OPTIMAL REGULATION OF SYSTEMIC RISK BY TAX

by

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Abstract

Financial systemic crisis could be broadly understood as the deterioration of the banking sector which results in damage to the real economy. From elementary accounting, a firm's financial position can be characterized by the value of its asset holdings versus the amount it borrow from others. If assets are not worth sufficiently more than the firm owes, it will be in distress, and will not be able to operate its business efficiently. In the case of a bank, this means the difference between the dollar amount it lends and the amount it receives from depositors is not sufficiently high or even worse, is negative. If either case happens to the aggregate banking sector, a systemic crisis will ensue, and there will be significant costs incurred by society. This M.Sc. thesis will concentrate on an existing economic model which incorporates the risk of systemic crisis, as defined above, at a future time. In the context of this model, a tax as a function of the banks' dollar value of investments, raised debt, and equity funding at present time will incentivize them to choose these quantities in the interest of social welfare. The thesis will provide mathematical explanations for this effect. Moreover, MATLAB codes are included to calculate the tax amounts charged to each bank when they behave in a socially optimal manner.

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Chapter 1

Introduction

In the simplest form, a bank's role in the economy is to distribute money from agents who possess funds to those who need to borrow. Banks can carry out this action efficiently because of their expertise in assessing the credibility of borrowers. When a bank lends money to a borrower, it is making an investment for which a return is expected. For example, when it offers a mortgage, the terms of the contract requires the home buyer to pay interest in addition to the borrowed amount. Funding for the bank to make these investments comes from its depositors or debtholders (i.e. those with money in the economy) and its equityholders (i.e. those who own shares of the bank). A stylized balance sheet for a bank at a fixed point in time is given in Figure 1.1. Basic accounting principles state that the equation Assets = Liabilities + Equity must hold at all times, and a firm is in financial distress when its equity is not above an acceptable threshold, or even more, if it is negative. The first situation means the firm is **undercapitalized** and the latter defines its **bankruptcy**, when asset holdings are worth less than liabilities, and the firm is unable to repay all its debt.

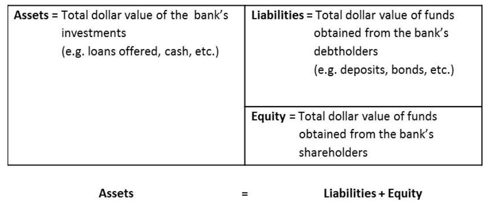

Figure 1.1: Stylized Bank Balance Sheet at a Fixed Point in Time

When a bank cannot perform the function of channeling funds, borrowers will have less opportunities of raising money to finance their productive activities. In view of the balance sheet, this will occur when a bank is bankrupt. Even if it is solvent (not bankrupt), if equity is sufficiently low, the bank will have reduced efficiency in performing this function, because its debtholders will lose confidence in making deposits in the future (or even start withdrawing). If undercapitalization happens to numerous banks, or to a few large banks which provide most of the lending in the economy, it will become costly, if not impossible, for agents to borrow from the banking sector. Thus, a financial crisis will materialize and as a result, productivity in the economy will be hampered.

To prevent the financial sector from collapsing, governments have historically bailed out failing banks that are major players in the industry. However, this hurts society's well-being, as spending for other areas in need is reduced (Honohan and Klingebiel (2000)). In the recent financial crisis of 2007-2009, the gross world product, which measures economic productivity of the world as a whole, contracted for the first time in decades (by 0.8%) and international trade reduced dramatically by 12% (Acharya, Pedersen, Philippon, and Richardson (2013)). Hoggarth, Reis, and Saporta (2002) have shown that a country's gross domestic product (GDP) drops by about 15% to 20% during crises. Moreover, Honohan and Klingebiel (2000) documented that national governments spend an average of 12.8% of GDP to support the financial sector in the event of a crisis.

The significant costs to social welfare suggest the necessity of effective financial regulation. This should be aimed at providing banks with incentives to make decisions (i.e. choices for types and quantities of assets, liabilities, and equity) while considering the potential consequence of contributing to a loss in financial intermediation in the banking sector at a future time.

1.1 Tools for Financial Regulation

This section will explore methods that governments may use to regulate systemic risk in the financial sector. Namely, the descriptions for capital requirement, contingent capital, and systemic taxation will be provided.

1.1.1 Capital Requirement

Traditionally, bank regulations have focused on the use of capital requirements, which are rules that set the minimum value of equity a bank must hold. The rationale for such an approach is intuitive. For any fixed amount of assets, the more reliant a bank is on debt funding (i.e. the less equity funding it uses), the higher its risk of default (unable to pay back debtholders in full). This is because assets will need to decrease by a smaller percentage in order for the bank to have insufficient funds to fully pay back its debt. To see this reasoning, consider an analogy of an individual who has no money on hand and borrows the full purchase price (say $100) from another person to buy a stock. Assume that the lender does not charge interest, but wants the loan repaid after one month. If at that time, the stock is worth $90, the borrower will only be able to sell it for this value and repay the lender in this amount. Of course, the lender will not be happy. This extreme case would be reflective of a bank which uses 100% debt and 0% equity funding. As long as the stock (asset) price is lower than $100 (its original value) in one month, the borrower will not be able to fully repay the loan. Now, suppose the stock buyer originally had $20 on hand and borrowed $80 to buy the $100 stock (80% debt, 20% equity), and assume again the stock's price is $90 in one month's time. In this case, the stock will be sold for $90, and the lender will be repaid in full ($80). Thus, requiring the dollar amount of equity (like the stock buyer's own money at the beginning) to be at least a percentage of the value of assets reduce the possibility of the bank's default in the future (like the stock buyer's inability to fully repay the loan), and to an extent, this is an assurance that its debtholders will be repaid (Tarullo (2008)).

Another reason for imposing capital requirements involve the relationship between deposit insurance and the risk of a bank's assets (Acharya and Richardson (2009)). In simple terms, deposit insurance is a guarantee that the government provide a bank's depositors up to a certain amount. For example, a person depositing up to CAD 100,000 into a bank in Canada can withdraw all of his/her money on demand (Canada Deposit Insurance Corporation (2014)). If the bank is in distress and cannot fulfill this request, the Canada Deposit Insurance Corporation (a crown corporation) will provide the money to the depositor. The intent of deposit insurance is to provide individuals the confidence to place funds into the country's banking sector, which is a necessary ingredient for financial intermediation. As explained by Acharya and Richardson (2009), the side effect, though, is that depositors will have less incentives to monitor risks taken by the bank's managers. For example, knowing that they can get some of their money back from the government if the bank shuts down, depositors will exercise less discipline (e.g. demand higher interest in savings accounts, or threaten a bank run) on management, even if very risky assets (e.g. mortgages earning high returns, but with high chances of default) are held on the balance sheet. To address this shortcoming, the riskier a bank's assets are, the more equity capital government rules would require it to hold. The reasoning comes from a concept in corporate finance (Berk, DeMarzo, and Stangeland (2012)): raising equity (issuing shares) is costly for a firm because it signals to the market that the company's performance will decline. Berk, DeMarzo, and Stangeland (2012) explains that this is because the market assumes that managers of the firm have inside knowledge, and if shares are being issued, they must believe firm value will decline in the future, and it is better to issue the stock for a higher value now. Due to this belief, the market will suppress the price of stock after an issuance. Hence, requiring more equity for holding assets bearing higher risk will incentivize banks to reduce excessive risk taking when making investments.

The Basel Committee on Banking Supervision at the Bank for International Settlements sets the guidelines for capital requirements, which are generally followed by jurisdictions around the world. For example, the general rule from the accords known as Basel I and II states that a bank's capital adequacy ratio (equity value divided by risk-weighted assets (RWA)) should be at least 8% (Acharya, Kulkarni, and Richardson (2011)). RWA is calculated by multiplying each asset's value by a weight according to the risk category it belongs to and summing the results together (Basel Committee on Banking Supervision (1998)). The weight for each category is fixed, but a category of greater risk will have a higher weight. For example, suppose the bank's assets consist of a \$100,000 residential mortgage and a \$50,000 AAA-rated (low default risk relative to the mortgage) government bond. Moreover, assume the risk weight of the mortgage is 35%, while that for the bond is 20%. Then, the bank's RWA is (\$100,000)(0.35) + (\$50,000)(0.20) = \$45,000.

It has been argued (Acharya and Richardson (2009), Acharya, Schnabl, and Suarez (2011), Slovik (2012)) that the structure of Basel I and II was insufficient to curb systemic risk and even more, encouraged banks to behave in a way that led to the recent financial crisis. Acharya and Richardson (2009) explains that banks issued risky sub-prime mortgages to borrowers with high chances of default and restructured (securitized) these assets into mortgage-backed securities (MBS), which were then deemed to carry low risk by rating agencies. The fallacy of rating agencies to properly assess the risk of these securities have also been blamed for contributing to the crisis (Altman, Öncü, Richardson, Schmeits, and White (2011)). If the securities were simply sold to outside investors, the bank would eliminate its exposure to risk from mortgages altogether. However, banks retained this risk in their asset holdings by, for example, buying these securities themselves (Acharya and Richardson (2009)). Due to the low risk rating attached to MBS, the corresponding risk weight for calculation of RWA was also low, so the bank's capital adequacy ratio will not decrease much, indicating that it is still well-capitalized and no additional equity needs to be raised. Thus, banks were engaged in regulatory arbitrage, meaning they were making profits by taking high risks, but escaped the capital requirements for doing so. When the housing market crashed, the original mortgages dropped in value (borrowers defaulted) and the MBS became worthless. This amounted to huge losses on the asset side of the balance sheet for many banks, and the financial crisis surfaced (Acharya and Richardson (2009)).

In addition to this shortcoming, Basel I and II capital requirements were focused on regulating individual banks, without consideration of the systemic risk posed by them (Acharya, Kulkarni, and Richardson (2011), Schwerter (2011)). For example, in calculating risk-weighted assets, one does not consider the systemic impact of the bank when assets are simply assigned to categories with different risk weights based on the possibility that they will become worthless. Hence, banks are not provided with incentives to operate with the well-being of the financial system in mind.

In response to these issues during the recent crisis, the Basel Committee devised the Basel III accord. One recommendation was that a bank's leverage ratio, which roughly equals equity divided by total assets, should be at least 3% (Basel Committee on Banking Supervision (2014)). With the denominator being total assets, the intention is that the minimum leverage ratio requirement would reduce opportunities for banks to manipulate assets to receive lower risk weights and not hold adequate capital against the exposure (like in securitization).

The Committee also recommends to impose additional capital requirements for systemically important banks (Basel Committee on Banking Supervision (2011c)), which will be described as follows. First, a sample of important banks in the global financial system is considered. Then, each bank should be assigned scores under the categories of size, interconnectedness, cross-jurisdictional activity, substitutability, and complexity, reflecting its relative significance in each of these factors. In particular, a bank's score for an attribute is calculated as its position in that area with respect to the sample total, in percentage terms. For example, suppose a bank's assets are worth \$50, while those for the sample (sum of assets over all the banks) are worth \$200. Then, the bank's score for size is $50/200 = 0.25$. A bank's final score is calculated as the average over its categorical scores, and would fall into a particular range (bucket) of values which then determines its additional capital requirement. Four buckets ($threshold_0 < threshold_1 < threshold_2 < threshold_3 < threshold_4$) with equal lengths (i.e. $threshold_1 - threshold_0 = threshold_2 - threshold_1$, etc.) are used, and a bank is deemed systemic if its final score exceeds $threshold_0$. The higher the final score is, the higher the range it belongs to, and the higher is the additional capital requirement. Within each bucket, the additional requirement represented as a percentage of risk-weighted assets is a constant value. For example, if it is 2.5%, the bank would need to have a capital adequacy ratio (equity/RWA) of at least $8\% + 2.5\% = 10.5\%$, where the 8% is the standard from Basel I and II. The goal is that a requirement to hold more equity (which again, is costly for a bank) will disincentivize a bank to possess systemic characteristics.

The following explanations were given by the Basel Committee for selecting the five categories to evaluate a bank's systemic importance.

- The larger the **size** of a bank (i.e. the more assets it holds, or in simple terms, the more it lends), the greater is its contribution to the banking sector's loss of financial intermediation, if it were to encounter distress.

- The more a bank is involved with interbank borrowing (lending), the more damage it will cause to assets of other banks (the more its assets will decline in value) if it were not able to pay them back (if other banks were not able to pay back their loans). In other words, the more

interconnected a bank is in the financial system, the more potential it has to threaten the sector's function of financial intermediation.

- The recent financial crisis caused disturbances to economies worldwide. Therefore, the more **cross-border** activities a bank is engaged in, the more opportunities it has to negatively impact international markets. The mechanisms (borrowing and lending) to explain this attribute is similar to those for interconnectedness.

- If a bank is specialized in providing a particular service, with few or no competitors, then in the event that it cannot operate, it will be costly or impossible for its customers to find another bank (**substitute**) to meet their needs.

- As explained before, the creation of mortgage-backed securities has been argued as a cause of the recent crisis. Assets like these add to the **complexity** of a bank's operations, and hence have the potential to create sector-wide damage.

The Financial Services Authority (2009) mentioned that using a cutoff score to define a bank as systemic, and assigning it a fixed percentage of additional capital requirement with respect to assets if it is so will induce the bank to carry out regulatory arbitrage. For example, the bank may find that the constant additional requirement is not too costly, but having itself labeled as systemic may incline the desire for the government to bail it out when it is distressed. To reduce this distortion of incentives, the Authority recommends to determine requirement percentage as a continuous and increasing function of a bank's contribution to systemic risk. Even though Basel III does tie higher additional requirement percentages to scores in higher buckets, the relationship between the two is not continuous (i.e. buckets are separated by thresholds, and within each bucket, the additional requirement is a constant percentage). Moreover, calculating a bank's final score as an arithmetic average over the categorical scores implies that each of the five systemic factors are weighted the same, and thus are equally important. However, the Basel Committee does not provide reasons for doing so (Sullivan and Cromwell LLP (2011)).

Other methods to set capital requirements in relation to systemic risk have been proposed. For example, Webber and Willison (2011) considers a network of banks that are capable of lending and borrowing with each other as well as with non-bank entities. They assume that banks choose their liability amounts which are fixed over time. Their value of initial asset holdings will be determined once equity capital requirements are known (due to Assets = Liabilities + Equity). Then, any (random) change in a bank's value of assets as time progress will only be met with a change in equity. Since holding equity is costly for the firm (explained earlier), higher capital requirements (imposed at present time) translate into more expensive loans for those who need to borrow from banks. In other words, having more capital in the aggregate financial system reduces the chance of a crisis, but could lead to inefficient credit markets. Knowing the fixed value of each bank's liabilities, the regulator's objective is to choose the levels of equity capital for each of them now such that the total (summed over all banks) is minimized, while the probability of a crisis at a future time is equal to an acceptable target. Crisis is defined as the event that the sum of liabilities over all banks is greater than the sum of asset values, at the future time. The levels of capital which solves the regulator's problem are then imposed on individual banks. Although the regulator's incentives are modeled, those of individual banks are not. In other words, in the presence of such a regulation scheme, it is not clear how banks will respond when choosing the fixed liability amounts that consequently determines their required equity capital. Moreover, the costs of government support and those to the real economy in the event of a crisis are not modeled.

Gauthier, Lehar, and Souissi (2012) also model the financial system as a network of banks which are allowed to borrow and lend with each other as well as with non-bank agents. It is assumed that banks have chosen their initial asset holdings, but at a future time, asset values will be random. Moreover, the aggregate level of initial equity capital (summed over all banks) is a given constant and the regulator needs to choose an allocation among banks to reflect each of their contribution to systemic risk. Again, due to the equation Assets = Liabilities + Equity, once this solution is found, banks will then know their corresponding values for liabilites as well. Finding this allocation is described as solving a fixed point problem in the following sense. Suppose the levels of initial equity capital for n banks are given as a vector $C = (C_1, ..., C_n)$. Base on C, one obtains the joint distribution of the n banks' future losses (in simple terms, loss is the value liabilities - assets), which is then used to determine the contribution of each bank to systemic risk. For example, if l_i is the future random loss for bank i, $l_p = \sum_{i=1}^{n} l_i$ is the loss of the banking system, and $\beta_i := \frac{\mathrm{cov}(l_i, l_p)}{\sigma^2(l_p)}$, then the amount of initial equity capital that bank i should hold, in reflection of its contribution to the system's losses, is $C_i^{\beta} = \beta_i \sum_{i=1}^{n} C_i$. Note that $\sum_{i=1}^{n} \beta_i = 1$, so it is a matter of reallocating the fixed total amount of initial capital in the system, $\sum_{i=1}^{n} C_i = \gamma$. The regulator needs to find $C^* = (C_1^*, ..., C_n^*)$ such that $\sum_{i=1}^{n} C_i^* = \gamma$ and $C_i^* = \beta_i^* \sum_{i=1}^{n} C_i^*$, where $\beta^* = (\beta_1^*, ..., \beta_n^*)$ is determined based on the joint loss distribution with

C^* as the initial equity capital allocation among the n banks. In this model, the incentives for both regulator and banks are not clearly defined. In particular, it is difficult to see whether imposing capital requirements as such will be optimal for society. Again, the costs of government support and those to the real economy are not considered.

1.1.2 Contingent Capital

Another tool that regulators may use to control a bank's level of capitalization is the requirement for it to hold contingent capital. Contingent capital is a debt (liability) for the bank which converts to equity (shares) when some trigger has been hit. The trigger is usually designed to define situations when a bank or banking system experience low levels of equity, so that the contingent capital's conversion will re-capitalize (i.e. reduce liabilities and increase equity) the bank, avoiding, or reducing the severity of its bankruptcy and associated costs to society.

For example, Flannery (2009) considers a bank to have hit the trigger when its market value of equity divided by book value of total assets is below a pre-specified value, emphasizing the importance of using market value of equity, as book value is more prone to the managers' manipulation in times of distress when they try to make a good appearance of the firm.

The Squam Lake Working Group (2009) propose to use a double trigger. First, the regulator have to declare that there is a financial crisis, and second, the bank itself would have to be in distress (e.g. its capital adequacy ratio = equity/risk-weighted assets is below a given value). The Group argues that the first trigger is required because conversion will reduce a bank's debt, but debtholders have the important role of providing discipline for the firm's managers in case they take excessive risks, while equityholders are not able to do so. Investing in risky assets may bring equityholders high dividends, and the worst case scenario is that the bank defaults, but the shares they hold will just drop to a value of zero, rather than become negative. That is, equityholders have unlimited gains but limited losses. On the other hand, even in the presence of deposit insurance, debtholders will still be more cautious than equityholders because the most they will receive is the bank's repayment, while the worst case is that they will have no repayment at all. In other words, debtholders have limited gains, so they may ask for higher returns or withdraw their funds if they view the bank's assets as too risky. Therefore, conversion of contingent capital should occur only if there is an absolute need to do so for the stability of the financial system. The requirement of the second trigger would avoid cases of conversion occurring to well-capitalized banks during a crisis.

McDonald (2013) calls for the use of a dual trigger as well, but with complete reliance on market, rather than accounting (book) quantities. In particular, conversion takes place when an index for financial institutions (like a stock index) and the stock price for the bank in question declines below certain values. The reasoning for the dual trigger is similar to that for the case provided by Squam Lake Working Group. However, the use of solely market-value triggers have the benefit that the decision to convert only depends on the market's view of the firm's financial position, rather than on the interference of the regulator.

Holding contingent capital only fix the problem of undercapitalization when its likelihood of occurring increases, as intended by the use of triggers. In other words, it is an ex-post solution, which does little to control banks to be risky (individually or to the system) in the first place (Acharya, Kulkarni, and Richardson (2011)). As explained by Acharya, Kulkarni, and Richardson (2011), banks will continue to invest in risky assets which are capable of generating both high returns (boom) and huge losses (bust). In the case of a boom, contingent capital will remain as a liability. If a bust occurs, its conversion to equity may still be insufficient to re-capitalize the firm because it is not likely that all the bank's liabilities are contingent capital, so there will be debt which the bank cannot repay in full.

The Basel Committe on Banking Supervision (2011c) also indicate that contingent capital could embed a new type of "event risk," which it describes as the market's loss of confidence in the financial sector when a bank hits the conversion trigger(s), however it is defined. Moreover, the Committee indicate that trigger mechanisms could negatively distort the incentives of a bank's shareholders and managers, because the conversion of contingent capital to equity effectively increase the number of shares outstanding, which reduces the bank's stock price. As such, when it is imminent that conversion will take place, shareholders/managers of a bank may want to reduce asset holdings (lending) to improve its capital adequacy ratio and avoid the conversion. The result is that less money is available for borrowers in the economy.

1.1.3 Systemic Taxation

Another method for regulating systemic risk is to use taxation. As mentioned before, when banks make investment and funding decisions, they have the potential to disturb the sector's financial intermediation ability in the future, which consequently impose costs on the real economy because funds available for borrowers will be reduced and government spending may have to be directed to compensate a failing bank's debtholders. The rationale for taxation, then, is that if the bank has to pay for taking risks which increase this potential, it will be disincentivized to do so excessively. Similar in spirit to Basel III's additional capital requirement, Doluca, Klüh, Wagner, and Weder di Mauro (2010) propose to use a scoring method based on a bank's balance sheet and market figures to determine its level of systemic importance. In particular, a bank is deemed systemically important if its score exceeds a given cutoff value. The tax rate is then a continuous and increasing function with respect to score values above the cutoff, and is zero for values below it. A bank's liabilities will be taxed based on the rate corresponding to its score. The idea is that for a fixed value of assets, the more liabilities it hold, the less equity capital the bank has, and the closer it is to bankruptcy. Therefore, liabilities is used as the base for taxation. Moreover, for a fixed amount of liabilities, the higher the bank's score is, the higher will be its tax rate, and this mechanism has the intention to reduce the bank's desire of increasing its systemic importance. The paper of Doluca et al. (2010) however, does not explicitly describe the incentives of the government and the banks. Hence, it does not explain whether social welfare will be optimal in the presence of such a tax scheme.

Another reason for use of taxation is that unless banks pay for the costs they impose on the economy, they will operate in a way which is not optimal for society's well-being (Acharya, Pedersen, Philippon, and Richardson (2011), Kocherlakota (2010), Schwerter (2011)). As Kocherlakota (2010) explains, one can consider an analogy of a production factory that causes pollution to the environment. If it does not have to pay for the pollution, it will operate solely on the basis of its own revenues and expenses. The result is that the production and pollution levels generated will not be socially optimal, as all costs associated with pollution are borne by society. Kocherlakota (2010) continues to argue that if the factory knows it will be charged exactly the cost of its pollution externality, then it will produce and pollute at levels which are best for social welfare. Even though systemic taxation in this sense does reduce the potential occurrence of a crisis due to the increased costs for banks relating to their risk-inducing behaviour, it does not necessarily eliminate this possibility altogether. Rather, taxes provide banks with incentives to operate in a socially optimal manner (Schwerter (2011)).

An economic model by Acharya, Pedersen, Philippon, and Richardson (2010) explicitly define the objectives of the government and banks in utility terms and argue that a tax designed in this line of reasoning will induce banks to make decisions which are optimal for the general economy. In particular, the model considers one time period (times $t = 0, 1$), one government, and $N \in \mathbb{N}$ banks. At time $t = 0$, a bank will make its investment and funding decisions. That is, it decides on the quantity of money and to whom it will lend to, but also on the dollars of funding it will obtain from debtholders and equityholders respectively. At time $t = 1$, it is obliged to pay debtholders a fixed quantity of money, but the amount of dollars it will receive from borrowers as they repay their loans is random (e.g. loan default may occur if economic conditions are bad). On the basis of these two amounts relative to each other, the market will judge on the bank's level of financial soundness, and provide its valuation of the bank's assets at that time (i.e. repayments from bank loan borrowers). In other words, the **market** value of assets that the bank invest in at time $t = 0$ will be worth a random amount at time $t = 1$. The difference between this random amount and the quantity of money that it needs to give debtholders is the bank's equity value at time $t = 1$. Again, in view of the balance sheet, this value will determine whether the bank is bankrupt (equity is negative) and/or undercapitalized (equity is low) at that time. The system's equity value is the sum of those over all banks and a crisis is defined as the event that this value is sufficiently low. Banks which are undercapitalized when the system's equity is sufficiently low, then, contribute to the formation of a crisis. The individual bank's objective is to maximize its shareholders' utility, but the government's goal is to maximize the sum of bank utilities net of expected costs arising from crisis and those incurred from bank insolvencies. When each bank knows it will be charged a tax at time $t = 0$ equal to the expected costs to the government due to its potential bankruptcy at time $t = 1$ plus the expected costs that it impose on society in the event of a systemic crisis (at time $t = 1$), the banks' incentives will align with those of the government.

1.2 Focus of this thesis

The contribution of this M.Sc. thesis is the rigorous mathematical interpretation of how taxes in the model of Acharya, Pedersen, Philippon, and Richardson (2010) will ensure bank and government incentives align. Based on this formulation, we also develop a MATLAB algorithm to calculate the banks' tax bills once they make decisions at time $t = 0$ which are optimal for society (the government).

Chapter 2

Model Description

To account for uncertainty between times $t = 0$ and $t = 1$, consider a discrete and finite probability space $\Omega = \{\omega_1, ..., \omega_K\}$ equipped with a probability measure P. Assume that $p_k = P[\omega_k] \in (0, 1)$ for all $k \in \{1, ..., K\}$ and $\sum_{k=1}^{K} p_k = 1$. The index for each bank is $i \in \{1, ..., N\}$ and that for each asset which a bank can invest in is $j \in \{1, ..., J\}$.

2.1 Random Variables

2.1.1 Bank i's Gross Asset Returns

- $r_j^i : \Omega \to (0, \infty)$ is the gross return that asset j earns for bank i from time $t = 0$ to $t = 1$. That is, \$1 invested at time $t = 0$ becomes $\$r_j^i(\omega)$ at time $t = 1$ if $\omega \in \Omega$ is realized. We restrict $r_j^i > 0$ because bank investments typically have collateral (e.g. real estate property as collateral for a mortgage).

- $\vec{r^i}(\omega) = \left(r_1^i(\omega), ..., r_J^i(\omega)\right)^T$ is the vector of asset returns for bank i, in scenario $\omega \in \Omega$.

2.2 Bank i Decision Variables

2.2.1 Dollar Amounts of Investment

- $x_j^i \in [0, \infty)$ is the quantity of dollars that bank i invest in asset j at time $t = 0$.
- $\vec{x}^i = (x_1^i, ..., x_J^i)^T$ is the vector with the jth entry equal to x_j^i, $j \in \{1, ..., J\}$.
- $a_i = \sum_{j=1}^{J} x_j^i$ is the total dollar amount of investment made by bank i at time $t = 0$.

2.2.2 Face Value of Debt $f_i \in [0, \infty)$ in dollars at Time $t = 1$

At time $t = 0$, bank i borrows money from a group of investors called debtholders. Bank i promises to pay this group a total of $f_i \in [0, \infty)$ dollars at time $t = 1$. The quantity f_i is called the face value of bank i's debt. As explained in the upcoming Section 2.3.3, this promise may not always be fulfilled. Hence, one of the government's role is to support debtholders in the event that bank i breaks this promise.

2.3 Functions

2.3.1 Dollar Cost of Financial Distress Φ

At time $t = 1$, bank i will realize a gross (book) value of assets of $\sum_{j=1}^{J} x_j^i r_j^i(\omega)$ and is liable to pay debtholders the face value of f_i, in scenario $\omega \in \Omega$. Based on these two quantities, the market will have its perception on the level of distress for bank i. The market value of bank i's assets will then be suppressed from its book value. This is modelled by the cost of financial distress, which is defined as follows.

Let $\Phi : \mathbb{R}^2 \to [0, \infty)$ be the dollar cost of financial distress at time $t = 1$ which applies to all banks. Assume $\Phi \in C^{2,2}$, $\Phi_1 < 0$, $\Phi_{11} > 0$, $\Phi_2 > 0$, and $\Phi_{22} > 0$ where Φ_l and Φ_{ll} are the first and second derivatives with respect to the lth variable.

For bank i, the first variable is gross dollar value of assets: $\sum_{j=1}^{J} x_j^i r_j^i(\omega)$, while the second variable is face value of debt f_i. Intuitively, this explains my assumption of $\Phi_1 < 0$ and $\Phi_2 > 0$ because higher asset value implies lower distress and higher promised debt payment implies higher distress. The convexity of Φ is assumed by convention.

The dollar quantity $\Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]$ reduce bank i's time $t = 1$ assets to a post-distress (market) value of $\sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]$.

2.3.2 Bank i's Dollar Value of Equity w_1^i at Time $t = 1$

Bank i's equity in dollars at time $t = 1$ is a function $w_1^i : \mathbb{R}^J \times \mathbb{R}^J \times \mathbb{R} \times \Omega \to \mathbb{R}$ defined by

$$w_1^i\left(\vec{x}^i, \vec{r}^i(\omega), f_i, \omega\right) = \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right] - f_i. \tag{2.1}$$

After assets earn the respective returns, financial distress costs and face value of debt are deducted to arrive at this value of net worth for bank i's equityholders at time $t = 1$. If $w_1^i < 0$, bank i is bankrupt.

2.3.3 Bank i's Dollar Value of Debt b_i Raised at Time $t = 0$

To fund its total dollar investment in assets at time $t = 0$ (a_i), bank i may raise b_i dollars of debt, with a promised payment of f_i dollars to its holders at time $t = 1$. However, depending on the realized $\omega \in \Omega$, and the resulting dollar value of equity at time $t = 1$ (w_1^i), bank i may not be able to keep its promise, and the quantity b_i will be affected. This is further described below.

At time $t = 1$, if bank i's equity is negative $(w_1^i < 0)$, it is considered bankrupt and will not have sufficient funds to pay its debtholders f_i dollars, because its value of assets net of distress costs, $\left(\sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]\right) = w_1^i + f_i$, will be less than f_i. Then, its debtholders will seize this value of post-distress assets and receive $\alpha_i \times \left(f_i - \left[\sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]\right]\right) = -\alpha_i \times w_1^i > 0$ dollars from the government, for some $\alpha_i \in [0,1]$. In other words, $(\alpha_i \times 100)$ is the percentage of bank i's debt which is covered by the government. The parameter α_i is assumed to be given. For example, it could represent an existing deposit insurance scheme provided by the government.

On the other hand, if bank i's equity is non-negative ($w_1^i \geq 0$) at time $t = 1$, its debtholders will receive f_i dollars.

The debtholders' payoff at time $t = 1$ can then be described by the following random variable.

$$Y(\omega) = \begin{cases} f_i; & \text{if } w_1^i(\omega) \geq 0 \\ \\ \alpha_i f_i \\ + (1-\alpha_i)\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]\right]; & \text{if } w_1^i(\omega) < 0 \end{cases}$$

Using the definition of w_1^i (Section 2.3.2), an equivalent notation for Y is

$$Y(\omega) = \alpha_i f_i + (1-\alpha_i)\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]\right).$$

As the authors assume, the quantity of dollars (b_i) that debtholders provide to bank i at time $t = 0$ is equal to their expected payoff at time $t = 1$.

Therefore,

$$b_i = \alpha_i f_i + (1-\alpha_i) E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right]. \tag{2.2}$$

2.3.4 Bank i's Dollar Value of Equity w_0^i used at Time $t = 0$

As an alternative to debt, bank i may also obtain money from its equityholders to fund the total dollar of investments in assets (a_i) at time $t = 0$.

The dollar value of equity funding to use is

$$
\begin{aligned}
w_0^i &= a_i - b_i \\
&= \sum_{j=1}^{J} x_j^i - \alpha_i f_i - (1-\alpha_i)\, E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right].
\end{aligned}
\tag{2.3}
$$

2.3.5 Utility Function u at Time $t = 1$

$u : \mathbb{R} \to \mathbb{R}$ is the utility function at time $t = 1$ which applies to every bank's equityholders. Assume that $u \in C^2$, $u^{'} > 0$, and $u^{''} < 0$, where $u^{'}$ and $u^{''}$ are the first and second derivatives.

In the model of the authors, if bank i realizes a time $t = 1$ equity of w_1^i dollars, the value of utility will be $u\left(w_1^i 1_{[w_1^i > 0]}\right)$.

Suppose that $w_1^i > 0$. As the dollar value of equity increase, bank i's equityholders will receive a higher utility ($u^{'} > 0$), but the rate of increase will decline ($u^{''} < 0$). That is, equityholders are risk-averse with respect to $w_1^i > 0$.

Whenever $w_1^i \leq 0$, utility takes the constant value $u(0)$. This is because if bank i is bankrupt at time $t = 1$, its equityholders have limited liability. Therefore, they will have a net worth of zero dollars regardless of how negative

w_1^i becomes.

2.4 Constraints

To guarantee that the time $t = 0$ dollar values of debt and equity (b_i and w_0^i) are non-negative, the following constraints are assumed to hold.

2.4.1 Minimum Dollar Invesment in Assets at Time $t = 0$

$$\forall i \in \{1, ..., N\} : \sum_{j=1}^{J} x_j^i \geq f_i \tag{2.4}$$

The total dollar amount that bank i invest in assets at time $t = 0$ needs to be at least the promised payment to debtholders at time $t = 1$.

If the above constraint holds, the time $t = 0$ dollar value of equity is non-negative ($w_0^i \geq 0$), since

$$\begin{aligned} b_i &= \alpha_i f_i + (1-\alpha_i)\, E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right] \\ &\leq \alpha_i f_i + (1-\alpha_i)\, f_i \\ &= f_i \end{aligned} \tag{2.5}$$

implies that

$w_0^i = a_i - b_i = \sum_{j=1}^{J} x_j^i - b_i \overset{(2.5)}{\geq} \sum_{j=1}^{J} x_j^i - f_i \overset{(2.4)}{\geq} 0.$

2.4.2 Dollar Value of Gross Assets Net of Financial Distress Cost at Time $t = 1$

$$\forall i \in \{1, ..., N\} \, \forall \omega \in \Omega : \sum_{j=1}^{J} x_j^i r_j^i (\omega) - \Phi \left[\sum_{j=1}^{J} x_j^i r_j^i (\omega), f_i \right] \geq 0 \qquad (2.6)$$

In all scenarios, for all banks, the time $t = 1$ dollar value of gross assets net of distress cost must be non-negative. This constraint will guarantee that the dollar value of debt raised at time $t = 0$ (b_i) is non-negative, because $f_i \in [0, \infty)$ and (2.6) implies that $E \left[\min \left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi \left[\sum_{j=1}^{J} x_j^i r_j^i, f_i \right] \right) \right]$ ≥ 0. Then, since $0 \leq \alpha_i \leq 1$ as well, from the expression (2.2), $b_i \geq 0$.

2.5 Given Parameters

2.5.1 Bank i's Dollar Value of Endowment $\bar{w}_0^i \in (0, \infty)$ at Time $t = 0$

At time $t = 0$, bank i's equityholders are endowed with $\bar{w}_0^i \in (0, \infty)$ dollars.

2.5.2 Total Dollars of Tax Revenue $\bar{\tau} \in (0, \infty)$

Let τ_i (described in a later Section 2.8.3) be the dollar value of tax paid by bank i at time $t = 0$. Then, $\bar{\tau} = \sum_{i=1}^{N} \tau_i$ should hold. That is, $\bar{\tau} \in (0, \infty)$ is the total dollar amount of taxes that the government wants to collect at time $t = 0$.

2.5.3 Rate of Utility $c \in (0, \infty)$ for Consumption at Time $t = 0$

For each dollar that bank i equityholders consume at time $t = 0$, they will experience a utility of $c \in (0, \infty)$. The parameter c applies to every bank $i \in \{1, ..., N\}$.

In the model of the authors, bank i equityholders' value of utility at time $t = 0$ is $c \times (\bar{w}_0^i - w_0^i - \tau_i)$.

If $\bar{w}_0^i - w_0^i - \tau_i > 0$, bank i's endowment at time $t = 0$ is sufficient to fund its desired dollar value of equity (w_0^i) and tax bill (τ_i). The remainder positive dollar amount $\bar{w}_0^i - w_0^i - \tau_i$ is consumed immediately (e.g. paid as dividends to its shareholders) at time $t = 0$, with a corresponding utility of $c \times (\bar{w}_0^i - w_0^i - \tau_i) > 0$.

On the other hand, if $\bar{w}_0^i - w_0^i - \tau_i < 0$, the bank's endowment is not enough to fund its desired equity amount and tax bill. Therefore it will need to raise an additional $-(\bar{w}_0^i - w_0^i - \tau_i) > 0$ dollars by issuing shares. From the perspective of bank i's equityholders, this is a cash outflow (negative consumption) at time $t = 0$, with a corresponding utility of $c \times (\bar{w}_0^i - w_0^i - \tau_i) < 0$.

2.5.4 Undercapitalization Threshold $z \in (0, 1]$

Bank i is deemed undercapitalized at time $t = 1$ if $w_1^i < za_i$, when its dollar value of equity at time $t = 1$ is less than $(z \times 100)$ percent of its initial total dollar value of assets (a_i).

2.5.5 Rate of Disutility $e \in (0, \infty)$ for Undercapitalization at Time $t = 1$

Using the notion of individual bank undercapitalization in Section 2.5.4, consider the case of that occurring system-wide (i.e. a crisis ensues), when $W_1 < zA$, where $W_1 = \sum_{i=1}^{N} w_1^i$ and $A = \sum_{i=1}^{N} a_i$.

The externality cost to social utility due to the possible event of a crisis is $e \times (W_1 - zA)\, 1_{[W_1 < zA]} \leq 0$. In other words, if there is indeed a crisis, for each dollar that W_1 falls below zA, society experiences a utility amount of $-e < 0$.

One example to justify this term is that when the banking system's equity W_1 is low at time $t = 1$, other agents in the economy will find it more costly to obtain funds (e.g. take out a loan) for their operations. As a result, overall productivity in the economy will decline, which is a cost to social utility.

2.5.6 Rate of Disutility $g \in (0, \infty)$ for bankruptcy at Time $t = 1$

If $w_1^i < 0$, bank i is considered bankrupt at time $t = 1$. When this happens, recall from Section 2.3.3 that the government will provide $-\alpha_i \times w_1^i > 0$ dollars to bank i's debtholders.

The cost to social utility due to the possible event of bank i's bankruptcy is $g \times \alpha_i \times w_1^i 1_{[w_1^i < 0]} \leq 0$. That is, for each dollar that bank i's equity at time $t = 1$ (w_1^i) falls below zero, society experiences a utility amount of $-g \times \alpha_i < 0$.

The reasoning is that when a bank is bankrupt, government funds used to guarantee its debtholders could have alternative uses, and therefore represent an opportunity cost (in monetary and utility terms) for other agents in the economy.

The parameter g applies to every bank $i \in \{1, ..., N\}$.

2.6 Events

2.6.1 Bank i Bankruptcy

The **bankruptcy** of bank i at time $t = 1$ is defined as $\{\omega \in \Omega : w_1^i < 0\}$.

2.6.2 Bank i Undercapitalization

The **undercapitalization** of bank i at time $t = 1$ is defined as $\{\omega \in \Omega : w_1^i < za_i\}$.

2.6.3 Systemic Crisis (Aggregate Undercapitalization)

Let $W_1 = \sum_{i=1}^N w_1^i$ and $A = \sum_{i=1}^N a_i$. Then, a **systemic crisis** at time $t = 1$ is defined as $\{\omega \in \Omega : W_1 < zA\}$.

2.7 Measures of Risk

2.7.1 Expected Shortfall

The expected shortfall measures bank i's bankruptcy risk, and is defined as $ES_i := -E\left[w_1^i \middle| w_1^i < 0\right]$.

2.7.2 Systemic Expected Shortfall

The systemic expected shortfall for bank i is defined as $SES_i = E\left[za_i - w_1^i \middle| W_1 < zA\right]$. This quantity measures the systemic risk of bank i, as its expected contribution to aggregate undercapitalization in the event of a crisis.

2.8 Incentives and Taxation

Recall from Section 2.2 the definitions for $\vec{x}^i = (x_1^i, ..., x_J^i)^T$ and f_i. Let $x = \left(\vec{x}^1, ..., \vec{x}^N\right)$ be the $(J \times N)$ matrix of the banks' dollar investment amounts, and $f = (f_1, ..., f_N)$ be the $(1 \times N)$ vector of their face value of debt.

2.8.1 Bank Incentives

Let τ_i be the dollar amount of taxes that bank i pays at time $t = 0$. Then, its objective function is as follows.

$$
\begin{aligned}
F_i &= c \times \left(\bar{w}_0^i - w_0^i\left(\vec{x}^i, f_i\right) - \tau_i\right) + E\left[u\left(w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i > 0\right]}\right)\right] \\
&= c \times \left(\bar{w}_0^i - \tau_i - \sum_{j=1}^{J} x_j^i + \alpha_i f_i\right. \\
&\quad \left. + \left(1 - \alpha_i\right) E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right]\right) \\
&\quad + E\left[u\left(w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i > 0\right]}\right)\right] \qquad (2.7)
\end{aligned}
$$

Bank i will choose $\vec{x}^i$ and f_i to solve the following problem.

$$\begin{cases} \text{Max}_{\vec{x}^i, f_i} F_i \\ \text{Subject To:} \\ \bullet \forall j \in \{1, ..., J\} : x_j^i \geq 0 \\ \bullet f_i \geq 0 \\ \bullet \sum_{j=1}^J x_j^i \geq f_i \\ \bullet \forall \omega \in \Omega : \sum_{j=1}^J x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^J x_j^i r_j^i(\omega), f_i\right] \geq 0 \end{cases} \tag{2.8}$$

Bank i will maximize its equityholders' time $t = 0$ utility plus time $t = 1$ expected utility by choosing the dollar amount of investment in each asset j $\left(\text{i.e. } \left\{x_j^i\right\}_{j \in \{1,...,J\}}\right)$ and the promised payment to debtholders (f_i).

The quantities decided for these variables will consequently determine the total dollars it invest in assets (a_i) from time $t = 0$ to time $t = 1$, as well as the corresponding dollar amounts of debt (b_i) and equity (w_0^i) funding.

2.8.2 Government Incentives

The government will maximize the sum of individual bank objectives, while accounting for

- expected disutility to society due to costs of government support when individual banks are bankrupt $\left(\text{i.e. } \sum_{i=1}^N \alpha_i g E\left[w_1^i 1_{\left[w_1^i < 0\right]}\right]\right)$ and
- expected disutility to society due to negative externalities when the banking system as a whole is undercapitalized $\left(\text{i.e. } eE\left[(W_1 - zA) 1_{[W_1 < zA]}\right]\right)$.

Applying the constraint $\bar{\tau} = \sum_{i=1}^{N} \tau_i$ from Section 2.5.2, its objective function is as follows.

$$
\begin{aligned}
F(x,f) = {} & \sum_{i=1}^{N} F_i + \sum_{i=1}^{N} \alpha_i g E\left[w_1^i 1_{\left[w_1^i<0\right]}\right] + eE\left[(W_1 - zA)\, 1_{[W_1<zA]}\right] \\
= {} & -c \times \bar{\tau} + \sum_{i=1}^{N} c \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + \alpha_i f_i\right. \\
& \left. + (1-\alpha_i)\, E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right]\right) \\
& + \sum_{i=1}^{N} E\left[u\left(w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i>0\right]}\right)\right] + \sum_{i=1}^{N} \alpha_i g E\left[w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i<0\right]}\right] \\
& + eE\left[\left(W_1(x,f) - zA(x)\right) 1_{[W_1<zA]}\right]
\end{aligned}
\tag{2.9}
$$

The government would like to achieve social optimality as described by the following problem.

$$
\begin{cases}
\mathrm{Max}_{x,f} F(x,f) \\
\text{Subject To:} \\
\bullet \forall i \in \{1,...,N\}\, \forall j \in \{1,...,J\} : x_j^i \geq 0 \\
\bullet \forall i \in \{1,...,N\} : f_i \geq 0 \\
\bullet \forall i \in \{1,...,N\} : \sum_{j=1}^{J} x_j^i \geq f_i \\
\bullet \forall i \in \{1,...,N\} \\
\quad \forall \omega \in \Omega : \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right] \geq 0
\end{cases}
\tag{2.10}
$$

2.8.3 Taxation

Let $\left(\tilde{x}, \tilde{f}\right)$ be an optimal solution to the government's problem (2.10). In the presence of regulation described below, banks will choose $\left(\hat{x}, \hat{f}\right)$ and pay taxes so that $F\left(\hat{x}, \hat{f}\right) = F\left(\tilde{x}, \tilde{f}\right)$.

1. Let

$$\hat{\tau}_i = \hat{\tau}_i\left(x, f\right)$$

$$= -\frac{\alpha_i g}{c} E\left[w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i<0\right]}\right] - \frac{e}{c} E\left[\left(w_1^i\left(\vec{x}^i, f_i\right) - z a_i\left(\vec{x}^i\right)\right) 1_{[W_1<zA]}\right]$$

$$= \frac{\alpha_i g}{c} P\left[w_1^i < 0\right] ES_i + \frac{e}{c} P\left[W_1 < zA\right] SES_i \tag{2.11}$$

be the dollars of taxes payable by bank i, in functional form. Note that while w_1^i and a_i only depends on bank i decisions $(\vec{x}^i, f_i)$, W_1 and A depends on those of every bank (x, f).

2. With $\hat{\tau}_i$, the objective function of bank i becomes

$$\hat{F}_i\left(x, f\right) = c \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + \alpha_i f_i\right.$$

$$+ \left(1-\alpha_i\right) E\left[\min\left(f_i, \sum_{j=1}^{J} x_j^i r_j^i - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i, f_i\right]\right)\right]\Bigg)$$

$$+E\left[u\left(w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i>0\right]}\right)\right] + \alpha_i g E\left[w_1^i\left(\vec{x}^i, f_i\right) 1_{\left[w_1^i<0\right]}\right]$$

$$+eE\left[\left(w_1^i\left(\vec{x}^i, f_i\right) - z a_i\left(\vec{x}^i\right)\right) 1_{[W_1<zA]}\right] \tag{2.12}$$

Then, bank i will face the following problem.

$$\begin{cases} \text{Max}_{\vec{x}^i, f_i} \hat{F}_i\left(x, f\right) \\ \text{Subject To:} \\ \bullet \forall j \in \{1, ..., J\} : x_j^i \geq 0 \\ \bullet f_i \geq 0 \\ \bullet \sum_{j=1}^{J} x_j^i \geq f_i \\ \bullet \forall \omega \in \Omega : \sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right), f_i\right] \geq 0 \end{cases} \tag{2.13}$$

However, the objective function $\hat{F}_i$, and hence optimal utility for its equityholders now depends on decisions of every bank (x, f).

Therefore, the banks will have the incentive to collectively solve the following multi-objective problem.

$$\begin{cases} \text{Max}_{x,f} \left\{\hat{F}_1\left(x, f\right), ..., \hat{F}_N\left(x, f\right)\right\} \\ \text{Subject To:} \\ \bullet \forall i \in \{1, ..., N\} \, \forall j \in \{1, ..., J\} : x_j^i \geq 0 \\ \bullet \forall i \in \{1, ..., N\} : f_i \geq 0 \\ \bullet \forall i \in \{1, ..., N\} : \sum_{j=1}^{J} x_j^i \geq f_i \\ \bullet \forall i \in \{1, ..., N\} \\ \quad \forall \omega \in \Omega : \sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right), f_i\right] \geq 0 \end{cases} \tag{2.14}$$

Suppose $\left(\bar{x}, \bar{f}\right)$ is a feasible point of problem (2.14). Then, we can make a **Pareto improvement** over $\left(\bar{x}, \bar{f}\right)$ if the following condition holds.

$$\begin{aligned} &\exists \left(x, f\right) \neq \left(\bar{x}, \bar{f}\right) : \\ &\Big[\forall i \in \{1, ..., N\} : \hat{F}_i\left(x, f\right) \geq \hat{F}_i\left(\bar{x}, \bar{f}\right) \\ &\text{ and } \exists i^{'} \in \{1, ..., N\} : \hat{F}_{i^{'}}\left(x, f\right) > \hat{F}_{i^{'}}\left(\bar{x}, \bar{f}\right)\Big] \end{aligned} \tag{2.15}$$

On the other hand, the point $\left(\bar{x}, \bar{f}\right)$ is **Pareto optimal** if it is impossible to make any Pareto improvement over it. That is, the following condition, which is the negation of (2.15), defines Pareto optimality of $\left(\bar{x}, \bar{f}\right)$.

$$\begin{aligned} &\forall \left(x, f\right) \neq \left(\bar{x}, \bar{f}\right) : \\ &\Big[\forall i \in \{1, ..., N\} : \hat{F}_i\left(x, f\right) \geq \hat{F}_i\left(\bar{x}, \bar{f}\right) \\ &\Rightarrow \forall i^{'} \in \{1, ..., N\} : \hat{F}_{i^{'}}\left(x, f\right) \leq \hat{F}_{i^{'}}\left(\bar{x}, \bar{f}\right)\Big] \\ \\ &\Leftrightarrow \\ \\ &\forall \left(x, f\right) \neq \left(\bar{x}, \bar{f}\right) : \\ &\Big[\exists i^{'} \in \{1, ..., N\} : \hat{F}_{i^{'}}\left(x, f\right) > \hat{F}_{i^{'}}\left(\bar{x}, \bar{f}\right) \\ &\Rightarrow \exists i \in \{1, ..., N\} : \hat{F}_i\left(x, f\right) < \hat{F}_i\left(\bar{x}, \bar{f}\right)\Big] \end{aligned} \tag{2.16}$$

In other words, suppose (x, f) is another feasible point of problem (2.14) for which a bank obtains a higher objective value than at $\left(\bar{x}, \bar{f}\right)$. Then, there must be another bank which experience a lower objective value at (x, f) than at $\left(\bar{x}, \bar{f}\right)$.

The solutions to the multi-objective problem (2.14) is defined as the set

S of $\left(\bar{x}, \bar{f}\right)$ for which Pareto optimality is achieved. That is,

$$S = \left\{\left(\bar{x}, \bar{f}\right) \text{ feasible for problem (2.14)} : \left(\bar{x}, \bar{f}\right) \text{ satisfies (2.16)}\right\}.$$

3. It is common to obtain a solution of a multi-objective problem, like (2.14), under assumptions about the decision maker, which in this case, is the collection of banks. Assume that banks believe, as a group, that each individual bank i's objective is equally important. Together, the banks will solve the following problem, which is scalarized from (2.14) with weights $\frac{1}{N}$ applied to the objective function of each bank i, $\hat{F}_i\left(x, f\right)$.

$$\begin{cases} \text{Max}_{x,f} \sum_{i=1}^{N} \frac{1}{N} \hat{F}_i\left(x, f\right) \\ \text{Subject To:} \\ \bullet \forall i \in \{1, ..., N\}\, \forall j \in \{1, ..., J\} : x_j^i \geq 0 \\ \bullet \forall i \in \{1, ..., N\} : f_i \geq 0 \\ \bullet \forall i \in \{1, ..., N\} : \sum_{j=1}^{J} x_j^i \geq f_i \\ \bullet \forall i \in \{1, ..., N\} \\ \quad \forall \omega \in \Omega : \sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega\right), f_i\right] \geq 0 \end{cases} \tag{2.17}$$

Let $\left(\hat{x}, \hat{f}\right)$ be a solution of (2.17). Then,

(a) $\left(\hat{x}, \hat{f}\right)$ is Pareto optimal. That is, $\left(\hat{x}, \hat{f}\right) \in S$, which implies it is a solution of (2.14).

Justification:

Since $\left(\hat{x}, \hat{f}\right)$ solves (2.17), it must be that $\sum_{i=1}^{N} \frac{1}{N} \hat{F}_i\left(\hat{x}, \hat{f}\right) \geq \sum_{i=1}^{N} \frac{1}{N} \hat{F}_i\left(x, f\right)$, for all $\left(x, f\right)$ in the feasible set of (2.17). Suppose that $\left(\hat{x}, \hat{f}\right)$ is not Pareto optimal. This means that there exists a feasible (x_{new}, f_{new}) such that for all $i \in \{1, ..., N\}$, $\hat{F}_i(x_{new}, f_{new}) \geq \hat{F}_i\left(\hat{x}, \hat{f}\right)$ and there exists $i' \in \{1, ..., N\}$ for which $\hat{F}_{i'}(x_{new}, f_{new}) > \hat{F}_{i'}\left(\hat{x}, \hat{f}\right)$. This implies that $\sum_{i=1}^{N} \frac{1}{N} \hat{F}_i(x_{new}, f_{new}) > \sum_{i=1}^{N} \frac{1}{N} \hat{F}_i\left(\hat{x}, \hat{f}\right)$, which contradicts the assumption that $\left(\hat{x}, \hat{f}\right)$ solves (2.17).

Therefore, it must be true that $\left(\hat{x}, \hat{f}\right) \in S$.

(b) $\left(\hat{x}, \hat{f}\right)$ solves the government's problem (2.10).

Justification:

Notice that the feasible set of problem (2.10) is the same as that for (2.17).

Let the objective function of (2.17) be denoted as $G(x, f) := \sum_{i=1}^{N} \frac{1}{N} \hat{F}_i(x, f)$. Then, the objective function of the government, expressed in (2.9), is related to $\left\{\hat{F}_i(x, f)\right\}_i$, written in (2.12), as follows.

$$F(x, f) = -c \times \bar{\tau} + \sum_{i=1}^{N} \hat{F}_i(x, f) = -c \times \bar{\tau} + N \times G(x, f)$$

Therefore, a solution of (2.17) coincides with a solution of (2.10). That is, $G\left(\hat{x}, \hat{f}\right) = G\left(\tilde{x}, \tilde{f}\right)$.

4. After the banks solve problem (2.17), they will report a solution $\left(\hat{x}, \hat{f}\right) \in S$ as their investment and funding decisions. Then, based on these reported quantities, bank i will be charged $\hat{\tau}_i\left(\hat{x}, \hat{f}\right)$ dollars by the govern-

ment. Moreover, each bank must pay a dollar amount equal to

$$\tau_0 = \frac{\bar{\tau} - \sum_{i=1}^{N} \hat{\tau}_i \left(\hat{x}, \hat{f}\right)}{N}. \tag{2.18}$$

This adjustment (τ_0) after banks make their decisions ensure that the government will collect a total of $\bar{\tau}$ dollars of tax revenue. Taking τ_0 into account, at $\left(\hat{x}, \hat{f}\right)$, bank i's objective becomes $-c \times \tau_0 + \hat{F}_i \left(\hat{x}, \hat{f}\right)$. Then, as in (2.9), the government's objective function takes the following value.

$$\begin{aligned}
F\left(\hat{x}, \hat{f}\right) = \sum_{i=1}^{N} \left[-c \times \tau_0 + \hat{F}_i \left(\hat{x}, \hat{f}\right)\right] + \sum_{i=1}^{N} \alpha_i g E \left[w_1^i \left(\hat{\vec{x}}^i, \hat{f}_i\right) 1_{\left[w_1^i < 0\right]}\right] \\
+ eE \left[\left(W_1 \left(\hat{x}, \hat{f}\right) - zA\left(\hat{x}\right)\right) 1_{\left[W_1 < zA\right]}\right] \\
= -c \times \bar{\tau} + \sum_{i=1}^{N} c \times \left(\bar{w}_0^i - \sum_{j=1}^{J} \hat{x}_j^i + \alpha_i \hat{f}_i\right. \\
\left. + \left(1 - \alpha_i\right) E \left[\min \left(\hat{f}_i, \sum_{j=1}^{J} \hat{x}_j^i r_j^i - \Phi \left[\sum_{j=1}^{J} \hat{x}_j^i r_j^i, \hat{f}_i\right]\right)\right]\right) \\
+ \sum_{i=1}^{N} E \left[u \left(w_1^i \left(\hat{\vec{x}}^i, \hat{f}_i\right) 1_{\left[w_1^i > 0\right]}\right)\right] \\
+ \sum_{i=1}^{N} \alpha_i g E \left[w_1^i \left(\hat{\vec{x}}^i, \hat{f}_i\right) 1_{\left[w_1^i < 0\right]}\right] \\
+ eE \left[\left(W_1 \left(\hat{x}, \hat{f}\right) - zA\left(\hat{x}\right)\right) 1_{\left[W_1 < zA\right]}\right]
\end{aligned} \tag{2.19}$$

By the above point 3(b), it follows that $F\left(\hat{x},\hat{f}\right)=F\left(\tilde{x},\tilde{f}\right)$, and the optimal objective value for the government's problem is achieved.

Chapter 3

Analysis of Objective Function

To understand how regulation described in Section 2.8.3 affect decisions made by banks, we shall investigate the objective function of the scalarized multi-bank problem (2.17).

3.1 Restricted Domains

For variables $f_i \in [0, \infty)$ and $x_l^i \in [0, \infty)$, the constraints from Section 2.4 will have an impact on the domain for which the objective function is defined.

3.1.1 Restricted Domain for f_i: $D(f_i)$

1. $\sum_{j=1}^{J} x_j^i \geq f_i$ provides an upper bound for f_i.

2. $\forall \omega : \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right] \geq 0$ also provides an upper bound $\bar{f}_i$ for f_i. Namely, $\bar{f}_i = \min\left\{\bar{f}_i(1), ..., \bar{f}_i(K)\right\}$, where $\bar{f}_i(k)$ is such that $\sum_{j=1}^{J} x_j^i r_j^i(\omega_k) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), \bar{f}_i(k)\right] = 0$.

   Let's denote $h(f_i, \omega) := \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right]$.

Then,

- $\frac{\partial h(f_i,\omega_k)}{\partial f_i} = -\Phi_2\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] < 0$ implies that $h(f_i,\omega_k)$ is decreasing in f_i, and
- $\frac{\partial^2 h(f_i,\omega_k)}{\partial (f_i)^2} = -\Phi_{22}\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] \leq 0$ implies that $h(f_i,\omega_k)$ is concave in f_i.

One must assume that for all $\omega_k \in \Omega$, $\left\{x_j^i\right\}_j$ and $\left\{r_j^i(\omega_k)\right\}_j$ are fixed such that $\exists f_i \geq 0 : h(f_i,\omega_k) \geq 0$. Otherwise, it must be that:

- $\bar{f}_i(k) < 0$, which together with the constraint $f_i \geq 0$ would lead to an empty domain for f_i (i.e. $D(f_i) = \emptyset$), or that
- $\forall f_i \in \mathbb{R} : h(f_i,\omega_k) < 0$, which violates the required constraint.

Therefore, assuming that $\exists f_i \geq 0 : h(f_i,\omega_k) \geq 0$ and using the fact that $h(f_i,\omega_k)$ is decreasing and concave in f_i, such $\bar{f}_i(k)$ exists and is non-negative $\left(\bar{f}_i(k) \geq 0\right)$, for all $\omega_k \in \Omega$.

Notice that for every $\omega_k \in \Omega$, $h(f_i,\omega_k) \geq 0$ for $f_i \leq \bar{f}_i(k)$.

Then, considering all $\omega_k \in \Omega$ simultaneously, $\bar{f}_i = \min\left\{\bar{f}_i(1), ..., \bar{f}_i(K)\right\}$ is an upper bound for f_i.

Therefore, one should restrict attention to the domain $D(f_i) = \left[0, \bar{\bar{f}}_i\right]$, where $\bar{\bar{f}}_i = \min\left\{\sum_{j=1}^{J} x_j^i, \bar{f}_i\right\}$.

3.1.2 Restricted Domain for x_l^i: $D\left(x_l^i\right)$

1. $\sum_{j=1}^{J} x_j^i \geq f_i \Leftrightarrow x_l^i \geq f_i - \sum_{j\neq l} x_j^i$ provides a lower bound for x_l^i.

2. $\forall\omega : \sum_{j=1}^{J} x_j^i r_j^i(\omega) - \Phi\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega), f_i\right] \geq 0$ also provides a lower bound for x_l^i.

   First, for a fixed $\omega_k \in \Omega$, denote
   $h(x_l^i, \omega_k) := x_l^i r_l^i(\omega_k) + \sum_{j \neq l} x_j^i r_j^i(\omega_k) - \Phi\left[x_l^i r_l^i(\omega_k) + \sum_{j \neq l} x_j^i r_j^i(\omega_k), f_i\right]$,
   and $\bar{x}_l^i(k)$ such that $h(\bar{x}_l^i(k), \omega_k) = 0$.

   Then, the lower bound for x_l^i is $\bar{x}_l^i = \max\{\bar{x}_l^i(1), ..., \bar{x}_l^i(K)\}$, and the reasoning is as follows.

   Notice that

   - $\frac{\partial h}{\partial x_l^i} = r_l^i(\omega_k) - r_l^i(\omega_k)\Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] > 0$ implies that h is increasing in x_l^i, and
   - $\frac{\partial^2 h}{\partial\left(x_l^i\right)^2} = -\left(r_l^i(\omega_k)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] < 0$ implies that h is concave in x_l^i.

   One must assume that $\left\{x_j^i\right\}_{j \neq l}, \left\{r_j^i(\omega_k)\right\}_j$ and f_i are fixed such that $\bar{x}_l^i(k)$ satisfying $h(\bar{x}_l^i(k), \omega_k) = 0$ exists. Otherwise, since $h(x_l^i, \omega_k)$ is increasing and concave, it must be that $h(x_l^i, \omega_k) < 0$, for all $x_l^i \in \mathbb{R}$, which makes it impossible for $h(x_l^i, \omega_k) \geq 0$ to be satisfied.

   Therefore, $\bar{x}_l^i(k)$ must exist, and it will be a lower bound for x_l^i, since $h(x_l^i, \omega_k) \geq 0$ for $x_l^i \geq \bar{x}_l^i(k)$.

   Considering all $\omega_k \in \Omega$ simultaneously, $\bar{x}_l^i = \max\{\bar{x}_l^i(1), ..., \bar{x}_l^i(K)\}$ is a lower bound for x_l^i which ensures that $\forall\omega : h(x_l^i, \omega) \geq 0$ will be satisfied.

Altogether, one can restrict attention to the domain of $D(x_l^i) = [\bar{\bar{x}}_l^i, \infty)$, where $\bar{\bar{x}}_l^i = \max\left\{f_i - \sum_{j \neq l} x_j^i, \bar{x}_l^i, 0\right\}$.

3.2 Objective for the Group of Banks when Taxes are Present

When banks as a group solve problem (2.17), the objective function is $\frac{1}{N}\sum_{i=1}^{N}\hat{F}_i(x,f)$, which equals the sum of individual bank objectives when taxes are accounted for, multiplied by $\frac{1}{N}$. Therefore, the time $t=0$ utility and $t=1$ expected utility for equityholders of an average bank will be captured.

Since banks are now determining investment and funding decisions $\left(\left\{x_j^i\right\}_{i,j}, \left\{f_i\right\}_i\right)$ as a group, we shall study how this objective depends on f_i, and x_l^i. For ease of notation, denote $\Psi = \frac{1}{N}\sum_{i=1}^{N}\hat{F}_i$.

3.2.1 Effect of $f_i \in D(f_i)$ on Ψ

Consider Ψ as a function of f_i only, with other variables taking fixed values.

(i) **Continuity**

Ψ is continuous for $f_i \in D(f_i)$, but is indifferentiable at $0 < \breve{f}_i^1 < ... < \breve{f}_i^L < \bar{\bar{f}}_i \in D(f_i)$, $0 \le L \le 2K$.

For values of f_i greater than that of $\breve{f}_i^l$, $l \in \{1,...,L\}$, there exists a scenario $\omega \in \Omega$ for which bank i is bankrupt ($w_1^i < 0$) or the system is in crisis ($W_1 < zA$).

(ii) **Increase/Decrease**

There is no generalization about the increasing/decreasing behaviour of Ψ with respect to $f_i \in D(f_i)$.

(iii) **Convexity**

For f_i within $\left(0, \breve{f}_i^1\right)$, $\left(\breve{f}_i^1, \breve{f}_i^2\right)$,...,$\left(\breve{f}_i^L, \bar{\bar{f}}_i\right)$, Ψ is concave.

Discussion

Consider the following as a function ψ of f_i, $i \in \{1, ..., N\}$ only, with $\omega_k \in \Omega$ and other variables fixed. Note that $\Psi = E[\psi]$.

$$\psi(f_i, \omega_k) = \sum_{m=1}^{N} \frac{c}{N} \times \left(\bar{w}_0^m - \sum_{j=1}^{J} x_j^m + \alpha_m f_m \right.$$

$$\left. + (1 - \alpha_m) \min \left(f_m, \sum_{j=1}^{J} x_j^m r_j^m(\omega_k) - \Phi\left[\sum_{j=1}^{J} x_j^m r_j^m(\omega_k), f_m \right] \right) \right)$$

$$+ \sum_{m=1}^{N} \frac{1}{N} u\left(w_1^m 1_{\left[w_1^m > 0\right]} \right) + \sum_{m=1}^{N} \frac{\alpha_m g}{N} w_1^m 1_{\left[w_1^m < 0\right]}$$

$$+ \frac{e}{N} (W_1 - zA) 1_{[W_1 < zA]}$$

$$= \psi_i(f_i, \omega_k)$$

$$+ \sum_{m \neq i} \frac{c}{N} \times \left(\bar{w}_0^m - \sum_{j=1}^{J} x_j^m + f_m + (1 - \alpha_m) w_1^m(\omega_k) 1_{\left[w_1^m < 0\right]} \right)$$

$$+ \sum_{m \neq i} \frac{1}{N} u\left(w_1^m(\omega_k) 1_{\left[w_1^m > 0\right]} \right) + \sum_{m \neq i} \frac{\alpha_m g}{N} w_1^m(\omega_k) 1_{\left[w_1^m < 0\right]}$$

where

$$\psi_i(f_i,\omega_k) = \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + f_i + (1-\alpha_i)\, w_1^i(f_i,\omega_k)\, 1_{\left[w_1^i<0\right]} \right)$$

$$+\frac{1}{N} u\left(w_1^i(f_i,\omega_k)\, 1_{\left[w_1^i>0\right]} \right) + \frac{\alpha_i g}{N} w_1^i(f_i,\omega_k)\, 1_{\left[w_1^i<0\right]}$$

$$+\frac{e}{N}\left(W_1(f_i,\omega_k) - zA\right) 1_{[W_1<zA]}.$$

In other words, $\psi(f_i,\omega_k)$ is the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders when taxes are considered, for a fixed $\omega_k \in \Omega$ (i.e. assuming that $\omega_k \in \Omega$ occurs with certainty).

For the moment, assume that $f_i \in \mathbb{R}$. Then, the relationship between $\psi(f_i,\omega_k)$ (equivalently $\psi_i(f_i,\omega_k)$) and f_i will depend on the behaviour of $w_1^i(f_i,\omega_k)$ and $W_1(f_i,\omega_k) - zA$, where $W_1(f_i,\omega_k) = w_1^i(f_i,\omega_k) + \sum_{m\neq i} w_1^m(\omega_k)$.

Notice that $w_1^i(f_i,\omega_k)$ and $W_1(f_i,\omega_k) - zA$ are continuous, decreasing, and concave since

- $\frac{\partial w_1^i}{\partial f_i} = \frac{\partial[W_1 - zA]}{\partial f_i} = -\Phi_2\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] - 1 < 0$, and
- $\frac{\partial^2 w_1^i}{\partial (f_i)^2} = \frac{\partial^2[W_1 - zA]}{\partial (f_i)^2} = -\Phi_{22}\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] \leq 0$.

As such, there is at most one $\tilde{f}_i^k$ such that $w_1^i\left(\tilde{f}_i^k,\omega_k\right) = 0$ and one $\tilde{\tilde{f}}_i^k$ such that $W_1\left(\tilde{\tilde{f}}_i^k,\omega_k\right) - zA = 0$. When $f_i > \tilde{f}_i^k$, bank i is bankrupt ($w_1^i < 0$) and when $f_i > \tilde{\tilde{f}}_i^k$, the system is in crisis ($W_1 < zA$) for scenario $\omega_k \in \Omega$.

Now, $\psi_i(f_i,\omega_k)$ can be studied in one of the five possible forms as follows, depending on the existence and values of $\tilde{f}_i^k$ and $\tilde{\tilde{f}}_i^k$.

(I) Suppose both $\tilde{f}_i^k$ and $\tilde{\tilde{f}}_i^k$ exists, and $\tilde{f}_i^k < \tilde{\tilde{f}}_i^k$. Then,

$$\psi_i\left(f_i, \omega_k\right) = \begin{cases} \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i\right) + \frac{1}{N} u\left(w_1^i\left(f_i, \omega_k\right)\right); \\ \text{if } -\infty < f_i \leq \tilde{f}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i + w_1^i\left(f_i, \omega_k\right)\left(1 - \alpha_i\right)\right) \\ + \frac{1}{N} u\left(0\right) + \frac{\alpha_i g}{N} w_1^i\left(f_i, \omega_k\right); \\ \text{if } \tilde{f}_i^k < f_i \leq \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i + w_1^i\left(f_i, \omega_k\right)\left(1 - \alpha_i\right)\right) \\ + \frac{1}{N} u\left(0\right) + \frac{\alpha_i g}{N} w_1^i\left(f_i, \omega_k\right) + \frac{e}{N}\left(W_1\left(f_i, \omega_k\right) - zA\right); \\ \text{if } \tilde{\tilde{f}}_i^k < f_i < \infty \end{cases}$$

In general, $\psi_i\left(f_i, \omega_k\right)$ is continuous, but not differentiable at $\tilde{f}_i^k$ or $\tilde{\tilde{f}}_i^k$. For other $f_i \in \mathbb{R}$, $\psi_i\left(f_i, \omega_k\right)$ is differentiable.

The first derivative is

$$\frac{\partial \psi}{\partial f_i} = \frac{\partial \psi_i}{\partial f_i}$$

$$= \begin{cases} \frac{c}{N} - \frac{1}{N} u'\left(w_1^i\left(f_i, \omega_k\right)\right)\left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right); \\ \text{if } -\infty < f_i < \tilde{f}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right)\left(\frac{c}{N}\left(1 - \alpha_i\right) + \frac{\alpha_i g}{N}\right); \\ \text{if } \tilde{f}_i^k < f_i < \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right)\left(\frac{c}{N}\left(1 - \alpha_i\right) + \frac{\alpha_i g}{N} + \frac{e}{N}\right); \\ \text{if } \tilde{\tilde{f}}_i^k < f_i < \infty \end{cases}$$

There is no generalization about the positivity/negativity of $\frac{\partial \psi}{\partial f_i}$, so no conclusions are made about the increasing/decreasing behaviour of $\psi(f_i, \omega_k)$. However, $\frac{\partial \psi}{\partial f_i}$ does reflect incentives faced by the group of banks when deciding on f_i, the promised payment to bank i's debtholders at time $t = 1$. Let's consider f_i increasing within each interval $\left(-\infty, \tilde{f}_i^k\right)$, $\left(\tilde{f}_i^k, \tilde{\tilde{f}}_i^k\right)$, and $\left(\tilde{\tilde{f}}_i^k, \infty\right)$.

(i) $-\infty < f_i < \tilde{f}_i^k$

In this region of f_i, bank i is not bankrupt ($w_1^i \geq 0$) and the system is not in crisis ($W_1 \geq zA$). Therefore, bank i's debtholders will receive f_i dollars and equityholders will obtain $w_1^i \geq 0$ dollars at time $t = 1$. Since $\omega_k \in \Omega$ is seen in isolation, f_i is also the amount of funding debtholders provide the bank at time $t = 0$.

As such, if f_i increase but does not exceed $\tilde{f}_i^k$, bank i equityholders' consumption at time $t = 0$, $\left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + f_i\right)$, and thus their utility, will increase at that time. This is captured by $\frac{c}{N} > 0$ in $\frac{\partial \psi}{\partial f_i}$, which means the time $t = 0$ utility of an average bank will also increase as a result.

However, since w_1^i decreases in f_i, equityholders of bank i will receive a lower utility at time $t = 1$. This is captured by $-\frac{1}{N} u'\left(w_1^i\right)(\Phi_2 + 1) < 0$ in $\frac{\partial \psi}{\partial f_i}$, which means the average bank's utility will also decrease.

Notice that since $w_1^i \geq 0$ and $W_1 \geq zA$ for $\omega_k \in \Omega$ if $-\infty < f_i < \tilde{f}_i^k$, an increase in f_i within this region has no effect on taxes, because bank i would not require government compensation for its debtholders or cause a crisis to occur at time $t = 1$.

(ii) $\tilde{f}_i^k < f_i < \tilde{\tilde{f}}_i^k$

In this region of f_i, bank i is bankrupt ($w_1^i < 0$) but the system is well-capitalized ($W_1 \geq zA$). At time $t = 1$, bank i's debtholders will receive its post-distress value of assets ($w_1^i + f_i$) plus an amount of government compensation $-\alpha_i w_1^i > 0$. Since $\omega_k \in \Omega$ is considered in isolation, this is also the amount of debt funding provided to bank i at time $t = 0$. The derivative of debt funding with respect to f_i is thus $\frac{\partial\left[w_1^i + f_i - \alpha_i w_1^i\right]}{\partial f_i} = 1 - (1 - \alpha_i)(\Phi_2 + 1)$, which may be positive or negative for different values of f_i. Therefore, debt funding for bank i may increase or decrease as f_i increases.

Moreover, the cost to social utility for the government compensation is $-\alpha_i g w_1^i > 0$, so bank i will pay taxes of $-\frac{\alpha_i g}{c} w_1^i > 0$ dollars at time $t = 0$. An increase in f_i will increase the bank's tax bill because $\frac{\partial\left[-\frac{\alpha_i g}{c} w_1^i\right]}{\partial f_i} = \frac{\alpha_i g}{c}(\Phi_2 + 1) > 0$.

Altogether, bank i equityholders' time $t = 0$ utility is $c \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i + w_1^i (1 - \alpha_i) + \frac{\alpha_i g}{c} w_1^i\right)$, and its first derivative with respect to f_i is $c - (\Phi_2 + 1)(c(1 - \alpha_i) + \alpha_i g)$, which may be positive or negative for different values of f_i. The derivative of an average bank's time $t = 0$ utility with respect to f_i is this expression divided by N, as captured by $\frac{\partial \psi}{\partial f_i}$. The result is that an increase in f_i may increase or decrease an average bank's time $t = 0$ utility.

Note that the time $t = 1$ utility for equityholders of bank i (and consequently, those of the average bank) will not be affected because when $w_1^i < 0$, it takes a constant value of $u(0)$.

(iii) $\tilde{\tilde{f}}_i^k < f_i < \infty$

The reasoning from (ii) regarding debt funding applies in this case to explain $\frac{c}{N}\left[1-(1-\alpha_i)(\Phi_2+1)\right]$ in $\frac{\partial\psi}{\partial f_i}$, because for $\tilde{\tilde{f}}_i^k < f_i < \infty$, bank i is bankrupt ($w_1^i < 0$).

However, the banking system is also undercapitalized ($W_1 < zA$) for $f_i > \tilde{\tilde{f}}_i^k$. Considering $\omega_k \in \Omega$ in isolation, the tax for bank m is $-\frac{\alpha_m g}{c} w_1^m(\omega_k) 1_{\left[w_1^m<0\right]} - \frac{e}{c}\left(w_1^m(\omega_k) - za_m\right) 1_{[W_1<zA]}$ dollars. So, the tax for an average bank is $-\frac{1}{N}\sum_{m=1}^{N} \frac{\alpha_m g}{c} w_1^m(\omega_k) 1_{\left[w_1^m<0\right]} - \frac{e}{cN}\left(W_1(f_i,\omega_k) - zA\right)$ dollars. A higher value of f_i will increase this tax amount, because its derivative with respect to f_i is $\left(\frac{\alpha_i g}{cN} + \frac{e}{cN}\right)(\Phi_2+1) > 0$. When a bank is charged τ dollars of taxes, its equityholders' obtains a utility of $-c\times\tau$. Therefore, for an average bank, the utility experienced due to taxes will decrease, because its derivative with respect to f_i is $-\left(\frac{\alpha_i g}{N} + \frac{e}{N}\right)(\Phi_2+1) < 0$, as reflected in $\frac{\partial\psi}{\partial f_i}$.

Again, the time $t=1$ utility for equityholders of bank i and for those of the average bank is unaffected because $u\left(w_1^i 1_{\left[w_1^i>0\right]}\right) = u(0)$ when $\tilde{\tilde{f}}_i^k < f_i < \infty$.

The second derivative of $\psi(f_i, \omega_k)$ is

$$\frac{\partial^2 \psi}{\partial (f_i)^2} = \frac{\partial^2 \psi_i}{\partial (f_i)^2}$$

$$= \begin{cases} -\frac{1}{N} u'\left(w_1^i(f_i, \omega_k)\right) \Phi_{22}\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right] \\ +\frac{1}{N}\left(-\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right] - 1\right)^2 u''\left(w_1^i(f_i, \omega_k)\right) < 0; \\ \text{if } -\infty < f_i < \tilde{f}_i^k \\ \\ -\Phi_{22}\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right]\left(\frac{c}{N}(1-\alpha_i) + \frac{\alpha_i g}{N}\right) < 0; \\ \text{if } \tilde{f}_i^k < f_i < \tilde{\tilde{f}}_i^k \\ \\ -\Phi_{22}\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right]\left(\frac{c}{N}(1-\alpha_i) + \frac{\alpha_i g}{N} + \frac{e}{N}\right) < 0; \\ \text{if } \tilde{\tilde{f}}_i^k < f_i < \infty \end{cases}$$

Therefore, $\psi(f_i, \omega_k)$ is concave within $\left(-\infty, \tilde{f}_i^k\right)$, $\left(\tilde{f}_i^k, \tilde{\tilde{f}}_i^k\right)$, and $\left(\tilde{\tilde{f}}_i^k, \infty\right)$.

(II) Suppose both $\tilde{f}_i^k$ and $\tilde{\tilde{f}}_i^k$ exists, and $\tilde{f}_i^k > \tilde{\tilde{f}}_i^k$. Then,

$$\psi_i\left(f_i,\omega_k\right) = \begin{cases} \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i\right) + \frac{1}{N} u\left(w_1^i\left(f_i,\omega_k\right)\right); \\ \text{ if } -\infty < f_i \leq \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i\right) + \frac{1}{N} u\left(w_1^i\left(f_i,\omega_k\right)\right) \\ +\frac{e}{N}\left(W_1\left(f_i,\omega_k\right) - zA\right); \\ \text{ if } \tilde{\tilde{f}}_i^k < f_i \leq \tilde{f}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i + w_1^i\left(f_i,\omega_k\right)\left(1-\alpha_i\right)\right) + \frac{1}{N} u\left(0\right) \\ +\frac{\alpha_i g}{N} w_1^i\left(f_i,\omega_k\right) + \frac{e}{N}\left(W_1\left(f_i,\omega_k\right) - zA\right); \\ \text{ if } \tilde{f}_i^k < f_i < \infty \end{cases}$$

$\psi_i\left(f_i,\omega_k\right)$ is continuous, but not differentiable at $\tilde{f}_i^k$ or $\tilde{\tilde{f}}_i^k$. For other $f_i \in \mathbb{R}$, $\psi_i\left(f_i,\omega_k\right)$ is differentiable.

$$\frac{\partial\psi}{\partial f_i} = \frac{\partial\psi_i}{\partial f_i}$$

$$= \begin{cases} \frac{c}{N} - \frac{1}{N} u'\left(w_1^i\left(f_i,\omega_k\right)\right)\left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right); \\ \text{ if } -\infty < f_i < \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right)\left(\frac{u'\left(w_1^i\left(f_i,\omega_k\right)\right)}{N} + \frac{e}{N}\right); \\ \text{ if } \tilde{\tilde{f}}_i^k < f_i < \tilde{f}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right)\left(\frac{c}{N}\left(1-\alpha_i\right) + \frac{\alpha_i g}{N} + \frac{e}{N}\right); \\ \text{ if } \tilde{f}_i^k < f_i < \infty \end{cases}$$

Similar to (I), there is no general conclusion about the positivity/negativity of $\frac{\partial \psi}{\partial f_i}$. Nonetheless, $\frac{\partial \psi}{\partial f_i}$ does show the tradeoff encountered by the group of banks when choosing f_i. Again, three regions of f_i: $\left(-\infty, \tilde{\tilde{f}}_i^k\right)$, $\left(\tilde{\tilde{f}}_i^k, \tilde{f}_i^k\right)$, $\left(\tilde{f}_i^k \infty\right)$ may be considered.

(i) $-\infty < f_i < \tilde{\tilde{f}}_i^k$

The explanation is equivalent to (i) of (I), since $w_1^i \geq 0$ and $W_1 \geq zA$ in this case.

(ii) $\tilde{\tilde{f}}_i^k < f_i < \tilde{f}_i^k$

In this region of f_i, bank i is not bankrupt ($w_1^i \geq 0$), but the system is in crisis ($W_1 < zA$).

The effects on the average bank's quantity of debt funding received at time $t = 0$ and on its equityholders' utility at time $t = 1$ are described in (i) of (I). This explains $\frac{c}{N} > 0$ and $-\frac{1}{N} u' \left(w_1^i\right) \left(\Phi_2 + 1\right) < 0$ in $\frac{\partial \psi}{\partial f_i}$.

However, since the system is in crisis for $\tilde{\tilde{f}}_i^k < f_i < \tilde{f}_i^k$, an increase in f_i will affect dollars of taxes paid by an average bank: $-\frac{1}{N} \sum_{m=1}^N \frac{\alpha_m g}{c} w_1^m \left(\omega_k\right) 1_{\left[w_1^m < 0\right]} - \frac{e}{cN} \left(W_1 \left(f_i, \omega_k\right) - zA\right)$. Since $w_1^i \geq 0$, the derivative of this amount with respect to f_i is $\frac{e}{cN} \left(\Phi_2 + 1\right) > 0$. That is, an average bank will pay more taxes. Similar to (iii) of (I), for an average bank, the utility experienced due to paying taxes will decrease, since its derivative is $-\frac{e}{N} \left(\Phi_2 + 1\right) < 0$, which is reflected in $\frac{\partial \psi}{\partial f_i}$.

(iii) $\tilde{f}_i^k < f_i < \infty$ The explanation is equivalent to (iii) of (I), since $w_1^i < 0$ and $W_1 < zA$ in this case.

Similar to (I), one may take the second derivative $\frac{\partial^2 \psi}{\partial (f_i)^2}$ and conclude that $\psi(f_i, \omega_k)$ is concave within $\left(-\infty, \tilde{\tilde{f}}_i^k\right)$, $\left(\tilde{\tilde{f}}_i^k, \tilde{f}_i^k\right)$, and $\left(\tilde{f}_i^k, \infty\right)$.

(III) Suppose $\tilde{f}_i^k$ exists, but $\tilde{\tilde{f}}_i^k$ does not exist.

This situation occurs when $W_1(f_i, \omega_k) < zA$ for all $f_i \in \mathbb{R}$ (i.e. the system is in crisis regardless of f_i), and bank i is not bankrupt ($w_1^i(f_i, \omega_k) > 0$) for $f_i < \tilde{f}_i^k$.

$$\psi_i(f_i, \omega_k) = \begin{cases} \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i\right) + \frac{1}{N} u\left(w_1^i(f_i, \omega_k)\right) \\ + \frac{e}{N}\left(W_1(f_i, \omega_k) - zA\right); \\ \text{if } -\infty < f_i \leq \tilde{f}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^J x_j^i + f_i + w_1^i(f_i, \omega_k)(1-\alpha_i)\right) + \frac{1}{N} u(0) \\ + \frac{\alpha_i g}{N} w_1^i(f_i, \omega_k) + \frac{e}{N}\left(W_1(f_i, \omega_k) - zA\right); \\ \text{if } \tilde{f}_i^k < f_i < \infty \end{cases}$$

$\psi_i(f_i, \omega_k)$ is continuous but not differentiable at $\tilde{f}_i^k$. For other values of f_i, $\psi_i(f_i, \omega_k)$ is differentiable.

$$\begin{aligned} \frac{\partial \psi}{\partial f_i} &= \frac{\partial \psi_i}{\partial f_i} \\ &= \begin{cases} \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right] + 1\right)\left(\frac{u'\left(w_1^i(f_i, \omega_k)\right)}{N} + \frac{e}{N}\right); \\ \text{if } -\infty < f_i < \tilde{f}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^J x_j^i r_j^i(\omega_k), f_i\right] + 1\right)\left(\frac{c}{N}(1-\alpha_i) + \frac{\alpha_i g}{N} + \frac{e}{N}\right); \\ \text{if } \tilde{f}_i^k < f_i < \infty \end{cases} \end{aligned}$$

Similar to (I) and (II), no generalizations are made about the positivity/negativity of $\frac{\partial \psi}{\partial f_i}$. However, it can still be used to explain incentives for the group of banks when deciding on f_i.

The explanation of $\frac{\partial \psi}{\partial f_i}$ for $-\infty < f_i < \tilde{f}_i^k$ is equivalent to (ii) of (II) ($w_1^i \geq 0$ and $W_1 < zA$), and that for $\tilde{f}_i^k < f_i < \infty$ is equivalent to (iii) of (I) ($w_1^i < 0$ and $W_1 < zA$).

As in (I) and (II), one can consider $\frac{\partial^2 \psi}{\partial (f_i)^2}$ and conclude that $\psi\left(f_i, \omega_k\right)$ is concave within $\left(-\infty, \tilde{f}_i^k\right)$, and $\left(\tilde{f}_i^k, \infty\right)$.

(IV) Suppose $\tilde{\tilde{f}}_i^k$ exists, but $\tilde{f}_i^k$ does not exist.

This situation occurs when $w_1^i\left(f_i, \omega_k\right) < 0$ for all $f_i \in \mathbb{R}$ (i.e. bank i is bankrupt regardless of f_i) and the system is not in crisis ($W_1\left(f_i, \omega_k\right) \geq zA$) for $f_i < \tilde{\tilde{f}}_i^k$.

$$\psi_i\left(f_i, \omega_k\right) = \begin{cases} \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + f_i + w_1^i\left(f_i, \omega_k\right)\left(1 - \alpha_i\right)\right) \\ + \frac{1}{N} u\left(0\right) + \frac{\alpha_i g}{N} w_1^i\left(f_i, \omega_k\right); \\ \text{ if } -\infty < f_i \leq \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + f_i + w_1^i\left(f_i, \omega_k\right)\left(1 - \alpha_i\right)\right) \\ + \frac{1}{N} u\left(0\right) + \frac{\alpha_i g}{N} w_1^i\left(f_i, \omega_k\right) + \frac{e}{N}\left(W_1\left(f_i, \omega_k\right) - zA\right); \\ \text{ if } \tilde{\tilde{f}}_i^k < f_i < \infty \end{cases}$$

$\psi_i\left(f_i, \omega_k\right)$ is continuous but not differentiable at $\tilde{\tilde{f}}_i^k$. For other values of

f_i, $\psi_i(f_i, \omega_k)$ is differentiable.

$$\frac{\partial \psi}{\partial f_i} = \frac{\partial \psi_i}{\partial f_i}$$
$$= \begin{cases} \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] + 1\right)\left(\frac{c}{N}(1-\alpha_i) + \frac{\alpha_i g}{N}\right); \\ \text{if } -\infty < f_i < \tilde{\tilde{f}}_i^k \\ \\ \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] + 1\right)\left(\frac{c}{N}(1-\alpha_i) + \frac{\alpha_i g}{N} + \frac{e}{N}\right); \\ \text{if } \tilde{\tilde{f}}_i^k < f_i < \infty \end{cases}$$

Again, there is no generalization about whether $\frac{\partial \psi}{\partial f_i}$ is positive or negative, but it still describe incentives for the group of banks when deciding on the value of f_i.

The explanation of $\frac{\partial \psi}{\partial f_i}$ for $-\infty < f_i < \tilde{\tilde{f}}_i^k$ is equivalent to (ii) of (I) ($w_1^i < 0$ and $W_1 \geq zA$), and that for $\tilde{\tilde{f}}_i^k < f_i < \infty$ is equivalent to (iii) of (I) ($w_1^i < 0$ and $W_1 < zA$).

$\frac{\partial^2 \psi}{\partial (f_i)^2}$ may be considered, and one will conclude that $\psi(f_i, \omega_k)$ is concave within $\left(-\infty, \tilde{\tilde{f}}_i^k\right)$, and $\left(\tilde{\tilde{f}}_i^k, \infty\right)$.

(V) Suppose that neither $\tilde{f}_i^k$ or $\tilde{\tilde{f}}_i^k$ exists.

This situation occurs when bank i is bankrupt ($w_1^i(f_i, \omega_k) < 0$) and the system is in crisis ($W_1(f_i, \omega_k) < zA$) for all $f_i \in \mathbb{R}$. Then,

$$\psi_i(f_i, \omega_k) = \frac{c}{N} \times \left(\bar{w}_0^i - \sum_{j=1}^{J} x_j^i + f_i + w_1^i(f_i, \omega_k)(1-\alpha_i)\right) + \frac{1}{N} u(0)$$
$$+ \frac{\alpha_i g}{N} w_1^i(f_i, \omega_k) + \frac{e}{N}(W_1(f_i, \omega_k) - zA)$$

for all $f_i \in \mathbb{R}$. This function is differentiable with first derivative as follows.

$$\begin{aligned}\frac{\partial \psi}{\partial f_i} &= \frac{\partial \psi_i}{\partial f_i} \\ &= \frac{c}{N} - \left(\Phi_2\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right] + 1\right)\left(\frac{c}{N}\left(1-\alpha_i\right) + \frac{\alpha_i g}{N} + \frac{e}{N}\right)\end{aligned}$$

There is no generalization about whether $\psi\left(f_i, \omega_k\right)$ is increasing or decreasing, but $\frac{\partial \psi}{\partial f_i}$ does describe incentives for choosing f_i when it takes values where bank i is bankrupt and the system is undercapitalized. The explanation of $\frac{\partial \psi}{\partial f_i}$ is equivalent to that of (iii) in (I).

Similar to previous situations, one can consider $\frac{\partial^2 \psi}{\partial \left(f_i\right)^2}$ and conclude that $\psi_i\left(f_i, \omega_k\right)$ is concave in $(-\infty, \infty)$.

Recall that so far, $\psi\left(f_i, \omega_k\right)$ is assumed to be defined for $f_i \in \mathbb{R}$. To understand the relationship between $\psi\left(f_i, \omega_k\right)$ and $f_i \in D\left(f_i\right)$, one has to take the following steps.

(a) For the fixed scenario $\omega_k \in \Omega$, determine which of (I) to (V) applies. Depending on the behaviour of $w_1^i\left(f_i, \omega_k\right)$ and $W_1\left(f_i, \omega_k\right) - zA$, only one of the five situations is relevant. Then, one will know whether $\tilde{f}_i^k$ and $\tilde{\tilde{f}}_i^k$ exist, and their corresponding values.

(b) Determine $\bar{\bar{f}}_i$, which defines $D\left(f_i\right) = \left[0, \bar{\bar{f}}_i\right]$. See Section 3.1.1.

(c) (a) and (b) implies that the values of 0, $\bar{\bar{f}}_i$, $\tilde{f}_i^k$ and $\tilde{\tilde{f}}_i^k$, relative to each other, are now known.

(d) One can now describe $\psi\left(f_i, \omega_k\right)$ for $f_i \in D\left(f_i\right)$, using information for $\frac{\partial \psi}{\partial f_i}$ and $\frac{\partial^2 \psi}{\partial \left(f_i\right)^2}$ for whichever case ((I) to (V)) was determined in (a).

In the restricted domain $f_i \in D(f_i)$, $\psi(f_i, \omega_k)$ is continuous but will exhibit at most two indifferentiable (corner) points. For values of f_i greater than that of an indifferentiable point, either bank i is bankrupt ($w_1^i(f_i, \omega_k) < 0$) or the system is in crisis ($W_1(f_i, \omega_k) < zA$) in scenario $\omega_k \in \Omega$. Although there is no generalization about its increasing/decreasing behaviour for $f_i \in D(f_i)$, $\psi(f_i, \omega_k)$ is concave within intervals separated by indifferentiable points.

As a convex combination of $\psi(f_i, \omega)$ over all $\omega \in \Omega$, $\Psi = E[\psi]$ $= \sum_{k=1}^{K} p_k \psi(f_i, \omega_k)$ is continuous with indifferentiable points at $0 < \breve{f}_i^1 < ... < \breve{f}_i^L < \bar{\bar{f}}_i \in D(f_i)$, $0 \leq L \leq 2K$ and is concave within $\left(0, \breve{f}_i^1\right)$,..., $\left(\breve{f}_i^L, \bar{\bar{f}}_i\right)$. The group of banks will consider the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders (ψ) in expected value, when f_i is being selected.

3.2.2 Effect of $x_l^i \in D\left(x_l^i\right)$ on Ψ

Consider Ψ as a function of x_l^i only, with other variables taking fixed values.

(i) **Continuity**

Ψ is continuous for $x_l^i \in D(x_l^i)$, but is indifferentiable at $\bar{\bar{x}}_l^i < \breve{x}_l^i(1) < ... < \breve{x}_l^i(M) \in D(x_l^i)$, $0 \leq M \leq 3K$.

(ii) **Increase/Decrease**

There is no generalization about the increasing/decreasing behaviour of Ψ with respect to $x_l^i \in D(x_l^i)$.

(iii) **Convexity**

For x_l^i within $(\bar{\bar{x}}_l^i, \breve{x}_l^i(1))$, $(\breve{x}_l^i(1), \breve{x}_l^i(2))$,...,$(\breve{x}_l^i(M), \infty)$, Ψ is concave.

<u>**Discussion**</u>

Consider the following as a function ψ of x_l^i, $i \in \{1, ..., N\}$, $l \in \{1, ..., J\}$ only, with $\omega_k \in \Omega$ and other variables fixed. Notice that $\Psi = E[\psi]$.

$$\psi\left(x_l^i, \omega_k\right) = \sum_{m=1}^{N} \frac{c}{N} \times \left(\bar{w}_0^m - \sum_{j=1}^{J} x_j^m + \alpha_m f_m \right.$$

$$\left. + \left(1 - \alpha_m\right) \min\left(f_m, \sum_{j=1}^{J} x_j^m r_j^m\left(\omega_k\right) - \Phi\left[\sum_{j=1}^{J} x_j^m r_j^m\left(\omega_k\right), f_m \right] \right) \right)$$

$$+ \sum_{m=1}^{N} \frac{1}{N} u\left(w_1^m 1_{\left[w_1^m > 0\right]} \right) + \sum_{m=1}^{N} \frac{\alpha_m g}{N} w_1^m 1_{\left[w_1^m < 0\right]}$$

$$+ \frac{e}{N}\left(W_1 - zA\right) 1_{\left[W_1 < zA\right]}$$

$$= \psi_i\left(x_l^i, \omega_k\right)$$

$$+ \sum_{m \neq i} \frac{c}{N} \times \left(\bar{w}_0^m - \sum_{j=1}^{J} x_j^m + f_m + \left(1 - \alpha_m\right) w_1^m\left(\omega_k\right) 1_{\left[w_1^m < 0\right]} \right)$$

$$+ \sum_{m \neq i} \frac{1}{N} u\left(w_1^m\left(\omega_k\right) 1_{\left[w_1^m > 0\right]} \right) + \sum_{m \neq i} \frac{\alpha_m g}{N} w_1^m\left(\omega_k\right) 1_{\left[w_1^m < 0\right]}$$

where

$$\psi_i\left(x_l^i, \omega_k\right) = \frac{c}{N} \times \left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i + \left(1 - \alpha_i\right) w_1^i\left(x_l^i, \omega_k\right) 1_{\left[w_1^i < 0\right]} \right)$$

$$+ \frac{1}{N} u\left(w_1^i\left(x_l^i, \omega_k\right) 1_{\left[w_1^i > 0\right]} \right) + \frac{\alpha_i g}{N} w_1^i\left(x_l^i, \omega_k\right) 1_{\left[w_1^i < 0\right]}$$

$$+ \frac{e}{N}\left(W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right)\right) 1_{\left[W_1 < zA\right]}.$$

As in Section 3.2.1, $\psi\left(x_l^i,\omega_k\right)$ is the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders when taxes are accounted for, with $\omega_k \in \Omega$ fixed (i.e. considering that $\omega_k \in \Omega$ occurs with certainty).

For the moment, assume that $x_l^i \in \mathbb{R}$. Then, the relationship between $\psi\left(x_l^i,\omega_k\right)$ (or equivalently $\psi_i\left(x_l^i,\omega_k\right)$) and x_l^i depends on the behaviour of $w_1^i\left(x_l^i,\omega_k\right)$ and $W_1\left(x_l^i,\omega_k\right) - zA\left(x_l^i\right)$, where $W_1\left(x_l^i,\omega_k\right) = w_1^i\left(x_l^i,\omega_k\right)$ $+\sum_{m\neq i} w_1^m\left(\omega_k\right)$ and $A\left(x_l^i\right) = x_l^i + \sum_{j\neq l} x_j^i + \sum_{m\neq i}\sum_{j=1}^J x_j^m$.

Notice that $w_1^i\left(x_l^i,\omega_k\right)$ is increasing and concave because $\frac{\partial w_1^i}{\partial x_l^i} = r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] > 0$ and $\frac{\partial^2 w_1^i}{\partial\left(x_l^i\right)^2} = -\left(r_l^i\left(\omega_k\right)\right)^2\Phi_{11}\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] < 0$. Hence, there is at most one $\tilde{x}_l^i\left(k\right)$ for which $w_1^i\left(\tilde{x}_l^i\left(k\right),\omega_k\right) = 0$.

However, $\frac{\partial\left[W_1 - zA\right]}{\partial x_l^i} = r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] - z$ and $\frac{\partial^2\left[W_1 - zA\right]}{\partial\left(x_l^i\right)^2} = -\left(r_l^i\left(\omega_k\right)\right)^2\Phi_{11}\left[\sum_{j=1}^J x_j^i r_j^i\left(\omega_k\right), f_i\right] < 0$. Therefore, $W_1\left(x_l^i,\omega_k\right) - zA\left(x_l^i\right)$ is concave, but its increasing/decreasing behaviour depends on whether $r_l^i\left(\omega_k\right) \geq z$ or $r_l^i\left(\omega_k\right) < z$.

(a) If $r_l^i\left(\omega_k\right) \geq z$, then $\frac{\partial\left[W_1 - zA\right]}{\partial x_l^i} > 0$.

$\Rightarrow$ $W_1\left(x_l^i,\omega_k\right) - zA\left(x_l^i\right)$ is increasing.

(b) If $r_l^i\left(\omega_k\right) < z$, then consider the following condition.

$$\exists \ddot{x}_l^i(k) : \frac{\partial [W_1 - zA]}{\partial x_l^i}\left(\ddot{x}_l^i(k)\right) = \left(r_l^i(\omega_k) - z\right)$$

$$-r_l^i(\omega_k) \times \Phi_1\left[\ddot{x}_l^i(k)\, r_l^i(\omega_k) + \sum_{j \neq l} x_j^i r_j^i(\omega_k), f_i\right]$$

$$= 0 \tag{3.1}$$

(b.1) Assume that (3.1) holds. Then, since $\Phi_{11} > 0$,

- for $x_l^i < \ddot{x}_l^i(k)$, $\frac{\partial [W_1 - zA]}{\partial x_l^i}(x_l^i) > 0$, and $W_1(x_l^i, \omega_k) - zA(x_l^i)$ is increasing.
- for $x_l^i > \ddot{x}_l^i(k)$, $\frac{\partial [W_1 - zA]}{\partial x_l^i}(x_l^i) < 0$, and $W_1(x_l^i, \omega_k) - zA(x_l^i)$ is decreasing.

Hence, if (3.1) holds, $W_1(x_l^i, \omega_k) - zA(x_l^i)$ reaches a maximum at $x_l^i = \ddot{x}_l^i(k)$. From the above two results, this $\ddot{x}_l^i(k)$ is unique.

(b.2) Assume that (3.1) does not hold. Then,

- if $\frac{\partial [W_1 - zA]}{\partial x_l^i}(x_l^i) > 0$ for all x_l^i, $W_1(x_l^i, \omega_k) - zA(x_l^i)$ is increasing.
- if $\frac{\partial [W_1 - zA]}{\partial x_l^i}(x_l^i) < 0$ for all x_l^i, $W_1(x_l^i, \omega_k) - zA(x_l^i)$ is decreasing.

Hence, $W_1(x_l^i, \omega_k) - zA(x_l^i)$ is either always increasing, always decreasing, or increasing then decreasing for $x_l^i \in \mathbb{R}$ and there could be at most two points $\tilde{\tilde{x}}_l^i(k, 1)$, $\tilde{\tilde{x}}_l^i(k, 2)$ for which $W_1\left(\tilde{\tilde{x}}_l^i(k, 1), \omega_k\right) - zA\left(\tilde{\tilde{x}}_l^i(k, 1)\right) = W_1\left(\tilde{\tilde{x}}_l^i(k, 2), \omega_k\right) - zA\left(\tilde{\tilde{x}}_l^i(k, 2)\right) = 0$.

Let's partition $\mathbb{R} = X_1(x_l^i, \omega_k) \cup X_2(x_l^i, \omega_k) \cup X_3(x_l^i, \omega_k) \cup X_4(x_l^i, \omega_k)$ as follows.

$$X_1\left(x_l^i, \omega_k\right) = \left\{x_l^i \in \mathbb{R} : w_1^i\left(x_l^i, \omega_k\right) \geq 0 \text{ and } W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right) \geq 0\right\}$$

$$X_2\left(x_l^i, \omega_k\right) = \left\{x_l^i \in \mathbb{R} : w_1^i\left(x_l^i, \omega_k\right) \geq 0 \text{ and } W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right) < 0\right\}$$

$$X_3\left(x_l^i, \omega_k\right) = \left\{x_l^i \in \mathbb{R} : w_1^i\left(x_l^i, \omega_k\right) < 0 \text{ and } W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right) \geq 0\right\}$$

$$X_4\left(x_l^i, \omega_k\right) = \left\{x_l^i \in \mathbb{R} : w_1^i\left(x_l^i, \omega_k\right) < 0 \text{ and } W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right) < 0\right\}$$

Then, consider the form of $\psi(x_l^i, \omega_k)$ (or equivalently, $\psi_i(x_l^i, \omega_k)$) for x_l^i in each of $X_1(x_l^i, \omega_k)$ to $X_4(x_l^i, \omega_k)$.

(I) Suppose $x_l^i \in X_1(x_l^i, \omega_k)$.

$$\psi_i\left(x_l^i, \omega_k\right) = \frac{c}{N} \times \left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i\right) + \frac{1}{N} u\left(w_1^i\left(x_l^i, \omega_k\right)\right)$$

Then, the first derivative of $\psi(x_l^i, \omega_k)$ is

$$\begin{aligned}
\frac{\partial \psi}{\partial x_l^i} &= \frac{\partial \psi_i}{\partial x_l^i} \\
&= -\frac{c}{N} \\
&\quad + \frac{1}{N} u'\left(w_1^i\left(x_l^i, \omega_k\right)\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right]\right)
\end{aligned}$$

For $x_l^i \in X_1(x_l^i, \omega_k)$, bank i is not bankrupt ($w_1^i \geq 0$) and there is no systemic crisis ($W_1 \geq zA$) for scenario $\omega_k \in \Omega$.

In particular, since $w_1^i \geq 0$, debtholders will receive f_i dollars at time $t = 1$, so an increase in x_l^i will not affect their payoff at that time. The funding they provide at time $t = 0$ is therefore unchanged. The increase of x_l^i (bank i invests more money in asset l) at time $t = 0$ will be funded fully by equityholders, which implies that their consumption dollars $\left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i\right)$ and hence utility will decrease at that time. Then, the time $t = 0$ utility for an average bank will also decline. This is captured by $-\frac{c}{N} < 0$ in $\frac{\partial \psi}{\partial x_l^i}$.

An increase in x_l^i, however, would increase bank i's time $t = 1$ equity (w_1^i) and thus the utility of its holders. This means the utility of an average bank's equityholders will also increase at time $t = 1$, which is reflected by $\frac{1}{N} u' \left(w_1^i\right) \left(r_l^i \left(\omega_k\right) - r_l^i \left(\omega_k\right) \Phi_1\right) > 0$ in $\frac{\partial \psi}{\partial x_l^i}$.

The second derivative of $\psi \left(x_l^i, \omega_k\right)$ is

$$\frac{\partial^2 \psi}{\partial \left(x_l^i\right)^2} = -\frac{1}{N} u' \left(w_1^i \left(x_l^i, \omega_k\right)\right) \left(r_l^i \left(\omega_k\right)\right)^2 \Phi_{11} \left[\sum_{j=1}^{J} x_j^i r_j^i \left(\omega_k\right), f_i\right]$$

$$+\frac{1}{N} u'' \left(w_1^i \left(x_l^i, \omega_k\right)\right) \times$$
$$\left(r_l^i \left(\omega_k\right) - r_l^i \left(\omega_k\right) \Phi_1 \left[\sum_{j=1}^{J} x_j^i r_j^i \left(\omega_k\right), f_i\right]\right)^2 < 0.$$

Therefore, $\psi \left(x_l^i, \omega_k\right)$ is concave for $x_l^i \in X_1 \left(x_l^i, \omega_k\right)$.

(II) Suppose $x_l^i \in X_2 \left(x_l^i, \omega_k\right)$.

$$\psi_i \left(x_l^i, \omega_k\right) = \frac{c}{N} \times \left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i\right) + \frac{1}{N} u \left(w_1^i \left(x_l^i, \omega_k\right)\right)$$
$$+ \frac{e}{N} \left(W_1 \left(x_l^i, \omega_k\right) - zA \left(x_l^i\right)\right)$$

Then, the first derivative of $\psi\left(x_l^i, \omega_k\right)$ is

$$\frac{\partial \psi}{\partial x_l^i} = \frac{\partial \psi_i}{\partial x_l^i} = -\frac{c}{N}$$

$$+\frac{1}{N} u^{'}\left(w_1^i\left(x_l^i, \omega_k\right)\right) \times$$
$$\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right]\right)$$

$$+\frac{e}{N}\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right] - z\right).$$

The explanations for the terms $-\frac{c}{N}$ and $\frac{1}{N} u^{'}\left(w_1^i\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1\right)$ are equivalent to those for $x_l^i \in X_1\left(x_l^i, \omega_k\right)$ (in (I)), since $w_1^i \geq 0$.

However, when $x_l^i \in X_2\left(x_l^i, \omega_k\right)$, the system is in crisis $\left(W_1 < zA\right)$ in scenario $\omega_k \in \Omega$, and an increase in x_l^i will affect taxes paid by all banks, because $W_1 - zA$ depends on x_l^i.

In particular, when $\omega_k \in \Omega$ is considered in isolation, the dollars of taxes charged to bank m is $-\frac{\alpha_m g}{c} w_1^m\left(\omega_k\right) 1_{\left[w_1^m < 0\right]} - \frac{e}{c}\left(w_1^m\left(\omega_k\right) - z a_m\right) 1_{\left[W_1 < zA\right]}$. The tax for an average bank is then $-\frac{1}{N} \sum_{m=1}^{N} \frac{\alpha_m g}{c} w_1^m\left(\omega_k\right) 1_{\left[w_1^m < 0\right]} - \frac{e}{cN}\left(W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right)\right)$. Since $w_1^i \geq 0$ for $x_l^i \in X_2\left(x_l^i, \omega_k\right)$, the derivative of this amount with respect to x_l^i is $-\frac{e}{cN}\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1 - z\right)$, which may be positive or negative for different values of x_l^i. When a bank is charged τ dollars of taxes, its equityholders obtains a utility of $-c \times \tau$. For an average bank, the utility experienced due to payment of taxes may increase or decrease for different values of x_l^i, because its derivative with respect to x_l^i is $\frac{e}{N}\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right) \Phi_1 - z\right)$, which may be positive or negative. This last expression is included in $\frac{\partial\psi}{\partial x_l^i}$.

The second derivative of $\psi\left(x_l^i, \omega_k\right)$ is

$$\frac{\partial^2\psi}{\partial\left(x_l^i\right)^2} = -\frac{1}{N}u'\left(w_1^i\left(x_l^i, \omega_k\right)\right)\left(r_l^i\left(\omega_k\right)\right)^2\Phi_{11}\left[\sum_{j=1}^{J}x_j^i r_j^i\left(\omega_k\right), f_i\right]$$

$$+\frac{1}{N}u''\left(w_1^i\left(x_l^i, \omega_k\right)\right)\times$$
$$\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\left[\sum_{j=1}^{J}x_j^i r_j^i\left(\omega_k\right), f_i\right]\right)^2$$

$$-\frac{e}{N}\left(r_l^i\left(\omega_k\right)\right)^2\Phi_{11}\left[\sum_{j=1}^{J}x_j^i r_j^i\left(\omega_k\right), f_i\right] < 0.$$

Therefore, $\psi\left(x_l^i, \omega_k\right)$ is concave for $x_l^i \in X_2\left(x_l^i, \omega_k\right)$.

(III) Suppose $x_l^i \in X_3\left(x_l^i, \omega_k\right)$.

$$\psi_i\left(x_l^i, \omega_k\right) = \frac{c}{N}\times\left(\bar{w}_0^i - x_l^i - \sum_{j\neq l}x_j^i + f_i + (1-\alpha_i)\,w_1^i\left(x_l^i, \omega_k\right)\right)$$
$$+\frac{1}{N}u(0) + \frac{\alpha_i g}{N}w_1^i\left(x_l^i, \omega_k\right)$$

Then, the first derivative of $\psi\left(x_l^i, \omega_k\right)$ is

$$\frac{\partial\psi}{\partial x_l^i} = \frac{\partial\psi_i}{\partial x_l^i} = -\frac{c}{N}$$

$$+\frac{c}{N}(1-\alpha_i)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\left[\sum_{j=1}^{J}x_j^i r_j^i\left(\omega_k\right), f_i\right]\right)$$

$$+\frac{\alpha_i g}{N}\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\left[\sum_{j=1}^{J}x_j^i r_j^i\left(\omega_k\right), f_i\right]\right).$$

If $x_l^i \in X_3\left(x_l^i, \omega_k\right)$, at time $t=1$, bank i is bankrupt $\left(w_1^i < 0\right)$, but there will not be a systemic crisis $\left(W_1 \geq zA\right)$ for scenario $\omega_k \in \Omega$.

Then, bank i's debtholders will have a payoff of $w_1^i + f_i - \alpha_i w_1^i$ at time $t=1$. Since $\omega_k \in \Omega$ is considered fixed, this is also the dollars of debt funding provided to the bank at time $t=0$. Moreover, bank i's tax is $-\frac{\alpha_i g}{c} w_1^i\left(x_l^i, \omega_k\right)$, so its time $t=0$ utility will be $c \times \left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i + \left(1-\alpha_i\right) w_1^i\left(x_l^i, \omega_k\right) + \frac{\alpha_i g}{c} w_1^i\left(x_l^i, \omega_k\right)\right)$ with first derivative $-c + c\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) + \alpha_i g\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right)$. The derivative of an average bank's time $t=0$ utility would be this expression divided by N, which is equal to $\frac{\partial \psi}{\partial x_l^i}$.

Notice that an increase in x_l^i will increase bank i's debt funding, since its derivative is positive: $\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) > 0$. Also, the dollars of taxes payable by bank i decreases, since its derivative with respect to x_l^i is negative: $-\frac{\alpha_i g}{c}\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) < 0$. In other words, from bank i equityholders' perspective, these two effects are cash inflows at time $t=0$ and will increase their utility at that time, since $c\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) > 0$ and $\alpha_i g\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) > 0$. However, these two cash increases may or may not be sufficient to cover the increased investment in asset l. Therefore, equityholders may need to provide, or may receive additional funds at time $t=0$, so their utility could increase or decrease at that time. That is, $-c + c\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right) + \alpha_i g\left(r_l^i\left(\omega_k\right) - r_l^i\left(\omega_k\right)\Phi_1\right)$ may be positive or negative, for different values of x_l^i.

Note that the time $t=1$ utility for bank i's equityholders, and thus for those of the average bank, is not affected because $u\left(w_1^i 1_{\left[w_1^i > 0\right]}\right) = u(0)$

when $x_l^i \in X_3(x_l^i, \omega_k)$.

The second derivative of $\psi(x_l^i, \omega_k)$ is

$$\frac{\partial^2 \psi}{\partial \left(x_l^i\right)^2} = -\frac{c}{N}(1-\alpha_i)\left(r_l^i(\omega_k)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right]$$

$$-\frac{\alpha_i g}{N}\left(r_l^i(\omega_k)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] < 0,$$

so it is concave for $x_l^i \in X_3(x_l^i, \omega_k)$.

(IV) Suppose $x_l^i \in X_4(x_l^i, \omega_k)$.

$$\psi_i\left(x_l^i, \omega_k\right) = \frac{c}{N} \times \left(\bar{w}_0^i - x_l^i - \sum_{j \neq l} x_j^i + f_i + (1-\alpha_i) w_1^i\left(x_l^i, \omega_k\right)\right)$$

$$+\frac{1}{N}u(0) + \frac{\alpha_i g}{N} w_1^i\left(x_l^i, \omega_k\right) + \frac{e}{N}\left(W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right)\right)$$

Then, the first derivative of $\psi(x_l^i, \omega_k)$ is

$$\frac{\partial \psi}{\partial x_l^i} = \frac{\partial \psi_i}{\partial x_l^i} = -\frac{c}{N}$$

$$+\frac{c}{N}(1-\alpha_i)\left(r_l^i(\omega_k) - r_l^i(\omega_k)\Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right]\right)$$

$$+\frac{\alpha_i g}{N}\left(r_l^i(\omega_k) - r_l^i(\omega_k)\Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right]\right)$$

$$+\frac{e}{N}\left(r_l^i(\omega_k) - r_l^i(\omega_k)\Phi_1\left[\sum_{j=1}^{J} x_j^i r_j^i(\omega_k), f_i\right] - z\right).$$

When $x_l^i \in X_4\left(x_l^i, \omega_k\right)$ bank i is bankrupt $\left(w_1^i < 0\right)$ and the system is in crisis $\left(W_1 < zA\right)$ at time $t = 1$ for scenario $\omega_k \in \Omega$. Therefore, the explanation for $-\frac{c}{N}+\frac{c}{N}\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right)-r_l^i\left(\omega_k\right)\Phi_1\right)+\frac{\alpha_i g}{N}\left(r_l^i\left(\omega_k\right)-r_l^i\left(\omega_k\right)\Phi_1\right)$ is the same as in (III) when $x_l^i \in X_3\left(x_l^i, \omega_k\right)$, and $\frac{e}{N}\left(r_l^i\left(\omega_k\right)-r_l^i\left(\omega_k\right)\Phi_1 - z\right)$ is described in (II), when $x_l^i \in X_2\left(x_l^i, \omega_k\right)$.

The second derivative of $\psi\left(x_l^i, \omega_k\right)$ is

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial\left(x_l^i\right)^2} = &-\frac{c}{N}\left(1-\alpha_i\right)\left(r_l^i\left(\omega_k\right)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right] \\
&-\frac{\alpha_i g}{N}\left(r_l^i\left(\omega_k\right)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right] \\
&-\frac{e}{N}\left(r_l^i\left(\omega_k\right)\right)^2 \Phi_{11}\left[\sum_{j=1}^{J} x_j^i r_j^i\left(\omega_k\right), f_i\right] < 0,
\end{aligned}$$

so it is concave for $x_l^i \in X_4\left(x_l^i, \omega_k\right)$.

So far, $\psi\left(x_l^i, \omega_k\right)$ is assumed to be defined for $x_l^i \in \mathbb{R}$. To understand the relationship between $\psi\left(x_l^i, \omega_k\right)$ and $x_l^i \in D\left(x_l^i\right)$, one has to take the following steps.

(a) For the fixed $\omega_k \in \Omega$, one can analyze $w_1^i\left(x_l^i, \omega_k\right)$ and $W_1\left(x_l^i, \omega_k\right) - zA\left(x_l^i\right)$ and determine the values of $\tilde{x}_l^i(k)$, $\tilde{\tilde{x}}_l^i(k,1)$, and $\tilde{\tilde{x}}_l^i(k,2)$, if they exist.

Then, the partitions of $\mathbb{R}$ ($X_1\left(x_l^i, \omega_k\right)$ to $X_4\left(x_l^i, \omega_k\right)$) are determined, and are separated by the points $\tilde{x}_l^i(k)$, $\tilde{\tilde{x}}_l^i(k,1)$, and $\tilde{\tilde{x}}_l^i(k,2)$.

From its definition, $\psi\left(x_l^i, \omega_k\right)$ is continuous, but in general, not differentiable at these points. For all other values of x_l^i, it is differentiable.

(b) Determine $\bar{\bar{x}}_l^i$, which defines $D\left(x_l^i\right) = [\bar{\bar{x}}_l^i, \infty)$. See Section 3.1.2.

(c) (a) and (b) implies that the values of $\bar{\bar{x}}_l^i$, $\tilde{x}_l^i(k)$, $\tilde{\tilde{x}}_l^i(k,1)$, and $\tilde{\tilde{x}}_l^i(k,2)$, relative to each other, are now known.

(d) One can now describe $\psi\left(x_l^i,\omega_k\right)$ for $x_l^i \in D\left(x_l^i\right) = [\bar{\bar{x}}_l^i,\infty) \cap (X_1\left(x_l^i,\omega_k\right) \cup ...$ $\cup X_4\left(x_l^i,\omega_k\right))$ using information for $\frac{\partial\psi}{\partial x_l^i}$ and $\frac{\partial^2\psi}{\partial\left(x_l^i\right)^2}$ described for each of $X_1\left(x_l^i,\omega_k\right)$ to $X_4\left(x_l^i,\omega_k\right)$, as determined in (a).

It follows that in the restricted domain $D\left(x_l^i\right)$, $\psi\left(x_l^i,\omega_k\right)$ is continuous, and can have at most three indifferentiable points. The meaning of an indifferentiable point depend on whether it is $\tilde{x}_l^i(k)$, $\tilde{\tilde{x}}_l^i(k,1)$, or $\tilde{\tilde{x}}_l^i(k,2)$. Moreover, the meaning of $\tilde{\tilde{x}}_l^i(k,1)$, and/or $\tilde{\tilde{x}}_l^i(k,2)$ depends on whether $W_1\left(x_l^i,\omega_k\right) - zA\left(x_l^i\right)$ is increasing or decreasing.

- To illustrate steps (a) to (d), let's consider an example.

  Let $\omega_k \in \Omega$ and values for all variables except x_l^i be fixed. Suppose that as a function of x_l^i, $W_1\left(x_l^i,\omega_k\right) - zA\left(x_l^i\right)$ is increasing (e.g. $r_l^i\left(\omega_k\right) \geq z$ would imply $W_1 - zA$ is increasing). As discussed previously, we also know that $w_1^i\left(x_l^i,\omega_k\right)$ is increasing, and that both w_1^i and $W_1 - zA$ are concave.

  Furthermore, suppose that $\tilde{x}_l^i(k)$ and $\tilde{\tilde{x}}_l^i(k,1)$ satisfying $w_1^i\left(\tilde{x}_l^i(k),\omega_k\right) = 0$ and $W_1\left(\tilde{\tilde{x}}_l^i(k,1),\omega_k\right) - zA\left(\tilde{\tilde{x}}_l^i(k,1)\right) = 0$ exists, and $0 < \tilde{x}_l^i(k) < \tilde{\tilde{x}}_l^i(k,1)$. Then, $X_4\left(x_l^i,\omega_k\right) = \left(-\infty,\tilde{x}_l^i(k)\right)$, $X_2\left(x_l^i,\omega_k\right) = \left[\tilde{x}_l^i(k),\tilde{\tilde{x}}_l^i(k,1)\right)$, $X_1\left(x_l^i,\omega_k\right) = \left[\tilde{\tilde{x}}_l^i(k,1),\infty\right)$, and $X_3\left(x_l^i,\omega_k\right) = \emptyset$.

  Assume that the lower bound $\bar{\bar{x}}_l^i$ of $D\left(x_l^i\right) = [\bar{\bar{x}}_l^i,\infty)$ is positive but less than $\tilde{x}_l^i(k)$.

Then, for $x_l^i \in D\left(x_l^i\right)$,

$$\psi_i\left(x_l^i,\omega_k\right)=\begin{cases}\frac{c}{N}\times\left(\bar{w}_0^i-x_l^i-\sum_{j\neq l}x_j^i+f_i\right.\\ \left.+\left(1-\alpha_i\right)w_1^i\left(x_l^i,\omega_k\right)\right)+\frac{1}{N}u\left(0\right)+\frac{\alpha_i g}{N}w_1^i\left(x_l^i,\omega_k\right)\\ +\frac{e}{N}\left(W_1\left(x_l^i,\omega_k\right)-zA\left(x_l^i\right)\right);\\ \text{ if }\bar{\bar{x}}_l^i\leq x_l^i<\tilde{x}_l^i\left(k\right)\\ \\ \frac{c}{N}\times\left(\bar{w}_0^i-x_l^i-\sum_{j\neq l}x_j^i+f_i\right)\\ +\frac{1}{N}u\left(w_1^i\left(x_l^i,\omega_k\right)\right)+\frac{e}{N}\left(W_1\left(x_l^i,\omega_k\right)-zA\left(x_l^i\right)\right);\\ \text{ if }\tilde{x}_l^i\left(k\right)\leq x_l^i<\tilde{\tilde{x}}_l^i\left(k,1\right)\\ \\ \frac{c}{N}\times\left(\bar{w}_0^i-x_l^i-\sum_{j\neq l}x_j^i+f_i\right)\\ +\frac{1}{N}u\left(w_1^i\left(x_l^i,\omega_k\right)\right);\\ \text{ if }\tilde{\tilde{x}}_l^i\left(k,1\right)\leq x_l^i<\infty\end{cases}.$$

So, $\psi_i\left(x_l^i,\omega_k\right)$ (equivalently, $\psi\left(x_l^i,\omega_k\right)$) is continuous, but indfferentiable at $\tilde{x}_l^i\left(k\right)$ and $\tilde{\tilde{x}}_l^i\left(k,1\right)$. For other $x_l^i \in D\left(x_l^i\right)$, it is differentiable. Within $\left(\bar{\bar{x}}_l^i,\tilde{x}_l^i\left(k\right)\right)$, $\left(\tilde{x}_l^i\left(k\right),\tilde{\tilde{x}}_l^i\left(k,1\right)\right)$ and $\left(\tilde{\tilde{x}}_l^i\left(k,1\right),\infty\right)$, the function is concave.

Finally, as a convex combination of $\psi\left(x_l^i,\omega\right)$ over all $\omega\in\Omega$, $\Psi=E\left[\psi\right]=\sum_{k=1}^K p_k\psi\left(x_l^i,\omega_k\right)$ is continuous with indifferentiable points at $\bar{\bar{x}}_l^i<\breve{x}_l^i\left(1\right)<\ldots<\breve{x}_l^i\left(M\right)\in D\left(x_l^i\right)$, $0\leq M\leq 3K$. Moreover, Ψ is concave within each of $\left(\bar{\bar{x}}_l^i,\breve{x}_l^i\left(1\right)\right)$,...,$\left(\breve{x}_l^i\left(M\right),\infty\right)$. When the group of banks choose x_l^i, it has to consider the time $t=0$ plus $t=1$ utilities of an average bank's equityholders (ψ), in expected value.

3.3 Remarks

When a scenario $\omega_k \in \Omega$ is fixed, bank i equityholders' dollar value of consumption at time $t = 0$ and net worth at time $t = 1$ (gross value of assets - distress cost - debt) will relate differently with respect to a change in a decision variable's (x_l^i or f_i) value, depending on whether its bankruptcy and/or a systemic crisis will occur at time $t = 1$. Therefore, the same is said about their utility at those two points in time.

It follows that the relationship between the utility ($t = 0$ plus $t = 1$) of an average bank and a bank i decision will have this dependence as well. In expectation, the objective function for the group of banks, $\Psi = E[\psi]$, will consider this dependence over all scenarios $\omega \in \Omega$. Although there is no generalization about whether the objective will increase or decrease, an analysis with a numerical example is possible, as presented in Appendix C.

Chapter 4

Conclusion

The recent financial crisis of 2007-2009 has again sparked the interest of the academia and industry practitioners to study regulation of systemic risk. As with any regulation, the entities being targeted (i.e. banks) should be given the incentives to operate in a way that is desired by society.

Besides the traditional tool of setting capital requirements, there are also suggestions to introduce taxation for banks so that they will consider their impact on the banking system and the resulting effects on social welfare when making investment and funding decisions. In particular, Acharya, Pedersen, Philippon, and Richardson (2010) creates a setting where each bank maximizes its shareholders' utility while the government maximizes the sum of those over all banks net of expected costs to social utility arising from potential bank insolvencies and systemic undercapitalization (crisis). The authors suggest that each bank should pay a tax equal to its contribution to these two expected costs. As a result of this tax, shareholders' utility of each bank will be interdependent with those of other banks. The mathematical formulation provided by this thesis assume that it is then reasonable for them to come together as a group and decide on each bank's investment and funding amounts to achieve Pareto optimality. That is, any deviation from the group's decision which increase one bank's utility can only occur at the sacrifice (decrease) of that for another bank, and thus will not be allowed. A further assumption that each bank's utility is equally important to the group implies that it will maximize the utility of the average bank. The resulting decisions of banks are Pareto and socially optimal, meeting the government's objective.

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Appendix A

MATLAB Code `tax.m` for Tax Calculation

`[t0,t1,ses,prob,w1,uc,sysuc] = tax(x,f,alpha,p,taubar,z,c,e,g,r1,...,rN)` will take various quantities from Chapter 2 as inputs and generate τ_0, $\{\hat{\tau}_i\}_i$, $\{SES_i\}_i$, and the probability of systemic crisis at time $t = 1$ as outputs. Moreover, some details regarding the dollar values of time $t = 1$ equity ($\{w_1^i\}_i$) and level of capitalization ($\{w_1^i - za_i\}_i$) will be provided.

The code is a MATLAB function and can handle any finite number of assets ($J \in \mathbb{N}$), scenarios ($K \in \mathbb{N}$), and banks ($N \in \mathbb{N}$).

A.1 Inputs of `tax.m`

- **Investment Decisions at Time** $t = 0$

  `x` $= \left(\vec{x}^1, ..., \vec{x}^N\right)$ is the $(J \times N)$ matrix containing the banks' dollar investment amounts at time $t = 0$. The ji^{th} entry of `x` corresponds to the amount for bank i in asset j. See Section 2.2.1 for a description of

$\vec{x^i}$.

- **Time $t = 1$ Debt Values**

  `f` $= (f_1, ..., f_N)$ is the $(1 \times N)$ vector containing the banks' face value of debt at time $t = 1$. See Section 2.2.2 for a description of f_i.

- **Government Support Proportions**

  `alpha` $= (\alpha_1, ..., \alpha_N)$ is the $(1 \times N)$ vector containing proportions of government support. See Section 2.3.3 for a description of α_i.

- **Probabilities**

  `p` $= (p_1, ..., p_K)$ is the $(1 \times K)$ vector containing probabilities for scenarios $\{\omega_1, ..., \omega_K\}$ at time $t = 1$.

- **Other Parameters**

  Scalar parameters `taubar` ($\bar{\tau}$), `z`, `c`, `e`, and `g`, are as described in Section 2.5.

- **Return Distributions**

  `ri` $= (\vec{r^i}(\omega_1), ..., \vec{r^i}(\omega_K))$ is the $(J \times K)$ matrix of return distributions for bank i. The j^{th} row of `ri` is the distribution of returns of asset j for bank i.

  A total of N such matrices (one for each bank) should be entered. See Section 2.1.1 for a description of $\vec{r^i}(\omega)$.

A.2 Outputs of `tax.m`

- **Tax**

- Based on the inputs to the code, the $(1 \times N)$ vector `t1` $= (\hat{\tau}_1, ..., \hat{\tau}_N)$, where $\hat{\tau}_i$ is defined by (2.11), is provided as an output.
- Using the results of `t1`, the additional dollar amount of tax charged to each bank, calculated as `t0` $= \tau_0 = \frac{\bar{\tau} - \sum_{i=1}^{N} \hat{\tau}_i}{N}$, is provided as an output. This amount charged to each bank ensures that the government will collect exactly $\bar{\tau}$ dollars in tax revenue at time $t = 0$.

- **Systemic Expected Shortfall (Measure of Systemic Risk)**

  The $(1 \times N)$ vector `ses` $= (SES_1, ..., SES_N)$, where SES_i is defined in Section 2.7.2, is calculated as an output.

- **Probability of Systemic Crisis at Time $t = 1$**

  `prob` $= P[W_1 < zA] = \sum_{k=1}^{K} p_k \times 1_{[W_1 < zA]}(\omega_k)$ is the probability of a systemic crisis at time $t = 1$.

- **Scenario Specific Details**

  - `w1` is a $(N \times K)$ matrix, where the ik^{th} entry is $w_1^i(\omega_k)$. See Section 2.3.2 for a description of w_1^i. A negative value in the ik^{th} component indicates that bank i is bankrupt in scenario $\omega_k \in \Omega$.
  - `uc` is a $(N \times K)$ matrix, where the ik^{th} entry is $w_1^i(\omega_k) - za_i$. Recall that $a_i = \sum_{j=1}^{J} x_j^i$.
  - `sysuc` is a $(1 \times K)$ vector, where the k^{th} entry is $W_1(\omega_k) - zA$. Notice that the k^{th} entry is equal to the sum of column k's components in matrix `uc`.

Suppose the k^{th} component of **sysuc** is negative ($W_1(\omega_k) < zA$). That is, $\omega_k \in \Omega$ is a crisis scenario. Then, the entries in column k of **uc** shows each bank's level of time $t = 1$ equity in excess of its undercapitalization threshold in scenario $\omega_k \in \Omega$. If the i^{th} entry of column k is negative ($w_1^i < za_i$), bank i contributes to the systemic crisis occurring in $\omega_k \in \Omega$.

A.3 Code of tax.m

The calculations in **tax.m** make use of indicator functions. For this purpose, the MATLAB built-in function **heaviside.m** $\left(\texttt{heaviside}(\mathsf{a}) = 1_{[\mathsf{a}>0]} + 0.5\times 1_{[\mathsf{a}=0]}\right)$ is used to create four functions, **heaviside1.m** to **heaviside4.m** as follows.

$\texttt{heaviside1}(\mathsf{a}) = 1_{[\mathsf{a}>0]}$

```
% Written by Kevin Wai, May 2014.
function Y = heaviside1(a)
% HEAVISIDE1(a) is 0 for a < 0, 1 for a > 0, and 0 for a == 0.
Y = heaviside(a)-2*heaviside(a).*heaviside(-a);
```

$\texttt{heaviside2}(\mathsf{a}) = 1_{[\mathsf{a}<0]}$

```
% Written by Kevin Wai, May 2014.
function Y = heaviside2(a)
% HEAVISIDE2(a) is 1 for a < 0, 0 for a > 0, and 0 for a == 0.
```

```
Y = heaviside(-a)-2*heaviside(a).*heaviside(-a);
```

$\texttt{heaviside3}(\mathsf{a}) = 1_{[\mathsf{a} \geq 0]}$

```
% Written by Kevin Wai, May 2014.
function Y = heaviside3(a)
% HEAVISIDE3(a) is 0 for a < 0, 1 for a > 0, and 1 for a == 0.
Y = heaviside(a)+2*heaviside(a).*heaviside(-a);
```

$\texttt{heaviside4}(\mathsf{a}) = 1_{[\mathsf{a} \leq 0]}$

```
% Written by Kevin Wai, May 2014.
function Y = heaviside4(a)
% HEAVISIDE4(a) is 1 for a < 0, 0 for a > 0, and 1 for a == 0.
Y = heaviside(-a)+2*heaviside(a).*heaviside(-a);
```

Moreover, one must define the cost of financial distress function (Section 2.3.1) as a separate MATLAB function. For example, $\texttt{phi}(\mathsf{a}, \mathsf{b}) = \Phi(a, b) = \exp(b) \times \exp(-a)$ is specified below.

```
% Written by Kevin Wai, May 2014.
function Y = phi(a,b)
    Y = exp(b).*exp(-a);
```

```
end
```

Then, the code for `tax.m` is the following.

```
% Written by Kevin Wai, May 2014.

function [t0,t1,ses,prob,w1,uc,sysuc] = tax(x,f,alpha,p,...
                                            taubar,z,c,...
                                            e,g,varargin)

u = size(x);
n_assets = u(1);
n_banks = u(2);
n_omega = length(p);

% Initialize matrices to organize input variables.

% The ik^th entry,
% A1(i,k) = \sum_{j=1}^{J} x_{j}^{i}*r_{j}^{i}(\omega_{k}),
% will be the gross dollar value of assets of bank i in
% scenario \omega_{k} at time t = 1.
A1 = ones(n_banks,n_omega);

% The ith row will be F(i,:) = (f(i),f(i),...,f(i)) k
% times, where f(i) is bank i's promised payment to its
% debholders (face value of debt) at time t = 1.
F = ones(n_banks,n_omega);

% The ik^th entry,
% Phi(i,k)
% = phi(\sum_{j=1}^{J} x_{j}^{i}*r_{j}^{i}(\omega_{k}), f(i)),
% will be the cost of financial distress for bank i in
% scenario \omega_{k} at time t = 1.
Phi = ones(n_banks,n_omega);

% The ik^th entry, w1(i,k) = A1(i,k) - Phi(i,k) - f(i),
% will be the dollar value of equity for bank i in scenario
% \omega_{k} at time t = 1.
w1 = ones(n_banks,n_omega);

% The ik^th entry,
% uc(i,k) = w1(i,k) - z*\sum_{j=1}^{J} x_{j}^{i}, will be the
% level of equity for bank i in excess of its
```

```
% undercapitalization threshold in scenario \omega_{k}
% at time t = 1.
uc = ones(n_banks,n_omega);

% The ik^th entry, IndW1(i,k) = Indicator(W1(i,k) < 0), will
% take a value of 1 if the bank i is bankrupt in scenario
% \omega_{k} at time t = 1 (i.e. if w1(i,k) < 0).
% Otherwise, the entry is zero.
IndW1 = ones(n_banks,n_omega);

% Enter values for matrices initialized above, based on
% values from function inputs.

for i = 1:n_banks

    A1(i,:) = x(1,i)*varargin{i}(1,:);

    if (n_assets >= 2)

        for j = 2:n_assets

            A1(i,:) = A1(i,:) + x(j,i)*varargin{i}(j,:);

        end

    end

    F(i,:) = f(i)*ones(1,n_omega);
    Phi(i,:) = phi(A1(i,:),F(i,:));
    w1(i,:) = A1(i,:) - Phi(i,:) - F(i,:);
    uc(i,:) = w1(i,:) - z*sum(x(:,i))*ones(1,n_omega);
    IndW1(i,:) = heaviside2(w1(i,:));

end

% Create indicator variables Feasible1Ind, Feasible2Ind,...
% ,Feasible13Ind to ensure that the function inputs satisfy
% constraints specified in the government's problem (2.10)
% (and equivalently, the banks' scalarized multi-objective
% problem (2.17)).

% Check if sum(x(:,i)) >= f(i), for all banks i = 1,...,N.
Feasible1 = heaviside3(sum(x)-f);
Feasible1Ind = prod(Feasible1);

% Check if banks will invest such that
% (assets - distress cost) is non-negative in all scenarios
% at time t = 1. That is, if A(i,k) - Phi(i,k) >= 0, for all
% i = 1,...,N, and all k = 1,...,K.
```

```
Feasible2 = heaviside3(A1 - Phi);
Feasible2Ind = prod(prod(Feasible2));

% Check if banks will take non-negative positions in asset
% investments to hold from times t = 0 to t = 1. That is,
% if x(j,i) >= 0, for all j = 1,...,J and all i = 1,...,N.
Feasible3 = heaviside3(x);
Feasible3Ind = prod(prod(Feasible3));

% Check if GROSS investment return is strictly positive.
% That is, if varargin{i}(j,k) > 0, for all i = 1,...,N,
% all j = 1,...,J, and all k = 1,...,K.
Feasible4Ind = 1;

for i = 1:n_banks

    Feasible4Ind = Feasible4Ind*...
                   prod(prod(heaviside1(varargin{i})));

end

% Check if face value of debt at time 1 is non-negative.
% That is, if f(i) >= 0, for all i = 1,...,N.
Feasible5Ind = prod(heaviside3(f));

% Check if the proportion of government support is
% between zero and one. That is, if 0 <= alpha(i) <= 1,
% for all i = 1,...,N.
Feasible6 = heaviside3(alpha).*...
            heaviside4(alpha-ones(1,n_banks));
Feasible6Ind = prod(Feasible6);

% Check if undercapitalization threshold is a proportion
% greater than zero and less than or equal to one.
% That is, if 0 < z <= 1.
Feasible7Ind = heaviside1(z)*heaviside4(z-1);

% Check if time t = 0 rate of utility
% c = (utility/dollar consumption) is strictly positive.
Feasible8Ind = heaviside1(c);

% Check if time t = 1 rate of disutility due to
% undercapitalization, e = (utility/dollars undercapitalized),
% is strictly positive.
Feasible9Ind = heaviside1(e);

% Check if time t = 1 rate of disutility due to government
% support in the event that a bank is bankrupt,
% g = (utility/dollars of negative equity), is
```

```
% strictly positive.
Feasible10Ind = heaviside1(g);

% Check if the sum of all probabilities is one.
Totalp = sum(p);
Feasible11Ind = heaviside3(Totalp - 1)*...
                heaviside4(Totalp - 1);

% Check if the probability of each scenario (omega) is
% strictly between zero and one.
Feasible12 = heaviside1(p).*heaviside2(p - ones(1,n_omega));
Feasible12Ind = prod(Feasible12);

% Check if the total dollars of taxes that the government
% wants to collect is strictly positive.
Feasible13Ind = heaviside1(taubar);

FeasibleInd = Feasible1Ind*Feasible2Ind*Feasible3Ind...
              *Feasible4Ind*Feasible5Ind*Feasible6Ind...
              *Feasible7Ind*Feasible8Ind*Feasible9Ind...
              *Feasible10Ind*Feasible11Ind...
              *Feasible12Ind*Feasible13Ind;

% If function inputs are valid
% (satisfy all required constraints), then the tax calculation
% proceeds. Otherwise, an error message will be displayed.

if FeasibleInd == 1

    sysuc = sum(uc);

    IndUC = heaviside2(sysuc + 0.0001);

    t1 = ones(1,n_banks);

    for i = 1:n_banks

        t1(i) = ((-alpha(i)*g/c)*sum(p.*w1(i,:).*IndW1(i,:))...
                + (-e/c)*sum(p.*uc(i,:).*IndUC));

    end

    t0 = ((taubar - sum(t1))/n_banks);

    ses = ones(1,n_banks);

    prob = sum(p.*IndUC);

    for i = 1:n_banks
```

```
        ses(i) = -(sum(p.*uc(i,:).*IndUC))/prob;
    end

 else
     error('Inputs are invalid.')
 end
end
```

Appendix B

MATLAB Code socopt_patternsearch.m for Solving Problems of the Government and the Group of Banks

[optobj,x,f,t0,t1,ses,prob,w1,uc,sysuc]=socopt_patternsearch(alpha,wbar,p,taubar,z,c,e,g,r1,...,rN) will determine the optimal solution $\left(\mathtt{x} = \tilde{x}, \mathtt{f} = \tilde{f}\right)$ to the government's problem (2.10) for the model defined by values of alpha, wbar, p, taubar, z, c, e, g, r1, ..., rN, which are parameters described in Chapter 2. Outputs of tax.m, as well as the objective value of problem (2.10), will be determined for $\left(\tilde{x}, \tilde{f}\right)$.

Because the objective function F of problem (2.10) have indifferentiable points, `socopt_patternsearch.m` will use `patternsearch.m`, a built-in function in the Global Optimization Toolbox of MATLAB, to obtain a solution. As the name suggests, `patternsearch.m` uses a metaheuristic method known as "pattern search" to determine a solution in an ad-hoc manner. According to MATLAB's documentation, `patternsearch.m` can find an optimal solution even if the objective function is discontinuous or indifferentiable.

As with `tax.m`, `socopt_patternsearch.m` can handle any finite number of assets ($J \in \mathbb{N}$), scenarios ($K \in \mathbb{N}$) and banks ($N \in \mathbb{N}$).

B.1 Inputs of `socopt_patternsearch.m`

- **Endowment at Time** $t = 0$

  `wbar` $= \left(\bar{w}_0^1, ..., \bar{w}_0^N\right)$ is the $(1 \times N)$ vector containing the dollar value of endowment for the banks' equityholders at time $t = 0$, as defined in Section 2.5.1.

- Descriptions for `alpha`, `p`, `taubar`, `z`, `c`, `e`, `g`, `r1`, ..., `rN` are the same as those in Section A.1 for `tax.m`.

B.2 Outputs of `socopt_patternsearch.m`

- **Optimal Investment Decisions at Time** $t = 0$

  `x` $= \tilde{x} = \left(\tilde{\vec{x}}^1, ..., \tilde{\vec{x}}^N\right)$ is the $(J \times N)$ matrix containing the banks' dollar amounts of investment at time $t = 0$ which is optimal for the government's problem (2.10) defined by input parameters in Section B.1. The ji^{th} entry of $\tilde{x}$ corresponds to the amount for bank i in asset j.

- **Optimal Time $t = 1$ Debt Values**

  `f` $= \tilde{f} = \left(\tilde{f}_1, ..., \tilde{f}_N\right)$ is the $(1 \times N)$ vector containing the banks' face value of debt (promised dollar payment to debtholders) which is optimal for the government (2.10), when parameters of the model are prescribed as inputs in Section B.1.

- **Optimal Objective Value for the Government**

  `optobj` $= F\left(\tilde{x}, \tilde{f}\right)$ is the objective value (in utility) of the government's problem (2.10) at the optimal solution $\left(\tilde{x}, \tilde{f}\right)$.

  As discussed in Section 2.8.3, the solution to problem (2.17) for the group of banks, $\left(\hat{x}, \hat{f}\right)$, coincides with that of the government's problem, $\left(\tilde{x}, \tilde{f}\right)$. The optimal objective value of problem (2.17) will be $\frac{1}{N}\sum_{i=1}^{N} \hat{F}_i\left(\hat{x}, \hat{f}\right) = \frac{1}{N}\sum_{i=1}^{N} \hat{F}_i\left(\tilde{x}, \tilde{f}\right) = \frac{1}{N}\left(F\left(\tilde{x}, \tilde{f}\right) + c\bar{\tau}\right)$.

- The outputs `t0`, `t1`, `ses`, `prob`, `w1`, `uc`, `sysuc` as described in Section A.2, are determined at the optimal solution $\left(\tilde{x}, \tilde{f}\right)$ of (2.10) (equivalently, of (2.17)) when the model is prescribed with the inputted parameter values of Section B.1.

B.3 Code of `socopt_patternsearch.m`

Note that functions `heaviside1.m`, `heaviside2.m`, and `phi.m` as defined in Section A.3, as well as `tax.m` are used by `socopt_patternsearch.m`.

Moreover, one must also define the utility function applicable to bank equityholders at time $t = 1$ (Section 2.3.5) as a MATLAB function. The function `u(a)` $= a + 1 - \exp(-a)$ is specified below.

```
% Written by Kevin Wai, May 2014.

function Y = u(a)

    Y = a + 1 - exp(-a);

end
```

Then, the code for `socopt_patternsearch.m` is as follows.

```
% Written by Kevin Wai, May 2014.

function [optobj,x,f,t0,t1,ses,prob,w1,uc,sysuc]...
               = socopt_patternsearch(alpha,wbar,p,taubar,z,...
                                      c,e,g,varargin)

% Handles for objective and constraint functions for the problem
% to be solved by patternsearch.m (defined below) are created.

ObjectiveFunction = @social_objective;
ConstraintFunction = @social_constraint;

v = size(varargin{1});
num_assets = v(1);
num_omega = v(2);
num_banks = length(alpha);

Y0 = zeros(1,num_banks*(num_assets + 1));
LB = zeros(1,num_banks*(num_assets + 1));
x = zeros(num_assets,num_banks);
f = zeros(1,num_banks);

options = psoptimset('CompletePoll','on');

% The built-in function patternsearch.m (from Global
% Optimization Toolbox) is called to solve a minimization
% problem with objective function equal to the negative of
% that for problem (2.10), and with the same constraints.
% The optimal objective value for problem (2.10) (optobj)
% is thus the negative of that obtained by
% patternsearch.m (obj), but the solution (soln) would be
% the same.
```

```
[soln,obj] = patternsearch(ObjectiveFunction,Y0,[],[],[],[],...
                           LB,[],ConstraintFunction,options);

optobj = -1*obj;

% Populating matrix x and vector f with values of the optimal
% solution.

for l = 1:num_banks

    for m = 1:num_assets

        x(m,l) = soln((num_assets + 1)*(l - 1) + m);

    end

    f(l) = soln((num_assets + 1)*l);

end

% With the optimal solution entered into x and f, the
% function tax.m is called to determine the taxes, ses,
% etc., at optimality.

[t0,t1,ses,prob,w1,uc,sysuc] = tax(x,f,alpha,p,taubar,z,c,...
                                   e,g,varargin{:});

% Defining the objective function to be used by
% patternsearch.m (called above), which is the negative
% of that in the government's problem (2.10).
% The reason is that patternsearch.m is designed to solve
% a minimization problem, and (2.10) is a maximization problem.
%
% The variable y is a vector with num_banks*(num_assets + 1)
% components. For example, suppose there are two banks and
% two assets, then y will have six components, where y(1) is
% for x_{1}^{1} (bank 1 asset 1), y(2) is for
% x_{2}^{1} (bank 1 asset 2), y(3) is for f_{1}, y(4) is for
% x_{1}^{2} (bank 2 asset 1), y(5) is for
% x_{2}^{2} (bank 2 asset 2), and y(6) is for f_{2}.
    function objfunc = social_objective(y)

        invdollars = ones(num_assets,num_banks);
        facedebt = ones(num_banks,num_omega);
        grossassets = ones(num_banks,num_omega);
        distress = ones(num_banks,num_omega);
        postdistressassets = ones(num_banks,num_omega);
        equity = ones(num_banks,num_omega);
```

```
undercap = ones(num_banks,num_omega);
indnegequity = ones(num_banks,num_omega);
indposequity = ones(num_banks,num_omega);
utility = ones(num_banks,num_omega);

for i = 1:num_banks

    for j = 1:num_assets

        invdollars(j,i)...
                  = y((num_assets + 1)*(i - 1) + j);

    end

    facedebt(i,:) = y((num_assets + 1)*i)...
                    *ones(1,num_omega);

end

for i = 1:num_banks

    grossassets(i,:) = invdollars(1,i)...
                       *varargin{i}(1,:);

    if (num_assets >= 2)

        for j = 2:num_assets

            grossassets(i,:) = grossassets(i,:)...
                               + invdollars(j,i)...
                                 *varargin{i}(j,:);

        end

    end

    distress(i,:) = phi(grossassets(i,:),facedebt(i,:));
    postdistressassets(i,:) = grossassets(i,:)...
                              - distress(i,:);
    equity(i,:) = postdistressassets(i,:)...
                  - facedebt(i,:);
    undercap(i,:) = equity(i,:)...
                    - z*sum(invdollars(:,i))...
                      *ones(1,num_omega);
    indnegequity(i,:) = heaviside2(equity(i,:));
    indposequity(i,:) = heaviside1(equity(i,:));
    utility(i,:) = u(equity(i,:).*indposequity(i,:));

end
```

```
        sysundercap = undercap(1,:);

        if (num_banks >= 2)

            for i = 2:num_banks

                sysundercap = sysundercap + undercap(i,:);

            end

        end

        indcrisis = heaviside2(sysundercap);

        BankObj = ones(1,num_banks);

        for i = 1:num_banks

            BankObj(i) = c*(wbar(i) - sum(invdollars(:,i))...
                         + alpha(i)*facedebt(i,1)...
                         + (1 - alpha(i))...
                           *sum(p.*min(facedebt(i,:),...
                                postdistressassets(i,:))))...
                         + sum(p.*utility(i,:))...
                         + alpha(i)*g...
                           *sum(p.*equity(i,:).*...
                                  indnegequity(i,:))...
                         + e*sum(p.*undercap(i,:).*indcrisis);

        end

        objfunc = c*taubar - sum(BankObj);

    end

% Defining the inequality constraint functions in the
% government's problem (2.10). The vector y have
% num_banks*(num_assets + 1) components, and have the same
% representation as described for the objective function
% above (social_objective(y)).
%
% As there are (num_banks*num_omega + num_banks) constraints
% to the problem (2.10), ineqcons will be a row vector
% consisting of this number of components. A constraint of
% the form (h(y) <= constant) will be entered as
% (h(y) - constant) into a component of ineqcons.
    function [ineqcons,eqcons] = social_constraint(y)
```

```
Invdollars = ones(num_assets,num_banks);
Facedebt = ones(num_banks,num_omega);
Grossassets = ones(num_banks,num_omega);
Distress = ones(num_banks,num_omega);
Postdistressassets = ones(num_banks,num_omega);

for i = 1:num_banks

    for j = 1:num_assets

        Invdollars(j,i)...
                 = y((num_assets + 1)*(i - 1) + j);

    end

    Facedebt(i,:) = y((num_assets + 1)*i)...
                    *ones(1,num_omega);

end

for i = 1:num_banks

    Grossassets(i,:) = Invdollars(1,i)...
                       *varargin{i}(1,:);

    if (num_assets >= 2)

        for j = 2:num_assets

            Grossassets(i,:) = Grossassets(i,:)...
                               + Invdollars(j,i)...
                                 *varargin{i}(j,:);

        end

    end

    Distress(i,:) = phi(Grossassets(i,:),Facedebt(i,:));
    Postdistressassets(i,:) = Grossassets(i,:)...
                              - Distress(i,:);

end

ineqcons = zeros(1,num_banks*num_omega + num_banks);

for i = 1:num_banks

    ineqcons(((i - 1)*num_omega + 1):(i*num_omega))...
                         = -1*Postdistressassets(i,:);
```

```
        end

        for i = 1:num_banks

            ineqcons(num_banks*num_omega + i)...
                       = Facedebt(i,1) - sum(Invdollars(:,i));

        end

        eqcons = [];

    end
end
```

Appendix C

Numerical Example

In this appendix, we will consider a numerical example and utilize the MATLAB code `socopt_patternsearch.m` from Appendix B to determine the socially optimal solution $\left(\text{i.e. } \left(\tilde{x}, \tilde{f}\right)\right.$ for problem (2.10), or equivalently, the solution $\left(\hat{x}, \hat{f}\right)$ to (2.17), the scalarized multi-bank problem in the presence $\left.\text{of taxes}\right)$. This will be presented in Section C.1. Then, in Section C.2, we will verify its optimality for problem (2.17) by considering the scalarized multi-bank objective $\Psi = \frac{1}{N}\sum_{i=1}^{N} \hat{F}_i$ (see Section 3.2) as a univariate function of each decision variable, while others take values of the optimal solution. For example, we will investigate $\Psi = \Psi(f_1)$, while other decisions $\left(\{f_i\}_{i\neq 1}, \{x_j^i\}_{i\in\{1,\ldots,N\}, j\in\{1,\ldots,J\}}\right)$ are fixed with numerical quantities of the solution from Section C.1. This process will be done by graphically studying $\{\psi(\bullet, \omega_k)\}_{k\in\{1,\ldots,K\}}$ and $\Psi(\bullet)$, where $\Psi(\bullet) = E[\psi] = \sum_{k=1}^{K} p_k \times \psi(\bullet, \omega_k)$. Economic interpretations of $\{\psi(\bullet, \omega_k)\}_{k\in\{1,\ldots,K\}}$ based on Sections 3.2.1 and 3.2.2 will also be provided for selected decision variables [1].

The discussions to follow are based on the model with $N = 2$ banks, $J = 2$

[1] This is not done for all variables due to the repetitive nature of explanations.

assets, and $K = 2$ scenarios, along with the following functions and parameter values.

- $u\left(w_1^i 1_{\left[w_1^i > 0\right]}\right) = w_1^i 1_{\left[w_1^i > 0\right]} + 1 - \exp\left(-w_1^i 1_{\left[w_1^i > 0\right]}\right)$
- $\Phi\left[x_1^i r_1^i(\omega) + x_2^i r_2^i(\omega), f_i\right] = \exp\{f_i\} \exp\left\{-\left(x_1^i r_1^i(\omega) + x_2^i r_2^i(\omega)\right)\right\}$
- $p_1 = p_2 = 0.5$
- $\alpha_1 = \alpha_2 = 0.8$
- $\bar{w}_0^1 = \bar{w}_0^2 = 200$
- $r_1^1(\omega_1) = r_1^1(\omega_2) = 1$

  $r_2^1(\omega_1) = 0.5$, $r_2^1(\omega_2) = 2$
- $r_1^2(\omega_1) = r_1^2(\omega_2) = 1$

  $r_2^2(\omega_1) = 2$, $r_2^2(\omega_2) = 1$
- $\bar{\tau} = 100$, $c = 1.7$, $e = 0.2$, $g = 0.625$, $z = 0.6$

C.1 Optimal Solution

By using `socopt_patternsearch.m` as described in Appendix B, an optimal solution for the government problem (2.10) is:

- $\tilde{x} = \begin{bmatrix} \tilde{x}_1^1 & \tilde{x}_1^2 \\ \tilde{x}_2^1 & \tilde{x}_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 352 & 711 \end{bmatrix}$
- $\tilde{f} = \left(\tilde{f}_1, \tilde{f}_2\right) = (176, 709)$

with corresponding objective value $F = 819.0413$.

As discussed in Section 2.8.3, the problem (2.17) that banks collectively solve will have the same solution, but with optimal objective value $\frac{F+c\bar{\tau}}{N} = \frac{819.0413+(1.7)(100)}{2} = 494.5207$.

Moreover, the following results are obtained.

- `t0` $= \tau_0 = 46.7519$
- `t1` $= (\hat{\tau}_1, \hat{\tau}_2) = (-18.4882, 24.9844)$
- `ses` $= (SES_1, SES_2) = (-316.8, 424.7353)$
- `prob` $= P\left[W_1 < zA\right] = 0.5$
- `w1` $= \begin{bmatrix} w_1^1(\omega_1) & w_1^1(\omega_2) \\ w_1^2(\omega_1) & w_1^2(\omega_2) \end{bmatrix} = \begin{bmatrix} -1 & 528 \\ 713 & 1.8647 \end{bmatrix}$
- `uc` $= \begin{bmatrix} w_1^1(\omega_1) - za_1 & w_1^1(\omega_2) - za_1 \\ w_1^2(\omega_1) - za_2 & w_1^2(\omega_2) - za_2 \end{bmatrix} = \begin{bmatrix} -212.2 & 316.8 \\ 286.4 & -424.7 \end{bmatrix}$
- `sysuc` $= \left[W_1(\omega_1) - zA, W_1(\omega_2) - zA\right] = [74.2, -107.9]$

From `sysuc`, $\omega_2 \in \Omega$ is a crisis scenario, with $W_1(\omega_2) - zA = -107.9$. Since $p_1 = p_2 = 0.5$, $P\left[W_1 < zA\right] = 0.5$. From `uc`, the undercapitalization of bank 2 ($w_1^2(\omega_2) - za_2 = -424.7$) caused a crisis to occur in $\omega_2 \in \Omega$. Bank 1 is well-capitalized ($w_1^1(\omega_2) - za_1 = 316.8$) in this case, so it actually makes aggregate undercapitalization (negativity of $W_1 - zA$) less severe.

Notice that in $\omega_1 \in \Omega$, even though bank 1 is undercapitalized, ($w_1^1(\omega_1) - za_1 = -212.2$), bank 2 is well-capitalized to a level such that a crisis does not occur. That is, $w_1^2(\omega_1) - za_2 = 286.4 > 0$ and $W_1(\omega_1) - zA = \sum_{i=1}^{2}\left(w_1^i(\omega_1) - za_i\right) = 74.2 > 0$.

The values in `ses` ($SES_1 = -316.8 < 424.7356 = SES_2$) shows that bank 2 is systemically riskier than bank 1. The reason is that bank 2 is, in expectation, undercapitalized during crisis scenarios, while bank 1 is not.

In `w1`, we see that the only case of bankruptcy occurs in $\omega_1 \in \Omega$, for bank 1 ($w_1^1(\omega_1) < 0$).

Finally, the tax bill for bank 1 will be $\tau_0 + \hat{\tau}_1 = 46.7519 - 18.4882 = 28.2637$ and that for bank 2 will be $\tau_0 + \hat{\tau}_2 = 46.7519 + 24.9844 = 71.7363$. Bank 2 pays a higher tax due to its higher level of systemic risk (SES_2), which resulted from it being undercapitalized to the extent that a crisis occurs in $\omega_2 \in \Omega$. On the other hand, bank 1 did not cause a crisis to materialize in any scenario, and it reduced the severity of the crisis in $\omega_2 \in \Omega$. Therefore, its tax bill is lower.

C.2 Objective Function at Optimality

Recall from Section 3.2 the notation $\Psi = \frac{1}{2}\sum_{i=1}^{2} \hat{F}_i$ for the objective function of problem (2.17) that banks collectively solve. This section will study the relationship between Ψ and f_1, f_2, x_1^1, x_2^1, x_1^2, and x_2^2 individually at the optimal solution obtained in Section C.1.

C.2.1 Relationship between Ψ and f_1

Consider Ψ as a function of f_1 only, with other variables $(f_2, x_1^1, x_2^1, x_1^2, x_2^2)$ taking values of the optimal solution in Section C.1.

Following the steps of Section 3.1.1, the domain of f_1 is determined to be $D(f_1) = [0, 181.17]$.

Recall from Section 3.2.1 the function $\psi(f_1, \omega_k)$, which is the sum of time

$t = 0$ and $t = 1$ utilities of an average bank's equityholders when taxes are considered, if $\omega_k \in \Omega$ is to occur for certain. We shall consider functions $\psi(f_1, \omega_1)$ and $\psi(f_1, \omega_2)$ in order to determine the relationship between $\Psi = E[\psi]$ and f_1.

1. **Function** $\psi(f_1, \omega_1)$

   The time $t = 1$ equity for bank 1 is $w_1^1(f_1, \omega_1) = 176 - \exp(f_1 - 176) - f_1$, and $\tilde{f}_1^1 \approx 175.43$ is such that $w_1^1\left(\tilde{f}_1^1, \omega_1\right) = 0$.

   Moreover, the time $t = 1$ aggregate level of bank equity in excess of the undercapitalization threshold is $W_1(f_1, \omega_1) - zA = \frac{1256}{5} - \exp(-713) - \exp(f_1 - 176) - f_1$, and $\tilde{\tilde{f}}_1^1 \approx 180.26$ is such that $W_1\left(\tilde{\tilde{f}}_1^1, \omega_1\right) - zA = 0$. This situation $\left(\tilde{f}_1^1 < \tilde{\tilde{f}}_1^1\right)$ coincides with that in (I) under the discussion of Section 3.2.1. Since $w_1^1(f_1, \omega_1)$ and $W_1(f_1, \omega_1) - zA$ are decreasing and concave in f_1, with restriction to $f_1 \in D(f_1)$, $\psi(f_1, \omega_1)$ is as follows, with the corresponding plot in Figure C.2.1.1. Note that for $f_1 > 175.43$, bank 1 is bankrupt ($w_1^1 < 0$) and for $f_1 > 180.26$, the system is under-

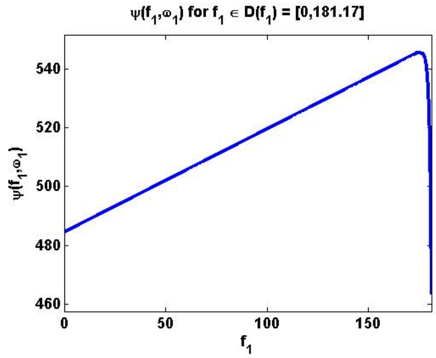

Figure C.2.1.1: $\psi\left(f_1, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $f_1 \in D\left(f_1\right)$

capitalized ($W_1 < zA$), in scenario $\omega_1 \in \Omega$.

$$\psi\left(f_1, \omega_1\right)=\begin{cases} \frac{7}{20} f_1-\frac{1}{2} \exp \{\exp (-713)-713\}-\frac{1}{2} \exp (-713) \\ -\frac{1}{2} \exp \left\{f_1+\exp \left(f_1-176\right)-176\right\} \\ -\frac{1}{2} \exp \left(f_1-176\right)+\frac{2423}{5}; \\ \text{ if } 0 \leq f_1 \leq 175.43 \\ \\ \frac{43}{100} f_1-\frac{1}{2} \exp \{\exp (-713)-713\} \\ -\frac{1}{2} \exp (-713)-\frac{21}{50} \exp \left(f_1-176\right)+\frac{23501}{50}; \\ \text{ if } 175.43<f_1 \leq 180.26 \\ \\ \frac{33}{10} f_1-\frac{1}{2} \exp \{\exp (-713)-713\} \\ -\frac{3}{5} \exp (-713)-\frac{13}{25} \exp \left(f_1-176\right)+\frac{24757}{50}; \\ \text{ if } 180.26<f_1 \leq 181.17 \end{cases}$$

- For $0 \leq f_1 \leq 175.43$, bank 1 is not bankrupt and there is no crisis for $\omega_1 \in \Omega$. That is, $w_1^1\left(f_1, \omega_1\right) \geq 0$ and $W_1\left(f_1, \omega_1\right)-zA \geq 0$.

  As described in the discussion of Section 3.2.1, (I), (i), an increase in f_1 will have an increasing effect on ψ due to an increased amount of debt funding provided to bank 1 at time $t=0$.

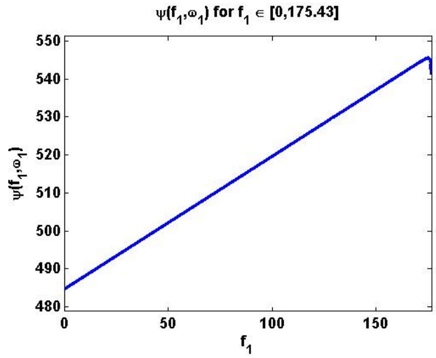

Figure C.2.1.2: $\psi(f_1, \omega_1)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $0 \leq f_1 \leq 175.43$

However, there will also be a decreasing effect on ψ due to reduced equity w_1^1 and hence utility of its holders at time $t = 1$. From the plot in Figure C.2.1.2, for $0 \leq f_1 \leq 174.7$, the benefit to an average bank due to increased debt funding for bank 1 outweighs the cost of its decreased equity at time $t = 1$, because ψ is increasing. For $174.7 < f_1 \leq 175.43$, ψ is decreasing, so the opposite holds (i.e. the cost to an average bank due to decreased time $t = 1$ equity for bank 1 outweighs the benefits of its increased debt funding at time $t = 0$).

- For $175.43 < f_1 \leq 180.26$, bank 1 is bankrupt but there is no crisis for $\omega_1 \in \Omega$: $w_1^1(f_1, \omega_1) < 0$ and $W_1(f_1, \omega_1) - zA \geq 0$. From the discussion of Section 3.2.1, (I), (ii), an increase in f_1 may increase or decrease debt funding provided to bank 1 at time $t = 0$ and hence may increase or decrease ψ. However, the taxes for bank 1 will definitely increase, because a higher f_1 reduces w_1^1 to become even more negative, and the government has to provide more funds to compensate debtholders in bank 1's bankruptcy. The result is a decreasing effect on ψ, because higher taxes reduce the dollars available for bank 1 equityholders to consume, which lower their

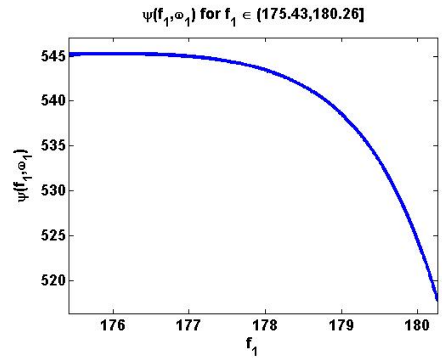

Figure C.2.1.3: $\psi\left(f_1, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $175.43 < f_1 \leq 180.26$

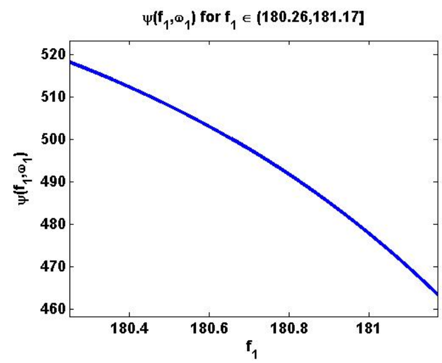

Figure C.2.1.4: $\psi\left(f_1, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $180.26 < f_1 \leq 181.17$

utility (and that of an average bank) at time $t=0$. From Figure C.2.1.3, the combination of the two effects will increase ψ when $175.43 < f_1 \leq 176$ and decrease ψ when $176 < f_1 \leq 180.26$ for our example.

Note that since $w_1^1\left(f_1, \omega_1\right) < 0$ in this region of f_1, the time $t=1$ utility of bank 1's equityholders (and those of the average bank) is unaffected because $u\left(w_1^1 1_{\left[w_1^1>0\right]}\right) = u\left(0\right)$.

- We have just seen that ψ will decrease in f_1 when $f_1 > 176$. Recall that bank 1 is bankrupt in this region of f_1. Now, when $180.26 < f_1 \leq 181.17$, bank 1 is bankrupt and there is a systemic crisis for $\omega_1 \in \Omega$. As per the discussion of Section 3.2.1, (I), (iii), an increase in f_1 will further increase taxes for the average bank, reducing its time $t=0$ utility and thus ψ. Therefore, ψ will decrease in f_1 for the example here, a shown in Figure C.2.1.4.

Again, since bank 1 is bankrupt ($w_1^1 < 0$) in this region of f_1, the time $t = 1$ utility of its equityholders (and those of an average bank) are unaffected.

2. **Function** $\psi(f_1, \omega_2)$

The time $t = 1$ equity for bank 1 is $w_1^1(f_1, \omega_2) = 704 - \exp(f_1 - 704) - f_1$, and $\tilde{f}_1^2 \approx 703.5$ is such that $w_1^1\left(\tilde{f}_1^2, \omega_2\right) = 0$.

The time $t = 1$ aggregate level of bank equity in excess of the undercapitalization threshold is $W_1(f_1, \omega_2) - zA = \frac{341}{5} - \exp(-2) - \exp(f_1 - 704) - f_1$ and $\tilde{\tilde{f}}_1^2 \approx 68.5$ is such that $W_1\left(\tilde{\tilde{f}}_1^2, \omega_2\right) - zA = 0$.

This situation $\left(\tilde{f}_1^2 > \tilde{\tilde{f}}_1^2\right)$ coincides with that in (II) under the discussion of Section 3.2.1. Again, since $w_1^1(f_1, \omega_2)$ and $W_1(f_1, \omega_2) - zA$ are decreasing and concave in f_1, with restriction to $f_1 \in D(f_1)$, $\psi(f_1, \omega_2)$ is as follows, with the corresponding plot in Figure C.2.1.5. Notice that for $f_1 > 68.5$, the system is in crisis ($W_1 < zA$), and for $f_1 > 703.5$ bank 1 is bankrupt ($w_1^1 < 0$) in scenario $\omega_2 \in \Omega$.

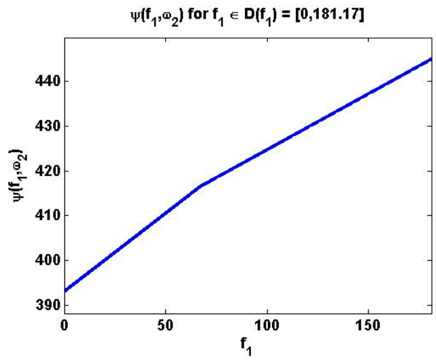

Figure C.2.1.5: $\psi\left(f_1, \omega_2\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_2, for $f_1 \in D\left(f_1\right)$

$$\psi\left(f_1, \omega_2\right)=\begin{cases} \frac{7}{20} f_1-\frac{1}{2} \exp \{\exp (-2)-2\}-\frac{1}{2} \exp (-2) \\ -\frac{1}{2} \exp \left\{\exp \left(f_1-704\right)+f_1-704\right\}-\frac{1}{2} \exp \left(f_1-704\right) \\ +\frac{3931}{10}; \\ \text{ if } 0 \leq f_1 \leq 68.5 \\ \\ \frac{1}{4} f_1-\frac{1}{2} \exp \{\exp (-2)-2\}-\frac{3}{5} \exp (-2) \\ -\frac{1}{2} \exp \left\{\exp \left(f_1-704\right)+f_1-704\right\}-\frac{3}{5} \exp \left(f_1-704\right) \\ +\frac{9998}{25}; \\ \text{ if } 68.5<f_1 \leq 181.17 \end{cases}$$

- When $0 \leq f_1 \leq 68.5$, bank 1 is not bankrupt and there is no systemic crisis for $\omega_2 \in \Omega$. The behaviour of ψ then depends on bank 1's tradeoff between increased debt funding at time $t=0$ and reduced equity at time $t=1$, when f_1 increases. This is explained in the discussion of Section 3.2.1, (II), (i). As shown in Figure C.2.1.6, ψ is increasing. This means that the utility benefit to equityholders of bank 1 due to increased debt funding outweighs the loss in utility due to reduced time $t=1$ equity w_1^1, for all

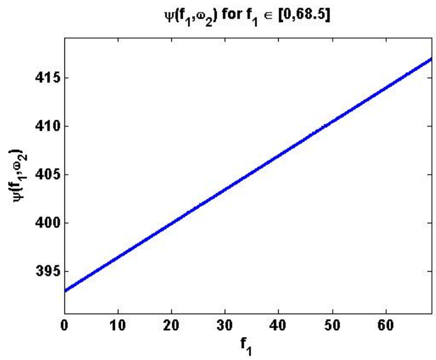

Figure C.2.1.6: $\psi\left(f_1, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $0 \leq f_1 \leq 68.5$

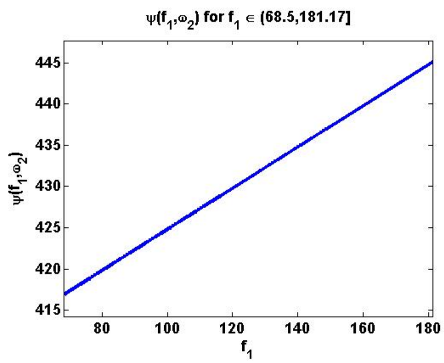

Figure C.2.1.7: $\psi\left(f_1, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $68.5 < f_1 \leq 181.17$

$0 \leq f_1 \leq 68.5$. Therefore, the same can be said for the utility of an average bank's equityholders, as captured by ψ.

- For $68.5 < f_1 \leq 181.17$, bank 1 is not bankrupt, but there is a systemic crisis for $\omega_2 \in \Omega$. Similar to the case when $0 \leq f_1 \leq 68.5$, when f_1 increases, more debt funding for bank 1 will be a benefit, but a reduced w_1^1 will be a cost to the utility of an average bank. However, since the system is undercapitalized ($W_1 < zA$), and $W_1\left(f_1, \omega_2\right) - zA$ decreases in f_1, an average bank will experience a higher tax bill as f_1 increases. This translates into an additional utility cost as well. These explanations are provided in the discussion of Section 3.2.1, (II), (ii). By Figure C.2.1.7, ψ is increasing for all $68.5 < f_1 \leq 181.17$, so the benefit due to increased debt funding for bank 1 outweighs both costs for f_1 in this region.

Finally, the objective function of problem (2.17), $\Psi(f_1) = E[\psi]$ $= 0.5\psi(f_1, \omega_1) + 0.5\psi(f_1, \omega_2)$, is:

$$\Psi(f_1) = \begin{cases} \frac{7}{20} f_1 - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{1}{4}\exp(-2) - \frac{1}{4}\exp(-713) - \frac{1}{4}\exp\{\exp(f_1 - 176) + f_1 - 176\} \\ -\frac{1}{4}\exp\{\exp(f_1 - 704) + f_1 - 704\} - \frac{1}{4}\exp(f_1 - 176) \\ -\frac{1}{4}\exp(f_1 - 704) + \frac{8777}{20}; \\ \text{if } 0 \le f_1 \le 68.5 \\ \\ \frac{3}{10} f_1 - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{3}{10}\exp(-2) - \frac{1}{4}\exp(-713) - \frac{1}{4}\exp\{\exp(f_1 - 176) + f_1 - 176\} \\ -\frac{1}{4}\exp\{\exp(f_1 - 704) + f_1 - 704\} - \frac{1}{4}\exp(f_1 - 176) \\ -\frac{3}{10}\exp(f_1 - 704) + \frac{22113}{50}; \\ \text{if } 68.5 < f_1 \le 175.43 \\ \\ \frac{17}{50} f_1 - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{3}{10}\exp(-2) - \frac{1}{4}\exp(-713) - \frac{1}{4}\exp\{\exp(f_1 - 704) + f_1 - 704\} \\ -\frac{21}{100}\exp(f_1 - 176) - \frac{3}{10}\exp(f_1 - 704) + \frac{43497}{100}; \\ \text{if } 175.43 < f_1 \le 180.26 \\ \\ \frac{29}{100} f_1 - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{3}{10}\exp(-2) - \frac{3}{10}\exp(-713) \\ -\frac{1}{4}\exp\{\exp(f_1 - 704) + f_1 - 704\} \\ -\frac{13}{50}\exp(f_1 - 176) - \frac{3}{10}\exp(f_1 - 704) + \frac{44753}{100}; \\ \text{if } 180.26 < f_1 \le 181.17 \end{cases},$$

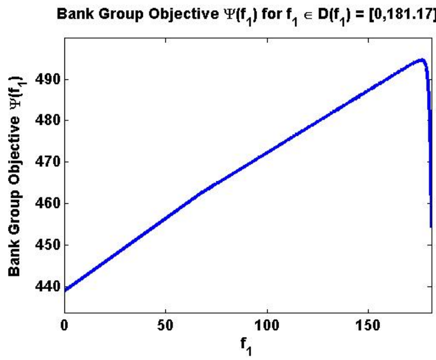

Figure C.2.1.8: $\Psi\left(f_1\right)$, the sum of time $t = 0$ utility and $t = 1$ expected utility of an average bank's equityholders, for $f_1 \in D\left(f_1\right)$

with the corresponding plot in Figure C.2.1.8.

The plot of Ψ shows that the maximum is obtained at around $f_1 = 176$, which coincides with that of the optimal solution in Section C.1.

C.2.2 Relationship between Ψ and f_2

Consider Ψ as a function of f_2 only, with other variables taking values of the optimal solution in Section C.1.

Following the steps of Section 3.1.1, the domain of f_2 is $D\left(f_2\right) = [0, 711]$.

As in Section C.2.1, we shall consider $\psi\left(f_2, \omega_1\right)$ and $\psi\left(f_2, \omega_2\right)$ in order to understand how $\Psi = E\left[\psi\right]$ depends on f_2.

Using results of Section 3.2.1, the explanations for the behaviour of $\psi\left(f_2, \omega_1\right)$ and $\psi\left(f_2, \omega_2\right)$ can be done similarly as for $\psi\left(f_1, \omega\right)$ in Section C.2.1. Therefore, only the plots and expressions of $\psi\left(f_2, \omega_1\right)$, $\psi\left(f_2, \omega_2\right)$, and $\Psi\left(f_2\right)$ will be provided below.

One can observe that $\Psi\left(f_2\right)$ reaches a maximum at about $f_2 = 709$, which is that of the optimal solution.

1. **Function** $\psi\left(f_2, \omega_1\right)$

   The time $t = 1$ equity for bank 2 is $w_1^2\left(f_2, \omega_1\right) = 1422 - \exp\left(f_2 - 1422\right) - f_2$, and $\tilde{f}_2^1 \approx 1421.5$ is such that $w_1^2\left(\tilde{f}_2^1, \omega_1\right) = 0$.

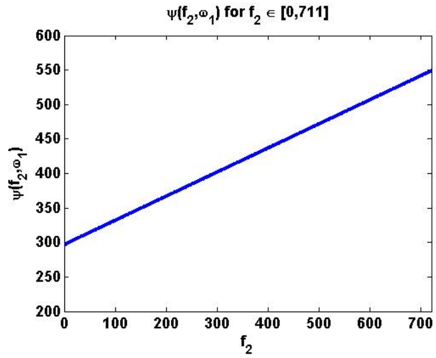

Figure C.2.2.1: $\psi(f_2, \omega_1)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $f_2 \in D(f_2)$

Moreover, the time $t = 1$ aggregate bank equity in excess of the undercapitalization threshold is $W_1(f_2, \omega_1) - zA = \frac{3916}{5} - \exp(f_2 - 1422) - f_2$, and $\tilde{\tilde{f}}_2^1 \approx 783.2$ is such that $W_1\left(\tilde{\tilde{f}}_2^1, \omega_1\right) - zA = 0$.

This situation $\left(\tilde{f}_2^1 > \tilde{\tilde{f}}_2^1\right)$ coincides with that in (II) under the discussion of Section 3.2.1. However, when attention is restricted to $D(f_2) = [0, 711]$, $\psi(f_2, \omega_1)$ can only take the following form, because $\tilde{f}_2^1, \tilde{\tilde{f}}_2^1 \notin D(f_2)$ (i.e. bank 2 is not bankrupt and there is no systemic crisis in $\omega_1 \in \Omega$, for all $f_2 \in D(f_2)$), and both functions $w_1^2(f_2, \omega_1)$ and $W_1(f_2, \omega_1) - zA$ are decreasing and concave. The plot of $\psi(f_2, \omega_1)$ is provided in Figure C.2.2.1.

$$\begin{aligned}\psi(f_2, \omega_1) = \frac{7}{20} f_2 - \frac{1}{2} \exp\{\exp(f_2 - 1422) + f_2 - 1422\} \\ - \frac{1}{2} \exp(f_2 - 1422) + \frac{29713}{100}\end{aligned}$$

2. **Function** $\psi(f_2, \omega_2)$

The time $t = 1$ equity for bank 2 is $w_1^2(f_2, \omega_2) = 711 - \exp(f_2 - 711) - f_2$, and $\tilde{f}_2^2 \approx 710.5$ is such that $w_1^2\left(\tilde{f}_2^2, \omega_2\right) = 0$.

The time $t = 1$ aggregate bank equity in excess of the undercapitalization

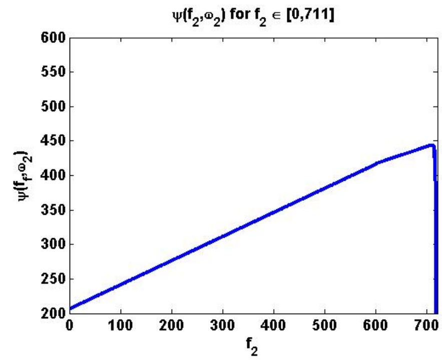

Figure C.2.2.2: $\psi\left(f_2, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $f_2 \in D\left(f_2\right)$

threshold is $W_1\left(f_2, \omega_2\right) - zA = \frac{3006}{5} - \exp\left(-528\right) - \exp\left(f_2 - 711\right) - f_2$, and $\tilde{\tilde{f}}_2^2 \approx 601$ is such that $W_1\left(\tilde{\tilde{f}}_2^2, \omega_2\right) - zA = 0$.

This situation $\left(\tilde{\tilde{f}}_2^2 < \tilde{f}_2^2\right)$ coincides with that of (II) under the discussion of Section 3.2.1.

Since $w_1^2\left(f_2, \omega_2\right)$ and $W_1\left(f_2, \omega_2\right) - zA$ are decreasing and concave, for $f_2 \in D\left(f_2\right)$, $\psi\left(f_2, \omega_2\right)$ is as follows, with the corresponding plot in Figure C.2.2.2. Moreover, for $f_2 > 601$, the system is undercapitalized $(W_1 < zA)$ and for $f_2 > 710.5$, bank 2 is bankrupt $(w_1^2 < 0)$ in $\omega_2 \in \Omega$.

$$\psi\left(f_2,\omega_2\right)=\begin{cases}\frac{7}{20}f_2-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{1}{2}\exp\left(-528\right)\\-\frac{1}{2}\exp\left\{\exp\left(f_2-711\right)+f_2-711\right\}-\frac{1}{2}\exp\left(f_2-711\right)\\+\frac{4131}{20};\\ \text{ if } 0\leq f_2\leq 601\\ \\ \frac{1}{4}f_2-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{5}\exp\left(-528\right)\\-\frac{1}{2}\exp\left\{\exp\left(f_2-711\right)+f_2-711\right\}-\frac{3}{5}\exp\left(f_2-711\right)\\+\frac{26667}{100};\\ \text{ if } 601< f_2\leq 710.5\\ \\ \frac{33}{100}f_2-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{5}\exp\left(-528\right)\\-\frac{13}{25}\exp\left(f_2-711\right)+\frac{20929}{100};\\ \text{ if } 710.5< f_1\leq 711\end{cases}$$

Finally, the objective function of problem (2.17), $\Psi\left(f_2\right)=E\left[\psi\right]$ $=0.5\psi\left(f_2,\omega_1\right)+0.5\psi\left(f_2,\omega_2\right)$, is as follows, with the corresponding plot in Figure C.2.2.3.

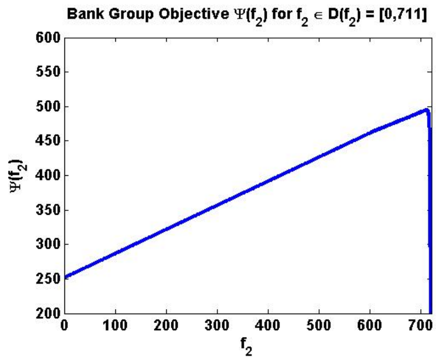

Figure C.2.2.3: $\Psi(f_2)$, the sum of time $t=0$ utility and $t=1$ expected utility of an average bank's equityholders, for $f_2 \in D(f_2)$

$$\Psi(f_2) = \begin{cases} \frac{7}{20}f_2 - \frac{1}{4}\exp\{\exp(-528)-528\} - \frac{1}{4}\exp(-528) \\ -\frac{1}{4}\exp\{\exp(f_2-711)+f_2-711\} \\ -\frac{1}{4}\exp\{\exp(f_2-1422)+f_2-1422\} - \frac{1}{4}\exp(f_2-711) \\ -\frac{1}{4}\exp(f_2-1422) + \frac{6296}{25}; \\ \text{ if } 0 \leq f_2 \leq 601 \\ \\ \frac{3}{10}f_2 - \frac{1}{4}\exp\{\exp(-528)-528\} - \frac{3}{10}\exp(-528) \\ -\frac{1}{4}\exp\{\exp(f_2-711)+f_2-711\} \\ -\frac{1}{4}\exp\{\exp(f_2-1422)+f_2-1422\} - \frac{3}{10}\exp(f_2-711) \\ -\frac{1}{4}\exp(f_2-1422) + \frac{2819}{10}; \\ \text{ if } 601 < f_2 \leq 710.5 \\ \\ \frac{17}{50}f_2 - \frac{1}{4}\exp\{\exp(-528)-528\} - \frac{3}{10}\exp(-528) \\ -\frac{1}{4}\exp\{\exp(f_2-1422)+f_2-1422\} - \frac{13}{50}\exp(f_2-711) \\ -\frac{1}{4}\exp(f_2-1422) + \frac{25321}{100}; \\ \text{ if } 710.5 < f_1 \leq 711 \end{cases}$$

C.2.3 Relationship between Ψ and x_2^1

Consider Ψ as a function of x_2^1 (quantity of dollars invested by bank 1 in asset 2) only, with other variables taking values of the optimal solution in Section C.1.

Following the steps of Section 3.1.2, the domain of x_2^1 is $D\left(x_2^1\right) = [341.7183, \infty)$.

Recall from Section 3.2.2 the function $\psi\left(x_2^1, \omega_k\right)$, which is the time $t=0$ plus $t=1$ utilities of an average bank's equityholders, if $\omega_k \in \Omega$ occurs with certainty. The following discussion will consider $\psi\left(x_2^1, \omega_1\right)$ and $\psi\left(x_2^1, \omega_2\right)$ in order to understand the relationship between $\Psi = E\left[\psi\right]$ and x_2^1.

1. **Function** $\psi\left(x_2^1, \omega_1\right)$

   The time $t=1$ equity of bank 1 is $w_1^1\left(x_2^1, \omega_1\right) = \frac{1}{2}x_2^1 - \exp\left(-\frac{1}{2}x_2^1 + 176\right) - 176$ and $\tilde{x}_2^1(1) \approx 353$ is such that $w_1^1\left(\tilde{x}_2^1(1), \omega_1\right) = 0$. $\frac{\partial w_1^1}{\partial x_2^1} = \frac{1}{2} + \frac{1}{2}\exp\left(-\frac{1}{2}x_2^1 + 176\right) > 0$ implies that $w_1^1\left(x_2^1, \omega_1\right)$ is increasing. Hence, for $x_2^1 < 353$, bank 1 is bankrupt $\left(w_1^1 < 0\right)$ in $\omega_1 \in \Omega$.

   The time $t=1$ aggregate level of bank equity in excess of the undercapitalization threshold is $W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right) = \frac{552}{5} - \exp\left(-713\right) - \exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{1}{10}x_2^1$ with derivative $\frac{\partial\left[W_1 - zA\right]}{\partial x_2^1} = \frac{1}{2}\exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{1}{10}$, which is positive for $x_2^1 < 355.22$ and is negative for $x_2^1 > 355.22$. That is, $W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)$ increases for $x_2^1 < 355.22$ and decreases for $x_2^1 > 355.22$. Moreover, $\tilde{\tilde{x}}_2^1(1,1) \approx 343.35$ and $\tilde{\tilde{x}}_2^1(1,2) \approx 1100$ are values of x_2^1 such that $W_1\left(\tilde{\tilde{x}}_2^1(1,1), \omega_1\right) - zA\left(\tilde{\tilde{x}}_2^1(1,1)\right) = W_1\left(\tilde{\tilde{x}}_2^1(1,2), \omega_1\right) - zA\left(\tilde{\tilde{x}}_2^1(1,2)\right) = 0$. Therefore, for $x_2^1 \in [0, 343.35) \cup (1100, \infty]$, the system is in crisis $\left(W_1 < zA\right)$ in $\omega_1 \in \Omega$.

Restricting to $D\left(x_2^1\right) = [341.7183, \infty)$, $\psi\left(x_2^1, \omega_1\right)$ is as follows, with the corresponding plot in Figure C.2.3.1

$$\psi\left(x_2^1, \omega_1\right) = \begin{cases} \frac{39101}{50} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} \\ -\frac{3}{5}\exp\left(-713\right) - \frac{13}{25}\exp\left(176 - \frac{1}{2}x_2^1\right) - \frac{13}{20}x_2^1; \\ \text{ if } 341.7183 \leq x_2^1 < 343.35 \\ \\ \frac{38549}{50} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} \\ -\frac{1}{2}\exp\left(-713\right) - \frac{21}{50}\exp\left(176 - \frac{1}{2}x_2^1\right) - \frac{16}{25}x_2^1; \\ \text{ if } 343.35 \leq x_2^1 < 353 \\ \\ \frac{3787}{5} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} - \frac{1}{2}\exp\left(-713\right) \\ -\frac{1}{2}\exp\left\{176 - \frac{1}{2}x_2^1 + \exp\left(176 - \frac{1}{2}x_2^1\right)\right\} \\ -\frac{1}{2}\exp\left(176 - \frac{1}{2}x_2^1\right) - \frac{3}{5}x_2^1; \\ \text{ if } 353 \leq x_2^1 \leq 1100 \\ \\ \frac{19211}{25} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} - \frac{3}{5}\exp\left(-713\right) \\ -\frac{1}{2}\exp\left\{176 - \frac{1}{2}x_2^1 + \exp\left(176 - \frac{1}{2}x_2^1\right)\right\} \\ -\frac{3}{5}\exp\left(176 - \frac{1}{2}x_2^1\right) - \frac{61}{100}x_2^1; \\ \text{ if } 1100 < x_2^1 < \infty \end{cases}$$

- For $341.7183 \leq x_2^1 < 343.35$, bank 1 is bankrupt ($w_1^1 < 0$) and the system is undercapitalized ($W_1 < zA$). That is, $x_2^1 \in X_4\left(x_2^1, \omega_1\right) \cap D\left(x_2^1\right)$, where $X_4\left(x_2^1, \omega_1\right)$ is described in (IV) of the discussion in Section 3.2.2.

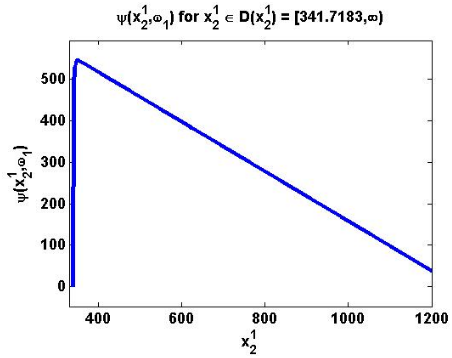

Figure C.2.3.1: $\psi\left(x_2^1, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $x_2^1 \in D\left(x_2^1\right)$

Recall from Section C.1 that at the optimal solution, bank 2's equity for scenario $\omega_1 \in \Omega$ is positive ($w_1^2\left(\omega_1\right) > 0$). Therefore, its debt funding at time $t=0$ is $\alpha_2 f_2 + \left(1-\alpha_2\right) \min\left(f_2, w_1^2\left(\omega_1\right) + f_2\right) = f_2$. Since both $w_1^1\left(x_2^1, \omega_1\right)$ and $W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)$ are increasing for $341.7183 \leq x_2^1 < 343.35$, the taxes paid by the average bank, $-\frac{1}{2}\left(\frac{\alpha_1 g}{c} w_1^1\left(x_2^1, \omega_1\right) + \frac{\alpha_2 g}{c} w_1^2\left(\omega_1\right) 1_{\left[w_1^2<0\right]}\right) - \frac{e}{2c}\left(W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)\right)$ $= -\frac{\alpha_1 g}{2c} w_1^1\left(x_2^1, \omega_1\right) - \frac{e}{2c}\left(W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)\right)$, decreases in x_2^1.

Furthermore, since bank 1 is bankrupt, its debtholders will receive $w_1^1\left(x_2^1, \omega_1\right) + f_1 - \alpha_1 w_1^1\left(x_2^1, \omega_1\right)$ dollars at time $t=1$. When $\omega_1 \in \Omega$ is considered in isolation, this is also the funding they provide bank 1 at time $t=0$. It follows that the funding for the average bank, $\frac{1}{2}\left(w_1^1\left(x_2^1, \omega_1\right) + f_1 - \alpha_1 w_1^1\left(x_2^1, \omega_1\right) + f_2\right)$ will increase with respect to x_2^1.

The dollars of consumption for the average bank's equityholders at time $t=0$ is $\frac{1}{2}\left(\bar{w}_0^1 + \bar{w}_0^2\right) - \frac{1}{2}\left(x_1^1 + x_2^1 + x_1^2 + x_2^2\right)$ $+\frac{1}{2}\left(w_1^1\left(x_2^1, \omega_1\right) + f_1 - \alpha_1 w_1^1\left(x_2^1, \omega_1\right) + f_2\right) + \frac{\alpha_1 g}{2c} w_1^1\left(x_2^1, \omega_1\right)$ $+\frac{e}{2c}\left(W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)\right)$. From the plot of ψ (Figure C.2.3.2), one can deduce that this consumption (equivalently, utility) increase when bank 1's investment in asset 2 increase within the region

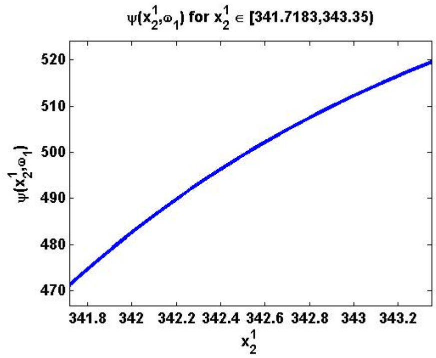

Figure C.2.3.2: $\psi\left(x_2^1, \omega_1\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $341.7183 \leq x_2^1 < 343.35$

$341.7183 \leq x_2^1 < 343.35$.

Note that the time $t = 1$ utility for equityholders of bank 1, and those of an average bank, are unaffected since $w_1^1 < 0$ in this region of x_2^1, and $u\left(w_1^1 1_{\left[w_1^1>0\right]}\right) = u\left(0\right)$.

- For $343.35 \leq x_2^1 < 353$, bank 1 is bankrupt ($w_1^1 < 0$) and the system is not in crisis ($W_1 \geq zA$). That is, $x_2^1 \in X_3\left(x_2^1, \omega_1\right) \cap D\left(x_2^1\right)$, where $X_3\left(x_2^1, \omega_1\right)$ is described in (III) of the discussion in Section 3.2.2.

  In this region of x_2^1, bank 1's dollars of tax payment is $-\frac{\alpha_1 g}{c} w_1^1\left(x_2^1, \omega_1\right)$ and debt raised is $w_1^1\left(x_2^1, \omega_1\right) + f_1 - \alpha_1 w_1^1\left(x_2^1, \omega_1\right)$. Since $w_1^1\left(x_2^1, \omega_1\right)$ is increasing, the dollars of debt funding will increase and the tax bill will decrease. However, the dollars of consumption, and hence utility, for bank 1's equityholders at time $t = 0$, $\bar{w}_0^1 - \left(x_1^1 + x_2^1\right) + f_1 + \left(1 - \alpha_1\right) w_1^1\left(x_2^1, \omega_1\right) + \frac{\alpha_1 g}{c} w_1^1\left(x_2^1, \omega_1\right)$, may increase or decrease. Therefore, the average bank's equityholders may experience an increase or decrease in time $t = 0$ utility if x_2^1 increase.

  As exhibited by the plot in Figure C.2.3.3, ψ increases for $343.35 \leq x_2^1 < 349.7713$. This means that if bank 1 increases investment in asset 2, its equityholders (thus those of an average bank) will have more money to consume at time $t = 0$, after considering benefits of

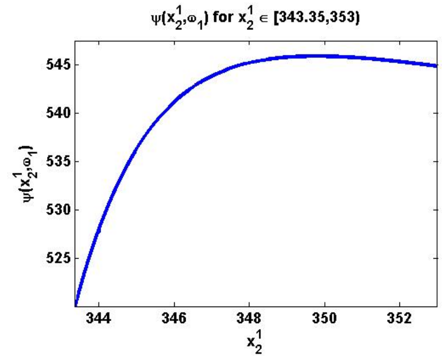

Figure C.2.3.3: $\psi\left(x_2^1, \omega_1\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $343.35 \le x_2^1 < 353$

increased debt funding and reduced taxes.

When $349.7713 < x_2^1 < 353$, however, investing more in asset 2 would require bank 1 equityholders to contribute additional dollars, even though debt funding increased and taxes decreased. This means they will have a lower consumption, and thus utility at time $t = 0$. Then, the same can be said about an average bank, as shown in the decreasing behaviour of ψ for x_2^1 in this region (Figure C.2.3.3).

Again, the time $t = 1$ utility of bank 1's equityholders, and those of an average bank, is unaffected for $343.35 \le x_2^1 < 353$, because bank 1 will be bankrupt ($w_1^1 < 0$) and $u\left(w_1^1 1_{\left[w_1^1 > 0\right]}\right) = u\left(0\right)$.

- For $353 \le x_2^1 \le 1100$, bank 1 is not bankrupt ($w_1^1 \ge 0$) and the system is well-capitalized ($W_1 \ge zA$). That is, $x_2^1 \in X_1\left(x_2^1, \omega_1\right) \cap D\left(x_2^1\right)$, where $X_1\left(x_2^1, \omega_1\right)$ is described in (I) of the discussion in Section 3.2.2.

  In this case, the dollars of debt raised by bank 1 would equal f_1, which is not affected by changes in x_2^1. So, any increase in investment in asset 2 will be fully funded by its equityholders, who will then experience a decline in time $t = 0$ utility. However, their util-

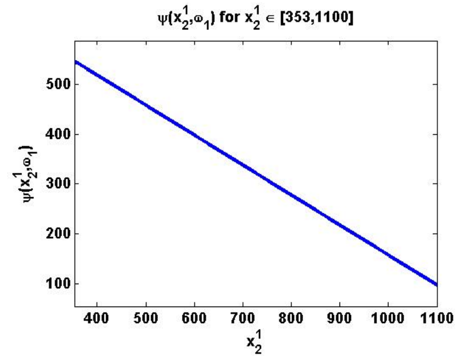

Figure C.2.3.4: $\psi\left(x_2^1, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $353 \leq x_2^1 \leq 1100$

ity at time $t=1$ will increase since equity w_1^1 will be higher at that time.

From the plot in Figure C.2.3.4, ψ is decreaseing for all $353 \leq x_2^1 \leq 1100$, which means that if bank 1 increase its investment in asset 2, the time $t=0$ utility of an average bank will reduce more than the amount that its time $t=1$ utility increase by. Equivalently, for bank 1, the utility cost of decreased consumption at time $t=0$ outweighs the benefit of increase equity at time $t=1$.

- For $1100 < x_2^1 < \infty$, bank 1 is not bankrupt $\left(w_1^1 \geq 0\right)$ but the system is in crisis $\left(W_1 < zA\right)$. That is, $x_2^1 \in X_2\left(x_2^1, \omega_1\right) \cap D\left(x_2^1\right)$, where $X_2\left(x_2^1, \omega_1\right)$ is described in (II) of the discussion in Section 3.2.2.

  Again, recall that $w_1^2\left(\omega_1\right) > 0$ at the optimal solution, which implies that debt funding for bank 2 at time $t=0$ is equal to f_2. The debt funding for the average bank, $\frac{1}{2}\left(f_1+f_2\right)$ is thus unaffected by x_2^1, but the time $t=1$ utility $\frac{1}{2}\left(u\left(w_1^1\left(x_2^1, \omega_1\right)\right)+u\left(w_1^2\left(\omega_1\right)\right)\right)$ increases in x_2^1.

  However, since $W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)$ is decreasing for $1100 < x_2^1 < \infty$, the dollars of taxes payable by an average bank,

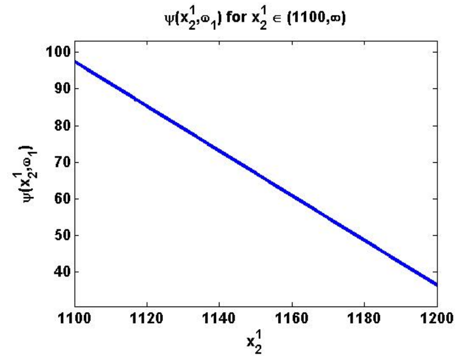

Figure C.2.3.5: $\psi\left(x_2^1, \omega_1\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $1100 < x_2^1 < \infty$

$-\frac{e}{2c}\left(W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)\right)$, increases.

The sum of time $t = 0$ and $t = 1$ utilities for the average bank,

$\frac{c}{2}\left(\bar{w}_0^1 + \bar{w}_0^2 - \left(x_1^1 + x_2^1 + x_1^2 + x_2^2\right) + f_1 + f_2\right)$
$+\frac{e}{2}\left(W_1\left(x_2^1, \omega_1\right) - zA\left(x_2^1\right)\right) + \frac{1}{2}\left(u\left(w_1^1\left(x_2^1, \omega_1\right)\right) + u\left(w_1^2\left(\omega_1\right)\right)\right)$, is decreasing for $1100 < x_2^1 < \infty$, as shown by the plot of ψ in Figure C.2.3.5.

This means that if bank 1 increases its investment in asset 2, the utility cost for the average bank's equityholders due to decreased consumption at time $t = 0$ (resulting from their complete funding of the investment increase and payment of higher taxes) outweighs the benefit of increased equity value at time $t = 1$.

2. **Function** $\psi\left(x_2^1, \omega_2\right)$

The time $t = 1$ equity of bank 1 is $w_1^1\left(x_2^1, \omega_2\right) = 2x_2^1 - \exp\left(-2x_2^1 + 176\right) - 176$ and $\tilde{x}_2^1(2) \approx 88.4$ is such that $w_1^1\left(\tilde{x}_2^1(2), \omega_2\right) = 0$. $w_1^1\left(x_2^1, \omega_2\right)$ increases in x_2^1 because $\frac{\partial w_1^1}{\partial x_2^1} = 2 + 2\exp\left(-2x_2^1 + 176\right) > 0$. Therefore, for $x_2^1 \geq 88.4$, bank 1 is not bankrupt ($w_1^1 \geq 0$) in $\omega_2 \in \Omega$

The time $t = 1$ aggregate bank equity in excess of the undercapitalization threshold is $W_1\left(x_2^1, \omega_2\right) - zA\left(x_2^1\right) = \frac{7}{5}x_2^1 - \exp(-2) - \exp\left(-2x_2^1 + 176\right) -$

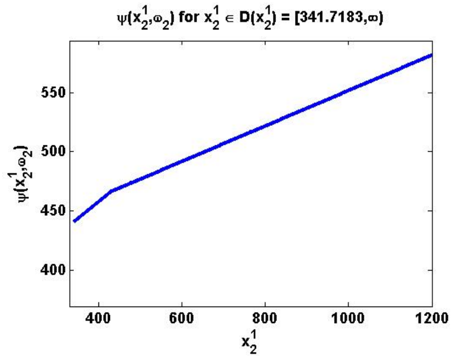

Figure C.2.3.6: $\psi\left(x_2^1, \omega_2\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_2, for $x_2^1 \in D\left(x_2^1\right)$

$\frac{3003}{5}$, with derivative $\frac{\partial[W_1 - zA]}{\partial x_2^1} = \frac{7}{5} + 2\exp\left(-2x_2^1 + 176\right) > 0$. That is, $W_1\left(x_2^1, \omega_2\right) - zA\left(x_2^1\right)$ increases with respect to x_2^1. Also, $\tilde{\tilde{x}}_2^1(2) \approx 429$ is such that $W_1\left(\tilde{\tilde{x}}_2^1(2), \omega_2\right) - zA\left(\tilde{\tilde{x}}_2^1(2)\right) = 0$. Hence, for $x_2^1 < 429$, the system is in crisis ($W_1 < zA$) in $\omega_2 \in \Omega$.

Restricting to $D\left(x_2^1\right) = [341.7183, \infty)$,

$$\psi\left(x_2^1, \omega_2\right) = \begin{cases} \frac{29}{100}x_2^1 - \frac{1}{2}\exp\left\{\exp(-2) - 2\right\} - \frac{3}{5}\exp(-2) \\ -\frac{1}{2}\exp\left\{\exp\left(-2x_2^1 + 176\right) - 2x_2^1 + 176\right\} \\ -\frac{3}{5}\exp\left(-2x_2^1 + 176\right) + \frac{8546}{25}; \\ \text{ if } 341.7183 \leq x_2^1 < 429 \\ \\ \frac{3}{20}x_2^1 - \frac{1}{2}\exp\left\{\exp(-2) - 2\right\} - \frac{1}{2}\exp(-2) \\ -\frac{1}{2}\exp\left\{\exp\left(-2x_2^1 + 176\right) - 2x_2^1 + 176\right\} \\ -\frac{1}{2}\exp\left(-2x_2^1 + 176\right) + \frac{4019}{10}; \\ \text{ if } 429 \leq x_2^1 < \infty \end{cases},$$

with the corresponding plot in Figure C.2.3.6.

- When $341.7183 \leq x_2^1 < 429$, bank 1 is not bankrupt ($w_1^1 \geq 0$) but the system is in crisis ($W_1 < zA$) for $\omega_2 \in \Omega$. This situation is

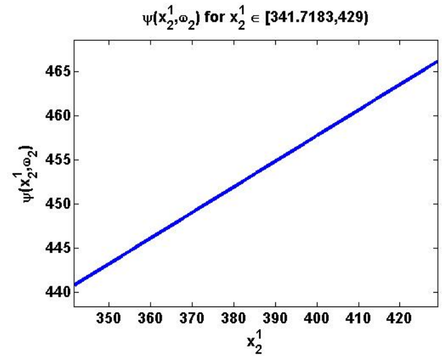

Figure C.2.3.7: $\psi\left(x_2^1, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $341.7183 \leq x_2^1 < 429$

described in (II) of the discussion in Section 3.2.2

(i.e. $x_2^1 \in X_2\left(x_2^1, \omega_2\right) \cap D\left(x_2^1\right)$).

At the optimal solution in Section C.1, bank 2's time $t = 1$ equity is $w_1^2\left(\omega_2\right) = 1.8647 > 0$, which implies that bank 2's debt funding at time $t = 0$ is f_2 (equals to its debtholders' payoff at time $t = 1$ since $\omega_2 \in \Omega$ is considered in isolation).

Then, the debt funding for the average bank at time $t = 0$, $\frac{1}{2}\left(f_1 + f_2\right)$ does not depend on x_2^1, but the time $t = 1$ utility,

$\frac{1}{2}\left(u\left(w_1^1\left(x_2^1, \omega_2\right)\right) + u\left(w_1^2\left(\omega_2\right)\right)\right)$ is increasing in that variable. Moreover, since $W_1\left(x_2^1, \omega_2\right) - zA\left(x_2^1\right)$ increases in x_2^1, an average bank's tax bill in dollars, $-\frac{e}{2c}\left(W_1\left(x_2^1, \omega_2\right) - zA\left(x_2^1\right)\right)$, is decreasing.

From the plot of ψ in Figure C.2.3.7, the time $t = 0$ plus $t = 1$ utilities of an average bank, $\frac{c}{2}\left(\bar{w}_0^1 + \bar{w}_0^2 - \left(x_1^1 + x_2^1 + x_1^2 + x_2^2\right) + f_1 + f_2\right)$ $+\frac{e}{2}\left(W_1\left(x_2^1, \omega_2\right) - zA\left(x_2^1\right)\right) + \frac{1}{2}\left(u\left(w_1^1\left(x_2^1, \omega_2\right)\right) + u\left(w_1^2\left(\omega_2\right)\right)\right)$, is increasing.

Therefore, if bank 1 increases its investment in asset 2, the utility benefit for the average bank's equityholders due to reduced taxes and increased time $t = 1$ equity value outweighs the cost of contributing money for funding the additional investment at time $t = 0$.

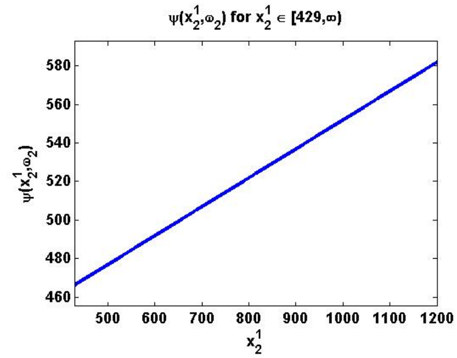

Figure C.2.3.8: $\psi\left(x_2^1, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $429 \leq x_2^1 < \infty$

- For $429 \leq x_2^1 < \infty$, bank 1 is not bankrupt ($w_1^1 \geq 0$) and the system is well-capitalized ($W_1 \geq zA$). This situation is described in (I) of the discussion under Section 3.2.2 (i.e. $x_2^1 \in X_1\left(x_2^1, \omega_2\right) \cap D\left(x_2^1\right)$).

  In this situation, bank 1 debtholders' payoff at time $t = 1$ (equivalently, the funding they provide at time $t = 0$) is f_1, which does not depend on x_2^1. Therefore, if bank 1 increases investment in asset 2, its equityholders will completely pay for it, and their dollars of consumption (hence utility) will decline at time $t = 0$. However, their time $t = 1$ utility will increase as a response to higher equity value w_1^1 at that time.

  The plot of ψ in Figure C.2.3.8 is increasing, which shows that this latter benefit to utility at time $t = 1$ outweighs the cost of reduced consumption at time $t = 0$ for the average bank, or equivalently, for bank 1.

Finally, $\Psi\left(x_2^1\right) = E\left[\psi\right] = 0.5\psi\left(x_2^1, \omega_1\right) + 0.5\psi\left(x_2^1, \omega_2\right)$, the objective function to (2.17) is as follows.

$$\Psi\left(x_2^1\right)=\begin{cases}\frac{56193}{100}-\frac{1}{4}\exp\left\{\exp\left(-2\right)-2\right\}-\frac{1}{4}\exp\left\{\exp\left(-713\right)-713\right\}\\ -\frac{3}{10}\exp\left(-2\right)-\frac{3}{10}\exp\left(-713\right)\\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^1+176\right)-2x_2^1+176\right\}\\ -\frac{3}{10}\exp\left(-2x_2^1+176\right)-\frac{13}{50}\exp\left(-\frac{1}{2}x_2^1+176\right)-\frac{9}{50}x_2^1;\\ \text{ if } 341.7183\leq x_2^1<343.35\\ \\ \frac{55641}{100}-\frac{1}{4}\exp\left\{\exp\left(-2\right)-2\right\}-\frac{1}{4}\exp\left\{\exp\left(-713\right)-713\right\}\\ -\frac{3}{10}\exp\left(-2\right)-\frac{1}{4}\exp\left(-713\right)\\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^1+176\right)-2x_2^1+176\right\}\\ -\frac{3}{10}\exp\left(-2x_2^1+176\right)-\frac{21}{100}\exp\left(-\frac{1}{2}x_2^1+176\right)-\frac{7}{40}x_2^1;\\ \text{ if } 343.35\leq x_2^1<353\\ \\ \frac{27481}{50}-\frac{1}{4}\exp\left\{\exp\left(-2\right)-2\right\}-\frac{1}{4}\exp\left\{\exp\left(-713\right)-713\right\}\\ -\frac{3}{10}\exp\left(-2\right)-\frac{1}{4}\exp\left(-713\right)\\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^1+176\right)-2x_2^1+176\right\}\\ -\frac{1}{4}\exp\left\{\exp\left(-\frac{1}{2}x_2^1+176\right)-\frac{1}{2}x_2^1+176\right\}-\frac{3}{10}\exp\left(-2x_2^1+176\right)\\ -\frac{1}{4}\exp\left(-\frac{1}{2}x_2^1+176\right)-\frac{31}{200}x_2^1;\\ \text{ if } 353\leq x_2^1<429\end{cases}$$

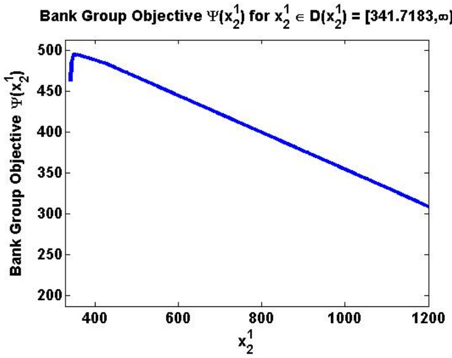

Figure C.2.3.9: $\Psi\left(x_2^1\right)$, the sum of time $t = 0$ utility and $t = 1$ expected utility of an average bank's equityholders, for $x_2^1 \in D\left(x_2^1\right)$

$$\Psi\left(x_2^1\right) \text{ Continued} \begin{cases} \frac{11593}{20} - \frac{1}{4}\exp\left\{\exp\left(-2\right) - 2\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-713\right) - 713\right\} \\ -\frac{1}{4}\exp\left(-2\right) - \frac{1}{4}\exp\left(-713\right) \\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^1 + 176\right) - 2x_2^1 + 176\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{1}{2}x_2^1 + 176\right\} \\ -\frac{1}{4}\exp\left(-2x_2^1 + 176\right) - \frac{1}{4}\exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{9}{40}x_2^1; \\ \text{ if } 429 \leq x_2^1 \leq 1100 \\ \\ \frac{58517}{100} - \frac{1}{4}\exp\left\{\exp\left(-2\right) - 2\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-713\right) - 713\right\} \\ -\frac{1}{4}\exp\left(-2\right) - \frac{3}{10}\exp\left(-713\right) \\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^1 + 176\right) - 2x_2^1 + 176\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{1}{2}x_2^1 + 176\right\} \\ -\frac{1}{4}\exp\left(-2x_2^1 + 176\right) - \frac{3}{10}\exp\left(-\frac{1}{2}x_2^1 + 176\right) - \frac{23}{100}x_2^1; \\ \text{ if } 1100 < x_2^1 < \infty \end{cases}.$$

From the plot of $\Psi\left(x_2^1\right)$ in Figure C.2.3.9, the maximum is reached at around $x_2^1 = 352$, coinciding with that of the optimal solution in Section C.1.

C.2.4 Relationship between Ψ and x_1^1

Consider Ψ as a function of x_1^1 (dollars invested by bank 1 in asset 1) only, with other variables taking values of the optimal solution in Section C.1.

Using the procedure in Section 3.1.2, the domain of x_1^1 is $D\left(x_1^1\right) = [0, \infty)$.

Similar to Section C.2.3, we shall consider $\psi\left(x_1^1, \omega_1\right)$ and $\psi\left(x_1^1, \omega_2\right)$ to determine how $\Psi = E\left[\psi\right]$ relates with x_1^1.

The explanations about the behaviour of $\psi\left(x_1^1, \omega_1\right)$ and $\psi\left(x_1^1, \omega_2\right)$ are done similarly as in Section C.2.3, so only their plots and expressions will be provided.

The maximum of $\Psi\left(x_1^1\right)$ is reached at $x_1^1 = 0$, which coincides with that of the optimal solution in Section C.1.

1. **Function** $\psi\left(x_1^1, \omega_1\right)$

   The time $t = 1$ equity for bank 1 is $w_1^1\left(x_1^1, \omega_1\right) = x_1^1 - \exp\left(-x_1^1\right)$ and $\tilde{x}_1^1\left(1\right) \approx 0.5$ is such that $w_1^1\left(\tilde{x}_1^1\left(1\right), \omega_1\right) = 0$. $w_1^1\left(x_1^1, \omega_1\right)$ is increasing, because $\frac{\partial w_1^1}{\partial x_1^1} = 1 + \exp\left(-x_1^1\right) > 0$. Therefore, in scenario $\omega_1 \in \Omega$, bank 1 is bankrupt for $x_1^1 < 0.5$.

   The time $t = 1$ aggregate bank equity in excess of the undercapitalization threshold is $W_1\left(x_1^1, \omega_1\right) - zA\left(x_1^1\right) = \frac{2}{5}x_1^1 - \exp\left(-713\right) - \exp\left(-x_1^1\right) + \frac{376}{5}$ with derivative $\frac{\partial\left[W_1 - zA\right]}{\partial x_1^1} = \frac{2}{5} + \exp\left(-x_1^1\right) > 0$. Moreover, $\tilde{\tilde{x}}_1^1\left(1\right) \approx -4.3$ is such that $W_1\left(\tilde{\tilde{x}}_1^1\left(1\right), \omega_1\right) - zA\left(\tilde{\tilde{x}}_1^1\left(1\right)\right) = 0$. Since $W_1\left(x_1^1, \omega_1\right) - zA\left(x_1^1\right)$ is increasing, the system is well-capitalized ($W_1 \geq zA$, no crisis) for $x_1^1 \geq -4.3$, in $\omega_1 \in \Omega$.

   Restricting to $D\left(x_1^1\right) = [0, \infty)$, $\psi\left(x_1^1, \omega_1\right)$ is as follows, with the corre-

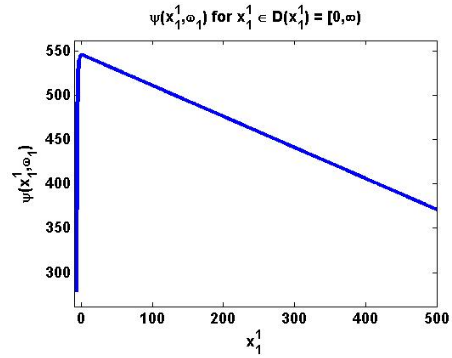

Figure C.2.4.1: $\psi\left(x_1^1, \omega_1\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $x_1^1 \in D\left(x_1^1\right)$

sponding plot in Figure C.2.4.1.

$$\psi\left(x_1^1, \omega_1\right) = \begin{cases} \frac{5457}{10} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} - \frac{1}{2}\exp\left(-713\right) \\ -\frac{21}{50}\exp\left(-x_1^1\right) - \frac{43}{100}x_1^1; \\ \text{ if } 0 \le x_1^1 < 0.5 \\ \\ \frac{2731}{5} - \frac{1}{2}\exp\left\{\exp\left(-713\right) - 713\right\} - \frac{1}{2}\exp\left(-713\right) \\ -\frac{1}{2}\exp\left\{\exp\left(-x_1^1\right) - x_1^1\right\} - \frac{1}{2}\exp\left(-x_1^1\right) \\ -\frac{7}{20}x_1^1; \\ \text{ if } 0.5 \le x_1^1 < \infty \end{cases}$$

Using the notations of Section 3.2.2, $D\left(x_1^1\right) \cap X_3\left(x_1^1, \omega_1\right) = [0, 0.5)$ and $D\left(x_1^1\right) \cap X_1\left(x_1^1, \omega_1\right) = [0.5, \infty)$. Therefore, the system is well-capitalized for all $x_1^1 \in D\left(x_1^1\right) = [0, \infty)$ and bank 1 is bankrupt for $0 \le x_1^1 < 0.5$.

2. **Function** $\psi\left(x_1^1, \omega_2\right)$

The time $t = 1$ equity for bank 1 is $w_1^1\left(x_1^1, \omega_2\right) = x_1^1 - \exp\left(-x_1^1 - 528\right) + 528$ and $\tilde{x}_1^1(2) \approx -527.5$ is such that $w_1^1\left(\tilde{x}_1^1(2), \omega_2\right) = 0$. $w_1^1\left(x_1^1, \omega_2\right)$ is increasing, because $\frac{\partial w_1^1}{\partial x_1^1} = 1 + \exp\left(-x_1^1 - 528\right) > 0$. Hence, bank 1 is not bankrupt in scenario $\omega_2 \in \Omega$, if $x_1^1 \ge -527.5$.

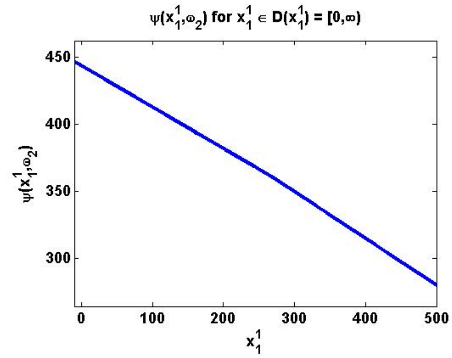

Figure C.2.4.2: $\psi\left(x_1^1, \omega_2\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_2, for $x_1^1 \in D\left(x_1^1\right)$

The time $t = 1$ aggregate level of bank equity in excess of the undercapitalization threshold is $W_1\left(x_1^1, \omega_2\right) - zA\left(x_1^1\right) = \frac{2}{5}x_1^1 - \exp\left(-2\right) - \exp\left(-x_1^1 - 528\right) - \frac{539}{5}$ with derivative $\frac{\partial\left[W_1 - zA\right]}{\partial x_1^1} = \frac{2}{5} + \exp\left(-x_1^1 - 528\right) > 0$. Furthermore, $W_1\left(x_1^1, \omega_2\right) - zA\left(x_1^1\right) = 0$ for $x_1^1 = \tilde{\tilde{x}}_1^1\left(2\right) \approx 270$. Because $W_1\left(x_1^1, \omega_2\right) - zA\left(x_1^1\right)$ is increasing, the system is in crisis ($W_1 < zA$) for $x_1^1 < 270$, in $\omega_2 \in \Omega$.

With restriction to $D\left(x_1^1\right) = [0, \infty)$, $\psi\left(x_1^1, \omega_2\right)$ is the following, and its plot is in Figure C.2.4.2.

$$\psi\left(x_1^1, \omega_2\right) = \begin{cases} \frac{11098}{25} - \frac{1}{2}\exp\left\{\exp\left(-2\right) - 2\right\} - \frac{3}{5}\exp\left(-2\right) \\ -\frac{1}{2}\exp\left\{\exp\left(-x_1^1 - 528\right) - x_1^1 - 528\right\} \\ -\frac{3}{5}\exp\left(-x_1^1 - 528\right) - \frac{31}{100}x_1^1; \\ \text{ if } 0 \leq x_1^1 < 270 \\ \\ \frac{4547}{10} - \frac{1}{2}\exp\left\{\exp\left(-2\right) - 2\right\} - \frac{1}{2}\exp\left(-2\right) \\ -\frac{1}{2}\exp\left\{\exp\left(-x_1^1 - 528\right) - x_1^1 - 528\right\} \\ -\frac{1}{2}\exp\left(-x_1^1 - 528\right) - \frac{7}{20}x_1^1; \\ \text{ if } 270 \leq x_1^1 < \infty \end{cases}$$

Again, using notations of Section 3.2.2, $D\left(x_1^1\right) \cap X_2\left(x_1^1, \omega_2\right) = [0, 270)$

and $D\left(x_1^1\right) \cap X_1\left(x_1^1, \omega_2\right) = [270, \infty)$.

That is, for all $x_1^1 \in D\left(x_1^1\right) = [0, \infty)$, bank 1 is not bankrupt, and the system is in crisis for $0 \leq x_1^1 < 270$.

Altogether, $\Psi\left(x_1^1\right) = E[\psi] = 0.5\psi\left(x_1^1, \omega_1\right) + 0.5\psi\left(x_1^1, \omega_2\right)$ is as follows, and its plot is provided in Figure C.2.4.3.

$$\Psi\left(x_1^1\right) = \begin{cases} \frac{49481}{100} - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{3}{10}\exp(-2) - \frac{1}{4}\exp(-713) \\ -\frac{1}{4}\exp\{\exp(-x_1^1 - 528) - x_1^1 - 528\} \\ -\frac{21}{100}\exp(-x_1^1) - \frac{3}{10}\exp(-x_1^1 - 528) - \frac{37}{100}x_1^1; \\ \text{ if } 0 \leq x_1^1 < 0.5 \\ \\ \frac{24753}{50} - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{3}{10}\exp(-2) - \frac{1}{4}\exp(-713) - \frac{1}{4}\exp\{\exp(-x_1^1) - x_1^1\} \\ -\frac{1}{4}\exp\{\exp(-x_1^1 - 528) - x_1^1 - 528\} \\ -\frac{1}{4}\exp(-x_1^1) - \frac{3}{10}\exp(-x_1^1 - 528) - \frac{33}{100}x_1^1; \\ \text{ if } 0.5 \leq x_1^1 < 270 \\ \\ \frac{10009}{20} - \frac{1}{4}\exp\{\exp(-2) - 2\} - \frac{1}{4}\exp\{\exp(-713) - 713\} \\ -\frac{1}{4}\exp(-2) - \frac{1}{4}\exp(-713) - \frac{1}{4}\exp\{\exp(-x_1^1) - x_1^1\} \\ -\frac{1}{4}\exp\{\exp(-x_1^1 - 528) - x_1^1 - 528\} \\ -\frac{1}{4}\exp(-x_1^1) - \frac{1}{4}\exp(-x_1^1 - 528) - \frac{7}{20}x_1^1; \\ \text{ if } 270 \leq x_1^1 < \infty \end{cases}$$

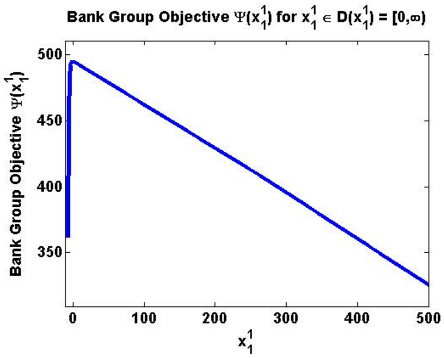

Figure C.2.4.3: $\Psi\left(x_1^1\right)$, the sum of time $t = 0$ utility and $t = 1$ expected utility of an average bank's equityholders, for $x_1^1 \in D\left(x_1^1\right)$

C.2.5 Relationship between Ψ and x_1^2

Consider Ψ as a function of x_1^2 (dollars invested by bank 2 in asset 1) only, with other variables taking values of the optimal solution in Section C.1.

The domain of x_1^2 is $D\left(x_1^2\right) = [0, \infty)$, as determined using the steps of Section 3.1.2.

Again, because the explanations for $\psi\left(x_1^2, \omega_1\right)$ and $\psi\left(x_1^2, \omega_2\right)$ are done as in Section C.2.3, only the plots and expressions will be provided.

The maximum of $\Psi\left(x_1^2\right)$ is reached at $x_1^2 = 0$, which coincides with the value of x_1^2 of the optimal solution.

1. **Function** $\psi\left(x_1^2, \omega_1\right)$

   The time $t = 1$ equity for bank 2 is $w_1^2\left(x_1^2, \omega_1\right) = x_1^2 - \exp\left(-x_1^2 - 713\right) + 713$ and $\tilde{x}_1^2(1) \approx -712.5$ is such that $w_1^2\left(\tilde{x}_1^2(1), \omega_1\right) = 0$. $w_1^2\left(x_1^2, \omega_1\right)$ is increasing, because $\frac{\partial w_1^2}{\partial x_1^2} = 1 + \exp\left(-x_1^2 - 713\right) > 0$. Hence, bank 1 is not bankrupt in scenario $\omega_1 \in \Omega$ for $x_1^2 \geq -712.5$.

   The aggregate level of bank equity in excess of the undercapitalization threshold at time $t = 1$, $W_1\left(x_1^2, \omega_1\right) - zA\left(x_1^2\right) = \frac{2}{5}x_1^2 - \exp\left(-x_1^2 - 713\right) + \frac{371}{5}$, is increasing because its derivative is $\frac{\partial [W_1 - zA]}{\partial x_1^2} = \frac{2}{5} + \exp\left(-x_1^2 - 713\right) > 0$. At $\tilde{\tilde{x}}_1^2(1) \approx -185$, $W_1\left(\tilde{\tilde{x}}_1^2(1), \omega_1\right) - zA\left(\tilde{\tilde{x}}_1^2(1)\right) = 0$. Therefore,

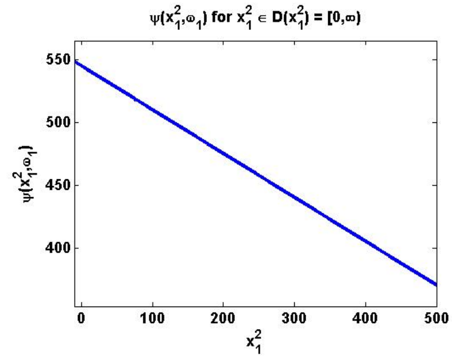

Figure C.2.5.1: $\psi\left(x_1^2, \omega_1\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_1, for $x_1^2 \in D\left(x_1^2\right)$

the system is well-capitalized ($W_1 \geq zA$, not in crisis) for $x_1^2 \geq -185$ in $\omega_1 \in \Omega$.

Using notation from Section 3.2.2, for all $x_1^2 \in D\left(x_1^2\right) = D\left(x_1^2\right) \cap X_1\left(x_1^2, \omega_1\right) = [0, \infty)$, bank 2 is not bankrupt and the system is well-capitalized, so $\psi\left(x_1^2, \omega_1\right)$ takes the following form, and is plotted in Figure C.2.5.1.

$$\begin{aligned}\psi\left(x_1^2, \omega_1\right) = \frac{13632}{25} - \frac{1}{2}\exp\left\{\exp\left(-x_1^2 - 713\right) - x_1^2 - 713\right\} \\ - \frac{1}{2}\exp\left(-x_1^2 - 713\right) - \frac{7}{20}x_1^2\end{aligned}$$

2. **Function** $\psi\left(x_1^2, \omega_2\right)$

The time $t=1$ equity for bank 2 is $w_1^2\left(x_1^2, \omega_2\right) = x_1^2 - \exp\left(-x_1^2 - 2\right) + 2$, and $\tilde{x}_1^2(2) \approx -1.5$ is such that $w_1^2\left(\tilde{x}_1^2(2), \omega_2\right) = 0$. Since $\frac{\partial w_1^2}{\partial x_1^2} = 1 + \exp\left(-x_1^2 - 2\right) > 0$, $w_1^2\left(x_1^2, \omega_2\right)$ is increasing. That is, for $x_1^2 \geq -1.5$, bank 2 is not bankrupt in $\omega_2 \in \Omega$.

The banking system's level of equity in excess of the undercapitalization threshold at time $t=1$ is $W_1\left(x_1^2, \omega_2\right) - zA\left(x_1^2\right) = \frac{2}{5}x_1^2 - \exp(-528) - \exp\left(-x_1^2 - 2\right) - \frac{539}{5}$. This function is increasing since $\frac{\partial\left[W_1 - zA\right]}{\partial x_1^2} = \frac{2}{5} +$

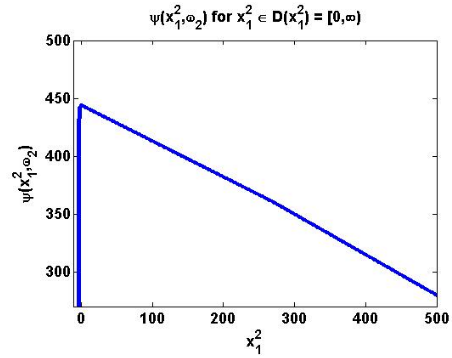

Figure C.2.5.2: $\psi\left(x_1^2, \omega_2\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_2, for $x_1^2 \in D\left(x_1^2\right)$

$\exp\left(-x_1^2-2\right) > 0$. Moreover, $\tilde{\tilde{x}}_1^2(2) \approx 270$ is such that $W_1\left(\tilde{\tilde{x}}_1^2(2), \omega_2\right) - zA\left(\tilde{\tilde{x}}_1^2(2)\right) = 0$. It follows that the system is in crisis ($W_1 < zA$) in scenario $\omega_2 \in \Omega$ for $x_1^2 < 270$.

Therefore, $\psi\left(x_1^2, \omega_2\right)$ is the following, for $x_1^2 \in D\left(x_1^2\right) = [0, \infty)$. The corresponding plot is provided in Figure C.2.5.2.

$$\psi\left(x_1^2, \omega_2\right) = \begin{cases} \frac{11098}{25} - \frac{1}{2}\exp\left\{\exp(-528) - 528\right\} - \frac{3}{5}\exp(-528) \\ -\frac{1}{2}\exp\left\{\exp\left(-x_1^2-2\right) - x_1^2 - 2\right\} \\ -\frac{3}{5}\exp\left(-x_1^2-2\right) - \frac{31}{100}x_1^2; \\ \text{if } 0 \leq x_1^2 < 270 \\ \\ \frac{4547}{10} - \frac{1}{2}\exp\left\{\exp(-528) - 528\right\} - \frac{1}{2}\exp(-528) \\ -\frac{1}{2}\exp\left\{\exp\left(-x_1^2-2\right) - x_1^2 - 2\right\} \\ -\frac{1}{2}\exp\left(-x_1^2-2\right) - \frac{7}{20}x_1^2; \\ \text{if } 270 \leq x_1^2 < \infty \end{cases}$$

With notations from Section 3.2.2, $D\left(x_1^2\right) \cap X_2\left(x_1^2, \omega_2\right) = [0, 270)$ and $D\left(x_1^2\right) \cap X_1\left(x_1^2, \omega_2\right) = [270, \infty)$. In other words, for all $x_1^2 \in D\left(x_1^2\right) = [0, \infty)$, bank 2 is not bankrupt and for $0 \leq x_1^2 < 270$, the system is in crisis for $\omega_2 \in \Omega$.

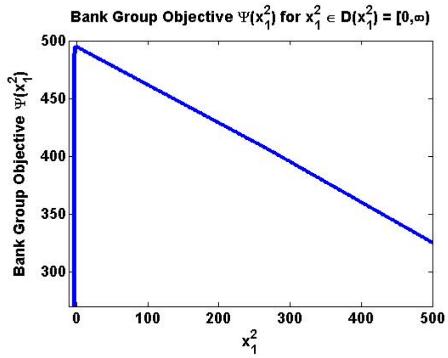

Figure C.2.5.3: $\Psi\left(x_1^2\right)$, the sum of time $t = 0$ utility and $t = 1$ expected utility of an average bank's equityholders, for $x_1^2 \in D\left(x_1^2\right)$

The bank's group objective $\Psi\left(x_1^2\right) = E\left[\psi\right] = 0.5\psi\left(x_1^2, \omega_1\right) + 0.5\psi\left(x_1^2, \omega_2\right)$ is as follows. Its plot is given in Figure C.2.5.3.

$$\Psi\left(x_1^2\right) = \begin{cases} \frac{2473}{5} - \frac{1}{4}\exp\left\{\exp\left(-528\right) - 528\right\} - \frac{3}{10}\exp\left(-528\right) & \\ -\frac{1}{4}\exp\left\{\exp\left(-x_1^2 - 2\right) - x_1^2 - 2\right\} & \\ -\frac{1}{4}\exp\left\{\exp\left(-x_1^2 - 713\right) - x_1^2 - 713\right\} & \\ -\frac{3}{10}\exp\left(-x_1^2 - 2\right) - \frac{1}{4}\exp\left(-x_1^2 - 713\right) & \\ -\frac{33}{100}x_1^2; & \text{if } 0 \leq x_1^2 < 270 \\ & \\ \frac{49999}{100} - \frac{1}{4}\exp\left\{\exp\left(-528\right) - 528\right\} - \frac{1}{4}\exp\left(-528\right) & \\ -\frac{1}{4}\exp\left\{\exp\left(-x_1^2 - 2\right) - x_1^2 - 2\right\} & \\ -\frac{1}{4}\exp\left\{\exp\left(-x_1^2 - 713\right) - x_1^2 - 713\right\} & \\ -\frac{1}{4}\exp\left(-x_1^2 - 2\right) - \frac{1}{4}\exp\left(-x_1^2 - 713\right) & \\ -\frac{7}{20}x_1^2; & \text{if } 270 \leq x_1^2 < \infty \end{cases}$$

C.2.6 Relationship between Ψ and x_2^2

Consider Ψ as a function of x_2^2 (dollars invested by bank 2 in asset 2) only, with other variables taking values of the optimal solution in Section C.1.

By the steps of Section 3.1.2, the domain of x_2^2 is $D\left(x_2^2\right) = [709, \infty)$.

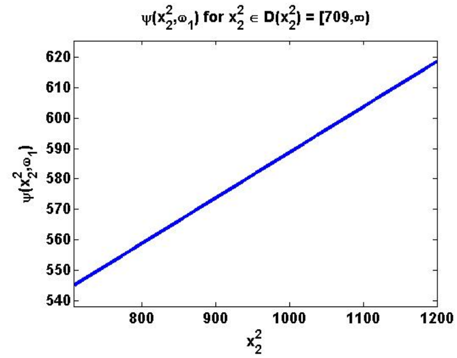

Figure C.2.6.1: $\psi\left(x_2^2, \omega_1\right)$, the sum of time $t = 0$ and $t = 1$ utilities of an average bank's equityholders in scenario ω_1, for $x_2^2 \in D\left(x_2^2\right)$

The explanations about $\psi\left(x_2^2, \omega_1\right)$ and $\psi\left(x_2^2, \omega_2\right)$, which determines how $\Psi = E\left[\psi\right]$ relates with x_2^2, are done similarly as in Section C.2.3. Therefore, only the plots and functional expressions will be provided here.

The maximum of $\Psi\left(x_2^2\right)$ is reached at $x_2^2 = 711$, which is the same as that of the optimal solution.

1. **Function** $\psi\left(x_2^2, \omega_1\right)$

   The time $t = 1$ equity for bank 2 is $w_1^2\left(x_2^2, \omega_1\right) = 2x_2^2 - \exp\left(-2x_2^2 + 709\right) - 709$ and $\tilde{x}_2^2\left(1\right) \approx 354.7$ is such that $w_1^2\left(\tilde{x}_2^2\left(1\right), \omega_1\right) = 0$. Since $\frac{\partial w_1^2}{\partial x_2^2} = 2 + 2\exp\left(-2x_2^2 + 709\right) > 0$, $w_1^2\left(x_2^2, \omega_1\right)$ is increasing. Therefore, for $x_2^2 \geq 354.7$, bank 2 is not bankrupt in scenario $\omega_1 \in \Omega$.

   The banking system's equity level in excess of the undercapitalization threshold at time $t = 1$ is $W_1\left(x_2^2, \omega_1\right) - zA\left(x_2^2\right) = \frac{7}{5}x_2^2 - \exp\left(-2x_2^2 + 709\right) - \frac{4606}{5}$, with derivative $\frac{\partial\left[W_1 - zA\right]}{\partial x_2^2} = \frac{7}{5} + 2\exp\left(-2x_2^2 + 709\right) > 0$. At $\tilde{\tilde{x}}_2^2\left(1\right) \approx 658.3$, $W_1\left(\tilde{\tilde{x}}_2^2\left(1\right), \omega_1\right) - zA\left(x_2^2\right) = 0$. Therefore, the system is not in crisis ($W_1 \geq zA$) for $x_2^2 \geq 658.3$ in $\omega_1 \in \Omega$.

   One can observe that for all $x_2^2 \in D\left(x_2^2\right) = [709, \infty)$, bank 2 is not bankrupt, and there is no systemic crisis. Then, using the notation of Section 3.2.2, for all $x_2^2 \in D\left(x_2^2\right) \cap X_1\left(x_2^2, \omega_1\right) = D\left(x_2^2\right)$,

$$\psi\left(x_2^2,\omega_1\right)=\frac{3}{20}x_2^2-\frac{1}{2}\exp\left\{\exp\left(-2x_2^2+709\right)-2x_2^2-709\right\}$$
$$-\frac{1}{2}\exp\left(-2x_2^2+709\right)+\frac{43863}{100}.$$

The corresponding plot is in Figure C.2.6.1.

2. **Function** $\psi\left(x_2^2,\omega_2\right)$

   The time $t=1$ equity for bank 2 is $w_1^2\left(x_2^2,\omega_2\right)=x_2^2-\exp\left(-x_2^2+709\right)-709$ and $\tilde{x}_2^2\left(2\right)\approx 709.3$ is such that $w_1^2\left(\tilde{x}_2^2\left(2\right),\omega_2\right)=0$. Since $\frac{\partial w_1^2}{\partial x_2^2}=1+\exp\left(-x_2^2+709\right)>0$, $w_1^2\left(x_2^2,\omega_2\right)$ is increasing. It follows that bank 2 is bankrupt in $\omega_2\in\Omega$ for $x_2^2<709.3$.

   The time $t=1$ level of aggregate bank equity in excess of the undercapitalization thershold is $W_1\left(x_2^2,\omega_2\right)-zA\left(x_2^2\right)=\frac{2}{5}x_2^2-\exp\left(-528\right)-\exp\left(-x_2^2+709\right)-\frac{1961}{5}$ with derivative $\frac{\partial\left[W_1-zA\right]}{\partial x_2^2}=\frac{2}{5}+\exp\left(-x_2^2+709\right)>0$. Moreover, at $\tilde{\tilde{x}}_2^2\left(2\right)\approx 981$, $W_1\left(\tilde{\tilde{x}}_2^2\left(2\right),\omega_2\right)-zA\left(\tilde{\tilde{x}}_2^2\left(2\right)\right)=0$. Therefore, the system is in crisis ($W_1<zA$) in $\omega_2\in\Omega$ for $x_2^2<981$.

   Restricting to the domain $D\left(x_2^2\right)=[709,\infty)$, $\psi\left(x_2^2,\omega_2\right)$ is the following, and its plot is given in Figure C.2.6.2.

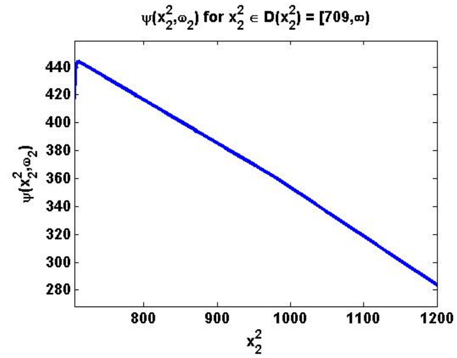

Figure C.2.6.2: $\psi\left(x_2^2,\omega_2\right)$, the sum of time $t=0$ and $t=1$ utilities of an average bank's equityholders in scenario ω_2, for $x_2^2 \in D\left(x_2^2\right)$

$$\psi\left(x_2^2,\omega_2\right)=\begin{cases}\frac{14411}{20}-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{5}\exp\left(-528\right)\\ -\frac{13}{25}\exp\left(-x_2^2+709\right)-\frac{39}{100}x_2^2;\\ \text{ if } 709\leq x_2^2<709.3\\ \\ \frac{66433}{100}-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{5}\exp\left(-528\right)\\ -\frac{1}{2}\exp\left\{\exp\left(-x_2^2+709\right)-x_2^2+709\right\}\\ -\frac{3}{5}\exp\left(-x_2^2+709\right)-\frac{31}{100}x_2^2;\\ \text{ if } 709.3\leq x_2^2<981\\ \\ \frac{14071}{20}-\frac{1}{2}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{1}{2}\exp\left(-528\right)\\ -\frac{1}{2}\exp\left\{\exp\left(-x_2^2+709\right)-x_2^2+709\right\}\\ -\frac{1}{2}\exp\left(-x_2^2+709\right)-\frac{7}{20}x_2^2;\\ \text{ if } 981\leq x_2^2<\infty\end{cases}$$

From the notations of Section 3.2.2, $D\left(x_2^2\right)\cap X_4\left(x_2^2,\omega_2\right)=[709,709.3)$ (bank 2 is bankrupt and the system is in crisis), $D\left(x_2^2\right)\cap X_2\left(x_2^2,\omega_2\right)=[709.3,981)$ (bank 2 is not bankrupt but the system is in crisis), and $D\left(x_2^2\right)\cap X_1\left(x_2^2,\omega_2\right)=[981,\infty)$ (bank 2 is not bankrupt and the system is not in crisis).

Then, the banks' group objective $\Psi\left(x_2^2\right)=E\left[\psi\right]=0.5\psi\left(x_2^2,\omega_1\right)$ $+0.5\psi\left(x_2^2,\omega_2\right)$ is

$$\Psi\left(x_2^2\right)=\begin{cases} \frac{57959}{100}-\frac{1}{4}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{10}\exp\left(-528\right) \\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^2+709\right)-2x_2^2+709\right\} \\ -\frac{13}{50}\exp\left(-x_2^2+709\right)-\frac{1}{4}\exp\left(-2x_2^2+709\right)-\frac{3}{25}x_2^2; \\ \text{ if } 709\leq x_2^2<709.3 \\ \\ \frac{13787}{25}-\frac{1}{4}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{3}{10}\exp\left(-528\right) \\ -\frac{1}{4}\exp\left\{\exp\left(-x_2^2+709\right)-x_2^2+709\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^2+709\right)-2x_2^2+709\right\} \\ -\frac{3}{10}\exp\left(-x_2^2+709\right)-\frac{1}{4}\exp\left(-2x_2^2+709\right)-\frac{2}{25}x_2^2; \\ \text{ if } 709.3\leq x_2^2<981 \\ \\ \frac{57109}{100}-\frac{1}{4}\exp\left\{\exp\left(-528\right)-528\right\}-\frac{1}{4}\exp\left(-528\right) \\ -\frac{1}{4}\exp\left\{\exp\left(-x_2^2+709\right)-x_2^2+709\right\} \\ -\frac{1}{4}\exp\left\{\exp\left(-2x_2^2+709\right)-2x_2^2+709\right\} \\ -\frac{1}{4}\exp\left(-x_2^2+709\right)-\frac{1}{4}\exp\left(-2x_2^2+709\right)-\frac{1}{10}x_2^2; \\ \text{ if } 981\leq x_2^2<\infty \end{cases}.$$

The plot of $\Psi\left(x_2^2\right)$ is in Figure C.2.6.3.

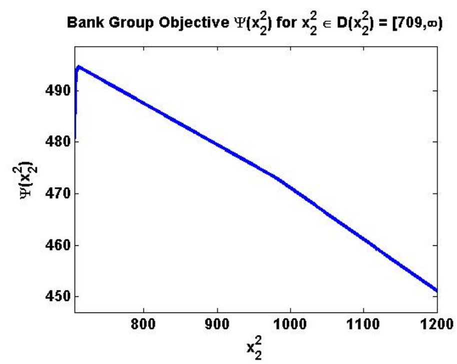

Figure C.2.6.3: $\Psi\left(x_2^2\right)$, the sum of time $t = 0$ utility and $t = 1$ expected utility of an average bank's equityholders, for $x_2^2 \in D\left(x_2^2\right)$