Risk Allocation and Risk Attribution in Static and Dynamic Settings

by

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Abstract

Risk can be decomposed along two dimensions: risk allocation and risk attribution. On the one hand, the total risk of a company can be allocated to its divisions, using that the company's profit/loss is the sum of the divisions' profits/losses. On the other hand, risk is attributed to risk drivers that may affect the company's profit/loss in a nonlinear way. This thesis deals with risk allocation and risk attribution by extending results from a single-period model to a dynamic setting. For risk allocation, we apply the Euler allocation principle while for risk attribution, we use a linear approximation of the profit/loss contributions of risk drivers and then apply risk allocation. We also show an example for risk allocation and risk attribution, using the entropic risk measure and simulating the risk drivers in MATLAB.

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Chapter 1

Introduction

Risk is the possibility of adverse events happening. In finance, risk is often understood as the probability of losses. There are the following four main types of financial risk:

- *credit risk:* the risk coming from a default of a borrower,
- *market risk:* the risk related to the performance of financial markets,
- *operational risk:* the risk incurred for breakdowns in internal processes, people, and systems,
- *liquidity risk:* the risk arising when an investment cannot be bought or sold quickly enough to minimize a loss.

Mathematics and statistics are used to model and quantify risk. A key concept is that of a risk measure, which quantifies the risk related to potential losses. Mathematically, a risk measure is a mapping from the set of random variables to the real numbers satisfying the properties of being normalized, translative, and monotone; we will discuss these properties in Definition 2.1.1. Why do we need to pay attention to risk measures? The answer is that risk measures can identify high-risk situations and predict potential losses to prevent insolvency, help manage risk by analyzing the efficiency of risk control measures, and assist companies to make decisions. Balzer [2] said that there is "no single universally acceptable risk measure". Common risk measures are value at risk (VaR), which is defined as a quantile of the loss distribution, and expected shortfall (ES), which is also known as conditional VaR or expected tail loss. ES is computed as the average loss over a percentage of worst-case scenarios.

To analyze risk in more detail, we focus on risk decomposition. Risk decomposition deals with questions, such as, how much do a company's divisions contribute to the total risk? Or, how can the total risk be attributed to different types of risk, for example, credit risk? Risk decomposition can be explained by using Table 1.1, which has two dimensions: risk allocation and risk attribution. For risk allocation, the company's profit/loss is considered as the sum

risk allocation

risk attribution

	Division 1	Division 2	...	Division K	Total company
Risk driver 1					
Risk driver 2					
Risk driver 3					
...					
Cross effects					
Total risk					

Table 1.1: Decomposition of risk along two dimensions: risk allocation and attribution. Illustration reproduced from Frei [8]

of the profits/losses of its division. The goal is to allocate the company's total risk to its divisions so that the allocated risks sum up to the total risk. Note that the risk allocated to a division is not equal to the risk that the division would have stand-alone because of diversification benefits between divisions. For risk attribution, differently from the situation with risk allocation, risk drivers can contribute to the total profit/loss in a nonlinear way, and the total profit/loss may not be the sum of profits/losses of different risk drivers. The purpose is to identify risk drivers and attribute risk to them while cross effects between risk drivers may remain. Therefore, the sum of the risk attributions may not be equal to the total risk.

The Euler principle is widely used in academia and industry for risk allocation; see, for instance, Li and Xing [14], McNeil et al. [15], and Tasche [20]. The methodology is to allocate to every division its marginal contribution to the total risk: the risk allocated to the division equals the instantaneous rate of change of the company's risk when a division's profit/loss contribution increases. The Euler principle has its name derived from Euler's theorem on homogeneous functions, which implies the full-allocation property (risks allocated to the divisions sum up to the total risk) if the risk measure is homogenous, which is satisfied for VaR and ES. The Euler principle has desirable economic properties, namely, it is compatible with return on risk adjusted capital (RORAC); compare Tasche [20] and Proposition 2.1.1 below. It also satisfies the property that it does not allocate more risk to a division than the risk that the division would have stand-alone; see Denault [6] and Kalkbrener [10]. A discussion about different risk allocation methodologies and how they are affected by the risk measure and loss distribution can be found in Koyluoglu and Stoker [12], Urban et al. [21], and Zhang and Rachev [22].

There is not as much literature for risk attribution as for risk allocation. The Shapley value is one of the possible methods to solve the risk attribution problem. Its idea is to share the diversification effects among the risk drivers by applying a concept from cooperative game theory that has been introduced by Shapley [19]. The Shapley value for a risk driver is computed as the average of the contribution of this risk driver when it enters at different stages. In the first round, the impact of only a single risk driver on the loss variable is considered and the corresponding value of the risk measure computed. In the second round, the effect is considered that a risk driver has when there are two drivers present, and the stand-alone contribution of the other driver is subtracted. In the third round, the impact of a risk driver is considered when there are three drivers present, and the joint contribution of the other two drivers is subtracted. The procedure continues until all drivers are considered. The average value over the different rounds gives the Shapley value; see Denault [6] or Powers [17] for details. The shortage of the Shapley value is that it is very computationally demanding when there is a large number of risk drivers.

As another approach to risk attribution presented in an insurance context, Boonen et al. [4] show that the Euler principle applied to an auxiliary linearized fuzzy game provides a plausible and easily implemented risk attribution. Rosen and Saunders [18] employ the Hoeffding decomposition to express the loss variable as a sum of functions of all subsets of risk factors and then apply the Euler principle to the loss decomposition. Another method for risk attribution has been proposed in Frei [8], using a linearization of the loss variable. It considers the risk drivers over time, computes the change of the total risk with respect to the change of one particular risk driver at each time step, and then sums up these changes to obtain the contribution of that risk driver. Therefore, iterating this step for each risk driver leads to a linear approximation of the loss variable. Applying the Euler principle to the approximation gives a risk attribution.

Other related literature includes Bauer and Zanjani [3], who use the reverse logic to identify the risk measure by calculating the marginal risk contribution first. Then they find risk measures that can deliver the correct risk allocation. While risk allocation and risk attribution go from the total risk to smaller entities or drivers, risk aggregation goes in the opposite direction, which computes the total risk from several risk components. Risk aggregation is particularly important when dealing with systemic and vector-valued risk measures, such as in Cousina and Di Bernardino [5], Feinstein et al. [7], Jouini et al. [9], and Landsman et al. [13].

This thesis aims to analyze risk allocation and risk attribution dynamically over time, but we also revisit the static setting. For the static risk allocation, we primarily base on Tasche [20], which we extend to the dynamic setting by using time-consistent dynamic risk measures, discussed in Acciaio and Penner [1]. Furthermore, because the risk attribution in Frei [8] considers risk drivers over time, we use this approach to generalize risk attribution from a static to a dynamic setting.

The remainder of this thesis is organized as follows. Chapter 2 discusses risk allocation in two sections, one section devoted to a static setting and the other to a dynamic setting, where the static setting is a one-period model and the dynamic setting is over an interval $[0, T]$ for a time horizon T. First, we introduce the RORAC of a portfolio and determine when risk allocations are RORAC compatible and when the full-allocation property is satisfied for both static and dynamic settings. Towards the end of Chapter 2, we analyze the time consistency of risk measures and risk allocations in the dynamic setting. Chapter 3 also has two sections about the static and dynamic settings, but deals with risk attribution rather than risk allocation. In both sections of Chapter 3, we start with a two-factor model and then apply the same logic to a multi-factor model. The methodology for risk attribution is to use a linearization of the profit/loss contributions and then apply risk allocation. We show an example for both risk allocation and risk attribution in Chapter 4. Firstly, we check the properties of a risk measure in the case of the entropic risk measure. Then we calculate its Euler risk contributions by using risk allocation and risk attribution. In addition, we simulate the risk attribution for this example, when VaR and ES are used as risk measures. Chapter 5 concludes, and Appendix A contains MATLAB code used for the computations in Chapter 4.

Chapter 2

Risk Allocation

In this chapter, we discuss risk allocation, which addresses the question of how to allocate the risk of some entities to different sub-entities. We start in Section 2.1 with a one-period model, following closely Tasche [20], but providing more details. In Section 2.2, we analyze the situation when the model is over a continuous time interval.

Throughout this master's thesis, we are working on a probability space $(\Omega, \mathcal{F}, P)$, and all equations and inequalities between random variables are understood to hold almost surely.

2.1 Static Setting

We consider a portfolio consisting of n assets. Equivalently, we can think of a company consisting of n divisions. Suppose that we describe the profit/loss of asset i in the portfolio by a real-valued random variable X_i, then the portfoliowide profit/loss is,

$$X = \sum_{i=1}^{n} X_i.$$

In the equivalent interpretation, X is the company's profit/loss while X_i is the profit/loss of division i.

Definition 2.1.1. *A risk measure ρ is a mapping from a set $\mathcal{L}$ of random variables to the real numbers. The risk measure $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ has the following properties:*

Normalized

$\rho(0) = 0$

Translative

If $a \in \mathbb{R}$ and $Z \in \mathcal{L}$, then $\rho(Z + a) = \rho(Z) - a$

Monotone

If $Z_1, Z_2 \in \mathcal{L}$ and $Z_1 \leq Z_2$ almost surely, then $\rho(Z_2) \leq \rho(Z_1)$

The economic capital (EC) is the amount of capital allocated for preventing insolvency. It depends on the profit/loss. Thus, we determine EC by a risk measure ρ,

$$EC = \rho(X). \tag{2.1}$$

Let the variable u_i be the weight of asset i, and the vector $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ be a portfolio. At present, the profit/loss of portfolio u is $X(u)$ given by

$$X(u) = \sum_{i=1}^{n} u_i X_i$$

and the risk of portfolio u is $\rho(X(u))$. We use a function $f_{\rho,X}(u)$ to present the same risk measure $\rho(X(u))$. When we assume that the distribution of X

is fixed, we can write $f_{\rho,X}$ as f_ρ. With that, we have

$$f_\rho(u) = \rho(X(u)). \tag{2.2}$$

After defining EC for the whole portfolio as $\rho(X)$ in (2.1), we still need to define the risk contribution of X_i to $\rho(X)$. It can help us get a better understanding of the risk contribution of asset i to the total risk measure. We denote the risk contribution of X_i to $\rho(X)$ by $\rho(X_i|X)$. With this notation, we can consider the following notions of returns.

Definition 2.1.2. *The total portfolio return on risk adjusted capital (RORAC) is defined by*

$$RORAC(X) = \frac{E[X]}{\rho(X)}.$$

The portfolio-related RORAC of the i^{th} asset is defined by

$$RORAC(X_i|X) = \frac{E[X_i]}{\rho(X_i|X)}.$$

We next give two desirable properties of risk allocations.

Definition 2.1.3. *Risk contributions $\rho(X_1|X), \ldots, \rho(X_n|X)$ have the following properties:*

- *They satisfy the full-allocation property if*

$$\sum_{i=1}^{n} \rho(X_i|X) = \rho(X).$$

- *They are RORAC compatible if there exists an* $\epsilon_i > 0$ *such that*

$$RORAC(X_i|X) > RORAC(X) \Rightarrow RORAC(X + hX_i) > RORAC(X)$$

for all $0 < h < \epsilon_i$.

RORAC compatibility says that if the portfolio-related RORAC of the i^{th} asset is greater than the portfolio RORAC, then increasing the weight of the i^{th} asset will increase the portfolio RORAC. The following result, taken from Tasche [20], states that for a smooth risk measure, the RORAC compatibility characterizes the risk contributions.

Proposition 2.1.1. *Assume that* f_ρ *given in* (2.2) *is continuously differentiable. Risk contributions* $\rho(X_1|X), \ldots, \rho(X_n|X)$ *are RORAC compatible for arbitrary expected values of* $X_1, \ldots, X_n$*, if and only if* $\rho(X_i|X)$ *is uniquely determined by*

$$\rho^{Euler}(X_i|X) = \frac{d\rho}{dh}(X + hX_i)\bigg|_{h=0} = \frac{\partial f_\rho}{\partial u_i}(1, \ldots, 1). \tag{2.3}$$

The risk allocation given by (2.3) is called the Euler allocation principle.

Proof. Firstly, we prove if $\rho(X_i|X)$ is uniquely determined by (2.3), then $\rho(X_i|X)$ is RORAC compatible. Set

$$M(u) = E[X(u)].$$

Notice that RORAC can be written as

$$RORAC(X(u)) = \frac{E[X(u)]}{\rho(X(u))} = \frac{M(u)}{\rho(X(u)) + M(u) - M(u)} = R(u).$$

Because the risk measure is translative, we know

$$\rho(X(u)) + M(u) = \rho(X(u) - M(u)).$$

Then we define

$$\rho^X(u) = \rho(X(u)),$$
$$\rho^Y(u) = \rho(X(u) - M(u)),$$

where $(X(u) - M(u))$ is the fluctuation of $X(u)$. So we have $\rho^X(u) = \rho^Y(u) - M(u)$. Thus,

$$RORAC(X(u)) = \frac{M(u)}{\rho^Y(u) - M(u)} = R(u).$$

Given the notion of a per-unit risk contribution of the profit fluctuation $a_i(u)$, we define it as

$$a_i(u) \stackrel{\text{def}}{=} \rho(X_i|X) + \frac{\partial M(u)}{\partial u_i}.$$

We get

$$RORAC(X_i|X) = \frac{\frac{\partial M(u)}{\partial u_i}}{a_i(u) - \frac{\partial M(u)}{\partial u_i}} = \frac{M_i'(u_i)}{a_i(u) - M_i'(u_i)}$$

where

$$M(u) = (M_1(u_1), \dots, M_n(u_n))'.$$

We get

$$\begin{aligned}
\frac{\partial R(u)}{\partial u_i} &= (\rho^Y(u) - M(u))^{-2}\bigg(M_i'(u_i)(\rho^Y(u) - M(u)) \\
&\qquad\qquad - M(u)\Big(\frac{\partial \rho^Y(u)}{\partial u_i} - M_i'(u_i)\Big)\bigg) \\
&= (\rho^Y(u) - M(u))^{-2}\bigg(M_i'(u_i)\rho^Y(u) - M(u)\frac{\partial \rho^Y(u)}{\partial u_i}\bigg).
\end{aligned}$$

If $a_i(u) = \frac{\partial \rho^Y(u)}{\partial u_i}$, then

$$\frac{\partial R(u)}{\partial u_i} = (\rho^Y(u) - M(u))^{-2}\big(M_i'(u_i)\rho^Y(u) - M(u)a_i(u)\big).$$

If

$$RORAC(X_i|X) > R(u)$$

we get

$$\begin{aligned}
\frac{M_i'(u_i)}{a_i(u) - M_i'(u_i)} &> \frac{M(u)}{\rho^Y(u) - M(u)}, \\
M_i'(u_i)(\rho^Y(u) - M(u)) &> (a_i(u) - M_i'(u_i))M(u), \\
M_i'(u_i)\rho^Y(u) &> a_i(u)M(u), \\
M_i'(u_i)\rho^Y(u) - a_i(u)M(u) &> 0.
\end{aligned}$$

Then $\frac{\partial R(u)}{\partial u_i} > 0$, such that R is an increasing function in the u_i-direction. So it is clear that $RORAC(X+hX_i) > RORAC(X)$, for all small enough $h > 0$. So we have proved that if $a_i(u) = \frac{\partial \rho^Y(u)}{\partial u_i}$, then $\rho(X_i|X)$ is RORAC compatible.

Next, we prove $a_i(u) = \frac{\partial \rho^Y(u)}{\partial u_i}$ is equivalent to (2.3) as follows. Since $\rho^X(u) = \rho^Y(u) - M(u)$,

$$\frac{\partial \rho^X(u)}{\partial u_i} = \frac{\partial \rho^Y(u)}{\partial u_i} - \frac{\partial M(u)}{\partial u_i} = a_i(u) - M_i'(u_i)$$

and $\rho^{Euler}(X_i|X) = \frac{d\rho}{dh}(X+hX_i)\big|_{h=0}$ = per-unit risk contribution of the total profit = per-unit risk contribution of profit fluctuation - per-unit risk contribution of expect profit $= a_i(u) - M_i'(u_i) = \frac{\partial \rho^X(u)}{\partial u_i} = \frac{\partial \rho(X(u))}{\partial u_i} = \frac{\partial f_\rho}{\partial u_i}$. Thus, if $\rho^{Euler}(X_i|X) = \frac{d\rho}{dh}(X+hX_i)\big|_{h=0} = \frac{\partial f_\rho}{\partial u_i}$, then $\rho(X_i|X)$ is RORAC compatible.

Secondly, we prove if $\rho(X_i|X)$ is RORAC compatible, then $\rho(X_i|X)$ is uniquely determined by (2.3). Let

$$M(u) = m^\top u = \sum_{i=1}^{n} m_i u_i$$

where m_i is the expected profit/loss of asset i. Thus,

$$\frac{\partial M(u)}{\partial u_i} = M_i'(u_i) = m_i.$$

Define $m(t) \in \mathbb{R}^d$ by

$$\begin{aligned} m_i(t) &\stackrel{\text{def}}{=} 1, \\ m_j(t) &\stackrel{\text{def}}{=} \frac{t}{u_j}\left(\frac{\rho^Y(u)}{a_i(u)} - u_i\right), \\ m_l(t) &\stackrel{\text{def}}{=} 0 \text{ for } l \neq i, j, \end{aligned}$$

then

$$m(t)^\top u = t\frac{\rho^Y(u)}{a_i(u)} + (1-t)u_i$$

and

$$m_i(t)\rho^Y(u) - a_i(u)m(t)^\top u = (1-t)(\rho^Y(u) - u_i a_i(u)).$$

We obtain

$$\begin{aligned}
&(1-t)(\rho^Y(u) - u_i a_i(u)) + \left(a_i(u) - \frac{\partial\rho^Y(u)}{\partial u_i}\right)\left(t\frac{\rho^Y(u)}{a_i(u)} + (1-t)u_i\right) \\
&= m_i(t)\rho^Y(u) - a_i(u)m(t)^\top u + \left(a_i(u) - \frac{\partial\rho^Y(u)}{\partial u_i}\right)m(t)^\top u \\
&= m_i(t)\rho^Y(u) - a_i(u)m(t)^\top u + a_i(u)m(t)^\top u - m(t)^\top u\frac{\partial\rho^Y(u)}{\partial u_i} \\
&= m_i(t)\rho^Y(u) - m(t)^\top u\frac{\partial\rho^Y(u)}{\partial u_i}.
\end{aligned}$$

We can choose a sequence t_k with $t_k \to 1$ such that $RORAC(X_i|X) > RORAC(X)$ and $\rho(X_i|X)$ are RORAC compatible, then we have $RORAC(X+hX_i|X) > RORAC(X)$, and we can get $\frac{\partial R(u)}{\partial u_i} \geq 0$ so that

$$M_i'(u_i)\rho^Y(u) - M(u)\frac{\partial\rho^Y(u)}{\partial u_i} \geq 0.$$

As $M(u) = m^\top u$ and $M_i'(u_i) = m_i$, taking t as t_k, we get $m_i(t_k)\rho^Y(u) - m(t_k)^\top u\frac{\partial\rho^Y(u)}{\partial u_i} \geq 0$, thus $(1-t_k)(\rho^Y(u) - u_i a_i(u)) + (a_i(u) - \frac{\partial\rho^Y(u)}{\partial u_i})(t_k\frac{\rho^Y(u)}{a_i(u)} + (1-t_k)u_i) \geq 0$.

Similarly, we can choose a sequence s_k with $s_k \to 1$ such that we deduce

$RORAC(X_i|X) < RORAC(X)$ and get $\frac{\partial R(u)}{\partial u_i} \leq 0$, then taking t as s_k, we get

$$m_i(s_k)\rho^Y(u) - m(s_k)^\top u \frac{\partial \rho^Y(u_i)}{\partial u_i} \leq 0.$$

Thus,

$$(1-s_k)(\rho^Y(u) - u_i a_i(u)) + \left(a_i(u) - \frac{\partial \rho^Y(u)}{\partial u_i}\right)\left(s_k \frac{\rho^Y(u)}{a_i(u)} + (1-s_k)u_i\right) \leq 0.$$

When $k \to \infty$, it follows that

$$0 \leq a_i(u) - \frac{\partial \rho^Y(u)}{\partial u_i} \leq 0$$

We get

$$a_i(u) = \frac{\partial \rho^Y(u)}{\partial u_i},$$

which is equivalent to $\rho^{Euler}(X_i|X) = \frac{d\rho}{dh}(X + hX_i)\big|_{h=0} = \frac{\partial f_\rho}{\partial u_i}$, as shown before. Consequently, if $\rho(X_i|X)$ is RORAC compatible, then we have $\rho^{Euler}(X_i|X) = \frac{d\rho}{dh}(X + hX_i)\big|_{h=0} = \frac{\partial f_\rho}{\partial u_i}$. $\square$

After having discussed when risk contributions are RORAC compatible, we now mention when the full-allocation property is satisfied. By Tasche [20], the full-allocation property for the Euler allocation (2.3) holds if and only if the risk measure is homogeneous of degree 1, which means $\rho(\tau X) = \tau\rho(X)$ for all $\tau > 0$. Examples of risk measures that are homogeneous of degree 1 include VaR and ES.

2.2 Dynamic Setting

We continue to consider a portfolio with n assets, but now we are interested in the risk allocation at time $t \in [0, T]$, where T is a fixed time horizon.

We define $\mathcal{F}_t$ for $t \in [0, T]$ as the information set which contains all the information up to time t. $\mathcal{F}_t$ is a σ-algebra, and it holds that $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$ for all $t_1 \leq t_2$. We assume that $\mathcal{F}_0$ is trivial in the sense that it consists of only sets of probability 0 or 1. We further assume that $\mathcal{F}_T = \mathcal{F}$.

Definition 2.2.1. *For a random variable X and $t \in [0, T]$,*

- *$X \in \mathcal{L}^\infty$ if there exists a real number $c > 0$ such that $|X| \leq c$.*
- *$X \in \mathcal{L}_t^\infty$ if there exists a real number $c > 0$ such that $|X| \leq c$ and X is $\mathcal{F}_t$-measurable.*

We note that $\mathcal{L}_0^\infty = \mathbb{R}$ and $\mathcal{L}_T^\infty = \mathcal{L}^\infty$.

Definition 2.2.2. *For $t \in [0, T]$, a map $\rho_t : \mathcal{L}^\infty \to \mathcal{L}_t^\infty$ is called a dynamic risk measure if it satisfies the following properties for all $X \in \mathcal{L}^\infty$:*

Normalized

$\rho_t(0) = 0$

Translative

If $a_t \in \mathcal{L}_t^\infty$, then $\rho_t(X + a_t) = \rho_t(X) - a_t$

Monotone

If $X_1, X_2 \in \mathcal{L}^\infty$ and $X_1 \leq X_2$ almost surely, then $\rho_t(X_2) \leq \rho_t(X_1)$

Comparing with Definition 2.1.1, a dynamic risk measure is a mapping from random variables to random variables, rather than from random variables to real numbers. In addition, ρ_t is measured at a fixed time $t \in [0, T]$. Definition 2.1.1 can be thought of as Definition 2.2.2 on the trivial σ-algebra.

At time t, the economic capital (EC_t) of the portfolio is $\rho_t(X)$. The risk of portfolio u is $\rho_t(X(u))$ at time t, where $X(u) = \sum_{i=1}^{n} u_i X_i$. We use a function $f_{\rho_t,X}(u)$ to present the same risk measure $\rho_t(X(u))$. When we assume that the distribution of X is fixed, we can write $f_{\rho_t,X}$ as f_{ρ_t}. With that, we have

$$f_{\rho_t}(u) = \rho_t(X(u)). \tag{2.4}$$

After defining EC_t for the whole portfolio at time t as $\rho_t(X)$, we still need to define the risk contribution of X_i to $\rho_t(X)$. It can help us get a better understanding of the risk contribution of asset i to the total risk measure at time t. We denote the risk contribution of X_i to $\rho_t(X)$ by $\rho_t(X_i|X)$.

Definition 2.2.3. *The total portfolio return on risk adjusted capital at time t ($RORAC_t$) is defined by*

$$RORAC_t(X) = \frac{E[X|\mathcal{F}_t]}{\rho_t(X)}.$$

The portfolio-related $RORAC_t$ of the i^{th} asset is defined by

$$RORAC_t(X_i|X) = \frac{E[X_i|\mathcal{F}_t]}{\rho_t(X_i|X)}.$$

We next give two desirable properties of risk allocations.

Definition 2.2.4. *Risk contributions $\rho_t(X_1|X), \ldots, \rho_t(X_n|X)$ at time t have the following properties:*

- *they satisfy the full-allocation property if*

$$\sum_{i=1}^{n} \rho_t(X_i|X) = \rho_t(X).$$

- *they are $RORAC_t$ compatible if there exists an $\mathcal{F}_t$-measurable random variable ϵ_i with $\epsilon_i > 0$ such that*

$$RORAC_t(X_i|X) > RORAC_t(X) \Rightarrow RORAC_t(X+hX_i) > RORAC_t(X)$$

almost surely, for any $\mathcal{F}_t$-measurable random variable h with $0 < h < \epsilon_i$.

Proposition 2.2.1. *For a fixed $t \in [0,T]$, assume that f_{ρ_t} given in (2.4) is continuously differentiable. Risk contributions $\rho_t(X_1|X), \ldots, \rho_t(X_n|X)$ are $RORAC_t$ compatible for arbitrary conditionally expected values $E[X_1|\mathcal{F}_t], \ldots,$ $E[X_n|\mathcal{F}_t]$ of $X_1, \ldots, X_n$ at time t, if and only if $\rho_t(X_i|X)$ is uniquely determined by*

$$\rho_t^{Euler}(X_i|X) = \frac{d\rho_t}{dh}(X+hX_i)\bigg|_{h=0} = \frac{\partial f_{\rho_t}}{\partial u_i}(1,\ldots,1). \tag{2.5}$$

Proof. Firstly, we prove if $\rho_t(X_i|X)$ is uniquely determined by (2.5), then $\rho(X_i|X)$ is $RORAC_t$ compatible. Set

$$M_t(u) = E[X(u)|\mathcal{F}_t].$$

Notice that $RORAC_t$ can be written as

$$RORAC_t(X(u)) = \frac{E[X(u)|\mathcal{F}_t]}{\rho_t(X)} = \frac{M_t(u)}{\rho_t(X(u)) + M_t(u) - M_t(u)} = R_t(u).$$

Because the dynamic risk measure is translative, we know

$$\rho_t(X(u)) + M_t(u) = \rho_t(X(u) - M_t(u)).$$

Then we define

$$\rho_t^X(u) = \rho_t(X(u)),$$
$$\rho_t^Y(u) = \rho_t(X(u) - M_t(u)),$$

where $(X(u) - M_t(u))$ is the fluctuation of $X(u)$ at time t.

So we have $\rho_t^X(u) = \rho_t^Y(u) - M_t(u)$. Thus,

$$RORAC_t(X(u)) = \frac{M_t(u)}{\rho_t^Y(u) - M_t(u)} = R_t(u).$$

Given the notion of a per-unit risk contribution of the profit fluctuation at time t as the notation $a_{ti}(u)$, we define it as

$$a_{ti}(u) \stackrel{\text{def}}{=} \rho_t(X_i|X) + \frac{\partial M_t(u)}{\partial u_i}.$$

We get

$$RORAC_t(X_i|X) = \frac{\frac{\partial M_t(u)}{\partial u_i}}{a_{ti}(u) - \frac{\partial M_t(u)}{\partial u_i}} = \frac{M'_{ti(}u_i)}{a_{ti}(u) - M'_{ti}(u_i)}$$

where $M_t(u) = (M_{t1}(u_1), \dots, M_{tn}(u_n))'$. We get

$$\frac{\partial R_t(u)}{\partial u_i} = (\rho_t^Y(u) - M_t(u))^{-2}\Bigg(M'_{ti}(u_i)(p_{ty}(u) - M_t(u)) - M_t(u)\bigg(\frac{\partial \rho_t^Y(u)}{\partial u_i} - M'_{ti}(u_i)\bigg)\Bigg)$$

so that

$$\frac{\partial R_t(u)}{\partial u_i} = (\rho_t^Y(u) - M_t(u))^{-2}\left(M'_{ti}(u_i)\rho_t^Y(u) - M_t(u)\frac{\partial \rho_t^Y(u)}{\partial u_i}\right).$$

If $a_{ti}(u) = \frac{\partial \rho_t^Y(u)}{\partial u_i}$, then

$$\frac{\partial R_t(u)}{\partial u_i} = (\rho_t^Y(u) - M_t(u))^{-2}(M'_{ti}(u_i)\rho_t^Y(u) - M_t(u)a_{ti}(u)).$$

If

$$RORAC_t(X_i|X) > R_t(u),$$

we get

$$\frac{M'_{ti}(u_i)}{a_{ti}(u) - M'_{ti}(u_i)} > \frac{M_t(u)}{\rho_t^Y(u) - M_t(u)},$$

$$M'_{ti}(u_i)(\rho_t^Y(u) - M_t(u)) > (a_{ti}(u) - M'_{ti}(u_i))M_t(u),$$

$$M'_{ti}(u_i)\rho_t^Y(u) > a_{ti}(u)M_t(u),$$

$$M'_{ti}(u_i)\rho_t^Y(u) - a_{ti}(u)M_t(u) > 0.$$

Then $\frac{\partial R_t(u)}{\partial u_i} > 0$, such that R_t is an increasing function. This implies that there exists an $\mathcal{F}_t$-measurable random variable ϵ_i with $\epsilon_i > 0$ such that $RORAC_t(X+hX_i) > RORAC_t(X)$ for all $\mathcal{F}_t$-measurable random variables h with $0 < h < \epsilon_i$. So we have proved that if $a_{ti}(u) = \frac{\partial \rho_t^Y(u)}{\partial u_i}$, then $\rho_t(X_i|X)$ is $RORAC_t$ compatible.

Next, we prove $a_{ti}(u) = \frac{\partial \rho_t^Y(u)}{\partial u_i}$ is equivalent to (2.5) as follows. Since

$\rho_t^X(u) = \rho_t^Y(u) - M_t(u)$,

$$\frac{\partial \rho_t^X(u)}{\partial u_i} = \frac{\partial \rho_t^Y(u)}{\partial u_i} - \frac{\partial M_t(u)}{\partial u_i} = a_{ti}(u) - M'_{ti}(u_i)$$

and $\rho_t^{Euler}(X_i|X) = \frac{d\rho_t}{dh}(X + hX_i)\big|_{h=0}$ = per-unit risk contribution of the total profit at time t = per-unit risk contribution of profit fluctuation at time t − per-unit risk contribution of expect profit at time $t = a_{ti}(u) - M'_{ti}(u_i) = \frac{\partial \rho_t^X(u)}{\partial u_i} = \frac{\partial \rho_t(X(u))}{\partial u_i} = \frac{\partial f_{\rho t}}{\partial u_i}$.

Thus, if $\rho_t^{Euler}(X_i|X) = \frac{d\rho_t}{dh}(X+hX_i)\big|_{h=0} = \frac{\partial f_{\rho t}}{\partial u_i}$, then $\rho_t(X_i|X)$ is $RORAC_t$ compatible.

Secondly, we prove if given $\rho_t(X_i|X)$ is $RORAC_t$ compatible, then $\rho_t(X_i|X)$ is uniquely determined by (2.5). Let

$$M_t(u) = m_t^\top u = \sum_{i=1}^n m_{ti} u_i$$

where m_{ti} is the expected profit/loss by asset i at time t. Thus,

$$\frac{\partial M_t(u)}{\partial u_i} = M'_{ti}(u_i) = m_{ti}.$$

Define $m_t(b)$ as a random vector valued in $\mathbb{R}^d$ by

$$\begin{aligned}
m_{ti}(b) &\stackrel{\text{def}}{=} 1, \\
m_{tj}(b) &\stackrel{\text{def}}{=} \frac{b}{u_j}\left(\frac{\rho_t^Y(u)}{a_{ti}(u)} - u_i\right), \\
m_{tl}(b) &\stackrel{\text{def}}{=} 0 \text{ for } l \neq i, j,
\end{aligned}$$

then

$$m_t(b)^\top u = b\frac{\rho_t^Y(u)}{a_{ti}(u)} + (1-b)u_i$$

and

$$m_{ti}(b)\rho_t^Y(u) - a_{ti}(u)m_t(b)^\top u = (1-b)(\rho_t^Y(u) - u_i a_{ti}(u)).$$

We obtain

$$\begin{aligned}
&(1-b)(\rho_t^Y(u) - u_i a_{ti}(u)) + \left(a_{ti}(u) - \frac{\partial \rho_t^Y(u)}{\partial u_i}\right)\left(b\frac{\rho_t^Y(u)}{a_{ti}(u)} + (1-b)u_i\right) \\
&= m_{ti}(b)\rho_t^Y(u) - a_{ti}(u)m_t(b)^\top u + \left(a_{ti}(u) - \frac{\partial \rho_t^Y(u)}{\partial u_i}\right)m_t(b)^\top u \\
&= m_{ti}(b)\rho_t^Y(u) - a_{ti}(u)m_t(b)^\top u + a_{ti}(u)m_t(b)^\top u - m_t(b)^\top u\frac{\partial \rho_t^Y(u)}{\partial u_i} \\
&= m_{ti}(b)\rho_t^Y(u) - m_t(b)^\top u\frac{\partial \rho_t^Y(u)}{\partial u_i}.
\end{aligned}$$

We can choose a sequence b_k with $b_k \to 1$ such that $RORAC_t(X_i|X) > RORAC_t(X)$ and $\rho_t(X_i|X)$ are $RORAC_t$ compatible, then $RORAC_t(X + hX_i|X) > RORAC_t(X)$ almost surely, we can get $\frac{\partial R_t(u)}{\partial u_i} \geq 0$, so $M'_{ti}(u_i)\rho_t^Y(u) - M_t(u)\frac{\partial \rho_t^Y(u)}{\partial u_i} \geq 0$.

As $M_t(u) = m_t^\top u$ and $M'_{ti}(u_i) = m_{ti}$, taking b as b_k, we get $m_{ti}(b_k)\rho_t^Y(u) - m_t(b_k)^\top u\frac{\partial \rho_t^Y(u)}{\partial u_i} \geq 0$, thus $(1-b_k)(\rho_t^Y(u) - u_i a_{ti}(u)) + (a_{ti}(u) - \frac{\partial \rho_t^Y(u)}{\partial u_i})(b_k\frac{\rho_t^Y(u)}{a_{ti}(u)} + (1-b_k)u_i) \geq 0$.

Similarly, we can choose a sequence v_k with $v_k \to 1$ so that we deduce $RORAC_t(X_i|X) < RORAC_t(X)$ and get $\frac{\partial R_t(u)}{\partial u_i} \leq 0$, then taking b as v_k, we get $m_{ti}(v_k)\rho_t^Y(u) - m_t(v_k)^\top u\frac{\partial \rho_t^Y(u_i)}{\partial u_i} \leq 0$. Thus,

$$(1-v_k)(\rho_t^Y(u) - u_i a_{ti}(u)) + (a_{ti}(u) - \frac{\partial \rho_t^Y(u)}{\partial u_i})(v_k\frac{\rho_t^Y(u)}{a_{ti}(u)} + (1-v_k)u_i) \leq 0.$$

When $k \to \infty$,

$$0 \leq a_{ti}(u) - \frac{\partial \rho_t^Y(u)}{\partial u_i} \leq 0.$$

We get

$$a_{ti}(u) = \frac{\partial \rho_t^Y(u)}{\partial u_i},$$

which is equivalent to $\rho_t^{Euler}(X_i|X) = \frac{d\rho_t}{dh}(X + hX_i)\big|_{h=0} = \frac{\partial f_{\rho t}}{\partial u_i}$, as shown before. So, if $\rho_t(X_i|X)$ is $RORAC_t$ compatible, then we have $\rho_t^{Euler}(X_i|X) = \frac{d\rho_t}{dh}(X + hX_i)\big|_{h=0} = \frac{\partial f_{\rho t}}{\partial u_i}$. □

Similarly to Section 2.1, the full-allocation property is satisfied when the risk measure is homogeneous of degree 1, which means $\rho_t(\tau X) = \tau \rho_t(X)$ for all $\tau > 0$.

We next analyze the relation between the time consistency of risk measures and the time consistency of the risk contributions. To this end, we recall the following definition; see for example, Acciaio and Penner [1].

Definition 2.2.5. *A dynamic risk measure is time consistent if*

$$\rho_t(X) > \rho_t(Y) \Rightarrow \rho_s(X) > \rho_s(Y) \quad \textit{for all } s \leq t \textit{ and } X, Y \in \mathcal{L}^\infty$$

Proposition 2.2.2. *Assume:*

- *ρ is time consistent*
- *risk contributions are $RORAC_t$ and $RORAC_s$ compatible for $t \geq s$*
- *f_{ρ_t} and f_{ρ_s} given in (2.4) are continuously differentiable,*

then $\rho_t(X_i|X) > \rho_t(X_j|X) \Rightarrow \rho_s(X_i|X) > \rho_s(X_j|X)$.

Proposition 2.2.2 shows that for $RORAC$ compatible risk contributions, time consistency of the risk measure translates to time consistency of risk contributions.

Proof. From Proposition 2.2.1, since risk contributions are $RORAC_t$ compatible, we get $\rho_t(X_i|X) = \frac{d\rho_t}{dh}(X + hX_i)\big|_{h=0}$ and $\rho_t(X_j|X) = \frac{d\rho_t}{dh}(X + hX_j)\big|_{h=0}$. Similarly, as risk contributions are $RORAC_s$ compatible, we obtain $\rho_s(X_i|X) = \frac{d\rho_s}{dh}(X + hX_i)\big|_{h=0}$ and $\rho_s(X_j|X) = \frac{d\rho_s}{dh}(X + hX_j)\big|_{h=0}$.

If $\rho_t(X_i|X) > \rho_t(X_j|X)$, we obtain $\frac{d\rho_t}{dh}(X + hX_i)\big|_{h=0} > \frac{d\rho_t}{dh}(X + hX_j)\big|_{h=0}$. Then we can conclude that $\rho_t(X + hX_i) > \rho_t(X + hX_j)$ for small h. Using Definition 2.2.5 and that ρ is time consistent by the first assumption, we get $\rho_s(X + hX_i) > \rho_s(X + hX_j)$ for all $s \leq t$ and small h. Therefore, we have $\frac{d\rho_s}{dh}(X+hX_i)\big|_{h=0} > \frac{d\rho_s}{dh}(X+hX_j)\big|_{h=0}$. This implies $\rho_s(X_i|X) > \rho_s(X_j|X)$. $\square$

Chapter 3

Risk Attribution

In this chapter, we introduce risk attribution, which is about identifying and quantifying risk drivers, risk classifications, and risk management features. We begin with a simple model which contains two risk factors. Then we analyze a model that has d risk factors.

3.1 Static Setting

In this section, we discuss the approximation of the total loss variable and loss variables of divisions at time 0, following the approach of Frei [8].

3.1.1 Two-Factor Model

Assume a portfolio has two risk factors. We use random variables R^1, R^2 to represent them. There is a function f that transforms the risk factors to the total loss variable $L = f(R^1, R^2)$. Thus, the risk measure of the portfolio is $\rho(-f(R^1, R^2))$.

For risk attribution, the risk drivers may contribute to a portfolio's profit/loss in a nonlinear way, so we need to linearize the loss $L = f(R^1, R^2) \approx A^1 + A^2$, where A^i is the loss contribution of the i^{th} risk factor ($i \in \{1, 2\}$). We next discuss how to determine A^1 and A^2.

We consider the risk factors in discrete time for T time steps. We use R^i_t to denote the value of the i^{th} risk factor at time t, where $i \in \{1, 2\}$ and $t \in \{0, 1, \ldots, T\}$. The loss at time T is $f(R^1_T, R^2_T) \approx A^1 + A^2 + f(R^1_t, R^2_t)$. We define A^1 as the losses arising from changes in R^1 while R^2 is fixed. This implies that $A^1 = f(R^1_T, R^2_0) - f(R^1_0, R^2_0)$. Simply taking $f(R^1_0, R^2_0) = 0$ and defining A^2 by using the same logic, we get

$$f(R^1_T, R^2_T) \approx f(R^1_T, R^2_0) + f(R^1_0, R^2_T).$$

To avoid a significant estimation error, we prefer computing the loss contributions by summing up the marginal changes at different time steps, rather than just one step, which leads to

$$A^1 = \sum_{t=0}^{T-1} \big(f(R^1_{t+1}, R^2_t) - f(R^1_t, R^2_t)\big) \quad \text{and} \quad A^2 = \sum_{t=0}^{T-1} \big(f(R^1_t, R^2_{t+1}) - f(R^1_t, R^2_t)\big).$$

The approximation error can be decreased by dividing the time horizon into more steps.

3.1.2 Multi-Factor Model

In this subsection, we discuss the general case with d risk factors. Applying a similar idea to sum up the marginal changes in losses resulting from R^j at different time steps while R^i for $i \neq j$ is fixed. This means the loss contribution A^j is defined by

$$A^j = \sum_{t=0}^{T-1} \big(f(R^j_{t+1}, (R^i_t)_{i \neq j}) - f((R^i_t)_i) \big), \tag{3.1}$$

where $f(R^j_t + 1, (R^i_t)_{i \neq j}) - f((R^i_t)_i)$ is the change in losses caused by R^j from time t to time $t+1$ while R^i for $i \neq j$ is fixed.

We obtain that A^j is the risk comes from the j^{th} risk driver while the other risk factors remain at the current values. Thus, the approximate change in total loss from time 0 to T is the sum of A^j for $j = 1, \dots, d$ whereas the real change is $f((R^i_T)_i) - f((R^i_0)_i)$. The overall residual, which is the difference between the approximate change and the real change, is

$$\begin{aligned}
&f((R^i_T)_i) - f((R^i_0)_i) - \sum_{j=1}^{d} A^j \\
&= \sum_{t=0}^{T-1} \big(f((R^i_{t+1})_i) - f((R^i_t)_i) \big) - \sum_{j=1}^{d} A^j \\
&= \sum_{t=0}^{T-1} \big(f((R^i_{t+1})_i) - f((R^i_t)_i) \big) - \sum_{j=1}^{d} \sum_{t=0}^{T-1} \big(f(R^j_t + 1, (R^i_t)_{i \neq j}) - f((R^i_t)_i) \big) \\
&= \sum_{t=0}^{T-1} \big(f((R^i_{t+1})_i) - f((R^i_t)_i) - \sum_{j=1}^{d} \big(f(R^j_t + 1, (R^i_t)_{i \neq j}) - f((R^i_t)_i) \big) \big).
\end{aligned}$$

For the following results, taken from Frei [8], we fix the time horizon T, assume that the risk factors are observable continuously on $[0, T]$, and consider the loss contributions on a more and more granular time grid. The result then shows that the sum of the loss contributions converges to the total losses under suitable conditions.

Proposition 3.1.1. *Assume $L = f(R_T^1, \ldots, R_T^d)$ for a twice continuously differentiable f and let $(R_t^1 \ldots, R_t^d)_{t\in[0,T]}$ be a continuous semimartingale on $[0,T]$ with zero quadratic covariation $\langle R^i, R^j\rangle_t = 0$ for all $t \in [0,T]$ and $i \neq j$. We set*

$$A_N^j = \sum_{n=0}^{N-1} \left(f(R_{t_{n+1}}^j, (R_{t_n}^i)_{i\neq j}) - f((R_{t_n}^i)_i) \right)$$

for $0 = t_0 \leq t \leq \cdots \leq t_N = T$. Then $\sum_{j=1}^d A_N^j + f(R_0^1, \ldots, R_0^d)$ converges to L almost surely as $N \to \infty$.

Proof. By Itô formula, we get

$$\begin{aligned} f(R_T^1, \ldots, R_T^d) = f(R_0^1, \ldots, R_0^d) &+ \sum_{j=1}^d \int_0^T f_{x^j}(R_t^1, \ldots, R_t^d)\, dR_t^j \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^T f_{x^i x^j}(R_t^1, \ldots, R_t^d)\, d\langle R^i, R^j\rangle_t. \end{aligned}$$

We can rewrite it as,

$$\begin{aligned} f(R_T^1, \ldots, R_T^d) - f(R_0^1, \ldots, R_0^d) = &\sum_{j=1}^d \int_0^T f_{x^j}(R_t^1, \ldots, R_t^d)\, dR_t^j \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^T f_{x^i x^j}(R_t^1, \ldots, R_t^d)\, d\langle R^i, R^j\rangle_t. \end{aligned}$$

Since $\langle R^i, R^j\rangle_t = 0$ for all $i \neq j$ by assumption, we obtain

$$\begin{aligned} f(R_T^1, \ldots, R_T^d) - f(R_0^1, \ldots, R_0^d) = &\sum_{j=1}^d \int_0^T f_{x^j}(R_t^1, \ldots, R_t^d)\, dR_t^j \\ &+ \frac{1}{2} \sum_{j=1}^d \int_0^T f_{x^j x^j}(R_t^1, \ldots, R_t^d)\, d\langle R^j, R^j\rangle_t. \end{aligned}$$

By the proof of Theorem 3.3 in Karatzas and Shreve [11], we conclude that A_N^j converges almost surely to

$$\int_0^T f_{x^j}(R_t^1,\dots,R_t^d)\,R_t^j + \frac{1}{2}\int_0^T f_{x^jx^j}(R_t^1,\dots,R_t^d)\,d\langle R^j,R^j\rangle_t$$

Thus,

$$\begin{aligned}
\lim_{N\to\infty}\sum_{j=1}^d A_N^j + f(R_0^1,\dots,R_0^d) &= \sum_{j=1}^d \int_0^T f_{x^j}(R_t^1,\dots,R_t^d)\,dR_t^j \\
&\quad + \frac{1}{2}\sum_{j=1}^d \int_0^T f_{x^jx^j}(R_t^1,\dots,R_t^d)\,d\langle R^j,R^j\rangle_t \\
&\quad + f(R_0^1,\dots,R_0^d) \\
&= f(R_T^1,\dots,R_T^d) \\
&= L \quad \text{almost surely.}
\end{aligned}$$

□

We assume there are K divisions in the company, and $L^k = f^k((R_T^i)_i)$ is the loss contribution of the k^{th} division to the total loss variable L, where $L = \sum_{k=1}^K L^k$. We next introduce A^{jk}, which is the change in loss due to the j^{th} risk factor in the k^{th} division. Similar to (3.1), we have

$$A^{jk} = \sum_{t=0}^{T-1}\big(f^k(R_{t+1}^j,(R_t^i)_{i\neq j}) - f^k((R_t^i)_i)\big).$$

Therefore, we can use $\sum_{j=1}^d A^{jk}$ to approximate L^k.

3.2 Dynamic Setting

In this section, we discuss the approximation of the total loss variable and loss variables of divisions at time t.

3.2.1 Two-Factor Model

Assume a portfolio has two risk factors, and we use random variables R^1_T, R^2_T at time T to represent them. There is a function f that transforms the risk factors to the total loss variable $L = f(R^1_T, R^2_T)$. Thus, the risk measure of the portfolio is $\rho_t(-f(R^1_T, R^2_T))$ at time t.

For risk attribution, the risk drivers may contribute to a portfolio's profit/loss in a nonlinear way, so we need to linearize the loss $L = f(R^1_T, R^2_T) \approx A^1 + A^2 + f(R^1_t, R^2_t)$, where A^i is the loss contribution of the i^{th} risk factor ($i \in \{1, 2\}$). We next discuss how to determine A^1 and A^2.

We consider the risk factors on the fixed time horizon T. Therefore, we use R^i_t to denote the value of the i^{th} risk factor at time t, where $i \in \{1, 2\}$. The loss at time T is $f(R^1_T, R^2_T) \approx A^1 + A^2 + f(R^1_t, R^2_t)$. We define A^1 as the losses arising from changes in R^1 while R^2 is fixed. This implies that $A^1 = f(R^1_T, R^2_t) - f(R^1_t, R^2_t)$. Simply taking $f(R^1_t, R^2_t) = X_t$ and defining A^2 by using the same logic, we get

$$\begin{aligned} f(R^1_T, R^2_T) &\approx f(R^1_T, R^2_t) - X_t + f(R^1_t, R^2_T) - X_t + X_t \\ &\approx f(R^1_T, R^2_t) + f(R^1_t, R^2_T) - X_t. \end{aligned}$$

To avoid a significant estimation error, we prefer computing the loss contributions by summing up the marginal changes at different time steps rather than just one step, which leads to

$$A^1 = \sum_{s=t}^{T-1} \big(f(R^1_{s+1}, R^2_s) - f(R^1_s, R^2_s)\big) \quad \text{and} \quad A^2 = \sum_{s=t}^{T-1} \big(f(R^1_s, R^2_{s+1}) - f(R^1_s, R^2_s)\big).$$

The approximation error can be decreased by dividing the time into more steps.

3.2.2 Multi-Factor Model

In this subsection, we discuss the general case with d risk factors. Applying the similar idea that summing up the marginal changes in losses result from R^j at different time steps while R^i for $i \neq j$ is fixed. This means the loss contribution A^j is defined by

$$A^j = \sum_{s=t}^{T-1} \big(f(R^j_{s+1}, (R^i_s)_{i\neq j}) - f((R^i_s)_i)\big), \tag{3.2}$$

where $f(R^j_s + 1, (R^i_s)_{i\neq j}) - f((R^i_s)_i)$ is the change in losses caused by R^j from time s to time $s+1$ while R^i for $i \neq j$ is fixed.

We obtain that A^j is the risk coming from the j^{th} risk driver while the other risk factors remain at the current values. Thus, the approximate change in total loss from time t to T is the sum of A^j for $j = 1, \ldots, d$ whereas the real change is $f((R^i_T)_i) - f((R^i_t)_i)$. Then the overall residual, which is the difference between the approximate change and the real change, is

$$f((R^i_T)_i) - f((R^i_t)_i) - \sum_{j=1}^{d} A^j = \sum_{s=t}^{T-1} \big(f((R^i_{s+1})_i) - f((R^i_s)_i)\big) - \sum_{j=1}^{d} A^j,$$

which can be written as

$$
\begin{aligned}
&\sum_{s=t}^{T-1}\big(f((R^i_{s+1})_i)-f((R^i_s)_i)\big)-\sum_{j=1}^{d}\sum_{s=t}^{T-1}\big(f(R^j_{s+1},(R^i_s)_{i\neq j})-f((R^i_s)_i)\big)\\
&=\sum_{s=t}^{T-1}\Bigg(f((R^i_{s+1})_i)-f((R^i_s)_i)-\sum_{j=1}^{d}\big(f(R^j_{s+1},(R^i_s)_{i\neq j})-f((R^i_s)_i)\big)\Bigg).
\end{aligned}
$$

The following result is the analogue to Proposition 3.1.1 for dynamic risk measures. We considered a fixed time interval $[t,T]$ and assume that the risk factors are observable continuously on $[t,T]$.

Proposition 3.2.1. *Assume $L=f(R^1_T,\dots,R^d_T)$ for a twice continuously differentiable f and let $(R^1_s\dots,R^d_s)_{s\in[t,T]}$ be a continuous semimartingale on $[t,T]$ with zero quadratic covariation $\langle R^i,R^j\rangle_s=0$ for all $s\in[t,T]$ and $i\neq j$. We set*

$$
A^j_N=\sum_{n=1}^{N-1}\big(f(R^j_{s_{n+1}},(R^i_{s_n})_{i\neq j})-f((R^i_{s_n})_i)\big)
$$

for $t=s_0\leq s_1\leq\cdots\leq s_N=T$. Then $\sum_{j=1}^d A^j_N+f(R^1_t,\dots,R^d_t)$ converges to L almost surely as $N\to\infty$.

Proof. Similar to the proof of Proposition 3.1.1, we can get

$$
\begin{aligned}
\lim_{N\to\infty}\sum_{j=1}^{d}A^j_N+f(R^1_t,\dots,R^d_t)&=\sum_{j=1}^{d}\int_t^T f_{x^j}(R^1_s,\dots,R^d_s)\,dR^j_s\\
&\quad+\frac{1}{2}\sum_{j=1}^{d}\int_t^T f_{x^jx^j}(R^1_s,\dots,R^d_s)\,d\langle R^j,R^j\rangle_s\\
&\quad+f(R^1_t,\dots,R^d_t)\\
&=f(R^1_T,\dots,R^d_T)\\
&=L\quad\text{almost surely.}
\end{aligned}
$$

□

We assume there are K divisions in the company and $L^k = f^k((R_T^i)_i)$ is the loss contribution of the k^{th} division to the total loss variable L, where $L = \sum_{k=1}^{K} L^k$. We next introduce A^{jk}, which is the change in loss due to j^{th} risk factor in the k^{th} division. Similar to (3.2), we have

$$A^{jk} = \sum_{s=t}^{T-1} \big(f^k(R_{s+1}^j, (R_s^i)_{i \neq j}) - f^k((R_s^i)_i) \big).$$

Therefore, we can use $\sum_{j=1}^{d} A^{jk}$ to approximate L^k.

From Proposition 3.2.1, we have $L = \lim_{N\to\infty} \sum_{j=1}^{d} A_N^j + f(R_t^1, \ldots, R_t^d)$. Assume that $f(R_t^1, \ldots, R_t^d) = 0$, then $L = \lim_{N\to\infty} \sum_{j=1}^{d} A_N^j$. Thus, we conclude that, for all $\epsilon > 0$, there exist an N_0 such that $\big| \sum_{j=1}^{d} A_N^j - L \big| < \epsilon$ for all $N \geq N_0$ almost surely. After letting $\sum_{j=1}^{d} A_N^j = A_N$, we apply $RORAC$ to A_N^j,

$$RORAC_t(A_N) \stackrel{\text{def}}{=} \frac{E[A_N|\mathcal{F}t]}{\rho_t(A_N)},$$

$$RORAC_t(A_N^j|A_N) \stackrel{\text{def}}{=} \frac{E[A_N^j|\mathcal{F}_t]}{\rho_t(A_N^j|A_N)}.$$

Risk contribution $\rho(A_N^1|A_N), \ldots, \rho(A_N^d|A_N)$ are $RORAC_t$ compatible if there exists an $\mathcal{F}_t$-measurable random variable ϵ_j with $\epsilon_j > 0$ such that

$$RORAC_t(A_N^j|A_N) > RORAC_t(A_N) \Rightarrow RORAC_t(A_N + hA_N^j) > RORAC_t(A_N)$$

almost surely, for any $\mathcal{F}_t$-measurable random variable h with $0 < h < \epsilon_j$. We define

$$\rho_t^{Euler}(A_N^j|A_N) = \frac{d\rho_t}{dh}(A_N + hA_N^j)\bigg|_{h=0}, \tag{3.3}$$

which is $RORAC_t$ compatible if (2.4) is continuously differentiable for $X_i =$

A_N^j; this can be shown similarly to Proposition 2.2.1.

Proposition 3.2.2. *If* $\left|\sum_{j=1}^d A_N^j - L\right| < \epsilon$ *almost surely, then* $\left|\rho_t(\sum_{j=1}^d A_N^j) - \rho_t(L)\right| < \epsilon$ *almost surely.*

Proof. $\left|\sum_{j=1}^d A_N^j - L\right| < \epsilon$ almost surely implies $-\epsilon < \sum_{j=1}^d A_N^j - L < \epsilon$ almost surely. Thus, $L - \epsilon < \sum_{j=1}^d A_N^j < L + \epsilon$ almost surely. Then we apply ρ_t on the both side of this inequality, by monotonicity, we obtain,

$$\rho_t(L+\epsilon) < \rho_t\left(\sum_{j=1}^d A_N^j\right) < \rho_t(L-\epsilon).$$

And by translativity,

$$\rho_t(L) - \epsilon < \rho_t\left(\sum_{j=1}^d A_N^j\right) < \rho_t(L) + \epsilon.$$

This gives

$$-\epsilon < \rho_t\left(\sum_{j=1}^d A_N^j\right) - \rho_t(L) < \epsilon$$

and thus,

$$\left|\rho_t\left(\sum_{j=1}^d A_N^j\right) - \rho_t(L)\right| < \epsilon.$$

□

Proposition 3.2.3. *Assume:*

- ρ *is time consistent,*
- *Risk contributions* $\rho_t(A_N^1|A_N), \ldots, \rho_t(A_N^d|A_N)$ *are both* $RORAC_t$ *and* $RORAC_s$ *compatible for* $t \geq s$*,*
- f_{ρ_t} *and* f_{ρ_s} *given in* (2.4) *are continuously differentiable,*

- $\frac{d\rho_t(A_N+hA_N^j)}{dh}\Big|_{h=0} - \frac{d\rho_t(L+hA_N^j)}{dh}\Big|_{h=0}$ *and* $\frac{d\rho_s(A_N+hA_N^j)}{dh}\Big|_{h=0} - \frac{d\rho_s(L+hA_N^j)}{dh}\Big|_{h=0}$ *converge to zero almost surely for all j,*

then $\liminf_{N\to\infty}(\rho_t(A_N^i|L) - \rho_t(A_N^j|L)) > 0 \Rightarrow \liminf_{N\to\infty}(\rho_s(A_N^i|L) - \rho_s(A_N^j|L)) \geq 0.$

Proof. Because of the convergence in the last assumption, for every $\epsilon > 0$, there exists N_0 such that

$$\left| \frac{d\rho_t(A_N + hA_N^j)}{dh}\Big|_{h=0} - \frac{d\rho_t(L + hA_N^j)}{dh}\Big|_{h=0} \right| < \epsilon$$

for all $N \geq N_0$. Therefore, if $\liminf_{N\to\infty}(\rho_t(A_N^i|L) - \rho_t(A_N^j|L)) > 0$, then

$$\liminf_{N\to\infty} \left(\frac{d\rho_t(L + hA_N^i)}{dh}\Big|_{h=0} - \frac{d\rho_t(L + hA_N^j)}{dh}\Big|_{h=0} \right) > 0$$

and

$$\liminf_{N\to\infty} \left(\frac{d\rho_t(A_N + hA_N^i)}{dh}\Big|_{h=0} - \frac{d\rho_t(A_N + hA_N^j)}{dh}\Big|_{h=0} \right) > 0,$$

which is equivalent to

$$\liminf_{N\to\infty}(\rho_t(A_N^i|A_N) - \rho_t(A_N^j|A_N)) > 0.$$

So we conclude that, for N large enough,

$$\rho_t(A_N^i|A_N) - \rho_t(A_N^j|A_N) > 0.$$

By Proposition 2.2.2, for N large enough,

$$\rho_s(A_N^i|A_N) - \rho_s(A_N^j|A_N) > 0.$$

Therefore, we get

$$\liminf_{N\to\infty}(\rho_s(A_N^i|A_N) - \rho_s(A_N^j|A_N)) \geq 0.$$

In conclusion, if we have the four assumptions, then

$$\liminf_{N\to\infty}(\rho_t(A_N^i|L) - \rho_t(A_N^j|L)) > 0$$

implies $\liminf_{N\to\infty}(\rho_s(A_N^i|L) - \rho_s(A_N^j|L)) \geq 0$. $\square$

Chapter 4

Example

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$. For a random variable X, let $\rho_t(X) = \frac{1}{\gamma}\ln E[e^{-\gamma X}|\mathcal{F}_t]$, where γ is a positive constant and ρ_0 is called the entropic risk measure. To prove that ρ is a dynamic risk measure, we check its normalization, transitivity and monotonicity:

Check normalization

$$\rho_t(0) = \frac{1}{\gamma}\ln E[e^0|\mathcal{F}_t] = \frac{1}{\gamma}\ln(1) = 0$$

Check transitivity

Let $a_t \in \mathcal{L}_t^\infty$, then

$$\begin{aligned}\rho_t(X + a_t) &= \frac{1}{\gamma}\ln E[e^{-\gamma(X+a_t)}|\mathcal{F}_t] \\ &= \frac{1}{\gamma}\ln(E[e^{-\gamma X}|\mathcal{F}_t]e^{-\gamma a_t}) \\ &= \frac{1}{\gamma}\ln E[e^{-\gamma X}|\mathcal{F}_t] + \frac{1}{\gamma}\ln(e^{-\gamma a_t})\end{aligned}$$

$$= \rho_t(X) + \frac{1}{\gamma}(-\gamma a_t)$$

$$= \rho_t(X) - a_t.$$

Check monotonicity

Let $X_1,\ X_2 \in \mathcal{L}^\infty$ and $X_1 \le X_2$ almost surely. Recall that in this example $\rho_t(X_1) = \frac{1}{\gamma} \ln E\big[e^{-\gamma X_1}\big|\mathcal{F}_t\big]$ and $\rho_t(X_2) = \frac{1}{\gamma} \ln E\big[e^{-\gamma X_2}\big|\mathcal{F}_t\big]$. And $X_1 \le X_2$ almost surely $\Rightarrow -X_1 \ge -X_2$ almost surely $\Rightarrow e^{-\gamma X_1} \ge^{-\gamma X_2}$, by the monotonicity of conditional expectation, we get $E\big[e^{-\gamma X_1}\big|\mathcal{F}_t\big] \ge E\big[e^{-\gamma X_2}\big|\mathcal{F}_t\big]$. Since ln is an increasing function and γ is a positive constant, we conclude that

$$\frac{1}{\gamma} \ln E\big[e^{-\gamma X_1}\big|\mathcal{F}_t\big] \ge \frac{1}{\gamma} \ln E\big[e^{-\gamma X_2}\big|\mathcal{F}_t\big]$$

$$\Rightarrow \rho_t(X_1) \ge \rho_t(X_2).$$

As ρ_t satisfies the three properties, it is a dynamic risk measure. Next, we check the time consistency.

Check time consistency

Let $\rho_t(X) \ge \rho_t(Y)$ and $s \le t$, then

$$\frac{1}{\gamma} \ln E\big[e^{-\gamma X}\big|\mathcal{F}_t\big] \ge \frac{1}{\gamma} \ln E\big[e^{-\gamma Y}\big|\mathcal{F}_t\big]$$

$$\Rightarrow E\big[e^{-\gamma X}\big|\mathcal{F}_t\big] \ge E\big[e^{-\gamma Y}\big|\mathcal{F}_t\big]$$

$$\Rightarrow E\big[E\big[e^{-\gamma X}\big|\mathcal{F}_t\big]\big|\mathcal{F}_s\big] \ge E\big[E\big[e^{-\gamma Y}\big|\mathcal{F}_t\big]\big|\mathcal{F}_s\big]$$

by the tower property of conditional expectation for $s \leq t$,

$$\Rightarrow E\left[e^{-\gamma X}\middle|\mathcal{F}_s\right] \geq E\left[e^{-\gamma Y}\middle|\mathcal{F}_s\right]$$

$$\Rightarrow \frac{1}{\gamma}\ln E\left[e^{-\gamma X}\middle|\mathcal{F}_s\right] \geq \frac{1}{\gamma}\ln E\left[e^{-\gamma Y}\middle|\mathcal{F}_s\right]$$

$$\Rightarrow \rho_s(X) \geq \rho_s(Y)$$

We proved that $\rho_t(X) \geq \rho_t(Y) \Rightarrow \rho_s(X) \geq \rho_s(Y)$ for all $s \leq t$.

Analyze risk allocation

Let $X_1 = \sigma_1 W_1(T)$, and $X_2 = \sigma_2 W_2(T)$, where W_1 and W_2 are Brownian motion with correlation ρ. We can write $W_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_3(t)$ for all t, where W_3 is a Brownian motion independent of W_1. The total loss is $X = X_1 + X_2 = \sigma_1 W_1(T) + \sigma_2 W_2(T) = \sigma_1 W_1(T) + \sigma_2(\rho W_1(T) + \sqrt{1-\rho^2}W_3(T)) = (\sigma_1 + \rho\sigma_2)W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T)$.

By definition, $\rho_t^{Euler}(X_1|X) = \frac{d\rho_t(X+hX_1)}{dh}\Big|_{h=0}$. Firstly, we calculate

$$\begin{aligned}
&\rho_t(X + hX_1)\\
&= \rho_t\Big((\sigma_1 + \rho\sigma_2)W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T) + h\sigma_1 W_1(T)\Big)\\
&= \rho_t\Big(((1+h)\sigma_1 + \rho\sigma_2)W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T)\Big)\\
&= \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma(((1+h)\sigma_1 + \rho\sigma_2)W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T))\Big)\Big|\mathcal{F}_t\Big]\\
&= \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma(((1+h)\sigma_1 + \rho\sigma_2)(W_1(T) - W_1(t) + W_1(t))\\
&\quad + \sqrt{1-\rho^2}\sigma_2(W_3(T) - W_3(t) + W_3(t)))\Big)\Big|\mathcal{F}_t\Big]\\
&= \frac{1}{\gamma}\ln E\Big[\exp(-\gamma((1+h)\sigma_1 + \rho\sigma_2)(W_1(T) - W_1(t)))\\
&\quad \times \exp(-\gamma((1+h)\sigma_1 + \rho\sigma_2)W_1(t)) \times \exp\Big(-\gamma\sqrt{1-\rho^2}\sigma_2(W_3(T) - W_3(t))\Big)
\end{aligned}$$

$$\times \exp\Big(- \gamma\sqrt{1-\rho^2}\sigma_2(W_3(t))\Big)\Big|\mathcal{F}_t\Big].$$

Since $W_1(T) - W_1(t)$ and $W_3(T) - W_3(t)$ are independent of $\mathcal{F}_t$, $W_1(t)$ and $W_3(t)$ are $\mathcal{F}_t$-measurable, we get

$$\begin{aligned}
\rho_t(X + hX_1) &= \frac{1}{\gamma} \ln E\Big[\exp(-\gamma((1+h)\sigma_1 + \rho\sigma_2)(W_1(T) - W_1(t))) \\
&\quad \times \exp(-\gamma((1+h)\sigma_1 + \rho\sigma_2)W_1(t)) \\
&\quad \times \exp\Big(- \gamma\sqrt{1-\rho^2}\sigma_2(W_3(T) - W_3(t))\Big) \\
&\quad \times \exp\Big(- \gamma\sqrt{1-\rho^2}\sigma_2(W_3(t))\Big)\Big|\mathcal{F}_t\Big] \\
&= \frac{1}{\gamma} \ln E[\exp(-\gamma((1+h)\sigma_1 + \rho\sigma_2)(W_1(T) - W_1(t)))] \\
&\quad + \frac{1}{\gamma}(-\gamma((1+h)\sigma_1 + \rho\sigma_2)W_1(t)) \\
&\quad + \frac{1}{\gamma} \ln E\Big[\exp(-\gamma\sqrt{1-\rho^2}\sigma_2(W_3(T) - W_3(t)))\Big] \\
&\quad + \frac{1}{\gamma}\Big(- \gamma\sqrt{1-\rho^2}\sigma_2 W_3(t)\Big).
\end{aligned}$$

Since $W_1(T) - W_1(t)$ and $W_3(T) - W_3(t) \sim \mathcal{N}(0, T-t)$, we continue the previous calculation with

$$\begin{aligned}
&\rho_t(X + hX_1) \\
&= \frac{1}{2\gamma}\gamma^2((1+h)\sigma_1 + \rho\sigma_2)^2(T-t) - ((1+h)\sigma_1 + \rho\sigma_2)W_1(t) \\
&\quad + \frac{1}{2\gamma}\gamma^2(1-\rho^2)\sigma_2^2(T-t) - \sqrt{1-\rho^2}\sigma_2 W_3(t) \\
&= \frac{\gamma}{2}(\sigma_1^2 + 2h\sigma_1^2 + 2\rho\sigma_1\sigma_2 + h^2\sigma_1^2 + 2h\rho\sigma_1\sigma_2 + \rho^2\sigma_2^2)(T-t)
\end{aligned}$$

$$
\begin{aligned}
&- ((1+h)\sigma_1 + \rho\sigma_2)W_1(t) + \frac{\gamma}{2}(\sigma_2^2 - \sigma_2^2\rho^2)(T-t) - \sqrt{1-\rho^2}\sigma_2 W_3(t)\\
&= \frac{\gamma}{2}(\sigma_1^2 + 2h\sigma_1^2 + 2\rho\sigma_1\sigma_2 + h^2\sigma_1^2 + 2h\rho\sigma_1\sigma_2 + \sigma_2^2)(T-t)\\
&- ((1+h)\sigma_1 + \rho\sigma_2)W_1(t) - \sqrt{1-\rho^2}\sigma_2 W_3(t).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\rho_t^{Euler}(X_1|X) &= \frac{d\rho_t(X+hX_1)}{dh}\Big|_{h=0}\\
&= \frac{d(\frac{\gamma}{2}(\sigma_1^2 + 2h\sigma_1^2 + 2\rho\sigma_1\sigma_2 + h^2\sigma_1^2 + 2h\rho\sigma_1\sigma_2 + \sigma_2^2)(T-t)}{dh}\Big|_{h=0}\\
&\quad + \frac{d(-((1+h)\sigma_1 + \rho\sigma_2)W_1(t) - \sqrt{1-\rho^2}\sigma_2 W_3(t))}{dh}\Big|_{h=0}\\
&= \left(\frac{\gamma}{2}(2\sigma_1^2 + 2h\sigma_1^2 + 2\rho\sigma_1\sigma_2)(T-t) - \sigma_1 W_1(t)\right)\Big|_{h=0}\\
&= \gamma(\sigma_1^2 + \rho\sigma_1\sigma_2)(T-t) - \sigma_1 W_1(t).
\end{aligned}
$$

Similarly, $\rho_t^{Euler}(X_2|X) = \frac{d\rho_t(X+hX_2)}{dh}\Big|_{h=0}$, where

$$
\begin{aligned}
&\rho_t(X+hX_2)\\
&= \rho_t\Big((\sigma_1 + \rho\sigma_2)W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T)\\
&\qquad + h\Big(\sigma_2\rho W_1(T) + \sqrt{1-\rho^2}\sigma_2 W_3(T)\Big)\Big)\\
&= \rho_t\Big((\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(T) + \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(T)\Big)\\
&= \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma((\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(T)\\
&\qquad + \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(T))\Big)\Big|\mathcal{F}_t\Big]\\
&= \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma((\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)(W_1(T) - W_1(t) + W_1(t)) +\\
&\quad (\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2})(W_3(T) - W_3(t) + W_3(t)))\Big)\Big|\mathcal{F}_t\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\gamma} E[\exp(-\gamma(\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)(W_1(T) - W_1(t)))] \\
&\quad - (\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(t) \\
&\quad + \frac{1}{\gamma} E\Big[\exp\Big(-\gamma\Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)(W_3(T) - W_3(t))\Big)\Big] \\
&\quad - \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(t) \\
&= \frac{\gamma}{2}(\sigma_1^2 + 2\sigma_1\sigma_2\rho + 2h\sigma_1\sigma_2\rho + \rho^2\sigma_2^2 + 2h\sigma_2^2\rho^2 + h^2\sigma_2^2\rho^2)(T-t) + \\
&\quad \frac{\gamma}{2}(\sigma_2^2 - \sigma_2^2\rho^2 + h^2\sigma_2^2 - h^2\sigma_2^2\rho^2 + 2h\sigma_2^2 - 2h\sigma_2^2\rho^2)(T-t) \\
&\quad - (\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(t) - \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(t) \\
&= \frac{\gamma}{2}(\sigma_1^2 + 2\sigma_1\sigma_2\rho + 2h\sigma_1\sigma_2\rho + \sigma_2^2 + h^2\sigma_2^2 + 2h\sigma_2^2)(T-t) \\
&\quad - (\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(t) - \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(t).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\rho_t^{Euler}(X_2|X) \\
&= \frac{d\rho_t(X + hX_2)}{dh}\bigg|_{h=0} \\
&= \frac{d(\frac{\gamma}{2}(\sigma_1^2 + 2\sigma_1\sigma_2\rho + 2h\sigma_1\sigma_2\rho + \sigma_2^2 + h^2\sigma_2^2 + 2h\sigma_2^2)(T-t)}{dh}\bigg|_{h=0} \\
&\quad + \frac{d(-(\sigma_1 + \rho\sigma_2 + h\sigma_2\rho)W_1(t) - \Big(\sigma_2\sqrt{1-\rho^2} + h\sigma_2\sqrt{1-\rho^2}\Big)W_3(t))}{dh}\bigg|_{h=0} \\
&= \Big(\frac{\gamma}{2}(2\sigma_1\sigma_2\rho + 2h\sigma_2^2 + 2\sigma_2^2)(T-t) - \sigma_2\rho W_1(t) - \sigma_2\sqrt{1-\rho^2}W_3(t)\Big)\bigg|_{h=0} \\
&= \frac{\gamma}{2}(2\sigma_1\sigma_2\rho + 2\sigma_2^2)(T-t) - \sigma_2\Big(\rho W_1(t) + \sqrt{1-\rho^2}W_3(t)\Big) \\
&= \gamma(\sigma_1\sigma_2\rho + \sigma_2^2)(T-t) - \sigma_2 W_2(t).
\end{aligned}
$$

The result of $\rho_t^{Euler}(X_2|X)$ follows directly from $\rho_t^{Euler}(X_1|X)$ by symmetry.

Analyze risk attribution

Let's consider a two-step and two-factor model. In this example, we assume $R_T^1 = \sigma_1 W_1(T)$, $R_T^2 = \sigma_2 W_2(T)$ and $L = f(R_T^1, R_T^2) = R_T^1 R_T^2$, where W_1 and W_2 are defined the same as the risk allocation example.

Then

$$
\begin{aligned}
A^1 &= \sum_{t=0}^{T-1} \big(f(R_{t+1}^1, R_t^2) - f(R_t^1, R_t^2)\big) \\
&= f(R_1^1, R_0^2) - f(R_0^1, R_0^2) + f(R_2^1, R_1^2) - f(R_1^1, R_1^2) \\
&= R_1^1 R_0^2 - R_0^1 R_0^2 + R_2^1 R_1^2 - R_1^1 R_1^2 \\
&= \sigma_1\sigma_2 W_1(2) W_2(1) - \sigma_1\sigma_2 W_1(1) W_2(1) \\
&= \sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1)).
\end{aligned}
$$

Similarly, $A_2 = \sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1))$. By definition, $\rho_t^{Euler}(A_1|A) = \frac{d\rho_t(A+hA_1)}{dh}\big|_{h=0}$, where $A = A_1 + A_2$, and $t \in \{0,1,2\}$. Firstly, let's calculate $\rho_t(A + hA_1)$ at $t = 1$.

$$
\begin{aligned}
&\rho_1(A + hA_1) \\
&= \rho_1\big(\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1)) + (1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1))\big) \\
&= \frac{1}{\gamma} \ln E\big[\exp(-\gamma(\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1)) \\
&\quad + (1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1))))\big|\mathcal{F}_1\big] \\
&= \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2 W_1(1)\big(\rho(W_1(2) - W_1(1)) \\
&\quad + \sqrt{1-\rho^2}(W_3(2) - W_3(1))\big) - \gamma(1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(W_3(2) - W_3(1))
\end{aligned}
$$

$$
\begin{aligned}
&- \gamma\sigma_1\sigma_2\big(\rho W_1(1) + (1+h)W_2(1)\big)(W_1(2) - W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(W_3(2) - W_3(1))\Big)\Big|\mathcal{F}_1\Big] \\
&\quad + \frac{1}{\gamma}\ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))(W_1(2) - W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma}\ln\left(\exp\left(\frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}\Big)^2\right)\right. \\
&\qquad\qquad \left.\times\exp\left(\frac{1}{2}\big(\gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))\big)^2\right)\right) \\
&= \frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}\Big)^2 + \frac{1}{2}\big(\gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))\big)^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\rho_1^{Euler}(A_1|A) &= \frac{d\rho_1(A + hA_1)}{dh}\bigg|_{h=0} \\
&= \frac{d\left(\frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}\Big)^2\right)}{dh}\Bigg|_{h=0} \\
&\quad + \frac{d\left(\frac{1}{2}\big(\gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))\big)^2\right)}{dh}\Bigg|_{h=0} \\
&= \gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))\gamma\sigma_1\sigma_2 W_2(1)\big|_{h=0} \\
&= \gamma^2\sigma_1^2\sigma_2^2\big(\rho W_1(1)W_2(1) + W_2(1)^2\big)
\end{aligned}
$$

Then we calculate $\rho_1(A + hA_2)$:

$$
\begin{aligned}
&\rho_1(A + hA_2) \\
&= \rho_1\big(\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1)) + (1+h)\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1))\big) \\
&= \frac{1}{\gamma}\ln E\big[\exp\big(-\gamma(\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1)) \\
&\quad + (1+h)\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1)))\big)\big|\mathcal{F}_1\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma(1+h)\sigma_1\sigma_2 W_1(1)\Big(\rho(W_1(2)-W_1(1)) \\
&\quad + \sqrt{1-\rho^2}(W_3(2)-W_3(1))\Big) - \gamma\sigma_1\sigma_2 W_2(1)(W_1(2)-W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(W_3(2)-W_3(1))(1+h) \\
&\quad - \gamma\sigma_1\sigma_2\big((1+h)\rho W_1(1) + W_2(1)\big)(W_1(2)-W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(W_3(2)-W_3(1))(1+h)\Big)\Big|\mathcal{F}_1\Big] \\
&\quad + \frac{1}{\gamma} \ln E\Big[\exp\Big(-\gamma\sigma_1\sigma_2((1+h)\rho W_1(1) + W_2(1))(W_1(2)-W_1(1))\Big)\Big|\mathcal{F}_1\Big] \\
&= \frac{1}{\gamma} \ln\left(\exp\left(\frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(1+h)\Big)^2 \right) \right. \\
&\qquad\qquad \left. \times \exp\left(\frac{1}{2}\big(\gamma\sigma_1\sigma_2((1+h)\rho W_1(1) + W_2(1))\big)^2 \right) \right) \\
&= \frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(1+h)\Big)^2 + \frac{1}{2}\big(\gamma\sigma_1\sigma_2((1+h)\rho W_1(1) + W_2(1))\big)^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\rho_1^{Euler}(A_2|A) &= \frac{d\rho_1(A+hA_2)}{dh}\bigg|_{h=0} \\
&= \frac{d\left(\frac{1}{2}\big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(1+h)\big)^2\right)}{dh}\Bigg|_{h=0} \\
&\quad + \frac{d\left(\frac{1}{2}\big(\gamma\sigma_1\sigma_2((1+h)\rho W_1(1) + W_2(1))\big)^2\right)}{dh}\Bigg|_{h=0} \\
&= \gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}(1+h)\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}\big|_{h=0} \\
&\quad + \gamma\sigma_1\sigma_2\big((1+h)\rho W_1(1) + W_2(1)\big)\gamma\sigma_1\sigma_2\rho W_1(1)\big|_{h=0} \\
&= \gamma^2\sigma_1^2\sigma_2^2(1-\rho^2)W_1(1)^2 + \gamma^2\sigma_1^2\sigma_2^2\big(\rho^2 W_1(1)^2 + \rho W_1(1)W_2(1)\big) \\
&= \gamma^2\sigma_1^2\sigma_2^2\big(W_1(1)^2 + \rho W_1(1)W_2(1)\big).
\end{aligned}
$$

At time $t = 0$, $\rho_t(A + hA_1)$ becomes

$$
\begin{aligned}
&\rho_0(A + hA_1)\\
&= \rho_0\big(\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1)) + (1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1))\big)\\
&= \frac{1}{\gamma}\ln E\big[\exp\big(-\gamma(\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1))\\
&\quad + (1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1)))\big)\big|\mathcal{F}_0\big]\\
&= \frac{1}{\gamma}\ln E\Big[E\big[\exp\big(-\gamma(\sigma_1\sigma_2 W_1(1)(W_2(2) - W_2(1))\\
&\quad + (1+h)\sigma_1\sigma_2 W_2(1)(W_1(2) - W_1(1)))\big)\big|\mathcal{F}_1\big]\Big]\\
&= \frac{1}{\gamma}\ln E\left[\exp\left(\frac{1}{2}\Big(\gamma\sigma_1\sigma_2 W_1(1)\sqrt{1-\rho^2}\Big)^2\right)\right.\\
&\qquad\qquad \left.\times\exp\left(\frac{1}{2}\Big(\gamma\sigma_1\sigma_2(\rho W_1(1) + (1+h)W_2(1))\Big)^2\right)\right]\\
&= \frac{1}{\gamma}\ln E\left[\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2\left(W_1(1)^2 + 2\rho W_1(1)W_2(1) + 2\rho h W_1(1)W_2(1)\right)}{2}\right)\right]\\
&\qquad\qquad \times\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)W_2(1)^2}{2}\right)\Bigg]\\
&= \frac{1}{\gamma}\ln E\left[E\left[\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2\left(W_1(1)^2 + 2\rho W_1(1)W_2(1) + 2\rho h W_1(1)W_2(1)\right)}{2}\right)\right.\right.\\
&\qquad\qquad\qquad \left.\left.\times\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)W_2(1)^2}{2}\right)\middle| W_1(1)\right]\right]\\
&= \frac{1}{\gamma}\ln E\left[\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2 W_1(1)^2}{2}\right)E\left[\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2\left(2\rho a + 2\rho h a\right)}{2}W_2(1)\right.\right.\right.\\
&\qquad\qquad\qquad\qquad\qquad \left.\left.\left.+\frac{\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}{2}W_2(1)^2\right)\right]\Big|_{a=W_1(1)}\right]\\
&= \frac{1}{\gamma}\ln E\left[\exp\left(\frac{\gamma^2\sigma_1^2\sigma_2^2 W_1(1)^2}{2}\right)E\left[\exp\left(kz + c_1 Z^2\right)\right]\Big|_{a=W_1(1)}\right],
\end{aligned}
$$

where

$$
k = \frac{\gamma^2\sigma_1^2\sigma_2^2\left(2\rho a + 2\rho h a\right)}{2},
$$

$$
\begin{aligned}
c_1 &= \frac{\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}{2},\\
Z &= W_2(1) \sim \mathcal{N}(0,1).
\end{aligned}
$$

We compute

$$
\begin{aligned}
E\left[\exp\left(kZ + c_1Z^2\right)\right] &= \int_{-\infty}^{\infty} e^{kz+c_1z^2}\frac{e^{\frac{-z^2}{2}}}{\sqrt{2\pi}}\,dz\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{z^2\left(c_1-\frac{1}{2}\right)+kz}\,dz\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(c_1-\frac{1}{2}\right)\left(z+\frac{k}{2\left(c_1-\frac{1}{2}\right)}\right)^2-\frac{k^2}{4\left(c_1-\frac{1}{2}\right)}}\,dz\\
&= e^{\frac{-k^2}{4\left(c_1-\frac{1}{2}\right)}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left(c_1-\frac{1}{2}\right)\left(z+\frac{k}{2\left(c_1-\frac{1}{2}\right)}\right)^2}\,dz.
\end{aligned}
$$

By assuming $c_1 < \frac{1}{2}$, we obtain that

$$
\begin{aligned}
E\left[\exp\left(kZ + c_1Z^2\right)\right] &= \frac{e^{\frac{-\gamma^4\sigma_1^4\sigma_2^4(2\rho W_1(1)+2\rho h W_1(1))^2/4}{4\left(\frac{\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}{2}-\frac{1}{2}\right)}}}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}\\
&= \frac{e^{\frac{-\gamma^4\sigma_1^4\sigma_2^4(\rho W_1(1)+\rho h W_1(1))^2}{2\left(\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)-1\right)}}}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\rho_0(A+hA_1) &= \frac{1}{\gamma}\ln E\left[e^{\left(\frac{\gamma^2\sigma_1^2\sigma_2^2W_1(1)^2}{2}\right)}\frac{e^{\frac{-\gamma^4\sigma_1^4\sigma_2^4(\rho W_1(1)+\rho h W_1(1))^2}{2\left(\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)-1\right)}}}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}\right]\\
&= \frac{1}{\gamma}\ln\left(\frac{1}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}E\left[\exp\left(\left(\frac{\gamma^2\sigma_1^2\sigma_2^2}{2}\right.\right.\right.\right.
\end{aligned}
$$

$$
\left.\left.\left. - \frac{-\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{2\left(\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)-1\right)}\right) W_1(1)^2\right)\right]\right).
$$

Let $E\left[\exp\left(\left(\frac{\gamma^2\sigma_1^2\sigma_2^2}{2} - \frac{-\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{2\left(\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)-1\right)}\right) W_1(1)^2\right)\right] = E\left[e^{c_2 Z_1^2}\right]$, where

$$
\begin{aligned}
c_2 &= \frac{\gamma^2\sigma_1^2\sigma_2^2}{2} + \frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{2\left(\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)-1\right)}, \\
Z_1 &= W_1(1) \sim \mathcal{N}(0,1).
\end{aligned}
$$

By the moment generating function of Chi-square distribution, we get that $E\left[e^{c_2 Z_1^2}\right] = \frac{1}{\sqrt{1-2c_2}}$ for $c_2 < 1/2$. For the following computations, we assume that $\gamma\sigma_1\sigma_2$ is small enough so that $c_2 < 1/2$. Then

$$
E\left[e^{c_2 Z_1^2}\right] = \frac{1}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2 - \frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2(1+2h+h^2)}{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}}
$$

and

$$
\begin{aligned}
\rho_0(A+hA_1) &= \frac{1}{\gamma}\ln\left(\frac{1}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}\right) \\
&\quad + \frac{1}{\gamma}\ln\left(\frac{1}{\sqrt{1-\gamma^2\sigma_1^2\sigma_2^2 - \frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2(1+2h+h^2)}{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}}}\right) \\
&= -\frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)\right) \\
&\quad - \frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2 - \frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{1-\gamma^2\sigma_1^2\sigma_2^2\left(1+2h+h^2\right)}\right).
\end{aligned}
$$

Thus,

$$
\rho_0^{Euler}(A_1|A) = \left.\frac{d\rho_0(A+hA_1)}{dh}\right|_{h=0}
$$

$$
\begin{aligned}
&= \frac{d\left(-\frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)\right)\right)}{dh}\Bigg|_{h=0} \\
&\quad + \frac{d\left(-\frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2-\frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}\right)\right)}{dh}\Bigg|_{h=0}
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{d\left(-\frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)\right)\right)}{dh}\Bigg|_{h=0} &= \frac{\gamma^2\sigma_1^2\sigma_2^2\,(2+2h)}{2\gamma\left(1-\gamma^2\sigma_1^2\sigma_2^2\left(1+2h+h^2\right)\right)}\Bigg|_{h=0} \\
&= \frac{\gamma^2\sigma_1^2\sigma_2^2}{\gamma\left(1-\gamma^2\sigma_1^2\sigma_2^2\right)} \\
&= \frac{\gamma\sigma_1^2\sigma_2^2}{1-\gamma^2\sigma_1^2\sigma_2^2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d\left(-\frac{1}{2\gamma}\ln\left(1-\gamma^2\sigma_1^2\sigma_2^2-\frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2\left(1+2h+h^2\right)}{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}\right)\right)}{dh}\Bigg|_{h=0} \\
&= -\frac{1}{2\gamma}\times\frac{1}{1-\gamma^2\sigma_1^2\sigma_2^2-\frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2(1+2h+h^2)}{1-\gamma^2\sigma_1^2\sigma_2^2(1+2h+h^2)}} \\
&\quad\times\frac{-\gamma^4\sigma_1^4\sigma_2^4\rho^2(2+2h)}{1-2\gamma^2\sigma_1^2\sigma_2^2\left(1+2h+h^2\right)+\gamma^4\sigma_1^4\sigma_2^4\left(1+2h+h^2\right)^2}\Bigg|_{h=0} \\
&= \frac{\gamma^4\sigma_1^4\sigma_2^4\rho^2}{\gamma\left(1-\gamma^2\sigma_1^2\sigma_2^2\right)\left(1-2\gamma^2\sigma_1^2\sigma_2^2+\gamma^4\sigma_1^4\sigma_2^4-\gamma^4\sigma_1^4\sigma_2^4\rho^2\right)} \\
&= \frac{\gamma^3\sigma_1^4\sigma_2^4\rho^2}{\left(1-\gamma^2\sigma_1^2\sigma_2^2\right)\left(1-2\gamma^2\sigma_1^2\sigma_2^2+\gamma^4\sigma_1^4\sigma_2^4-\gamma^4\sigma_1^4\sigma_2^4\rho^2\right)}.
\end{aligned}
$$

We obtain that

$$\begin{aligned}\rho_0^{Euler}(A_1|A) &= \frac{\gamma\sigma_1^2\sigma_2^2}{1-\gamma^2\sigma_1^2\sigma_2^2} \\ &\quad+\frac{\gamma^3\sigma_1^4\sigma_2^4\rho^2}{(1-\gamma^2\sigma_1^2\sigma_2^2)\,(1-2\gamma^2\sigma_1^2\sigma_2^2+\gamma^4\sigma_1^4\sigma_2^4-\gamma^4\sigma_1^4\sigma_2^4\rho^2)} \\ &= \frac{\gamma\sigma_1^2\sigma_2^2-2\gamma^3\sigma_1^4\sigma_2^4+\gamma^5\sigma_1^6\sigma_2^6-\gamma^5\sigma_1^6\sigma_2^6\rho^2+\gamma^3\sigma_1^4\sigma_2^4\rho^2}{(1-\gamma^2\sigma_1^2\sigma_2^2)\,(1-2\gamma^2\sigma_1^2\sigma_2^2+\gamma^4\sigma_1^4\sigma_2^4-\gamma^4\sigma_1^4\sigma_2^4\rho^2)},\end{aligned}$$

where we recall that we assumed for this computation that $\gamma\sigma_1\sigma_2$ is small enough. The computation of $\rho_0^{Euler}(A_2|A)$ goes analogously, with W_1 and W_2 interchanged.

The above derivation has been done for two time steps. When there are more than two steps, we using a numerical simulation. In MATLAB we calculate the Euler contributions for two risk factors and their summation; see Appendix A for the MATLAB code. To simulate Brownian Motion, we use a discrete-time approximation with N steps and independent normally distributed increments:

$$W_0 = 0, \quad W_{j\frac{T}{N}} = W_{(j-1)\frac{T}{N}} + \sqrt{\frac{T}{N}}Z_j \text{ for } j = 1, \dots, N,$$

where $Z_1, \dots, Z_N$ are independent and standard normally distributed and T is the time horizon, which we choose $T = 1$. We set the maximal number of steps to be 20, and compute the values of the risk measures and contributions using 10,000,000 simulations. For numerical tractability, we first simulate 1,000,000 sample paths and then repeat this 10 times before taking the average. For this example, we choose $\gamma = 1/2$ as the value of the coefficient of absolute risk aversion and choose both σ_1 and σ_2 as 1. To approximate the Euler contribution, we use

$$\rho_0^{Euler}(A_1|A) = \frac{d\rho_0(A + hA_1)}{dh}\bigg|_{h=0} \approx \frac{\rho_0(A + hA_1) - \rho_0(A)}{h}$$

for $h = 1/100$.

We plot the risk contributions of two risk factors for the different number of steps as Figure 4.1. It shows that the risk contributions for the two risk factors are almost the same for different numbers of steps as expected because of symmetry.

We also plot the risk measure of the total risk and its approximation for different numbers of steps in Figure 4.2. It illustrates that $\rho(A_1+A_2)$ converges to $\rho(L)$ as the number of steps increasing, which consistent with Propositions 3.2.1 and 3.2.2. Note that the condition of zero quadratic variation in Proposition 3.2.1 is satisfied because the correlation parameter ρ between the two Brownian motions has been set to zero.

We also analyze numerically the underlying risk measure is given by VaR or ES at the confidence level 95%. The choices of the parameters are unchanged, and the computational procedure is analogous. We deduce from Figures 4.3–4.6 the same conclusions as before. However, we observe that the Euler contributions (two curves in each of Figure 4.3 and Figure 4.5) sum up to $\rho(A_1 + A_2)$ (red curves in Figure 4.4 and Figure 4.6). By contrast, the sum of the Euler contributions in Figure 4.1 is not equal to $\rho(A_1 + A_2)$ (red curve in Figure 4.2). The reason is as follows. The risk measures VaR and ES are homogeneous of degree 1. As mentioned in Section 2.1, this implies that the Euler contributions satisfy the full-allocation property by Tasche [20]. By contrast, the entropic risk measure is not homogeneous, which is the reason why its Euler contribution does not satisfy the full-allocation property.

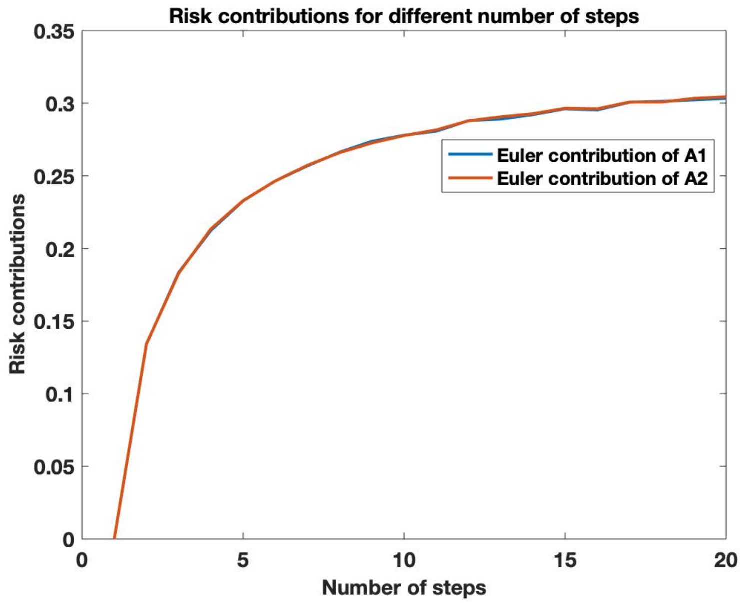

Figure 4.1: Risk contributions of two risk factors for different numbers of steps, using the entropic risk measure

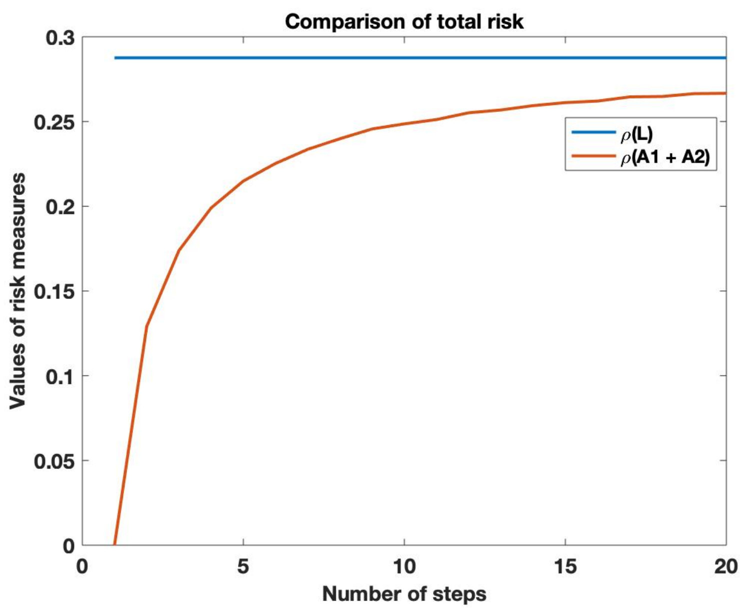

Figure 4.2: Risk measure of total risk and its approximation for different numbers of steps, using the entropic risk measure

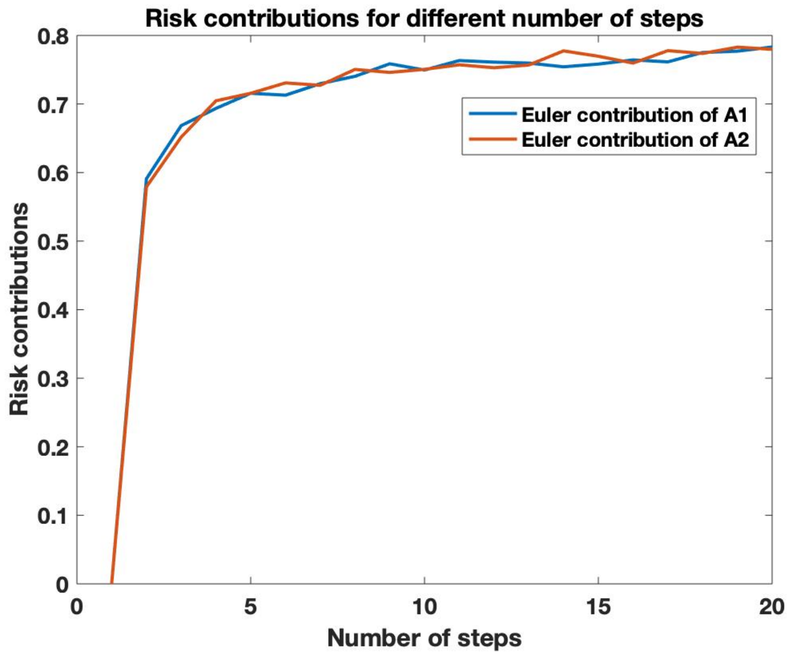

Figure 4.3: Risk contributions of two risk factors for different numbers of steps, using VaR as the risk measure

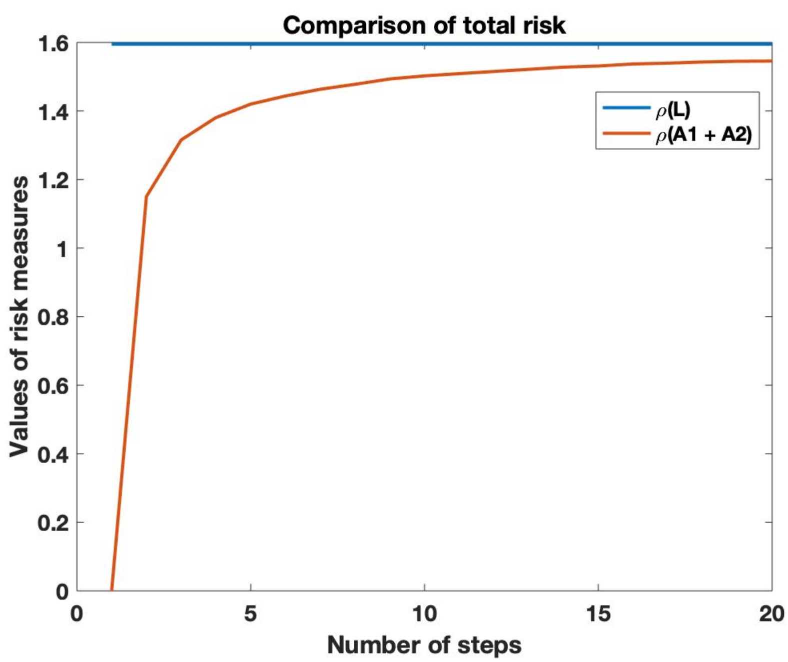

Figure 4.4: Risk measure of total risk and its approximation for different numbers of steps, using VaR as the risk measure

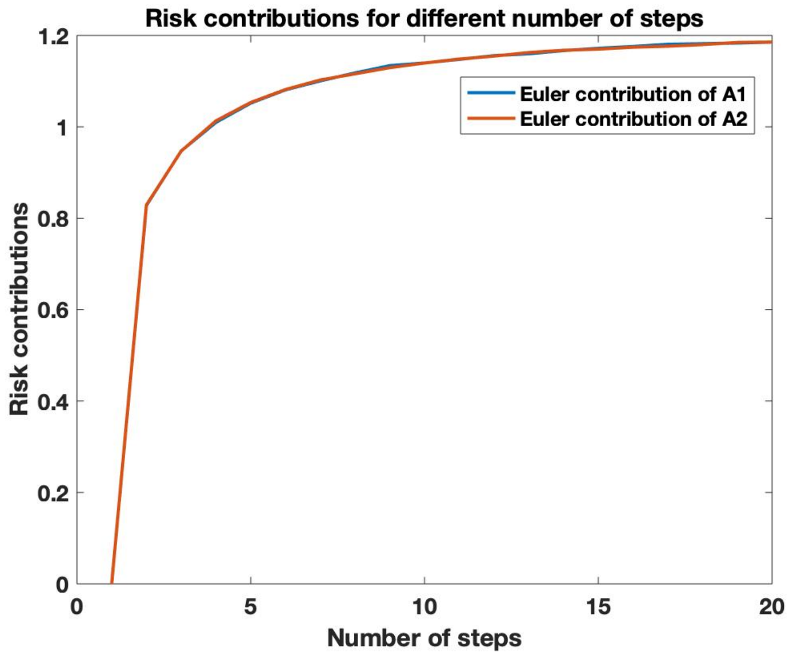

Figure 4.5: Risk contributions of two risk factors for different numbers of steps, using ES as the risk measure

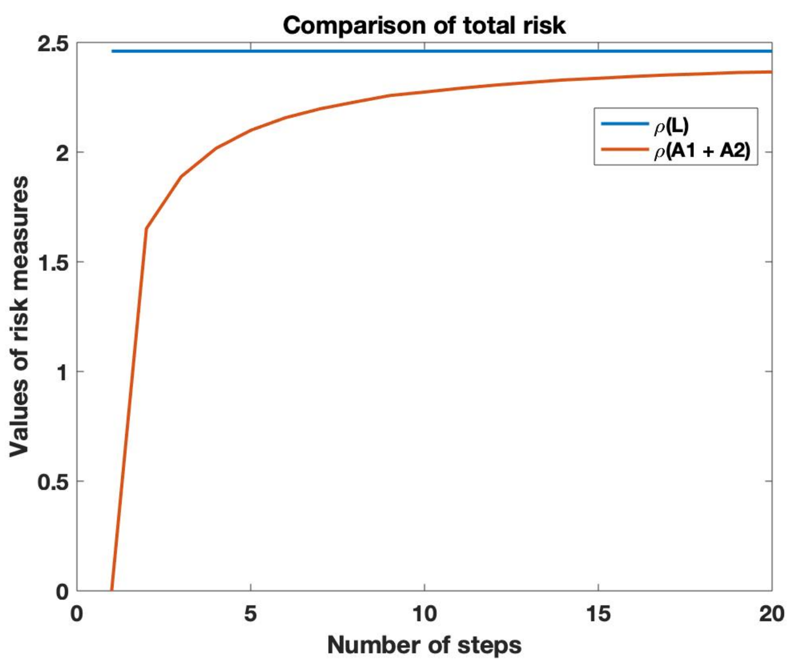

Figure 4.6: Risk measure of total risk and its approximation for different numbers of steps, using ES as the risk measure

Chapter 5

Conclusion

Risk measures have been a popular topic for decades in the financial risk management field. A classical question is how to aggregate different sources of risk by taking diversification effects into account. However, the converse question of how to decompose risk is also relevant, which is the topic of this thesis. We focus on two dimensions of risk decomposition, which are risk allocation and risk attribution. Since a company's total profit/loss is the sum of the profits/losses of its divisions, this linear relationship is used to allocate risk to different divisions. By contrast, risk drivers may contribute to losses in a nonlinear way, so that additional techniques need to be used to attribute risk to different risk drivers.

We use the economically justified Euler allocation principle to compute the risk allocations and discuss when the risk allocations are RORAC compatible, when they satisfy the full-allocation property, and when the dynamic risk measure is time consistent. The methodology for risk attribution is to construct a linear approximation for the loss random variable and then apply the Euler principle. This approach has two main advantages. Firstly, it is computationally more efficient than other methods, such as the Shapley value. Secondly, it relies on the Euler principle, which has a solid economic justification in the literature.

The contributions of this thesis are as follows. Firstly, we extend both risk allocation and risk attribution to dynamic settings by computing the risk contributions at time t for $t \in [0, T]$. Secondly, we show that for RORAC compatible risk contributions, time consistency translates from risk measures to risk contributions. Thirdly, we illustrate the computation of risk contributions based on an example for the entropic risk measure where we computed and simulated risk allocation and risk attribution.

Interesting questions for future research are about the relation of risk allocation and risk attribution to stochastic differential equations (SDEs). The link between risk measures and backward SDEs is well known in the literature; see for example Øksendal and Sulem [16]. Using this link could help find new results for risk allocation and risk attribution.

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Appendix A

MATLAB Code

The following code was used to create Figures 4.1 and 4.2 in the example of Chapter 4.

```
rng('default')
T = 1;
Npaths = 1000000;
rho = 0;
gamma = 1/2;
Nruns = 10;
rhoEuler = zeros(Nruns,20);
rhoEuler2 = zeros(Nruns,20);
rhoA12 = zeros(Nruns,20);
rhoL = zeros(Nruns,20);

for j = 1:Nruns
    for Nsteps = 1:20
        s = (T/Nsteps)^.5;
        clear incr incr3
        incr(1,:) = zeros(1,Npaths);
        incr(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths = cumsum(incr);
        incr3(1,:) = zeros(1,Npaths);
        incr3(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths3 = cumsum(incr3);
        W1 = paths;
        W3 = paths3;
        W2 = rho*W1 + sqrt(1-rho^2)*W3;
        A1 = sum(W1(2:end,:).*W2(1:end-1,:)-W1(1:end-1,:)
        .*W2(1:end-1,:));
        A2 = sum(W2(2:end,:).*W1(1:end-1,:)-W1(1:end-1,:)
        .*W2(1:end-1,:));
        h = 1/100;
```

```
        A = A1 + A2;
        B = A + h*A1;
        C = exp(-gamma*B);
        D = mean(C);
        rho0 = (1/gamma)*log(D);
        E = exp(-gamma*A);
        F = mean(E);
        rhoa = (1/gamma)*log(F);
        rhoEuler(j,Nsteps) = (rho0 - rhoa)/h;

        B2 = A + h*A2;
        C2 = exp(-gamma*B2);
        D2 = mean(C2);
        rho02 = (1/gamma)*log(D2);
        rhoEuler2(j,Nsteps) = (rho02 - rhoa)/h;
        rhoA12(j,Nsteps) = rhoa;

        L = W1(end,:).*W2(end,:);
        G = exp(-gamma*L);
        H = mean(G);
        rhoL(j,Nsteps) = (1/gamma)*log(H);
    end
end
figure, plot(1:20,mean(rhoEuler),1:20,mean(rhoEuler2),
'linewidth',2);
legend('Euler contribution of A1','Euler contribution of A2');
set(gca,'fontsize',14,'FontWeight','bold');
title('Risk contributions for different number of ...
    steps','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Risk contributions','fontsize',14);

figure, plot(1:20,mean(mean(rhoL))*ones(20,1),
1:20,mean(rhoA12),'linewidth',2);
legend('\rho(L)','\rho(A1 + A2)');
set(gca,'fontsize',14,'FontWeight','bold');
title('Comparison of total risk','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Values of risk measures','fontsize',14);
```

The following code was used to create Figures 4.3 and 4.4 in the example of Chapter 4.

```
rng('default')
confidence_level = 0.95;
T = 1;
Npaths = 1000000;
rho = 0;
```

```
Nruns = 10;
rhoEuler = zeros(Nruns,20);
rhoEuler2 = zeros(Nruns,20);
rhoA12 = zeros(Nruns,20);
rhoL = zeros(Nruns,20);
for j = 1:Nruns
    for Nsteps = 1:20
        s = (T/Nsteps)^.5;
        clear incr incr3
        incr(1,:) = zeros(1,Npaths);
        incr(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths = cumsum(incr);
        incr3(1,:) = zeros(1,Npaths);
        incr3(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths3 = cumsum(incr3);
        W1 = paths;
        W3 = paths3;
        W2 = rho*W1 + sqrt(1-rho^2)*W3;
        A1 = sum(W1(2:end,:).*W2(1:end-1,:)-W1(1:end-1,:)
        .*W2(1:end-1,:));
        A2 = sum(W2(2:end,:).*W1(1:end-1,:)-W1(1:end-1,:)
        .*W2(1:end-1,:));
        h = 1/100;
        A = A1 + A2;
        B = A + h*A1;
        sorted_returns = sort(B);
        num_returns = numel(B);
        VaR_index = ceil((1-confidence_level)*num_returns);
        rho0 = - sorted_returns(VaR_index);
        sorted_returns_a = sort(A);
        num_returns_a = numel(A);
        VaR_index_a = ceil((1-confidence_level)*num_returns_a);
        rhoa = - sorted_returns_a(VaR_index_a);
        rhoEuler(j,Nsteps) = (rho0 - rhoa)/h;

        B2 = A + h*A2;
        sorted_returns_b2 = sort(B2);
        num_returns_b2 = numel(B2);
        VaR_index_b2 = ...
            ceil((1-confidence_level)*num_returns_b2);
        rho02 = - sorted_returns_b2(VaR_index_b2);
        rhoEuler2(j,Nsteps) = (rho02 - rhoa)/h;
        rhoA12(j,Nsteps) = rhoa;

        L = W1(end,:).*W2(end,:);
        sorted_returns_L = sort(L);
        num_returns_L = numel(L);
        VaR_index_L = ceil((1-confidence_level)*num_returns_L);
        rhoL(j,Nsteps) = - sorted_returns_L(VaR_index_L);
```

```
    end
end
figure, plot(1:20,mean(rhoEuler),
1:20,mean(rhoEuler2),'linewidth',2);
legend('Euler contribution of A1','Euler contribution of A2');
set(gca,'fontsize',14,'FontWeight','bold');
title('Risk contributions for different number of ...
    steps','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Risk contributions','fontsize',14);

figure, plot(1:20,mean(mean(rhoL))*ones(20,1),
1:20,mean(rhoA12),'linewidth',2);
legend('\rho(L)','\rho(A1 + A2)');
set(gca,'fontsize',14,'FontWeight','bold');
title('Comparison of total risk','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Values of risk measures','fontsize',14);
```

The following code was used to create Figures 4.5 and 4.6 in the example of Chapter 4.

```
rng('default')
confidence_level = 0.95;
T = 1;
Npaths = 1000000;
rho = 0;
Nruns = 10;
rhoEuler = zeros(Nruns,20);
rhoEuler2 = zeros(Nruns,20);
rhoA12 = zeros(Nruns,20);
rhoL = zeros(Nruns,20);
for j = 1:Nruns
    for Nsteps = 1:20
        s = (T/Nsteps)^.5;
        clear incr incr3
        incr(1,:) = zeros(1,Npaths);
        incr(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths = cumsum(incr);
        incr3(1,:) = zeros(1,Npaths);
        incr3(2:Nsteps+1,:) = s*randn(Nsteps,Npaths);
        paths3 = cumsum(incr3);
        W1 = paths;
        W3 = paths3;
        W2 = rho*W1 + sqrt(1-rho^2)*W3;
        A1 = sum(W1(2:end,:).*W2(1:end-1,:)-W1(1:end-1,:)
        .*W2(1:end-1,:));
        A2 = sum(W2(2:end,:).*W1(1:end-1,:)-W1(1:end-1,:)
```

```
        .*W2(1:end-1,:));
        h = 1/100;
        A = A1 + A2;
        B = A + h*A1;
        sorted_returns = sort(B);
        num_returns = numel(B);
        VaR_index = ceil((1-confidence_level)*num_returns);
        rho0 = - mean(sorted_returns(1:VaR_index));
        sorted_returns_a = sort(A);
        num_returns_a = numel(A);
        VaR_index_a = ceil((1-confidence_level)*num_returns_a);
        rhoa = - mean(sorted_returns_a(1:VaR_index_a));
        rhoEuler(j,Nsteps) = (rho0 - rhoa)/h;

        B2 = A + h*A2;
        sorted_returns_b2 = sort(B2);
        num_returns_b2 = numel(B2);
        VaR_index_b2 = ...
            ceil((1-confidence_level)*num_returns_b2);
        rho02 = - mean(sorted_returns_b2(1:VaR_index_b2));
        rhoEuler2(j,Nsteps) = (rho02 - rhoa)/h;
        rhoA12(j,Nsteps) = rhoa;

        L = W1(end,:).*W2(end,:);
        sorted_returns_L = sort(L);
        num_returns_L = numel(L);
        VaR_index_L = ceil((1-confidence_level)*num_returns_L);
        rhoL(j,Nsteps) = - ...
            mean(sorted_returns_L(1:VaR_index_L));
    end
end
figure, plot(1:20,mean(rhoEuler),
1:20,mean(rhoEuler2),'linewidth',2);
legend('Euler contribution of A1','Euler contribution of A2');
set(gca,'fontsize',14,'FontWeight','bold');
title('Risk contributions for different number of ...
    steps','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Risk contributions','fontsize',14);

figure, plot(1:20,mean(mean(rhoL))*ones(20,1)
,1:20,mean(rhoA12),'linewidth',2);
legend('\rho(L)','\rho(A1 + A2)');
set(gca,'fontsize',14,'FontWeight','bold');
title('Comparison of total risk','fontsize',14);
xlabel('Number of steps','fontsize',14);
ylabel('Values of risk measures','fontsize',14);
```