GINDIKIN-KARPELEVICH FINITENESS FOR LOCAL KAC-MOODY GROUPS

by

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Abstract

One of the main difficulties in extending Macdonald's theory of spherical functions from p-adic Chevalley groups to p-adic Kac-Moody groups is the absence of Haar measure in the infinite dimensional case. Related to this problem is the question of how to generalize the integral defining Harish-Chandra's $\mathbf{c}$-function to the p-adic Kac-Moody setting. Finding answers to these questions is the key objective of this thesis.

Our main results, proven in the setting of p-adic Kac-Moody groups, are the finiteness of formal analogues of the spherical function (Spherical Finiteness), the $\mathbf{c}$-function (Gindikin-Karpelevich Finiteness), and a formal analogue of Harish-Chandra's limit (Approximation Theorem) relating spherical and $\mathbf{c}$-function.

These results have been proven by A. Braverman, H. Garland, D. Kazhdan and M. Patnaik for untwisted affine Kac-Moody groups using algebraic and representation theoretic techniques. In this thesis, we prove these results for p-adic Kac-Moody groups by using a method motivated by Braverman et. al. but distinct even in the affine case.

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Chapter 1

Introduction

1.1 Origin of the Problem

The central object of this thesis "the Gindikin-Karplevich formula" originated from the theory of spherical functions on the real semi-simple Lie groups. The study of these functions was initiated by V. Bargman [2], I. M. Gelfand [25], R. Godment [28], and significantly advanced by Harish-Chandra's work [30, 31]. In the following, we introduce these functions and give an account of their relationship with the Gindikin-Karpelvich formula.

1.1.1 Spherical Functions

Let G be a connected, semisimple Lie group with finite center and $\mathfrak{g} := \mathrm{Lie}(G)$ be its Lie algebra. Let K be a maximal compact subgroup of G, and $\theta : \mathfrak{g} \to \mathfrak{g}$ the corresponding Cartan involution, which has eigenvalues ± 1. Letting $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$, we have a *Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Pick $\mathfrak{a} \subset \mathfrak{p}$ a maximal, abelian subalgebra and let A be the corresponding Lie subgroup of G; writing $\exp$ for the exponential map, we have $\exp(\mathfrak{a}) = A$.

Under the adjoint action, the elements of $\mathfrak{a}$ are (jointly) diagonalizable and the non-zero eigenvalues are given by the roots of $\mathfrak{g}$. Pick a set of positive roots, and let $\mathfrak{n}^+$ denote the sum of the corresponding eigenspaces; it is a nilpotent subalgebra. Let U^+ be the corresponding Lie subgroup of G. The Iwasawa decomposition then states that

$$G = KAU^+ = K\exp(\mathfrak{a})U^+ \tag{1.1}$$

that is, each $g \in G$ can be written uniquely as $g = kau = k\exp(h)u$ for $k \in K$,$u \in U^+$, and $a = \exp(h)$, $h \in \mathfrak{a}$. Let $\mathfrak{a}_{\mathbb{C}}^*$ be the dual of the complexification $\mathfrak{a}_{\mathbb{C}} := \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$. Let $\Delta_0 \subset \mathfrak{a}_{\mathbb{C}}^*$ be the set of roots, $\Delta_{0,+}$ be the set of positive roots and Π_0 be set of simple roots. Let $A^+ \subset A$ be the cone of dominant elements of A. We denote by $D_K(G)$ the set of all K-bi-invariant differential operators on G. Let us now define the main notion of this subsection.

Definition 1.1.1. *A continuous function*

$$f \colon G \longrightarrow \mathbb{C} \tag{1.2}$$

is called a spherical function if: (i) $f(1) = 1$; (ii) $f(kgk') = f(g)$ for all $k, k' \in K$ and $g \in G$; (iii) f is an eigenfunction for each operator in $D_K(G)$.

Let dk be the invariant Haar measure on K, normalized by $\int_K dk = 1$.

1.1.2 Harish-Chandra's c-function

In his seminal work [30], Harish-Chandra introduced the now well known $\mathbf{c}$-function. This was the starting point of the Gindikin-Karpelevich formula. For more details on the $\mathbf{c}$-function, let $v \colon \mathfrak{a} \longrightarrow \mathbb{C}$ be a function. Associated with this function, we define a character $\phi_v\colon G \longrightarrow \mathbb{C}^*$ as $\phi_v(g) = e^{\langle v,h\rangle}$, where $g \in G$ has the Iwasawa decomposition $g = k\exp(h)u$ for some $k \in K$, $u \in U^+$ and $\exp(h) \in A$ with $h \in \mathfrak{a}$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_{0,+}} \alpha$. Harish-Chandra parametrized the set of spherical functions on G by the following theorem (see corollary on page 61 in [30].)

Theorem 1.1.2. *As v runs through $\mathfrak{a}^*_{\mathbb{C}}$, the functions*

$$f_v(g) \quad = \quad \int_K \phi_{v+\rho}(gk)dk, \;\; g \in G, \tag{1.3}$$

exhaust the class of spherical functions on G and $f_v = f_{v'}$ if and only if $v' \in Wv$, where Wv is the orbit of v under the action of W on $\mathfrak{a}^*_{\mathbb{C}}$.

Let U^- be the unipotent group opposite to U^+. In Theorem 4 of *op. cit.* the asymptotic behaviour of the spherical function f_v was studied and Harish-Chandra showed

Theorem 1.1.3. *Let $Re(iv) \in \mathfrak{a}^*$, then the limit $\mathbf{c}_v := \lim_{a \xrightarrow{+} \infty} \frac{f_v(a)}{\phi_{v+\rho}(a)}$ exists and it is equal to*

$$\mathbf{c}_v \quad = \quad \int_{U^-} \phi_{v+\rho}(u^-)du^-, \tag{1.4}$$

where du^- is the Haar measure on U^- normalized such that $\int_{U^-} \phi_{-2\rho}(u^-)du^- = 1$ and $a \xrightarrow{+} \infty$ means a is made increasingly dominant in the cone of dominant elements A^+.

The function $\mathbf{c}_v$ is known as the Harish-Chandra's $\mathbf{c}$-*function*.

1.1.3 Bhanu Murti's Solution

F. Karpelevich started working on the **c**-*function* and computed it for $SL_3(\mathbb{R})$ in 1959, but he never made his solution public [27]. F. A. Berezin proposed this problem to his PhD student T. S. Bhanu Murti, who found a product formula for $\mathrm{SL}_n(\mathbb{R})$ in [5]. After this, Karpelevich suggested to Bhanu Murti the problem for the symplectic group $\mathrm{Sp}_n(\mathbb{R})$. He solved this problem (see [4]) and obtained these solutions by using an inductive method. In this subsection, we outline Bhanu Murti's strategy for $\mathrm{SL}_n(\mathbb{R})$.

Let $U^- \subset \mathrm{SL}_n(\mathbb{R})$ be the group of lower unipotent matrices and $u^- \in U^-$ be such that

$$u^- = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ u_{21} & 1 & 0 & \dots & 0 \\ u_{31} & u_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ u_{n1} & u_{n2} & u_{n3} & \dots & 1 \end{pmatrix}. \tag{1.5}$$

Next, let $u_1, u_2, u_3, \dots, u_n$ be columns of matrix u^- and for $1 \leq i, j \leq n$, $\langle u_i, u_j \rangle$ denotes the inner product of two columns. For $1 \leq r \leq n$, suppose

$$u_r^- = \begin{pmatrix} \langle u_1, u_1 \rangle & \langle u_1, u_2 \rangle & \dots & \langle u_1, u_r \rangle \\ \langle u_2, u_1 \rangle & \langle u_2, u_2 \rangle & \dots & \langle u_2, u_r \rangle \\ \vdots & \vdots & \dots & \vdots \\ \langle u_r, u_1 \rangle & \langle u_r, u_2 \rangle & \dots & \langle u_r, u_r \rangle \end{pmatrix} \tag{1.6}$$

and $D_r = \det(u_r^-)$. Bhanu Murti [5, P. 862] proved the following result.

Theorem 1.1.4. *Harish-Chandra's* $\mathbf{c}$*-function for* $\mathrm{SL}_n(\mathbb{R})$ *satisfies*

$$\mathbf{c}_v = NC_n(v), \tag{1.7}$$

where

$$C_n(v) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} D_1^{-i\frac{v_1-v_2}{2}-\frac{1}{2}} \dots D_{n-1}^{-i\frac{v_{n-1}-v_n}{2}-\frac{1}{2}} du^-, \tag{1.8}$$

$$N = \int_{U^-} \frac{1}{D_1 D_2 \dots D_{n-1}} du^-, \tag{1.9}$$

$v = (v_1, v_2, \dots, v_n)$ *and* du^- *is the product measure* $\prod_{i>j} du_{ij}^-$.

Moreover, he showed that the integral $C_n(v)$ can be solved inductively and the problem reduces to rank one computations. He obtained a formula for $\mathbf{c}_v$ as a product of Beta functions (cf. Section 1.1.4).

1.1.4 General Formula

Bhanu Murti could not extend and generalize his work for all semi-simple Lie groups. Then, Gindikin and Karpelevich took up this problem. By following Bhanu Murti's inductive method, they evaluated the integral on the right hand side of (1.4) for any semi-simple Lie group and gave the following solution in [26].

With the notations of Subsection 1.1.1, let

$$\tilde{\Delta}_{0,+} = \{\alpha \in \Delta_{0,+} \mid \frac{1}{2}\alpha \notin \Delta_{0,+}\}. \tag{1.10}$$

Theorem 1.1.5. *For each* $\alpha \in \tilde{\Delta}_{0,+}$ *set* $r_{\alpha,v} = 2^{-1}\langle v, \alpha^\vee \rangle$. *The Harish-Chandra* $\mathbf{c}_v$

has the following formula

$$\mathbf{c}_v = c_0 \prod_{\alpha\in\tilde{\Delta}_{0,+}} \frac{2^{r_{\alpha,v}}\Gamma(r_{\alpha,v})}{\Gamma(\frac{1}{2}(\frac{1}{2}m(\alpha)+1+r_{\alpha,v}))\Gamma(\frac{1}{2}(\frac{1}{2}m(\alpha)+m(2\alpha)+1+r_{\alpha,v}))}, \quad (1.11)$$

where for a positive integer n*,* $\Gamma(n)$ *is the gamma function,* c_0 *is a constant such that* $\mathbf{c}_\rho = 1$*, and for a root* β*,* $m(\beta)$ *denotes the multiplicity of* β*.*

This expression for the $\mathbf{c}$-function is now known as the *Gindikin-Karpelevich formula*.

1.2 Non-Archimedean Case

1.2.1 Macdonald's Work

The theory of spherical functions for real semi-simple groups was extended to p-adic groups by F. T. Mautner [48], T. Tamagawa [59], F. Bruhat [9, 10], I. Satake [56], and others. To describe the non-archimedean analogue of the construction of the previous section, let $\mathcal{K}$ be a non-archimedean local field with ring of the integers $\mathcal{O}$. Pick π be a uniformizing element and $\mathrm{k} = \mathcal{O}/\pi\mathcal{O}$ be the finite residue field of cardinality q. Suppose $G = G(\mathcal{K})$ is a split, simply-connected Chevalley group over $\mathcal{K}$. We denote the integral subgroup $G(\mathcal{O})$ by K; this group is a non-archimedean analogue of maximal compact subgroup. Let H be a Cartan subgroup and $H_\mathcal{O} = H \cap K$. The quotient group $A := H/H_\mathcal{O}$ can be identified with the coweight lattice $\Lambda^\vee$ (which is equal to the coroot lattice $Q^\vee$ since G is simply-connected) via the map $\mu^\vee \mapsto \pi^{\mu^\vee}$. The Iwasawa decomposition in this context states that $G = \cup_{\mu^\vee\in\Lambda^\vee} K\pi^{\mu^\vee}U^+$, that is, every $g \in G$ can be written as $g = k\pi^{\mu^\vee}u$ with $\mu^\vee$ uniquely determined by g (note that $k \in K$ and $u \in U^+$ are not uniquely determined). For a map $v\colon A \longrightarrow \mathbb{C}^*$ we can define a function

$$\phi_v\colon G \longrightarrow \mathbb{C}^*$$

as $\phi_v(g) = q^{\langle v,\mu^\vee\rangle}$, if $g \in G$ has Iwasawa decomposition as above $g \in K\pi^{\mu^\vee}U^+$, $\mu^\vee \in \Lambda^\vee$. The spherical function on G can be defined as in (1.3)

$$f_v(g) = \int_K \phi_{v+\rho}(gk)dk, \tag{1.12}$$

where dk is the normalized Haar measure on K so that $\int_K dk = 1$. In an analogy with the archimedean case, by taking the limit $\lambda^\vee \to \infty$ in the dominant cone, the integral on the right hand side of (1.12) over K can be shifted to an integral over U^-, that is

$$\mathbf{c}_v := \lim_{\lambda^\vee \xrightarrow{+} \infty} \frac{f_v(\pi^{\lambda^\vee})}{\phi_{v+\rho}(\pi^{\lambda^\vee})} = \int_{U^-} \phi_{v+\rho}(u^-)du^-, \tag{1.13}$$

where du^- is an appropriately normalized Haar measure on U^-. Following the same inductive method of Gindikin and Karpelevich, Macdonald in [43, P. 77] found the following formula for the above integral.

$$\mathbf{c}_v = \prod_{\alpha^\vee \in \Delta^\vee_{0,+}} \frac{1 - q^{-1-v(\alpha^\vee)}}{1 - q^{-v(\alpha^\vee)}}, \tag{1.14}$$

where $\Delta^\vee_{0,+}$ is the set of positive coroots.

1.2.2 Constant Term Formula

In Langlands' notes [40, P 25], the integrals on the right hand side of (1.4) and (1.13) also appear in his computation of the constant term of the Fourier series expansion of Eisenstein series on certain adelic groups. A full account on the Fourier series expansion can be found in Langlands' notes and [21, Ch. 9].

1.3 Non-Archimedean Case: Formal Analogues

Since Haar measures do not exist in general Kac-Moody groups over non-archimedean local fields, we need an algebraic formulation for the constructions given in the Subsection 1.2.1. Let us rephrase the integrals (1.12) and (1.13) as follows. The function f_v is bi-invariant with respect to K and hence constant on each Cartan cell $K\pi^{\lambda^\vee}K$, $\lambda^\vee \in \Lambda_+^\vee$. So, if $g \in G$ and $k \in K$ are such that $gk \in K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K$ for some $\mu^\vee \in \Lambda^\vee$ and $\lambda^\vee \in \Lambda_+^\vee$ then $\phi_{v+\rho}(gk) = q^{\langle v+\rho,\mu^\vee\rangle}$. Thus, the integral (1.12) becomes equal to

$$f_v(g) = \sum_{\mu^\vee \in \Lambda^\vee} q^{\langle v+\rho,\mu^\vee\rangle} Vol(K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K). \tag{1.15}$$

The volume $Vol(K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K)$ with Haar measure dk (which is normalized so that K has volume 1) is equal to $|K\backslash K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K|$, where $|X|$ denotes the cardinality of a set X.

By the Iwasawa decomposition, $\mu^\vee$ runs through all the coweights. So, one obtains the following formal version (i.e. valued in $\mathbb{C}[\Lambda^\vee]$ rather than $\mathbb{C}$) of the spherical function f_v:

$$\mathcal{S}_{\lambda^\vee} \quad := \quad \sum_{\mu^\vee \in \Lambda^\vee} |K\backslash K\pi^{\mu^\vee} U^+ \cap K\pi^{\lambda^\vee} K| q^{\langle \rho, \mu^\vee \rangle} e^{\mu^\vee}, \tag{1.16}$$

such that the value of function $f_v(g)$ for $g \in K\pi^{\lambda^\vee} K$ satisfying $gk \in K\pi^{\mu^\vee} U^+$ is equal to $f_v(g) = \mathrm{ev}_v(\mathcal{S}_{\lambda^\vee})$, where $\mathrm{ev}_v \colon \mathbb{C}[\Lambda^\vee] \longrightarrow \mathbb{C}$ is the map defined as $e^{\mu^\vee} \mapsto q^{\langle v, \mu^\vee \rangle}$. Similarly, a formal version of the Gindikin-Karpelevich integral (1.13) is written as:

$$\mathscr{G}_{\lambda^\vee} \quad := \quad \sum_{\mu^\vee \in \Lambda^\vee} |K\backslash K\pi^{\lambda^\vee - \mu^\vee} U^+ \cap K\pi^{\lambda^\vee} U^-| q^{\langle \rho, \lambda^\vee - \mu^\vee \rangle} e^{\lambda^\vee - \mu^\vee}. \tag{1.17}$$

Let us immediately notice the following "homogenity" property of the sum $\mathscr{G}_{\lambda^\vee}$, which shows that it suffices to obtain a formula for $\mathscr{G}_0$.

Lemma 1.3.1. *The sum $\mathscr{G}_{\lambda^\vee}$ satisfies*

$$\mathscr{G}_{\lambda^\vee} = q^{\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee} \mathscr{G}_0 \tag{1.18}$$

The proof of Lemma 1.3.1 follows from the following bijection of the sets

$$K\pi^{\lambda^\vee - \mu^\vee} U^+ \cap K\pi^{\lambda^\vee} U^- \leftrightarrow K\pi^{\lambda^\vee - \mu^\vee} U^+ \pi^{-\lambda^\vee} \cap K\pi^{\lambda^\vee} U^- \pi^{-\lambda^\vee},$$

by the equality

$$K\pi^{\lambda^\vee - \mu^\vee} U^+ \pi^{-\lambda^\vee} \cap K\pi^{\lambda^\vee} U^- \pi^{-\lambda^\vee} \quad = \quad K\pi^{-\mu^\vee} U^+ \cap KU^- \tag{1.19}$$

and some simple algebra.

The sum $\mathcal{S}_{\lambda^\vee}$ is connected with the *Satake map*,

$$\mathcal{S} : \mathscr{H} \longrightarrow \mathbb{C}[\Lambda^\vee]^W,$$

where the notation is as follows: $\mathscr{H}$ is the space of complex valued, compactly supported K-bi-invariant functions on G with basis consisting of the characteristic functions $h_{\lambda^\vee} = \chi_{K\pi^{\lambda^\vee}K}$ of $K\pi^{\lambda^\vee}K$ for all $\lambda^\vee \in \Lambda^\vee_+$; W is the Weyl group; $\mathbb{C}[\Lambda^\vee]$ is the group algebra of $\Lambda^\vee$; and $\mathbb{C}[\Lambda^\vee]^W$ is its W-invariant subspace.

For $\lambda^\vee \in \Lambda^\vee_+$, the Satake map $\mathcal{S}$ sends $h_{\lambda^\vee}$ to $\mathcal{S}_{\lambda^\vee}$. In [43], I. G. Macdonald determined an explicit formula for $\mathcal{S}_{\lambda^\vee}$

$$\mathcal{S}_{\lambda^\vee} = \frac{q^{\langle \rho, \lambda^\vee \rangle}}{W_{\lambda^\vee}(q^{-1})} \sum_{w \in W} w(\Upsilon) e^{w\lambda^\vee}, \tag{1.20}$$

where $\Upsilon = \prod_{\alpha^\vee \in \Delta^\vee_{0,+}} \frac{1-q^{-1}e^{-\alpha^\vee}}{1-e^{-\alpha^\vee}}$ is a rational expression from $\mathbb{C}_q[\Lambda^\vee] := \mathbb{C}[q, q^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda^\vee]$; and $W_{\lambda^\vee}(q^{-1}) = \sum_{\sigma \in W_{\lambda^\vee}} q^{-\ell(\sigma)}$ is the *Poincare polynomial* of the stabilizer $W_{\lambda^\vee} \subset W$ of $\lambda^\vee$, where $\ell \colon W \longrightarrow \mathbb{Z}$ denotes the length function on W. The formal analogue of the limit (1.13) can be stated as:

Theorem 1.3.2 (Approximation Theorem)**.** *For each $\mu^\vee \in \Lambda^\vee$, there exists $\lambda^\vee_0 \in \Lambda^\vee_+$ regular such that for all $\lambda^\vee > \lambda^\vee_0$, we have*

$$K\pi^{\lambda^\vee - \mu^\vee} U^+ \cap K\pi^{\lambda^\vee} U^- = K\pi^{\lambda^\vee - \mu^\vee} U^+ \cap K\pi^{\lambda^\vee} K. \tag{1.21}$$

For finite dimensional groups, a proof of this result can be found in [3, Proposition 3.6 (ii)]. By using Lemma 1.3.1 and Theorem 1.3.2, $\mathscr{G}_0$ can be expressed as

$$\mathscr{G}_0 = \lim_{\lambda^\vee \xrightarrow{+} \infty} \frac{\mathcal{S}_{\lambda^\vee}}{q^{\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee}}, \tag{1.22}$$

where $\lambda^\vee \xrightarrow{+} \infty$ indicates that the regular dominant coweight $\lambda^\vee$ is approaching to infinity while remaining within the regular dominant cone. By using the expression (1.20), one can compute the limit on the right hand side of (1.22) to obtain

$$\mathscr{G}_0 = \Upsilon. \tag{1.23}$$

This strategy to compute the Gindikin-Karpelevich formula is stated and generalized for affine Kac-Moody groups in [6].

1.4 General Setting

Suppose, now G is a general Kac-Moody group over a non-archimedean local field $\mathcal{K}$. To consider $\mathscr{G}_{\lambda^\vee}$ and $\mathcal{S}_{\lambda^\vee}$ associated with G and compute a formula for $\mathscr{G}_{\lambda^\vee}$, the first challenge is to show that these sums are well defined when G is not of finite type, and an infinite dimensional version of Theorem 1.3.2 holds.

For $\mathscr{G}_{\lambda^\vee}$, one needs to prove the following,

Theorem 1.4.1 (Gindikin-Karpelevich Finiteness)**.** *For $\lambda^\vee, \mu^\vee \in \Lambda^\vee$, the set $K\backslash K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}U^-$ is finite. Moreover, it is empty unless $\mu^\vee \leq \lambda^\vee$.*

For $\mathcal{S}_{\lambda^\vee}$, one needs to prove

Theorem 1.4.2 (Spherical Finiteness)**.** *For $\lambda^\vee, \mu^\vee \in \Lambda^\vee$ with $\lambda^\vee$ dominant, the coset space $K\backslash K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K$ is finite. Moreover, it is empty unless $\mu^\vee \leq \lambda^\vee$.*

For (untwisted) *affine* Kac-Moody groups, Braverman et. al. obtain Theorems 1.3.2, 1.4.1 and 1.4.2. Note that their proofs for the second part of the Spherical and Gindikin-Karpelevich Finiteness (the inequalities $\mu^\vee \leq \lambda^\vee$) extend to general Kac-Moody groups without any change (see Lemma 8.2.1 and the proof of Theorem 1.4.1 in Section 9.2.3 for the details about these two proofs).

The first part of the (untwisted) *affine* Gindikin-Karplevich Finiteness was proven for $\lambda^\vee = 0$ by showing that:

(a) $K\pi^{\mu^\vee}U^+ \cap KU^- = \cup_{w \in W} K\pi^{\mu^\vee}U^+ \cap K\mathcal{V}_w^-$, where for each $w \in W$, $\mathcal{V}_w^-$ is a certain subset of U^- defined in Section 3 of [6] (or see Subsection 4.2.1 of this thesis)

(b) A corollary of the Kac-Moody generalization [7, Lemma 18.2] of a representation theoretic construction due to A. Joseph [29, 33] implies that there are finitely many w which appear in the above union.

(c) By using the completions, it is then proved that for each such w, $K \backslash K\pi^{\mu^\vee}U^+ \cap K\mathcal{V}_w^-$ is finite.

Next, the Gindikin-Karpelevich finiteness is used to get the Approximation Theorem as well as Spherical Finiteness. Finally, by combining these results with an affine generalization of the Macdonald's formula for $\mathcal{S}_{\lambda^\vee}$ from [8], the following affine version of the Gindikin-Karpelvich formula is obtained

$$\mathscr{G}_0 = \frac{1}{\mathfrak{m}} \prod_{\alpha^\vee \in \Delta_+^\vee} \left(\frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}} \right)^{m(\alpha^\vee)}, \tag{1.24}$$

where $m(\alpha^\vee)$ is the multiplicity of the coroot $\alpha^\vee$ and $\mathfrak{m}$ is a W-invariant factor which depends on the Langlands-dual root system of given affine Lie algebra. An exposition on the affine construction is given in Chapter 4.

1.5 Our Strategy

As stated in previous section, our key objective in this thesis is to obtain the proofs of Theorems 1.3.2, 1.4.1 and 1.4.2 for general Kac-Moody groups over non-archimedean local fields. Though Braverman-Garland-Kazhdan-Patnaik's work [6] is the main motivation of this project, our approach to attack this problem is a little different. Unlike the affine case, we prove Theorem 1.3.2 independently of the finiteness result Theorem 1.4.1. Theorem 1.3.2 has also been proven by A. Hébert in [32, Theorem 6.1], but our proof is perhaps more elementary and can also be used to obtain the Iwahori version of the assertion (see Proposition 5.2.1).

Next, we turn to Theorem 1.4.1. For the first part of the assertion, our method of proof restricts us to put a further condition on $\lambda^\vee$. Namely, we first prove

Theorem 1.5.1 (Weak Spherical Finiteness)**.** *Let* $\mu^\vee \in \Lambda^\vee$*. For* $\lambda^\vee \in \Lambda^\vee_+$ *regular and sufficiently dominant, the set* $K\backslash K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K$ *is finite.*

This theorem is proven by getting the finiteness at the Iwahori level. The Iwahori level questions are indexed by the Weyl group W. So, first we show that there are finitely many elements of the Weyl group which contribute (Section 7.1), each indexing an Iwahori piece of our sum. In Chapter 6, we introduce a certain integral and show that it satisfies a recursion relation in terms of certain Demazure-Lusztig operators. Our objective in so doing is to obtain the finiteness of the Iwahori piece. In Chapter 7, we establish a relation between an Iwahori piece and a level set of the integral which allows us to complete the proof of the Weak Spherical Finiteness.

In Chapter 8, we discuss the applications of the results proven in the previous three chapters. First, in Section 8.1, we apply the Approximation Theorem and the Weak Spherical Finiteness to show that if $\mu^\vee \in Q^\vee_-$ is very small as compared to $\lambda^\vee = 0$, then $K\backslash K\pi^{\mu^\vee}U^+ \cap KU^-$ is finite and this together with Lemma 1.3.1

imply the Gindikin-Karplevich Finiteness. Then, as in the affine case, we use the Gindikin-Karplevich Finiteness to get the proof of the Spherical Finiteness.

We also tried to obtain the Gindikin-Karpelevich Finiteness (Theorem 1.4.1) independently of Spherical Finiteness (Theorem 1.4.2) but our efforts did not succeed. This incomplete solution is discussed in the last chapter of our thesis. Our approach uses the completion $\mathbb{U}^-$ of U^- and the local analogues of the geometric embeddings of certain subgroups and subsets of $\mathbb{U}^-$ into its finite dimensional quotient [39, Section 7.3]. We construct this completion and prove the results about these embeddings by using the representation theory in Subsection 9.2.2.

Our method of proof is motivated by the affine case; one needs to prove a certain bounded condition satisfied by a finitely generated subgroup of U^- (see Subsection 4.2.3 for detail). In the affine setting, this became possible because of the presence of: (a) a natural order on the set of roots corresponding to the finite set of generators, and (b) a set of coordinates on the elements of that finitely generated subgroup when identified in $\mathbb{U}^-$. This order and the system of coordinates do not exist in general settings; therefore, we can not proceed further beyond this step and state this bounded condition as a Conjecture 9.2.7.

1.6 Alternative Approach

These finiteness theorems for general Kac-Moody settings have also appeared in some other publications. In a recent paper [32, Theorem 6.1], Hébert has obtained the proof of Theorem 1.4.2 for general Kac-Moody settings. Theorem 1.3.2 is shown to be true by S. Gaussent and G. Rousseau in [24]. Both of these proofs involve the techniques based on the use of geometric objects known as *masures*, introduced by Gaussent and Rousseau in [23]. These are an analogue of the Bruhat-

Tits *buildings* for groups over local fields. On the other hand, our strategy for proving the assertions of these theorems is elementary, algebraic in nature and relies on the use of the representation theory. It would be interesting to compare these two techniques in more detail.

Chapter 2

Preliminaries

2.1 Local Fields

Let $\mathbb{K}$ be a field. An *absolute value* on $\mathbb{K}$ is a function

$$|.|: \mathbb{K} \longrightarrow \mathbb{R}, \tag{2.1}$$

such that:

(i) $|x| \geq 0$ for all $x \in \mathbb{K}$ and $|x| = 0$ if and only if $x = 0$,

(i) $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$,

(iii) $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{K}$.

Axiom (iii) is known as the *triangle inequality*. An absolute value $|.|$ is called *non-archimedean* if it satisfies a stronger version of the triangle inequality

$$|x + y| \leq max\{|x|, |y|\} \tag{2.2}$$

for all $x, y \in \mathbb{K}$. Otherwise, $|.|$ is called an *archimedean* absolute value. Associated with a non-archimedean absolute value $|.|$, there is a function

$$val\colon \mathbb{K} \longrightarrow \mathbb{R} \cup \{\infty\} \tag{2.3}$$

defined as $val(x) = -log(|x|)$. This function satisfies the following properties: (1) $val(xy) = val(x) + val(y)$; (2) $val(x) = \infty$ if and only if $x = 0$, and (3) $val(x + y) \geq min\{val(x), val(y)\}$, for all $x, y \in \mathbb{K}$.

Conversely, given a function (2.3) satisfying (1), (2) and (3) we obtain an absolute value by putting $|x| = q^{-val(x)}$ for all $x \in \mathbb{K}$, where $q \in \mathbb{R}_{>1}$. The function val is called a *valuation* on $\mathbb{K}$ and a field $\mathbb{K}$ equipped with a valuation is called a *valuation field*. Let $\mathbb{K}$ be a valuation field with an absolute value $|.|$. The subset

$$\mathcal{O} = \{x \in \mathbb{K} \mid |x| \leq 1\} = \{x \in \mathbb{K} \mid val(x) \geq 0\} \tag{2.4}$$

is a ring with the set of units

$$\mathcal{O}^* = \{x \in \mathbb{K} \mid |x| = 1\} = \{x \in \mathbb{K} \mid val(x) = 0\} \tag{2.5}$$

and a unique maximal ideal

$$\mathfrak{p} = \{x \in \mathbb{K} \mid |x| < 1\} = \{x \in \mathbb{K} \mid val(x) > 0\}. \tag{2.6}$$

The ring $\mathcal{O}$ is called a *valuation ring* and $\mathbb{K}$ is the field of fraction of $\mathcal{O}$. The field $\mathrm{k} = \mathcal{O}/\mathfrak{p}$ is called the *residue field* of $\mathcal{O}$. The ideal $\mathfrak{p}$ is a principal ideal, we fix a generator π of $\mathfrak{m}$ and shall refer to it as a *uniformizer*. The valuation val is called *discrete* if $val(\mathbb{K}^*) = s\mathbb{Z}$ for some real number s, in the case of which we can chose

$s = 1$ and then the valuation is said to be *normalized*.

A valued field $\mathbb{K}$ is complete with respect to the absolute value $|.|$ if every *Cauchy sequence* of elements of $\mathbb{K}$ converges. A complete field $\mathbb{K}$ is called a *local field* if the valuation is discrete and its residue field is finite. If the absolute value is archimedean, then $\mathbb{K}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. The non-archimedean local fields have also been classified. In characteristic zero, $\mathbb{K}$ is either $\mathbb{Q}_p$ for some prime p or a finite extension of $\mathbb{Q}_p$. If the characteristic of $\mathbb{K}$ is positive, then $\mathbb{K}$ is the field of formal Laurent series $F_q((t))$ or its finite extension, where q is a power of p.

2.2 Kac-Moody Algebra

2.2.1 Generalized Cartan Matrices

Let I be a finite set of cardinality l and $A = (a_{ij})_{i,j \in I}$ be a square matrix such that for all $i, j \in I$,

(i) $a_{ii} = 2$; for $i \neq j$, a_{ij} are non positive integers; and $a_{ij} = 0$ if and only if $a_{ji} = 0$,

(ii) there exist a diagonal matrix D and a positive definite matrix P such that $A = DP$.

A matrix A is said to be a *Cartan matrix* if it satisfies (i) & (ii) and if A satisfies only (i), then it is called a *generalized Cartan matrix* (GCM).

A GCM A is said to be equivalent to another matrix B, if B is obtained by reordering the indices i and j, and vice versa. If a GCM A is equivalent to a block diagonal matrix with more than one block, we say it is a *decomposable* GCM; otherwise, it is known as an *indecomposable* GCM. An indecomposable GCM A can be classified into the following three types,

(a) **Finite:** If A is positive-definite. In this case the determinant of A is positive.

(b) **Affine:** If A is positive-semidefinite. In this case $det(A) = 0$.

(c) **Indefinite:** If A is neither finite nor affine type.

As in [38], suppose $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of GCM A, that is, $\mathfrak{h}$ is a complex vector space of finite dimension; $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}$, $\Pi = \{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$ are two linearly independent sets, such that for all $i, j \in I$, $\alpha_j(\alpha_i^\vee) = a_{ij}$; and, $dim(\mathfrak{h}) = 2l - rank(A)$. The elements of Π are called *simple roots* and those of $\Pi^\vee$ are known as *simple coroots*. A realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ is said to be a *decomposable realization* if $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $\Pi^\vee = (\Pi_1^\vee \times \{0\}) \cup (\{0\} \times \Pi_2^\vee)$ and $\Pi = (\Pi_1 \times \{0\}) \cup (\{0\} \times \Pi_2)$, where $(\mathfrak{h}_1, \Pi_1, \Pi_1^\vee)$ and $(\mathfrak{h}_2, \Pi_2, \Pi_2^\vee)$ are realizations themselves. A realization is called *indecomposable realization* if it is not a decomposable realization. In [38, Proposition 1.1], Kac asserts that an indecomposable GCM A corresponds to an indecomposable realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ which is unique up to equivalence in the following sense:

A realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ is said to be equivalent to another realization $(\mathfrak{h}', \Pi', \Pi'^\vee)$, if there exists an isomorphism $\phi \in \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathfrak{h}')$, such that $\phi(\Pi^\vee) = \Pi'^\vee$ and $\phi^*(\Pi) = \Pi'$, where ϕ^* is the induced isomorphism of the dual spaces of $\mathfrak{h}$ and $\mathfrak{h}'$. Furthermore, this isomorphism is unique if $det(A) \neq 0$.

2.2.2 Presentation of Kac-Moody Algebras

A GCM of finite type, up to equivalence, corresponds to a unique finite dimensional complex semisimple Lie algebra $\mathfrak{g}$ up to isomorphism. In [18], Chevalley showed that a finite dimensional complex semisimple Lie algebra admits a finite presentation as a set of generators satisfying relations in terms of entries of a Cartan matrix. Later, Serre in [57] proved that these relations are the defining relations of $\mathfrak{g}$. Inspried by this, Victor G. Kac [34] and R. Moody [49] independently introduced a new class of Lie algebras by giving a finite presentation in terms of entries of a GCM A. These Lie algebras are infinite dimensional generalizations of finite dimensional complex semisimple Lie algebras.

To describe these Lie algebras, let $A = (a_{ij})_{i,j \in I}$ be a GCM and $(\mathfrak{h}, \Pi, \Pi^{\vee})$ the associated realization. We define $\mathfrak{g}$ as the Lie algebra generated by $\mathfrak{h}$ and the $2n$ generators $\{e_i, f_i\}_{i \in I}$ subject to the following relations:

(1) $[\mathfrak{h}, \mathfrak{h}] = 0$. For all $i, j \in I$,

(2) $[e_i, h] = \alpha_i(h) e_i, \ h \in \mathfrak{h}$,

(3) $[f_i, h] = -\alpha_i(h) f_i, \ h \in \mathfrak{h}$,

(4) $[e_i, f_j] = \delta_{ij} \alpha_i^{\vee}$, where δ_{ij} is the *Kronecker delta* ,

(5) For $i \neq j$, $\mathrm{ad}_{e_i}{}^{(-a_{ij}+1)}(e_j) = 0$ and $\mathrm{ad}_{f_i}{}^{(-a_{ij}+1)}(f_j) = 0$, where ad is the adjoint representation of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ is known as a Kac-Moody algebra. The subalgebra $\mathfrak{h}$ is called *Cartan subalgebra* and $\{e_i, f_i\}_{i \in I}$ are known as the *Chevalley generators* of $\mathfrak{g}$. Let $\mathfrak{n}^+$ and $\mathfrak{n}^-$ be the subalgebras generated by $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$, respectively. Then $\mathfrak{g}$ has a vector subspace decomposition

$$\mathfrak{g} \quad = \quad \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+. \tag{2.7}$$

This decomposition is known as the *triangular decomposition* of $\mathfrak{g}$.

2.2.3 Classification of Kac-Moody Algebras

A Kac-Moody algebra $\mathfrak{g}$ falls in one of the three categories finite, affine or indefinite, according to the type of GCM A (cf. Section 2.2). A finite type Kac-Moody algebra is a complex semisimple Lie algebra. There are four infinite families of complex semisimple Lie algebras $A_n,\ n \geq 1$; $B_n,\ n \geq 2$; $C_n,\ n \geq 3$; $D_n,\ n \geq 4$. These four families are known as *classical Lie algebras*. In addition to the classical Lie algebras there are five so-called *exceptional Lie algebras* Lie algebras, G_2, F_4, E_6, E_7 and E_8. The classes of the Kac-Moody algebras can be described by particular graphs called the *Dynkin diagrams* associated with the corresponding GCMs. The Dynkin diagrams of finite types are given below.

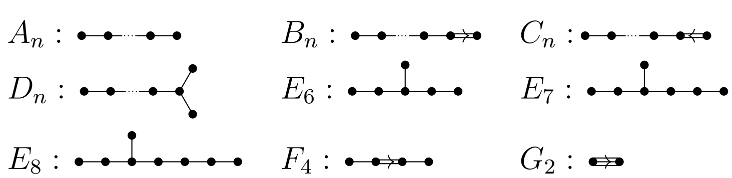
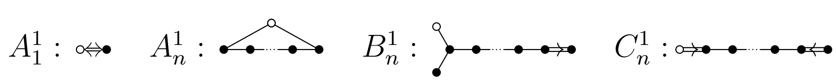


Table 2.1: Finite Type Dynkin Diagram

An affine Kac-Moody algebra is an extensions of loop algebra of finite dimensional semisimple Lie algebra. So, once the classification in finite dimension is known, the classification of affine Kac-Moody algebras becomes possible. These Lie algebras are described with the symbol X^r with $X = A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$ and G_2, and $r = 1, 2, 3$. When $r = 1$, the corresponding Lie algebras are called the *untwisted affine Lie algebras*. The following table contains the affine Dynkin diagrams of untwisted type:



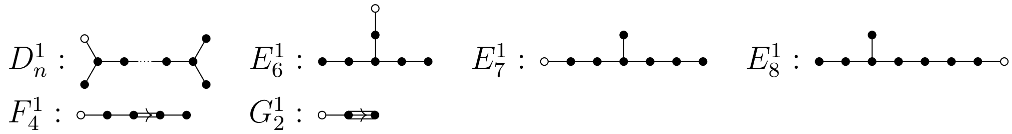


Table 2.2: Untwisted Affine Type Dynkin Diagram

When $r = 2, 3$, the corresponding Lie algebras are called the *twisted affine Lie algebras*. The following is the list of affine diagrams of twisted type:

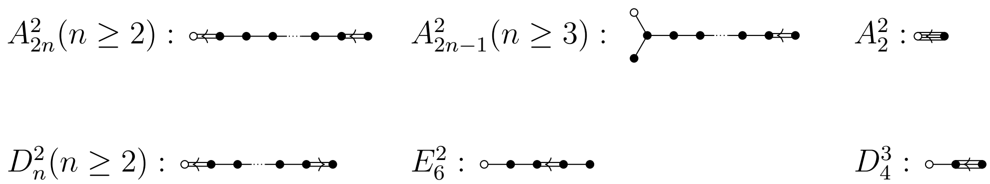


Table 2.3: Twisted Affine Type Dynkin Diagram

The Kac-Moody algebras of indefinite type are the least understood and the classification of the corresponding root systems has not yet been achieved. However, a subclass known as the *hyperbolic Kac-Moody algebras* is well known. These correspond to the GCMs A such that every proper, indecomposable principal submatrix of A is either of finite or affine type. In this case, we have $det(A) < 0$. These Lie algebras and the related data have been studied in the literature, for example, see [11, 12, 14, 41, 64].

2.2.4 Roots and Weyl Group

Let $Q = \oplus_{i \in I} \mathbb{Z}\alpha_i$ and $Q^\vee = \oplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ be the root and coroot lattice, respectively. The Lie algebra $\mathfrak{g}$ admits another decomposition $\mathfrak{g} = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$ under the adjoint action of $\mathfrak{h}$ where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x; \forall\, h \in \mathfrak{h}\}.$$

This is known as the *root space decomposition*. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$, the subspace $\mathfrak{g}_\alpha$ is known as the *root space* of α; its dimension $m(\alpha)$ is called the multiplicity of α; and, α is called a *root*. We denote the set of roots by Δ. Every root can be written as an integral combination of the simple roots, with the coefficients, either all positive or all negative integers; a root is called positive or negative, accordingly. We denote the set of positive roots by Δ_+, the set of negative roots by Δ_- one has a disjoint union,

$$\Delta = \Delta_+ \sqcup \Delta_-.$$

In the study of finite dimensional complex semisimple Lie algebras, a symmetric, nondegenerate and invariant bilinear form known as the *Killing form* plays a significant role. An analogue of the Killing form in the general setting can be defined if the GCM A is *symmetrizable*. That is, $A = DB$, where D is a non-singular diagonal matrix and B is a symmetric matrix. Restriction of this bilinear form to $\mathfrak{h}$ is also nondegenerate; therefore it induces an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$. The natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$ is denoted by

$$\langle -, - \rangle : \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{C}.$$

Let $Q_+ = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. The space $\mathfrak{h}^*$ can be equipped with a partial order $\leq$ defined as: $\mu \leq \lambda$ if and only if $\lambda - \mu \in Q_+$, for all $\lambda, \mu \in \mathfrak{h}^*$. Similarly, we can define a partial order on $\mathfrak{h}$, which we denote by the same symbol $\leq$, by setting $Q_+^\vee = \oplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^\vee$ and imposing the same defining condition as above. An element $\lambda \in \mathfrak{h}^*$ is *integral* if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$, is *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$, and is called *regular* if $\langle \lambda, \alpha_i^\vee \rangle \neq 0$, for all $i \in I$. Let

$$\Lambda := \{\lambda \in \mathfrak{h}^* \mid \lambda(h) \in \mathbb{Z},\ \forall h \in \mathfrak{h}\} \tag{2.8}$$

and $\Lambda^\vee = \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the weight and coweight lattice, respectively. We denote by Λ_+ the set of dominant weights and Λ_{reg} the set of regular weights. Similarly we define the sets $\Lambda_+^\vee$ and $\Lambda_{reg}^\vee$. For $i \in I$, let us define a map $w_i = w_{\alpha_i}$ on $\mathfrak{h}^*$ by setting

$$w_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \tag{2.9}$$

for all $\lambda \in \mathfrak{h}^*$. This map is a reflection in the hyperplane

$$(\alpha_i^\vee)^\perp = \{\mu \in \mathfrak{h} \mid \langle \mu, \alpha_i^\vee \rangle = 0\}. \tag{2.10}$$

The reflection w_i is called a *simple root reflection* or the *Weyl reflection* and the group $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ generated by the simple root reflections w_i for all $i \in I$, is called the *Weyl group*. The Weyl group also acts on $\mathfrak{h}$ via the following formula,

$$w_i(\lambda^\vee) = \lambda^\vee - \langle \alpha_i, \lambda^\vee \rangle \alpha_i^\vee, \tag{2.11}$$

for all $\lambda^\vee \in \mathfrak{h}$. If the Kac-Moody algebra $\mathfrak{g}$ is of affine or indefinite type, the set Δ of roots admits another partition

$$\Delta = \Delta^{re} \cup \Delta^{im}, \tag{2.12}$$

where $\Delta^{re} = W\Pi$ and $\Delta^{im} = \Delta - \Delta^{re}$. The elements of Δ^{re} are called *real roots* and those of Δ^{im} are known as the *imaginary roots*. The transpose of GCM A corresponds to the realization $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ and a dual root system $\Delta^\vee \subset \mathfrak{h}$, which is called the set of coroots. This set also admits the disjoint decompositions

$$\Delta^\vee = \Delta_+^\vee \cup \Delta_-^\vee, \quad \Delta^\vee = \Delta^{\vee,re} \cup \Delta^{\vee,im} \tag{2.13}$$

and there is a bijection $\alpha \mapsto \alpha^\vee$ between Δ and $\Delta^\vee$.

2.3 Highest Weight Representation

Let $\mathfrak{g}$ be a Kac-Moody algebra, $\mathcal{U} = \mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. The triangular decomposition (2.7) of $\mathfrak{g}$ yields the triangular decomposition

$$\mathcal{U} = \mathcal{U}(\mathfrak{n}^+) \oplus \mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{n}^-)$$

of $\mathcal{U}$. For $\lambda \in \Lambda_+$, a $\mathfrak{g}$ representation $V = V^\lambda$ over $\mathbb{C}$ is a *highest weight representation* with the highest weight $\lambda \in \mathfrak{h}^*$ and a highest weight vector v_λ if:

(i) $\mathfrak{n}^+ v_\lambda = 0$,

(ii) $h.v_\lambda = \lambda(h)v$ for all $h \in \mathfrak{h}$,

(iii) $V = \mathcal{U} v_\lambda$.

Moreover, if

(iv) for all $i \in I$, e_i and f_i act as locally nilpotent operators on V, that is, for each $v \in V$ there exist integers M and N such that $e_i^M v = f_i^N v = 0$,

then the space V is said to be an *integrable* highest weight representation.

The space V has a *weight space decomposition*

$$V = \oplus_{\mu \in \mathfrak{h}^*} V_\mu, \tag{2.14}$$

where $V_\mu = \{v \in V \mid hv = \mu(h)v, \ \forall \ h \in \mathfrak{h}\}$. Let us denote by P_λ the set of weights of V. For $\mu \in P_\lambda$

$$\eta_\mu : V \longrightarrow V_\mu$$

denotes the projection map. Unless otherwise specified, throughout this thesis our highest weight representation shall be integrable.

Now, suppose

$$\rho = \sum_{i\in I} \omega_i, \tag{2.15}$$

where for $i \in I$, $\omega_i \in \mathfrak{h}^*$ is a *fundamental weight* defined by $\omega_i(\alpha_j^\vee) = \delta_{ij}$ for all $i, j \in I$ and $\omega_i = 0$ outside $Q^\vee \otimes_{\mathbb{Z}} \mathbb{C}$ (where δ_{ij} is the Kronecker delta).

Let $w \in W$ and F^m be the canonical filtration on the universal enveloping algebra of $\mathcal{U}(\mathfrak{n}^+)$. The following lemma from [7, Section 18] relates the weight vectors v_ρ and $v_{w\rho} := wv_\rho$ in V^ρ,

Lemma 2.3.1 (Joseph's Lemma)**.** *Suppose $v_\rho \in F^m(\mathcal{U}(\mathfrak{n}^+))v_{w\rho}$. Then we must have $\ell(w) < 2m$, where $\ell(w)$ denotes the length of w.*

We end this section with the following construction of subspaces and finite dimensional quotients of V, which will be used to construct completion of certain groups in Chapter 9. The set P_λ inherits the partial order from $\mathfrak{h}^*$ and each $\mu \in P_\lambda$ satisfies $\mu \leq \lambda$ which implies $\lambda - \mu = \sum_{i\in I} n_i\alpha_i$ with $n_i \in \mathbb{Z}_{\geq 0}$ for all $i \in I$. For $\mu \in P_\lambda$, we define the *depth* of μ as $depth(\mu) = \sum_{i\in I} n_i$. For $m \geq 0$, let

$$P_\lambda(m) = \{\mu \in P_\lambda \mid depth(\mu) > m\}$$

and set $V^\lambda(m) = \oplus_{\mu\in P_\lambda(m)} V_\mu$. The quotient $V^m = V/V^\lambda(m)$, isomorphic to a direct sum of finitely many weight spaces, is a finite dimensional vector space.

Chapter 3

Kac-Moody Groups

Let $\mathfrak{g}$ be complex Kac-Moody algebra of finite type, that is, $\mathfrak{g}$ is a finite dimensional complex semisimple Lie algebra. Associated with $\mathfrak{g}$, there are finitely many complex Lie groups. This list includes a unique simple group G^{ad}, which is known as the group of *adjoint type*; a unique simply connected semisimple Lie group $\tilde{G}$, which is said to be of *simply connected* type; and intermediate forms between G^{ad} and $\tilde{G}$.

By using the $\mathbb{Z}$-form of $\mathfrak{g}$, Chevalley in [17] proved that an analogous situation holds for algebraic groups over an arbitrary field $\mathbb{K}$. He found a uniform procedure to associate an algebraic group G with $\mathfrak{g}$, which is semisimple if $\mathbb{K}$ is algebraically closed. These groups are known as the *Chevalley groups*. A detailed exposition and complete construction of the Chevalley groups can be found in [58].

Chevalley further extended the study of these groups and introduced *Chevalley group schemes* in [19]. Chevalley's scheme-theoretic treatment of these groups was generalized to all reductive algebraic groups by M. Demazure in his thesis [20]. These schemes are known as the *Chevalley-Demazure group schemes*.

The Chevalley-Demazure group schemes can be classified with a combinatorial data known as the *root datum*. A Chevalley-Demazure group scheme corresponding to a root datum $\mathcal{D}$ is a group functor $\mathcal{G}_{\mathcal{D}}$ over the category of commutative rings such that a reductive algebraic group G over an algebraically closed field $\mathbb{K}$ is precisely equal to $\mathcal{G}_{\mathcal{D}}(\mathbb{K})$ for some root datum $\mathcal{D}$. So, there are two equivalent ways to associate Chevalley groups with the Kac-Moody algebras of finite type, namely, the Chevalley-Demazure group schemes or via Steinberg's construction.

In the general setting, the association of a group with a Kac-Moody algebra corresponding to a GCM of affine or indefinite type is a complex problem. R. Moody and K. Teo initiated work on this problem in [50] and subsequent contributions were made in [46, 22, 36, 35, 37] and [60].

In 1987, J. Tits [61] defined a group functor over the category of commutative rings associated with a Kac-Moody root datum corresponding to a GCMs of affine and indefinite type. He presented this functor as a set of axioms and this generalizes the constructions done by Steinberg, Chevalley and Demazure. The groups obtained by Tits are now known as the *Kac-Moody groups*. In the next sections of this chapter, we are going to discuss all these notions in more details.

3.1 Functorial Construction

3.1.1 Kac-Moody Root Datum and Algebra

Let I be a finite set of cardinality ℓ and $A = (a_{ij})_{i,j\in I}$ be a GCM as introduced earlier.

Definition 3.1.1. *A Kac-Moody root datum associated with the pair (I, A) is a quadruple $\mathcal{D} = (X, X^\vee, \{c_i\}_{i\in I}, \{h_i\}_{i\in I})$ such that:*

(D1) *X is a free $\mathbb{Z}$-module with a free rank, $X^\vee$ is its $\mathbb{Z}$ dual, these come equipped with a perfect pairing $\langle\cdot,\cdot\rangle \colon X \times X^\vee \longrightarrow \mathbb{Z}$.*

(D2) *we have* $c_i \in X$, $h_i \in X^\vee$ *such that* $\langle c_i, h_j \rangle = a_{ij}$ *for all* $i, j \in I$.

A Kac-Moody root datum $\mathcal{D}$ is called *free* (resp. *cofree*) if the set $\{c_i\}_{i\in I}$ (resp. $\{h_i\}_{i\in I}$) is $\mathbb{Z}$-linearly independent; $\mathcal{D}$ is *adjoint* (resp. *coadjoint*) if the sets $\{c_i\}_{i\in I}$ (resp. $\{h_i\}_{i\in I}$) spans X (resp. $X^\vee$); $\mathcal{D}$ is said to be *simply connected* if for every $i \in I$ there exists $x_i \in X$ such that $\langle x_i, h_j \rangle = \delta_{ij}$, for all $j \in I$. Given a Kac-Moody root datum $\mathcal{D}$, set $\mathfrak{h}_{\mathcal{D}} := X^\vee \otimes \mathbb{C}$.

3.1.2 Some Group Functors

Let R be a commutative ring with identity and R^* be its group of units. A *split torus scheme* associated with a Kac-Moody root datum $\mathcal{D}$ is a group functor

$$\mathfrak{T}_X \colon \mathbb{Z}\textbf{-alg} \longrightarrow \mathbf{Grp},$$

from the category $\mathbb{Z}$**-alg** of $\mathbb{Z}$-algebras to the category **Grp** of groups, defined as

$$\mathfrak{T}_X(R) := Hom_{\mathbb{Z}}(X, R^*) = X^\vee \otimes_{\mathbb{Z}} R^*. \tag{3.1}$$

Thus, if n is the rank of X, the group $\mathfrak{T}_X(R)$ is isomorphic to $\mathbb{G}_m^n = (R^*)^n$. For $r \in R$ and $\lambda \in X$, $r^\lambda \in \mathfrak{T}_X(R)$ is the map $\mu \mapsto r^{\langle \lambda, \mu \rangle}$, where $\mu \in X$.

The action of the Weyl group W on $\mathfrak{h}_{\mathcal{D}}$ restricts to one on $X^\vee$ and X via:

$$w_i(\lambda^\vee) = \lambda^\vee - \langle c_i, \lambda^\vee \rangle h_i \text{ and } w_i(\lambda) = \lambda - \langle \lambda, h_i \rangle c_i, \tag{3.2}$$

respectively, where $i \in I$, $c_i, \lambda \in X$, and $h_i, \lambda^\vee \in X^\vee$. This action induces an action of W on $\mathfrak{T}_X(R)$ which is defined as: for a simple root reflection w_i and $t \in \mathfrak{T}_X(R)$,

$$w_i.t = (t)^{w_i} \colon X \longrightarrow R^* \tag{3.3}$$

such that $t^{w_i}(\lambda) = t(w_i\lambda)$, for all $\lambda \in X$.

Let $\mathbb{G}_+$ be one-dimensional additive group scheme. For $\alpha \in \Delta^{re}$, we denote by $\mathfrak{U}_\alpha$ an affine group scheme over $\mathbb{Z}$ isomorphic to $\mathbb{G}_+$ such that $Lie(\mathfrak{U}_\alpha) := \mathfrak{g}_{\alpha,\mathbb{Z}} = \mathbb{Z}X_\alpha$, where X_α is a non-zero vector in $\mathfrak{g}_\alpha$. Using the choice of a double basis ([61, Section 3.3, Section 3.6]), we obtain an isomorphism

$$x_\alpha \colon \mathbb{G}_+ \longrightarrow \mathfrak{U}_\alpha, \tag{3.4}$$

then $\mathfrak{U}_\alpha(R) = \{x_\alpha(r) \mid r \in R\}$ and for any $r, s \in R$,

$$x_\alpha(r+s) = x_\alpha(r)x_\alpha(s). \tag{3.5}$$

To define a commutation relation between the elements of different groups $\mathfrak{U}_\alpha(R)$, $\alpha \in \Delta^{re}$, we introduce the following notions on the set of roots.

Definition 3.1.2. *A pair of roots $\alpha, \beta \in \Delta$ is called a prenilpotent pair if there exist $w, w' \in W$ such that $w\alpha,\ w\beta \in \Delta_+$ and $w'\alpha,\ w'\beta \in \Delta_-$.*

A subset $\Psi \subset \Delta$ is said to be prenilpotent if there exists $w, w' \in W$ such that $w\Psi \subset \Delta_+$ and $w'\Psi \subset \Delta_-$; $\Psi \subset \Delta \cup \{0\}$ is said to be a *closed set* or an *ideal* if $\alpha, \beta \in \Psi$ and $\alpha + \beta \in \Delta \cup \{0\}$ then $\alpha + \beta \in \Psi$; Ψ is referred to as a *nilpotent set* if it is both prenilpotent and closed.

For a prenilpotent pair α, β, set

$$[\alpha, \beta] := (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Delta, \quad]\alpha, \beta[:= [\alpha, \beta] \setminus \{\alpha, \beta\}.$$

One can check that $[\alpha, \beta]$ and $]\alpha, \beta[$ are nilpotent sets. To any subset Ψ of the set of roots, we write $\mathfrak{g}_\Psi := \oplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$. If Ψ is a closed subset then $\mathfrak{g}_\Psi$ is a Lie subalgebra and if Ψ is a nilpotent subset then $\mathfrak{g}_\Psi$ is a nilpotent Lie subalgebra.

Proposition 3.1.3 ([47, P. 134])**.** *Let α, β be a prenilpotent pair and the set $]\alpha, \beta[$ be equipped with any order. Then there exist integers $C_{ij}^{\alpha\beta}$ depending on α, β and the order on $]\alpha, \beta[$, such that*

$$[x_\alpha(r), x_\beta(s)] = \prod_\gamma x_\gamma(C_{ij}^{\alpha\beta} r^i s^j) \tag{3.6}$$

where $r, s, \in R$ and $\gamma = i\alpha + j\beta$ runs through the elements of $]\alpha, \beta[$ in the prescribed order.

Definition 3.1.4. *The Steinberg group functor*

$$\mathbf{St} \colon \mathbb{Z}\text{-}\mathbf{alg} \longrightarrow \mathbf{Grp}$$

sends any ring R to $\mathbf{St}(R)$, the free product of the groups $\mathfrak{U}_\alpha(R)$ for all $\alpha \in \Delta^{re}$ modulo the relations (3.5) and (3.6).

For $i \in I$ and $t \in R^*$, let

$$w_i^*(t) = w_{\alpha_i}^*(t) = x_{\alpha_i}(t) x_{-\alpha_i}(-t^{-1}) x_{\alpha_i}(t), \tag{3.7}$$

$$w_i^* := w_i^*(1) \text{ and } h_i(t) = h_{\alpha_i}(t) = w_i^*(t)(w_i^*)^{-1}. \tag{3.8}$$

Now, we introduce the last group functor of this subsection.

Definition 3.1.5. *Associated with $\mathcal{D}$, we define a group functor $\mathfrak{G}_{\mathcal{D}}$ over the category $\mathbb{Z}$-**alg**, such that its value over the ring R is a free product* **St**$(R) * \mathfrak{T}_X(R)$ *modulo the following relations:*

(R1) $tx_{\alpha_i}(r)t^{-1} = x_{\alpha_i}(t(c_i)r)$, $i \in I$, $t \in \mathfrak{T}_X(R)$,

(R2) $w_i^* t(w_i^*)^{-1} = w_i(t)$, $i \in I$, $t \in \mathfrak{T}_X(R)$,

(R3) $w_i^*(r^{-1}) = w_i r^{h_i}$, *for* $r \in R^*$,

(R4) $w_i^* x_\beta(r)(w_i^*)^{-1} = x_{w_i\beta}(\eta_{\alpha_i,\beta} r)$, $r \in R$, $\beta \in \Delta^{re}$ *and* $\eta_{\alpha_i,\beta} \in \{\pm 1\}$.

The above construction of the group functor can be found in Section 3 of Tits' paper [61]. He also gave an abstract definition of this functor and presented it a set of axioms. This abstract definition is now known as the *Tits' Group Functor*, which is going to be discussed in the next subsection.

3.1.3 Tits Group Functor

Let $\mathcal{D}$ be a Kac-Moody root datum associated with (A, I) as defined earlier. Let R be a commutative ring, $\mathfrak{T} := \mathfrak{T}_X$ be a *split torus scheme* as introduced in the previous section and $\mathfrak{SL}_2$ is a group functor defined as:

$$\mathfrak{SL}_2(R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R; ad - bc = 1\}.$$

The Tits' functor is a system $(\mathfrak{G}, (\phi_i)_{i\in I}, \eta)$, which consists of a group functor

$$\mathfrak{G} \colon \mathbb{Z}\textbf{-alg} \longrightarrow \textbf{Grp}$$

and homomorphisms of functors

$$\phi_i : \mathfrak{SL}_2 \longrightarrow \mathfrak{G} \text{ and } \eta : \mathfrak{T} \longrightarrow \mathfrak{G},$$

that is, for a ring R the maps $\phi_{i,R}\colon \mathfrak{SL}_2(R) \longrightarrow \mathfrak{G}(R)$ and $\eta_R : \mathfrak{T}(R) \longrightarrow \mathfrak{G}(R)$ are group homomorphisms and this data satisfies the following axioms: for all $i \in I$

(a) *If R is a field, $\mathfrak{G}(R)$ is generated by $\phi_{i,R}(\mathfrak{SL}_2(R))$ and $\eta_R(\mathfrak{T}(R))$.*

(b) *For all R, the homomorphism $\eta_R : \mathfrak{T}(R) \longrightarrow \mathfrak{G}(R)$ is injective.*

(c) *For $r \in R^*$ and $i \in I$, let $r^{\alpha_i^\vee} \in \mathfrak{T}(R)$ be defined as $r^{\alpha_i^\vee}(\lambda) = r^{\langle \lambda, \alpha_i^\vee \rangle}$, then*

$$\phi_{i,R}\left(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}\right) = \eta_R(r^{\alpha_i^\vee}).$$

(d) *If i is an an embedding of a ring R in a field K, then $\mathfrak{G}(i)$ is an embedding of group $\mathfrak{G}(R)$ in $\mathfrak{G}(K)$.*

(e) *There is a homomorphism $\mathrm{Ad} : \mathfrak{G}(\mathbb{C}) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ such that:*

  (i) *we have $\ker \mathrm{Ad} \subset \eta_{\mathbb{C}}(\mathfrak{T}(\mathbb{C}))$.*

  (ii) *For $c \in \mathbb{C}$ and $i \in I$*

$$\mathrm{Ad}(\phi_{i,\mathbb{C}}(u^+(c))) = e^{\mathrm{ad}_{ce_i}}, \quad \mathrm{Ad}(\phi_{i,\mathbb{C}}(u^-(c))) = e^{\mathrm{ad}_{cf_i}},$$

where $u^+(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ *and* $u^-(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$.

(iii) *For* $t \in \mathfrak{T}(R)$ *and* $i \in I$

$$\mathrm{Ad}(\eta(t))(e_i) = t(\alpha_i)e_i \quad \mathrm{Ad}(\eta(t))(f_i) = -t(\alpha_i)f_i\,.$$

The main result in Tits' paper [61, Theorem 1] asserts that for any system $(\mathfrak{G}, (\phi_i)_{i\in I}, \eta)$ satisfying the above axioms, the group $\mathfrak{G}(\mathbb{K})$ over a field $\mathbb{K}$ is defined up to canonical isomorphism. The group $\mathfrak{G}(\mathbb{K})$ is known as a *minimal* or *incomplete* Kac-Moody group.

3.2 Carbone-Garland Construction

Steinberg's construction of Chevalley groups is also a natural candidate for an infinite dimensional generalization. L. Carbone and H. Garland extended this construction to Kac-Moody root systems over arbitrary fields in [13]. Recently this construction has been generalized to define Kac-Moody groups over $\mathbb{Z}$ and arbitrary rings by Carbone et al. in [1, 15].

3.2.1 $\mathbb{Z}$-Forms: Pathway to Arbitrary Fields

We retain the notation of Kac-Moody algebra $\mathfrak{g}$ and the related data from Section 2.2. Let $\mathcal{U}$, $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{n}^-)$ be the universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. Let $S(\mathfrak{h})$ be the symmetric algebra of $\mathfrak{h}$, Tits in [62] asserts that the canonical map

$$\mathcal{U}(\mathfrak{n}^+) \otimes S(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}^-) \longrightarrow \mathcal{U} \tag{3.9}$$

is a bijection. Next, we introduce some notions on an associative algebra $\mathcal{A}$ over $\mathbb{C}$ which will be used later.

Definition 3.2.1. *A $\mathbb{Z}$ form of $\mathcal{A}$ is a $\mathbb{Z}$ subalgebra $\mathcal{A}_{\mathbb{Z}}$ of $\mathcal{A}$ such that the canonical map $\mathcal{A}_{\mathbb{Z}} \otimes \mathbb{C} \longrightarrow \mathcal{A}$ is a bijection.*

Definition 3.2.2. *For $a \in \mathcal{A}$ and $n \in \mathbb{Z}_{\geq 0}$, we define the following elements of $\mathcal{A}$,*

$$a^{(n)} := \frac{a^n}{n!} \tag{3.10}$$

$$\binom{a}{n} := \frac{a(a-1)(a-2)\dots(a-n+1)}{n!}. \tag{3.11}$$

Let us denote by $\mathfrak{t}$ and $\mathfrak{t}^\vee$ the linear span of α_i and $\alpha_i^\vee$ for $i \in I$, respectively. For $i \in I$ and $n \in \mathbb{Z}_{\geq 0}, \mathcal{U}_{i,+}$ (resp. $\mathcal{U}_{i,-}$) be the subring $\sum_n \mathbb{Z} e_i^{(n)}$ (resp. $\sum_n \mathbb{Z} f_i^{(n)}$) of $\mathcal{U}$. Let $\mathcal{U}_{\mathbb{Z},+}$ (resp. $\mathcal{U}_{\mathbb{Z},-}$) be a subring of $\mathcal{U}(\mathfrak{n}^+)$ (resp. of $\mathcal{U}(\mathfrak{n}^-)$) generated by $\mathcal{U}_{i,+}$ (resp. $\mathcal{U}_{i,-}$) for all $i \in I$. Then $\mathcal{U}_{\mathbb{Z},+}$ and $\mathcal{U}_{\mathbb{Z},-}$ are the $\mathbb{Z}$-subalgebras of $\mathcal{U}(\mathfrak{n}^+)$ and $\mathcal{U}(\mathfrak{n}^-)$, respectively [61, p.556]. Let $\mathcal{U}_{\mathbb{Z},0}$ be the $\mathbb{Z}$-subalgebra of the universal enveloping algebra $S(\mathfrak{h})$ generated by $\binom{\lambda}{n}$ ($\lambda \in \mathfrak{t}^\vee$) and $\mathcal{U}_{\mathbb{Z}}$ be the $\mathbb{Z}$-subalgebra of the universal enveloping algebra $\mathcal{U}$ generated by $\mathcal{U}_{i,+}$, $\mathcal{U}_{i,-}$ and $\binom{\lambda}{n}$ for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathfrak{t}^\vee$.

We state the following result from [47, p.106] and [63] without giving its proof.

Proposition 3.2.3. *We have the following*

(i) *$\mathcal{U}_{\mathbb{Z},+}$, $\mathcal{U}_{\mathbb{Z},-}$ and $\mathcal{U}_{\mathbb{Z},0}$ are the $\mathbb{Z}$-forms of $\mathcal{U}(\mathfrak{n}^+)$, $\mathcal{U}(\mathfrak{n}^-)$ and $S(\mathfrak{h})$, respectively.*

(ii) *$\mathcal{U}_{\mathbb{Z}}$ is the $\mathbb{Z}$-form of $\mathcal{U}$.*

(iii) *The product map*

$$\mathcal{U}_{\mathbb{Z},-} \otimes \mathcal{U}_{\mathbb{Z},0} \otimes \mathcal{U}_{\mathbb{Z},+} \longrightarrow \mathcal{U}_{\mathbb{Z}} \tag{3.12}$$

is an isomorphism of $\mathbb{Z}$-modules.

The $\mathbb{Z}$-forms of $\mathcal{U}$ and its subalgebras will be used to define the Kac-Moody algebra $\mathfrak{g}_{\mathbb{K}}$ over an arbitrary field $\mathbb{K}$. Let $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{g} \cap \mathcal{U}_{\mathbb{Z}}$, $\mathfrak{n}^{\pm}_{\mathbb{Z}} = \mathfrak{n}^{\pm} \cap \mathcal{U}_{\mathbb{Z}}$ and $\mathfrak{t}^{\vee}_{\mathbb{Z}} = \mathfrak{t}^{\vee} \cap \mathcal{U}_{\mathbb{Z},0}$. The following proposition on page 78 of [47] implies that $\mathfrak{g}_{\mathbb{Z}}$ is $\mathbb{Z}$-form of $\mathfrak{g}$.

Proposition 3.2.4. *The sum map*

$$\mathfrak{n}^{+}_{\mathbb{Z}} \oplus \mathfrak{t}^{\vee}_{\mathbb{Z}} \oplus \mathfrak{n}^{-}_{\mathbb{Z}} \longrightarrow \mathfrak{g}_{\mathbb{Z}}$$

is a bijection.

For a field $\mathbb{K}$, set

$$\begin{aligned} \mathfrak{n}^{\pm}_{\mathbb{K}} &:= \mathfrak{n}^{\pm}_{\mathbb{Z}} \otimes \mathbb{K},\ \mathfrak{t}^{\vee}_{\mathbb{K}} := \mathfrak{t}^{\vee}_{\mathbb{Z}} \otimes \mathbb{K},\ \mathfrak{g}_{\mathbb{K}} := \mathfrak{g}_{\mathbb{Z}} \otimes \mathbb{K}, \\ \mathcal{U}_{\mathbb{K},\pm} &:= \mathcal{U}_{\mathbb{Z},\pm} \otimes \mathbb{K},\ \mathcal{U}_{\mathbb{K},0} := \mathcal{U}_{\mathbb{Z},0} \otimes \mathbb{K},\ \mathcal{U}_{\mathbb{K}} := \mathcal{U}_{\mathbb{Z}} \otimes \mathbb{K}. \end{aligned}$$

Then $\mathfrak{g}_{\mathbb{K}}$ is a Kac-Moody algebra over $\mathbb{K}$ and it admits the root spaces decomposition

$$\mathfrak{g}_{\mathbb{K}} = \mathfrak{t}^{\vee}_{\mathbb{K}} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha,\mathbb{K}}), \tag{3.13}$$

where for each $\alpha \in \Delta$, $\mathfrak{g}_{\alpha,\mathbb{K}} = (\mathfrak{g}_{\alpha} \cap \mathfrak{g}_{\mathbb{Z}}) \otimes \mathbb{K}$.

3.2.2 Minimal Kac-Moody Group

As in Steinberg's presentation of Chevalley groups, the second essential ingredient in Carbone and Garland's construction of Kac-Moody groups is an integrable representation having a stable lattice. We begin this subsection with a description of this lattice.

Let $V = V^{\lambda}$ be an integrable highest weight representation with the highest weight λ and the highest weight vector v_{λ}. As in the finite dimensional case [58], a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ is constructed by setting $V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} v_{\lambda}$. The following lemma gives a more concrete description of the lattice $V_{\mathbb{Z}}$.

Lemma 3.2.5. *We have*

$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z},-}(v_{\lambda}). \tag{3.14}$$

Proof. Since $X_{\alpha} v_{\lambda} = 0$ for all $\alpha \in \Delta_+$, $\mathcal{U}_{\mathbb{Z},+} \setminus \{1\}$ annihilates v_{λ}. Moreover, for $n \geq 1$ and $\mu^{\vee} \in \mathfrak{t}^{\vee}$

$$\frac{\lambda(\mu^{\vee})(\lambda(\mu^{\vee}) - 1)(\lambda(\mu^{\vee}) - 2) \ldots (\lambda(\mu^{\vee}) - n + 1)}{n!} \in \mathbb{Z}$$

which gives $\mathcal{U}_{\mathbb{Z},0} v_{\lambda} = \mathbb{Z} v_{\lambda}$. Finally, by Propositon 3.2.3 (iii) we have

$$V_{\mathbb{Z}} = \mathcal{U}_{\mathbb{Z}} v_{\lambda} = \mathcal{U}_{\mathbb{Z},-} \mathbb{Z} v_{\lambda} = \mathcal{U}_{\mathbb{Z},-} v_{\lambda}.$$

Corollary 3.2.6. *The space $V_{\mathbb{Z}}$ is a $\mathbb{Z}$-form and an admissible lattice of V, that is, for $i \in I$ and for some $n \geq 0$*

$$e_i^{(n)} V_{\mathbb{Z}} \subset V_{\mathbb{Z}}; \quad f_i^{(n)} V_{\mathbb{Z}} \subset V_{\mathbb{Z}}.$$

For each weight μ of V and the corresponding weight space V_{μ}, we set $V_{\mu,\mathbb{Z}} = V_{\mu} \cap V_{\mathbb{Z}}$. Then $V_{\mathbb{Z}} = \oplus_{\mu \in P_{\lambda}} V_{\mathbb{Z},\mu}$. For a field $\mathbb{K}$, let $V_{\mathbb{K}} := V_{\mathbb{Z}} \otimes \mathbb{K}$, $\;V_{\mathbb{K},\mu} := V_{\mathbb{Z},\mu} \otimes \mathbb{K}$ and for $t \in \mathbb{K}$

$$\chi_{\alpha_i}(t) := \sum_{n \geq 0} t^n e_i^{(n)} = e^{te_i}, \;\; \chi_{-\alpha_i}(t) := \sum_{n \geq 0} t^n f_i^{(n)} = e^{tf_i}. \tag{3.15}$$

Since e_i and f_i act as locally nilpotent operators, $\chi_{\pm\alpha_i}(t)$ are well defined automorphisms of $V_{\mathbb{K}}$. The minimal Kac-Moody group $G(\mathbb{K})$ of Carbone and Garland is the subgroup of $\mathrm{Aut}(V_{\mathbb{K}})$ generated by the elements $\chi_{\pm\alpha_i}(t)$ with $t \in \mathbb{K}$ and $i \in I$ [13, Section 5]. Similar to the Chevalley groups [58, Lemma 27], the group constructed above depends on the integrable highest weight representation V and the choice of an admissible lattice in V. Though we do not need the completed version of G for our results, for the sake of completion we briefly discuss this notion in the last subsection.

3.2.3 Completion of G

Intuitively, a minimal or incomplete Kac-Moody group is constructed by exponentiating the root spaces of real roots. If the root spaces corresponding to the imaginary roots are also used in the construction, the resulting group is called a *complete* or *maximal* Kac-Moody group. There are three completions of minimal Kac-Moody groups which can be found in the literature, a representation theoretic completion by L. Carbone and H. Garland in [13], a completion by using the *building topology* by B. Remy and M. Ronan in [54] and a scheme theoretic completion over $\mathbb{C}$ by S. Kumar in [39] and over algebraically closed field by G. Rousseau in [55].

3.3 Tits Axioms and BN-Pairs

In what follows, we shall consider Carbone-Garland's Kac-Moody group $G(\mathbb{K})$ over a field $\mathbb{K}$ and denote it by G, by dropping $\mathbb{K}$ from our notation. For $i \in I$ and $t \in \mathbb{K}^*$, set

$$\begin{aligned}\tilde{w}_i(t) &:= \chi_\alpha(t)\chi_{-\alpha}(-t^{-1})\chi_\alpha(t), \quad \tilde{w}_i := \tilde{w}_i(1) && (3.16)\\ \tilde{h}_i(t) &:= \tilde{w}_i(t)\tilde{w}_i^{-1}. && (3.17)\end{aligned}$$

Let H be the subgroup generated by the elements $h_i(t)$ for all $i \in I$ and $t \in \mathbb{K}^*$. Let $\alpha \in \Delta^{re}$ then $\alpha = w\alpha_i$ form some $w \in W$ and simple root α_i, $i \in I$. For $t \in \mathbb{K}$, we set

$$\chi_\alpha(t) = w\chi_{\alpha_i}(t)w^{-1} \tag{3.18}$$

One can check that for $t \in \mathbb{K}$, we have $\chi_\alpha(t) \in \mathrm{Aut}(V_{\mathbb{K}})$. Associated with $\alpha \in \Delta^{re}$, a *root group* is defined as,

$$U_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{K}\}.$$

Continuing with $\alpha \in \Delta^{re}_{\pm}$, let $B^{\pm}_\alpha$ be the group generated by H and U_α; G_α be the group generated by $B^{\pm}_\alpha$; and $B^{\pm}$ be the group generated by B_α for all $\alpha \in \Delta^{re}_{\pm}$. The following properties of these subgroups can be verified easily.

(RD1) *Let α, β be a prenilpotent pair then $[U_\alpha, U_\beta] \subset \langle\, U_\gamma \mid \gamma \in]\alpha, \beta[\,\rangle$.*

(RD2) *For each $i \in I$, $B^+_{\alpha_i} \cap B^-_{\alpha_i} = H$.*

(RD3) *The group $B^+_{\alpha_i}$ has two double cosets in G_{α_i}.*

(RD4) *For each $i \in I$ and $\beta \in \Delta^{re}$, there exists an element $s_i \in G_{\alpha_i}$ such that $s_i B_\alpha s_i^{-1} = B_{w_i\alpha}$.*

(RD5) *For each $i \in I$, $B^+_{\alpha_i}$ is not contained in B^- and $B^-_{\alpha_i}$ is not contained in B^+.*

Let N be the subgroup generated by H and $\tilde{w}_i$ for all $i \in I$. For a proof of the next result, we refer readers to [61, Section 5].

Theorem 3.3.1. *The axioms RD1-RD5 imply the following important consequences,*

(a) *The pairs (B^+, N) and (B^-, N) form a Tits system.*

(b) *There exists a unique homomorphism $\phi \colon N \longrightarrow W$ with $Ker\phi = H$, and for all $n \in N$ and $\beta \in \Delta^{re}$, $nB_\beta n^{-1} = B_{\phi(n)\beta}$.*

(c) *Group G has Bruhat decompositions*

$$\begin{aligned} G &= BNB = B^- NB^- \\ &= BNB^- = B^- NB. \end{aligned}$$

3.4 Subgroups and Decompositions

Now, we consider $\mathcal{K}$ to be a non-archimedean local field (See Section 2.1 for notations) and $G = G(\mathcal{K})$. The group G has an integral subgroup which is defined as

$$K := \{g \in G \mid gV_{\mathcal{O}} \subset V_{\mathcal{O}}\}, \tag{3.19}$$

where

$$V_{\mathcal{O}} = V_{\mathbb{Z}} \otimes \mathcal{O}. \tag{3.20}$$

The group K is an analogue of maximal compact subgroup from the finite dimensional theory. There is a pair of *unipotent subgroups*

$$U^{\pm} = \langle U_{\alpha} \mid \alpha \in \Delta^{re,\pm} \rangle$$

Let $U_{\mathcal{O}}^{\pm} = U^{\pm} \cap K$ be the integral subgroup and $U_{\pi}^{\pm}$ be the level one congruence subgroups of $U^{\pm}$. Let $H_{\mathcal{O}} = H \cap K$. The weight lattice $\Lambda^{\vee}$ can be identified with $H^{'} = H/H_{\mathcal{O}}$ via the map $\lambda^{\vee} \mapsto \pi^{\lambda^{\vee}}$, and G has an *Iwasawa decomposition*

$$G = \cup_{\mu^{\vee} \in \Lambda^{\vee}} K\pi^{\mu^{\vee}} U^{+} = \cup_{\nu^{\vee} \in \Lambda^{\vee}} U^{+}\pi^{\nu^{\vee}} K \tag{3.21}$$

with respect to U^{+} and

$$G = \cup_{\gamma^{\vee} \in \Lambda^{\vee}} K\pi^{\gamma^{\vee}} U^{-} = \cup_{\delta^{\vee} \in \Lambda^{\vee}} U^{-}\pi^{\delta^{\vee}} K \tag{3.22}$$

with respect to U^{-}. Let $G(\mathrm{k})$ be the Kac-Moody group over the finite residue field k and $\varpi : K \longrightarrow G(k)$ be the reduction mod π map. The group K has a pair of subgroups defined as

$$I^{\pm} = \varpi^{-1}(B^{\pm}(\mathrm{k})). \tag{3.23}$$

These groups are known as the *Iwahori subgroups* and admit the following direct product decompositions,

$$\begin{aligned} I^{+} &= U_{\mathcal{O}}^{+} U_{\pi}^{-} H_{\mathcal{O}} \\ I^{-} &= U_{\mathcal{O}}^{-} U_{\pi}^{+} H_{\mathcal{O}}, \end{aligned}$$

which are known as the *Iwahori-Matsumoto decompositions*. The group K admits the following decompositions

$$\begin{aligned} K &= \cup_{w \in W} I^+ w I^+ = \cup_{w \in W} I^- w I^+ \\ &= \cup_{w \in W} I^+ w I^- = \cup_{w \in W} I^- w I^- \end{aligned}$$

known as the *Iwahori decompositions*.

For $w \in W$, we define the following two subsets of the set of roots Δ.

$$\begin{aligned} S_w^+ &:= \{\alpha \in \Delta_+ \mid w\alpha \in \Delta_-\} = \Delta_+ \cap w^{-1}(\Delta_-), \\ S_w^- &:= \{\alpha \in \Delta_- \mid w\alpha \in \Delta_+\} = \Delta_- \cap w^{-1}(\Delta_+). \end{aligned}$$

Similarly, we define the subsets $S_w^{\pm,\vee} \subset \Delta^\vee$. By using $S_w^\pm$, we introduce finitely generated subgroups

$$U_w^\pm = \langle U_\alpha \mid \alpha \in S_w^\pm \rangle.$$

Let $U^{\pm,w} = U^\pm - U_w^\pm$. Set

$$U_{w,\mathcal{O}}^\pm = U_w^\pm \cap U_{\mathcal{O}}^\pm, \quad U_{w,\pi}^\pm = U_w^\pm \cap U_\pi^\pm \tag{3.24}$$

$$U_{\mathcal{O}}^{\pm,w} = U^{\pm,w} \cap U_{\mathcal{O}}^\pm, \quad U_\pi^{\pm,w} = U^{\pm,w} \cap U_\pi^\pm. \tag{3.25}$$

Chapter 4

Affine Construction and Formula

As stated earlier, our thesis project is motivated by a desire to extend certain construction and results from the affine to general Kac-Moody setting. Our methods for proving the main theorems are not exactly the same, even in the affine case. We present a review of the previous work in the affine case here to make these differences clear.

4.1 Realization of Affine Kac-Moody Data

Affine Kac-Moody algebras and groups are the most investigated objects of the infinite dimensional Kac-Moody theory. A concrete description of these algebraic structures make them suitable for both theoretical purposes and applications to other branches of mathematics and physics. In this section, we briefly describe affine Lie algebras and groups and explain how they arise as extensions of the finite dimensional Lie theoretic data.

4.1.1 Affine Kac-Moody Data

Affine Generalized Cartan Matrix

As discussed in Subsection 2.2.3, an affine GCM is classified into categories: untwisted and twisted type; and can be constructed from a GCM of finite type. For this chapter we will consider untwisted and symmetrizable affine GCM A. First we describe how A is obtained from a GCM of finite type. For this, let $\mathring{A}$ be an $l \times l$ indecomposable GCM of finite type, $\mathring{\mathfrak{g}}$ be the associated finite dimensional simple Lie algebra, and $\mathring{\mathfrak{h}}$ be its Cartan subalgebra. Let $\mathring{\Delta}$, $\mathring{\Delta}^{\vee}$, $\mathring{\Pi} = \{\alpha_i\}_{1 \le i \le l}$, $\mathring{\Pi}^{\vee} = \{\alpha_i^{\vee}\}_{1 \le i \le l}$ be the set of roots, coroots, simple roots and simple coroots, respectively. Let $\kappa(-,-)$ be the killing form on $\mathring{\mathfrak{g}}$, $\mathring{\mathfrak{h}}$ and $\mathring{\mathfrak{h}}^*$, and θ be the highest root with the corresponding coroot $\theta^{\vee}$. Set $\alpha_0 = -\theta$ and $A = (a_{ij})_{1 \le i,j \le l+1}$ where $a_{ij} = \frac{2\kappa(\alpha_i,\alpha_j)}{\kappa(\alpha_i,\alpha_i)}$, $i, j = 1, 2, \ldots l+1$. The matrix A is called an affine GCM associated with A as defined in [22, p. 204].

Affine Kac-Moody Algebra

Let t be an indeterminate and $\mathbb{C}[t, t^{-1}]$ be the *ring of polynomials* in t and t^{-1}. Suppose $\mathbb{C}((t))$ and $\mathbb{C}[[t]]$ denote the *field of Laurent series* and the *ring of formal power series* over $\mathbb{C}$, respectively.

Let $\mathfrak{g}$ and $\mathring{\mathfrak{g}}$ be the Kac-Moody algebras over $\mathbb{C}$ associated A and $\mathring{A}$, respectively. These two Kac-Moody algebras are related as follows. Let

$$\tilde{\mathring{\mathfrak{g}}} := \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathring{\mathfrak{g}}. \tag{4.1}$$

With the Lie bracket defined as $[u \otimes x, v \otimes y] = uv \otimes [x, y]$, for $u \otimes x, v \otimes y \in \tilde{\mathring{\mathfrak{g}}}$ and $[x, y]$ the Lie bracket of $x, y \in \mathring{\mathfrak{g}}$, the space $\tilde{\mathring{\mathfrak{g}}}$ becomes a Lie algebra known as the *loop algebra* of $\mathring{\mathfrak{g}}$. We fix a symmetric, non-degenerate, invariant bilinear form $(-,-)$ on $\mathring{\mathfrak{g}}$ which exists by Theorem 2.2 of [38] and is a scalar multiple of $\kappa(-,-)$. The untwisted affine Kac-Moody algebra constructed on the top of $\mathring{\mathfrak{g}}$ is the following double 1-dimensional extension of the loop algebra

$$\hat{\mathfrak{g}} := \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{g}} \oplus \mathbb{C}c \oplus \mathbb{C}d \tag{4.2}$$

with the Lie bracket defined as,

$$[X_1, X_2] = t^{m_1+m_2} \otimes [x_1, x_2] + \mu_1 m_2 t^{m_2} \otimes x_2 - \mu_2 m_1 t^{m_1} \otimes x_1 \\ + m_1 \delta_{m_1,m_2} (x_1, x_2) c, \tag{4.3}$$

for all $X_1 = t^{m_1} \otimes x_1 + \lambda_1 c + \mu_1 d$, $X_2 = t^{m_2} \otimes x_2 + \lambda_2 c + \mu_2 d \in \hat{\mathfrak{g}}$ with $x_1, x_2 \in \mathfrak{g}$, $m_1, m_2 \in \mathbb{Z}$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Set

$$\mathfrak{h} := \mathring{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

then $\mathfrak{h}$ is an $l+2$ dimensional subalgebra of $\hat{\mathfrak{g}}$ and $\mathring{\mathfrak{h}} \hookrightarrow \mathfrak{h}$. As in [38, p. 100], the dual $\mathring{\mathfrak{h}}^*$ of Cartan subalgebra imbeds in $\mathfrak{h}^*$ through an extension of each element $\lambda \in \mathring{\mathfrak{h}}^*$ to $\mathfrak{h}^*$ by setting $\lambda(c) = \lambda(d) = 0$. Theorem 7.4 in [38] and Theorem 13.1.3 in [39] assert that the Lie algebra $\hat{\mathfrak{g}}$ is isomorphic to the Kac-Moody algebra $\mathfrak{g}$ corresponding to A defined by a set of generators and relation in Subsection 2.2.2.

Roots and the Weyl Group

Let $\delta \in \mathfrak{h}^*$ be defined as

$$\delta|_{\mathring{\mathfrak{h}} \oplus \mathbb{C}c} = 0, \quad \delta(d) = 1,$$

where $\delta|_{\mathring{\mathfrak{h}}\oplus\mathbb{C}c}$ is the restriction of δ on $\mathring{\mathfrak{h}} \oplus \mathbb{C}c$. Then $\Pi^\vee = \{\alpha_0^\vee = c - \theta^\vee\} \cup \mathring{\Pi}^\vee$ are the simple coroots corresponding to the simple roots $\Pi = \{\alpha_0 = \delta - \theta\} \cup \mathring{\Pi}$. The triplet $(\mathfrak{h}, \Pi, \Pi^\vee)$ constructed above is a realization associated with the GCM A. The corresponding abstractly defined Kac-Moody algebra $\mathfrak{g}$ (as defined Subsection 2.2.2) is isomorphic to the Lie algebra $\hat{\mathfrak{g}}$. For the rest of this chapter we shall use the notation $\mathfrak{g}$ to denote the affine Kac-Moody algebra $\hat{\mathfrak{g}}$.

The set of roots and positive roots of $\mathfrak{g}$ are given by

$$\Delta = \{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\} \cup \{n\delta + \beta \mid n \in \mathbb{Z}, \beta \in \mathring{\Delta}\}. \tag{4.4}$$

and

$$\Delta^+ = \{n\delta \mid n > 0\} \cup \{n\delta + \beta \mid n > 0, \beta \in \mathring{\Delta}\}. \tag{4.5}$$

Let $\mathring{W}$ be the Weyl group associated with $\mathring{\mathfrak{g}}$ and $Q^\vee = \oplus_{i=1}^{l} \mathbb{Z}\alpha_i$ be coroot lattice then $\mathring{W}$ acts on $Q^\vee$ through the restriction on $\mathring{\mathfrak{h}}$. We denote by

$$\widetilde{W} = \mathring{W} \ltimes Q^\vee. \tag{4.6}$$

The group $\widetilde{W}$ is called the *affine Weyl group* and the Weyl group W associated with $\mathfrak{g}$ is isomorphic to $\widetilde{W}$.

4.1.2 Affine Kac-Moody Group

Loop Groups and Extension

Let $\mathring{\mathbf{G}}$ be the simple, simply-connected algebraic group over $\mathbb{Z}$ with Lie algebra $\mathring{\mathfrak{g}}$. We choose a pair of opposite Borel subgroups $\mathring{\mathbf{B}}^+, \mathring{\mathbf{B}}^-$ with unipotent radicals

$\mathring{\mathbf{U}}^{+}, \mathring{\mathbf{U}}^{+}$, respectively. The intersection $\mathring{\mathbf{H}} = \mathring{\mathbf{B}}^{+} \cap \mathring{\mathbf{B}}^{-}$ is a maximal torus of $\mathring{\mathbf{G}}$. The *polynomial loop group* $\mathring{\mathbf{G}}[t, t^{-1}]$ is a functor whose points over a ring R are given by $\mathring{\mathbf{G}}(R[t, t^{-1}])$. By a theorem of Pressley and Segal [53, Theorem 4.4.1], the loop group admits a central extension

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \widetilde{\mathbf{G}} \longrightarrow \mathring{\mathbf{G}}[t, t^{-1}] \longrightarrow 1. \tag{4.7}$$

The multiplicative group $\mathbb{G}_m$ acts on $\mathring{\mathbf{G}}[t, t^{-1}]$ and this action lifts to $\widetilde{\mathbf{G}}$. The affine Kac-Moody group $\mathbf{G}$ is the semidirect product $\mathbb{G}_m \ltimes \widetilde{\mathbf{G}}$ under this action. One can associate a Lie algebra $\mathfrak{g}$ with $\mathbf{G}$ which is isomorphic to the untwisted affine Kac-Moody algebra corresponding to the affine GCM A. The affine Kac-Moody group $\mathbf{G}$ and the Lie algebra $\mathfrak{g}$ can be described by the affine root system introduced earlier in the previous subsection. .

Subgroups

Let $\mathring{\mathbf{G}}$, $\mathring{\mathbf{B}}^{\pm}$, $\mathring{\mathbf{U}}^{\pm}$ and $\mathring{\mathbf{H}}$ be as introduced earlier. Let $\mathbf{H} = \mathbb{G}_m \times \mathring{\mathbf{H}} \times G_m$. Let $\mathring{\mathbf{G}}[t]_{\mathbf{B}^+}$ denote the preimage of $\mathring{\mathbf{B}}^{+}$ under the natural map $\mathring{\mathbf{G}}[t] \longrightarrow \mathring{\mathbf{G}}$. We let $\mathbf{B}^{+}$ to be the preimage in $\mathbf{G}$ of $\mathring{\mathbf{G}}[t]_{\mathbf{B}} \ltimes \mathbb{G}_m$. This is a group-scheme, which is endowed with a natural map to $\mathbf{H}$. We denote by $\mathbf{U}^{+}$ the kernel of this map. This is the pro-unipotent radical of $\mathbf{B}^{+}$. Similarly, let $\mathring{\mathbf{G}}[t^{-}]_{\mathbf{B}^-}$ be the preimage of $\mathring{\mathbf{B}}^{-}$ under the map $\mathring{\mathbf{G}}[t^{-}] \rightarrow \mathring{\mathbf{G}}$ coming from evaluating t to ∞. We let $\mathbf{B}^{-} \subset \mathbf{G}$ to be the preimage in $\mathbf{G}$ of $\mathring{\mathbf{G}}[t^{-1}]_{B} \ltimes \mathbb{G}_m$. This is a group ind-scheme, which (similarly to $\mathbf{B}^{+}$) is endowed with a natural map to $\mathbf{H}$ and we denote its kernel by $\mathbf{U}^{-}$. In addition, the intersection $\mathbf{B}^{+} \cap \mathbf{B}^{-}$ is naturally isomorphic to $\mathbf{H}$.

Let $\mathcal{K}$ be a non-arcimedean local field as before and $G = \mathbf{G}(\mathcal{K})$. This group is different than the affine Kac-Moody group constructed by Carbone and Garland (see Subsection 3.2.2) or Tits group (constructed in Subsection 3.1.2) as it does not depend on the representation or on the choice of Kac-Moody data $\mathfrak{D}$. Let G_k and K be the values of $\mathbf{G}$ over the residue field k and the ring of integers $\mathcal{O}$, respectively. Let I^+ and I^- be the pair of Iwahori subgroups of K which are preimages of B_{k}^+ and B_{k}^-, respectively, under the natural map $\varpi\colon K \longrightarrow G_{\mathrm{k}}$ (reduction mod π). Let $U^{\pm} = \mathbf{U}^{\pm}(\mathcal{K})$, and $H = \mathbf{H}(\mathcal{K})$ be the subgroups of G, and $U^{\pm}_{\mathcal{O}} = \mathbf{U}^{\pm}(\mathcal{O})$ and $H_{\mathcal{O}} = \mathbf{H}(\mathcal{O})$ be the subgroups of K. Let $G_\pi := \{g \in K \mid \varpi(g) = 1\}$. We denote by $U^{\pm}_{\pi} = U^{\pm} \cap G_\pi$ and $H_\pi = H \cap G_\pi$. The group G admits the Iwasawa decompositions, K admits the Iwahori decompsoitions and $I^{\pm}$ admit the Iwahori-Matsumoto decompositions as given in Subsection 3.4.

4.1.3 Representation Theoretic Norm

Let $V = V^\lambda$ be the integrable highest weight $\mathfrak{g}$-representation of highest weight λ defined over the local field $\mathcal{K}$ and $V_{\mathcal{O}}$ be the integral lattice in V as defined in (3.20). For $v \in V$, set

$$Ord(v) = min_{n \in \mathbb{Z}}\ \pi^n v \in V_{\mathcal{O}} \tag{4.8}$$

and define a norm on V as

$$||v|| := q^{Ord(v)}, \tag{4.9}$$

for all $v \in V$. An element $v \in V$ is said to be a *primitive element* if $||v|| = 1$; we shall always choose the highest weight vector v_λ to be a primitive element.

We will choose a *coherently ordered basis* $\mathcal{B} = \{v_1, v_2, \ldots\}$ consisting of primitive elements, that is, $\mathcal{B}$ consists of weight vectors; if $v_i \in V_\mu$, $v_j \in V_\delta$ and

$depth(\delta) > depth(\mu)$ then $j > i$; and, $V_\mu \cap \mathcal{B}$ consists of an interval $v_r, v_{r+1}, \ldots, v_{r+m}$.

We end this section with the following lemma without giving its proof. This lemma will be used frequently during the course of representation theoretic arguments.

Lemma 4.1.1. *If $v, w \in V$ belong to different weight spaces then*

$$||v + w|| \geq ||v||. \tag{4.10}$$

4.2 Affine Gindikin-Karpelevich Finiteness

In this section, we elaborate the three steps mentioned in Section 1.4, which Braverman et al. followed in [6] to obtain the affine version of the Gindikin-Karpelevich Finiteness. The first step was to decompose U^- as a disjoint union of certain subsets.

4.2.1 Step 1: Decomposition of U^-

Let $Iw_K : U^- \longrightarrow K/K \cap B$ be a function defined by

$$Iw_K(u^-) = k(K \cap B) \tag{4.11}$$

for all $u^- \in U^-$ such that K-component of the Iwasawa decomposition of u^- is equal to k. It is straightforward to check that Iw_K is an embedding. Let ϖ be the reduction mod π map defined earlier. Combining these two functions with the natural projections of the quotient spaces, we get the following commutative diagram,

$$\begin{tikzcd}
U^- \arrow[r, two heads, "\pi_1"] & U^-_{\mathcal{O}}\backslash U^- \arrow[r, hook, "\psi"] \arrow[dr, "\phi"] & U^-_{\mathcal{O}}\backslash K/K\cap B \arrow[d, "\varpi"] \\
& & U^-(k)\backslash G(k)/B(k) \simeq W
\end{tikzcd}$$

For $w \in W$, the subset $\mathcal{V}^-_w$ is defined as

$$\mathcal{V}^-_w := \pi_1^{-1}(\phi^{-1}(w)). \tag{4.12}$$

Thus U^- decomposes as a disjoint union of its subsets,

$$U^- \;=\; \bigsqcup_{w\in W} \mathcal{V}^-_w. \tag{4.13}$$

As explained on page 51 of [6], the set $\mathcal{V}^-_w$ can be described more explicitly as an intersection

$$\mathcal{V}^-_w = U^- \cap U^-_{\mathcal{O}} U^+_{w,\pi} wB, \tag{4.14}$$

where $U^+_{w,\pi}$ is the subgroup introduced in (3.24) and each element $u^- \in \mathcal{V}^-_w$ has the following form

$$u^- = u^-_{\mathcal{O}} u^+_{w,\pi} w h_{\mathcal{O}} \pi^{\mu^\vee} u^+ \tag{4.15}$$

for some $u^-_{\mathcal{O}} \in U^-_{\mathcal{O}}$, $u^+_{w,\pi} \in U^+_{w,\pi}$, $h_{\mathcal{O}} \in H_{\mathcal{O}}$, $u^+ \in U^+$, and $\mu^\vee \in \Lambda^\vee$. The quotient $U^-_{\mathcal{O}}\backslash U^-$ is isomorphic to $K\backslash KU^-$ and hence

$$K\backslash KU^- = \bigsqcup_{w\in W} U^-_{\mathcal{O}}\backslash \mathcal{V}^-_w. \tag{4.16}$$

4.2.2 Step 2: Finiteness I

Due to the homogeneity of the Gindikin-Karpelevich formal sum (see Lemma 1.3.1), it suffices to obtain the finiteness of the coset space $K\backslash KU^- \cap K\pi^{\mu^\vee}U^+$. Now by using the right hand side of (4.13), we can write

$$\begin{aligned} K\backslash KU^- \cap K\pi^{\mu^\vee}U^+ &= \bigsqcup_{w\in W} K\backslash K\mathcal{V}_w^- \cap K\pi^{\mu^\vee}U^+ \\ &= \bigsqcup_{w\in W} U_{\mathcal{O}}^-\backslash\mathcal{V}_w^- \cap K\pi^{\mu^\vee}U^+ \\ &:= \bigsqcup_{w\in W} U_{\mathcal{O}}^-\backslash\mathcal{V}_w^-(\mu^\vee). \end{aligned} \tag{4.17}$$

The second step towards the Gindikin-Karpelevich finiteness is to show that there are finitely many w's which contribute in the union on the right hand side of (4.17). It follows from the next proposition which is a consequence of Joseph's Lemma 2.3.1.

Proposition 4.2.1. *If $w \in W$ and $\mu^\vee \in Q_-^\vee$ be such that $\mathcal{V}_w^-(\mu^\vee) \neq \emptyset$ then $\ell(w) \leq -2\langle\rho, \mu^\vee\rangle$.*

The assertion of above proposition is proven by Braverman *et. al.* for affine Kac-Moody groups on page 51 of [6] but its generalization is straightforward. Since we are using this result for our proofs, we sketch its proof in the following.

Poof of Proposition 4.2.1. Let $u^- \in \mathcal{V}_w^- \cap K\pi^{\mu^\vee}U^+$, then

$$u^- = u_{w,\pi}^+ w h_{\mathcal{O}} \pi^{\mu^\vee} u_2^+, \tag{4.18}$$

for some $u_{w,\pi}^+ \in U_{w,\pi}^+$, $h_{\mathcal{O}} \in H_{\mathcal{O}}$, $u_1^+, u_2^+ \in U^+$. We let this element act on the highest weight vector v_ρ and get the following equations,

$$u^- v_\rho = v_\rho + \text{ weight vectors of lower weights,} \tag{4.19}$$

$$u^+_{w,\pi} w h_{\mathcal{O}} \pi^{\mu^\vee} u_2^+ v_\rho \quad = \quad \delta^* \pi^{\langle \rho, \mu^\vee \rangle} u^+_{w,\pi} v_{w\rho} \tag{4.20}$$

for some $\delta^* \in \mathcal{O}^*$. Now, if

$u^+_{w,\pi} = \sum_{n_1,n_2,\ldots,n_r} \sigma_1^{n_1} \sigma_2^{n_2} \ldots \sigma_r^{n_r} \zeta_{\alpha_1}^{(n_1)} \zeta_{\alpha_2}^{(n_2} \ldots \zeta_{\alpha_r}^{(n_r)}$

where $\alpha_i \in S^+_{w^{-1}}$ and $\zeta_{\alpha_i}^{(n_i)}$ is the divided power of the Chevalley generator corresponding to α_i. Now, by Joseph's Lemma 2.3.1 to get the highest weight vector from the action $u^+_{w,\pi} v_{w\rho}$, we must have $\frac{\ell(w)}{2} \leq n_1 + n_2 + \ldots n_r$. Moreover, to preserve the primitivity we must have $n_1 + n_2 + \ldots n_r = -\langle \rho, \mu^\vee \rangle$. Combining these two facts we get the assertion. □

4.2.3 Completion and Coordinates

The third step for Gindikin-Karpelevich finiteness is to show that for each $w \in W$, the coset $U^-_{\mathcal{O}} \backslash \mathcal{V}^-_w(\mu^\vee)$ is finite. The key point in getting this finiteness is to realize the elements of $\mathcal{V}^-_w(\mu^\vee)$ as uniformly bounded operators (with respect to the norm (4.9)) on a finite dimensional subspace of V. These notions are going be made more precise in Subsection 4.2.4. The main tool used to get this realization is a set of coordinates, which exists in the completion. So, we first discuss the completion and this coordinate system as given on page 53-54 of [6].

Let $\mathring{\mathbf{G}}[[t^{-1}]]$ be the formal loop group functor in the variable t^{-1}. The group ind-scheme $\mathbf{U}^-$ is the preimage of $\mathring{\mathbf{U}}^-$ under the natural map $\mathring{\mathbf{G}}[t^{-1}] \longrightarrow \mathring{\mathbf{G}}$ of evaluation at ∞, where $\mathring{\mathbf{G}}[t^{-1}]$ be the polynomial algebra in the variable t^{-1} (see Subsection 4.1.2). Let $\mathbb{U}^-$ be the preimage of $\mathring{\mathbf{U}}^-$ in $\mathring{\mathbf{G}}[[t^{-1}]]$ as defined in Subsection. This is a group ind-scheme which comes equipped with the natural injection between the sets of $\mathcal{K}$ points

$$\iota \colon U^- \hookrightarrow \mathbb{U}^-,$$

For $m \geq 0$, suppose

$$U^{-}(m) := \{u^{-} \in U^{-} \mid u^{-} \equiv \mathrm{Id}(\bmod\, t^{-m})\} \tag{4.21}$$

and $U^{-}[m] := U^{-}/U^{-}(m)$. We shall denote by ϕ_m the projection

$$\phi_m \colon \mathbb{U}^{-} \longrightarrow U^{-}[m]. \tag{4.22}$$

Next, we define the following elements of $\mathbb{U}^{-}$, if $m = 0$

$$u^{-}[0] := \prod_{\beta \in \mathring{\Delta}_{-}} u_{\beta}(y_{0,\beta}) \tag{4.23}$$

and for $m \geq 1$,

$$u^{-}[m] := \prod_{\alpha \in \mathring{\Delta}_{+}} u_{\alpha}(t^{m} x_{m,\alpha}) \prod_{i=1}^{l} h_i(1 + c_{i,m} t^{m}) \prod_{\beta \in \mathring{\Delta}_{-}} u_{\beta}(t^{m} y_{m,\beta}), \tag{4.24}$$

where $\mathring{\Delta}_{+}$ and $\mathring{\Delta}_{-}$ are the underlying finite dimensional set of positive and negative roots, respectively, and $y_{0,\beta}, c_{i,m}, x_{m,\alpha}, y_{m,\beta} \in \mathcal{K}$.

Definition 4.2.2. *An element $u^{-}[j]$ is said to be componentwise bounded by a positive constant C if:*

1. *$j = 0$ and $u^{-}[j]$ has expression (4.23), we have $|y_{0,\beta}| < C$ for all $\beta \in \mathring{\Delta}_{-}$.*

2. *$j \geq 1$ and $u^{-}[j]$ has expression (4.24), we have $|x_{j,\alpha}| < C$, $|y_{j,\beta}| < C$ and $|c_{i,j}| < C$ for all $\alpha, \beta \in \mathring{\Delta}_{-}$.*

The last result of this subsection asserts that the elements of U^{-} in $\mathbb{U}^{-}$ can be expressed as the products of the above coordinates.

Proposition 4.2.3. *By identifying U^- inside $\mathbb{U}^-$, every element u^- of U^- has the following unique form as a product*

$$u^- = \prod_{j \geq 0} u^-[j]. \tag{4.25}$$

Moreover, for $m \geq 0$

$$\phi_m(u^-) = \phi_m(\prod_{j=0}^{m} u^-[j]), \tag{4.26}$$

where ϕ_m is the projection (4.22).

4.2.4 Step 2: Finiteness II

Being a subset of U^-, the elements of the set $\mathcal{V}_w^-(\mu^\vee)$ admit a decomposition (4.25). Moreover,

Lemma 4.2.4. (a) *For each element $u^- \in \mathcal{V}_w^-(\mu^\vee)$ there exists $n \geq 0$, such that $u^- = \prod_{j=0}^{n} u^-[j]$.*

(b) *There exists a constant $C > 0$ such that $||u^- v_\lambda|| \leq C$, for all $u^- \in \mathcal{V}_w^-(\mu^\vee)$.*

Lemma 4.2.4 implies that the set $\mathcal{V}_w^-(\mu^\vee)$ fulfills the conditions of the family $\mathcal{F}$ given in the next propositions.

Proposition 4.2.5. *Let $\mathcal{F}$ be a subset U^- such that,*

(a) The family $\mathcal{F}$ is bounded by a constant $C > 0$,

(b) every $u^- \in \mathcal{F}$ is a finite product of the coordinates $u^- = \prod_{j=1}^{n} u^-[j]$.

Then there exists D such that each $u^-[j]$ is componentwise bounded by D.

Thus, the elements of $\mathcal{V}_w^-(\mu^\vee)$ can be written as a product of finitely many coordinates and each coordinate is componentwise bounded by the same constant.

Moreover, it can be viewed as embedded inside the automorphism group of the finite dimensional space $V[m]$. This gives rise to a family $\mathcal{F}$ of equivalence classes of elements $\mathcal{V}_w^-(\mu^\vee)$. By using the properties of finitely ordered matrices consisting of uniformly bounded entries from Section 7 of [6], one can show that $\mathcal{F}$ is finite, and for each $u^- \in \mathcal{V}_w^-(\mu^\vee)$, there exists $v^- \in \mathcal{F}$ such that $u^- \in U_{\mathcal{O}}^- v^-$. This proves $K\backslash\mathcal{V}_w^-(\mu^\vee)$ is finite.

4.3 Other Results

The affine Spherical Finiteness and Approximation theorems are proven by using the Gindikin-Karpelevich Finiteness. For instance, one of containments

$$K\pi^{\lambda^\vee}U^- \cap K\pi^{\lambda^\vee-\mu^\vee}U^+ \subset K\pi^{\lambda^\vee}K$$

to prove the Approximation Theorem is shown by using the fact that there exists a finite subset $\Omega \subset U^-$ such that

$$K\backslash K\pi^{\lambda^\vee-\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}U^- = \cup_{u^-\in\Omega}K\backslash K\pi^{\lambda^\vee-\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}u^- \qquad (4.27)$$

and corresponding to these finitely many elements, $\lambda^\vee$ can be chosen sufficiently big, such that we get $\Omega \subset K$. For the Spherical Finiteness, we refer readers to Subsection 8.2.

4.4 Computation of Limit

With the finiteness theorems proven, the formal analogues $\mathscr{G}_{\lambda^\vee}$ and $\mathcal{S}_{\lambda^\vee}$ of the Gindikin-Karpelevich integral and image of Satake isomorphism respectively, make sense in the infinite dimensional settings. In [8], Braverman et. al. obtained the generalized version of the Macdonald's formula for $\mathcal{S}_{\lambda^\vee}$ and used it to compute a formula for $\mathscr{G}_{\lambda^\vee}$ where $\lambda^\vee = 0$. We describe this process of computation in the following. Let $\Sigma \subset W$, $\Sigma(q) = \sum_{w\in\Sigma} q^{-l(w)}$ be the *Poincare Polynomial*. For $\lambda^\vee \in \Lambda^\vee$, let $W_{\lambda^\vee} = \{w \in W \mid w\lambda^\vee = \lambda^\vee\}$ be the stabilizer of $\lambda^\vee$. For each $\alpha \in \Delta_+$ with multiplicity $m(\alpha)$, we define an element

$$\Upsilon_{\alpha^\vee} = \left(\frac{1 - q^{-1}e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}}\right)^{m(\alpha)}$$

of $\mathbb{C}[q, q^{-1}][[Q_-^\vee]]$. Set

$$\Gamma := \prod_{\alpha\in\Delta_+} \Upsilon_{\alpha^\vee}.$$

The rational function Γ is also an element of of $\mathbb{C}[q, q^{-1}][[Q_-^\vee]]$. For each $w \in W$, let

$$\Gamma^w := \prod_{\alpha\in\Delta_+} \Upsilon_{w\alpha^\vee}.$$

For $\lambda^\vee \in \Lambda^\vee$, let

$$H_{\lambda^\vee} = \frac{q^{\langle\rho,\lambda^\vee\rangle}}{W_{\lambda^\vee}(q^{-1})} \sum_{w\in W} \Gamma^w e^{w\lambda^\vee}.$$

Let $\mathbb{C}_{\leq}[\Lambda^\vee]$ be a completion of the group algebra $\mathbb{C}[\Lambda^\vee]$ as defined in Section 2.1.5 of *op. cit.* Proof of the following theorem can be found in Subsection 7.2 of *op. cit.*

Theorem 4.4.1. *Let $\lambda^\vee \in \Lambda^\vee$ be dominant. The ratio $\frac{H_{\lambda^\vee}}{H_0}$ is an element of $\mathbb{C}[q, q^{-1}] \otimes_{\mathbb{C}} \mathbb{C}_{\leq}[\Lambda^\vee]$ and is equal to $\mathcal{S}_{\lambda^\vee}$.*

By Theorem 4.4.1, for $\lambda^\vee$ dominant,

$$\mathcal{S}_{\lambda^\vee} = (H_0)^{-1}\frac{q^{\langle\rho,\lambda^\vee\rangle}}{W_{\lambda^\vee}(q^{-1})} \sum_{w\in W} \Gamma^w e^{w\lambda^\vee}. \tag{4.28}$$

Theorem 1.3.2 and Lemma 1.3.1 imply that for $\xi^\vee \in \Lambda^\vee$ if we choose λ sufficiently big as compare to $\xi^\vee$, then

$$[e^{\xi^\vee}]\mathscr{G}_0 \;=\; \frac{[e^{\lambda^\vee-\xi^\vee}]\mathcal{S}_{\lambda^\vee}}{q^{\langle\rho,\lambda^\vee\rangle}}, \tag{4.29}$$

where for any $f \in \mathbb{C}[\Lambda^\vee]$, such that $f = \sum_{\nu^\vee\in\Lambda^\vee} c_{\nu^\vee}e^{\nu^\vee}$, $[e^{\mu^\vee}]f = c_{\mu^\vee}$. Next, if $\lambda^\vee$ is regular, $W_{\lambda^\vee} = \{1\}$ and hence $W_{\lambda^\vee}(q^{-1}) = 1$. So, (4.28) becomes equal to

$$\mathcal{S}_{\lambda^\vee} \;=\; (H_0)^{-1}q^{\langle\rho,\lambda^\vee\rangle}\sum_{w\in W}\Gamma^w e^{w\lambda^\vee}. \tag{4.30}$$

Now, if we choose $\lambda^\vee$ very large as compare to $\xi^\vee$, then only the term with $w = 1$ in the sum $\sum_{w\in W}\Gamma^w e^{w\lambda^\vee}$ can contribute to the coefficient of $e^{\lambda^\vee-\xi^\vee}$. Indeed, if $\lambda^\vee$ is chosen very large such that $w\lambda^\vee$ becomes very small as compare to $\lambda^\vee - \xi^\vee$ for $w \neq 1$ and the presence of $(H_0)^{-1}$ in (4.30) (which can be expanded in the negative powers of e^{α_i}, $i \in I$) force $w = 1$. As a consequene of (4.29), by choosing λ regular and sufficiently big, we get

$$\begin{aligned}\mathscr{G}_0 &= \frac{1}{H_0}\Gamma\\ &= \frac{1}{H_0}\prod_{\alpha\in\Delta_+}\Upsilon_{\alpha^\vee}\\ &= \frac{1}{H_0}\prod_{\alpha\in\Delta_+}\left(\frac{1-q^{-1}e^{-\alpha^\vee}}{1-e^{-\alpha^\vee}}\right)^{m(\alpha)}.\end{aligned} \tag{4.31}$$

The equality $H_0 = \frac{\sum_{w\in W}\Gamma^w}{\sum_{w\in W}q^{-l(w)}}$ follows from the fact that the Satake Isomorphism $\mathcal{S}$ being a homomorphism of algebras must satisfy $\mathcal{S}(h_0) = \mathcal{S}(\chi_K) = 1$. In the finite dimensional case $H_0 = 1$ but this equality does not hold in affine settings. The W-invariant factor H_0 is under investigation for infinite dimensional Kac-Moody groups. One of the few known cases is when G is affine and the underlying group G_0 is of simply laced type. Then it has an infinite product decomposition

$$H_0 = \prod_{i=1}^{l} \prod_{j=1}^{\infty} \frac{1 - q^{-m_i} e^{-jc}}{1 - q^{-(m_i+1)} e^{-jc}}, \tag{4.32}$$

where c is the minimal imaginary coroot and for $1 \leq i \leq l$, m_i are the exponents of G_0. This value of H_0 was conjectured by Macdonald in [45]. It is known as constant term conjecture and has been proven by I. Cherednik in [16]. Recently, for general Kac-Moody settings this factor was studied and various properties were listed by D. Muthiah, A. Puskás and I. Whitehead in [51].

Chapter 5

Approximation Theorem

In this chapter, we shall prove Theorem 1.3.2. This theorem establishes a link between the image $\mathcal{S}_{\lambda^\vee}$ of the Satake isomorphism and the Gindikin-Karpelevich sum $\mathscr{G}_{\lambda^\vee}$ when $\lambda^\vee$ is very large.

5.1 Proof of Main Theorem

Let $\rho \in \Lambda$ be the element as defined in (2.15). Throughout this chapter we fix our highest weight module V^ρ with highest weight ρ and equipped with the norm $||.||$ given in (4.9). We also fix a primitive highest weight vector v_ρ in V^ρ, that is, $||v_\rho|| = 1$. Let P_ρ be the set of weights of the representation V^ρ and for $\nu \in P_\rho$, η_ν be the projection map as introduced in Section 2.3. First, we prove the following lemma which will be used to prove the statement of the theorem.

Lemma 5.1.1. *Let $\mu^\vee$ be fixed and $\lambda^\vee \in \Lambda_+$ be regular. There exists a finite subset $\Xi = \Xi(\lambda^\vee, \mu^\vee) \subset P_\rho$ such that if $u^- \in U^-$ satisfies*

$$\pi^{\lambda^\vee} u^- \in K\pi^{\lambda^\vee - \mu^\vee} U^+, \tag{5.1}$$

and $\eta_\nu(u^- v_\rho) \notin V_{\mathcal{O}}^\rho$ for some $\nu \in P_\rho$, then $\nu \in \Xi$.

Proof. If $u^- \in U_{\mathcal{O}}^-$ then $u^- v_\rho \in V_{\mathcal{O}}^\rho$ and hence there is nothing to prove. Let $U^-(\lambda^\vee, \mu^\vee)$ be the set of elements $u^- \in U^- \setminus U_{\mathcal{O}}^-$ which satisfy (5.1), and set

$$\Sigma = \Sigma(\lambda^\vee, \mu^\vee) = \{\gamma \in Q_+ \mid \eta_{\rho-\gamma}(u^- v_\rho) \notin V_{\mathcal{O}}^\rho,\ u^- \in U^-(\lambda^\vee, \mu^\vee)\}.$$

It suffices to show that Σ is a finite set. Let $u^- \in U^-(\lambda^\vee, \mu^\vee)$ with a corresponding $\gamma \in \Sigma$ with $\gamma = \sum_{i=1}^{l} k_i \alpha_i$, for $k_i \in \mathbb{Z}_{>0}$ and $\alpha_i \in \Pi$. By assumption

$$\pi^{\lambda^\vee} u^- = k \pi^{\lambda^\vee - \mu^\vee} u^+, \tag{5.2}$$

for some $k \in K$ and $u^+ \in U^+$. We apply both sides of (5.2) to the highest weight vector v_ρ. The action of the left hand side of (5.2), Lemma 4.1.1 and the fact $\eta_{\rho-\gamma}(u^- v_\rho) \notin V_{\mathcal{O}}^\rho$ give,

$$||\pi^{\lambda^\vee} u^- v_\rho|| \;\; \geq \;\; ||\pi^{\lambda^\vee} v_{\rho-\gamma}|| > q^{-\langle \rho-\gamma, \lambda^\vee \rangle}. \tag{5.3}$$

The right hand side of (5.2) acts as,

$$||k \pi^{\lambda^\vee - \mu^\vee} u^+ v_\rho|| \;\; = \;\; q^{-\langle \rho, \lambda^\vee - \mu^\vee \rangle}. \tag{5.4}$$

So, (5.3) and (5.4) imply $q^{-\langle \rho-\gamma, \lambda^\vee \rangle} < q^{-\langle \rho, \lambda^\vee - \mu^\vee \rangle}$. Which shows $q^{\langle \gamma, \lambda^\vee \rangle} < q^{\langle \rho, \mu^\vee \rangle}$ and hence $\langle \gamma, \lambda^\vee \rangle < \langle \rho, \mu^\vee \rangle$. Consequently,

$$\sum_{i=1}^{l} k_i \langle \alpha_i, \lambda^\vee \rangle \;\; < \;\; \langle \rho, \mu^\vee \rangle. \tag{5.5}$$

Since $\mu^\vee$ is fixed, and $\lambda^\vee$ is dominant and regular, $\langle \alpha_i, \lambda^\vee \rangle$ are fixed positive numbers for $1 \le i \le l$, the bound $\langle \rho, \mu^\vee \rangle$ in (5.5) on the coefficients k_i appearing in the simple root decomposition of γ implies that we have only finitely many choices for $\gamma \in Q_+$. Therefore, the set $\Sigma \subset Q^\vee$ is finite and this completes the proof.

Now, we give the proof of the Approximation Theorem.

Proof of Theorem 1.3.2. To prove the assertion of the theorem, we show that for $\mu^\vee \in Q_+^\vee$ and $\lambda^\vee$ sufficiently dominant the following set theoretic inclusions hold

$$K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ \quad \subset \quad K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ \tag{5.6}$$

$$K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ \quad \subset \quad K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee - \mu^\vee} U^+. \tag{5.7}$$

The first containment follows exactly as it does in affine case [6, Subsection 6.3], we only sketch its (slightly modified) proof here. First note that if $\lambda^\vee$ is dominant then

$$\pi^{\lambda^\vee} U_{\mathcal{O}}^+ \pi^{-\lambda^\vee} \subset U_{\mathcal{O}}^+ \tag{5.8}$$

and $K\pi^{\lambda^\vee} I^+ = K\pi^{\lambda^\vee} U_\pi^- \subset K\pi^{\lambda^\vee} U^-$.

Thus, it suffices to show that for $\mu^\vee \in Q_+^\vee$ and $\lambda^\vee$ as above

$$K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ \subset K\pi^{\lambda^\vee} I^+. \tag{5.9}$$

For this, let $k_1 \in K$ be such that

$$\pi^{\lambda^\vee} k_1 \in K\pi^{\lambda^\vee - \mu^\vee} U^+ \tag{5.10}$$

and suppose $k_1 \in I^+ w I^+$ for some $w \in W$. For any $w \in W$, $U_\pi^- w I^+ \subseteq w I^+$.

Then by using this fact and (5.8), we have

$$\begin{aligned}\pi^{\lambda^\vee} k_1 \in \pi^{\lambda^\vee} I^+ w I^+ &= \pi^{\lambda^\vee} U_{\mathcal{O}}^+ U_\pi^- w I^+ \\ &\subseteq U_{\mathcal{O}}^+ \pi^{\lambda^\vee} w I^+ \\ &= U_{\mathcal{O}}^+ \pi^{\lambda^\vee} w U_\pi^- U_{\mathcal{O}}^+. \end{aligned} \tag{5.11}$$

So, (5.10) and (5.11) imply that for $u_1, u_2 \in U_{\mathcal{O}}^+$, $u^- \in U_\pi^-$, $u_3 \in U^+$ and $k' \in K$

$$\pi^{\lambda^\vee} k_1 = u_1 \pi^{\lambda^\vee} w u^- u_2 = k' \pi^{\lambda^\vee - \mu^\vee} u_3 \tag{5.12}$$

Thus, by taking $(u_1)^{-1} k' = k \in K$ and $u_3 u_2^{-1} = u^+ \in U^+$, we have

$$\pi^{\lambda^\vee} w u^- = k \pi^{\lambda^\vee - \mu^\vee} u^+. \tag{5.13}$$

Next, we choose $\lambda^\vee \in \Lambda^\vee$ sufficiently dominant such that if $\sigma \in W$, $\sigma \neq 1$ and $\sigma\lambda^\vee = \lambda^\vee - \beta^\vee$ for some $\beta^\vee \in Q_+$, then

$$\langle \rho, \beta^\vee \rangle > \langle \rho, \mu^\vee \rangle. \tag{5.14}$$

If $w \neq 1$, by letting the both sides of (5.13) act on the highest weight vector v_ρ, we compute

$$\begin{aligned} q^{-\langle \rho, \lambda^\vee - \mu^\vee \rangle} &= ||k \pi^{\lambda^\vee - \mu^\vee} u^+ v_\rho|| = ||\pi^{\lambda^\vee} w u^- v_\rho|| \\ &\geq ||\pi^{\lambda^\vee} v_{w\rho}|| = q^{-\langle w\rho, \lambda^\vee \rangle} = q^{-\langle \rho, w^{-1}\lambda^\vee \rangle}. \end{aligned} \tag{5.15}$$

This implies $\langle \rho, \lambda^\vee - \mu^\vee \rangle \leq \langle \rho, w^{-1}\lambda^\vee \rangle$ and this results in a contradiction of the inequality (5.14).

For the containment (5.7), let $u^- \in U^-$ be such that $\pi^{\lambda^\vee} u^- \in K\pi^{\lambda^\vee - \mu^\vee} U^+$. If $u^- \in U^-_{\mathcal{O}}$, then our theorem follows. If $u^- \notin U^-_{\mathcal{O}}$ then by Lemma 5.1.1, there exists a finite subset $\Xi \subset P_\rho$ such that if $\nu \in \Xi$ then $\eta_\nu(u^- v_\rho) \notin V^\rho_{\mathcal{O}}$. Moreover, as in the proof of the above lemma, if we write $\nu = \rho - \gamma$ for some $\gamma \in Q_+$, we get $q^{\langle \gamma, \lambda^\vee \rangle} \leq q^{\langle \rho, \mu^\vee \rangle}$. Thus by choosing $\lambda^\vee$ sufficiently dominant corresponding to the finitely many elements in Ξ, we can arrange

$$q^{\langle \rho, \mu^\vee \rangle} < q^{\langle \gamma, \lambda^\vee \rangle}, \tag{5.16}$$

leading us to a contradiction.

5.2 Iwahori Refinement

The following proposition is an Iwahori analogue of the Approximation Theorem.

Proposition 5.2.1. *Let $w \in W$ and $\mu^\vee \in \Lambda^\vee$ be fixed. Then for all sufficiently dominant $\lambda^\vee = \lambda^\vee(\mu^\vee, w)$ (that is, sufficiently dominant depending on $\mu^\vee$ and w), we have*

$$I^- \pi^{\lambda^\vee} U^- \cap I^- w \pi^{\lambda^\vee - \mu^\vee} U^+ \;=\; I^- \pi^{\lambda^\vee} U^-_{\mathcal{O}} \cap I^- w \pi^{\lambda^\vee - \mu^\vee} U^+.$$

Proof. One inclusion is straightforward. So, we prove the other

$$I^- \pi^{\lambda^\vee} U^- \cap I^- w \pi^{\lambda^\vee - \mu^\vee} U^+ \;\subset\; I^- \pi^{\lambda^\vee} U^-_{\mathcal{O}} \cap I^- w \pi^{\lambda^\vee - \mu^\vee} U^+. \tag{5.17}$$

For this, let $v^- \in U^-$ be such that $\pi^{\lambda^\vee} v^- \in I^- w \pi^{\lambda^\vee - \mu^\vee} U^+$. Then using the

Iwahori-Matsumoto decomposition $I^- = U_\pi^+ U_\mathcal{O}^- H_\mathcal{O}$, we have

$$\pi^{\lambda^\vee} v^- \in U_\mathcal{O}^- U_{w,\pi}^+ w \pi^{\lambda^\vee - \mu^\vee} U^+.$$

Hence, for the containment (5.17), it suffices to show that:

(**P1**) Let $w \in W$, $\mu^\vee \in \Lambda^\vee$ be fixed. Then for sufficiently dominant $\lambda^\vee = \lambda^\vee(\mu^\vee, w)$, if $\pi^{\lambda^\vee} u^- \in U_{w,\pi}^+ w \pi^{\lambda^\vee - \mu^\vee} U^+$ with $u^- \in U^-$, then $u^- \in U_\mathcal{O}^-$.

To prove (**P1**), let $u^- \in U^-$ be such that

$$\pi^{\lambda^\vee} u^- \quad = \quad u_w^+ w \pi^{\lambda^\vee - \mu^\vee} u^+, \tag{5.18}$$

for some $u_w^+ \in U_{w,\pi}^+$ and $u^+ \in U^+$. We apply both sides of (5.18) to the highest weight vector v_ρ. The right hand side gives us,

$$\begin{aligned} u_w^+ w \pi^{\lambda^\vee - \mu^\vee} u^+ v_\rho &= u_w^+ w \pi^{\lambda^\vee - \mu^\vee} v_\rho \\ &= \pi^{\langle \rho, \lambda^\vee - \mu^\vee \rangle} u_w^+ w v_\rho. \end{aligned}$$

Since $u_w^+ \in K$, $||u_w^+|| = 1$ and hence

$$\begin{aligned} ||u_w^+ w \pi^{\lambda^\vee - \mu^\vee} u^+ v_\rho|| &= ||\pi^{\langle \rho, \lambda^\vee - \mu^\vee \rangle} w v_\rho|| \\ &= q^{-\langle \rho, \lambda^\vee - \mu^\vee \rangle}. \end{aligned} \tag{5.19}$$

On the other hand, the element on the left hand side of (5.18) acts as

$$\pi^{\lambda^\vee} u^- v_\rho \quad = \quad \pi^{\lambda^\vee} (\sum_{\nu \in P_\rho} v_\nu), \tag{5.20}$$

If $u^- \notin U^-_{\mathcal{O}}$, then there exists at least one weight $\xi := \rho - \gamma \in P_\rho$, with $\gamma \in Q_+$ such that the corresponding weight vector $v_{\rho-\gamma}$ on the right hand side of (5.20) is not integral, i.e. $\eta_{\rho-\gamma}(u^- v_\rho) \notin V_{\rho,\mathcal{O}}$. This gives

$$\begin{aligned} ||\pi^{\lambda^\vee} u^- v_\rho|| &\geq ||\pi^{\lambda^\vee} v_{\rho-\gamma}|| \\ &\geq q^{-\langle \rho-\gamma, \lambda^\vee \rangle}. \end{aligned} \tag{5.21}$$

So, (5.19) and (5.21) imply

$$q^{\langle \rho, \mu^\vee \rangle} \quad \geq \quad q^{\langle \gamma, \lambda^\vee \rangle}. \tag{5.22}$$

Claim 1. *There are finitely many $\gamma \in Q_+$ such that $\eta_{\rho-\gamma}(u^- v_\rho) \neq 0$ for all u^- satisfying (5.18).*

Proof. The subgroup $U^+_{w,\pi}$ is generated by the finite number of root subgroups $U_{\alpha,\pi}$, $\alpha \in S^+_{w^{-1}}$. So, for each element $u^+_w \in U^+_{w,\pi}$, when $u^+_w w$ acts on the highest weight vector v_ρ, there are a finite number of choices of weights that can appear in the weight vector decomposition of $u^+_w w v_\rho$. The same is true for u^- appearing on the left hand side of (5.18). Hence our claim holds.

Thus, by choosing $\lambda^\vee$ sufficiently dominant corresponding to these finitely many γ from the claim, we may arrange

$$q^{\langle \rho, \mu^\vee \rangle} < q^{\langle \gamma, \lambda^\vee \rangle}.$$

Thus our assumption $u^- \notin U^-_{\mathcal{O}}$ leads to a contradiction of (5.22), and consequently the statement (**P1**) and the proposition follow.

Chapter 6

An Integral and Recursion Relation

In this chapter we introduce two propotional integrals $I_{w,\lambda^\vee}$ and $\tilde{I}_{w,\lambda^\vee}$, and prove the convergence of $I_{w,\lambda^\vee}$ by showing that it satisfies a recursion relation in terms of a certain operator. This will imply the convergence of $\tilde{I}_{w,\lambda^\vee}$ as well. The finiteness of the level set of $\tilde{I}_{w,\lambda^\vee}$ obtained as a consequence of this convergence will be used to obtain the proof of the Weak Spherical Finiteness in Chapter 7.

6.1 The Integral

Let ρ be the sum of fundamental weights as introduced earlier (see (2.15)).

Definition 6.1.1. *We define a function*

$$\Phi_\rho : G \longrightarrow \mathbb{C}[\Lambda^\vee]$$

by the formula $\Phi_\rho(g) = q^{-\langle \rho, \mu^\vee \rangle} e^{\mu^\vee}$, *where* $g \in G$ *has an Iwasawa decomposition* $g \in U \pi^{\mu^\vee} K$.

For $w \in W$, recall the subset $S^-_{w^{-1}}$ and the corresponding subgroups $U^-_{w^{-1}}$,

$U^-_{w^{-1},\mathcal{O}}$ and $U^-_{w^{-1},\pi}$ from Subsection 3.4. The group $U^-_{w^{-1}}$ is finite dimensional and carries a natural Haar measure du^-_w which is normalized such the group $U^-_{w^{-1},\mathcal{O}}$ has volume 1 with respect to this measure.

Definition 6.1.2. *The integral $I_{w,\lambda^\vee}$ is defined by the following equation,*

$$I_{w,\lambda^\vee} \quad = \quad \int_{U^-_{w^{-1},\pi}} \Phi_\rho(u^-_w \pi^{w\lambda^\vee}) du^-_w.$$

We also need another integral which we define as follows: The group $U^-_{w^{-1},\pi}$ has finite volume with respect to the measure du^-_w. We normalize the restriction of this measure on $U^-_{w^{-1},\pi}$ so that the volume of $U^-_{w^{-1},\pi}$ becomes equal to 1 and call this measure $\tilde{du^-_w}$.

Definition 6.1.3. *The integral $\tilde{I}_{w,\lambda^\vee}$ is defined by the following equation,*

$$\tilde{I}_{w,\lambda^\vee} \quad = \quad \int_{U^-_{w^{-1},\pi}} \Phi_\rho(u^-_w \pi^{w\lambda^\vee}) \tilde{du^-_w}.$$

6.2 Demazure-Lusztig operator

Let v be a formal variable and $\mathbb{C}_v := \mathbb{C}[v]$ be the ring of polynomial in v. Let $\mathscr{R} := \mathbb{C}[[Q\vee_-]]$. Set

$$\mathscr{L} := \mathbb{C}_v \otimes_{\mathbb{C}} \mathscr{R} \tag{6.1}$$

and

$$\mathscr{L}[W] := \{\sum_{w\in W} a_w[w] \mid a_w \in \mathscr{L}\}. \tag{6.2}$$

Now, we consider another formal variable X and define the following rational functions

$$\mathbf{b}(X) := \frac{v-1}{1-X} \ \text{ and } \ \mathbf{c}(X) := \frac{1-vX}{1-X}.$$

By expanding $\mathbf{b}(X)$ and $\mathbf{c}(X)$ in X^{-1} and using $X = e^{\alpha^\vee}$ for some positive coroot $\alpha^\vee$ it follows that $\mathbf{b}(X), \mathbf{c}(X) \in \mathscr{L}$. For a coroot $\alpha^\vee$, we shall denote

$$\mathbf{b}(\alpha^\vee) := \mathbf{b}(e^{\alpha^\vee}) \text{ and } \ \mathbf{c}(\alpha^\vee) := \mathbf{c}(e^{\alpha^\vee}). \tag{6.3}$$

Definition 6.2.1. *For $i \in I$, let $\alpha_i^\vee$ be the simple coroot and $w_i = w_{\alpha_i}$ be the simple root reflection. A Demazure-Lusztig operator on $\mathscr{L}$ is defined by*

$$\mathbf{T}_{w_i} := \mathbf{c}(\alpha_i^\vee)[w_i] + \mathbf{b}(\alpha_i^\vee)[1], \tag{6.4}$$

which, by expanding the rational functions, can be seen to be an element of $\mathscr{L}[W]$.

The operator $\mathbf{T}_{w_i}$ satisfies the following properties.

Proposition 6.2.2. *For $i \in I$,*

(1) *For $i \in I$, $\mathbf{T}_{w_i}^2 = (v-1)\mathbf{T}_{w_i} + v$.*

(2) *The operators $\mathbf{T}_{w_i}$ satisfy the braid relations. So, if $w \in W$ has a reduced decomposition $w = w_{i_1} w_{i_2} \dots w_{i_n}$ then*

$$\mathbf{T}_w = \mathbf{T}_{w_{i_1}} \mathbf{T}_{w_{i_2}} \dots \mathbf{T}_{w_{i_n}}$$

and this expansion is independent of the chosen reduced decomposition.

Proof. For the proof of the last part, we refer readers to [52, §6] or from the references given there. Here, we only sketch the proof of the first part. By using

$$[w_i]\mathbf{c}(\alpha_i^\vee) = \mathbf{c}(-\alpha_i^\vee)[w_i], \quad [w_i]\mathbf{b}(\alpha_i^\vee) = \mathbf{b}(-\alpha_i^\vee)[w_i],$$

and

$$\mathbf{b}(-\alpha_i^\vee) + \mathbf{b}(\alpha_i^\vee) = v - 1$$

we write

$$\begin{aligned}
\mathbf{T}_{w_i}^2 &= \mathbf{c}(\alpha_i^\vee)\mathbf{c}(-\alpha_i^\vee)[1] + \mathbf{c}(\alpha_i^\vee)\mathbf{b}(-\alpha_i^\vee)[w_i] + \mathbf{b}(\alpha_i^\vee)\mathbf{c}(\alpha_i^\vee)[w_i] + \mathbf{b}(\alpha_i^\vee)^2[1] \\
&= \left(\mathbf{c}(\alpha_i^\vee)\mathbf{c}(-\alpha_i^\vee) + \mathbf{b}(\alpha_i^\vee)^2\right)[1] + \mathbf{c}(\alpha_i^\vee)(\mathbf{b}(-\alpha_i^\vee) + \mathbf{b}(\alpha_i^\vee))[w_i] \\
&= (v-1)\mathbf{c}(\alpha_i^\vee)[w_i] + \left(\mathbf{c}(\alpha_i^\vee)\mathbf{c}(-\alpha_i^\vee) + \mathbf{b}(\alpha_i^\vee)^2\right)[1]. \qquad (6.5)
\end{aligned}$$

Now, we use $(v-1)\mathbf{c}(\alpha_i^\vee)[w_i] = (v-1)\mathbf{T}_{w_i} - (v-1)\mathbf{b}(\alpha_i^\vee)[1]$ in (6.5) and simplify the resulting expression to get

$$\mathbf{c}(\alpha_i^\vee)\mathbf{c}(-\alpha_i^\vee) + \mathbf{b}(\alpha_i^\vee)^2 - (v-1)\mathbf{b}(\alpha_i^\vee) = v. \qquad (6.6)$$

This is what we want to show.

Now, we state and prove the main result of this section. This is an analogue of the results previously proven in [43, Theorem 4.4.5], [6, Proposition 7.3.7] and [52, Proposition 2.10].

Proposition 6.2.3. *Let* $\lambda^\vee$ *be a dominant and regular,* $w \in W$ *be such that* $w = w_\alpha w'$ *and* $l(w) = 1 + l(w')$. *Then*

$$I_{w,\lambda^\vee} = \mathbf{T}_{w_\alpha}(I_{w',\lambda^\vee}). \qquad (6.7)$$

The above proposition has the following corollaries.

Corollary 6.2.4. *For $w \in W$ and $\lambda \in \Lambda_+$ regular, the value of $\mathbf{T}_w(e^{\lambda^\vee})$ at $v = q^{-1}$ is equal to a constant multiple of the integral $I_{w,\lambda^\vee}$. More precisely,*

$$I_{w,\lambda^\vee} \quad = \quad q^{-\langle \rho, \lambda^\vee \rangle} \mathbf{T}_w(e^{\lambda^\vee}). \tag{6.8}$$

Proof. Let $w = w_{\alpha_m} w_{\alpha_{m-1}} \dots w_{\alpha_1}$ be a reduced decomposition of w, then by Proposition 6.2.3

$$I_{w,\lambda^\vee} \quad = \quad \mathbf{T}_{w_{\alpha_m}} \mathbf{T}_{w_{\alpha_{m-1}}} \dots \mathbf{T}_{w_{\alpha_2}}(I_{w_1,\lambda^\vee}). \tag{6.9}$$

The rank 1 computation implies that $I_{w_1,\lambda^\vee} = q^{-\langle \rho, \lambda^\vee \rangle} \mathbf{T}_{w_1}(e^{\lambda^\vee})$. Finally, this corollary follows by part (2) of Proposition 6.2.2.

Corollary 6.2.5. *With the same assumptions as above, the integral $I_{w,\lambda^\vee}$ converges in the following sense: there exists a finite subset $\mathfrak{B} \subset \Lambda^\vee$ such that*

$$I_{w,\lambda^\vee} = \sum_{\mu^\vee \in \mathfrak{B}} c_{\mu^\vee} e^{\mu^\vee}$$

with $c_{\mu^\vee} \in \mathbb{C}$ for all $\mu^\vee \in \mathfrak{B}$.

Proof. By Proposition 6.2.3, the assertion follows by showing $\mathbf{T}_w(e^{\lambda^\vee}) \in \mathbb{C}[\Lambda^\vee]$ at $v = q^{-1}$. An affine version of this statement is given in [44, Section 4.3] which extends to general Kac-Moody root systems as well. The proof is obtained by combining the rank 1 computations with Proposition 6.2.2 (2). In the following we give the rank 1 computation. For $\lambda^\vee \in \Lambda_+^\vee$

$$\begin{aligned}
\mathbf{T}_{w_\alpha}(e^{\lambda^\vee}) &= \mathbf{c}(\alpha^\vee)e^{w_\alpha\lambda^\vee} + \mathbf{b}(\alpha_i^\vee)e^{\lambda^\vee} \\
&= \frac{1-ve^{\alpha^\vee}}{1-e^{\alpha^\vee}}e^{w_\alpha\lambda^\vee} + \frac{v-1}{1-e^{\alpha^\vee}}e^{\lambda^\vee} \\
&= (e^{w_\alpha\lambda^\vee} + (1-v)e^{w_\alpha\lambda^\vee+\alpha^\vee} + (1-v)e^{w_\alpha\lambda^\vee+2\alpha^\vee} + \dots) \\
&- ((1-v)e^{\lambda^\vee} + (1-v)e^{\lambda^\vee+\alpha^\vee} + \dots) \\
&= e^{w_\alpha\lambda^\vee} + (1-v)e^{w_\alpha\lambda^\vee+\alpha^\vee} + \dots + (1-v)e^{\lambda^\vee-\alpha^\vee}. \qquad (6.10)
\end{aligned}$$

The way we defined Haar measure, it is easy to verify that there exists a constant $C > 0$ such that

$$\tilde{I}_{w,\lambda^\vee} = CI_{w,\lambda^\vee}. \qquad (6.11)$$

Corollary 6.2.5 and (6.11) imply that the integral $\tilde{I}_{w,\lambda^\vee}$ also converges. The proof of Proposition 6.2.3 will be carried out in next two sections.

6.3 Rank 1 Proof

First, we prove the proposition in rank one where the computations are very similar to what Macdonald did in Proposition 4.3.1 from [42] for a slightly different integral.

Step 1: Decomposition

The integral $I_{w_\alpha,\lambda^\vee}$ can be split into two parts

$$I_{w_\alpha,\lambda^\vee} = \int_{U^-_{w_\alpha,\pi}} \Phi_\rho(u^-_{w_\alpha}\pi^{w_\alpha\lambda^\vee})du^-_{w_\alpha} = I^1_{w_\alpha,\lambda^\vee} - I^2_{w_\alpha,\lambda^\vee}, \qquad (6.12)$$

where

$$I^1_{w_\alpha,\lambda^\vee} = \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee})du_{-\alpha}, \tag{6.13}$$

$$I^2_{w_\alpha,\lambda^\vee} = \int_{U_{-\alpha}[\leq 0]} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee})du_{-\alpha}, \tag{6.14}$$

and where $U_{-\alpha}[\leq 0] = \cup_{n\leq 0}U_{-\alpha}[n]$ and for $n \leq 0$,

$$U_{-\alpha}[n] = \{u_{-\alpha}(t) : val(t) = n\}. \tag{6.15}$$

Step 2: Evaluation of $I^2_{w_\alpha,\lambda^\vee}$

We start by evaluating $I^2_{w_\alpha,\lambda^\vee}$, which can be written as

$$I^2_{w_\alpha,\lambda^\vee} = \sum_{t=0}^{-\infty} \int_{U_{-\alpha}[t]} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee})du_{-\alpha}. \tag{6.16}$$

Let $s \in \mathcal{K}$ with $val(s) = t$, $t \leq 0$ and $s = \pi^t u$ for some $u \in \mathcal{O}^*$. We use the following identity which can be proven by using the relations (3.7)

$$u_{-\alpha}(s) = u_\alpha(s^{-1})\pi^{-t\alpha^\vee}w_\alpha u_\alpha(s^{-1}) \tag{6.17}$$

and write

$$\begin{aligned} u_{-\alpha}(s)\pi^{w_\alpha\lambda^\vee} &= u_\alpha(s^{-1})\pi^{-t\alpha^\vee}w_\alpha u_\alpha(s^{-1})\pi^{w_\alpha\lambda^\vee} \\ &= u_\alpha(s^{-1})\pi^{\lambda^\vee - t\alpha^\vee}w_\alpha u_\alpha(\pi^{\langle\alpha,-w_\alpha\lambda^\vee\rangle}s^{-1}) \\ &= u_\alpha(s^{-1})\pi^{\lambda^\vee - t\alpha^\vee}w_\alpha u_\alpha(\pi^{\langle\alpha,\lambda^\vee\rangle}s^{-1}). \end{aligned}$$

Since $\lambda^\vee$ is dominant and regular, $\langle \alpha, \lambda^\vee \rangle > 0$. Therefore,

$$\begin{aligned} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee}) &= q^{-\langle \rho, \lambda^\vee - t\alpha^\vee \rangle} e^{\lambda^\vee - t\alpha^\vee} \\ &= q^{-\langle \rho, \lambda^\vee \rangle} q^t e^{\lambda^\vee - t\alpha^\vee}. \end{aligned}$$

The integral in the sum (6.16) solves as,

$$\begin{aligned} \int_{U_{-\alpha}[t]} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee}) du_{-\alpha} &= \int_{U_{-\alpha}[t]} q^{-\langle \rho, \lambda^\vee \rangle} q^t e^{\lambda^\vee - t\alpha^\vee} du_{-\alpha} \\ &= q^{-\langle \rho, \lambda^\vee \rangle} q^t e^{\lambda^\vee - t\alpha^\vee} Vol(U_{-\alpha}[t]) \\ &= q^{-\langle \rho, \lambda^\vee \rangle} q^t e^{\lambda^\vee - t\alpha^\vee} (q^{-t} - q^{-t-1}) \\ &= q^{-\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee} (1 - q^{-1}) e^{-t\alpha^\vee}. \end{aligned}$$

Hence,

$$\begin{aligned} I^2_{w_\alpha, \lambda^\vee} &= \sum_{t=0}^{-\infty} q^{-\langle \rho, \lambda^\vee \rangle} e^{\lambda^\vee} (1 - q^{-1}) e^{-t\alpha^\vee} \\ &= q^{-\langle \rho, \lambda^\vee \rangle} \frac{1 - q^{-1}}{1 - e^{\alpha^\vee}} e^{\lambda^\vee}. \end{aligned} \tag{6.18}$$

Step 3: Evaluation of $I^1_{w_\alpha, \lambda^\vee}$

Next, we compute $I^1_{w_\alpha, \lambda^\vee}$. By the change of variables $u_{-\alpha} \mapsto \pi^{w_\alpha\lambda^\vee} u_{-\alpha} \pi^{-w_\alpha\lambda^\vee}$, we get

$$\begin{aligned} I^1_{w_\alpha, \lambda^\vee} &= \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(u_{-\alpha}\pi^{w_\alpha\lambda^\vee}) du_{-\alpha} \\ &= \jmath(u_{-\alpha}\pi^{w_\alpha\lambda^\vee}) \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(\pi^{w_\alpha\lambda^\vee} u_{-\alpha}) du_{-\alpha}, \end{aligned} \tag{6.19}$$

where $\jmath(u_{-\alpha}\pi^{w_\alpha\lambda^\vee})$ is a *Jacobian* factor that is equal to $q^{-\langle \alpha, \lambda^\vee \rangle}$. Now, we have two cases:

Case 1: if $u_{-\alpha} \in U_{-\alpha}(\mathcal{O})$. Then $\Phi_\rho(\pi^{w_\alpha \lambda^\vee} u_{-\alpha}) = q^{-\langle \rho, w_\alpha \lambda^\vee - \rangle} e^{w_\alpha \lambda^\vee}$ and hence

$$\int_{U_{-\alpha}(\mathcal{O})} \Phi_\rho(\pi^{w_\alpha \lambda^\vee} u_{-\alpha}) du_{-\alpha} = q^{-\langle \rho, w_\alpha \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee} Vol(U_{-\alpha}(\mathcal{O})) = q^{-\langle \rho, w_\alpha \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee}.$$

Case 2: If $u_{-\alpha} \notin U_{-\alpha}(\mathcal{O})$, then

$$\begin{aligned} \pi^{w_\alpha \lambda^\vee} u_{-\alpha}(s) &= \pi^{w_\alpha \lambda^\vee} u_\alpha(s^{-1}) \pi^{-val(s)\alpha^\vee} w_\alpha u_\alpha(s^{-1}) \\ &= u_\alpha(\pi^{-\langle \alpha, \lambda^\vee \rangle} s^{-1}) \pi^{w_\alpha \lambda^\vee - val(s)\alpha^\vee} w_\alpha u_\alpha(s^{-1}). \end{aligned}$$

So,
$\Phi_\rho(\pi^{w_\alpha \lambda^\vee} u_{-\alpha}(s)) = q^{-\langle \rho, w_\alpha \lambda^\vee - val(s)\alpha^\vee \rangle} e^{w_\alpha \lambda^\vee - val(s)\alpha^\vee} = q^{-\langle \rho, w_\alpha \lambda^\vee \rangle} q^{val(s)} e^{w_\alpha \lambda^\vee} e^{-val(s)\alpha^\vee}$
$= q^{-\langle \rho, \lambda^\vee \rangle} q^{\langle \alpha, \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee} q^{val(s)} e^{-val(s)\alpha^\vee}$.

Putting the values of the function for both cases in (6.19), we obtain

$$\begin{aligned} I^1_{w_\alpha, \lambda^\vee} &= \jmath(u_{-\alpha} \pi^{w_\alpha \lambda^\vee}) q^{-\langle \rho - \alpha, \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee} [1 + (1 - q^{-1}) e^{\alpha^\vee} + (1 - q^{-1}) e^{2\alpha^\vee} + \dots] \\ &= q^{-\langle \rho, \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee} [1 + \frac{(1 - q^{-1}) e^{\alpha^\vee}}{1 - e^{\alpha^\vee}}] \\ &= q^{-\langle \rho, \lambda^\vee \rangle} e^{w_\alpha \lambda^\vee} \frac{1 - q^{-1} e^{\alpha^\vee}}{1 - e^{\alpha^\vee}}. \end{aligned} \tag{6.20}$$

Step 4: Conclusion

Using (6.20) and (6.18) in (6.12), we get

$$\begin{aligned} I_{w_\alpha, \lambda^\vee} &= q^{-\langle \rho, \lambda^\vee \rangle} \frac{1 - q^{-1} e^{\alpha^\vee}}{1 - e^{\alpha^\vee}} e^{-w_\alpha \lambda^\vee} - q^{-\langle \rho, \lambda^\vee \rangle} \frac{1 - q^{-1}}{1 - e^{\alpha^\vee}} e^{\lambda^\vee} \\ &= q^{-\langle \rho, \lambda^\vee \rangle} [\mathbf{c}(\alpha_i^\vee) e^{w_i \lambda^\vee} + \mathbf{b}(\alpha_i^\vee) e^{\lambda^\vee}] \\ &= q^{-\langle \rho, \lambda^\vee \rangle} \mathbf{T}_{w_{\alpha_i}}(e^{\lambda^\vee}). \end{aligned} \tag{6.21}$$

6.4 Higher Rank Proof

The assertion in higher rank is proven below.

Step 1: Preliminary Reduction:

We will use the following description of the elements of $S^-_{w^{-1}} = \Delta^- \cap w\Delta^+$ which can be verified easily.

Lemma 6.4.1. *Let $w \in W$ be such that $w = w_\alpha w'$ and $l(w) = 1 + l(w')$. Then*

$$S^-_{w^{-1}} \;=\; \{-\alpha\} \cup \{w_\alpha \beta \mid \beta \in S^-_{(w')^{-1}}\}. \tag{6.22}$$

The above lemma implies the following decomposition of $U^-_{w^{-1}}$.

Lemma 6.4.2. *By assuming the conditions on $w \in W$ from the above lemma, each $u^-_w \in U^-_{w^{-1}}$ can be written as $u^-_w = u_{-\alpha} w_\alpha u^-_{w'} w_\alpha$, where $u_{-\alpha} \in U_{-\alpha}$ and $u^-_{w'} \in U^-_{(w')^{-1}}$.*

Lemma 6.4.2 yields the following splitting of the integral $I_{w,\lambda^\vee}$:

$$\begin{aligned} I_{w,\lambda^\vee} &= \int_{U^-_{w^{-1},\pi}} \Phi_\rho(u^-_w \pi^{w\lambda^\vee}) du^-_w \\ &= \int_{U_{-\alpha,\pi}} \int_{U^-_{(w')^{-1},\pi}} \Phi_\rho(u_{-\alpha} w_\alpha u^-_{w'} w_\alpha \pi^{w\lambda^\vee}) du_{-\alpha} du^-_{w'} \\ &= I^1_{w,\lambda^\vee} - I^2_{w,\lambda^\vee}, \end{aligned}$$

where

$$I^1_{w,\lambda^\vee} = \int_{U_{-\alpha}(\mathcal{K})} \int_{U^-_{(w')^{-1},\pi}} \Phi_\rho(u_{-\alpha} w_\alpha u^-_{w'} w_\alpha \pi^{w\lambda^\vee}) du_{-\alpha} du^-_{w'}$$

and

$$I^2_{w,\lambda^\vee} = \int_{U_{-\alpha}[\leq 0]} \int_{(U^-_{w'})^{-1},\pi} \Phi_\rho(u_{-\alpha} w_\alpha u^-_{w'} w_\alpha \pi^{w\lambda^\vee}) du_{-\alpha} du^-_{w'}.$$

Step 2: Evaluation of $I^1_{w,\lambda^\vee}$:

To simplify the integrand, we define a map.

Definition 6.4.3. *Let A be the quotient group as introduced in Subsection 3.4. The function*

$$Iw_A : G \longrightarrow A$$

is defined by setting the formula $Iw_A(g) = \pi^{\mu^\vee}$ for all $g \in U\pi^{\mu^\vee}K$ and $\mu^\vee \in \Lambda^\vee$.

Following Kumar [39, P. 77], for a simple root α, we denote a subset by U^α of U^+ which is equal to $w_\alpha U^+ w_\alpha \cap U^+$. This subset is normalized by the root subgroups U_α and $U_{-\alpha}$ and each element u of U^+ can be written as $u = u_\alpha u^\alpha$ for some $u_\alpha \in U_\alpha$ and $u^\alpha \in U^\alpha$. Writing $u^-_w \in U^-_{w^{-1},\pi}$ as in Lemma 6.4.2, we have

$$\begin{aligned} u^-_w \pi^{w\lambda^\vee} &= u_{-\alpha} w_\alpha u^-_{w'} w_\alpha \pi^{w\lambda^\vee} \\ &= u_{-\alpha} w_\alpha u^-_{w'} \pi^{w'\lambda^\vee} w_\alpha, \end{aligned} \tag{6.23}$$

for some $u_{-\alpha} \in U_{-\alpha}$ and $u^-_{w'} \in U^-_{(w')^{-1},\pi}$. Next assume $u^-_{w'}\pi^{w'\lambda^\vee} w_\alpha = u\pi^{\mu^\vee}k$ be an Iwasawa decomposition, $u = x_\alpha u^\alpha$ for some $x_\alpha \in U_\alpha$ and $u^\alpha \in U^\alpha$, and let $n_{-\alpha} \in U_{-\alpha}$ be defined as $n_{-\alpha} = w_\alpha x_\alpha w_\alpha$. The right hand side of (6.23) becomes

$$\begin{aligned} u_{-\alpha} w_\alpha u^-_{w'} \pi^{w'\lambda^\vee} w_\alpha &= u_{-\alpha} w_\alpha u \pi^{\mu^\vee} k w_\alpha \\ &= u^\alpha_1 \pi^{w_\alpha \mu^\vee} (\pi^{-w_\alpha\mu^\vee} u_{-\alpha} n_{-\alpha} \pi^{w_\alpha\mu^\vee}) w_\alpha k w_\alpha. \end{aligned} \tag{6.24}$$

Let $\tilde{n}_{-\alpha} = \pi^{-w_\alpha\mu^\vee} u_{-\alpha} n_{-\alpha} \pi^{w_\alpha\mu^\vee}$. Summarizing, we have

Lemma 6.4.4. *In the above notations,*

$$Iw_A(u^-_w) = Iw_A(u^-_{w'})^{w_\alpha} Iw_A(\tilde{n}_{-\alpha}).$$

So, the integral $I^1_{w,\lambda^\vee}$ takes the form

$$I^1_{w,\lambda^\vee} = \int_{U_{-\alpha}(\mathcal{K})} \int_{U^-_{(w')^{-1},\pi}} \Phi_\rho(Iw_A(u^-_{w'})^{w_\alpha} Iw_A(\tilde{n}_{-\alpha}))d\tilde{n}_{-\alpha} du^-_{w'}$$

$$= \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(Iw_A(\tilde{n}_{-\alpha}))d\tilde{n}_{-\alpha} \int_{U^-_{(w')^{-1},\pi}} \Phi_\rho(Iw_A(u^-_{w'})^{w_\alpha})du^-_{w'}.$$

The integral defined with measure $d\tilde{n}_{-\alpha}$ can be related to the integral defined with measure $du_{-\alpha}$ by a change of variables contain a Jacobian factor $q^{-\langle\alpha,\mu^\vee\rangle}$. So, we obtain

$$\begin{aligned}
I^1_{w,\lambda^\vee} &= \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(Iw_A(\tilde{n}_{-\alpha}))d\tilde{n}_{-\alpha} \int_{U^-_{(w')^{-1},\pi}} \Phi_\rho(Iw_A(u^-_{w'})^{w_\alpha})du^-_{w'} \\
&= \int_{U_{-\alpha}(\mathcal{K})} q^{-\langle\alpha,\mu^\vee\rangle}\Phi_\rho(Iw_A(u_{-\alpha}))du_{-\alpha} \int_{U^-_{(w')^{-1},\pi}} q^{\langle\alpha,\mu^\vee\rangle}(\Phi_\rho(Iw_A(u^-_{w'})))^{w_\alpha}du^-_{w'} \\
&= \int_{U_{-\alpha}(\mathcal{K})} \Phi_\rho(Iw_A(u_{-\alpha}))du_{-\alpha} \int_{U^-_{(w')^{-1},\pi}} (\Phi_\rho(Iw_A(u^-_{w'})))^{w_\alpha}du^-_{w'}
\end{aligned}$$

where in the second integral the following fact is used

$$\Phi_\rho(\pi^{w_\alpha\mu^\vee}) = q^{-\langle\rho,w_\alpha\mu^\vee\rangle}e^{w_\alpha\mu^\vee} = q^{-\langle\rho-\alpha,\mu^\vee\rangle}e^{w_\alpha\mu^\vee} = q^{\langle\alpha,\mu^\vee\rangle}\Phi_\rho(\pi^{\mu^\vee})^{w_\alpha}.$$

The rank 1 computation for the first integral now implies,

$$\begin{aligned}
I^1_{w,\lambda^\vee} &= \mathbf{c}[\alpha^\vee] \int_{U^-_{(w')^{-1},\pi}} (\Phi_\rho(Iw_A(u^-_{w'})))^{w_\alpha}du^-_{w'} \\
&= \mathbf{c}[\alpha^\vee](I_{w',\lambda^\vee})^{w_\alpha}.
\end{aligned} \tag{6.25}$$

Step 3: Evaluation of $I^2_{w,\lambda^\vee}$:

For $t \in \mathcal{K}$ and $val(t) \leq 0$, we write the integrand of $I^2_{w,\lambda^\vee}$ as

$u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha\pi^{w\lambda^\vee}$

$= \pi^{w\lambda^\vee}(\pi^{-w\lambda^\vee}u_{-\alpha}(t)\pi^{w\lambda^\vee})(\pi^{-w\lambda^\vee}w_\alpha u^-_{w'}w_\alpha\pi^{w\lambda^\vee}).$

Thus

$$\begin{aligned} u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha\pi^{w\lambda^\vee} &= \pi^{w\lambda^\vee}u_{-\alpha}(\pi^{\langle-\alpha,-w_\alpha w'\lambda^\vee\rangle}t)(w_\alpha\pi^{-w'\lambda^\vee}u^-_{w'}\pi^{w'\lambda^\vee}w_\alpha) \\ &= \pi^{w\lambda^\vee}u_{-\alpha}(\pi^{-\langle\alpha,w'\lambda^\vee\rangle}t)(w_\alpha\pi^{-w'\lambda^\vee}u^-_{w'}\pi^{w'\lambda^\vee}w_\alpha). \end{aligned} \tag{6.26}$$

Set $n_{w'\alpha} = -\langle\alpha, w'\lambda^\vee\rangle$. Since $\lambda^\vee$ is dominant $n_{w'\alpha}$ is a non positive integer. We define the subset $U^-_{w'}[\lambda] \subset U^-_{w',\pi}$ by setting $U^-_{w'}[\lambda] = \pi^{-w'\lambda}U^-_{(w')^{-1},\pi}\pi^{w'\lambda}$. We use (6.26) and the above notation to write,

$$\begin{aligned} I^2_{w,\lambda^\vee} &= \jmath_1\jmath_2\int_{U_{-\alpha}[\leq n_{w'\alpha}]}\int_{U^-_{w'}[\lambda]}\Phi_\rho(\pi^{w\lambda^\vee}u_{-\alpha}w_\alpha u^-_{w'}w_\alpha)du_{-\alpha}du^-_{w'} \\ &= \jmath_1\jmath_2 q^{-\langle\rho,w\lambda^\vee\rangle}e^{w\lambda^\vee}\int_{U_{-\alpha}[\leq n_{w'\alpha}]}\int_{U^-_{w'}[\lambda]}\Phi_\rho(u_{-\alpha}w_\alpha u^-_{w'}w_\alpha)du_{-\alpha}du^-_{w'}, \end{aligned} \tag{6.27}$$

where $\jmath_1 := \jmath(u_{-\alpha}\pi^{w\lambda^\vee}) = q^{-\langle w'\lambda,\alpha^\vee\rangle}$ and $\jmath_2 := \jmath(u^-_{w'}\pi^{w'\lambda^\vee})$ are the Jacobian factors. Suppose

$$J^2_{w,\lambda^\vee} := \int_{U_{-\alpha}[\leq n_{w'\alpha}]}\int_{U^-_{w'}[\lambda]}\Phi_\rho(\pi^{w\lambda^\vee}u_{-\alpha}w_\alpha u^-_{w'}w_\alpha)du_{-\alpha}du^-_{w'}. \tag{6.28}$$

The following lemma will be used to write $J^2_{w,\lambda^\vee}$ as a product of two integrals.

Lemma 6.4.5. *Let* $u_{-\alpha}(t) \in U_{-\alpha}[\leq n_{w'\alpha}]$ *and* $u^-_{w'} \in U^-_{w'}[\lambda]$, *then* $Iw_A(u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha) = Iw_A(u_{-\alpha}(t)w_\alpha)Iw_A(u_{-\alpha}(t^{-1})u^-_{w'}u_{-\alpha}(t^{-1})^{-1})$.

Proof. Let $\tilde{u}^-_{w'} := (u_{-\alpha}(t^{-1})u^-_{w'}u_{-\alpha}(t^{-1})^{-1})$. We have

$$\begin{aligned} u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha &= u_\alpha(t^{-1})\pi^{-val(t)\alpha^\vee}u_{-\alpha}(t^{-1})u^-_{w'}w_\alpha \\ &= u_\alpha(t^{-1})\pi^{-val(t)\alpha^\vee}\tilde{u}^-_{w'}u_{-\alpha}(t^{-1})w_\alpha. \end{aligned} \tag{6.29}$$

Let $\tilde{u}^-_{w'} = u_{-\alpha}(t^{-1})u^-_{w'}u_{-\alpha}(t^{-1})^{-1} = u'\pi^{\nu^\vee}k'$, for some $u' \in U$ and $k' \in K$. Using this Iwasawa decomposition in (6.29), we get

$$\begin{aligned} u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha &= u_\alpha(t^{-1})\pi^{-val(t)\alpha^\vee}(u^{'}\pi^{\nu^\vee}k^{'})u_{-\alpha}(t^{-1})w_\alpha \\ &= u^{''}\pi^{\nu^\vee-val(t)\alpha^\vee}k^{'}u_{-\alpha}(t^{-1})w_\alpha, \end{aligned} \tag{6.30}$$

for some $u^{''}\in U^+$. Thus

$$u_{-\alpha}(t)w_\alpha u^-_{w'}w_\alpha \in U\pi^{\nu^\vee-val(t)\alpha^\vee}K$$

and the assertion follows.

By following the above lemma integral $J^2_{w,\lambda^\vee}$ can be split,

$$J^2_{w,\lambda^\vee} = \int_{U_{-\alpha}[\leq n_{w'\alpha}]}\Phi_\rho(u_{-\alpha}w_\alpha)du_{-\alpha}\int_{U^-_{w'}[\lambda]}\Phi_\rho(u_{-\alpha}(t^{-1})u^-_{w'}u_{-\alpha}(-t^{-1}))du^-_{w'}.$$

For $t\in\mathcal{K}$ with $val(t)=n_{w'\alpha}$, $u_{-\alpha}(t^{-1})\in U_{-\alpha}(\mathcal{O})$ therefore $U^-_{w'}[\lambda]$ and $u_{-\alpha}(t^{-1})U^-_{w'}[\lambda]u_{-\alpha}(-t^{-1})$ have the same measure and we can write

$$J^2_{w,\lambda^\vee} = \int_{U_{-\alpha}[\leq n_{w'\alpha}]}\Phi_\rho(u_{-\alpha}w_\alpha)du_{-\alpha}\int_{U^-_{w'}[\lambda]}\Phi_\rho(u^-_{w'})du^-_{w'}.$$

Since $\jmath_1=\jmath(u_{-\alpha}\pi^{w\lambda^\vee})=q^{-\langle w'\lambda,\alpha^\vee\rangle}$ and $n_{w'\alpha}=\langle\alpha,w'\lambda^\vee\rangle$, therefore

$\jmath_1 q^{-\langle\rho,w\lambda^\vee\rangle}e^{w\lambda^\vee}=\jmath_1 q^{-\langle\rho,w_\alpha w'\lambda^\vee\rangle}e^{w_\alpha w'\lambda^\vee}=\jmath_1 q^{-\langle\rho-\alpha,w'\lambda^\vee\rangle}e^{w'\lambda^\vee-\langle\alpha,w'\lambda^\vee\rangle\alpha^\vee}$

$=\jmath_1 q^{-\langle-\alpha,w'\lambda^\vee\rangle}q^{-\langle\rho,w'\lambda^\vee\rangle}e^{w'\lambda^\vee}e^{-\langle\alpha,w'\lambda^\vee\rangle\alpha^\vee}=q^{-\langle\rho,w'\lambda^\vee\rangle}e^{w'\lambda^\vee}e^{-n_{w'\alpha}\alpha^\vee}$.

Also,

$e^{-n_{w'\alpha}\alpha^\vee}\int_{U_{-\alpha}[\leq n_{w'\alpha}]}\Phi_\rho(u_{-\alpha}w_\alpha)du_{-\alpha}$

$=e^{-n_{w'\alpha}}[(1-q^{-1})e^{n_{w'\alpha}\alpha^\vee}+(1-q^{-1})e^{(n_{w'\alpha}+1)\alpha^\vee}+\dots]$

$=[(1-q^{-1})+(1-q^{-1})e^{\alpha^\vee}+\dots]=\mathbf{b}[\alpha^\vee]$.

By putting these pieces back in (6.27), we obtain

$$
\begin{aligned}
I^1_{w,\lambda^\vee} &= \jmath_2 \mathbf{b}[\alpha^\vee] \int_{U^-_{w'}[\lambda]} q^{-\langle \rho, w'\lambda^\vee \rangle} e^{w'\lambda^\vee} \Phi_\rho(u^-_{w'} w_\alpha) du^-_{w'} \\
&= \jmath_2 \mathbf{b}[\alpha^\vee] \int_{U^-_{w'}} \Phi_\rho(\pi^{w'\lambda^\vee} u^-_{w'}) du^-_{w'} \\
&= \mathbf{b}[\alpha^\vee] I_{w',\lambda^\vee}. \qquad (6.31)
\end{aligned}
$$

The solutions (6.25) and (6.31) imply that

$$
I_{w,\lambda^\vee} = \mathbf{T}_{w_\alpha}(I_{w',\lambda^\vee}). \qquad (6.32)
$$

Chapter 7

Weak Spherical Finiteness

7.1 Iwahori Level Decomposition

Now, we initiate the proof of the Theorem 1.5.1. For $\lambda^\vee, \mu^\vee \in \Lambda^\vee$, set

$$M(\lambda^\vee, \mu^\vee) = K\backslash K\pi^{\mu^\vee}U^+ \cap K\pi^{\lambda^\vee}K. \tag{7.1}$$

We begin by establishing a bijective correspondence between the coset space $M(\lambda^\vee, \mu^\vee)$ and a disjoint union of so-called Iwahori pieces. This disjoint union is indexed by the Weyl group W and the main result of this section asserts that there are finitely many elements of W which contribute in this union.

7.1.1 Iwahori Pieces

Let $\Gamma \leq G$, X be a right-Γ and Y be a left-Γ set. We need the following relation on the set $X \times Y$ from [8, Section 4].

Definition 7.1.1. *Let $(x, y), (x', y') \in X \times Y$, $(x, y) \sim (x', y')$ if and only if there exists some $r \in \Gamma$ such that, $x' = xr$ and $y' = r^{-1}y$.*

One can check that $\sim$ is an equivalence relation. We denote by $X \times_\Gamma Y$, the quotient space $(X \times Y)/\sim$. For $\mu^\vee, \lambda^\vee \in \Lambda^\vee$, we take $X = U^+K$, $Y = K\pi^{\lambda^\vee}K$, $\Gamma = K$ and consider the following map induced by the multiplication

$$m_{\lambda^\vee} \quad : \quad U^+K \times_K K\pi^{\lambda^\vee}K \longrightarrow G. \tag{7.2}$$

For $w \in W$ and $\lambda^\vee \in \Lambda^\vee$, we take $X = U^+wI^-$, $Y = I^-\pi^{\lambda^\vee}K$, $\Gamma = I^-$ and consider also the following map induced by multiplication:

$$m_{\lambda^\vee,w} \quad : \quad U^+wI^- \times_{I^-} I^-\pi^{\lambda^\vee}K \longrightarrow G. \tag{7.3}$$

As in the affine case from [52, Section 4.4], it can be shown that for $\mu^\vee \in \Lambda^\vee$, the fiber $m_{\lambda^\vee}^{-1}(\pi^{\mu^\vee})$ is in bijective correspondence with $M(\lambda^\vee, \mu^\vee)$. Also, the following lemma can be proven more generally along the same lines as those of [8, Lemma 7.3.3], which again was written in the affine context.

Lemma 7.1.2. *For all $w \in W$ and $\lambda^\vee \in \Lambda_+^\vee$ regular, the fibers $m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee})$ are disjoint and there is a bijection*

$$m_{\lambda^\vee}^{-1}(\pi^{\mu^\vee}) \simeq \sqcup_{w\in W} m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee}). \tag{7.4}$$

From now on, for $w \in W$ and $\mu^\vee \in \Lambda^\vee$, the fiber $m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee})$ will be referred to as an Iwahori piece of $M(\lambda^\vee, \mu^\vee)$. For each $w \in W$ and fixed $\lambda^\vee, \mu^\vee \in \Lambda^\vee$, by definition

$$m_{\lambda^\vee,w}^{-1}(\pi^{\mu^\vee}) \quad = \quad I^-\backslash I^-w^{-1}\pi^{\mu^\vee}U^+ \cap I^-\pi^{\lambda^\vee}K. \tag{7.5}$$

Theorem 1.5.1 will follow if we prove: (a) for fixed $\mu^\vee \in \Lambda^\vee$ and $\lambda^\vee$ regular and sufficiently dominant, there are finitely many elements of W contribute in the union on the right hand side of (7.4); and (b) for each such $w \in W$, the Iwahori piece $m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee})$ is finite. Part (b) is discussed in §$5-6$ and part (a) is a consequence of

Proposition 7.1.3. *For a fixed $\mu^\vee \in \Lambda^\vee$, $\lambda^\vee \in \Lambda^\vee$ regular and sufficiently large with respect to $\mu^\vee$, there exists a finite subset $\Omega = \Omega(\lambda^\vee, \mu^\vee) \subset W$ such that*

$$m_{\lambda^\vee}^{-1}(\pi^{\mu^\vee}) \simeq \sqcup_{w\in\Omega} m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee}).$$

Proof. By (7.5), it suffices to show the following: for $\mu^\vee$ fixed and $\lambda^\vee$ regular and sufficiently large, there exist finitely many $w \in W$ such that

$$I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} K \neq \emptyset. \tag{7.6}$$

We replace K by $K = \cup_{\sigma\in W} I^+ \sigma I^-$ and then use the Iwahori-Matsumoto decomposition $I^+ = U_{\mathcal{O}}^+ U_\pi^- H_{\mathcal{O}}$ on the left hand side of (7.6) to obtain

$$\begin{aligned} I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} K &= \cup_{\sigma\in W} I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} U_{\mathcal{O}}^+ U_\pi^- \sigma I^- \\ &= \cup_{\sigma\in W} I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} \sigma I^-, \end{aligned} \tag{7.7}$$

where in the last step we use the fact that if $\lambda^\vee$ is dominant and regular then $\pi^{\lambda^\vee} U_{\mathcal{O}}^+ \pi^{-\lambda^\vee} \subset U_\pi^+$. Consider $\sigma \in W$ such that

$$I^- w\pi^{\mu^\vee} U^+ \cap I^- \sigma \pi^{\sigma\lambda^\vee} U_{\mathcal{O}}^- \neq \emptyset. \tag{7.8}$$

Now,

$$I^- w\pi^{\mu^\vee} U^+ \cap I^- \sigma \pi^{\sigma\lambda^\vee} U_{\mathcal{O}}^- \subset K\pi^{\mu^\vee} U^+ \cap K\pi^{\sigma\lambda^\vee} U^-. \tag{7.9}$$

By the second part of Theorem 1.4.1 we get,

$$\mu^\vee \leq \sigma\lambda^\vee. \tag{7.10}$$

Since $\lambda^\vee$ is regular, if we choose $\lambda^\vee$ very large compared to $\mu^\vee$ then (7.10) holds only for $\sigma = 1$. Hence (7.7) implies that

$$I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} K = I^- w\pi^{\mu^\vee} U^+ \cap I^- \pi^{\lambda^\vee} U^-_{\mathcal{O}}.$$

By (7.6), we also have:

$$U^- \cap U^+_{w^{-1},\pi} w\pi^{\mu^\vee - \lambda^\vee} H_{\mathcal{O}} U^+ \neq \emptyset.$$

Finally, Corollary 4.2.1 implies

$$l(w) \leq 2\langle \rho, \lambda^\vee - \mu^\vee \rangle. \tag{7.11}$$

The bound (7.11) proves the Proposition.

7.2 Finiteness of Fiber

In this section, we fix $w \in \Omega$, $\mu^\vee \in \Lambda$ and $\lambda \in \Lambda_+$ regular, where $\Omega \subset W$ is the finite set obtained in Proposition 7.1.3, and prove the finiteness of the Iwahori piece $m^{-1}_{w,\lambda^\vee}(\mu^\vee)$ for $w \in \Omega$. We introduce the following terminology which will be used in this section.

Definition 7.2.1. *Let $f = \sum_{\mu^\vee \in \Lambda^\vee} c_{\mu^\vee} e^{\mu^\vee}$ be a formal sum, we write*

$$[e^{\xi^\vee}]f := c_{\xi^\vee}. \tag{7.12}$$

Let $Z := \{\mu^\vee \in \Lambda^\vee \mid m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee}) \neq \emptyset\}$.

Lemma 7.2.2. *For $w \in W$ and $\lambda^\vee \in \Lambda_+^\vee$ regular, $Z \subset supp(I_{w,\lambda^\vee})$, where $supp(I_{w,\lambda^\vee}) = \{\mu^\vee \in \Lambda^\vee \mid [e^{\mu^\vee}]I_{w,\lambda^\vee} \neq 0\}$.*

Proof. Let $\mu^\vee$ is such that $m_{w,\lambda^\vee}^{-1}(\pi^{\mu^\vee}) \neq \emptyset$, then

$$wI^-\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K \quad \neq \quad \emptyset$$

which implies

$$wU_\pi^+U_\mathcal{O}^-\pi^{\lambda^\vee}K \cap U\pi^{\mu^\vee}K \quad \neq \quad \emptyset.$$

Since $\lambda^\vee$ is dominant and regular, $\pi^{-\lambda^\vee}U_\mathcal{O}^-\pi^{\lambda^\vee} \subset K$ and this gives

$$U_{w^{-1}\pi}^-\pi^{w\lambda^\vee} \cap U\pi^{\mu^\vee}K \neq \emptyset.$$

and thus $\mu^\vee \in supp(I_{w,\lambda^\vee})$.

7.2.1 Quotient Space and Surjection

We equip the group $U_{w,\pi}^+$ with the following relation

Definition 7.2.3. *Let $u_w, z_w \in U_{w,\pi}^+$. We say $u_w \sim z_w$ if and only if*

$$u_w = z_w\pi^{\lambda^\vee}U_{w,\pi}^+\pi^{-\lambda^\vee}. \tag{7.13}$$

It can be easily verified that $\sim$ is an equivalence relation. For $u_w \in U^+_{w,\pi}$, $[u_w]$ will denote the equivalence class of u_w with respect to the relation $\sim$. Next, set

$$U^+_{w,\pi}(\mu^\vee) := \{u_w \in U^+_{w,\pi} \mid w u_w \pi^{\lambda^\vee} \in U\pi^{\mu^\vee} K\}, \tag{7.14}$$

and

$$\mathcal{X}^+_{w,\pi}(\mu^\vee) := \{[u_w] \mid u_w \in U^+_{w,\pi}(\mu^\vee)\}. \tag{7.15}$$

If $[u_w] \in \mathcal{X}^+_{w,\pi}(\mu^\vee)$ with $u_w \in U^+_{w,\pi}(\mu^\vee)$, the relation

$$w u_w \pi^{\lambda^\vee} = u\pi^{\mu^\vee} k, \tag{7.16}$$

implies

$$u^{-1} w u_w \pi^{\lambda^\vee} k^{-1} = \pi^{\mu^\vee}, \tag{7.17}$$

for some $k \in K$ and $u \in U^+$. Thus $[u_w] \in \mathcal{X}^+_{w,\pi}(\mu^\vee)$ gives rise to an element in $m^{-1}_{w,\lambda^\vee}(\mu^\vee)$.

Definition 7.2.4. *Let*

$$\phi\colon \mathcal{X}^+_{w,\pi}(\mu^\vee) \longrightarrow m^{-1}_{w,\lambda^\vee}(\mu^\vee) \tag{7.18}$$

be a map defined as $\phi([u_w])\colon = (u^{-1} w u_w, \pi^{\lambda^\vee} k^{-1})$.

Lemma 7.2.5. *The function* ϕ *is well defined and onto.*

Proof. To show that ϕ is a well defined, let $u_w, z_w \in U^+_{w,\pi}(\mu^\vee)$ and $u_w \sim z_w$ then there exists $u^+ \in U^+_{w,\pi}$ such that

$$u_w = z_w \pi^{\lambda^\vee} u^+ \pi^{-\lambda^\vee}. \tag{7.19}$$

Also, for some $k \in K$ and $u_1 \in U^+$,

$$w u_w \pi^{\lambda^\vee} = u_1 \pi^{\mu^\vee} k, \tag{7.20}$$

which implies

$$\begin{aligned} \pi^{\mu^\vee} &= u_1^{-1} w u_w \pi^{\lambda^\vee} k^{-1}, \\ &= u_1^{-1} w u_w (\pi^{\lambda^\vee} u^+ \pi^{-\lambda^\vee})^{-1} (\pi^{\lambda^\vee} u^+ \pi^{-\lambda^\vee}) \pi^{\lambda^\vee} k^{-1} \\ &= u_1^{-1} w z_w \pi^{\lambda^\vee} u^+ k^{-1} \\ &= u_1^{-1} w z_w \pi^{\lambda^\vee} k', \end{aligned} \tag{7.21}$$

where $u^+ k^{-1} = k' \in K$. Thus by taking $(\pi^{\lambda^\vee} u^+ \pi^{-\lambda^\vee})^{-1} = i^- \in I^-$

$$(u_1^{-1} w z_w, \pi^{\lambda^\vee} k') = (u_1^{-1} w u_w i^-, (i^-)^{-1} \pi^{\lambda^\vee} k), \tag{7.22}$$

and hence

$$\phi([u_w]) = \phi([z_w]). \tag{7.23}$$

Next, we show that ϕ is onto. Let $(x, y) \in m^{-1}_{w,\lambda^\vee}(\mu^\vee)$ with $x = uwi_1^-$ and $y = i_2^- \pi^{\lambda^\vee} k$ for some $u \in U^+$, $k \in K$ and $i_1^-, i_2^- \in I^-$. Then there exists $i^- \in I^-$

such that

$$uwi^{-}\pi^{\lambda^{\vee}}k=\pi^{\mu^{\vee}}. \tag{7.24}$$

Suppose i^{-} has the following decomposition

$$i^{-}=u_{\pi}^{+}u_{\mathcal{O}}^{-}h_{\mathcal{O}} \tag{7.25}$$

for some $u_{\pi}^{+}\in U_{\pi}^{+}$, $u_{\mathcal{O}}^{-}\in U_{\mathcal{O}}^{-}$ and $h_{\mathcal{O}}\in H_{\mathcal{O}}$. By putting it into (7.24), we have

$$\begin{aligned}\pi^{\mu^{\vee}} &= uwi^{-}\pi^{\lambda^{\vee}}k\\ &= uwu_{\pi}^{+}u_{\mathcal{O}}^{-}h_{\mathcal{O}}\pi^{\lambda^{\vee}}k\\ &= uwu_{w,\pi}\pi^{\lambda^{\vee}}k'' \end{aligned} \tag{7.26}$$

for some $u_{w,\pi}\in U_{w,\pi}^{+}$ and $k''=\pi^{-\lambda^{\vee}}u_{\mathcal{O}}^{-}h_{\mathcal{O}}\pi^{\lambda^{\vee}}k\in K$. So, we get an element $u_{w,\pi}\in U_{w,\pi}^{+}(\mu^{\vee})$ such that

$$\phi([u_w])=(x,y). \tag{7.27}$$

This completes the proof.

7.2.2 Finiteness of Level Sets

Let $U_{w^{-1}}^{-}[\lambda^{\vee}]:=\pi^{-w\lambda^{\vee}}U_{w^{-1},\pi}^{-}\pi^{w\lambda^{\vee}}$ and $U_{w^{-1}}^{-}[\lambda^{\vee},\mu^{\vee}]:=U_{w^{-1}}^{-}[\lambda^{\vee}]\cap U\pi^{\mu^{\vee}}K.$

Remark 7.2.6. *If $u_w\in U_{w,\pi}^{+}$ satisfies $wu_w\pi^{\lambda^{\vee}}\in U\pi^{\mu^{\vee}}K$, then*

$$\pi^{-w\lambda^{\vee}}u_w\pi^{w\lambda^{\vee}}\in U\pi^{\mu^{\vee}-w\lambda^{\vee}}K\cap U_{w^{-1}}^{-}[\lambda^{\vee}]. \tag{7.28}$$

Set

$$\widehat{Y} := \{\mu^\vee - w\lambda^\vee \mid \mu^\vee \in Y\},$$

where $Y = Supp(\tilde{I}_{w,\lambda^\vee})$.

Lemma 7.2.7. *For each $\xi \in \widehat{Y}$, the coset space $U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]/U^-_{w^{-1},\pi}$ is finite.*

Proof. The integral $\tilde{I}_{w,\lambda^\vee}$ which is defined in Section 6.1 can be written as

$$\begin{aligned}
\tilde{I}_{w,\lambda^\vee} &= \int_{U^-_{w^{-1},\pi}} \Phi_\rho(u^-_w \pi^{w\lambda^\vee}) d\tilde{u^-_w} \\
&= \int_{U^-_{w^{-1}}[\lambda^\vee]} \Phi_\rho(\pi^{w\lambda^\vee} u^-_w) d\tilde{u^-_w} \\
&= q^{-\langle\rho, w\lambda^\vee\rangle} e^{w\lambda^\vee} \int_{U^-_{w^{-1}}[\lambda^\vee]} \Phi_\rho(u^-_w) d\tilde{u^-_w} \\
&= q^{-\langle\rho, w\lambda^\vee\rangle} e^{w\lambda^\vee} \sum_{\xi^\vee \in \widehat{Y}} Vol(U^-_{w^{-1}}[\lambda^\vee] \cap U\pi^{\xi^\vee} K) q^{-\langle\rho,\xi^\vee\rangle} e^{\xi^\vee} \\
&= q^{-\langle\rho, w\lambda^\vee\rangle} e^{w\lambda^\vee} \sum_{\xi^\vee \in \widehat{Y}} |U^-_{w^{-1}}[\lambda^\vee] \cap U\pi^{\xi^\vee} K/U^-_{w^{-1},\pi}| q^{-\langle\rho,\xi^\vee\rangle} e^{\xi^\vee} \\
&= \sum_{\xi^\vee \in \widehat{Y}} |U^-_{w^{-1}}[\lambda^\vee] \cap U\pi^{\xi^\vee} K/U^-_{w^{-1},\pi}| q^{-\langle\rho, w\lambda^\vee + \xi^\vee\rangle} e^{w\lambda^\vee + \xi^\vee}. \qquad (7.29)
\end{aligned}$$

By Theorem 6.2.4, for each $\mu^\vee \in Y$, there exists a constant D such that

$$[e^{\mu^\vee}]\tilde{I}_{w,\lambda^\vee} = D[e^{\mu^\vee}]T_w(e^{\lambda^\vee}). \qquad (7.30)$$

Since the right hand side of (7.30) is finite, so is the left hand side and hence the lemma follows.

7.2.3 Main Result

In this subsection we complete the proof of the Weak Spherical Finiteness by showing that the fiber $m^{-1}_{w,\lambda^\vee}(\mu^\vee)$ is finite.

Proposition 7.2.8. *There exists a one-to-one map from $\mathcal{X}^+_{w,\pi}(\mu^\vee)$ to $U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]/U^-_{w^{-1},\pi}$.*

Proof. By using the fact

$$w\pi^{-\lambda}U^+_{w.\pi}(\mu^\vee)\pi^\lambda w^{-1} \subset U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]$$

we define a map

$$\psi\colon \mathcal{X}^+_{w,\pi}(\mu^\vee) \longrightarrow U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]/U^-_{w^{-1},\pi}$$

as,

$$\psi([u_w]) = (w\pi^{-\lambda}u_w\pi^\lambda w^{-1})U^-_{w^{-1},\pi}. \tag{7.31}$$

We prove that ψ is our required one-to-one map. First, we show that

ψ **is well defined:** Let $u_w, z_w \in U^+_{w,\pi}(\mu^\vee)$ and $u_w \sim z_w$ then there exists $u^+ \in U^+_{w,\pi}$ such that

$$u_w = z_w\pi^{\lambda^\vee}u^+\pi^{-\lambda^\vee}. \tag{7.32}$$

Hence $(z_w)^{-1}u_w = \pi^{\lambda^\vee}u^+\pi^{-\lambda^\vee}$ and

$$w\pi^{-\lambda^\vee}(z_w)^{-1}u_w\pi^{\lambda^\vee}w^{-1} = w\pi^{-\lambda^\vee}(\pi^{\lambda^\vee}u^+\pi^{-\lambda^\vee})\pi^{\lambda^\vee}w^{-1}. \tag{7.33}$$

Since $u^- = w(u^+)^{-1}w^{-1} \in U^-_{w^{-1},\pi}$, we get

$$w\pi^{-\lambda^\vee}u^-_{w^{-1}}\pi^{\lambda^\vee}w^{-1}U^-_{w,\pi} \quad = \quad w\pi^{-\lambda^\vee}z^-_{w^{-1}}\pi^{\lambda^\vee}w^{-1}U^-_{w,\pi} \tag{7.34}$$

and hence ψ is well defined.

ψ **is injective:** To show that ψ is one-one, suppose $u_w, z_w \in U^+_{w,\pi}(\mu^\vee)$ be such that

$$\psi([u_w]) = \psi([z_w])$$

that is,

$$\begin{aligned} w\pi^{-\lambda^\vee} z_w^{-1} u_w \pi^{w\lambda^\vee} w^{-1} &\in U^-_{w^{-1},\pi} \\ \pi^{-\lambda^\vee} z_w^{-1} u_w \pi^{\lambda^\vee} &\in w^{-1} U^-_{w^{-1},\pi} w. \end{aligned}$$

Consequently, $z_w^{-1} u_w \in \pi^{\lambda^\vee} U^+_{w,\pi} \pi^{-\lambda^\vee}$ and $z_w \sim u_w$. Hence ψ is a one-one map.

We now state and prove the main result of this section.

Proposition 7.2.9. *For $w \in \Omega$, $\mu^\vee \in \Lambda^\vee$ and $\lambda^\vee \in \Lambda^\vee_+$ regular, the fibers $m^{-1}_{w,\lambda^\vee}(\pi^{\mu^\vee})$ is finite.*

Proof. By Proposition 7.2.8 the set $\mathcal{X}^+_{w,\pi}(\mu^\vee)$ is embedded in $U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]/U^-_{w^{-1},\pi}$. The quotient $U^-_{w^{-1}}[\lambda^\vee, \xi^\vee]/U^-_{w^{-1},\pi}$ is finite by Lemma 7.2.7 and hence $\mathcal{X}^+_{w,\pi}(\mu^\vee)$ is finite. Finally, by Lemma 7.2.5 the finite set $\mathcal{X}^+_{w,\pi}(\mu^\vee)$ is mapped onto the fiber $m^{-1}_{w,\lambda^\vee}(\mu^\vee)$ which implies the finiteness of $m^{-1}_{w,\lambda^\vee}(\mu^\vee)$. So, the Weak Spherical Finiteness follows.

Chapter 8

Proof of Main Finiteness Theorems

Our aim in this chapter is to prove the following diagram of implications by applying the finiteness results we obtained so far.

$$\begin{array}{ccc} \text{Weak Spherical Finiteness+Approximation Theorem} & & \\ \Downarrow & & \\ \text{Gindikin-Karpelevich Finiteness} & \Longrightarrow & \text{Spherical Finiteness} \end{array}$$

8.1 Gindikin-Karpelevich Finiteness

Proof. By using the Approximation Theorem and Weak Spherical Finiteness, we give a proof of Theorem 1.4.1. The Approximation Theorem implies that for a fixed $\mu^\vee \in \Lambda^\vee$, $\lambda^\vee \in \Lambda^\vee$ regular and sufficiently large, we have an equality

$$K \backslash K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ = K \backslash K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U^+ \tag{8.1}$$

of coset spaces. With the same assumption on $\mu^\vee$ and considering $\lambda^\vee$ sufficiently large, the Weak Spherical Finiteness implies the finiteness of the left hand side of

(8.1). Now, there is a bijection of sets

$$K\pi^{\lambda^\vee}U^- \cap K\pi^{\lambda^\vee - \mu^\vee}U^+ \leftrightarrow KU^- \cap K\pi^{-\mu^\vee}U^+, \tag{8.2}$$

which implies the finiteness of the set $K\backslash KU^- \cap K\pi^{-\mu^\vee}U^+$. As $\mu^\vee$ was chosen arbitrarily, the Gindikin-Karpelevich finiteness follows.

8.2 Spherical Finiteness

For the implication

$$\text{Gindikin-Karpelevich Finiteness} \implies \text{The Spherical Finiteness},$$

we will use the second part of Theorem 1.4.1. An affine version of this result was proven in [6, P. 60], which generalizes to indefinite type Kac-Moody groups as well. However, for the sake of completion we rewrite its proof as the following lemma.

Lemma 8.2.1. *For $\lambda^\vee \in \Lambda^\vee$ dominant and for any $\mu^\vee \in \Lambda^\vee$*

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- = \emptyset \textit{ unless } \mu^\vee \leq \lambda^\vee. \tag{8.3}$$

Proof. Let $y \in K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^-$, there exists $k_1, k_2, k_3 \in K$, and $u^- \in U^-$ such that

$$y = k_1\pi^{\lambda^\vee}k_2 = k_3\pi^{\mu^\vee}u^-. \tag{8.4}$$

We apply the both decompositions of the above element on the highest weight vector

v_ρ of the highest weight module V^ρ and compute the norms of the resulting vectors

$$\begin{aligned} ||y^{-1}v_\rho|| &= q^{\langle \rho, \lambda^\vee \rangle} = ||k_2^{-1}\pi^{-\lambda^\vee}k_1^{-1}v_\rho|| \\ &= ||(u^-)^{-1}\pi^{\mu^\vee}k_3^{-1}v_\rho|| \geq q^{\langle \rho, \mu^\vee \rangle}, \end{aligned} \tag{8.5}$$

to get the proof of the lemma.

By combining this lemma with the last part of Theorem 1.4.1, we get the following corollary.

Corollary 8.2.2. *For $\lambda^\vee \in \Lambda_+^\vee$ and for any $\nu^\vee, \mu^\vee \in \Lambda^\vee$*

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- \cap K\pi^{\nu^\vee}U^+ = \emptyset$$

unless $\nu^\vee \leq \mu^\vee \leq \lambda^\vee$.

So, for $\lambda^\vee \in \Lambda^\vee$ dominant, we can write

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^+ = \bigcup_{\nu^\vee \leq \mu^\vee \leq \lambda^\vee} K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- \cap K\pi^{\nu^\vee}U^+. \tag{8.6}$$

The Spherical Finiteness follows by the following two facts:

(a) *For fixed $\lambda^\vee, \nu^\vee, \in \Lambda^\vee$, the set $\{\mu^\vee \in \Lambda^\vee \mid \nu^\vee \leq \mu^\vee \leq \lambda^\vee\}$ is finite.*

(b) *For $\lambda^\vee \in \Lambda^\vee$ dominant and for any $\mu, \nu^\vee \in \Lambda^\vee$ the containment*

$$K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- \cap K\pi^{\nu^\vee}U^+ \subset K\pi^{\mu^\vee}U^- \cap K\pi^{\nu^\vee}U^+,$$

implies that $K\backslash K\pi^{\lambda^\vee}K \cap K\pi^{\mu^\vee}U^- \cap K\pi^{\nu^\vee}U^+$ is finite by the Gindikin-Karpelevich Finiteness.

Chapter 9

An Open Problem

In this chapter, we present an incomplete proof of the Gindikin-Karpelevich Finiteness independent of the Spherical Finiteness. By Lemma 1.3.1, it suffices to get this finiteness for $\lambda^\vee = 0$, we restate the assertion as the following proposition.

Proposition 9.0.1. *Let $\mu^\vee \in \Lambda^\vee$ be fixed and $U^-(\mu^\vee) = U^- \cap K\pi^{\mu^\vee}U^+$. Then the coset space $U^-_{\mathcal{O}} \backslash U^-(\mu^\vee)$ has finite cardinality.*

To obtain the proof of this proposition, we use an unproven bounded condition, which we state as Conjecture 9.2.7. As indicated in Subsection 4.2.4, the existence of a coordinate system on U^- was the key to prove the affine version of this finiteness. This coordinate system is used to obtain a particular product representation of the certain elements of U^- and to prove certain bounded conditions satisfied by them. This construction can not be generalized to arbitrary Kac-Moody setting. However, we believe that these bounded conditions holds true in general settings as well, but our method of proof does not work. We begin this chapter with the properties of finitely ordered matrices.

9.1 A Finite Dimensional Result

9.1.1 Properties of Finite Matrices

Let $A = (a_{ij}) \in GL_r(\mathcal{K})$ be a unipotent lower triangular matrix such that a_{ij} are uniformly bounded by some constant C, for all $1 \leq i, j \leq r$. For each $l \geq 0$, such a matrix can be written as

$$A = A_0 + \epsilon, \tag{9.1}$$

where A_0 is an $r \times r$ lower unipotent matrix and ϵ is an $r \times r$ strictly lower triangular matrix such that $\epsilon \equiv 0_{r\times r} (\bmod\ \pi^l)$. Moreover, the entries of A^{-1} are also uniformly bounded by some constant which depends on C and r and thus it has an expression similar to (9.1). More precisely,

Lemma 9.1.1. *Let A and A_0 be $r \times r$ lower unipotent matrices with entries which are uniformly bounded by a constant C. Then given any $m \geq 0$, there exists $l = l(m, C, r)$ such that if*

$$A = A_0 + \epsilon_A, \quad \textit{with} \quad \epsilon_A \equiv 0_{r\times r} (\bmod\ \pi^l) \tag{9.2}$$

then

$$A^{-1} = A_0^{-1} + \epsilon_{A^{-1}}, \quad \textit{with} \quad \epsilon_{A^{-1}} \equiv 0_{r\times r} (\bmod\ \pi^m). \tag{9.3}$$

Proof. The above statement is implied by the following facts:

(i) *If A is an $r \times r$ lower unipotent matrix with entries from $\pi^l\mathcal{O}$, then there exists some m which depends on l and r such that A^{-1} is is a lower unipotent matrix* *with entries from* $\pi^m\mathcal{O}$*.*

(ii) *Given* $p_1, p_2 \geq 0$*, there exists positive integer* $s = s(C, r, p)$ *such that if*

$$\epsilon_A \equiv 0_{r\times r}(\text{mod } \pi^s)$$

then $A^{-1}\epsilon_A \equiv 0_{r\times r}(\text{mod } \pi^{p_1})$ *and* $\epsilon_A A^{-1} \equiv 0_{r\times r}(\text{mod } \pi^{p_2})$*, and*

(iii) *if* $A = A_0 + \epsilon_A$ *then* $A = A_0(\mathbb{I}_r + A_0^{-1}\epsilon_A)$ *and hence*

$$A^{-1} = (\mathbb{I}_r + A_0^{-1}\epsilon_A)^{-1}A_0^{-1},$$

where $\mathbb{I}_r + A_0^{-1}\epsilon_A$ is a lower unipotent matrix with entries from $\pi^j\mathcal{O}$ for some j.

Proposition 9.1.2. *Let* r *be a positive integer and* $C > 0$*. There exists* $l = l(r, C)$ *such that for any* $A, B \in GL_r(\mathcal{K})$ *satisfying,*

(a) *entries of* A *and* B *are bounded by* C*.*

(b) $A - B \equiv 0 \ (\text{mod } \pi^l)$*,*

then $AB^{-1} \in GL_r(\mathcal{O})$*.*

Proof. By using Lemma 9.1.1, we write $A = A_0 + \epsilon_A$ and $B^{-1} = A_0^{-1} + \epsilon_{B^{-1}}$ such that $\epsilon_A \equiv 0_{r\times r}(\text{mod } \pi^{m_1})$ and $\epsilon_{B^{-1}} \equiv 0_{r\times r}(\text{mod } \pi^{m_2})$, then

$$AB^{-1} = \mathbb{I}_r + A_0\epsilon_{B^{-1}} + \epsilon_A A_0^{-1} + \epsilon_A\epsilon_{B^{-1}}.$$

The integers m_1 and m_2 can be chosen sufficiently large such that $A_0\epsilon_{B^{-1}}, \epsilon_A A_0^{-1} \in GL_r(\mathcal{O})$ and thus $AB^{-1} \in GL_r(\mathcal{O})$.

9.1.2 Bounded Conditions

Let V_0 be a vector space over $\mathcal{K}$ of finite dimension r with a basis $\mathcal{B} = \{v_i\}_{i=1}^r$. Let $V_0(\mathcal{O}) = \oplus_{i=1}^r \mathcal{O}v_i$ the integral lattice in V_0. Let $||\cdot||$ be a norm on V_0 as defined earlier in Subsection 4.1.3. With respect to this norm, we assume that $||v_i|| = 1$, for all $1 \leq i \leq r$. Next, suppose

$$G_0 := \mathrm{Aut}(V_0)$$

and set

$$K_0 := \{g \in G_0 \mid gV_0(\mathcal{O}) \subset V_0(\mathcal{O})\}.$$

Let U_0^- (resp. U_0^+) be the subgroup of G_0 consisting of lower (resp. upper) triangular unipotent matrices with respect to $\mathcal{B}$ and

$$U_0^\pm(\mathcal{O}) = U_0^\pm \cap K_0.$$

We equip G_0 with a norm $||\cdot||_0$, by setting

$$||g||_0 := \max_{1\leq i\leq r} ||gv_i|| \tag{9.4}$$

for all $g \in G_0$. Let b be a constant and

$$G_{0,b} := \{g \in G \mid ||g||_0 \leq b\}. \tag{9.5}$$

Set $U_{0,b}^- := G_{0,b} \cap U_0^-$. Note that $U_0^-(\mathcal{O}) \subset U_{0,b}^-$. Our aim in this subsection is to show the following.

Proposition 9.1.3. *For a fixed constant b, the coset space $U_0^-(\mathcal{O})\backslash U_{0,b}^-$ is finite.*

Proof. By the constructions, elements of $U_{0,b}^-$ are lower unipotent matrices with entries which are uniformly bounded by the constant b. Then, for $l > 0$, each $X^- \in U^-_{0,b}$ can be written as $X^- = X^-_0 + \epsilon_{X^-}$, where X^-_0 is an $r \times r$ lower unipotent matrix and ϵ_{X^-} is a strictly lower triangular matrix such that $\epsilon_{X^-} \equiv 0_{r\times r}(\mathrm{mod}\ \pi^l)$. Furthermore, there are finitely many choices for X^-_0's (say n). We denote by $\mathcal{A}_l = \{X^{-,j}_0\}^n_{j=1}$ the set of these matrices. Let $U^-_{0,b,fin}$ be a finite subset of $U^-_{0,b}$ consisting of elements u^-_j such that $u^-_j = X^{-,j}_0 + \epsilon_j$, with $\epsilon_j \equiv 0_{r\times r}(\mathrm{mod}\ \pi^l)$, for $1 \leq j \leq n$. We choose l sufficiently big to satisfy the condition of Proposition 9.1.2. Then, for every $u^- \in U^-_{0,b}$ there exists $u^-_{fin} \in U^-_{0,b,fin}$ such that

$$u^-(u^-_{fin})^{-1} \in U^-_0(\mathcal{O})$$

and this implies the proposition.

9.2 General Settings

To prove Proposition 9.0.1, we shall use the embeddings and projections of certain subsets of the completion of U^-. A geometric version of this completion and these mappings can be found in Chapter 7 of Kumar's book [39]. We give a representation theoretic completion of U^- and prove the existence of analogous embeddings and projections in this completion.

9.2.1 Completion of U^-

As before, let $\lambda \in \Lambda_+$ and $V = V^\lambda$ an irreducible highest weight representation. For $m \geq 0$, suppose $V(m) \subset V$ be the subspace and its finite dimensional quotient space $V[m] := V/V(m)$ as introduced in Subsection 2.3. For $0 \leq m_1 \leq m_2$, the containment $V(m_2) \subseteq V(m_1)$ induces a projection

$$V[m_2] \twoheadrightarrow V[m_1]. \tag{9.6}$$

The action of U^- on V preserves $V(m)$ and descends to give,

$$\omega_m \colon U^- \longrightarrow \mathrm{Aut}(V[m]).$$

Let $U^-(m) := Ker\omega_m$ and $U^-[m] := \omega_m(U^-) \simeq U^-/U^-(m)$. If $0 \leq m_1 \leq m_2$, the map (9.6) implies $U^-(m_2) \subseteq U^-(m_1)$ which gives a projection

$$\pi_{m_2}^{m_1} \colon U^-[m_2] \longrightarrow U^-[m_1].$$

This allows us to consider the projective family $\{U^-[m]\}_{m\geq 0}$ of groups with the maps $\cdots \to U^-[3] \xrightarrow{\pi_3^2} U^-[2] \xrightarrow{\pi_2^1} U^-[1] \xrightarrow{\pi_1^0} U^-[0]$.

We define the completion of U^- as the projective limit

$$\mathbb{U}^- := \varprojlim U^-[m].$$

This completion comes equipped with an inclusion

$$\iota \colon U^- \longrightarrow \mathbb{U}^-, \tag{9.7}$$

and projections

$$\phi_m \colon \mathbb{U}^- \longrightarrow U^-[m] \tag{9.8}$$

for all $m \geq 0$. This construction yields the following straightforward fact.

Lemma 9.2.1. *For any integrable highest weight $\mathfrak{g}$-representation $V = V^\lambda$ and $m \geq 1$, the following diagram commutes*

$$\begin{tikzcd} U^- \arrow[r, hook, "i"] \arrow[d, "\omega_m"] & \mathbb{U}^- \arrow[dl, "\phi_m"] \\ \mathrm{Aut}(V[m]) & \end{tikzcd} \tag{9.9}$$

9.2.2 Some Embeddings and Surjections

For $m, n \geq 1$, set

$$X_n := \cup_{\ell(\sigma)\leq n} B^+ \sigma B^+ \cap U^-,$$

$$X_n(\mu^\vee) := X_n \cap U^-(\mu^\vee)$$

and

$$\vartheta_{n,m} \colon X_n \longrightarrow U^-[m] \tag{9.10}$$

be the restriction of composition of the maps

$$U^- \xhookrightarrow{i} \mathbb{U}^- \xrightarrow{\phi_m} U^-[m] \tag{9.11}$$

on X_n. Then we have,

Lemma 9.2.2. *For a fixed $n \geq 1$, there exists $k(n)$ such that for all $m > k(n)$ the map $\vartheta_{n,m}$ is an embedding.*

Proof. For n and m as above, we define a map

$$\phi_{n,m} \colon X_n \longrightarrow V[m]$$

as $x \mapsto \overline{xv_\lambda}$, where for $v \in V$, $\overline{v} = v + V(m)$.

Claim 2. *There exists $k(n)$ such that for all $m > k(n)$, $\phi_{n,m}$ is an injection.*

Proof. Let $\sigma_1, \sigma_2, \ldots, \sigma_p \in W$ be such that $X_n = U^- \cap (\cup_{1 \le i \le p} B\sigma_i B)$. Next, suppose $\nu_i := \sigma_i \lambda$ and $k(n) = \max_{1 \le i \le p}\{depth(\nu_i)\}$. Then

$$X_n v_\lambda \subseteq V_\lambda \oplus \cdots \oplus V_{\nu_p},$$

that is, each element of $X_n v_\lambda$ can be written as a sum of weight vectors with weights of depths up to $k(n)$. This implies $X_n v_\lambda \cap V(m) = \emptyset$ for all $m > k(n)$ and hence $\phi_{n,m}$ is one-one.

For $n \ge 1$, $k(n)$ as chosen above and by assuming $m > k(n)$, we get the following commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "\vartheta_{n,m}"] \arrow[d, "\phi_{n,m}"] & U^-[m] \arrow[dl, "\theta_m"] \\ V[m] & \end{tikzcd} \tag{9.12}$$

where θ_m is the map $g \mapsto g\overline{v_\lambda}$. Let $x, y \in X_n$ be such that $x \neq y$. If

$$\vartheta_{n,m}(x) = \vartheta_{n,m}(y), \tag{9.13}$$

then

$$\theta_m(\vartheta_{n,m}(x)) = \theta_m(\vartheta_{n,m}(y)). \tag{9.14}$$

Commutativity of the diagram (9.12) implies

$$\phi_{n,m}(x) = \phi_{n,m}(y) \tag{9.15}$$

which contradicts that $\phi_{n,m}$ is one-one. Thus $\vartheta_{n,m}$ is one-one.

Lemma 9.2.3. *There exists a sufficiently large n such that $X_n(\mu^\vee)$ is mapped onto $U^-_{\mathcal{O}}\backslash U^-(\mu^\vee)$.*

Proof. For every $r \geq 1$, by definition we have $X_r(\mu^\vee) \subset U^-(\mu^\vee)$. We will show that there exists a sufficiently large n such that the restriction π of the projection

$$U^-(\mu^\vee) \twoheadrightarrow U^-_{\mathcal{O}}\backslash U^-(\mu^\vee)$$

to $X_n(\mu^\vee)$ is our required surjective map. The last part of the Theorem 1.4.1 implies that we may choose $\mu^\vee \in -Q^\vee_+$. We replace K by its Iwahori decomposition $K = \cup_{w\in W} I^- w I^+$ to obtain

$$\begin{aligned} U^- \cap K\pi^{\mu^\vee}U^+ &= \cup_{w\in W} U^- \cap I^- w I^+ \pi^{\mu^\vee} U^+ \\ &= \cup_{w\in W} U^- \cap I^- w U^-_{\pi} U^+_{\mathcal{O}} \pi^{\mu^\vee} U^+ \\ &= \cup_{w\in W} U^- \cap I^- w \pi^{\mu^\vee} U^+ \\ &= \cup_{w\in W} U^- \cap U^-_{\mathcal{O}} U^+_{\pi} w \pi^{\mu^\vee} H_{\mathcal{O}} U^+. \end{aligned} \tag{9.16}$$

By Corollary 4.2.1, there are finitely many $\sigma_1, \sigma_2, \ldots, \sigma_s \in W$ such that

$$U^- \cap K\pi^{\mu^\vee}U^+ = \cup_{i=1}^s U^- \cap U^-_{\mathcal{O}} U^+_{\pi} \sigma_i \pi^{\mu^\vee} H_{\mathcal{O}} U^+.$$

Set $n = Max\{\ell(\sigma_i)\}_{1\leq i\leq s}$.

Let $\bar{y} = U_{\mathcal{O}}^{-} y \in U_{\mathcal{O}}^{-} \backslash U^{-}(\mu^{\vee})$, where $y = u_{\mathcal{O}}^{-} u_{\pi} \sigma_i \pi^{\mu^{\vee}} h_{\mathcal{O}} u$ for some $u_{\mathcal{O}}^{-} \in U_{\mathcal{O}}^{-}$, $u_{\pi} \in U_{\pi}^{+}, h_{\mathcal{O}} \in H_{\mathcal{O}}$, $u \in U^{+}$ and $\sigma_i \in W$ with $1 \leq \ell(\sigma_i) \leq n$. Suppose $z = u_{\pi} \sigma_i \pi^{\mu^{\vee}} h_{\mathcal{O}} u$ then $z \in U^{-}(\mu^{\vee}) \cap B^{+} \sigma_i B^{+} \subset X_n(\mu^{\vee})$ and $\pi(z) = \bar{y}$. Thus, the restriction of π which we also denote by π is the required onto map.

Definition 9.2.4. *For $m \geq 1$, set*

$$U^{-}[m]_{\mathcal{O}} := \{u^{-} \in U^{-}[m] \mid u^{-}(V[m]_{\mathcal{O}}) \subset V[m]_{\mathcal{O}}\}. \tag{9.17}$$

(**F**) For the rest of this section, we fix $n \geq 1$ as given in Lemma 9.2.3 and m be as given in Lemma 9.2.2.

Definition 9.2.5. *We define a relation $\sim$ on X_n as: for $x, y \in X_n$, $x \sim y$ if and only if there exists some $u^{-} \in U^{-}[m]_{\mathcal{O}}$ such that*

$$\vartheta_{n,m}(x) = u^{-} \vartheta_{n,m}(y),$$

where $\vartheta_{n,m}$ is the map (9.10).

It can be verified easily that $\sim$ is an equivalence relation on X_n. Let $\widehat{X}_n := X_n/\sim$ be the quotient space under this equivalence relation. Then

$$\widehat{X}_n(\mu^{\vee}) \subset \widehat{X}_n$$

Lemma 9.2.6. *Let $\bar{\pi} : \widehat{X}_n(\mu^{\vee}) \longrightarrow U_{\mathcal{O}}^{-} \backslash U^{-}(\mu^{\vee})$ be defined as*

$$\bar{\pi}([x]) := \pi(x); \quad [x] \in \widehat{X}_n(\mu^{\vee}),$$

where π is the projection obtained in Lemma 9.2.3. Then $\bar{\pi}$ is an onto map.

Proof. It is enough to show that $\bar{\pi}$ is well defined since surjectivity of π implies the surjectivity of $\bar{\pi}$. Let $x \in X_n(\mu^\vee)$ be such that $x \sim 1$. We shall show that $x \in U^-_{\mathcal{O}}$. By the assumption

$$\vartheta_{n,m}(x) = u^- \vartheta_{n,m}(1) \tag{9.18}$$

for some $u^- \in U^-[m]_{\mathcal{O}}$. It implies $\vartheta_{n,m}(x) \in U^-[m]_{\mathcal{O}}$ and hence

$$\vartheta_{n,m}(x) V[m]_{\mathcal{O}} \subset V[m]_{\mathcal{O}}. \tag{9.19}$$

In particular

$$\theta(\vartheta_{n,m}(x)) = \overline{\vartheta_{n,m}(x)(v_\lambda)} \in V[m]_{\mathcal{O}}, \tag{9.20}$$

where θ is the map defined in Lemma 9.2.2. The diagram (9.12) commutes, so we get

$$\phi_{n,m}(x) = x\bar{v}_\lambda \in V[m]_{\mathcal{O}}. \tag{9.21}$$

Now, since m is chosen such that $yv_\lambda \cap V(m) = \{0\}$ for all $y \in X_n$. Therefor $xv_\lambda \in V_{\mathcal{O}}$ and by Lemma 5.16 of [6], $x \in U^-_{\mathcal{O}}$. This completes the proof.

9.2.3 Proof of Main Results

For a positive integer b, let $U^-[m]_b$ be the set of uniformly bounded elements (bounded by b) inside $\mathrm{Aut}(V[m])$ as in the previous section.

Conjecture 9.2.7. *There exists a positive integer b such that the set $\widehat{X}_n(\mu^\vee)$ is* *embedded in the quotient* $U^-[m]_{\mathcal{O}}\backslash U^-[m]_b$

Remark 9.2.8. *We believe this conjecture is true and tried to prove the assertion using the strategy from [6]. By this method, one can prove the statement by showing:*

S1: $\omega_m(\widehat{X}_n(\mu^\vee))$ *is finitely generated.*

S2: *Express the elements of* $\omega_m(\widehat{X}_n(\mu^\vee))$ *as an ordered product.*

We only succeeded to realize $\omega_m(\widehat{X}_n(\mu^\vee))$ *inside* $U^-[m]$ *as a set of elements generated by the root subgroups corresponding to the roots of heights less than or equal to* m. *The set of such roots is finite. But we don't know how to get the ordered presentation of elements of* $\omega_m(\widehat{X}_n(\mu^\vee))$.

Proof of the Proposition 9.0.1. Let n and m be as chosen in (**F**). The group $U^-[m]$ is a finitely generated group; Proposition 9.1.3 implies that the quotient space $U^-[m]_{\mathcal{O}}\backslash U^-[m]_b$ is finite. By assuming the Conjecture 9.2.7 and combining it with Lemma 9.2.6, we obtain the following diagram of maps,

$$\begin{tikzcd} X_n(\mu^\vee) \arrow[r, two heads, "\pi"] \arrow[d, two heads, "\pi_1"] & U^-_{\mathcal{O}}\backslash U^-(\mu^\vee) \\ \widehat{X}_n(\mu^\vee) \arrow[r, hook] \arrow[ur, "\bar{\pi}"] & U^-[m]_{\mathcal{O}}\backslash U^-[m]_b \end{tikzcd} \tag{9.22}$$

So, we get an onto map from a subset of the finite set $U^-[m]_{\mathcal{O}}\backslash U^-[m]_b$ to the the quotient $U^-_{\mathcal{O}}\backslash U^-(\mu^\vee)$, which implies the finiteness of $U^-_{\mathcal{O}}\backslash U^-(\mu^\vee)$.

Proof of Theorem 1.4.1 . Proposition 9.0.1 implies that the coset space $K\backslash KU^- \cap K\pi^{\mu^\vee}U$ is finite for any $\mu^\vee \in \Lambda^\vee$. For any $\lambda^\vee \in \Lambda^\vee$, there is a bijection of the sets

$$KU^- \cap K\pi^{\mu^\vee}U = K\pi^{\lambda^\vee}U^- \cap K\pi^{-\lambda^\vee+\mu^\vee}U. \tag{9.23}$$

Thus for any $\mu^\vee$ and $\lambda^\vee$, the set $K\backslash K\pi^{\lambda^\vee}U^- \cap K\pi^{-\lambda^\vee+\mu^\vee}U$ is finite. Though the second part of Theorem 1.4.1 follows exactly as it does in affine, for the sake of completion we rewrite it. Let $u^- \in U^-$ be such that

$$\pi^{\lambda^\vee}u^- = k\pi^{\mu^\vee}u \tag{9.24}$$

for some $k \in K$ and $u \in U^+$. We let the both sides act on the highest weight vector v_ρ of the highest weight module V^ρ and compute their norms. The right hand side of (9.24) gives

$$||k\pi^{\mu^\vee}uv_\rho|| = q^{-\langle\rho,\mu^\vee\rangle}, \tag{9.25}$$

whereas the action of the left hand side of (9.24) and Lemma 4.1.1 give

$$||\pi^{\lambda^\vee}u^-v_\rho|| \geq q^{-\langle\rho,\lambda^\vee\rangle}. \tag{9.26}$$

Comparing both norms, we get $\langle\rho,\lambda^\vee\rangle \geq \langle\rho,\mu^\vee\rangle$ and hence $\lambda^\vee \geq \mu^\vee$.

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