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Numerical Analysis of Multidimensional Euler Equations

By
Houshi Li

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Permanent Address:
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Numerical Analysis of Multidimensional Euler Equations** submitted by **Houshi Li** in partial fulfilment of the requirements for the degree of **Doctor of Philosophy** in Applied Mathematics.

Y. S. Wong

Y. P. Lin

R. J. Tait

W. Allegretto

B. Joe

J. So

S. Sivaloganathan (Univ. of Waterloo)

Date: 30/1/95

To my

Parents(Zhong-Xiang Li, Zong-Xiu He)

Wife(Xueqin Li)

Daughter(Mandi Li)

Son(Adam Jinhua Li)

献给我的

父母(李忠祥，何宗秀)

妻子(李雪琴)

女儿(李曼迪)

儿子(李锦华)!

ABSTRACT

This thesis is devoted to the numerical analysis of multidimensional Euler equations. The physics of one-dimensional system of Euler equations is simple and well established, and many efficient numerical procedures are available for their solutions. Two-dimensional problems however are much more complex, in particular, acoustic waves can propagate in infinitely many directions rather than just two as in a one-dimensional problem. In dealing with multidimensional systems, the commutativity of the coefficient matrices plays a very important role. Unfortunately, these coefficient matrices do not commute for multidimensional Euler equations.

The main contributions of this thesis are summarised as follows. Exponential numerical algorithms are derived for one-dimensional systems. The concept of a "weakly coupled system" is then introduced for the multidimensional hyperbolic systems. It is shown that the system of two-dimensional Euler equations is a weakly coupled system if and only if the flow conditions are supersonic, which implies that for the two-dimensional Euler equations, the weakly coupling property is the characteristic for supersonic flows. A preconditioning technique is presented, in which the coefficient matrices of the corresponding preconditioned two-dimensional Euler systems are commutative. For a preconditioned system of Euler equations, we performed the stability analysis for the upwinding , the Lax-Friedrichs and the fractional step methods. Numerical experiments are reported for steady solutions of Euler equations.

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Chapter I

INTRODUCTION

The fact that mathematics is useful in gas dynamics has now been widely accepted. In the late 19th century, it was stated in the 14th Annual Report of the ASGB[1] that *"Mathematics up to the present day have been quite useless to us in regard to flying"*. Nevertheless considerable progress has been made in the past fifty years. In 1954, Theodore von Karman wrote: *"Mathematical theories from the happy hunting grounds of pure mathematicians are found suitable to describe the airflow produced by aircraft with such excellent accuracy that they can be applied directly to airplane design* "[1]. Moreover, with the advance in modern computer technology, it has been shown that mathematics is a basic tool in the study of problems in gas dynamics. Computational fluid dynamics (CFD) is an active research topic attracting many mathematicians, physicists, engineers, and computer scientists.

The basic model in CFD is concerned with multidimensional Euler and Navier-Stokes systems. There are, however, no rigorous stability analysis, error estimate or convergence proof available for these equations. In CFD, researchers depend heavily on rigorous mathematical analysis for simpler and linearised problems.

In establishing numerical simulations for a real physical problem, there are usually four steps:

A) Set up a mathematical model, i.e. present a partial differential equation (PDE) or a system of partial differential equations (PDEs) with appropriate boundary conditions. Depending on the properties of a given problem, it can be classified as linear or nonlinear problem, scalar or system, and one dimensional (1D) or multidimensional problem. The most challenging problem is certainly that of solving multidimensional nonlinear systems.

B) Design numerical schemes for the solutions of PDEs, and if possible provide the analysis of the proposed schemes. Usually one needs to consider routine questions such as "consistency", "stability", and "convergence".

C) Develop efficient linear solvers to solve the discrete equations resulting from Step B; Multigrid methods, conjugate-gradient like methods (including the CGS method [31]) and domain decomposition methods[32] are very often applied.

D) Interpret the numerical results for the corresponding real physical problem.

For a given problem, both theoretical and experimental results are important, but they are often very hard or impossible to obtain. Thus, it is necessary to consider numerical simulations. This Ph.D. thesis is devoted to the study of solving PDEs numerically. Our goal is to study multidimensional systems of Euler equations. There are three parts to this thesis:

(1) study artificial viscosity methods;

(2) introduce the concept of a *weakly coupled system*;

(3) study Euler solutions numerically.

If the model is either a one-dimensional system or a multidimensional scalar equation, there are many successful discretization schemes([9], [13], [14], [15], [18], [19], [21], [26], [27], [34], [36], [40], [43], and etc.). This is because of their simplicity and the availability of relatively complete theoretical results on their solution. Comparatively much less is known about multidimensional systems of PDEs, e.g. two-dimensional or three-dimensional Euler and Navier-Stokes equations. Even for a simple two-dimensional linear system with constant coefficients, we do not know very much about its solutions.

There are two approaches to consider multidimensional systems: (1) generalise results which are available from one-dimensional system or scalar equations; (2) develop genuinely multidimensional methods directly . There are many sophisticated methods available for one-dimensional systems, e.g. total variation diminishing (TVD), the characteristic method, flux splitting, etc.([22], [40], [44], [48], [54]). It will be useful to study and understand them thoroughly when we consider the development of multidimensional methods. Hence we begin by listing some of the facts regarding numerical methods for one-dimensional systems and scalar equations.

One popular approach in the recent development of CFD is the use of "upwinding". To study upwinding methods we have to deal with artificial viscosity methods which were first developed by von Neumann and Richtmyer[51]. Adding an artificial term into a numerical scheme provides a smoothing effect on the solutions (it is well-known that nonlinear hyperbolic equations can develop discontinuous solutions) and it also leads to an easy design for a stable numerical algorithm. On this topic, we will study an artificial viscosity method for the Burgers equation and the shock tube problems. Some of these numerical schemes are called exponential algorithms.

Multidimensional systems are essentially different from one dimensional systems. The physics of one-dimensional flows is especially simple and well understood, and they can be easily simulated by numerical processes. Two dimensional flows however are more complex; in particular, acoustic waves can propagate in infinitely many directions rather than just two as in a one-dimensional problem. Moreover, the existence of vorticity presents a new phenomenon. On the other hand, from the mathematical point of view, one-dimensional systems are much simpler than dealing with multidimensional systems with respect to the analysis of the equations ([13], [14], [15], and [56], etc.). It has been conjectured that solving a two-dimensional problem is blocked by the complexity of the "two-dimensional Riemann solver", by which we mean an algorithm for computing the breakdown of initial conditions which are piecewise constant in two-dimensional cells[41]. Due to the results of Rauch[39], the commutativity of the coefficient matrices for quasi-linear hyperbolic systems in dimensions greater than one plays a very important role. Unfortunately for the Euler equations, their coefficient matrices do not satisfy such commutativity. Therefore simply extending the one-dimensional work does not guarantee convergence. There are some works available for developing genuinely multidimensional methods([41], [49], and [55]).

In this thesis, we try to make contributions to the understanding of multidimensional Euler solutions. In this respect, we first introduce a concept called a "weakly coupled system". For a general weakly coupled system, we develop a semi-discretization scheme for which the stability condition is proved. We then show that Euler equations are weakly coupled if and only if the flow conditions are supersonic. For the numerical simulations, we compute steady state solutions of the shock reflection problems and supersonic channel flows.

The main contributions of this thesis on the numerical analysis of multidimensional Euler equations are as follows:

(1) The idea is new;

(2) The method is genuinely multidimensional;

(3) The implementation of the resulting numerical schemes are much easier and simpler than the upwinding methods. It works very well for very large Courant-Friedrichs-Levy (CFL) numbers when it is used in implicit computations.

(4) The analysis is simple. The stability conditions for the well-known Lax-Friedrichs scheme, the upwinding scheme and the fractional step scheme can be easily derived. It is shown numerically that the Lax-Friedrichs scheme works when using our algorithm but it fails when simply extending the one-dimensional results to the original two-dimensional quasi-linear Euler equations.

Finally we will make some comments and discuss some issues regarding possible future investigations.

Chapter II

MODEL PROBLEMS

2.1. Introduction

Model problems play a crucial role in numerical simulations of physical problems. There are several reasons:

1. They are the standard problems which describe some real physical problems under special circumstances;

2. Their solution's structures are known and theoretical and numerical analysis are available;

3. They are used as guidelines to design and test numerical schemes for more complicated problems.

Our model problems are the hyperbolic wave equations, and they include the following model problems:

(1) Linear scalar equation;

(2) Nonlinear scalar equation: Burgers' equation;

(3) Linear one-dimensional systems;

(4) Nonlinear one-dimensional system: 1D Euler equations.

2.2. Linear scalar equation

The simplest wave equation is the linear scalar equation:

$$u_t + cu_x = 0, \tag{2.2.1}$$

$$u(x,0) = f(x). \tag{2.2.2}$$

It is well-known th ; the the solution of the Cauchy problem (2.2.1) and (2.2.2) is given by

$$u(x,t) = f(x - ct). \tag{2.2.3}$$

The physical inter;,retation for this solution is that a wave propagates with speed c in the sign$(c)x$-direction. Note that the smoothness of f is required. If f does not have the required smoothness, e.g. $f \in L^2(R)$, one may not be comfortable with the expression given in (2.2.3). Modified equations of (2.2.1) and (2.2.2) can be derived by the following two regularizations, i.e. either regularising equation (2.2.1) or smoothing the initial condition (2.2.2). Here we consider the first kind of regularization, and it leads to a parabolic perturbation of (2.2.1) and (2.2.2):

$$u_t^\epsilon + cu_x^\epsilon = \epsilon u_{xx}^\epsilon, \tag{2.2.4}$$

$$u^\epsilon(x,0) = u_0(x). \tag{2.2.5}$$

The solution is well-known [42] and is given by the following theorem.

Theorem 2.2.1. *If* $\int_0^x u_0(s)\,ds = o(x^2)$, *then the solution of (2.2.4) and (2.2.5) is*

$$u^\epsilon(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-y^2} u_0(x - ct - \sqrt{4\epsilon t}\,y)\,dy. \tag{2.2.6}$$

Proof. This is derived from the solution of a parabolic equation.

Corollary. *If* $u_0 \in L^p(R)$, $1 \le p < \infty$, *then*

$$\begin{aligned} &u^\epsilon(\cdot\,,t) \in L^p(R), \\ &\|u^\epsilon(\cdot\,,t)\|_p \le \|u_0\|_p, \end{aligned} \tag{2.2.7}$$

and

$$\|u^\epsilon(\cdot\,,t) - u(\cdot\,,t)\|_p \to 0, \qquad \textit{as } \epsilon \to 0. \tag{2.2.8}$$

For a multidimensional scalar equation

$$u_t + au_x + bu_y = 0, \tag{2.2.9}$$

$$u(x,y,0) = u_0(x,y), \tag{2.2.10}$$

we have the following theorem.

Theorem 2.2.2. *The solution of (2.2.9) and (2.2.10) is*

$$u(x,y,t) = u_0(x - at, y - bt), \tag{2.2.11}$$

and

$$(2.2.12)\quad u^\epsilon(x,y,t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\xi^2-\eta^2} u_0(x-at-\sqrt{4\epsilon t}\xi, y-bt-\sqrt{4\epsilon t}\eta)\,d\xi\,d\eta$$

is the solution of

$$(2.2.14)\quad \begin{aligned} &u_t^\epsilon + au_x^\epsilon + bu_y^\epsilon = \epsilon(u_{xx}^\epsilon + u_{yy}^\epsilon),\\ &u^\epsilon(x,y,0) = u_0(x,y), \end{aligned}$$

where u_0 *is such that* $\int_0^x u_0(s,y)\,ds = o(x^2)$ *and* $\int_0^y u_0(x,s)\,ds = o(y^2)$.

2.3 Linear System with Constant Coefficients

Consider the linear one-dimensional system with constant coefficients:

$$(2.3.1)\quad U_t + AU_x = 0,$$

$$(2.3.2)\quad U(x,0) = U_0(x).$$

Here, we assume A is hyperbolic, i.e. A has a complete eigen-system, which means that all eigenvalues are real and the eigenvectors are linearly independent. So there exists a nonsingular matrix P such that $A = P\Lambda P^{-1}$, where Λ is a diagonal real matrix. Then $V = P^{-1}U$ satisfies the relation

$$(2.3.3)\quad V_t + \Lambda V_x = 0,$$

which is a decoupled system. The corresponding parabolic regularization is

$$(2.3.4)\quad V_t + \Lambda V_x = \epsilon V_{xx}.$$

Another parabolic regularization is one applied directly to (2.3.1) which is equivalent to (2.3.4) and is given by:

$$U_t + AU_x = \epsilon U_{xx}. \tag{2.3.5}$$

2.4 Nonlinear Scalar Equation: Burgers' Equation

Using the linear scalar equation as a model is too simple to explore more sophisticated phenomena such as shock waves. A well-known nonlinear model problem is described by the Burgers' equation:

$$u_t + uu_x = 0. \tag{2.4.1}$$

With a small viscous term, this becomes

$$u_t + uu_x = \epsilon u_{xx}. \tag{2.4.2}$$

This is the simplest model that includes the nonlinear and viscous effects of fluid dynamics. Burgers' equation is a special case of the nonlinear scalar conservation law

$$u_t + f(u)_x = 0, \tag{2.4.3}$$

where $f(u)$ is a nonlinear function of u. Usually we assume that $f(u)$ is a convex function of u, i.e. $f''(u) > 0$ for all u.

There is another reason why Burgers' equation has received a lot of attention. If we are only considering a smooth solution of (2.4.3) and set

$$v = f'(u), \tag{2.4.4}$$

then

$$v_t + vv_x = 0,$$

i.e. every nonlinear scalar conservation law with a convex flux can always be changed to the Burgers' equation at least in the smooth region of the solutions.

The Burgers' equation allows discontinuous solutions. Using the Hopf-Cole transformation one can use (2.4.2) to approximate (2.4.1). The solution of (2.4.2) [24] is

$$u_\epsilon(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{t}\, exp[-\frac{1}{\epsilon}F(x,\xi,t)]\, d\xi}{\displaystyle\int_{-\infty}^{\infty} exp[-\frac{1}{\epsilon}F(x,\xi,t)]\, d\xi}, \tag{2.4.6}$$

where

$$F(x,\xi,t) = \frac{(x-\xi)^2}{2t} + \int_0^{\xi} f(\eta)\, d\eta, \tag{2.4.7}$$

and

$$f(x) = u(x,0).$$

2.5 Nonlinear System: 1D Euler Equations

Now consider a nonlinear system in conservation form:

$$U_t + F(U)_x = 0, \tag{2.5.1}$$

where U and F are vectors in R^n. Its matrix form can be written as

$$U_t + A(U)U_x = 0, \tag{2.5.2}$$

where $A = \frac{\partial F}{\partial U} = (\frac{\partial F_i}{\partial U_j})_{n\times n}$. Here, we assume (2.5.1) is a hyperbolic system. The main difference between nonlinear and linear systems, i.e. (2.5.1) and (2.3.1) is their eigen-systems; one depends on the variable vector U while the other does not. For a nonlinear system it is not necessary to get the characteristic variables or the Riemann invariants, i.e. one may not be able to find P and V, such that $A(U) = P\Lambda(U)P^{-1}$, and $\frac{\partial V}{\partial U} = P^{-1}$. Even though we can successfully find the characteristic variables,

$$V_t + \Lambda(U)V_x = 0\ , \tag{2.5.3}$$

there is another difference, namely the following two systems

$$U_t + A(U)U_x = \epsilon U_{xx}, \tag{2.5.4}$$

and

$$V_t + \Lambda(U)V_x = \epsilon V_{xx} \tag{2.5.5}$$

are equivalent if $A(U)$ is constant and different if $A(U)$ is varying, because in the later case, P is not a constant matrix. Under the assumption of the equivalence of (2.5.2) and (2.5.3), we could construct a new artificial viscosity for (2.5.1).

We will consider the well-known shock tube problem below. The one-dimensional Euler equations can be expressed in the form[42]:

(2.5.6)

$$\begin{aligned} \rho_t + (\rho u)_x &= 0\,, \quad \text{(conservation of mass)},\\ (\rho u)_t + (\rho u^2 + P)_x &= 0\,, \quad \text{(conservation of momentum)},\\ [\rho(\frac{u^2}{2} + \rho)]_t + [\rho u(\frac{1}{2}u^2 + e + \gamma/\rho)]_x &= 0\,, \quad \text{(conservation law of energy)} \end{aligned}$$

where ρ is the density, u the velocity, P the pressure, e the energy per unit mass and γ the ratio of specific heats. The shock tube problem is the Riemann problem for one-dimensional Euler equations. Their weak solution is well-known ([42], [43]) and they serve as a good test problem for testing numerical methods.

Chapter III

Finite Difference Methods

3.1. Introduction

To solve PDEs numerically many numerical methods are available, e.g., finite difference, finite element, and finite volume methods. According to the accuracy of the methods, they can be classified as first-order, second-order, and higher-order schemes, including TVD(total variation diminishing), ENO (essentially non-oscillatory) and monotone methods. We may also construct conservation and non-conservation methods. A naive objective of developing numerical methods is to find a reliable convergent method. The fundamental theorem on numerical approximations of PDEs is given by the Lax equivalence theorem[22]:

Theorem 3.1.1. *For a well-posed linear initial value problem and a consistent discretisation scheme, stability is the necessary and sufficient condition for convergence.*

Hence, to study numerical schemes, one needs to perform the following tasks:

(1) Analyse the consistency condition;

(2) Analyse the stability conditions.

The first task can be achieved easily through space discretisations. The second task is hard but it is an important consideration in numerical analysis.

In this chapter we list some of the facts about finite difference approximations and some well-known schemes.

3.2. Finite Difference Discretisation Methods

In this thesis we only consider finite difference schemes. The concept of finite difference approximations is based on the properties of Taylor expansions. Even though we are mainly interested in two-dimensional problems, for simplicity we first consider the finite difference approximations for functions of one variable. Applying Taylor series expansion to $u(x+\Delta x)$ we get

$$u(x+\Delta x) = u(x) + \Delta x u_x(x) + \frac{(\Delta x)^2}{2} u_{xx}(x) + \dots. \tag{3.2.1}$$

Therefore,

$$u_x(x) = \frac{u(x+\Delta x) - u(x)}{\Delta x} + O(\Delta x). \tag{3.2.2}$$

Denoting $x_i = i\Delta x$, $i = 0, \pm 1, \pm 2, \dots$, and $u_i = (u)_{x=x_i}$, $(u_x)_i = (\partial u/\partial x)_{x=x_i}$, we can derive

$$(u_x)_i = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x), \tag{3.2.3}$$

for the forward difference:

(3.2.4) $$(u_x)_i = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x).$$

for the backward difference: and

(3.2.5) $$(u_x)_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x)^2,$$

for the central difference. The forward and backward difference formulae for $(u_x)_i$ can be considered as a central difference with respect to the midpoint

(3.2.6) $$x_{i+1/2} = \frac{x_i + x_{i+1}}{2} \text{ and } x_{i-1/2} = \frac{x_{i-1} + x_i}{2}.$$

So

(3.2.7) $$(u_x)_{i+1/2} = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x)^2,$$

and

(3.2.8) $$(u_x)_{i-1/2} = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x)^2.$$

This is a key idea in developing numerical algorithms.

The following one-sided formulas are also used and they are second order accurate:

(3.2.9)
$$(u_x)_i = \frac{3u_i - 4u_{i-1} + u_{i-2}}{2\Delta x} + O(\Delta x)^2,$$
$$(u_x)_i = \frac{-3u_i + 4u_{i+1} - u_{i+2}}{2\Delta x} + O(\Delta x)^2.$$

These formulae are used especially in treating the boundary conditions.

For higher order derivatives finite difference approximations can also be derived in a similar way, e.g. a second-order approximation to the second derivative $(u_{xx})_i$:

$$(u_{xx})_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x)^2. \tag{3.2.10}$$

which can be viewed as

$$(u_{xx})_i = \frac{(u_x)_{i+1/2} - (u_x)_{i-1/2}}{\Delta x} + O(\Delta x)^2. \tag{3.2.11}$$

The same technique can be extended to two dimensional cases. For example, a central difference for (u_x) and (u_{xx}) at (x_i, y_j) are

$$(u_x)_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x)^2, \tag{3.2.12}$$

$$(u_{xx})_{ij} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x)^2. \tag{3.2.13}$$

3.3. Upwinding Methods

The family of upwind schemes, which was first proposed by Courant, Isaacson and Reeves [8], is directed towards introducing physical properties of the flow equations into the discretized formulations. They have many applications in CFD. The family of techniques known as upwinding covers a variety of approaches, such as flux vector splitting, flux difference splitting and various 'flux controlling' methods.

The original scheme of Courant et al. [S] was based on the characteristic form of the equation $u_t + au_x = 0$ and a discretisation depending on the sign of the eigenvalue a.

With a first-order forward difference in time, it has been noted that the central difference of u_x leads to an unstable scheme. However, with a one-sided differencing the following scheme can be considered for $a > 0$:

$$u_i^{n+1} = u_i^n - \sigma(u_i^n - u_{i-1}^n) \tag{3.3.1}$$

where $\sigma = \frac{a\Delta t}{\Delta x}$. It is stable for values of the CFL number σ:

$$0 < \sigma \leq 1 . \tag{3.3.2}$$

The truncation error ϵ_T is

$$\epsilon_T = \frac{a\Delta x}{2}(1 - \sigma)u_{xx} . \tag{3.3.3}$$

which indicates that the scheme is first-order accurate.

For a negative propagation speed, $a < 0$, the following one-sided scheme is stable

$$u_i^{n+1} = u_i^n - \sigma(u_{i+1}^n - u_i^n), \tag{3.3.4}$$

for $-1 \leq \sigma < 0$. Therefore an upwind scheme such as (3.3.1) or (3.3.4) cannot be simultaneously stable for both positive and negative eigenvalues. But if we introduce positive and negative projections of the eigenvalues, the above two equations can be combined into a compact form. Let

$$a^+ = \max(a, 0) = \frac{1}{2}(a + |a|)\,, \qquad a^- = \min(a, 0) = \frac{1}{2}(a - |a|)\,. \tag{3.3.5}$$

The general form for the *first-order accurate upwind scheme written for the linearised scalar form of the wave equation* can be written as

$$u_i^{n+1} = u_i^n - \tau[a^+(u_i^n - u_{i-1}^n) + a^-(u_{i+1}^n - u_i^n)]. \tag{3.3.6}$$

The stability limit is

$$|\sigma| = \tau|a| \le 1\ , \tag{3.3.7}$$

$\tau = \frac{\Delta t}{\Delta x}$. The general form (3.3.6) can be used to develop the upwinding schemes for 1D system or 2D system by replacing a^+ by A^+, etc.

For one-dimensional Euler equations we can always diagonalise the coefficient matrix A. Suppose $A = L\Lambda L^{-1}$. We define

$$|A| = L|\Lambda|L^{-1}, \qquad A^+ = \frac{1}{2}(A + |A|), \qquad A^- = \frac{1}{2}(A - |A|). \tag{3.3.8}$$

$|\Lambda|$ is a diagonal matrix with absolute value of Λ. So for 1D quasi-linear system

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0\ , \tag{3.3.9}$$

the corresponding upwinding scheme is

$$U_i^{n+1} = U_i^n - \tau[A^+(U_i^n - U_{i-1}^n) + A^-(U_{i+1}^n - U_i^n)]. \tag{3.3.10}$$

Note that it can be written as

$$(3.3.10') \qquad U_i^{n+1} = U_i^n - \frac{\Delta t}{2\Delta x} A(U_{i+1}^n - U_{i-1}^n) + \frac{\Delta t}{2\Delta x} |A|(U_{i+1}^n - 2U_i^n + U_{i-1}^n).$$

The last term $\frac{\Delta t}{2\Delta x}|A|(U_{i+1}^n - 2U_i^n + U_{i-1}^n)$ can thus be viewed as a numerical artificial viscosity. The stability condition is that

$$(3.3.11) \qquad \frac{\Delta t}{\Delta x} \max_{1 \le i \le n} |\lambda_i| \le 1,$$

where λ_i are the eigenvalues of $A(U)$.

For a two-dimensional quasi-linear system

$$(3.3.12) \qquad \frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + B(U)\frac{\partial U}{\partial y} = 0,$$

we can diagonalise A and B separately but in general not simultaneously. Like the one-dimensional upwinding scheme, the two-dimensional upwinding scheme is

$$(3.3.13) \qquad \begin{aligned} U_{ij}^{n+1} = U_{ij}^n &- \tau_x[A^+(U_{ij}^n - U_{i-1,j}^n) + A^-(U_{i+1,j}^n - U_{ij}^n)] \\ &- \tau_y[B^+(U_{ij}^n - U_{i,j-1}^n) + B^-(U_{i,j+1}^n - U_{ij}^n)]. \end{aligned}$$

The stability condition is more complicated than in the 1D case. But if A and B commute (which is not the case for the Euler equations), we have the following stability condition:

$$(3.3.14) \qquad \frac{\Delta t}{\Delta x} \max_{1 \le i \le n} |\lambda_i^x| + \frac{\Delta t}{\Delta y} \max_{1 \le j \le n} |\lambda_j^y| \le 1,$$

where λ_i^x and λ_j^y are the eigenvalues of $A(U)$ and $B(U)$, respectively. Note that, for 2D Euler equations A and B do not commute. Hence even if the condition (3.3.14) is satisfied, convergence is not guaranteed.

We now go back to the scheme (3.3.6), and for simplicity rewrite it as

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{\tau}{2}|a|(u_{i+1}^n - 2u_i^n + u_{i-1}^n), \tag{3.3.15}$$

which shows the presence of a numerical viscosity term of the form $\Delta x^2|\sigma|u_{xx}/2$ added to the central discretized scheme. Hence, an upwind finite difference scheme is equivalent to the combination of a central difference and a dissipation operator which is similar to the viscosity terms in the Navier-Stokes equations. Hence there are two possible interpretations corresponding to the two forms (3.3.6) and (3.3.15). In order to properly take into account the propagation properties of a hyperbolic equation, either one applies an upwind, directionally biased space discretisation, or one uses a central difference discretisation without paying attention to the direction of propagation of the wave, but introduces a numerical artificial viscosity term. It has been noted that the second interpretation is much broader. Many schemes also share this interpretation, e.g., the Lax-Friedrichs scheme.

3.4. Lax-Friedrichs Scheme

The schemes of Lax or Lax-Friedrichs [27] are very important and interesting. They are simple and many theoretical results are available([4, 6, and 7]). Here we state some well-known results of the Lax-Friedrichs schemes.

The basic idea behind the one-dimensional Lax-Friedrichs scheme is to stabilize the explicit, unstable central scheme obtained from a central differencing applied to the first derivative of the flux term.

It is known that

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n), \tag{3.4.1}$$

is unstable for the linearised convection equation $u_t + au_x = 0$, where σ is the Courant number, also called the CFL number:

$$\sigma = \frac{a\Delta t}{\Delta x}. \tag{3.4.2}$$

But if we replace u_i^n in the right-hand side by the average value $(u_{i+1}^n + u_{i-1}^n)/2$, we get a stable Lax-Friedrichs scheme:

$$u_i^{n+1} = \frac{1}{2}(u_{i+1}^n + u_{i-1}^n) - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n). \tag{3.4.3}$$

Comparing with (3.4.1), we can see that the Lax-Friedrichs scheme is nothing more than adding a linear stabiliser to the explicit, unstable central scheme.

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}(u_{i+1}^n - u_{i-1}^n) + \frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n). \tag{3.4.4}$$

The linear stabiliser $\frac{1}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$ is equivalent to $\frac{(\Delta x)^2}{2}(u_{xx})_i$. Therefore the Lax-Friedrichs scheme introduces a numerical artificial viscosity term.

The generalization to a one-dimensional quasi-linear system (3.3.9) is

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{\Delta t}{2\Delta x}A(U_{i+1}^n - U_{i-1}^n). \tag{3.4.5}$$

Also, for a two-dimensional quasi-linear system

$$U_t + AU_x + BU_y = 0, \tag{3.4.6}$$

we have

$$U_{i,j}^{n+1} = \frac{1}{4}(U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^n) - \frac{\Delta t}{2\Delta x} A(U_{i+1,j}^n - U_{i-1,j}^n) - \frac{\Delta t}{2\Delta y} B(U_{i,j+1}^n - U_{i,j-1}^n). \tag{3.4.7}$$

The stability of the Lax-Friedrichs scheme is given in the following well-known theorem.

Theorem 3.4.1. *(1) The scheme (3.4.3) is stable if*

$$-1 \leq \sigma \leq 1; \tag{3.4.8}$$

(2) The scheme (3.4.5) is stable if

$$-1 \leq \min_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i \leq \max_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i \leq 1; \tag{3.4.9}$$

(3) If A and B commute, the scheme (3.4.7) is stable if

$$\begin{aligned} -\frac{1}{2} \leq \min_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i(A) \leq \max_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i(A) \leq \frac{1}{2}; \\ -\frac{1}{2} \leq \min_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i(B) \leq \max_{1 \leq i \leq n} \frac{\Delta t}{\Delta x} \lambda_i(B) \leq \frac{1}{2}. \end{aligned} \tag{3.4.10}$$

The proof is straight forward and will not be presented here. See [22].

Chapter IV

Artificial Viscosity Methods

4.1. Introduction

It is well-known that the entropy condition plays a crucial role in the study of conservation laws. Lax[27] applied the artificial viscosity method to get an entropy condition. Another example is due to Krujkov[26] in which the existence and uniqueness of the Cauchy problem for a quasi-linear hyperbolic equation in several variables were proved by using the artificial viscosity method.

It was von Neumann and Richtmyer[51] who first developed the concept of the artificial viscosity method. The goal of the artificial viscosity is to reduce the oscillation while allowing the shock transition to occupy only a few mesh points and having negligible effect in the smooth regions.

In many numerical schemes, e.g. the methods of Godunov, MacCormack, and Lax-Friedrichs, an artificial viscosity term was added. W[illegible] Lax-Friedrichs as an example.

The Lax-Friedrichs scheme for

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0 \tag{4.1.1}$$

is given by

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{\Delta t}{2\Delta x}A(U_{i+1}^n - U_{i-1}^n). \tag{4.1.2}$$

It can be rewritten as

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + A\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x} = \frac{(\Delta x)^2}{2\Delta t}\frac{(U_{i-1}^n - 2U_i^n + U_{i+1}^n)}{(\Delta x)^2}. \tag{4.1.3}$$

The right hand side of the above equation is a numerical viscosity term. A similar numerical viscosity term is also included in the 2D Lax-Friedrichs scheme.

The viscosity method has its theoretical importance in the study of solvability of transonic flow problems, conservation laws and nonlinear hyperbolic systems,(see [16]).

4.2. Linear Scalar Wave Equation – Exponential Schemes

Consider the linear scalar wave equation

$$u_t + au_x = 0. \tag{4.2.1}$$

It can be approximated by

$$u_t + au_x = \epsilon u_{xx}. \tag{4.2.2}$$

This parabolic regularization is largely considered because its solutions converge to the original equation ([14], [15] and [26]). Also from the numerical computation point of view this parabolic regularization is useful in analysing a particular numerical scheme. Here the idea behind our considerations is to discretize (4.2.2) directly. Now rewrite (4.2.2) into a compact form:

$$u_t = \epsilon \exp(\frac{ax}{\epsilon})[\exp(-\frac{ax}{\epsilon})u_x]_x. \tag{4.2.3}$$

This form, called the exponential form, is useful because it is easily shown that the solution is decreasing in a weighted L_2 norm.

Theorem 4.2.1. *Let u be a nonzero solution of (4.2.3). If*

$$\lim_{x \to \pm\infty} \exp(-\frac{ax}{\epsilon})u_x u = 0,$$

then

$$\frac{d}{dt}(\exp(-\frac{ax}{\epsilon})u, u) < 0, \tag{4.2.4}$$

where the inner product is defined as $(u, v) = \int_{-\infty}^{\infty} u(x,t)v(x,t)\,dx$.

Proof. From (4.2.3) we have

$$\exp(-\frac{ax}{\epsilon})u_t = \epsilon[\exp(-\frac{ax}{\epsilon})u_x]_x. \tag{4.2.5}$$

So

$$\begin{aligned}(\exp(-\frac{ax}{\epsilon})u_t, u) &= \epsilon([\exp(-\frac{ax}{\epsilon})u_x]_x, u)\\ &= -\epsilon(\exp(-\frac{ax}{\epsilon})u_x, u_x).\end{aligned} \tag{4.2.6}$$

Therefore

$$\frac{d}{dt}(\exp(-\frac{ax}{\epsilon})u, u) < 0. \tag{4.2.7}$$

Let u_i^n denote the approximation of $u(x_i, t^n)$. We consider the discretized form of (4.2.3).

(4.2.8)

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \epsilon \frac{\exp(-\frac{ah}{2\epsilon})(u_{i+1}^{n+1/2} - u_i^{n+1/2}) - \exp(\frac{ah}{2\epsilon})(u_i^{n+1/2} - u_{i-1}^{n-1/2})}{h^2},$$

where

$$u^{n+1/2} = \frac{u^{n+1} + u^n}{2}. \tag{4.2.9}$$

The discretized form of Theorem 4.2.1 is also true.

Theorem 4.2.2. *Let u_i^n be a non-constant solution of (4.2.8). If*

$$\lim_{i\to\infty} \exp(-\frac{aih}{\epsilon})\frac{u_{i+1}^{n+1/2} - u_i^{n+1/2}}{h}u_i^{n+1/2} = 0,$$

and

$$\lim_{i\to-\infty} \exp(-\frac{aih}{\epsilon})\frac{u_i^{n+1/2} - u_{i-1}^{n+1/2}}{h}u_i^{n+1/2} = 0,$$

then

$$\sum \exp(-\frac{aih}{\epsilon})\frac{(u_i^{n+1})^2 - (u_i^n)^2}{2\tau} < 0. \tag{4.2.10}$$

Proof:.

$$\sum \exp(-\frac{aih}{\epsilon})\frac{(u_i^{n+1})^2 - (u_i^n)^2}{2\tau}$$

$$= \sum \exp(-\frac{aih}{\epsilon})\frac{u_i^{n+1} - u_i^n}{\tau} u_i^{n+1/2}$$

$$= \epsilon \sum [\frac{\exp(-\frac{a(i+1/2)h}{\epsilon})(u_{i+1}^{n+1/2} - u_i^{n+1/2})u_i^{n+1/2}}{h^2}$$

$$- \frac{\exp(-\frac{a(i-1/2)h}{\epsilon})(u_i^{n+1/2} - u_{i-1}^{n+1/2})u_i^{n+1/2}}{h^2}]$$

$$= -\epsilon \sum \frac{\exp(-\frac{a(i-1/2)h}{\epsilon})(u_i^{n+1/2} - u_{i-1}^{n+1/2})^2}{h^2}$$

$$<0.$$

4.3. Burgers Equation

Apply the idea discussed in the previous section to the Burgers equation:

$$u_t + uu_x = \epsilon u_{xx} \tag{4.3.1}$$

We get the following scheme:

$$
(4.3.2) \qquad \frac{u_i^{n+1} - u_i^n}{\tau} = \epsilon\Big[\frac{\exp(-\frac{(3u_i^n + u_{i+1}^n)h}{8\epsilon})(u_{i+1}^n - u_i^n)}{h^2} - \frac{\exp(\frac{(3u_i^n + u_{i-1}^n)h}{8\epsilon})(u_i^n - u_{i-1}^n)}{h^2}\Big].
$$

Suppose

$$
m \le \min_i u_i^0 < \max_i u_i^0 \le M.
$$

Then we have the following theorem for the stability of the algorithm (4.3.2).

Theorem 4.3.1. *The algorithm (4.3.2) is stable if*

$$
(4.3.3) \qquad \tau < \frac{h^2}{\epsilon(\exp(-\frac{mh}{2\epsilon}) + exp(\frac{Mh}{2\epsilon}))}.
$$

For $\epsilon = h$, $m = 0$ *and* $M = 1$, *(4.3.4) becomes*

$$
\tau < \frac{h}{1 + \exp(1/2)}.
$$

Proof. The proof is straightforward. We can write (4.3.2) into

$$
(4.3.4) \qquad u_i^{n+1} = au_{i-1}^n + bu_i^n + cu_{i+1}^n,
$$

with

$$a = \frac{\epsilon\tau}{h^2}\exp(\frac{(3u_i^n + u_{i-1}^n)h}{8\epsilon}).$$

$$b = 1 - a - c.$$

$$c = \frac{\epsilon\tau}{h^2}\exp(-\frac{(3u_i^n + u_{i+1}^n)h}{8\epsilon}).$$

Then we only need to note that the condition (4.3.3) implies $a > 0$, $b > 0$ and $c > 0$ (see [29]).

Next, we implement numerically the algorithm (4.3.2) with the initial condition:

$$u(x,t) = \begin{cases} 1, & x \leq 0; \\ 0, & x > 0. \end{cases}$$

The solutions at $t = 0.25$ and 1.0 are given in Figure 4.1 and 4.2, where $\tau = 0.0025$, $h = 0.01$, $\epsilon = h$. The theoretical shock locations for $t = 0.5$ and $t = 1.0$ are $x = 0.25$ and $x = 0.5$, respectively.

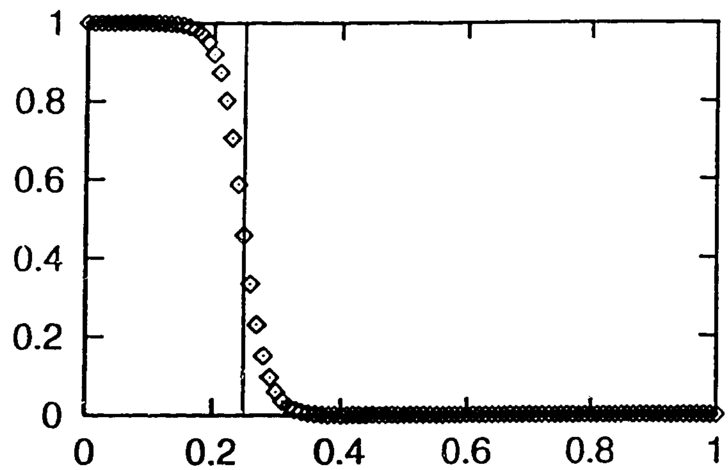

FIGURE 4.1. At t=0.5

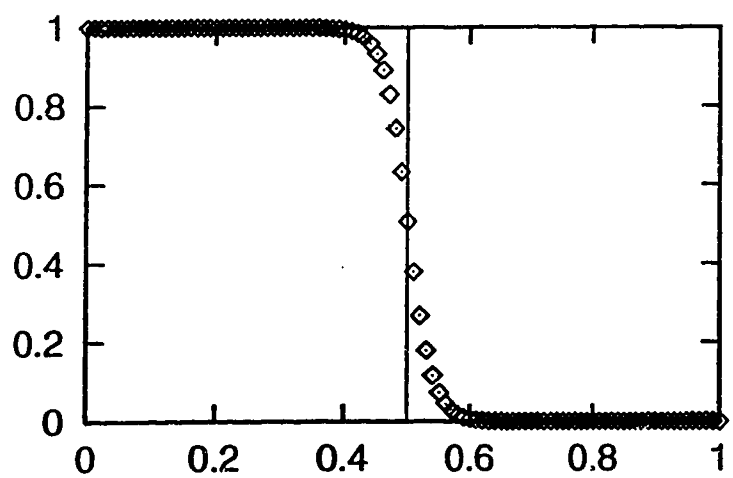

FIGURE 4.2. At t=1.0

4.4. 1D Euler Equation–Shock Tube Problem

4.4.1. The Shock Tube Problem

The shock tube problem of gas dynamics is a simple example that illustrates the interesting behaviour of the solutions to a system of conservation laws. The physical set up is a tube filled with gas, initially divided by a membrane into two sections. The gas has a higher density and pressure in one half of the tube than in the other half, with zero velocity everywhere. At time $t = 0$, the membrane is suddenly removed or broken, and the gas allowed to flow. We expect a net motion in the direction of lower pressure. Assuming the flow is uniform across the tube, there is variation in only one direction.

4.4.2. 1D Euler Equations

The mathematical set up is known as the Riemann problem, because it was Riemann who first studied this problem. The corresponding PDEs are

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + P \\ u(E+P) \end{pmatrix} = 0, \tag{4.4.1}$$

where $\rho = \rho(x,t)$ is the density, u is the velocity, ρu is the momentum, E is the energy per unit mass, and P is the pressure.

The initial conditions at $t = 0$ are

$$\begin{aligned} \rho &= \rho_L, \\ u &= u_L, \\ P &= P_L, \end{aligned} \tag{4.4.2.a}$$

for $x < x_0$, and

(4.4.2.b)
$$\rho = \rho_R,\quad u = u_R,\quad P = P_R,$$

for $x > x_0$, with $P_R < P_L$.

4.4.3 Finite Difference Schemes

We generalise the idea used in section 4.2 to the system of 1D Euler equations. Eq. (4.4.1) can be rewritten as

$$U_t = AU_x,$$

where $U = \begin{pmatrix} \rho \\ u \\ P \end{pmatrix}$, $A = A(U) = \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u \end{pmatrix}$ *and* c is the speed of sound.
Considering a system analogue of (4.2.8), we have the following schemes:

(4.4.3)
$$\frac{U_i^{n+1} - U_i^n}{\tau} = \epsilon \frac{\alpha(U_{i+1}^n - U_i^n) - \beta(U_i^n - U_{i-1}^n)}{h^2}.$$

with

(4.4.4)
$$\alpha = \exp\left(-\frac{(3A_i^n + A_{i+1}^n)h}{8\epsilon}\right),$$
$$\beta = \exp\left(\frac{(3A_i^n + A_{i-1}^n)h}{8\epsilon}\right).$$

The implicit form is

(4.4.5)
$$\frac{U_i^{n+1} - U_i^n}{\tau} = \epsilon \frac{\alpha(U_{i+1}^{n+1/2} - U_i^{n+1/2}) - \beta(U_i^{n+1/2} - U_{i-1}^{n+1/2})}{h^2}.$$

with

$$(4.4.6) \qquad U^{n+1/2} = \frac{U^n + U^{n+1}}{2} .$$

If we use the first two terms of the Taylor expansions for α and β in the expressions (4.4.3) and (4.4.5), we obtain the following algorithms,

$$(4.4.7) \qquad \begin{aligned} \frac{U_i^{n+1} - U_i^n}{\tau} =& \epsilon\left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}\right) \\ & - \frac{3A_i^n + A_{i+1}^n}{8}\frac{U_{i+1}^n - U_i^n}{h} - \frac{3A_i^n + A_{i-1}^n}{8}\frac{U_i^n - U_{i-1}^n}{h}. \end{aligned}$$

and

(4.4.8)

$$\begin{aligned} \frac{U_i^{n+1} - U_i^n}{\tau} =& \epsilon\left(\frac{U_{i+1}^{n+1/2} - 2U_i^{n+1/2} + U_{i-1}^{n+1/2}}{h^2}\right) - \frac{3A_i^n + A_{i+1}^n}{8}\frac{U_{i+1}^{n+1/2} - U_i^{n+1/2}}{h} \\ & - \frac{3A_i^n + A_{i-1}^n}{8}\frac{U_i^{n+1/2} - U_{i-1}^{n+1/2}}{h}. \end{aligned}$$

More general form for α and β in (4.4.6) are

$$(4.4.9) \qquad \begin{aligned} \alpha &= \exp\left(-\frac{((2+\theta)A_i^n + (2-\theta)A_{i+1}^n)h}{8\epsilon}\right), \\ \beta &= \exp\left(\frac{((2+\theta)A_i^n + (2-\theta)A_{i-1}^n)h}{8\epsilon}\right), \end{aligned}$$

where $-2 \le \theta \le 2$.

4.4.4. Numerical Results

We applied the above two algorithms (4.4.3) and (4.4.7) to the shock tube problem:

$$u = u_L, P = P_L, \rho = \rho_L, x < x_0, t = 0;$$
(4.4.10)
$$u = u_R, P = P_R, \rho = \rho_R, x > x_0, t = 0.$$

with $P_R < P_L$. Here we consider the test problem with the following data:

$$P_L = 1.0, \rho_L = 1; P_R = 0.1, \rho_R = 0.125;$$
(4.4.11)
$$u_L = u_R = 0.$$

When implementing the implicit algorithm, we use exact tri-diagonal block LU factorisation.

The theoretical shock location [43] is $x = 0.75$ at the time 0.14154. Taking $h = 0.00125$, Fig.4.3 give the numerical results using the algorithm (4.4.3) with CFL = 0.65. Fig. 4.4 are obtained by (4.4.7) with CFL = 0.90. Fig.4.5 illustrate the implicit solution of (4.4.5) with CFL = 2.14, Fig. 4.6 are the results using (4.4.8) with CFL = 2.14.

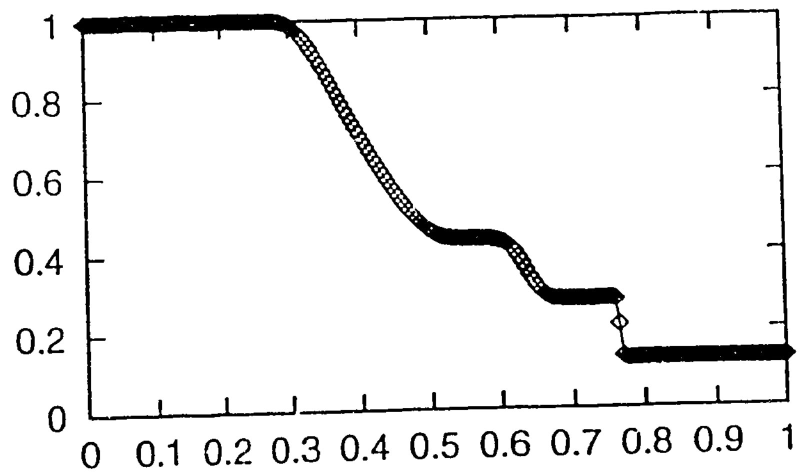

FIGURE 4.3A. Density at t=0.14154

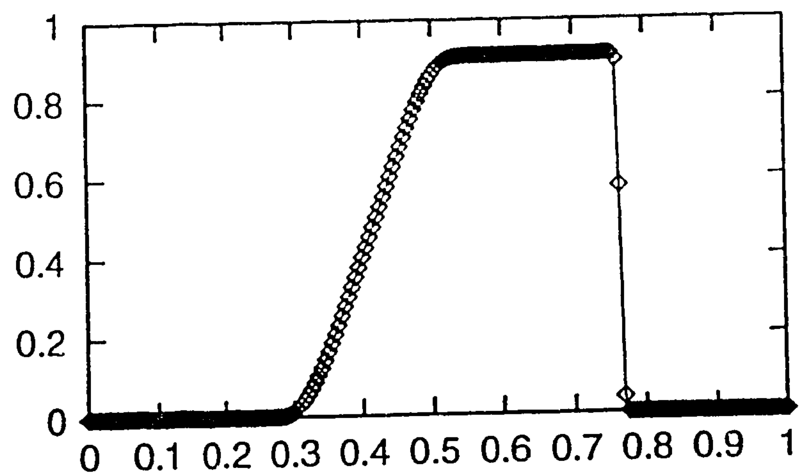

FIGURE 4.3B. Velocity at t=0.14154

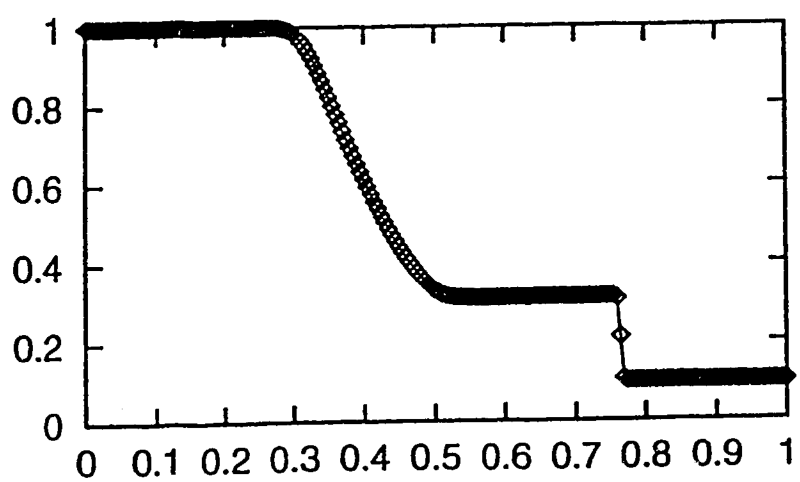

FIGURE 4.3C. Pressure at t=0.14154

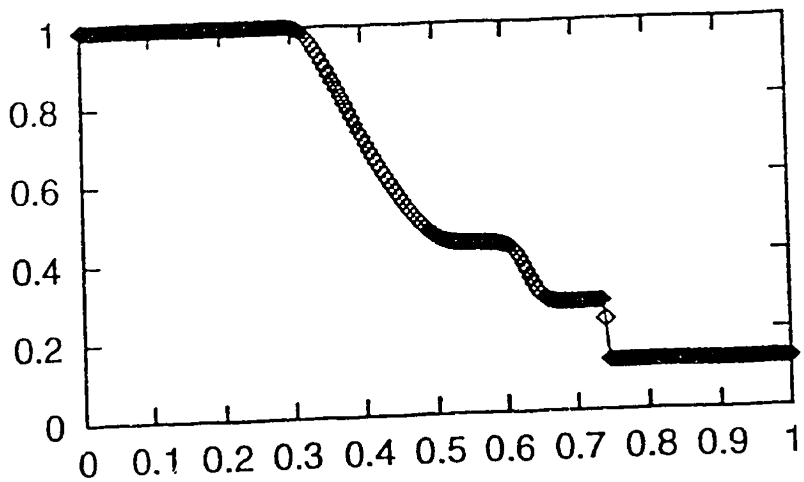

FIGURE 4.4A. Density at t=0.14154

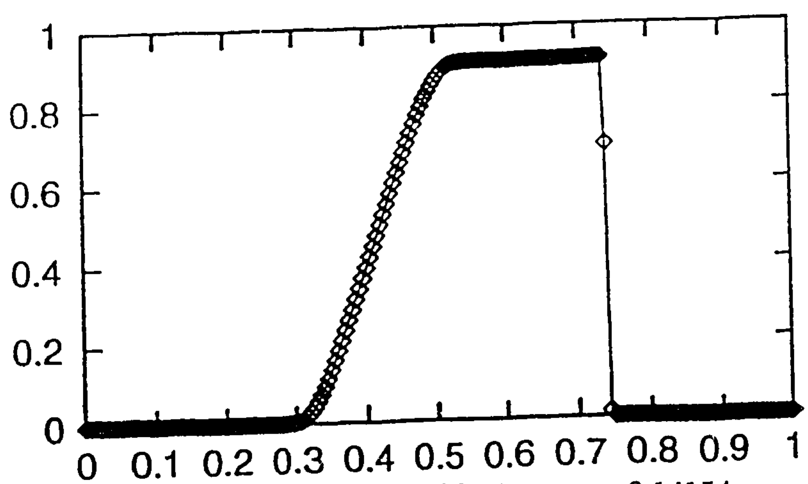

FIGURE 4.4B. Velocity at t=0.14154

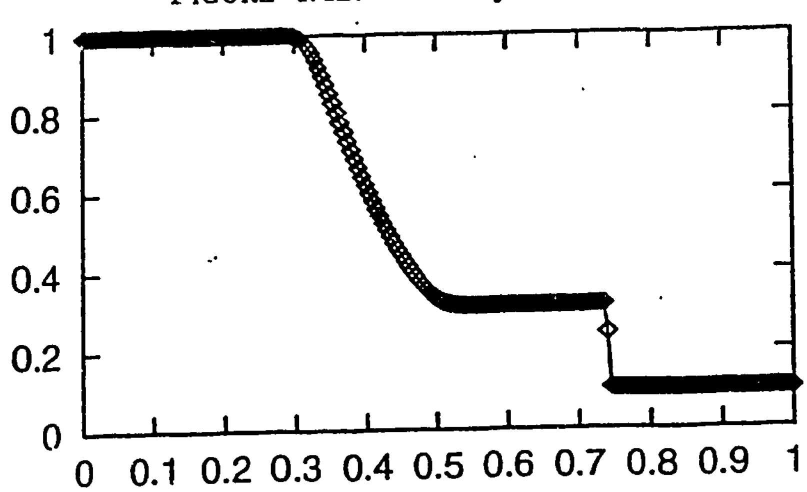

FIGURE 4.4C. Pressure at t=0.14154

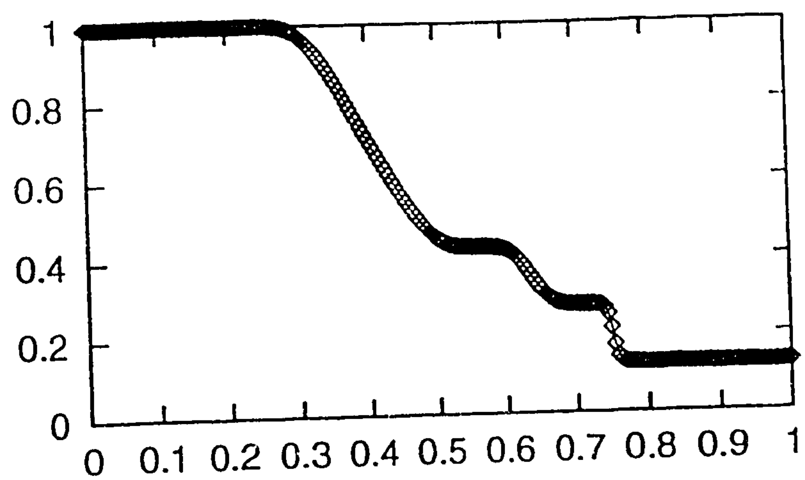

FIGURE 4.5A. Density at t=0.14154

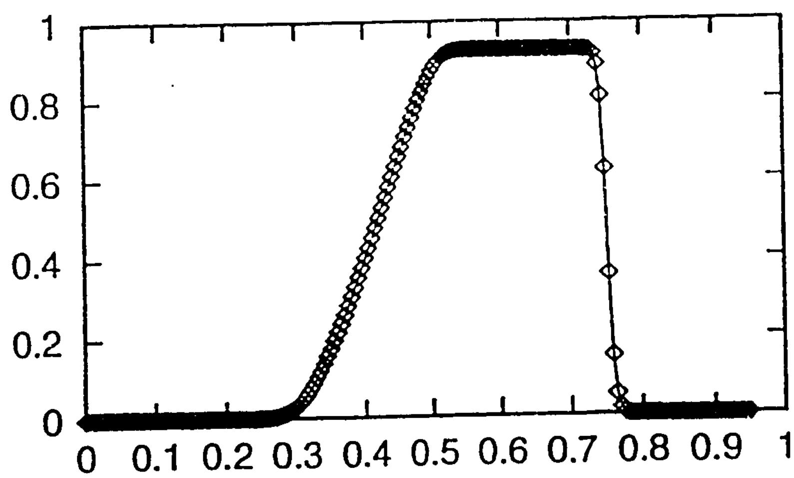

FIGURE 4.5B. Velocity at t=0.14154

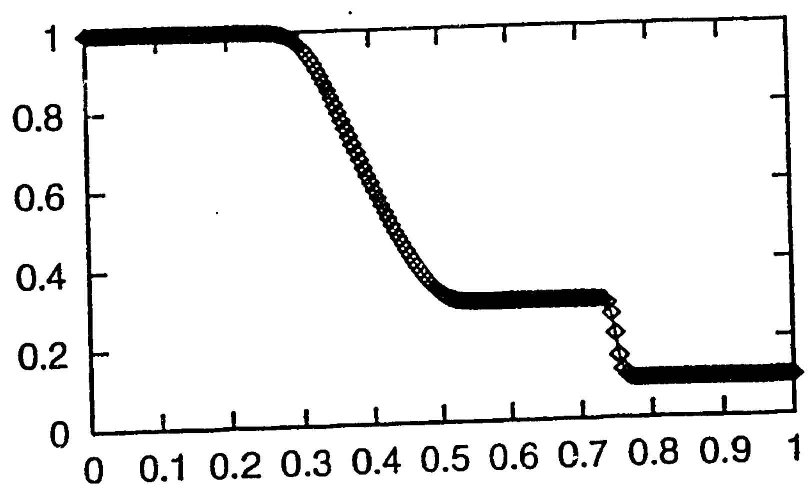

FIGURE 4.5C. Pressure at t=0.14154

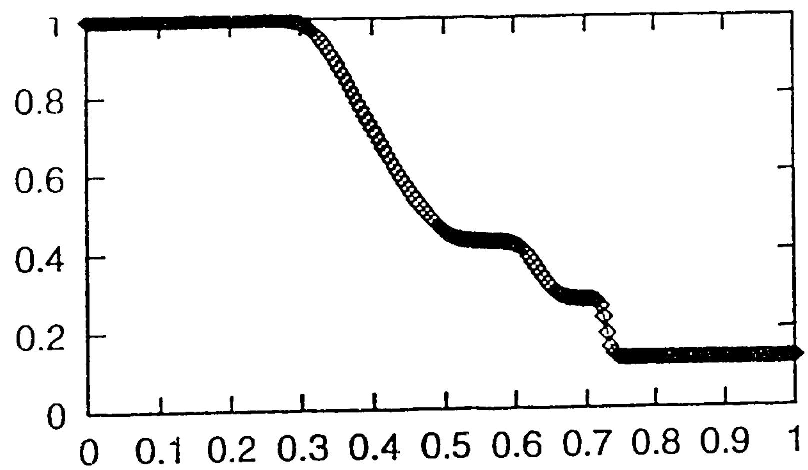

FIGURE 4.6A. Density at t=0.14154

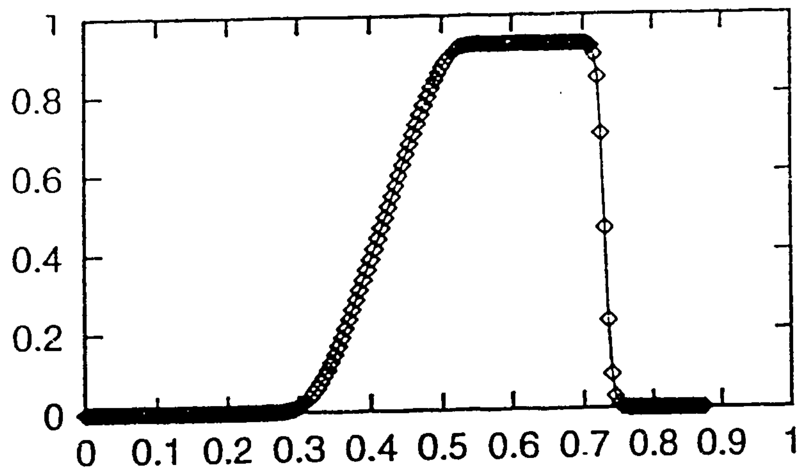

FIGURE 4.6B. Velocity at t=0.14154

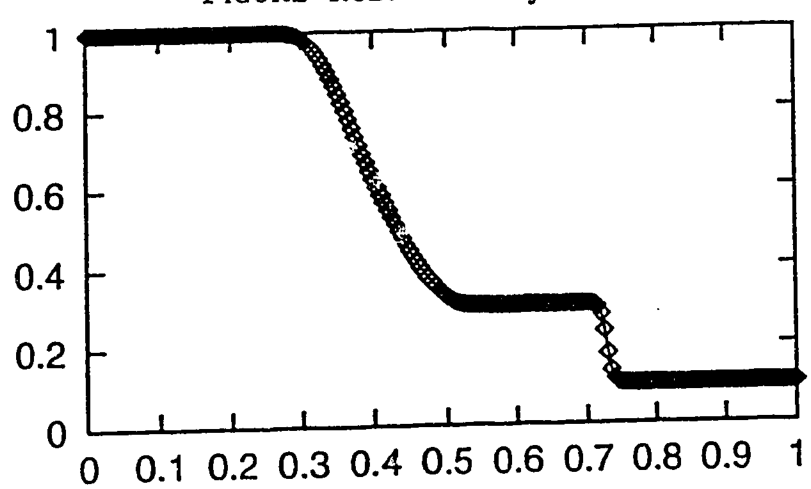

FIGURE 4.6C. Pressure at t=0.14154

Chapter V

Weakly Coupled Hyperbolic Systems

5.1. Introduction

The majority of unsolved problems in CFD research are governed by nonlinear systems of partial differential equations. One of the great challenges in CFD is to solve a multidimensional nonlinear system:

$$U_t + F(U)_x + G(U)_y = 0, \tag{5.1.1}$$

or

$$U_t + A(U)U_x + B(U)U_y = 0. \tag{5.1.2}$$

Because of the lack of mathematical analysis, there has been little progress as yet. From the mathematical point of view some negative results on multidimensional systems are known. *"Bounded variation estimates fail for most quasi-linear hyperbolic systems in dimensions greater than one"*, according to

Rauch[39]. Note that the commutativity of the coefficients of the quasi-linear system plays a key role, because for commutative matrices we can diagonalise them simultaneously. Hence the main problem or difficulty is the "coupling", which is caused by non-commutativity of A and B. In fact, for 2D Euler equations the commutator $[A(\bar{U}), B(\bar{U})] = AB - BA \neq 0$ for every state $\bar{U}$.

In this chapter we consider 2D linear or quasi-linear system

$$U_t + AU_x + BU_y = 0, \tag{5.1.3}$$

where A and B are $n \times n$ matrices. Here the main idea is to introduce a preconditioning approach applied to the PDEs, so that the coefficient matrices of the preconditioned system are commutative.

5.2. Definition and Examples

Our goal is to study multidimensional Euler equations, and we first consider a general 2-D linear system. The following definition then is not restricted to 2-D Euler equations. In order to help our understanding of this new concept—"weakly coupled system", we provide some examples of 2-D linear systems with 2×2 constant coefficient matrices.

Definition 1. *Given two $n \times n$ matrices A and B, they are said to be weakly coupled if there exists an $n \times n$ matrix K such that*

(1) K is positive definite;

(2) KA and KB are commutative.

Definition 2. *The system (5.1.3) is said to be hyperbolic if $\forall \alpha, \alpha A + (1 -$*

has a complete real eigen-system. i.e. it has n real eigenvalues, which are not necessary distinct, and n dimensional eigenspaces.

Definition 3. *The system (5.1.3) is said to be weakly coupled if it is hyperbolic and* $\exists K$ *such that*

(1) K *is positive definite;*

(2) KA *and* KB *are hyperbolic and commutative.*

For a quasi-linear system, it is said to be weakly coupled if it is weakly coupled for every frozen state.

The condition of hyperbolicity for the system in our definition is due to the following result on Euler equations.

Theorem 5.2.1. *The 2-D Euler equations are hyperbolic.*

Proof. This result is well-known. For the proof see Spekreijse[45].

A detailed discussion of the Euler equations is given in the next chapter. We present some simple examples here.

Regardless of the hyperbolicity condition, the concept of a *weakly coupled system* produces the following algebraic question:

Given two $n \times n$ *matrices* A *and* B, *determine whether there exists an* $n \times n$ *positive definite matrix* K *such that* $[KA, KB] = 0$.

The solution is in fact not trivial. Generally speaking the analysis of the couple-matrices (A, B) will be helpful for the understanding of general 2-D systems such as the Navier-Stokes equations.

Here we give some examples of *weakly coupled systems* with 2×2 constant coefficient matrices.

Theorem 5.2.2. *Given* $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. *If there exists* δ *and*

γ, *such that*

$$\delta a_1 + \gamma b_1 > 0.$$

(5.2.4)

$$\text{and} \quad 4(\delta a_1 + \gamma b_1)(\delta a_4 + \gamma b_4) > (\delta(a_2 + a_3) + \gamma(b_2 + b_3))^2.$$

then A and B are weakly coupled.

Proof. Define

$$K = \begin{pmatrix} \delta a_4 + \gamma b_4 & -(\delta a_2 + \gamma b_2) \\ -(\delta a_3 + \gamma b_3) & \delta a_1 + \gamma b_1 \end{pmatrix}.$$

Then we can show that $AKB = BKA$. and K is a positive definite matrix.

The conditions given in Theorem 5.2.2 are only sufficient. For the following simple matrices, we give the necessary and sufficient conditions.

Theorem 5.2.3. *Given* $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$. *if* $b_2^2 + b_3^2 \neq 0$, *they are weakly coupled if and only if*

$$(b_1 + b_4)^2 > (b_2 + b_3)^2. \tag{5.2.5}$$

Proof. Without lose of generality, we assume that $b_2 \neq 0$. By solving $AKB = BKA$ for the 2×2 matrix K, we get

$$K = \begin{pmatrix} -m_4 - \dfrac{b_1 + b_4}{b_2} m_2 & m_2 \\ \dfrac{b_3}{b_2} m_2 & m_4 \end{pmatrix}.$$

It is easy to see that K can be chosen to be positive definite if and only if the condition (5.2.5) holds.

Using the above two theorems we can construct many examples of 2×2 systems which are either weakly coupled or not weakly coupled. This concept is important in the applications to 2D Euler systems.

Chapter VI

2D Euler Equations

6.1. Introduction

The two-dimensional Euler equations can be expressed as

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ (E+P)u\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ (E+P)v\end{pmatrix}=0, \tag{6.1.1}$$

where ρ is the density, u and v are the velocities, E is the total energy per unit mass, and P is the pressure. Over the past four decades, numerous methods have been devised for the solutions of these equations. Relatively speaking, theory for one dimensional problems is essentially completed (see [27], [42], [43], [14] and [15]). Because of their physical complexity and the lack of mathematical analysis, our understanding of two-dimensional nonlinear systems is still very limited. Simply extending one dimensional methods is not enough and it is necessary to develop genuinely multidimensional methods([6], [12], [41] and [49]).

From the mathematical point of view, one essential multidimensional characteristic is as follows. If we rewrite (6.1.1) in its quasi-linear form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{6.1.2}$$

then in general, A and B do not commute, i.e., $AB \neq BA$, for every state U.

In this cha[illegible] consider the mechanism of commutativity and we show that the two dime[illegible]nal Euler system for supersonic flows is a weakly coupled system. We also consider the Euler equations in conservation form and the special case of constant total enthalpy.

6.2. 2D Euler Equations

The two-dimensional Euler equations consists of four conservation laws, namely, the conservation of mass, the conservation of momentums (in the x- and y-directions), and the conservation of energy. These equations are valid for a non-viscous, non-heat-conducting fluid without body forces. There are five unknowns in the four equations, and in order to complete the system we need an equation called the state equation, which can be written in general as

$$P = P(\rho, e), \tag{6.2.1}$$

where e is the internal energy per unit mass. For a perfect gas the thermodynamic equation of state gives

$$
\begin{aligned}
P &= \frac{R}{c_v}\rho e \\
&= (\gamma - 1)\rho e \\
&= (\gamma - 1)(E - \frac{1}{2}\rho(u^2 + v^2)),
\end{aligned}
\tag{6.2.2}
$$

where R is gas constant, c_v is the specific heat at constant volume, and γ is a constant, the ratio of specific heats, and $\gamma = 1.4$ for air.

In the primitive variables $U = (\rho, u, v, P)^T$, the two dimensional Euler equations take the following form:

$$
\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0\,, \tag{6.2.3}
$$

where

$$
\begin{aligned}
A &= \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & u \end{pmatrix}, \\
B &= \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \rho c^2 & v \end{pmatrix},
\end{aligned}
\tag{6.2.4}
$$

where $c = \sqrt{\frac{\gamma P}{\rho}}$ is the speed of sound.

Theorem 6.2.1. *A system of the two dimensional Euler equations is a weakly coupled system if and only if* $\dfrac{u^2 + v^2}{c^2} > 1$, *i.e. it is supersonic.*

Proof. The proof will be completed by using the following several lemmas.

Lemma 1. *Let $K = (k_{ij})_{4\times 4}$, and consider the matrix equation*

$$AKB = BKA. \tag{6.2.5}$$

The general solution of (6.2.5) is

$$K = \begin{pmatrix} k_{11} & t_1 & \frac{v}{u}t_1 - \frac{\rho^2 v}{u}t_3 + \rho^2 t_6 & t_2 \\ \frac{u}{v}t_4 & k_{22} & k_{23} & t_3 \\ t_4 & k_{23} & t_5 & t_6 \\ 0 & \rho^2 c^2 t_3 + \frac{\rho^2 u}{v}t_4 & \rho^2 t_4 + \rho^2 c^2 t_6 & k_{44} \end{pmatrix} \tag{6.2.6}$$

where

$$k_{11} = -c^2 t_2 - \rho u t_3 - \frac{\rho(u^2+v^2)}{c^2 v}t_4 - \rho v t_6,$$

$$k_{22} = -\frac{\rho u(c^2 - u^2 + v^2)}{v^2}t_3 + \frac{\rho(u^2 - v^2)(c^2 - u^2 - v^2)}{c^2 v^3}t_4 - \frac{u^2}{v^2}t_5,$$

$$k_{23} = \frac{\rho(u^2 - c^2)}{v}t_3 + \frac{\rho u(u^2 + v^2 - c^2)}{c^2 v^2}t_4 + \frac{u}{v}t_5 + \rho u t_6,$$

$$k_{44} = -\rho u t_3 - \frac{\rho(u^2+v^2)}{c^2 v}t_4 - \rho v t_6,$$

in which $t_1, \ldots, t_6$ are parameters.

We need only consider the first two eigenvalues of KA and KB in order to establish the necessary condition for Theorem 6.2.1.

Lemma 2. *(I) The first two eigenvalues of KA are*

$$(6.2.7)\quad \begin{aligned}\lambda_{1,2}(KA) = & \rho(c^2 - u^2)t_3 - \frac{\rho u(u^2 + v^2 - c^2)t_4}{vc^2} - \rho uvt_6 \\ & \pm \frac{\rho\sqrt{u^2 + v^2 - c^2}}{c}(t_4 + c^2 t_6).\end{aligned}$$

(II) The first two eigenvalues of KB are

$$(6.2.8)\quad \begin{aligned}\lambda_{1,2}(KB) = & -\rho uvt_3 + \rho c^{-2}(c^2 - u^2 - v^2)t_4 + \rho(c^2 - v^2)t_6 \\ & \pm \frac{\rho\sqrt{u^2 + v^2 - c^2}}{vc}(vc^2 t_3 + ut_4).\end{aligned}$$

The remaining three lemmas are used to establish the sufficient condition for Theorem 6.2.1. The following K is a particular choice of K defined by (6.2.6).

Lemma 3. *Let*

$$(6.2.9)\quad K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} & s\frac{uv}{q^2} & -\frac{u}{\rho q^2} \\ 0 & s\frac{uv}{q^2} & 1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2} & -\frac{v}{q^2} \\ 0 & -\rho c^2\frac{u}{q^2} & -\rho c^2\frac{v}{q^2} & 1 \end{pmatrix}$$

where $q^2 = u^2 + v^2$ and $c^2 = \dfrac{\lambda P}{\rho}$. If $M = \sqrt{\dfrac{u^2 + v^2}{c^2}} > 1$, then K is positive-definite for some s.

Lemma 4. *The K defined by (6.2.9) commutes A and B, i.e.* $[KA, KB] = 0$.

Lemma 5. *Let K be defined by (6.2.9). Then the eigen-systems for KA and KB are as follows*

(1) The eigenvalue matrix of KA is

(6.2.10)
$$\Lambda_{KA} = \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & u(1-\frac{2c^2}{q^2}+s) & 0 & 0 \\ 0 & 0 & u(1-\frac{c^2}{q^2})-\frac{c^2 v\alpha}{q^2} & 0 \\ 0 & 0 & 0 & u(1-\frac{c^2}{q^2})+\frac{c^2 v\alpha}{q^2} \end{bmatrix};$$

(2) The eigenvalue matrix of KB is

(6.2.11)
$$\Lambda_{KB} = \begin{bmatrix} v & 0 & 0 & 0 \\ 0 & v(1-\frac{2c^2}{q^2}+s) & 0 & 0 \\ 0 & 0 & v(1-\frac{c^2}{q^2})+\frac{c^2 u\alpha}{q^2} & 0 \\ 0 & 0 & 0 & v(1-\frac{c^2}{q^2})-\frac{c^2 u\alpha}{q^2} \end{bmatrix};$$

(3) Their eigenvectors are, in either case

$$L = \begin{bmatrix} 1 & \frac{1}{sq^2-2c^2} & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & \frac{u}{\rho q^2} & -\frac{u+v\alpha}{2\rho q^2} & \frac{v\alpha-u}{2\rho q^2} \\ 0 & \frac{v}{\rho q^2} & \frac{u\alpha-v}{2\rho q^2} & -\frac{u\alpha+v}{2\rho q^2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{6.2.12}$$

where $\alpha = \sqrt{M^2 - 1} = \sqrt{\frac{u^2 + v^2}{c^2} - 1}$. L^{-1} *is given by*

$$(6.2.13) \qquad L^{-1} = \begin{pmatrix} 1 & -\frac{\rho u}{sq^2 - 2c^2} & -\frac{\rho v}{sq^2 - 2c^2} & \frac{c^2 - sq^2}{c^2(sq^2 - 2c^2)} \\ 0 & \rho u & \rho v & 1 \\ 0 & -\frac{\rho v}{\alpha} & \frac{\rho u}{\alpha} & 1 \\ 0 & \frac{\rho v}{\alpha} & -\frac{\rho u}{\alpha} & 1 \end{pmatrix}.$$

First note the above five lemmas implies Theorem 6 2. From Lemma 2, if KA and KB are hyperbolic, a necessary condition is that both $\lambda_1(KA)$ and $\lambda_1(KB)$ must be real. If $u^2 + v^2 - c^2 < 0$, we have to have

$$(6.2.14) \qquad \begin{aligned} t_4 + c^2 t_6 &= 0, \\ vc^2 t_3 + ut_4 &= 0, \end{aligned}$$

from (6.2.9) and (6.2.10). But this condition (6.2.14) implies that all coefficients in the last row of the matrix K defined by (6.2.7) are zero and hence K is singular. Therefore we conclude that $u^2 + v^2 - c^2 > 0$. The 'if' part of Theorem 6.2.1 is proved by using Lemma 3 to Lemma 5.

Now we give the proofs for the above five lemmas.

Proof of Lemma 1. Denote

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix}.$$

Consider $AKB = BKA$ where A and B are defined by (6.2.3). We get a linear homogeneous system for k_{ij} as follows.

$$\rho v k_{21} - \rho u k_{31} = 0,$$
$$-\rho v k_{11} - \rho v c^2 k_{14} + \rho v k_{22} - \rho^2 k_{31} - \rho u k_{32} - \rho^2 c^2 k_{34} = 0,$$
$$\rho u k_{11} + \rho u c^2 k_{14} + \rho^2 k_{21} + \rho v k_{23} + \rho^2 c^2 k_{24} - \rho u k_{33} = 0,$$
$$-\frac{v}{\rho} k_{12} + \frac{u}{\rho} k_{13} + k_{23} + \rho v k_{24} - k_{32} - \rho u k_{34} = 0,$$
$$k_{41} = 0,$$
$$-\rho v k_{21} - \rho c^2 v k_{24} + \frac{v}{\rho} k_{42} = 0,$$
$$\rho u k_{21} + \rho c^2 u k_{24} + k_{41} + \frac{v}{\rho} k_{43} + c^2 k_{44} = 0,$$
$$-\frac{v}{\rho} k_{22} + \frac{u}{\rho} k_{23} + \frac{1}{\rho^2} k_{43} + \frac{v}{\rho} k_{44} = 0,$$
$$-\rho v k_{31} - \rho c^2 v k_{34} - k_{41} - \frac{u}{\rho} k_{42} - c^2 k_{44} = 0,$$
$$\rho u k_{31} + \rho c^2 u k_{34} - \frac{u}{\rho} k_{43} = 0,$$
$$\frac{v}{\rho} k_{32} + \frac{u}{\rho} k_{33} - \frac{1}{\rho^2} k_{42} - \frac{u}{\rho} k_{44} = 0,$$
$$v k_{21} - u k_{31} = 0,$$
$$\rho c^2 v k_{22} - \rho^2 c^2 k_{31} - \rho c^2 u k_{32} - \rho^2 c^4 k_{34} - \rho v k_{41} - \rho c^2 v k_{44} = 0,$$
$$\rho^2 c^2 k_{21} + \rho c^2 v k_{23} + \rho^2 c^4 k_{24} - \rho c^2 u k_{33} + \rho u k_{41} + \rho c^2 u k_{44} = 0,$$
$$c^2 k_{23} + \rho c^2 v k_{24} - c^2 k_{32} - \rho c^2 u k_{34} - \frac{v}{\rho} k_{42} + \frac{u}{\rho} k_{43} = 0.$$

With the aid of the software *Maple*, we find the general solutions K which are given by (6.2.6).

Proof of Lemma 2. Let $KA = (\alpha_{ij})_{4\times 4}$. Then

$$(6.2.15) \qquad KA = \begin{pmatrix} \alpha_{11} & \alpha_{12} & vt_1 - \rho^2 vt_3 + \rho^2 ut_6 & \frac{t_1}{\rho} + ut_2 \\ \frac{u^2 t_4}{v} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ ut_4 & \alpha_{32} & ut_5 & \alpha_{34} \\ 0 & -\rho^2 v(t_4 + c^2 t_6) & \rho^2 u(t_4 + c^2 t_6) & \alpha_{44} \end{pmatrix},$$

where $\alpha_{11} = (-c^2 t_2 - \rho u t_3 - \dfrac{\rho(u^2+v^2)t_4}{vc^2} - \rho v t_6)u$

$$
\begin{aligned}
\alpha_{12} &= ut_1 - \rho^2 ut_3 - \frac{\rho^2(u^2+v^2)t_4}{vc^2} - \rho^2 vt_6, \\
\alpha_{22} &= \frac{\rho(u^4 + u^2v^2 - u^2c^2 + v^2c^2)t_3}{v^2} + \frac{\rho u(-u^4 - u^2 - v^4 + 2v^2c^2)t_4}{v^3c^2} \\
&\quad + \frac{u^3 t_5}{v^2} + \frac{\rho u(u^2 - v^2 + c^2)t_6}{v} \\
\alpha_{23} &= \frac{\rho u(u^2 - c^2)t_3}{v} + \frac{\rho u^2(u^2 + v^2 - c^2)t_4}{v^2c^2} + \frac{u^2 t_5}{v^2} + \rho u^2 t_6 \\
\alpha_{24} &= \frac{u(u^2 - c^2)t_3}{v^2} + \frac{(u^2 - c^2)(u^2 + v^2 - c^2)t_4}{v^3c^2} \\
&\quad + \frac{u^2 t_5}{\rho v^2} + \frac{(u^2 + v^2 - c^2)t_6}{v} \\
\alpha_{32} &= \frac{\rho u(u^2 - c^2)t_3}{v} + \frac{\rho(u^4 + u^2(v^2 - c^2) + v^2c^2)t_4}{v^2c^2} \\
&\quad + \frac{u^2 t_5}{v} + \rho(u^2 + c^2)t_6 \\
\alpha_{34} &= \frac{(u^2 - c^2)t_3}{v} + \frac{u(u^2 + v^2 - c^2)t_4}{v^2c^2} + \frac{ut_5}{v^2c^2} + 2ut_6 \\
\alpha_{44} &= \rho(c^2 - u^2)t_3 - \frac{u(u^2 + v^2 - c^2)t_4}{vc^2} - uvt_6.
\end{aligned}
$$

Again with aid of the software *Maple*, we can show that $\lambda_{1,2}(KA)$ given by (6.2.7) satisfy

$\det(\lambda I - KA) = 0.$

Similarly we can show that $\lambda_{1,2}(KB)$ given by (6.2.8) satisfy

$\det(\lambda I - KB) = 0.$

Proof of Lemma 3. Consider

$$
K_1 = \frac{K + K^T}{2}
$$

(6.2.16)

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} & s\frac{uv}{q^2} & -\frac{1+\rho^2c^2}{2\rho q^2}u \\
0 & s\frac{uv}{q^2} & 1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2} & -\frac{1+\rho^2c^2}{2\rho q^2}v \\
0 & -\frac{1+\rho^2c^2}{2\rho q^2}u & -\frac{1+\rho^2c^2}{2\rho q^2}v & 1
\end{pmatrix}
$$

By the definition, in order to show K is positive-definite we need to show that

K_1 is positive-definite and that is

(i) $1 > 0$;

(ii) $det \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} \end{pmatrix} > 0$;

(iii) $det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} & s\frac{uv}{q^2} \\ 0 & s\frac{uv}{q^2} & 1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2} \end{pmatrix} > 0$;

(iv) $det(K_1) > 0$.

Note, (i) is [illegible]d. If $M > 1$, (ii) is satisfied provided that $s > 0$. Now

$$\begin{aligned} det &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} & s\frac{uv}{q^2} \\ 0 & s\frac{uv}{q^2} & 1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2} \end{pmatrix} \\ &=(1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2})(1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2}) - s^2\frac{u^2v^2}{q^4} \\ &=(1 - \frac{c^2}{q^2})^2 + s(1 - \frac{c^2}{q^2}). \end{aligned}$$

So again if we choose $s > 0$, (iii) is satisfied.

$$\begin{aligned} det(K_1) = &(1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2})(1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2}) + 2(\frac{1+\rho^2c^2}{2\rho q^2})^2\frac{su^2v^2}{q^2} \\ &- (\frac{1+\rho^2c^2}{2\rho q^2})^2u^2(1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2}) - s^2\frac{u^2v^2}{q^4} \\ &- (\frac{1+\rho^2c^2}{2\rho q^2})^2v^2(1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2}) \\ = &(1 - \frac{c^2}{q^2})^2 + (1 - \frac{c^2}{q^2})(s - q^2(\frac{1+\rho^2c^2}{2\rho q^2})^2) \\ = &(1 - \frac{c^2}{q^2})(s - \frac{(1+\rho^2c^2)^2}{4\rho^2q^2} + 1 - \frac{c^2}{q^2}). \end{aligned}$$

If

$$s \geq \frac{(1+\rho^2 c^2)^2}{4\rho^2 q^2} - 1 + \frac{c^2}{q^2},$$

(iv) is satisfied. Since $M > 1$, we could take

$$s = \frac{(1+\rho^2 c^2)^2}{4\rho^2 c^2}.$$

Therefore the proof is completed.

Proof of Lemma 4. By choosing

$$t_1 = 0;$$

$$t_2 = 0;$$

$$t_3 = -\frac{u}{\rho q^2};$$

$$t_4 = 0;$$

$$t_5 = \frac{q^2 - c^2 + sv^2}{q^2};$$

$$t_6 = -\frac{v}{\rho q^2},$$

the matrix K is given in (6.2.6). Using Lemma 1, $AKB = BKA$. Since K is positive-definite, K is also nonsingular. One can check directly if $AKB = BKA$ instead of checking $[KA, KB] = 0$. Here A and B can be rewritten as

$A = uI + A_1$ and $B = vI + B_1$ respectively, where

(6.2.7)
$$A_1 = \begin{pmatrix} 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 1/\rho \\ 0 & 0 & 0 & 0 \\ 0 & \rho c^2 & 0 & 0 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\rho \\ 0 & 0 & \rho c^2 & 0 \end{pmatrix}.$$

Therefore, it is equivalent to show that

(6.2.8)
$$uKB_1 + vA_1K + A_1KB_1 = vKA_1 + uB_1K + B_1KA_1.$$

Since KA_1, KB_1, A_1K, B_1K, A_1KB_1, and B_1KA_1 are given by

$$KA_1 = \begin{bmatrix} 0 & \rho & 0 & 0 \\ 0 & -\dfrac{c^2u}{q^2} & 0 & \dfrac{(1-\frac{c^2}{q^2})}{\rho} + \dfrac{su^2}{\rho q^2} \\ 0 & -\dfrac{c^2v}{q^2} & 0 & \dfrac{suv}{\rho q^2} \\ 0 & \rho c^2 & 0 & -\dfrac{c^2u}{\rho q^2} \end{bmatrix},$$

$$
KB_1 = \begin{bmatrix}
0 & 0 & \rho & 0 \\
0 & 0 & -\dfrac{c^2 u}{q^2} & \dfrac{suv}{\rho q^2} \\
0 & 0 & -\dfrac{c^2 v}{q^2} & \dfrac{(1 - \dfrac{c^2}{q^2})}{\rho} + \dfrac{sv^2}{\rho q^2} \\
0 & 0 & \rho c^2 & -\dfrac{c^2 v}{\rho q^2}
\end{bmatrix},
$$

$$
A_1 K = \begin{bmatrix}
0 & \rho(1 - c^2/q^2) + s\rho u^2/q^2 & s\rho uv/q^2 & -u/q^2 \\
0 & -c^2 u/q^2 & -c^2 v/q^2 & 1/\rho \\
0 & 0 & 0 & 0 \\
0 & \rho c^2(1 - c^2/q^2) + s\rho c^2 u^2/q^2 & s\rho c^2 uv/q^2 & -c^2 u/q^2
\end{bmatrix},
$$

$$
B_1 K = \begin{bmatrix}
0 & s\rho uv/q^2 & \rho(1 - c^2/q^2) + s\rho v^2/q^2 & -uv/q^2 \\
0 & 0 & 0 & 0 \\
0 & -c^2 u/q^2 & -c^2 v/q^2 & 1/\rho \\
0 & s\rho c^2 uv/q^2 & \rho c^2(1 - c^2/q^2) + s\rho c^2 uv/q^2 & -c^2 v/q^2
\end{bmatrix},
$$

$$
A_1 K B_1 = \begin{bmatrix}
0 & 0 & -\rho c^2 u/q^2 & suv/q^2 \\
0 & 0 & c^2 & -c^2 v/\rho q^2 \\
0 & 0 & 0 & 0 \\
0 & 0 & -\rho c^4 u/q^2 & sc^2 uv/q^2
\end{bmatrix},
$$

and

$$B_1KA_1 = \begin{bmatrix} 0 & -\rho c^2 v/q^2 & 0 & suv/q^2 \\ 0 & 0 & 0 & 0 \\ 0 & c^2 & 0 & -c^2 u/\rho q^2 \\ 0 & -\rho c^4 v/q^2 & 0 & sc^2 uv/q^2 \end{bmatrix}.$$

$$uKB_1 + vA_1K + A_1KB_1$$

$$= \begin{bmatrix} 0 & \rho(1-\frac{c^2}{q^2})v + \frac{s\rho u^2 v}{q^2} & \rho(1-\frac{c^2}{q^2})u + \frac{s\rho uv^2}{q^2} & \frac{(s-\rho)uv}{\rho q^2} \\ 0 & -\frac{c^2uv}{q^2} & 0 & (1-\frac{c^2}{q^2})\frac{v}{q^2} + \frac{su^2v}{\rho q^2} \\ 0 & 0 & -\frac{c^2uv}{q^2} & (1-\frac{c^2}{q^2})\frac{u}{q^2} + \frac{suv^2}{\rho q^2} \\ 0 & \rho c^2(1-\frac{c^2}{q^2})v + \frac{s\rho c^2u^2v}{q^2} & \rho c^2(1-\frac{c^2}{q^2})u + \frac{s\rho c^2uv^2}{q^2} & \frac{(s-2)c^2uv}{q^2} \end{bmatrix}$$

$$= vKA_1 + uB_1K + B_1KA_1.$$

Proof of Lemma 5. The proof is straightforward. Since

(6.2.17)

$$KA = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u - \frac{u(q^2-2c^2+su^2)}{q^2} & \frac{su^2v}{q^2} & \frac{q^2-c^2+(s-1)u^2}{\rho q^2} \\ 0 & \frac{su^2v-c^2v}{q^2} & \frac{u(q^2-c^2+sv^2)}{q^2} & \frac{(s-1)uv}{\rho q^2} \\ 0 & \rho c^2(1-\frac{u^2}{q^2}) & -\rho c^2\frac{uv}{q^2} & \frac{u(q^2-c^2)}{q^2}, \end{pmatrix}$$

and

(6.2.18)

$$KB = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & \dfrac{v(q^2 - c^2 + su^2)}{q^2} & \dfrac{u(sv^2 - c^2)}{q^2} & \dfrac{(s-1)uv}{\rho q^2} \\ 0 & \dfrac{suv^2}{q^2} & v - \dfrac{v(q^2 - 2c^2 + sv^2)}{q^2} & \dfrac{q^2 - c^2 + (s-1)v^2}{\rho q^2} \\ 0 & -\rho c^2 uv/q^2 & \rho c^2(1 - \dfrac{v^2}{q^2}) & \dfrac{u(q^2 - c^2)}{q^2}. \end{pmatrix}.$$

With L given in (6.2.12), we have

$$KAL = \begin{pmatrix} u & \dfrac{u((1+s)q^2 - 2c^2)}{q^2(sq^2 - 2c^2)} & \dfrac{um - c^2 v\alpha}{2c^2 q^2} & \dfrac{um + c^2 v\alpha}{2c^2 q^2} \\ 0 & \dfrac{u^2((1+s)q^2 - 2c^2)}{\rho q^4} & -\dfrac{(u + v\alpha)[um - c^2 v\alpha]}{2\rho q^4} & \dfrac{(v\alpha - u)[um + c^2 v\alpha]}{2\rho q^4} \\ 0 & \dfrac{uv((1+s)q^2 - 2c^2)}{\rho q^4} & \dfrac{(u\alpha + v)[um - c^2 v\alpha]}{2\rho q^4} & -\dfrac{(u\alpha + v)[um + c^2 v\alpha]}{2\rho q^4} \\ 0 & 0 & \dfrac{um - c^2 v\alpha}{2q^2} & \dfrac{um + c^2 v\alpha}{2q^2} \end{pmatrix}$$

$$= L\Lambda_{KA};$$

and

$$KBL = \begin{pmatrix} v & \dfrac{v(q^2 - 2c^2 + sq^2)}{q^2(sq^2 - 2c^2)} & \dfrac{vm + c^2 u\alpha}{2c^2 q^2} & \dfrac{vm - c^2 u\alpha}{2c^2 q^2} \\ 0 & \dfrac{uv(q^2 - 2c^2 + sq^2)}{\rho q^4} & -\dfrac{(u + v\alpha)[vm + c^2 u\alpha]}{2\rho q^4} & \dfrac{(v\alpha - u)[vm - c^2 u\alpha]}{2\rho q^4} \\ 0 & \dfrac{v^2((1+s)q^2 - 2c^2)}{\rho q^4} & \dfrac{(u\alpha + v)[vm + c^2 u\alpha]}{2\rho q^4} & -\dfrac{(u\alpha + v)[vm - c^2 u\alpha]}{2\rho q^4} \\ 0 & 0 & \dfrac{vm + c^2 u\alpha}{2q^2} & \dfrac{vm - -c^2 u\alpha}{2q^2} \end{pmatrix}$$

$$= L\Lambda_{KB},$$

where $m = q^2 - c^2$.

From the proof of Theorem 6.2.1, we get the following result.

Corollary. *For the system of two-dimensional Euler equations (6.2.2), there exists a nonsingular matrix K such that KA and KB commute and are hyperbolic if and only if $M > 1$ (i.e. for supersonic flows).*

The complete eigen-systems of K defined by (6.2.9) is given in the following theorem.

Theorem 6.2.2. *The eigenvalues of K are*

$$\lambda_1(K) = 1,$$

$$\lambda_2(K) = \frac{u^2+v^2-c^2}{u^2+v^2} = 1 - \frac{c^2}{q^2},$$

(6.2.18)
$$\begin{aligned}\lambda_3(K) &= \frac{(2+s)(u^2+v^2) - c^2 + \sqrt{4c^2(u^2+v^2) + (s(u^2+v^2)-c^2)^2}}{2(u^2+v^2)} \\ &= 1 + \frac{s}{2} - \frac{c^2}{2q^2} + \sqrt{\frac{c^2}{q^2} + 0.25(s - \frac{c^2}{q^2})^2},\end{aligned}$$

$$\begin{aligned}\lambda_4(K) &= \frac{(2+s)(u^2+v^2) - c^2 - \sqrt{4c^2(u^2+v^2) + (s(u^2+v^2)-c^2)^2}}{2(u^2+v^2)} \\ &= 1 + \frac{s}{2} - \frac{c^2}{2q^2} - \sqrt{\frac{c^2}{q^2} + 0.25(s - \frac{c^2}{q^2})^2}\end{aligned}$$

and the four corresponding eigenvectors are

(6.2.19)
$$\begin{aligned}L_1 &= (1,0,0,0)^T, \\ L_2 &= (0,-v,u,0)^T, \\ L_3 &= (0,u,v,\rho((s+1-\lambda_1(K))(u^2+v^2)-c^2))^T, \\ L_4 &= (0,u,v,\rho((s+1-\lambda_2(K))(u^2+v^2)-c^2))^T.\end{aligned}$$

Proof. Denote $l_1 = \rho((s+1-\lambda_3)q^2 - c^2)$ and $l_2 = \rho((s+1-\lambda_4)q^2 - c^2)$. Then

$$KL = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{c^2}{q^2} + s\frac{u^2}{q^2} & s\frac{uv}{q^2} & -\frac{u}{\rho q^2} \\ 0 & s\frac{uv}{q^2} & 1 - \frac{c^2}{q^2} + s\frac{v^2}{q^2} & -\frac{v}{q^2} \\ 0 & -\rho c^2\frac{u}{q^2} & -\rho c^2\frac{v}{q^2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & u & u \\ 0 & u & v & v \\ 0 & 0 & l_1 & l_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -v(1 - \frac{c^2}{q^2}) & \lambda_3 u & \lambda_4 u \\ 0 & u(1 - \frac{c^2}{q^2}) & \lambda_3 v & \lambda_4 v \\ 0 & 0 & \lambda_3 l_1 & \lambda_4 l_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -v & u & u \\ 0 & u & v & v \\ 0 & 0 & l_1 & l_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - \frac{c^2}{q^2} & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

$$= L\Lambda_K.$$

The inverse matrix K^{-1} is given in Theorem 6.2.3.

Theorem 6.2.3. *The inverse of K is given by*

$$K^{-1} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{q^2(q^2+sv^2)-c^2(q^2+v^2)}{x} & -\dfrac{u\,v\,(sq^2-c^2)}{x} & \dfrac{u}{y\,\rho} \\ 0 & \dfrac{u\,v\,(sq^2-c^2)}{x}, & \dfrac{q^2(su^2+q^2)-c^2\,(u^2+q^2)}{x} & \dfrac{v}{2\,\rho} \\ 0 & \dfrac{u\,\rho\,c^2}{y} & \dfrac{\rho\,c^2\,v}{y} & \dfrac{(s+1)q^2-c^2}{y} \end{pmatrix}$$

$$x := u^4 + u^4\,s + 2\,s\,v^2\,u^2 - 3\,c^2\,u^2 - u^2\,s\,c^2 + 2\,u^2\,v^2 - 3\,v^2\,c^2 + v^4 + 2\,c^4$$
$$+ s\,v^4 - c^2\,s\,v^2$$

$$y := u^2 + s\,u^2 + s\,v^2 - 2\,c^2 + v^2$$

Proof. It is easy to check $KK^{-1} = I$.

6.3 Euler Equations in Conservative Variables

The two-dimensional Euler equations in conservation form are

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ \rho u^2 + P \\ \rho uv \\ (E+P)u \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + P \\ (E+P)v \end{pmatrix} = 0. \tag{6.3.1}$$

The Jacobian matrix of the transformation from the non-conservative variables to the conservative variables is given by

$$M = \frac{\partial V}{\partial U} \tag{6.3.2}$$

where $V = \begin{pmatrix} \rho \\ u \\ v \\ P \end{pmatrix}$ and $U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}$.

Hence

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -u/\rho & 1/\rho & 0 & 0 \\ -v/\rho & 0 & 1/\rho & 0 \\ \frac{(\gamma-1)}{2}(u^2+v^2) & -(\gamma-1)u & -(\gamma-1)v & \gamma-1 \end{pmatrix}. \tag{6.3.3}$$

If we define $A_c = M^{-1}AM$, $B_c = M^{-1}BM$, we get

$$\frac{\partial U}{\partial t} + A_c\frac{\partial U}{\partial x} + B_c\frac{\partial U}{\partial y} = 0, \tag{6.3.4}$$

which is the quasi-linear form of the two-dimensional Euler equations in conservation form. The matrices M^{-1}, A_c, and B_c are defined as follows.

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & \rho & 0 & 0 \\ v & 0 & \rho & 0 \\ \frac{u^2+v^2}{2} & \rho u & \rho v & \frac{1}{\gamma-1} \end{pmatrix}; \tag{6.3.5}$$

(6.3.6)

$$A_c = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 + \frac{\gamma-1}{2}v^2 & (3-\gamma)u & -(\gamma-1)v & \gamma-1 \\ -uv & v & u & 0 \\ -\frac{\gamma u E}{\rho} + (\gamma-1)u(u^2+v^2) & \frac{\gamma E}{\rho} - \frac{\gamma-1}{2}(v^2+3u^2) & -(\gamma-1)uv & \gamma u \end{pmatrix};$$

and

(6.3.7)

$$B_c = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -uv & v & u & 0 \\ \frac{\gamma-3}{2}v^2 + \frac{\gamma-1}{2}u^2 & -(\gamma-1)u & (3-\gamma)v & \gamma-1 \\ -\frac{\gamma v E}{\rho} + (\gamma-1)v(u^2+v^2) & -(\gamma-1)uv & \frac{\gamma E}{\rho} - \frac{\gamma-1}{2}(u^2+3v^2) & \gamma v \end{pmatrix}.$$

Corollary. *Let $K_c = M^{-1}KM$, where K is defined in (6.2.9), i.e.*

(6.3.8)

$$K_c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u[\frac{c^2}{q^2} - t_1] & 1 + \frac{t_2u^2 - c^2}{q^2} & \frac{t_2uv}{q^2} & \frac{(1-\gamma)u}{q^2} \\ v[\frac{c^2}{q^2} - t_1] & \frac{t_2uv}{q^2} & 1 + \frac{t_2v^2 - c^2}{q^2} & \frac{(1-\gamma)v}{q^2} \\ (\frac{3-\gamma}{2} - (1+s-t_3)\rho)q^2 & u(t_2 - t_3) & v(t_2 - t_3) & 2-\gamma \end{pmatrix},$$

where $t_1 = \frac{(\gamma-1)}{2} - s$, $t_2 = s + \gamma - 1$ and $t_3 = \frac{\gamma c^2}{(\gamma-1)q^2}$. Then K_cA_c and K_cB_c are commutative.

Proof. Since K commutes A and B, i.e. $KAKB = KBKA$, hence

$$\begin{aligned} K_cA_cK_cB_c &= M^{-1}KAKBM \\ &= M^{-1}KBKAM \\ &= K_cB_cK_cA_c. \end{aligned}$$

6.4 Two-Dimensional Steady Euler Equations

Recall that the 2D Euler equations (6.1.1) is:

$$(6.1.1) \qquad \frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ (E+P)u\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ (E+P)v\end{pmatrix}=0.$$

If we are only interested in the steady state solutions, it becomes

$$(6.4.1) \qquad \frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ \rho u^2+P\\ \rho uv\\ (E+P)u\end{pmatrix}+\frac{\partial}{\partial y}\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+P\\ (E+P)v\end{pmatrix}=0.$$

To solve this directly is difficult because we do not know the exact information on the boundary. Time-marching methods are often applied. In this section, we first show that the system (6.1.1) can be reduced to a system with only three variables. Then we consider weak coupling for the resulting reduced system.

6.4.1 Total Enthalpy

Let the total enthalpy H be

$$(6.4.2) \qquad H=\frac{E+P}{\rho},$$

the energy equation then can be written as

$$(6.4.3) \qquad \frac{\partial}{\partial t}(\rho H)+\frac{\partial}{\partial x}(\rho uH)+\frac{\partial}{\partial y}(\rho vH)=\frac{\partial P}{\partial t}.$$

Combining this equation with the continuity equation, Equation (6.4.3) can be written as

(6.4.4)
$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = \frac{1}{\rho}\frac{\partial P}{\partial t}.$$

For steady flows, we get

$$\frac{DH}{Dt} = 0,$$

with $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$.

Thus, in the case of steady flows, the total enthalpy H is constant along the streamlines. One can show that the total enthalpy remains also constant when a streamline passes a discontinuity (*shock wave*)([45]). When we consider steady Euler equations with uniform inflow, the total enthalpy H is uniformly constant.

Theorem 6.4.1. *The Euler equations with the total enthalpy H uniformly constant can be written in the following quasi-linear form:*

(6.4.4)
$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ u\\ v\end{pmatrix} + \begin{bmatrix} u & \rho & 0\\ \frac{c^2}{\gamma\rho} & \frac{u}{\gamma} & \frac{1-\gamma}{\gamma}v\\ 0 & 0 & u\end{bmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho\\ u\\ v\end{pmatrix} + \begin{bmatrix} v & 0 & \rho\\ 0 & v & 0\\ \frac{c^2}{\gamma\rho} & \frac{1-\gamma}{\gamma}u & \frac{v}{\gamma}\end{bmatrix}\frac{\partial}{\partial y}\begin{pmatrix}\rho\\ u\\ v\end{pmatrix} = 0.$$

Proof. Suppose $H = H_0$, i.e.

$$\frac{\gamma P}{(\gamma-1)\rho} + \frac{1}{2}(u^2+v^2) \equiv H_0,$$

from which we get a relation

$$P = \frac{\gamma-1}{\gamma}\rho(H_0 - \frac{1}{2}(u^2+v^2)).$$

Applying this relation to the Euler equation (6.4.1), (6.4.4) is derived.

When one calculates the steady solutions for the 2D Euler equations, it is very often to apply this idea to reduce the number of independent variables(e.g. Ni [38]). We will now discuss the "weak coupledness" of the system (6.4.4).

6.4.2 3×3 system (6.4.4)

Our main theorem in his section is given in the following theorem.

Theorem 6.4.2. *The system (6.4.4) is weakly coupled only if $u^2+v^2>c^2$, i.e., only if the flow condition for system (6.4.4) is supersonic.*

Proof. The proof given by proving the following three lemmas.

Lemma 1. *Suppose $K=(k_{ij})_{3\times 3}$ and $AKB=BKA$, then K has the following form:*

$$K = \begin{pmatrix} -\dfrac{\rho(ut_1+v_t2)}{c^2} & k_{12} & k_{13} \\ t_1 & k_{22} & k_{23} \\ t_2 & k_{23} & t_3 \end{pmatrix}$$

where t_1, t_2 and t_3 are parameters and

$$k_{12} = \frac{\rho^2((\gamma-1)u^2m^2+v^2c^2)}{c^4v^2}t_1 + \frac{\rho^2(\gamma-1)u(\gamma q^2+c^2)}{c^4v}t_2$$

$$+ \frac{(\gamma-1)\rho u q^2}{c^2v^2}t_3,$$

$$k_{13} = \frac{(\gamma-1)\rho^2 u m^2}{c^4 v}t_1 + \frac{\gamma\rho^2(\gamma q^2 - m^2)}{c^4}t_2 + \frac{(\gamma-1)\rho q^2}{c^2 v}t_3,$$

$$k_{22} = \frac{\rho u(m^2-(\gamma+1)v^2)}{c^2v^2}t_1 - \frac{\rho(q^2-(\gamma-1)u^2)}{c^2 v}t_2 + \frac{u^2}{v^2}t_3,$$

$$k_{23} = \frac{\rho m^2 - \gamma\rho v^2}{c^2 v}t_1 + \frac{\gamma\rho u}{c^2}t_2 + \frac{u}{v}t_3,$$

$$k_{32} = \frac{\rho(u^2-c^2)}{c^2v}t_1 + \frac{\rho u}{c^2}t_2 + \frac{u}{v}t_3.$$

and $q^2 = u^2+v^2$ and $m^2 = q^2 - c^2$.

Lemma 2. *The eigenvalues of KA are*

$$\lambda_1(KA) = \frac{\rho((u^2-c^2)t_1 - (uv - c\sqrt{u^2+v^2-c^2})t_2)}{c^2},$$

$$\lambda_2(KA) = \frac{\rho((u^2-c^2)t_1 - (uv + c\sqrt{u^2+v^2-c^2})t_2)}{c^2},$$

$$\lambda_3(KA) = \frac{\rho((u^2-c^2)t_1 + uv(\gamma(u^2+v^2)-(v^2-c^2))t_2) + uc^2(u^2+v^2)t_3}{c^2v^2};$$

and the eigenvalues of KB are

$$\lambda_1(KB) = \frac{\rho(-(uv - c\sqrt{u^2 + v^2 - c^2})t_1 - (v^2 - c^2)t_2}{c^2},$$

$$\lambda_2(KB) = \frac{\rho(-(uv + c\sqrt{u^2 + v^2 - c^2})t_1 - (v^2 - c^2)t_2}{c^2},$$

$$\lambda_3(KB) = \frac{\rho(u(u^2 - c^2)t_1 + v(\gamma(u^2 + v^2) - (v^2 - c^2))t_2) + c^2(u^2 + v^2)t_3}{c^2 v}.$$

Lemma 3. *If $\lambda_i(KA)$ and $\lambda_i(KB)$ are all real for $i = 1, 2, 3$, then K can be chosen to be non-singular provided that $u^2 + v^2 - c^2 > 0$.*

The proofs are similar to the proofs of Theorem 6.2.1. However we are unable to show that K is a positive definite matrix.

6.5 On Symmetric Preconditionning Matrices

Now consider the quasi-linear systems (6.2.3) and (6.4.4). First recall that

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{6.2.3}$$

where A and B are given by (6.2.4). The problem under consideration here is that if there exists a symmetric positive definite matrix K such that $[KA, KB] = 0$ and KA and KB are hyperbolic, then the stability analysis can be established.

Theorem 6.5.1. *The system (6.2.3) is weakly coupled with a positive symmetric matrix K if and only if it is supersonic.*

Proof. Let

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{pmatrix},$$

and solve the matrix equation $AKB = BKA$. Applying Lemma 1 of section 6.4 to determine the parameters $t_1, \ldots, t_6$ we get

$$t_1 = \frac{u}{v} t_4,$$

$$t_4 = \frac{v}{u} t_1 - \frac{\rho^2 v}{u} t_3 + \rho^2 t_6,$$

(6.5.1) $$t_2 = 0,$$

$$t_3 = \rho^2 c^2 t_3 + \frac{\rho^2 u}{v} t_4,$$

$$t_6 = \rho^2 t_4 + \rho^2 c^2 t_6.$$

There are six unknowns and five equations. By choosing t_6 free, we have

$$
(6.5.2)\qquad
\begin{aligned}
t_1 &= \frac{u(1-\rho^2c^2)}{v\rho^2}t_6, \\[2ex]
t_2 &= 0; \\[2ex]
t_3 &= \frac{u}{v}t_6, \\[2ex]
t_4 &= \frac{1-\rho^2c^2}{\rho^2}t_6.
\end{aligned}
$$

Then without loss of generality, we may assume

$$
\begin{aligned}
k_{11} &= 1, \\
k_{33} &= t_5 = \frac{u^2+v^2-c^2}{u^2+v^2} + sv^2.
\end{aligned}
$$

Thus

$$
(6.5.3)\qquad K = \begin{pmatrix}
1 & \dfrac{uc^2r}{\rho q^2} & \dfrac{vc^2r}{\rho q^2} & 0 \\[2ex]
\dfrac{uc^2r}{\rho q^2} & \dfrac{m^2}{q^2}+su^2 & suv & -\dfrac{\rho uc^2}{q^2} \\[2ex]
\dfrac{vc^2r}{\rho q^2} & suv & \dfrac{m^2}{q^2}+sv^2 & -\dfrac{\rho vc^2}{q^2} \\[2ex]
0 & -\dfrac{\rho uc^2}{q^2} & -\dfrac{\rho vc^2}{q^2} & 1
\end{pmatrix},
$$

where $r = \rho^2c^2 - 1$, $q^2 = u^2 + v^2$, and $m^2 = q^2 - c^2$. We can find s such that K is symmetric positive-definite. The value of s is determined by assuring

$\det(K_i) > 0$ for all i.

$$\det(K_2) = \det\begin{pmatrix} 1 & \frac{uc^2r}{\rho q^2} \\ \frac{uc^2r}{\rho q^2} & \frac{m^2}{q^2} + su^2 \end{pmatrix} > 0.$$

(6.5.4)

$$\det(K_3) = \det\begin{pmatrix} 1 & \frac{uc^2r}{\rho q^2} & \frac{vc^2r}{\rho q^2} \\ \frac{uc^2r}{\rho q^2} & \frac{m^2}{q^2} + su^2 & suv \\ \frac{vc^2r}{\rho q^2} & suv & \frac{m^2}{q^2} + sv^2 \end{pmatrix} > 0.$$

and $\det(K) > 0$.

$$\det(K_2) = \frac{u^2 + v^2 - c^2}{u^2 + v^2} + u^2(s - \frac{(\rho^2c^2 - 1)^2c^4}{\rho^2(u^2 + v^2)^2}),$$

$$\det(K_3) = \frac{(u^2 + v^2 - c^2)^2}{(u^2 + v^2)^2} + (u^2 + v^2 - c^2)(s - \frac{(\rho^2c^2 - 1)^2c^4}{\rho^2(u^2 + v^2)^2}),$$

$$\det(K) = \frac{(u^2 + v^2 - c^2)^2}{(u^2 + v^2)^2} + (u^2 + v^2 - c^2)(s - \frac{\rho^2c^4}{(u^2 + v^2)^2} - \frac{(\rho^2c^2 - 1)^2c^4}{\rho^2(u^2 + v^2)^2}).$$

Hence let s be

(6.5.5)

$$s > \frac{\rho^2c^4}{(u^2 + v^2)^2} + \frac{(\rho^2c^2 - 1)^2c^4}{\rho^2(u^2 + v^2)^2}$$

$$= \frac{\rho^2}{M^4}[1 + (c^2 - \frac{1}{\rho^2})^2].$$

Then K becomes positive definite.

Next we need to show that we can find $s > \frac{\rho^2c^4}{(u^2 + v^2)^2} + \frac{(\rho^2c^2 - 1)^2c^4}{\rho^2(u^2 + v^2)^2}$ and KA and KB are hyperbolic. With the aid of the softerware *Maple*, the eigenvalues of KA are given by

(6.5.6)

$$\lambda_1(KA) = u - \frac{uc^2}{u^2+v^2} + \frac{vc\sqrt{u^2+v^2-c^2}}{u^2+v^2},$$

$$\lambda_2(KA) = u - \frac{uc^2}{u^2+v^2} - \frac{vc\sqrt{u^2+v^2-c^2}}{u^2+v^2},$$

$$\lambda_3(KA) = u(1+\frac{s}{2}-\frac{c^2}{q^2}) + \frac{u}{2q^2}\sqrt{(sq^2-2c^2-2cq)(sq^2-2c^2+2cq)+4c^4q^2(\frac{(\rho^2c^2-1)^2}{\rho^2}+1)};$$

$$\lambda_4(KA) = u(1+\frac{s}{2}-\frac{c^2}{q^2}) - \frac{u}{2q^2}\sqrt{(sq^2-2c^2-2cq)(sq^2-2c^2+2cq)+4c^4q^2(\frac{(\rho^2c^2-1)^2}{\rho^2}+1)}.$$

Similarly the eigenvalues of KB are

(6.5.7)

$$\lambda_1(KB) = v - \frac{vc^2}{u^2+v^2} + \frac{uc\sqrt{u^2+v^2-c^2}}{u^2+v^2},$$

$$\lambda_2(KB) = v - \frac{vc^2}{u^2+v^2} - \frac{uc\sqrt{u^2+v^2-c^2}}{u^2+v^2},$$

$$\lambda_3(KB) = v(1+\frac{s}{2}-\frac{c^2}{q^2}) + \frac{v}{2q^2}\sqrt{(sq^2-2c^2-2cq)(sq^2-2c^2+2cq)+4c^4q^2(\frac{(\rho^2c^2-1)^2}{\rho^2}+1)};$$

$$\lambda_4(KB) = v(1+\frac{s}{2}-\frac{c^2}{q^2}) - \frac{v}{2q^2}\sqrt{(sq^2-2c^2-2cq)(sq^2-2c^2+2cq)+4c^4q^2(\frac{(\rho^2c^2-1)^2}{\rho^2}+1)}.$$

Therefore if

$$u^2+v^2-c^2 \ge 0;$$

(6.5.8)

$$s > 2(\frac{1}{M^2}+\frac{1}{M}).$$

Then $\lambda_i(KA)$ and $\lambda_i(KB)$ are all real for $i = 1,2,3,4$. So combining the conditions (6.5.5) and (6.5.8), for the supersonic flows,

(6.5.9) $$s > \max(2(\frac{1}{M^2}+\frac{1}{M}), \frac{\rho^2}{M^4}[1+(c^2-\frac{1}{\rho^2})^2]).$$

Therefore we have completed the proof.

Next for (6.4.4) we have the following theorem.

Theorem 6.5.2. *The 3×3 system (6.4.4) is weakly coupled with a symmetric positive preconditioning matrix K only if*

$$u^2 + v^2 - c^2 > 0.$$

Proof. The proof can be completed by proving the following two lemmas.

Lemma 1. *Let*

$$K = \begin{pmatrix} 1 & -\frac{uc^2}{\rho q^2} & -\frac{vc^2}{\rho q^2} \\ -\frac{uc^2}{\rho q^2} & 1 - \frac{c^2}{q^2} + \frac{su^2}{q^2} & \frac{suv}{q^2} \\ -\frac{vc^2}{\rho q^2} & \frac{suv}{q^2} & 1 - \frac{c^2}{q^2} + \frac{sv^2}{q^2} \end{pmatrix}, \tag{6.5.10}$$

where $s = \gamma - 1 + \frac{2\gamma - 1}{(\gamma - 1)M^2} - \frac{c^4}{(\gamma - 1)\rho^2 q^2}$.
Then K is the symmetric solution of $AKB = BKA$ with the first entry 1.

Proof. Using Lemma 1 of §6.4 and solving

$$k_{11} = 1\ ,$$

$$k_{12} = k_{21},$$

$$k_{13} = k_{31}.$$

We get

$$t_1 = -\frac{uc^2}{\rho(u^2 + v^2)},$$

$$t_2 = -\frac{vc^2}{\rho(u^2 + v^2)},$$

$$t_3 = 1 + \frac{sv^2 - c^2}{u^2 + v^2},$$

where s is given in the lemma. And also we get $k_{23} = k_{32} = \frac{suv}{u^2 + v^2}$, which is a solution of $AKB = BKA$ due to Lemma 1 in §6.4.

Lemma 2. *KA and KB are hyperbolic if and only if*

$$q^2 = u^2 + v^2 > 0.$$

Proof. This is a consequence of Theorem 6.4.2.

6.6 On the Three-Dimensional Euler Equations

Here, we attempt to extend the results established for 2D Euler equations to 3D Euler equations. Unfortunately, the following theorem indicates that a non-singular matrix K does not exist which makes all KA, KB, and KC commute.

Theorem 6.6.1. *Consider the 3D Euler equations in the quasi-linear form.*

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} + C\frac{\partial U}{\partial z} = 0,$$

with $U = (\rho, u, v, w, P)^T$.

$$A = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & 0 & 0 & 1/\rho \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & 0 & u \end{pmatrix},$$

$$B = \begin{pmatrix} v & 0 & \rho & 0 & 0 \\ 0 & v & 0 & 0 & 0 \\ 0 & 0 & v & 0 & 1/\rho \\ 0 & 0 & 0 & v & 0 \\ 0 & 0 & \rho c^2 & 0 & v \end{pmatrix}, \qquad (6.6.2)$$

$$C = \begin{pmatrix} w & 0 & 0 & \rho & 0 \\ 0 & w & 0 & 0 & 0 \\ 0 & 0 & w & 0 & 0 \\ 0 & 0 & 0 & w & 1/\rho \\ 0 & 0 & 0 & \rho c^2 & w \end{pmatrix}.$$

There does not exist a nonsingular matrix K *making* A, B *and* C *all commu-*

tative, *i.e.*

$$[KA, KB] = 0,$$

(6.6.3) $$[KA, KC] = 0,$$

$$[KB, KC] = 0.$$

Proof. If K is nonsingular and (6.6.3) holds, then K must satisfy the following relations

$$AKB = BKA,$$

(6.6.4) $$AKC = CKA,$$

$$BKC = CKB.$$

Suppose $K = (k_{ij})_{5\times 5}$. Expanding (6.6.4) one gets a 75×25 linear homogeneous system of k_{ij}. From the first equation of (6.6.4) we get

$$k_{51} = 0 \text{ and } k_{54} = 0.$$

From the second equation of (6.6.4)

$$k_{51} = 0 \text{ and } k_{53} = 0.$$

Similarly, from the last equation of (6.6.4)

$$k_{51} = 0 \text{ and } k_{52} = 0.$$

The last two relations from (6.6.4) lead to $k_{55} = 0$. Hence K is singular.

However, a nonsingular matrix K can be found to satisfy two of the three conditions in (6.6.4) but not all three.

6.7 On Stability Analysis

It is very important to study the stability analysis for a given numerical algorithm. Because of the non-commutativity of the coefficient matrices, there is no stability analysis for the initial value problems of the 2D or 3D Euler equations(see Bernner [3], [4]). However some classical works on this topic regarding numerical schemes are available (Kreiss et al [20], Lax [29]). The von Neumann stability analysis is often used. It is successful for some problems with certain restrictions. For two-dimensional or three-dimensional quasi-linear equations, we usually need the coefficient matrices A and B to be commutative, i.e., $AB = BA$. But it is in general not possible for the Euler equations. For two-dimensional problems, if we are only concerned with the steady solutions, we can apply the results obtained in the sections 6.2 to 6.5 and perform the stability analysis as follows.

Consider

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0, \tag{6.7.1}$$

where A and B commute. This system can be viewed from a time-marching scheme applied to the preconditioned steady Euler equations obtained in §6.2.

6.7.1 Semi-Discretization

Suppose the system (6.7.1) is a weakly coupled system. Freezing the coefficient matrices and using the characteristic variables $W = L^{-1}U$, we have the following form

$$W_t + \Lambda_A W_x + \Lambda_B W_y = 0,$$
(6.7.2)
$$W(x,0) = W_0(x).$$

where L is the common eigenvector matrix of A and L Using an upwinding technique to discretize the equation, we get the following linear system of ODEs:

(6.7.3) $$\frac{dW}{dt} = HW + b.$$

The dimensions of vector W and b are nN_xN_y, where n is the dimensions of A and B, and N_x and N_y are numbers of grids in the x – and y – directions respectively.

Theorem 6.7.1.

1. The operator H is non-positive,

(6.7.4) $$H \le 0, \quad i.e.(HU,U) \le 0, \forall\, U \in R^{nN_xN_y}.$$

2. If $b = 0$, we have the monotonicity property for the solutions of (6.7.3):

(6.7.5) $$(W(t), W(t)) \le (W(s), W(s)) \; for \; t > s \ge 0.$$

Therefore, the following estimates for U

(6.7.6) $$\|U(t)\| \le C\|U(0)\|,$$

is obtained, where $U(t)$ is an approximation of the solution of (6.7.2).

3. The general solution has the form

$$W(t) = S(t)W_0 + \int_0^t S(t-s)b\,ds. \tag{6.7.8}$$

where $S(t)$ is the semigroup e^{Ht}.

The proof is straightforward and will not be presented here.

6.7.2 The von Neumann Method for Stability Analysis

In this section we apply the von Neumann stability analysis to the upwinding scheme, the Lax-Friedrichs scheme and the fractional step method.

A. Upwinding Method

For the upwinding method, we require A and B to be hyperbolic and commute, so that there exists a matrix L which simultaneously diagonalizes A and B.

Let $|A| = L|\Lambda_A|L^{-1}, A^+ = L\Lambda_A^+L^{-1},\ A^- = L\Lambda_A^-L^-$, and similar definition for $|B|$, B^+, and B^-. Then

$$\begin{aligned} \frac{\Delta U}{\tau} &+ \frac{A^+(U_{i,j}-U_{i-1,j}) + A^-(U_{i+1,j}-U_{i,j})}{h_x} \\ &+ \frac{B^+(U_{i,j}-U_{i,j-1}) + B^-(U_{i,j+1}-U_{i,j})}{h_y} = 0. \end{aligned} \tag{6.7.9}$$

Equation (6.7.9) can be rewritten into the following form

$$
\begin{aligned}
U_{k,j}^{n+1} &= (I - \sigma_x|A| - \sigma_y|B|)U_{k,j}^n + \sigma_y B^+ U_{k,j-1}^n + \sigma_x A^+ U_{k-1,j}^n \\
&\quad - \sigma_x A^- U_{k+1,j}^n - \sigma_y B^- U_{k,j+1}^n \\
&= (I - \sigma_x|A| - \sigma_y|B| + \sigma_y B^+ S_-^y + \sigma_x A^+ S_-^x \\
&\quad - \sigma_x A^- S_+^x - \sigma_y B^- S_+^y)U_{i,j}^n,
\end{aligned} \tag{6.7.10}
$$

where $\sigma_x = \dfrac{\tau}{h_x}, \sigma_y = \dfrac{\tau}{h_y}$, and $S_\pm^x$, $S_\pm^y$ are the shift operators, e.g.,

$$S_+^x U_{k,j} = U_{k+1,j}, \quad S_-^x U_{k,j} = U_{k-1,j}$$

. The amplification matrix is give by

$$
\begin{aligned}
G = &I - \sigma_x|A|(1 - \cos\phi_x) - \sigma_y|B|(1 - \cos\phi_y) \\
&- i(A\sin\phi_x + B\sin\phi_y).
\end{aligned} \tag{6.7.11}
$$

The von Neumann stability condition requires that $||G^n||$ is bounded, hence

$$
\begin{aligned}
&(1-\sigma_x|\lambda_k(A)|(1 - \cos\phi_x) - \sigma_y|\lambda_k(B)|(1 - \cos\phi_y))^2 \\
&\quad + (\sigma_x\lambda_k(A)\sin\phi_x + \sigma_y\lambda_k(B)\sin\phi_y)^2 \le 1.
\end{aligned} \tag{6.7.12}
$$

Theorem 6.7.2. *The condition (6.7.12) holds if and only if*

$$\max_{1\le k\le 4}\left(\sigma_x|\lambda_k(A)| + \sigma_y|\lambda_k(B)|\right) \le 1. \tag{6.7.13}$$

Proof. It is sufficient to consider only the following case:

$a = \lambda_k(A)$ *and* $b = \lambda_k(B)$ *and* $\cos\phi_x$, $\cos\phi_y$, $\sin\phi_x$ *and* $\sin\phi_y$ *are all positive.*

Then the proof is completed by the following lemma.

Lemma. *Let*

$$f(x,y) = (1 - a - b + ax + by)^2 + (a\sqrt{1-x^2} + b\sqrt{1-y^2})^2.$$

Then $f(x,y) \leq 1$ *iff* $a + b \leq 1$.

Proof.

$$\frac{\partial f}{\partial x} = 2a(1 - a - b + ax + by) - 2ax\frac{a\sqrt{1-x^2} + b\sqrt{1-y^2}}{\sqrt{1-x^2}}$$
$$\frac{\partial f}{\partial y} = 2b(1 - a - b + ax + by) - 2by\frac{a\sqrt{1-x^2} + b\sqrt{1-y^2}}{\sqrt{1-y^2}}.$$

Let $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$. Then $x = y$ and $\frac{\partial f}{\partial x} = 2a(1-a-b) \geq 0$. Therefore $f(x,y) \leq f(1,1) = 1$.

On the other hand, if $a + b = 1 + \epsilon$ for some $\epsilon > 0$. Then $f(1,1) = (1-(a+b))^2 + (a+b)^2 \geq (a+b)^2 \geq 1 + \epsilon > 1$.

The implicit form of (6.7.9) is

(6.7.14)

$$U_{k,j}^{n+1} = (I + \sigma_x|A| + \sigma_y|B| - \sigma_y B^+ S_+^y - \sigma_x A^+ S_+^x + \sigma_x A^- S_+^x + \sigma_y B^- S_+^y)^{-1} U_{i,j}^n,$$

which is unconditionally stable because

$$\frac{1}{(1 + \sigma_x|\lambda_k^A|(1-\cos\phi_x) + \sigma_y|\lambda_k^B|(1-\cos\phi_y))^2 + (\sigma_x\lambda_k^A \sin\phi_x + \sigma_y\lambda_k^B\sin\phi_y)^2} \leq 1,$$

where $\lambda_k^A = \lambda_k(A)$, and $\lambda_k^B = \lambda_k(B)$.

B. Lax-Friedrichs Method

For the Lax-Friedrichs scheme, recall that (3.4.7)

$$\begin{aligned} U_{i,j}^{n+1} =& \frac{1}{4}(U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^n) \\ & - \frac{\Delta t}{2\Delta x} A(U_{i+1,j}^n - U_{i-1,j}^n) - \frac{\Delta t}{2\Delta y} B(U_{i,j+1}^n - U_{i,j-1}^n). \end{aligned}$$

The amplification matrix becomes

$$G = \frac{1}{2}(\cos\phi_x + \cos\phi_y) - i(\sigma_x A \sin\phi_x + \sigma_y B \sin\phi_y). \tag{6.7.15}$$

The von Neumann stability condition is then equivalent to

(6.7. $\quad + \cos\phi_y)^2 + (\sigma_x \rho(A)\sin\phi_x + \sigma_y \rho(B)\sin\phi_y)^2 \le 1.$

...cessary and sufficient condition but with an indirect

... elementary proof.

Th... ..7.3. *The condition (6.7.16) holds if and only if*

$$(\sigma_x \rho(A))^2 + (\sigma_y \rho(B))^2 \le \frac{1}{2}. \tag{6.7.17}$$

Proof. The proof is completed by proving the following lemma.

Lemma 2. *Let*

$$f(x,y) = \frac{1}{4}(\sqrt{1-x^2} + \sqrt{1-y^2})^2 + (ax+by)^2.$$

Then $f(x,y) \leq 1$ *iff* $a^2 + b^2 \leq \frac{1}{2}$.

Proof. First we show that $f(0,y) \leq 1$. Note that

$$\begin{aligned} f(0,y) &= \frac{1}{4}(1+\sqrt{1-y^2})^2 + b^2y^2 \\ &\leq \frac{1}{4}(1+\sqrt{1-y^2})^2 + \frac{y^2}{2} \\ &= \frac{1}{4}(1+2\sqrt{1-y^2}+1-y^2) + \frac{y^2}{2} \\ &= \frac{1}{4}(4-(1-y^2)+2\sqrt{1-y^2}-1) \\ &= \frac{1}{4}(4-(\sqrt{1-y^2}-1)^2) \\ &\leq 1. \end{aligned}$$

Next, $g(y) = f(1,y) \leq 1$.

$$\begin{aligned} g(y) &= \frac{1}{4}(1-y^2) + (a+by)^2 \\ &= \frac{1}{4}(1-y^2) + a^2 + 2aby + b^2y^2 \\ &= \frac{1}{4} + a^2 + 2aby + (b^2 - \frac{1}{4})y^2 \\ g'(y) &= 2ab + 2(b^2 - \frac{1}{4})y. \end{aligned}$$

There are two cases, i.e. $b^2 \geq 1/4$ and $b^2 < 1/4$. If $b^2 \geq 1/4$, $g'(y) \geq 0$ and hence $g(y) \leq g(1) = (a+b)^2 \leq 2(a^2+b^2) \leq 1$. On the other hand, if $b^2 < 1/4$, then

$$g'(y) = 0 \text{ at } y_0 = \frac{ab}{1/4 - b^2}.$$

If $y_0 \geq 1$, we have $g(y) \leq g(1) \leq 1$. So assuming $y_0 < 1$, then

$$\begin{aligned} g(y_0) &= \frac{1}{4} + a^2 + 2ab\frac{ab}{1/4 - b^2} + (b^2 - \frac{1}{4})(\frac{ab}{1/4 - b^2})^2 \\ &= \frac{1}{4} + a^2 + \frac{a^2b^2}{1/4 - b^2} \\ &\leq \frac{1}{4} + a^2 + ab \\ &\leq \frac{1}{4} + \frac{a^2}{2} + \frac{(a+b)^2}{2} \\ &\leq \frac{1}{4} + \frac{a^2+b^2}{2} + \frac{(a+b)^2}{2} \\ &\leq 1. \end{aligned}$$

Similarly we can show that $f(x,0) \leq 1$ and $f(x,1) \leq 1$. Now consider $(x,y) \in (0,1)^2$.

$$\frac{\partial f}{\partial x} = -\frac{x(\sqrt{1-x^2} + \sqrt{1-y^2})}{2\sqrt{1-x^2}} + 2a(ax+by);$$

$$\frac{\partial f}{\partial y} = -\frac{y(\sqrt{1-x^2} + \sqrt{1-y^2})}{2\sqrt{1-y^2}} + 2b(ax+by).$$

Let $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, then

$$y = \frac{bx}{\sqrt{a^2 + (b^2 - a^2)x^2}}.$$

Substituting it into $\frac{\partial f}{\partial x} = 0$ we have

$$\frac{a}{\sqrt{a^2 + (b^2 - a^2)x^2}} = \frac{1/4 - a^2}{b^2 - 1/4}. \tag{6.7.18}$$

If $\frac{1/4 - a^2}{b^2 - 1/4} \leq 0$, then (6.7.18) is violated and hence $\frac{\partial f}{\partial x} \neq 0$ and $\frac{\partial f}{\partial y} \neq 0$. Therefore f can never reach its maxima in the interior of D. So assuming

$\frac{1/4 - a^2}{b^2 - 1/4} > 0$, using the relation (6.7.18), we have

$$\text{if } b^2 - a^2 > 0, \text{ then } \frac{1/4 - a^2}{b^2 - 1/4} < 1,$$

$$\text{if } b^2 - a^2 < 0, \text{ then } \frac{1/4 - a^2}{b^2 - 1/4} > 1.$$

Each case will lead to $a^2 + b^2 > 1/2$ which is a contradiction. The other possibility is $b^2 - a^2 = 0$ which will imply that $\frac{\partial f}{\partial x} < 0$. Thus we get

$$f(x, y) \leq \max\{f(0, y), f(1, y), f(x, 0), f(x, 1)\} \leq 1.$$

On the other hand, if $a^2 + b^2 > 1/2$, say $a^2 + b^2 = 1/2 + \epsilon$ for some $\epsilon > 0$, then $\max f(x, y) > 1$. Note that $f(\sqrt{2\epsilon - \epsilon^2}, \sqrt{2\epsilon - \epsilon^2}) \geq 1 + 2\epsilon^2$. Hence we complete the proof.

Note that for the Lax-Friedrichs scheme we do not require that A and B are hyperbolic. For the general Euler equations, A and B are not commute, and our numerical experiments in §7.4 verify that even if the condition (6.7.16) is satisfied, the Lax-Friedrichs scheme fails to converge.

C. Fractional Step Method

Fractional methods are applied widely in the area of CFD([44]). It is interesting to note that in [25] an approximate factorisation method for the unpreconditioned Euler equations is unstable for flows with Mach number between 0.8 to 2 and with large CFL numbers [25]. The only explanation is that A

and B do not commute for the 2D Euler equations. In this section, we consider (6.7.1) when A and B are commutative.

The fractional step method is given as follows. Solving

(6.7.19)
$$\frac{\partial U}{\partial t} = -2A\frac{\partial U}{\partial x}, \text{ for } t \in [n\Delta t, (n+\frac{1}{2})\Delta t),$$
$$\frac{\partial U}{\partial t} = -2B\frac{\partial U}{\partial y}, \text{ for } t \in [(n+\frac{1}{2})\Delta t, (n+1)\Delta t).$$

The explicit forms are

(6.7.20)
$$\frac{U_{i,j}^{n+1/2} - U_{i,j}^{n}}{\Delta t/2} = -2(A_+\frac{U_{i,j}^n - U_{i-1,j}^n}{\Delta x} + A_-\frac{U_{i+1,j}^n - U_{i,j}^n}{\Delta x}),$$
$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+1/2}}{\Delta t/2} = -2(B_+\frac{U_{i,j}^{n+1/2} - U_{i,j-1}^{n+1/2}}{\Delta y} + B_-\frac{U_{i+1,j}^{n+1/2} - U_{i,j}^{n+1/2}}{\Delta y}).$$

Denote $S_\pm^x$ and $S_\pm^y$ be the shift operators. Then (6.7.20) can be rewritten as

(6.7.21)
$$\begin{aligned} U_{i,j}^{n+1} =&(I - \sigma_y|B| + \sigma_y B_+ S_-^y - \sigma_y B_- S_+^y) \\ &(I - \sigma_x|A| + \sigma_x A_+ S_-^x - \sigma_x A_- S_+^x)U_{i,j}^n. \end{aligned}$$

The corresponding amplification matrix is given by

$$G = (I - \sigma_y|B|(1-\cos\phi_y) + \sigma_y B\sin\phi_y)(I - \sigma_x|A|(1-\cos\phi_x) + \sigma_x A\sin\phi_x).$$

Because A and B are commutative and hyperbolic, it can be shown that $||G^n||$ is bounded if and only if $\max(\sigma_x|\lambda_k(A)|, \sigma_y|\lambda_k(B)|) \leq 1$. Note that if A and B do not commute, one cannot obtain the stability condition.

The implicit form is

(6.7.22)
$$\begin{aligned} U_{i,j}^{n+1} =&(I + \sigma_y|B| - \sigma_y B_+ S_-^y + \sigma_y B_- S_+^y)^{-1} \\ &(I + \sigma_x|A| - \sigma_x A_+ S_+^x - \sigma_x A_- S_+^x)^{-1}U_{i,j}^n. \end{aligned}$$

The algorithm (6.7.22) is unconditionally stable because

$$\frac{1}{(1+\sigma_x|\lambda_k(A)|(1-\cos\phi_x))^2+(\sigma_x\lambda_k(A)\sin\phi_x)^2}\leq 1;$$

$$\frac{1}{(1+\sigma_y|\lambda_k(B)|(1-\cos\phi_y))^2+(\sigma_y\lambda_k(B)\sin\phi_y)^2}\leq 1.$$

Chapter VII

Numerical Solutions for 2D Steady Euler Equations

7.1 Introduction

Supersonic steady flow problems are important in CFD applications([35], [50]). As early as 1947, Theodore von Karman wrote [1] that *"I believe we have now arrived at the stage where knowledge of supersonic aerodynamics should be considered by the aeronautical engineer as a necessary pre-requisite to his art."* In 1948, Courant and Friedrichs' book [7] on supersonic flow was published. Today the aerospace world is seeing renewed interest in the utilisation of supersonic and hypersonic systems. Multiple activities are ongoing in the United States, Europe, Russia, and Japan to explore new vehicle technologies and systems [50].

From section 6.1 the two-dimensional Euler equations in quasi-linear form are

$$\frac{\partial U}{\partial t} + AU_x + BU_y = 0. \tag{7.1.1}$$

Introducing a preconditioner K, (7.1.1) is then equivalent to

$$K\frac{\partial U}{\partial t} + KAU_x + KBU_y = 0. \tag{7.1.2}$$

If we are only interested in steady-state computations, we may consider the preconditioned PDE:

$$\frac{\partial U}{\partial t} + KAU_x + KBU_y = 0. \tag{7.1.3}$$

The above three equations (7.1.1), (7.1.2) and (7.1.3) have the same steady state solutions if the systems with boundary conditions are well-posed. Our computations will be based on the equations (7.1.3), and for simplicity we just write (7.1.3) as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0. \tag{7.1.4}$$

If we can successfully define (for the linearised equations we can do that) the characteristic variables W [22]:

$$\partial W = L^{-1}\partial U, \tag{7.1.5}$$

then

$$W_t + \Lambda_A W_x + \Lambda_B W_y = 0. \tag{7.1.6}$$

This is a fully decoupled system if we freeze the eigenvalues. Therefore one can easily develop a stable numerical algorithm by using an upwinding technique.

Let $|A| = L|\Lambda_A|L^{-1}, A^+ = L\Lambda_A^+L^{-1}, A^- = L\Lambda_A^-L^{-1}$, etc., a general numerical algorithm for the solution of (7.1.4) can be expressed as

$$\begin{aligned}\frac{\Delta U}{\tau} &+ \frac{A^+(U^*_{i,j} - U^*_{i-1,j}) + A^-(U^*_{i+1,j} - U^*_{i,j})}{h_x} \\ &+ \frac{B^+(U^*_{i,j} - U^*_{i,j-1}) + B^-(U^*_{i,j+1} - U^*_{i,j})}{h_y} = 0 ,\end{aligned} \tag{7.1.7}$$

where $\Delta U = U^{n+1}_{i,j} - U^n_{i,j}$. The algorithm is explicit if $* = n$, and becomes implicit if $* = n + 1$.

In this chapter we present numerical experiments for some supersonic steady flow problems. One is the *shock reflection problem* and another is the *4% bump channel problem*.

7.2 Shock Reflection Problem

The physical domain for the shock reflection problem is $[0, 4.1] \times [0, 1]$. The pressures at the freestream and upper boundary are prescribed as follows

$$\begin{aligned} P_\infty &= 0.714286, \\ P_{y=1} &= 1.52819. \end{aligned} \tag{7.2.1}$$

Moreover the Mach number at free stream is $M_\infty = 2.9$. According to the jump conditions and the incident angle $\Phi = 29^0$, we get

$$\begin{pmatrix} \rho_\infty \\ u_\infty \\ v_\infty \\ P_\infty \end{pmatrix} = \begin{pmatrix} 1 \\ 2.9 \\ 0.0 \\ 0.714286 \end{pmatrix} ; \begin{pmatrix} \rho_{y=1} \\ u_{y=1} \\ v_{y=1} \\ P_{y=1} \end{pmatrix} = \begin{pmatrix} 1.6996 \\ 2.619343 \\ -0.506178 \\ 1.52819 \end{pmatrix} . \tag{7.2.2}$$

Numerical experiments are performed by using both explicit and implicit algorithms. The numerical results show that large CFL numbers in the range of 100 to 1000 can be used in implicit computations. Therefore the theoretical results in section 6.7 are numerically verified.

The resulting linear system is solved by conjugate gradient method applied to the normal equation.

The computational grids are 60×20 (*Fig. 7.1*) and 120×40 (*Fig. 7.2*). Figures 7.3 to 7.6 are the numerical results performed on the grid 60×20, while Figures 7.7 and 7.8 are on the grid 120×40.

Fig.7.3 are the numerical results using the explicit version of (7.1.7) with $h_x = 4.1/60$, $h_y = 1.0/20$, and $\tau = 0.25h_x$. Fig.7.4 to 7.6 are the numerical results using implicit version of (7.1.7) with $\tau = 100h_x$, $\tau = 500h_x$ and $\tau = 1000h_x$, respectively. Results for a finer mesh are given in Figs. 7.7 and 7.8, where Fig.7.7 gives the solutions using explicit scheme with $\tau = 0.25h_x$. The solutions using implicit scheme with $\tau = 100h_x$ are illustrated in Fig.7.8.

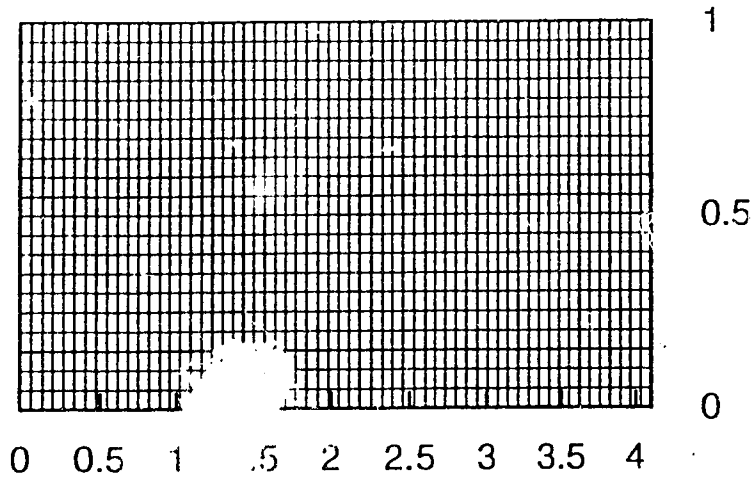

FIGURE 7.1 THE COMPUTATIONAL GRID 60 × 20

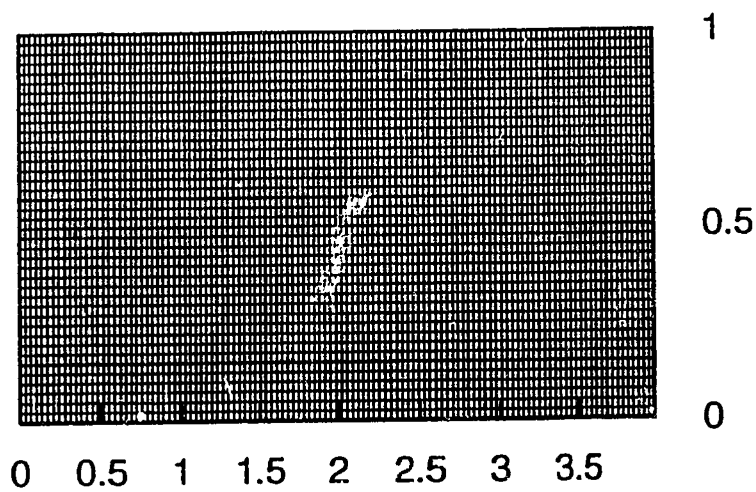

FIGURE 7.2 THE COMPUTATIONAL GRID 120 × 40

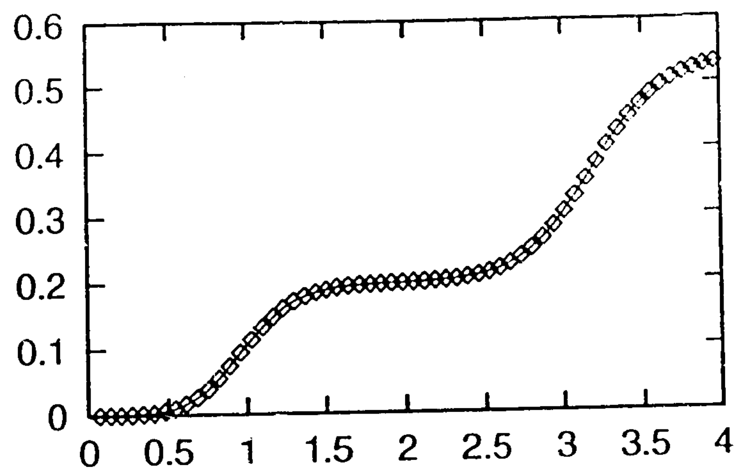

FIGURE 7.3A COEFFICIENT OF PRESSURE AT $y = 0.5$

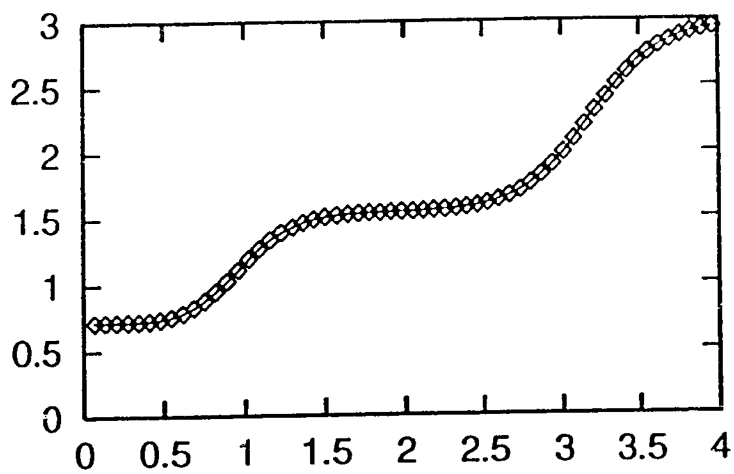

FIGURE 7.3B PRESSURE DISTRIBUTION AT $y = 0.5$

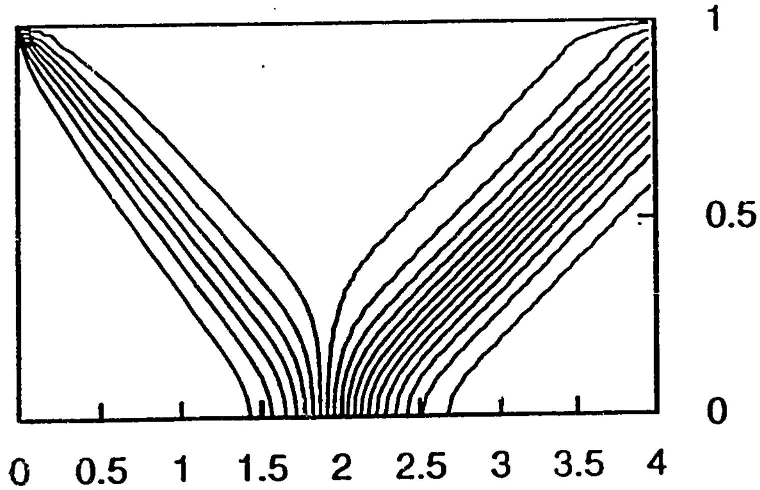

FIGURE 7.3C PRESSURE CONTOUR

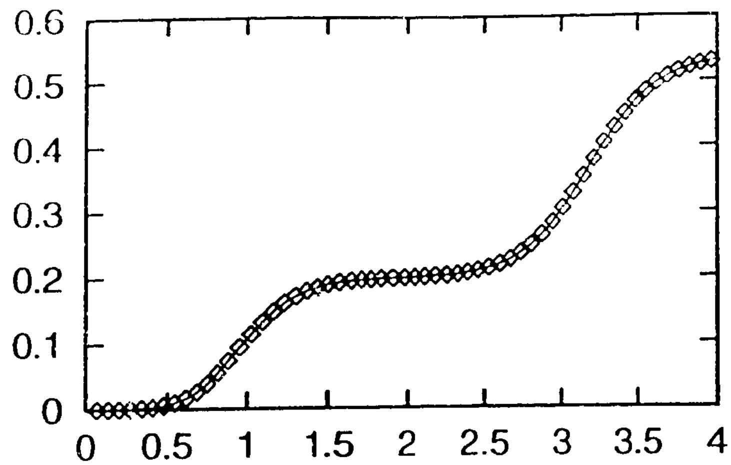

FIGURE 7.4A COEFFICIENT OF PRESSURE AT $y = 0.5$

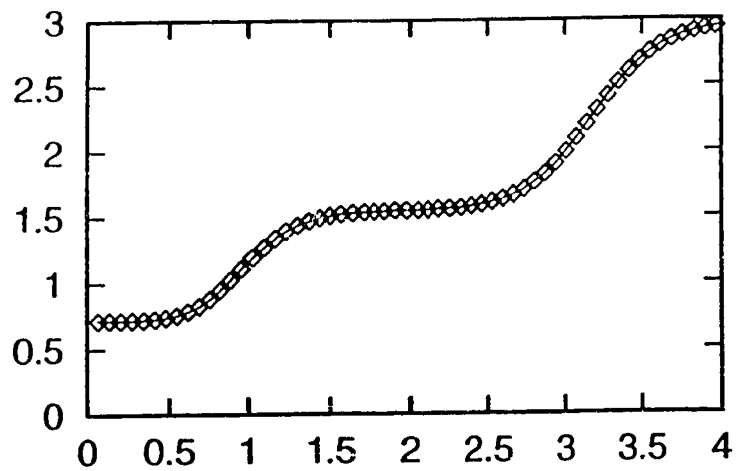

FIGURE 7.4B PRESSURE DISTRIBUTION AT $y = 0.5$

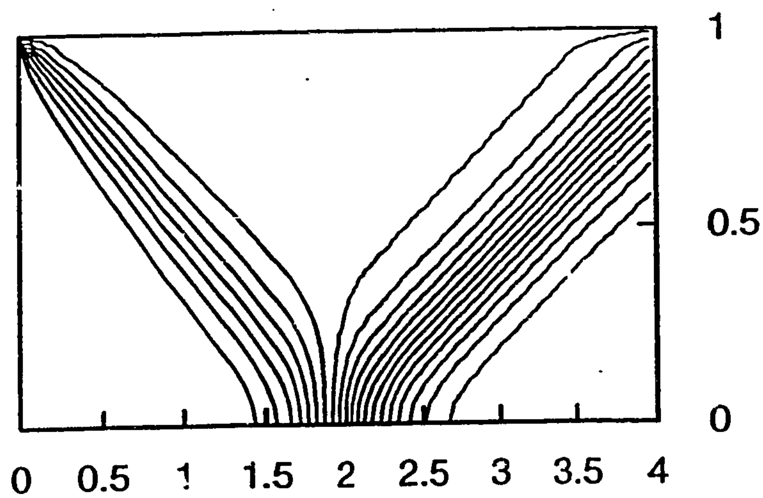

FIGURE 7.4C PRESSURE CONTOUR

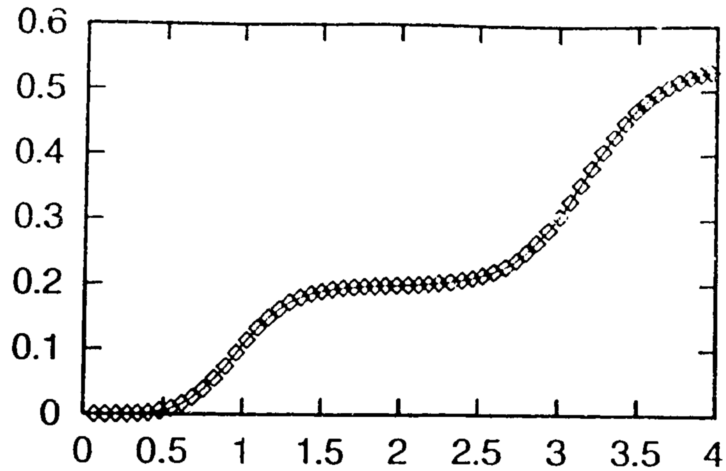

FIGURE 7.5A COEFFICIENT OF PRESSURE AT $y = 0.5$

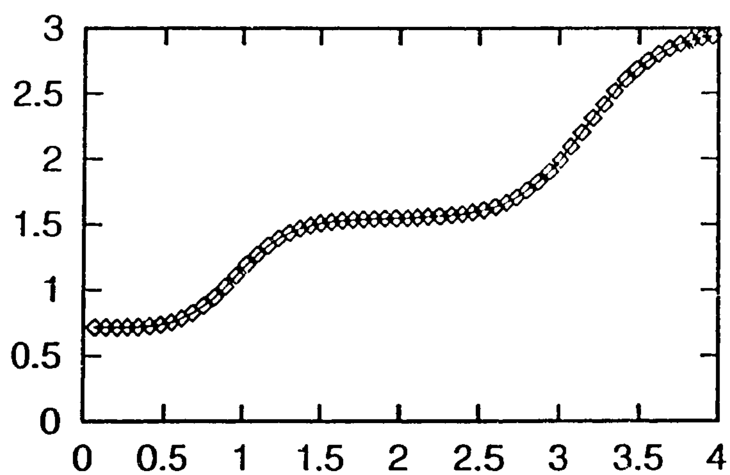

FIGURE 7.5B PRESSURE DISTRIBUTION AT $y = 0.5$

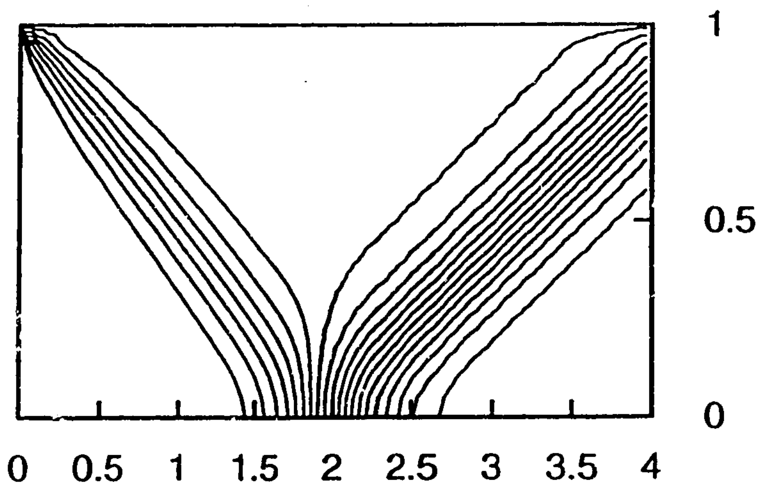

FIGURE 7.5C PRESSURE CONTOUR

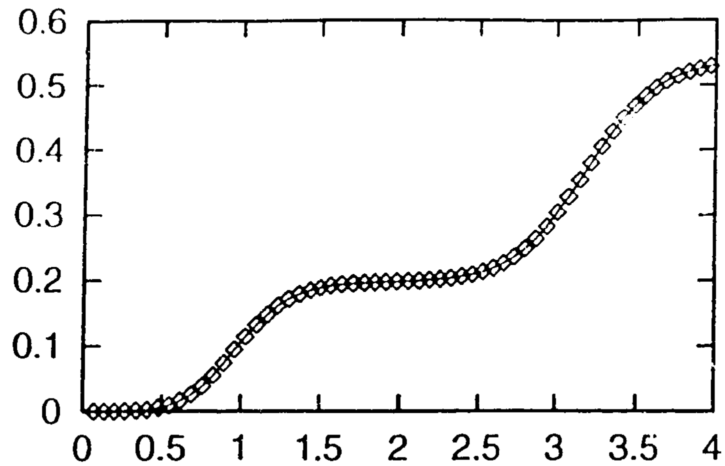

FIGURE 7.6A COEFFICIENT OF PRESSURE AT $y = 0.5$

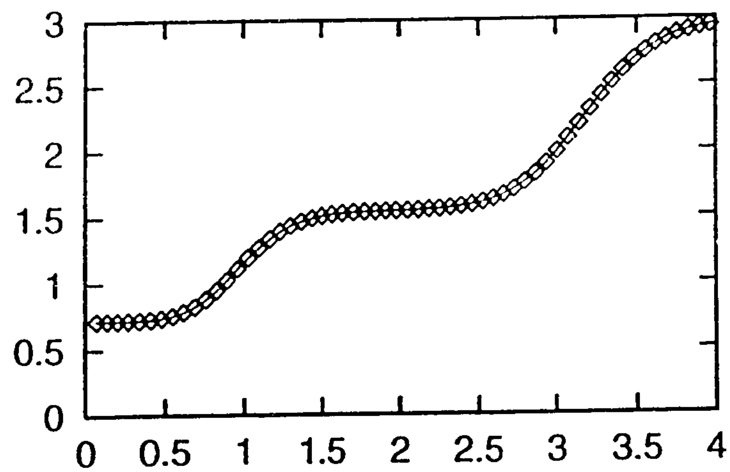

FIGURE 7.6B PRESSURE DISTRIBUTION AT $y = 0.5$

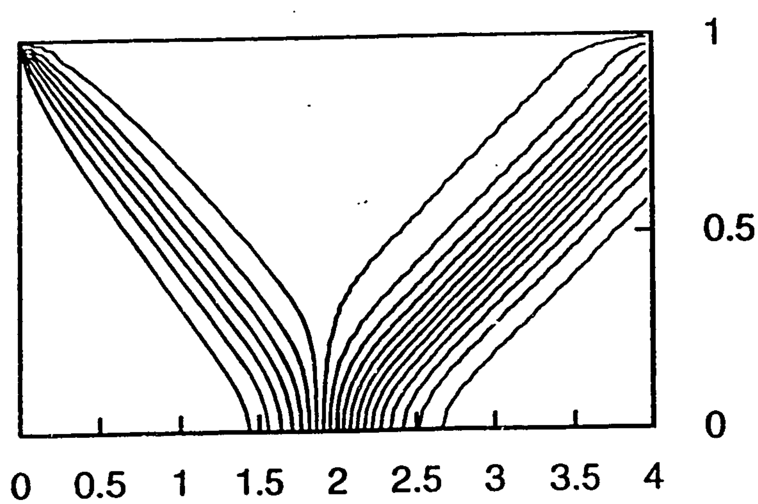

FIGURE 7.6C PRESSURE CONTOUR

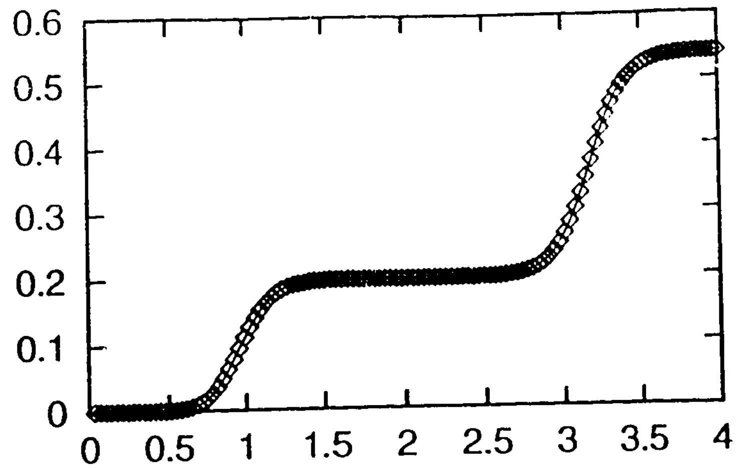

FIGURE 7.7A COEFFICIENT OF PRESSURE AT $y = 0.5$

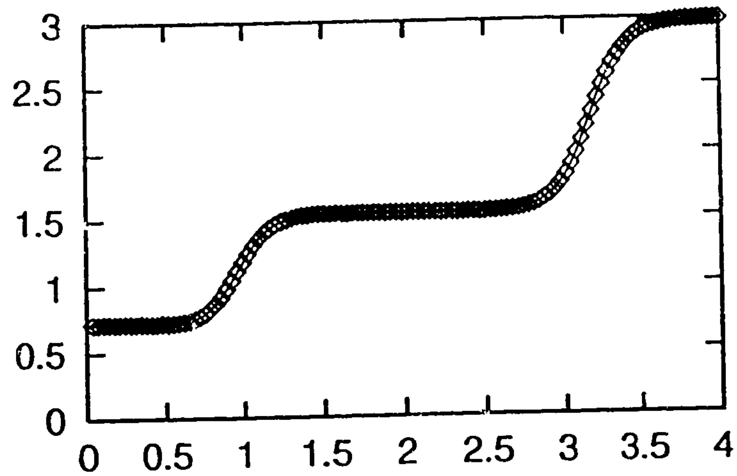

FIGURE 7.7B PRESSURE DISTRIBUTION AT $y = 0.5$

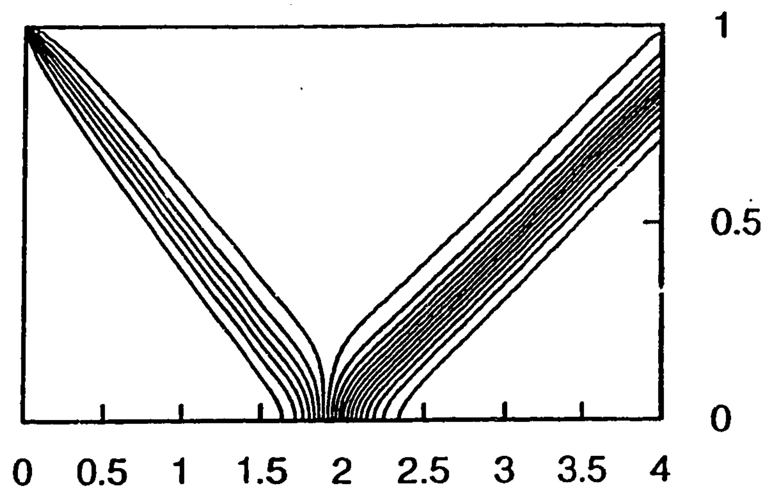

FIGURE 7.7C PRESSURE CONTOUR

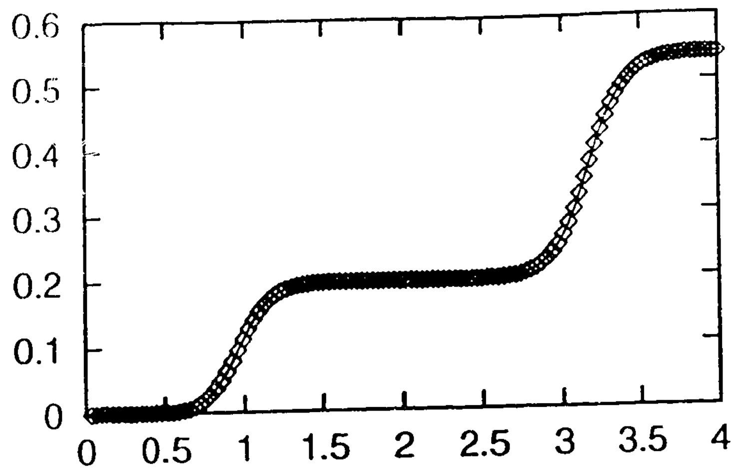

FIGURE 7.8A COEFFICIENT OF PRESSURE AT $y = 0.5$

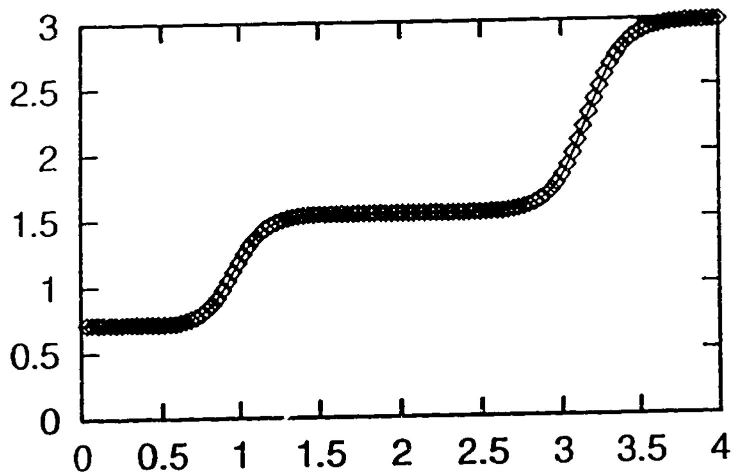

FIGURE 7.8B PRESSURE DISTRIBUTION AT $y = 0.5$

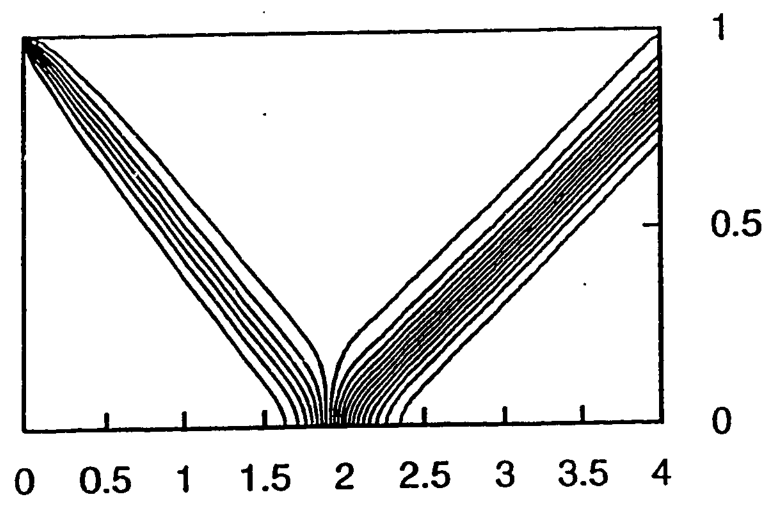

FIGURE 7.8C PRESSURE CONTOUR

7.3 Supersonic Channel Steady Flow

Next consider supersonic flows in a channel with a 4% thick circular arc bump. This is a standard test problem considered in [45]. The geometry of the channel is given by the following mapping from the (ξ, η) – computational space to the (x, y) – physical space with $(\xi, \eta) \in [-1, 2] \times [0, 1]$. The mapping is given by

(7.3.1)
$$-1 \le \xi \le -\frac{1}{4}, \tilde{\xi} = (4\xi + 1)/3,$$
$$-\frac{1}{4} \le \xi \le \frac{5}{4}, \tilde{\xi} = (4\xi + 1)/6,$$
$$\frac{5}{4} \le \xi \le 2, \tilde{\xi} = (4\xi - 2)/3.$$

(7.3.2)
$$-1 \le \tilde{\xi} \le 0, x = -1 + \frac{e^{-\beta_1(\tilde{\xi}+1)} - 1}{e^{-\beta_1} - 1},$$
$$0 \le \tilde{\xi} \le 1, x = \tilde{\xi},$$
$$1 \le \tilde{\xi} \le 2, x = 2 - \frac{e^{\beta_1(\tilde{\xi}-2)} - 1}{e^{-\beta_1} - 1}.$$

(7.3.3)
$$\tilde{\eta} = \frac{e^{\beta_2 \eta} - 1}{e^{\beta_2} - 1}.$$

$$y = \hat{\eta} + (1 - \hat{\eta})(\sqrt{9.89105 - (x - \frac{1}{2})^2} - 3.105) \text{ for } 0 \le x \le 1,$$

(7.3.4)

$$y = \hat{\eta} \text{ for} x \le 0 \text{ or } x \ge 1.$$

with $\beta_1 = 1.26$ and $\beta_2 = 1.01$.

Initial Conditions

At the inflow boundary ($x = -1$) we prescribe $M_{inlet} = 1.4$. Then we use

(7.3.5) $$\rho_{inlet} = 1.4,\ u_{inlet} = M_{inlet},\ P_{inlet} = 1.0.$$

Boundary Conditions

The outflow boundary conditions are derived from the first order interpolation. At the lower and upper rigid boundaries, a normal condition is imposed, i.e., $\vec{U}\vec{n} = 0$, where $\vec{U} = (u, v)$, the velocity vector, and $\vec{n}$ is the normal vector to the rigid boundary.

We apply a first-order upwinding method and Lax-Friedrichs method for the supersonic channel flow problem.

7.3.1 First Order Upwinding Method

Here, the first order upwinding method (7.1.7) is applied to the supersonic channel steady flow problem. First the problem is solved with the computational grid 48×16 (*Fig. 7.9*). The method works very well. *Fig. 7.11* shows the computational results using the explicit method with $\frac{\tau}{h_x} = 0.25$ while *Fig. 7.12* uses the implicit method with $\frac{\tau}{h_x} = 100$. When the computational grid is refined to 96×32 (*Fig. 7.10*) there is some problem for both explicit and implicit methods due to $M \approx 1$. But using the results from the coarse grid 48×16 as an initial approximation, no difficulty is observed in the computations (*Fig. 7.13*) using implicit method with $\frac{\tau}{h_x} = 100$.

The same problems with $M_{inlet} = 1.5$ or 2 can easily be computed by applying implicit method with grid 96×32 and $\frac{\tau}{h_x} = 100$ (*Fig. 7.14*, *Fig. 7.15*).

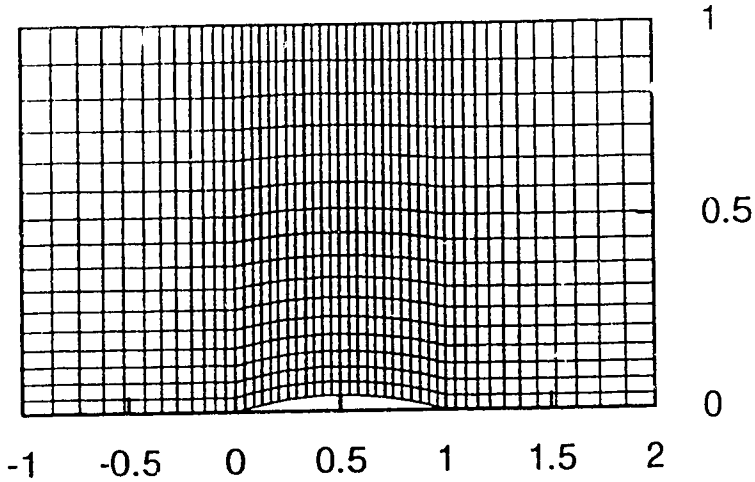

FIGURE 7.9. The Computational Grid 48 × 16

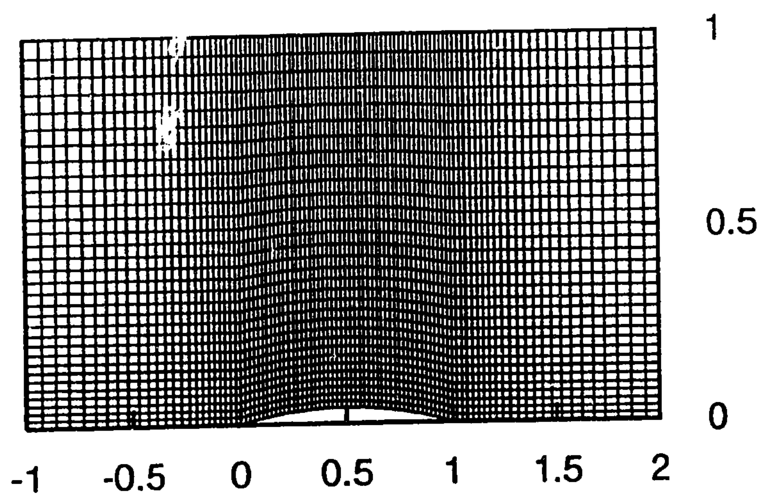

FIGURE 7.10. The Computational Grid 96 × 32

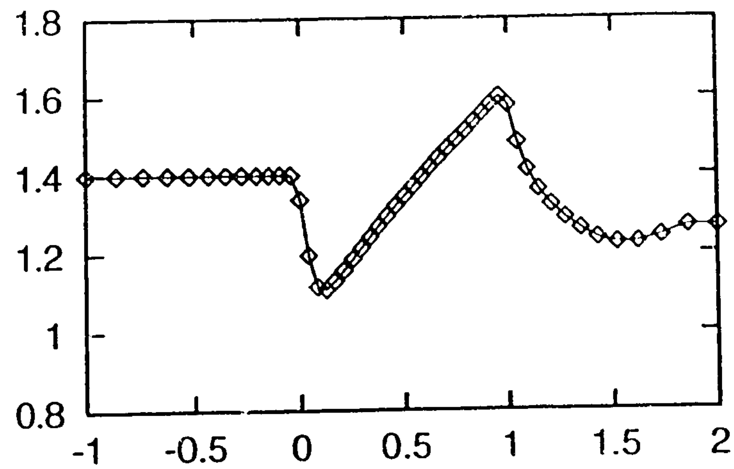

FIGURE 7.11A. Mach Number Distribution Along Lower Surface

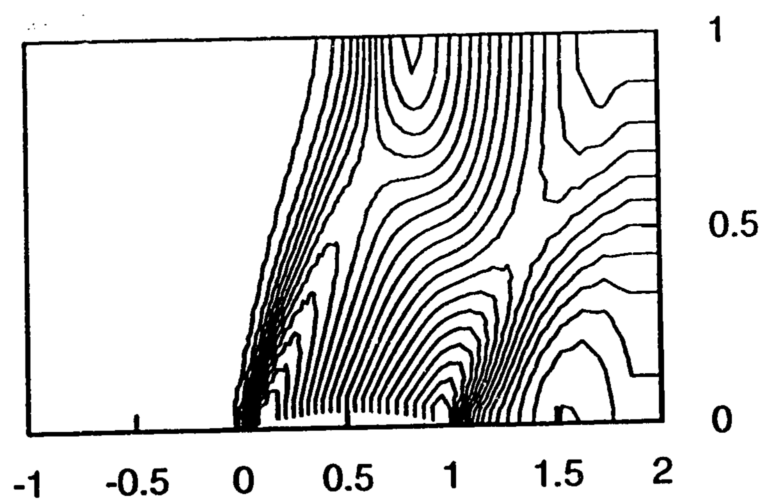

FIGURE 7.11B. Mach Contour

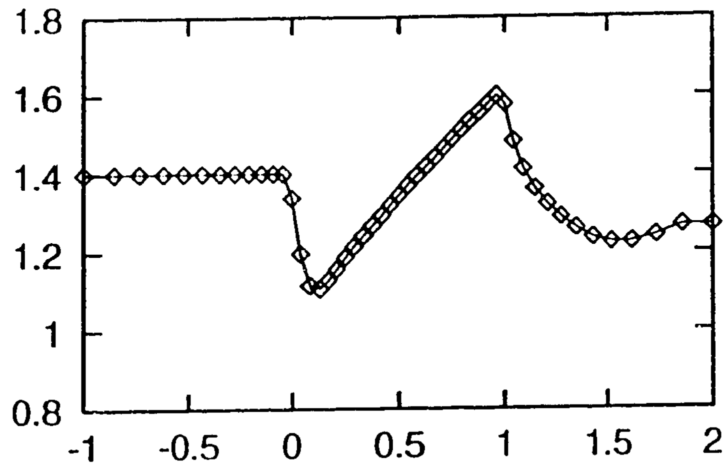

FIGURE 7.12A. Mach Number Distribution Along Lower Surface

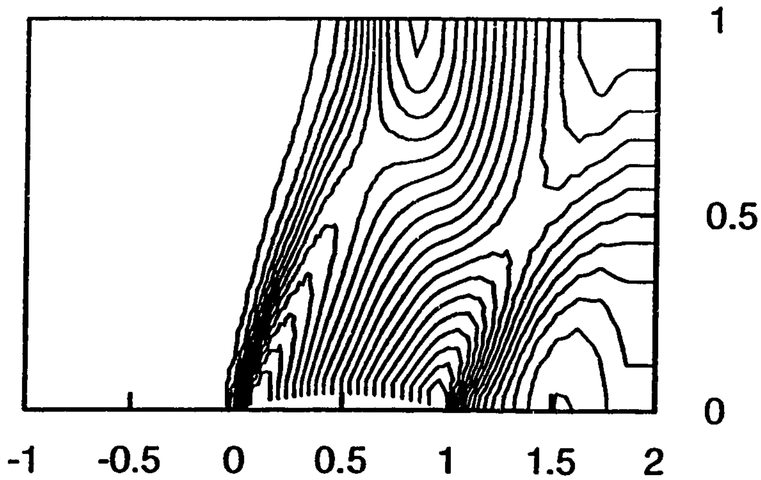

FIGURE 7.12B. Mach Contour

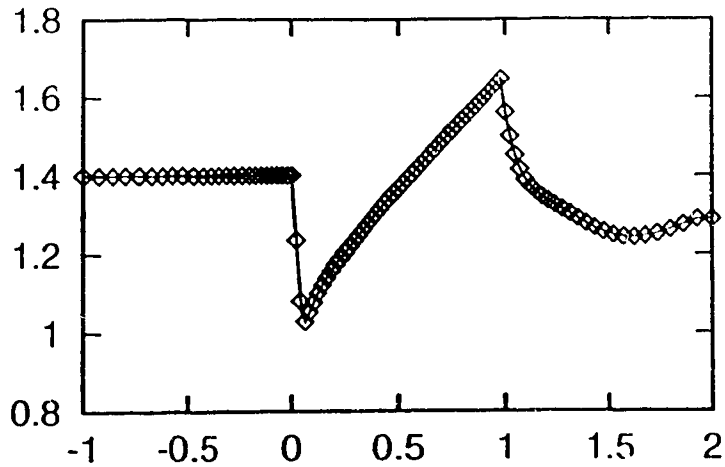

FIGURE 7.13A. Mach Number Distribution Along Lower Surface

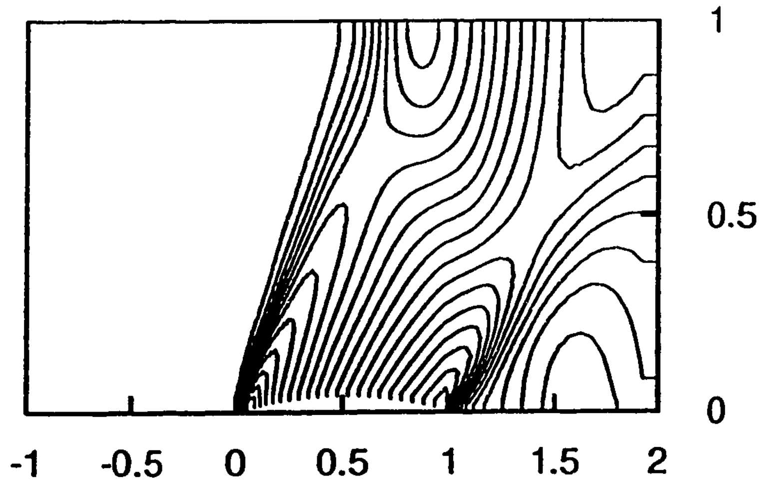

FIGURE 7.13B. Mach Contour

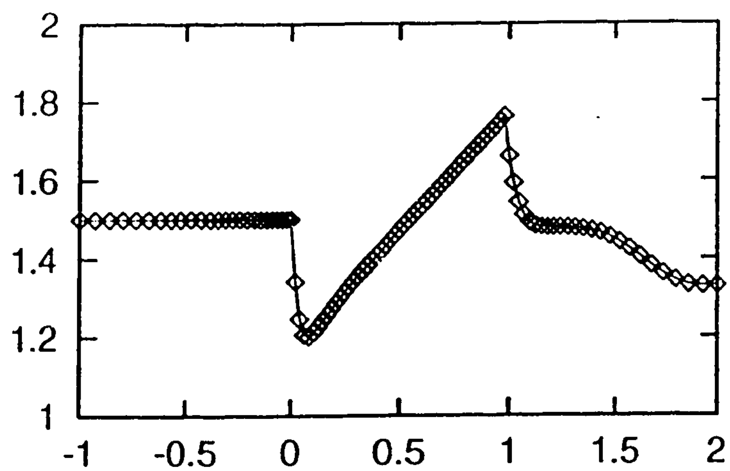

FIGURE 7.14A. Mach Number Distribution Along Lower Surface

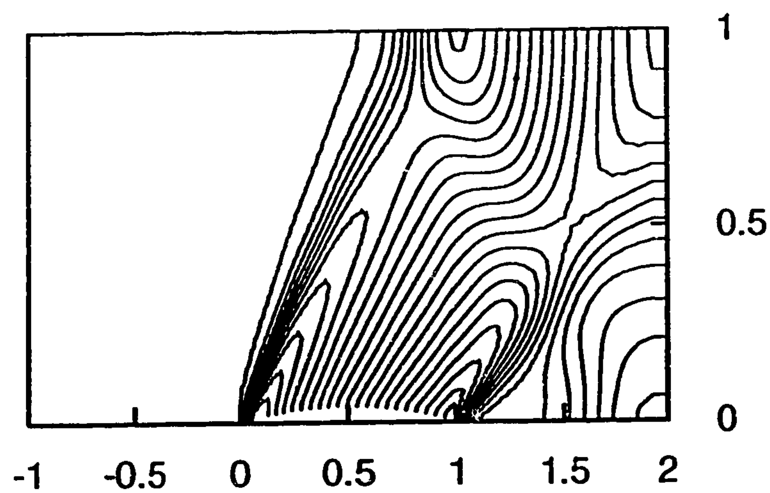

FIGURE 7.14B. Mach Contour

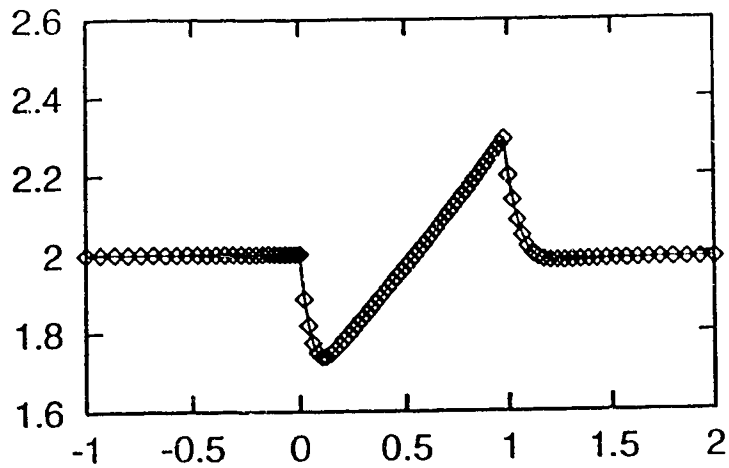

FIGURE 7.15A. Mach Number Distribution Along Lower Surface

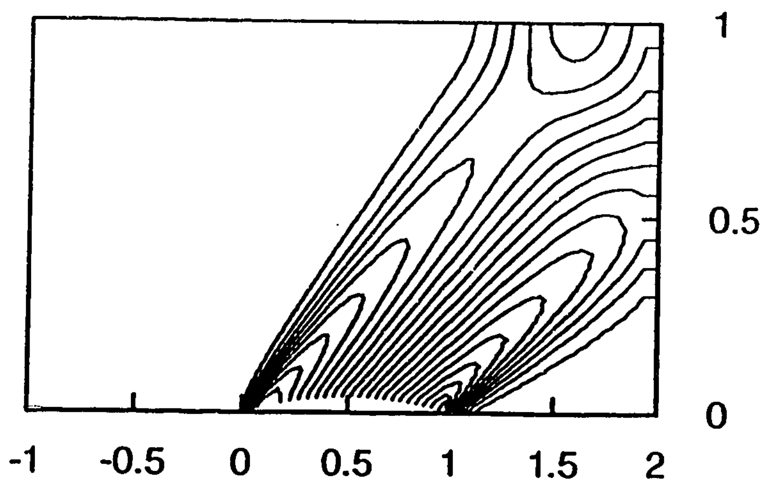

FIGURE 7.15B. Mach Contour

7.3.2 Lax-Friedrichs Method

Now the Lax-Friedrichs scheme (3.4.7) is applied to the modified equations (7.1.3) of the 2D Euler equations (7.1.1). We recall that (3.4.7)

$$
\begin{aligned}
U_{i,j}^{n+1} =& \frac{1}{4}(U_{i+1,j}^n + U_{i-1,j}^n + U_{i,j+1}^n + U_{i,j-1}^n) \\
& - \frac{\Delta t}{2\Delta x}A(U_{i+1,j}^n - U_{i-1,j}^n) - \frac{\Delta t}{2\Delta y}B(U_{i,j+1}^n - U_{i,j-1}^n)
\end{aligned} \tag{3.4.7}
$$

The interesting observation is that (3.4.7) fails to converge when applied to the original system (7.1.1). The numerical results with $M_{inlet} = 1.4$, grid 96×32 and $\frac{\tau}{h_x} = 0.25$ and $\frac{\tau}{h_x} = 0.275$ are shown in *Fig. 7.16*. *Fig. 7.16a* are residues for (7.1.1) and *Fig. 7.16b* are residues for our preconditioned system (7.1.3). Here the residue is defined as $||U^n - U^{n-1}||_\infty$.

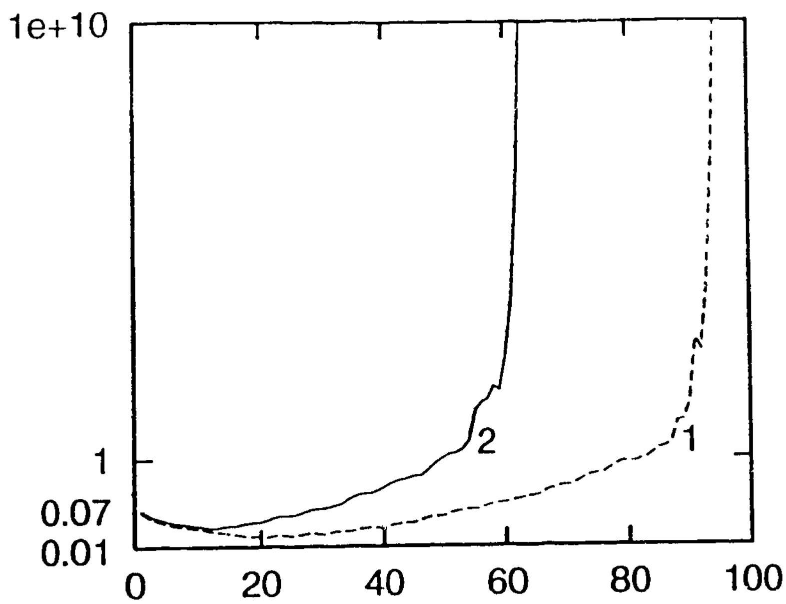

FIGURE 7.16A LOGORITHMS OF RESIDUES VS TIME STEPS

(1) $\frac{\tau}{h_x} = 0.25$ (2) $\frac{\tau}{h_x} = 0.275$

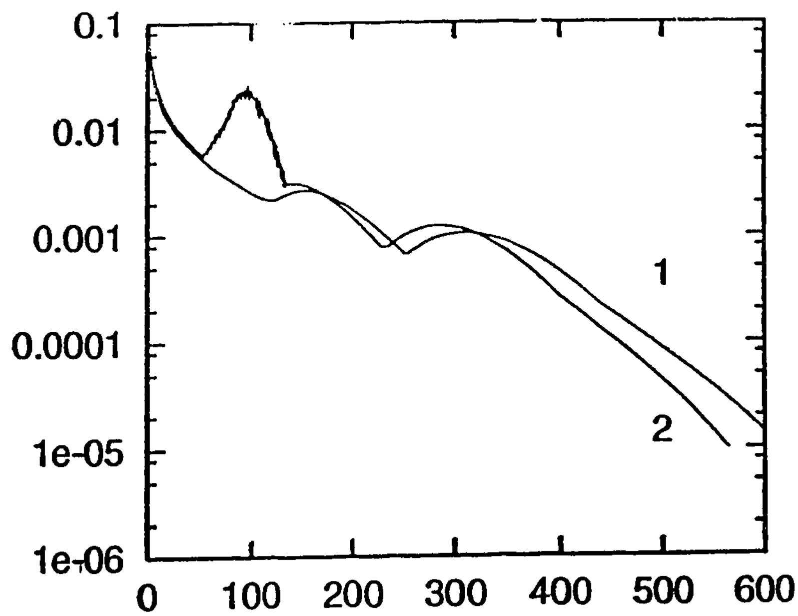

FIGURE 7.16B LOGORITHMS OF RESIDUES VS TIME STEPS

(1) $\frac{\tau}{h_x} = 0.25$ (2) $\frac{\tau}{h_x} = 0.275$

Chapter VIII

Conclusion

In this chapter, we summarise our work and make some remarks. In summary, the following contributions are made in the thesis.

1 Numerical schemes (see (4.3.2), (4.4.3) and (4.4.4)) which are called the exponential schemes are derived for one-dimensional hyperbolic systems. For the Burgers' equation, the stability analysis (see Theorem 4.3.1) is presented for the algorithm (4.3.2). The numerical results for the Burgers' equation show that the speed of the wave propagation is equal to the one obtained theoretically.

2 The concept of a *weakly coupled system* is introduced for multidimensional hyperbolic systems. Theoretical results show that the *coupledness* property of the coefficient matrices for the two-dimensional and three-dimensional Euler equations causes difficulties in developing numerical methods for their solutions.

3 We have shown that *the system of the two-dimensional Euler equations is a weakly coupled system if and only if the flow conditions are supersonic*(Theorem 6.2.1). The interesting fact is that if KA and KB are

commutative where K is nonsingular, then KA and KB are hyperbolic if and only if $u^2 + v^2 > c^2$. This fact implies that for the two dimensional Euler equations, the *weakly coupling* property is characteristic for supersonic flows. Moreover, it is proven that K can be chosen to be symmetric and positive definite.

4 From the Theorem 6.6.1, it is shown that the system of the three dimensional Euler equations can not be weakly coupled. Hence, there does not exist a K such that KA, KB, and KC are mutually commutative.

5 For the following system

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0.$$

with A and B commutative, we performed the stability analysis for the upwinding method (Theorem 6.7.2), for the Lax-Friedrichs method (Theorem 6.7.3) and for the fractional step method. The implicit form of the upwinding method and the fractional step method are unconditionally stable.

6 Numerical experiments are shown in Chapter 7 for several steady Euler solutions.

7 Although the numerical experiments are performed using a simple first order scheme, higher order numerical algorithms can be applied in a straightforward manner to the preconditioned Euler equations.

Next, we list several problems which require further investigations.

1 Well-posedness problem. This is the most important open problem for the mathematical analysis of multidimensional Euler equations. We hope to make some contributions to the study of steady supersonic flow problems.

2 Unsteady solutions of the two-dimensional Euler equations. It is of interest to find a preconditioning matrix K such that LKL^{-1} is symmetric and positive-definite, where L is the common eigenvector matrix of KA and KB. If such K exists, a stable semi-discretization (like Theorem 6.7.1) scheme can be established.

3 Genuinely multidimensional methods. We need to explore multidimensional properties in order to develop genuinely multidimensional methods. We considered the commutativity of the coefficient matrices in the thesis. Considering the wave decompositions could be another direction of investigation.

4 Effects of boundary conditions. We can not hope to prove the well-posedness of the Cauchy problem for a general system ([3], [4]). Considering the effects of boundary conditions on well-posedness is very important. I.e. it is important to study what boundary conditions imposed on a given problem make the problem well-posed.

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