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SINGULARITY ANALYSIS OF AN INTERFACE CRACK BETWEEN PIEZOELECTRIC AND ORTHOTROPIC HALF-SPACES

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Abstract — In this paper, an interface crack between piezoelectric and orthotropic half-spaces has been studied in detail. By using integral transformation techniques, the present mixed boundary value problem was reduced to the solution of singular integral equations, which can be further reduced to solving a Riemann-Hilbert problem with a closed form solution. The crack-tip singularities of the interface crack have been investigated for possible combinations of the piezoelectric and orthotropic materials; a criterion based on the *g*-parameter of the Riemann-Hilbert problem is introduced to study the possible singularity behavior of the interface crack. It is shown that there can be either oscillatory or non-oscillatory singularity for the interface crack depending on the particular combinations of the bi-materials. A closed form solution for stresses, electric field and electric displacement in the cracked bi-materials is given, and of particular interests, the analytical expression of the stresses and electric displacements along the interface has been obtained.

Keywords- interface crack; piezoelectric and orthotropic materials; singular integral equations; Riemann-Hilbert problem; oscillatory or non-oscillatory singularity

I. INTRODUCTION

The problem of interfacial cracks in dissimilar isotropic materials has been extensively studied, and the characteristic oscillating stress singularity was determined by Williams [1]. Investigations of interfacial cracks between dissimilar anisotropic media can be found in the works of Clements [2], among many others. Interfacial fracture in adhesively bonded structures presents an important concern in multilayer devices, in which interfacial cracks is often observed. In view of the wide application of piezoelectric materials in smart structures, interface crack problems in dissimilar piezoelectric materials have received considerable attention. Stress singularities for the interfacial cracks in bonded piezoelectric half-spaces have been investigated by Kuo and Barnett [3]. Suo et al. [4] considered impermeable interface cracks between two dissimilar anisotropic piezoelectric materials and the solution was found in an exact analytical form, which showed that the crack tip singularity can be oscillatory and/or non-oscillatory, and the non-oscillatory singularity is different from the classical square root singularity for cracks in homogeneous materials. Qin and Mai [5] established a closed crack-tip model for an interface crack between two thermo-piezoelectric materials by using the extended Stroh formalism and the method of singular integral equations; and the effects of the combined thermal, mechanical and electrical loads on the stress intensity factors and size of contact zone have been analyzed.

In many practical applications, piezoelectric materials are bonded to non-piezoelectric (conducting or insulating) materials. Parton [6] analyzed interfacial cracks between piezoelectric and conducting isotropic materials. Liu and Hsia [7] studied an interfacial external crack between piezoelectric and orthotropic half-spaces using the method of singular integral equations and Riemann boundary value problems. Ou and Chen [8] have investigated interfacial cracks between piezoelectric material and non-piezoelectric isotropic elastic materials by employing Stroh formalism and assuming that elastic materials have extremely low but non-zero piezoelectric and dielectric constants. A hybrid complex-variable solution for piezoelectric/isotropic elastic interfacial cracks has been obtained by combining the Stroh's method of piezoelectric materials with Muskhelishvili's method for isotropic elastic materials, and a simple explicit condition is given for the absence of the oscillating singularity for interfacial cracks by Ru [9]. It is noted that the majority of existing works on interfacial cracks in piezoelectric materials have been based on the Stroh formalism, while when a non-piezoelectric elastic material is being treated as a special case of piezoelectric materials with vanishing piezoelectric constants, the solution becomes complicated due to the appearance of repeated eigenvalues [10].

In this work, we develop an exact method for a conducting interface crack between piezoelectric and orthotropic materials by using the method of Integral transforms and singular integral equations, and closed-form solutions have been obtained by solving the corresponding Riemann-Hilbert problem. Exact solution of the fields in the cracked bi-materials has been obtained, and analytical expression of the stresses and electric displacement on the bonded interface is provided, and the singularity behavior of the interface crack is analyzed in detail. A criterion based on the *g*-parameter of the Riemann-Hilbert problem is introduced to study the possible singularity behavior of the interface crack, which shows that the interface crack can have either oscillatory or non-oscillatory singularities depending on the value of the *g*-parameter. The extend Dunders parameter β for the interface crack between piezoelectric and orthotropic materials has been obtained and the relation between β and the g-parameter was given.

II. Problem Statement

We study an interface crack of length $2c$ between piezoelectric and orthotropic half-spaces, with the poling direction of the piezoelectric medium perpendicular to the crack plane, as shown in Fig. 1. For convenience, a set of Cartesian coordinate system (x, y) is attached to the crack. Assume that a uniform normal stress, P_0 , is applied on the crack faces.

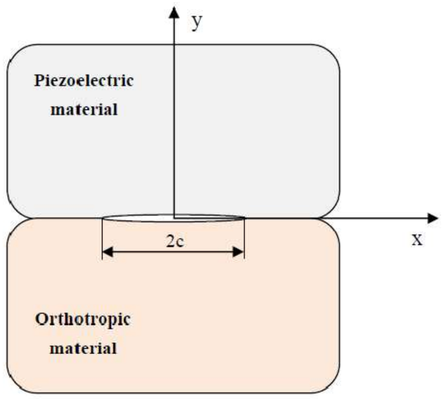

Figure 1. An interface crack between piezoelectric and orthotropic materials.

Consider a transversely isotropic, linear elastic piezoelectric half-space and denote the rectangular coordinates of a point by (x, y) . In the absence of body forces and electric charge density, the equilibrium equations for plane strain piezoelectricity can be expressed as [11]

$$\begin{aligned} &C_{11}u_{x,xx} + C_{44}u_{x,yy} + (C_{13} + C_{44})u_{y,xy} + (e_{31} + e_{15})\phi_{,xy} = 0 \\ &(C_{13} + C_{44})u_{x,xy} + C_{44}u_{y,xx} + C_{33}u_{y,yy} + e_{15}\phi_{,xx} + e_{33}\phi_{,yy} = 0 \\ &(e_{31} + e_{15})u_{x,xy} + e_{15}u_{y,xx} + e_{33}u_{y,yy} - \lambda_{11}\phi_{,xx} - \lambda_{33}\phi_{,yy} = 0 \end{aligned} \tag{1}$$

where u_x, u_y are components of the displacement vector, ϕ is the electric potential, $C_{11}, C_{13}, C_{33}, C_{44}$ are elastic constants, e_{15}, e_{31}, e_{33} are piezoelectric constants, and $\lambda_{11}, \lambda_{33}$ are dielectric permittivities.

The constitutive equations of the piezoelectric media are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{13} & 0 \\ C_{13} & C_{33} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} - \begin{bmatrix} 0 & e_{31} \\ 0 & e_{33} \\ e_{15} & 0 \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} D_x \\ D_y \end{Bmatrix} = \begin{bmatrix} 0 & 0 & e_{15} \\ e_{31} & e_{33} & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix} + \begin{bmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{33} \end{bmatrix} \begin{Bmatrix} E_x \\ E_y \end{Bmatrix} \tag{3}$$

where σ_{ij} , ε_{ij} , D_i and E_i ($i, j = x, y$) are components of stress, strain, electric displacement and electric field, respectively.

The gradient equations are

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \;\; E_i = -\phi_{,i} \quad (i, j = x, y) \tag{4}$$

For an orthotropic elastic half-space under in-plane loading, the constitutive equations and the equilibrium equations are

$$\begin{aligned} \sigma_{xx}^E &= C_{11}^E \frac{\partial u_x^E}{\partial x} + C_{13}^E \frac{\partial u_y^E}{\partial y} \\ \sigma_{yy}^E &= C_{13}^E \frac{\partial u_x^E}{\partial x} + C_{33}^E \frac{\partial u_y^E}{\partial y} \\ \sigma_{xy}^E &= C_{44}^E \left(\frac{\partial u_x^E}{\partial y} + \frac{\partial u_y^E}{\partial x} \right) \end{aligned} \tag{5}$$

$$\begin{aligned} &C_{11}^E \frac{\partial^2 u_x^E}{\partial x^2} + C_{44}^E \frac{\partial^2 u_x^E}{\partial y^2} + \left(C_{13}^E + C_{44}^E\right) \frac{\partial^2 u_y^E}{\partial x \partial y} = 0 \\ &C_{44}^E \frac{\partial^2 u_z^E}{\partial x^2} + C_{33}^E \frac{\partial^2 u_y^E}{\partial y^2} + \left(C_{13}^E + C_{44}^E\right) \frac{\partial^2 u_x^E}{\partial x \partial y} = 0 \end{aligned} \tag{6}$$

where the superscript "*E*" denotes the quantities of the elastic orthotropic half-spaces, and the same definitions are applied through the paper. u_x^E and u_y^E are the displacement vectors, and C_{11}^E, C_{13}^E, C_{33}^E, C_{44}^E are the elastic stiffness constants of the orthotropic half-spaces.

The continuity conditions along the interface between the piezoelectric and orthotropic half-spaces at $y = 0$ are

$$\begin{aligned} &\sigma_{yy}(x, 0^+) = \sigma_{yy}(x, 0^-) \\ &\sigma_{xy}(x, 0^+) = \sigma_{xy}(x, 0^-) \qquad (|x| \geq 0) \\ &\phi(x, 0) = 0 \end{aligned} \tag{7}$$

It is noted that the orthotropic half-space is considered an ideal conductor that a zero electric potential holds at the interface $y = 0$ between the piezoelectric and orthotropic materials. The boundary conditions of the mixed boundary value problem for the interface crack can be expressed as

$$\sigma_{yy}(x,0) = -P_0\,, \qquad (|x| < c) \tag{8}$$

$$u_y(x,0^+) = u_y^E(x,0^-)\,, \qquad (|x| \ge c) \tag{9}$$

$$\sigma_{xy}(x,0) = 0\,, \qquad (|x| < c) \tag{10}$$

$$u_x(x,0^+) = u_x^E(x,0^-)\,, \qquad (|x| \ge c) \tag{11}$$

III. Method Of Solution

For the piezoelectric medium, by using the technique of integral transform to (1), the displacements and electric potential in the piezoelectric half-space can be expressed as

$$u_y(x,y) = \sum_{j=1}^{3} \int_0^\infty A_j(\xi)\exp(-\gamma_j \xi y)\cos(\xi x)d\xi \tag{12}$$

$$u_x(x,y) = \sum_{j=1}^{3} a_j\gamma_j \int_0^\infty A_j(\xi)\exp(-\gamma_j \xi y)\sin(\xi x)d\xi \tag{13}$$

$$\phi(x,y) = -\sum_{j=1}^{3} b_j \int_0^\infty A_j(\xi)\exp(-\gamma_j \xi y)\cos(\xi x)d\xi \tag{14}$$

where a_j, b_j ($j = 1-3$) are constants related to material properties, and the roots γ_j ($j = 1-3$) are determined from the following characteristic equation

$$\begin{vmatrix} C_{11} - C_{44}\gamma^2 & (C_{13}+C_{44})\gamma & (e_{31}+e_{15})\gamma \\ (C_{13}+C_{44})\gamma & C_{33}\gamma^2 - C_{44} & e_{33}\gamma^2 - e_{15} \\ (e_{31}+e_{15})\gamma & e_{33}\gamma^2 - e_{15} & \lambda_{11} - \lambda_{33}\gamma^2 \end{vmatrix} = 0 \tag{15}$$

in which the notation "$|\ |$" denotes determinant.

For the orthotropic elastic medium, the displacements inside the orthotropic half-space may be given as

$$\begin{aligned} u_y^E(x,y) &= \sum_{j=1}^{2} \int_0^\infty A_j^E(\xi)\exp(-\gamma_j^E \xi y)\cos(\xi x)d\xi \\ u_x^E(x,y) &= -\sum_{j=1}^{2} a_j^E \gamma_j^E \int_0^\infty A_j^E(\xi)\exp(-\gamma_j^E \xi y)\sin(\xi x)d\xi \end{aligned} \tag{16}$$

where a_j^E ($j = 1,2$) are constants, and the roots γ_j^E ($j = 1,2$) are defined by the characteristic equation as following

$$\begin{vmatrix} C_{11}^E - C_{44}^E \gamma^{E^2} & (C_{13}^E + C_{44}^E)\gamma^E \\ (C_{13}^E + C_{44}^E)\gamma^E & C_{33}^E\gamma^{E^2} - C_{44}^E \end{vmatrix} = 0 \tag{17}$$

Note that the sixth-order characteristic equation (15) has six roots which occur in pairs with the same magnitude but opposite signs for the real ones, and in conjugate pairs for the complex roots. Similarly, the bi-quadratic equation (17) has four roots which occur in pair with the same magnitude but opposite signs for the real roots and in conjugate pairs for the complex roots, respectively. The roots γ_j $(j = 1-3)$ with $\mathrm{Re}(\gamma_j) > 0$ and γ_j^E $(j = 1,2)$ with $\mathrm{Re}(\gamma_j^E) < 0$ are chosen by requiring a positive internal energy for the system to be in a steady state, where "Re" denotes the real part of a complex number.

The stress and electric displacement components of the piezoelectric medium can be obtained by using the constitutive equations (2, 3), and the stresses in the orthotropic elastic half-space can be obtained from (5). The detailed expression is omitted here.

The application of the continuity condition in (7) leads to the result that the unknown functions $A_3(\xi)$ and $A_1^E(\xi)$, $A_2^E(\xi)$ can be expressed as functions of the independent unknowns $A_j(\xi)$ $(j = 1,2)$.

Define the following displacement jump across the crack face as

$$\begin{aligned} w(x) &= u_y(x,0^+) - u_y^E(x,0^-) \\ u(x) &= u_x(x,0^+) - u_x^E(x,0^-) \end{aligned} \tag{18}$$

The satisfaction of the mixed boundary conditions (8-11) lead to the following dual integral equations for the unknowns $A_1(\xi)$ and $A_2(\xi)$:

$$\frac{d}{dx}\int_0^\infty \sum_{j=1}^{2} \delta_j A_j(\xi)\cos(\xi x)d\xi = 0, \quad (0 \le x < c) \tag{19}$$

$$\frac{d}{dx}\int_0^\infty \sum_{j=1}^{2} \delta_{j+2} A_j(\xi)\sin(\xi x)d\xi = -P_0, \quad (0 \le x < c) \tag{20}$$

$$\int_0^\infty \sum_{j=1}^{2} \delta_{j+4} A_j(\xi)\cos(\xi x)d\xi = 0, \quad (x \ge c) \tag{21}$$

$$\int_0^\infty \sum_{j=1}^{2} \delta_{j+6} A_j(\xi)\sin(\xi x)d\xi = 0, \quad (x \ge c) \tag{22}$$

where the constants δ_j ($j = 1-8$) are related to material properties and are omitted here.

Following the procedure in [6], we can obtain the following simultaneous singular integral equations to solve the functions $u(x)$ and $w(x)$

$$q_1 w(x) + q_2 \frac{1}{\pi}\int_{-c}^{c} \frac{u(t)}{t-x}dt = -C_0 \tag{23}$$

$$q_3 u(x) - q_4 \frac{1}{\pi}\int_{-c}^{c} \frac{w(t)}{t-x}dt = P_0 x \tag{24}$$

where C_0 is a constant which can be determined by the far-field boundary conditions, and the real constants q_j $(j = 1-4)$ are material properties related and are defined as:

$$\begin{aligned} &q_1 = (\delta_2\delta_7 - \delta_1\delta_8)/\Delta, \; q_2 = (\delta_1\delta_6 - \delta_2\delta_5)/\Delta \\ &q_3 = (\delta_3\delta_6 - \delta_4\delta_5)/\Delta, \; q_4 = (\delta_4\delta_7 - \delta_3\delta_8)/\Delta \\ &\Delta = \delta_5\delta_8 - \delta_6\delta_7 \end{aligned} \tag{25}$$

By introducing the function of a complex variable $z = x + iy$ as [12]

$$F(z) = \frac{1}{2\pi i}\int_{-c}^{c} \frac{f(t)}{t - z} dt \tag{26}$$

which is analytic in the whole complex plane with a cut along the segment $-c \le x \le c$ of the real axis, and the boundary values of the continuous extension on this segment to the left and right are determined by the Sokhotskii-Plemelj formulas

$$\begin{aligned} F^+(x) + F^-(x) &= \frac{1}{\pi i}\int_{-c}^{c} \frac{f(t)}{t - x} dt \\ F^+(x) - F^-(x) &= f(x) \end{aligned} \tag{27}$$

where the signs "+" and "-" denote the limiting values of the function $F(z)$ at $y = 0$ from the positive and negative y-axis, respectively. It is noted that the function $f(t)$ in (26) is defined as

$$\begin{aligned} f(t) &= w(t) + iq_0 u(t) \\ q_0 &= \sqrt{\frac{q_2 q_3}{q_1 q_4}},\ q = \frac{q_2}{q_0 q_1} = \frac{q_0 q_4}{q_3} \end{aligned} \tag{28}$$

The following Riemann-Hilbert problem can be obtained

$$F^+(x) = g \cdot F^-(x) + \frac{1}{q+1}\left(\frac{iq_0}{q_3} P_0 x - \frac{C_0}{q_1}\right) \tag{29}$$

where the parameter g is defined as $g = (1-q)/(1-q)$. It will be shown later that this parameter is related to the extended Dunders parameter β of the interface crack and the singularity behavior of the interface crack is dependent on the value of the parameter g.

3.1 when g is real and negative, i.e., $g < 0$

The solution of the Riemann-Hilbert problem can be obtained as [12]

$$F(z) = \frac{X(z)}{2\pi i(q+1)}\int_{-c}^{c} \frac{1}{X^+(t)(t-z)}\left(\frac{iq_0}{q_3} P_0 t - \frac{C_0}{q_1}\right) dt \tag{30}$$

where $X(z)$ is the particular solution of the homogeneous Riemann-Hilbert problem which is bounded near the ends $x = \pm c$, $X^+(t)$ is the value of $X(z)$ on the left boundary of the discontinuity, and

$$X(z) = \sqrt{z^2 - c^2}\left(\frac{z-c}{z+c}\right)^{i\varepsilon} \tag{31}$$

$$\begin{aligned} \varepsilon &= \frac{-1}{2\pi}\log(-g) = -\frac{1}{\pi}\tanh^{-1}(\beta) \\ \beta &= \frac{g+1}{g-1} = -\frac{1}{q} \end{aligned} \tag{32}$$

where $\tanh^{-1}(X)$ is the inverse hyperbolic tangent of the element X, ε is a real constant related to the material properties of the piezoelectric and orthotropic elastic materials and is known as the oscillatory index of the interface crack problem. It is noted that the extended Dundurs parameter β is obtained which is a function of the material properties of the bi-materials and the roots of the characteristic equations for the corresponding bi-materials. The relations between the extended Dundurs parameter β and the q-parameter and the g-parameter in the Riemann-Hilbert equation (29) is clearly shown in Equation (32).

Considering that the differences of the displacements w and u vanish at infinity, it requires that $F(\infty) = 0$, which results in the condition

$$\int_{-c}^{c} \frac{1}{X^+(t)}\left(\frac{iq_0}{q_3} P_0 t - \frac{C_0}{q_1}\right) dt = 0 \tag{33}$$

and by using the methods of evaluating integrals [12], the solution of (33) can be obtained as:

$$C_0 = -2c\varepsilon \frac{q_0 q_1}{q_3} P_0 \tag{34}$$

The general solution of the Riemann-Hilbert problem (29) can be given as

$$F(z) = \frac{q_0 P_0}{2 q_3 q}\{2c\varepsilon + i[z - X(z)]\} \tag{35}$$

The functions $w(x)$ and $u(x)$ can be obtained as

$$w(x) = \frac{P_0}{q_4}\cosh(\varepsilon\pi)\sqrt{c^2 - x^2}\cos\left(\varepsilon \log\left|\frac{x+c}{x-c}\right|\right),\ (|x| < c) \tag{36}$$

$$u(x) = \frac{-P_0}{q_0 q_4}\cosh(\varepsilon\pi)\sqrt{c^2 - x^2}\sin\left(\varepsilon \log\left|\frac{x+c}{x-c}\right|\right),\ (|x| < c) \tag{37}$$

and it is obvious that $w(x) = u(x) = 0$ for $|x| \ge c$ (due to the continuity conditions on the bonded interface).

It can be observed that the dislocation functions $w(x)$ and $u(x)$ are of oscillating nature and change their sign persistently when $x \to \pm c$. Intervals of sign changing of $w(x)$ and $u(x)$ are located within rather small regions of the crack tips which is dependent on the value of the oscillatory index ε.

Of particular interest are the stresses and electric displacements along the bonded interface between the piezoelectric and orthotropic elastic half-spaces, and these quantities can be easily obtained. The asymptotic distribution of stresses and electric displacements near the crack tip can be expressed in terms of the distance from the crack, $r = x - a$, as

$$\sigma_{yy}(r,0) = \frac{1}{\sqrt{2\pi r}}\mathrm{Re}\left[(K_1 + iK_2)\left(\frac{2c}{r}\right)^{-i\varepsilon}\right] \tag{38}$$

$$\sigma_{xy}(r,0) = -\frac{Q_0}{\sqrt{2\pi r}}\mathrm{Im}\left[(K_1 + iK_2)\left(\frac{2c}{r}\right)^{-i\varepsilon}\right] \quad (39)$$

$$D_y(r,0) = \frac{d_1}{q_4\sqrt{2\pi r}}\mathrm{Re}\left[(K_1 + iK_2)\left(\frac{2c}{r}\right)^{-i\varepsilon}\right] \quad (40)$$

$$D_x(r,0) = -\frac{d_2 Q_0}{q_2\sqrt{2\pi r}}\mathrm{Im}\left[(K_1 + iK_2)\left(\frac{2c}{r}\right)^{-i\varepsilon}\right] \quad (41)$$

where $Q_0 = q_2/q_3 q$, "Im" denotes the imaginary part of a complex number, and the mode-I and mode-II stress intensity factors (SIFs) K_1 and K_2 are defined respectively as

$$K_1 = P_0\sqrt{\pi c},\ \ K_2 = 2\varepsilon P_0\sqrt{\pi c} \quad (42)$$

It can be observed from (38-41) that all asymptotic fields of the stresses and electric displacements have the square root singularity and the oscillating term $(2c/r)^{-i\varepsilon}$, which is in agreement with the result for interface cracks in dissimilar isotropic materials [13]. It is observed that the mode-II stress intensity factor K_2 is non-zero when the material-related parameter ε is not zero. In the special case of a crack in a homogeneous material, i.e., $\varepsilon = 0$, the material dissimilarity vanishes, the value of K_2 is zero, and the quasi Mode-I degenerate to Mode-I crack problem.

3.2 when g is complex and $|g| = 1$

In this case, q_0 and q are imaginary numbers, and the solutions of the Riemann-Hilbert problem (29) are

$$F(z) = -i\frac{q_0}{2q_3 q}P_0[X(z) - z - 2ck] \quad (43)$$

$$w(x) + iq_0 u(x) = \frac{q_0}{q_3 q}P_0\cos(k\pi)\sqrt{c^2 - x^2}\left(\frac{c+x}{c-x}\right)^k \quad (44)$$

where the real constant k is defined as

$$k = \frac{1}{2} - \frac{1}{2\pi i}\log\left(\frac{1-q}{1+q}\right) \quad (45)$$

with $\tan^{-1}(X)$ being the inverse tangent of the element X.

$$w(x) - iq_0 u(x) = \frac{q_0}{q_3 q}P_0\cos(k\pi)\sqrt{c^2 - x^2}\left(\frac{c-x}{c+x}\right)^k \quad (46)$$

The functions $w(x)$ and $u(x)$ can be obtained as

$$w(x) = \frac{q_0}{2q_3 q}P_0\cos(k\pi)\sqrt{c^2 - x^2}\left[\left(\frac{c+x}{c-x}\right)^k + \left(\frac{c-x}{c+x}\right)^k\right] \quad (47)$$

$$u(x) = \frac{-i}{2q_3 q}P_0\cos(k\pi)\sqrt{c^2 - x^2}\left[\left(\frac{c+x}{c-x}\right)^k - \left(\frac{c-x}{c+x}\right)^k\right] \quad (48)$$

The asymptotic stress field near the right crack tip can be obtained as

$$\sigma_{yy}(r,0) = \frac{P_0}{2}\left[\left(\frac{2c}{r}\right)^{\frac{1}{2}+k}\left(\frac{1}{2} - k\right) + \left(\frac{2c}{r}\right)^{\frac{1}{2}-k}\left(\frac{1}{2} + k\right)\right] - P_0 \quad (49)$$

$$\sigma_{xy}(r,0) = \frac{iq_2 P_0}{2q_3 q}\left[\left(\frac{2c}{r}\right)^{\frac{1}{2}+k}\left(\frac{1}{2} - k\right) - \left(\frac{2c}{r}\right)^{\frac{1}{2}-k}\left(\frac{1}{2} + k\right)\right] \quad (50)$$

where $r = x - a$ is the distance from the right crack tip along the interface.

It can be observed that there is no singularity oscillating behavior occurs in this case for the interface crack between piezoelectric and orthotropic elastic half-spaces. The stress fields near the crack-tip have the singularity of order of $-1/2 \pm k$. These results are in agreement with the results of Kuo and Barnett [3] and Suo et al. [4] for interface cracks between dissimilar piezoelectric materials. It is noted that this type of singularity does not occur in purely elastic solids and is believed due to the piezoelectric effect.

IV. RESULTS AND DISCUSSION

The solution obtained in the previous section can be used to evaluate the distribution of the stresses, electric displacements near the interface crack between the piezoelectric and orthotropic elastic materials. The material constants of three kinds of piezoelectric materials (PZT-4, P-7, PCM-80) and three kinds of orthotropic elastic materials (SMC, Beryllium, Magnesium) are used in numerical calculation [14].

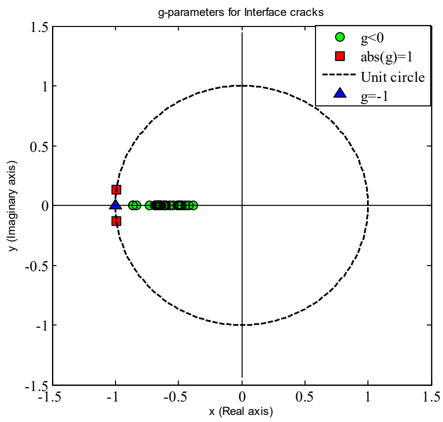

Figure 2. g-parameter for the interface crack between piezoelectric and orthotropic materials.

Fig. 2 graphically displays the distribution of the g-parameter for the interface crack between piezoelectric and orthotropic materials. For most material combinations, the values of g are negative and $-1 < g < 0$, and the interface crack has oscillatory singularity order $-1/2 \pm i\varepsilon$. If $g = -1$, the classical singularity order $-1/2$ is obtained for the crack problem; otherwise if g is a complex number with the magnitude of 1, the singularity order of the corresponding interface crack will be $-1/2 \pm k$.

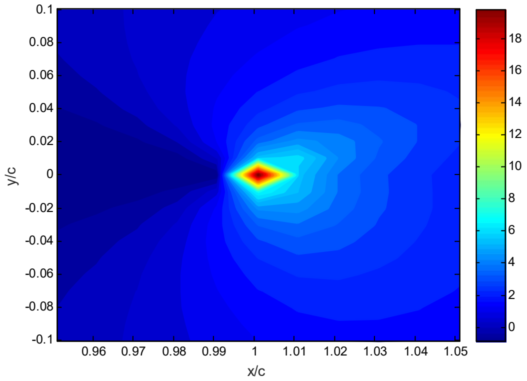

Figure 3. Normalized stress σ_{yy}/P_0 around the tip of an interface crack between PZT-4 and SMC.

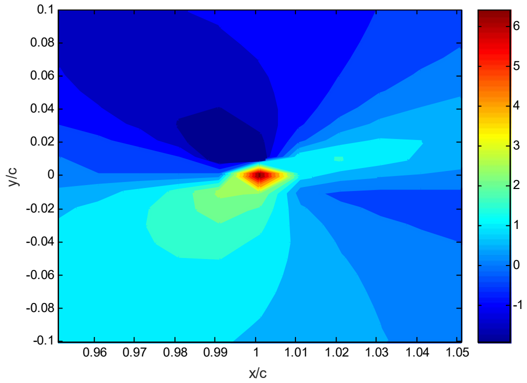

Figure 4. Normalized shear stress σ_{xy}/P_0 around the tip of an interface crack between PZT-4 and SMC.

The normalized stress σ_{yy}/P_0 around the tip of an interface crack between PZT-4 and SMC is plotted in Fig. 3, and the distribution of normalized shear stress σ_{xy}/P_0 around the tip is plotted in Fig. 4. Singular stresses near crack tip are observed and the normal stress is asymmetric about the interface due to the mismatch of the piezoelectric and orthotropic materials. It is shown that non-zero shear stress exist on the interface even though the applied loading is tension, which indicates that the mode-II stress intensity factor K_2 is non-zero when the oscillatory parameter ε is not zero.

V. CONCLUSIONS

An interface crack between piezoelectric and orthotropic elastic materials under in-plane loading has been studied using integral transform method and singular integral equations. The mixed boundary value problem for the interface crack was reduced to solving singular integral equations, which can be further reduced to the solution of a Riemann-Hilbert problem. The *g*-parameter of the Riemann-Hilbert problem is introduced to investigate the singularity behavior of the interface crack, and it is found that oscillatory and/or non-oscillatory singularity may exist for the interface crack between piezoelectric and orthotropic materials, depending on the particular material combinations of the bi-materials. Full-field solutions for the stresses and electric displacements in the cracked bi-materials have been obtained in a closed form, and an analytical solution for stresses and electric displacement on the interfaces have been provided. The crack tip singularity order may be $-1/2 \pm i\varepsilon$ or $-1/2 \pm k$, depending on the combination of the piezoelectric and orthotropic materials.

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