prove that MP×OA = BC×OQ.

Solution



We have $\angle AOQ = \angle ABO + \angle BAO = ½(180° - \angle C) = \angle HMC = \angle MOC$ and $\angle OMQ = \angle MCO$ (2 sides perpendicular to each other), or $\angle AOQ + \angle AMQ = \angle HMC + 90° + \angle MCO = 180°$

Therefore, AMQO is cyclic and $\angle AQO = \angle AMO = 90°$ and triangles AQO and CHM and MHO are all similar.
Similarly, $\angle APB = 90°$.

These similarities give us $\dfrac{OA}{OQ} = \dfrac{CM}{MH} = \dfrac{CN}{MH} = \dfrac{OM}{OH}$ (i)

On the other hand because $\angle APB = 90°$, APNB is cyclic and $\angle OBN + \angle OPN = 180°$, or $\angle OBN = \angle OPH$, or the two triangles OBN and OPH are also similar.

We have $\frac{BN}{PH} = \frac{ON}{OH} = \frac{OM}{OH}$ (ii)

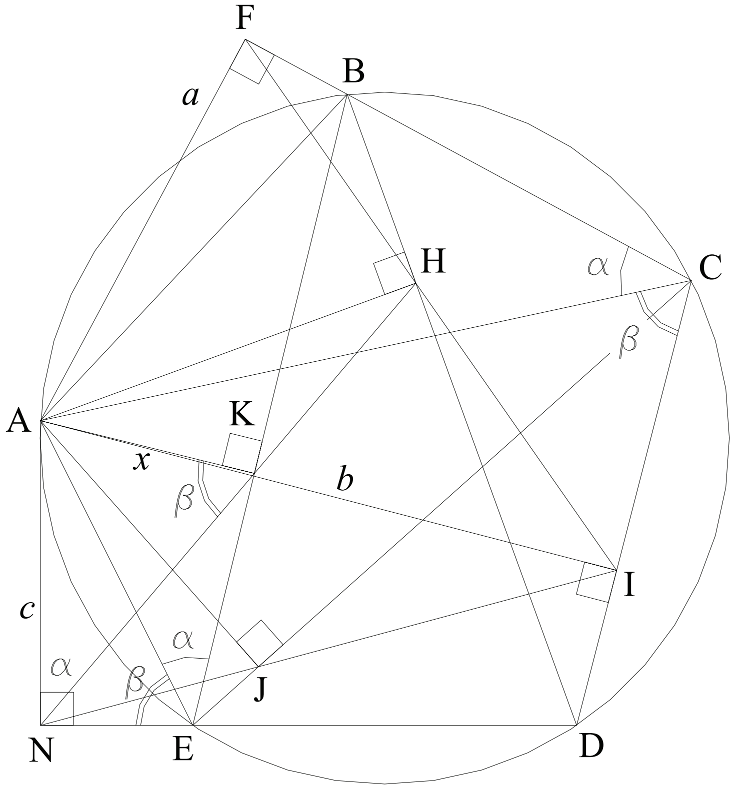
Combining (i) and (ii), we obtain

$\frac{OA}{OQ} = \frac{CN}{MH} = \frac{BN}{PH} = \frac{CN + BN}{MH + PH} = \frac{BC}{MP}$, or $MP \times OA = BC \times OQ$.

Problem 6 of Austria Mathematical Olympiad 1990

A convex pentagon ABCDE is inscribed in a circle. The distances of A from the lines BC, CD, DE are a, b, c, respectively. Compute the distance of A from the line BE.

Solution

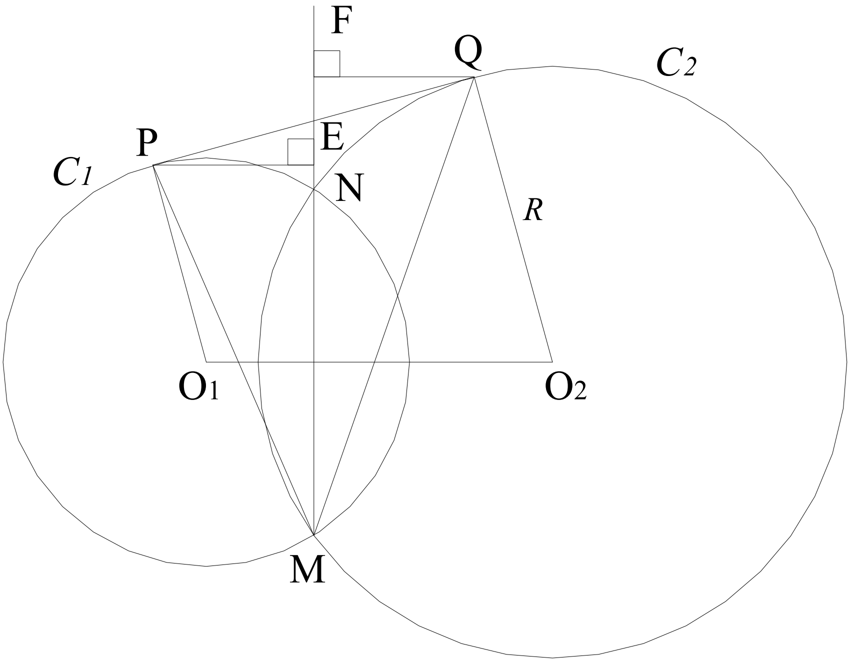


Let F, I, N, K be the feet of A onto BC, CD, DE and BE, respectively, $\alpha = \angle AEK = \angle ANK$ (because ANEK is cyclic) $= \angle ACB$ (subtends same small arc AB), $\beta = \angle AKN = \angle AEN$ (also because ANEK is cyclic) $= \angle ACD$ (subtends same arc AED), AK $= x$ which is the unknown we need to find. The law of sine gives us $\dfrac{x}{\sin\alpha} = \dfrac{c}{\sin\beta}$, or $x = \dfrac{c\sin\alpha}{\sin\beta}$. But in the right ΔACF and ΔACI, $\sin\alpha = \dfrac{AF}{AC} = \dfrac{a}{AC}$, $\sin\beta = \dfrac{AI}{AC} = \dfrac{b}{AC}$, or $x = \dfrac{ac}{b}$.

Problem 6 of the British Mathematical Olympiad 2000

Two intersecting circles C_1 and C_2 have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. Prove that the triangles MNP and MNQ have equal areas.

Solution



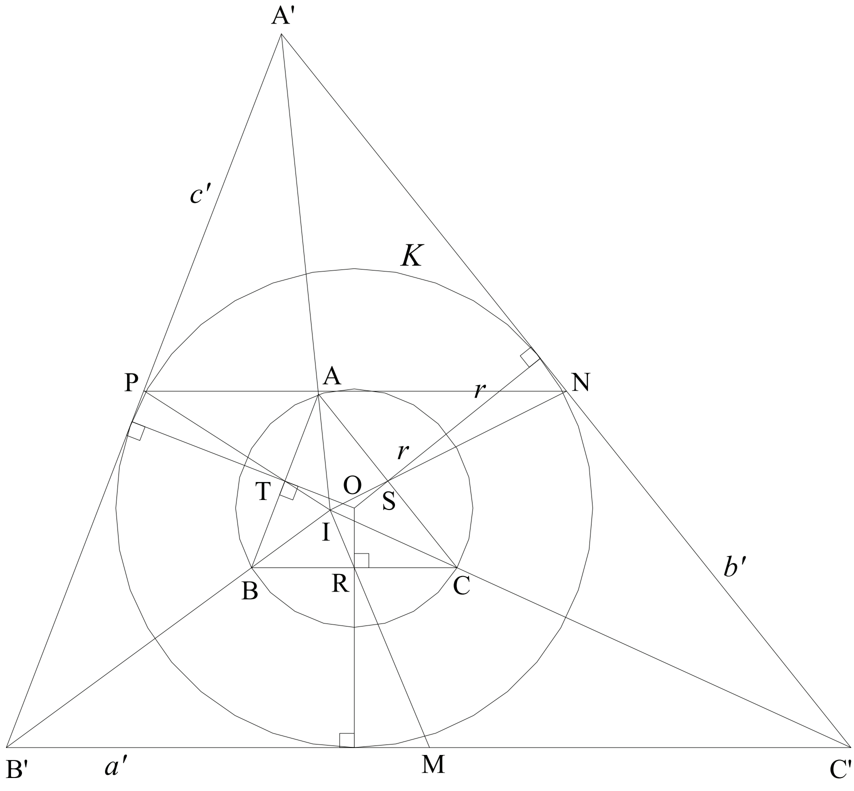
Extending MN to meet PQ at I, From P and Q draw the perpendicular lines to MI to meet it at E and F, respectively.

We have $IP^2 = IN \times M = IQ^2$, or IP = IQ. The two triangles PIE and QIF have all their respective angles equaled and also have the same length for sides IP = IQ; therefore, they are congruent.
As a result PE = QF. The two triangles MNP and MNQ have equal areas because they have the equal altitude PE = QF dropping down to the same base MN.

Problem 3 of Austria Mathematical Olympiad 2001

We are given a triangle ABC and its circumcircle with center O and radius *r*. Let *K* be the circle with midpoint O and radius 2*r*, and let *c*' be the tangent to *K* that is parallel to *c* = AB and has the property that C lies between *c* and *c*'. Analogously, the tangents *a*' and *b*' are determined. The resulting triangle with sides *a*', *b*', *c*' is called triangle A'B'C'. Prove that the lines joining the midpoints of corresponding sides of the triangles ABC and A'B'C' pass through a common point.

Solution



Since AB || A'B', BC || B'C' and AC || A'C', the two triangles ABC and A'B'C' are similar which gives us

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} \qquad \text{(i)}$$

Link A’A, B’B, and let their extensions meet at I. Let T and P be the midpoints of AB and A’B’, respectively. The three points I, T and P are, therefore, collinear.

And we then have $\frac{IB}{IB'} = \frac{AB}{A'B'}$.

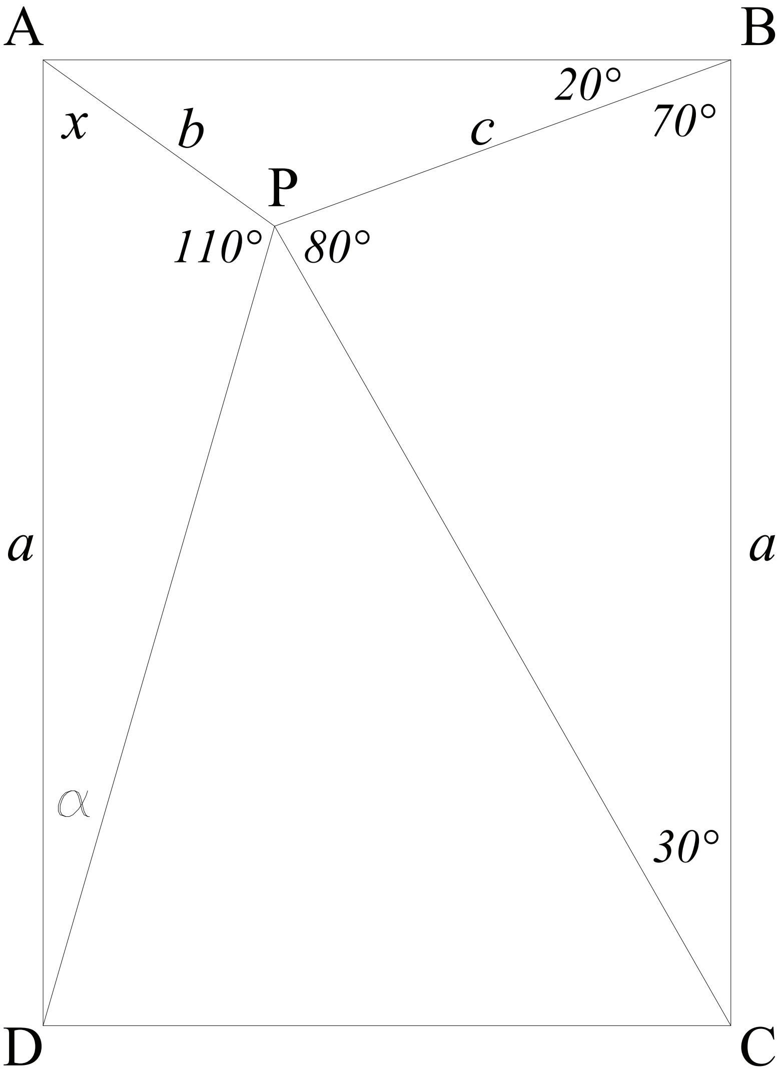
Combining with (i), we now have $\frac{IB}{IB'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$, or the three points I, C and C’ collinear. Therefore, if R, M, S and N are the midpoints of BC, B’C’, AC and A’C’, respectively, I, R, and M are collinear. The same conclusion can also be made for the three points I, S and N.

Thus the lines joining the midpoints of corresponding sides of the triangles ABC and A’B’C’ pass through a common point I.

Problem at Art Of the Problem Solving website 2011

There is a point P inside a rectangle ABCD such that $\angle$APD = 110°, $\angle$PBC = 70°, $\angle$PCB = 30°. Find $\angle$PAD.

Solution



Let a = AD = BC, b = AP, c = BP, $\angle$PAD = x, $\angle$ADP = α.

Applying the law of sines for ΔADP, we get $\dfrac{a}{\sin 110°} = \dfrac{b}{\sin\alpha}$ (i)

and for triangle BCP, $\dfrac{a}{\sin 80°} = \dfrac{c}{\sin 30°}$, or $\dfrac{c}{a} = \dfrac{1}{2\sin 80°}$ (ii)

But for triangle ABP, $\dfrac{b}{\sin 20°} = \dfrac{c}{\sin(90° - x)} = \dfrac{c}{\cos x}$, and equation (i) becomes $\dfrac{a}{\sin 110°} = \dfrac{c\sin 20°}{\sin\alpha\cos x}$, or $\cos x = \dfrac{c}{a} \times \dfrac{\sin 20° \sin 110°}{\sin\alpha}$.

Substituting the ratio $\dfrac{c}{a}$ from (ii) and $\sin 110° = \cos 20°$ to the previous equation, we get $\cos x = \dfrac{1}{2\sin 80°} \times \dfrac{\sin 40°}{2\sin\alpha} = \dfrac{1}{4}$

$\times \dfrac{\sin 40°}{2\sin\alpha \sin 40° \cos 40°} = \dfrac{1}{8} \times \dfrac{1}{\sin\alpha\cos 40°}$, or

$\sin\alpha = \dfrac{1}{8} \times \dfrac{1}{\cos 40° \cos x}$ and

$\cos\alpha = \dfrac{1}{8} \times \dfrac{1}{\cos 40° \cos x} \sqrt{64\cos^2 40° \cos^2 x - 1}$

On the other hand in triangle ADP, $\cos x = \cos(180° - \angle APD - \alpha)$ $= \cos(70° - \alpha) = \cos 70° \cos\alpha + \sin 70° \sin\alpha$.

Substituting $\sin\alpha$ and $\cos\alpha$ above into this equation, we get

$64\cos^2 40° \cos^4 x - 16\cos 40°(4\cos^2 70° \cos 40° + \sin 70°)\cos^2 x + 1 = 0.$

The only acceptable angle is $x = 53.92°$.

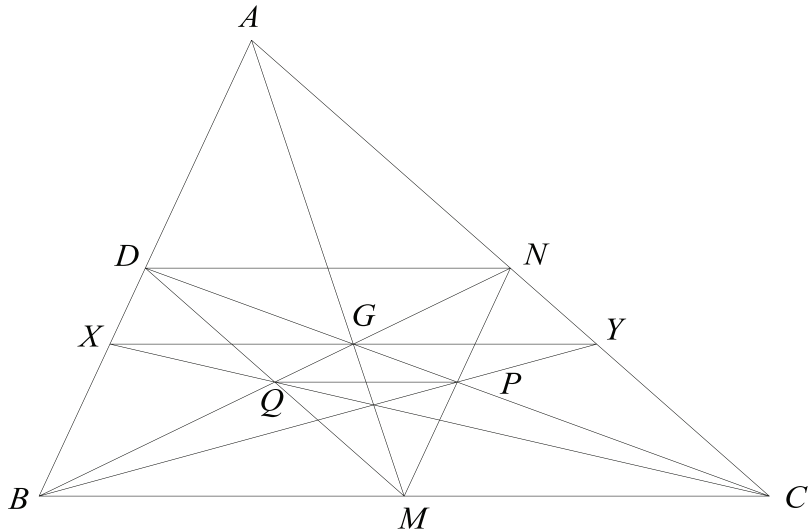
<u>Further observation</u>

This method applies to any angle measurements for all the given angles in the problem.

Problem 1 of the Asian Pacific Mathematical Olympiad 1991

Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and Y B and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Solution



Let N and D be midpoints of AC and AB, respectively. Since XY || BC, we have YC/YN = GB/GN, or (YC/YN)×(GN/GB) = 1. Also because M is midpoint of BC, we have (MB/MC)×(YC/YN)×(GN/GB) = 1.

Per Ceva's theorem, the three segments MN, GC and BY are then concurrent and meet at Q. Since MN || AB and D is midpoint of AB, P is then midpoint of MN. We have MP || AB.

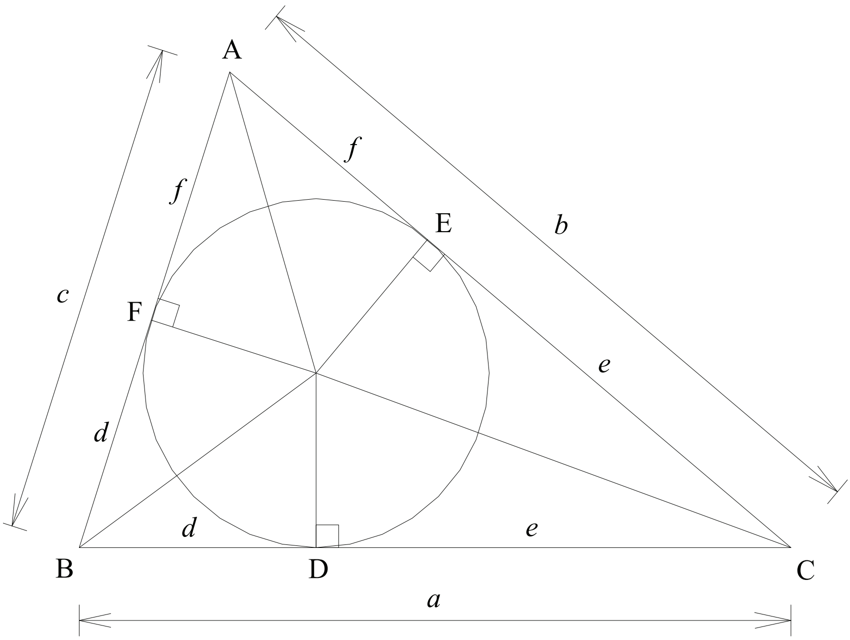
With the same argument, Q is midpoint of MD and MQ || AC and PQ || DN. In addition with DN || BC, we have PQ || BC.

Triangle MPQ has the three sides parallel to those of triangle ABC; therefore, they are similar.

Problem 1 of the Asian Pacific Mathematical Olympiad 1992

A triangle with sides a, b, and c is given. Denote by s the semi-perimeter, that is $s = \dfrac{a+b+c}{2}$. Construct a triangle with sides $s - a$, $s - b$, and $s - c$. This process is repeated until a triangle can no longer be constructed with the side lengths given. For which original triangles can this process be repeated indefinitely?

Solution



Draw the incircle of triangle ABC to tangent with the sides BC, AC and AB at D, E and F, respectively.

Let BC = a, AC = b, AB = c,
BD = BF = d,
CD = CE = e,
AE = AF = f.

We have $s = \dfrac{a+b+c}{2} = d + e + f = a + f = b + d = c + e$.
So now we have $s - a = f$, $s - b = d$, and $s - c = e$

For the three sides to form a non-degenerate triangle, the sum of any two has to be greater than the third. So we must have

$f + d = c > e$, $f + e = b > d$, or $d + e = a > f$.

For $c > e$, $\Rightarrow c > s - c \Rightarrow 2c > s \Rightarrow 4c > a + b + c \Rightarrow$
$3c > a + b$.

Similarly, for $b > d \Rightarrow 3a > b + c$, and $a > f \Rightarrow 3b > a + c$.

If one of those conditions is met, the process can be repeated, and the triangle can be constructed.

To construct the triangle draw a segment with the length of distance e; the ends of this segment are the vertices of the triangle that is under construction. Then from each end draw the circles with radii of d and f. These two circles intercept at another vertex of the triangle.

If the original triangle is equilateral, it will meet those conditions indefinitely since for an equilateral triangle $a = b = c$ and $d = e = f$ making the subsequent triangle also equilateral and the process keeps repeating forever.

Problem 1 of the British Mathematical Olympiad 2008

Find all solutions in non-negative integers a, b to $\sqrt{a} + \sqrt{b} =$

$\sqrt{2009}$.

Solution 1

Squaring both sides, we get $a + b + 2\sqrt{ab} = 2009$; rearranging and squaring them again, we have $(a + b - 2009)^2 = 4ab$, or $a^2 - 2(b + 2009)a + b^2 + 2009^2 - 4018b = 0$. Solving for a, we obtain $a = b +$

$2009 \pm \sqrt{2 \times 4018b} = b + 2009 \pm 14\sqrt{41b}$.

For a to be an integer, $41b$ has to be the square of an integer, or $b = 41n^2$ where n is an integer. Now $a = 41n^2 + 2009 \pm 14 \times 41n = 41n^2 + 2009 \pm 574n$.
Note that a or b can not exceed 2009 and must not be negative, we have the following solutions when $n = 1, 2, 3, 4, 5, 6$ and 7.

$(b, a) = (41, 41\times36), (41\times4, 41\times25), (41\times9, 41\times16), (41\times16, 41\times9), (41\times25, 41\times4), (41\times36, 41)$, and $(41\times49, 0)$, and since $\sqrt{a}$ and $\sqrt{b}$ are commutative, another series of solutions are

$(a, b) = (41, 41\times36), (41\times4, 41\times25), (41\times9, 41\times16), (41\times16, 41\times9), (41\times25, 41\times4), (41\times36, 41)$, and $(41\times49, 0)$.

Solution 2

Let's write $\sqrt{a} + \sqrt{b} = \sqrt{2009}$ as $\sqrt{a} + \sqrt{b} = 7\sqrt{41}$. From here $\sqrt{a}$

takes on the values 0, $\sqrt{41}$, $2\sqrt{41}$, $3\sqrt{41}$, $4\sqrt{41}$, $5\sqrt{41}$, $6\sqrt{41}$

and $7\sqrt{41}$ whereas $\sqrt{b}$ takes on the corresponding values $7\sqrt{41}$,

$6\sqrt{41}$, $5\sqrt{41}$, $4\sqrt{41}$, $3\sqrt{41}$, $2\sqrt{41}$, $\sqrt{41}$ and 0.

The same results as above are drawn.

Problem 1 of the Canadian Mathematical Olympiad 1969

Show that if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ and p_1, p_2, p_3 are not all zero, then $(\frac{a_1}{b_1})^n$ $= \frac{p_1 a_1^{\ n} + p_2 a_2^{\ n} + p_3 a_3^{\ n}}{p_1 b_1^{\ n} + p_2 b_2^{\ n} + p_3 b_3^{\ n}}$ for every positive integer n.

Solution

Adding $\frac{a_1}{b_1}$ ratio to the left onto the already existing equation $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$ to get $\frac{a_1}{b_1} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

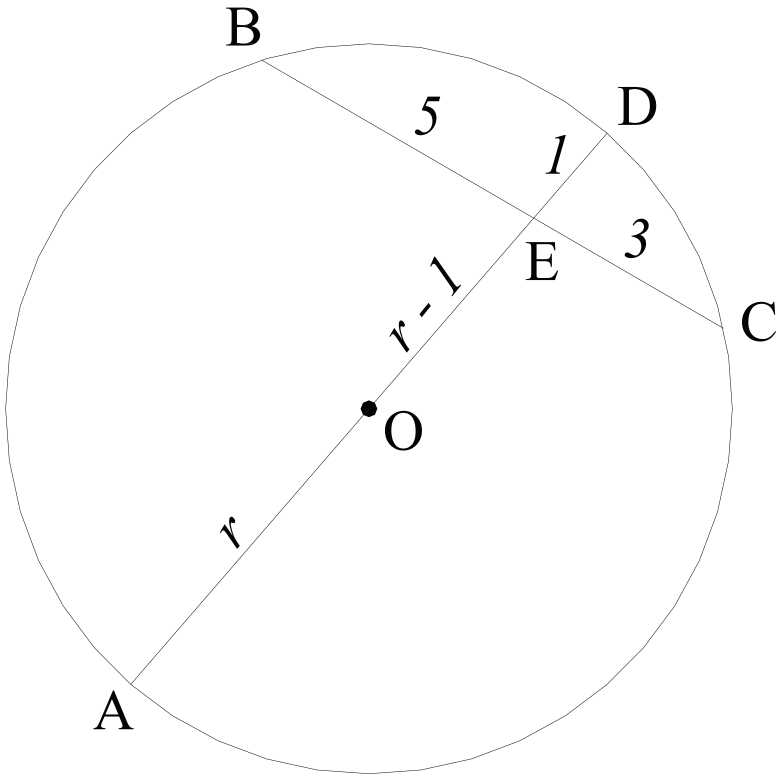
Now raising to the n power for all, we get
$(\frac{a_1}{b_1})^n = (\frac{a_1}{b_1})^n = (\frac{a_2}{b_2})^n = (\frac{a_3}{b_3})^n$.

Multiply both sides of different ratios with equal numbers p's
$(\frac{a_1}{b_1})^n = \frac{p_1 a_1^{\ n}}{p_1 b_1^{\ n}} = \frac{p_2 a_2^{\ n}}{p_2 b_2^{\ n}} = \frac{p_3 a_3^{\ n}}{p_3 b_3^{\ n}} = \frac{p_1 a_1^{\ n} + p_2 a_2^{\ n} + p_3 a_3^{\ n}}{p_1 b_1^{\ n} + p_2 b_2^{\ n} + p_3 b_3^{\ n}}$.

Problem 1 of the Canadian Mathematical Olympiad 1971

DEB is a chord of a circle such that DE = 3 and EB = 5. Let O be the center of the circle. Join OE and extend OE to cut the circle at C. Given EC = 1, find the radius of the circle.

Solution



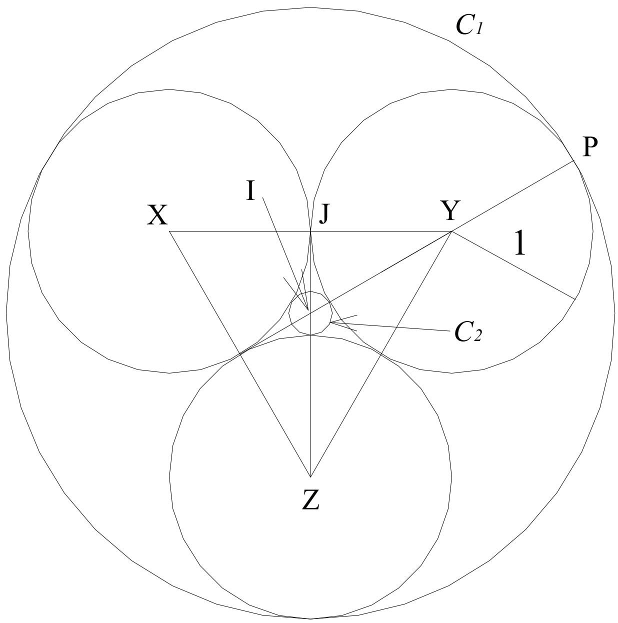
Extend CO to intercept the circle at A. Let r be the radius of the circle; we get OE = $r - 1$, OA = r.

Applying the intersecting chord theorem to get BE×EC = AE×ED, or $15 = (2r - 1)\times 1$, or $r = 8$.

<u>Problem 1 of the Canadian Mathematical Olympiad 1972</u>

Given three distinct unit circles, each of which is tangent to the other two, find the radii of the circles which are tangent to all three circles.

<u>Solution</u>



The three distinct unit circles are congruent with radii equal to 1. It's easily seen that the three centers X, Y and Z make an equilateral triangle since its lengths XY = YZ = ZX = 1.
The same point incircle, centroid and circumcenter I of this triangle will be the centers of the two circles which tangent to all three circles. For the larger circle C_1 that tangents all three circle, its radius is R = IP = YP + IY = 1 + IY. But IY = IZ = 2/3 altitude of triangle XYZ = 2/3 ZJ = $2\sqrt{ZY^2 - JY^2}/3 = 2/\sqrt{3}$, so $R = 1 + 2/\sqrt{3}$. For the small C_2 that tangents all three circle, its radius is $r = R - 2YP = 1 + 2/\sqrt{3} - 2 = 2/\sqrt{3} - 1$.

Problem 1 of the Canadian Mathematical Olympiad 1975

Simplify

$$\sqrt[3]{\frac{1.2.4 + 2.4.8 + \ldots + n.2n.4n}{1.3.9 + 2.6.18 + \ldots + n.3n.9n}}$$

Solution

We have

$1.2.4 + 2.4.8 + \ldots + n.2n.4n = 2.4\,(1^3 + 2^3 + 3^3 + \ldots + n^3)$ and

$1.3.9 + 2.6.18 + \ldots + n.3n.9n = 3.9\,(1^3 + 2^3 + 3^3 + \ldots + n^3)$,

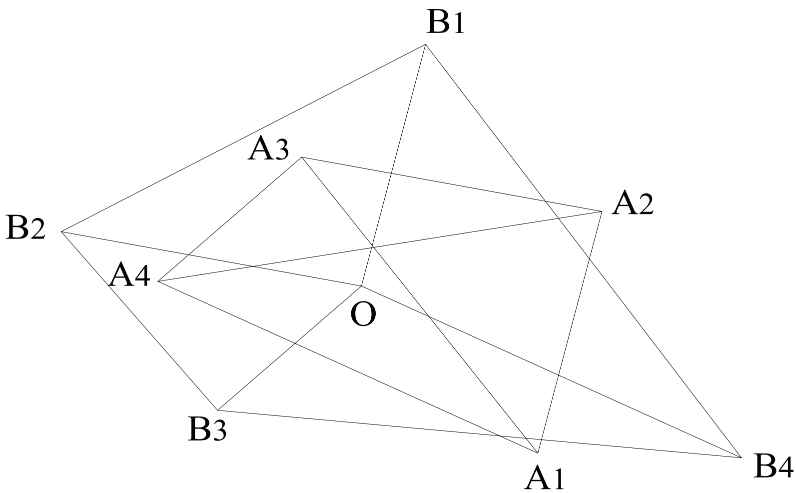
and the ratio becomes $\frac{2.4}{3.9} = \frac{2^3}{3^3}$, or

$$\sqrt[3]{\frac{1.2.4 + 2.4.8 + \ldots + n.2n.4n}{1.3.9 + 2.6.18 + \ldots + n.3n.9n}} = \frac{2}{3}.$$

<u>*Problem 1 of Canadian Mathematical Olympiad 1982*</u>

In the diagram, OBi is parallel and equal in length to A_iA_{i+1} for $i = 1, 2, 3$ and 4 ($A_5 = A_1$). Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.

<u>Solution</u>



Let (Ω) denote the area of shape Ω. If we move the triangle OB_1B_2 with $B_2 \rightarrow A_3$ and $O \rightarrow A_2$, $(OB_1B_2) = (A_1A_2A_3)$ since they have the equal base $A_1A_2 = OB_1$ and the same altitude from A_3 (or B_2 after the move).

We will see the same effect if we move triangle OB_3B_4 ($B_4 \rightarrow A_1$ and $O \rightarrow A_4$), $(OB_3B_4) = (A_1A_3A_4)$.
Now adding the two areas

$$(A_1A_2A_3) + (A_1A_3A_4) = (OB_1B_2) + (OB_3B_4) \qquad \text{(i)}$$

Next, move the triangle $A_1A_2A_4$ ($A_1 \rightarrow O$ and $A_2 \rightarrow B_1$),

$(A_1A_2A_4) = (OB_1B_4)$, and move the triangle $A_2A_3A_4$ ($A_3 \rightarrow O$ and $A_4 \rightarrow B_3$), $(A_2A_3A_4) = (OB_2B_3)$.

Adding the previous two areas

$$(A_1A_2A_4) + (A_2A_3A_4) = (OB_1B_4) + (OB_2B_3) \qquad \text{(ii)}$$

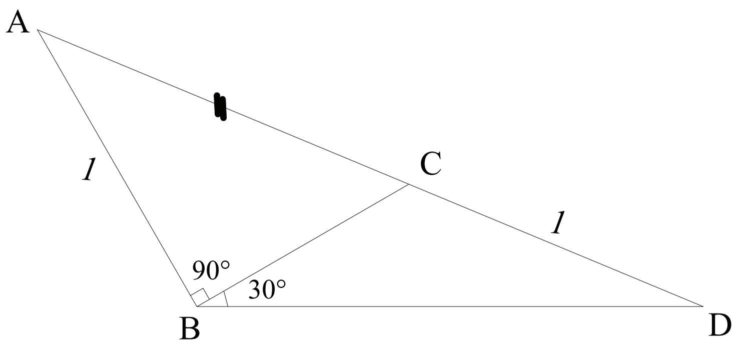
Now adding the sides of (i) and (ii) to get

$$2(A_1A_2A_3A_4) = (B_1B_2B_3B_4).$$

Problem 1 of the Canadian Mathematical Olympiad 1986

In the diagram line segments AB and CD are of length 1 while angles ABC and CBD are 90° and 30°, respectively. Find AC.

Solution



We have $AC^2 = 1 + BC^2$ and $\dfrac{BC}{\sin\angle D} = \dfrac{CD}{\sin 30°} = 2$, or

$\sin\angle D = \dfrac{1}{2}BC$. We also have $\dfrac{AC + 1}{\sin 120°} = \dfrac{AB}{\sin\angle D} = \dfrac{1}{\sin\angle D} = \dfrac{2}{BC}$.

or $\dfrac{AC + 1}{\sqrt{3}} = \dfrac{1}{\sqrt{AC^2 - 1}}$, or $(AC + 1)^3(AC - 1) = 3$, or

$AC^4 + 2AC^3 - 2AC + AC^2 - 4 = 0$, or

$AC(AC^3 - 2) + 2(AC^3 - 2) = 0$, or

$(AC^3 - 2)(AC + 2) = 0$, but $AC + 2 > 0$.

Therefore, $AC^3 - 2 = 0$, and $AC = \sqrt[3]{2}$.

Further observation

This problem is the same as problem 3 of the Irish Mathematical Olympiad 2010.

Problem 1 of the Irish Mathematical Olympiad 2007

Let r, s and t be the roots of the cubic polynomial

$$p(x) = x^3 - 2007x + 2002$$

Determine the value of $\frac{r-1}{r+1} \cdot \frac{s-1}{s+1} \cdot \frac{t-1}{t+1}$.

Solution

Expanding

$$\frac{r-1}{r+1} \cdot \frac{s-1}{s+1} \cdot \frac{t-1}{t+1} = \frac{3rst - 3 + rt + st + rs - s - r - t}{rst + rt + st + rs + s + r + t + 1} \qquad \text{(i)}$$

Since r, s and t are the roots, we can write $p(x)$ as
$(x - r)(x - s)(x - t) = x^3 - (s + r + t)x^2 + (rt + st + rs)x + rst =$
$x^3 - 2007x + 2002$, or
$s + r + t = 0$, $rt + st + rs = -2007$, and $rst = -2002$.

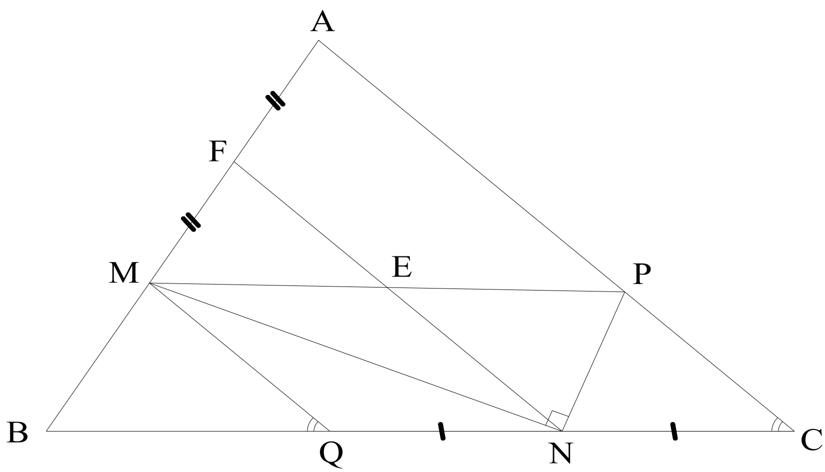
Substituting them into (i), we have

$$\frac{r-1}{r+1} \cdot \frac{s-1}{s+1} \cdot \frac{t-1}{t+1} = \frac{-3 \times 2002 - 3 - 2007}{-2002 - 2007 + 1} = \frac{-8016}{-4008} = 2.$$

Problem 1 of Romanian Mathematical Olympiad 2006

Let ABC be a triangle and the points M and N on the sides AB and BC, respectively, such that 2CN/BC = AM/AB. Let P be a point on the line AC. Prove that the lines MN and NP are perpendicular if and only if PN is the interior angle bisector of $\angle$MPC.

Solution



a) Assume MN and NP are perpendicular.

Since $2\times\frac{CN}{BC}=\frac{AM}{AB}$, pick point Q on BC such that QN = CN and MQ || AC. Let E and F be the midpoints of MP and MA, respectively. We have EF || AC but N is also the midpoint of QC and MQ || AC; therefore, FN || AC and F, E and N are collinear. We then get $\angle$ENP = $\angle$NPC.

But EN = EP = EM (E is midpoint of MP and $\angle$MNP is right angle) causing $\angle$ENP = $\angle$EPN, or $\angle$EPN = $\angle$NPC, and PN is the interior bisector of $\angle$MPC.

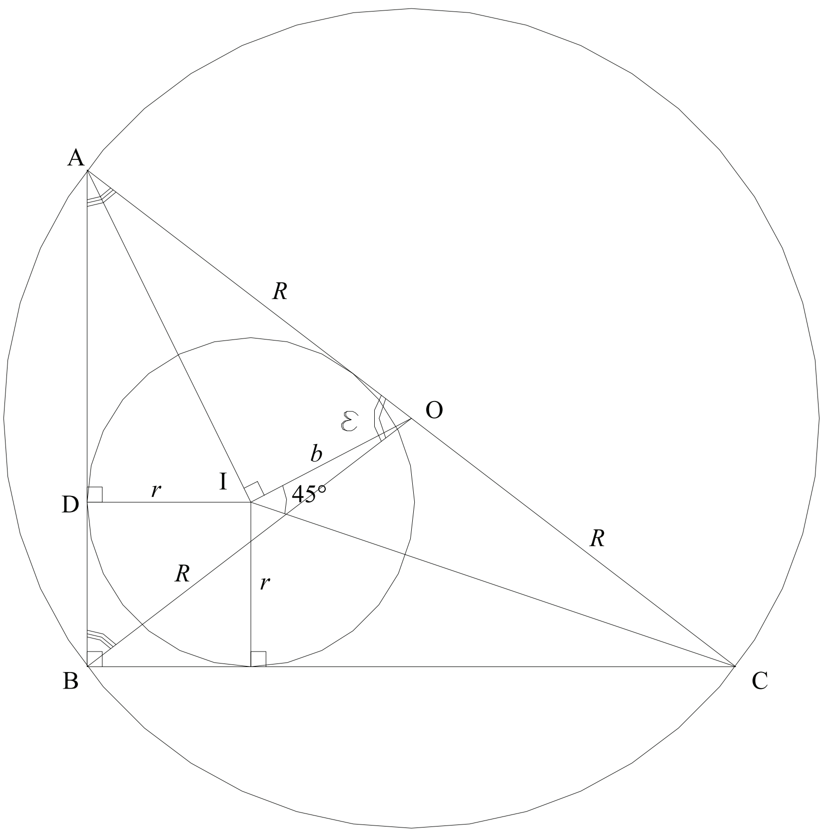
b) Assume PN is interior bisector of $\angle$MPC.

$\angle$EPN = $\angle$NPC and since MQ || AC and F and N are the midpoints of MA and QC, respectively, we have FN || AC. Therefore, $\angle$FNP = $\angle$NPC, and E is the midpoint of MP. It follows that $\angle$EPN = $\angle$ENP and EN = EP = EM or $\angle$MNP = 90° and MN is orthogonal to NP.

<u>Problem 2 of the British Mathematical Olympiad 2007</u>

Let triangle ABC have incenter I and circumcenter O. Suppose that $\angle AIO = 90°$ and $\angle CIO = 45°$. Find the ratio AB : BC : CA.

<u>Solution</u>



Let the incircle tangent AB at D and r be its radius, $\alpha = \frac{1}{2}\angle A$, $\beta = \frac{1}{2}\angle C$. We have $\alpha + \beta + \angle B = \angle AIC = 90° + 45° = 135°$ and $\angle A + \angle B + \angle C = 180°$, or $\angle B = 90°$ and the circumcenter O of triangle ABC is the midpoint of AC. Now let $\frac{1}{2}AC = OA = OC = OB = R$ be the radius of the circumcircle, $OI = b$ and $\angle AOB = \varepsilon$.

Applying the law of the sines for triangle OIC, we obtain $\frac{R}{\sin 45°} = \frac{b}{\sin\beta}$, but in triangle AOI, $R = \frac{b}{\sin\alpha}$, and the previous expression becomes $\frac{R}{\sin 45°} = \frac{R\sin\alpha}{\sin\beta}$, or $\sin\alpha = \sqrt{2}\sin\beta$ (i)

We also have $\alpha + \beta = 45°$, or $\sin(\alpha + \beta) = \sin 45°$.

Now expand it, $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{\sqrt{2}}{2}$.

Substituting $\sin\alpha$ from (i), we have

$\sin\alpha\cos(45° - \alpha) + \frac{\sqrt{2}}{2}\cos\alpha\sin\alpha = \frac{\sqrt{2}}{2}$, or

$\sin\alpha(\cos\alpha + \sin\alpha) + \cos\alpha\sin\alpha = 1$, or

$2\sin\alpha\cos\alpha = \cos^2\alpha$, or $2\sin\alpha = \cos\alpha$, or

$\tan\alpha = \frac{1}{2} = \frac{\mathrm{DI}}{\mathrm{AD}}$, but $\mathrm{DI} = r$, and $\mathrm{AD} = 2r$, $\mathrm{AI} = r\sqrt{5}$, $\cos\angle\mathrm{A} = \cos^2\alpha - \sin^2\alpha = (\frac{\mathrm{AD}}{\mathrm{AI}})^2 - (\frac{\mathrm{DI}}{\mathrm{AI}})^2 = \frac{3}{5}$. However, $\cos\angle\mathrm{A} = \frac{\mathrm{AB}}{\mathrm{CA}} = \frac{3}{5}$.

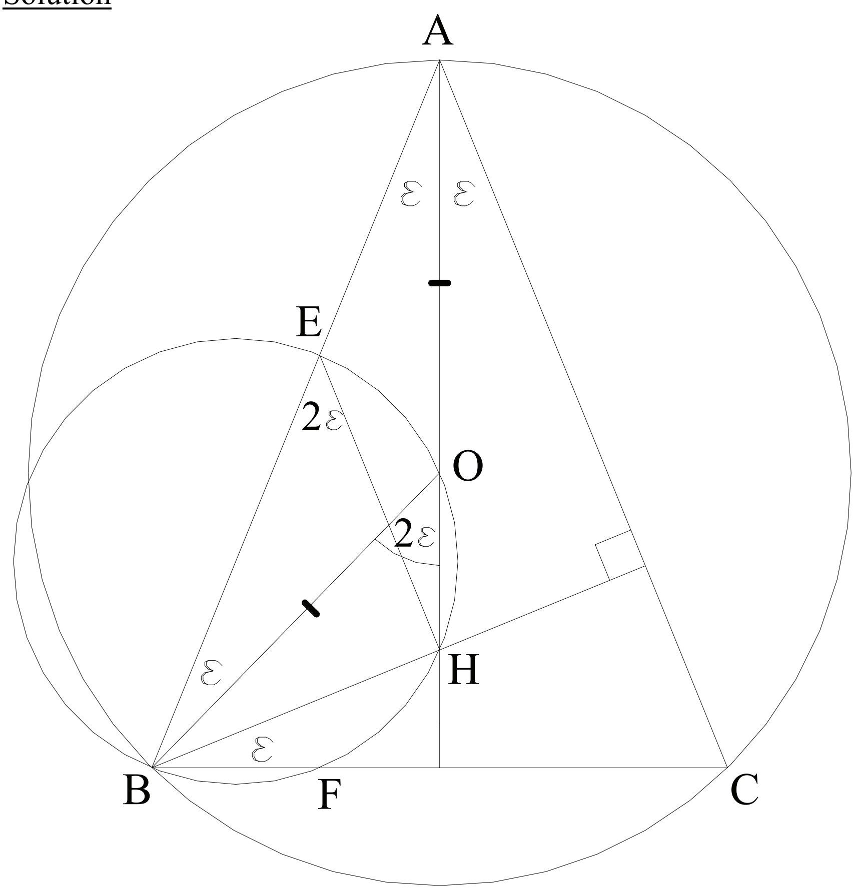
Now applying the Pythagorean formula $\mathrm{CA}^2 = \mathrm{AB}^2 + \mathrm{BC}^2$, we have $\mathrm{CA}^2 = (\frac{3}{5})^2\times\mathrm{CA}^2 + \mathrm{BC}^2$, or $\frac{\mathrm{BC}}{\mathrm{CA}} = 0.8$.

Finally, AB : BC : CA = 3 : 4 : 5.

Problem 2 of the British Mathematical Olympiad 2008

Let ABC be an acute-angled triangle with $\angle B = \angle C$. Let the circumcenter be O and the orthocenter be H. Prove that the center of the circle BOH lies on the line AB. The circumcenter of a triangle is the center of its circumcircle. The orthocenter of a triangle is the point where its three altitudes meet.

Solution



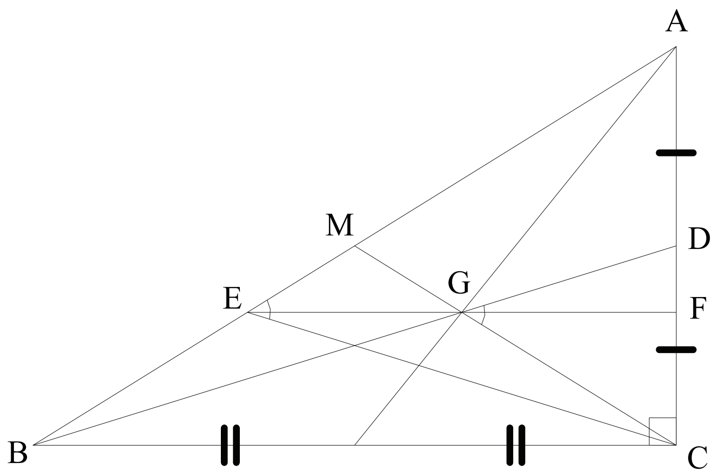
Let $\varepsilon = \angle HAB$, we then also have $\varepsilon = \angle HAC = \angle ABO$ (since O is center of circle), and $\angle BOH = \angle BAO + \angle ABO = 2\varepsilon$. Now let the circumcircle of triangle BOH intercept AB at E.
We then have $\angle BEH = \angle BOH = 2\varepsilon$ (these angles subtend the same arc BH)

Since $\angle$HBC + $\angle$ACB = 90° = $\angle$HAC + $\angle$ACB, we have $\angle$HBC = $\angle$HAC = ε, and $\angle$HAB + $\angle$ABC = 90° or $\angle$HAB + $\angle$ABO + $\angle$OBH + $\angle$HBC = 3ε + $\angle$HBO = 90° = $\angle$EBO + $\angle$HBO + $\angle$BEH, or $\angle$EBH + $\angle$BEH = 90°, and $\angle$BHE = 180° − $\angle$EBH − $\angle$BEH = 90°, or BE is the diameter of the circle BOH, or the center of the circle BOH lies on the line AB.

Problem 2 of the British Mathematical Olympiad 2009

In triangle ABC the centroid is G and D is the midpoint of CA. The line through G parallel to BC meets AB at E. Prove that $\angle AEC = \angle DGC$ if, and only if, $\angle ACB = 90°$. The centroid of a triangle is the intersection of the three medians, the lines which join each vertex to the midpoint of the opposite side.

Solution



If $\angle ACB = 90°$

Let EG intercept AC at F. Since EF || BC, $\angle AEF = \angle ABC$, $\angle DGF = \angle DBC$, and $\angle FEC = \angle ECB$.
We have $\angle AEC = \angle AEF + \angle FEC = \angle ABC + \angle ECB$

But since M is the midpoint of AB and $\angle ACB = 90°$, M is also the center of the circumcircle of triangle ABC and AM = MC = MB and $\angle ABC = \angle MCB$

Therefore, $\angle AEC = \angle ABC + \angle ECB = \angle MCB + \angle ECB$
But since EG || BC and MB = MC, we have $\angle MBG = \angle MCE$, or $\angle ECB = \angle DBC$, or $\angle AEC = \angle MCB + \angle ECB = \angle MCB + \angle DBC = \angle DGC$,

<u>If $\angle$AEC = $\angle$DGC</u>

We have $\angle$AEC = $\angle$ABC + $\angle$ECB = $\angle$EBG + $\angle$GBC + $\angle$ECB, and $\angle$DGC = $\angle$GBC + $\angle$GCB = $\angle$GBC + $\angle$GCE + $\angle$ECB, or $\angle$EBG = $\angle$GCE, or EGCB is cyclic.

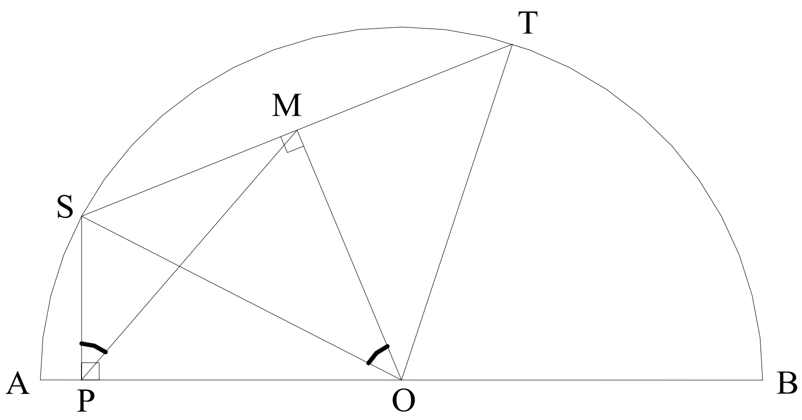
Combining with EG || BC, we have EB = GC, and EGCB is an isosceles trapezoid, and $\angle$EBC = $\angle$GCB, or MBC is an isosceles triangle and MB = MC = MA.

Therefore, $\angle$MBC = $\angle$MCB and $\angle$MAC = $\angle$MCA, or $\angle$MCB + $\angle$MCA = ½180° = 90°.

Problem 3 of Canadian Mathematical Olympiad 1986

A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

Solution



As the chord ST slides around AB, we note that triangle SOT rotates around O, and its shape remains constant, so is $\angle$SOM. Since M is the midpoint of ST, SM = MT.

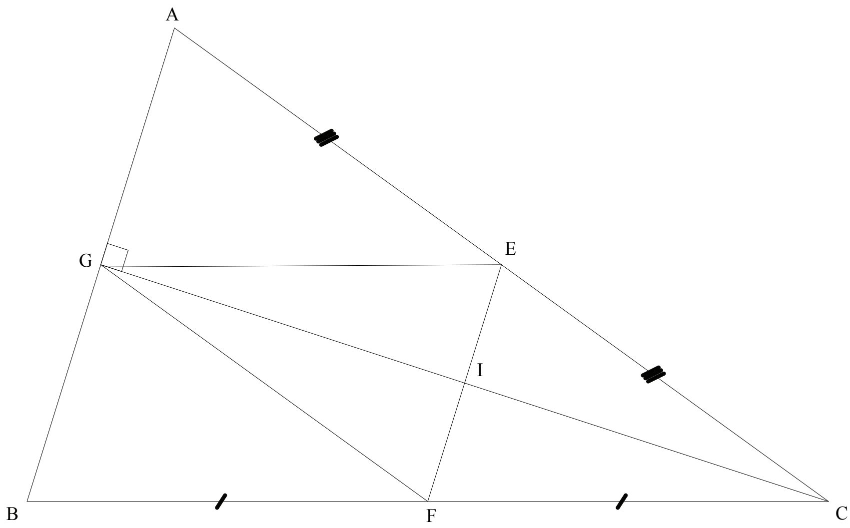
We also have OS = OT equals the radius of the semicircle. The two triangles SOM and TOM are congruent because of their three sides are equal. Therefore, $\angle$SMO = $\angle$TMO = ½180° = 90°.

So the quadrilateral SPOM is cyclic since $\angle$SPO + $\angle$SMO = 180°. Therefore, $\angle$SPM = $\angle$SOM is constant.

Problem 4 of Austria Mathematical Olympiad 2008

In a triangle ABC let E be the midpoint of the sides AC and F the midpoint of the side BC. Furthermore let G be the foot of the altitude through C on the side AB (or its extension). Show that the triangle EFG is isosceles if and only if ABC is isosceles.

Solution



Let CG intercept EF at I. We only solve the problem with one geometrical configuration. Other configurations can also be solved similarly.

First assume that the triangle EFG is isosceles and GE = GF, $\angle$GEF = $\angle$GFE.

Since E and F are the midpoints of AC and BC, respectively, EF || AB, $\angle$GEF = $\angle$AGE and $\angle$GFE = $\angle$BGF. Combining with CG $\perp$ AB, we have $\angle$EGC = $\angle$FGC.

The two triangles EGC and FGC are then congruent since they also share GC. It follows that EC = FC and AC = BC or triangle ABC is isosceles.

Now assume triangle ABC is isosceles, AC = BC and EC = FC, $\angle$BAC = $\angle$ABC.

Since $\angle$AGC = $\angle$BGC = 90°, the two triangles AGC and BGC are congruent and AG = BG which leads us to $\angle$ACG = $\angle$BCG. The two triangles EGC and FGC are then congruent since they also share GC.

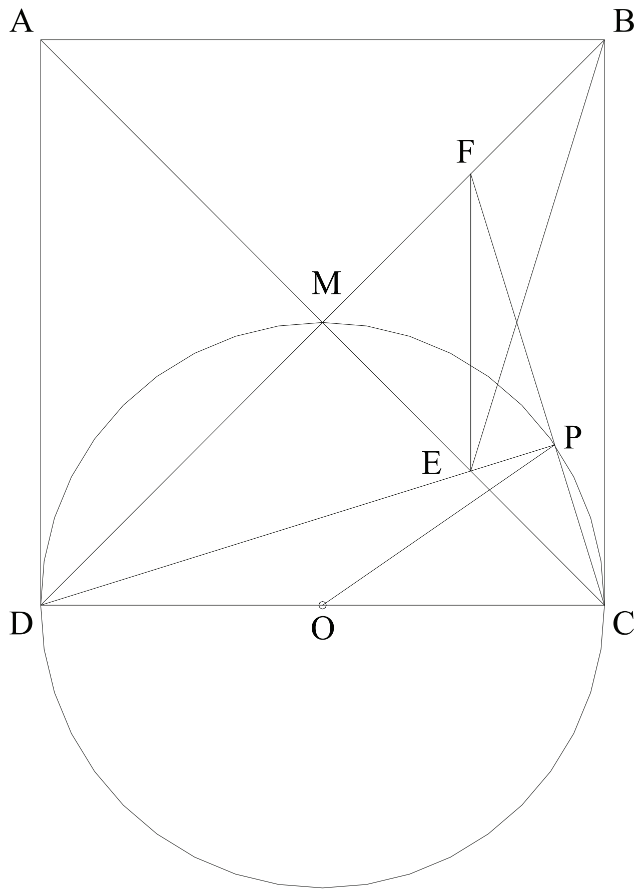
It follows that EG = FG and triangle EFG is also isosceles.

Problem 6 of Austria Mathematical Olympiad 2008

We are given a square ABCD. Let P be different from the vertices of the square and from its center M. For a point P for which the line PD intersects the line AC, let E be this intersection. For a point P for which the line PC intersects the line DB, let F be this intersection. All those points P for which E and F exist are called acceptable points.

Determine the set of acceptable points for which the line EF is parallel to AD.

Solution



Link EB. Since ABCD is a square and AC is the perpendicular bisector of BD, and E is on AC, ED = EB and $\angle$EDB = $\angle$EBD.

Furthermore, since EF || BC, EFBC is an isosceles trapezoid and $\angle$ECF = $\angle$EBD, or $\angle$ECF = $\angle$EDB.

We also have $\angle$FCB + $\angle$ECF = 45°, or
$\angle$FCB + $\angle$EDB = 45°, or
$\angle$FCB = $\angle$EDC, and
FC $\perp$ DP, or $\angle$DPC = 90°.

So all the acceptable points form a circle with center O being the midpoint of DC and the diameter equals to the side length of the square ABCD.

Problem 6 of Australia Mathematical Olympiad 2010

Prove that $\sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}}=$ $\sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}$.

Solution

Observe that $845 = 13^2\times5$, $325 = 5^2\times13$,
$847 = 11^2\times7$, $539 = 7^2\times11$,
$245 = 7^2\times5$, $175 = 5^2\times7$,
$1859 = 13^2\times11$, $1573 = 11^2\times13$

and $6+\sqrt[3]{845}+\sqrt[3]{325} = (\sqrt[3]{\frac{13}{3}})^3+3(\sqrt[3]{\frac{13}{3}})^2\times\sqrt[3]{\frac{5}{3}}+3\sqrt[3]{\frac{13}{3}}\times(\sqrt[3]{\frac{5}{3}})^2+(\sqrt[3]{\frac{5}{3}})^3=[\sqrt[3]{\frac{13}{3}}+\sqrt[3]{\frac{5}{3}}]^3$,

or $\sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}=\sqrt[3]{\frac{13}{3}}+\sqrt[3]{\frac{5}{3}}$.

Similarly, $\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}}=\sqrt[3]{\frac{11}{3}}+\sqrt[3]{\frac{7}{3}}$,

$$\sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}=\sqrt[3]{\frac{7}{3}}+\sqrt[3]{\frac{5}{3}},$$

$$\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}=\sqrt[3]{\frac{13}{3}}+\sqrt[3]{\frac{11}{3}}.$$

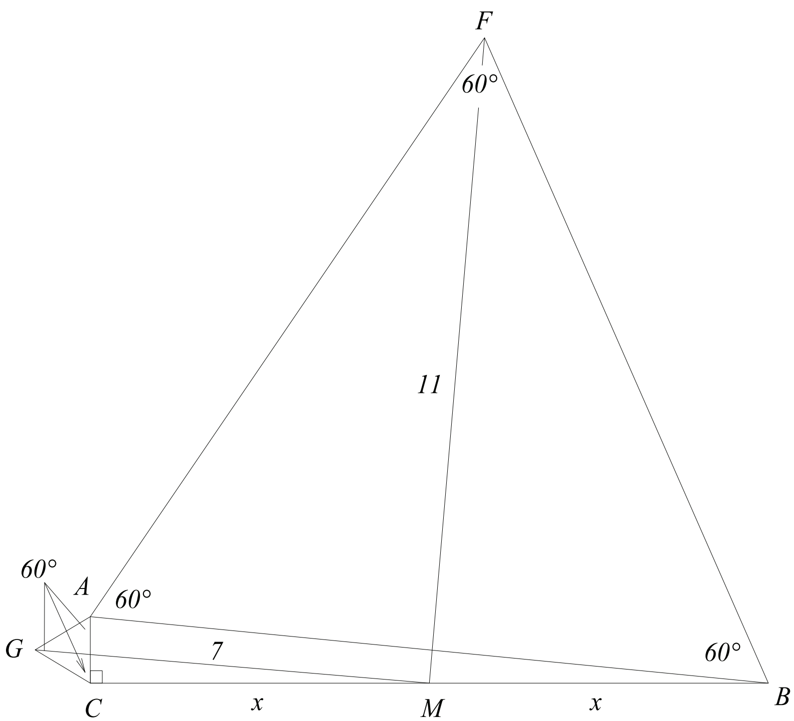
Therefore, $\sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}}=$

$$\sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}.$$

Problem 6 of Belarus Mathematical Olympiad 2000

The equilateral triangles ABF and CAG are constructed in the exterior of a right-angled triangle ABC with $\angle C = 90^\circ$. Let M be the midpoint of BC. Given that MF = 11 and MG = 7, find the length of BC.

Solution



Let $x = \dfrac{BC}{2}$, AB = b and AC = c.

Applying the law of cosines, we have
$GM^2 = x^2 + c^2 - 2xc \times \cos(\angle ACB + 60^\circ)$, or
$GM^2 = x^2 + c^2 - 2xc \times \cos 150^\circ$, and $FM^2 = x^2 + b^2 - 2xb \times \cos(\angle ABC + 60^\circ)$.

Expanding those two equations with MG = 7 and MF = 11,

$\cos(\angle ABC + 60°) = \cos\angle ABC \times \cos 60° - \sin\angle ABC \times \sin 60°$,

$\cos 60° = \frac{1}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$, and observing the

Pythagorean theorem, we have $49 = c^2 + x^2 + xc\sqrt{3}$.

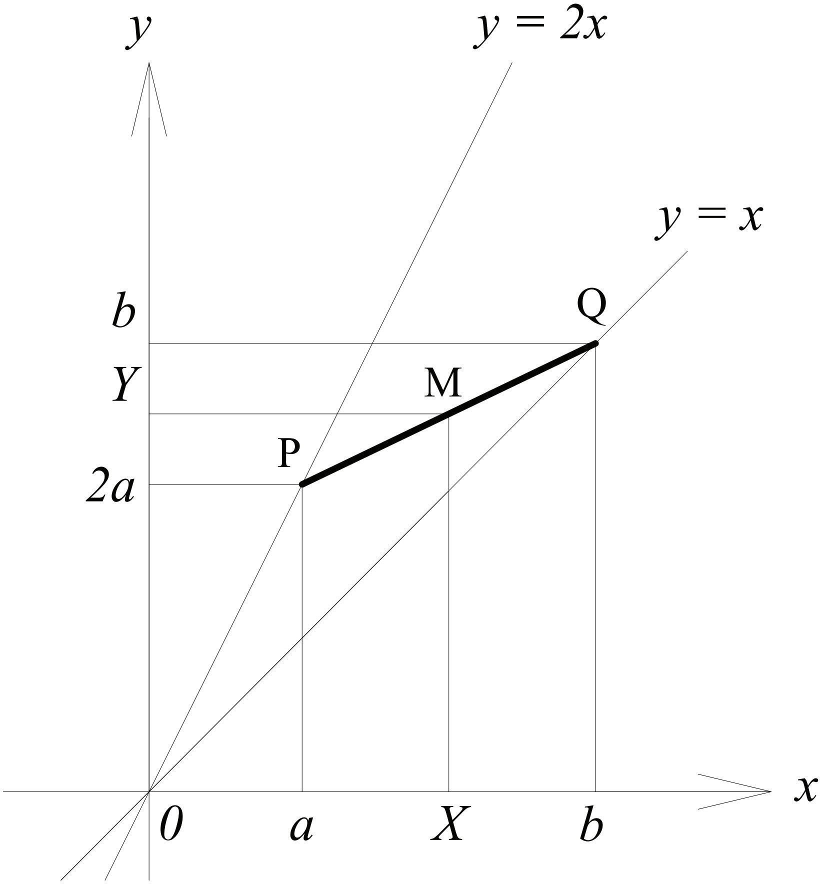
$121 = b^2 - x^2 + xc\sqrt{3}$.
$b^2 = c^2 + 4x^2$.

Solving for x, we obtain $x = 6$ or BC = 12.

Problem 8 of the Canadian Mathematical Olympiad 1970

Consider all line segments of length 4 with one end-point on the line $y = x$ and the other end-point on the line $y = 2x$. Find the equation of the locus of the midpoints of these line segments.

Solution



Let the segment be PQ, P on $y = 2x$ and Q on $y = x$, the midpoint of PQ be M. We have PQ = 4. Because point Q is on line $y = x$, its coordinates is Q(b, b), and the coordinates of P is P(a, $2a$) because it's on $y = 2x$ line. To use the least numbers of unknowns possible, let's pick the half segment MQ for the calculation. We have

$$MQ^2 = (b - X)^2 + (b - Y)^2 = (\frac{PQ}{2})^2 = 4 \qquad \text{(i)}$$

Besides, $X = \frac{a+b}{2}$, or $\frac{a}{2} = X - \frac{b}{2}$, and $Y = \frac{2a+b}{2} = X + \frac{a}{2} = 2X - \frac{b}{2}$ or $\frac{b}{2} = 2X - Y$, or $b = 4X - 2Y$.

Substituting b into (i), we have $13Y^2 - 36XY + 25X^2 - 4 = 0$, or

$$Y = \frac{18X}{13} \pm \frac{\sqrt{52 - X^2}}{13}.$$

When $X \leq 0$, $Y = \frac{18X}{13} + \frac{\sqrt{52 - X^2}}{13}$.

When $X > 0$, $Y = \frac{18X}{13} - \frac{\sqrt{52 - X^2}}{13}$.

Note that the locus only goes from point $N(\sqrt{52}, \frac{18\sqrt{52}}{13})$ to

$N'(-\sqrt{52}, -\frac{18\sqrt{52}}{13})$.

Problem 2 of the Canadian Mathematical Olympiad 1971

Let x and y be positive real numbers such that $x + y = 1$. Show that $(1 + \frac{1}{x})(1 + \frac{1}{y}) \geq 9$.

Solution

Applying the AM-GM inequality, we have $x + y \geq 2\sqrt{xy}$ (i)
or $(x + y)^2 \geq 4xy$, or

$1 \geq 4xy$, or $xy \geq 4x^2y^2$, or $\sqrt{xy} \geq 2xy$, or $4\sqrt{xy} \geq 8xy$.

Since $x + y = 1$, (i) can also be written as $1 \geq 2\sqrt{xy}$, or $2 \geq 4\sqrt{xy} \geq 8xy$.

$x + y + 1 \geq 8xy$, or $\frac{x + y + 1}{xy} \geq 8$.

$\frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \geq 8$.

$1 + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy} \geq 9$, or $(1 + \frac{1}{x})(1 + \frac{1}{y}) \geq 9$.

<u>Problem 2 of the Canadian Mathematical Olympiad 1973</u>

Find all the real numbers which satisfy the equation $|x + 3| - |x - 1| = x + 1$. (Note: $|a| = a$ if $a \geq 0$; $|a| = -a$ if $a < 0$.)

<u>Solution</u>

For $x \in (-\infty, -3]$ (including point -3), the equation can be written as $-x - 3 + x - 1 = x + 1$, or $x = -5$.

For $x \in (-3, 1]$ (including point 1), the equation can be written as $x + 3 + x - 1 = x + 1$, or $x = -1$.

For $x \in (1, +\infty)$ (excluding point 1), the equation can be written as $x + 3 - x + 1 = x + 1$, or $x = 3$.

All the real numbers which satisfy the equation $|x + 3| - |x - 1| = x + 1$ are -5, -1 and 3.

Problem 2 of the Canadian Mathematical Olympiad 1969

Determine which of the two numbers $\sqrt{c+1} - \sqrt{c}$, $\sqrt{c} - \sqrt{c-1}$ is greater for any $c \geq 1$.

Solution

Since $c \geq 1$, we always have $c > \sqrt{c^2 - 1}$, or $2c > 2\sqrt{c^2 - 1}$, or

$4c > 2c + 2\sqrt{c^2 - 1} = (\sqrt{c+1} + \sqrt{c-1})^2$, or $2\sqrt{c} > \sqrt{c+1} + \sqrt{c-1}$,

or $\sqrt{c} - \sqrt{c-1} > \sqrt{c+1} - \sqrt{c}$.

Problem 2 of the Auckland Mathematical Olympiad 2009

Is it possible to write the number $1^2 + 2^2 + 3^2 + \ldots + 12^2$ as a sum of 11 distinct squares?

Solution

We note that $5^2 + 12^2 = 25 + 144 = 169 = 13^2$ and
$1^2 + 2^2 + 3^2 + \ldots + 12^2 = 1^2+ 2^2+ 3^2+ 4^2+ 6^2+ 7^2+ 8^2+ 9^2+ 10^2+ 11^2+ 13^2$ which is the sum of 11 distinct squares.

Also note that $3^2 + 4^2 = 5^2$, the expression can be written as a sum of 10 distinct squares $1^2 + 2^2 + 3^2 + \ldots + 12^2 = 1^2 + 2^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 13^2$.

Problem 3 of Austria Mathematical Olympiad 2004

In a trapezoid ABCD with circumcircle K the diagonals AC and BD are perpendicular. Two circles Ka and Kc are drawn whose diameters are AB and CD respectively.

Calculate the circumference and the area of the region that lies within the circumcircle K, but outside of the circles Ka and Kc.

Solution

Let (Ω) denote the area of shape **Ω**, AC intercept BD at I. Since AB || CD and ABCD is cyclic and also a trapezoid, it is an isosceles trapezoid. Triangles ABD and ABC are then congruent and $\angle ABD = \angle BAC$, AD = BC.

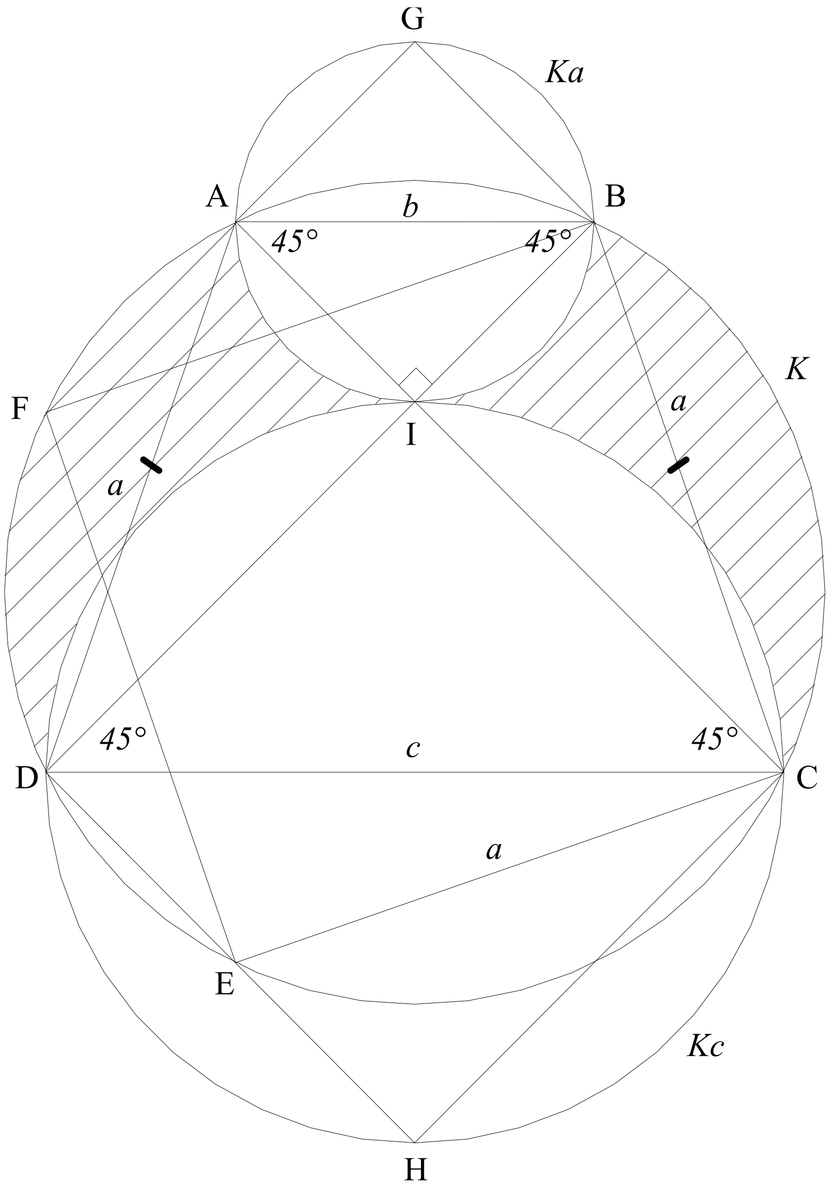
Let a = AD = BC, b = AB, c = CD. Because AC $\perp$ BD, $\angle BAC = \angle ABD = \angle BDC = \angle ACD = 45°$.

Since $\angle BAC = 45°$ it subtends a quarter of the circumference of circle K or arc BC = arc AB = ¼(π×diameter of K).

Similarly, $\angle BAC = \angle ABI = 45°$ and each subtends a quarter of the circumference of circle Ka, and $\angle IDC = \angle ICD = 45°$ and each subtends a quarter of the circumference of circle Kc.

Construct squares BCEF, AIBG and DICH with E and F on circle K, G on circle Ka and H on circle Kc, respectively. BE is the diameter of K. We then have $BE^2 = 2a^2$, or $BE = a\sqrt{2}$.

The circumference in question = ½ circumference of K + ½ circumference of Ka + ½ circumference of Kc = ½ the sum of all circumferences of three circles = $\frac{1}{2}(a\pi\sqrt{2} + b\pi + c\pi) = \pi(a\sqrt{2} + b + c)$.



Applying Pythagorean's theorem, we have $a^2 = AI^2 + DI^2 = BI^2 + CI^2$ or $2a^2 = b^2 + c^2$, or $a\sqrt{2} = \sqrt{b^2 + c^2}$.

Therefore, the circumference is $\pi(b + c + \sqrt{b^2 + c^2})$.

Now the area of the region in question, let's call it A, has been shaded. One half of its area is equal to the area covered by BC and the smaller arc BD of circle *K*, this part equals to $\frac{1}{4}$ [the area of circle *K* – (FBCE)] + (BIC) – $\frac{1}{4}$ [the area of circle *Ka* – (AIBG)] – $\frac{1}{4}$ [the area of circle *Ka* – (CIDH)] = $\frac{1}{4}$ [(radius of *K*)$^2\times\pi - a^2$] + ½BI×CI – $\frac{1}{4}$ [(radius of *Ka*)$^2\times\pi$ – BI^2] – $\frac{1}{4}$ [(radius of *Kc*)$^2\times\pi$ – CI^2].

But we also have $2\mathrm{BI}^2 = b^2$, and $2\mathrm{CI}^2 = c^2$, or $\mathrm{BI} = \frac{b}{\sqrt{2}}$, and $\mathrm{CI} = \frac{c}{\sqrt{2}}$.

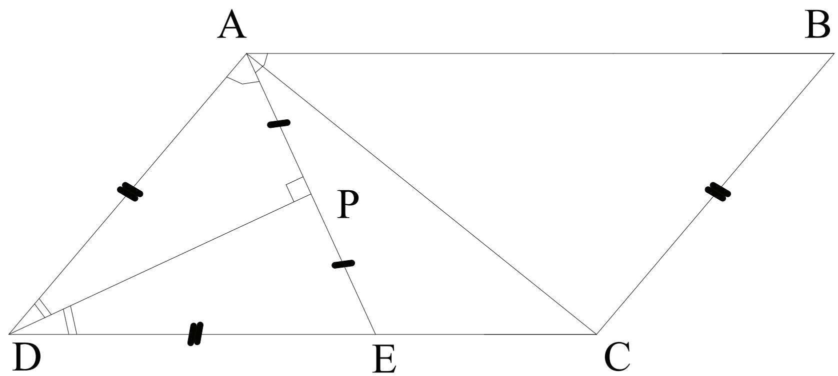
Therefore, half the area of the region is $\frac{1}{2}\mathrm{A} = \frac{1}{4}(\pi\frac{a^2}{2} - a^2) + \frac{bc}{4} - \frac{1}{4}(\pi\frac{b^2}{4} - \frac{b^2}{2}) - \frac{1}{4}(\pi\frac{c^2}{4} - \frac{c^2}{2}) = \frac{(\pi - 2)(2a^2 - b^2 - c^2)\ + 4bc}{16}$

But again $2a^2 = b^2 + c^2$, and $\frac{1}{2}\mathrm{A} = \frac{bc}{4}$, or $\mathrm{A} = \frac{bc}{2}$.

Problem 4 of Austria Mathematical Olympiad 2002

We are given three mutually distinct points A, C and P in the plane. A and C are opposite corners of a parallelogram ABCD, the point P lies on the bisector of the angle DAB, and the angle APD is a right angle. Construct all possible parallelograms ABCD that satisfy these conditions.

Solution



We know that point D is on the line Px perpendicular to AP at P. Since AP is the bisector of $\angle$DAB, $\angle$DAP = $\angle$BAP. The sum of two consecutive angles of a parallelogram is 180°, or $\angle$BAD + $\angle$ADC = 180°, or 2$\angle$DAP + $\angle$ADP + $\angle$PDC = 180°. But $\angle$DAP + $\angle$ADP = 90°; therefore, $\angle$DAP + $\angle$PDC = 90°.

It follows that $\angle$ADP = $\angle$PDC. Now extend AP a segment PE to equal itself AP = PE. Since the two right triangles DAP and DEP are congruent, point E is on DC. Now link and extend CE to meet the perpendicular line Px at D.

Link AD and draw other sides of the parallelogram ABCD.

Problem 3 of the Canadian Mathematical Olympiad 1977

N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

Solution

Let's write N = $7b^2 + 7b + 7 = 7(b^2 + b + 1) = n^4$,
Or $b^2 + b + 1 = 7^3 \times m^4$ where m is a positive integer.

The smallest positive integer b for which N is the fourth power of an integer is when $m = 1$, or $b^2 + b + 1 = 7^3$, or $b(b + 1) = 342$.

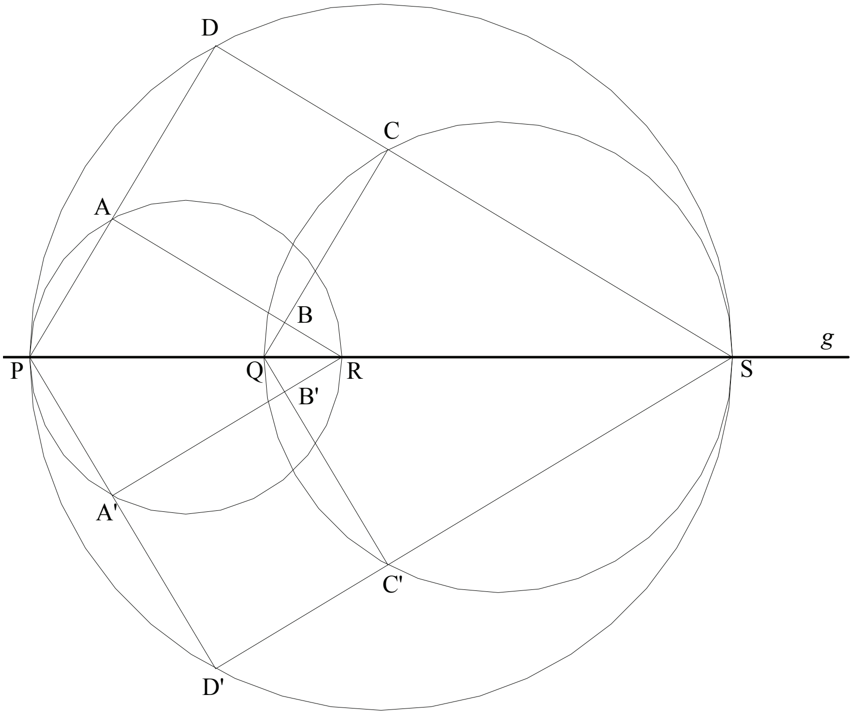
But $18 \times 19 = 342$, or $b = 18$, and then N = $7(18^2 + 18 + 1) = 7^4$.

Problem 3 of Austria Mathematical Olympiad 2008

The line g is given, and on it lie the four points P, Q, R, and S (in this order from left to right).

Construct all squares ABCD with the following properties:
P lies on the line through A and D.
Q lies on the line through B and C.
R lies on the line through A and B.
S lies on the line through C and D.

Solution



It is easily recognized that A must lie on the circle with diameter PR, D on circle with diameter PS, and C on circle with diameter QS. Let DS = d, CS = c, AP = a and DP = b.

For ABCD to be a square, AD || BC, DC || AB, and $b - a = d - c$.

We then have $\frac{d}{\mathrm{PS}} = \frac{c}{\mathrm{QS}} = \frac{d - c}{\mathrm{PS} - \mathrm{QS}} = \frac{\mathrm{CD}}{\mathrm{PS} - \mathrm{QS}}$ and $\frac{b}{\mathrm{PS}} = \frac{a}{\mathrm{PR}} = \frac{b - a}{\mathrm{PS} - \mathrm{PR}} = \frac{\mathrm{AD}}{\mathrm{PS} - \mathrm{PR}}$.

Now CD = AD gives us $\frac{d(\mathrm{PS} - \mathrm{QS})}{\mathrm{PS}} = \frac{b(\mathrm{PS} - \mathrm{PR})}{\mathrm{PS}}$, or

$d = \frac{b(\mathrm{PS} - \mathrm{PR})}{\mathrm{PS} - \mathrm{QS}}$.

We also have $b^2 + d^2 = \mathrm{PS}^2$. From those two equations, we have

$b^2 + \frac{b^2(\mathrm{PS} - \mathrm{PR})^2}{(\mathrm{PS} - \mathrm{QS})^2} = \mathrm{PS}^2$ or

$b^2 \times \frac{(\mathrm{PS} - \mathrm{QS})^2 + (\mathrm{PS} - \mathrm{PR})^2}{(\mathrm{PS} - \mathrm{QS})^2} = \mathrm{PS}^2$, or

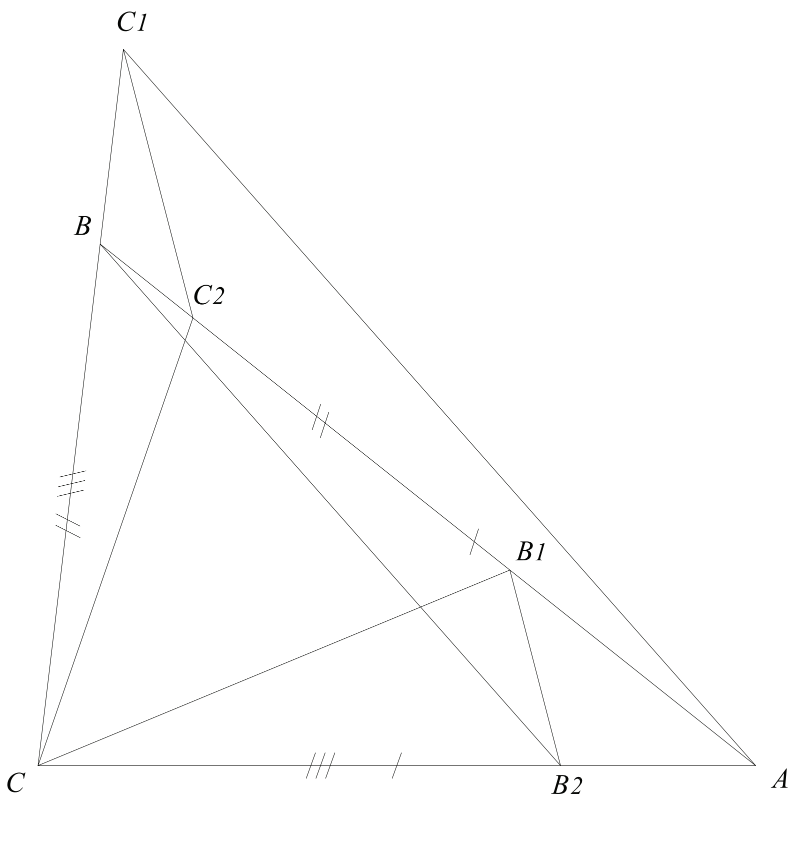
$b = \frac{\mathrm{PS}(\mathrm{PS} - \mathrm{QS})}{\sqrt{(\mathrm{PS} - \mathrm{QS})^2 + (\mathrm{PS} - \mathrm{PR})^2}}$.

The square is then defined. Its mirror image A'B'C'D' across g is also a solution.

Problem 7 of Belarus Mathematical Olympiad 2000

On the side AB of a triangle ABC with BC < AC < AB, points B_1 and C_2 are marked so that AC_2 = AC and BB_1 = BC. Points B_2 on side AC and C_1 on the extension of CB are marked so that CB_2 = CB and CC_1 = CA. Prove that the lines C_1C_2 and B_1B_2 are parallel.

Solution



Observe that BB_2 and C_1A are parallel as given by the problem which makes $\angle B_2BB_1 = \angle C_1AC_2$. It suffices to prove that the two triangles B_2BB_1 and C_1AC_2 are similar which makes $\angle C_2C_1A = \angle B_1B_2B$ and $C_1C_2 \parallel B_1B_2$.

Given $\angle B_2BB_1 = \angle C_1AC_2$ as mentioned, it only suffices to prove
$\frac{BB_2}{AC_1} = \frac{BB_1}{AC_2}$.

But since $BB_2 \parallel AC_1$, we have $\frac{BB_2}{AC_1} = \frac{CB}{CC_1}$.

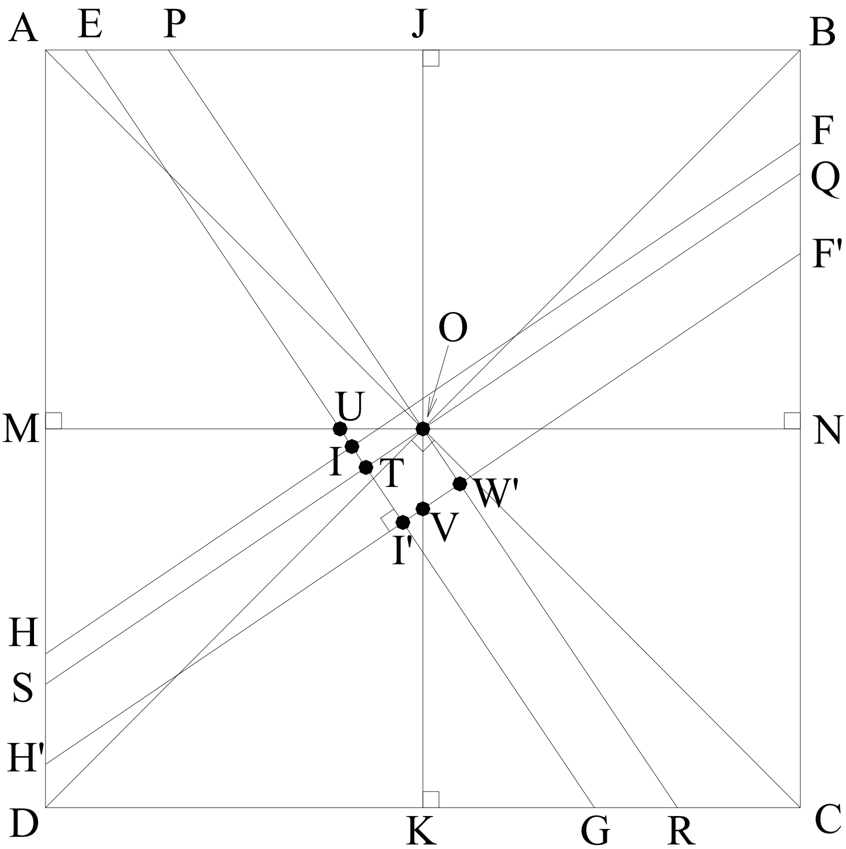
The problem also gives $CB = BB_1$ and $CC_1 = AC = AC_2$.

Therefore, $\frac{CB}{CC_1} = \frac{BB_1}{AC_2}$, or $\frac{BB_2}{AC_1} = \frac{BB_1}{AC_2}$, and the proof is complete.

<u>*Problem 2 of the Ibero-American Mathematical Olympiad 1991*</u>

Two perpendicular lines divide a square in four parts; three of them have area equal to 1. Show that the area of the full square is four.

<u>Solution</u>



Let the square be ABCD and the two perpendicular lines dividing it into four parts intercept each other at I and let them divide the square into AEIH, DHIG, CGIF and BFIE whereas the areas of AEIH, DHIG, CGIF are all equal to 1.

Divide the square into four equal squared quadrants AJOM, MOKD, JBNO and ONCK. We only prove the problem with the location of I in a single quadrant MOKD. Note that there are four possible distinct locations for I to cover all the scenarios of the problem:

(1) I is located above line SQ or on MO (I on line HF)
(2) I is located on segment SO (I coincides with T)
(3) I is located below line SQ (I on line H'F')

Case (1) I is located above line SQ or on MO (I on line HF)

From O draw a line parallel to EG to intercept AB and DC at P and R, respectively and draw another line parallel to HF to intercept AD and BC at S and Q, respectively.

In two quadrilaterals APRD and CRPB, $\angle$APR = $\angle$CRP and mid-segments OM = ON; therefore, the two quadrilaterals APRD and CRPB are congruent and their areas are equal.

The two quadrilaterals DSQC and BQSA are congruent and their areas are equal.

These two perpendicular lines PR and SQ divide the square into four equal areas since AP= CR=DS=BQ. Now let SQ intercept EG at T, MN intercept EG at U and let (Ω) denotes the area of Ω, we have

(AEIH) = (APOS) – (EPOU) – (UOT) – (HITS) (i)
(HIGD) = (SORD) – (UORG) + (UOT) + (HITS) (ii)

Since (AEIH) = (HIGD), (APOS) = (SORD) and (EPOU) = (UORG), subtract (i) from (ii) we have (UOT) + (HITS) = 0. This is impossible; therefore, this case is not possible.

Case (2) I is located on SQ (I coincides T and H coincides S, F coincides Q)

(AETS) = (APOS) – (EPOU) – (UOT) (iii)
(STGD) = (SORD) – (UORG) + (UOT) (iv)

Subtract (iii) from (iv) we have (UOT) = 0; this is also impossible; therefore, this case is also not possible.
Case (3) I is located below line SQ (I on line H'F')

Let V and W be the intersections of H'F' and OK, and H'F' and PR, respectively. We have

(H'I'GD) = (SORD) – (SOVH') – (VOW) – (GI'WR) (v)
(CGI'F') = (CROQ) – (VOQF') + (VOW) + (GI'WR) (vi)

Since (H'I'GD) = (CGI'F'), (SORD) = (CROQ) and (SOVH') = (VOQF'), subtracting (v) from (vi) yields (VOW) + (GI'WR) = 0 which is also impossible; therefore, this case is also not possible.

Therefore, the only prevailing scenario is for I to coincide with O and the four parts to be equal as proven earlier and the total area is 3 + 1 = 4.

Further observation

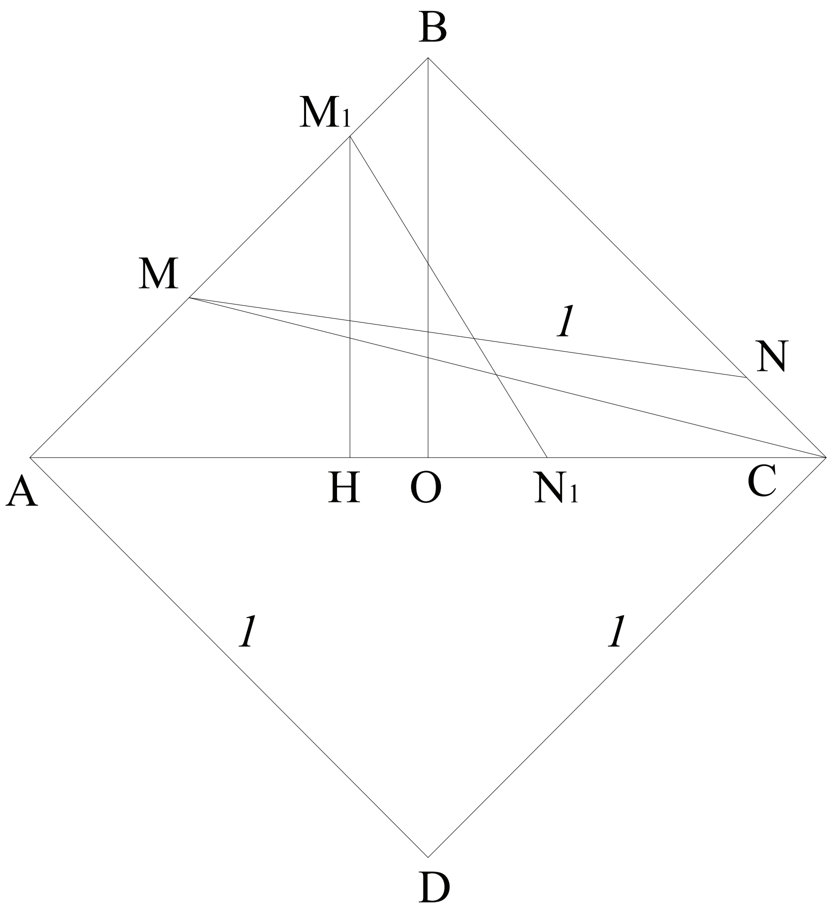
The problem below is derived from the above problem:

Two perpendicular lines divide a square into four parts, three of them have equal areas. Prove that all four parts have equal areas.

<u>Problem 2 of the British Mathematical Olympiad 1993</u>

A square piece of toast ABCD of side length 1 and center O is cut in half to form two equal pieces ABC and CDA. If the triangle ABC has to be cut into two parts of equal area, one would usually cut along the line of symmetry BO. However, there are other ways of doing this. Find, with justification, the length and location of the shortest straight cut which divides the triangle ABC into two parts of equal area.

<u>Solution</u>



There are two ways that we can consider. One way is to cut across the two lines AB and BC. The other is to cut through AB and AC.

a) For the former way, let's use letters M and N on the figure. Let's denote (Ω) the area of the shape Ω. For the two areas to be equal, (MBN) = ½(ABC) = $\frac{1}{4}$, or BM×BN = $\frac{1}{2}$.

But $MN^2 = BM^2 + BN^2 = BM^2 + \frac{1}{4BM^2}$.

The derivative of $MN^2 = [BM^2 + \frac{1}{4BM^2}]' = 2BM - ½BM^{-3} = 0$

when $4BM^4 - 1 = 0$, or when $BM = \frac{1}{\sqrt[4]{4}} = 0.707$.

So then BN = 0.707, and MN = 1.

b) For the latter way, we denote M_1 and N_1 in the figure.

Cut through AB at M_1 and AC at N_1. Let H be the foot of M_1 onto AC. We have

$\frac{M_1H}{BO} = \frac{AM_1}{AB}$, or $M_1H\sqrt{2} = AM_1$.

And the area of AM_1N_1 is $(AM_1N_1) = \frac{1}{4}$, or $AN_1 \times M_1H = \frac{1}{2}$, or

$AN_1 \times AM_1/\sqrt{2} = \frac{1}{2}$, or $AN_1 \times AM_1 = \frac{1}{\sqrt{2}}$, or $AN_1 = \frac{1}{\sqrt{2}AM_1}$.

Now applying the law of cosines, we have

$M_1N_1^2 = AN_1^2 + AM_1^2 - 2\times AN_1 \times AM_1 \cos 45° = AM_1^2 + \frac{1}{2AM_1^2} - 1.$

To find the minimum value of M_1N_1, we can take the derivative of its square, and it is

$(M_1N_1^2)' = 2AM_1 - \frac{1}{AM_1^3} = 0$ when $AM_1 = \frac{1}{\sqrt[4]{2}} = 0.84$.

And the value of the square of M_1N_1 is

$M_1N_1^2 = 0.41$, or $M_1N_1 = 0.64$.

The value of M_1N_1 in this case is smaller than its previous value of 1 in the previous case, so the shortest straight cut which divides the triangle ABC into two parts of equal area has M_1 on AB at a distance of 0.84 away from A, and $M_1N_1 = 0.64$.

Problem 9 of the Irish Mathematical Olympiad 1998

The year 1978 was “peculiar" in that the sum of the numbers formed with the first two digits and the last two digits is equal to the number formed with the middle two digits, i.e., 19 + 78 = 97. What was the last previous peculiar year, and when will the next one occur?

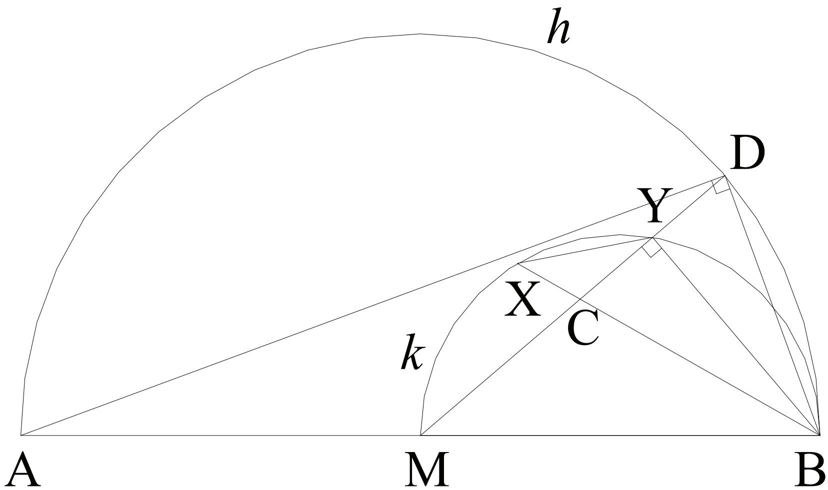
Solution

Let the “peculiar” year be *abcd* where *a, b, c*, and *d* are the integer digits from 0 to 9. We then have $10a + b + 10c + d = 10b + c$, or $a = \frac{9(b-c)-d}{10}$. Observe that a is an integer from 0 to 9; therefore, $9(b-c) - d$ must be divisible by 10. To satisfy this, we must have
$b - c = 0$, or $b = c$, $d = 0$, $a = 0$, or
$b - c = 1$, $d = 9$, $a = 0$, or
$b - c = 2$, $d = 8$, $a = 1$, or
$b - c = 3$, $d = 7$, $a = 2$, or
$b - c = 4$, $d = 6$, $a = 3$, or
$b - c = 5$, $d = 5$, $a = 4$, or
$b - c = 6$, $d = 4$, $a = 5$, or
$b - c = 7$, $d = 3$, $a = 6$, or
$b - c = 8$, $d = 2$, $a = 7$, or
$b - c = 9$, $d = 1$, $a = 8$.
The actual values for *abcd* are (Was 0000 used as a year?)
0000, 0110, 0220, 0330, 0440, 0550, 0660, 0770, 0880, 0990,
0109, 0219, 0329, 0439, 0549, 0659, 0769, 0879, 0989,
1208, 1318, 1428, 1538, 1648, 1758, 1868, 1978,
2307, 2417, 2527, 2637, 2747, 2857, 2967,
3406, 3516, 3626, 3736, 3846, 3956,
4505, 4615, 4725, 4835, 4945,
5604, 5714, 5824, 5934,
6703, 6813, 6923,
7802, 7912,
8901.
The last previous peculiar year is 1868, and the next one occurs on 2307.

Problem 2 of Austria Mathematical Olympiad 2005

A semicircle *h* with diameter AB and center M is drawn. A second semicircle *k* with diameter MB is drawn on the same side of the line AB. Let X and Y be points on *k* such that the arc BX is one and a half times as long as the arc BY. The line MY intersects the line BX at C and the larger semicircle *h* at D. Show that Y is the midpoint of the line segment CD.

Solution



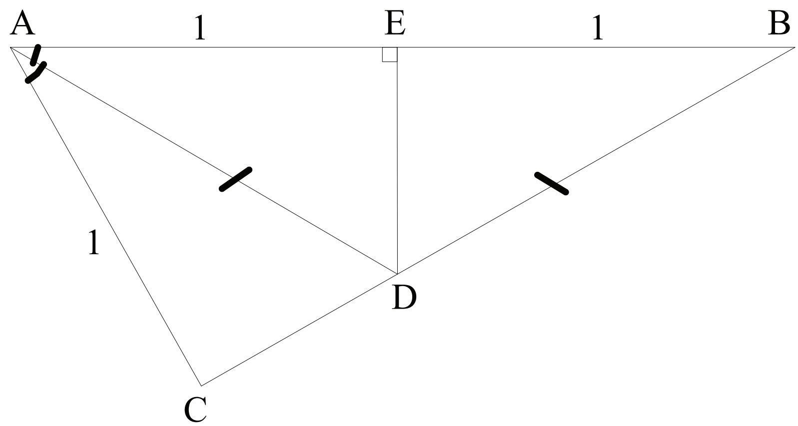
Since arc BX is one and a half that of arc BY, ∠YMB = 2∠XBY, and since Y is on the semicircle k, ∠MYB = 90° and ∠YCB = 90° – ∠XBY.

Also, since D is on the semicircle h, ∠ADB = 90° and ∠MDB = 90° – ∠ADM = 90° – ½∠YMB = 90° – ∠XBY = ∠YCB, or triangle CBD is isosceles with CB = DB.

Therefore, BY is also the perpendicular bisector of ∠CBD, or Y is the midpoint of the line segment CD.

<u>Problem 1 of Japan's Kyoto University Entrance Exam 2010</u>

Given a ΔABC such that $AB = 2$, $AC = 1$. A bisector of $\angle BAC$ intersects BC at D. If $AD = BD$, then find the area of ΔABC.



<u>Solution 1</u> (Applying Stewart's theorem)

Let $AD = BD = x$, $CD = \frac{1}{2}x$. Applying Stewart's theorem to the problem, we get $x^2 = 2 - \frac{1}{2}x^2$, or $\frac{3x^2}{2} = 2$, $3x^2 = 4$, or $BD = AD = x = \frac{2\sqrt{3}}{3}$, and $BC = \sqrt{3}$. We now have $AB^2 = AC^2 + BC^2$, or $\angle ACB$ is a right angle, and the area of $\Delta ABC = \frac{\sqrt{3}}{2}$.

<u>Solution 2</u> (Using perpendicular bisector)

From D draw its altitude DE to AB. Since $AD = BD$, DE is perpendicular bisector of AB and thus $AE = BE = 1 = AC$.

Combining with $\angle CAD = \angle EAD$, the two triangles CAD and EAD are congruent which implies $\angle ACD = \angle AED = 90°$, and we end up with the same result as above.

<u>Problem 1 of the Canadian Mathematical Olympiad 1977</u>

If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.

<u>Solution</u>

The problem gives us $4a^2 + 4a - b^2 - b = 0$.

Solving for a, we have $a = ½(-1 \pm \sqrt{b^2 + b + 1}\ )$

We now need to prove that $b^2 + b + 1$ is not a perfect square. Assuming it is a perfect square, we have

$b^2 + b + 1 = m^2$ where m is an integer, or

$$b^2 + b = b(b+1) = m^2 -1 = m(m +1) - m -1 \qquad \text{(i)}$$

But $1 + 2 + \ldots + n = \dfrac{n(n + 1)}{2}$, and we can write equation (i) as

$$2(1 + 2 + \ldots + b) = 2(1 + 2 + \ldots + m) - m -1 \qquad \text{(ii)}$$

Since $b^2 + b + 1 = m^2 \Rightarrow m > b$, and (ii) becomes

$0 = 2[(b + 1) + (b + 2) + \ldots. + m] - m - 1$, or
$0 = 2[(b + 1) + (b + 2) + \ldots. + m - 1] + m - 1$, or
$1 = 2[(b + 1) + (b + 2) + \ldots. + (m - 1)] + m$.

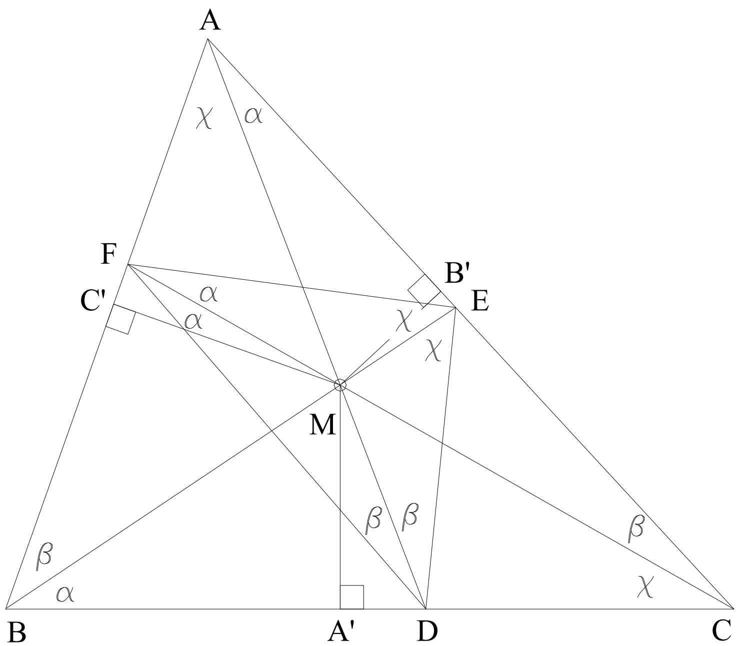
b is positive as required by the problem; therefore, the right side of the above equation is greater than 1 (even when $m = b + 1$), and our assumption of $b^2 + b + 1$ being a perfect square is not possible.

Therefore, the equation $44f(a) = f(b)$ has no solutions in positive integers a and b.

<u>Problem 4 of the Vietnamese Mathematical Olympiad 1990</u>

A triangle ABC is given in the plane. Let M be a point inside the triangle and A’, B’, C’ be its projections on the sides BC, CA, AB, respectively. Find the locus of M for which MA×MA’ = MB×MB’ = MC×MC’.

<u>Solution</u>

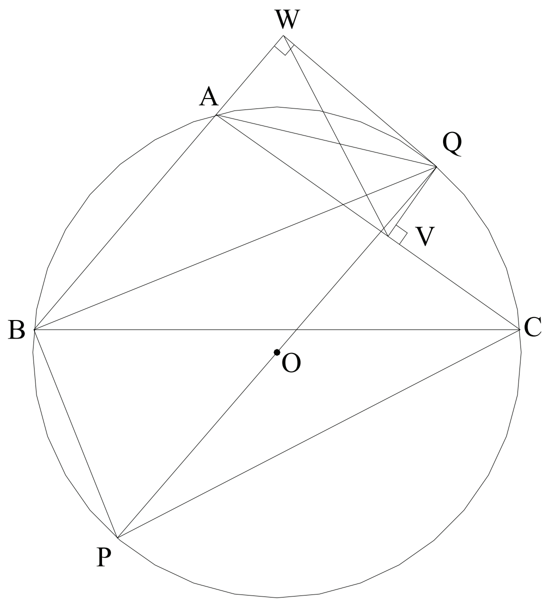


Let α = ∠MAB’, β = ∠MBC’, χ = ∠MCA’. Extending AM, BM and CM to meet BC, AC and AB at D, E and F, respectively.
From MA×MA’ = MB×MB’ = MC×MC’, we have
$\frac{MA'}{MB} = \frac{MB'}{MA} = \sin\alpha$, $\frac{MB'}{MC} = \frac{MC'}{MB} = \sin\beta$, $\frac{MA'}{MC} = \frac{MC'}{MA} = \sin\chi$.
Therefore, ∠MBA’ = α, ∠MCB’ = β, ∠MAC’ = χ.
Since the three angles of ΔABC = 2(α + β + χ), α + β + χ = 90°, and ∠ADC = ∠BEA = ∠CFB = 90°, or AD, BE and CF are the three altitudes of ΔABC, and M is its orthocenter. Therefore, the locus of M for which MA×MA’ = MB×MB’ = MC×MC’ is just the orthocenter of ΔABC.

<u>Problem 3 of the British Mathematical Olympiad 2007</u>

Let ABC be a triangle, with an obtuse angle at A. Let Q be a point (other than A, B or C) on the circumcircle of the triangle, on the same side of chord BC as A, and let P be the other end of the diameter through Q. Let V and W be the feet of the perpendiculars from Q onto CA and AB, respectively. Prove that the triangles PBC and AWV are similar. *Note: The circumcircle of the triangle ABC is the circle which passes through the vertices A, B and C.*

<u>Solution</u>

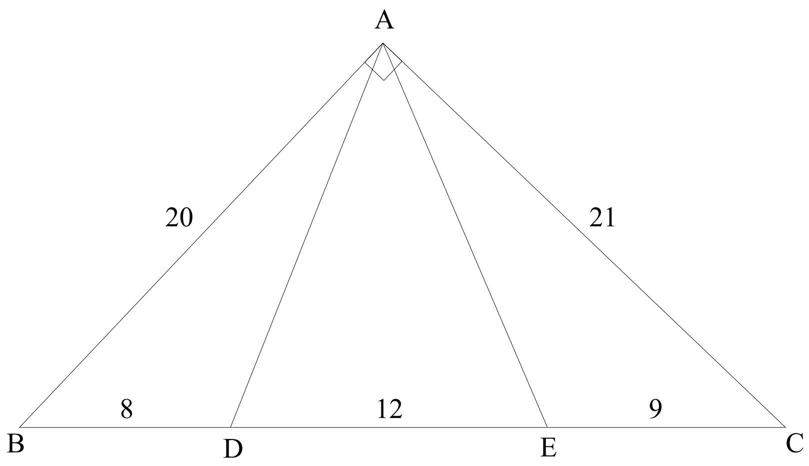


Since ABPC (with four vertices on the circle) and AWQV (with opposite right angles) are cyclic, we have $\angle$BPC = $\angle$WAV (because each added to $\angle$BAC to be 180°).
Furthermore, since AWQV is cyclic, $\angle$AVW = $\angle$AQW = 90° – $\angle$WAQ = 90° – (180° – $\angle$BAQ) = $\angle$BAQ – 90°. Since $\angle$BAQ subtends arc BPQ and PQ is the diameter, angle $\angle$BAQ – 90° subtends arc BPQ – arc PQ = small arc BP and is equal to $\angle$PCB, or $\angle$AVW = $\angle$PCB.

<u>Problem 1 of the Irish Mathematical Olympiad 2001</u>

In a triangle ABC, AB = 20, AC = 21 and BC = 29. The points D and E lie on the line segment BC, with BD = 8 and EC = 9. Calculate the angle $\angle$DAE.

<u>Solution</u>



Observe that $BC^2 = AB^2 + AC^2$, or $\angle BAC = 90°$.

Since BD = 8 and EC = 9, DE = 12. The two triangles BAE and CDA are isosceles with BA = BE and CA = CD, and $\angle BAE = \angle BEA$ and $\angle CAD = \angle CDA$, or $2\angle BEA + \angle B = 180°$, and $2\angle CDA + \angle C = 180°$.

Adding these two equations, we have $2(\angle BEA + \angle CDA) + \angle B + \angle C = 360°$.
Now combining with $\angle B + \angle C = 90°$, we have $\angle BEA + \angle CDA = 135°$, or $\angle DAE = 45°$.

<u>Problem 1 of the Irish Mathematical Olympiad 1997</u>

Find, with proof, all pairs of integers (x, y) satisfying the equation $1 + 1996x + 1998y = xy$.

<u>Solution</u>

Let $x = y + n$ where n is an integer. The given equation can now be written as $1 + 1996(y + n) + 1998y = (y + n)y$, or

$y^2 + (n - 1997\times2)y - 1996n - 1 = 0$
Solving for y, we have

$$y_{1\&2} = 1997 - \frac{n}{2} \pm \frac{1}{2}\sqrt{n^2 - 4n + 4(1997^2 + 1)}$$

So now $n^2 - 4n + 4(1997^2 + 1)$ has to be a perfect square. Let $m^2 = n^2 - 4n + 4(1997^2 + 1)$, or $m^2 = (n - 2)^2 + 4\times1997^2$, or
$m^2 - (n - 2)^2 = (2\times1997)^2$, or
$(m + n - 2)(m - n + 2) = (2\times1997)^2$

The possible combinations of values for $(m + n - 2, m - n + 2)$ are $(m + n - 2, m - n + 2) = (1, 4\times1997^2), (2, 2\times1997^2), (4, 1997^2),$ $(1997, 4\times1997), (2\times1997, 2\times1997), (4\times1997, 1997), (1997^2, 4),$ $(2\times1997^2, 2), (4\times1997^2, 1)$.

But observe that $(m + n - 2) + (m - n + 2) = 2m$ is an even number. The above combinations reduce to
$(m + n - 2, m - n + 2) = (2, 2\times1997^2), (2\times1997, 2\times1997),$ $(2\times1997^2, 2)$.

Solving for m and n, we have

For $m = 1 + 1997^2$, $n = 3 - 1997^2$, and for $m = 1997\times2$, $n = 2$.

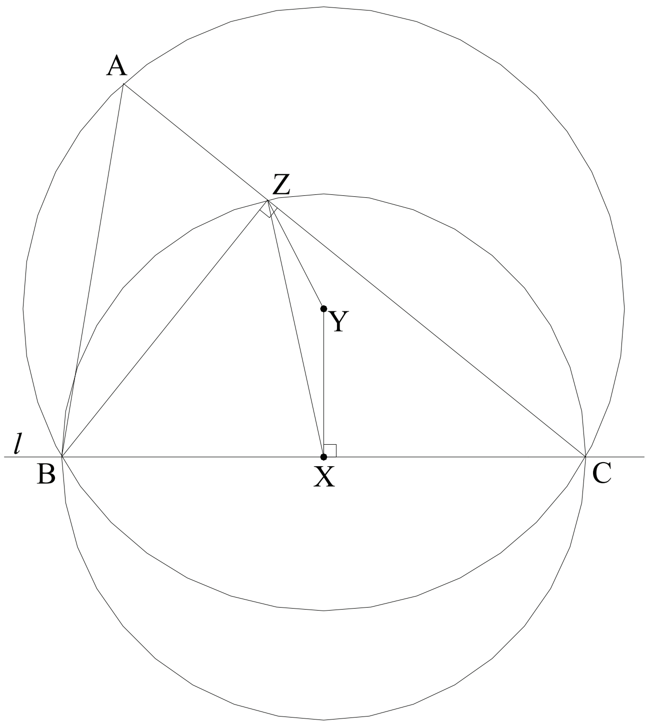
The corresponding x and y values are

$(x, y) = (1999, 1996 + 1997^2), (1998 - 1997^2, 1995), (3995, 3993),$ $(1, -1)$.

<u>Problem 1 of the Irish Mathematical Olympiad 1991</u>

Three points Y, X and Z are given that are, respectively, the circumcenter of a triangle ABC, the midpoint of BC, and the foot of the altitude from B on AC. Show how to reconstruct the triangle ABC.

<u>Solution</u>



Link XY and YZ.

Draw a line *l* perpendicular to XY through X. We know that B and C are on line *l*. However, since ∠BZC is a right angle, the circumcircle of triangle BZC will have the diameter BC and circumcenter at X.

Draw the circumcircle with center X and radius XZ; it will meet line *l* at B and C as shown. We next draw the circumcircle with center Y and radius BY to meet the extension of CZ at A.

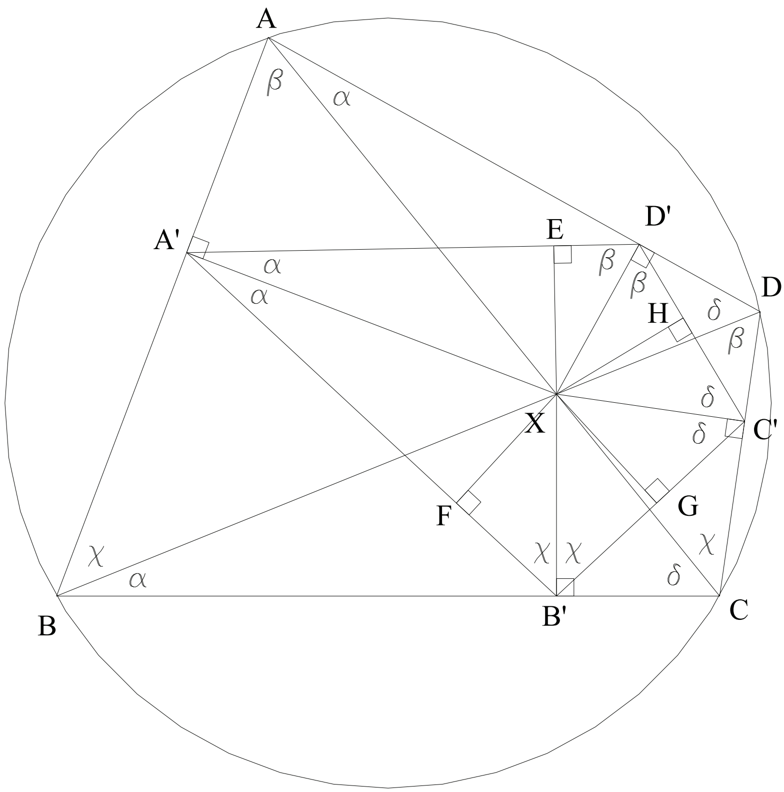
<u>Problem 3 of the Canadian Mathematical Olympiad 1990</u>

Let ABCD be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D', respectively. Prove that

|A'B'| + |C'D'| = |A'D'| + |B'C'|.

(|A'B'| is the length of line segment A'B', etc.)

<u>Solution</u>



Let $\angle XAD = \alpha$, $\angle XDC = \beta$, $\angle XBA = \chi$ and $\angle XCB = \delta$. Because the following quadrilaterals ABCD, AA'XD', A'BB'X, B'CC'X and DD'XC' are cyclic, we then also have

$\angle$XAD = $\angle$XBC = $\angle$XA'B' = $\angle$XA'D' = α, $\angle$XDC = $\angle$XAB = $\angle$XD'A' = $\angle$XD'C' = β, $\angle$XBA = $\angle$XCD = $\angle$XB'C' = $\angle$XB'A' = χ and $\angle$XCB = $\angle$XDA = $\angle$XC'D' = $\angle$XC'B' = δ. In other words, XA', XD', XC' and XB' are the angle bisectors of $\angle$B'A'D', $\angle$A'D'C', $\angle$B'C'D' and $\angle$A'B'C', respectively.

From X draw the altitudes XE, XF, XG and XH to A'D', A'B', B'C' and C'D'. We then have XE = XF = XG = XH = *h* as a result by the angle bisectors.

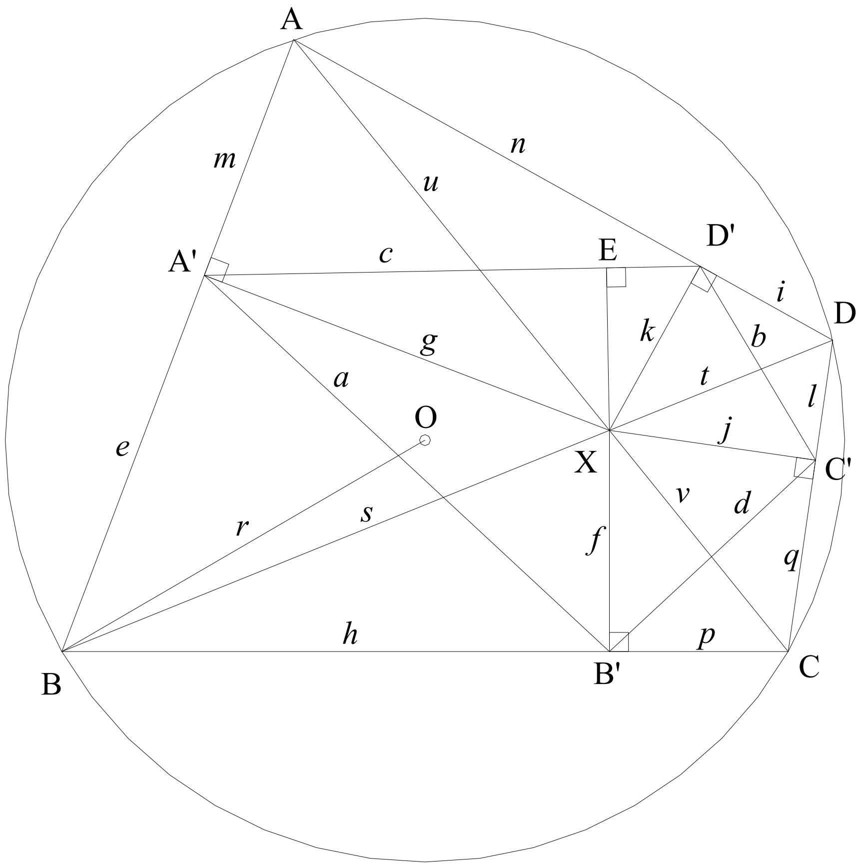
Let (Ω) denote the area of shape Ω. We then have
(A'XE) = (A'XF), (B'XF) = (B'XG), (C'XG) = (C'XH) and
(D'XH) = (D'XE), or
(A'XF) + (B'XF) + (C'XH) + (D'XH) = (A'XE) + (D'XE) + (B'XG) + (C'XG), or
(A'XB') + (C'XD') = (A'XD') + (B'XC'), or
$h\times$A'B' + $h\times$C'D' = $h\times$A'D' + $h\times$B'C', or
|A'B'| + |C'D'| = |A'D'| + |B'C'|.

<u>Further observation</u>

Let a = A'B', b = C'D', c = A'D', d = B'C', e = A'B, f = B'X, g = A'X, h = BB', i = DD', j = C'X, k = D'X, l = DC', m = AA', n = AD', p = B'C, q = CC', s = BX, t = DX, u = AX and v = CX as shown in the graph on the next page.
By Ptolemy's theorem
$a\times s = e\times f + g\times h$, *or* $a = (e\times f + g\times h)/s$,
$b\times t = i\times j + k\times l$, *or* $b = (i\times j + k\times l)/t$,
$c\times u = m\times k + n\times g$, *or* $c = (m\times k + n\times g)/u$,
$d\times v = f\times q + j\times p$, *or* $d = (f\times q + j\times p)/v$.

The problem confirms that |A'B'| + |C'D'| = |A'D'| + |B'C'|, or
$a + b = c + d$, *or*
$(e\times f + g\times h)/s + (i\times j + k\times l)/t = (m\times k + n\times g)/u + (f\times q + j\times p)/v$, *or*
$[t(e\times f + g\times h) + s(i\times j + k\times l)]/(s\times t) = [v(m\times k + n\times g) + u(f\times q + j\times p)]/(u\times v)$.

But since ABCD is cyclic, we have $s\times t = u\times v$*, and the above equation becomes*
$t(e\times f + g\times h) + s(i\times j + k\times l) = v(m\times k + n\times g) + u(f\times q + j\times p)$



It's also beneficial to know that the sum of the radii of the incircles of ΔABC and ΔADC equals to the sum of the radii of the incircles of ΔABD and ΔCBD.

Problem 1 of International Mathematical Talent Search Round 7

In trapezoid ABCD, the diagonals intersect at E. The area of triangle ABE is 72, and the area of triangle CDE is 50. What is the area of trapezoid ABCD?

Solution

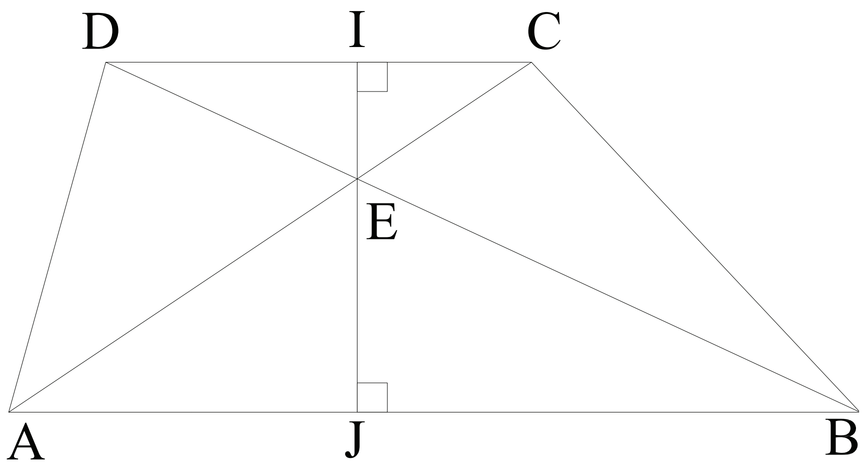


Figure (not to scale)

Let (Ω) denote the area of shape Ω, I and J be the feet of E onto CD and AB, respectively. We have

$2(CDE) = EI\times CD = 100$ and $2(ABE) = EJ\times AB = 144$.

Since ABCD is a trapezoid, AB || CD and the two triangles AEB and CED are similar resulting in the ratio

$$\frac{EI}{EJ} = \frac{CD}{AB}, \text{ or } \frac{(CDE)}{(ABE)} = \frac{EI^2}{EJ^2} = \frac{100}{144}, \text{ or } \frac{EI}{EJ} = \frac{CD}{AB} = \frac{10}{12} = \frac{5}{6}.$$

The area of the trapezoid is $(EI + EJ)\times(CD + AB)/2 = (EJ + \frac{5}{6}\times EJ)(AB + \frac{5}{6}\times AB)/2 = (\frac{11}{6})^2\times EJ\times AB/2 = (\frac{11}{6})^2\times 144/2 = 242$.

Problem 2 of the British Mathematical Olympiad 2005

Let x and y be positive integers with no prime factors larger than 5. Find all such x and y which satisfy $x^2 - y^2 = 2k$ for some non-negative integer k.

Solution

Since x and y are positive integers with no prime factors larger than 5, we can express them as follows
$x = 2^a \times 3^b \times 5^c$, and $y = 2^d \times 3^e \times 5^f$ where all the values a, b, c, d, e and f take on the values of either 0 or 1.

Therefore, the possible values for x^2 and y^2 are
$x^2 = 1, 4, 9, 25, 36, 100, 225, 900.$
$y^2 = 1, 4, 9, 25, 36, 100, 225, 900.$

The problem requires $x > y$ and the difference of $x^2 - y^2$ to be an even number. Therefore,

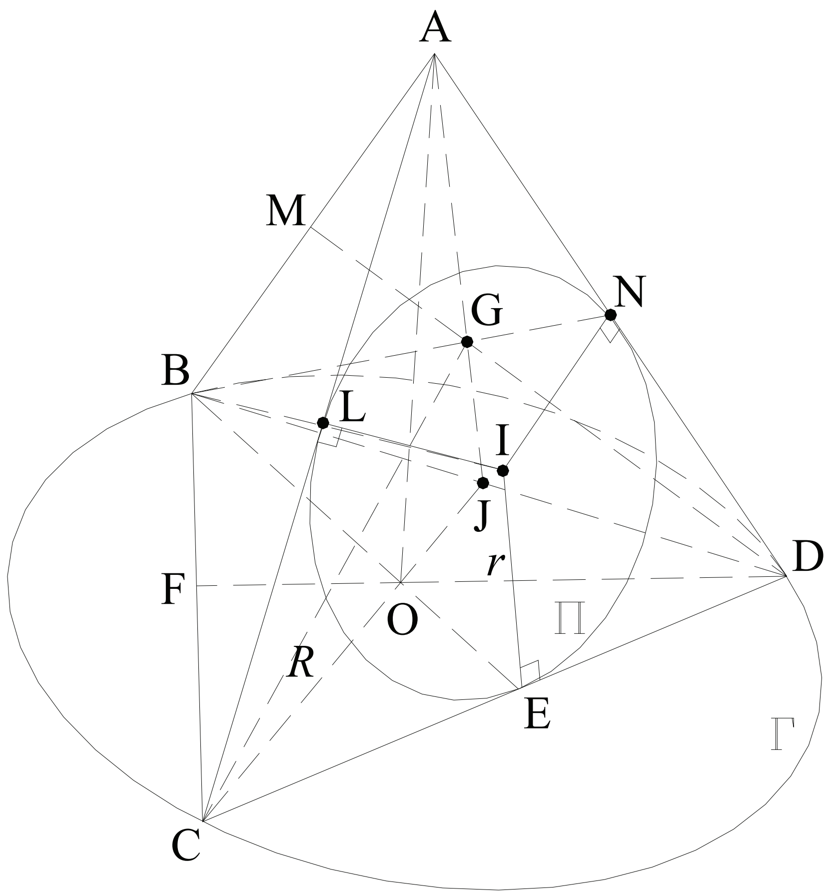
$(x^2, y^2) =$ (9, 1), (25, 1), (225, 1),
(25, 9), (225, 9), (225, 25),
(36, 4), (100, 4), (900, 4),
(100, 36), (900, 36),
(900, 100), and

$(x, y) =$ (3, 1), (5, 1), (15, 1), (5, 3), (15, 3), (15, 5), (6, 2), (10, 2), (30, 2), (10, 6), (30, 6), (30, 10).

Problem 2 of Poland Mathematical Olympiad 2010

The orthogonal projections of the vertices A, B, C of the tetrahedron ABCD on the opposite faces are denoted by O, I, G, respectively. Suppose that point O is the circumcenter of the triangle BCD, point I is the incenter of the triangle ACD and G is the centroid of the triangle ABD. Prove that tetrahedron ABCD is regular.

Solution



Let the circumcircle of triangle BCD be Γ, the incircle of triangle ACD be Π, *R* and *r* be the radii of Γ and Π, respectively. We then have R = OB = OC = OD. Since OA is perpendicular to the plane containing triangle BCD, apply the Pythagorean theorem to get $AB^2 = OA^2 + OB^2 = OA^2 + R^2 = AC^2 = AD^2$, or **AB = AC = AD**.

Now let AC, AD and CD touch Π at L, N and E, respectively. We have AL = AN, r = IL = IN = IE, AN = AL and $BL^2 = r^2 + BI^2 = BN^2 = BE^2$, or BL = BN = BE. The two triangles ABL and ABN are congruent to give us ∠BAC = ∠BAD. Combining this with AC = AD and the common segment AB, the two triangles ABC and ABD are also congruent which implies **BC = BD**.

Similarly, respectively let M and N’ (not shown on graph, but N’ will be proven to coincide N) be the midpoints of AB and AD. We get AM = AN’. Now the two triangles ABN’ and ADM are congruent because AM = AB/2 = AD/2 = AN’, AD = AB and they share angle BAD. This gives us BN’ = DM or $\frac{1}{3}$BN’ = $\frac{1}{3}$DM, or MG = N’G. Next, $CM^2 = CG^2 + MG^2 = CG^2 + N'G^2 = C'N^2$, or CM = CN’. This directly causes the two triangles ACM and CAN’ to be congruent and we then get ∠BAC = ∠DAC. With this additional requirement, the two triangles ABC and ADC are congruent and **BC = CD**. In addition to **BC = BD** that was obtained earlier, BCD is now an equilateral triangle and ∠BCD = ∠BDC = ∠CBD = 60°.

The two triangles BDE and BDN are now congruent because BE = BN, DE = DN and a common segment BD. This gives us ∠BDA = ∠BDC = 60°.

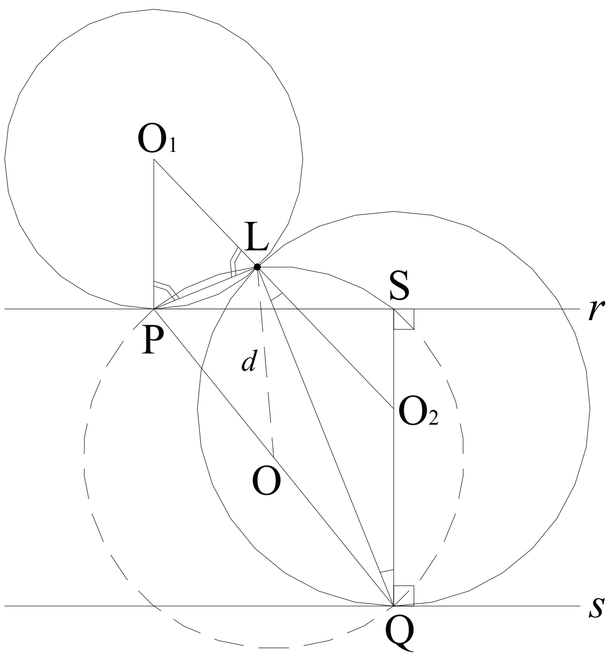
Combining ∠BDA = 60° with the fact that ABD is already an isosceles triangle with AB = AD, ABD is now also an equilateral triangle, or **AB = AD = BD**.

We finally have AB = AC = AD = BC = BD = CD and ABCD is a regular tetrahedron.

Problem 2 of Italy Mathematical Olympiad 2004

Two parallel lines r, s and two points $P \in r$ and $Q \in s$ are given in a plane. Consider all pairs of circles (C_P, C_Q) in that plane such that C_P touches r at P and C_Q, touches s at Q and which touch each other externally at some point T. Find the locus of T.

Solution



Let O_1 and O_2 be the centers of the circles touching r and s, respectively, L be the common point of these circle. Since triangles O_1PL and O_2QL are both isosceles, $\angle O_1LP = \angle O_1PL = (180° - \angle PO_1L)/2$, and $\angle O_2LQ = \angle O_2QL = (180° - \angle QO_2L)/2$, or $\angle O_1LP + \angle O_2LQ = 180° - (\angle PO_1L + \angle QO_2L)/2$. But because $O_1P \parallel O_2Q$ and O_1LO_2 are collinear, $\angle PO_1L + \angle QO_2L = 180°$. Successively, $\angle O_1LP + \angle O_2LQ = 90°$ and $\angle PLQ = 90°$.

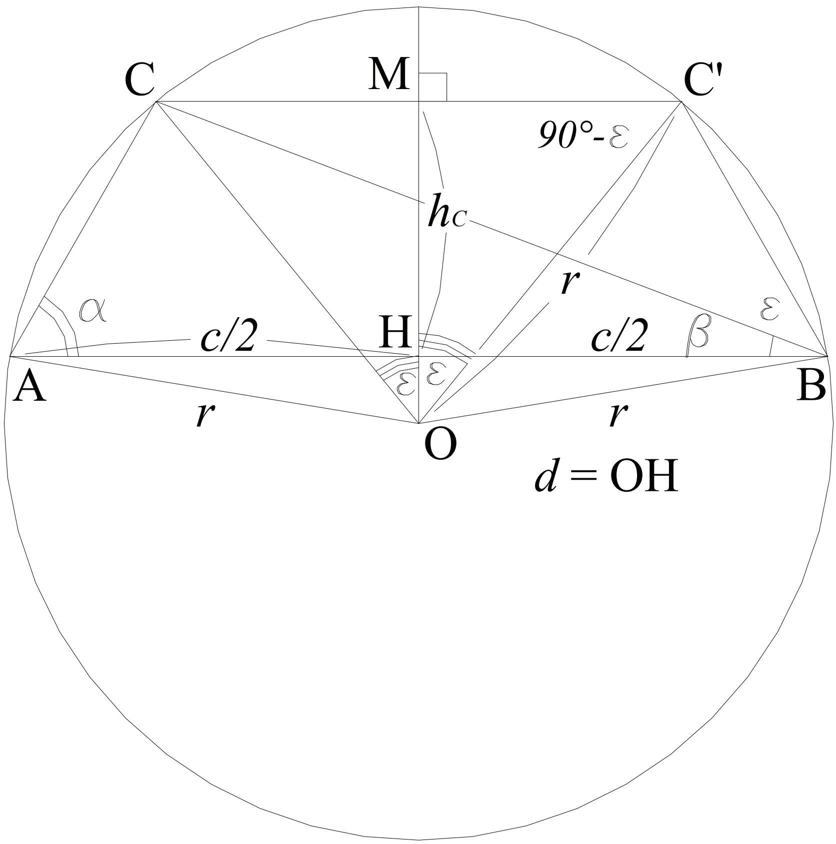
Therefore, the locus is part of the circle above line r from point P to point S that has diameter PQ. Both points P and S are on line r.

<u>*Problem 4 of Germany Mathematical Olympiad 1996*</u>

A pupil wants to construct a triangle ABC, given the length c = AB, the altitude h_c from C and the angle $\varepsilon = \alpha - \beta$. Here c and h_c are arbitrary and satisfies $0 < \varepsilon < 90$.

a) Is there such a triangle for any c, h_c and ε?
b) Is this triangle unique up to the congruence?
c) Show how to construct one such triangle, if it exists.

<u>Solution</u>



a) Draw the circumcircle of triangle ABC with center O and radius r and pick point C' on it such that CC' $\|$AB. Let $\alpha = \angle$BAC, $\beta = \angle$ABC. The angle ε subtends arc BC'C less smaller arc AC.

However, arc AC = arc BC' because CC' || AB, $\varepsilon = \alpha - \beta = \angle CBC'$. Now respectively let M and H be the midpoints of CC' and AB. We then have BH = $\frac{c}{2}$ and $\varepsilon = \angle COM = \angle C'OM$.

So for any c, h_c and ε, there always exists a circumcircle with arc CC'. Any angle subtending arc CC' will have the value ε.

b) This triangle ABC, as we have seen, is not unique. There are many such triangles since there are an infinite number of angles ε.

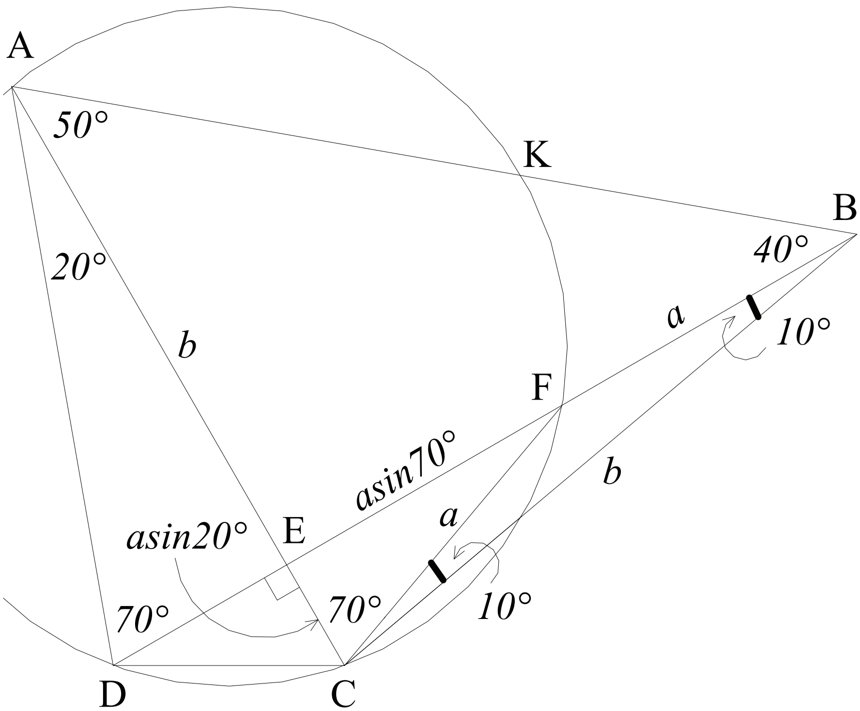
c) Let d = OH, $\cos\varepsilon = \frac{OM}{OC'} = \frac{h_c + d}{r}$, or $r = \frac{h_c + d}{\cos\varepsilon}$.
On the other hand, by applying the Pythagorean theorem we get $r^2 = d^2 + \frac{c^2}{4}$, or $\frac{(h_c + d)^2}{\cos^2\varepsilon} = d^2 + \frac{c^2}{4}$, or $(h_c + d)^2 = \cos^2\varepsilon(d^2 + \frac{c^2}{4})$, or
$4\sin^2\varepsilon d^2 + 8h_c \times d + 4h_c^2 - c^2\cos^2\varepsilon = 0$.

Solving for d to get $d = \frac{1}{2\sin^2\varepsilon}(-2h_c \pm \cos\varepsilon\sqrt{4h_c^2 + c^2\sin^2\varepsilon})$.
These values for d are constant, and we can find the values for the radius r. From there we can construct such triangles ABC.

Problem 3 of Germany Mathematical Olympiad 1997

In a convex quadrilateral ABCD we are given that $\angle CBD = 10°$, $\angle CAD = 20°$, $\angle ABD = 40°$, $\angle BAC = 50°$. Determine the angles $\angle BCD$ and $\angle ADC$.

Solution



We have $\angle ADB = 180° - \angle DAB - \angle DBA = 70°$ and $\angle ACB = 180° - \angle ABC - \angle BAC = 80°$. Let AC and BD intersect at E. $\angle AEB = 180° - \angle ABD - \angle BAC = 90°$.

Draw a segment CF such that F is on BD and BF = CF; BFC is isosceles and $\angle CFD = 20°$, $\angle ECF = 70°$. Also because $\angle BAC = \angle ABC = 50°$, AC = BC. Now let a = BF = CF and b = AC = BC. Applying the law of sines, $\dfrac{b}{\sin\angle BFC} = \dfrac{b}{\sin(180° - \angle CFD)} =$

$$\frac{b}{\sin\angle \mathrm{CFD}} = \frac{b}{\sin 20°} = \frac{a}{\sin 10°}, \text{ or } b = \frac{a\sin 20°}{\sin 10°} \qquad \text{(i)}$$

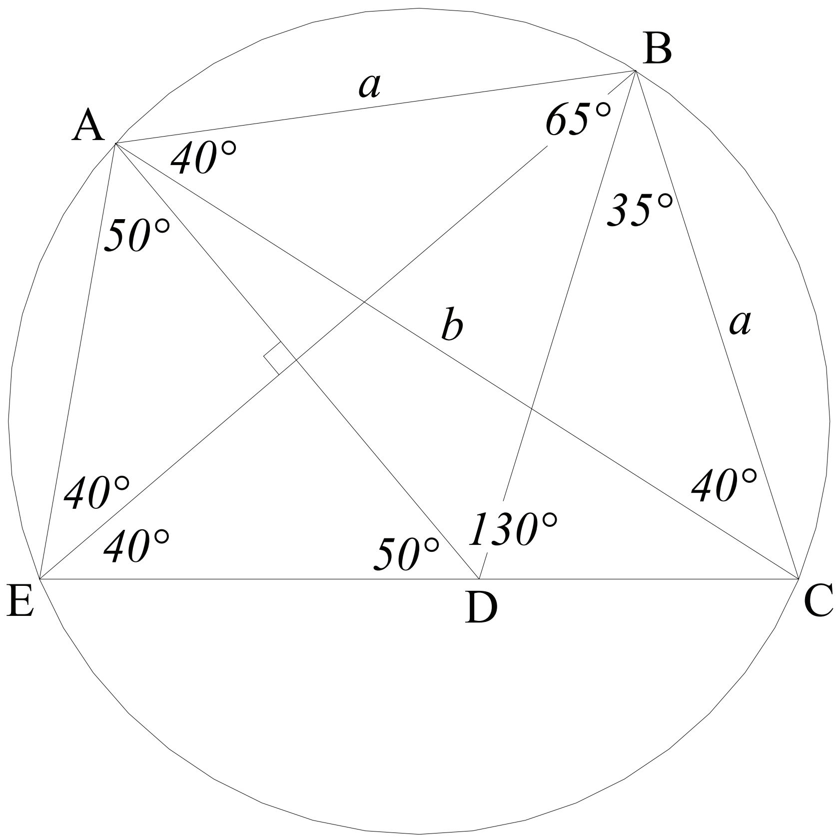
Now in the right triangle CEF, CE = $a\sin 20°$ and EF = $a\sin 70°$. Since $\angle$ACF = $\angle$ADF = 70°, ADCF is cyclic and we have $\frac{\mathrm{DE}}{\mathrm{CE}} =$
$\frac{\sin\angle \mathrm{ACD}}{\sin\angle \mathrm{BDC}} = \frac{\mathrm{EA}}{\mathrm{EF}} = \frac{b - a\sin 20°}{a\sin 70°} = \frac{\sin 20°(1 - \sin 10°)}{\sin 70°\sin 10°} =$
$\frac{\sin 20°(\sin 90° - \sin 10°)}{\sin 70°\sin 10°} = \frac{\sin 20°(2\cos 50°\sin 40°)}{\sin 70°\sin 10°} =$
$\frac{4\sin 10°\cos 10°\cos 50°\sin 40°}{\sin 70°\sin 10°} = \frac{8\sin 10°\cos 10°\cos 50°\sin 20°\cos 20°}{\sin 70°\sin 10°} =$
$\frac{8\cos 10°\cos 50°\sin 20°\cos 20°}{\cos 20°} = 8\cos 10°\cos 50°\sin 20° =$
$8\cos 10°\cos 50°\cos 70° = 8\cos 50°\times\frac{\cos 80° + \cos 60°}{2} =$
$4\cos 50°(\cos 80° + \frac{1}{2}) = 4\cos 50°\cos 80° + 2\cos 50° =$
$4(\frac{\cos 130° + \cos 30°}{2}) + 2\cos 50° = 2(\cos 130° + \frac{\sqrt{3}}{2}) + 2\cos 50° =$
$2\cos 130° + \sqrt{3} + 2\cos 50° = \sqrt{3} + 2(\cos 130° + \cos 50°) = \sqrt{3} +$
$4\cos 90°\cos 40° = \sqrt{3} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sin 60°}{\sin 30°}.$

Hence, $\sin\angle$ACD = $\sin 60°$ and $\sin\angle$BDC = $\sin 30°$, or $\angle$ACD = 60° and $\angle$BDC = 30°, and $\angle$BCD = $\angle$BCA + $\angle$ACD = 80° + 60° = 140° while $\angle$ADC = $\angle$ADB + $\angle$BDC = 70° + 30° = 100°.

Problem 1 of Mongolia Teacher Level 1999

In a convex quadrilateral ABCD, $\angle$ABD = 65°, $\angle$CBD = 35°, $\angle$ADC = 130°, and BC = AB. Find the angles of ABCD.

Solution

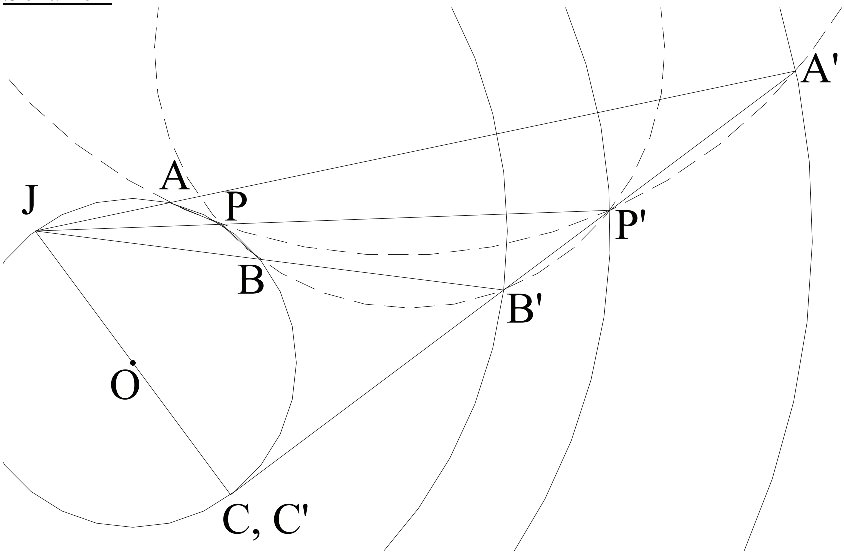


Draw the circumcircle of ΔABC and extend CD to meet it at E. Angles $\angle$AEB and $\angle$BEC subtend same 40° arc length. Since $\angle$ADC = 130°, $\angle$EDA = $\angle$EAD = 50° and AD $\perp$BE. Therefore, EB is the perpendicular bisector of both $\angle$AED and $\angle$ABD, or $\angle$BAD = $\angle$BDA = (180° – 65°)/2 = 57.5°. This causes $\angle$CAD = 57.5° – 40° = 17.5°, $\angle$ACD = 50° – 17.5° = 32.5°, or $\angle$BCD = 72.5° to go with $\angle$ABC = 100° and $\angle$ADC = 130°.

Problem 3 of Germany Mathematical Olympiad 2001

Let be given a circle of radius 1 and points J, A, B on it. We denote by k the arc AB of the circle not containing J. For every point P on k, point P′ on the ray JP is such that JP×JP′ = 4. Describe and draw the locus of points P′.

Solution



Solution 1

Apply the principal of inversion. Point A on the circle is inverted to point A’, B inverted to point B’ and point P on the circle between arc AB is inverted to point P’ while point C such that JC’ = 2 or JC’ is the diameter of the circle is inverted to itself, C ≡ C’. All points P on the circle will be inverted to a line and the locus is segment A’B’ as shown.

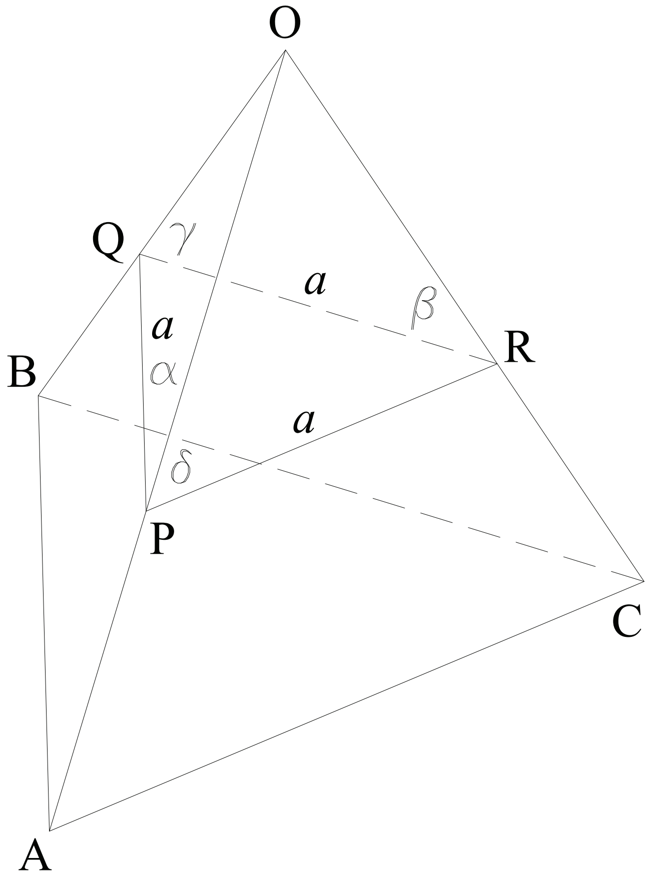
Solution 2

With JA×JA′ = JP×JP′ = JB×JB′ = 4, the quadrilaterals AA’P’P, BB’P’P and ABB’A’ are all cyclic. We then have ∠JA’P’ = ∠JPA which subtends arc JA while ∠JP’B’ = ∠JBP which subtends arc JB = arc JA + arc AP = ∠JA’P’ + ∠A’JP, or the three points A’, P and B’ are collinear. Therefore, the locus is segment A’B’.

Problem 2 of Kyoto University Entrance Exam 2012

Given a regular tetrahedron OABC and points P, Q, R on the sides OA, OB, OC, respectively. Note that P, Q, R are different from the vertices of the tetrahedron OABC. If triangle PQR is an equilateral triangle, then prove that three sides PQ, QR, RP are parallel to three sides AB, BC, CA, respectively.

Solution



Let a = PQ = PR = QR, $\alpha = \angle$OPQ, $\beta = \angle$ORQ, $\gamma = \angle$OQR and $\delta = \angle$OPR. Applying the law of sines, $a/\sin\angle$AOB $= a/\sin 60°$ = OQ$/\sin\alpha = a/\sin\angle$BOC = OQ$/\sin\beta$. However, because both α and β are in the range of (0, 120°), $\alpha = \beta$. Similarly, $\beta = \gamma = \delta$ and OQ = OP = OR. Therefore, PQ || AB, QR || BC and RP || CA.

Problem 3 of Kyoto University Entrance Exam 2012

When real numbers x, y moves in the constraint with $x^2 + xy + y^2 = 6$. Find the range of $x^2y + xy^2 - x^2 - 2xy - y^2 + x + y$.

Solution

$x^2y + xy^2 - x^2 - 2xy - y^2 + x + y = xy(x + y) - (x + y)^2 + x + y = (x + y)[xy - (x + y) + 1]$.

However, $x^2 + xy + y^2 = 6$ gives us $(x + y)^2 = 6 + xy$, or $xy = (x + y)^2 - 6$, and the expression $(x + y)[xy - (x + y) + 1]$ becomes $(x + y)[(x + y)^2 - (x + y) - 5]$.

Now let $z = x + y$, the expression is equivalent to $z(z^2 - z - 5) = z^3 - z^2 - 5z$. This expression attains its extreme values when its derivative equals zero, or $(z^3 - z^2 - 5z)' = 3z^2 - 2z - 5 = 0$.
Solving for z, we get $z = \frac{5}{3}$, -1.
Therefore, $-\frac{175}{27} \leq z^3 - z^2 - 5z \leq 3$, or $-\frac{175}{27} \leq x^2y + xy^2 - x^2 - 2xy - y^2 + x + y \leq 3$.

The author's previous books are now at many college and city libraries around the world. Below is a partial list of these libraries:

The World Cat libraries
http://www.worldcat.org/title/how-to-solve-the-worlds-mathematical-olympiad-problems-volume-1/oclc/693533166&referer=brief_results

http://www.worldcat.org/title/hard-mathematical-olympiad-problems-and-their-solutions/oclc/747808929&referer=brief_results

Hong Kong City libraries
http://libcat.hkpl.gov.hk/webpac_eng/wgbroker.exe?2011101400230500011335+-access+top.all-materials-page+search+open+T+how%20to%20solve%20the%20world's%20mathematical%20olympiad%20problems%23%23A:NONE%23NONE:NONE::%23%23

City libraries of Auckland, New Zealand
http://search.aucklandlibraries.govt.nz/?q=how%20to%20solve%20the%20world's%20mathematical%20olympiad%20problems&refx=&uilang=en

City libraries of Dublin, Ireland
http://libcat.dublincity.ie/02_Catalogue/02_004_TitleResults.aspx?page=1&searchTerm=How+to+solve+the+world's+Mathematical+Olympiad+problems%2c+Steve+D&searchType=1&media=&referrer=02_001_Search.aspx

Technical University of Munich, Germany
Universitätsbibliothek TU München University Library TUM München, D-80333 Germany

https://opac.ub.tum.de/InfoGuideClient.tumsis/start.do?Login=wotum&Query=-1=%22steve%20dinh%22

http://www.worldcat.org/title/hard-mathematical-olympiad-problems-and-their-solutions/oclc/747808929&referer=brief_results

The Indian Institute of Technology library, Mumbai, India
http://www.library.iitb.ac.in/newsearchbook/ca_det.php?m_doc_no=299557

The Michael Schwartz library at the Cleveland State University, Cleveland, Ohio, U.S.A.
http://scholar.csuohio.edu/search~/a?searchtype=t&searcharg=how+to+solve+the+world%27s+mathematical+olympiad+problems&SORT=D

The Auburn University at Montgomery, U.S.A.
http://ehis.ebscohost.com/eds/results?sid=b606fa0e-5fc4-43f5-a88b-d984fcf2327b%40sessionmgr14&vid=1&hid=5&bquery=((how+AND+to+AND+solve+AND+the+AND+world%26%2339%3bs+AND+mathematical+AND+olympiad+AND+problems))&bdata=JnR5cGU9MCZzaXRlPWVkcy1saXZl

http://aum.lib.auburn.edu/cgi-bin/Pwebrecon.cgi?DB=local&BOOL1=all+of+these&FLD1=Keyword+Anywhere+(GKEY)&CNT=50+records+per+page&SAB1=?693533166

Buffalo State College E. H. Butler library, U.S.A.
http://bsc.sunyconnect.suny.edu:4380/F?func=find-b&find_code=035&request=747808929

The National library of Australia
http://trove.nla.gov.au/result?q=how+to+solve+the+world%27s+mathematical+olympiad+problems

City libraries of Sydney, Australia
http://library.cityofsydney.nsw.gov.au/opac/default.aspx

The city libraries of Santa Cruz, California, U.S.A.
http://aqua.santacruzpl.org/default.ashx?q=How+to+solve+the+world%27s+mathematical+olympiad+problems

The city libraries of San Jose, California, U.S.A.
http://catalog.sjlibrary.org/search~/a?searchtype=X&searcharg=How+to+solve+the+world%27s+mathematical+olympiad+problems&search-submit=Go&SORT=D&searchscope=1

The Santa Clara County libraries, California, U.S.A.
http://sccl.bibliocommons.com/search?q=How+to+solve+the+world%27s+mathematical+olympiad+problems&submit=Search&t=keyword

http://sccl.bibliocommons.com/item/show/1475137016_hard_mathematical_olympiad_problems_and_their_solutions

Multnomah County Library, Portand, Oregon, U.S.A.
http://catalog.multcolib.org/search/a?searchtype=Y&searcharg=how+to+solve+the+world%27s+mathematical+olympiad+problems&SORT=R&searchscope=1&submit=Search+catalog

About the Author

Steve Dinh, a.k.a. Vo Duc Dien, is a prolific math problem solver. He has solved many difficult mathematical problems and has written many math books. At the age of 18 he left Vietnam and became a refugee living in a refugee camp in Hong Kong.

His other hobbies include designing software, writing poetry and piloting airplanes. He currently lives in the Silicon Valley, California.