Spherical h-Harmonic Analysis and Related Topics

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

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University of Alberta

Abstract

This thesis contains the following three parts:

Part 1(Chapters 1-5): Spherical h-harmonic analysis.

Part 2: Reverse Hölder's inequality for spherical harmonics.

Part 3: Multivariate Lagrange and Hermite approximation and pointwise limits of interpolants.

The main results of Part 1 are included in two journal papers, one long joint paper with Prof. F. Dai submitted to Adv. Math., and one single-authored paper to appear in Bull. Can. Math. Soc. Results of Part 2 are contained in a joint paper with Prof. F. Dai and Prof. S. Tikhonov to appear in Pro. AMS, and results of Part 3 are from a joint paper with Prof. M. Buhmann submitted to J. Math. Anal. Appl.

Part 1 consists of 5 chapters and is organized as follows. Chapter 1 is devoted to a brief description of some background information and main results for Part 1. Chapter 2 contains some preliminary materials on the Dunkl spherical h-harmonic analysis. After that in Chapter 3 the analogues of the classical

Hardy-Littlewood-Sobolev (HLS) inequality for the spherical h-harmonics with respect to general reflection groups on the sphere is established. A critical index for the validity of the HLS inequality is obtained and is expressed explicitly involving in the multiplicity function and the structure of the reflection group, which allows us to compute the critical indexes for most known examples of reflection groups. One of the main difficulties in our proofs lies in the fact that an explicit formula for the Dunkl intertwining operator is unknown in the case of general reflection groups, and therefore, closed forms of the reproducing kernels for the spaces of spherical h-harmonics are not available. A novel feature in our argument is to apply weighted Christoffel functions to establish new sharp pointwise estimates of some highly localized kernel functions associated to the spherical h-harmonic expansions. In Chapter 4, we introduce Riesz transforms for the spherical h-harmonic expansions, which are motivated by a new elegant decomposition of the Dunkl-Laplace-Beltrami operator involving the tangent gradient and the difference operators. These Riesz transforms are shown to have properties similar to those of the classical Riesz means. In particular, the L^p boundedness of these operators is proved. The proof of the main result in this chapter uses the Calderon-Zygmund decomposition, but the main difficulty is to establish some sharp kernel estimates related to the Riesz transforms. Finally, it is worthwhile to point out that the decomposition of the Dunkl-Laplace-Beltrami operator, discovered in this thesis, seems to be of in-

dependent interest. Indeed, as an application of this decomposition, in the last section of this chapter we establish the uncertainty principle with respect to the spherical h-harmonic expansions on the weighted spheres. Finally, we close this part by extending the results in preceding chapters to the corresponding weighted orthogonal expansions on the unit balls and the simplices. These results, in particular, generalize a classical inequality of Muckenhoupt and Stein [*Trans. Amer. Math. Soc.* **118**(1965), 17–92] on conjugate ultraspherical polynomial expansions.

In Part 2 our aim is to determine the sharp asymptotic order of the following reverse Hölder inequality for spherical harmonics Y_n of degree n on the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$ as $n \to \infty$:

$$\|Y_n\|_{L^q(\mathbb{S}^{d-1})} \leq C n^{\alpha(p,q)} \|Y_n\|_{L^p(\mathbb{S}^{d-1})}, \quad 0 < p < q \leq \infty.$$

It is shown that, in many cases, these sharp estimates are significantly better than the corresponding estimates in the Nikolskii inequality for spherical polynomials. These inequalities allow us to improve a result on the restriction conjecture of Fourier transform, as well as the sharp constant in the Pitt inequalities on $\mathbb{R}^d$.

Finally, Part 3 studies various approaches to multivariate interpolation. Precisely, we analyse interpolation and the reproduction of polynomials and other functions by linear combinations of shifts of radial basis functions and cardinal interpolants. We also consider gridded data Hermite interpolation. Of particular interest in practice is a class of radial basis functions which contains the celebrated multiquadrics and inverse multiquadrics for instance. For those, we provide new results on the asymptotic limits of the aforementioned cardinal interpolants when the parameter in the generalised multiquadric function $(r^2+c^2)^\gamma$ diverges.

Acknowledgement

It is a pleasure to acknowledge the support I have received during my study.

My sincere gratitude goes to my supervisor, Professor Feng Dai for his generous support, constant encouragement, guidance and friendship. Throughout my graduate study, I have profited greatly from his enthusiasm, insight and precision, which have enormously influenced my development. Especially, his continuous guidance, hand-on teaching, suggestions and wise comments made the task of finishing my graduate program under control. This thesis would not have been possible without his scholarly guidance.

My special thanks also go to Professor Kunyang Wang, who introduced me to the beautiful theory of harmonic analysis and approximation, and provided generous support during my study at Beijng Normal University.

Next, I want to thank Professor Sergey Tikhonov and Professor Martin Buhmann for providing me golden opportunities to collaborate with them. Furthermore, Professor Martin Buhmann have invited and sponsored my visit in Germany, and have invariably provided hospitality and kindness. It is sincere to express my gratitude to him.

I also would like to appreciate Professor Anthony T-M Lau, Professor Bin Han for their constant kind help and encouragement throughout my study at the University of Alberta and their recommend letters for my postdoctoral application.

There are many unmentioned people to whom I owe thanks. Certainly this work would not have been possible without the help of my friends and family. In particular, I thank my parents and grandparents. This dissertation is dedicated to the memory of my grandfather.

Contents

Symbols and Notation

$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{S}^{d-1}$	$(d-1)$-dimensional unite sphere in $\mathbb{R}^d$
$\mathbb{B}^d$	d-dimensional unite ball in $\mathbb{R}^d$
$\mathbb{T}^d$	d-dimensional simplex in $\mathbb{R}^d$
$\langle\cdot,\cdot\rangle$	Euclidean inner product in $\mathbb{R}^d$
$\\|\cdot\\|$	Euclidean norm in $\mathbb{R}^d$
$\|\cdot\|$	ℓ_1 norm in $\mathbb{R}^d$, $\|x\| = \|x_1\| + \|x_2\| + \cdots + \|x_d\|$
$\bar{x}$	absolute form in $\mathbb{R}^d$, $\bar{x} = (\|x_1\|, \cdots, \|x_d\|)$
$L^p(w;\mathbb{S}^{d-1})$	L^p space on $\mathbb{S}^{d-1}$ with respect to the weighted Lebegue measure $w(x)d\sigma(x)$
$\\|\cdot\\|_{w,p}$	L^p norm on $L^p(w;\mathbb{S}^{d-1})$
$\mathbb{Z}_2^d$	the Abelian reflection group identified with the set $\{\pm 1\}^d$
σ_v	reflection in $\mathbb{R}^d$, $\sigma_v : x \mapsto x - 2\frac{\langle x,v\rangle}{\\|v\\|^2}v$, $v \in \mathbb{R}^d$
$A \sim B$	asymptotical equivalence, $c_1 A \le B \le c_2 A$ for some constants $c_1, c_2 > 0$
$A \lesssim B$	asymptotically less relationship, $A \le cB$ for some constant $c > 0$

Part I

Spherical h-Harmonic analysis

Chapter 1

Introduction

The classical Hardy-Littlewood-Sobolev (HLS) fractional integration theorem states that if $0 < \alpha < d$ and $1 < p \le q < \infty$, then the HLS inequality,

$$\|(-\Delta)^{-\alpha/2} f\|_{L^q(\mathbb{R}^d)} \le C\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall f \in L^p(\mathbb{R}^d), \tag{1.0.1}$$

holds if and only if $\alpha = d(\frac{1}{p} - \frac{1}{q})$ (see [St1, Chapter V]), where $\partial_j = \frac{\partial}{\partial x_j}$ and $(-\Delta)^{\beta}$ denotes the fractional power of the Laplacian $\Delta = \sum_{j=1}^{d} \partial_j^2$. This theorem implies the Sobolev embedding theorem essentially by the relationship between the Riesz transforms $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$, $j = 1, 2 \cdots, d$ and the fractional integral operators $(-\Delta)^{-\alpha/2}$ (i.e. the Riesz potentials). The HLS inequality and the Riesz transforms on $\mathbb{R}^d$ have been extended to many different settings with fractional integration being mostly defined via orthogonal expansions or distributional Fourier transform (see, for instance, [AsWa, ArLi, AuHoLa, BoTh, ChWh, NoSt, SaWh, St1, SaSuTa, ThXu]).

In this part, we will study the HLS inequality and the Riesz transforms for fractional integration associated to weighted orthogonal polynomial expansions (WOPEs) on the sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$, on the ball $\mathbb{B}^d := \{x \in \mathbb{R}^d : \|x\| \le 1\}$ and on the simplex $\mathbb{T}^d := \{x \in \mathbb{R}^d : x_1, \cdots, x_d \ge 0, |x| \le 1\}$ with weights being invariant under a general finite reflection group on $\mathbb{R}^d$. Here and throughout the part, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$, and $|x| := \sum_{j=1}^{d} |x_j|$ denotes the ℓ^1-norm of $\mathbb{R}^d$. In this introduction we shall describe our main results for WOPEs on the sphere $\mathbb{S}^{d-1}$ with a " minimum" of definitions. Necessary details and appropriate definitions will be given in the next section.

Let $G \subset O(d)$ be a finite reflection group on $\mathbb{R}^d$. For $v \in \mathbb{R}^d \setminus \{0\}$, we denote by σ_v the reflection with respect to the hyperplane perpendicular to v; that is,

$$\sigma_v x = x - \frac{2\langle x, v\rangle}{\|v\|^2} v, \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$. Let $\mathcal{R}$ be the root system of G, normalized so that $\langle v, v\rangle = 2$ for all $v \in \mathcal{R}$, and fix a positive subsystem $\mathcal{R}_+$ of $\mathcal{R}$. It is known that (see, for instance, [Ro2]) the set of reflections in G coincides with the set $\{\sigma_v : \ v \in \mathcal{R}_+\}$, which also generates the group G. The dimension of the linear subspace of $\mathbb{R}^d$ spanned by all elements from the root system $\mathcal{R}$ is called the rank of $\mathcal{R}$ and is denoted by $\operatorname{rank}(\mathcal{R})$. Let $\kappa : \ \mathcal{R} \to [0, \infty)$, $v \mapsto \kappa_v = \kappa(v)$ be a nonnegative multiplicity function on $\mathcal{R}$ (i.e., a nonnegative G-invariant function on $\mathcal{R}$). Let h_κ denote the weight function on $\mathbb{R}^d$ defined by

$$h_\kappa(x) := \prod_{v \in \mathcal{R}_+} |\langle x, v\rangle|^{\kappa_v}, \quad x \in \mathbb{R}^d. \tag{1.0.2}$$

The function h_κ is G-invariant and homogeneous of degree $|\kappa| := \sum_{v \in \mathcal{R}_+} \kappa_v$.

The weight function we shall consider on the sphere $\mathbb{S}^{d-1}$ is $h_\kappa^2(x)$, which can also be written as $h_\kappa^2(x) = \prod_{v \in \mathcal{R}} |\langle x, v\rangle|^{\kappa_v}$. We denote by $d\sigma(x)$ the usual Haar measure on $\mathbb{S}^{d-1}$, $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ the L^p-space defined with respect to the measure $h_\kappa^2(x)\, d\sigma(x)$ on $\mathbb{S}^{d-1}$, and $\|\cdot\|_{\kappa,p}$ the norm of $L^p(h_\kappa^2; \mathbb{S}^{d-1})$. A spherical polynomial on $\mathbb{S}^{d-1}$ is the restriction to $\mathbb{S}^{d-1}$ of an algebraic polynomial in d variables, whereas a spherical h-harmonic of degree n on $\mathbb{S}^{d-1}$ is a spherical polynomial of degree n that is orthogonal to spherical polynomials of lower degree with respect to the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. We denote by $\mathcal{H}_n^d(h_\kappa^2)$ the space of all spherical h-harmonic polynomials of degree n on $\mathbb{S}^{d-1}$. Each $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$ then has an orthogonal expansion in spherical h-harmonics,

$$f = \sum_{n=0}^{\infty} \operatorname{proj}_n(h_\kappa^2; f), \tag{1.0.3}$$

converging in the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$, where $\operatorname{proj}_n(h_\kappa^2; f)$ denotes the orthogonal projection of f onto $\mathcal{H}_n^d(h_\kappa^2)$.

The theory of spherical h-harmonics was developed by Dunkl in [Du1, Du2, Du4]. It has applications in physics for the analysis of quantum many body systems of Calogero-Moser-Sutherland type (see, for instance, [Ro2] and [DuXu, pp. 360-370]). From the mathematical analysis point of view, the importance of spherical h-harmonics lies in the fact that they generalize the theory of ordinary spherical harmonics. There is a vast literature related to spherical h-harmonics and Dunkl analysis, see for instance [BoTh, BoRoTh, DaXu, Du1, Du2, Du6, Du4, DuXu, deJ, Ro2, Ro1, RoCo, ThXu, Xu, Xu2].

The spaces $\mathcal{H}_n^d(h_\kappa^2)$ of spherical h-harmonics can also be characterized as eigenfunction spaces of a second order differential-difference operator $\Delta_{\kappa,0}$ on $\mathbb{S}^{d-1}$, which we shall call the Dunkl-Laplace-Beltrami operator. Indeed,

$$\mathcal{H}_n^d(h_\kappa^2) = \Big\{ f \in C^2(\mathbb{S}^{d-1}) : \quad \Delta_{\kappa,0} f = -n(n+2\lambda_\kappa) f \Big\}, \quad n = 0, 1, \cdots ,$$

where

$$\lambda_\kappa := \frac{d-2}{2} + |\kappa| = \frac{d-2}{2} + \sum_{v \in \mathcal{R}_+} \kappa_v. \tag{1.0.4}$$

As a matter of fact, we may define the fractional power $(-\Delta_{\kappa,0})^\alpha$ of $(-\Delta_{\kappa,0})$ for $\alpha \in \mathbb{R}$ in a distributional sense by

$$\operatorname{proj}_n(h_\kappa^2; (-\Delta_{\kappa,0})^\alpha f) = (n(n+2\lambda_\kappa))^\alpha \operatorname{proj}_n(h_\kappa^2; f), \quad n = 0, 1, \cdots . \tag{1.0.5}$$

Our first main result determines the optimal power α of the operator $\Delta_{\kappa,0}$ for which the following HLS inequality holds:

$$\|(-\Delta_{\kappa,0})^{-\alpha/2} f\|_{\kappa,q} \le C_{p,q,\kappa} \|f\|_{\kappa,p}, \quad 1 < p < q < \infty, \tag{1.0.6}$$

where $C_{p,q,\kappa} > 0$ is a constants depending only on p, q, κ.p

Theorem 1.0.1. *Let $1 < p < q < \infty$ and $\alpha > 0$. Then the inequality (1.0.6) holds for all $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$ if and only if $\alpha \ge s_\kappa(\frac{1}{p} - \frac{1}{q})$, where*

$$s_\kappa = \begin{cases} d - 1 + 2|\kappa|, & \text{if } \operatorname{rank}(\mathcal{R}) \le d-1; \\ d - 1 + 2 \max\limits_{X_{d-1}} \sum\limits_{v \in \mathcal{R}_+ \cap X_{d-1}} \kappa_v, & \text{if } \operatorname{rank}(\mathcal{R}) = d \end{cases} \tag{1.0.7}$$

with the maximum being taken over all $(d-1)$-dimensional subspaces X_{d-1} of $\mathbb{R}^d$ spanned by $d-1$ elements from $\mathcal{R}_+$.

It turns out that the optimal index s_κ in (1.0.7) can be written explicitly for many typical examples of finite reflection groups. Below we include the results for the examples given in [DaXu2, p.168], with details of calculations being sketched in the appendix. Throughout the part , we set

$$e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1) \in \mathbb{R}^d.$$

Example 1.0.2. *The case $G = \mathbb{Z}_2^d$ (the Abelian group). Here the group G has a positive root system $\mathcal{R}_+ = \{e_1, \cdots, e_d\}$, the associated weight function can be written in the form*

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad \kappa_{e_j} = \kappa_j \geq 0,$$

and the index s_κ is given by $s_\kappa = 2\sigma_\kappa + 1$ with

$$\sigma_\kappa := \lambda_\kappa - \min_{1\leq j\leq d} \kappa_j = \frac{d-2}{2} + \sum_{j=1}^{d} \kappa_j - \min_{1\leq j\leq d} \kappa_j. \tag{1.0.8}$$

It is worthwhile to point out that in this case, $\sigma_\kappa = \frac{s_\kappa - 1}{2}$ corresponds to the critical index for the Cesàro summability of the spherical h-harmonic expansions, see [DaXu1, DaXu5, LiXu].

Example 1.0.3. *The case $G = A_{d-1}$ (the symmetric group on d elements). Here the group G has a positive root system $\mathcal{R}_+ = \{e_i - e_j : \ 1 \leq i < j \leq d\}$, the weight function can be written as*

$$h_\kappa(x) = \prod_{1\leq i<j\leq d} |x_i - x_j|^{\kappa_0}, \quad \kappa_0 \geq 0,$$

and the associated index s_κ is given by

$$s_\kappa = d - 1 + d(d-1)\kappa_0.$$

Example 1.0.4. *The case $G = B_d$ (the hyperoctahedral group). Here the group G is the symmetric group of $\{\pm e_1, \cdots, \pm e_d\}$, for which*

$$\mathcal{R}_+ = \{e_i \pm e_j : \ 1 \leq i < j \leq d\} \cup \{e_i : \ 1 \leq i \leq d\}$$

and

$$h_\kappa(x) = \Big(\prod_{i=1}^{d} |x_i|^{\kappa_1}\Big)\Big(\prod_{1\le i<j\le d} |x_i^2 - x_j^2|^{\kappa_2}\Big), \ \kappa_1, \kappa_2 \ge 0.$$

The associated index s_κ *is given by*

$$s_\kappa = \begin{cases} 2 + \max\{6\kappa_2, 4\kappa_1 + 4\kappa_2\}, & \text{if } d = 3; \\ d - 1 + 2\kappa_1(d-1) + 2\kappa_2(d-1)(d-2), & \text{if } d \ge 4. \end{cases} \tag{1.0.9}$$

Of particular interest is the case when $\alpha = 1$, where Theorem 1.0.1 can be formulated equivalently as follows: if $f \in C^1(\mathbb{S}^{d-1})$ and $\int_{\mathbb{S}^{d-1}} f(y)h_\kappa^2(y)\,d\sigma(y) = 0$, then for $1 < p < q < \infty$ with $(2s_k + 1)(\frac{1}{p} - \frac{1}{q}) \le 1$,

$$\|f\|_{\kappa,q} \le C\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p}. \tag{1.0.10}$$

However, $(-\Delta_{\kappa,0})^{1/2}$ here is not a local operator, and hence, more difficult to compute in practice. Our next main result gives an equivalent estimate of the norm $\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p}$ in terms of the tangential gradient ∇_0 on $\mathbb{S}^{d-1}$,

$$\nabla_0 f = \nabla F\Big|_{\mathbb{S}^{d-1}} \ \text{ with } \ F(x) = f(x/\|x\|), \quad x \in \mathbb{R}^d \setminus \{0\},$$

and the operators

$$E_v f(x) = \frac{f(x) - f(\sigma_v x)}{\langle x, v\rangle}, \qquad v \in \mathbb{R}^d \setminus \{0\}. \tag{1.0.11}$$

Theorem 1.0.5. *If* $1 < p < \infty$ *and* $f \in C^1(\mathbb{S}^{d-1})$*, then*

$$\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,p} \sim \|\nabla_0 f\|_{\kappa,p} + \max_{v\in\mathcal{R}_+} \kappa_v \|E_v f\|_{\kappa,p}, \tag{1.0.12}$$

where $A \sim B$ *means that there exists an inessential positive constants* c *such that* $cA \le B \le c^{-1}B$*. Furthermore, if* $p = 2$*, then we have the following equality:*

$$\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,2}^2 = \|\nabla_0 f\|_{\kappa,2}^2 + \sum_{v\in\mathcal{R}_+} \kappa_v \|E_v f\|_{\kappa,2}^2. \tag{1.0.13}$$

Combining Theorem 1.0.1 with Theorem 1.0.5, we obtain

Corollary 1.0.6. *If* $1 < p < q < \infty$, $s_\kappa(\frac{1}{p} - \frac{1}{q}) \le 1$, $f \in C^1(\mathbb{S}^{d-1})$ *and*

$\int_{\mathbb{S}^{d-1}} f(y)\,d\sigma(y) = 0$, *then*

$$\|f\|_{\kappa,q} \le C\|\nabla_0 f\|_{\kappa,p} + C \max_{v\in\mathcal{R}_+} \kappa_v \|E_v f\|_{\kappa,p}. \tag{1.0.14}$$

Remark 1.0.1. A straightforward calculation shows that if $p > \max_{\alpha\in\mathcal{R}_+} 2\kappa_\alpha + 1$, then the weight function $h_\kappa^2(x)$ satisfies the A_p condition on $\mathbb{S}^{d-1}$. Hence, by the Poincaré inequality on the sphere it follows that for $p > \max_{\alpha\in\mathcal{R}_+} 2\kappa_\alpha + 1$,

$$\|E_v f\|_{\kappa,p} \le C\|\nabla_0 f\|_{\kappa,p}.$$

This means that the second term $\max_{v\in\mathcal{R}_+} \kappa_v \|E_v f\|_{\kappa,p}$ on the right hand sides of (1.0.12) and (1.0.14) can be dropped when $p > \max_{\alpha\in\mathcal{R}_+} 2\kappa_\alpha + 1$.

The proof of Theorem 1.0.5 requires delicate pointwise estimates of certain kernel functions in spherical h-harmonic expansions, which turn out to be rather involved. A main difficulty comes from the fact that explicit integral representations of the reproducing kernels for the spaces of spherical h-harmonics are not available except in the case of the Abelian group, $G = \mathbb{Z}_2^d$ (see [Xu2]).

As an application of Theorem 1.0.5, we shall introduce and study the Riesz transforms for spherical h-harmonic expansions on $\mathbb{S}^{d-1}$. Indeed, by Equation (2.2.11) in Section 2, we can rewrite the formula (1.0.13) in Theorem 1.0.5 equivalently as the following new decomposition of the Dunkl-Laplace-Beltrami operator $-\Delta_{\kappa,0}$:

$$-\Delta_{\kappa,0} = \sum_{1\le i<j\le d} D_{i,j}^* D_{i,j} + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha E_\alpha^* E_\alpha, \tag{1.0.15}$$

where $D_{i,j} = x_i\partial_j - x_j\partial_i$ denotes the angular derivative in the x_ix_j-plane, $D_{i,j}^*$ and E_α^* denote the adjoint operators of $D_{i,j}$ and E_α in the space $L^2(h_\kappa^2; \mathbb{S}^{d-1})$ respectively. The operators $D_{i,j}^* D_{i,j}$ and $E_\alpha^* E_\alpha$ can be expressed explicitly as follows (see Section 7 for details):

$$D_{i,j}^* D_{i,j} = -(h_\kappa^2(x))^{-1} D_{i,j} h_\kappa^2(x) D_{i,j}, \quad E_\alpha^* E_\alpha = 2E_\alpha/\langle \alpha, x\rangle. \tag{1.0.16}$$

It is worthwhile to point out that the angular derivatives $D_{i,j}$ play an important role in ordinary spherical harmonic analysis (see, for instance, [DaXu2, Chapter 1] and [DaXu4]). By the decomposition (1.0.15), we may define the

Riesz transforms for the spherical h-harmonic expansions as follows:

Definition 1.0.7. *For $1 \le i < j \le d$ and $v \in \mathcal{R}_+$, define*

$$R_{i,j}f = D_{i,j}(-\Delta_{\kappa,0})^{-1/2}f, \quad R_v = \sqrt{\kappa_v}E_v(-\Delta_{\kappa,0})^{-1/2}f. \tag{1.0.17}$$

In the unweighted case (i.e., $\kappa = 0$), the Riesz transforms for the ordinary spherical harmonic expansions were introduced and studied in [ArLi].

As a consequence of (1.0.15), we have the following identity that is well-known for the classical Riesz transform on $\mathbb{R}^d$:

$$\sum_{1\le i<j\le d} R_{i,j}^* R_{i,j} + \sum_{v\in\mathcal{R}_+} R_v^* R_v = I, \tag{1.0.18}$$

where I is the identity operator on the space $\{f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}) : \int_{\mathbb{S}^{d-1}} f(y)h_\kappa^2(y)\, d\sigma(y) = 0\}$.

The L^p-boundedness of these Riesz transforms follows directly from Theorem 1.0.5:

Corollary 1.0.8. *For $1 < p < \infty$, there exists a constant $C_p > 0$ such that for all $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$,*

$$\max_{i,j} \|R_{i,j}f\|_{\kappa,p} + \max_{v\in\mathcal{R}_+} \|R_v f\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}.$$

If, in addition, $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\, d\sigma(x) = 0$, then

$$\max_{i,j} \|R_{i,j}f\|_{\kappa,p} + \max_{v\in\mathcal{R}_+} \|R_v f\|_{\kappa,p} \sim \|f\|_{\kappa,p}.$$

Finally, we will also establish similar results for WOPEs with respect to the weight function

$$W_{\kappa,\mu}^B(x) := h_\kappa^2(x)(1 - \|x\|^2)^{\mu-1/2}, \qquad \mu \ge 0, \quad x \in \mathbb{B}^d \tag{1.0.19}$$

on the unit ball $\mathbb{B}^d$, and for WOPEs with respect to the weight function

$$W_{\kappa,\mu}^T(x) := \frac{h_\kappa^2(\sqrt{x_1}, \cdots, \sqrt{x_d})}{\sqrt{x_1 \cdots x_d}}(1 - |x|)^{\mu-1/2}, \quad \mu \ge 0, \quad x \in \mathbb{T}^d, \tag{1.0.20}$$

on the simplex $\mathbb{T}^d$, where in the case of $\mathbb{T}^d$ we assume additionally that the weight $h_\kappa^2(x)$ is also $\mathbb{Z}_2^d$-invariant (see, for instance, the weights in Examples

1.2 and 1.4). These results, in particular, extend a classical inequality of Muckenhoupt and Stein [MuSt, p. 43, Corollary 1] on conjugate ultraspherical polynomial expansions.

Throughout this part, all functions will be assumed real-valued and Lebesgue measurable, and the letter $C, C_1, \ldots$ denotes generic (positive) constants, which may differ on each occurrence, even within the same formula.

Chapter 2

Preliminaries

To better describe our results, in the chapter we shall introduce some needed preliminaries and standard notion which will be valid throughout the rest of this thesis.

2.1 The Jacobi polynomials

For parameters $\alpha, \beta > -1$, the Jacobi polynomials are defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right), \quad (2.1.1)$$

where $x \in [-1, 1]$ and $n = 0, 1, \cdots$. They are mutually orthogonal with respect to the weight function $w_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ on $[-1, 1]$ and satisfy that ([Sz, (7.32.5) and (4.1.3)])

$$\left|P_n^{(\alpha,\beta)}(\cos\theta)\right| \leq cn^{-\frac{1}{2}}(n^{-1}+\theta)^{-\alpha-\frac{1}{2}}(n^{-1}+\pi-\theta)^{-\beta-\frac{1}{2}}, \quad \theta \in [0, \pi]. \quad (2.1.2)$$

For a smooth function $\varphi : [0, \infty) \to \mathbb{C}$, we define

$$B_{N,\varphi}^{(\alpha,\beta)}(t) := \sum_{k=0}^{\infty} \varphi(\frac{k}{N})P_k^{(\alpha,\beta)}(t). \quad (2.1.3)$$

We will use the following known estimates of the kernels $B_{N,\varphi}^{(\alpha,\beta)}$ and their derivatives (see, for instance, [BrDa, Lemma 3.3] and [IvPe, Theorem 2.6]):

Lemma 2.1.1. *Let φ be a C^∞-function on $[0, \infty)$ with compact support that* *is constant in a neighborhood of* 0*, and let* $B_N \equiv B_{N,\varphi}^{(\alpha,\beta)}$ *be the function defined by* (2.1.3) *with* $\alpha \ge \beta \ge -1/2$*. Then for any* $\ell \in \mathbb{N}$ *and* $\theta \in [0, \pi]$*,*

$$|B_N^{(i)}(\cos\theta)| \le C_{\ell,i,\alpha} \|\varphi^{(3\ell-1)}\|_{L^\infty[0,\infty)} N^{2\alpha+2i+2}(1+N\theta)^{-\ell}, \quad i = 0, 1, \cdots, \tag{2.1.4}$$

where $N \in \mathbb{N}$*,* $B_N^{(0)}(t) = B_{N,\varphi}^{(\alpha,\beta)}(t)$ *and* $B_N^{(i)}(t) = \left(\frac{d}{dt}\right)^i \{B_{N,\varphi}^{(\alpha,\beta)}(t)\}$ *for* $i \ge 1$*.*

For $\lambda > 0$, the ultraspherical polynomials C_n^λ are defined by

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{\left(\lambda + \frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x). \tag{2.1.5}$$

They satisfy

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!} \tag{2.1.6}$$

where $(a)_n = \prod_{j=0}^{n-1}(a - j)$.

2.2 Dunkl operators, intertwining operator and angular derivatives

A finite set $\mathcal{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system if $\sigma_v \mathcal{R} = \mathcal{R}$ and $\mathcal{R} \cap \{tv : t \in \mathbb{R}\} = \{\pm v\}$ for all $v \in \mathcal{R}$. The subgroup $G \subset O(d)$ that is generated by the reflections σ_v, $v \in \mathcal{R}$ is called the reflection group associated with $\mathcal{R}$. The dimension of the subspace of $\mathbb{R}^d$ that is spanned by all elements in $\mathcal{R}$ is called the rank of $\mathcal{R}$ and is denoted by $\text{rank}(\mathcal{R})$. Each root system $\mathcal{R}$ can be written as a disjoint union $\mathcal{R} = \mathcal{R}_+ \cup (-\mathcal{R}_+)$, where $\mathcal{R}_+$ and $-\mathcal{R}_+$ are separated by a hyperplane through the origin. Such a set $\mathcal{R}_+$ is called a positive subsystem of $\mathcal{R}$. A function $\kappa : \mathcal{R} \to [0, \infty)$ on the root system $\mathcal{R}$ is called a multiplicity function on $\mathcal{R}$ if it is invariant under the action of G; that is, $\kappa_{gv} = \kappa_v$ for all $v \in \mathcal{R}$ and $g \in G$, where $\kappa_v = \kappa(v)$.

Let $\mathcal{R}$ be a fixed root system in $\mathbb{R}^d$ normalized so that $\langle v, v\rangle = 2$ for all $v \in \mathcal{R}$, and G the associated reflection group. Let $\kappa : \mathcal{R} \to [0, \infty)$ be a multiplicity function on $\mathcal{R}$ and h_κ the weight function defined by (1.0.2). We denote by $\mathbb{P}_n^d$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$, and $\Pi^d := \Pi(\mathbb{R}^d)$ the algebra of algebraic polynomials on $\mathbb{R}^d$.

The Dunkl operators associated with G and κ are defined by

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v\in\mathcal{R}_+} \kappa_v \langle v, e_i\rangle E_v f(x), \quad i = 1, \cdots, d, \quad f \in C^1(\mathbb{R}^d), \quad (2.2.1)$$

where $\mathcal{R}_+$ is a fixed positive subsystem of $\mathcal{R}$ and E_v is given by (1.0.11). This definition does not depend on the special choice of the positive subsystem $\mathcal{R}_+$, thanks to the G-invariance of κ. The operators $\mathcal{D}_i$ were introduced and first studied by C. F. Dunkl [Du1, Du2, Du4, Du5], and can be considered as perturbations of the usual partial derivatives by reflection parts. They enjoy properties similar to those of partial derivatives. In particular, they mutually commute and map $\mathbb{P}_n^d$ to $\mathbb{P}_{n-1}^d$.

One of the most important results in the Dunkl theory states that associated with a reflection group G and multiplicity κ there exists a unique linear operator $V_\kappa : \Pi^d \to \Pi^d$, called the Dunkl intertwining operator, such that

$$V_\kappa(\mathbb{P}_n^d) = \mathbb{P}_n^d, \quad V_\kappa(1) = 1, \quad \text{and} \quad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad 1 \le i \le d. \qquad (2.2.2)$$

The intertwining operator V_κ commutes with the G-action; that is, $g^{-1}\circ V_\kappa \circ g = V_\kappa$ for all $g \in G$. Here and throughout, we use the notation $g \circ f(x) := f(gx)$ for $g \in G$, $f \in C(\mathbb{S}^{d-1})$ and $x \in \mathbb{S}^{d-1}$. An explicit "closed" form for the intertwining operator is known so far only in the case of $G = \mathbb{Z}_2^d$ (see [Du4, Xu2]) and the case of $G = S_3$ (see [Du6]). However, the explicit integral formula of V_κ given in [Du6] for $G = S_3$ does not seem to be in a form strong enough for carrying out further analysis. At the moment, little information is known on the intertwining operator for general finite reflection groups other than $\mathbb{Z}_2^d$, except the following important result of Rösler [Ro1]:

Theorem 2.2.1. *[Ro1, Th. 1.2 and Cor. 5.3] For every $x \in \mathbb{R}^d$, there exists a unique probability measure μ_x^κ on the Borel σ-algebra of $\mathbb{R}^d$ such that*

$$V_\kappa P(x) = \int_{\mathbb{R}^d} P(\xi)\, d\mu_x^\kappa(\xi), \quad P \in \Pi^d. \qquad (2.2.3)$$

Furthermore, the representing measures μ_x^κ are compactly supported in the convex hull $\widehat{G}_x := co\{gx : \ g \in G\}$ of the orbit of x under G, and satisfy

$$\mu_{rx}^\kappa(E) = \mu_x^\kappa(r^{-1}E), \quad \textit{and } \mu_{gx}^\kappa(E) = \mu_x^\kappa(g^{-1}E) \qquad (2.2.4)$$

for all $r > 0$, $g \in G$ and each Borel subset E of $\mathbb{R}^d$.

In particular, Theorem 2.2.1 shows that the intertwining operator V_κ is positive, and can be extended to the space $C(\mathbb{R}^d)$ of continuous functions on $\mathbb{R}^d$, which we denote again by V_κ. The intertwining operator also has the following property:

$$V_\kappa\Big[f(\langle x,\cdot\rangle)\Big](y) = V_\kappa\Big[f(\langle y,\cdot\rangle)\Big](x), \quad x, y \in \mathbb{S}^{d-1}, \quad f \in C[-1,1]. \tag{2.2.5}$$

The Dunkl κ-Laplacian on $\mathbb{R}^d$ is defined by $\Delta_\kappa := \sum_{j=1}^d \mathcal{D}_j^2$. It is G-invariant; that is, $g \circ \Delta_\kappa = \Delta_\kappa \circ g$ for all $g \in G$, and has the following explicit expression (see [Ro2, pp. 99] and [DuXu, Theorem 4.4.9])

$$\Delta_k = \Delta + 2\sum_{v\in\mathcal{R}_+} \kappa_v \delta_v \quad \text{with} \quad \delta_v f(x) = \frac{\langle \nabla f(x), v\rangle}{\langle v, x\rangle} - \frac{E_v f(x)}{\langle v, x\rangle}, \tag{2.2.6}$$

where $\Delta = \sum_{j=1}^d \partial_j^2$. It is worthwhile to recall the normalization $\langle v, v\rangle = 2$, $v \in \mathcal{R}_+$ in this last formula.

Finally, we record the following useful identity on the operators E_v (see, for instance, [Ro2, Lemma 2.3]):

$$\sum_{v,v'\in\mathcal{R}_+} \kappa_v \kappa_{v'} \langle v, v'\rangle E_v E_{v'} = 0. \tag{2.2.7}$$

Particularly, we will focus on its restriction $\Delta_{\kappa,0}$ on the sphere, which is called Dunkl-Laplace-Beltrami operator. The precise definition of $\Delta_{\kappa,0}$ is given as follows:

$$\Delta_{\kappa,0} f(x) := \Delta_\kappa F(z)|_{z=x}, \qquad \forall\, x \in \mathbb{S}^{d-1} \tag{2.2.8}$$

where $F(z) = f(\frac{z}{\|z\|})$.

We end this subsection with a brief description of the angular derivatives $D_{i,j} = x_i\partial_j - x_j\partial_i$, $1 \le i < j \le d$, which play a very important role in the ordinary spherical harmonic analysis ([DaXu2, pp. 23-27], [DaXu4]). For simplicity, in the case of $\kappa \equiv 0$, we write $\mathcal{H}_n^d$ for $\mathcal{H}_n^d(h_\kappa^2)$ and Δ_0 for $\Delta_{\kappa,0}$. Thus, $\mathcal{H}_n^d$ is the space of ordinary spherical harmonics of degree n on $\mathbb{S}^{d-1}$, and Δ_0 is the usual Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We collect some useful facts on the operators $D_{i,j}$ in the following lemma, whose proof can be found in [DaXu2, pp. 23-27].

Lemma 2.2.2. *The following statements hold true:*

- $D_{i,j}f$ *is independent of the* C^1*-extension of* $f \in C^1(\mathbb{S}^{d-1})$*; that is, if* F_1, F_2 *are two* C^1*-functions in an open neighborhood of* $\mathbb{S}^{d-1}$ *that coincide on* $\mathbb{S}^{d-1}$*, then*

$$D_{i,j}F_1\Big|_{\mathbb{S}^{d-1}} = D_{i,j}F_2\Big|_{\mathbb{S}^{d-1}}.$$

- *The Laplace-Beltrami operator* Δ_0 *on* $\mathbb{S}^{d-1}$ *can be decomposed as*

$$\Delta_0 = \sum_{1\le i<j\le d} D_{i,j}^2 \tag{2.2.9}$$

- *For* $f, g \in C^1(\mathbb{S}^{d-1})$*,*

$$\int_{\mathbb{S}^{d-1}} f(x)D_{i,j}g(x)d\sigma(x) = -\int_{\mathbb{S}^{d-1}} \big(D_{i,j}f(x)\big)g(x)d\sigma(x). \tag{2.2.10}$$

- *The operator* $D_{i,j}$ *is invariant on the space* $\mathcal{H}_n^d$*; that is, it maps* $\mathcal{H}_n^d$ *to itself.*

- *For* $f, g \in C^1(\mathbb{S}^{d-1})$*,*

$$\langle \nabla_0 f(\xi), \nabla_0 g(\xi)\rangle = \sum_{1\le i<j\le d} D_{i,j}f(\xi)D_{i,j}g(\xi), \quad \xi \in \mathbb{S}^{d-1}. \tag{2.2.11}$$

2.3 Spherical h-harmonic expansions

Recall that

$$\|f\|_{\kappa,p} := \Big(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y)d\sigma(y)\Big)^{1/p}, \quad 1 < p < \infty,$$

where h_κ given in (1.0.2). We denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$, and $\mathcal{H}_n^d(h_\kappa^2)$ the space of all spherical h-harmonics of degree n on $\mathbb{S}^{d-1}$. $\mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal complement of Π_{n-1}^d in the space Π_n^d with respect to the inner product

$$\langle f, g\rangle_{L^2(h_\kappa^2)} := \int_{\mathbb{S}^{d-1}} f(x)\overline{g(x)}h_\kappa^2(x)\, d\sigma(x),$$

and each function $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$ has a spherical h-harmonic expansion $f = \sum_{n=0}^{\infty} \operatorname{proj}_n(h_\kappa^2; f)$ converging in the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$. Here, $\operatorname{proj}_n(h_\kappa^2)$: $L^2(h_\kappa^2; \mathbb{S}^{d-1}) \to \mathcal{H}_n^d(h_\kappa^2)$ is the orthogonal projection which has an integral representation

$$\operatorname{proj}_n(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} f(y) P_n^\kappa(x, y) h_\kappa^2(y)\, d\sigma(y), \; x \in \mathbb{S}^{d-1}, \tag{2.3.1}$$

where $P_n^\kappa(x, y)$ is the reproducing kernel of $\mathcal{H}_n^d(h_\kappa^2)$. A crucial point in the theory of spherical h-harmonics is the fact that $P_n^\kappa(x, y)$ can be expressed in terms of the intertwining operator V_κ as (see [Xu, Theorem 3.2, (3.1)]):

$$P_n^\kappa(x, y) = \frac{n + \lambda_k}{\lambda_\kappa} V_\kappa \left[C_n^{\lambda_k}(\langle x, \cdot \rangle) \right](y), \qquad x, y \in \mathbb{S}^{d-1} \tag{2.3.2}$$

with $\lambda_\kappa := \frac{d-2}{2} + |\kappa|$. By means of (2.3.1) and (2.3.2), the projection $\operatorname{proj}_n(h_\kappa^2; f)$ can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

The space $\mathcal{H}_n^d(h_\kappa^2)$ can also be seen as an eigenfunction space of the Dunkl Laplace-Beltrami operator $\Delta_{\kappa,0}$(defined as (2.2.8)), corresponding to the eigenvalue $-n(n + 2\lambda_\kappa)$; that is,

$$\mathcal{H}_n^d(h_\kappa^2) = \Big\{ f \in C^2(\mathbb{S}^{d-1}) : \quad \Delta_{\kappa,0} f = -n(n + 2\lambda_\kappa) f \Big\}, \quad n = 0, 1, \dots.$$

Definition 2.3.1. *Given a compactly supported continuous function $\theta : [0, \infty) \to \mathbb{R}$, we define a sequence of operators $L_{\theta,j}$, $j = 0, 1, \cdots$, by $L_{\theta,0}(f) = \operatorname{proj}_0(h_\kappa^2; f)$, and*

$$L_{\theta,j}(f) := \sum_{n=0}^{\infty} \theta\left(\frac{n}{2^j}\right) \operatorname{proj}_n(h_\kappa^2; f), \quad j = 1, 2, \cdots.$$

The following Littlewood-Paley type inequality is a direct consequence of the Marcinkiewitcz multiplier theorem for spherical h-harmonic expansions, which was proved in [DaXu] (see also [DaXu2, pp. 67-71]):

Theorem 2.3.2. *If θ is a compactly supported function in $C^\infty[0, \infty)$ with* $\operatorname{supp} \theta \subset (a, b)$ *for some $0 < a < b < \infty$, then for all $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$ with $1 < p < \infty$,*

$$\left\| \left(\sum_{j=0}^{\infty} \left| L_{\theta,j} f \right|^2 \right)^{1/2} \right\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}, \tag{2.3.3}$$

where the constant C_p is independent of f. If, in addition,

$$0 < A_1 \le \sum_{j=0}^{\infty} |\theta(2^{-j}t)|^2 \le A_2 < \infty, \quad \forall t > 0, \tag{2.3.4}$$

for some positive constants A_1, A_2, then for $f \in L^p(h_\kappa^2; \mathbb{S}^{d-1})$ with

$$\int_{\mathbb{S}^{d-1}} f(x) h_\kappa^2(x)\, d\sigma(x) = 0,$$

we have that

$$\left\| \left(\sum_{j=0}^{\infty} \left| L_{\theta,j} f \right|^2 \right)^{1/2} \right\|_{\kappa,p} \sim \|f\|_{\kappa,p}, \quad 1 < p < \infty. \tag{2.3.5}$$

Besides, the following Nikolskii type inequality holds true and will be needed in our proof.

Theorem 2.3.3 (Weighted Nikolskii's Inequalities [DaWa],Lemma 2.3)**.** *Let $0 < p < q \le \infty$, then for any $g \in \prod_n^d$*

$$\|g\|_{\kappa,q} \le C n^{(2\sigma_\kappa+1)(\frac{1}{p}-\frac{1}{q})} \|g\|_{\kappa,p},$$

where C depends only on p, q and κ.

2.3.1 Cesàro means

In this subsection, we will talk about some facts about Cesàro means in terms of the spherical h-harmonic, which will be a key tool of our following proof. For more details, one can refer to [DaXu2].

Definition 2.3.4. *For $\delta > 0$, the Cesàro means of the spherical function f are defined by*

$$S_n^\delta(h_\kappa^2; f) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \mathcal{P}_j^\kappa f,$$

where A_j^δ denotes as

$$A_j^\delta = \begin{pmatrix} \delta + j \\ j \end{pmatrix} = \frac{(\delta+j)(\delta+j-1)\cdots(\delta+1)}{j!}.$$

Theorem 2.3.5 ([DaXu2],corollary 8.1.2). *If $\delta > \sigma_\kappa$, then for $f \in L^p(h_\kappa, \mathbb{S}^{d-1})$ and $1 \le p < \infty$, or $f \in C(\mathbb{S}^{d-1})$ when $p = \infty$,*

$$\sup_n \|S_n^\delta(h_\kappa^2; f)\|_{p,\kappa} \le c\|f\|_{p,\kappa}.$$

Consider $S_n^\delta(h_\kappa^2; f)$ as a convolution:

$$S_n^\delta(h_\kappa^2; f) = f * K_n^\delta(h_\kappa^2),$$

then the kernel $K_n^\delta(h_\kappa^2; x, y)$ is the Cesàro means of $Z_j^{\lambda_\kappa}(x, y)$

$$K_n^\delta(h_\kappa^2; x, y) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-k}^\delta Z_j^{\lambda_\kappa}(x, y),$$

which has the following pointwise estimate.

Theorem 2.3.6 ([DaXu2], Theorem 8.1.1). *For any $x, y \in \mathbb{S}^{d-1}$,*

$$|K_n^\delta(h_\kappa^2; x, y)h_\kappa^2(y)| \le cn^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-\beta(\delta)},$$

where $\beta(\delta) := \min\{d + 1, \delta - \sigma_\kappa + d\}$.
Further more, for any $\delta > \sigma_\kappa$

$$\int_{\mathbb{S}^{d-1}} |K_n^\delta(h_\kappa^2; x, y)|h_\kappa^2(y)d\sigma(y) \le C, \tag{2.3.6}$$

where C is a constant independent of n.

Theorem 2.3.7 ([DaXu2],B.1.13). *Let*

$$S_n^\ell(u) := \frac{1}{A_n^\delta} \sum_{j=0}^{n} A_{n-j}^\delta \frac{j + \lambda}{\lambda} C_j^\lambda(u),$$

then for $\ell > 2\lambda + 1$,

$$0 \le S_n^\ell(u) \le cn^{-1}(1 - u + n^{-2})^{\lambda+1}.$$

2.4 Singular integrals on homogeneous Spaces

In this section, we shall extent some well-known classical results of harmonic analysis to the more general setting of homogeneous spaces, which are guaranteed on the weighted unit sphere as a consequence. For more detail of proof below, one can refer to [St1] and [Da].

Definition 2.4.1. *Given a measure space* $(X, \mathbb{B}, \mu)$ *with a metric* ρ*, it is called homogeneous space, if all open balls* $B(x,r) := \{y \in X : \rho(x,y) < r\}, x \in X, r > 0$ *are measurable with positive finite measure, and that one has the doubling property there exists a positive constant* C *such that*

$$\mu(B(x,2r)) \leq C\mu(B(x,r)),$$

for any $x \in X$*,* $r > 0$*. In addition, the best constant* C *for which this last inequality holds is called the doubling constant of* μ*.*

Theorem 2.4.2. *Let* T *be an operator in the form*

$$(Tf)(x) = \int_X K(x,y)f(y)d\mu(y),$$

and bounded on $L^q(X)$ *with norm* A*; that is*

$$\|Tf\|_{L^q(X)} \leq A\|f\|_{L^q(x)}, \quad \forall\, f \in L^q(X).$$

Moreover, if K *satisfies that for some constant* $c > 1$*,*

$$\int_{B(z,c\delta)^c} |K(x,y) - K(x,z)| d\mu(x) \leq A, \quad \forall\, y \in B(z,\delta), \tag{2.4.1}$$

for all $y, z \in X$*,* $\delta > 0$*. Then the operator* T *is bounded in* L^p *norm on* $L^p \cap L^q$ *for* $1 < p < q$*; that is*

$$\|Tf\|_p \leq A\|f\|_p, \quad \textit{for} \quad f \in L^p \cap L^q.$$

In addition, it is necessary to point out the following remarks.

(i) T can be extended to L^q uniquely and keep the boundedness, since $L^p \cap L^q$ is dense in L^q;

(ii) If there is an upper bound for the radius of all of balls in X, then the condition
" for any $\delta > 0$" can be deduced to " for $0 < \delta < \delta_0$ with some $\delta_0 > 0$";

(iii) The domain in the integral of (2.4.1) can be replaced as well by a measurable set D^c with $\mu(D) \le c\sum_j \mu(B_j)$.

The sphere $\mathbb{S}^{d-1}$ is a metric space with geodesic metric $\rho(x,y) := \arccos\langle x,y\rangle$, $x, y \in \mathbb{S}^{d-1}$. We denote by $B_r(x)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : \quad \rho(x,y) < r\}$ with center $x \in \mathbb{S}^{d-1}$ and radius $r \in (0,\pi)$, and write $\operatorname{meas}_\kappa(E) := \int_E h_\kappa^2(x)\, d\sigma(x)$ for a set $E \subset \mathbb{S}^{d-1}$. Given a spherical cap $B = B_r(x) \subset \mathbb{S}^{d-1}$ and a scaling $c > 0$, we write cB for the spherical cap $B_{cr}(x)$ with the same center as that of B but c times the radius of B. More generally, given a weight function w on $\mathbb{S}^{d-1}$, we write $w(E) := \int_E w(x)\, d\sigma(x)$ for $E \subset \mathbb{S}^{d-1}$. A weight function w on $\mathbb{S}^{d-1}$ is called a doubling weight if there exists a constant $L > 0$ such that

$$w(2B) \le Lw(B) \quad \text{for all spherical caps } B \subset \mathbb{S}^{d-1}, \tag{2.4.2}$$

where the least constant L is called the doubling constant of w. The following lemma collects some useful properties on doubling weights (see [Da]):

Lemma 2.4.3. *Let w be a doubling weight on $\mathbb{S}^{d-1}$ with the doubling constant L. Then the following statements hold:*

- *There exists a positive number s such that*

  $$w(2^m B) \le C2^{ms}w(B), \quad \forall m \in \mathbb{N}, \quad \forall \text{ spherical caps } \ B \subset \mathbb{S}^{d-1}, \tag{2.4.3}$$

  where the constant C is independent of m and B.

- *For $0 < r < t$, and $x \in \mathbb{S}^{d-1}$,*

  $$w(B_t(x)) \le C\Big(\frac{t}{r}\Big)^s w(B_r(x)), \tag{2.4.4}$$

  where s is a positive number satisfying (2.4.3).

- *For $0 < r < \pi$, and $x, y \in \mathbb{S}^{d-1}$,*

  $$w(B_r(x)) \le C(1 + r^{-1}\rho(x,y))^s w(B_r(y)), \tag{2.4.5}$$

  where s is a positive number satisfying (2.4.3).

Many of the weights on $\mathbb{S}^{d-1}$ that appear in analysis satisfy the doubling condition (2.4.2); in particular, all weights of the form

$$w_{\alpha,\mathbf{v}}(x) = \prod_{j=1}^{m} |\langle x, v_j\rangle|^{\alpha_j}, \quad x \in \mathbb{S}^{d-1}, \, m \in \mathbb{N}, \tag{2.4.6}$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\alpha_j > -1$, $\mathbf{v} = (v_1, v_2, \cdots, v_m)$ and $v_j \in \mathbb{S}^{d-1}$. Indeed, a slight modification of the proof in [Da, (5.3)] shows that for $t \in (0, \pi)$ and $x \in \mathbb{S}^{d-1}$,

$$\int_{B_t(x)} w_{\alpha,\mathbf{v}}(x)\, d\sigma(x) \sim t^{d-1} \prod_{j=1}^{m} (|\langle x, v_j\rangle| + t)^{\alpha_j}. \tag{2.4.7}$$

The weighted Hardy-Littlewood (HL) maximal function M_w with respect to a weight function w on $\mathbb{S}^{d-1}$ is defined by

$$M_w g(x) = \sup_{0<r\le\pi} \frac{1}{w(B_r(x))} \int_{B_r(x)} |g(y)| w(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}. \tag{2.4.8}$$

As is well known, if w has the doubling property (2.4.2), then M_w satisfies that

$$\|M_w g\|_{p,w} \le c_p \|g\|_{p,w}, \quad 1 < p \le \infty, \tag{2.4.9}$$

where $\|\cdot\|_{p,w}$ denotes the L^p norm defined with respect to the measure $w(x)d\sigma(x)$ on $\mathbb{S}^{d-1}$, and the following Fefferman-Stein inequalty:

Theorem 2.4.4 (Fefferman-Stein)**.** *If $1 < p, q < \infty$ and $\{f_j\}$ is a sequence of functions on X, then*

$$\| \left(\sum_{j=0}^{\infty} |M_\mu f_j|^q \right)^{1/q} \|_p \lesssim \| \left(\sum_{j=0}^{\infty} |f_j|^q \right)^{1/q} \|_p,$$

where $(X, d\mu)$ is a measurable space and M_μ is the Hardy-Littlewood maximum function correspondingly.

Chapter 3

Hardy-Littlewood-Sobolev inequality on unite sphere

In this chapter, we shall first formulate auxiliary results, including a characterization of the critical index s_κ and pointwise estimates of kernel functions, that will be indispensable in much of our future work. Next, the first main result, HLS inequality and its necessary conditions, on the weighted unit sphere will be introduced and proved in detail.

3.1 A characterization of the critical index s_κ

Recall that, (2.4.6) and (2.4.7) give that with h_κ as in (1.0.2), for a spherical cap $B := B_\theta(x)$ with center $x \in \mathbb{S}^{d-1}$ and radius $\theta \in (0, 1)$,

$$\mathrm{meas}_\kappa(B) \sim \theta^{d-1} \prod_{v \in \mathcal{R}_+} (|\langle x, v\rangle| + \theta)^{2\kappa_v}, \tag{3.1.1}$$

where $\mathrm{meas}_\kappa(B) = \int_B h_\kappa^2(x)\, d\sigma(x)$. This, in particular, implies that h_κ^2 is a doubling weight on $\mathbb{S}^{d-1}$.

The main purpose of this section is to give an equivalent characterization of the index s_κ given in (1.0.7) in terms of the doubling property of the weight function h_κ^2. Such a characterization will be used repeatedly in the next two sections.

Our main result in this section can be stated as follows.

Theorem 3.1.1. *The index s_κ given in (1.0.7) is the smallest positive number s for which there exists a general constant $C > 0$ such that for all spherical caps $B \subset \mathbb{S}^{d-1}$ and all $m \in \mathbb{N}$,*

$$\mathrm{meas}_\kappa(2^m B) \le C 2^{ms}\,\mathrm{meas}_\kappa(B). \tag{3.1.2}$$

Furthermore,

$$s_\kappa = \lim_{m\to\infty} \frac{1}{m} \log_2 \sup_B \frac{\mathrm{meas}_\kappa(2^m B)}{\mathrm{meas}_\kappa(B)}, \tag{3.1.3}$$

where the supremum $\sup_B$ is taken over all spherical caps $B \subset \mathbb{S}^{d-1}$ with radius $\le 2^{-m}$.

Proof. Our proof will be under the aid of following two terms.

$$s'_\kappa = \limsup_{m\to\infty} \frac{1}{m} \log_2 \sup_B \frac{\mathrm{meas}_\kappa(2^m B)}{\mathrm{meas}_\kappa(B)},$$
$$s''_\kappa = \liminf_{m\to\infty} \frac{1}{m} \log_2 \sup_B \frac{\mathrm{meas}_\kappa(2^m B)}{\mathrm{meas}_\kappa(B)},$$

where the supremums $\sup_B$ are still taken over all spherical caps $B \subset \mathbb{S}^{d-1}$ with radius $\le 2^{-m}$. Then if s is a positive number such that (3.1.2) holds for all spherical caps B and all $m \in \mathbb{N}$, then $s'_\kappa \le s$. To complete the proof, it suffices to show that $s_\kappa \le s''_\kappa$ and (3.1.2) holds with $s = s_\kappa$ for all spherical caps.

Notice that by (3.1.1),

$$s''_\kappa = d - 1 + \liminf_{m\to\infty} \frac{1}{m} \log_2 \Big[\sup_{x\in\mathbb{S}^{d-1}} \sup_{\theta\in(0,2^{-m})} \frac{\prod_{v\in\mathcal{R}_+} (|\langle x, v\rangle| + 2^m\theta)^{2\kappa_v}}{\prod_{v\in\mathcal{R}_+} (|\langle x, v\rangle| + \theta)^{2\kappa_v}} \Big]. \tag{3.1.4}$$

When rank $(\mathcal{R}) \le d-1$, $s_\kappa = 2\lambda_\kappa + 1$ with $\lambda_k = \frac{d-2}{2} + \sum_{v\in\mathcal{R}_+} k_v$ and we can take $x_0 \in \mathbb{S}^{d-1}$ such that $\langle x_0, v\rangle = 0$ for all $v \in \mathcal{R}_+$. Then, making x to be x_0 in (3.1.4), we have that

$$s''_\kappa \ge d - 1 + \lim_{m\to\infty} \frac{1}{m} \log_2 \Big[\sup_{\theta\in(0,2^{-m})} \frac{\prod_{v\in\mathcal{R}_+} (2^m\theta)^{2\kappa_v}}{\prod_{v\in\mathcal{R}_+} \theta^{2\kappa_v}} \Big] = 2\lambda_\kappa + 1 = s_\kappa.$$

On the other hand, (3.1.2) holds for s_κ can be easily verified by using (3.1.1).

When $\operatorname{rank}(\mathcal{R}) = d$, recall that

$$s_\kappa = d - 1 + 2 \max_{v \in X_{d-1} \cap \mathcal{R}_+} k_v,$$

where X_{d-1} is taken over all $d-1$ dimensional subspaces. Given any fixed $x \in \mathbb{S}^{d-1}$ we define

$$c_1 := \min_{v_1, \dots, v_d} \min_{x \in \mathbb{S}^{d-1}} \Big(\sum_{j=1}^{d} |\langle x, v_j \rangle|^2 \Big)^{1/2},$$

where the first minimum on the right hand side is taken over all d linearly independent elements $v_1, \cdots, v_d$ from $\mathcal{R}_+$. Clearly, c_1 is well defined and positive since $\operatorname{rank}(\mathcal{R}) = d$. Next, let $v_1, \dots, v_N$ be all the distinct elements of $\mathcal{R}_+$ ordered so that

$$|\langle x, v_N \rangle| \ge \dots \ge |\langle x, v_1 \rangle|.$$

If we let $n = n_x \ge d-1$ be the largest integer such that the linear space $X_{d-1} := \operatorname{span}\{v_1, \cdots, v_n\}$ has dimension $d-1$, then

$$|\langle x, v_j \rangle| \ge c_1/\sqrt{d} > 0, \quad \text{for}\, n+1 \le j \le N,$$

and hence for $2^m \theta \in (0,1)$

$$\frac{\prod_{v \in \mathcal{R}_+} (|\langle x, v \rangle| + 2^m \theta)^{2\kappa_v}}{\prod_{v \in \mathcal{R}_+} (|\langle x, v \rangle| + \theta)^{2\kappa_v}} \sim \frac{\prod_{j=1}^{n} (|\langle x, v_j \rangle| + 2^m \theta)^{2\kappa_{v_j}}}{\prod_{j=1}^{n} (|\langle x, v_j \rangle| + \theta)^{2\kappa_{v_j}}} \lesssim 2^{2m \sum_{v \in X_{d-1} \cap \mathcal{R}_+} k_v}$$

which combining with (3.1.4) implies that (3.1.2) holds with $s = s_\kappa$.

Finally, it remains to show that $s_\kappa \le s''_k$. Let Y denote the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ spanned by certain elements from $\mathcal{R}_+$ such that

$$s_\kappa = d - 1 + 2 \sum_{v \in \mathcal{R}_+ \cap Y} \kappa_v,$$

and a point $x_0 \in \mathbb{S}^{d-1}$ such that $\langle x_0, v \rangle = 0$ for all $v \in Y \cap \mathcal{R}_+$. Then for any

$m \in \mathbb{N}$ and $0 < \theta \le 2^{-m}$,

$$s''_\kappa \ge d - 1 + \liminf_{m\to\infty} \frac{1}{m} \log_2 \sup_{\theta\in(0,2^{-m})} \frac{\prod_{v\in\mathcal{R}_+}(|\langle x_0, v\rangle| + 2^m\theta)^{2\kappa_v}}{\prod_{v\in\mathcal{R}_+}(|\langle x_0, v\rangle| + \theta)^{2\kappa_v}}$$
$$\ge d - 1 + 2\sum_{v\in Y\cap\mathcal{R}_+} \kappa_v = s_\kappa,$$

which completes the proof. □

3.2 Kernel estimates and weighted Christoffel Functions

In this section, we shall establish some pointwise estimates of a class of kernel functions concerning Dunkl intertwining operator.

Theorem 3.2.1. *Let Ψ_n, $n = 1, 2, \cdots$ be a sequence of continuous functions on $[-1, 1]$ satisfying that*

$$|\Psi_n(\cos\theta)| \le Cn^{2\lambda_\kappa+1}(1 + n\theta)^{-\ell}, \quad \theta \in [0, \pi] \tag{3.2.1}$$

for some positive number $\ell > 2\lambda_\kappa + 1$. Let ℓ_0 be an arbitrarily given positive number smaller than $\ell - 2\lambda_\kappa - 1$. Then for any $x, y \in \mathbb{S}^{d-1}$,

$$\left|V_\kappa\Big[\Psi_n(\langle y, \cdot\rangle)\Big](x)\right| \le C\frac{n^{d-1}(1 + n\widetilde{\rho}(x, y))^{-\ell_0+3s_\kappa/2-d+1}}{\prod_{v\in\mathcal{R}_+}(|\langle x, v\rangle| + |\langle g_0 y, v\rangle| + \widetilde{\rho}(x, y) + n^{-1})^{2\kappa_v}}, \tag{3.2.2}$$

where V_κ is Dunkl intertwining operator defined as (2.2.2), $\widetilde{\rho}(x, y) := \min_{g\in G} \rho(gx, y)$, $x, y \in \mathbb{S}^{d-1}$, and $g_0 \in G$ is the one such that $\rho(g_0 x, y) = \widetilde{\rho}(x, y)$.

Remark that

$$\widetilde{\rho}(g_1 x, g_2 y) = \widetilde{\rho}(x, y) = \widetilde{\rho}(y, x), \quad \forall g_1, g_2 \in G, \quad \forall x, y \in \mathbb{S}^{d-1}.$$

Together with Lemma 2.1.1, we deduce the following useful corollary by Theorem 3.2.1.

Corollary 3.2.2. *Let η be a compactly supported C^∞-function on $\mathbb{R}$ which is* *constant near the origin, and let*

$$\Phi_n(x,y) := \sum_{j=0}^{\infty} \eta(\frac{j}{n}) \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa \Big[C_j^{\lambda_\kappa}(\langle x, \cdot \rangle) \Big](y), \quad x, y \in \mathbb{S}^{d-1}, \quad n \in \mathbb{N}.$$

Then for any $\ell \in \mathbb{N}$ and $x, y \in \mathbb{S}^{d-1}$,

$$|\Phi(x,y)| \le C_\ell \|\eta^{(3\ell-1)}\|_{L^\infty(\mathbb{R})} \frac{n^{d-1}(1+n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle| + |\langle g_0 y, v\rangle| + n^{-1} + \widetilde{\rho}(x,y))^{2\kappa_v}},$$

where $g_0 \in G$ is such that $\rho(g_0 x, y) = \widetilde{\rho}(x,y)$ and the constant C_ℓ only depends on ℓ.

The main tool for the proof of Theorem 3.2.1 is the weighted Christoffel function defined for a weight function w on $\mathbb{S}^{d-1}$ by

$$\lambda_n(w,x) := \inf_{P_n(x)=1} \int_{\mathbb{S}^{d-1}} |P_n(z)|^2 w(z)\, d\sigma(z), \quad n = 0, 1, 2, \cdots,$$

where the infimum is taken over all spherical polynomials of degree n on $\mathbb{S}^{d-1}$ that take the value 1 at the point $x \in \mathbb{S}^{d-1}$. The following lemma illustrates the connection between weighted Christoffel functions and weighted orthogonal polynomial expansions.

Lemma 3.2.3. *Let $P_{n,1}, \cdots, P_{n,a_n}$ be an orthornormal basis of the space Π_n^d of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$, with respect to the inner product*

$$\langle f, g \rangle_w := \int_{\mathbb{S}^{d-1}} f(x) g(x) w(x)\, d\sigma(x).$$

Then

$$\lambda_n(w,x) = \Big(\sum_{j=1}^{a_n} |P_{n,j}(x)|^2 \Big)^{-1}, \quad x \in \mathbb{S}^{d-1}.$$

Lemma 3.2.3 is a well known result in approximation theory (see, for instance, [DuXu]), but for the sake of completeness, we include a proof here.

Proof. For $P \in \Pi_n^d$ with $P(x) = 1$, we have

$$\begin{aligned}
1 = P(x) &= \int_{\mathbb{S}^{d-1}} \Big(\sum_{j=1}^{a_n} P_{n,j}(x)P_{n,j}(y)\Big) P(y)w(y)\,d\sigma(y) \\
&\le \Big(\int_{\mathbb{S}^{d-1}} |\sum_{j=1}^{a_n} P_{n,j}(x)P_{n,j}(y)|^2 w(y)\,d\sigma(y)\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^{d-1}} |P(y)|^2 w(y)\,d\sigma(y)\Big)^{\frac{1}{2}} \\
&= \Big(\sum_{j=1}^{a_n} |P_{n,j}(x)|^2\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{S}^{d-1}} |P(y)|^2 w(y)\,d\sigma(y)\Big)^{\frac{1}{2}}. \qquad (3.2.3)
\end{aligned}$$

This, in particular, implies that

$$\sum_{j=1}^{a_n} |P_{n,j}(x)|^2 > 0, \quad \forall x \in \mathbb{S}^{d-1}.$$

Thus, taking infimum over all $P \in \Pi_n^d$ with $P(x) = 1$ in (3.2.3) , we obtain the lower estimate

$$\lambda_n(w,x) \ge \Big(\sum_{j=1}^{a_n} |P_{n,j}(x)|^2\Big)^{-1}.$$

To obtain the desired upper estimate, we fix a vector $x \in \mathbb{S}^{d-1}$, and set

$$P_n(y) := \frac{\sum_{j=1}^{a_n} P_{n,j}(x)P_{n,j}(y)}{\sum_{j=1}^{a_n} |P_{n,j}(x)|^2}.$$

Clearly, $P_n \in \Pi_n^d$, and $P_n(x) = 1$. Thus,

$$\lambda_n(w,x) \le \int_{\mathbb{S}^{d-1}} |P_n(y)|^2 w(y)\,d\sigma(y) = \frac{1}{\sum_{j=1}^{a_n} |P_{n,j}(x)|^2}.$$

□

In the case when w is a doubling weight on $\mathbb{S}^{d-1}$, we have the following pointwise estimate of $\lambda_n(w,x)$:

Lemma 3.2.4. *If w is a doubling weight on $\mathbb{S}^{d-1}$, then for $x \in \mathbb{S}^{d-1}$ and $n \in \mathbb{N}$,*

$$\lambda_n(w,x) \sim \int_{B_{n^{-1}}(x)} w(y)\,d\sigma(y), \qquad (3.2.4)$$

where $B_{n^{-1}}(x)$ is the spherical cap with center x and radius $1/n$, and the constant of equivalence is independent of x and n.

In the case of $d = 2$, Lemma 3.2.4 for doubling weights on the unit circle was first established in the work of Mastroianni and Totik [MT]. Our proof here is however different from that of [MT].

Proof. We start with the lower estimate of (3.2.4). Let Λ be a finite subset of $\mathbb{S}^{d-1}$ which contains the point $x \in \mathbb{S}^{d-1}$ and has the properties $\min\{\rho(\omega, \omega') : \omega, \omega' \in \Lambda, \omega \neq \omega'\} \geq \delta/n$ and $\mathbb{S}^{d-1} = \bigcup_{\omega\in\Lambda} B_{\delta/n}(\omega)$ for some $\delta > 0$. By Theorem 4.1 of [Da], we may find a constant $\delta \in (0, 1)$ depending only on the doubling weight of w for which there exists a sequence of positive numbers ν_ω, $\omega \in \Lambda$ such that $\nu_\omega \sim \int_{B_{n^{-1}}(\omega)} w(x)\, d\sigma(x)$, and

$$\int_{\mathbb{S}^{d-1}} g(y)w(y)\, d\sigma(y) = \sum_{\omega\in\Lambda} \nu_\omega g(\omega), \quad \forall g \in \Pi^d_{2n}.$$

Since $x \in \Lambda$, it follows that for $f \in \Pi^d_n$ with $f(x) = 1$,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}} |f(y)|^2 w(y)\, d\sigma(y) &= \sum_{\omega\in\Lambda} \nu_\omega |f(\omega)|^2 \geq \nu_x |f(x)|^2 \\ &= \nu_x \sim \int_{B_{n^{-1}}(x)} w(y)\, d\sigma(y).\end{aligned}$$

Taking infimum over all $f \in \Pi^d_n$ with $f(x) = 1$ yields the desired lower estimate:

$$\lambda_n(w, x) \geq c \int_{B_{1/n}(x)} w(y)\, d\sigma(y).$$

Next, we show the upper estimate of (3.2.4). Set

$$w_n(x) := n^{d-1} \int_{B_{1/n}(x)} w(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad n = 1, 2, \cdots .$$

By the doubling property of w, we have that ([Da, 2.3])

$$w_n(y) \leq C(1 + n\rho(x, y))^s w_n(x), \quad \forall x, y \in \mathbb{S}^{d-1}, \quad \forall n \in \mathbb{N}, \tag{3.2.5}$$

where $s = \log_2 L$ with L being the doubling constant of w. We will also use the following result from [Da, Cor. 3.4]: If $1 \leq p < \infty$ and $f \in \Pi^d_n$, then

$$\|f\|_{p,w} \sim \|f\|_{p,w_n}, \tag{3.2.6}$$

where $\|f\|_{p,w}$ denotes the L^p-norm defined with respect to the measure $w(x)\,d\sigma(x)$ on $\mathbb{S}^{d-1}$.

Now by Lemma 2.1.1, there exists an algebraic polynomial g of degree at most n on $[-1,1]$ such that $g(1)=1$ and

$$|g(\cos\theta)| \le C(1+n\theta)^{-\ell}, \quad \forall \theta \in [0,\pi], \quad \forall \ell > 0.$$

For a fixed point $x \in \mathbb{S}^{d-1}$, let $f(y) := g(\langle x, y\rangle)$ for $y \in \mathbb{S}^{d-1}$. Clearly, $f \in \Pi_n^d$ and $f(x) = 1$. Hence, by (3.2.5) and (3.2.6),

$$\begin{aligned}\lambda_n(w,x) &\le \int_{\mathbb{S}^{d-1}} |g(\langle x,y\rangle)|^2 w(y)\,d\sigma(y) \sim \int_{\mathbb{S}^{d-1}} |g(\langle x,y\rangle)|^2 w_n(y)\,d\sigma(y) \\ &\le C w_n(x) \int_{\mathbb{S}^{d-1}} (1+n\rho(x,y))^{-\ell+s} d\sigma(y) \sim \int_{B_{1/n}(x)} w(z) d\sigma(z),\end{aligned}$$

where ℓ is taken to be greater than $s+d$. This shows the desired upper estimate of $\lambda_n(w,x)$. □

The proof of Theorem 3.2.1 also relies on the following lemma.

Lemma 3.2.5. *Let $\delta > 2\lambda_\kappa + 1$. Then for each positive integer n, there exists a nonnegative algebraic polynomial of degree n of the form*

$$P_n(t) = \sum_{j=0}^{n} c_{n,j} \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(t), \quad t \in [-1,1], \tag{3.2.7}$$

which satisfies that $\sup_{n,j} |c_{n,j}| \le C < \infty$, and

$$P_n(\cos\theta) \sim n^{2\lambda_\kappa+1}(1+n\theta)^{-\delta}, \quad \theta \in (0,\pi). \tag{3.2.8}$$

Proof. By Lemma 4.6 of [Da], there exists a nonnegative algebraic polynomial P_n of degree at most n which satisfies (6.1.2) . Let the ultraspherical polynomial expansion of P_n be given by (3.2.7). It remains to show that $\sup_{n,j} |c_{n,j}| \le C < \infty$. Recall that (see [Sz])

$$\|C_j^{\lambda_\kappa}\|_{2,\lambda_\kappa}^2 = \frac{c}{j+\lambda_\kappa} C_j^{\lambda_\kappa}(1) = \frac{c'}{j+\lambda_\kappa} \frac{\Gamma(2\lambda_\kappa + j)}{\Gamma(j+1)} \sim j^{2\lambda_\kappa - 2},$$

and

$$\max_{\theta\in[0,\pi]} |C_j^{\lambda_\kappa}(\cos\theta)| = C_j^{\lambda_\kappa}(1) \sim j^{2\lambda_\kappa - 1},$$

here $\|\cdot\|_{2,\lambda_\kappa}$ denotes the L^2-norm computed with respect to the measure $(1-t^2)^{\lambda_\kappa-\frac{1}{2}}\,dt$ on $[-1,1]$. By orthogonality of the ultraspherical polynomials, we have

$$j^{2\lambda_\kappa-1}|c_{n,j}| \sim |c_{n,j}|\frac{j+\lambda_\kappa}{\lambda_\kappa}\|C_j^{\lambda_\kappa}\|_{2,\lambda_\kappa}^2 = c\Big|\int_0^\pi P_n(\cos\theta)C_j^{\lambda_\kappa}(\cos\theta)(\sin\theta)^{2\lambda_\kappa}\,d\theta\Big|$$
$$\leq Cn^{2\lambda_\kappa+1}j^{2\lambda_\kappa-1}\int_0^\pi (1+n\theta)^{-\delta}(\sin\theta)^{2\lambda_\kappa}\,d\theta \leq Cj^{2\lambda_\kappa-1},$$

provided that $\delta > 2\lambda_\kappa+1$. It then follows that $|c_{j,k}| \leq C$. □

We are now in a position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. Using Theorem 2.2.1, we have that

$$V_\kappa\Big[\Psi_n(\langle\cdot,y\rangle)\Big](x) = \int_{\widehat{G}_x}\Psi_n(\langle y,z\rangle)\,d\mu_x(z), \tag{3.2.9}$$

where $\widehat{G}_x$ denotes the convex hull of the orbit $G_x := \{gx : \ g\in G\}$ of x under the group G. Since the group G has finite order, it follows that every element $z\in\widehat{G}_x$ can be written in the form $z = \sum_{g\in G} t_{g,z}\cdot gx$ for some $t_{g,z}\in[0,1]$ satisfying $\sum_{g\in G} t_{g,z} = 1$. This implies that

$$\langle z,y\rangle = \sum_{g\in G} t_{g,z}\langle gx,y\rangle \leq \max_{g\in G}\langle gx,y\rangle, \quad \forall z\in\widehat{G}_x,$$

and hence

$$\rho(z,y) \geq \min_{g\in G}\rho(gx,y) =: \widetilde{\rho}(x,y), \quad \forall z\in\widehat{G}_x. \tag{3.2.10}$$

Thus, using (3.2.1), (3.2.9) and (3.2.10), we deduce that for $\ell\geq\delta>2\lambda_\kappa+1$,

$$\Big|V_\kappa\Big[\Psi_n(\langle y,\cdot\rangle)\Big](x)\Big| \leq C(1+n\widetilde{\rho}(x,y))^{-\ell+\delta}n^{2\lambda_\kappa+1}\int_{\widehat{G}_x}(1+n\rho(y,z))^{-\delta}\,d\mu_x(z)$$
$$\leq C(1+n\widetilde{\rho}(x,y))^{-\ell+\delta}V_\kappa\Big[P_n(\langle y,\cdot\rangle)\Big](x), \tag{3.2.11}$$

where P_n is the polynomial as given in Lemma 3.2.5, and the last step uses the positivity of V_κ (i.e., Theorem 2.2.1).

We claim that

$$V_\kappa\Big[P_n(\langle y,\cdot\rangle)\Big](x) \leq C\frac{(1+n\widetilde{\rho}(x,y))^{s_\kappa/2}}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}, \tag{3.2.12}$$

which, combined with (3.2.11), will show that

$$\left|V_\kappa\Big[\Psi_n(\langle y,\cdot\rangle)\Big](x)\right| \le C\frac{(1+n\widetilde{\rho}(x,y))^{-\ell+\delta+s_\kappa/2}}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}. \tag{3.2.13}$$

To show (3.2.12), let $p_{j,1},\cdots,p_{j,a_j}$ be an orthonormal basis of the space $\mathcal{H}_j^d(h_\kappa^2)$ with respect to the inner product of $L^2(h_\kappa^2;\mathbb{S}^{d-1})$. Then (2.3.2) implies

$$\sum_{k=1}^{a_j} p_{j,k}(x)p_{j,k}(y) = \frac{\lambda_\kappa+j}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle\cdot,y\rangle)\Big](x), \quad x,y\in\mathbb{S}^{d-1}. \tag{3.2.14}$$

It, combining with Lemma 3.2.5 and positivity of V_κ, yields that

$$\begin{aligned}0 \le & V_\kappa\Big[P_n(\langle\cdot,y\rangle)\Big](x) = \sum_{j=0}^n c_{n,j}\sum_{k=1}^{a_j} p_{j,k}(x)p_{j,k}(y) \le C\sum_{j=0}^n\sum_{k=1}^{a_j}|p_{j,k}(x)p_{j,k}(y)| \\ & \le C\Big(\sum_{j=0}^n\sum_{k=1}^{a_j}|p_{j,k}(x)|^2\Big)^{1/2}\Big(\sum_{j=0}^n\sum_{k=1}^{a_j}|p_{j,k}(y)|^2\Big)^{\frac12},\end{aligned}$$

which is bounded above by a constant multiple of

$$\Big(\int_{B_{n^{-1}}(x)} h_\kappa^2(z)\,d\sigma(z)\Big)^{-\frac12}\Big(\int_{B_{n^{-1}}(y)} h_\kappa^2(z)\,d\sigma(z)\Big)^{-\frac12} \tag{3.2.15}$$

by Lemma 3.2.3 and Lemma 3.2.4. However, since the weight h_κ^2 is invariant under the group G,

$$\begin{aligned}\int_{B_{n^{-1}}(x)} h_\kappa^2(z)\,d\sigma(z) &= \int_{B_{n^{-1}}(g_0x)} h_\kappa^2(z)\,d\sigma(z) \\ &\le C(1+n\rho(g_0x,y))^{s_\kappa}\int_{B_{n^{-1}}(y)} h_\kappa^2(z)\,d\sigma(z),\end{aligned}$$

where the last step follows from (2.4.5). Hence, (3.2.15) can be dominated by a constant multiple of

$$(1+n\rho(g_0x,y))^{s_\kappa/2}\Big(\int_{B_{n^{-1}}(x)} h_\kappa^2(z)\,d\sigma(z)\Big)^{-1}$$

and thereby this shows (3.2.12).

Next, we show that

$$\left|V_\kappa\Big[\Psi_n(\langle y,\cdot\rangle)\Big](x)\right| \le C\frac{n^{d-1}(1+n\widetilde{\rho}(x,y))^{-\ell+\delta+3s_\kappa/2-d+1}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle|+\widetilde{\rho}(x,y)+n^{-1})^{2\kappa_v}}. \tag{3.2.16}$$

To see this, we let $m\in\mathbb{N}$ be such that $2^m n^{-1}\sim n^{-1}+\widetilde{\rho}(x,y)$, and then use (3.1.2) to obtain

$$\begin{aligned}\operatorname{meas}_k(B_{n^{-1}+\widetilde{\rho}(x,y)}(x)) &\le C2^{ms_\kappa}\operatorname{meas}_\kappa(B_{n^{-1}}(x))\\ &\le C(1+n\widetilde{\rho}(x,y))^{s_\kappa}\operatorname{meas}_\kappa(B_{n^{-1}}(x)).\end{aligned}$$

The estimate (3.2.16) then follows from (3.2.13) and (3.1.1).

Finally, (3.2.2) follows from (3.2.16). To see this, note that

$$|\langle x,v\rangle-\langle g_0y,v\rangle|\le 2\|x-g_0y\|\le 2\widetilde{\rho}(x,y),\quad \forall v\in\mathcal{R}_+.$$

Hence, if $|\langle x,v\rangle|\le 4\widetilde{\rho}(x,y)$, then $|\langle g_0y,v\rangle|\le c\widetilde{\rho}(x,y)$ and

$$|\langle x,v\rangle|+|\langle g_0y,v\rangle|+\widetilde{\rho}(x,y)+n^{-1}\sim\widetilde{\rho}(x,y)+n^{-1}\sim\widetilde{\rho}(x,y)+n^{-1}+|\langle x,v\rangle|;$$

if $|\langle x,v\rangle|\ge 4\widetilde{\rho}(x,y)$, then $|\langle x,v\rangle|\sim|\langle g_0y,v\rangle|$ and

$$|\langle x,v\rangle|+|\langle g_0y,v\rangle|+\widetilde{\rho}(x,y)+n^{-1}\sim\widetilde{\rho}(x,y)+n^{-1}+|\langle x,v\rangle|.$$

Thus, the RHS of (3.2.2) is equivalent to the RHS of (3.2.16). □

3.3 Proof of HLS inequality

This section is devoted to the proof of Theorem 1.0.1. The proof relies on pointwise estimates of the kernel function K_α, defined by

$$K_\alpha(x,y):=\sum_{j=1}^\infty(j(j+2\lambda_\kappa))^{-\alpha/2}\frac{\lambda_\kappa+j}{\lambda_\kappa}V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x,\cdot\rangle)\Big](y), \tag{3.3.1}$$

for $\alpha>0$ and $x,y\in\mathbb{S}^{d-1}$. Recalling that fractional power of $(-\Delta_{\kappa,0})$ given by (1.0.5), we have that

$$(-\Delta_{\kappa,0})^{-\alpha/2}f(x)=\int_{\mathbb{S}^{d-1}}f(y)K_\alpha(x,y)h_\kappa^2(y)\,d\sigma(y),\quad x\in\mathbb{S}^{d-1}. \tag{3.3.2}$$

Lemma 3.3.1. *For any $x, y \in \mathbb{S}^{d-1}$ and $\alpha > 0$, we have*

$$|K_\alpha(x,y)| \le C \frac{\widetilde{\rho}(x,y)^{\alpha-d+1}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle| + |\langle g_0 y, v\rangle| + \widetilde{\rho}(x,y))^{2\kappa_v}}, \tag{3.3.3}$$

where $g_0 \in G$ is such that $\rho(g_0 x, y) = \tilde{\rho}(x,y)$.

Proof. Let θ be a C^∞-function on $[0,\infty)$ which is supported in $[\frac{1}{2}, 2]$ and has the property that $\sum_{n=0}^\infty \theta(2^{-n}x) = 1$ for all $x \ge 1$. We decompose the kernel K_α as follows:

$$K_\alpha(x,y) = \sum_{n=0}^\infty D_{n,\alpha}(x,y), \tag{3.3.4}$$

where

$$\begin{aligned} D_{n,\alpha}(x,y) &= \sum_{j=0}^\infty \theta(\frac{j}{2^n}) \frac{1}{(j(j+2\lambda_\kappa))^{\alpha/2}} \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x,\cdot\rangle)\Big](y) \\ &= 2^{-n\alpha} \sum_{2^{n-1}\le j\le 2^n} \varphi_n(\frac{j}{2^n}) \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x,\cdot\rangle)\Big](y) \end{aligned} \tag{3.3.5}$$

with

$$\varphi_n(t) = \frac{\theta(t)}{(t(t+2^{-n+1}\lambda_\kappa))^{\alpha/2}}. \tag{3.3.6}$$

Clearly,

$$D_{n,\alpha}(x,y) = 2^{-n\alpha} V_\kappa\Big[\Psi_{\varphi_n,n}(\langle x,\cdot\rangle)\Big](y),$$

with

$$\Psi_{\varphi_n,n}(t) = \sum_{j=0}^\infty \varphi_n(2^{-n}j) \frac{j+\lambda_\kappa}{\lambda_\kappa} C_j^{\lambda_\kappa}(t).$$

Since φ_n is a C^∞ function supported in $[\frac{1}{2}, 2]$ and satisfying

$$\sup_n \|\varphi_n^{(j)}\|_\infty \le C_j < \infty, \quad j = 0, 1, \cdots,$$

it follows by Lemma 2.1.1 that

$$|\Psi_{\varphi_n,n}(\cos t)| \le C 2^{n(2\lambda_\kappa+1)}(1+2^n t)^{-\ell}, \quad \forall \ell > 0, \quad \forall t \in [0,\pi].$$

By Theorem 3.2.1, this yields that for any $\ell > 0$,

$$|D_{n,\alpha}(x,y)| \le C \frac{2^{n(d-1-\alpha)}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle| + |\langle g_0 y, v\rangle| + 2^{-n} + \widetilde{\rho}(x,y))^{2\kappa_v}}. \tag{3.3.7}$$

Thus, by (3.3.4), it follows that for $\ell > d$,

$$|K_\alpha(x,y)| \le \sum_{n=0}^\infty |D_{n,\alpha}(x,y)| \le C \sum_{n=0}^\infty \frac{2^{n(d-1-\alpha)}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle| + |\langle g_0 y, v\rangle| + \widetilde{\rho}(x,y))^{2\kappa_v}}$$
$$\le \frac{C\widetilde{\rho}(x,y)^{\alpha-d+1}}{\prod_{v\in\mathcal{R}_+}(|\langle x,v\rangle| + |\langle g_0 y, v\rangle| + \widetilde{\rho}(x,y))^{2\kappa_v}}.$$

□

Proof of Theorem 1.0.1. Sufficiency. Assume that $\alpha \ge s_\kappa$ and set

$$U(x,y) := \text{meas}_\kappa\Big(B_{\widetilde{\rho}(x,y)}(x)\Big), \qquad x, y \in \mathbb{S}^{d-1}.$$

It is easily seen that $U(gx,y) = U(x,y)$ for all $x, y \in \mathbb{S}^{d-1}$ and $g \in G$. Noticing (2.4.7), by (3.3.2) and Lemma 3.3.1, we have

$$\Big|(-\Delta_{\kappa,0})^{-\alpha/2} f(x)\Big| \le C \int_{\mathbb{S}^{d-1}} |f(y)| \frac{\widetilde{\rho}(x,y)^\alpha}{U(x,y)} h_\kappa^2(y)\, d\sigma(y). \tag{3.3.8}$$

Let $\delta \in (0,\pi]$ be a temporarily fixed positive constant to be specified later. We split the integral in (3.3.8) into two parts: $I_1(x) + I_2(x)$, where

$$I_1(x) := \int_{\tilde{\rho}(x,y)<\delta} \frac{|f(y)|\widetilde{\rho}(x,y)^\alpha}{U(x,y)} h_\kappa^2(y) d\sigma(y),$$
$$I_2(x) := \int_{\tilde{\rho}(x,y)\ge\delta} \frac{|f(y)|\widetilde{\rho}(x,y)^\alpha}{U(x,y)} h_\kappa^2(y) d\sigma(y).$$

A straightforward calculation shows that

$$I_1(x) \le \sum_{k=0}^\infty \frac{(2^{-k}\delta)^\alpha}{\text{meas}_\kappa(B_{2^{-k-1}\delta}(x))} \int_{2^{-k-1}\delta \le \tilde{\rho}(x,y) < 2^{-k}\delta} |f(y)| h_\kappa^2(y) d\sigma(y)$$
$$\le C\delta^\alpha \max_{g\in G} M_\kappa f(gx), \tag{3.3.9}$$

where M_κ denotes the weighted HL maximal function defined as (2.4.7) with weight $w = h_\kappa^2$.

For the term $I_2(x)$, we use Hölder's Inequality to obtain

$$I_2(x) \le \|f\|_{\kappa,p} \left\{ \int_{\tilde{\rho}(x,y)\ge\delta} \frac{\tilde{\rho}(x,y)^{\alpha p'}}{U(x,y)^{p'}} h_\kappa^2(y)\, d\sigma(y) \right\}^{\frac{1}{p'}}. \tag{3.3.10}$$

where $p' = \frac{p}{p-1}$. Let m be the integer such that $2^m\delta \le \pi \le 2^{m+1}\delta$. Splitting the integral $\int_{\tilde\rho(x,y)\ge\delta}\cdots$ in (3.3.10) into a sum $\sum_{k=0}^m \int_{2^k\delta\le\tilde\rho(x,y)\le 2^{k+1}\delta}\cdots$, we obtain

$$\int_{\tilde\rho(x,y)\ge\delta} \frac{\tilde\rho(x,y)^{\alpha p'}}{U(x,y)^{p'}} h_\kappa^2(y)\,d\sigma(y) \le \sum_{k=0}^m \frac{(2^{k+1}\delta)^{\alpha p'}}{\mathrm{meas}_\kappa(B_{2^k\delta}(x))^{p'}} \sum_{g\in G} \mathrm{meas}_\kappa(B_{2^{k+1}\delta}(gx)).$$

Using the G-invariance and doubling property of $h_\kappa^2(x)$, we have

$$\mathrm{meas}_\kappa(B_{2^{k+1}\delta}(gx)) = \mathrm{meas}_\kappa(B_{2^{k+1}\delta}(x)) \le C\,\mathrm{meas}_\kappa(B_{2^k\delta}(x)),$$

and

$$0 < c = \mathrm{meas}_\kappa(\mathbb{S}^{d-1}) = \mathrm{meas}_k(B_\pi(x)) \le C(2^k\delta)^{-s_\kappa}\,\mathrm{meas}_\kappa(B_{2^k\delta}(x)).$$

These yield that when $\alpha \ge s_\kappa(\frac{1}{p} - \frac{1}{q})$,

$$\int_{\tilde\rho(x,y)\ge\delta} \frac{\tilde\rho(x,y)^{\alpha p'}}{U(x,y)^{p'}} h_\kappa^2(y)\,d\sigma(y) \le C\sum_{k=0}^m (2^k\delta)^{\alpha p' - s_\kappa(p'-1)} \le C'\delta^{-s_\kappa p'/q}$$

where C, C' are constants independent of δ, x, y. It follows from (3.3.10) that

$$I_2(x) \le C\|f\|_{\kappa,p}\delta^{-\frac{s_\kappa}{q}}. \tag{3.3.11}$$

Now combining the estimates (3.3.9) and (3.3.11), we obtain that for any $0 < \delta \le \pi$ and $x \in \mathbb{S}^{d-1}$,

$$\left|(-\Delta_{\kappa,0})^{-\alpha/2} f(x)\right| \le C\delta^{s_\kappa(\frac{1}{p}-\frac{1}{q})} \max_{g\in G} M_\kappa f(gx) + C\|f\|_{\kappa,p}\delta^{-\frac{s_\kappa}{q}}. \tag{3.3.12}$$

By optimizing the parameter δ, for instance, setting

$$\delta := \min\Big\{\Big(\frac{\|f\|_{\kappa,p}}{\max_{g\in G} M_\kappa f(gx)}\Big)^{\frac{p}{s_\kappa}}, \pi\Big\},$$

we obtain that,

$$\left|(-\Delta_{\kappa,0})^{-\alpha/2} f(x)\right| \le C\|f\|_{\kappa,p}^{1-\frac{p}{q}} \left(\max_{g\in G} M_\kappa f(gx)\right)^{\frac{p}{q}} + C\|f\|_{\kappa,p},$$

for $x \in \mathbb{S}^{d-1}$. This, using the boundedness of the operator M_κ, yields that

$$\|(-\Delta_{\kappa,0})^{-\alpha/2} f\|_{q,\kappa}^q \le C\|f\|_{\kappa,p}^q \tag{3.3.13}$$

with constant $C > 0$ being independent of f.

Necessity. We now turn to the proof of the optimality of the index s_κ. By (2.3.2), (3.2.4), Lemma 3.2.3, we have that

$$\sum_{j=0}^{n} \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}\langle x, \cdot\rangle\Big](x) \sim \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}, \quad x \in \mathbb{S}^{d-1}. \tag{3.3.14}$$

Thus, if $m = [\varepsilon n] + 1$ for some $\varepsilon \in (0,1)$, then for some absolute constants $c_1, c_2 > 0$,

$$\begin{aligned}\sum_{j=m+1}^{n} \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x, \cdot\rangle)\Big](x) &\ge c_1 \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))} - c_2 \frac{1}{\mathrm{meas}_\kappa(B_{m^{-1}}(x))} \\ &= c_1 \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}\Big[1 - \frac{c_2}{c_1}\frac{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}{\mathrm{meas}_\kappa(B_{m^{-1}}(x))}\Big],\end{aligned} \tag{3.3.15}$$

which, by (3.1.1), is not smaller than

$$c_1 \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}\Big[1 - c'\Big(\frac{m}{n}\Big)^{d-1}\Big] \ge \frac{c_1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}(1 - c'\varepsilon^{d-1}).$$

Thus, there exists a general constant $\delta_0 \in (0,1)$ such that for $n \in \mathbb{N}$,

$$\sum_{\delta_0 n \le j \le n} \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x, \cdot\rangle)\Big](x) \ge c' \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}, \quad x \in \mathbb{S}^{d-1}. \tag{3.3.16}$$

Now let Y denote the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ spanned by certain elements from $\mathcal{R}_+$ such that

$$s_\kappa = d - 1 + 2\sum_{\alpha \in \mathcal{R}_+ \cap Y} \kappa_\alpha.$$

Then there exists a vector $x_0 \in \mathbb{S}^{d-1}$ such that $\langle x_0, \alpha\rangle = 0$ for all $\alpha \in Y \cap \mathcal{R}_+$. This also implies that

$$\min\{|\langle x_0, v\rangle| : \quad v \in \mathcal{R}_+, \quad v \notin Y\} = c > 0,$$

and hence

$$\mathrm{meas}_\kappa(B_t(x_0)) \sim t^{d-1} \prod_{v\in\mathcal{R}_+\cap Y} t^{2\kappa_v} = t^{s_\kappa}, \quad t \in (0,1).$$

Let $\eta \in C^\infty(\mathbb{R})$ be such that $\chi_{[\delta_0,1]} \le \eta \le \chi_{[\delta_0/2,2]}$, and define

$$f_n(x) = \sum_{j=1}^\infty \eta(\frac{j}{n}) \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x_0, \cdot\rangle)\Big](x).$$

Clearly, $f_n \in \Pi_{2n}^d$, by (3.3.14) and (3.3.16),

$$f_n(x_0) \sim \frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x_0))} \sim n^{s_\kappa}$$

and by Cauchy-Swarchz inequality and (3.3.14),

$$\begin{aligned} |f_n(x)| &\le \Big(\sum_{j=1}^\infty \eta(\frac{j}{n}) \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x_0, \cdot\rangle)\Big](x_0)\Big)^{\frac12} \\ &\quad \times \Big(\sum_{j=1}^\infty \eta(\frac{j}{n}) \frac{j+\lambda_\kappa}{\lambda_\kappa} V_\kappa\Big[C_j^{\lambda_\kappa}(\langle x, \cdot\rangle)\Big](x)\Big)^{\frac12} \\ &\le C n^{s_\kappa/2} \Big(\frac{1}{\mathrm{meas}_\kappa(B_{n^{-1}}(x))}\Big)^{\frac12}, \end{aligned}$$

which yields that $|f_n(x)| \le C n^{s_\kappa}$ for $x \in \mathbb{S}^{d-1}$. Thus, by the Bernstein inequality for trigonometric polynomials, there exists $\delta_1 \in (0,1)$ such that

$$|f_n(x)| \ge \frac12 |f_n(x_0)| \ge c n^{s_\kappa}, \qquad \forall x \in B_{\delta_1 n^{-1}}(x_0).$$

This implies that for $1 < p < \infty$,

$$\|f_n\|_{\kappa,p} \ge C n^{s_\kappa} \big(\mathrm{meas}_\kappa(B_{n^{-1}}(x_0))\big)^{\frac1p} \sim n^{s_\kappa(1-\frac1p)}.$$

On the other hand, the Cesàro summability of the spherical h-harmonics also implies that

$$\|f_n\|_{\kappa,1} \le C,$$

which by the log-convexity of the L^p-norm leads to

$$\|f_n\|_{\kappa,p} \le \|f_n\|_{\kappa,1}^{\frac1p} \|f_n\|_{\kappa,\infty}^{1-\frac1p} \le C n^{s_\kappa(1-\frac1p)}.$$

Summing up, we obtain that

$$\|f_n\|_{\kappa,p} \sim n^{s_\kappa(1-\frac{1}{p})}. \tag{3.3.17}$$

By summation by parts and the Cesàro summability of the spherical h-harmonics, this also implies that

$$\|(-\Delta_{\kappa,0})^{-\frac{\alpha}{2}} f_n\|_{\kappa,p} \geq Cn^{-\alpha}\|f_n\|_{\kappa,p} \sim n^{s_\kappa(1-\frac{1}{p})-\alpha}. \tag{3.3.18}$$

Thus, if the HLS inequality (1.0.6) holds for some $\alpha > 0$ and $1 < p < q < \infty$, then

$$C' n^{-\alpha} n^{s_\kappa(1-\frac{1}{q})} \leq \|(-\Delta_{\kappa,0})^{-\frac{\alpha}{2}} f_n\|_{\kappa,q} \leq C\|f_n\|_{\kappa,p} \sim n^{s_\kappa(1-\frac{1}{p})}, \quad \forall n \in \mathbb{N},$$

which implies $-\alpha + s_\kappa(1-\frac{1}{q}) \leq s_\kappa(1-\frac{1}{p})$, and hence $\alpha \geq s_\kappa(\frac{1}{p}-\frac{1}{q})$.

3.4 Examples of the critical index s_κ

In this section, we show how to calculate the index s_κ for the examples of reflection groups G and weights $h_\kappa^2(x)$ given in the first section. We will not consider the simpler case of $G = \mathbb{Z}_2^d$ here. Recall that s_κ is the smallest positive number s such that (3.1.2) holds.

Example 3.4.1. *The case $G = B_d$, the hyperoctahedral group.* In this case, G has a positive root system

$$\mathcal{R}_+ = \{e_i \pm e_j : \quad 1 \leq i < j \leq d\} \cup \{e_i : \quad 1 \leq i \leq d\},$$

and the root system R has a full rank d and the following two orbits under the group G:

$$\begin{aligned} \text{orbit}_G(e_1) &:= \{ge_1 : \quad g \in G\} = \{\pm e_i, i = 1, 2, \cdots, d\}, \\ \text{orbit}_G(e_2 - e_1) &:= \{g(e_2 - e_1) : \quad g \in G\} = \{\pm(e_i \pm e_j) : \quad 1 \leq i \neq j \leq d\}. \end{aligned}$$

This means that each product weight h_κ^2 invariant under the group G is of the form

$$h_\kappa^2(x) = \Big(\prod_{i=1}^d |x_i|^{2\kappa_1}\Big)\Big(\prod_{1\leq i<j\leq d} |x_i^2 - x_j^2|^{2\kappa_2}\Big)$$

for some nonnegative constants κ_1, κ_2. We claim that

$$s_\kappa = \begin{cases} 2 + \max\{6\kappa_2, 4\kappa_1 + 4\kappa_2\}, & \text{if } d = 3; \\ d - 1 + 2\kappa_1(d-1) + 2\kappa_2(d-1)(d-2), & \text{if } d \geq 4. \end{cases} \tag{3.4.1}$$

Proof. It is enough to show that s_κ given in (3.4.1) is the smallest number s for which the following inequality holds at all $x \in \mathbb{S}^{d-1}$ with the constant C independent of x, θ and n:

$$\operatorname{meas}_\kappa(B(x, 2^n\theta)) \leq C 2^{ns} \operatorname{meas}_\kappa(B(x,\theta)), \quad \forall \theta \in (0, \pi), \quad n \in \mathbb{N}, \quad 2^n\theta \leq \pi. \tag{3.4.2}$$

Let $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$ and assume that $\{x_j^*\}_{j=1}^d$ is a decreasing rearrangement of $\{|x_j|\}_{j=1}^d$:

$$x_1^* \geq x_2^* \geq \cdots \geq x_d^*.$$

By (3.1.1), we have, for $\theta \in (0, \pi)$,

$$\begin{aligned} \operatorname{meas}_\kappa(B(x,\theta)) &\sim \theta^{d-1}\Big(\prod_{j=1}^d (|x_j| + \theta)^{2\kappa_1}\Big) \times \\ &\times \Big(\prod_{1\leq i<j\leq d} (|x_i| + |x_j| + \theta)^{2\kappa_2}\Big)\Big(\prod_{1\leq i<j\leq d} \big||x_i| - |x_j| + \theta\big|^{2\kappa_2}\Big) \\ &\sim \theta^{d-1}\Big(\prod_{i=1}^d (x_i^* + \theta)^{2\kappa_1 + 2(d-i)\kappa_2}\Big)\Big(\prod_{1\leq i<j\leq d} \big|x_i^* - x_j^* + \theta\big|^{2\kappa_2}\Big). \end{aligned} \tag{3.4.3}$$

Fix for the moment $x \in \mathbb{S}^{d-1}$ and consider the following two cases:

Case 1. $x_d^* \geq \frac{1}{2} x_1^*$.

In this case, $x_j^* \sim 1$ for all $1 \leq j \leq d$, and hence, (3.4.3) implies that (3.4.2) holds at the point x with

$$s = d - 1 + \kappa_2 d(d-1),$$

and this index s is sharp if $|x_1| = \cdots = |x_d| = \frac{1}{\sqrt{d}}$.

Case 2. $x_d^* \le \frac{1}{2}x_1^*$.

In this case,

$$x_1^* - x_d^* = \sum_{j=1}^{d-1}(x_j^* - x_{j+1}^*) \ge \frac{1}{2}x_1^* \ge \frac{1}{2\sqrt{d}},$$

hence, there exists $1 \le j_0 \le d-1$ such that

$$x_{j_0}^* - x_{j_0+1}^* \ge c_d := \frac{1}{2(d-1)\sqrt{d}}. \tag{3.4.4}$$

We denote by m the largest integer $j_0 \in [1, d-1]$ for which (3.4.4) holds. Then

$$x_1^* \ge \cdots \ge x_m^* \ge c_d > 0$$

and for $1 \le i \le m$ and $m < j \le d$,

$$x_i^* - x_j^* \ge x_m^* - x_{m+1}^* \ge c_d.$$

Thus, (3.4.3) implies that the inequality (3.4.2) holds at the point x with

$$\begin{aligned} s &= d - 1 + 2\sum_{j=m+1}^{d}(\kappa_1 + \kappa_2(d-j)) + \kappa_2 d(d-1) - 2\kappa_2(d-m)m \\ &= d - 1 + \kappa_2 d(d-1) + (d-m)\Big[(d-1-3m)\kappa_2 + 2\kappa_1\Big]. \end{aligned} \tag{3.4.5}$$

Furthermore, this index sharp at x for any $1 \le m \le d$ if

$$x_1^* = \cdots = x_m^* = \frac{1}{\sqrt{m}}, \quad x_{m+1}^* = \cdots = x_d^* = 0.$$

Now, combining the results proved in the above two cases, and setting $n = d - m$ in (4.4.7), we obtain

$$\begin{aligned} s_\kappa &= d - 1 + \kappa_2 d(d-1) + \max_{0 \le n \le d-1} n[2\kappa_1 + (3n - 2d - 1)\kappa_2] \\ &= d - 1 + \kappa_2 d(d-1) + \max_{0 \le n \le d-1}\Big[3\kappa_2 n^2 + n\big(2\kappa_1 - (2d+1)\kappa_2\big)\Big]. \end{aligned}$$

Since $3\kappa_2 \ge 0$, the above maximum is attained at either $n = 0$ or $n = d - 1$.

Thus,

$$s_\kappa = d - 1 + \max\Big\{\kappa_2 d(d-1), 2\kappa_1(d-1) + 2\kappa_2(d-1)(d-2)\Big\}.$$

A straightforward calculation then shows that for $d \geq 4$,

$$s_\kappa = d - 1 + 2\kappa_1(d-1) + 2\kappa_2(d-1)(d-2),$$

while for $d = 3$,

$$s_\kappa = 2 + \max\{6\kappa_2, 4\kappa_1 + 4\kappa_2\}.$$

□

Example 3.4.2. *The case $G = A_{d-1}$ (the symmetric group on d elements). Here the group G has a positive root system $\mathcal{R}_+ = \{e_i - e_j : \ \ 1 \leq i < j \leq d\}$, and the root system $\mathcal{R}$ has rank $d-1$ and one orbit $\mathrm{orbit}_G(e_2 - e_1) = \mathcal{R}$. Hence, every product weight $h_\kappa^2(x)$ in this case can be written in the following form for some $\kappa_0 \geq 0$:*

$$h_\kappa^2(x) = \prod_{1 \leq i < j \leq d} |x_i - x_j|^{2\kappa_0}.$$

Furthermore, by (1.0.7),

$$s_\kappa = d - 1 + 2\sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha = d - 1 + d(d-1)\kappa_0.$$

Chapter 4

Riesz transforms

In the HLS theory, it is most concerned to people when $\alpha = 1$. In particular, at this moment, the inequality can be rewritten as

$$\|f\|_{\kappa,p} \le C\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,q},$$

for certain proper p, q. Motivated by this discussion, in this chapter, we shall introduce two versions of decomposition of Laplace-Beltramic operator, $\Delta_{\kappa,0}$. These lead to a practical replacement of $(-\Delta_{\kappa,0})^{1/2}$ in the sense of the equivalence of the $L^p(h_\kappa^2)$ norm.

4.1 Weighted analogue of the angular derivatives

The angular derivatives $D_{i,j} = x_i\partial_j - x_j\partial_i$, $1 \le i < j \le d$ described in Section 2 have been playing an important role in the theory of ordinary spherical harmonics. These operators are invariant on spaces of ordinary spherical harmonics, and commute with the Laplace-Beltrami operator Δ_0 (see Lemma 2.2.2). Recently, Yuan Xu [Xu3] considered a weighted analogue of these angular derivatives in the Dunkl setting, replacing the partial derivatives ∂_i with the Dunkl operators $\mathcal{D}_j$:

$$\mathcal{D}_{i,j} = x_i\mathcal{D}_j - x_j\mathcal{D}_i = D_{i,j} + E_{i,j}, \qquad 1 \le i < j \le d, \tag{4.1.1}$$

where

$$E_{i,j} = \sum_{v \in \mathcal{R}_+} \Big[x_i \langle v, e_j \rangle - x_j \langle v, e_i \rangle \Big] \kappa_v E_v \tag{4.1.2}$$

and e_j is the jth standard vector. These operators were used to study the uncertainty principle for the spherical h-harmonic expansions and to decompose the Dunkl Laplace-Beltrami operator $\Delta_{\kappa,0}$ in [Xu3]. The decomposition of $\Delta_{\kappa,0}$ in [Xu3, Lemma 3.2] can be written equivalently as follows:

$$\Delta_{\kappa,0} = \sum_{1 \le i < j \le d} \mathcal{D}_{i,j}^2 + \mathcal{T}, \tag{4.1.3}$$

with

$$\mathcal{T} := (d-2) \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha (I - \sigma_\alpha) + \sum_{\alpha,\beta \in \mathcal{R}_+} \kappa_\alpha \kappa_\beta (I - \sigma_\alpha \sigma_\beta). \tag{4.1.4}$$

Here we recall that $\sigma_\alpha f(x) = f(\sigma_\alpha x)$ and I denotes the identity operator. Unlike the decomposition (2.2.9) of the classical Laplace-Beltrami operator Δ_0 on $\mathbb{S}^{d-1}$, the decomposition (4.1.3) contains an extra difference term $\mathcal{T}$, which causes difficulties in applications (see, for instance, [Xu3]).

It was shown in [Xu3] that the operators $\mathcal{D}_{i,j}$ enjoy several important properties similar to those of $D_{i,j}$, including the following useful formula of integration by parts:

$$\int_{\mathbb{S}^{d-1}} \mathcal{D}_{i,j} f(x) g(x) h_\kappa^2(x)\, d\sigma(x) = -\int_{\mathbb{S}^{d-1}} f(x) \mathcal{D}_{i,j} g(x) h_\kappa^2(x)\, d\sigma(x), \tag{4.1.5}$$

for $f, g \in C^1(\mathbb{S}^{d-1})$, and $1 \le i < j \le d$.

One of the most important properties of the operators $\mathcal{D}_{i,j}$ is the fact that they are invariant on each space of spherical h-harmonics, that is, $\mathcal{D}_{i,j} \mathcal{H}_n^d(h_\kappa^2) \subset \mathcal{H}_n^d(h_\kappa^2)$ for each n, which, in particular, implies that the $\mathcal{D}_{i,j}$ commute with all multiplier operators for the spherical h-harmonic expansions. This property is a simple consequence of (4.1.1), (4.1.2) and (4.1.5). Indeed, it is easily seen from (4.1.1) and (4.1.2) that $\mathcal{D}_{i,j} \Pi_n^d \subset \Pi_n^d$. Thus, by (4.1.5) and the orthogonality of spherical h-harmonics, we have that for any $f \in \mathcal{H}_n^d(h_\kappa^2)$ and

$g \in \Pi_{n-1}^d$,

$$\int_{\mathbb{S}^{d-1}} \mathcal{D}_{i,j} f(x) g(x) h_\kappa^2(x)\, d\sigma(x) = -\int_{\mathbb{S}^{d-1}} f(x) \mathcal{D}_{i,j} g(x) h_\kappa^2(x)\, d\sigma(x) = 0,$$

which implies that $\mathcal{D}_{i,j} f$ is in the space $\mathcal{H}_n^d(h_\kappa^2)$, the orthogonal complement of Π_{n-1}^d in the Hilbert space Π_n^d with the inner product of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$.

Our goal in this section is to show the following result, which will be needed in the next section.

Theorem 4.1.1. *If $1 < p < \infty$ and $f \in C^1(\mathbb{S}^{d-1})$, then*

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} \sim \max_{1 \le i < j \le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p}. \tag{4.1.6}$$

The proof of Theorem 4.1.1 relies on several lemmas.

Lemma 4.1.2. *Let $f \in C^1[-1,1]$, and define, for a fixed $y \in \mathbb{S}^{d-1}$,*

$$F(x) := V_\kappa[f(\langle \cdot, y \rangle)](x), \quad x \in \mathbb{S}^{d-1}.$$

Then for $1 \le i < j \le d$,

$$\mathcal{D}_{i,j} F(x) = (x_i y_j - x_j y_i) V_\kappa\Big[f'(\langle \cdot, y \rangle)\Big](x).$$

Proof. Setting $\varphi(z) = f(\langle z, y \rangle)$, we deduce from (2.2.2) that

$$\begin{aligned} \mathcal{D}_{i,j} F(x) &= x_i \mathcal{D}_j V_\kappa \varphi(x) - x_j \mathcal{D}_i V_\kappa \varphi(x) = x_i V_\kappa \partial_j \varphi(x) - x_j V_\kappa \partial_i \varphi(x) \\ &= x_i y_j V_\kappa\Big[f'(\langle \cdot, y \rangle)\Big](x) - x_j y_i V_\kappa\Big[f'(\langle \cdot, y \rangle)\Big](x). \end{aligned}$$

□

Lemma 4.1.3. *Let $f_n \in C^1[-1,1]$ be a sequence of functions satisfying that*

$$|f_n'(\cos t)| \le C_\ell n^{2\lambda_\kappa + 1} (1 + nt)^{-\ell}, \quad t \in [0, \pi] \qquad \forall \ell > 0.$$

If we let

$$F_n(x, y) := V_\kappa\Big[f_n(\langle \cdot, y \rangle)\Big](x), \quad x, y \in \mathbb{S}^{d-1},$$

then for $x, y \in \mathbb{S}^{d-1}$ and $1 \le i < j \le d$,

$$|\mathcal{D}_{i,j}^{(x)} F_n(x,y)| \le C_\ell \frac{n^{d-1}\rho(x,y)(1+n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{v\in\mathcal{R}_+}\Big(|\langle x,v\rangle| + \langle g_0 y, v\rangle| + \widetilde{\rho}(x,y) + n^{-1}\Big)^{2\kappa_v}}, \qquad \forall \ell > 0.$$

where $g_0 \in G$ is such that $\rho(x, g_0 y) = \widetilde{\rho}(x,y)$. Here and throughout, $\mathcal{D}_{i,j}^{(x)}$ means that the operator $\mathcal{D}_{i,j}$ is acting on the variable x.

Proof. The stated estimate follows from Lemma 4.1.2, Theorem 3.2.1 and the fact that

$$|x_i y_j - x_j y_i| \le 2\rho(x,y).$$

□

Lemma 4.1.4. *For $f \in C^1(\mathbb{S}^{d-1})$,*

$$\begin{aligned}&\sum_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,2}^2 &(4.1.7)\\ =&\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,2}^2 + \frac{d-2}{2}\sum_{\alpha\in\mathcal{R}_+}\kappa_\alpha \int_{\mathbb{S}^{d-1}} |f(x)-f(\sigma_\alpha x)|^2 h_\kappa^2(x)\,d\sigma(x)\\ &+\frac{1}{2}\sum_{\alpha,\beta\in\mathcal{R}_+}\kappa_\alpha\kappa_\beta \int_{\mathbb{S}^{d-1}} \Big|f(x) - f(\sigma_\beta\sigma_\alpha x)\Big|^2 h_\kappa^2(x)\,d\sigma(x).\end{aligned}$$

Proof. Using (4.1.5) and the decomposition (4.1.3), we have that

$$\begin{aligned}\|(-\Delta_{\kappa,0})^{1/2} f\|_{\kappa,2}^2 &= \int_{\mathbb{S}^{d-1}} (-\Delta_{\kappa,0} f)(x) f(x) h_\kappa^2(x)\,d\sigma(x)\\ &= \sum_{1\le i<j\le d}\|\mathcal{D}_{i,j} f\|_{\kappa,2}^2 - \int_{\mathbb{S}^{d-1}} \mathcal{T} f(x) f(x) h_\kappa^2(x)\,d\sigma(x). \qquad (4.1.8)\end{aligned}$$

Since the measure $h_\kappa^2(x)\,d\sigma(x)$ is G-invariant, and since for each $g \in G$, we may write

$$f(x) = \frac{f(x)-f(gx)}{2} + \frac{f(x)+f(gx)}{2},$$

it follows that for each $g \in G$,

$$\int_{\mathbb{S}^{d-1}} f(x)(f(x)-f(gx)) h_\kappa^2(x)\,d\sigma(x) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} |f(x)-f(gx)|^2 h_\kappa^2(x)\,d\sigma(x).$$

Thus,

$$\int_{\mathbb{S}^{d-1}} \Big(\sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha (I-\sigma_\alpha) f(x)\Big) f(x)\, h_\kappa^2(x)\, d\sigma(x)$$

$$= \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha \int_{\mathbb{S}^{d-1}} |f(x)-f(\sigma_\alpha x)|^2 h_\kappa^2(x)\, d\sigma(x). \tag{4.1.9}$$

On the other hand, clearly,

$$\int_{\mathbb{S}^{d-1}} \Big(\sum_{\alpha\in\mathcal{R}_+}\sum_{\beta\in\mathcal{R}_+} \kappa_\alpha\kappa_\beta (I-\sigma_\alpha\sigma_\beta) f(x)\Big) f(x) h_\kappa^2(x)\, d\sigma(x)$$

$$=\frac{1}{2}\sum_{\alpha\in\mathcal{R}_+}\sum_{\beta\in\mathcal{R}_+} \kappa_\alpha\kappa_\beta \int_{\mathbb{S}^{d-1}} \Big|f(x)-f(\sigma_\beta\sigma_\alpha x)\Big|^2 h_\kappa^2(x)\, d\sigma(x). \tag{4.1.10}$$

Now substituting (4.1.9) and (4.1.10) into (4.1.4) and (4.1.8) yield the desired identity (4.1.7). This completes the proof. □

We are now in a position to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Let $\theta \in C^\infty[0,\infty)$ be supported in the interval $[\frac{1}{2}, 2]$ and satisfy $\sum_{j=0}^\infty \theta(2^{-j}x) = 1$ for all $x \geq 1$. Let $\widetilde{\theta} \in C^\infty[0,\infty)$ be such that $\chi_{[\frac{1}{2},2]}(t) \leq \widetilde{\theta}(t) \leq \chi_{[\frac{1}{4},4]}(t)$ for $t \geq 0$. Define $L_n = L_{\theta,n}$ and $\widetilde{L}_n = L_{\widetilde{\theta},n}$ as in Definition 2.3.1. Clearly, $L_n = L_n\widetilde{L}_n = \widetilde{L}_n L_n$, and the operators $L_n, \widetilde{L}_n$, $(-\Delta_{\kappa,0})^\gamma$ and $\mathcal{D}_{i,j}$ are all commutative.

Next, we show that for $1 \leq i < j \leq d$,

$$\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} f\|_{\kappa,p} \leq C\|f\|_{\kappa,p}, \quad 1 < p < \infty, \tag{4.1.11}$$

which will imply the inequality

$$\|\mathcal{D}_{i,j} f\|_{\kappa,p} \leq C\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p}.$$

Indeed, using the Littlewood-Paley inequality (2.3.5), and setting $f_n = \widetilde{L}_n f$, we have

$$\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} f\|_{\kappa,p} \sim \Big\|\Big(\sum_{n=0}^{\infty} |L_n \mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} f|^2\Big)^{1/2}\Big\|_{\kappa,p}$$
$$= \Big\|\Big(\sum_{n=0}^{\infty} |\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} L_n f_n|^2\Big)^{1/2}\Big\|_{\kappa,p}.$$

For each nonnegative integer n, we may write

$$\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} L_n f_n(x) := \int_{\mathbb{S}^{d-1}} f_n(y) \mathcal{D}_{i,j}^{(x)} \Big(V_\kappa \Big[G_n(\langle \cdot, y\rangle)\Big](x)\Big) h_\kappa^2(y)\, d\sigma(y), \tag{4.1.12}$$

with

$$G_n(t) = 2^{-n} \sum_{k=0}^{\infty} \varphi_n(2^{-n}k) \frac{k+\lambda_\kappa}{\lambda_\kappa} C_k^{\lambda_\kappa}(t)$$

and $\varphi_n(s) = \theta(s)(s(s+2^{-n+1}\lambda_\kappa))^{-\frac{1}{2}}$. Since $\sup_n \|\varphi_n^{(j)}\|_\infty \le C_j < \infty$, it follows by Lemma 2.1.1 that

$$|G_n'(\cos u)| \le 2^{n(2\lambda_\kappa+2)}(1+2^n u)^{-\ell}, \quad \forall \ell > 0, \quad u \in (0,\pi),$$

which, using Lemma 4.1.3, in turn implies that for any $\ell > 0$,

$$\Big|\mathcal{D}_{i,j} V_\kappa \Big[G_n(\langle \cdot, y\rangle)\Big](x)\Big| \le C \frac{2^{n(d-1)}(2^n \rho(x,y))(1+2^n \widetilde{\rho}(x,y))^{-\ell}}{\prod_{v\in\mathcal{R}_+} \big(|\langle x,v\rangle| + \widetilde{\rho}(x,y) + 2^{-n}\big)^{2\kappa_v}}. \tag{4.1.13}$$

Thus, combining (4.1.13) with (4.1.12), we obtain by a straightforward calculation that

$$\Big|(\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} L_n f_n(x)\Big| \le C \max_{g\in G} M_\kappa f_n(gx),$$

where M_κ denotes the weighted HL maximal function given in (2.4.8) with $w = h_\kappa^2$. Since $d\mu(x) = h_\kappa^2(x) d\sigma(x)$ is a doubling Radon measure on $\mathbb{S}^{d-1}$, it follows by (2.3.5) and the Fefferman-Stein inequality that

$$\Big\|\Big(\sum_{n=0}^{\infty} |(\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}} L_n f_n)|^2\Big)^{1/2}\Big\|_{\kappa,p} \le C \Big\|\Big(\sum_{n=0}^{\infty} |M_\kappa f_n|^2\Big)^{1/2}\Big\|_{\kappa,p}$$
$$\le C \Big\|\Big(\sum_{n=0}^{\infty} |f_n|^2\Big)^{1/2}\Big\|_{\kappa,p} \le C\|f\|_{\kappa,p}.$$

Finally, we show the inverse inequality

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}}f\|_{\kappa,p} \le C \max_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,p}. \tag{4.1.14}$$

This can be deduced from (4.1.11) and (4.1.7) via a duality argument. Indeed, let $g \in L^{p'}(h_\kappa^2;\mathbb{S}^{d-1})$ be such that $\|g\|_{\kappa,p'} \le C$ and

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}}f\|_{\kappa,p} = \int_{\mathbb{S}^{d-1}} \Big[(-\Delta_{\kappa,0})^{\frac{1}{2}}f(y)\Big] g(y)h_\kappa^2(y)\,d\sigma(y).$$

Here and elsewhere, $\frac{1}{p}+\frac{1}{p'}=1$. Without loss of generality, we may assume that $\int_{\mathbb{S}^{d-1}} g(y)h_\kappa^2(y)\,d\sigma(y)=0$, replacing g with $\widetilde{g}(x)=g(x)-\int_{\mathbb{S}^{d-1}} g(y)h_\kappa^2(y)\,d\sigma(y)$ otherwise. It then follows by (4.1.3) and (4.1.5) that

$$\begin{aligned}\|(-\Delta_{\kappa,0})^{\frac{1}{2}}f\|_{\kappa,p} &= \int_{\mathbb{S}^{d-1}} \Big[(-\Delta_{\kappa,0})f(y)\Big]\Big[(-\Delta_{\kappa,0})^{-\frac{1}{2}}g(y)\Big] h_\kappa^2(y)\,d\sigma(y)\\ &= \sum_{1\le i<j\le d} \int_{\mathbb{S}^{d-1}} \mathcal{D}_{i,j}f(y)\Big[\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}g(y)\Big] h_\kappa^2(y)\,d\sigma(y)\\ &\quad - \int_{\mathbb{S}^{d-1}} \mathcal{T}f(y)\Big[(-\Delta_{\kappa,0})^{-\frac{1}{2}}g(y)\Big] h_\kappa^2(y)\,d\sigma(y),\end{aligned}$$

which, using Hölder's inequality, (4.1.11) and that fact that $\|(-\Delta_{\kappa,0})^{-\frac{1}{2}}g\|_{\kappa,p'} \le C\|g\|_{\kappa,p'} \le C$, is bounded above by

$$\begin{aligned}&\sum_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,p}\|\mathcal{D}_{i,j}(-\Delta_{\kappa,0})^{-\frac{1}{2}}g\|_{\kappa,p'} + \|\mathcal{T}f\|_{\kappa,p}\|(-\Delta_{\kappa,0})^{-\frac{1}{2}}g\|_{\kappa,p'}\\ &\le C \max_{1\le i<j\le d} \|\mathcal{D}_{i,j}f\|_{\kappa,p} + C\|\mathcal{T}f\|_{\kappa,p}.\end{aligned}$$

To estimate the term $\|\mathcal{T}f\|_{\kappa,p}$, let $\eta \in C^\infty[0,\infty)$ be such that $\eta(x)=1$ for $x\in[0,1]$ and $\eta(x)=0$ for $x\ge 2$, and define

$$\mathcal{L}_n^\kappa f := \sum_{j=0}^{2n} \eta(n^{-1}j)\,\mathrm{proj}_j(h_\kappa^2;f).$$

It is well known that (see [DaXu4, Lemma 10.2.4])

$$\|f-\mathcal{L}_n^\kappa f\|_{\kappa,p} \le Cn^{-r}\|(-\Delta_{\kappa,0})^{r/2}f\|_{\kappa,p}, \quad r>0. \tag{4.1.15}$$

Without loss of generality, we may assume that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\,d\sigma(x) = 0$.

Since $\mathcal{T}$ is bounded on $L^p(h_\kappa^2;\mathbb{S}^{d-1})$, it follows that for each $n_0 \in \mathbb{N}$,

$$\|\mathcal{T}f\|_{\kappa,p} \le C\|f - \mathcal{L}_{n_0}^\kappa f\|_{\kappa,p} + C\|\mathcal{L}_{n_0}^\kappa f\|_{\kappa,p} \le Cn_0^{-1}\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} + C\|\mathcal{L}_{n_0}^\kappa f\|_{\kappa,p}$$
$$\le Cn_0^{-1}\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} + C_{n_0}\|\mathcal{L}_{n_0}^\kappa f\|_{\kappa,2},$$

where we used the equivalence of different norms in a finite-dimensional vector space in the last step. To estimate the term $\|\mathcal{L}_{n_0}^\kappa f\|_{\kappa,2}$, we use Lemma 4.1.4 and obtain

$$\|\mathcal{L}_{n_0}^\kappa f\|_{\kappa,2} \le \|(-\Delta_{\kappa,0})^{\frac{1}{2}}\mathcal{L}_{n_0}^\kappa f\|_{\kappa,2} \le \sum_{1\le i<j\le d} \|\mathcal{D}_{i,j}\mathcal{L}_{n_0}^\kappa f\|_{\kappa,2}$$
$$\le C \max_{1\le i<j\le d} \|\mathcal{L}_{n_0}^\kappa \mathcal{D}_{i,j} f\|_{\kappa,2} \le C \max_{1\le i<j\le d} \|\mathcal{L}_{n_0}^\kappa \mathcal{D}_{i,j} f\|_{\kappa,p} \le C \max_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p},$$

where we used the equivalence of different norms in the finite-dimensional space $\Pi_{2n_0}^d$ in the fourth step, and the boundedness of the operator $\mathcal{L}_{n_0}^\kappa$ in the last step.

Putting the above together, we deduce

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} \le Cn_0^{-1}\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,p} + C_{n_0} \max_{1\le i<j\le d} \|\mathcal{D}_{i,j} f\|_{\kappa,p}.$$

Now choosing $n_0 \in \mathbb{N}$ large enough so that $\frac{1}{4} \le Cn_0^{-1} \le \frac{1}{2}$, we obtain the desired inverse inequality (4.1.14).

4.2 A new decomposition of Dunkl-Laplace-Beltrami operator

It turns out that the decomposition (4.1.3) of $\Delta_{\kappa,0}$ obtained in [Xu3] is not enough for the proof of our main result, Theorem 1.0.5. Our main goal in this section is to prove the following new decomposition of $\Delta_{\kappa,0}$, which will play a crucial role in the next section when we prove Theorem 1.0.5.

Theorem 4.2.1. *For $x \in \mathbb{S}^{d-1}$,*

$$\Delta_{\kappa,0} = \sum_{1\le i<j\le d} \frac{D_{i,j} h_\kappa^2(x) D_{i,j}}{h_\kappa^2(x)} - 2\sum_{\alpha\in\mathcal{R}_+} \frac{\kappa_\alpha}{\langle \alpha, x\rangle} E_\alpha. \tag{4.2.1}$$

Furthermore, if $f, g \in C^2(\mathbb{S}^{d-1})$, then

$$\langle(-\Delta_{\kappa,0})f, g\rangle_{L^2(h_\kappa^2)} = \int_{\mathbb{S}^{d-1}} \langle\nabla_0 f, \nabla_0 g\rangle h_\kappa^2(x)\, d\sigma(x) + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha \langle E_\alpha f, E_\alpha g\rangle_{L^2(h_\kappa^2)}, \tag{4.2.2}$$

where $\langle\cdot,\cdot\rangle_{L^2(h_\kappa^2)}$ denotes the inner product of $L^2(h_\kappa^2;\mathbb{S}^{d-1})$. In particular, this implies

$$\|(-\Delta_{\kappa,0})^{\frac{1}{2}} f\|_{\kappa,2}^2 = \|\nabla_0 f\|_{\kappa,2}^2 + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha \|E_\alpha f\|_{\kappa,2}^2. \tag{4.2.3}$$

The significance of the decomposition (4.2.1) lies in the fact that each term on the right hand side of (4.2.1) is self-adjoint in $L^2(h_\kappa^2;\mathbb{S}^{d-1})$ and relatively easier to deal with.

Proof. We start with the proof of the decomposition (4.2.1). A straightforward calculation shows that

$$\begin{aligned}\frac{D_{i,j} h_\kappa^2(x) D_{i,j}}{h_\kappa^2(x)} &= \frac{(D_{i,j} h_\kappa^2(x)) D_{i,j}}{h_\kappa^2(x)} + D_{i,j}^2 \\ &= D_{i,j}^2 + \sum_{\alpha\in\mathcal{R}_+} \frac{2\kappa_\alpha}{\langle x, \alpha\rangle}\Big[x_j^2\alpha_i\partial_i - x_i x_j \alpha_i \partial_j - x_i x_j \alpha_j \partial_i + x_i^2 \alpha_j \partial_j\Big].\end{aligned}$$

Hence, by (2.2.9), it follows that for $x \in \mathbb{S}^{d-1}$

$$\begin{aligned}\sum_{1\le i<j\le d} \frac{D_{i,j} h_\kappa^2(x) D_{i,j}}{h_\kappa^2(x)} &= \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \frac{D_{i,j} h_\kappa^2(x) D_{i,j}}{h_\kappa^2(x)} \\ &= \Delta_0 + 2\sum_{\alpha\in\mathcal{R}_+} \frac{\kappa_\alpha \langle\alpha, \nabla\rangle}{\langle x, \alpha\rangle} - 2|\kappa|\langle x, \nabla\rangle,\end{aligned}$$

where $\nabla = (\partial_1, \cdots, \partial_d)$, and $|\kappa| = \sum_{v\in\mathcal{R}_+} \kappa_v$.

Now let $f \in C^2(\mathbb{S}^{d-1})$ and define $F(z) = f(z/\|z\|)$ for $\|z\| > 0$. Since F is a radial function,

$$\langle x, \nabla\rangle F(x) = D_x F(x) = 0, \quad x \in \mathbb{S}^{d-1},$$

where D_x denotes the directional derivative in the direction of $x \in \mathbb{S}^{d-1}$. Thus, by Lemma 2.2.2 (i),

$$\sum_{1\le i<j\le d}\frac{D_{i,j}h_\kappa^2(x)D_{i,j}f(x)}{h_\kappa^2(x)}=\Delta_0 f(x)+2\sum_{\alpha\in\mathcal{R}_+}\frac{\kappa_\alpha\langle\alpha,\nabla\rangle F(x)}{\langle x,\alpha\rangle},\quad x\in\mathbb{S}^{d-1}. \tag{4.2.4}$$

Now using (2.2.6) and (2.2.8), we obtain that for $x\in\mathbb{S}^{d-1}$,

$$\Delta_{\kappa,0}f(x)=\Delta_\kappa F(x)=\Delta_0 f(x)+2\sum_{\alpha\in\mathcal{R}_+}\kappa_\alpha\Big[\frac{\langle\nabla F(x),\alpha\rangle}{\langle x,\alpha\rangle}-\frac{E_\alpha f(x)}{\langle x,\alpha\rangle}\Big],$$

which, together with (4.2.4), yields the desired decomposition (4.2.1).

Finally, we show the identity (4.2.2). First, we observe that by (4.4),

$$\langle h_\kappa^{-2}D_{i,j}h_\kappa^2 D_{i,j}f,g\rangle_{L^2(h_\kappa^2)}=-\langle D_{i,j}f,D_{i,j}g\rangle_{L^2(h_\kappa^2)}. \tag{4.2.5}$$

Next, writing

$$g(x)=\frac{g(x)+g(\sigma_\alpha x)}{2}+\frac{g(x)-g(\sigma_\alpha x)}{2},$$

and using symmetry, we get that for any $\varepsilon\in(0,1)$,

$$\begin{aligned}&\int_{\{x\in\mathbb{S}^{d-1}:|\langle x,\alpha\rangle|\ge\varepsilon\}}\frac{f(x)-f(\sigma_\alpha x)}{\langle x,\alpha\rangle^2}g(x)h_\kappa^2(x)\,d\sigma(x)\\&=\frac12\int_{\{x\in\mathbb{S}^{d-1}:|\langle x,\alpha\rangle|\ge\varepsilon\}}\frac{f(x)-f(\sigma_\alpha x)}{\langle x,\alpha\rangle}\frac{g(x)-g(\sigma_a x)}{\langle x,\alpha\rangle}h_\kappa^2(x)\,d\sigma(x).\end{aligned}$$

Letting $\varepsilon\to 0$ yields

$$\int_{\mathbb{S}^{d-1}}\frac{2E_\alpha f(x)}{\langle\alpha,x\rangle}g(x)h_\kappa^2(x)\,d\sigma(x)=\langle E_\alpha f,E_\alpha g\rangle_{L^2(h_\kappa^2)}. \tag{4.2.6}$$

Substituting (4.2.5) and (4.2.6) into (4.2.1), and using (2.2.11), we deduce the identity (4.2.2).

□

4.3 Riesz transform operators and their boundedness

Throughout this section, the letter v denotes a fixed element in $\mathcal{R}_+$ with $\kappa_v>0$. For simplicity, we write $T=E_v(-\Delta_{\kappa,0})^{-\frac12}$. To prove Theorem 1.0.5, by (4.1.1), and Theorem 4.1.1, it suffices to show that for $1 < p < \infty$,

$$\|Tf\|_{\kappa,p} \le C_p \|f\|_{\kappa,p}. \tag{4.3.1}$$

We start with the case of $1 < p \le 2$. For $p = 2$, the inequality (4.3.1) follows directly from (4.2.3). Thus, to prove (4.3.1) for $1 < p < 2$, by the Riesz-Thorin theorem, it suffices to show that

$$\mathrm{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad |Tf(x)| > \alpha \Big\} \le C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{4.3.2}$$

The proof of (4.3.2) relies on the Calderon-Zygmund decomposition. Indeed, by the integral representation (3.3.2) , we may write

$$Tf(x) = \int_{\mathbb{S}^{d-1}} f(y) K(x,y) h_\kappa^2(y) \, d\sigma(y), \tag{4.3.3}$$

where

$$K(x,y) = \frac{K_1(x,y) - K_1(\sigma_v x, y)}{2\langle x, v\rangle}, \quad x, y \in \mathbb{S}^{d-1} \tag{4.3.4}$$

and $K_1(x,y)$ is given in (3.3.1) with $\alpha = 1$. Recall that $\widetilde{\rho}(x,y) = \min_{g\in G} \rho(gx, y)$ for $x, y \in \mathbb{S}^{d-1}$. For a spherical cap $B(x,t) \subset \mathbb{S}^{d-1}$, we write

$$\widetilde{B}(x,t) = \bigcup_{g \in G} B(gx, t),$$

whereas for $E \subset \mathbb{S}^{d-1}$, write ${}^cE := \mathbb{S}^{d-1} \setminus E$.

The following integral estimates of the kernel $K(x,y)$ will play a crucial role in the proof of (4.3.1) and (4.3.2) :

Proposition 4.3.1. *Let $K(x,y)$ be the kernel given in (7.1.6) with $v \in \mathcal{R}_+$ and $\kappa_v > 0$. Then for all $y \in \mathbb{S}^{d-1}$ and $t \in (0, \pi)$,*

$$\int_{{}^c\widetilde{B}(y,2t)} |K(x,y) - K(x,y')| h_\kappa^2(x) d\sigma(x) \le A, \quad \forall y' \in B(y,t) \tag{4.3.5}$$

and

$$\int_{{}^c\widetilde{B}(y,2t)} |K(y,x) - K(y',x)| h_\kappa^2(x) d\sigma(x) \le A, \quad \forall y' \in B(y,t), \tag{4.3.6}$$

where A is a constant independent of y, y' and t.

The proof of Proposition 4.3.1 is long and technical, so we postpone it until next two subsections.

Once Proposition 4.3.1 is proved, then (4.3.2) can be deduced by slightly modifying the standard technique of the Calderon-Zygmund decomposition. For completeness, we sketch the proof of (4.3.2) as follows. Without loss of generality, we may assume that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\,d\sigma(x) = 0$. We then apply the Calderon-Zygmund decomposition of f at the height $\alpha > 0$:

$$f = g + b = g + \sum_{j=1}^{\infty} b_j,$$

where the following conditions are satisfied:

- $|g(x)| \le C\alpha$ for a.e. $x \in \mathbb{S}^{d-1}$;
- For $j = 1, 2, \cdots$, we have that supp $b_j \subset B_j := B(y_j, t_j)$, $\int_{B_j} b_j(y)h_\kappa^2(y)\,d\sigma(y) = 0$ and
$$\frac{1}{\text{meas}_\kappa(B_j)} \int_{B_j} |b_j(y)|h_\kappa^2(y)\,d\sigma(y) \le C\alpha;$$
- $$\sum_{j=1}^{\infty} \text{meas}_\kappa(B_j) \le C\frac{\|f\|_{\kappa,1}}{\alpha}.$$

Set

$$B_j^* = \widetilde{B}(y_j, 2t_j) = \bigcup_{g\in G} B(gy_j, 2t_j), \quad j = 1, 2, \cdots,$$

and let $\Omega := \bigcup_{j=1}^{\infty} B_j^*$. Then by the G-invariance and the doubling property of the weight $h_\kappa^2(x)$, we have

$$\text{meas}_\kappa(\Omega) \le C\sum_{j=1}^{\infty} \text{meas}_\kappa(B_j) \le C\frac{\|f\|_{\kappa,1}}{\alpha}.$$

On the other hand, however, using (4.3.5), we obtain

$$\begin{aligned}\int_{{}^cB_j^*} |Tb_j(x)|h_\kappa^2(x)\,d\mu(x) &= \int_{{}^cB_j^*} \Big|\int_{B_j} b_j(y)\big(K(x,y) - K(x,y_j)\big)h_\kappa^2(y)\,d\sigma(y)\Big|h_\kappa^2(x)\,d\sigma(x)\\ &\le \int_{B_j} |b_j(y)|\Big[\int_{{}^cB_j^*} \big|K(x,y) - K(x,y_j)\big|h_\kappa^2(x)\,d\sigma(x)\Big]h_\kappa^2(y)\,d\sigma(y)\\ &\le C\alpha\,\text{meas}_\kappa(B_j).\end{aligned}$$

Thus,

$$\begin{aligned}&\text{meas}_\kappa\{x\in\mathbb{S}^{d-1}:\quad |Tf(x)|>\alpha\}\le \text{meas}_\kappa\{x\in\mathbb{S}^{d-1}:\quad |Tg(x)|>\alpha/2\}\\ &+\text{meas}_\kappa\{x\in {}^c\Omega:\quad |Tb(x)|>\alpha/2\}+\text{meas}_\kappa(\Omega)\\ &\le C\frac{\|g\|_{\kappa,2}^2}{\alpha^2}+C\frac{1}{\alpha}\sum_{j=1}^\infty\int_{{}^cB_j^*}|Tb_j(x)|h_\kappa^2(x)\,d\sigma(x)+C\frac{\|f\|_{\kappa,1}}{\alpha}\le C\frac{\|f\|_{\kappa,1}}{\alpha}.\end{aligned}$$

This proves (4.3.2), and hence (4.3.1) for $1<p\le 2$.

Finally, we show (4.3.1) for $2<p<\infty$. Let T^* be the dual operator of T. Then by (4.3.3),

$$T^*f(x)=\int_{\mathbb{S}^{d-1}}K(y,x)f(y)h_\kappa^2(y)\,d\sigma(y).$$

Repeating the above argument, and using (4.3.6) rather than (4.3.5), we obtain

$$\|T^*f\|_{\kappa,p}\le C\|f\|_{\kappa,p},\quad 1<p<2,$$

which, by duality, implies (4.3.1) for $2<p<\infty$.

4.3.1 Proof of Proposition 4.3.1: the estimate (4.3.6)

This subsection is devoted to the proof of (4.3.6). Let φ be a C^∞-function on $[0,\infty)$ supported in $[\frac{1}{2},2]$ and satisfying $\sum_{n=0}^\infty\varphi(2^{-n}x)=1$ for all $x\ge 1$. Set

$$A_n(t)=2^{-n}\sum_{j=1}^\infty\varphi_n(\frac{j}{2^n})\frac{\lambda_\kappa+j}{\lambda_\kappa}C_j^{\lambda_\kappa}(t)\quad\text{with}\quad\varphi_n(x)=\frac{\varphi(x)}{\sqrt{x(x+2^{-n+1}\lambda_\kappa)}}.$$

We then decompose the kernel $K(x,y)$ as follows:

$$K(x,y)=\sum_{n=0}^\infty K_{n,v}(x,y),\tag{4.3.7}$$

where

$$K_{n,v}(x,y)=\frac{V_\kappa\big[A_n(\langle y,\cdot\rangle)\big](x)-V_\kappa\big[A_n(\langle y,\cdot\rangle)\big](\sigma_v x)}{2\langle x,v\rangle}.\tag{4.3.8}$$

Obviously, by (2.2.4),

$$K_{n,v}(x,y) = \frac{V_\kappa\Big[A_n(\langle x, \cdot\rangle) - A_n(\langle \sigma_v x, \cdot\rangle)\Big](y)}{2\langle x, v\rangle}, \tag{4.3.9}$$

whereas by Lemma 2.1.1,

$$|A_n^{(j)}(\cos\theta)| \le C2^{n(2\lambda_\kappa+2j)}(1+2^n\theta)^{-\ell}, \quad j=0,1,2, \quad t\in[0,\pi], \quad \forall \ell>0, \tag{4.3.10}$$

The proof of (4.3.6) relies on several lemmas.

Lemma 4.3.2. *For $x, y \in \mathbb{S}^{d-1}$,*

$$|K_{n,v}(x,y)| \le C\frac{2^{n(d-1)}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \widetilde{\rho}(x,y) + 2^{-n})^{2\kappa_\alpha}}, \quad \forall \ell > 0.$$

Proof. We consider the following three cases.

Case 1. $|\langle x, v\rangle| \le \frac{\widetilde{\rho}(x,y)}{12\pi^2}$.

In this case, we first claim that for $z \in \widehat{G}_y$, $\theta := \arccos\langle x, z\rangle$ and $\theta' := \arccos\langle \sigma_v x, z\rangle$, one has

$$|\cos\theta - \cos\theta'| \le 4\theta|\langle x, v\rangle|, \quad \text{and} \quad \theta\sim\theta'. \tag{4.3.11}$$

For the moment, we take the claim (4.3.11) for granted and proceed with the proof of (4.3.6). By the mean value theorem, for each fixed $z \in \widehat{G}_y$, there exists a number t between $\theta := \arccos\langle x, z\rangle$ and $\theta' := \arccos\langle \sigma_v x, z\rangle$, such that

$$\frac{A_n(\langle x, z\rangle) - A_n(\langle \sigma_v x, z\rangle)}{\langle x, v\rangle} = \frac{A_n(\cos\theta) - A_n(\cos\theta')}{\langle x, v\rangle}$$
$$= A_n'(\cos t)\frac{\cos\theta - \cos\theta'}{\langle x, v\rangle} = 2A_n'(\cos t)\langle z, v\rangle.$$

It then follows by (4.3.10) and (4.3.11) that for any $\ell > 0$,

$$|A_n(\langle x, z\rangle) - A_n(\langle \sigma_v x, z\rangle)|$$
$$\le C2^{n(2\lambda_\kappa+2)}(1+2^n\theta)^{-\ell-1}\theta|\langle x, v\rangle| \le C2^{n(2\lambda_\kappa+1)}(1+2^n\theta)^{-\ell}|\langle x, v\rangle|.$$

In other words, for each $z \in \widehat{G}_y$, we have

$$\frac{|A_n(\langle x, z\rangle) - A_n(\langle \sigma_v x, z\rangle)|}{|\langle x, v\rangle|} \le C N_{n,\ell}(\langle x, z\rangle), \quad \forall \ell > 0.$$

where

$$N_{n,\ell}(\cos\theta) = 2^{n(2\lambda_\kappa+1)}(1 + 2^n\theta)^{-\ell}.$$

The stated estimate in this case then follows by (3.2.2) and Theorem 2.2.1.

It remains to show the claim (4.3.11). Since

$$\max\{\cos\theta, \cos\theta'\} \le \max_{g\in G} \langle x, gy\rangle = \cos\widetilde{\rho}(x, y),$$

we have $\theta, \theta' \ge \widetilde{\rho}(x, y)$. Without loss of generality, we may assume that $\theta, \theta' > 0$, since otherwise, $\langle x, v\rangle = \widetilde{\rho}(x, y) = 0$, $\theta = \theta'$ and (4.3.11) holds trivially. Since

$$\|z - x\|^2 = 1 + \|z\|^2 - 2\langle z, x\rangle \le 2(1 - \cos\theta) \le \theta^2,$$

it follows that

$$|\langle z, v\rangle| \le |\langle x, v\rangle| + \|z - x\| \le 2\theta. \tag{4.3.12}$$

Thus,

$$\begin{aligned} |\cos\theta - \cos\theta'| &= |\langle x, z\rangle - \langle \sigma_v x, z\rangle| \\ &= 2|\langle x, v\rangle||\langle z, v\rangle| \le 4\theta|\langle x, v\rangle|. \end{aligned}$$

This proves the first part of (4.3.11). Finally, to show that $\theta \sim \theta'$, without loss of generality, we may assume that $\theta + \theta' \le \frac{3\pi}{2}$, since otherwise $\theta, \theta' \ge \frac{\pi}{2}$ and there's nothing to prove. Using the inequality $\sin t \ge \frac{2t}{3\pi}$ for $0 \le t \le \frac{3\pi}{4}$, we have

$$\begin{aligned} |\cos\theta - \cos\theta'| &= 2\sin\frac{\theta + \theta'}{2}\Big|\sin\frac{\theta - \theta'}{2}\Big| \\ &\ge \frac{2\theta}{3\pi^2}|\theta - \theta'|. \end{aligned}$$

This combined with (4.3.11) yields that

$$|\theta - \theta'| \le 6\pi^2|\langle x, v\rangle| \le \frac{1}{2}\widetilde{\rho}(x, y) \le \frac{1}{2}\theta.$$

It then follows that $\theta \sim \theta'$.

Case 2. $|\langle x, v\rangle| \geq \frac{\widetilde{\rho}(x,y)}{12\pi^2}$ and $|\langle x, v\rangle| \geq 2^{-n}$.

In this case, we apply Theorem 3.2.1 directly to obtain that

$$\begin{aligned}
|K_{n,v}(x,y)| :\leq & C\frac{\Big|V_\kappa\big[A_n(\langle x,\cdot\rangle)\big](y)\Big| + \Big|V_\kappa\big[A_n(\langle \sigma_v x,\cdot\rangle)\big](y)\Big|}{\widetilde{\rho}(x,y)+2^{-n}} \\
\leq & C\frac{1}{\widetilde{\rho}(x,y)+2^{-n}}\frac{2^{(d-2)n}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}} \\
\leq & C\frac{2^{(d-1)n}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}}.
\end{aligned}$$

Case 3. $\frac{\widetilde{\rho}(x,y)}{12\pi^2} \leq |\langle x, v\rangle| \leq 2^{-n}$.

Following the notation in Case 1, we set $\theta^* = \min\{\theta, \theta'\}$ for a fixed $z \in \widehat{G}_y$, where $\theta = \arccos(x\cdot z)$ and $\theta' = \arccos(\sigma_v x \cdot z)$. By the mean value theorem, there exists t between θ and θ' such that

$$\begin{aligned}
&|A_n(\langle x,z\rangle) - A_n(\langle \sigma_v x, z\rangle)| = |A_n'(\cos t)||\cos\theta - \cos\theta'| \\
&\leq C2^{n(2\lambda_\kappa+2)}(1+2^n\theta^*)^{-\ell-1}|\langle z,v\rangle||\langle x,v\rangle|. \qquad (4.3.13)
\end{aligned}$$

Since $|\langle z,v\rangle| \leq |\langle x,v\rangle| + \theta$ and $|\langle z,v\rangle| \leq |\langle \sigma_v x,v\rangle| + \theta' = |\langle x,v\rangle| + \theta'$, it follows that $|\langle z,v\rangle| \leq |\langle x,v\rangle| + \theta^*$. Thus, the term on the right hand side of (4.3.13) is bounded above by a constant multiple of

$$2^{n(2\lambda_\kappa+2)}(1+2^n\theta^*)^{-\ell-1}(2^{-n}+\theta^*)|\langle x,v\rangle| \leq 2^{n(2\lambda_\kappa+1)}(1+2^n\theta^*)^{-\ell}|\langle x,v\rangle|.$$

Putting the above together, we obtain in this case that for any $z \in \widehat{G}_y$,

$$\frac{|A_n(\langle x,z\rangle) - A_n(\langle \sigma_v x, z\rangle)|}{|\langle x,v\rangle|} \leq CN_n(\langle x,z\rangle) + CN_n(\langle \sigma_v x,z\rangle).$$

The stated estimate in this case follows again from Theorem 3.2.1 and Theorem 2.2.1.

□

In the sequel, we use the notation $D^{(x)}$ to mean that an operator D is acting on the variable x.

Lemma 4.3.3. *For $x, y \in \mathbb{S}^{d-1}$, and any $\ell > 0$,*

$$\left|\nabla_0^{(x)}\Big[V_\kappa\Big[A_n(\langle x,\cdot\rangle)\Big](y)\right| \le C\frac{2^{(d-1)n}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}}. \quad (4.3.14)$$

Proof. By (2.2.11), it suffices to show the estimate (4.3.14) with the tangential gradient $\nabla_0^{(x)}$ being replaced by the angular derivatives $D_{i,j}^{(x)}$, $1 \le i < j \le d$. Without loss of generality, we may assume that $i = 1$ and $j = 2$.

Using Theorem 2.2.1, we have

$$D_{1,2}^{(x)}\Big[V_\kappa\Big[A_n(\langle x,\cdot\rangle)\Big](y) = \int_{\widehat{G}_y} A_n'(\langle x,z\rangle)(x_2z_1 - x_1z_2)\,d\mu_y(z). \quad (4.3.15)$$

However, for each fixed $z \in \widehat{G}_y$,

$$|x_2z_1 - x_1z_2| \le 2\|x - z\| \le 2\theta := 2\arccos\langle x, z\rangle,$$

and hence, by (4.3.10),

$$\begin{aligned}&\Big|A_n'(\langle x,z\rangle)(x_2z_1 - x_1z_2)\Big| \le C2^{n(2\lambda_\kappa+2)}(1+2^n\theta)^{-\ell-1}\theta\\ &\le C2^{n(2\lambda_\kappa+1)}(1+2^n\theta)^{-\ell} = CN_{n,\ell}(\langle x,z\rangle).\end{aligned}$$

It follows that

$$\left|D_{1,2}^{(x)}\Big[V_\kappa\Big[A_n(\langle x,\cdot\rangle)\Big](y)\right| \le CV_\kappa\Big[N_{n,\ell}(\langle x,\cdot\rangle)\Big](y), \quad \forall \ell > 0,$$

which, by Theorem 3.2.1, implies the stated estimate. □

Lemma 4.3.4. *Let $x \in \mathbb{S}^{d-1}$ and $z \in \mathbb{B}^d$. If $f \in C^2[-1,1]$, then*

$$\begin{aligned}&D_{1,2}^{(x)}\Big[\frac{f(\langle x,z\rangle) - f(\langle \sigma_v x, z\rangle)}{2\langle x,v\rangle}\Big]\\ &\quad = \langle z,v\rangle\int_0^1 f''\Big(\langle x, \alpha(z,s,v)\rangle\Big)\Big[x_1\alpha_2(z,s,v) - x_2\alpha_1(z,s,v)\Big]\,ds, \quad (4.3.16)\end{aligned}$$

where $\alpha_j(z,s,v)$ denotes the j-th component of the vector

$$\alpha(z,s,v) := sz + (1-s)\sigma_v z. \quad (4.3.17)$$

Proof. A straightforward calculation shows that for $f \in C^1[-1,1]$,

$$\frac{f(\langle x,z\rangle)-f(\langle \sigma_v x,z\rangle)}{2\langle x,v\rangle}=\langle z,v\rangle\int_0^1 f'\Big(\langle x,\alpha(z,s,v)\rangle\Big)\,ds.$$

(4.3.16) then follows directly from this last equation and the definition of $D_{1,2}$. $\square$

Lemma 4.3.5. *For* $x,y\in\mathbb{S}^{d-1}$,

$$|\nabla_0^{(x)}K_{n,v}(x,y)|\le C\frac{2^{nd}(1+2^n\widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}},\quad \forall \ell>0.$$

Proof. Again, by (2.2.11), it suffices to show the stated estimate for the angular derivative $D_{1,2}^{(x)}$ instead of the tangential gradient $\nabla_0^{(x)}$. By (4.3.16) and (4.3.9),

$$\begin{aligned} D_{1,2}^{(x)}K_{n,v}(x,y)&=\int_{\widehat{G}_y}D_{1,2}^{(x)}\Big(\frac{A_n(\langle x,z\rangle)-A_n(\langle\sigma_v x,z\rangle)}{2\langle x,v\rangle}\Big)\,d\mu_y(z)\\ &=\int_{\widehat{G}_y}S(A_n'')(x,z)\,d\mu_y(z), \end{aligned}\tag{4.3.18}$$

where

$$S(A_n'')(x,z):=\langle z,v\rangle\int_0^1 A_n''\Big(\langle x,\alpha(z,s,v)\rangle\Big)\big[x_1\alpha_2(z,s,v)-x_2\alpha_1(z,s,v)\big]\,ds.$$

As in the proof of Lemma 4.3.2, we consider the following three cases:

Case 1. $|\langle x,v\rangle|\le\frac{\widetilde{\rho}(x,y)}{12\pi^2}$.

For a fixed $z\in\widehat{G}_y$, we set $\theta=\arccos\langle x,z\rangle$, and $\theta'=\arccos\langle\sigma_v x,z\rangle$. From the proof of Lemma 4.3.2, we know that $|\langle z,v\rangle|\le 2\theta$, $\|x-z\|\le\theta$, and $\theta\sim\theta'$. Let $\theta''=\theta(x,z,s)=\arccos\langle x,\alpha(z,s,v)\rangle$. Then

$$\cos\theta''=s\langle x,z\rangle+(1-s)\langle x,\sigma_v z\rangle=s\cos\theta+(1-s)\cos\theta'.$$

This means that $\cos\theta''$ is between $\cos\theta$ and $\cos\theta'$. It follows that $\theta\sim\theta''$, and $\|x-\alpha(z,s,v)\|\le\theta''\le C\theta$. Thus, using (4.3.10), we obtain that for any $\ell>0$,

$$|SA_n''(x,z)|\le C2^{n(2\lambda_\kappa+4)}(1+2^n\theta)^{-\ell-2}\theta^2\le C2^{n(2\lambda_\kappa+2)}(1+2^n\theta)^{-\ell}.$$

The stated estimate in this case then follows by (4.3.18) and (3.2.2).

Case 2. $|\langle x, v\rangle| \geq \frac{\widetilde{\rho}(x,y)}{12\pi^2}$ and $|\langle x, v\rangle| \geq 2^{-n}$.

For simplicity, we set

$$B_n(x, y) = V_\kappa\Big[A_n(\langle x, \cdot\rangle) - A_n(\langle \sigma_v x, \cdot\rangle)\Big](y).$$

By product rule, it follows that

$$|D_{1,2}^{(x)} K_{n,v}(x, y)| \leq |K_{n,v}(x, y)| \frac{|x_2 v_1 - x_1 v_2|}{|\langle x, v\rangle|} + \frac{|D_{1,2}^{(x)} B_n(x, y)|}{|\langle x, v\rangle|}. \tag{4.3.19}$$

By Lemma 4.3.2, the first term on the right hand side of (4.3.19) is bounded above by a constant multiple of

$$\frac{2^{(d-1)n}(1 + 2^n \widetilde{\rho}(x, y))^{-\ell-1}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x, \alpha\rangle| + \widetilde{\rho}(x, y) + 2^{-n})^{2\kappa_\alpha}} \frac{1}{\widetilde{\rho}(x, y) + 2^{-n}},$$

whereas the second term on the right hand side of (4.3.19) is bounded above by

$$\frac{\Big|D_{1,2}^{(x)} V_\kappa\Big[A_n(\langle x, \cdot\rangle)\Big](y)\Big| + \Big|D_{1,2}^{(x)} V_\kappa\Big[A_n(\langle x, \cdot\rangle)\Big](\sigma_v y)\Big|}{|\langle x, v\rangle|},$$

which, using Lemma 4.3.3, is in turn estimated above by

$$C \frac{2^{(d-1)j}(1 + 2^n \widetilde{\rho}(x, y))^{-\ell}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x, \alpha\rangle| + \widetilde{\rho}(x, y) + 2^{-n})^{2\kappa_\alpha}} \frac{1}{\widetilde{\rho}(x, y) + 2^{-n}}.$$

Putting the above together, we deduce the desired estimate in this second case.

Case 3. $\frac{\widetilde{\rho}(x,y)}{12\pi^2} \leq |\langle x, v\rangle| \leq 2^{-n}$.

Given a fixed $z \in \widehat{G}_y$, let $\theta := \arccos\langle x, z\rangle$, $\theta' =: \arccos\langle \sigma_v x, z\rangle$, and set $\theta^* = \min\{\theta, \theta'\}$. Let $\theta'' = \arccos\langle x, \alpha(z, s, v)\rangle$ for $s \in [0, 1]$. Then $\cos\theta'' = s\cos\theta + (1-s)\cos\theta'$, and hence $\theta'' \geq \theta^*$. Since

$$\Big|x_1\alpha_2(z, s, v) - x_2\alpha_1(z, s, v)\Big| \leq 2\|x - \alpha(z, s, v)\| \leq 2\theta'',$$

it follows that

$$\begin{aligned}&|A_n''(\langle x, \alpha(z,s,v)\rangle)|\big|x_1\alpha_2(z,s,v) - x_2\alpha_1(z,s,v)\big|\\ &\le C2^{n(2\lambda_\kappa+4)}(1+2^n\theta'')^{-\ell-2}\theta'' \le C2^{n(2\lambda_\kappa+3)}(1+2^n\theta'')^{-\ell-1}\\ &\le C2^{n(2\lambda_\kappa+3)}(1+2^n\theta^*)^{-\ell-1}.\end{aligned}$$

Also, note that in this case

$$|\langle z, v\rangle| \le 2^{-n} + \theta^*.$$

Therefore, we conclude in this case that for any $z \in \widehat{G}_y$, and any $\ell > 0$,

$$\begin{aligned}|SA_n''(x,z)| &\le C(2^{-n}+\theta^*)2^{n(2\lambda_\kappa+3)}(1+2^n\theta^*)^{-\ell-1}\\ &\le C2^{n(2\lambda_\kappa+2)}(1+2^n\theta^*)^{-\ell}\\ &\le C2^n\Big[N_{n,\ell}(\langle x,z\rangle) + CN_{n,\ell}(\langle\sigma_v x, z\rangle)\Big].\end{aligned}$$

The stated estimate in this case then follows by (4.3.18) and Theorem 3.2.1. □

Now substituting the estimates in Lemma 4.3.6 into the decomposition (4.3.7) , we deduce the following estimate:

Lemma 4.3.6. *If $x, y \in \mathbb{S}^{d-1}$ and $\widetilde{\rho}(x,y) \neq 0$, then*

$$|\nabla_0^{(x)} K(x,y)| \le \frac{C}{\widetilde{\rho}(x,y)^d \prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \widetilde{\rho}(x,y))^{2\kappa_\alpha}}.$$

We are now in a position to prove the estimate (4.3.6).

Proof of (4.3.6). Assume that $y \in \mathbb{S}^{d-1}$, $y' \in B(y,t)$ and $x \in \mathbb{S}^{d-1}\backslash\widetilde{B}(y,2t)$. By the mean value theorem, there exists $y'' \in B(y,t)$ such that

$$|K(y,x) - K(y',x)| \le |\nabla_0^{(y'')}K(y'',x)|\rho(y,y'),$$

which, using Lemma 4.3.6, is estimated above by a constant multiple of

$$\frac{\rho(y,y')}{\widetilde{\rho}(x,y'')^d\prod_{\alpha\in\mathcal{R}_+}(|\langle y'',\alpha\rangle| + \widetilde{\rho}(x,y''))^{2\kappa_\alpha}} \sim \frac{\rho(y,y')}{\widetilde{\rho}(x,y'')^d\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \widetilde{\rho}(x,y''))^{2\kappa_\alpha}}.$$

However, since $x \notin \widetilde{B}(y, 2t)$, we have

$$\rho(x, gy'') \sim \rho(x, gy) \geq 2t, \quad \forall g \in G.$$

Thus,

$$\widetilde{\rho}(x, y'') \sim \widetilde{\rho}(x, y),$$

and

$$|K(y,x) - K(y',x)| \leq \frac{Ct}{\widetilde{\rho}(x,y)^d \prod_{\alpha \in \mathcal{R}_+} (|\langle x, \alpha \rangle| + \widetilde{\rho}(x,y))^{2\kappa_\alpha}}. \tag{4.3.20}$$

It follows that the integral on the left hand side of (4.3.6) can be estimated above by a constant multiple of

$$\begin{aligned} &t \sum_{g \in G} \int_{\rho(x,gy) \geq 2t} \frac{h_\kappa^2(x)\, d\sigma(x)}{\rho(x,gy)^d \prod_{\alpha \in \mathcal{R}_+} (|\langle x, \alpha \rangle| + \rho(x,gy))^{2\kappa_\alpha}} \\ &\leq C(\#G) t \int_{\rho(x,y) \geq 2t} \frac{1}{\rho(x,y)^d}\, d\sigma(x) \leq A < \infty. \end{aligned}$$

4.3.2 Proof of Proposition 4.3.1: the estimate (4.3.5)

This subsection is devoted to the proof of (4.3.5). We will keep the notations of the last subsection.

Lemma 4.3.7. *If $x, y \in \mathbb{S}^{d-1}$, then*

$$|\nabla_0^{(y)} K_{n,v}(x,y)| \leq C \frac{2^{(d-1)n}(1 + 2^n \widetilde{\rho}(x,y))^{-\ell}}{|\langle x, v \rangle| \prod_{\alpha \in \mathcal{R}_+} (|\langle x, \alpha \rangle| + \widetilde{\rho}(x,y) + 2^{-n})^{2\kappa_\alpha}}. \tag{4.3.21}$$

If , in addition, $|\langle x, v \rangle| \geq c2^{-n}$, then for any $\ell > 0$,

$$|\nabla_0^{(y)} K_{n,v}(x,y)| \leq C \frac{2^{nd}(1 + 2^n \widetilde{\rho}(x,y))^{-\ell}}{\prod_{\alpha \in \mathcal{R}_+} (|\langle x, \alpha \rangle| + \widetilde{\rho}(x,y) + 2^{-n})^{2\kappa_\alpha}}, \tag{4.3.22}$$

Proof. Without loss of generality, we may assume $\widetilde{\rho}(x,y) = \rho(x,y)$, since otherwise we may replace x by $g_0 x$ for some $g_0 \in G$.

First, assuming $|\langle x, v \rangle| \geq c2^{-n}$, we prove the estimate (4.3.22). It is enough to show this estimate with the tangential gradient ∇_0 being replaced by the angular derivative $D_{1,2}$. Using Lemma 4.3.6 and the fact that $K_{n,v}(x,y) =$

$K_{n,v}(y,x)$, we have

$$\Big|D_{1,2}^{(y)}\Big(\frac{K_{n,v}(x,y)\langle x,v\rangle}{\langle y,v\rangle}\Big)\Big| \le C\frac{2^{nd}(1+2^n\widetilde{\rho}(x,y))^{-\ell-1}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}}. \quad (4.3.23)$$

It then follows by the product rule and Lemma 4.3.2 that

$$\begin{aligned}
&|D_{1,2}^{(y)}K_{n,v}(x,y)|\\
\le&\Big|D_{1,2}^{(y)}\Big(\frac{K_{n,v}(x,y)\langle x,v\rangle}{\langle y,v\rangle}\Big)\Big|\frac{|\langle y,v\rangle|}{|\langle x,v\rangle|}+\Big|\frac{K_{n,v}(x,y)\langle x,v\rangle}{\langle y,v\rangle}\Big|\frac{|y_2v_1-y_1v_2|}{|\langle x,v\rangle|}\\
\le&\frac{C2^{n(d-1)}(1+2^n\widetilde{\rho}(x,y))^{-\ell-1}}{\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y)+2^{-n})^{2\kappa_\alpha}}\Big[2^n\frac{|\langle y,v\rangle|}{|\langle x,v\rangle|}+\frac{|y_2v_1-y_1v_2|}{|\langle x,v\rangle|}\Big].
\end{aligned}$$

On the other hand, however, recalling that $\widetilde{\rho}(x,y)=\rho(x,y)$ and $|\langle x,v\rangle|\ge c2^{-n}$, we have

$$\frac{|\langle y,v\rangle|}{|\langle x,v\rangle}\le\frac{|\langle x,v\rangle|+\widetilde{\rho}(x,y)}{|\langle x,v\rangle|}\le C(1+2^n\widetilde{\rho}(x,y)),$$

and

$$\frac{|y_1v_2-y_2v_1|}{|\langle x,v\rangle|}\le C\frac{1}{|\langle x,v\rangle|}\le C2^n.$$

Therefore, putting the above estimates together, we obtain the desired estimate (4.3.22) under the assumption $|\langle x,v\rangle|\ge c2^{-n}$.

Finally, we prove (4.3.21). Indeed, by (4.3.8), we have

$$|D_{1,2}^{(y)}K_{n,v}(x,y)|\le\frac{\Big|D_{1,2}^{(y)}V_\kappa\big[A_n(\langle y,\cdot\rangle)\big](x)\Big|+\Big|D_{1,2}^{(y)}V_\kappa\big[A_n(\langle y,\cdot\rangle)\big](\sigma_v x)\Big|}{2|\langle x,v\rangle|},$$

which, using Lemma 4.3.3, yields the desired estimate (4.3.21). □

As a direct consequence of Lemma 4.3.7 , we have

Lemma 4.3.8. *For $x,y\in\mathbb{S}^{d-1}$,*

$$|\nabla_0^{(y)}K(x,y)|\le\frac{C}{\widetilde{\rho}(x,y)^{d-1}|\langle x,v\rangle|\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y))^{2\kappa_\alpha}}. \quad (4.3.24)$$

If, in addition, $|\langle x,v\rangle|\ge c\widetilde{\rho}(x,y)$ for some $c>0$, then

$$|\nabla_0^{(y)}K(x,y)|\le\frac{C}{\widetilde{\rho}(x,y)^{d}\prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle|+\widetilde{\rho}(x,y))^{2\kappa_\alpha}}. \quad (4.3.25)$$

We are now in a position to prove (4.3.5) .

Proof of (4.3.5) . Assume for the moment that $x, y \in \mathbb{S}^{d-1}$, $y' \in B(y,t)$ and $x \in \mathbb{S}^{d-1} \setminus \widetilde{B}(y,2t)$. By the mean value theorem, there exists $y'' \in B(y,t)$ such that

$$|K(x,y) - K(x,y')| \le \|\nabla_0^{(y)} K(x,y'')\| \rho(y,y').$$

Since $x \notin \widetilde{B}(y,2t)$, $\rho(x,y) \sim \widetilde{\rho}(x,y'')$. Thus, if $|\langle x, v\rangle| \ge c\widetilde{\rho}(x,y)$, then $|\langle x,v\rangle| \ge c\widetilde{\rho}(x,y'')$, and hence, (4.3.25) is applicable to obtain

$$|K(x,y) - K(x,y')| \le \frac{C\rho(y,y')}{\widetilde{\rho}(x,y)^d \prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \widetilde{\rho}(x,y))^{2\kappa_\alpha}}. \tag{4.3.26}$$

Similarly, if $|\langle x,v\rangle| \le c\widetilde{\rho}(x,y)$, then we may use (4.3.24) to get

$$|K(x,y) - K(x,y')| \le \frac{C\rho(y,y')}{\widetilde{\rho}(x,y)^{d-1}|\langle x,v\rangle| \prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \widetilde{\rho}(x,y))^{2\kappa_\alpha}}. \tag{4.3.27}$$

Now write

$$\begin{aligned}
&\int_{{}^c\widetilde{B}(y,2t)} |K(x,y) - K(x,y')| h_\kappa^2(x) d\sigma(x) \\
&= \int_{\left\{x\in {}^c\widetilde{B}(y,2t):\quad |\langle x,v\rangle| \ge c\widetilde{\rho}(x,y)\right\}} \cdots + \int_{\left\{x\in {}^c\widetilde{B}(y,2t):\quad |\langle x,v\rangle| < c\widetilde{\rho}(x,y)\right\}} \cdots \\
&= I + II.
\end{aligned}$$

For the first integral I, as in the proof of (4.3.6), it is straightforward to deduce from (4.3.26) that $I \le A$. For the second integral II, we use (4.3.27) to obtain

$$\begin{aligned}
II &\le Ct \sum_{g\in G} \int_{\rho(x,gy)\ge \max\{2t, c|\langle x,v\rangle|\}} \frac{\rho(x,gy)^{-d+1} h_\kappa^2(x)\, d\sigma(x)}{|\langle x,v\rangle| \prod_{\alpha\in\mathcal{R}_+}(|\langle x,\alpha\rangle| + \rho(x,gy))^{2\kappa_\alpha}} \\
&\le Ct \sup_{z\in\mathbb{S}^{d-1}} \int_{\rho(x,z)\ge\max\{2t, c|\langle x,v\rangle|\}} \frac{|\langle x,v\rangle|^{2\kappa_v - 1}}{\rho(x,z)^{d-1+2\kappa_v}}\, d\sigma(x).
\end{aligned} \tag{4.3.28}$$

Here, we recall that $\kappa_v > 0$. Fix for the moment $z \in \mathbb{S}^{d-1}$, and set

$$J_j := \{x \in \mathbb{S}^{d-1} : \quad 2^j t \le \rho(x,z) \le 2^{j+1}t, \quad |\langle x, v\rangle| \le C2^j t\}, \quad j = 1, 2, \cdots .$$

Note that if $J_j \neq \emptyset$ and $x \in J_j$, then

$$|\langle z, v\rangle| \le |\langle x, v\rangle| + \rho(x,z) \le C2^j t,$$

and hence, by (2.4.7),

$$\int_{J_j} |\langle x, v\rangle|^{2\kappa_v - 1}\, d\sigma(x) = \int_{B(z, 2^{j+1}t)\cap J_j} |\langle x, v\rangle|^{2\kappa_v - 1}\, d\sigma(x) \le C(2^j t)^{2\kappa_v + d - 2}.$$

It follows that for each $z \in \mathbb{S}^{d-1}$,

$$\int_{\rho(x,z)\ge \max\{2t, c|\langle x,v\rangle|\}} \frac{|\langle x, v\rangle|^{2\kappa_v-1}}{\rho(x,z)^{d-1+2\kappa_v}}\, d\sigma(x) \le C\sum_{j=1}^{\infty} (2^j t)^{-(d-1+2\kappa_v)} \int_{J_j} |\langle x, v\rangle|^{2\kappa_v-1}\, d\sigma(x)$$

$$\le C\sum_{j=1}^{\infty} (2^j t)^{-(d-1+2\kappa_v)} (2^j t)^{2\kappa_v + d - 2} \le Ct^{-1},$$

which together with (4.3.28) implies that $II \le A$. This competes the proof of (4.3.5).

4.4 Uncertainty principle on the weighted sphere

[1] In this section, motivated by the new decomposition of the Dunkl-Laplace-Beltrami operator in section 4.2 , we study the uncertainty principle for spherical h-harmonic expansions, which is in full analogy with the classical Heisenberg inequality.

The uncertainty principle is a fundamental result in quantum mechanics, and it can be formulated in the Euclidean space $\mathbb{R}^d$, in the form of the classical Heisenberg inequality, as

$$\inf_{a\in\mathbb{R}^d} \int_{\mathbb{R}^d} \|x - a\|^2 |f(x)|^2 dx \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx \ge \frac{d^2}{4}\left(\int_{\mathbb{R}^d} |f(x)|^2 dx\right)^2, \quad (4.4.1)$$

where ∇ is the gradient operator. There are many papers devoted to the

[1] A version of this section has been accepted for publication [Fe].

study of this inequality and its various generalizations, for instance [FoSi], [Ro1], [DaXu6].

In particular, on the unit sphere, F. Dai and Y. Xu [DaXu6] established the analogue result, which states that: if $f : \mathbb{S}^{d-1} \to \mathbb{R}$ satisfying $\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = 0$ and $\int_{\mathbb{S}^{d-1}} |f(x)|^2 d\sigma(x) = 1$,

$$\Big(\min_{y\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} (1-\langle x,y\rangle)|f(x)|^2\, d\sigma(x)\Big)\Big(\int_{\mathbb{S}^{d-1}} |\nabla_0 f|^2 d\sigma(x)\Big) \geq C_d > 0, \quad (4.4.2)$$

where ∇_0 is the tangential gradient operator as before.

In another paper by [Xu3], with a weight function $h_\kappa^2(x)$ invariant under a group G, Y. Xu studied the uncertainty principle on the unit sphere $\mathbb{S}^{d-1}$. By introducing a weighted analogue $\nabla_{\kappa,0}$ of the tangential gradient ∇_0, he [Xu3, Theorem 4.1] proved that if $f : \mathbb{S}^{d-1} \to \mathbb{R}$ is invariant under the group G and satisfies that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\, d\sigma(x) = 0$ and $\int_{\mathbb{S}^{d-1}} |f(x)|^2 h_\kappa^2(x) d\sigma(x) = 1$, then

$$\Big(\min_{1\leq i\leq d} \int_{\mathbb{S}^{d-1}} (1-\langle x,e_i\rangle)|f(x)|^2 h_\kappa^2(x)\, d\sigma(x)\Big)\Big(\int_{\mathbb{S}^{d-1}} |\nabla_{\kappa,0} f|^2 h_\kappa^2(x) d\sigma(x)\Big) \geq C_{\kappa,d} > 0. \quad (4.4.3)$$

where e_i, $i = 1, \cdots, d$, is the standard vector,namely only the ith coordinate is nonzero 1, and $C_{\kappa,d}$ is a constant only depends on parameter κ, d, and $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^d$.

Rather than the finite subset $\{e_1, \cdots, e_d\}$, we shall show that the inequality (4.4.3) with minimum being taken over all $y \in \mathbb{S}^{d-1}$ remains true without the extra assumption that f is G-invariant. Precisely, our main result can be stated as follows:

Theorem 4.4.1. *Let $f \in C^1(\mathbb{S}^{d-1})$ be such that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\, d\sigma(x) = 0$ and $\int_{\mathbb{S}^{d-1}} |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) = 1$. Then*

$$\Big[\min_{y\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} (1-\langle x,y\rangle)|f(x)|^2 h_\kappa^2(x)\, d\sigma(x)\Big]\times$$
$$\times\Big[\int_{\mathbb{S}^{d-1}} |\sqrt{-\Delta_{\kappa,0}} f(x)|^2 h_\kappa^2(x)\, d\sigma(x)\Big] \geq C_{\kappa,d} > 0. \quad (4.4.4)$$

As a direct corollary , we obtain the following improvement of Theorem 4.1 and Theorem 4.2 of [Xu3]:

Corollary 4.4.2. *If $f \in C^1(\mathbb{S}^{d-1})$ satisfies that $\int_{\mathbb{S}^{d-1}} f(x)h_\kappa^2(x)\, d\sigma(x) = 0$ and*

$\int_{\mathbb{S}^{d-1}} |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) = 1$, *then*

$$\Big(\min_{y\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} (1-\langle x,y\rangle)|f(x)|^2 h_\kappa^2(x)\, d\sigma(x)\Big)\big(\int_{\mathbb{S}^{d-1}} |\nabla_{\kappa,0} f|^2 h_\kappa^2 d\sigma(x)\big) \geq C_{\kappa,d} > 0. \tag{4.4.5}$$

For the moment, we take Theorem 4.4.1 for granted and proceed with the proof of Corollary 4.4.2.

Proof. By (4.4.4), it suffices to show

$$\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2} \leq \|\nabla_{\kappa,0} f\|_{\kappa,2}. \tag{4.4.6}$$

Indeed, noticing (3.15), (3.13) of [Xu3], we have that

$$\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2}^2 = \|\nabla_{h,0} f\|_{\kappa,2}^2 - \frac{2\lambda_\kappa}{\omega_d^\kappa} \int_{\mathbb{S}^{d-1}} (\xi \cdot \nabla_{h,0} f(\xi)) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi), \tag{4.4.7}$$

where $\omega_d^\kappa = \int_{\mathbb{S}^{d-1}} h_\kappa^2(x)\, d\sigma(x)$. Here it should be pointed that the last two terms in (3.15) of [Xu3] in fact can be cancelled out by realising that

$$(I-\sigma_v)^2 = 2(I-\sigma_v), \quad \forall v \in \mathcal{R}_+.$$

Furthermore, by (3.3) of [Xu3], we obtain

$$\int_{\mathbb{S}^{d-1}} (\xi \cdot \nabla_{h,0} f(\xi)) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi) = \sum_{v\in\mathcal{R}_+} \kappa_v \int_{\mathbb{S}^{d-1}} (f(\xi) - f(\sigma_v \xi)) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi).$$

However, by the Cauchy-Schwartz inequality,

$$\int_{\mathbb{S}^{d-1}} f(x) f(\sigma_v x) h_\kappa^2(x)\, d\sigma(x) \leq \|f\|_{\kappa,2}^2, \quad \forall v \in \mathcal{R}_+.$$

Thus,

$$\int_{\mathbb{S}^{d-1}} \Big(\xi \cdot \nabla_{h,0}) f(\xi)\Big) f(\xi) h_\kappa^2(\xi)\, d\sigma(\xi) \geq 0.$$

The desired inequality (4.4.6) then follows by (4.4.7). □

Our proof crucially relies on the following lemma.

Lemma 4.4.3. *If $f \in C^1(\mathbb{S}^{d-1})$ and $y \in \mathbb{S}^{d-1}$, then*

$$\Big(\frac{d-1}{2} + |\kappa|\Big) \int_{\mathbb{S}^{d-1}} \langle x, y\rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) = \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \langle y, \alpha\rangle \int_{\mathbb{S}^{d-1}} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha\rangle}\, d\sigma(x)$$
$$- \int_{\mathbb{S}^{d-1}} \Big[\sum_{i=1}^d \sum_{j=1}^d x_j y_i D_{i,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x), \qquad (4.4.8)$$

where $x_j = \langle x, e_j\rangle$ and $y_j = \langle y, e_j\rangle$.

Proof. By noticing that for $f, g \in C^1(\mathbb{S}^{d-1})$ and $i \neq j$,

$$\int_{\mathbb{S}^{d-1}} f(x) D_{i,j} g(x) d\sigma(x) = -\int_{\mathbb{S}^{d-1}} D_{i,j} f(x) g(x) d\sigma(x),$$

we obtain that for $2 \le j \le d$,

$$\int_{\mathbb{S}^{d-1}} \Big[x_j D_{1,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x) = -\int_{\mathbb{S}^{d-1}} f(x) \Big[D_{1,j} f(x)\Big] x_j h_\kappa^2(x)\, d\sigma(x)$$
$$- \int_{\mathbb{S}^{d-1}} |f(x)|^2 \Big[D_{1,j}\big(x_j h_\kappa^2(x)\big)\Big]\, d\sigma(x).$$

A straightforward calculation shows that

$$D_{1,j}\big(x_j h_\kappa^2(x)\big) = \Big(x_1 + x_1 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha x_j \alpha_j}{\langle x, \alpha\rangle} - x_j^2 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha \alpha_1}{\langle x, \alpha\rangle}\Big) h_\kappa^2(x),$$

where $\alpha_j = \langle \alpha, e_j\rangle$. Thus,

$$2\int_{\mathbb{S}^{d-1}} \Big[x_j D_{1,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} |f(x)|^2 x_j^2 \Big(\sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha \alpha_1}{\langle x, \alpha\rangle}\Big) h_\kappa^2(x)\, d\sigma(x)$$
$$- \int_{\mathbb{S}^{d-1}} |f(x)|^2 \Big[x_1 + x_1 \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha x_j \alpha_j}{\langle x, \alpha\rangle}\Big] h_\kappa^2(x)\, d\sigma(x)$$

Summing this last equation over $j = 2, \cdots, d$ yields

$$\int_{\mathbb{S}^{d-1}} \Big[\sum_{j=2}^d x_j D_{1,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} |f(x)|^2 \sum_{\alpha \in \mathcal{R}_+} \frac{\kappa_\alpha \alpha_1}{\langle x, \alpha\rangle} h_\kappa^2(x)\, d\sigma(x)$$
$$- \Big(|\kappa| + \frac{d-1}{2}\Big) \int_{\mathbb{S}^{d-1}} x_1 |f(x)|^2 h_\kappa^2(x)\, d\sigma(x).$$

In general, for $1 \le i \le d$, recalling $D_{i,i} = 0$, and using symmetry, we obtain

$$\int_{\mathbb{S}^{d-1}} \Big[\sum_{j=1}^d x_j D_{i,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} |f(x)|^2 \sum_{\alpha\in\mathcal{R}_+} \frac{\kappa_\alpha \alpha_i}{\langle x, \alpha\rangle} h_\kappa^2(x)\, d\sigma(x)$$
$$- \Big(|\kappa| + \frac{d-1}{2}\Big) \int_{\mathbb{S}^{d-1}} x_i |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) d\sigma(x). \tag{4.4.9}$$

Multiplying both sides of (4.4.9) by y_i and summing the resulting equation over $i = 1, \cdots, d$ yield the desired identity (4.4.8). □

We are now in a position to prove Theorem 4.4.1 .

Proof of Theorem 4.4.1. Let $\varepsilon \in (0, 1)$ be a small absolute constant to be specified later. If

$$\int_{\mathbb{S}^{d-1}} \langle x, y\rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) \le 1 - \varepsilon,$$

then

$$\int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y\rangle) h_\kappa^2(x)\, d\sigma(x) \ge \varepsilon,$$

and (4.4.4) holds trivially as $\|\sqrt{-\Delta_{\kappa,0}} f\|_{\kappa,2} \ge \|f\|_{\kappa,2} = 1$. Thus, without loss of generality, we may assume that

$$\int_{\mathbb{S}^{d-1}} \langle x, y\rangle |f(x)|^2 h_\kappa^2(x)\, d\sigma(x) > 1 - \varepsilon. \tag{4.4.10}$$

We will use the identity (4.4.8). Indeed, it will be shown that

$$J_1 := \Big|\int_{\mathbb{S}^{d-1}} \Big[\sum_{i=1}^d \sum_{j=1}^d y_i x_j D_{i,j} f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x)\Big|$$
$$\le C \|\nabla_0 f\|_{\kappa,2} \Big(\int_{\mathbb{S}^{d-1}} |f(x)|^2 (1 - \langle x, y\rangle) h_\kappa^2(x)\, dx\Big)^{\frac{1}{2}} \tag{4.4.11}$$

and that for each $\alpha \in \mathcal{R}_+$ with $\kappa_\alpha > 0$,

$$J_2(\alpha) := \Big|\langle y, \alpha\rangle \int_{\mathbb{S}^{d-1}} \frac{|f(x)|^2 h_\kappa^2(x)}{\langle x, \alpha\rangle}\, d\sigma(x)\Big| \\ \le \frac{1}{1-\varepsilon} + \frac{C}{\varepsilon}\|E_\alpha f\|_{\kappa,2}\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\, d\sigma(x)\Big)^{\frac{1}{2}}. \tag{4.4.12}$$

Once (4.4.11) and (4.4.12) are proven, then using (4.4.8), (4.4.10) and (1.0.13), we obtain

$$(1-\varepsilon)\Big(|\kappa| + \frac{d-1}{2}\Big) \le \frac{C|\kappa|}{\varepsilon}\|\sqrt{-\Delta_{\kappa,0}}f\|_{\kappa,2}\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\, d\sigma(x)\Big)^{\frac{1}{2}} \\ + \frac{|\kappa|}{1-\varepsilon}.$$

Thus, choosing $\varepsilon \in (0,1)$ small enough so that

$$(1-\varepsilon)\Big(|\kappa| + \frac{d-1}{2}\Big) - \frac{1}{1-\varepsilon}|\kappa| \ge C_{d,\kappa} > 0,$$

we deduce the desired inequality (4.4.4).

It remains to show (4.4.11) and (4.4.12). For the proof of (4.4.11), we first note that for $x \in \mathbb{S}^{d-1}$,

$$\sum_{i=1}^d \sum_{j=1}^d x_i x_j D_{i,j} = \sum_{i=1}^d \sum_{j=1}^d (x_i^2 x_j \partial_j - x_i x_j^2 \partial_i) = 0.$$

Thus,

$$J_1 = \Big|\int_{\mathbb{S}^{d-1}} \Big[\sum_{i=1}^d \sum_{j=1}^d (y_i - x_i)x_j D_{i,j}f(x)\Big] f(x) h_\kappa^2(x)\, d\sigma(x)\Big| \\ \le \Big(\int_{\mathbb{S}^{d-1}} \frac{|\sum_{i,j=1}^d (y_i - x_i)x_j D_{i,j} f(x)|^2}{1-\langle x,y\rangle} h_\kappa^2(x)\, d\sigma(x)\Big)^{\frac{1}{2}} \times \\ \times \Big(\int_{\mathbb{S}^{d-1}} |f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\, d\sigma(x)\Big)^{\frac{1}{2}}.$$

But, by the Cauchy-Schwartz inequality,

$$\Big|\sum_{i=1}^{d}\sum_{j=1}^{d}(y_i-x_i)x_jD_{i,j}f(x)\Big|^2 \le \Big[\sum_{i,j=1}^{d}|x_j|^2(y_i-x_i)^2\Big]\Big[\sum_{i,j=1}^{d}|D_{i,j}f(x)|^2\Big]$$
$$=4(1-\langle x,y\rangle)\Big[\sum_{1\le i<j\le d}|D_{i,j}f(x)|^2\Big]$$

It follows that

$$J_1 \le 2\Big(\sum_{1\le i<j\le d}\int_{\mathbb{S}^{d-1}}|D_{i,j}f(x)|^2h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^2(1-\langle x,y\rangle)h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac{1}{2}},$$

which proves (4.4.11).

Finally, we prove (4.4.12). Splitting the integral $\int_{\mathbb{S}^{d-1}}\cdots$ into two parts, we get

$$J_2(\alpha) \le J_{2,1}(\alpha)+J_{2,2}(\alpha), \tag{4.4.13}$$

where

$$J_{2,1}(\alpha) := \Big|\langle y,\alpha\rangle\int_{|\langle x,\alpha\rangle|>(1-\varepsilon)|\langle y,\alpha\rangle|}\frac{|f(x)|^2h_\kappa^2(x)}{\langle x,\alpha\rangle}\,d\sigma(x)\Big|,$$
$$J_{2,2}(\alpha) := \Big|\langle y,\alpha\rangle\int_{|\langle x,\alpha\rangle|\le(1-\varepsilon)|\langle y,\alpha\rangle|}\frac{|f(x)|^2h_\kappa^2(x)}{\langle x,\alpha\rangle}\,d\sigma(x)\Big|.$$

A straightforward calculation shows that

$$J_{2,1}(\alpha) \le \frac{1}{1-\varepsilon}\int_{\mathbb{S}^{d-1}}|f(x)|^2h_\kappa^2(x)\,d\sigma(x) = \frac{1}{1-\varepsilon}. \tag{4.4.14}$$

To estimate the term $J_{2,2}(\alpha)$, we first note that for any $t\in(0,1)$ and $\alpha\in\mathcal{R}_+$,

$$\int_{|\langle x,\alpha\rangle|\le t}\frac{|f(x)|^2}{\langle x,\alpha\rangle}h_\kappa^2(x)\,d\sigma(x) = \int_{|\langle x,\alpha\rangle|\le t}\Big(E_\alpha f(x)\Big)f(x)h_\kappa^2(x)\,d\sigma(x).$$

Thus,

$$\begin{aligned} J_{2,2}(\alpha) &= \Big| |\langle y, \alpha\rangle \int_{|\langle x,\alpha\rangle|\leq(1-\varepsilon)|\langle y,\alpha\rangle|} \Big(E_\alpha f(x)\Big) f(x) h_\kappa^2(x)\,d\sigma(x)\Big| \\ &\leq \frac{1}{\varepsilon}\Big|\int_{\mathbb{S}^{d-1}} \|x-y\| \Big(E_\alpha f(x)\Big) f(x) h_\kappa^2(x)\,d\sigma(x)\Big| \\ &\leq \frac{\sqrt{2}}{\varepsilon}\|E_\alpha f\|_{\kappa,2}\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^2(1-\langle x,y\rangle) h_\kappa^2(x)\,d\sigma(x)\Big)^{\frac{1}{2}}, \qquad (4.4.15) \end{aligned}$$

where the second step uses the fact that if $|\langle x,\alpha\rangle| \leq (1-\varepsilon)|\langle y,\alpha\rangle|$, then

$$\varepsilon|\langle y,\alpha\rangle| \leq |\langle y,\alpha\rangle| - |\langle x,\alpha\rangle| \leq \|x-y\|.$$

Now a combination of (4.4.13), (4.4.14) and (4.4.15) yields the estimate (4.4.12). This completes the proof of Theorem 4.4.1. □

Chapter 5

Corresponding results on unit balls and simplices

5.1 Results on unit balls

In this section, we shall show how to deduce similar results on the unit ball $\mathbb{B}^d$ from those already proven results on the unit sphere. Our argument is based on close connections between WOPEs on $\mathbb{B}^d$ and spherical h-harmonic expansions on the sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, as observed by Y. Xu [Xu5, Xu7].

Recall that G is a finite reflection group on $\mathbb{R}^d$ with a root system $\mathcal{R} \subset \mathbb{R}^d$; $\kappa : \mathcal{R} \to [0, \infty)$ is a nonnegative multiplicity function on $\mathcal{R}$; the weight functions h_κ on $\mathbb{S}^{d-1}$ and $W^B_{\kappa,\mu}$ on $\mathbb{B}^d$ are given in (1.0.2) and (1.0.19) respectively. For $1 \le p \le \infty$, we denote by $L^p(W^B_{\kappa,\mu}; \mathbb{B}^d)$ the L^p-space defined with respect to the measure $W^B_{\kappa,\mu}(x)dx$ on $\mathbb{B}^d$, and $\| \cdot \|_{L^p(W^B_{\kappa,\mu})}$ the norm of $L^p(W^B_{\kappa,\mu}; \mathbb{B}^d)$.

Let $\widetilde{G}$ be the finite reflection group on $\mathbb{R}^{d+1}$ associated with the root system

$$\widetilde{\mathcal{R}} := \{\tilde{v} = (v, 0) \in \mathbb{R}^{d+1} : v \in \mathcal{R}\} \cup \{\pm e_{d+1}\},$$

and define $\widetilde{\kappa} : \widetilde{\mathcal{R}} \to [0, \infty)$ by $\tilde{\kappa}(\tilde{v}) = \kappa(v)$ for $v \in \mathcal{R}$ and $\tilde{\kappa}(\pm e_{d+1}) = \mu$. Clearly, $\widetilde{\kappa}$ is a $\widetilde{G}$-invariant nonnegative multiplicity function on $\widetilde{\mathcal{R}}$. Let $h_{\widetilde{\kappa}}$ be the $\widetilde{G}$-invariant weight function on $\mathbb{R}^{d+1}$ associated with the root system $\widetilde{\mathcal{R}}$ and the multiplicity function $\widetilde{\kappa}$ as defined in (1.0.2); that is,

$$h_{\widetilde{\kappa}}(x, x_{d+1}) = |x_{d+1}|^\mu \prod_{v \in \mathcal{R}_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d, \quad x_{d+1} \in \mathbb{R}.$$

The weight $h_{\widetilde{\kappa}}$ on $\mathbb{S}^d$ is related to the weight function $W^B_{\kappa,\mu}$ on $\mathbb{B}^d$ by

$$h^2_{\widetilde{\kappa}}(x, \sqrt{1-\|x\|^2}) = W^B_{\kappa,\mu}(x)\sqrt{1-\|x\|^2}, \quad x \in \mathbb{B}^d. \tag{5.1.1}$$

Furthermore, a change of variables $y = \phi(x)$ with

$$\phi : \ \mathbb{B}^d \to \mathbb{S}^d, \ \ x \in \mathbb{B}^d \mapsto (x, \sqrt{1-\|x\|^2}) \in \mathbb{S}^d \tag{5.1.2}$$

shows that

$$\begin{aligned} &\int_{\mathbb{S}^d} f(y) h^2_{\widetilde{\kappa}}(y) d\sigma(y) \\ =& \int_{\mathbb{B}^d} \left[f(x, \sqrt{1-\|x\|^2}\,) + f(x, -\sqrt{1-\|x\|^2}\,) \right] W^B_{\kappa,\mu}(x)\, dx. \end{aligned} \tag{5.1.3}$$

Given a function $f : \ \mathbb{B}^d \to \mathbb{R}$, define $\widetilde{f} : \mathbb{S}^d \to \mathbb{R}$ by

$$\widetilde{f}(x, x_{d+1}) = f(x), \ \ x \in \mathbb{B}^d, \ \ (x, x_{d+1}) \in \mathbb{S}^d.$$

Then, $\widetilde{f} \circ \phi = f$, and by (5.1.3), the mapping $f \to \widetilde{f}$ is an isometry from $L^p(W^B_{\kappa,\mu}; \mathbb{B}^d)$ to $L^p(\mathbb{S}^d; h^2_{\widetilde{\kappa}}/2)$. More importantly, the orthogonal structure on the weighted ball $\mathbb{B}^d$ is preserved under the mapping $\phi : \mathbb{B}^d \to \mathbb{S}^d$. To be precise, let $\nu^d_n(W^B_{\kappa,\mu})$ denote the space of weighted orthogonal polynomials of degree n with respect to the measure $W^B_{\kappa,\mu}(x)\, dx$ on $\mathbb{B}^d$, and let $\mathrm{proj}_n(W^B_{\kappa,\mu}; f)$ denote the orthogonal projection of f onto the space $\nu^d_n(W^B_{\kappa,\mu})$. Then a function f on $\mathbb{B}^d$ belongs to the space $\nu^d_n(W^B_{\kappa,\mu})$ if and only if $\widetilde{f} \in \mathcal{H}^{d+1}_n(h^2_{\widetilde{\kappa}})$, and moreover (see [DuXu, Xu5, Xu7]),

$$\mathrm{proj}_n(W^B_{\kappa,\mu}; f, x) = \mathrm{proj}_n(W^B_{\kappa,\mu}; \widetilde{f} \circ \phi, x) = \mathrm{proj}_n(h^2_{\widetilde{\kappa}}; \widetilde{f}, \phi(x)), \quad x \in \mathbb{B}^d. \tag{5.1.4}$$

The second order differential-difference operator $\Delta^B_{\kappa,\mu}$ on $\mathbb{B}^d$ is defined by

$$\begin{aligned} \Delta^B_{\kappa,\mu} =& \Delta - (d + 2|\kappa| + 2\mu)(x \cdot \nabla) - (x \cdot \nabla)^2 \\ &+ 2 \sum_{\alpha \in \mathcal{R}_+} \frac{\kappa_\alpha}{\langle x, \alpha \rangle} (\alpha \cdot \nabla) - 2 \sum_{\alpha \in \mathcal{R}_+} \frac{\kappa_\alpha}{\langle \alpha, x \rangle} E_\alpha, \end{aligned}$$

where $\Delta = \sum_{j=1}^d \partial_j^2$, and $\nabla = (\partial_1, \cdots, \partial_d)$. The operator $-\Delta^B_{\kappa,\mu}$ is self-adjoint, semi-positive definite on $L^2(W^B_{\kappa,\mu}; \mathbb{B}^d)$, and more importantly, the s-pace $\nu^d_n(W^B_{\kappa,\mu})$ coincides with the eigenfunction space of $\Delta^B_{\kappa,\mu}$ corresponding to the eigenvalue $\mu_n := -n(n+d-1+2|\kappa|+2\mu)$ (see [DuXu]); that is,

$$\nu_n^d(W_{\kappa,\mu}^B) = \Big\{ f \in C^2(\mathbb{B}^d) : \quad \Delta_{\kappa,\mu}^B f = \mu_n f \Big\}.$$

As a matter of fact, we may define the fractional power of $-\Delta_{\kappa,\mu}^B$ in a distributional sense by

$$\operatorname{proj}_n\big(W_{\kappa,\mu}^B; (-\Delta_{\kappa,\mu}^B)^\alpha f\big) = (-\mu_n)^\alpha \operatorname{proj}_n(W_{\kappa,\mu}^B; f), \quad n = 0, 1, \cdots .$$

Finally, the operator $\Delta_{\kappa,\mu}^B$ is related to the Dunkl-Laplace-Beltrami operator $\Delta_{\tilde{\kappa},0}$ on $\mathbb{S}^d$ by

$$(-\Delta_{\kappa,\mu}^B)^\alpha f(x) = (-\Delta_{\tilde{\kappa},0})^\alpha \widetilde{f}(\phi(x)), \quad x \in \mathbb{B}^d, \quad \alpha \in \mathbb{R}. \tag{5.1.5}$$

The HLS inequality for the fractional integration $(-\Delta_{\kappa,\mu}^B)^{-\alpha/2}$ on the weighted ball can be stated as follows:

Theorem 5.1.1. *Let $1 < p < q < \infty$ and $\alpha > 0$. Then the inequality*

$$\|(-\Delta_{\kappa,\mu}^B)^{-\alpha/2} f\|_{L^q(W_{\kappa,\mu}^B)} \le C \|f\|_{L^p(W_{\kappa,\mu}^B)}, \quad f \in L^p(W_{\kappa,\mu}^B; \mathbb{B}^d) \tag{5.1.6}$$

holds if and only if $\alpha \ge s_{\kappa,\mu}(\frac{1}{p} - \frac{1}{q})$, where

$$s_{\kappa,\mu} = s_{\tilde{\kappa}} := \max\{2|\kappa| + d, s_\kappa + 2\mu + 1\} \tag{5.1.7}$$

with s_κ being given in (1.0.7).

Proof. The sufficiency part of Theorem 5.1.1 follows directly from Theorem 1.0.1, (5.1.3) and (5.1.5), whereas the proof of the necessity part runs along the same line as that of Theorem 1.0.1. □

Our next result on the ball gives a very useful new decomposition of the operator $\Delta_{\kappa,\mu}^B$. Recall that the tangential gradient operator on $\mathbb{B}^d$ is defined by

$$\nabla_0 f(r\xi) = \nabla_0^{(\xi)} f(r\xi), \quad f \in C^1(\mathbb{B}^d), \quad 0 \le r \le 1, \quad \xi \in \mathbb{S}^{d-1},$$

where $\nabla_0^{(\xi)}$ means that the tangential gradient ∇_0 is acting on the variable $\xi \in \mathbb{S}^{d-1}$. Also, note that $E_v f(x)$, given in (1.0.11), is well-defined for each function f on $\mathbb{B}^d$ and each $v \in \mathbb{R}^d \setminus \{0\}$ as $\mathbb{B}^d$ is rotation-invariant. We shall use the notation $\langle \cdot, \cdot \rangle_{L^2(W_{\kappa,\mu}^B)}$ to denote the inner product of the space $L^2(W_{\kappa,\mu}^B; \mathbb{B}^d)$.

Theorem 5.1.2. *For $f, g \in C^2(\mathbb{B}^d)$,*

$$\begin{aligned}\langle(-\Delta^B_{\kappa,\mu})f, g\rangle_{L^2(W^B_{\kappa,\mu})} =& \langle\nabla_0 f, \nabla_0 g\rangle_{L^2(W^B_{\kappa,\mu})} + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha \langle E_\alpha f, E_\alpha g\rangle_{L^2(W^B_{\kappa,\mu})} \\ &+ \int_{\mathbb{B}^d} (1-\|x\|^2)\big(\nabla f\cdot\nabla g\big) W^B_{\kappa,\mu}(x)dx.\end{aligned} \tag{5.1.8}$$

Furthermore,

$$\begin{aligned}\Delta^B_{\kappa,\mu} =& \sum_{1\le i<j\le d} W^B_{\kappa,\mu}(x)^{-1} D_{i,j} W^B_{\kappa,\mu}(x) D_{i,j} + \\ &+ \sum_{i=1}^d W^B_{\kappa,\mu}(x)^{-1}\sqrt{1-\|x\|^2}\partial_i W^B_{\kappa,\mu}(x)\sqrt{1-\|x\|^2}\partial_i - \sum_{\alpha\in\mathcal{R}_+} \frac{2\kappa_\alpha E_\alpha}{\langle x,\alpha\rangle}.\end{aligned} \tag{5.1.9}$$

The significance of the decomposition (5.1.9) lies in the fact that each term in the sums on the right hand side of (5.1.9) is self-adjoint with respect to the inner product of $L^2(W^B_{\kappa,\mu};\mathbb{B}^d)$. In the case when $\kappa = 0$ and $W^B_{\kappa,\mu}(x) = (1-\|x\|^2)^{\mu-\frac{1}{2}}$, (5.1.9) was previously obtained in [DaXu4, (7.1)].

Proof. For simplicity, we denote by $\langle\cdot,\cdot\rangle_{\widetilde{\kappa}}$ the inner product of $L^2(h^2_{\widetilde{\kappa}};\mathbb{S}^d)$. Recall that given a function f on $\mathbb{B}^d$, $\widetilde{f}$ is a function on $\mathbb{S}^d$ given by

$$\widetilde{f}(x, x_{d+1}) = f(x), \quad x\in\mathbb{B}^d, \quad (x, x_{d+1})\in\mathbb{S}^d.$$

By (5.1.3) and (5.1.5), it follows that

$$\langle(-\Delta_{\widetilde{\kappa},0})\widetilde{f}, \widetilde{g}\rangle_{\widetilde{\kappa}} = 2\langle(-\Delta^B_{\kappa,\mu})f, g\rangle_{L^2(W^B_{\kappa,\mu})}.$$

On the other hand, using (4.2.2), (2.2.11) and (5.1.3), we obtain

$$\begin{aligned}\langle(-\Delta_{\widetilde{\kappa},0})\widetilde{f}, \widetilde{g}\rangle_{\widetilde{\kappa}} &= \sum_{1\le i<j\le d} \langle D_{i,j}f, D_{i,j}g\rangle_{\widetilde{\kappa}} + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha\langle E_\alpha f, E_\alpha g\rangle_{\widetilde{\kappa}} + \sum_{i=1}^d \langle D_{i,d+1}\widetilde{f}, D_{i,d+1}\widetilde{g}\rangle_{\widetilde{\kappa}} \\ &= 2\langle\nabla_0 f, \nabla_0 g\rangle_{L^2(W^B_{\kappa,\mu})} + 2\sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha\langle E_\alpha f, E_\alpha g\rangle_{L^2(W^B_{\kappa,\mu})} + \Sigma_1.\end{aligned}$$

where $\Sigma_1 := \sum_{i=1}^d \langle D_{i,d+1}\widetilde{f}, D_{i,d+1}\widetilde{g}\rangle_{\widetilde{\kappa}}$. Thus, to complete the proof of (5.1.8), we just need to observe that

$$\Sigma_1 = \sum_{i=1}^{d} \int_{\mathbb{S}^d} x_{d+1}^2 \partial_i \widetilde{f}(x) \partial_i \widetilde{g}(x) h_{\widetilde{\kappa}}^2(x)\, d\sigma(x) = 2\int_{\mathbb{B}^d} (1-\|x\|)^2 (\nabla f \cdot \nabla g) W_{\kappa,\mu}^B(x)\, dx.$$

Next, we turn to the proof of the decomposition (5.1.9). Setting $X = (x, x_{d+1}) = (x, \sqrt{1-\|x\|^2})$ for $x \in \mathbb{B}^d$, and using (4.2.1) and (5.1.5), we obtain

$$\Delta_{\kappa,\mu}^B f(x) = \Delta_{\widetilde{\kappa},0} \widetilde{f}(X) =: S_1 + S_2 - \sum_{\alpha \in \mathcal{R}_+} \frac{2\kappa_\alpha E_\alpha f(x)}{\langle x, \alpha\rangle},$$

where

$$S_1 := \sum_{1 \le i < j \le d} h_\kappa^{-2}(x) D_{i,j} h_\kappa^2(x) D_{i,j} f(x)$$

$$S_2 := \sum_{i=1}^{d} h_{\widetilde{\kappa}}^{-2}(X) D_{i,d+1} h_{\widetilde{\kappa}}^2(X) D_{i,d+1} \widetilde{f}(X).$$

For the first sum S_1, recalling that the $D_{i,j} = x_i \partial_j - x_j \partial_i$ are tangential derivatives,

$$S_1 = \sum_{1 \le i < j \le d} W_{\kappa,\mu}^B(x)^{-1} D_{i,j} W_{\kappa,\mu}^B(x) D_{i,j} f(x).$$

To handle the sum S_2, we note that for each $1 \le i \le d$,

$$h_{\widetilde{\kappa}}^{-2}(X) D_{i,d+1} h_{\widetilde{\kappa}}^2(X) D_{i,d+1} \widetilde{f}(X) = x_{d+1}^2 h_\kappa^{-2}(x) \partial_i \big[h_\kappa^2(x) \partial_i f(x)\big] -$$
$$- x_i \partial_i f(x) x_{d+1}^{-2\mu} \partial_{d+1} \big[x_{d+1}^{2\mu+1}\big] = x_{d+1}^2 h_\kappa^{-2}(x) \partial_i \big[h_\kappa^2(x) \partial_i f(x)\big] - (2\mu+1) x_i \partial_i f(x).$$

Thus, to complete the proof of (5.1.9), it remains to verify that for all $1 \le i \le d$,

$$(W_{\kappa,\mu}^B(x))^{-1} \partial_i W_{\kappa,\mu}^B(x)(1-\|x\|^2)\partial_i = (1-\|x\|^2) h_\kappa^{-2}(x) \partial_i h_\kappa^2(x) \partial_i - (2\mu+1) x_i \partial_i,$$

which follows directly by a straightforward calculation. $\square$

Remark 5.1.1. By (5.1.8), it follows that

$$-\Delta^B_{\kappa,\mu} = \sum_{1\le i<j\le d} D^*_{i,j} D_{i,j} + \sum_{i=1}^d \left[\sqrt{1-\|x\|^2}\partial_i\right]^* \left[\sqrt{1-\|x\|^2}\partial_i\right] \tag{5.1.10}$$
$$+ \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha E^*_\alpha E_\alpha,$$

where T^* denotes the adjoint operator of T in the space $L^2(W^B_{\kappa,\mu};\mathbb{B}^d)$. However, a straightforward calculation shows that

$$D^*_{i,j}D_{i,j} = -h_\kappa^{-2}(x) D_{i,j} h^2_\kappa(x) D_{i,j}, \quad E^*_\alpha E_\alpha = \frac{2E_\alpha}{\langle x,\alpha\rangle}$$
$$[\sqrt{1-\|x\|^2}\partial_i]^*[\sqrt{1-\|x\|^2}\partial_i] = -W^B_{\kappa,\mu}(x)^{-1}\sqrt{1-\|x\|^2}\partial_i W^B_{\kappa,\mu}(x)\sqrt{1-\|x\|^2}\partial_i.$$

This means that (5.1.10) and (5.1.9) are in fact equivalent.

Our third result gives an analogue of Theorem 1.0.5 on the ball $\mathbb{B}^d$.

Theorem 5.1.3. *If* $1<p<\infty$ *and* $\int_{\mathbb{B}^d} f(x)W^B_{\kappa,\mu}(x)\,dx = 0$, *then*

$$\|\sqrt{-\Delta^B_{\kappa,\mu}} f\|_{L^p(W^B_{\kappa,\mu})} \sim \|\nabla_0 f\|_{L^p(W^B_{\kappa,\mu})} + \max_{v\in\mathcal{R}_+} \kappa_v \|E_v f\|_{L^p(W^B_{\kappa,\mu})} \tag{5.1.11}$$
$$+ \sum_{i=1}^d \|\varphi\partial_i f\|_{L^p(W^B_{\kappa,\mu})},$$

where $\varphi(x) = \sqrt{1-\|x\|^2}$.

Proof. First, using (5.1.3) and (5.1.5), we have

$$2\|(-\Delta^B_{\kappa,\mu})^{\frac12} f\|^p_{L^p(W^B_{\kappa,\mu})} = \|(-\Delta_{\widetilde{\kappa},0})^{\frac12}\widetilde{f}\|^p_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)}.$$

Second, it follows by Theorem 1.0.5 that

$$\|(-\Delta_{\widetilde{\kappa},0})^{\frac12}\widetilde{f}\|_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)} \sim \sum_{1\le i<j\le d+1} \|D_{i,j}\widetilde{f}\|_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)} + \sum_{\alpha\in\mathcal{R}_+} \kappa_\alpha \|E_\alpha f\|_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)},$$

which, using (5.1.3), equals

$$2^{1/p}\sum_{1\le i<j\le d} \|D_{i,j} f\|_{L^p(W^B_{\kappa,\mu})} + 2^{1/p}\sum_{\alpha\in\mathcal{R}_+}\kappa_\alpha\|E_\alpha f\|_{L^p(W^B_{\kappa,\mu})} + \sum_{i=1}^d \|D_{i,d+1}\widetilde{f}\|_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)}.$$

Finally, we note that for each $1 \le i \le d$, and $X = (x, x_{d+1}) \in \mathbb{S}^d$,

$$|D_{i,d+1}\widetilde{f}(X)| = |x_{d+1}\partial_i f(x)| = |\varphi(x)\partial_i f(x)|,$$

which, using (5.1.3), implies that

$$\|D_{i,d+1}\widetilde{f}\|^p_{L^p(h^2_{\widetilde{\kappa}};\mathbb{S}^d)} = 2\|\varphi\partial_i f\|^p_{L^p(W^B_{\kappa,\mu})}.$$

Putting the above together, we obtain the desired equation (5.1.11). □

The decomposition (5.1.10) and Theorem 5.1.3 allow us to introduce the following Riesz transforms for WOPEs on $\mathbb{B}^d$:

Definition 5.1.4. *Define the Riesz transforms for the WOPEs with respect to the weight $W^B_{\kappa,\mu}$ on $\mathbb{B}^d$ by*

$$R^B_{i,j} = D_{i,j}(-\Delta^B_{\kappa,\mu})^{-1/2}, \quad 1 \le i < j \le d, \quad R^B_v = \sqrt{\kappa_v}E_v(-\Delta^B_{\kappa,\mu})^{-1/2}, \quad v \in \mathcal{R}_+,$$
$$R^B_{i,i} = \sqrt{1-\|x\|^2}\partial_i(-\Delta^B_{\kappa,\mu})^{-1/2}, \quad i = 1, \cdots, d.$$

It follows by (5.1.10) that

$$\sum_{1\le i\le j\le d}(R^B_{i,j})^* R^B_{i,j} + \sum_{v\in\mathcal{R}_+}(R^B_v)^* R^B_v = I, \tag{5.1.12}$$

where I is the identity operator on the space

$$\Big\{f \in L^1(W^B_{\kappa,\mu}) : \int_{\mathbb{B}^d} f(x)W^B_{\kappa,\mu}(x)\,dx = 0\Big\}.$$

Furthermore, the L^p-boundedness of these Riesz transforms follows directly from Theorem 5.1.3:

Corollary 5.1.5. *If $1 < p < \infty$ and $\int_{\mathbb{B}^d} f(x)W^B_{\kappa,\mu}(x)\,dx = 0$, then*

$$\|f\|_{L^p(W^B_{\kappa,\mu})} \sim \sum_{1\le i\le j\le d}\|R^B_{i,j}f\|_{L^p(W^B_{\kappa,\mu})} + \sum_{v\in\mathcal{R}_+}\|R^B_v f\|_{L^p(W^B_{\kappa,\mu})}.$$

Remark 5.1.2. In the case when $d = 1$ and $\kappa = 0$, WOPEs with respect to the weight $W^B_{\kappa,\mu}$ on $\mathbb{B}^1 = [-1, 1]$ become the classical ultraspherical polynomial expansions on $[-1,1]$:

$$f(x) \simeq \sum_{k=0}^{\infty} a_k(f) C_k^{\mu}(x), \quad x \in [-1,1].$$

In this case, $R_v^B = 0$, and since $\frac{d}{dx} C_k^{\mu}(x) = 2\mu C_{k-1}^{\mu+1}(x)$, the Riesz transform $R_{1,1}^B f = Rf$ can be written explicitly as

$$Rf(\cos\theta) = \sin\theta \sum_{k=1}^{\infty} a_k(f) \frac{2\mu}{(k(k+2\mu))^{1/2}} C_{k-1}^{\mu+1}(\cos\theta), \quad \theta \in [0,\pi],$$

which is essentially equivalent to the conjugate of f introduced by Muckenhoupt and Stein [MuSt] (see also[Mu]).

Finally, similar argument also guarantees the uncertainty principle Theorem 5.1.1 on unit sphere can be extended to the unit ball immediately by using the facts (5.1.3) and (5.1.5).

Theorem 5.1.6. *Let $f \in C^1(\mathbb{B}^d)$ be such that $\int_{\mathbb{B}^d} f(x) W_{\kappa,\mu}^B(x)\, dx = 0$ and $\int_{\mathbb{B}^d} |f(x)|^2 W_{\kappa,\mu}^B(x) dx = 1$. Then*

$$\Big[\min_{y\in\mathbb{B}^d} \int_{\mathbb{B}^d} (1-\langle x,y\rangle)|f(x)|^2 W_{\kappa,\mu}^B(x)\, d(x)\Big] \times \\ \times \Big[\int_{\mathbb{B}^d} |\sqrt{-\Delta_{\kappa,\mu}^B} f(x)|^2 W_{\kappa,\mu}^B(x)\, d(x)\Big] \geq C_{d,\kappa,\mu} > 0. \tag{5.1.13}$$

5.2 Results on simplices

In this chapter, we shall show how to deduce similar results in the previous chapters on the simplex $\mathbb{T}^d$ from the already proven results on the ball $\mathbb{B}^d$. Our argument is based on the connections between WOPEs on $\mathbb{B}^d$ and WOPEs on $\mathbb{T}^d$, as observed by Y. Xu [Xu7] (see also [DaXu2]).

The weight function $W_{\kappa,\mu}^T$ we consider on the simplex $\mathbb{T}^d$ is given in (1.0.20) with $h_\kappa^2(x)$ being invariant under both G and $\mathbb{Z}_2^d$. It is related to the weight $W_{\kappa,\mu}^B$ on $\mathbb{B}^d$ through the mapping

$$\psi : (x_1, \ldots, x_d) \in \mathbb{B}^d \mapsto (x_1^2, \ldots, x_d^2) \in \mathbb{T}^d \tag{5.2.1}$$

by

$$W^T_{\kappa,\mu}(\psi(x)) = \frac{W^B_{\kappa,\mu}(x)}{|x_1 \cdots x_d|}, \quad x \in \mathbb{B}^d.$$

Furthermore, a change of variables shows that

$$\int_{\mathbb{B}^d} g\big(\psi(x)\big) W^B_{\kappa,\mu}(x)dx = \int_{\mathbb{T}^d} g(x) W^T_{\kappa,\mu}(x)\, dx. \tag{5.2.2}$$

For $1 \le p \le \infty$, we denote by $L^p(W^T_{\kappa,\mu}; \mathbb{T}^d)$ the L^p-space defined with respect to the measure $W^T_{\kappa,\mu}(x)dx$ on $\mathbb{T}^d$, and by $\|\cdot\|_{L^p(W^T_{\kappa,\mu})}$ the norm of $L^p(W^T_{\kappa,\mu}; \mathbb{T}^d)$. Note that (5.2.2) particularly implies that the mapping

$$L^p(W^T_{\kappa,\mu}; \mathbb{T}^d) \to L^p(W^B_{\kappa,\mu}; \mathbb{B}^d), \quad f \mapsto f \circ \psi$$

is an isometry.

Let $\nu^d_n(W^T_{\kappa,\mu})$ denote the space of weighted orthogonal polynomials of degree n with respect to the weight $W^T_{\kappa,\mu}$ on $\mathbb{T}^d$. The orthogonal structure is preserved under the mapping (5.2.1) in the sense that $R \in \mathcal{V}^d_n(W^T_{\kappa,\mu})$ if and only if $R \circ \psi \in \mathcal{V}^d_{2n}(W^B_{\kappa,\mu})$. Furthermore, the orthogonal projection, $\mathrm{proj}_n(W^T_{\kappa,\mu}; f)$, of f onto $\mathcal{V}^d_n(W^T_{\kappa,\mu})$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}^d_{2n}(W^B_{\kappa,\mu})$ as follows (see [DuXu] and [DWY, 5.2]):

$$\mathrm{proj}_n(W^T_{\kappa,\mu}; f, \psi(x)) = \mathrm{proj}_{2n}(W^B_{\kappa,\mu}; f \circ \psi, x), \quad x \in \mathbb{B}^d. \tag{5.2.3}$$

The space $\nu^d_n(W^T_{\kappa,\mu})$ can also be seen as the eigenfunction space of a self-adjoint, semi-negative definite operator $\Delta^T_{\kappa,\mu}$ on $L^2(W^B_{\kappa,\mu}; \mathbb{B}^d)$ corresponding to the eigenvalue $\mu^T_n := -n(n + \frac{d-1}{2} + |\kappa| + \mu)$ (see [DuXu]); that is,

$$\nu^d_n(W^T_{\kappa,\mu}) = \Big\{ f \in C^2(\mathbb{T}^d) : \quad \Delta^T_{\kappa,\mu} f = \mu^T_n f \Big\}, \quad n = 0, 1, \cdots.$$

Thus, we may also define the fractional power of $-\Delta^T_{\kappa,\mu}$ in a distributional sense by

$$\mathrm{proj}_n\big(W^T_{\kappa,\mu}; (-\Delta^T_{\kappa,\mu})^\alpha f\big) = (-\mu^T_n)^\alpha \,\mathrm{proj}_n(W^T_{\kappa,\mu}; f), \quad n = 0, 1, \cdots.$$

The operator $(-\Delta^T_{\kappa,\mu})^\alpha$ is related to the operator $(-\Delta^B_{\kappa,\mu})^\alpha$ on $\mathbb{B}^d$ by

$$\big((-\Delta^T_{\kappa,\mu})^\alpha f\big) \circ \psi(x) = 4^{-\alpha}(-\Delta^B_{\kappa,\mu})^\alpha (f \circ \psi)(x), \quad x \in \mathbb{B}^d, \quad \alpha \in \mathbb{R}, \tag{5.2.4}$$

where the constant $4^{-\alpha}$ justifies the fact that ψ is quadratic.

The requirement that the G-invariant weight h_κ is also $\mathbb{Z}_2^d$-invariant implies the reflection group G is a semi-product of $\mathbb{Z}_2^d$ and another reflection group. In the indecomposable case this limits G to two classes: $\mathbb{Z}_2^d$ itself and the hyperoctahedral group B_d (see [DuXu]). As a matter of fact, we will restrict our attention to the cases of $G = \mathbb{Z}_2^d$ and $G = B_d$ for the rest of this section. According to Example 1.2 and Example 1.4, the weight functions $W_{\kappa,\mu}^T(x) \equiv W_{\kappa,\mu}^T(x; G)$ in (1.0.20) can be written explicitly as follows:

$$W_{\kappa,\mu}^T(x; \mathbb{Z}_2^d) = x_1^{\kappa_1 - 1/2} \cdots x_d^{\kappa_d - 1/2}(1 - |x|)^{\mu - 1/2}, \tag{5.2.5}$$

$$W_{\kappa,\mu}^T(x; B_d) = (1 - |x|)^{\mu - 1/2}\Big(\prod_{i=1}^{d} x_i^{\kappa_1 - 1/2}\Big)\Big(\prod_{1 \le i < j \le d} |x_i - x_j|^{\kappa_2}\Big), \tag{5.2.6}$$

where $|x| = x_1 + \cdots + x_d$ for $x = (x_1, \cdots, x_d) \in \mathbb{T}^d$, and $\mu, \kappa_1, \cdots, \kappa_d \ge 0$. Note that $|\kappa| = \sum_{j=1}^d \kappa_j$ in the case of $G = \mathbb{Z}_2^d$, and $|\kappa| = \frac{d(d-1)}{2}\kappa_2 + d\kappa_1$ in the case of $G = B_d$.

The HLS inequality for the fractional integration $(-\Delta_{\kappa,\mu}^T)^{-\alpha/2}$ on the weighted simplex $\mathbb{T}^d$ can now be stated as follows:

Theorem 5.2.1. *Let $1 < p < q < \infty$ and $\alpha > 0$. Then the inequality*

$$\|(-\Delta_{\kappa,\mu}^T)^{-\alpha/2} f\|_{L^q(W_{\kappa,\mu}^T)} \le C\|f\|_{L^p(W_{\kappa,\mu}^T)}, \quad f \in L^p(W_{\kappa,\mu}^T; \mathbb{T}^d) \tag{5.2.7}$$

holds if and only if $\alpha \ge s_{\kappa,\mu}(\frac{1}{p} - \frac{1}{q})$, where $s_{\kappa,\mu} = \max\{2|\kappa| + d, s_\kappa + 2\mu + 1\}$, and s_κ is given in Example 1.2 for the case of $G = \mathbb{Z}_2^d$ and in (1.0.9) for the case of $G = B_d$.

Proof. The sufficiency part of Theorem 5.2.1 follows directly from Theorem 5.1.1, (5.2.2) and (5.2.4), whereas the proof of the necessity part runs along the same line as that of Theorem 1.0.1. □

Next, set $\varphi_i(x) = \sqrt{x_i(1 - |x|)}$ for $i = 1, \cdots, d$, and

$$\varphi_{i,j}(x) = \sqrt{x_i x_j}, \quad \partial_{i,j} = \partial_j - \partial_i, \quad 1 \le i < j \le d.$$

For simplicity, we denote by $\sigma_{i,j}$, $1 \le i < j \le d$, the reflection $\sigma_{e_i - e_j}$ given by

$$(x_1, \cdots, x_i \cdots, x_j, \cdots, x_d) \mapsto \sigma_{e_i - e_j} x = (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_d),$$

and define

$$S_{i,j}f(x) := \frac{f(x) - f(\sigma_{i,j}x)}{x_i - x_j}\sqrt{x_i + x_j}, \quad x = (x_1, \cdots, x_d) \in \mathbb{T}^d. \tag{5.2.8}$$

Also, we write

$$\langle f, g \rangle_{L^2(W^T_{\kappa,\mu})} := \int_{\mathbb{T}^d} f(x)g(x)W^T_{\kappa,\mu}(x)\,dx, \quad f, g \in L^2(W^T_{\kappa,\mu}; \mathbb{T}^d).$$

Finally, we define

$$a(G) = \begin{cases} 0, & \text{if } G = \mathbb{Z}_2^d, \\ 1, & \text{if } G = B_d. \end{cases}$$

Theorem 5.2.2. *For any $f, g \in C^2(\mathbb{T}^d)$,*

$$\begin{aligned} \langle -\Delta^T_{\kappa,\mu} f, g \rangle_{L^2(W^T_{\kappa,\mu})} = \sum_{j=1}^{d} \langle \varphi_i \partial_i f, \varphi_i \partial_i g \rangle_{L^2(W^T_{\kappa,\mu})} + \sum_{1 \le i < j \le d} \langle \varphi_{i,j} \partial_{i,j} f, \varphi_{i,j} \partial_{i,j} g \rangle_{L^2(W^T_{\kappa,\mu})} \\ + a(G)2^{-1}\kappa_2 \sum_{1 \le i < j \le d} \langle S_{i,j} f, S_{i,j} g \rangle_{L^2(W^T_{\kappa,\mu})}. \end{aligned} \tag{5.2.9}$$

Furthermore,

$$\Delta^T_{\kappa,\mu} = \sum_{1 \le i \le j \le d} U_{i,j;\kappa,\mu} - a(G)\kappa_2 \sum_{1 \le i < j \le d} \frac{x_i + x_j}{(x_i - x_j)^2}(I - \sigma_{i,j}), \tag{5.2.10}$$

where I denotes the identity operator, $\sigma_{i,j} f(x) = f(\sigma_{i,j}x)$, and

$$\begin{aligned} U_{i,i;\kappa,\mu} &= W^T_{\kappa,\mu}(x)^{-1}\partial_i\Big(x_i(1 - |x|)W^T_{\kappa,\mu}(x)\Big)\partial_i, \quad 1 \le i \le d, \\ U_{i,j;\kappa,\mu} &= W^T_{\kappa,\mu}(x)^{-1}\partial_{i,j}(x_i x_j W^T_{\kappa,\mu}(x))\partial_{i,j}, \quad 1 \le i < j \le d. \end{aligned}$$

Proof. We use the notation $\langle \cdot, \cdot \rangle_{L^2(W^B_{\kappa,\mu})}$ to denote the inner product of $L^2(W^B_{\kappa,\mu}; \mathbb{B}^d)$. We then use (5.1.8) and (5.2.2) to obtain

$$\begin{aligned} & 4\langle -\Delta^T_{\kappa,\mu} f, g \rangle_{L^2(W^T_{\kappa,\mu})} = \langle -\Delta^B_{\kappa,\mu}(f \circ \psi), g \circ \psi \rangle_{L^2(W^B_{\kappa,\mu})} \\ &= \sum_{1 \le i < j \le d} \langle D_{i,j}(f \circ \psi), D_{i,j}(g \circ \psi) \rangle_{L^2(W^B_{\kappa,\mu})} + \sum_{i=1}^{d} \langle \varphi \partial_i (f \circ \psi), \varphi \partial_i (g \circ \psi) \rangle_{L^2(W^B_{\kappa,\mu})} \\ & \quad + \sum_{\alpha \in \mathcal{R}_+} \kappa_\alpha \langle E_\alpha (f \circ \psi), E_\alpha (g \circ \psi) \rangle_{L^2(W^B_{\kappa,\mu})} \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

where $\varphi(x) = \sqrt{1 - \|x\|^2}$ as above. A straightforward calculation shows that

$$D_{i,j}(f \circ \psi)(x) = 2\varphi_{i,j}(\psi(x))\partial_{i,j} f(\psi(x)), \quad x \in \mathbb{B}^d,$$

and

$$(1 - \|x\|^2)\partial_i(f \circ \psi)(x)\partial_i(g \circ \psi)(x) = 4(\varphi_i(\psi(x)))^2 \partial_i f(\psi(x))\partial_i g(\psi(x)).$$

It follows that

$$\Sigma_1 = 4 \sum_{1 \le i < j \le d} \langle \varphi_{i,j}\partial_{i,j} f, \varphi_{i,j}\partial_{i,j} g\rangle_{L^2(W^T_{\kappa,\mu})}$$

and

$$\Sigma_2 = 4 \sum_{i=1}^{d} \langle (\varphi_i \partial_i f) \circ \psi, (\varphi_i \partial_i g) \circ \psi \rangle_{L^2(W^B_{\kappa,\mu})} = 4 \sum_{i=1}^{d} \langle \varphi_i \partial_i f, \varphi_i \partial_i g) \rangle_{L^2(W^T_{\kappa,\mu})}.$$

Next, it is clear that $\Sigma_3 = 0$ if $G = \mathbb{Z}_2^d$. Thus, to complete the proof of (5.2.9), it remains to show that for $G = B_d$ and $W^T_{\kappa,\mu}$ in (5.2.6),

$$\Sigma_3 = 2\kappa_2 \sum_{1 \le i < j \le d} \langle S_{i,j} f, S_{i,j} g \rangle_{L^2(W^T_{\kappa,\mu})}. \tag{5.2.11}$$

Indeed, if $G = B_d$, then

$$\begin{aligned} \Sigma_3 = \kappa_2 &\sum_{1 \le i < j \le d} \langle E_{e_i - e_j}(f \circ \psi), E_{e_i - e_j}(g \circ \psi) \rangle_{L^2(W^B_{\kappa,\mu})} \\ &+ \kappa_2 \sum_{1 \le i < j \le d} \langle E_{e_i + e_j}(f \circ \psi), E_{e_i + e_j}(g \circ \psi) \rangle_{L^2(W^B_{\kappa,\mu})} \end{aligned} \tag{5.2.12}$$

Note that for $1 \le i < j \le d$, $\sigma_{e_i - e_j} x = \sigma_{i,j}(x)$ and

$$\sigma_{e_i + e_j}(x_1, \cdots, x_i, \cdots, x_j, \cdots, x_d) = (x_1, \cdots, -x_j, \cdots, -x_i, \cdots, x_d).$$

This implies that for $x \in \mathbb{B}^d$,

$$\psi(\sigma_{e_i - e_j} x) = \sigma_{i,j}(\psi(x)), \quad \psi(\sigma_{e_i + e_j} x) = \sigma_{i,j}(\psi(x)).$$

It follows that

$$E_{e_i - e_j}(f \circ \psi)(x) = \frac{(I - \sigma_{i,j}) f(\psi(x))}{x_i - x_j}$$

and

$$E_{e_i+e_j}(f\circ\psi)(x) = \frac{(I-\sigma_{i,j})f(\psi(x))}{x_i+x_j}.$$

Thus,

$$\begin{aligned}
&\langle E_{e_i-e_j}(f\circ\psi), E_{e_i-e_j}(g\circ\psi)\rangle_{L^2(W^B_{\kappa,\mu})} + \langle E_{e_i+e_j}(f\circ\psi), E_{e_i+e_j}(g\circ\psi)\rangle_{L^2(W^B_{\kappa,\mu})}\\
&= \int_{\mathbb{B}^d} \big[(I-\sigma_{i,j})f(\psi(x))\big]\big[(I-\sigma_{i,j})g(\psi(x))\big]\Big[\frac{1}{(x_i+x_j)^2}+\frac{1}{(x_i-x_j)^2}\Big]W^B_{\kappa,\mu}(x)\,dx\\
&= 2\int_{\mathbb{B}^d} \big[(I-\sigma_{i,j})f(\psi(x))\big]\big[(I-\sigma_{i,j})g(\psi(x))\big]\frac{x_i^2+x_j^2}{(x_i^2-x_j^2)^2}W^B_{\kappa,\mu}(x)\,dx\\
&= 2\int_{\mathbb{T}^d} \big[(I-\sigma_{i,j})f(x)\big]\big[(I-\sigma_{i,j})g(x)\big]\frac{x_i+x_j}{(x_i-x_j)^2}W^T_{\kappa,\mu}(x)\,dx\\
&= 2\langle S_{i,j}f, S_{i,j}g\rangle_{L^2(W^T_{\kappa,\mu})}.
\end{aligned}$$

This together with (5.2.12) implies the desired equation (5.2.11).

Finally, we prove the decomposition (5.2.10). For simplicity, we define

$$A_{i,i}f(x) := \varphi_i(x)\partial_i f(x),\quad 1\le i\le d,\quad x\in\mathbb{T}^d, \tag{5.2.13}$$

and

$$A_{i,j}f(x) := \varphi_{i,j}(x)\partial_{i,j} f(x),\quad 1\le i<j\le d,\quad x\in\mathbb{T}^d. \tag{5.2.14}$$

Then (5.2.9) implies that

$$-\Delta^T_{\kappa,\mu} = \sum_{1\le i\le j\le d} A^*_{i,j}A_{i,j} + a(G)2^{-1}\kappa_2 \sum_{1\le i<j\le d} S^*_{i,j}S_{i,j}, \tag{5.2.15}$$

where $A^*_{i,j}$ and $S^*_{i,j}$ denote the adjoint operators $A_{i,j}$ and $S_{i,j}$ in the space $L^2(W^T_{\kappa,\mu};\mathbb{T}^d)$ respectively. However, integration by parts yields that $A^*_{i,j}A_{i,j} = -U_{i,j;\kappa,\mu}$, whereas a straightforward calculation shows that

$$S^*_{i,j}S_{i,j}f(x) = 2\Big[f(x)-f(\sigma_{i,j}x)\Big]\frac{x_i+x_j}{(x_i-x_j)^2}.$$

The decomposition (5.2.10) then follows. This completes the proof. □

The decomposition (5.2.15) together with (5.2.2) and Theorem 5.1.3 implies

Theorem 5.2.3. *If $1 < p < \infty$ and $\int_{\mathbb{T}^d} f(x) W^T_{\kappa,\mu}(x)\, dx = 0$, then*

$$\left\| \sqrt{-\Delta^T_{\kappa,\mu}} f \right\|_{L^p(W^T_{\kappa,\mu})} \sim \sum_{1 \le i \le j \le d} \|A_{i,j} f\|_{L^p(W^T_{\kappa,\mu})} + a(G)\kappa_2 \sum_{1 \le i < j \le d} \|S_{i,j} f\|_{L^p(W^T_{\kappa,\mu})},$$

where the $A_{i,j}$ are given in (5.2.13) and (5.2.14), and the $S_{i,j}$ are defined by (5.2.8).

We point out that in the case of $G = \mathbb{Z}_2^d$, the decomposition (5.2.10) was previously obtained in [BX, BSX, Dit], whereas Theorem 5.2.3 was proved by a different method in [DHH]. To the best of our knowledge, our results for the case of $G = B_d$ are new on $\mathbb{T}^d$.

The decomposition (5.2.15) allows us to introduce the following definition of the Riesz transforms for WOPEs on $\mathbb{T}^d$.

Definition 5.2.4. *Define the Riesz transforms for the WOPEs on $\mathbb{T}^d$ by*

$$R^T_{1;i,j} f(x) := A_{i,j}(-\Delta^T_{\kappa,\mu})^{-\frac{1}{2}} f(x), \quad 1 \le i \le j \le d,$$
$$R^T_{2;i,j} f(x) := \sqrt{\kappa_2/2}\, S_{i,j}(-\Delta^T_{\kappa,\mu})^{-\frac{1}{2}} f(x), \quad 1 \le i < j \le d.$$

By the decomposition (5.2.15), we have

$$\sum_{1 \le i \le j \le d} (R^T_{1;i,j})^* R^T_{1;i,j} + a(G) \sum_{1 \le i < j \le d} (R^T_{2;i,j})^* R^T_{2;i,j} = I,$$

where I denotes the identity operator on $\left\{ f \in L^1(W^T_{\kappa,\mu}; \mathbb{T}^d) : \int_{\mathbb{T}^d} f(x) W^T_{\kappa,\mu}(x)\, dx = 0 \right\}$, and U^* denotes the adjoint operator of U on the space $L^2(W^T_{\kappa,\mu}; \mathbb{T}^d)$. Furthermore, according to Theorem 5.2.3, we have

Corollary 5.2.5. *If $1 < p < \infty$ and $\int_{\mathbb{T}^d} f(x) W^T_{\kappa,\mu}(x)\, dx = 0$, then*

$$\|f\|_{L^p(W^T_{\kappa,\mu})} \sim \sum_{1 \le i \le j \le d} \|R^T_{1;i,j} f\|_{L^p(W^T_{\kappa,\mu})} + a(G) \sum_{1 \le i < j \le d} \|R^T_{2;i,j} f\|_{L^p(W^T_{\kappa,\mu})}.$$

Next, as what we applied to unit ball in preceding section, we also derive the uncertainty principle on simplex from Theorem 5.1.6 using (5.2.2) and (5.2.4).

Theorem 5.2.6. *Let* $f \in C^1(\mathbb{T}^d)$ *be such that* $\int_{\mathbb{T}^d} f(x) W^T_{\kappa,\mu}(x)\, dx = 0$ *and* $\int_{\mathbb{T}^d} |f(x)|^2 W^T_{\kappa,\mu}(x) dx = 1$*. Then*

$$\Big[\min_{y\in\mathbb{T}^d} \int_{\mathbb{T}^d} (1 - \langle \psi^{-1}(x), \psi^{-1}(y)\rangle) |f(x)|^2 W^T_{\kappa,\mu}(x)\, d(x)\Big] \times \\ \times \Big[\int_{\mathbb{T}^d} |\sqrt{-\Delta^T_{\kappa,\mu}} f(x)|^2 W^T_{\kappa,\mu}(x)\, d(x)\Big] \geq C_{d,\kappa,\mu} > 0, \tag{5.2.16}$$

where we recall that $\psi^{-1}(x) = (\sqrt{x_1}, \sqrt{x_2}, \cdots, \sqrt{x_d})$*.*

Part II

Reverse Hölder's inequality for spherical harmonics [1]

[1] A version of this part has been accepted for publication [DaFeTi].

6.1 Introduction

Let $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere of $\mathbb{R}^d$ endowed with the usual Haar measure $d\sigma(x)$, where $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^d$. Given $0 < p \le \infty$, we denote by $L^p(\mathbb{S}^{d-1})$ the usual Lebesgue L^p-space defined with respect to the measure $d\sigma(x)$ on $\mathbb{S}^{d-1}$, and by $\|\cdot\|_p$ the norm of $L^p(\mathbb{S}^{d-1})$. Throughout the chapter, unless otherwise stated, all functions on $\mathbb{S}^{d-1}$ will be assumed to be real-valued and measurable, and the notation $A \sim B$ means that there exists an inessential constant $c > 0$, called the constant of equivalence, such that $c^{-1}A \le B \le cA$.

Let Π_n^d denote the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$ (i.e., restrictions on $\mathbb{S}^{d-1}$ of polynomials in d variables of total degree at most n), and $\mathcal{H}_n^d$ the space of all spherical harmonics of degree n on $\mathbb{S}^{d-1}$. As is well known (see, for instance, [DaXu2, chapter 1]), $\mathcal{H}_n^d$ and Π_n^d are all finite dimensional spaces with $\dim \mathcal{H}_n^d \sim n^{d-2}$ and $\dim \Pi_n^d \sim n^{d-1}$ as $n \to \infty$. Furthermore, the spaces $\mathcal{H}_k^d$, $k = 0, 1, \cdots$ are mutually orthogonal with respect to the inner product of $L^2(\mathbb{S}^{d-1})$, and each space Π_n^d can be written as a direct sum $\Pi_n^d = \bigoplus_{j=0}^n \mathcal{H}_j^d$. Since the space of spherical polynomials is dense in $L^2(\mathbb{S}^{d-1})$, each $f \in L^2(\mathbb{S}^{d-1})$ has a spherical harmonic expansion, $f = \sum_{k=0}^\infty \operatorname{proj}_k f$, where proj_k is the orthogonal projection of $L^2(\mathbb{S}^{d-1})$ onto the space $\mathcal{H}_k^d$ of spherical harmonics. The orthogonal projection proj_k has an integral representation:

$$\operatorname{proj}_k f(x) = C_{k,d} \int_{\mathbb{S}^{d-1}} f(y) P_k^{(\frac{d-3}{2}, \frac{d-3}{2})}(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{6.1.1}$$

where

$$C_{k,d} := \frac{\Gamma(\frac{d}{2})\Gamma(\frac{d-1}{2})}{2\pi^{d/2}\Gamma(d-1)} \frac{(2k+d-2)\Gamma(k+d-2)}{\Gamma(k+\frac{d-1}{2})},$$

and $P_k^{(\alpha,\beta)}$ denotes the usual Jacobi polynomial of degree k and indices α, β, as defined in [Sz, Chapter IV].

Our goal in this chapter is to find a sharp asymptotic order of the quantity $\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p}$ for $0 < p < q \le \infty$ as $n \to \infty$. The background of this problem is as follows. In 1986, Sogge [Sog] proved that for $d \geq 3$

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^2(\mathbb{S}^{d-1})}} \sim \begin{cases} n^{\frac{d-2}{2}(\frac{1}{2}-\frac{1}{q})}, & 2 \leq q \leq \frac{2d}{d-2}, \\ n^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}, & \frac{2d}{d-2} \leq q \leq \infty, \end{cases} \tag{6.1.2}$$

which confirms a conjecture of Stanton–Weinstein [Sta] in the case of $d = 3$ and $q = 4$. Here and throughout the chapter, it is agreed that $0/0 = 0$. De Carli and Grafakos [DeGr, Section 6] proved that if $1 \leq p \leq q \leq 2$ and $Y_n \in \mathcal{H}_n^d$ can be written in the form

$$Y_n(x) = e^{im_{d-2}x_{d-1}} \prod_{k=0}^{d-2} (\sin x_{k+1})^{m_{k+1}} P_{m_k - m_{k+1}}^{(m_{k+1}+\frac{d-2-k}{2}, m_{k+1}+\frac{d-2-k}{2})}(\cos x_{k+1}), \tag{6.1.3}$$

with $n = m_0 \geq m_1 \geq \cdots m_{d-2} \geq 0$ being integers, then

$$\frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}} \leq C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})}, \quad 1 \leq p < q \leq 2, \tag{6.1.4}$$

which was further applied in [DeGr] to prove the restriction conjecture for the Fourier transform for the class of functions consisting of products of radial functions and spherical harmonics that are in the form (6.1.3), (see Section 4.1 for more details). Note that the set of functions Y_n in (6.1.3) with $n = m_0 \geq m_1 \geq \cdots m_{d-2} \geq 0$ forms a linear basis of the space $\mathcal{H}_n^d$. It is therefore natural to ask whether or not (6.1.4) holds for all spherical harmonics Y_n of degree n. A related work in this direction was done recently by De Carli, Gorbachev and Tikhonov in [DeGoTi], where the following weaker estimate of [Duo] for spherical harmonics was applied to study a sharp Pitt inequality for the Fourier transform on $\mathbb{R}^d$:

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_{p'}}{\|Y_n\|_p} \leq C n^{(d-1)(\frac{1}{p}-\frac{1}{2})}, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \leq p \leq 2, \tag{6.1.5}$$

Finally, let us recall the following well-known result of Kamzolov [Kam] on the Nikolskii inequality for spherical polynomials:

$$\|P_n\|_q \leq C n^{(d-1)(\frac{1}{p}-\frac{1}{q})} \|P_n\|_p, \qquad \forall P_n \in \Pi_n^d, \;\; 0 < p < q \leq \infty. \tag{6.1.6}$$

Since $\mathcal{H}_n^d \subset \Pi_n^d$, the Nikolskii inequality (6.1.6) is applicable to every spherical harmonics $Y_n \in \mathcal{H}_n^d$. It turns out, however, that the resulting estimates are not sharp for spherical harmonics in many cases (see, for instance, (6.1.2), (6.1.5) and (6.1.4)).

We will prove the following result, which, in particular, shows that (6.1.4) holds for all spherical harmonics $Y_n \in \mathcal{H}_n^d$, and the upper bound on the right hand side of (6.1.5) can be improved to be $Cn^{(d-2)(\frac{1}{p}-\frac{1}{2})}$.

Theorem 6.1.1. *Assume that $d \geq 3$ and $\frac{1}{p}+\frac{1}{p'}=1$ if $p \geq 1$.*

(i) *If either $0<p\leq 1$ and $p<q\leq\infty$, or $1\leq p\leq 2$ and $p<q\leq \frac{dp'}{d-2}$, then*

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})}. \tag{6.1.7}$$

(ii) *If either $1\leq p\leq 2$ and $q\geq\frac{dp'}{d-2}$, or $2\leq p<\frac{2d-2}{d-2}$ and $q>\frac{2d-2}{d-2}$, then*

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}.$$

(iii) *If $\frac{2d-2}{d-2}<p<q\leq\infty$, then*

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{(d-1)(\frac{1}{p}-\frac{1}{q})}.$$

(iv) *If $d=3$ and $2\leq p<4$, then for $q\geq 3p'$,*

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\frac{1}{2}-\frac{2}{q}},$$

whereas for $p<q\leq 3p'$,

$$\sup_{Y_n\in\mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \sim n^{\frac{1}{2}(\frac{1}{p}-\frac{1}{q})}.$$

Of particular interest is the case when $1\leq p\leq 2$ and $q=p'$, where our result can be stated as follows:

Corollary 6.1.2. *If $Y_n\in\mathcal{H}_n^d$, $1\leq p\leq 2$ and $\frac{1}{p}+\frac{1}{p'}=1$, then*

$$\|Y_n\|_{p'} \leq Cn^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{p'})}\|Y_n\|_p, \quad 1\leq p\leq 2. \tag{6.1.8}$$

Furthermore, this estimate is sharp.

Several remarks are in order.

Remark 6.1.1. Estimate (6.1.8) for $p = p_d := \frac{2d}{d+2}$ follows directly from the well-known result of Sogge [Sog] on the orthogonal projection $\operatorname{proj}_n : L^2(\mathbb{S}^{d-1}) \to \mathcal{H}_n^d$. However, for $1 \le p < 2$ and $p \neq p_d$, the sharp estimate (6.1.8) in Corollary 6.1.2 is nontrivial and cannot be deduced from the result of Sogge [Sog], who proved that for $1 \le p \le p_d$,

$$\|\operatorname{proj}_n f\|_2 \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{(d+2)p}(p_d-p)} \|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}), \tag{6.1.9}$$

and this estimate is sharp. Since $\operatorname{proj}_n f = f$ for $f \in \mathcal{H}_n^d$, this leads to the inequality

$$\|Y_n\|_2 \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{(d+2)p}(p_d-p)} \|Y_n\|_p, \quad \forall\, Y_n \in \mathcal{H}_n^d, \ \ 1 \le p \le p_d,$$

which, according to Corollary 6.1.2, is not sharp unless $p = p_d$.

Remark 6.1.2. Interesting reverse Hölder inequalities for spherical harmonics,

$$\sup_{Y_n \in \mathcal{H}_n^d} \frac{\|Y_n\|_q}{\|Y_n\|_p} \le C(n, q)$$

with the constant $C(n, q)$ being independent of the dimension d but dependent on the degree n of spherical harmonics, were obtained in [Duo] for some pairs of (p, q), $0 < p < q < \infty$. The general constants C in this chapter are dependent on the dimension d, but independent of the degree n.

Remark 6.1.3. For $d \ge 4$, it remains open to find the asymptotic estimate of the supremum on the left hand side of (6.1.7) for $2 < p < \frac{d}{d-2}$ and $p < q < \frac{2d}{d-2}$.

This chapter is organized as follows. In Section 2, we construct a sequence of convolution operators $\{T_n\}_{n=0}^\infty$ on $L^1(\mathbb{S}^{d-1})$ with the properties that $T_n f = f$ for $f \in \mathcal{H}_n^d$, $|T_n f| \le C \sup_{0 \le j \le d} |\operatorname{proj}_{n+2j} f|$ and $\|T_n f\|_\infty \le C n^{\frac{d-2}{2}} \|f\|_1$ for all $f \in L^1(\mathbb{S}^{d-1})$. These operators play an indispensable role in the proof of Theorem 6.1.1, which is given in the third section. Finally, in Section 4, we give two applications of our main result, improving a recent result of [DeGr] on restriction conjecture and a result of [DeGoTi] on sharp Pitt's inequality.

6.2 A sequence of convolution operators

We start with the following well-known result of Sogge [Sog] on the operator norms of the orthogonal projections $\mathrm{proj}_n : L^2(\mathbb{S}^{d-1}) \to \mathcal{H}_n^d$.

Lemma 6.2.1. *[Sog] Let $n \in \mathbb{N}$ and $d \geq 3$. Then the following statements hold:*

(i) *If $1 \leq p \leq p_d := \frac{2d}{d+2}$, then*

$$\| \mathrm{proj}_n f\|_2 \leq C n^{(d-1)(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}} \|f\|_p.$$

(ii) *If $p_d \leq p \leq 2$, then*

$$\| \mathrm{proj}_n f\|_2 \leq C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{2})} \|f\|_p.$$

(iii) *If $\frac{2d}{d-2} \leq q \leq \infty$, then*

$$\| \mathrm{proj}_n f\|_q \leq C n^{(d-1)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \|f\|_2.$$

(iv) *If $2 \leq q \leq \frac{2d}{d-2}$, then*

$$\| \mathrm{proj}_n f\|_q \leq C n^{\frac{d-2}{2}(\frac{1}{2}-\frac{1}{q})} \|f\|_2.$$

Here, the letter C denotes a general positive constant independent of n and f.

As was pointed out in the introduction, Lemma 6.2.1 will not be enough for the proof of our main result. The crucial step in the proof of Theorem 6.1.1 is to construct a sequence of linear operators $\{T_n\}_{n=0}^\infty$ with the properties that $T_n f = f$ for $f \in \mathcal{H}_n^d$, $|T_n f| \leq C \sup_{0\leq j \leq d} |\mathrm{proj}_{n+2j} f|$ and $\|T_n f\|_\infty \leq C n^{\frac{d-2}{2}} \|f\|_1$ for all $f \in L^1(\mathbb{S}^{d-1})$.

To define the operators T_n, we need to recall several notations. First, given $h \in \mathbb{N}$, and a sequence $\{a_n\}_{n=0}^\infty$ of real numbers, define (see, for instance, [DeLo, 7.1])

$$\triangle_h a_n = a_n - a_{n+h}, \quad \triangle_h^{\ell+1} = \triangle_h \triangle_h^\ell, \quad \ell = 1, 2, \ldots .$$

Next, let

$$R_n(\cos\theta) := \frac{P_n^{(\frac{d-3}{2},\frac{d-3}{2})}(\cos\theta)}{P_n^{(\frac{d-3}{2},\frac{d-3}{2})}(1)}, \quad \theta\in[0,\pi]$$

denote the normalized Jacobi polynomial, and for a step $h\in\mathbb{N}$, define

$$\triangle_h^\ell R_n(\cos\theta) := \triangle_h^\ell a_n = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}R_{n+hj}(\cos\theta),\ \ \ell=1,2,\ldots,\ \ n=0,1,\cdots,$$

with $a_n := R_n(\cos\theta)$. Here and throughout, the difference operator in $\triangle_h^\ell R_n(\cos\theta)$ is always acting on the integer n. In the case when the step $h=1$, we have the following estimate ([DaXu2, Lemma B.5.1], [DaDi]):

$$\left|\triangle_1^\ell R_n(\cos\theta)\right| \le C\theta^\ell(1+n\theta)^{-\frac{d-2}{2}}, \quad \theta\in[0,\pi/2], \quad \ell\in\mathbb{N}. \tag{6.2.1}$$

On the other hand, however, the ℓ-th order difference $\triangle_1^\ell R_n(\cos\theta)$ with step $h=1$ does not provide a desirable upper estimate when θ is close to π, and as will be seen in our later proof, estimate (6.2.1) itself will not be enough for our purpose.

To overcome this difficulty, instead of the difference with step 1, we consider the ℓ-th order difference $\triangle_2^\ell R_n(\cos\theta)$ with step $h=2$. Since $\triangle_2^\ell a_n = \sum_{j=0}^{\ell}\binom{\ell}{j}\triangle_1^\ell a_{n+j}$, on one hand, (6.2.1) implies that

$$\left|\triangle_2^\ell R_n(\cos\theta)\right| \le C\theta^\ell(1+n\theta)^{-\frac{d-2}{2}}, \ \ \theta\in[0,\pi/2].$$

On the other hand, however, since

$$\triangle_2^\ell R_n(\cos\theta) = \sum_{j=0}^{\ell}(-1)^j\binom{\ell}{j}R_{n+2j}(\cos\theta),$$

and since $R_{n+2j}(-z)=(-1)^nR_{n+2j}(z)$, we have $\triangle_2^\ell R_n(\cos(\pi-\theta)) = (-1)^n\triangle_2^\ell R_n(\cos\theta)$. It follows that

$$\left|\triangle_2^\ell R_n(\cos\theta)\right| \le C\begin{cases}\theta^\ell(1+n\theta)^{-\frac{d-2}{2}}, & \theta\in[0,\pi/2],\\ (\pi-\theta)^\ell(1+n(\pi-\theta))^{-\frac{d-2}{2}}, & \theta\in[\pi/2,\pi].\end{cases} \tag{6.2.2}$$

By (6.1.1), we obtain that for every $P \in \mathcal{H}_n^d$,

$$P(x) = c_n \int_{\mathbb{S}^{d-1}} P(y) R_n(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1},$$

where

$$c_n := \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2}} \frac{d+2n-2}{d+n-2} \frac{\Gamma(d+n-1)}{\Gamma(n+1)\Gamma(d-1)} \sim n^{d-2},$$

and $x \cdot y$ denotes the dot product of $x, y \in \mathbb{R}^d$. Since $R_j(x \cdot) \in \mathcal{H}_j^d$ for any fixed $x \in \mathbb{S}^{d-1}$, it follows by the orthogonality of spherical harmonics that for any $P \in \mathcal{H}_n^d$, and any $\ell \in \mathbb{N}$,

$$\begin{aligned} P(x) &= c_n \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \int_{\mathbb{S}^{d-1}} P(y) R_{n+2j}(x \cdot y)\, d\sigma(y) \\ &= c_n \int_{\mathbb{S}^{d-1}} P(y) \triangle_2^{\ell} R_n(x \cdot y)\, d\sigma(y). \end{aligned} \tag{6.2.3}$$

For the rest of the section, we will choose ℓ to be an integer bigger than $\frac{d-2}{2}$ (for instance, we may set $\ell = d-2$), so that by (6.2.2), we have

$$\left| \triangle_2^{\ell} R_n(\cos\theta) \right| \le C n^{-\frac{d-2}{2}}. \tag{6.2.4}$$

Now we are in a position to define the operators T_n.

Definition 6.2.2. *For $f \in L^1(\mathbb{S}^{d-1})$, we define*

$$T_n f(x) := \int_{\mathbb{S}^{d-1}} f(y) \Phi_n(x \cdot y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \tag{6.2.5}$$

where

$$\Phi_n(\cos\theta) := c_n \sum_{j=0}^{d-2} (-1)^j \binom{d-2}{j} R_{n+2j}(\cos\theta).$$

By (6.2.4), we have

$$|\Phi_n(\cos\theta)| \le C n^{\frac{d-2}{2}}, \quad \theta \in [0, \pi], \tag{6.2.6}$$

whereas by (6.2.3)

$$T_n P(x) = P(x), \quad \forall P \in \mathcal{H}_n^d, \quad \forall x \in \mathbb{S}^{d-1}. \tag{6.2.7}$$

The main result of this section can now be stated as follows.

Theorem 6.2.3. *If $1 \le p \le 2$ and $p' \le q \le \frac{dp'}{d-2}$, then*

$$\|T_n f\|_q \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}). \tag{6.2.8}$$

If $1 \le p \le 2$ and $q \ge \frac{dp'}{d-2}$, then

$$\|T_n f\|_q \le C n^{\frac{d-2}{2}-\frac{d-1}{q}} \|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}).$$

Proof. First, we prove the assertion (i). Note that by definition, for each $f \in L^2(\mathbb{S}^{d-1})$,

$$T_n f = \sum_{j=0}^{d-2} (-1)^j \binom{d-2}{j} \frac{c_n}{c_{n+2j}} \operatorname{proj}_{n+2j} f, \tag{6.2.9}$$

which implies that

$$\|T_n f\|_2 \le C \|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{6.2.10}$$

On the other hand, however, using (6.2.6), we have

$$\|T_n f\|_\infty \le C n^{\frac{d-2}{2}} \|f\|_1, \quad \forall f \in L^1(\mathbb{S}^{d-1}). \tag{6.2.11}$$

Thus, applying the Riesz-Thorin interpolation theorem, and using (6.2.10) and (6.2.11), we deduce that for $1 \le p \le 2$,

$$\|T_n f\|_{p'} \le C n^{(d-2)(\frac{1}{p}-\frac{1}{2})} \|f\|_p, \quad \forall f \in L^p(\mathbb{S}^{d-1}). \tag{6.2.12}$$

Next, by (iv) of Lemma 6.2.1, and using (6.2.9), we obtain that for $2 \le r \le \frac{2d}{d-2}$,

$$\|T_n f\|_r \le C n^{\frac{d-2}{2}(\frac{1}{2}-\frac{1}{r})} \|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{6.2.13}$$

Assume that $1 \le p \le 2$ and $p' \le q \le \frac{dp'}{d-2}$. Let $\theta = \frac{2}{p'} \in [0,1]$, and let $r = \theta q = \frac{2}{p'} q$. Then $2 \le r \le \frac{2d}{d-2}$, and

$$\frac{1}{p} = 1 - \theta + \frac{\theta}{2}, \quad \frac{1}{q} = \frac{1-\theta}{\infty} + \frac{\theta}{r}.$$

Thus, by (6.2.12), (6.2.13) and applying the Riesz-Thorin interpolation theorem, we obtain that

$$\|T_n f\|_q \le C n^{\frac{d-2}{2}(1-\theta)} n^{\frac{d-2}{2}(\frac{1}{2}-\frac{1}{r})\theta} \|f\|_p = C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_p.$$

This completes the proof of the assertion (i).

Assertion (ii) can be proved similarly. Indeed, using (6.2.9) and (iii) of Lemma 6.2.1, we have that for $r \geq \frac{2d}{d-2}$,

$$\|T_n f\|_r \leq C n^{(d-2)(\frac{1}{2}-\frac{1}{r})-\frac{1}{r}} \|f\|_2, \quad \forall f \in L^2(\mathbb{S}^{d-1}). \tag{6.2.14}$$

Assume that $1 \leq p \leq 2$ and $q \geq \frac{dp'}{d-2}$. Let $\theta = \frac{2}{p'}$ and $r = \theta q = \frac{2}{p'} q$. Then $r \geq \frac{2d}{d-2}$. Using (6.2.14), (6.2.12) and applying the Riesz-Thorin interpolation theorem, we deduce that

$$\begin{aligned} \|T_n f\|_q &\leq C n^{\frac{d-2}{2}(1-\theta)} n^{(d-2)\theta(\frac{1}{2}-\frac{1}{r})-\frac{\theta}{r}} \|f\|_p = C n^{\frac{d-2}{2}-\frac{d-1}{q}} \|f\|_p \\ &= C n^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}} \|f\|_p. \end{aligned}$$

This completes the proof of (ii).

□

6.3 Proof of Theorem 6.1.1

The stated lower estimates of Theorem 6.1.1 follow directly from the following two known lemmas.

Lemma 6.3.1. *[Sog] Let*

$$f_n(x) = (x_1 + i x_2)^n$$

for $x = (x_1, x_2, \dots, x_d) \in \mathbb{S}^{d-1}$. Then $f \in \mathcal{H}_n^d$ and

$$\|f_n\|_p \sim n^{-\frac{d-2}{2p}}, \quad 0 < p < \infty.$$

Lemma 6.3.2. *[Sz, p.391] Let*

$$g_n(x) = P_n^{(\frac{d-3}{2}, \frac{d-3}{2})}(x \cdot e)$$

for a fixed point $e \in \mathbb{S}^{d-1}$. Then $g_n \in \mathcal{H}_n^d$, and

$$\|g_n\|_p \sim \begin{cases} n^{\frac{d-3}{2}} n^{-\frac{d-1}{p}}, & p > \frac{2(d-1)}{d-2}, \\ n^{-\frac{1}{2}} (\log n)^{\frac{1}{p}}, & p = \frac{2(d-1)}{d-2}, \\ n^{-\frac{1}{2}}, & p < \frac{2(d-1)}{d-2}. \end{cases}$$

For the proof of the upper estimates, we let $P \in \mathcal{H}_n^d$. The crucial tool in our proof is Theorem 6.2.3, where we recall that $T_n P = P$ for all $P \in \mathcal{H}_n^d$. We consider the following cases:

Case 1. $1 \le p \le q \le p'$.

In this case, $1 \le p \le 2 \le p'$, and the stated upper estimate for $q = p'$ follows directly from Theorem 6.2.3. In general, for $p \le q \le p'$, let $\theta \in [0,1]$ be such that $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p'}$. Then by the log-convexity of the L^p-norm, we have

$$\|P\|_q \le \|P\|_p^{\theta} \|P\|_{p'}^{1-\theta} \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{p'})(1-\theta)} \|P\|_p \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \|P\|_p,$$

which is as desired in this case.

Case 2. $0 < p \le 1$ and $p < q$.

In this case, note that

$$\|P\|_1 \le \|P\|_p^p \|P\|_\infty^{1-p} \le C n^{\frac{d-2}{2}(1-p)} \|P\|_p^p \|P\|_1^{1-p}.$$

It follows that

$$\|P\|_1 \le C n^{\frac{d-2}{2}(\frac{1}{p}-1)} \|P\|_p, \quad 0 < p \le 1,$$

which, in turn, implies that for $p < q$ and $\frac{1}{q} = \frac{1-\theta}{p}$,

$$\|P\|_q \le \|P\|_\infty^{\theta} \|P\|_p^{1-\theta} \le C n^{\frac{d-2}{2}\theta} \|P\|_1^{\theta} \|P\|_p^{1-\theta} \le C n^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \|P\|_p.$$

Case 3. $1 \le p \le 2$ and $q \ge p'$.

The desired estimate in this case follows directly from the first and the second parts of Theorem 6.2.3 since $T_n P = P$ for all $P \in \mathcal{H}_n^d$.

Case 4. $2 \le p \le \frac{2d-2}{d-2}$ and $q \ge \frac{2d}{d-2}$.

For $P \in \mathcal{H}_n^d$, by the already proven cases it follows that

$$\|P\|_q \le Cn^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}\|P\|_2 \le Cn^{(d-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{q}}\|P\|_p.$$

Case 5. $\frac{2d-2}{d-2} < p < q \le \infty$.

The reverse Hölder inequality in this case follows directly from the corresponding Nikolskii inequality for spherical polynomials given by (6.1.6).

Case 6. $d = 3$ and $2 \le p < 4$.

The proof in this case relies on the following result of Sogge [Sog]:

Lemma 6.3.3. *If* $d = 3$, $\frac{4}{3} < p < 4$ *and* $q = 3p'$, *then*

$$\|\operatorname{proj}_n f\|_q \le Cn^{\frac{1}{2}-\frac{2}{q}}\|f\|_p.$$

Now we return to the proof in Case 6. Again, in view of Lemmas 6.3.1 and 6.3.2, it is enough to prove the upper estimates. Assume first that $q \ge 3p'$. Let $2 \le p < p_1 < 4$ and let $\theta \in [0,1]$ be such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{2}.$$

Set $q_1 = 3p_1'$. Then by Lemma 6.3.3,

$$\|Tf\|_{q_1} \le Cn^{\frac{1}{2}-\frac{2}{q_1}}\|f\|_{p_1}. \tag{6.3.1}$$

For $q \ge 3p' > 3p_1' = q_1$, let $q_2 \ge q$ be such that

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Then

$$\frac{1}{3} \ge \frac{1}{3p} + \frac{1}{q} = \theta(\frac{1}{6} + \frac{1}{q_2}) + \frac{1}{3}(1-\theta) = \theta(\frac{1}{q_2} - \frac{1}{6}) + \frac{1}{3}.$$

This implies that $q_2 \ge 6$, hence by (ii) of Theorem 6.2.3,

$$\|T_n f\|_{q_2} \le Cn^{\frac{1}{2}-\frac{2}{q_2}}\|f\|_2. \tag{6.3.2}$$

Thus, using (6.3.1), (6.3.2), and the Riesz-Thorin theorem, we obtain

$$\|T_n f\|_q \le C n^{\frac{1}{2}-\frac{2}{q}} \|f\|_p,$$

which implies the desired estimate for the case of $q \ge 3p'$.

The case of $p < q < 3p'$ can be treated similarly. In fact, let p_1, q_1 and θ be as above. Observing that $\frac{1}{2} - \frac{2}{q_1} = \frac{1}{2}(\frac{1}{p_1} - \frac{1}{q_1})$, we may rewrite (6.3.1) as

$$\|Tf\|_{q_1} \le C n^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{q_1})} \|f\|_{p_1}.$$

Furthermore, we may choose $p_1 > p$ to be very close to p so that $q < q_1 = 3p_1' < 3p'$. Let $q_3 \le q$ be such that

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_3}.$$

Then

$$\frac{1}{3} < \frac{1}{3p} + \frac{1}{q} = \theta(\frac{1}{6} + \frac{1}{q_3}) + \frac{1}{3}(1-\theta) = \theta(\frac{1}{q_3} - \frac{1}{6}) + \frac{1}{3}.$$

Hence $2 < q_3 < 6$, and using (i) of Theorem 6.2.3, we deduce

$$\|T_n f\|_{q_3} \le C n^{\frac{1}{2}(\frac{1}{2}-\frac{1}{q_3})} \|f\|_2.$$

The stated estimate for $p < q < 3p'$ then follows by the Riesz-Thorin interpolation theorem. □

6.4 Applications: Fourier inequalities

6.4.1 The restriction conjecture.

One of the most challenging problems in classical Fourier analysis is the restriction conjecture, which states that if $1 \le p < \frac{2d}{d+1}$ and $q \le \frac{d-1}{d+1}p'$, then there exists a constant C depending only on p, q, d such that

$$\frac{\|\widehat{F}\|_{L^q(\mathbb{S}^{d-1})}}{\|F\|_{L^p(\mathbb{R}^d)}} \le C, \quad \forall F \in C_0^\infty(\mathbb{R}^d), \tag{6.4.1}$$

where $\hat{F}(\xi) := \int_{\mathbb{R}^d} F(x)e^{-2\pi i x\cdot\xi}\,dx$, $\xi \in \mathbb{R}^d$. This conjecture has been completely proved only in the case of $d = 2$. We refer to the book [St2, Chapter IX] for more background information of this problem.

De Carli and Grafakos [DeGr] recently proved that the restriction conjecture is valid for all functions F that can be expressed in the form

$$F(x) = f(\|x\|)\|x\|^n g_n(\frac{x}{\|x\|}), \quad n = 0, 1, \cdots$$

with $f(\|\cdot\|) \in C_0^\infty(\mathbb{R}^d)$ and $g_n \in \mathcal{H}_n^d$ being given in (6.1.3) . Using Theorem 6.1.1 (i), and following the argument of [DeGr], we may conclude here that the restriction conjecture holds for a wider class of functions

$$F \in \bigcup_{n=0}^{\infty}\Big\{ f(\|x\|)\|x\|^n Y_n(\frac{x}{\|x\|}) : \quad f(\|\cdot\|) \in C_0^\infty(\mathbb{R}^d), \quad Y_n \in \mathcal{H}_n^d \Big\}.$$

Indeed, it was shown in [DeGr] that for $F(x) = f(\|x\|)\|x\|^n Y_n(x/\|x\|)$ with $f \in C_0^\infty(\mathbb{R}^d)$ and $Y_n \in \mathcal{H}_n^d$,

$$\begin{aligned}\frac{\|\widehat{F}\|_{L^q(\mathbb{S}^{d-1})}}{\|F\|_{L^p(\mathbb{R}^d)}} &= \frac{\Big| \int_0^\infty f(r) J_{\frac{d}{2}-1+n}(r) r^{\frac{d}{2}+n} dr \Big|}{\Big(\int_0^\infty |f(r)|^p r^{d-1+np} dr \Big)^{1/p}} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}} \\ &\le C n^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}} \frac{\|Y_n\|_{L^q(\mathbb{S}^{d-1})}}{\|Y_n\|_{L^p(\mathbb{S}^{d-1})}}, \end{aligned} \tag{6.4.2}$$

where $J_n(r)$ is the Bessel function of the first kind. However, according to (i) of Theorem 6.1.1 , we obtain that for $1 \le p < \frac{2d}{d+1}$ and $q \le \frac{d-1}{d+1}p'$,

$$\text{RHS of (6.4.2)} \le C \sup_{m\ge 1} m^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}+\frac{d-2}{2}(\frac{1}{p}-\frac{1}{q})} \le C.$$

6.4.2 The sharp Pitt inequality

The following sharp Pitt inequality has been recently proved in [DeGoTi]:

Theorem 6.4.1. *If $1 \le p \le 2$ and $s = (d-1)\left(\frac{1}{2} - \frac{1}{p}\right)$, then for every $Y_k \in \mathcal{H}_k^d$ and every radial $f \in \mathcal{S}(\mathbb{R}^d)$, the Pitt inequality*

$$\|\, |y|^{-s}\widehat{fY_k}(y)\|_{L^{p'}(\mathbb{R}^d)} \le C\|\, |x|^s f(x) Y_k(x)\|_{L^p(\mathbb{R}^d)} \tag{6.4.3}$$

holds with the best constant

$$C = (2\pi)^{\frac{d}{2}} 2^{\frac{1}{2}-\frac{1}{p'}} \frac{p^{\frac{(2k+d-1)p+2}{4p}} \Gamma\left(\frac{(2k+d-1)p'+2}{4}\right)^{\frac{1}{p'}}}{(p')^{\frac{(2k+d-1)p'+2}{4p'}} \Gamma\left(\frac{(2k+d-1)p+2}{4}\right)^{\frac{1}{p}}} \sup_{Y_k \in \mathcal{H}_k^d} \frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}}. \quad (6.4.4)$$

According to Theorem 6.1.1, we have

$$\sup_{Y_k \in \mathcal{H}_k^d} \frac{\|Y_k\|_{L^{p'}(\mathbb{S}^{n-1})}}{\|Y_k\|_{L^p(\mathbb{S}^{n-1})}} \sim k^{(d-2)(\frac{1}{p}-\frac{1}{2})},$$

whereas only the weaker estimate (6.1.5) was obtained in [DeGoTi].

Part III

On the convergence of cardinal interpolations thought parametered radial basis functions[2]

[2]A version of this part is submitted for publication.

7.1 introduction

Approximation and interpolation in multiple (here: d) dimensions of functions and data by computationally simpler expressions is a task that is often addressed for instance by using linear combinations of shifts of a single kernel function. This is because the computation of the aforementioned approximant or interpolant is greatly simplified in this way especially when the said kernel function has certain symmetries for example. Especially in high dimensions $d \gg 1$, one type of symmetry is resulting from using a radially symmetric kernel $\varphi(\|\cdot\|) : \mathbb{R}^d \to \mathbb{R}$; here and anywhere else the norm $\|\cdot\|$ is Euclidean and the radial part $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is called the *radial basis function.*

Various different approaches to approximate the approximand f may be taken; when going back to the radial basis functions, for instance one may work by varying on the positions of the shifts – here called centres because of the radial symmetry about them – and among them we wish to study cardinal interpolation on equally spaced data. Indeed, the problem of interpolating to a multivariate function on an integer grid using the radial basis function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is formulated classically in the following way: given the continuous function $f : \mathbb{R}^d \to \mathbb{R}$ (the approximand), find a set of real coefficients $\{d_k\}_{k\in\mathbb{Z}^d}$ such that

$$If(x) = \sum_{k\in\mathbb{Z}^d} d_k \varphi(\|x-k\|), \qquad x \in \mathbb{R}^d,$$

is well-defined (the sum converges at a minimum quadratically, thus we may not in certain cases evaluate pointwise everywhere) and agrees with f everywhere on $\mathbb{Z}^d$. Alternatively, and this is our approach here, we may initially try to find coefficients $\{c_k\}_{k\in\mathbb{Z}^d}$ such that the so-called cardinal function

$$\chi(x) = \sum_{k\in\mathbb{Z}^d} c_k \varphi(\|x-k\|), \qquad x \in \mathbb{R}^d, \tag{7.1.1}$$

is an absolutely convergent sum with the cardinality conditions $\chi(j) = \delta_{0,j}$ for all multi-integers $j \in \mathbb{Z}^d$, where δ is the Dirac functional, that is, $\delta_{s,t} = 1$ if $s = t$ and $\delta_{s,t} = 0$ if $s \neq t$. We then set

$$If(x) = \sum_{k\in\mathbb{Z}^d} f(k)\chi(x-k), \qquad x \in \mathbb{R}^d, \tag{7.1.2}$$

whenever the approximant's sum (7.1.2) converges absolutely or at a minimum in an L^2-sense. In the latter case we may be unable to evaluate pointwise but may consider the error

$$\|f - If\|_2$$

nontheless.

This approach provides a useful and flexible family of approximants for many choices of φ. For instance, the famous multiquadric radial basis function (MQ) $\varphi(r) = \varphi_c(r) = \sqrt{r^2 + c^2}$, further inverse multiquadrics (IM)

$$\varphi(r) = \frac{1}{\sqrt{r^2 + c^2}},$$

inverse quadratics (IQ)

$$\varphi(r) = \frac{1}{r^2 + c^2},$$

which all unify and generalise in

$$\varphi_{c\gamma}(r) = \left(r^2 + c^2\right)^\gamma, \qquad \gamma \notin \mathbb{Z}_+;$$

nonnegative integers are forbidden because they force the radial function composed with the Euclidean norm to be simply a polynomial of degree 2γ in d unknowns. Finally, the popular Gaussians (GA) $\varphi(r) = \exp(-(cr)^2)$, the Poisson kernel $\varphi(r) = \exp(-cr)$ and shifted thin-plate spline radial basis function $\varphi(r) = (r^2 + c^2)\log(r^2 + c^2)$.

However, in this article we will focus mostly on the multiquadrics $\varphi_c(r) = \sqrt{r^2 + c^2}$ with real parameter c and its aforementioned generalisation for γ not a nonnegative integer

$$\varphi_{c\gamma}(r) = \left(r^2 + c^2\right)^\gamma.$$

In this case, the existence of the cardinal function $\chi = \chi_c$ defined by (7.1.1) was confirmed for example by the first author [Buh2], where it is furthermore proved that for instance beginning in one dimension and for the multiquadrics proper it is true that at a minimum

$$|\chi_c(x)| = O(\|x\|^{-5}) = O_c(\|x\|^{-5}) \qquad \text{as} \quad \|x\| \to \infty,$$

with the constant absorbed in $O = O_c$ being dependent on c but not on x. This is a first indication that the convergence of the infinite series for the cardinal interpolants may also be hoped for in the context of some polynomially increasing approximands f or indeed polynomials $p = f$ of certain degrees themselves.

Continuing now, from the broad theory in Chapter 4 in [Buh1], and when c is not zero, it follows that for the generalised multiquadrics function we get further decay estimates of

$$|\chi_c(x)| = O_c(\|x\|^{-4\gamma-3d}), \qquad \text{as} \quad \|x\| \to \infty, \tag{7.1.3}$$

for $x \in \mathbb{R}^d$ so long as $2\gamma + d$ is an even positive integer, and in all other cases

$$|\chi_c(x)| = O_c(\|x\|^{-2\gamma-2d}), \qquad \text{as} \quad \|x\| \to \infty. \tag{7.1.4}$$

Then, a frequently occurring question is whether the limits of interpolants (7.1.2) will recover the original function on the whole space either immediately or indeed asymptotically when the parameter c tends to infinity – which makes the radial basis functions "increasingly flat" in a term coined by Fornberg and Larsson [FoLa]. This aspect of radial basis function interpolation and its numerical solution is useful because it also concerns the numerical problem with ill-conditioned matrices when solving the mentioned interpolation problems for extreme parameters and how to solve the interpolation problems for the interpolation coefficients efficiently in the face of this ill-conditioning.

An earlier paper [Bax] by Baxter gave out certain sufficient conditions on functions f such that (7.1.2) uniformly converges to f on $\mathbb{R}^d$ when the parameter c tends to infinity. More precisely, the result is stated in the following theorem.

Theorem 7.1.1. *[Bax] Given a continuous function $f \in L^2(\mathbb{R}^d)$, whose square-integrable Fourier transform $\hat{f}$ is compactly supported in $[-\pi, \pi]^d$, so that it is band-limited, then the interpolant*

$$I_c f(x) = \sum_{k \in \mathbb{Z}^d} f(k)\chi_c(x-k), \quad x \in \mathbb{R}^d, \tag{7.1.5}$$

is well-defined in $L^2(\mathbb{R}^d)$, where χ_c denotes the cardinal function for the integer grid using the classical multiquadric radial function ($\gamma = 1/2$) with parameter *c. Furthermore, it is true that*

$$\lim_{c\to\infty} I_c f(x) = f(x) \tag{7.1.6}$$

uniformly for all arguments on $\mathbb{R}^d$.

In another recent article [**?**] by Ledford, the author established the similar result (see [**?**, Theorem2]) with respect to a relatively general family of basis functions. But in [**?**] it is still required all approximand functions satisfying the same conditions. However, Powell [Pow, Section 5] had pointed out that (7.1.6) holds for $f(x) = x^2$, which, obviously, as an approximand does not in fact satisfy the conditions of Theorem 7.1.1. Therefore, the central purpose of this paper is to extend the uniform approximation property (7.1.6) by relaxing the requirements on the approximands much further.

Our first main result establishes the uniform convergence of (7.1.6) for L^p-integrable functions, $1 < p < \infty$, with limited support of Fourier transforms. Also, it is shown that such approximation is true under the corresponding derivatives.

Theorem 7.1.2. *Let* $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, *with a Fourier transform* $\hat{f}$ *in the distributional sense. If the radial basis function in use is the generalised multiquadric function and* $\hat{f}$ *is supported in* $[-\pi, \pi]^d$, *we have that*

$$\lim_{c\to\infty} I_c f(x) = f(x) \tag{7.1.7}$$

uniformly on $\mathbb{R}^d$. *More generally, for any* $\alpha \in \mathbb{Z}_+^d$,

$$\lim_{c\to\infty} \partial^\alpha I_c f(x) = \partial^\alpha f(x) \tag{7.1.8}$$

uniformly on $\mathbb{R}^d$, *where* ∂^α *is a short notation for the partial derivative*

$$\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}$$

of order $\alpha \in \mathbb{Z}_+^d$.

Remarks.

In this sense, χ_c can be seen as a generalisation of the sinc function which provided the famous sampling theorem (see [Jer]). However, the sinc function decays far too slowly, so it is not very well localised, and it has to be used employing the tensor product form in the high dimensional case.

By Paley-Wiener's theorem, the functions satisfying the conditions in Theorem 7.1.2 can be extended to entire functions of exponential type at most π. For details, one can refer to [St3] and [PlPo].

The conclusions of Theorem 7.1.2 are still justified for any radial basis function with its Fourier transform using the modified Bessel functions K_{v_j} in the form of

$$\hat{\phi}_c(r) = \sum_{j=1}^{m} g_j(r) c^{s_j} \frac{K_{v_j}(cr)}{r^{v_j}},$$

where for each $j = 1, \ldots, m$, v_j being always positive, $s_j \in \mathbb{R}_+$, and g_j are univariate functions which have continuous derivatives with g_j and g_j' possessing at most polynomial growth.

Notice that when $p = \infty$, (7.1.6) may not be true. To see this, one can consider $f(x) = \sin \pi x$ as an example, which is nonzero but vanishes at every integer. In this view, we turn to establish (7.1.6) as well for approximand functions, which are in some special forms-Fourier transform of Borel measure, Fourier-Stieltjes integral and multivariate polynomials, respectively.

Theorem 7.1.3. *Let f be a multivariate function on $\mathbb{R}^d$ which is band-limited and defined by a Fourier transform of any Borel measure, that is*

$$f(x) = \int_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\mu(u), \tag{7.1.9}$$

where μ is a Borel measure on $\mathbb{R}^d$ with $\mu([-\pi,\pi]^d) < \infty$. The $\cdot$ denotes the usual inner product. Then we still have for the generalised multiquadric radial basis function

$$\lim_{c\to\infty} I_c f(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

Theorem 7.1.4. *Let f be a multivariate function on $\mathbb{R}^d$ defined by a Fourier-*

Stieltjes integral, that is

$$f(x) = \int_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\alpha_1(u_1) \cdots d\alpha_d(u_d), \qquad x \in \mathbb{R}^d,\ u = (u_1, \ldots, u_d), \tag{7.1.10}$$

where each $\alpha_j(u_j)$, $j = 1, \ldots, d$, *is of bounded variation in* $[-\pi, \pi]$ *with* $\alpha_j(-\pi + 0) - \alpha_j(-\pi) = \alpha_j(\pi) - \alpha_j(\pi - 0)$. *The cardinal interpolation in multiple dimensions using the aforementioned cardinal function* χ_c *with radial basis functions* $\varphi_{c\gamma} = (r^2 + c^2)^\gamma$ *will then in fact satisfy for all* γ *that are not non-negative integers*

$$\lim_{c \to \infty} I_c f(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

Theorem 7.1.5. *If* f *is a multivariate polynomial on* $\mathbb{R}^d$ *of degree componentwise less than* $4\gamma + 3d - 1$ *when* $2\gamma + d$ *is even or* $2\gamma + 2d - 1$ *for all other cases, it enjoys for the generalised multiquadric function the identity* (7.1.6) *pointwise with an absolutely convergent infinite sum. For a polynomial of degree componentwise less than* $4\gamma + 3d - 1/2$ *when* $2\gamma + d$ *is even or* $2\gamma + 2d - 1/2$ *for all other cases, the same is true in the sense of* L^2 *with a square summable series. So the* L^2*-error of the difference between approximand and approximant vanishes.*

We remark that the generalisation also could be seen easily by applying Theorem 7.1.4 to the example $f(x) = \cos \pi x$ as approximand for which therefore Theorem 7.1.1, Theorem 7.1.3 and Theorem 7.1.5 are not applicable. Also the observation of Powell [Pow, Section 5] about $f(x) = x^2$ is justified by Theorem 7.1.5.

In the papers [Bax] and [**?**], the authors essentially accomplished their proofs by applying the limit behaviour of $\widehat{\chi}_c$, the Fourier transform of cardinal function χ_c. However, in our cases it is no longer enough for the proofs. Hence, in the next section after recalling some well known facts we will first establish some pointwise estimate of χ_c. Then, in particular, taking into account special properties of the modified Bessel functions we gave an estimate of a sum of χ_c and its derivatives, which are crucial for the proofs of our main results. Finally, we will complete that section by proving Theorem 7.1.2, Theorem 7.1.3, Theorem 7.1.4 and Theorem 7.1.5.

7.2 limit for the parameter of cardinal interpolation with RBF

Again, the radial basis function we consider is called the generalised multiquadric with a parameter $c > 0$ and a nonzero parameter γ, not a positive integer, where incidentally for positive exponent γ also $c = 0$ is explicitly allowed,

$$\varphi_{c\gamma}(r) = \left(r^2 + c^2\right)^{\gamma}, \quad r > 0.$$

As it is well known, the Fourier transform preserves the radial symmetry property; that is, if f is a radial function on $\mathbb{R}^d$, its Fourier transform satisfies that

$$\widehat{f}(\xi) = \widehat{f}(\eta), \qquad \text{if } \|\xi\| = \|\eta\|,\ \xi, \eta \in \mathbb{R}^d.$$

So for convenience, given a fixed dimension d, we define

$$\widehat{\varphi_{c\gamma}}(r) := \widehat{\Phi_{c\gamma}}(x), \qquad r = \|x\|, \quad x \in \mathbb{R}^d,$$

with $\Phi_{c\gamma}(x) = \varphi_{c\gamma}(\|x\|)$. Here and in what follows, we specify the Fourier transform normalised incidentally as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) \exp(-ix \cdot \xi)\, dx, \qquad \xi \in \mathbb{R}^d. \tag{7.2.1}$$

So long as we have the classical case $\gamma = \frac{1}{2}$, $\widehat{\varphi_{c\gamma}}$ can be formulated as

$$\widehat{\varphi_{c\gamma}}(r) = \widehat{\varphi_{c,1/2}}(r) = -\frac{(2\pi c)^{(d+1)/2} K_{(d+1)/2}(cr)}{\pi r^{(d+1)/2}},$$

where $K_{(d+1)/2}$ is modified Bessel function with degree $(d+1)/2$. In particular, for the one-dimensional case with $\gamma = 1/2$, we have the simple expression

$$\widehat{\varphi_{c\gamma}}(\|x\|) = \widehat{\varphi_{c,1/2}}(\|x\|) = -\frac{2cK_1(c\|x\|)}{\|x\|} = -2\int_1^{\infty} \exp(-c\|x\|t)(t^2-1)^{\frac{1}{2}} dt. \tag{7.2.2}$$

Now, in the general d-dimensional case for γ not a nonnegative integer and $c > 0$,

$$\widehat{\varphi_{c\gamma}}(r) = -2\Gamma(\gamma+1)\pi^{d/2-1}(2c/r)^{\gamma+d/2} \sin(\pi\gamma) K_{\gamma+d/2}(cr)$$

which has an integral representation as

$$-2\pi^{(d-1)/2}c^{2\gamma+d}\frac{\Gamma(\gamma+1)\sin(\pi\gamma)}{\Gamma\Big(\gamma+\frac{d+1}{2}\Big)}\int_1^\infty \exp(-crt)(t^2-1)^{\gamma+\frac{d-1}{2}}dt, \tag{7.2.3}$$

and for the case $c = 0$, $\gamma > 0$, not integral,

$$\widehat{\varphi_{0\gamma}}(r) = -\Gamma\left(\gamma+\frac{d}{2}\right)\Gamma(1+\gamma)\sin(\pi\gamma)2^{2\gamma+d}\pi^{d/2-1}r^{-2\gamma-d}.$$

For further details of above formulae, one can refer to [Jon] for instance.

Especially the exponential decay of $\widehat{\varphi_{c\gamma}}$ for large argument is essential for our proofs, that is, for $0 < \|\xi\| < \|\eta\|$ in particular

$$|\widehat{\varphi_{c\gamma}}(\|\eta\|)| \le \exp\Big[-c(\|\eta\|-\|\xi\|)\Big]\,|\widehat{\varphi_{c\gamma}}(\|\xi\|)|\,, \tag{7.2.4}$$

which is in fact a slight generalisation of Lemma 2.1 in [Bax] and is guaranteed by an asymptotic behavior of modified Bessel functions (see [AbSt, 9.7.2]); that is, for any degree $v \in \mathbb{R}_+$,

$$K_v(x) \sim \frac{e^{-x}}{\sqrt{x}}, \quad x \to +\infty, \tag{7.2.5}$$

where $A \sim B$ means there is a constant θ independent of x such that $\theta^{-1}A \le B \le \theta A$. Apart from this, we need two more facts on modified Bessel functions. Namely,

$$K_v(x) \ge \sqrt{\frac{\pi}{2}}\frac{e^{-x}}{\sqrt{x}}, \quad x > 0, \;\; |v| \ge \frac{1}{2}, \tag{7.2.6}$$

and the formulas for derivatives (see for instance [AbSt, 9.6.28]), that is

$$\frac{d}{dz}\frac{K_v(z)}{z^v} = -\frac{K_{v+1}(z)}{z^v}, \;\; z \in \mathbb{C}. \tag{7.2.7}$$

Furthermore, due to [Buh2], with respect to the generalised multiquadric radial function again, the cardinal function defined by (7.1.1) in $\mathbb{R}^d$ exists, and its Fourier transform is given by

$$\widehat{\chi}_c(x) = \frac{\widehat{\varphi_{c\gamma}}(\|x\|)}{\sum_\ell \widehat{\varphi_{c\gamma}}(\|x+2\pi\ell\|)}, \tag{7.2.8}$$

where the sum is taken over all d-dimensional multi-integers ℓ. Based on this, the following two lemmas provide us with further details about the cardinal function χ_c and its Fourier transform.

Lemma 7.2.1. *For any* $u \in (-\pi, \pi)$,

$$|1 - \widehat{\chi}_c(u)| \leq e^{-c|\pi - u|}, \tag{7.2.9}$$

and for $u \in \mathbb{R} \setminus [-\pi, \pi]$*, say* $u = \zeta + 2\pi k$ *with* $k \geq 1$ *and* $\zeta \in (-\pi, \pi)$,

$$|\widehat{\chi}_c(u)| \leq e^{-c\pi k} + e^{-c|\pi - \zeta|}. \tag{7.2.10}$$

Remarks.

(i) Lemma 7.2.1 can be seen as a deeper characterisation of Proposition 2.2 in [Bax]. For the clarity of presentation, it is convenient to rewrite it as a lemma.

(ii) This result can be easily extended to any high dimensional case $\mathbb{R}^d$ by replacing $|\pi - \zeta|$ and k in (7.2.9), (7.2.10) by $\sigma_d(\zeta)$ and $|\kappa|_\infty$ respectively, here $|k|_\infty = \max |k_j|$ and

$$\sigma_d(\zeta) = \min\{|\pi\underline{\varepsilon} - \zeta| : \underline{\varepsilon} \in \{-1, 0, 1\}^d, \underline{\varepsilon} \neq 0\}, \quad \zeta \in (-\pi, \pi)^d. \tag{7.2.11}$$

(iii) Through the inverse Fourier transform, this lemma immediately implies that

$$|\partial^\alpha \chi_c(x)| \leq A, \qquad \alpha \in \mathbb{Z}_+^d, \ x \in \mathbb{R}^d, \tag{7.2.12}$$

where A is a constant independent of c and x.

Proof. By using (7.2.8),

$$|\widehat{\chi}_c(2\pi - u)| \leq \frac{\widehat{\varphi_{c\gamma}}(|2\pi - u|)}{\widehat{\varphi_{c\gamma}}(|u|)}.$$

Since $|2\pi - u| - |u| \geq |\pi - u|$ for $u \in [0, \pi)$, by (7.2.4) we have (7.2.9) immediately.

Similarly, when $k \geq 2$, notice that

$$|2k\pi - u| = |(2k-1)\pi + \pi - u| \geq (2k-1)\pi,$$

which means $|2k\pi - u| - |u| \geq (2k-2)\pi \geq k\pi$, and therefore (7.2.10) holds by (7.2.8) and (7.2.4). $\square$

Lemma 7.2.2. *For any* $\varepsilon > 0$,

$$\sum_{j\in\mathbb{Z}^d} |\chi_c(x+j)|^{1+\varepsilon} < A < \infty, \tag{7.2.13}$$

where A *is a constant independent of* c *and* x*. Furthermore, for any* $\alpha \in \mathbb{Z}_+^d$,

$$\sum_{j\in\mathbb{Z}^d} |\partial^\alpha \chi_c(x+j)|^{1+\varepsilon} < A' < \infty, \tag{7.2.14}$$

where A' *is a constant independent of* c *and* $x \in \mathbb{R}^d$.

Proof. It will be instructive to consider first the one dimensional case where the arguments can be transferred to the higher dimensional situation easily under a slight change.

By combining (7.2.8), (7.2.2) and (7.2.7), after a straightforward calculation, we have that

$$\begin{aligned}
\widehat{\chi}_c'(\xi) &= \frac{c\left[\frac{K_1(c|\xi|)}{|\xi|}\sum_\ell \frac{K_2(c|\xi+2\pi\ell|)H(\xi+2\pi\ell)}{|\xi+2\pi\ell|} - \frac{K_2(c|\xi|)}{|\xi|}\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]}{\left[\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]^2} \\
&= \frac{c\left[\frac{K_1(c|\xi|)}{|\xi|}\sum_{\ell\neq 0} \frac{K_2(c|\xi+2\pi\ell|)H(\xi+2\pi\ell)}{|\xi+2\pi\ell|} - \frac{K_2(c|\xi|)}{|\xi|}\sum_{\ell\neq 0} \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]}{\left[\sum_\ell \frac{K_1(c|\xi+2\pi\ell|)}{|\xi+2\pi\ell|}\right]^2},
\end{aligned}$$

where $H(x) = 1$ for $x \geq 0$ and $H(x) = -1$ otherwise. Now, suppose that the parameter c is sufficiently large, by using (7.2.5) and (7.2.6), we have that

$$|\widehat{\chi}_c'(\xi)| \lesssim \begin{cases} ce^{-c|\pi-\zeta|}, & |k| \leq 1; \\ cke^{-c\pi k}, & |k| > 1, \end{cases} \tag{7.2.15}$$

for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi, \pi)$ and $k \in \mathbb{Z}$ and $|\xi| > \varepsilon$. Here and in what follows we use $\lesssim$ to denote that there is an extra constant independent of c in the proposed upper bound.

Note that in case of choosing $\xi = \pi$ for example, the first infinite sum in the numerator in the pen-ultimate display cancels, that is

$$\sum_{\ell} \frac{K_2(c|\xi + 2\pi\ell|)H(\xi + 2\pi\ell)}{|\xi + 2\pi\ell|}$$

vanishes, which results in a nonzero numerator, because the two series no longer annul each other asymptotically, and explains the c factor for $\pm\pi$ as arguments in $\widehat{\chi}_c'(\xi)$.

Therefore, by symmetry,

$$\int_{-\infty}^{\infty} |\widehat{\chi}_c'(\xi)| d\xi = 2\sum_{k=0}^{\infty} \int_{[-\pi,\pi]} |\widehat{\chi}_c'(\zeta + 2\pi k)| d\zeta < B < \infty, \tag{7.2.16}$$

where $B > 0$ is independent of the parameter c. It turns out that

$$|\chi_c(x)| \leq \frac{1}{2\pi|x|} \left| \int_{-\infty}^{\infty} e^{ix\xi} \widehat{\chi}_c'(\xi) d\xi \right| \leq \frac{B}{|x|} \tag{7.2.17}$$

which, by combining with (7.2.12), implies the desired (7.2.13) for $d = 1$.

Then, for the general dimensional case, using the same argument we can obtain that for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi, \pi)^d$ and $k \in \mathbb{Z}^d$,

$$|\partial_\xi^{\mathbb{1}} \widehat{\chi}_c(\xi)| \lesssim \begin{cases} ce^{-c\sigma_d(\zeta)}, & |k|_\infty \leq 1; \\ ce^{-c\pi k}, & |k|_\infty > 1, \end{cases}$$

where $\partial_\xi^{\mathbb{1}} = \frac{\partial^d}{\partial\xi_1 \cdots \partial\xi_d}$ and σ_d is as defined in (7.2.11) . This immediately implies that

$$|\chi_c(x)| \leq \frac{1}{2\pi \prod_{j=1}^d |x_j|} \left| \int_{\mathbb{R}^d} e^{ix\cdot\xi} \partial_\xi^{\mathbb{1}} \widehat{\chi}_c(\xi) d\xi \right| \leq \frac{B'}{\prod_{j=1}^d |x_j|}$$

with B' independent of c, x and thus (7.2.13) is justified.

Finally, to prove (7.2.14), when $d = 1$ we notice that (7.2.15) implies the analogues of (7.2.15), (7.2.16) and (7.2.17); that is, for any $a \in \mathbb{Z}_+$, we have that

$$|\xi^a \widehat{\chi}_c'(\xi)| \lesssim \begin{cases} ce^{-c|\pi-\zeta|}, & |k| \leq 1; \\ ck^a e^{-c\pi k}, & |k| > 1, \end{cases}$$

for $\xi = \zeta + 2\pi k$ with $\zeta \in (-\pi, \pi)$, $k \in \mathbb{Z}$, and there is a constant B'' independent of c such that

$$\int_{-\infty}^{\infty} |\xi^a \widehat{\chi}_c'(\xi)| d\xi < B'' \tag{7.2.18}$$

and

$$\left|\frac{d^a}{dx^a}\chi_c(x)\right| \le \frac{1}{2\pi|x|}\left|\int_{-\infty}^{\infty}(i\xi)^a e^{ix\xi}\widehat{\chi}_c'(\xi)d\xi\right| \le \frac{B''}{2\pi|x|}, \quad |x|>0. \tag{7.2.19}$$

Consequently, with (7.2.12) we can conclude (7.2.14) for one dimension and indeed for any higher dimension. □

Now we are in the position to prove Theorem 7.1.2.

Proof of Theorem 7.1.2: Suppose that $f \in L^p(\mathbb{R}^d)$, $1 < p < \infty$, with its Fourier transform supported in $[-\pi,\pi]^d$. Let $f_n \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ such that $\operatorname{supp}\hat{f} \subset [-\pi,\pi]$ and $f_n \to f$ in $L^p(\mathbb{R}^d)$. Here $n \in \mathbb{N}$. For instance, one can set

$$f_n(x) = f * g_n(x)$$

with $g_n(x) = (n/\pi)^{d/2}e^{-n\|x\|^2}$. Here, the star denotes the classical convolution by integrals. Noticing the Nikolskii type inequality for exponential type obtained by Nessel and Wilmes [NeWi, Theorem 3], f_n also converges to f uniformly as $n \to \infty$.

Then by Hölder's inequality, for any $x \in \mathbb{R}^d$ and $p > 1$ with $p' = p/(p-1)$, we have

$$|I_c(f_n)(x) - I_c(f)(x)| \le \left(\sum_j |f_n(j) - f(j)|^p\right)^{1/p}\left(\sum_j |\chi_c(x-j)|^{p'}\right)^{1/p'},$$

which, with Lemma 7.2.2 and Plancherel-Pólya's theorem (see, for instance [PlPo]), implies that

$$|I_c(f_n)(x) - I_c(f)(x)| \le C\|f_n - f\|_p,$$

where C is a constant dependent on p but not dependent on x, n and c. Therefore, since

$$|I_c(f)(x) - f(x)| \le |I_c(f)(x) - I_c(f_n)(x)| + |I_c(f_n)(x) - f_n(x)| + |f_n(x) - f(x)|,$$

by applying Theorem 7.1.1 to f_n, we conclude (7.1.7), and the same is true for (7.1.8) by a similar argument using the statement about the sum of partial derivatives from the previous lemma. □

Next we turn to prove Theorem 7.1.3, Theorem 7.1.4 and Theorem 7.1.5, which will be essentially relying on the following Lemma 7.2.4. However, for more clarity of the presentation, before that we state a corollary of Lemma 7.2.1 since it will be used many times in the proof of Lemma 7.2.4.

Corollary 7.2.3. *Let m_1, m_2 be any two nonnegative integers with $m_1+m_2 = d$. Then for $u \in (-\pi,\pi)^d$, the series*

$$\sum_{k_1\in\mathcal{A}_1 \ \ or \ \ k_2\in\mathcal{A}_2} \widehat{\chi_c}\big(2\pi k_1 + u, \pi k_2\big) \lesssim e^{-c\sigma_d(u)}, \quad as \quad c\to\infty, \tag{7.2.20}$$

where $\mathcal{A}_1 = \mathbb{Z}^{m_1}\setminus\{0\}$ and $\mathcal{A}_2 = \mathbb{Z}^{m_2}\setminus\{0,-1,1\}^{m_2}$.

Lemma 7.2.4. *Let χ_c be the cardinal interpolation function as above, employing the said generalised multiquadric function $\varphi_{c\gamma}$. Then for any $x = (\underline{x}_1, \underline{x}_2) \in \mathbb{R}^d$ with $\underline{x}_1 \in \mathbb{R}^{m_1}$, $\underline{x}_2 \in \mathbb{R}^{m_2}$, and $m_1 + m_2 = d$, m_1, m_2 being nonnegative integers, if $u \in (-\pi,\pi)^{m_1}$,*

$$\left|\sum_{j_1\in\mathbb{Z}^{m_1}}\sum_{j_2\in\mathbb{Z}^{m_2}} e^{ij_1\cdot u}(-1)^{j_2}\chi_c(\underline{x}_1 - j_1, \underline{x}_2 - j_2) - e^{i\underline{x}_1\cdot u}\cos(\pi\underline{x}_2)\right| \lesssim e^{-c\sigma_d(u)}, \tag{7.2.21}$$

as $c\to\infty$, where $\sigma_d(u)$ is as defined in (7.2.11). Here and anywhere else we adopt the convention that for some nonnegative integer m we have $(-1)^\alpha := (-1)^{\alpha_1}\cdots(-1)^{\alpha_m}$ if $\alpha = (\alpha_1,\ldots,\alpha_m) \in \mathbb{Z}^m$ and $x\cdot y$, for $x, y \in \mathbb{R}^m$, is the inner product as before, and $\cos x$ denotes the componentwise product $\cos x_1 \cdots \cos x_{\hat{d}}$ if $x = (x_1,\ldots,x_{\hat{d}}) \in \mathbb{R}^{\hat{d}}$, any $\hat{d}\in\mathbb{N}$.

In particular, when m_2 vanishes, one can simplify (7.2.21) as the estimate

$$\left|\sum_{j\in\mathbb{Z}^d} e^{ij\cdot u}\chi_c(\underline{x} - j) - e^{i\underline{x}\cdot u}\right| \lesssim e^{-c\sigma_d(u)}, \quad u\in(-\pi,\pi)^d,\ \underline{x}\in\mathbb{R}^d. \tag{7.2.22}$$

Proof. Recall that with the specification of the Fourier transform (7.2.1), the Poisson summation formula states that, if for example – see e.g. [?] also for weaker requirements –

$$|f(x)| + \left|\widehat{f}(x)\right| = O\Big(1 + \|x\|^{-d-\epsilon}\Big) \ \text{ with some } \ \epsilon > 0, \tag{7.2.23}$$

then it is true that

$$\sum_{j\in\mathbb{Z}^d} f(x-j) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(2\pi k)e^{-2\pi i x\cdot k}, \qquad x\in\mathbb{R}^d. \tag{7.2.24}$$

Now, the proof is the same in all dimensions, but the description is simpler for $\mathbb{R}^2$, so first our proof is carried out for $\mathbb{R}^2$, and next we indicate the necessary changes to the desired generalisation to higher dimensions.

Obviously, the decay properties (7.1.3), (7.1.4) and (7.2.4) guarantee the requirement (7.2.23). Therefore, for fixed $x_1, x_2 \in \mathbb{R}$, since

$$\begin{aligned}
&\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} e^{ij_1u}(-1)^{j_2}\chi_c(x_1-j_1, x_2-j_2) \\
&= e^{ix_1u}\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} e^{-i(x_1-j_1)u}\Big[\chi_c\Big(x_1-j_1, 2\Big(\frac{x_2}{2}-j_2\Big)\Big)-\chi_c\Big(x_1-j_1, 2\Big(\frac{x_2-1}{2}-j_2\Big)\Big)\Big],
\end{aligned}$$

and by using the stated Poisson summation formula (7.2.24) with

$$f(x) = e^{-ix_1u}\chi_c(x_1, 2x_2),$$

we have that

$$\begin{aligned}
&\sum_{j_1\in\mathbb{Z}}\sum_{j_2\in\mathbb{Z}} e^{ij_1u}(-1)^{j_2}\chi_c(x_1-j_1, x_2-j_2) \\
&= \frac{e^{ix_1u}}{2}\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} \widehat{\chi}_c(2\pi k_1+u, \pi k_2)e^{-2i\pi k_1x_1}\left[e^{-i\pi k_2x_2}-e^{-i\pi k_2(x_2-1)}\right] \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = e^{ix_1u}\Big[\widehat{\chi}_c(u,\pi)e^{-i\pi x_2}+\widehat{\chi}_c(u,-\pi)e^{i\pi x_2}\Big]$$

and

$$I_2 = \frac{e^{ix_1u}}{2}\sum_{k_1\neq 0 \text{ or } |k_2|>1} \widehat{\chi}_c(2\pi k_1+u, \pi k_2)e^{-2i\pi k_1x_1}\left[e^{-i\pi k_2x_2}-e^{-i\pi k_2(x_2-1)}\right].$$

Notice that by using the symmetry of (7.2.8),

$$
\begin{aligned}
&\left|\widehat{\chi}_c(u,\pi)-\frac{1}{2}\right| \\
=&\left|\widehat{\chi}_c(u,-\pi)-\frac{1}{2}\right| \\
=&\left|\frac{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}{\sum_{\ell_1,\ell_2\in\mathbb{Z}}\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi+2\pi\ell_2)\|\big)}-\frac{1}{2}\right| \qquad (7.2.25)\\
=&\left|\left(2+\sum_{\substack{|\ell_1|\geq 1\\ \ell_2\neq 0,-1}}\frac{\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi+2\pi\ell_2)\|\big)}{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}\right)^{-1}-\frac{1}{2}\right| \\
\leq&\left|\left(2+\sum_{\substack{|\ell_1|\geq 1\\ \ell_2\neq 0,\pm 1}}\frac{\widehat{\varphi_{c\gamma}}\big(\|(u+2\pi\ell_1,\pi\ell_2)\|\big)}{\widehat{\varphi_{c\gamma}}\big(\|(u,\pi)\|\big)}\right)^{-1}-\frac{1}{2}\right| \\
=&o(1), \qquad (7.2.26)
\end{aligned}
$$

which uniformly approaches zero as $c\to\infty$ after a straightforward calculation by using (7.2.4).

For I_2, using (7.2.8) again, we have that

$$|I_2|\leq\sum_{k_1\neq 0\ \text{or}\ |k_2|>1}\widehat{\chi}_c\big(2\pi k_1+u,\pi k_2\big)\lesssim e^{-c\sigma_1(u)}, \qquad (7.2.27)$$

where the last step follows from Corollary 7.2.3.

Then, with a slight modification, the proof works equally well for the remaining cases when for instance $m_1=2$ and $m_1=0$ and therefore we have completed the proof now for 2-dimensional case.

In the general $\mathbb{R}^d$ case, with $m_1+m_2=d$, m_1,m_2 being both nonnegative integers, and $u \in (-\pi, \pi)^{m_1}$,

$$\begin{aligned}
&\sum_{\substack{j_1 \in \mathbb{Z}^{m_1} \\ j_2 \in \mathbb{Z}^{m_2}}} e^{ij_1 \cdot u} (-1)^{j_2} \chi_c(\underline{x}_1 - j_1, \underline{x}_2 - j_2) \\
&= e^{i\underline{x}_1 \cdot u} \sum_{s_2 \in \{0,1\}^{m_2}} \sum_{\substack{j_1 \in \mathbb{Z}^{m_1} \\ j_2 \in \mathbb{Z}^{m_2}}} e^{-i(\underline{x}_1 - j_1)\cdot u} (-1)^{s_2} \chi_c(\underline{x}_1 - j_1, \underline{x}_2 - s_2 - 2j_2) \\
&= \frac{e^{i\underline{x}_1 \cdot u}}{2^{m_2}} \sum_{\substack{k_1 \in \mathbb{Z}^{m_1} \\ k_2 \in \mathbb{Z}^{m_2}}} \widehat{\chi}_c(2\pi k_1 + u, \pi k_2) e^{i2\pi \underline{x}_1 \cdot k_1} e^{i\pi \underline{x}_2 \cdot k_2} \sum_{s_2 \in \{0,1\}^{m_2}} (-1)^{s_2} e^{-i\pi s_2 \cdot k_2}.
\end{aligned} \tag{7.2.28}$$

Then one can check that for any nonnegative integer m, if $k \in \{1, -1\}^m$,

$$\sum_{s \in \{0,1\}^m} (-1)^s e^{i\pi k \cdot s} = \sum_{s \in \{0,1\}^m} (-1)^{2s} = 2^m, \tag{7.2.29}$$

and if $k \in \{0, 1, -1\}^m \setminus \{1, -1\}^m$, say $k_{i_1} = k_{i_2} = \cdots = k_{i_t} = 0$, $1 \le i_1 < \cdots < i_t \le m$ with a positive integer $0 < t \le m$,

$$\sum_{s \in \{0,1\}^m} (-1)^s e^{i\pi k \cdot s} = \sum_{s_{i_1}, \ldots, s_{i_t} \in \{0,1\}} (-1)^{s_{i_1} + \cdots + s_{i_t}} = 0.$$

By applying these to (7.2.28), it implies that

$$\begin{aligned}
\sum_{j_1 \in \mathbb{Z}^{m_1}, j_2 \in \mathbb{Z}^{m_2}} e^{ij_1 \cdot u} (-1)^{j_2} \chi_c(\underline{x}_1 - j_1, \underline{x}_2 - j_2) &= e^{i\underline{x}_1 \cdot u} \widehat{\chi}_c(u, \pi \underline{e}) \sum_{k_2 \in \{-1,1\}^{m_2}} e^{i\pi \underline{x}_2 \cdot k_2} + J_2 \\
&= 2^{m_2} e^{i\underline{x}_1 \cdot u} \cos(\pi \underline{x}_2) \widehat{\chi}_c(u, \pi \underline{e}) + J_2.
\end{aligned} \tag{7.2.30}$$

Here recall that $\underline{e} = (1, 1, \ldots, 1) \in \mathbb{R}^{m_2}$ and

$$J_2 = \frac{e^{i\underline{x}_1 \cdot u}}{2^{m_2}} \sum_{k_1 \in \mathcal{A}_1 \text{or } k_2 \in \mathcal{A}_2} \left[\widehat{\chi}_c(2\pi k_1 + u, \pi k_2) e^{i2\pi \underline{x}_1 \cdot k_1} e^{i\pi \underline{x}_2 \cdot k_2} \sum_{s_2 \in \{0,1\}^{m_2}} (-1)^{s_2} e^{-i\pi s_2 \cdot k_2} \right]$$

satisfying that with $\mathcal{A}_1 = \mathbb{Z}^{m_1} \setminus \{0\}$ and $\mathcal{A}_2 = \mathbb{Z}^{m_2} \setminus \{0, 1, -1\}^{m_2}$ as before

$$|J_2| \le \sum_{k_1 \in \mathcal{A}_1 \quad \text{or} \quad k_2 \in \mathcal{A}_2} \widehat{\chi}_c\big(2\pi k_1 + u, \pi k_2\big) \lesssim e^{-c\sigma_d(u)}, \;\; c \to \infty,$$

by using Corollary 7.2.3. Moreover, in a similar way as in (7.2.26), we obtain that uniformly in u,

$$\lim_{c\to\infty} \widehat{\chi}_c(u, \pi \underline{e}) = 2^{-m_2}.$$

Then consequently (7.2.30) yields the desired (7.2.21). $\square$

Now we are in the position to prove Theorem 7.1.3, Theorem 7.1.4 and Theorem 7.1.5. Beginning by using Lemma 7.2.4 and the dominated convergence theorem, we can obtain Theorem 7.1.3 directly, basically in the same way as in the following

Proof of Theorem 7.1.4: For the sake of convenience and being concise, we shall carry out the proof only for $d = 2$, while the general case follows in a most similar way.

For each $j = 1, 2$, let

$$\alpha_{j,0}(u) = \begin{cases} \alpha_j(-\pi + 0), & \text{if } u = -\pi, \\ \alpha_j(u), & \text{if } -\pi < u < \pi, \\ \alpha_j(\pi - 0), & \text{if } u = \pi; \end{cases}$$

and define furthermore

$$A_j = \alpha_j(-\pi + 0) - \alpha_j(-\pi), \quad B_j = \alpha_j(\pi) - \alpha_j(\pi - 0) \quad \text{and} \quad C_j = A_j + B_j.$$

Then noticing that $A_j = B_j$, $j = 1, 2$,

$$f(x) = \prod_{j=1}^{2} \left[\int_{-\pi}^{\pi} e^{ix_j u_j}\, d\alpha_{j,0}(u_j) + C_j \cos(\pi x_j) \right]$$

where $u = (u_1, u_2)$, $x = (x_1, x_2)$.

Then, by expanding $f(k)$ for each $k = (k_1, k_2) \in \mathbb{Z}^2$, we have

$$
\begin{aligned}
&|I_c(f)(x) - f(x)| \\
=&|\sum_{k\in\mathbb{Z}^2} f(k)\chi_c(x-k) - f(x)| \\
\le &\int_{[-\pi,\pi]^2} \left| \sum_{k\in\mathbb{Z}^2} e^{ik\cdot u}\chi_c(x-k) - e^{ix\cdot u} \right| d\alpha_{1,0}(u_1)d\alpha_{2,0}(u_2) \\
&+ |C_1| \int_{-\pi}^{\pi} \left| \sum_{k\in\mathbb{Z}^2} (-1)^{k_1} e^{ik_2u_2}\chi_c(x-k) - e^{ix_2\cdot u_2}\cos\pi x_1 \right| d\alpha_{2,0}(u_2)+ \\
&+ |C_2| \int_{-\pi}^{\pi} \left| \sum_{k\in\mathbb{Z}^2} (-1)^{k_2} e^{ik_1u_1}\chi_c(x-k) - e^{ix_1u_1}\cos\pi x_2 \right| d\alpha_{1,0}(u_1)+ \\
&+ |C_1C_2| |\sum_{k\in\mathbb{Z}^2} (-1)^k \chi_c(x-k) - \cos(\pi x)|,
\end{aligned}
$$

which, by applying Lemma 7.2.4 and the continuity of $\alpha_{1,0}, \alpha_{2,0}$, allows us to claim that

$$\lim_{c\to\infty} |f(x) - I_c(f)(x)| = 0$$

uniformly on $x \in \mathbb{R}^2$ and therefore – using the analogous arguments in the general multivariate case – conclude the proof of Theorem 7.1.4. □

In order to prove Theorem 7.1.5, it is sufficient to notice that for example

$$x^2 = 2\lim_{u\to 0^+} \frac{1-\cos(xu)}{u^2} \tag{7.2.31}$$

and

$$x^3 = \lim_{u\to 0^+} \frac{2\sin(xu) - \sin(2xu)}{u^3}.$$

Moreover,

$$x^4 = \lim_{u\to 0^+} \frac{6 - 8\cos(xu) + 2\cos(2xu)}{u^4},$$

and similarly for all other powers. Then, by applying Lemma 7.2.4 again and choosing certain linear combinations, we arrive directly at Theorem 7.1.5. □

We remark that the idea of Theorem 7.1.4 using Fourier-Stieltjes integrals follows from the work of I.J. Schoenberg in [Sch], where it concerns the spline interpolation. Moreover, in [RiSc], he also proved the necessity of the condition (7.1.10). Nonetheless, this problem is still open for our case.

We also remark a straightforward generalisation of the Corollary 7.2.3, where the decay property and the existence of the Lagrange functions are needed and guaranteed by the work in Chapter 4 in [Buh1], and the remaining part of the proof follows the same lines as above.

Corollary 7.2.5. *Let $\underline{\varphi}_c$ be any radial basis function, depending on a positive parameter c, that possesses a generalised Fourier transform $\widehat{\underline{\varphi}}_c$ which is positive, decays exponentially with*

$$\widehat{\underline{\varphi}}_c(r) = O\Big(\exp(-\overline{\alpha}cr)\Big), \qquad c, r \to \infty, \tag{7.2.32}$$

and

$$1/\widehat{\underline{\varphi}}_c(r) = O\Big(\exp(\underline{\alpha}cr)\Big), \qquad c, r \to \infty,$$

for some positive $\underline{\alpha}, \overline{\alpha}$, and has a singularity of positive order $\overline{\mu}$ at the origin. Then the identities of the previous Lemmas 2.1 and 2.2 hold. If moreover, the standard conditions in [Buh1], p. 59, are satisfied, namely for $M > d + \overline{\mu}$ that $\widehat{\underline{\varphi}}_c \in C^M(\mathbb{R}_+)$ with all its derivatives satisfying (7.2.32) and having singularities

$$\widehat{\underline{\varphi}^{(\ell)}}(r) \sim r^{-\overline{\mu}-\ell}$$

at the origin, $\ell = 0, 1, \ldots, M$, then the cardinal function satisfies the decay estimate that at a minimum

$$|\chi_c(x)| = O(\|x\|^{-d-\overline{\mu}})$$

for large argument. Therefore in particular

$$\sum_{j\in\mathbb{Z}^d} \Big|\chi_c(x-j)\Big|$$

is uniformly convergent and bounded for all arguments.

Note that the proof of Theorem 7.1.4 essentially only relies on the decay property of radial basis function $\underline{\varphi}_c$ given in Corollary 7.2.5. Naturally we extend our results to this more general class of radial basis functions. A typical example is the generalised shifted thin-plate spline radial basis function

$$\underline{\varphi}_c(r) = (r^2 + c^2)\log(r^2 + c^2)$$

with Fourier transform

$$2(2\pi)^{d/2}\frac{d}{d\beta}2^{\beta/2}/\Gamma(\beta/2)\bigg|_{\beta=2}K_{d/2+1}(cr)(c/r)^{d/2+1},$$

see for example [BuDa] Example 2.7.

Corollary 7.2.6. *Let f be an entire multivariate function on $\mathbb{C}^d$ defined by a Fourier-Stieltjes integral, that is*

$$f(x) = \int_{[-\pi,\pi]^d} \exp(ix \cdot u)\, d\alpha_1(u_1)\cdots d\alpha_d(u_d), \qquad x \in \mathbb{R}^d,\ u = (u_1,\ldots,u_d),$$

where each $\alpha_j(u_j)$, $j = 1,\ldots,d$, is of bounded variation in $[-\pi,\pi]$ with $\alpha_j(-\pi+0) - \alpha_j(-\pi) = \alpha_j(\pi) - \alpha_j(\pi-0)$. The cardinal interpolation in d dimensions using the aforementioned cardinal function χ_c with radial basis functions $\underline{\varphi}_c$ as given in Corollary 7.2.5 will then in fact satisfy

$$\lim_{c\to\infty} I_c f(x) = f(x)$$

uniformly for all $x \in \mathbb{R}^d$.

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