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On the Origins of Gaggle Theory

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Abstract: The *generalized Galois logic* approach (i.e., *gaggle theory*), introduced by Dunn, provides a systematic way to define semantics for many substructural logics in the form of a relational representation of their Lindenbaum algebras. We provide an overview of some conceptual antecedents that we think that likely contributed to the creation of gaggle theory. In our reconstruction, we rely on Dunn's publications and some materials deposited in the Archives of Indiana University, Bloomington, IN, U.S.A.

Keywords: intuitionistic logic, Meyer–Routley semantics, modal logic, possible world semantics, relevance logic, residuation, $\mathbf{R}$-mingle, tense logic

1 Introduction

Dunn published a series of papers in the 1990s, in which he presented *gaggle theory*. The name "gaggle" intends to serve as a convenient pronunciation of the acronym "gGl," which abbreviates *generalized Galois logic*. Gaggle theory is, perhaps, better viewed as an *approach* to substructural and intensional logics rather than a motley collection of definitions and theorems. As a first approximation, gaggle theory aims to bring under a common theoretical umbrella the ways in which concrete set-theoretical semantics are defined for a range of logics that exceed 2-valued logic ($\mathbf{TV}$) in some way. A slightly more precise description would mention two steps in this process. First, the Lindenbaum algebra of a logic is formed; second, that algebra is represented using a relational structure in the sense of algebraic representation theory. Moreover, gaggle theory does not simply amount to an aggregation of set-theoretic constructions for various logics. It *generalizes* existing semantics and furnishes new semantics for logics in *a systematic way* based on the algebraic properties of the logics.

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This paper traces the *emergence of gaggle theory* in Dunn's work to the late 1970s, and points at some set-theoretical semantics that probably contributed to the formulation of the theory. The semantics we mention are—in the order of their appearance—Kripke-style semantics for modal and tense logics, **BAO**'s, Kripke's semantics for intuitionistic logic, Dunn's semantics for $\mathbf{R}$-mingle ($\mathbf{RM}$) and the Meyer–Routley semantics for relevance logics.

2 Some semantics as motivations for gaggle theory

2.1 Semantics for some normal modal logics

Kripke's semantics for some *normal modal logics* was first described in Kripke (1959), and then in Kripke (1963). This set-theoretical semantics is widely known now, and it was surely known in the 1960s by Dunn whose Ph.D. thesis supervisor was Nuel D. Belnap. Alan R. Anderson, Belnap and Kripke corresponded in the late 1950s. Indeed, Belnap pointed out to Kripke the decidability problem of $\mathbf{E}_{\rightarrow}$ in a letter dated May 31st, 1959, which Kripke solved within a few months.[2]

Let us consider the modal logic $\mathbf{S4}$ to illustrate an idea and a puzzlement.[3] The language of $\mathbf{S4}$ contains a denumerable sequence of sentence letters $\langle p_i \rangle_{i \in \omega}$. Formulas are generated by $\neg$ (negation), $\supset$ (conditional) and $\Box$ (necessity) as usual, and $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ range over formulas. An axiomatic system for $\mathbf{S4}$ may be defined by adding to an axiomatization of 2-valued logic (with detachment as a rule) the following axioms and rule

(K) $\Box(\mathcal{A} \supset \mathcal{B}) \supset (\Box\mathcal{A} \supset \Box\mathcal{B})$ (T) $\Box\mathcal{A} \supset \mathcal{A}$ (4) $\Box\mathcal{A} \supset \Box\Box\mathcal{A}$

(nec) $\vdash \mathcal{A}$ implies $\vdash \Box\mathcal{A}$

The notions of a *proof* and a *theorem* are defined as usual. We limit our considerations to this simple notion of consequence in this example.

A possible world semantics for $\mathbf{S4}$ is based on a *structure*, which is a pre-ordered (or possibly, weakly partially ordered) non-empty set of worlds, $\mathfrak{F} = \langle W, R \rangle$. ($R \subseteq W \times W$, and it is a reflexive and transitive (and possibly, anti-symmetric) relation.) A *model* $\mathfrak{M}$ adds a valuation v that assigns a set of worlds to each p, that is, $v(p) \subseteq W$. The meaning of $w \in v(p)$ is that p

[2]Belnap's letter is preserved within Kripke's correspondence in the Kripke Archives; cf. Bimbó (2020).

[3]We harmonize the notation in our presentation of ideas from several semantics, somewhat along the lines of Bimbó and Dunn (2008). Accordingly, we might not follow the notation or the terminology of the publications we refer to.

is true in the world w. The interpretation of all the formulas is given by an extension of v, which we denote by $[\![\ ]\!]$ (omitting decorations to indicate the model, which is fixed by the context). To start with, $[\![p]\!] = \{\, w \colon w \in v(p)\,\}$.
1. $[\![\neg\mathcal{A}]\!] = W \setminus [\![\mathcal{A}]\!]$ 2. $[\![\mathcal{A} \supset \mathcal{B}]\!] = (W \setminus [\![\mathcal{A}]\!]) \cup [\![\mathcal{B}]\!]$
3. $[\![\Box\mathcal{A}]\!] = \{\, w \colon \forall w'(Rww' \Rightarrow w' \in [\![\mathcal{A}]\!])\,\}$

The notions of $\mathcal{A}$ being *true at* w (i.e., $w \in [\![\mathcal{A}]\!]$), $\mathcal{A}$ being *valid* in $\mathfrak{M}$ ($\forall w\, w \in [\![\mathcal{A}]\!]$), and $\mathcal{A}$ being *valid in a class of models* $\mathbb{C}$ ($\forall\mathfrak{M} \in \mathbb{C}$, $\mathcal{A}$ is valid in $\mathfrak{M}$) are defined as usual. We may write $\vDash_{\mathfrak{F}} \mathcal{A}$, as customary, to indicate validity in $\mathbb{C}$, where $\mathbb{C}$ is the class of models on $\mathfrak{F}$.

Theorem 1 *For any formula $\mathcal{A}$, $\vdash_{\mathbf{S4}} \mathcal{A}$ iff $\vDash_{\mathfrak{F}} \mathcal{A}$, where $\mathfrak{F}$ is as above.*

The proof of this soundness and completeness theorem is fairly routine, and there are multiple published versions of it. Instead of spelling out the details, we note that the definition of $\Diamond$ (possibility) is straightforward. $\neg\Box\neg\mathcal{A}$ expresses that $\mathcal{A}$ is possible, because it is not that not-$\mathcal{A}$ is necessary. Having worked through the definition of the truth of $\neg\Box\neg\mathcal{A}$ step by step, we obtain the following clause.
4. $[\![\Diamond\mathcal{A}]\!] = \{\, w \colon \exists w'(Rww' \wedge w' \in [\![\mathcal{A}]\!])\,\}$

This means that having both $\Box$ and $\Diamond$ when they are *definable* via $\neg$ is unproblematic in the semantics of normal modal logic—in the sense that a *single accessibility relation* is sufficient to model these connectives. We may note that 3 and 4 are neatly in line with Leibnitz's readings of modalities through $\forall$ and $\exists$, although he seems to have assumed that all worlds are accessible from any world (cf. Look (2016)).

Tense logics, more precisely some versions of them, introduce modality-like operators for reasoning about the *past* and the *future*. A minimal tense logic, denoted here by $\mathbf{K}_t$, has two $\Diamond$-type connectives, namely, P (sometime in the past) and F (sometime in the future). Their $\Box$-like duals are H (always in the past) and G (always in the future). The tense axioms are two instances of (K) and two instances of (4) with H and G, respectively, and a pair of axioms (B$\downarrow$) and (B$\uparrow$) that tie the future and the past together, so to speak. These axioms are analogs of the axiom (B) $\mathcal{A} \supset \Box\Diamond\mathcal{A}$, which is a theorem of the normal modal logics $\mathbf{B}$ and $\mathbf{S5}$.

(B$\downarrow$) $\mathsf{PG}\mathcal{A} \supset \mathcal{A}$ (B$\uparrow$) $\mathsf{FH}\mathcal{A} \supset \mathcal{A}$

Finally, the (nec) rule is stipulated for both G and H. The (4) axioms will force the transitivity of the accessibility relation moving into the future and moving into the past. However, what is interesting from our point of view is that (B$\downarrow$) and (B$\uparrow$) create a very close relationship between R_{G} and R_{H}.

In particular, $R_{\mathsf{G}} = R_{\mathsf{H}}^{\smile}$, that is, the accessibility relations are each other's *converses*. Then, we may omit the subscripts and state the satisfiability conditions for G and H—reusing a previous clause for $\Box$—as follows.

5. $[\![\mathsf{G}\mathcal{A}]\!] = \{\, w\colon \forall w'(Rww' \Rightarrow w' \in [\![\mathcal{A}]\!])\,\}$
6. $[\![\mathsf{H}\mathcal{A}]\!] = \{\, w\colon \forall w'(Rw'w \Rightarrow w' \in [\![\mathcal{A}]\!])\,\}$

Rww' means informally that w' is in the future of w. If we swap the arguments as in $Rw'w$, then w' is in the past of w.[4] From the point of view of gaggle theory, it is interesting that G and H are *not definable* from each other in the context of $\mathbf{K}_t$, yet they are modeled from the same accessibility relation. However, F and H as well as P and G are each other's *residuals*. For example, for P and G, this means that $\mathsf{P}\mathcal{A} \supset \mathcal{B}$ is a theorem of $\mathbf{K}_t$ iff $\mathcal{A} \supset \mathsf{G}\mathcal{B}$ is a theorem.

2.2 Boolean algebras with operators

Boolean algebras are important beyond their basic role as the class of algebras into which the Lindenbaum algebra of $\mathbf{TV}$ falls. For instance, elementary probability theory adds a countably additive normal function on the event space and relation algebras add further operations such as relational composition and converse. Jónsson and Tarski (1951–52) introduced a lesser representation for relation algebras as part of a general representation theory for $\mathbf{BAO}$'s. The preferred representation of a relation algebra is by binary relations, whereas their focus is on representing the operations on binary relations (the elements of a relation algebra) by relations of appropriate (i.e., one larger) arity. For example, the composition of a pair of relations S_1 and S_2, usually denoted by $S_1; S_2$, is represented by a three-place relation, $R^3_{;}$.

Definition 1 $\mathfrak{A} = \langle A; -, \vee, o_{i\in I}^{n_i}\rangle$ *is a* Boolean algebra with operators *(a* $\mathbf{BAO}$*) when the equations* (a1)–(a5$_i$) *hold* ($\forall i \in I$ *and* $\forall j\; 1 \leq j \leq n_i$). ($\bot$ *abbreviates* $-(a \vee -a)$*, and* $o_i(\vec{a}, [\;\;]_j)$ *indicates that the* j*th argument has been singled out, while* $\vec{a}$ *fills the other argument places.* $a, b, c, \ldots \in A$.)

(a1) $a \vee b = b \vee a$ (a2) $(a \vee b) \vee c = a \vee (b \vee c)$
(a3) $-(-a \vee -b) \vee -(-a \vee --b) = a$
(a4$_i$) $o_i(\vec{a}, [b \vee c]_j) = o_i(\vec{a}, [b]_j) \vee o_i(\vec{a}, [c]_j)$
(a5$_i$) $o_i(\vec{a}, [\bot]_j) = \bot$

[4]We stress that these are intuitive renderings only. For instance, Rww reads as w is in its own future and in its own past.

Any operation satisfying (a4_i) is *additive* in each argument, and with (a5_i) added, o_i is *normal*. Of course, in concrete cases, the operators of a $\mathbf{BAO}$ may interact with each other or may have further properties. For a concrete example, we take an operator $\circ$ that is binary and satisfies two inequations (a6) $(a \circ b) \circ c \leq a \circ (b \circ c)$ and (a7) $(a \circ b) \circ c \leq b \circ (a \circ c)$ (which could be written as equations in a $\mathbf{BAO}$). We denote this sample $\mathbf{BAO}$ as $\mathfrak{A}^{\circ}$.

Definition 2 *A* structure *is* $\mathfrak{F} = \langle W, R^3_\circ \rangle$*, and a* model *is* $\mathfrak{M} = \langle W, R^3_\circ, v \rangle$*, where* $W \neq \emptyset$*,* $R^3_\circ \subseteq W^3$*,* $v(a) \subseteq W$*,* $v(a) = [\![a]\!]$ *and* (f1)–(m3) *hold.*

(f1) $\forall w_1, w_2, w_3, w_4 \big(\exists w_5\, (Rw_1w_2w_5 \wedge Rw_5w_3w_4) \Rightarrow \exists w_5\, (Rw_2w_3w_5 \wedge Rw_1w_5w_4)\big)$

(f2) $\forall w_1, w_2, w_3, w_4 \big(\exists w_5\, (Rw_1w_2w_5 \wedge Rw_5w_3w_4) \Rightarrow \exists w_5\, (Rw_1w_3w_5 \wedge Rw_2w_5w_4)\big)$[5]

(m1) $[\![-a]\!] = W \setminus [\![a]\!]$ (m2) $[\![a \vee b]\!] = [\![a]\!] \cup [\![b]\!]$

(m3) $[\![a \circ b]\!] = \{\, w_3 \colon \exists w_1, w_2\, (Rw_1w_2w_3 \wedge w_1 \in [\![a]\!] \wedge w_2 \in [\![b]\!])\,\}$

What we have so far will only guarantee the existence of a *homomorphic representation* for $\mathfrak{A}^{\circ}$. Thus, we accumulate some further notions.

Definition 3 *Let* $\mathfrak{A}^{\circ} = \langle A; -, \vee, \circ \rangle$ *be the* $\mathbf{BAO}$ *above.* $U \subseteq A$ *is an* ultrafilter *if (i)* $a, b \in U$ *iff* $a \wedge b \in U$ *(i.e.,* U *is a filter) and (ii)* $-a \in U$ *iff* $a \notin U$*. The set of ultrafilters is denoted by* $\mathcal{U}$*. (iii)* $R^3_\circ u_1u_2u_3$ *iff* $\forall a_1, a_2((a_1 \in u_1 \wedge a_2 \in u_2) \Rightarrow a_1 \circ a_2 \in u_3)$ *(where the* u*'s are from* $\mathcal{U}$*).*

We only state the following lemmas and a theorem, which are well known, and their proofs are easy or may be found in various publications.

Lemma 1 *For any* $a, b \in A \setminus \{\, \bot \,\}$ *in a* $\mathbf{BAO}$*, if* $a \nleq b$*, then* there are $u_a, u_b \in \mathcal{U}$ *such that* $a \in u_a$*,* $b \notin u_a$ *and* $b \in u_b$*.*

Lemma 2 *Let* $R'_\circ$ *be defined as* $R_\circ$ *above, with* $u_1, u_2 \in \mathcal{F}$ *(where* $\mathcal{F}$ *is the set of proper filters). If* $R'_\circ u_1u_2u_3$ *holds, then* there are u'_1, u'_2 *such that* $u_1 \subseteq u'_1$*,* $u_2 \subseteq u'_2$ *and* $R_\circ u'_1u'_2u_3$ *(where the* u'*'s are from* $\mathcal{U}$*).*

Lemma 3 *If* $\mathfrak{A}^{\circ}$ *is a* $\mathbf{BAO}$ *as above, then* $R_\circ$ satisfies (f1) *and* (f2)*.*

Theorem 2 *Let* $\mathfrak{A}^{\circ} = \langle A; -, \vee, \circ \rangle$ *be a* $\mathbf{BAO}$ *as above, and let* $h(a) = \{\, U \in \mathcal{U} \colon a \in U \,\}$*.* $h[A]$ *is a* concrete $\mathbf{BAO}$ *with the operations defined as in* (m1)–(m3) *that is* isomorphic *to* $\mathfrak{A}^{\circ}$*.*

[5] We could have written (f1) and (f2) using usual notation for composition of R^3. (f1) would turn into $R(w_1w_2)w_3w_4 \Rightarrow Rw_1(w_2w_3)w_4$ with the universal closure tacitly assumed.

This isomorphic representation theorem may be viewed as a *completeness theorem* for a logic that has a binary fusion connective of a certain kind on the basis of $\mathbf{TV}$. The homomorphic representation theorem similarly parallels the *soundness theorem* for a logic.

The limitation to operators may appear a drastic restriction even if the operators may have further properties. But it is not, because $\mathbf{BA}$'s are overly abundant in definable operations. For example, $\Diamond$ which is a unary operator in the Lindenbaum algebra of a normal modal logic, allows one to define $\Box$, but also $\neg\Diamond$ (impossible) and $\Diamond\neg$ (possibly not). Obviously, every operation that can be expressed by a contextual definition in a $\mathbf{BAO}$ falls under the scope of Theorem 2. However, we have seen in §2.1 that there are operators (or $\Diamond$-like connectives) that cannot be defined from each other on the basis of a $\mathbf{BA}$, yet they can be modeled from one relation by switching arguments. $\mathbf{BAO}$'s are an archetypical example, where a wide range of operations can be captured by a sole representation theorem. We note that all the operations have *distribution types* and *respect the bounds* too—in the terminology of gaggle theory. (Definitions of these and related notions for different kinds of gaggles may be found in Bimbó and Dunn (2008). See e.g., Definitions 1.3.2, 1.3.18, 2.4.1 and 4.3.13.) At the same time, $\mathbf{BAO}$'s are an example where the potential interactions of operations (e.g., through residuation) are not fully exploited in the representation, and in this sense, $\mathbf{BAO}$'s are *not* completely general.

2.3 Semantics for intuitionistic logic

Intuitionistic logic ($\mathbf{J}$) differs from normal modal logics and $\mathbf{BAO}$'s, because its Lindenbaum algebra does not have a $\mathbf{BA}$ reduct, or in other words, $\mathbf{J}$ is not simply an extension of $\mathbf{TV}$. We only mention one of the several interpretations that have been introduced for $\mathbf{J}$, namely, the semantics that originated in Kripke (1965). We assume that the reader is familiar with some formalization of propositional $\mathbf{J}$—as an axiomatic system, a sequent calculus, a tableau system or such.

A *frame* (a structure) for $\mathbf{J}$ is $\mathfrak{F} = \langle U, \sqsubseteq \rangle$, where $U \neq \emptyset$, $\sqsubseteq\, \subseteq U^2$ and $\sqsubseteq$ is a weak partial order. A *model* is $\mathfrak{M} = \langle U, \sqsubseteq, v \rangle$, where $v(p) = X$ and $X \in \mathcal{P}(U)^{\uparrow}$. ($X \in \mathcal{P}(U)^{\uparrow}$ iff $u' \in X$, whenever $u \sqsubseteq u'$ and $u \in X$.) That is, a propositional variable is mapped by v into a *cone of situations*. Formulas are interpreted by extending v according to clauses (j1)–(j5).

(j1) $[\![\mathcal{A} \wedge \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cap [\![\mathcal{B}]\!]$ (j2) $[\![\mathcal{A} \vee \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cup [\![\mathcal{B}]\!]$

(j3) $[\![\mathcal{A} \to \mathcal{B}]\!] = \{\, u \colon \forall u'(u \sqsubseteq u' \Rightarrow (u' \notin [\![\mathcal{A}]\!] \vee u' \in [\![\mathcal{B}]\!]))\,\}$

(j4) $[\![\neg\mathcal{A}]\!] = \{\, u \colon \forall u'(u \sqsubseteq u' \Rightarrow u' \notin [\![\mathcal{A}]\!]) \,\}$ (j5) $[\![\bot]\!] = \emptyset$

$\mathcal{A}$ is *true at* u in $\mathfrak{M}$ if $u \in [\![\mathcal{A}]\!]$; $\mathcal{A}$ is *true in* $\mathfrak{M}$, when $U \subseteq [\![\mathcal{A}]\!]$. Lastly, $\mathcal{A}$ is *valid*, if it is true in all models on all frames for $\mathbf{J}$.

The relationship between $\neg\mathcal{A}$ and $\mathcal{A} \to \bot$ is quite clear semantically (and it matches the syntactic definition of $\neg\mathcal{A}$). The crucial clause is (j3), which views $\to_{\mathbf{J}}$ *almost* as $\supset$, but only in the set of situations that are accessible from the current situation. To facilitate comparison with (j3), we may rewrite 2, the condition for $\supset$, as $[\![\mathcal{A} \supset \mathcal{B}]\!] = \{\, w \colon w \notin [\![\mathcal{A}]\!] \vee w \in [\![\mathcal{B}]\!] \,\}$. Of course, it is well known that $\to_{\mathbf{J}}$ is very close to $\supset$. The implicational theorems of $\mathbf{TV}$ that go beyond the implicational theorems of $\mathbf{J}$ (after the $\to_{\mathbf{J}}$'s are rewritten into $\supset$'s) are not principal simple type schemas of proper combinators.[6] Or we may note that the sequent calculus $L\mathbf{J}$ results from $L\mathbf{K}$ by an uncomplicated structural restriction.

Nevertheless, we may observe that the pattern in the possible world semantics for normal modal logics, which is explicit and general in the representation of $\mathbf{BAO}$'s is infringed upon by (j3). We have a binary sentential connective $\to_{\mathbf{J}}$, but we do not have a ternary relation. Dunn (1995) provided a semantics for $\mathbf{J}$ along the lines of gaggle theory without taking into consideration potential simplifications. Before recalling how to move back and forth between the two types of semantics, we state the soundness and completeness theorem for the semantics outlined.

Theorem 3 *A formula $\mathcal{A}$ is a theorem of* $\mathbf{J}$ *iff $\mathcal{A}$ is* valid *on all models.*

We will not give a proof of this theorem here; rather, we sketch the components of the canonical model that would be used for showing the "if" direction of the claim; they also figure into Dunn's completeness theorem.

The Lindenbaum algebra of $\mathbf{J}$ is a residuated distributive lattice with bottom, and it does not need to be a $\mathbf{BA}$. Accordingly, ultrafilters (or equivalently maximally consistent sets of sentences) cannot be used in the representation of such a lattice (or in a model of $\mathbf{J}$). The *canonical frame* is $\mathfrak{F}_c = \langle \mathcal{P}, \subseteq \rangle$, where $\mathcal{P}$ is the set of (proper) prime filters. A prime filter P satisfies (i) from Definition 3, (iv) $a, b \in P$ iff $a \vee b \in P$ and (v) $P \neq A$. $\subseteq$ is set inclusion (the prototypical partial order), which may hold between distinct prime filters. The *canonical valuation* is defined as $v(p) = \{\, P \in \mathcal{P} \colon [\, p \,] \in P \,\}$. We can decipher all the P's by saying that $v(p)$ is the set of prime filters, in which the equivalence class of p is an element. The following lemmas are helpful in the proof of the completeness theorem.

[6] This observation of H. B. Curry is well known; see, e.g., Hindley (1997), Hindley and Seldin (2008) and Bimbó (2012).

Lemma 4 *For any* $[\mathcal{A}], [\mathcal{B}]$ *in the Lindenbaum algebra of* $\mathbf{J}$*, if* $[\mathcal{B}] \neq [\bot]$ *and* $\mathcal{A} \to \mathcal{B}$ *is not a theorem of* $\mathbf{J}$*, then* there are $P_a, P_b \in \mathcal{P}$ *such that* $[\mathcal{A}] \in P_a$*,* $[\mathcal{B}] \notin P_a$ *and* $[\mathcal{B}] \in P_b$*. Therefore,* $P_b \nsubseteq P_a$*.*

Lemma 5 *For any formula* $\mathcal{A}$ *and prime filter* P*,* $P \in [\![\mathcal{A}]\!]$ *iff* $[\mathcal{A}] \in P$*.*

We briefly recall from Dunn (1995) how a ternary relational semantics for $\mathbf{J}$ is obtained. A Heyting algebra (exemplified by the algebra of $\mathbf{J}$) is residuated, where $\to$ is a residual (indeed, *the* residual) of $\wedge$. The truth condition for $\to$—using a ternary relation—is (j6), and that for $\wedge$ is (j7).

(j6) $[\![\mathcal{A} \to \mathcal{B}]\!] = \{\, u \colon \forall u', u''((Ruu'u'' \wedge u' \in [\![\mathcal{A}]\!]) \Rightarrow u'' \in [\![\mathcal{B}]\!]) \,\}$
(j7) $[\![\mathcal{A} \wedge \mathcal{B}]\!] = \{\, u'' \colon \exists u, u'(Ruu'u'' \wedge u \in [\![\mathcal{A}]\!] \wedge u' \in [\![\mathcal{B}]\!]) \,\}$

The latter clause suggests that u'' should be a superset of both u and u' on the analogy of $[a), [b) \subseteq [a \wedge b)$. Thus, the ternary relation is defined as $Ruu'u''$ iff $u \sqsubseteq u''$ and $u' \sqsubseteq u''$. Looking at $\wedge$ once more, the identity element is $[\top]$ (i.e., $[\neg\bot]$), which is an element of every prime filter. Thus, a ternary relational frame for $\mathbf{J}$ is $\mathfrak{F} = \langle U, \sqsubseteq, I, R \rangle$, where $U \neq \emptyset$, $I = U$ and $Ruu'u''$ is defined from the pre-order $\sqsubseteq$ as above. It is easy to see that the truth conditions (j1) and (j7) are equivalent. We quickly run through the proof that (j3) and (j6) are equivalent too. If $u \in [\![\mathcal{A} \to \mathcal{B}]\!]$, and also $Ruu'u''$ and $u' \in [\![\mathcal{A}]\!]$, then by $u' \sqsubseteq u''$, $u'' \in [\![\mathcal{A}]\!]$ follows, because propositions are cones of situations. But $u \sqsubseteq u''$ and $u'' \in [\![\mathcal{A}]\!]$ imply, by (j3), that $u'' \in [\![\mathcal{B}]\!]$, that is, (j6) holds. Now, if we assume $u \in [\![\mathcal{A} \to \mathcal{B}]\!]$, and $u \sqsubseteq u'$ and $u' \in [\![\mathcal{A}]\!]$, then using $u' \sqsubseteq u'$, we have that $Ruu'u'$, and by (j6), $u' \in [\![\mathcal{B}]\!]$.

The ternary modeling may seem only to complicate things. However, that $\to$ is the residual of conjunction $\wedge$ explains the properties of $\to$, and in turn, the properties of R. The residuation between $\wedge$ and $\to$ also implies that all the theorems are equivalent, hence, any and all of them are typified by $\top$.

2.4 Semantics for R-mingle

The logic $\mathbf{RM}$ is obtained from $\mathbf{R}$ by adding the mingle axiom (cf. Anderson, Belnap, and Dunn (1992, §R)). $\mathbf{RM}$ was introduced by Dunn adapting a suggestion of S. McCall (see Dunn (2021)). This logic is often called *semi-relevant* because it has theorems of the form $\mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ is the negation of a theorem and $\mathcal{B}$ is a theorem. Two instances of a theorem with disjoint sets of propositional variables easily let us create a theorem with implication as its main connective, but no variable partaking in both the antecedent and the consequent. If the variable sharing property is taken to be the hallmark of a relevance logic, then $\mathbf{RM}$ falls short, because it only satisfies the *weak relevance principle* (cf. Anderson and Belnap (1975, §29.4)). However, $\mathbf{R}$-mingle has many pleasant features; in particular, it has a linearly ordered infinite characteristic matrix. The only (non-trivial) linearly ordered $\mathbf{BA}$ is $\mathbf{2}$, the two-element Boolean algebra, and residuated distributive lattices with a least element do not need to be linearly ordered. The semantics that Dunn designed for $\mathbf{RM}$ toward the end of the 1960s (cf. Dunn (1976b)) seems to follow closely Kripke's terminology and notation; however, those similarities turn out to be quite superficial. Some of the novel properties of Dunn's semantics include: (1) The semantic uses a *generated model* (in the contemporary sense of the term in the modal logic literature). (2) The semantics is 3-*valued*, moreover, the three truth values are $\{\,T\,\}$, $\{\,F\,\}$ and $\{\,T, F\,\}$, that is, the non-empty subsets of the "usual" set of truth values. (3) The semantic uses a *distinguished situation*—like Kripke's semantics, but the distinguished situation cannot be an arbitrary situation—unlike in Kripke's semantics. (4) The frame is *linearly ordered*, which is not stipulated in the semantics for normal modal logics in general or in the semantics for $\mathbf{J}$.

Definition 4 *A* frame for $\mathbf{RM}$ *is* $\mathfrak{F} = \langle U, \iota, \leq \rangle$*, where* $\iota \in U$*,* $\leq\, \subseteq U^2$ *and* $\leq$ *is reflexive, transitive and connected. For ease of use,* $\leq$ *is stipulated to be anti-symmetric with* ι *being the least element in* U*. A* model *is* $\mathfrak{M} = \langle U, \iota, \leq, v \rangle$*, where* $v \colon \mathbb{P} \times U \longrightarrow \{\, \{T\}, \{F\}, \{T, F\}\,\}$ *satisfying* hereditariness*, that is,* (h) *if* $u \leq u'$*, then* $v(p, u) \subseteq v(p, u')$*.* v *is extended to compound formulas according to* (1)–(4) *(below).*

The condition (h) is stipulated for $p \in \mathbb{P}$ (i.e., propositional variables). To contrast this with the condition in a model for $\mathbf{J}$, we express the former from page 24 in a way similar to (h). Thus, ($\mathrm{h}_\mathbf{J}$) says that if $u \sqsubseteq u'$ then $u \in v(p)$ implies $u' \in v(p)$, or in other words, if $u \sqsubseteq u'$ then $v'(p, u) = \{\,T\,\}$ implies $v'(p, u') = \{\,T\,\}$ (where using v', we transformed v into a binary function in an obvious way). In the semantics of $\mathbf{J}$, $\{\,T, F\,\}$ cannot be the value for any p in any situation; hence, $v'(p, u') = \{\,T, F\,\}$ in the consequent is not possible. Essentially for the same reason, $v'(p, u') = \{\,F\,\}$ is not stipulated when $u \sqsubseteq u'$ and $v'(p, u) = \{\,F\,\}$. To put it concisely, for $\mathbf{J}$, the perpetuation of truth is required along the accessibility relation, whereas for $\mathbf{RM}$ both truth and falsity are upheld moving forward along the linear order.

The view of the three truth values as sets allows for a straightforward extension of v as follows.

(1) $T \in v({\sim}\mathcal{A}, u)$ iff $F \in v(\mathcal{A}, u)$;

$F \in v(\sim\mathcal{A}, u)$ iff $T \in v(\mathcal{A}, u)$;

(2) $T \in v(\mathcal{A} \wedge \mathcal{B}, u)$ iff $T \in v(\mathcal{A}, u)$ and $T \in v(\mathcal{B}, u)$;
$F \in v(\mathcal{A} \wedge \mathcal{B}, u)$ iff $F \in v(\mathcal{A}, u)$ or $F \in v(\mathcal{B}, u)$;

(3) $T \in v(\mathcal{A} \vee \mathcal{B}, u)$ iff $T \in v(\mathcal{A}, u)$ or $T \in v(\mathcal{B}, u)$;
$F \in v(\mathcal{A} \vee \mathcal{B}, u)$ iff $F \in v(\mathcal{A}, u)$ and $F \in v(\mathcal{B}, u)$;

(4) $T \in v(\mathcal{A} \to \mathcal{B}, u)$ iff $\forall u'(u \leq u' \Rightarrow$
$(T \in v(\mathcal{A}, u') \Rightarrow T \in v(\mathcal{B}, u') . \wedge . F \in v(\mathcal{B}, u') \Rightarrow F \in v(\mathcal{A}, u')))$;
$F \in v(\mathcal{A} \to \mathcal{B}, u)$ iff $T \notin v(\mathcal{A} \to \mathcal{B}, u)$ or
$T \in v(\mathcal{A}, u)$ and $F \in v(\mathcal{B}, u)$.

First, we note that the T and F clauses are *independent* from each other, and even in the case of the extensional connectives ($\sim$, $\wedge$ and $\vee$), neither line may be omitted. Second, the T clause for $\to$ starts similarly to the $\to$ clause in **J**, but here the falsity of the consequent must imply the falsity of the antecedent too. It may be also notable that only the T condition for $\to$ can shift the evaluation to a new situation.

The *truth of* $\mathcal{A}$ at a situation u means that the value of the formula is $\{ T \}$ or $\{ T, F \}$, that is, $T \in v(\mathcal{A}, u)$. *Truth in a model* means truth at ι, and *validity* obtains when a formula is true in all models. Dunn (1976b) proved the following soundness and completeness theorems.

Theorem 4 *If $\mathcal{A}$ is a theorem of* $\mathbf{RM}$*, then $\mathcal{A}$ is* valid*, and* vice versa*.*

We outline the canonical model and point out some of its properties. The *canonical frame* is defined with respect to a prime theory T_0 (or in the Lindenbaum algebra, a prime filter) that contains every theorem of $\mathbf{RM}$. U_c is the set of all prime theories that extend T_0 (and contain all $\mathbf{RM}$ theorems). T_0 is ι_c and $\leq_c$ is $\subseteq$. Obviously, $\langle U_c, \iota_c, \leq_c \rangle$ is a frame for $\mathbf{RM}$. (Connectedness follows from the chain theorem $(\mathcal{A} \to \mathcal{B}) \vee (\mathcal{B} \to \mathcal{A})$.) The canonical valuation is defined by a conjunctive condition, namely, $T \in v_c(p, u)$ iff $p \in u$, and $F \in v_c(p, u)$ iff $\sim p \in u$.

The selection of a prime theory ι_c that contains all the theorems of $\mathbf{RM}$ is motivated by the definition of the frame. But it also shows that in $\mathbf{RM}$—as in relevance logics typically—not all theorems are equal, which means that the top element of the Lindenbaum algebra of the logic (if there is one) does not stand for (or implies) all theorems.

Now, we turn to sketching another semantics for $\mathbf{RM}$, which is a step closer to what would result from an application of gaggle theory, and more in line with the Meyer–Routley semantics (though we diverge from the usual presentation of the latter). A *frame* is $\mathfrak{F} = \langle U, \iota, R, {}^* \rangle$, where $\iota \in U$, ${}^* \colon U \longrightarrow U$, $u^{**} = u$, $Ru'us \Rightarrow Ru's^*u^*$ and R satisfies (f3)–(f8).

(f3) $R(uu')ss' \Leftrightarrow Ru'(us)s'$; (f4) $R(uu')ss' \Rightarrow R(us)u's'$;
(f5) $R\iota uu$; (f6) $(R\iota u'u \wedge R\iota s's \wedge R\iota u''s'' \wedge Rusu'') \Rightarrow Ru's's''$;
(f7) $Russ' \Rightarrow R(us)ss'$; (f8) $Ruu's \Rightarrow (R\iota us \vee R\iota u's)$.

A pre-order relation can be recovered as $u \sqsubseteq u'$ iff $R\iota uu'$. Then the last condition (which "matches" the mingle axiom) may be written as $Ruu's \Rightarrow (u \sqsubseteq s \vee u' \sqsubseteq s)$. We note that taking a single logical situation, namely, ι is justified by the fact that the Lindenbaum algebra of $\mathbf{RM}$ is linearly ordered. A model is obtained by adding v, which maps p into a cone of situations (with respect to $\sqsubseteq$). It should be immediately clear that this semantics is 2-valued, because we have not mentioned any truth values. v is extended to compound formulas by intersection and union for $\wedge$ and $\vee$, respectively. $\mathcal{A} \to \mathcal{B}$ is evaluated as in (j6) (with $\to$ taken to be the implication of $\mathbf{RM}$). The remaining clauses are (m4) and (m5).

(m4) $[\![\boldsymbol{t}]\!] = [\, \iota)$ (m5) $[\![{\sim}\mathcal{A}]\!] = \{\, u \colon u^* \notin [\![\mathcal{A}]\!] \,\}$

It seems fair to say that Dunn's 3-valued semantics is more *elegant*. The frame is a kind of structure that is quite familiar to us; e.g., it is exemplified by $\mathbb{N}$ (which of course, brings additional properties with itself). The truth and falsity conditions for $\sim$, $\wedge$ and $\vee$ surely look familiar. The conditions for $\to$ are ingenious, but straightforward—except perhaps, when the implication is assigned F, because it's not true. (Dunn expressed a certain dissatisfaction with this disjunct and called it the "escape clause.") On the other hand, the ternary relational semantics treats $\mathbf{RM}$ as yet another intensional logic. The multiple conditions on R stem from the fact that $\mathbf{RM}$ extends $\mathbf{R}$; only the last condition is specific to mingle (over $\mathbf{R}$). Indeed, Meyer, as soon as he specified the conditions for R, commented with vexation on their number and somewhat complicated character. However, this complexity is the price for approximating $\supset$ as much as relevantly possible (and perhaps, it is a price for generality too). The *flexibility* of the ternary relational semantics for various relevance logics such as $\mathbf{B}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{R}$ was undoubtedly an impetus for Dunn's formulation of gaggle theory.

2.5 Semantics for relevance logics

The 3-valued semantics for $\mathbf{RM}$ does not seem to be easily adaptable to some other relevance logics, especially, to the main relevance logics we have just mentioned (i.e., $\mathbf{B}$, $\mathbf{T}$, $\mathbf{E}$ and $\mathbf{R}$) that do not contain the mingle axiom.[7] Another way to think about a concrete semantics is by having operations on a set of objects.[8] However, the latter idea does not work for the semantics of relevance logics just as it did not work for the semantics of normal modal logics; the natural operation on prime filters does not yield a prime filter.

A semantics that uses a ternary relation for the modeling of $\to$ and $\circ$ was worked out in detail and published by Routley and Meyer (1972a, 1972b, 1973). A leftover from the operational approach is the modeling of $\sim$ from an operation (cf. Dunn (1966, 1986)). A different combination of operations and relations is used in the semantics in Fine (1974), which in effect, turns out to be equivalent to the Meyer–Routley semantics.[9]

The formulation of the ternary relational semantics seems to have propelled the creation of gaggle theory. Dunn and Meyer both worked at Indiana University (in Bloomington, IN), when Meyer—inspired by an idea in Routley's big manuscript—worked out the relational semantics for $\mathbf{R}^\circ$ in a form that is very close to its later presentations (cf. Bimbó, Dunn, and Ferenz (2018)). Logics in which there is a conjunction and disjunction that distribute over each other are well behaved (from the point of view of their semantics), and the main relevance logics (in their full vocabulary) are among those (just as $\mathbf{J}$). The first notion of a *gaggle* introduced in Dunn (1991) incorporates a *distributive lattice* as the living quarters for a family of operations, and has been called a *distributive gaggle* afterward (cf. Bimbó and Dunn (2008)). To illustrate both the Meyer–Routley semantics and a concrete (multi-)gaggle, we will use $\mathbf{T}^{\circ t}$ and its algebra. An axiomatization of ticket entailment may be found in Anderson et al. (1992, §R); we assume familiarity with this logic.

Definition 5 *A* $\mathfrak{G}_{\mathbf{T}}$ gaggle *is an algebra* $\langle A; \wedge, \vee, \sim, \boldsymbol{t}, \circ, \to\rangle$ *of similarity type* $\langle 2,2,1,0,2,2\rangle$*, where* (g1)–(g6) *hold.*

(g1) $\langle A; \wedge, \vee, \sim\rangle$ *is a De Morgan lattice;*

(g2) $\langle A; \wedge, \vee, \boldsymbol{t}, \circ, \to\rangle$ *is a lattice ordered groupoid (with* $\circ$*) with left identity (*$\boldsymbol{t} \circ a = a$*) and with right residual (*$a \circ b \leq c$ *iff* $a \leq b \to c$*);*

(g3) $(a \circ b) \circ c \leq a \circ (b \circ c)$*;*

(g4) $(a \circ b) \circ c \leq b \circ (a \circ c)$*;*

(g5) $a \circ b \leq (a \circ b) \circ b$*;*

(g6) $a \circ b \leq c$ *iff* $a \circ {\sim}c \leq {\sim}b$*.*

[7] Some adaptations work well though. See Dunn (1976a) and Bimbó and Dunn (in press).

[8] Bimbó and Dunn (2017) provides an overview of some of the early work toward a set-theoretical semantics for relevance logics by several logicians, e.g., Urquhart (1972). We will not repeat that history here; rather, we focus on the Meyer–Routley semantics.

[9] Semantics that are duals of the Meyer–Routley semantics were defined for $\mathbf{T}$ and $\mathbf{E}$ in Bimbó (2007) and Bimbó (2009); the latter also includes a topological characterization.

We called $\mathfrak{G}_{\mathbf{T}}$ a gaggle, but it is really two gaggles and a constant integrated into one algebra. The constant $\boldsymbol{t}$ is connected to the $\circ$ gaggle (which includes $\rightarrow$) and this interacts with the $\sim$ gaggle (which is a component of the De Morgan lattice). For any $\mathcal{A}$ that is a theorem of $\mathbf{T}^{\circ \boldsymbol{t}}$, the formula $\boldsymbol{t} \rightarrow \mathcal{A}$ is provable. Thus, we may think of $\mathfrak{G}_{\mathbf{T}}$ as a matrix, with $D = \{\, a \colon \boldsymbol{t} \leq a \,\}$. In the modeling of $\circ$ and $\rightarrow$, we follow the Meyer–Routley semantics, but for $\boldsymbol{t}$ and $\sim$ we make some modifications.

Definition 6 *A* frame for $\mathfrak{G}_{\mathbf{T}}$ *is* $\mathfrak{F} = \langle U, \sqsubseteq, I, R_\circ, R_\sim \rangle$*, where* $I \neq \emptyset$*,* $I \subseteq U$*,* $\sqsubseteq$ *is a preorder on* U*,* $R_\circ \subseteq U^3$*,* $R_\sim \subseteq U^2$ *and* (f1)–(f7) *also hold.*

(f1) $(R_\circ uu'u'' \wedge s \sqsubseteq u \wedge s' \sqsubseteq u' \wedge u'' \sqsubseteq s'') \Rightarrow R_\circ ss's''$ *(i.e.,* $R_\circ{\downarrow}{\downarrow}{\uparrow}$*);*
(f2) $u \sqsubseteq u' \Leftrightarrow \exists \iota \in I\, R_\circ \iota uu'$*;* $I \in \mathcal{P}(U)^{\uparrow}$*;* $R_\sim{\uparrow}{\uparrow}$*;*
(f3) $\exists u\, (\neg R_\sim uu' \wedge \forall u''(\neg R_\sim u''u \Rightarrow u'' \sqsubseteq u'))$*;*
(f4) $(R_\circ uu'u'' \wedge \neg R_\sim su'') \Rightarrow \exists s', s''(R_\circ uss' \wedge \neg R_\sim s''s' \wedge u' \sqsubseteq s'')$*;*
(f5) $\neg R_\sim u'u'' \Rightarrow \exists s, s', s''\, (R_\circ u''ss' \wedge u' \sqsubseteq s \wedge u' \sqsubseteq s'' \wedge \neg R_\sim s''s')$*;*
(f6) $R_\circ(uu')ss' \Rightarrow R_\circ u(u's)s'$*;* $R_\circ(uu')ss' \Rightarrow R_\circ u'(us)s'$*;*
(f7) $R_\circ uu's \Rightarrow R_\circ(uu')u's$*.*

The frame is defined to have a pre-order, which makes this frame somewhat similar to that for $\mathbf{J}$. But now $\sqsubseteq$ is definable from $R_\circ$ and I, rather than R being definable from $\sqsubseteq$ as in the case of $\mathbf{J}$. This is explained by the fact that in $\mathbf{T}^{\circ \boldsymbol{t}}$ implication is not a residual of $\wedge$, and $\sqsubseteq$ is linked to provable implications. The ternary relation in the semantics of $\mathbf{J}$ seemed almost like a vapid complication, though we made some pertinent observations using R. Here the use of a binary relation $R_\sim$ instead of a unary operation is a similar intricacy; the operation could be denoted by *, as at the end of §2.4. If we let u^* to be s, then R^*us is definable as $\neg R_\sim us \wedge \neg \exists s'\, (s \neq s' \wedge \neg R_\sim us' \wedge s \sqsubseteq s')$. It so happens that in the Lindenbaum algebra of $\mathbf{T}^{\circ \boldsymbol{t}}$ such an s always exists, moreover, it is unique; these properties support the use of an operation. (The inequations in (g3) and (g4) are the previous (a6) and (a7) in §2.4, and the two conditions in (f6) are the same as (f1) and (f2) in Definition 2.)

Definition 7 *A* model for $\mathfrak{G}_{\mathbf{T}}$ *is* $\mathfrak{M} = \langle U, \sqsubseteq, I, R_\circ, R_\sim, v \rangle$*, where the frame is as above and* $v \colon \mathbb{P} \longrightarrow \mathcal{P}(U)^{\uparrow}$*, which is extended to all formulas according to* (m1)–(m7).

(m1) $[\![p]\!] = v(p)$ (m2) $[\![\boldsymbol{t}]\!] = \{\, u \colon \exists \iota \in I\ \iota \sqsubseteq u \,\}$
(m3) $[\![\mathcal{A} \wedge \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cap [\![\mathcal{B}]\!]$ (m4) $[\![\mathcal{A} \vee \mathcal{B}]\!] = [\![\mathcal{A}]\!] \cup [\![\mathcal{B}]\!]$
(m5) $[\![{\sim}\mathcal{A}]\!] = \{\, u \colon \forall u'(u' \in [\![\mathcal{A}]\!] \Rightarrow R_\sim u'u) \,\}$

(m6) $[\![\mathcal{A}\circ\mathcal{B}]\!]=\{\,u''\colon \exists u,u'(R_\circ uu'u''\wedge u\in[\![\mathcal{A}]\!]\wedge u'\in[\![\mathcal{B}]\!])\,\}$
(m7) $[\![\mathcal{B}\to\mathcal{C}]\!]=\{\,u\colon \forall u',u''((R_\circ uu'u''\wedge u'\in[\![\mathcal{B}]\!])\Rightarrow u''\in[\![\mathcal{C}]\!])\,\}$

A formula $\mathcal{A}$ is *true at the situation* u (in some $\mathfrak{M}$), when $u\in[\![\mathcal{A}]\!]$. The *truth* of $\mathcal{A}$ in a model means that $\forall\iota\in I\ \iota\in[\![\mathcal{A}]\!]$, that is, $[\![\,\boldsymbol{t}\,]\!]\subseteq[\![\mathcal{A}]\!]$. In Dunn's three-valued semantics for $\mathbf{RM}$, stipulating that ι was the least element of U had the effect of limiting U to *logical situations*, which is similar to requiring $U=I$ in the case of $\mathbf{J}$. However, here we did not assume that all situations are logical, neither have we stated that I is the principal cone generated by a particular logical situation. (In some presentations of the Meyer–Routley semantics occasionally one distinguished situation is selected, which is denoted by 0; we do not follow that track here.) *Validity* means truth in every model of a frame for $\mathfrak{G}_{\mathbf{T}}$. The proof of the following is easy and we do not include the details here.

Lemma 6 (Hereditary property) *For all formulas $\mathcal{A}$, $[\![\mathcal{A}]\!]\in\mathcal{P}(U)^{\uparrow}$.*

This lemma means that truth is retained along the $\sqsubseteq$ relation (which is not the accessibility relation as in $\mathbf{J}$, only a special part of it). The import of the lemma is that propositions (i.e., interpretations of formulas) are located among the upward closed sets of situations. To this extent the lemma is similar to the hereditariness lemma in the semantics of $\mathbf{J}$.

What we have so far suffices for soundness. For completeness, we outline the definition of the canonical frame and that of the canonical model.

Definition 8 *The* canonical frame for $\mathfrak{G}_{\mathbf{T}}$ *is* $\mathfrak{F}_c=\langle U_c,\subseteq,I_c,R_\circ,R_\sim\rangle$, *where* $U_c=\mathcal{P}$, $I_c=\{\,P\in\mathcal{P}\colon[\,\boldsymbol{t}\,)\subseteq P\,\}$ *and* $R_\sim,R_\circ$ *are as in* (c1)–(c2).

(c1) $R_\sim uu'\Leftrightarrow\exists a\,(a\in u\wedge{\sim}a\in u')$
(c2) $R_\circ uu'u''\Leftrightarrow\forall a,b\,((a\in u\wedge b\in u')\Rightarrow a\circ b\in u'')$

The canonical model for $\mathfrak{G}_{\mathbf{T}}$ *is* $\mathfrak{M}_c=\langle\mathfrak{F}_c,v_c\rangle$, *where* $\mathfrak{F}_c$ *is the canonical frame and* $v_c([\mathcal{A}])=\{\,P\in\mathcal{P}\colon[\mathcal{A}]\in P\,\}$.

To get to Theorem 5, it is convenient to prove certain claims as lemmas, which we only list here. First, the components of $\mathfrak{F}_c$ are of the declared types, however, it is far from obvious that they have all the required properties, especially, that $R_\circ$ and $R_\sim$ satisfy (f3)–(f7). In establishing the properties of $R_\sim$ and $R_\circ$, it is useful to prove versions of the *squeeze lemma*. The latter then may be utilized in the proof that v_c is a homomorphism. The canonical situations are prime filters, hence, it is sufficient to appeal to a well-known result from lattice theory about separation to see that v_c is injective, that is, an isomorphism. We simply state adequacy. The proofs may be found or can be pieced together from results in some of the publications cited.

Theorem 5 *For any formula $\mathcal{A}$, $\vdash_{\mathbf{T}^{\circ t}} \mathcal{A}$ iff on any frame $\mathfrak{F}$ for $\mathfrak{G}_{\mathbf{T}}$, $\vDash_{\mathfrak{F}} \mathcal{A}$.*

3 Concluding remarks

I attempted to reconstruct the conceptual components that likely influenced the formulation of gaggle theory. The sources for the reconstruction were Dunn's publications related to gaggle theory and a more comprehensive view of Dunn's research including his other publications and research talks.

Dunn (1966) algebraized $\mathbf{R}^t$ (and $\mathbf{E}$ too). Although much of the research in relevance logics was guided by Anderson (1963) at the time, with a focus on proving (propositional) $\mathbf{R}$ and $\mathbf{E}$ decidable, Dunn formulated the first relational semantics for an intensional logic (other than modal logics and $\mathbf{J}$) in the late 1960s (published as Dunn (1976b, 1976c)). He continued to publish on algebraic semantics and results for intensional logics (propositional and quantified) as well as on other aspects of intensional logics. However, after the invention of the Meyer–Routley semantics for relevance logics, Dunn published Dunn (1976d) and half a decade later Dunn (1982), which are alternative relational semantics for some logics.

Dunn gave over 200 research talks in his career; it seems that the first *gaggle talk* was delivered in Canada, in 1979, at the University of Victoria that was entitled "Generalized Representation and Completeness Results." In 1983, Dunn toured Australia, and gave several talks on relevance logic and other topics. The following year he gave a talk at the Carnegie–Mellon University, which mentions Galois in its title "A Uniform Treatment of Implication and Negation through Residuation and Galois Connections." Already in 1983, in a research proposal, Dunn stated that most of the representation results for a range of logics had been obtained.

In sum, the emergence of *Generalized Galois Logics* (or *gaggle theory*) can be safely dated to about a decade or so earlier than the publication of Dunn (1991), which started a series of papers on gaggle theory. The delay can be attributed to the abundance of publications by Dunn during this period—such as a series of papers on relevant predication, a co-authored book, a co-edited book, a chapter on relevance logic in a handbook (which became a standard reading in the area), and work on another co-authored book. Dunn also co-authored a short paper Dunn and Hellman (1986) on probability theory during the decade. Somewhat surprisingly, this paper—which is not a paper in logic per se, rather an application of logic—turned out to be Dunn's most widely known and read publication at the time. He received requests for offprints of this paper from all over the world and from people way outside of academia.

Gaggle theory can be seen as an *overarching approach* to propositional intensional logics that starts with an axiomatic calculus or some other proof system, then moves through algebraization to a set-theoretic semantics. Generalized Galois logics, including its development in Dunn and Hardegree (2001), Bimbó and Dunn (2008) and many other publications, proved to be exceptionally fruitful. However, it is worth mentioning that gaggle theory is a relatively modest part of Dunn's overall logical research.

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