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Diagonalizable subalgebras of the first Weyl algebra

By

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Abstract

Let A_1 denote the first Weyl algebra over a field K of characteristic 0; that is, A_1 is generated over K by elements p, q that satisfy the relation $pq - qp = 1$. One can view A_1 as an algebra of differential operators by setting $q = X$, $p = d/dX$.

The basic questions which are addressed in this paper is what are all the maximal diagonalizable subalgebras of A_1 and if K is not algebraically closed, what conditions should be placed on the element $x \in A_1$ so that x is diagonalizable on A_1. Thus, we use these diagonalizable elements to verify the Jacobian conjecture for $n = 1$.

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Contents

Chapter 1

Introduction

Let K be a field of characteristic 0. The first Weyl algebra A_1 is an associative algebra generated over the field K by elements p and q which satisfy the defining relation $pq - qp = 1$. The Weyl algebra A_1 is a simple, Noetherian domain of Gelfand-Kirillov dimension 2. It is canonically isomorphic to the ring of differential operators $K[X][\frac{d}{dX}]$ with coefficients from the polynomial algebra $K[X]$. The n-th Weyl algebra A_n is the tensor product $A_1 \otimes_K A_1 \otimes_K A_1 \otimes_K \ldots \otimes_K A_1$ of n copies of the first Weyl algebra.

Before 1968, not much was known about the first Weyl algebra: the commutativity of the centralizer $C(x)$ of an arbitrary nonzero element x of the first Weyl algebra had been proved by Amitsur [A] (see the paper of Goodearl [G] for generalization), the global dimension of the first Weyl algebra, which is 1, had been calculated by Rinehart[R], and Dixmier had proved that each derivation of the Weyl algebra is an inner derivation [D1]. After 1968, more progress was made in the study of the higher Weyl algebras A_n, $n \geq 2$, due to new techniques associated with the Gelfand-Kirillov dimension, introduced by Gelfand and Kirillov [GK1, GK2] that were created at that time. For more details, the interested reader is referred to the following books [BJ, BA, GW, KL, MR].

The importance of the Weyl algebra has grown steadily in the last 30 years; The work on noncommutative Noetherian ring that followed A. Goldie's famous theorems on quotient rings of Noetherian rings and the fact that the Weyl algebra is the simplest after finite-dimensional ones ring of differential operators has only added to its importance. In the fundamental paper [D2], Dixmier started a systematic study of the structure of A_1. The key idea of [D2] is that one can study properties of elements via properties of the corresponding inner derivations. So, for an arbitrary element x of the Weyl algebra A_1, one can attach the inner derivation $\mathrm{ad}(x)$ of the algebra A_1. The main result of [D2] is the description of the automorphism group of A_1: Let $\lambda \in K$, and n be an integer ≥ 0. The derivation $\triangle = \mathrm{ad}(\frac{\lambda}{n+1}p^{n+1})$ on A_1 is locally nilpotent, so that $\Phi_{n,\lambda} = \exp\triangle$ is a well defined automorphism of A_1 with $\Phi_{n,\lambda}(p) = p$ and $\Phi_{n,\lambda}(q) = q + \lambda p^n$. He also defined the automorphism $\Phi'_{n,\lambda}$ of A_1 such that $\Phi'_{n,\lambda}(q) = q$ and $\Phi'_{n,\lambda}(p) = p + \lambda q^n$. Then he showed that the automorphism group G of A_1 is generated by $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ for all integers $n \geq 0$ and $\lambda \in K$.

Let V be a finite-dimensional linear space over a field K and T be a linear transformation of V. The transformation T is called *diagonalizable* if there exist a basis of V that consists of eigenvectors. We are interested in maximal diagonalizable subalgebras of A_1. In Chapter 4, we show that the subalgebra $Kpq + K$ is such an algebra (see Theorem 4.5). Then, up to automorphisms of A_1, $Kpq + K$ is the only maximal diagonalizable subalgebras of A_1. But if K is not algebraically closed, we need more conditions to restrict to the semisimple elements of A_1 so that they are also diagonalizable in A_1 (see Lemma 4.7). The solution to Problem 4.1 shows how to use the semisimple elements to connect the Dixmier conjecture with the Jacobian conjecture if K is algebraically closed. Using this, we can verify that the Jacobian conjecture for $n = 1$ is true. The Jacobian conjecture is an old and interesting problem, that has inspired a lot of great mathematics, but so far has been resistant to any attempt at proving it, even if $n = 2$.

Chapter 2

Properties of the Weyl algebra

In this chapter, we will introduce the Weyl algebra. We begin with an account of the history of the Weyl algebra and then describe the main structure of the Weyl algebra.

2.1 The history of the Weyl algebra

Interest in the Weyl algebra began when a number of people like Heisenberg, Dirac or Born(1882) were trying to develop the principles of quantum mechanics used to explain the behavior of the atom, using dynamical variables that do not commute. One is interested in polynomial expressions in the dynamical variables momentum, denoted by p, and position, denoted by q. It is assumed that the variables satisfy the (normalized) relation $pq-qp=1$. This is what we now call the first Weyl algebra. The Weyl algebras of higher index appear when one considers systems with several degrees of freedom. Weyl's pioneer book *The theory of groups and quantum mechanics*[WH] was perhaps their first appearance in print. Then Littlewood (1903) [L] used the language of infinite dimensional algebras to describe the objects. In his paper Littlewood established many of the basic properties of the Weyl algebra. He showed that any element in the Weyl algebra has a canonical form (Lemma 2.3) and that the algebra is an integral domain (Proposition 2.8). He also showed that the relation $pq - qp = 1$ is not compatible with any other relation, or, as we would now say, the only proper ideal of this algebra is zero (Proposition 2.9). Dixmier (1924) introduced the notation A_n for the algebra that corresponds to the physicist's system with n degrees of freedom. The name Weyl algebra was used by Dixmier as the title of [D2]. He connected the Weyl algebra with the theory of Lie algebras, and in [D2], he described the automorphism group of the first Weyl algebra. At the end of this paper, he listed six problems and some of them are still open. Of course, the Weyl algebra is often used in the study of systems of differential equations (in this context the theory is often called *Algebraic Analysis*). This approach comes from people like Malgrange and Kashiwara (see for example [M, K, KK]) and, at the same time, from Bernstein (see [B]).

2.2 Basic properties of A_n

The Weyl algebra is a ring of operators on a vector space of infinite dimension. Let $K[X]$ be the ring of polynomials $K[X_1, ..., X_n]$ in n commuting indeterminates over K. The ring $K[X]$ is a vector space of infinite dimension over K. Its algebra of linear operators is denoted by $End_K(K[X])$, and we define the operators p_i on $K[X]$ by

$$p_i \cdot f(X) = \partial f/\partial X_i,\ q_i \cdot f(X) = X_i f(X),\ \forall\ f \in K[X].$$

Actually, the Weyl algebra is defined as a subalgebra of $End_K(K[X])$, then we introduce the definition of A_n:

Definition 2.1 The *n-th Weyl algebra* A_n is the K-subalgebra of $End_K(K[X])$ generated by the elements $p_1, ..., p_n$ and $q_1, ..., q_n$.

Consider the operator $p_i \cdot q_i$ and apply it to a polynomial $f \in k[X]$. Using the rule for the differentiation of a product, we get $p_i \cdot q_i(f) = X_i \partial f/\partial X_i + f$. In other words,

$$p_i \cdot q_i = q_i \cdot p_i + 1$$

where 1 stands for the identity operator. It is more convenient to rewrite the formula using commutators. If $x, y \in A_n$, then their *commutator* is defined as $[x, y] = x \cdot y - y \cdot x$. The formula above becomes $[p_i, q_i] = 1$. Similar calculations allow us to obtain formulae for the commutators of the other generators of A_n. These are summed up below:

$$[p_i, q_j] = \delta_{ij},$$

$$[p_i, p_j] = [q_i . q_j] = 0,$$

where $1 \leq i, j \leq n$. (δ_{ij} is the Kronecker delta symbol: it equals 1 if $i = j$ and zero otherwise)

Remark 2.2 From this definition of A_n, it is easy to see that $A_n \cong K[X_1, ..., X_{2n}]/J$ where $K[X_1, ..., X_{2n}]$ is a free algebra, and J is the two-sided ideal of $K[X_1, ..., X_{2n}]$ generated by the elements $[X_{i+n}, X_i] - 1$ ($i = 1, ..., n,$) and $[X_i, X_j]$ ($j \neq i + n$ and $1 \leq i, j \leq 2n$). Then this implies that $A_n \cong A_1 \otimes_K A_1 \otimes_K A_1 \otimes_K ... \otimes_K A_1$. That is the definition we mentioned in the introduction.

We now construct a basis for the Weyl algebra as a K-vector space. It is easier to describe the basis if we use a multi-index notation. A *multi-index* α is an element of $\mathbb{N}^n$ ($0 \in \mathbb{N}$), say $\alpha = (\alpha_1, ..., \alpha_n)$. By p^α we mean the monomial $p_1^{\alpha_1} ... p_n^{\alpha_n}$ and likewise $q^\alpha = q_1^{\alpha_1} ... q_n^{\alpha_n}$. The *degree* of this monomial is the *length* $|\alpha| = \alpha_1 + ... + \alpha_n$.

Remark 2.3 *We have*

$$p_i^{\alpha_i} q_j^{\beta_j} = \sum_{k=0}^{min\{\alpha_i, \beta_j\}} \frac{\alpha_i(\alpha_i - 1) \cdots (\alpha_i - k + 1)\beta_j(\beta_j - 1) \cdots (\beta_j - k + 1)}{k!} q_j^{\beta_j - k} p_i^{\alpha_i - k}$$

Lemma 2.4 *The elements* $\{p^\alpha q^\beta \colon \alpha, \beta \in \mathbb{N}^n\}$ *constitute a basis of* A_n *as a vector space over* K.

Proof. See [C] Proposition 2.1. □

This basis is known as the *canonical basis.* If an element of A_n is written as a linear combination of this basis then we say that it is in *canonical form.*

The degree of an operator of A_n behaves like the degree of a polynomial. The differences are accounted for by the noncommutativity of A_n.

Definition 2.5 Let $a \in A_n$. The *degree* of a is the largest length of the multi-indices $(i, j) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $p^i q^j$ appears with non-zero coefficient in the canonical form of a. It is denoted by $deg(a)$ ($deg(0) := -\infty$).

For example: the degree of $3p_1^2q_2 + p_1^3p_2^2q_1^4q_2$ is 10.

Lemma 2.6 *For $a, b \in A_n$, we have*

- $deg(ab) = deg(a) + deg(b)$,
- $deg(a + b) \leq max\{deg(a), deg(b)\}$,
- $deg[a, b] \leq deg(a) + deg(b) - 2$.

For more details and the proof of these lemmas, see [C] Chapters 1 and 2.

As in the case of polynomial rings over a field, Lemma 2.6 may be used to prove the following result.

Proposition 2.7 *The Weyl algebra A_n is a domain.*

Proof. Let $a, b \in A_n$. If $ab = 0$, then $deg(ab) = deg(0) = -\infty$. Then

$$deg(a) + deg(b) = -\infty.$$

So either $deg(a)$ or $deg(b)$ is $-\infty$, thus either a or b is 0. □

If we are familiar with commutative rings, we may find A_n very peculiar. Commutative rings have many two-sided ideals, but not so A_n. A ring whose only proper two-sided ideal is zero is called *simple*. A commutative simple ring must be a field, but this is not true of noncommutative rings. The Weyl algebra is a simple ring, but it is very far from being even a division ring.

Proposition 2.8 *A_n is a simple algebra with centre K. In particular, every endomorphism of A_n is injective and there are no non-trivial two-sided ideals.*

Proof. Let I be a non-zero two-sided ideal of A_n. Choose $a \neq 0$ of smallest degree in I. If $deg(a) = 0$, then $a \in K$. Then $I = A_n$, since $A_n \cdot a \subseteq I$. Now assume $deg(a) = t > 0$. Suppose that (i, j) is a multi-index of length t. If p^iq^j is a summand of a with non-zero coefficient and $i_s \neq 0$, then $[q_s, p^iq^j] \neq 0$, since $[q_s, p_s] = -1$. Hence $[q_s, a] \neq 0$. So $deg[q_s, a] \leq t - 1$, by Lemma 2.6. Since I is a two-sided ideal of A_n, it follows that

$[q_s, a] \in I$. But this contradicts the minimality of a. Thus $j = (0, ..., 0)$. Since $t > 0$, we have $i_s \neq 0$, for some $s = 1, 2, ..., n$. Hence $[p_s, a]$ is a non-zero element of I of degree $s - 1$, and again we have a contradiction. $\square$

Definition 2.9 The *filtration* of A_n is the increasing sequence F of vector subspaces F_i of A_n:

$$F_i = \left\{ \sum k_{\alpha\beta} p^\alpha q^\beta \text{ such that } |\alpha| + |\beta| \leq i \ \text{ for } i \in \mathbb{Z}. \right\}$$

Clearly, the filtration satisfies the following properties:

- $\bigcup_{i \geq 0} F_i = A_n$,
- For every $i, j \geq 0$, $F_i F_j \subset F_{i+j}$.

In addition, $F_i = \{0\}$ if $i < 0$, and $F_0 = K$. Furthermore all F_i have finite dimension. From the filtration we can define the corresponding graded algebra $gr_F(A_n)$,

$$gr_F(A_n) = \bigoplus_{i \geq 0} F(i) = \bigoplus_{i \geq 0} \frac{F_i}{F_{i-1}}.$$

Moreover, $gr_F(A_1) \cong K[X_1, X_2]$.

The Weyl algebra A_n is not a left principal ideal ring either. For example, the left ideal generated by p_1, p_2 in A_2 is not principal. However, every left ideal of A_n can be generated by two elements, the proof of which may be found in the original paper of [S].

2.3 Some useful results about A_1

Now we turn our attention to the main focus of this thesis, the first Weyl algebra A_1. We will dispense with the subscripts for the generators of A_1, and write them simply as p and q. Here we will discuss some useful properties and results of A_1.

We have introduced the basis of A_n above. Now we give the formula of the multiplication of the basis elements in A_1:

$$\begin{aligned}(p^iq^j)(p^kq^l) &= p^{i+k}q^{j+l} + jkp^{i+k-1}q^{j+l-1} + \frac{1}{2!}j(j-1)k(k-1)p^{i+k-2}p^{j+l-2} + \\ &\quad +\frac{1}{3!}j(j-1)(j-2)k(k-1)(k-2)p^{i+k-3}q^{j+l-3} + \cdots \\ &= \sum_{t=0}^{min\{j,k\}} \frac{1}{t!}\begin{pmatrix} j \\ t \end{pmatrix}\begin{pmatrix} k \\ t \end{pmatrix} p^{i+k-t}q^{j+l-t}\end{aligned}$$

.

Remark 2.10 We have

$$(pq)p = p(qp - pq) + p^2q = -p + p^2q = p(pq - 1).$$

Then, if $f \in K[T]$,

$$f(pq)p = pf(pq - 1),$$

and by induction, if n is an integer ≥ 0,

$$f(pq)p^n = p^n f(pq - n).$$

Similarly

$$q^n f(pq) = f(pq - n)q^n.$$

In particular,

$$p^nq^n = p^{n-1}(pq)q^{n-1} = p^{n-1}q^{n-1}(pq + n - 1)$$

and, by repeating the above, we obtain,

$$p^nq^n = pq(pq + 1)(pq + 2)\cdots(pq + n - 1)$$

Definition 2.11 Let x be an element of the Weyl algebra A_1. Its *centralizer* is defined as $C(x) = \{y \in A_1 \mid xy = yx\}$

The centralizer plays a very important role: it connects many of the classifications of elements in A_1 (see Section 3.2). Now we consider some nice properties of the centralizers of non-scalar elements of A_1.

Theorem 2.12 ([A]) *Let $x \in A_1 - K$. The centralizer $C(x)$ is a commutative subalgebra of A_1 which is a finitely generated free $K[x]$-module.*

Before proving Theorem 2.12, we consider A_1 as the ring of all differential polynomials in the variable p with coefficients in $K[q]$, i.e., any element in A_1 is of the form $x = x(p) = \alpha_0 + \alpha_1 p + ... + \alpha_n p^n$ where $\alpha_i \in K[q]$ with multiplication defined by the relation $pa = ap + a'$ for $a \in K[q]$. Here it suffices to prove the theorem for $n \geq 1$, since if $n = 0$, then $C(x) = K[x]$ which is a polynomial ring. Here we define the *order* of x as the exponent of a non-zero term with the highest exponent. Clearly, the order is well defined. Before proving the theorem, we introduce a definition: for the largest n such that $\alpha_n \neq 0$, α_n is called the *leading coefficient* of x. We shall use the following two lemmas.

Lemma 2.13 ([F], 10.1) *If α_n, β_n are respectively the leading coefficients of two polynomials $f(p), g(p)$ of order m which commute with x then $\alpha_m = c\beta_m$ for some $c \in K$.*

Proof. Let $order(x) = n$ and $x = \tau_0 + \tau_1 p + ... + \tau_n p^n$ where $\tau_i \in K[q]$, $n \geq 1$. Since $x(p)f(p) = f(p)x(p)$, then by comparing the coefficient of p^{n+m-1} on both sides we obtain:

$$m\tau_n'\alpha_m + \tau_n\alpha_{m-1} + \tau_{n-1}\alpha_m = n\alpha_m'\tau_n + \alpha_m\tau_{n-1} + \alpha_{m-1}\tau_n,$$

Thus, the leading coefficient α_m satisfies the homogeneous linear equation: $m\tau_n'\alpha_m - n\alpha_m'\tau_n = 0$. Similarly, the leading coefficient β_m of $g(p)$ satisfies the same equation and, therefore $\alpha_m = c\beta_m$ for some nonzero constant c. $\square$

Lemma 2.14 ([F], 10.2) *The set of elements in $C(x)$ of order $\leq m$ is a finite dimensional vector space over K.*

Proof. This follows immediately from Lemma 2.13, by induction on the order m. $\square$

We proceed now with the proof of Theorem 2.12:

We first show that $C(x)$ is finitely generated: Let $Z_x = \{z \in \mathbb{Z} | \ z = order(y) \text{ for } y \in C(x)\}$. Since $C(x)$ is a ring and $order(fg) = order(f) + order(g)$, it follows that Z_x

is closed under addition. Let $\bar{Z}_x = \{\bar{z} \in \mathbb{Z}/n\mathbb{Z} | \ z \in Z_x\}$. Then $\bar{Z}_x$ is a subgroup of the additive cyclic group of all residue classes mod n, so $\bar{Z}_x$ is cyclic of order t and t is a divisor of n. Let $\bar{0} = \bar{z}_1, ..., \bar{z}_t$ be the t classes mod n of $\bar{Z}_x$ and let z_i be the minimal nonnegative integer of its class $\bar{z}_i$. Choose $f_i \in C(x)$ to be a polynomial of order z_i and clearly we can choose $f_1 = 1$. Such f_i's exist, since Z_x is closed under addition. Now it suffices to show that these f_i are free generators of $C(x)$ over $K[x]$. Let $f_1g_1 + \cdots + f_tg_t = 0$ for some polynomials $g_i \in K[x]$. If $g_j \neq 0$ for some j, then $order(f_kg_k) = order(f_jg_j)$ for some $k \neq j$. But

$$order(f_kg_k) \equiv order(f_k) \equiv z_k (mod\ n) \not\equiv order(f_jg_j) \equiv z_j (mod\ n).$$

We have a contradiction. Consequently $g_i = 0$ for all i. It remains to show that any element $f \in C(x)$ can be written as $f = f_1g_1 + \cdots + f_tg_t$ for some $g_i \in K[x]$. This is obtained by induction on the order of f. If $order(f) = 0$, then $f = c \in K$ by Lemma 2.13, and hence $f = cf_1$. Let $order(f) = m$. Since $m \in Z_x$, $m = z_i + sn$ for some integer $s \geq 0$, so $order(f) = order(f_ix^s)$. Then by Lemma 2.13, $g = f - cf_ix^s \in C(x)$ for some constant c, and $order(g) < order(f)$. Thus, by induction $f - cf_ix^s = f_1g_1 + \cdots f_tg_t$.

We turn now to prove that $C(x)$ is commutative. Let $f \in C(x)$ be a polynomial whose residue class of $order(f)$ mod n generates the cyclic group $\bar{Z}_x$. Then the set of all orders of the polynomials of the form

$$H(f, x) = y_0 + fy_1 + \cdots + f^{t-1}y_{t-1}, \ y_i \in K[x],$$

contains all but finitely many integers of Z_x (t is the order of $\bar{Z}_x$). So we can assume this contain all integers $z \in Z_x$ for which $z \geq r$, for some fixed r. Hence any $h \in C(x)$ can be written in the form $h = H_0(f, x) + h_0$, where $h_0 \in C(x)$ and $order(h_0) \leq r$. From Lemma 2.14, we know the set of all polynomials h_0 is finite dimensional, we say the dimension is l. Let $x^\lambda h = H_\lambda(f, x) + h_\lambda$, where $\lambda = 0, 1, ..., l$ and $order(h_\lambda) \leq r$. The polynomials h_λ are K-dependent, so $\Sigma k_\lambda h_\lambda = 0$, for $k_\lambda \in K$ where not all $k_\lambda = 0$. This yield that $(\sum k_\lambda x^\lambda)h = \sum k_\lambda H_\lambda(f, x)$. This proves that for every $h \in C(x)$ there exist $H(f, x)$ and $K(x)$ ($K(x) \neq 0$) with constant coefficients such that $K(x)h = H(f, x)$.

Then the set of all polynomials $H(f, x)$ commute with each other, and we know the polynomials of $C(x)$ commute with the polynomial of $K[x]$, so if $K_i(x)h_i = H_i(f, x)$ for $h_i \in C(x)$ $i = 1, 2$, then

$$K_1(x)K_2(x)h_1h_2 = (K_1h_1)(K_2h_2) = H_1H_2 = H_2H_1 = (K_2h_2)(K_1h_1) = K_2K_1h_2h_1.$$

Since A_1 is domain by Prop 2.6, then $h_1h_2 = h_2h_1$. □

From this theorem, we obtain the useful corollaries about the centralizer $C(x)$ stated below.

Corollary 2.15 ([D2], 4.3) *Let $x \in A_1 - K$. $C(x)$ is a maximal commutative subalgebra in A_1, and any maximal commutative subalgebra is the centralizer of each of its non-scalar elements.*

Proof. Clearly, $C(x)$ is a commutative subalgebra, so we just need to show it is maximal. Let $y \in A_1$ such that y commutes with $C(x)$. Then y commutes with x, and so $y \in C(x)$.

Suppose B is any maximal commutative subalgebra in A_1. Clearly, $B \neq K$. So there is $x \in B - K$ such that $B \subseteq C(x)$. By maximality, $B = C(x)$. □

Definition 2.16 Let B be a subalgebra of A_1. The *centre* of B denoted $Z(B)$ is defined as $Z(B) = \{y \in A_1 \mid xy = yx \text{ for all } x \in B\}$.

Corollary 2.17 ([D2], 4.4) *Let B be a non-scalar subalgebra of A_1, and $B' = Z(B)$.*

(1) If B is not commutative, then $B' = K$.

(2) If B is commutative, then B' is a maximal commutative subalgebra in A_1.

Proof. (1) Suppose $B' \neq K$. There exists a non-scalar element x in B'. Then $xy = yx$ for all $y \in B$, thus $B \subseteq C(y)$ and so B is commutative. Contradiction.

(2) Suppose B is commutative, then $B \subseteq B'$. Let $B'' = Z(B')$. Any $y \in B''$ commutes with B, so $y \in B'$ and hence $B'' \subseteq B'$. Since $B \neq K$, B' is commutative by Lemma 2.13. Then $B' \subseteq B''$ and hence $B' = B''$. Thus B' is a maximal commutative subalgebra in A_1. □

Remark 2.18 B is maximal commutative. $\Leftrightarrow Z(B) = B$.

Corollary 2.19 ([D2], 4.5) *Let $x, y \in A_1 - K$. $xy = yx \Leftrightarrow C(x) = C(y)$.*

Proof. Follows from Corollary 2.15. □

Corollary 2.20 ([D2], 4.6) *Let $x, y \in A_1 - K$. $xy \neq yx \Leftrightarrow C(x) \cap C(y) = K$.*

Proof. ($\Rightarrow$) Suppose $C(x) \cap C(y) \neq K$. If we have a non-scalar element $z \in C(x) \cap C(y)$, then $C(x) = C(z) = C(y)$.

($\Leftarrow$) This follows from Corollary 2.15. □

The next proposition describes certain elements in A_1 and what their centralizer look like.

Proposition 2.21 ([D2], 5.3) *Let i, j be positive integers such that $i \geq j$. Let $d = gcd(i, j)$, $i = i'd$ and $j = j'd$. Then,*

(1) $C(p) = K[p]$, $C(q) = K[q]$.

(2) $C(p^n) = K[p]$, $C(q^n) = K[q]$ for every positive integer n.

(3) If $i = j$, then $C(p^i q^j) = K[pq]$.

(4) If $i \neq j$ and $i' \neq j' + 1$, then $C(p^i q^j) = K[p^i q^j]$.

(5) If $i' = j' + 1$, then

$$p^i q^j = (p(pq + d - 1)(pq + 2d - 1) \cdots (pq + j'd - 1))^d \text{ and}$$

$$C(p^i q^j) = K[p(pq + d - 1)(pq + 2d - 1) \cdots (pq + j'd - 1)]$$

Proof. See [D2] Lemma 5.3. □

Chapter 3

Automorphisms of A_1

In this chapter we will describe what the automorphism group of A_1 looks like. In 2008, Belov-Kanel and Kontsevich [BK1] conjectured that the automorphism group of $A_n(\mathbb{C})$ is isomorphic to the group of the polynomial symplectomorphisms of a $2n$-dimensional affine space

$$Aut(A_n(\mathbb{C})) \simeq Aut(P_n(\mathbb{C})),$$

where $P_n(\mathbb{C})$ is the Poisson algebra over $\mathbb{C}$ which is the usual polynomial algebra $\mathbb{C}[x_1, ..., x_{2n}]$ endowed with the Poisson bracket:

$$\{x_i, x_j\} = \omega_{ij},\ 1 \leq i, j \leq 2n,$$

where $(\omega_{ij})_{1\leq i,j\leq 2n}$ is the standard skew-symmetric matrix:

$$\omega = \delta_{i,n+j} - \delta_{n+i,j}.$$

3.1 Morphisms of A_1

Since A_1 is non-commutative, when we give it the structure of a Lie algebra, it has many non-trivial Lie algebra properties. In this section, we consider A_1 as a Lie algebra. We will construct the automorphisms of the Lie algebra A_1. The following definition should come as no surprise.

Remark 3.1 A linear transformation ϕ: $L \to L'$ (L, L' Lie algebras over K) is called a *homomorphism* if $\phi([x, y]) = [\phi(x), \phi(y)]$, for all $x, y \in L$. Also, ϕ is called a *monomorphism* if $Ker\ \phi = 0$, an *epimorphism* if $Im\ \phi = L'$, and an *isomorphism* if it is both a monomorphism and an epimorphism. An *automorphism* of L is an isomorphism of L onto itself. We write $\mathfrak{gl}(V)$ for $End(V)$ viewed as a Lie algebra and call it the *general linear algebra* (because it is closely associated with the *general linear group* $GL(V)$ consisting of all invertible endomorphisms of V), where V is a finite dimensional vector space over K. Any subalgebra of a Lie algebra $\mathfrak{gl}(V)$ is called a *linear Lie algebra.*

Definition 3.2 A *representation* of a Lie algebra L is a homomorphism ϕ: $L \to \mathfrak{gl}(V)$ (V is a vector space over K). The *adjoint representation* ad: $L \to \mathfrak{gl}(L)$ sends x to $\mathrm{ad}(x)$, where $\mathrm{ad}(x)(y) = [x, y]$.

It is clear that ad is a linear transformation and preserves the bracket operation. We calculate:

$$\begin{aligned}
[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= \mathrm{ad}(x)\mathrm{ad}(y)(z) - \mathrm{ad}(y)\mathrm{ad}(x)(z) \\
&= \mathrm{ad}(x)([y, z]) - \mathrm{ad}(y)([x, z]) \\
&= [x, [y, z]] - [y, [x, z]] \\
&= [x, [y, z]] + [[x, z], y] \\
&= [[x, y], z] \\
&= \mathrm{ad}[x, y](z)
\end{aligned}$$

The kernel of ad consists of all $x \in L$ for which $\mathrm{ad}(x) = 0$, i.e., for which $[x, y] = 0$ for all $y \in L$. So $Ker\,\mathrm{ad} = Z(L)$ (the centre of L). We now consider L as the first Weyl algebra A_1.

Remark 3.3 $\mathrm{ad}(x)$ is a *derivation* of A_1, i.e., $\mathrm{ad}(x)(yz) = y(\mathrm{ad}(x)(z)) + (\mathrm{ad}(x)(y))z$. Since

$$[x, yz] = xyz - yzx = y(xz - zx) + (xy - yx)z = y[x, z] + [x, y]z$$

Remark 3.4 Suppose $x \in L$ is an element for which $\mathrm{ad}(x)$ is *nilpotent*, i.e., $\mathrm{ad}^n(x) = \mathrm{ad}(x)^n = 0$ for some $n > 0$. Then the usual exponential power series for a linear transformation over $\bar{K}$ (the algebraic closure of K) makes sense over K, because it has only finitely many terms: $\exp(\mathrm{ad}(x)) = 1 + \mathrm{ad}x + \frac{(\mathrm{ad}x)^2}{2!} + \frac{(\mathrm{ad}x)^3}{3!} + \cdots + \frac{(\mathrm{ad}x)^{n-1}}{(n-1)!}$. We claim that $\exp(\mathrm{ad}(x)) \in Aut(L)$. For this, we use the familiar Leibniz rule:

$$\frac{\mathrm{ad}^n(x)}{n!}(yz) = \sum_{i=0}^{n}(\frac{\mathrm{ad}^i(x)(y)}{i!})(\frac{\mathrm{ad}^{n-i}(x)(z)}{(n-i)!}).$$

Then we have:

$$\begin{aligned}
\exp(\mathrm{ad}(x))(y)\exp(\mathrm{ad}(x))(z) &= (\sum_{i=0}^{k-1}(\frac{\mathrm{ad}^i(x)(y)}{i!})(\sum_{j=0}^{k-1}(\frac{\mathrm{ad}^j(x)(z)}{j!}) \\
&= \sum_{t=0}^{2k-2}(\sum_{i=0}^{t}(\frac{\mathrm{ad}^i(x)(y)}{i!})(\frac{\mathrm{ad}^{n-i}(x)(z)}{(t-i)!})) \\
&= \sum_{t=0}^{2k-2}\frac{\mathrm{ad}^t(x)(yz)}{t!} \quad \text{(Leibniz rule)} \\
&= \sum_{t=0}^{k-1}\frac{\mathrm{ad}^t(x)(yz)}{t!} \quad (\mathrm{ad}^n(x) = 0) \\
&= \exp(\mathrm{ad}(x))(yz)
\end{aligned}$$

Therefore, $\exp(\mathrm{ad}(x))$ is invertible with inverse $\exp(-\mathrm{ad}(x))$.

Here we will give the key definition of this paper.

Definition 3.5 (1) (V a finite dimensional K-vector space) Any $x \in End(V)$ is called *semisimple* if the roots of its minimal polynomial over K are all distinct.

(2) (V an infinite dimensional K-vector space) Let $x \in End(V)$. Let $F(x) = \{v \in V| \dim V_v < \infty\}$ where $V_v = \sum_{n\geq 0} Kx^n(v)$ ($n \in \mathbb{Z}$). We say that x is *semisimple* if

- $F(x) = V$ and
- $x|_{V_v}$ is semisimple for all $v \in V$.

Moreover if K is algebraically closed, x is semisimple if and only if x is diagonalizable.

For more information about Lie algebras, the interested reader is referred to [HU].

3.2 Classification of elements in A_1 and the Dixmier partition

Next we will recall the partition of A_1 into different classes. The elements in different classes have different properties. Later we will find a relation among different classes (Refer to [D2] 6.1).

Let $\bar{A}_1$ be the algebra $A_1 \otimes_K \bar{K}$. Let $x \in A_1$ and let $y \in \bar{A}_1$. Set $V_y := \sum_{n\geq 0} K(\mathrm{ad}^n(x))y$ and put $F(x; A_1) := \{y \in A_1 \mid dimV_y < +\infty\}$, and $F(x; \bar{A}_1) := \{y \in \bar{A}_1 \mid dimV_y < +\infty\}$, then it follows that $F(x; \bar{A}_1) = F(x; A_1) \otimes_K \bar{K}$. Let $\lambda \in \bar{K}$ and let $F(x, \lambda; \bar{A}_1) := \{y \in \bar{A}_1 \mid (\mathrm{ad}_{\bar{A}_1}(x) - \lambda)^n y = 0, \text{ for some positive integer n}\}$. Now we have,

$$F(x; \bar{A}_1) = \bigoplus_{\lambda \in \bar{K}} F(x, \lambda; \bar{A}_1).$$

Let $N(x) = N(x; A_1) := \{y \in A_1 \mid \mathrm{ad}^n(x)(y) = 0 \text{ for some positive integer n}\} = F(x, 0; A_1)$, and $D(x, \lambda; \bar{A}_1) := \{y \in \bar{A}_1 \mid \mathrm{ad}_{\bar{A}_1}(x)(y) = \lambda y\}$. Also $D(x, \lambda; \bar{A}_1) \subset F(x, \lambda; \bar{A}_1)$ and

$$F(x, \lambda; \bar{A}_1) \neq 0 \quad \Leftrightarrow \quad D(x, \lambda; \bar{A}_1) \neq 0.$$

Let $D(x; \bar{A}_1) := \bigoplus_{\lambda \in \bar{K}} D(x, \lambda; \bar{A}_1)$, and $D(x) = D(x, A_1) = D(x; \bar{A}_1) \cap A_1$.
It is immediate that $N(x) \cap D(x) = C(x)$.

We will state some useful results about $F(x)$, $N(x)$ and $D(x)$ for some $x \in A_1$.

Theorem 3.6 ([D2], 6.5) *Let λ be a non-zero element in $\bar{K}$. Let $x \in \bar{A}_1$. Then $D(x, \lambda) = F(x, \lambda)$.*

Proof. See [D2] Lemma 6.5. □

Corollary 3.7 ([D2], 6.6) *Let $x \in A_1$, then either $F(x) = D(x)$ or $F(x) = N(x)$.*

Proof. We could suppose $K = \bar{K}$, then

$$F(x) = \sum_{\lambda \in K} F(x, \lambda) = N(x) + \sum_{\lambda \in K, \lambda \neq 0} D(x, \lambda) = N(x) + D(x).$$

Suppose $F(x) \neq N(x)$ and $F(x) \neq D(x)$. Then there exist a non-zero $\lambda \in K$, a non-zero $y \in D(x, \lambda)$, and a $z \in A_1$ such that $\mathrm{ad}(x)(z) \neq 0$ and $\mathrm{ad}^2(x)(z) = 0$. Then

$$\begin{aligned}(\mathrm{ad}(x) - \lambda)(yz) &= \mathrm{ad}(x)(yz) - \lambda yz \\ &= y(\mathrm{ad}(x)(z)) + (\mathrm{ad}(x)(y))z - \lambda yz \\ &= y(\mathrm{ad}(x)(z)) + \lambda yz - \lambda yz \\ &= y(\mathrm{ad}(x)(z)) \neq 0,\end{aligned}$$

and similarly

$$(\mathrm{ad}(x) - \lambda)^2(yz) = y(\mathrm{ad}^2(x)(z)) = 0.$$

Thus, $F(x, \lambda) \neq D(x, \lambda)$ which contradicts Theorem 3.6. $\square$

Theorem 3.8 *(Dixmier partition) The set $A_1 \setminus K$ is a disjoint union of the following non-empty subsets.*

$$\begin{aligned}\Delta_1 &= \{x \in A_1 \setminus K : D(x) = C(x),\ N(x) \neq C(x),\ N(x) = A_1\} \\ \Delta_2 &= \{x \in A_1 \setminus K : D(x) = C(x),\ N(x) \neq C(x),\ N(x) \neq A_1\} \\ \Delta_3 &= \{x \in A_1 \setminus K : D(x) \neq C(x),\ N(x) = C(x),\ D(x) = A_1\} \\ \Delta_4 &= \{x \in A_1 \setminus K : D(x) \neq C(x),\ N(x) = C(x),\ D(x) \neq A_1\} \\ \Delta_5 &= \{x \in A_1 \setminus K : D(x) = C(x),\ N(x) = C(x),\ C(x) \neq A_1\}\end{aligned}$$

Elements of Δ_1 are locally nilpotent and elements of Δ_3 are semisimple.

3.3 The Automorphism Group of A_1

What is the automorphism group of A_1? Dixmier answered this question in [D2]. We will summarize some of the key results of this paper. Let n be an integer ≥ 0 and let $\lambda \in K$. In the introduction, we introduced two endomorphisms $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ of A_1 such that $\Phi_{n,\lambda}(p) = p$, $\Phi_{n,\lambda}(q) = q + \lambda p^n$ and $\Phi'_{n,\lambda}(q) = q$, $\Phi'_{n,\lambda}(p) = p + \lambda q^n$. By Remark

3.4, we know that $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ are automorphisms of A_1. Let $G =< \Phi_{n,\lambda}, \Phi'_{n,\lambda} |\ n \in \mathbb{Z}^+,\ \lambda \in K >$. We will prove that $G = Aut(A_1)$ (see Theorem 3.16).

We now introduce a new definition to prove the main theorem: Theorem 3.16. Let $x = \sum \alpha_{ij} p^i q^j \in A_1$, the set $E(x)$ consists of pairs (i, j) such that $\alpha_{ij} \neq 0$. Let t, s be real numbers, then we set

$$\chi_{t,s}(x) = \sup_{(i,j) \in E(x)} (ti + sj),$$

(we agree that $\chi_{t,s}(0) = -\infty$). Define the set $E(x, t, s) \subseteq E(x)$ as the pairs $(i, j) \in E(x)$ such that $ti + sj = \chi_{t,s}(x)$. The polynomial $\sum_{(i,j) \in E(x,t,s)} \alpha_{ij} X^i Y^j$ is called the (t, s)-associated polynomial of x.

Lemma 3.9 ([D2], 7.2) *Let $x \in A_1$. Consider that $F(x)$ is finitely generated as $C(x)$-module. Then $F(x) = C(x)$.*

Proof. Refer to [D2] Lemma 7.2. □

Lemma 3.10 ([D2], 7.3) *Let t, s be positive integers. Let $x \in A_1$, $y \in F(x)$, $\upsilon = \chi_{t,s}(x)$, $\omega = \chi_{t,s}(y)$, and f and g be the (t, s)-associative polynomials of x and y respectively. We suppose that $\upsilon > t + s$ and that f not is a monomial. Then we have the following cases:*

(a) f^ω is proportional to g^υ;

(b) $s > t$, s is a multiple of t, and $f(X, Y)$ has the form $\lambda X^m (X^{s/t} + \mu Y)^n)$, where $\lambda, \mu \in K$ and m, n are integers ≥ 0;

(c) $t > s$, t is a multiple of s, and $f(X, Y)$ has the form $\lambda Y^m (Y^{s/t} + \mu X)^n)$, where $\lambda, \mu \in K$ and m, n are integers ≥ 0;

(d) $t = s$ and $f(X, Y)$ has the form $\lambda(\mu X + \nu Y)^m (\mu' X + \nu' Y)^n)$, where $\lambda, \mu, \nu, \mu', \nu' \in K$ and m, n are integers ≥ 0.

Proof. Refer to [D2] Lemma 7.3. □

Proposition 3.11 ([D2], 7.4) *Let t, s be positive integers, $x \in A_1$, $\upsilon = \chi_{t,s}(x)$, and f the (t,s)-associative polynomials of x. We suppose that:*

1. *$\upsilon > t + s$;*
2. *f is not a monomial;*
3. *we are not in one of cases (b), (c), (d) in Lemma 3.9.*

Then $F(x) = C(x)$.

Proof. Let $\Omega = \{\omega \in \mathbb{Z} \mid \exists y \in F(x) \text{ such that } \chi_{t,s}(y) = \omega\}$. Then $\Omega + \Omega \subseteq \Omega$ and, in particular, for each $\omega \in \Omega$ the set $\{0, \omega, 2\omega, ...\} \subset \Omega$. Let Ω' be the canonical image of Ω over $\mathbb{Z}/\upsilon\mathbb{Z}$, and since Ω' is finite, let $\Omega' = \{0, \omega_1, \omega_2, ..., \omega_r\}$. So the elements of Ω are:

0, υ, 2υ, 3υ, ...;

ω_1, $\omega_1 + \upsilon$, $\omega_1 + 2\upsilon$, $\omega_1 + 3\upsilon$, ...;

$\vdots$

ω_r, $\omega_r + \upsilon$, $\omega_r + 2\upsilon$, $\omega_r + 3\upsilon, ...$

Let $y_i \in F(x)$ such that $\chi_{t,s}(y_i) = \omega_i$. It suffices to show that $\forall y \in F(x)$, $y \in K[x]y_0 + K[x]y_1 + ... + K[x]y_r$. It is obvious for $\chi_{t,s}(y) = 0$. Now assume it is true for y with $\chi_{t,s} < n$. Suppose $n = \omega_i + m\upsilon$, then $\chi_{t,s}(x^m y_i) = n$. Let g, h be the (t,s)-associative polynomial of y, $x^m y_i$ respectively. By Lemma 3.10, g^υ and h^υ are proportional to f^n, thus g and h are proportional. Hence there exists $\zeta \in K$ such that $\chi_{t,s}(y - \zeta x^m y_i) < n$. We have $y - \zeta x^m y_i \in F(x)$ and $y - \zeta x^m y_i \in K[x]y_0 + K[x]y_1 + ... + K[x]y_r$. So $F(x) = \Sigma_i k[x]y_i$. By Lemma 3.9, we have $F(x) = C(x)$. ☐

For example: let $x = p^2 + q^3 \in A_1$. We have $\chi_{3,2}(x) = 6$ and the $(3,2)$-associative polynomial of x is $X^2 + Y^3$. Therefore, $F(x) = C(x)$.

Remark 3.12 Let V be the vector space $Kp + Kq$. Any element of the special linear group $SL(V)$ of V can be extended uniquely to an automorphism of A_1. Let $G' \subset G$ be the subgroup of $Aut(A_1)$ generated by $\Phi_{1,\lambda}$ and $\Phi'_{1.\lambda}$, for $\lambda \in K$. $G' \cong SL(V)$, since $\Phi_{1,\lambda} \mid V$ and $\Phi'_{1.\lambda} \mid V$ generate the group $SL(V)$. In particular, there is an element Ψ of G' such that $\Psi(p) = q$, $\Psi(q) = -p$. ($\Psi = \Phi'_{1,1} \circ \Phi_{1,-1} \circ \Phi'_{1,1}$)

Now we want to consider what kind of elements x of A_1 satisfy $N(x) = A_1$ or $D(x) = A_1$.

Lemma 3.13 ([D2], 8.3 and 8.4) *(1) If $x \in K[p]$, then $N(x) = A_1$.*

(2) If $x = \lambda p^2 + \mu q^2 + \nu$, where $\lambda, \mu, \nu \in K$, $\lambda \neq 0, \mu \neq 0$, then $D(x) = A_1$.

Proof. (1) We have $p \in N(x)$ and $[x, q] \in K[p]$, so $q \in A_1$. (2) We can suppose $K = \bar{K}$. Since $x = (\sqrt{\lambda}p + i\sqrt{\mu}q)(\sqrt{\lambda}q - i\sqrt{\mu}p) + \nu$, we have $m = \frac{1}{\sqrt{8i\sqrt{\lambda\mu}}} \begin{pmatrix} \frac{1}{2\sqrt{\lambda}} & \frac{1}{2\sqrt{\lambda}} \\ \frac{1}{2i\sqrt{\mu}} & \frac{1}{-2i\sqrt{\mu}} \end{pmatrix} \in SL(V)$ such that $\frac{1}{\sqrt{8i\sqrt{\lambda\mu}}} \begin{pmatrix} \frac{1}{2\sqrt{\lambda}} & \frac{1}{2\sqrt{\lambda}} \\ \frac{1}{2i\sqrt{\mu}} & \frac{1}{-2i\sqrt{\mu}} \end{pmatrix} \begin{pmatrix} \sqrt{\lambda}p + i\sqrt{\mu}q \\ \sqrt{\lambda}p - i\sqrt{\mu}q \end{pmatrix} = \frac{1}{\sqrt{8i\sqrt{\lambda\mu}}} \begin{pmatrix} p \\ q \end{pmatrix}$.

This means that there exists $\Phi \in G'$ such that $\Phi(x) = pq + \xi$ where $\xi \in K$. In the introduction, we observed that $[pq, p^\alpha q^\beta] = (\alpha - \beta)p^\alpha q^\beta$, so $D(x) = A_1$. ☐

The following theorem is the key theorem for the proof our goal for this chapter.

Theorem 3.14 ([D2], 8.8) *Let $x \in A_1$, such that $F(x) = A_1$. Then there exists $\Phi \in G$ such that either $\Phi(x) \in K[p]$ or $\Phi(x)$ has the form $\lambda p^2 + \mu q^2 + \nu$, where $\lambda, \mu, \nu \in K$, $\lambda \neq 0, \mu \neq 0$.*

Before proving this theorem, we first state the following technical lemma.

Lemma 3.15 ([D2], 8.7) *Let $x = \Sigma \alpha_{ij} p^i q^j \in A_1$. Let ρ be the smallest integer ≥ 0 such that $\alpha_{i0} = 0$ for $i > \rho$. Let σ be the smallest integer ≥ 0 such that $\alpha_{0j} = 0$ for $j > \sigma$. We suppose there exists integer $i_0 \geq 0$, $j_0 \geq 0$ such that $\alpha_{i_0 j_0} \neq 0$, $(i_0, j_0) \neq (1, 1)$, and $\sigma i_0 + \rho j_0 > \rho\sigma$. Then $F(x) \neq A_1$.*

Proof. If $i_0 = 0$, then $\rho j_0 > \rho\sigma$, and so $j_0 > \sigma$, which contradicts the definition of σ. So $i_0 > 0$ and $j_0 > 0$. There exist irrational numbers $t, s > 0$ such that $si_0 + tj_0 > t\sigma$ and $si_0 + tj_0 > \rho s$. Then there exist i', j' such that $\alpha_{i'j'} \neq 0$ and $si' + tj' = \chi_{s,t}(x)$. Then $si' + tj' > t\sigma$ and $si' + tj' > \rho s$. By a similar argument used to prove $i_0, j_0 > 0$, we have $i' > 0$ and $j' > 0$. If $i' = j' = 1$, then $s + t \leq si_0 + tj_0 \leq si' + tj' = s + t$, and hence $i_0 = i'$, $j_0 = j'$. Therefore $(i_0, j_0) = (i', j') = (1, 1)$. Contradiction. So $i' > 1$ and $j' > 1$. The (s, t)-associative polynomial f of x is $\alpha_{i'j'}X^{i'}Y^{j'}$. This is a monomial since $(i_1, j_1), (i_2, j_2) \in E(f)$. We have $si_1 + tj_1 = si_2 + tj_2$, so $s(i_1 - i_2) = t(j_1 - j_2)$, but s and t are linearly independent, so $i_1 = i_2$ and $j_1 = j_2$. Suppose $i' \geq j'$ and let $y_n = \mathrm{ad}^n(x)(p)$.

Claim: the (s, t)-associative polynomial g of y_n is

$$\beta X^{1+n(i'-1)}Y^{n(j'-1)}$$

where $\beta \in K$, and $\beta \neq 0$. For $n = 0$, it is clear. Now assume it is true for n, then the (s, t)-associative polynomial of $y_{n+1} = [x, y_n]$ is

$$\begin{aligned}
& \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X} \\
= \; & i'\alpha_{i'j'}X^{i'-1}Y^{j'}n(j'-1)\beta X^{n(i'-1)}Y^{n(j'-1)-1} \\
& -j'\alpha_{i'j'}X^{i'}Y^{j'-1}(n(i'-1)+1)\beta X^{n(i'-1)}Y^{n(j'-1)} \\
= \; & (-j' + nj' - ni')\alpha_{i'j'}\beta X^{i'+1+n(i'-1)-1}Y^{j'+n(j'-1)-1}
\end{aligned}$$

where $-j' + nj' - ni' \leq -j' \neq 0$. Then

$$\chi_{s,t}(y_n) = s(1 + n(i'-1)) + t(n(j'-1)).$$

Since $i', j' > 1$, $\chi_{s,t}(y_n)$ tends to ∞ as n tends to ∞. So p is not in $F(x)$ and hence $F(x) \neq A_1$. If $i' \leq j'$, then q is not in $F(\Psi(X))$ and hence $F(\Psi(X)) \neq A_1$. $\square$

Now it is time to prove Theorem 3.14:

Let ρ, σ be the integers of Lemma 3.15. We prove it by induction on $\rho + \sigma$.

(i) If both $\rho \leq 2$ and $\sigma \leq 2$, then $\sigma i_0 + \rho j_0 \leq \rho\sigma \leq 4$ and hence $\chi_{\sigma,\rho} \leq 2$. Then x has the form

$$\alpha p^2 + 2\beta pq + \gamma q^2 + \delta p + \varepsilon q + \zeta \qquad (\alpha, \beta, ..., \zeta \in K).$$

If $\beta^2 - \alpha\gamma = 0$, then $\alpha p^2 + 2\beta pq + \gamma q^2 = \frac{1}{\gamma}(\beta p + \gamma q)^2 + \beta$. Thus there exists

$\begin{pmatrix} \beta+1/\beta & -\alpha \\ -\gamma & \beta \end{pmatrix} \in SL(V)$ such that

$$\begin{pmatrix} \beta+1/\beta & -\alpha \\ -\gamma & \beta \end{pmatrix} \begin{pmatrix} \beta p+\gamma q \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix}$$

and so there exists $\Phi \in G'$ such that $\Phi_1(x) = \alpha' p^2 + \delta' p + \varepsilon' q + \zeta'$. If $\varepsilon' = 0$, then we are done. If $\varepsilon' \neq 0$, we assume $\varepsilon' = 1$ so that

$$\Phi_{2,-\alpha'}(\Phi_1(x)) = \alpha' p^2 + \delta' p + q - \alpha' p^2 + \zeta' = \delta' p + q + \zeta'.$$

Thus $\Phi'_{1,-\frac{1}{\zeta'}} \circ \Phi_{2,-\alpha'} \circ \Phi_1$ is the required automorphism.

If $\beta^2 - \alpha\gamma \neq 0$, then

$$\begin{aligned} \Phi_{1,-\frac{\beta}{\gamma}}(x) &= \alpha p^2 + 2\beta p(q - \frac{\beta}{\gamma}p) + \gamma(q - \frac{\beta}{\gamma}p)^2 + \delta p + \varepsilon(q - \frac{\beta}{\gamma}p) + \zeta \\ &= \alpha' p^2 + \gamma' q^2 + \delta' p + \varepsilon' q + \zeta \end{aligned}$$

where $\alpha' \neq 0$, $\gamma' \neq 0$. Now we have

$$\begin{aligned} y &= \Phi_{0,-\frac{\varepsilon'\gamma'^{-1}}{2}}(\Phi_{1,-\frac{\beta}{\gamma}}(x)) \\ &= \alpha' p^2 + \delta' p + \zeta' + \gamma'(q - \frac{1}{2}\varepsilon'\gamma'^{-1})^2 + \varepsilon'(q - \frac{1}{2}\varepsilon'\gamma'^{-1}) \\ &= \alpha' p^2 + \delta' p + \gamma' q^2 + \zeta''. \end{aligned}$$

So $\Phi'_{0,-\frac{\delta'}{2\alpha'}} \circ \Phi_{0,-\frac{\varepsilon'}{2\gamma'}} \circ \Phi_{1,-\frac{\beta}{\gamma}}$ is the required automorphism in G.

(ii) If ρ is an arbitrary nonnegative integer, then using $\Psi \in G'$, we can assume $\rho \geq \sigma$. If $\sigma \leq 1$, then we can write

$$x = \alpha_{00} + \alpha_{10}p + ... + \alpha_{\rho 0}p^\rho + \alpha_{01}q + \alpha_{11}.$$

It is clear for $\rho \leq 1$. Assume the thorem is true for $\rho - 1$. If $\alpha_{11} \neq 0$, we can assume $\alpha_{11} = 1$. Then

$$\begin{aligned} \Phi_{\rho-1,-\alpha_{\rho 0}}(x) &= \alpha_{00} + \alpha_{10}p + ... + \alpha_{\rho 0}p^\rho + \alpha_{01}(p - \alpha_{\rho 0}p^{\rho-1}) + p(q - \alpha_{\rho 0}p^{\rho-1}) \\ &= \alpha_{00} + \alpha_{10}p + ... + \alpha_{\rho-2,0}p^{\rho-2} - \alpha_{01}\alpha_{rho0}p^{\rho-1} + \alpha_{01}q + pq \end{aligned}$$

and we are done by the induction hypothesis. If $\alpha_{11} = 0$ and $\alpha_{01} \neq 0$, we can assume $\alpha_{01} = 1$. Then

$$\begin{aligned}\Phi_{\rho,-\alpha_{\rho 0}}(x) &= \alpha_{00} + \alpha_{10}p + ... + \alpha_{\rho 0}p^{\rho} + q - \alpha_{\rho 0}p^{\rho} \\ &= \alpha_{00} + \alpha_{10}p + ... + \alpha_{\rho-1,0}p^{\rho-1} + q\end{aligned}$$

and again we are done by the induction hypothesis.

(iii) Now we suppose $\rho > 2$ and $\sigma > 2$, and we assume the thorem is true for $\rho + \sigma < n$. We want to show it for $\rho + \sigma = n$. So suppose $\rho \geq \sigma \geq 2$ and $\rho > 2$, then $\rho + \sigma < \rho\sigma$. If $(i, j) \in E(x)$, then by Lemma 3.15, either $\sigma i + \rho j \leq \rho\sigma$ or $i = j = 1$, so that $\sigma i + \rho j = \sigma + \rho < \rho\sigma$. So $\chi_{\sigma,\rho}(x) = \rho\sigma$ and the (σ, ρ)-associative polynomial of x has the form

$$(*) \quad f(X, Y) = \alpha_{\rho 0}X^{\rho} + ... + \alpha_{0\sigma}Y^{\sigma} \quad \text{where } \alpha_{\rho 0} \neq 0, \alpha_{0\sigma} \neq 0$$

By Lemma 3.11, when $t = \rho$ and $s = \sigma$, we see that we are in one of cases (b), (c) or (d) of Lemma 3.10. As $\rho \geq \sigma$, we are in either case (b) or (d). Suppose we have case (b). Then ρ is a multiple of σ and f is proportional to $(X^{\rho/\sigma} + \mu Y)^{\sigma}$, where $\mu \in K$, $\mu \neq 0$. If we multiply x by a scalar, we can suppose

$$x = (p^{\rho/\sigma} + \mu q)^{\sigma} + \sum_{(i,j)\in E} \alpha_{ij}p^i q^j,$$

where $\sigma i + \rho j < \rho\sigma$, when $(i, j) \in E$. Then

$$y = \Phi_{\rho/\sigma,-1/\mu}(x) = \mu^{\sigma}q^{\sigma} + \sum_{(i,j)\in E} \alpha_{ij}p^i(q - \mu^{-1}p^{\rho/\sigma}).$$

We have

$$\chi_{\sigma,\rho}(q - \mu^{-1}p^{\rho/\sigma}) = \rho \quad \text{and } \chi_{\sigma,\rho}(p) = \sigma,$$

so

$$\chi_{\sigma,\rho}(\sum_{(i,j)\in E} \alpha_{ij}p^i(q - \mu^{-1}p^{\rho/\sigma})^j) < \rho\sigma.$$

Let $\sigma_1 = \sigma$ and $\rho_1 < \rho$, then by the induction hypothesis, there exists $\Phi \in G$ such that $\Phi(y)$ is as described in the theorem. Since $\Phi(y) = \Phi \circ \Phi_{\rho/\sigma,-1/\mu}(x)$, the lemma holds for this case. Suppose x has the form (d). Then $\rho = \sigma$ and f is proportional to $(X + \mu Y)^t (X + \nu Y)^{\rho - t}$, where $\mu, \nu \in K$, and t is an integer such that $0 \leq t \leq r$. If we multiply x by a scalar, we suppose

$$x = (p + \mu q)^t (p + \nu q)^{\rho - t} + \sum_{(i,j) \in E(x)} \alpha_{ij} p^i q^j,$$

where $i + j < \rho$, when $(i, j) \in E(x)$. We can assume $t > 0$, otherwise we could consider $x = (p + \nu q)(p + \nu q)^{\rho - 1} + \sum_{(i,j) \in E(x)} \alpha_{ij} p^i q^j$. Then

$$y = \Phi_{1, -1/\mu}(x) = \mu^t q^t ((1 - \nu \mu^{-1}) p + \nu q)^{\rho - t} + \sum_{(i,j) \in E(x)} \alpha_{ij} p^i (q - \mu^{-1})^j.$$

Let $\sigma_1 = \sigma = \rho$ and $\rho_1 < \rho$, and use the same argument as in case (b). $\square$

Now that all the preparation work is done, we will prove the main theorem of this chapter.

Theorem 3.16 ([D2], 8.10) *The automorphism group of A_1 is generated by automorphisms $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ for all integers $n \geq 0$ and $\lambda \in K$ ($Aut(A_1) = G$).*

Proof. Let Θ be any automorphism of A_1, we will show $\Theta \in G$. We know $N(p) = A_1$, so that $N(\Theta(p)) = A_1$ and thus by Lemmas 3.13 and 3.14, we have $\Theta(p) \in K[p]$. Since $p \in C(\Theta(p)) = K[\Theta(p)]$(Prop 2.21), we have $\Theta(p) = \alpha + \beta p$ where $\alpha, \beta \in K$. Then $\frac{1}{\beta}\Phi_{0,-\frac{\alpha}{\beta}}(\Theta(p)) = (\alpha + \beta p + \beta(-\frac{\alpha}{\beta}))\frac{1}{\beta} = p$. So we can assume $\Theta(p) = p$ and thus

$$[p, \Theta(q) - q] = \Theta[p, q] - [p, q] = 1 - 1 = 0.$$

Hence $\Theta(q) - q \in K[p] = C(p)$ and so $\Theta(q) \in q + K[p]$. Thus, Θ is a composition of automorphisms $\Phi_{n,\lambda}$'s. $\square$

Chapter 4

Diagonalizable elements of A_1

4.1 Maximal diagonalizable subalgebras of A_1

In this section, we will talk about the maximal diagonalizable subalgebra of A_1. In the previous chapter, we studied the automorphism group of A_1. If we can find a maximal diagonalizable subalgebra, then we can describe the other maximal diagonalizable subalgebras.

Remark 4.1 Let L be a Lie algebra. A subalgebra T of L is called *diagonalizable* if we can write $L = \bigoplus L_\alpha$, where $L_\alpha = \{x \in L \mid [t, x] = \alpha(t)x \text{ for all } t \in T\}$ (α is a function $T \to K$). A *maximal diagonalizable subalgebra* H of L is a diagonalizable subalgebra that is not properly included in any other diagonalizable subalgebras.

Remark 4.2 Any diagonalizable subalgebra T of L is abelian. $T = \bigoplus(T \cap L_\alpha)$, so we can assume any element of T is an eigenvector of $\mathrm{ad}(T)$. We will show $\mathrm{ad}(T)(x) = 0$, for all $x \in T$. Suppose, on the contrary, that $[x, y] = ay$ ($a \neq 0 \in K$) for some nonzero $y \in T$. Then $[y, x] = -ay$. Since x is an eigenvector of $\mathrm{ad}_T\, y$, $[y, x] = -ay = bx$ ($b \in K$). If $b \neq 0$, then $x = (-a/b)y$ and hence $[x, y] = 0$. Contradiction. So $a = b = 0$.

We now introduce some technical definitions.

Definition 4.3 A *totally ordered set* is a set S with a binary relation $\geq$ on it such that the following hold for all $a, b, c \in S$:

- $a \geq a$.
- If $a \geq b$ and $b \geq a$, then $a = b$.

- If $a \geq b$ and $b \geq c$, then $a \geq c$.
- Either $a \geq b$ or $b \geq a$.

Definition 4.4 Let S be a set equipped with a total order $\geq$, and let $S^n = S \times \cdots \times S$ be the $n-$ fold product of S. Then the *lexicographic order* $\geq$ on S^n is defined as follows:

If $a = (a_1, ..., a_n) \in S^n$ and $b = (b_1, ..., b_n) \in S^n$, then $a \geq b$ if $a_1 \geq b_1$ or $a_1 = b_1$, $a_2 = b_2$,..., $a_k = b_k$, and $a_{k+1} \geq b_{k+1}$ for some $k = 1, 2, ..., n-1$.

For example: let $S = \mathbb{N}$. $(5, 1, 0) > (4, 9, 9)$ and $(3, 3, 5) > (3, 3, 3)$.

Now we have

$$[pq, p^i q^j] = pqp^i q^j - p^i q^j pq = -ip^i q^j + p^{i+1} q^{j+1} + jp^i q^j - p^{i+1} q^{j+1} = (j-i)p^i q^j.$$

From this we conclude the subalgebra $Kpq + K$ is a maximal diagonalizable subalgebra of A_1.

Theorem 4.5 *$\mathfrak{h} = Kpq + K$ is a maximal commutative diagonalizable subalgebra of A_1.*

Proof. Step (1): $\mathfrak{h}$ is diagonalizable.

Let $\sum_{i,j} k_{ij} \, p^i q^j \in A_1$, where $k_{ij} \in K$, and let $\alpha pq + \beta \in \mathfrak{h}$, where $\alpha, \beta \in K$.

We have

$$[\alpha pq + \beta, \sum_{i,j} k_{ij} \, p^i q^j] = \sum_{i,j} k_{ij} \, \alpha [pq, p^i q^j] = \sum_{i,j} \alpha (j-i) k_{ij} p^i q^j.$$

By lemma 2.4, we know $\{p^i q^j : i, j \in \mathbb{N}^n\}$ is a basis of A_1 and hence A_1 has a basis of eigenvectors.

Step (2): Any diagobalizable subalgebra of A_1 containing $\mathfrak{h}$ is in $K[pq]$.

Suppose $\mathfrak{h}$ is not maximal, then there exits an $\mathfrak{h}'$ which is a commutative diagonalizable subalgebra of A_1 properly containing $\mathfrak{h}$. First, we show that $\mathfrak{h}'$ is contained in $K[pq]$. Clearly, $[z, \alpha pq] = 0$ for all $z \in \mathfrak{h}'$ and $\alpha pq \in \mathfrak{h}$. This implies $z \in C[pq]$. Therefore $\mathfrak{h}' \subseteq C[pq] = K[pq]$ (Prop 2.21).

Step (3): Claim: For $i \neq j$, $[(pq)^n, p^iq^j] = n(j-i)p^{i+n-1}q^{j+n-1}+$ lower terms (by lexicographical order).

For n=1, $[pq, p^iq^j] = (j-i)p^iq^j$

By induction, assume the claim is true for n, then

$$
\begin{aligned}
& [(pq)^{n+1}, p^iq^j] = (pq)^n pqp^iq^j - p^iq^jpq(pq)^n \\
= \; & (pq)^n(p^{i+1}q^{j+1} - ip^iq^j) - (p^{i+1}q^{j+1} - jp^iq^j)(pq)^n \\
= \; & [(pq)^n, p^{i+1}q^{j+1}] - i(pq)^np^iq^j + jp^iq^j(pq)^n \\
= \; & n(j-i)p^{i+n}q^{j+n} - ip^{i+n}q^{j+n} + jp^{i+n}q^{j+n} + \text{lower terms} \;\; \text{(by hypothesis)} \\
= \; & (n+1)(j-i)p^{i+n}q^{j+n} + \text{lower terms.}
\end{aligned}
$$

Step (4): show $\mathfrak{h}' = \mathfrak{h}$.

Let $b \in \mathfrak{h}'$, we can write $b = \sum_n \alpha_n(pq)^n + \alpha_{n_0}(pq)^{n_0}$, where $\alpha_{n_0}(pq)^{n_0}$ is the leading term. Since b is diagonalizable, there exists a basis $\{e_s\}$ of A_1 such that $[b, e_s] = c_se_s$ for some $c_s \in K$. We can write $e_s = e + f$ where $e = \sum_t \beta_t p^t q^t$ and $f = \sum_{i \neq j} \beta_{ij}p^iq^j$. Let $\beta_{i_0j_0}p^{i_0}q^{j_0}$ be the leading term of f, where $\beta_{i_0j_0} \neq 0$. Assume $f \neq 0$.
By step (3), we have

$$
\begin{aligned}
& [b, e_s] = [b, e+f] = [b, e] + [b, f] = [b, f] \\
= \; & [\sum_n \alpha_n(pq)^n + \alpha_{n_0}(pq)^{n_0}, \sum_{i,j} \beta_{ij}p^iq^j + \beta_{i_0j_0}p^{i_0}q^{j_0}] \\
= \; & \beta_{i_0j_0}\alpha_{n_0}n_0(j_0 - i_0)p^{i_0+n_0-1}q^{j_0+n_0-1} + \text{lower terms} \\
= \; & c_se_s = c_se + c_sf.
\end{aligned}
$$

Since $\beta_{i_0j_0}\alpha_{i_0}n_0(j_0 - i_0) \neq 0$, $c_s \neq 0$. Suppose b is not in $\mathfrak{h}$, then the degree of the leading term n_0 is larger than 1 and therefore $(i_0 + n_0 - 1) + (j_0 + n_0 - 1) > i_0 + j_0$. Therefore, $\beta_{i_0j_0}\alpha_{i_0}n_0(j_0 - i_0)$ is not in f. It follows that $f = 0$ and $e_s = e \in C[pq]$. So the basis $\{e_s\} \in C[pq]$, and hence $C[pq] = A_1$. But this is not possible, since for example $p \in A_1 \backslash C[pq]$. Contradiction. $\square$

We now know many of the maximal commutative diagonalizable subalgebras, but we want to know all such subalgebras of A_1. We will consider two situations: (1) K is algebraically closed, (2) K is not algebraically closed.

Theorem 4.6 ([D2], 9.2) *Let $x \in A_1 - K$. Then the following are equivalent:*

(1) x is semisimple,

(2) There exist an automorphism Φ of A_1 such that $\Phi(x)$ has the form $\lambda p^2 + \mu q^2 + \nu$ where $\lambda, \mu, \nu \in K$, $\lambda \neq 0$,$\mu \neq 0$,

Moreover, If K is algebraically closed, there exists $\Phi \in G'$ such that $\Phi(x) = \delta pq + \zeta$ where $\delta, \zeta \in K$ and $\delta \neq 0$.

Proof. This result follows from Lemmas 3.13 and 3.14. □

From this theorem, we obtain that if K is algebraically closed then the subalgebra $Kpq + K$ under the automorphism group G is the only maximal commutative diagonalizable subalgebra of A_1.

If K is not algebraically closed, the above statement is not ture. Hence we want to know what conditions should be placed on λ and μ so that $x = \lambda p^2 + \mu q^2 + \nu$ is diagonalizable.

Lemma 4.7 *Let $x = \lambda p^2 + \mu q^2 + \nu$ where $\lambda, \mu, \nu \in K$, $\lambda \neq 0$ and $\mu \neq 0$. If $\sqrt{-\frac{\mu}{\lambda}} \in K$, then x is diagonalizable.*

Proof. If $\sqrt{-\frac{\mu}{\lambda}} \in K$, then we can write $x = \lambda(p - \sqrt{-\frac{\mu}{\lambda}}q)(p + \sqrt{-\frac{\mu}{\lambda}}q)$.
Let $M = \begin{pmatrix} 1 & -\sqrt{-\frac{\mu}{\lambda}} \\ 1 & \sqrt{-\frac{\mu}{\lambda}} \end{pmatrix}$, so $M \cdot \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} P - \sqrt{-\frac{\mu}{\lambda}}q \\ p + \sqrt{-\frac{\mu}{\lambda}}q \end{pmatrix}$. Thus, $\det M = 2\sqrt{-\frac{\mu}{\lambda}}$.
Let $M' = \begin{pmatrix} \det^{-1} M & -\sqrt{-\frac{\mu}{\lambda}} \\ \det^{-1} M & \sqrt{-\frac{\mu}{\lambda}} \end{pmatrix}^{-1}$ then $M' = \begin{pmatrix} \sqrt{-\frac{\mu}{\lambda}} & \sqrt{-\frac{\mu}{\lambda}} \\ -\frac{1}{2}\sqrt{-\frac{\lambda}{\mu}} & \frac{1}{2}\sqrt{-\frac{\lambda}{\mu}} \end{pmatrix}$. Since $\det M'$= 1, hence $M' \in SL(V)$ and

$$M' \cdot \begin{pmatrix} P - \sqrt{-\frac{\mu}{\lambda}}q \\ p + \sqrt{-\frac{\mu}{\lambda}}q \end{pmatrix} = \begin{pmatrix} \sqrt{-\frac{\mu}{\lambda}} & \sqrt{-\frac{\mu}{\lambda}} \\ -\frac{1}{2}\sqrt{-\frac{\lambda}{\mu}} & \frac{1}{2}\sqrt{-\frac{\lambda}{\mu}} \end{pmatrix} \cdot \begin{pmatrix} P - \sqrt{-\frac{\mu}{\lambda}}q \\ p + \sqrt{-\frac{\mu}{\lambda}}q \end{pmatrix} = \begin{pmatrix} 2\sqrt{-\frac{\mu}{\lambda}}p \\ q \end{pmatrix}.$$

So there exists $\Phi \in G'$ such that $\Phi(x) = \delta pq + \zeta$, where $\delta \neq 0$. □

4.2 Semisimple elements of A_1

Let K be an algebraically closed field. First, we consider the following problem:

Problem 4.1 (1) Suppose $x \in A_1$ is semisimple and let Φ be any algebra endomorphism of A_1. Is $\Phi(x)$ also semisimple?

(2) If $\Phi(x)$ is semisimple for all semisimple x, is Φ an automorphism of A_1?

Actually, if true, this would show the Dixmier's conjecture which says that any endomorphism of A_1 is an automorphism(see [D2] Problem 11.1), but it has been shown that if the Dixmier conjecture holds for A_n then the Jacobian conjecture holds ([C] Theorem 4.2). We don't know whether the converse is true.

Remark 4.8 Let $F : K^n \to K^n$ be a polinomial map. Let $\Delta F = detJ(F)$ where $J(F)$ is its Jacobian matrix. The *Jacobian conjecture* states that:

If ΔF is a non-zero constant on K^n, then F has an inverse polynomial map on the whole of K^n.

To answer Problem 4.1, we can assume $x = pq \in A_1$, since K is algebraically closed and semisimple elements of A_1 are all conjugate under the automorphism group G. For (2), since A_1 is simple, it suffices to show that Φ is surjective. If $\Phi(pq)$ is semisimple, then we could assume $\Phi(pq) = pq$, so that $\Phi(p)\Phi(q) = pq$. By Lemma 2.6, $deg(\Phi(p)\Phi(q)) = deg(\Phi(p)) + deg(\Phi(q)) = 2$. If $deg(\Phi(p)) = 2$ and $deg(\Phi(q)) = 0$, let $\Phi(p) = \alpha p^2 + \beta pq + \gamma q^2 + \delta$ where α, β, γ and $\delta \in K$ and let $\Phi(q) = \delta'$. Then $(\alpha p^2 + \beta pq + \gamma q^2)\delta' = pq$ and it follows that $\alpha = \gamma = \delta = 0$ and $\delta' = \frac{1}{\beta}$. But, then $\Phi(pq) = \Phi(p) = pq$, which contradicts the fact that A_1 is simple. So $deg(\Phi(p)) = deg(\Phi(q)) = 1$. Now assume $\Phi(p) = \alpha p + \beta q + \delta$ and $\Phi(q) = \alpha' p + \beta' q + \delta'$. Since $(\alpha p + \beta q + \delta)(\alpha' p + \beta' q + \delta') = pq$, we have four cases to consider: $\Phi(p) = kp$ and $\Phi(q) = \frac{1}{k}q$, $\Phi(p) = -kp$ and $\Phi(q) = -\frac{1}{k}q$, $\Phi(p) = kq$ and $\Phi(q) = \frac{1}{k}p$, $\Phi(p) = -kq$ and $\Phi(q) = -\frac{1}{k}p$ for some $k \in K$. Since K is algebraically closed, we can assume $k = 1$. In all cases, Φ is surjective, and hence Φ is a automorphism of A_1.

For (1), if Φ is an arbitrary endomorphism of A_1, it seems to be hard to answer this problem in general. To prove the Jacobian conjecture, however, it suffices to show that particular endomorphisms of A_1 are automorphisms, namely, those of the form $\phi : A_1 \to A_1$ such that $\phi(q) = F$ and $\phi(p) = D$, where $F \in K[q]$, D is a derivation of $K[q]$ and $[D, F] = 1$. Note that for $a \in A_1$,

$$deg([p, a]) \le deg(a) - 1.$$

There exists an $n \in \mathbb{N}$ such that $\mathrm{ad}^n(p)(a) = 0$. Since

$$\phi(\mathrm{ad}(p)(a)) = \mathrm{ad}(D)(\phi(a)),$$

we have that $(\mathrm{ad})^n(D)(\phi(a)) = 0$. If ϕ is an automorphism, then D is locally nilpotent. It follows that $K[F] = K[q]$, which is the Jacobian conjecture (for more details see [C]).

Now it suffices to prove that ϕ preserves semisimple elements. Suppose D is a derivation of $K[q]$. Then $D(q^i) = iq^{i-1}D(q)$ and hence

$$(D - D(q)p)(q^j) = 0.$$

Since $\{q^j\}$ forms a basis of $K[q]$, we have that $D = D(q)p = Hp$, where $H \in K[q]$. Since $D(F) = 1$, we have $F = \alpha q + \beta$ where α, $\beta \in K$, $\alpha \neq 0$. Also, if $[D, F] = D(F) = 1$, then $H = \frac{1}{\alpha}$ and $D = \frac{1}{\alpha}p$. So $\phi(pq) = pq + \frac{1}{\alpha}p$, which is semisimple.

It is almost trivial to prove the Jacobian conjecture for $n = 1$.

To say that $x \in A_1$ is diagonalizable has two possible interpretations: one is that x is diagonalizable on A_1, as considered above. The other is that x is diagonalizable on $K[X]$. So we want to consider another problem:

Problem 4.2 Let $x \in A_1$. Are the two notions equivalent?

The answer is NO. For example, $p^2 + q^2$ is diagonalizable on A_1, but not diagonalizable on $K[X]$. Also $p^n q^n$ (n is an integer > 1) is diagonalizable on $K[X]$, but not diagonalizable on A_1.

By Remark 2.10, we know $pq \in C(p^n q^n)$, but pq is diagonalizable on A_1. We want to find some relations between these two notions, now we can ask:

Problem 4.3 Let $x \in A_1$ be an element which is diagonalizable on $K[X]$. Does there exist an element in $C(x)$ which is diagonalizable on A_1?

Here we will assume that $K = \mathbb{C}$. Since x is diagonalizable on $\mathbb{C}[X]$, we can write $\mathbb{C}[X] = \bigoplus_\lambda V_\lambda$, where the $\lambda' s$ are the eigenvalues of x. The centralizer $C(x)$ acts on V_λ naturally, we would like to show this action is diagonalizable. First step, we will show that V_λ is finite-dimensional for every λ.

We now give the following technical theorem, this is the standard theorem on Ordinary Differential Equations.

Theorem 4.9 *Let matrices $A(t) = (a_{ij}(t))$ be given with elements depending on t. If $A(t)$ is real-valued (complex-valued) and continuous on the (arbitrary) interval J, then the set of real (complex) solutions $y(t)$ of the homogeneous equation $y' = A(t)y$ forms an n-dimensional real (complex) linear space.*

For fixed $\tau \in J$, the mapping

$$\eta \to y(t; \tau, \eta) \text{ for every } \tau \in \mathbb{R}^n(\mathbb{C}^n)$$

defines an isomorphism (a linear, bijective mapping) between $\mathbb{R}^n$ ($\mathbb{C}^n$) and the space of solutions.

Proof. See [WW] Theorem 15.1. □

Lemma 4.10 *Let $x \in A_1 - K$. The space V_λ is finite-dimensional.*

Proof. Let $D \neq 0$. and let $D = \sum_{i=1}^n f_i p^i \in A_1$. Write $D = \sum_{i=1}^{n-1} f_i p^i + f_n p^n$, and divide both sides by f_n, let $D' = \frac{D}{f_n} = \sum_{i=1}^{n-1} \frac{f_i}{f_n} p^i + p^n$ where $f_n \neq 0$. Since the zero set of f_n is finite, there is an open neighborhood U in $\mathbb{C}$ on which all $\frac{f_i}{f_n}$ are continuous. By Theorem 4.9, the dimension of $\{g \in K[X] | D'(g) = 0\}$ is n.

For every $y \in V_\lambda$, we have $x(y) = \lambda y$. Let $D = x - \lambda$, then $V_\lambda = \ker D$ and hence V_λ is finite-dimensional for every λ. $\square$

Next, we think that the elements which are diagonalizable on $\mathbb{C}[X]$ are in $C(p(q-c))$ where $c \in \mathbb{C}$, but we do not have a complete answer.

First, let $x = \sum_{\alpha,\beta} c_{\alpha\beta} p^\alpha q^\beta \in A_1$ be diagonalizable on $\mathbb{C}[X]$. Let D_m be the sum of terms of x such that $m = \beta - \alpha$ is maximal. Applying D_m to the leading term of an eigenvector, we have

$$D_m(X^n) = \sum_{\alpha,\beta} \frac{c_{\alpha\beta}(n+\beta)!}{(n+m)!} X^{n+m}.$$

If $m > 0$, then $\sum_{\alpha,\beta} c_{\alpha\beta}(n+\beta)! = 0$. This holds for infinitely many $n \in \mathbb{N}$, hence all $c_{\alpha\beta}$ are 0 and $D_m = 0$, so $m = 0$. Thus x has the form

$$\sum_{m \leq 0} \sum_{\beta - \alpha = m} c_{\alpha\beta} p^\alpha q^\beta.$$

We can write $x = D_m + D_s$, where D_m is the terms of x such that $\alpha = \beta$ and D_s is the terms with $\alpha > \beta$. Thus, $D_m(X^n) = \lambda X^n$ for all $X^n \in \mathbb{C}[X]$. We observe that λ only depends on n (the degree of X^n), and defines a function F from $\mathbb{N}$ to $\mathbb{C}$ by $F(n) = \lambda$. Let $f \in \mathbb{C}[X]$ and $D_m(f) = \lambda f$. Then $F(deg(f)) = \lambda$. For any $g \in \mathbb{C}[X]$, we can write $X^n = \Sigma_\lambda g_\lambda$ since $\mathbb{C}[X] = \bigoplus_\lambda V_\lambda$, where the $\lambda' s$ are the corresponding eigenvalues. There is one and only one λ such that

$$deg(X^n) = deg(g_\lambda) \text{ for some } g_\lambda \in V_\lambda.$$

I find this problem particularly interesting. As part of my further research, I would like to continue working on it and hopefully I can solve Problem 4.3.

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