Topological Recursion and Quantum Airy Structures: Titans of Geometry and Physics

by

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Abstract

Topological Recursion began its life as a series of recursive equations aimed at solving constraints which occur in matrix models of Quantum Field Theory. After its inception, Topological Recursion was given a more abstract formulation in terms of Quantum Airy Structures and has since been of help to Gromov-Witten theory, the study of Quantum Curves, enumerative geometry, integrable systems, Hurwitz Theory, and Knot Theory, for example, by revealing a common structure within the solutions to all of these varied problems. We recount the history of Quantum Airy Structures, present the cornerstone theorems upon which their theory depends, and show that these theorems remain true when passing to increasingly broad generalizations. Many of the interesting connections which ignited and maintain engagement with Quantum Airy Structures are put on display.

For my family.

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I am indebted to many. This is more certain than any of the mathematical results that follow. I must thank Vincent for his guidance, answers, questions, patience, enthusiasm, and outlook. Steve and Jacek, as well, as it has been their endorsement that has gotten me along this far. My colleagues at the University of Alberta deserve a mention, for their insights, but more importantly for all the time we spent recuperating from trying to be insightful. I have had so many friends and family around me that have been shockingly supportive, and this has been crucial, especially as my life has become new and unknown. My parents are among these, of course. I hope that others can tell by knowing me just how large and positive their impact has been on my life. Considering all of these people and all of their help, I begin to wonder what role, if any, I myself had in any of this.

I don't have the time to repay all of these debts, so I'll just do a little good and we'll call it even.

Table of Contents

Chapter 1

Introduction

1.A Motivation

In enumerative geometry, one frequently encounters *Virasoro constraints*. Whatever sequence of invariants $F_n[\alpha_1, \dots, \alpha_n]$ is in question, it is worthwhile to consider its generating series $F = \sum_{n=0}^{\infty} \sum_{\alpha_i=0}^{\infty} F_n[\alpha_1, \dots, \alpha_n] x_{\alpha_1} \cdots x_{\alpha_n}$ or the exponential $Z := e^F$. Whatever recursion exists among the F_n, or whatever relations they satisfy in virtue of their combinatorial meaning, can often be written as a list of differential equations $\mathcal{H}_i Z = 0$. Generically, the operators $\mathcal{H}_i$ form a representation of some subalgebra of the famous *Virasoro* algebra.

One may turn this around, asking the following question: supposing instead that you had in hand a collection of operators $\mathcal{H}_i$, do there exist some conditions on them that would guarantee a solution Z of $\mathcal{H}_i Z = 0$ to exist and to be unique? If so, would the resulting Z have coefficients with any special meaning? Topological Recursion, in the form developed by Kontsevich and Soibelman [31], can be viewed as answering in the positive. The collection $\{\mathcal{H}_i\}$ must form a *Quantum Airy Structure*, which is (along with some other data) a Lie algebra represented as differential operators. Then Topological Recursion is an iterative algorithm that begins with any Quantum Airy Structure and produces the unique solution Z.

Asking such a question, and being hopeful about its answer, had been motivated by Witten's Conjecture. Through his belief that two different approaches to two-dimensional quantum gravity must be equivalent, Witten conjectured [40] a relation between the hierarchy of Korteweg-deVries equations and the intersection theory on a moduli space of Riemannian manifolds. The Korteweg-deVries hierarchy is a series of differential equations, all of which are integrable, meaning that solutions admit descriptions in terms of a tau-function [17]. The content of Witten's Conjecture is that at least one tau-function for the KdV hierarchy is also the generating function for intersection numbers on $\overline{\mathcal{M}}_{g,n}$, the [compactified] moduli space of genus-g Riemann surfaces with n marked points. This conjecture was proven by Kontsevich in 1992 [30]. The more general problem outlined above, then, solved by Topological Recursion, can be seen as a generalization of Witten's Conjecture.

For the original Quantum Airy Structures the operators were restricted to have degree at most two, but this requirement was lifted in a recent paper that named the resulting possibilities "Higher Quantum Airy Structures" [8]. Higher Quantum Airy Structures can arise in connection with larger so-called $\mathcal{W}$-algebras [4, 25], extensions of the Virasoro algebra, and once again the function Z can be constructed. The additional generality allows for the coefficients of such a Z to capture a wider array of enumerative specimens. The story of Higher Quantum Airy Structures is related to the study of Vertex Operator Algebras (VOAs) [8, 33], objects that are implicated in Borcherd's proof of the Monstrous Moonshine conjecture [7, 26] and have made their way into the foundations of String Theory [35]. When Borot et al defined the Higher Quantum Airy Structures [8] they constructed a number of examples, all of them arising in the same way: from modules of a VOA. In fact, all examples descended from modules of one: the Heisenberg VOA.[1]

Topological Recursion and Higher Quantum Airy Structures are at the centre of broad questions concerning models of gravity [24], mirror symmetry [20], the passage from classical to quantum physics [31], integrable systems, matrix integrals [21], and enumerative geometry of all kinds [23]. The additional generality afforded by them over Quantum Airy Structures has begun to see its need in many new contexts.

1.B Outline

The most important theorem concerning Quantum Airy Structures is the existence and uniqueness of a power series Z, the Partition Function, which is annihilated by all its members. To be precise, we have:

Theorem 1. *Let* $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$ *be a Quantum Airy Structure. Then, among all* $Z = e^F$ *with* $F \in \mathcal{S}^\hbar$ *of the form* (1.1)*, there is a unique solution to the system of equations* $\hat{H}_i Z = 0$ *for* $i \in I$*.*

In the theorem statement, $\mathcal{S}^\hbar$ is simply a space of formal power series and "Quantum Airy Structure" is made precise by Definition 4. Our paper reviews alternative proofs of Theorem 1, with increasing degrees of generality. There are two main proofs. One is conceptual [31], using symplectic methods and making contact with the geometric notions at the heart of classical physics. The other is a more blindly computational proof. We present the computational proof twice, both a simpler version [3] which only proves a special case, and also a more complicated version [8] for full generality.

[1]Strictly speaking, from modules of orbifolds of a Heisenberg VOA.

A novel step that we take is to introduce a purely notational alteration. This simplifies some expressions, but most importantly it makes apparent a massive simplification of the conceptual proof, which in fact is also a generalization. Our new convention is a rescaling of variables. The Partition Function, Z, is a formal Laurent series in $n+1$ variables $\{\hbar, x_1, x_2, \ldots, x_n\}$, one of which is singled out. In its connection to geometry, each term admits visualization as a certain marked Riemann surface. The special variable's ($\hbar$'s) duty is to bear as its exponent the genus of that surface, while the other variables are related to marked points. The partition function can be written:

$$Z := e^F := \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{g-1}}{n!} F_{g,n}[\alpha] x_{\alpha_1}\cdots x_{\alpha_n}\right]$$

For some coefficients $F_{g,n}[\alpha] \in \mathbb{C}$. Naturally enough, when we begin solving the constraints upon this F we find that $\chi := 2-2g-n$, the Euler characteristic of the corresponding surface, emerges as extremely important. Knowing this, that χ is more important than g or n separately, we opt to rewrite F so that $\hbar$ instead bears $-\chi$ in its exponent. We write:

$$Z := e^F := \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_{\alpha_1}\cdots x_{\alpha_n}\right] \tag{1.1}$$

Expressions of this form do not have the same extension as expressions of the earlier form without also re-defining the variables, but they are in correspondence, and we do the work of carrying all things through the correspondence. Because the parameters g and n have been mixed together in this way, what in [31] had been a double-induction over both becomes, in our presentation, a single induction over $-\chi$.

Not only this, but our adaptation skips entirely all mention of symplectic geometry, Lagrangian manifolds, and their cronies. The presence of those objects is important to Kontsevich and Soibelman, whose aim in part is to draw tight connections between quantum and classical. The absence of them, however, is important as well because our proof can establish its result for a wider class of Quantum Airy Structures than those in Definition 4.

The Quantum Airy Structures presented in Definition 4 are collections $\{\hat{H}_i\}_{i\in I}$ of power series over $\mathbb{C}$ in $\hbar$, x_i, and $\partial/\partial x_i$ (for $i \in I$). Among other requirements, they must have a certain form:

$$\hat{H}_k = -\hbar\frac{\partial}{\partial x_i} + \sum_{m\geq 2} \hbar^m P_{m,k} \tag{1.2}$$

The constant and linear terms [in $\hbar$] are completely prescribed. The $P_{m,k}$ are power series over $\mathbb{C}$ in x_i, ∂_i having degree less than or equal to m, so that all terms can be realized as polynomials in $\hbar x_i, \hbar\partial_i$ over $\mathbb{C}[[\hbar]]$. The significance of this requirement will emerge when we discuss the classical limit, but it is exactly what we have lifted in our generalization. If "Quantum Airy Structure" is relaxed so that the $P_{m,k}$ of (1.2) have no limitations at all on their degree, allowing even that they have infinite degree, we can still prove the following:

Theorem 2. *Suppose we have a Quantum Airy Structure $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$. Then, among all*

$$Z = \exp\left[\sum_{g\in\frac{1}{2}\mathbb{Z}} \sum_{n\in\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha\right]$$

there is a unique solution to the system of equations $\hat{H}_i Z = 0$, $i \in I$.

Unsurprisingly, the additional structures falling under our broadened scope all have a trivial classical limit. This original result occurs in Chapter 4, and its consequences are taken up in Chapter 5. In Chapter 2 we review some background, including a history of Quantum Airy Structures, their existence/uniqueness theorem, their significance to other areas, and a few methods of their construction. Chapter 3 reviews what is known about the special case of quadratic Quantum Airy Structures. Chapter 6 concludes.

Chapter 2

Background

2.A History

A Quantum Airy Structure is a collection of differential operators with a mild prescription as to their form as well as a closure property regarding their commutators. Their utility inheres in their relationship to Topological Recursion, which was discovered initially as a way to solve the loop equations of a matrix model [19, 21–23].

Matrix models can be viewed variously as zero-dimensional Quantum Field Theories, as realizations of quantum gravity via random surfaces, or as representing other stochastic processes. Shorn of context, one common type of matrix model boils down to the study of some integral with the form [34]:

$$Z_N := \int_{N\times N \text{ Hermitian } M} e^{-N\mathrm{tr}V(M)/t}dM \tag{2.1}$$

for some analytic function V (the *potential*), which we take to be a polynomial.[1] When the integral converges, its integrand is taken as giving the relative probability that its argument, M, be drawn from the urn. When the integral does not converge one may take it to define, instead of a number, a formal power series in N/t [34].[2][3] In this case the connection to probability is strained, despite its being the case generically for quantum field theories. Either way, it has been recognized as the generating function which enumerates closed, discrete surfaces. We will see an example of this in section 2.D.1. As is quite usual in combinatorics, the logarithm $F_N := -\log(Z_N)$ enumerates *connected* discrete surfaces. This logarithm admits what is known as a *topological expansion*, $F_N = \sum_{g=0}^{\infty}(N/t)^{2-2g}F_g$, so called because the F_g count the connected, discrete surfaces of genus g.

In Quantum Field Theories one encounters the Schwinger-Dyson equations, an infinite family of constraints that relate the theory's correlation functions to one another. In the case of zero-dimensional Quantum Field Theory – i.e. a matrix model – these equations are instead known as the Loop equations [18, 34]. We define first the correlation functions:

$$W_n(x_1, \dots, x_n) := \left\langle \prod_i \mathrm{Tr}(Ix_i - M)^{-1} \right\rangle$$

in which the expected value is taken using the probability measure described earlier.[4] Giving names to the coefficients of W_n's N-expansion,

$$W_n(x_1, \dots, x_n) := \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-n} W_{g,n}(x_1, \dots, x_n)$$

[1]The measure, dM, is that induced by the Haar measure on $U(N)$ when realizing the Hermitian matrices as a symmetric space.

[2]This is accomplished by non-rigorously commuting the integration with the Taylor expansion.

[3]At best, it can be taken to be an asymptotic series.

[4]This "expected value" may be purely formal.

and defining also:

$$P_n(x_1,\ldots,x_n) := \left\langle \mathrm{Tr}\left[(V'(x_1I)-V'(M))(x_1I-M)^{-1}\right] \prod_{i=2}^{n} \mathrm{Tr}(x_iI-M)^{-1} \right\rangle$$
$$:= \sum_{g=0}^{\infty} \left(\frac{N}{t}\right)^{2-2g-n} P_{g,n}(x_1,\ldots,x_n)$$

we can express the Loop Equations as:

$$V'(x)W_{g,n+1}(x,J) = P_{g,n+1}(x,J) + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \frac{W_{g,n}(x,J\backslash x_j) - W_{g,n}(J)}{x-x_j}$$
$$+ \sum_{h=0}^{g} \sum_{I\subset J} W_{h,|I|+1}(x,I) W_{g-h,1+n-|I|}(x,J\backslash I) \qquad (2.2)$$

These equations result as consistency conditions after demanding that the matrix integral (2.1) retains a constant value throughout a continuous change in integration parameter. This explanation leaves us without reason to think that (2.2) holds when it describes a purely formal power series, and yet this is indeed true (Theorem 3.1 in [34]).

For $n = 0$ and $g = 0$, this reads $(W_{0,1}(x))^2 = V'(x)W_{0,1}(x) - P_{0,1}(x)$. Since $W_{0,1}$ is a function it represents only one solution to that quadratic equation. We may instead construe $W_{0,1}$ and x as bearing a two-to-one relation described by the algebraic equation $y^2 - (V'(x)/2)^2 + P_{0,1}(x) = 0$, which is implied by the previous equation on $W_{0,1}$ if we take $y = \frac{1}{2}V'(x) - W_{0,1}(x)$. The natural setting for $W_{0,1}$, and all the later correlations, is in fact a two-sheeting Riemann surface $x : \mathcal{L} \to \mathbb{CP}^1$ covering the Riemann sphere. This surface is known as the *Spectral Curve* associated to the matrix model. This is preferable, since the Loop Equations do not uniquely constrain any of the $W_{g,n}$. Each $W_{g,n}$ – if taken to be defined *only* by the Loop Equations – will end up being a multi-valued function of x unless we reinterpret it as a function on $\mathcal{L}$ instead. One can show ([34], section 4.5) that each $W_{g,n}$ gives a well-defined meromorphic differential on $\mathcal{L}^n$ according to:

$$\omega_{g,n}(z_1,\ldots,z_n) := W_{g,n}(x(z_1),\ldots,x(z_n))dx(z_1)\cdots dx(z_n) \\ + \delta_{n,2}\delta_{g,0}\frac{1}{(x(z_1)-x(z_2))^2}dx(z_1)dx(z_2)$$

with a minor correction that is necessary for $\omega_{0,2}$. We can solve the Loop Equations if we cast them as a recursion for the $\omega_{g,n}$, which will themselves be uniquely determined thereby.[5] The equations which result, now called Topological Recursion, are: [34]

$$\omega_{g,n}(z_0,J) = \sum_i \mathrm{Res}_{z\to a_i} K(z_0,z)\left[\omega_{g-1,n+2}(z,\overline{z},J) + \sum_{h=0}^{g}\sum_{I\subset J}{}' \omega_{h,|I|+1}(z,I)\omega_{g-h,1+n-|I|}(\overline{z},J\backslash I)\right] \tag{2.3}$$

This requires some information about $\mathcal{L}$. The i-sum is taken over the ramification points a_i of $x:\mathcal{L}\to\overline{\mathbb{C}}$; these are the points where sheets cross, or equivalently $dx(a_i)=0$. The automorphism $z\mapsto\overline{z}$ is the non-trivial operation such that $x(z)=x(\overline{z})$; in words, it swaps the sheets. Finally, with $y:\mathcal{L}\to\mathbb{CP}^1$ such that $\{x,y\}$ generate the function field on $\mathcal{L}$,[6] the *recursion kernel* is:

$$K = \frac{-\int_{z'=\overline{z}}^{z'=z} B(z_0,z')}{2(y(z)-y(\overline{z}))dx(z)}$$

with B the Bergmann kernel.[7] The recursion is initialized via $\omega_{0,1}(z) = -y(z)dx(z)$

[5] Relative to a choice of basis for $\mathcal{L}$'s first homology [23].

[6] If $\mathcal{L}$ is realized within $\mathbb{C}^2$ as the Riemann surface associated to some multi-valued function, then y is simply the horizontal projection (i.e., the function itself).

[7] The Bergmann kernel is a bidifferential that is idiosyncratic to $\mathcal{L}$, and depends upon a choice of basis for its first homology, with respect to which it is "normalized" [23]. It is, among other things, a Reproducing Kernel for the space of multi-differentials on $\mathcal{L}$ – with a suitable inner product, the linear functional it defines coincides with point-evaluation.

and $\omega_{0,2}(z_1, z_2) = B(z_1, z_2)$.

Part of this formalism's massive utility is that it easily delivers symplectic invariants of the spectral curve $\mathcal{L}$. For each $g \geq 2$, and for any local primitive Φ of ydx, the following *Free Energies*

$$\omega_{g,0}(\mathcal{L}) := \frac{1}{2-2g} \sum_i \text{Res}_{z \to a_i} \Phi(z) \omega_{g,1}(z)$$

are invariants of any conformal mapping $(\mathcal{L}, x, y) \mapsto (\tilde{\mathcal{L}}, \tilde{x}, \tilde{y})$ as long as $dx \wedge dy = d\tilde{x} \wedge d\tilde{y}$ [19, 23, 34].

Topological Recursion, however, has much broader scope. Many Riemann surfaces $\mathcal{E}$ may be inserted into it, even those that are not the spectral curve of any matrix model. The $\omega_{g,0}$ are still symplectic invariants, and the $\omega_{g,n}$ often compute important geometric quantities such as intersection numbers [23], Hurwitz numbers [9, 10], and Weil-Petersson volumes [20]. The only requirement we ask is that $\mathcal{E}$ has a realization as a branched cover $x : \mathcal{E} \to \mathbb{CP}^1$ such that only two sheets meet at any ramification point. This allows us to define the involution $z \mapsto \overline{z}$, but only locally near each ramification. In fact, this restriction has been lifted [13]. Also, the Topological Recursion has been given a manifestly global formulation [11]. The generalization resulting from these works is known as the Bouchard-Eynard topological recursion, in contrast to the earlier Checkhov-Eynard-Orantin recursion.

We do not have to work with differentials, if we prefer. It can be seen inductively that each $\omega_{g,n}$ has poles only at the ramification points, a_i. This gives a privileged basis of differentials, with poles at the ramifications, into which the $\omega_{g,n}$ can be expanded with coefficients $F_{g,n}[k_1, \ldots, k_n]$. With them, we construct the *Partition*

Function:

$$Z := \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0}\ \sum_{k\in I^n}\frac{\hbar^{2g-2+n}}{n!}F_{g,n}[k_1,\ldots,k_n]x_{k_1}\cdots x_{k_n}\right]$$

which, in this case, does not come from any matrix model.[8] The topological recursion for the $\omega_{g,n}$ can be recast equivalently as a system of differential equations, $\mathcal{V}_a^i Z = 0$, in which a indexes the ramification points and i is a natural number. As shown in [3, 8, 23, 31], it turns out that for each a the collection $\{\mathcal{V}_a^i\}$ forms a representation of a subalgebra of the Virasoro algebra.

This is all very non-trivial, although the general outline is easy to believe. Stringing up recursive sequences into power series and giving their recursion the guise of a differential constraint is a standard combinatorial gambit, and Virasoro constraints have been endemic to the study of matrix models for a very long time. (However, it is worth noting that the Virasoro constraints obtained here are not the same as the traditional constraints satisfied by partition functions of matrix models.[9]) This last formulation, though, is the one that the remainder of our paper will most closely resemble. To get, at last, to Quantum Airy Structures, we need only ask a question inverse to the one just contemplated: if you happened to have on hand a collection of differential operators $\{\mathcal{V}_i\}$, what would it take to guarantee that there exists a solution to the equations $\mathcal{V}_i Z = 0$ for all i? What would it take for that solution to be unique? Thanks to the Existence and Uniqueness theorem (Theorem 2.4.2, [31]), it suffices that $\{\mathcal{V}_i\}$ form a Quantum Airy Structure. Fantastically, just as Topological Recursion continues to prove invaluable outside of its connection to

[8]In fact, if you began with a partition function defining a matrix model, derived the corresponding $F_{g,n}$ via topological recursion on its spectral curve, and placed them into the above Z, you *would not* have the partition function that you began with.

[9]We have already noted that our present usage of “partition function" is distinct from that in matrix models.

matrix models, Quantum Airy Structures continue to produce solutions Z that are rich with geometric content despite, a priori, there being no longer a reason for this.

Recently, a significant generalization of Quantum Airy Structure has been introduced by Borot et al in [8]. One boon of this innovation is that their "Higher Quantum Airy Structures" bear the same relationship as did the original QASs to Virasoro constraints, although they also make contact with more general $\mathcal{W}$-constraints [4, 25]. A $\mathcal{W}$-algebra is an extension of the Virasoro algebra, finding purpose in multiple areas of enumerative geometry as well as physics [27]. The paper [8] gives constructions of several $\mathcal{W}$-algebras and provides examples of their significance, which include applications to Fan-Jarvis-Ruan theories, r-spin intersection numbers, and Brezin-Gross-Witten theory (see section 6 therein).

2.B Lemmas and Definitions

In this paper we always take $\mathbb{N}$ to be the strictly positive integers. We will use $\mathbb{N}_0$ to denote the non-negative integers.

Definition 1. *Fix a finite or infinite indexing set, $I := \{1, 2, 3, \ldots, N\}$ or $I = \mathbb{N}$, for the remainder of the paper and set:*

$$\mathcal{S} := \mathbb{C}[[\{x_i\}_{i\in I}]]$$
$$\mathcal{S}^{\hbar} := \mathbb{C}[[\hbar]][[\{x_i\}_{i\in I}]]$$

Define the operators x_i and ∂_i on $\mathcal{S}, \mathcal{S}^{\hbar}$ as follows:

- $x_i \cdot F := x_i F$
- $\partial_i \cdot x_j := \delta_{i,j}$*, and each ∂_i satisfies the Leibniz Law along with $\mathbb{C}[[\hbar]]$-linearity*

Lastly, set:

$$\mathcal{O} := \mathbb{C}[[\{x_i, \partial_i\}_{i\in I}]]$$
$$\mathcal{O}^\hbar := \mathbb{C}[[\hbar]][[\{x_i, \partial_i\}_{i\in I}]]$$

in which it is understood that x_i and ∂_i are not commuting variables, but instead satisfy $[x_i, \partial_j] = \delta_{ij}$. The rings $\mathcal{O}$, $\mathcal{O}^\hbar$ are free otherwise.

These distinctions will allow us to leave $\hbar$-dependence implicit or make it explicit. The role that $\hbar$ plays in this paper is that which was played by (N/t) in the previous section: it is the formal variable utilized in our generating functions to hang the genus on. Additionally, in the following, for any multi-index $J \in I^n$ we will write $x_J := \prod_{j\in J} x_j$ and $\partial_J := \prod_{j\in J} \partial_j$.

Definition 2. *For polynomials $F \in \mathcal{S}, \mathcal{S}^\hbar, \mathcal{O}, \mathcal{O}^\hbar$ let $\hbar$-deg(F), x-deg(F), and ∂-deg(F) refer to the degree in $\hbar$, the degree in all x, and the degree in all ∂, respectively.*[10] *For $O \in \mathcal{O}, \mathcal{O}^\hbar$, the "degree" or deg$(O)$ with no qualifications is the total degree in all x and ∂, or x-deg $+$ ∂-deg.*

Remark 1. *Note that for $O \in \mathcal{O}, \mathcal{O}^\hbar$ the degree does not coincide with its degree as a homogeneous operator on $\mathcal{S}, \mathcal{S}^\hbar$.*

Definition 3. *The degree on polynomials in $\mathcal{O}$ gives them a natural $\mathbb{Z}_2$-grading, $\mathcal{O}_0 \oplus \mathcal{O}_1 \subset \mathcal{O}$, with $\mathcal{O}_0$ and $\mathcal{O}_1$ being spanned by the monomials of odd and even degree, respectively.*[11]

We will also make use of operators $[\hbar^m] : F \mapsto [\hbar^m]F$ on $\mathcal{S}^\hbar$ which extracts the

[10]The ∂-degree is not well-defined. In actuality, it refers to the number of ∂ factors in any representation of F for which all ∂ occur to the right of all x in all terms.

[11]The indexes here may be any integer, but are taken modulo 2.

$\hbar$-degree m term. When we want to extract the x-constant term of F we will write $F|_0$.

Definition 4. *A* ***Quantum Airy Structure in Normal Form*** *is a collection* $\mathbb{H} = \{\hat{H}_k\}_{k\in I} \subset \mathcal{O}^\hbar$ *of polynomial*[12] *operators such that:*

1. $\hat{H}_k = -\hbar\partial_k + \sum_{m=2}^{D_k<\infty} \hbar^m P_{m,k}$ *for each* k*, where* $P_{m,k} \in \mathcal{O}_m$ *has degree* $\leq m$

2. *With* $\mathcal{O}^\hbar \cdot \mathbb{H}$ *the left ideal generated by* $\mathbb{H}$ *and* $[\mathcal{O}^\hbar \cdot \mathbb{H}, \mathcal{O}^\hbar \cdot \mathbb{H}]$ *the collection of all* $[s, s']$ *for* $s, s' \in \mathcal{O}^\hbar \cdot \mathbb{H}$*, we have* $[\mathcal{O}^\hbar \cdot \mathbb{H}, \mathcal{O}^\hbar \cdot \mathbb{H}] \subset \hbar^2\mathcal{O}^\hbar \cdot \mathbb{H}$

We may refer to the former of these as the "degree condition" and the latter as the "subalgebra condition." What we call a Quantum Airy Structure is what [8] introduced as a Higher Quantum Airy Structure. This is not the standard definition; for the correspondence, see section 2.E.

Definition 5. *We collect a few generalizations and special cases.*

1. *A* ***Quantum Airy Structure*** *(QAS) with "Normal Form" omitted refers to any collection* $\{\hat{K}_i\}_{i\in I} \subset \mathcal{O}^\hbar$ *which can be "diagonalized" into normal form, which is to say that there exists a QAS in Normal Form* $\{\hat{H}_i\}_{i\in I}$ *generating the same left-ideal in* $\mathcal{O}^\hbar$*.*

2. *The special case that our* $\hat{H}_i$ *contain only* $P_{m,k}$ *with* $m = 2$ *will be referred to as the* ***Quadratic Case****. This is the only case that was originally defined [31], and was referred to simply as a Quantum Airy Structure.*

3. *If we drop the requirement that* $P_{m,k}$ *belongs to* $\mathcal{O}_m$ *(that being equivalent to* $P_{m,k}$*'s degree having the same parity as* m*), we obtain what are known as*

[12] Not power series.

Cross-Capped Airy Structures. *In the original conventions (refer to section 2.E), this amounts to the allowance of half-integer* $\hbar$ *powers.* [13]

Definition 6. *The solution* $Z = e^F$ *to* $\hat{H}_i \cdot Z = 0$ *(if it exists) is known as the* ***Partition Function*** *of the Quantum Airy Structure, and its exponent* F *is known as the* ***Free Energy***. *The only restriction we place on the Free Energy when seeking solutions is that* $F \in \mathcal{S}^\hbar$ *and that it have no* $\hbar$*-deg* $= 0$ *term or* x*-deg* $= 0$ *term. The most general form, then, for* F *is*

$$F = \sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha \tag{2.4}$$

for some coefficients $F_{g,n}[\alpha] \in \mathbb{C}$.

2.C Existence and Uniqueness Theorems

The existence and uniqueness theorem (Theorem 2.4.2, [31]) asserts that any Quantum Airy Structure has a partition function, and tells also when that partition function is unique. Precisely,

Theorem 3. *Suppose we have a Quantum Airy Structure* $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$. *Then, among all* $Z = e^F$ *with* $F \in \mathcal{S}^\hbar$ *having the form* (2.4)*, there is a unique solution to the system of equations* $\hat{H}_i Z = 0$ *for* $i \in I$.

The interest in Quantum Airy Structures comes primarily from this theorem, as it provides an association between QASs and formal power series. Formal power series bearing interesting enumerative coefficients will satisfy a collection

[13] The diagrammatic representation of topological recursion in this case necessarily involves the "cross-cap", a quotient surface with a fundamental group of odd rank (and thus containing "half of a hole").

of differential constraints, often Virasoro constraints. The reverse is also true: the partition functions assigned to Quantum Airy Structures frequently come with important quantities as their coefficients.

2.D Significance

As mentioned, the significance of matrix models, Topological Recursion and Quantum Airy Structures ranges over many disparate regions of geometry and physics. In enumerative geometry, instances of Topological Recursion can formulate the relations holding of the Weil-Petersson volumes [20, 23], or of the Hurwitz numbers [9, 10], or the intersection numbers on $\overline{\mathcal{M}}_{g,n}$[14] [23], as well as the quantities of discrete surfaces at a given genus [23].

The enumeration of discrete surfaces on its own has a natural application to physics, since "random surfaces" constitutes one approach to quantum gravity [28,29]. Topological Recursion can aid physics in other ways, however. The WKB method was developed by physicists (Wentzel, Kramers, and Brillouin) to solve Schrödinger's Equation with an asymptotic series in $\hbar$ [15, 32, 39], and in some cases this solution can be solved for order-by-order using a suitable Topological Recursion [12]. Quantum Airy Structures have been interpreted as a "quantization" of Classical Airy Structures [31], with quantization being a large and ongoing project for physicists and mathematicians alike. The partition function of a QAS has a WKB form, as it provides an asymptotic solution in $\hbar$ to the equations $\hat{H}_i \cdot Z = 0$.

[14]The compactification of the moduli space of genus-g Riemann surfaces with n marked points.

2.D.1 Enumeration of Surfaces

Here we intend to unearth Topological Recursion within the enumeration of discrete surfaces. This material can be found in section 7 of [23].

Consider producing a surface of genus g by gluing together n_3 triangles, n_4 squares, ..., and n_i i-sided polygons. We also include n polygons, the i-th having l_i sides, which are called boundaries. One can imagine that the boundaries represent holes in the surface, as if we had instead added a cycle that bounds no face. One of the edges in each boundary is considered marked. For each $v \in \mathbb{N}$, $\mathbb{M}_{g,n}(v)$ is the set of connected orientable surfaces of genus g produced by gluing polygons as mentioned, with n marked points and v vertices. We have, then (theorem 7.1, pg. 88 of [23]):

Theorem 4. *There are finitely many members of* $\mathbb{M}_{g,n}(v)$.

Proof. Computing the Euler characteristic using the number of vertices, edges, and faces will give the relation:

$$\frac{1}{2}\sum_{j\geq 3}(j-2)n_j + \frac{1}{2}\sum_{i=1}^{n} l_i = 2g-2+n+v$$

which forces all of the n_j and the l_i to be bounded. □

If we write:

$$W_{g,n}(x,t) := \frac{t}{x_1}\delta_{n,1}\delta_{g,0} + \sum_{v=1}^{\infty} t^v \sum_{S\in\mathbb{M}_{g,n}(v)} \frac{1}{|\mathrm{Aut}(S)|}\frac{t_3^{n_3(S)}\cdots t_i^{n_i(S)}}{x_1^{l_1(S)+1}\cdots x_n^{l_n(S)+1}}$$

then the generating function for genus-g surfaces that contain n marked faces that are l_i-gons is:

$$T^g_{l_1,\ldots,l_n}(t) = (-1)^n \mathrm{Res}_{x_1\to\infty} \cdots \mathrm{Res}_{x_n\to\infty} x_1^{l_1} \cdots x_n^{l_n} W_{g,n}(x,t) dx_1 \cdots dx_n$$

It is not entirely hard to obtain a recursion between the coefficients. Tutte [37, 38] observed that the elimination of a marked edge could result in one of three things:

1. If the marked edge bordered a non-boundary face, we simply have removed the latter and increased the side-count of this boundary. There is the same number of boundaries and the same genus.

2. If the marked edge belonged to two boundaries, we now have one fewer boundary and a boundary that has increased it's side-count. The genus is the same.

3. If a single marked face lies on both sides of our marked edge (i.e., the edge was a bridge across two blobs) then the boundary has become disconnected. This can disconnect the surface, giving two surfaces that share the total genus amongst themselves, or the surface can remain connected but goes down in genus (if the blobs had been connected by another bridge).

Formulating these possibilities more carefully, as Tutte did, would give a recursion for the $T^g_{l_1,\ldots,l_n}(t)$. Casting this recursion in terms of the $W_{g,n}(x,t)$ gives:

$$\begin{aligned} V'(x)W_{g,n}(x_1, L) = \\ & P_{g,n}(x_1, L) + \sum_{j=2}^{n} \frac{\partial}{\partial x_j} \frac{W_{g,n-1}(x_1, L\backslash j) - W_{g,n-1}(L)}{x_1 - x_j} \\ & + \sum_{h=0}^{g} \sum_{J\subset L} W_{h,|J|+1}(x_1, J) W_{g-h,n-|J|}(x_1, L\backslash J) + W_{g-1,n+1}(x_1, x_1, L) \end{aligned}$$

in which $P_{g,n}$ is a specific but complicated polynomial in x (see eq. 7-10 of [23]) and $V'(x) = x - \sum t_j x^{j-1}$. But these are the Loop Equations of a particular matrix model! Topological Recursion, applied to the corresponding spectral curve (which, recall, is nothing but the Riemann surface affiliated with the $n = 0$, $g = 0$ Loop Equation), would produce meromorphic differentials $\omega_{g,n}(x) = W_{g,n}(x)dx$ that completely solve these relations.

2.D.2 Quantization and WKB

In physics, the problem of *quantization* is that of providing a quantum description for a physical system with the knowledge of its classical description. An incredible amount of beautiful mathematics has emerged from this pursuit, which is perhaps surprising since there is not and cannot ever be a 1-to-1 function from classical models to quantum models. The methods that we have often provide a remarkably good best-guess, which may require slight modification in some cases or none in others. The first of these was Dirac's heuristic [16].

The state of the art in classical physics is to describe the passage of time as a foliation of some Poisson manifold $(\mathcal{P}, \{-,-\})$, often a cotangent bundle $T^*\mathcal{M}$, by dimension-one curves [1]. Each such curve gives the past and future history of one possible physical arrangement. This foliation must be of the form $\frac{d}{dt} = \{-, H\}$, with t the parameter along a curve and H some function $\mathcal{P} \to \mathbb{R}$ (the *Hamiltonian*). The Darboux theorem gives us, locally, coordinates $\{x_i, y_i, z_i\}$ such that the Poisson brackets are $\{x_i, y_j\} = \delta_{i,j}$ and all others zero. Following Dirac, we seek a complex representation (ρ, V) of the algebra of functions on $\mathcal{M}$,[15] which is to say that we

[15]Or, at least, the polynomial algebra generated by those special coordinate functions. Or, at least, some Poisson sub-algebra thereof.

desire $[\rho(f), \rho(g)] = i\hbar\{f, g\}\mathrm{id}_V$.[16] Ordinarily V is taken to be some Hilbert space of functions. The quantum dynamics, now, is given by $\hbar\frac{d}{dt}Z = \rho(H)Z$ for all $Z \in V$. If one tries to find a Z which is stationary in time, one looks to solve $\rho(H)Z = 0$. If Z is homogeneous in other ways, e.g. across space or through rotation, then one demands $\rho(O)Z = 0$ for additional functions, O. This problem is very similar to the problem of finding a Quantum Airy Structure's partition function.

Indeed, one perspective on Quantum Airy Structures is that they are a quantization of *Classical Airy Structures*.

Definition 7. *Let V be an $|I|$-dimensional vector space, and $\{x_i, y_i\}_{i\in I}$ a set of coordinates adapted to the canonical symplectic structure on $V \oplus V^*$. A* ***Classical Airy Structure*** *is a Lagrangian subspace of $V \oplus V^*$ given by $|I|$ equations $L_i(x, y) = 0$ such that:*

- *$L_i = y_i + P_i(x, y)$ for some polynomial P_i without linear or constant terms.*
- *The $\mathbb{C}$-span of the L_i is closed under $\{-,-\}$.*

Definition 8. *The* ***classical limit*** *of a Quantum Airy Structure is a Classical Airy Structure. Fixing V as well as adapted coordinates $\{x_i, y_i\}_{i\in I}$, it is defined as the zero locus of equations $H_i(x, y) = 0$ for $i \in I$. Here H_i is the polynomial obtained from $\hat{H}_i$ via the $\mathbb{C}$-ring homomorphism $\hbar x_i \mapsto x_i$ and $\hbar\partial_i \mapsto y_i$ (and excess factors of $\hbar$ mapped to zero).*

Given a classical Airy Structure, one important question is whether it can be realized as the image of any Quantum Airy Structure under its classical limit. As expected, should such a QAS exist, it cannot be unique.

[16]We have modified somewhat the bracket that is native to $\mathcal{P}$.

The significance of the partition function Z in this setting is that it gives a WKB solution to the differential equations $\hat{H}_i \cdot Z$. In others' conventions, to be in WKB form is to be written $Z(x) = \exp\left[\frac{1}{\hbar}\sum_{n=0}^{\infty}\hbar^n S_n(x)\right]$ and the classical limit is equivalently expressed as the zero-locus of $\lim_{\hbar\to 0} Z^{-1}\hat{H}_i \cdot Z = H(x, \partial S_0)$.[17] This explains the factor of $1/\hbar$: if a higher power were taken, the limit would be trivial; if a lower power, the limit would not exist.

2.E Dictionary to Standard Definitions

The notation that we use does not coincide with any of the existing literature. In other works the operators $\hat{H}_k$ of a Quantum Airy Structure are defined as certain polynomials in the operators x_i and $\hbar\partial_i$ ($i \in I$). This mirrors the "position-space representation" of the Canonical Commutation Relations, those being the equations[18] $[\rho(x_i), \rho(y_i)] = i\hbar\delta_{ij}$ from quantum mechanics, which serve to constrain a choice of unirep $\rho : \mathbb{C} \otimes_{\mathbb{R}} C^\infty(T^*\mathcal{M}, \mathbb{R}) \to \mathrm{End}(\mathcal{H})$. Here $\mathcal{M}$ is a finite-dimensional differentiable manifold and $\mathcal{H}$, a Hilbert space, is taken to be $L^2(\mathcal{M})$ in the position rep. This choice can be convenient, as the values[19] of $|f(x)|^2$ for $f \in L^2(\mathcal{M})$ have interpretation as probability densities over the outcomes – spatial locations – of interventions which localize the modeled system. However, any choices of representation for the Canonical Commutation Relations are ultimately equivalent[20]

[17]This yields that the classical limit, a Lagrangian submanifold, is given by the image of $x \mapsto (x, dS_x)$ for some function $S : V \to \mathbb{C}$. Proceeding backwards, from submanifolds of this form to functions solving relevant equations, is the starting point of Geometric Quantization [5].

[18]In canonical, or Darboux, coordinates

[19]There is not such a thing as a value for an element of an L^2 space. We mean the values that integration functionals take on f.

[20]One prefers to work with the Weyl Relations, non-rigorously the exponentiation of the commutation relations, but unburdened by issues of unboundedness. In this setting, one can make a precise statement: all strongly continuous unitary irreps of the Weyl Relations are unitarily equivalent, and their corresponding infinitesimal reps will implement the commutator relations. This is the content of the Stone von Neumann Theorem [36].

and so we have chosen instead $\hat{x}_i = \sqrt{\hbar} x_i$, $\hat{y}_i = \sqrt{\hbar}\partial_i$. This brings a degree of simplicity to certain expressions and theorems. To avoid many radicals, we also relabel $\hbar \mapsto \hbar^2$, giving $\hbar x_i$ and $\hbar\partial_i$. Our $\hat{H}_i$ are indeed polynomials (over $\mathbb{C}[[\hbar]]$) in these two.

We have also adjusted the free energy to match. In other literature the free energy is written [31]:

$$F = \hbar^{-1} \sum_{n\geq 3} \sum_{\alpha \in I^n} \frac{1}{n!} F_{0,n}[\alpha] x_\alpha + \sum_{(g,n)\in \frac{1}{2}\mathbb{N}\times\mathbb{N}} \sum_{\alpha\in I^n} \frac{\hbar^{g-1}}{n!} F_{g,n}[\alpha] x_\alpha$$

So that -1 is the only allowed negative $\hbar$ power. What prevents us from combining the two sums is that, to ensure unique solutions, we must search only amongst series whose $\hbar^{-1}$ part ($g = 0$) have no linear or quadratic terms.

Applying our conventions, we first send $\hbar \mapsto \hbar^2$ and then $x_i \mapsto \hbar x_i$, yielding:

$$\begin{aligned} F &= \hbar^{n-2} \sum_{n\geq 3} \sum_{\alpha \in I^n} \frac{1}{n!} F_{0,n}[\alpha] x_\alpha + \sum_{(g,n)\in \frac{1}{2}\mathbb{N}\times\mathbb{N}} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha \\ &= \sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha \end{aligned}$$

which has a more uniform appearance. The necessary requirement that $n \geq 3$ when $g = 0$ has become something more natural: the stipulation that $\hbar$'s powers be strictly positive. One result of this is that $Z = e^F$ is now a genuine power series in $\hbar$, rather than a Laurent series, and so no awkward exceptions need be made on its behalf.

2.F Examples

A multitude of Quantum Airy Structures have been constructed and studied. Section 7 of [3] contains a full list of quadratic Abelian Quantum Airy Structures in dimensions two and three, as well as a full list of non-Abelian quadratic QASs in dimension two, and a strong showing of non-trivial non-Abelian quadratic QASs in dimension three. Section 8 produces Quantum Airy Structures from Froebenius algebras, again all of them quadratic. As far as non-quadratic structures, one systematic way of producing them was offered in [8], which locates the $\hat{H}_i$ within a module of a Vertex Operator Algebra (VOA). Another work [14] constructs an additional class of non-quadratic Quantum Airy Structures, although we will not summarize that method.

2.F.1 From Frobenius Algebras

We follow section 8 of [3].

A Frobenius algebra $\mathcal{F}$ is a finite-dimensional $\mathbb{C}$-vector space with an associative product and a linear map $\phi : \mathcal{F} \to \mathbb{C}$ such that $\langle a, b\rangle := \phi(ab)$ is a non-degenerate pairing and $\phi([a, b]) = 0$. They are implicated in, among other things, Topological Quantum Field Theories [2]. These are theories, germane to pure mathematics as well as to physics, that study functors into the category of Frobenius algebras from that of n-cobordisms. The cobordisms, n-manifolds interpolating (having as their oriented boundary) two other $(n - 1)$-manifolds, are taken as some sort of process occurring in spacetime, the affiliated algebras representing a quantum-mechanical state space and implementing the dynamics thereon.

Let $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ be the commutator and anti-commutator, respectively. Pick a

Frobenius algebra, $\mathcal{F}$, and name any orthonormal basis, $\{e_i\}$. Then there is straightforward recipe for quadratic Quantum Airy Structures. We write, without loss of generality, $\hat{H}_i := -\hbar\partial_i + \hbar^2(A^i_{jk}x_jx_k + B^i_{jk}x_j\partial_k + C^i_{jk}\partial_i\partial_k + D^i)$.

Proposition 1 (Prop. 8.7, page 53, [3]). *Let θ_A, θ_B, and θ_C be central elements in $\mathcal{F}$ satisfying $\theta_B^2 + \theta_A\theta_C = 0$. Take any $D \in \mathcal{F}$ such that $\theta_B D$ lies in the orthogonal complement of $[\mathcal{F}, \mathcal{F}]$. Then:*

$$A^i_{jk} := \phi(\theta_A\{e_j, e_k\}e_i)$$
$$B^i_{jk} := \phi(\theta_B[e_i, e_j]e_k)$$
$$C^i_{jk} := \phi(\theta_C\{e_i, e_j\}e_k)$$

along with the components D^i of D defines a Quantum Airy Structure.

Proof. We note that equivalent conditions (3.11)-(3.16) will later be derived in Lemma 5. Those conditions are, for all i and p, with implicit sums on all unquantified variables,

$$\begin{aligned}
A^p_{ik} - A^i_{pk} &= 0 && \forall k \\
B^p_{ik} - B^i_{pk} &= f^k_{ip} && \forall k \\
2(A^p_{j'k}B^i_{jj'} - A^i_{j'k}B^p_{jj'}) + 2(A^p_{jj'}B^i_{kk'} - A^i_{jj'}B^p_{kk'}) &= f^a_{ip}A^a_{jk} && \forall j, k \\
2(A^p_{jk}C^i_{jk} - A^i_{jk}C^p_{jk}) &= f^a_{ip}D^a && \\
4(A^p_{j'j}C^i_{kj'} - A^i_{j'j}C^p_{kj'}) + (B^i_{jk'}B^p_{k'k} - B^p_{jk'}B^i_{k'k}) &= f^a_{ip}B^a_{jk} && \forall j, k \\
2(B^p_{j'k}C^i_{j'j} - B^i_{j'k}C^p_{j'j}) + 2(B^p_{j'j}C^i_{j'k} - B^i_{j'j}C^p_{j'k}) &= f^a_{ip}C^a_{jk} && \forall j, k
\end{aligned}$$

which, eliminating f^k_{ip} everywhere, amount to the i, p-symmetry of (with repeated indices summed over I):

$$A^p_{ik}$$

$$2A^p_{j'k}B^i_{jj'} + 2A^p_{jj'}B^i_{kk'} - B^p_{ia}A^a_{jk}$$

$$2A^p_{jk}C^i_{jk} - B^p_{ia}D^a$$

$$4A^p_{j'j}C^i_{kj'} + B^i_{jk'}B^p_{k'k} - B^p_{ia}B^a_{jk}$$

$$2B^p_{j'k}C^i_{j'j} + 2B^p_{j'j}C^i_{j'k} - B^p_{ia}C^a_{jk}$$

Each of these can be borne out by direct computation. Just recall the basic linear algebra fact that $v = \langle v, e_i\rangle e_i = \phi(ve_i)e_i$ for all vectors v, as well as the cyclic permutation symmetry of ϕ implied by the condition $\phi([a,b]) = 0$. For example,

$$\begin{aligned}
A^p_{jk}C^i_{jk} &= \phi(\theta_A\{e_j, e_k\}e_p)\phi(\theta_C\{e_i, e_j\}e_k) \\
&= \phi(\theta_A e_j e_k e_p)\phi(\theta_C e_i e_j e_k) + \phi(\theta_A e_j e_k e_p)\phi(\theta_C e_j e_i e_k) \\
&\qquad + \phi(\theta_A e_j e_k e_p)\phi(\theta_C e_i e_k e_j) + \phi(\theta_A e_j e_k e_p)\phi(\theta_C e_k e_i e_j) \\
&= \langle e_j, \theta_A e_k e_p\rangle\phi(\theta_C e_i e_j e_k) + \langle e_j, \theta_A e_k e_p\rangle\phi(\theta_C e_j e_i e_k) \\
&\qquad + \langle e_j, \theta_A e_k e_p\rangle\phi(\theta_C e_i e_k e_j) + \langle e_j, \theta_A e_k e_p\rangle\phi(\theta_C e_k e_i e_j) \\
&= \phi(\langle e_j, \theta_A e_k e_p\rangle\theta_C e_i e_j e_k) + \phi(\langle e_j, \theta_A e_k e_p\rangle\theta_C e_j e_i e_k) \\
&\qquad + \phi(\langle e_j, \theta_A e_k e_p\rangle\theta_C e_i e_k e_j) + \phi(\langle e_j, \theta_A e_k e_p\rangle\theta_C e_k e_i e_j) \\
&= \phi(\theta_A\theta_C e_i e_k e_p e_k) + \phi(\theta_A\theta_C e_k e_p e_i e_k) \\
&\qquad + \phi(\theta_A\theta_C e_i e_k e_k e_p) + \phi(\theta_A\theta_C e_k e_i e_k e_p)
\end{aligned}$$

which is symmetric. Since our hypothesis on D gives $B^p_{ia}D^a = 0$, we have the third condition. Total symmetry of A is immediate. The other three are similar; two of them are symmetric identically, and the last is symmetric save for a term involving

$\theta_B^2 + \theta_A\theta_C$, which is why we ask that it be zero. $\square$

2.F.2 From Vertex Operator Algebras

We follow chapter 3 of [8].

Vertex Operator Algebras can be introduced as a distillation of the central features common to Conformal Field Theories [35].[21] In particular, Conformal Field Theories allow for an Operator Product Expansion [6]: for two operator-valued functions $A(x)$ and $B(y)$ on $\mathcal{M}$, their product $A(x)B(y)$ on $\mathcal{M} \times \mathcal{M}$ has a Laurent expansion[22] in powers of x (or y) near the diagonal $x = y$. This allows a detailed understanding of singular behavior as evaluation points collide.[23] Another major upshot of Conformal Field Theories is that they posses conformal symmetries; in two dimensions the conformal group is infinite-dimensional, which strongly constrains the possible field theories with those symmetries.

One now abstracts the algebra of these Laurent expansions $A(x)B(y) \xrightarrow[x,y\to z]{} \sum_k O_k z^{-k}$, taken formally. The central role of the conformal group suggests that representations of the Virasoro algebra on the space of states belong to the essential core we are extracting. Foregoing motivation of a VOA's other aspects, we record the definition:

Definition 9 (Vertex Operator Algebra)**.** *A **Vertex Operator Algebra** is a tuple* $(V, Y, \mathbf{0}, \mathbf{w})$*, with the following properties:*

- $V := \bigoplus_{k\in K} V_k$ *is a* $\mathbb{Z}$*-graded vector space, with* $K \subset \mathbb{Z}$ *having finitely many*

[21]More precisely, the chiral sector of Conformal Field Theories.

[22]The reason that this is a bigger deal in Conformal Field Theories than generally is that, in that setting, one can prove a positive radius of convergence for these expansions.

[23]This much success is valuable, as properly defining products of distributions evaluated at the same point has been a perennial thorn in the side of Quantum Field Theories.

negative members and dim(V_k) being finite for all $k \in K$.

- $Y(-, z) : V \to$ *End*$(V)[[z, z^{-1}]]$ *is linear. The endomorphisms v_k defined by $Y(v, z) := \sum_{k \in K} v_k z^{-k-1}$ are called the **modes** of $v \in V$.*

- $Y(v, z)\mathbf{0} - v$ *belongs to* $zV[[z]]$*, and* $Y(\mathbf{0}, z) = id$*. We refer to* $\mathbf{0}$ *as the **vacuum state**.*

- *The **conformal state** $\mathbf{w} \in V$ has four properties:*

  1. *For all $v \in V$, the series $Y(v, z)\mathbf{w}$ has finitely many terms.*
  2. *The modes $\mathbf{w}_k$ of $\mathbf{w}$ form a representation of the Virasoro algebra, i.e.*

     $$[\mathbf{w}_{\ell+1}, \mathbf{w}_{m+1}] = (\ell - m)\mathbf{w}_{\ell+m+1} + c\delta_{\ell,-m}\frac{\ell^3 - \ell}{12}$$

     for some "central charge" $c \in \mathbb{C}$.
  3. *All homogeneous $v \in V_n$ are eigenvectors of $\mathbf{w}_1$ with eigenvalue n.*
  4. *For all vectors v, we have $Y(\mathbf{w}_0 v, z) = \frac{d}{dz}Y(v, z)$.*

- *For every $u, v \in V$ there is an $N \in \mathbb{N}$ such that $(z - y)^N [Y(u, z), Y(v, y)] = 0$.*[24]

Only one family of VOAs will concern us: the Heisenberg VOAs. First we define some auxiliary paraphernalia. Take any lattice L with an inner product $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$, and produce a Lie algebra $\mathcal{H}_L$ out of the vector space:

$$\mathcal{H} := \left(\bigoplus_{\ell \in \mathbb{Z}} (L \otimes_{\mathbb{Z}} \mathbb{C}) \otimes t^{\ell} \right) \oplus \mathbb{C}\mathbf{1}$$

[24]Note the nontriviality of this axiom, as rings of formal power series are not integral domains.

using the bracket:

$$[\xi \otimes t^\ell, \eta \otimes t^m] = \langle \xi, \eta \rangle \ell \delta_{m,-\ell} \mathbf{1} \qquad [K, \eta \otimes t^\ell] = 0 \qquad \forall\, \xi, \eta \in L \otimes_{\mathbb{Z}} \mathbb{C},\, \ell, m \in \mathbb{Z}$$

Taking $\mathcal{H}_L^- := \{\xi \otimes t^\ell : \ell < 0\}$ to be the negative elements of this Lie algebra, the Heisenberg VOA's underlying vector space will be the symmetric algebra $V :=$ Sym$(\mathcal{H}_L^-)$. We will also want a linear representation of $\mathcal{H}_L$ on V. Any member of $\mathcal{H}_L^-$ acts in a natural way, by concatenation. For any non-negative element, $\xi \otimes t^\ell$ with $\ell \geq 0$, we define its action on V by declaring that it annihilates the constant polynomials. This, along with the fact that the Lie bracket is taken onto the commutator by the representation, is sufficient. From here on, elements $\xi \otimes t^\ell$ should be regarded in their role as linear operators on V. The *normal-ordered product* $\mathcal{N}(A_1, \dots, A_n)$ of series:

$$A_i := A_i^+ + A_i^- := \sum_{\ell > 0} A_i^\ell z^{-\ell-1} + \sum_{\ell \leq 0} A_i^\ell z^{-\ell-1} \in \mathrm{End}(V)[[z, z^{-1}]]$$

can be defined via $\mathcal{N}(A_1, A_2, \dots, A_n) := A_1^- \mathcal{N}(A_2, \dots, A_n) + \mathcal{N}(A_2, \dots, A_n) A_1^+$ as well as $\mathcal{N}(A) = A$. Effectively, in all monomials resulting from the product $A_1 A_2 \cdots A_n$, the factors are reordered so that negative modes occur to the left of positive modes.

Definition 10. *The* ***Heisenberg VOA*** *is comprised of the following:*

- *The vector space* $V :=$ *Sym*$(\mathcal{H}_L^-)$*,*

- *The linear map Y, defined by:*

$$Y(1, z) := \textit{id}$$
$$Y(\xi \otimes t^{-1}, z) := \sum_{m \in \mathbb{Z}} (\xi \otimes t^m) z^{-m-1}$$
$$Y(\Pi_i(\xi^i \otimes t^{-k_i}), z) := \mathcal{N}\left(\ldots, \frac{1}{(k_i-1)!} \frac{d^{k_i-1}}{dz^{k_i-1}} Y(\xi^i \otimes t^{-1}, z), \ldots\right)$$

- *The vaccum state $\mathbf{0} := 1$, which is simply the unit polynomial, and*

- *The conformal state, $\mathbf{w} := \frac{1}{2}\sum_i (\xi^i \otimes t^{-1})(\xi^i \otimes t^{-1})$, where $\{\xi^i\}$ is any orthonormal basis for $L \otimes_{\mathbb{Z}} \mathbb{C}$.*

Our interest in the Heisenberg VOA is due to the fact that a certain $\mathcal{W}$-algebra, called $\mathcal{W}(\mathfrak{gl}_n)$, can be realized as a sub-VOA of the Heisenberg VOA.[25] With $\mathcal{W}(\mathfrak{gl}_n)$ so realized, we plan to find a representation of it within a module of the Heisenberg VOA. The represented images of its strong generators[26] will have modes forming a Quantum Airy Structure – up to a certain corrective conjugation.

The Heisenberg VOA module we will define happens to be a *twisted* module, although we will not need to know that this is so. It will become untwisted once we restrict it to the $\mathcal{W}(\mathfrak{gl}_n)$ representation. Pick an idempotent automorphism σ of $(L \otimes_{\mathbb{Z}} \mathbb{C})$ with order r, as well as a basis $\{v^a\}$ which diagonalizes it. Then (W, Y_σ) is an example of a σ-twisted Heisenberg module, in which:

- $W := \mathbb{C}[[\hbar]][[\{x_i\}_{i \in I}]]$

[25] We will soon see a set of strong generators for $\mathcal{W}(\mathfrak{gl}_n)$ in its presentation as a Heisenberg sub-VOA, however see [8] page 26 for its general definition.

[26] A set of vectors $v^i \in V$ *strongly generate* a VOA if the underlying vector space, V, is spanned by vectors of the form $v^1_{-k_1} \cdots v^n_{-k_n}\mathbf{0}$, with all $k_i > 0$.

- $Y_\sigma(\cdot, z) : V \to \mathrm{End}(W)[[z^{1/r}, z^{-1/r}]]$ satisfies

$$Y_\sigma(v^a \otimes t^\ell, z) = \sum_{n \in \frac{a}{r} + \mathbb{Z}} Q_{rk} z^{-k-1}$$

  with those modes being (for $\ell > 0$) $Q_\ell = \hbar \partial_{x_\ell}$ and $Q_{-\ell} = \hbar \ell x_\ell$, and $Q_0 = 0$. Also, Y_σ satisfies conditions perfectly analogous to those (the first and third) of Y in the Heisenberg VOA.

The Quantum Airy Structures that we seek are in sight. The steps are simple, although cannot be easily motivated in the time we will dedicate to them (see Theorem 4.9, page 39, [8]):

1. Define a new basis, $\chi^b := \frac{1}{r} \sum_{a=0}^{r-1} e^{-2\pi i b a / r} v^a$
2. Let $E_i \in V$ be the elementary symmetric polynomials in the vectors $\chi^b \otimes t^{-1}$. These are strong generators for $\mathcal{W}(\mathfrak{gl}_n)$.
3. Define $W^i(z) := r^{i-1} Y_\sigma(E_i, z)$ for i between 1 and r. These form the representation of $\mathcal{W}(\mathfrak{gl}_n)$'s strong generators.
4. Expand into modes: $W^i(z) := \sum_{n \in \mathbb{Z}} W^i_n z^{-n-1}$ [27]
5. For any $r \geq 2$, any $s \in \{1, \ldots, r+1\}$ such that $r \equiv \pm 1 \mod s$, the operators $\hat{H}^i_k := e^{-J_s/s\hbar} W^i_k e^{J_s/s\hbar}$ indexed by $i \in I$ and $k \geq i - 1 - \lfloor \frac{s(i-1)}{r} \rfloor + \delta_{i,1}$ form a Quantum Airy Structure.

A few comments are worth making. Although the effort to get to this point may seem great, it has many benefits. Firstly, this constructs non-quadratic QASs, which

[27] Although, ostensibly, this sum should be taken over integer multiples of $1/r$, the vectors that we are putting through Y_σ are suitably restricted that these fractional powers do not occur. This is the meaning of the sub-module having become untwisted.

was a limitation of all other mentioned methods. Secondly, the attention paid to this strange parameter r pays off in that the operators $\hat{H}^i_k$ are of degree r, allowing us complete control over, say, returning to the quadratic case, or ascending arbitrarily high. From the geometrical point of view outlined in Section 2.1 the automorphism σ replaces $z \mapsto \overline{z}$, and so this procedure corresponds to cycling the sheets of a covering that has r of them meeting at its branches. The subalgebra condition of a QAS is already fulfilled by the W^i_k, and the remaining step of conjugation is required only to get to degree condition right. This requirement that arises at the end, allowing us to choose s only among those such that $r \equiv \pm 1 \mod s$, remains rather mysterious. Lastly, this construction makes explicit the connection Quantum Airy Structures have to $\mathcal{W}$-algebras.

Chapter 3

Quadratic Case

In this section we discuss two proofs of the Existence and Uniqueness theorems for Quantum Airy Structures, in the special case of a quadratic Quantum Airy Structure. One proof is deferred to Chapter 4, and the other is described in detail. The latter proof is broken up into existence and uniqueness results.

The assumption of quadratic $\hat{H}_i$ amounts to assuming, without loss of generality, the form:

$$\hat{H}_i = -\hbar\partial_i + \hbar^2(A^i_{jk}x_ix_j + B^i_{jk}x_j\partial_k + C^i_{jk}\partial_j\partial_k + D_i) \tag{3.1}$$

with implied summations on j and k. We are presently omitting the cross-capped Quantum Airy Structures, which forces the degree of all operators in parentheses to have the same parity as 2, i.e. even. We take these $\hat{H}_i$ to be a Quantum Airy Structure in Normal Form, which places several relations upon the A, B, C, D that will later be important. For now, it is clear that A and C can be taken as symmetric in their lower indices.

3.A A Few Lemmas

We will often be seeing expressions such as $e^{-F}Q \cdot e^F$ with $Q \in \mathcal{O}^\hbar$, and so the following lemmas will be very handy. Firstly, three definitions.

Definition 11. *For any multi-index $L := (L_1, L_2, \ldots, L_m) \in I^m$, the notation $\mathbb{J} \models L$ denotes that $\mathbb{J}$ is an* ***index-partition*** *of L. By this we mean that $\mathbb{J}$ is an ordered collection of multi-indices $((L_j)_{j\in J_\ell})_{\ell=1}^{|\mathbb{J}|}$ in which $\{J_\ell\}_{\ell=1}^{|\mathbb{J}|}$ are pair-wise disjoint* ***possibly empty*** *sets whose union is $\{1, \ldots, m\}$. All this amounts to is that the concatenation of the $(L_j)_j$ recovers [a permutation of] L. The $(L_j)_j$ are themselves ordered such that j is increasing.*

The first lemma is very easy to see, but convenient to keep in mind.

Lemma 1. *For any multi-index L and any $\Xi, \Psi \in \mathcal{S}$, we have*

$$\partial_L(\Xi\Psi) = \sum_{(A,B)\models L} \partial_A(\Xi)\partial_B(\Psi)$$

In general, of course, for $\{\Psi_k\}_{k\in K}$,

$$\partial_L\left(\prod_{k\in K}\Psi_k\right) = \sum_{(A_k)_{k\in K}\models L}\ \prod_{k\in K}\partial_{A_k}(\Psi_k)$$

Remark 2. *Lemma 1 holds even for $K = \varnothing$, because the product and sum become empty.*

Lemma 2. *For any multi-index $\beta \in I^m$ say that $n_i(\beta) \in \mathbb{N}_0$ of its entries are $i \in I$, so that $P(\beta) := \frac{m!}{\prod_i n_i(\beta)!}$ counts the permutations of β. Then we have $\partial_\beta(x_\beta) = \frac{m!}{P(\beta)}$.*

Proof. Notice that $\partial_\beta = \prod_i \partial_i^{n_i(\beta)}$, so that

$$\partial_\beta(x_\beta) = \prod_i [\partial_i^{n_i(\beta)} x_i^{n_i(\beta)}] = \prod_i [n_i(\beta)!] = \frac{m!}{P(\beta)}$$

□

Definition 12. *Let $\alpha \sim \beta$ denote the fact that α is a permutation of β.*

Lemma 3. *Consider $F := \sum_{n\in\mathbb{N}_0} \frac{1}{n!} \sum_{\alpha\in I^n} F_n[\alpha] x_\alpha \in \mathbb{C}[[\{x_i\}_i]]$, a formal power series with each $F_n[\alpha] \in \mathbb{C}$ completely symmetric under permutations in α. For any multi-index $\beta \in I^m$, we have $\partial_\beta(F)|_0 = F_m[\beta]$.*

Proof. Applying $|_0$ will kill any term not constant in the x, so we retain only those terms for which α is a permutation of β:

$$\begin{aligned}\partial_\beta \cdot F|_0 &= \frac{1}{m!} \sum_{\alpha:\alpha\sim\beta} F_m[\alpha] \partial_\beta(x_\alpha) \\ &= \frac{1}{m!} \sum_{\alpha:\alpha\sim\beta} F_m[\alpha] \frac{m!}{P(\beta)}\end{aligned}$$

Each of these terms are equal to one another, and there are $P(\beta)$ of them. □

Remark 3. *We can replace $\mathbb{C}$ with any commutative ring.*

Definition 13. *For $L := (L_1, \ldots, L_{|L|}) \in I^{|L|}$, we write $\mathbb{J} \vdash L$ to indicate that $\mathbb{J}$ is an **un-ordered** collection of multi-indices $\{(L_j)_{j\in J_\ell}\}_{\ell=1}^{|\mathbb{J}|}$ in which $\{J_\ell\}_{\ell=1}^{|\mathbb{J}|}$ are pair-wise disjoint **non-empty** sets whose union is $\{1, \ldots, |L|\}$. Again, the concatenation of the $(L_j)_j$ recovers [a permutation of] L, and they are each ordered such that j is increasing.*

The main difference between this definition and that of $\models$ is that none of the sub-indices $(L_j)_j$ can be empty here. Additionally, $\mathbb{J}$ is un-ordered here; two partitions $\mathbb{J}_1 \vdash L$ and $\mathbb{J}_2 \vdash L$ are the same if they have the same members.

Lemma 4. *For any finite multi-index $L \in I^{|L|}$ and any $F \in \mathcal{S}^\hbar$, we have:*

$$e^{-F}\partial_L \cdot e^F = \sum_{\mathbb{J}\vdash L}\prod_{J\in\mathbb{J}}\partial_J F$$

Example 1. *If we take $L = (k, j, i)$, then:*

$$\begin{aligned} e^{-F}\partial_L \cdot e^F = [\partial_k F][\partial_j F][\partial_i F] \\ &+ [\partial_k\partial_j F][\partial_i F] \\ &+ [\partial_j F][\partial_k\partial_i F] \\ &+ [\partial_k F][\partial_j\partial_i F] \\ &+ [\partial_k\partial_j\partial_i F] \end{aligned}$$

Proof. It can be proven by induction. Suppose that it holds for all L of length n, and consider (WLOG) the set $L \cup \{b\}$. Then:

$$\begin{aligned} \partial_b\partial_L \cdot e^F &= \partial_b e^F e^{-F}\partial_L \cdot e^F = \partial_b\left[e^F\sum_{\mathbb{J}\vdash L}\prod_{J\in\mathbb{J}}\partial_J F\right] \\ &= (e^F\partial_b \cdot F)\left(\sum_{\mathbb{J}\vdash L}\prod_{J\in\mathbb{J}}\partial_J F\right) + e^F\sum_{\mathbb{J}\vdash L}\partial_b\cdot\left(\prod_{J\in\mathbb{J}}\partial_J F\right) \\ &= (e^F\partial_b \cdot F)\left(\sum_{\mathbb{J}\vdash L}\prod_{J\in\mathbb{J}}\partial_J F\right) + e^F\sum_{\mathbb{J}\vdash L}\sum_{J'\in\mathbb{J}}(\partial_b\partial_{J'}\cdot F)\prod_{J\in\mathbb{J}\setminus J'}\partial_J F \\ &= e^F\sum_{\mathbb{J}\vdash L\cup\{b\}}\prod_{J\in\mathbb{J}}\partial_J F \end{aligned}$$

The second and third lines are merely Leibniz Law. In the final line, we've recognized that line three presents all possible partitions of $L \cup \{b\}$: in a partition $\mathbb{K} \vdash L \cup \{b\}$, either:

- The new member b is by itself (the lefthand term),

- Or b occurs grouped with some other indices, collectively K. But then K, with the remaining members of $\mathbb{K}$, form a partition of L. Each case occurs exactly once in the righthand term.

$\square$

This lemma is much stronger than we have any right to use in this chapter, although its additional generality will be necessary later on.

3.B Conceptual Proof

In the paper of Kontsevich and Soibelman [31] they prove both the existence and the uniqueness of a formal power series Z which is annihilated simultaneously by all operators $\hat{H}_i$, subject to the $\hat{H}_i$ forming a Quantum Airy Structure. Their theorem (Theorem 2.4.2 on page 13) is equivalent to the following:

Theorem 5. *Suppose we have a quadratic Quantum Airy Structure* $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$. *Then, among all* Z *having the form* (3.2)*, there is a unique solution to the system of equations* $\hat{H}_i Z = 0$ *for* $i \in I$.

This proof does not actually assume that $\mathbb{H}$ is quadratic. Also, it is much more convenient to prove if one allows $\mathbb{H}$ to be possibly cross-capped. For both of those reasons, we defer it to Chapter 4.

3.C Computational Proof

We now present an alternative proof of Theorem 5, which can be found in [3]. It makes use of naught but combinatorics and raw computation. In the following, repeated indices (usually j and k) will be summed over with range I. The expression

$\gamma\backslash j$, if j is a component of γ, refers to any of the multi-indices which could result after deleting a copy of j from γ; the expression is 0 otherwise.

What will be proven is in fact Theorem 5 along with this corollary:

Corollary 1. *Writing*

$$Z = \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0}\ \sum_{\alpha\in I^n}\frac{\hbar^{2g-2+n}}{n!}F_{g,n}[\alpha]x_\alpha\right] \tag{3.2}$$

for the solution referenced in Theorem 5, the coefficients $F_{g,n}[\alpha]\in\mathbb{C}$ are related to one another by the ***Quadratic Topological Recursion****:*

$$\begin{aligned} F_{\ell,\rho+1}[i,\gamma] = 2A^i_{\gamma_1,\gamma_2}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \\ + n_j(\gamma)B^i_{jk}F_{\ell,\rho}[\gamma\backslash j,k] + C^i_{jk}F_{\ell-1,\rho+2}[\gamma,j,k] \\ + C^i_{jk}\sum_{(J,K)\models\gamma}\ \sum_{g+g'=\ell}F_{g,|J|+1}[j,J]F_{g',|K|+1}[k,K] \end{aligned}$$

where, as earlier, $n_j(\gamma)$ is the number of components in γ that are equal to j.

Theorem 5 as well as this Corollary follow from two propositions (the first of which is Proposition 2.1, [3] pg. 7):

Proposition 2 (Uniqueness)**.** *For any quadratic Quantum Airy Structure $\{\hat{H}_i\}_{i\in I}\subset\mathcal{O}^\hbar$, among all Z having the form* (3.2)*, the system of equations $\hat{H}_iZ=0$ has either zero or one solution.*

Proposition 3 (Existence)**.** *For any quadratic Quantum Airy Structure $\{\hat{H}_i\}_{i\in I}\subset\mathcal{O}^\hbar$, the system $\hat{H}_iZ=0$ does have solutions Z having the form* (3.2)*.*

The first of these is nothing more than applying $\hat{H}_i$ to Z and then sorting out the mess in terms of both $\hbar$-degree and x-degree. One obtains a recursive formula for the coefficients $F_{g,n}[\alpha]$ of Z's exponent, and so a would-be solution cannot do other than obey the recursion. The reason that this does not also provide existence is subtle, and stems from the fact that the recursion does not necessarily produce symmetric $F_{g,n}[\alpha]$s without the aid of further assumptions. We will see how this all turns out. For now, the proof of Proposition 2:

Proof. If we simply compute the action of each $\hat{H}_i$ on Z and separate the resulting terms by both $\hbar$-deg and by x-deg, we obtain equations expressing $F_{g,n}[\alpha]$ in terms of other $F_{g',n'}[\alpha']$s with strictly smaller values of the quantity $2g-2+n$; i.e., a recursion. This will guarantee uniqueness, as we will also see the theorem statement has removed all choice of initial data.

Let's do that. Recalling:

$$\hat{H}_i = -\hbar\partial_i + \hbar^2(A^i_{jk}x_jx_k + B^i_{jk}x_j\partial_k + C^i_{jk}\partial_j\partial_k + D^i)$$

and that Z is given by (3.2), we get:

$$Z^{-1}\hat{H}_iZ = -\hbar\partial_iF + A^i_{jk}\hbar^2x_jx_k + B^i_{jk}\hbar^2x_j\partial_kF + C^i_{jk}\hbar^2((\partial_jF)(\partial_kF) + \partial_j\partial_kF) + \hbar^2D^i$$

If we sift these terms by x-deg, then at x-deg $= \rho$ we find:

$$
\begin{aligned}
&-\sum_{g\in\frac{1}{2}\mathbb{N}} \hbar^{2g+\rho} \sum_{\alpha\in I^{\rho+1}} \frac{1}{(\rho+1)!} F_{g,\rho+1}[\alpha]\partial_i x_\alpha + \hbar^2 A^i_{jk} x_j x_k \delta_{\rho,2} \\
&\qquad + \sum_{g\in\frac{1}{2}\mathbb{N}} \hbar^{2g+\rho} B^i_{jk} \sum_{\alpha\in I^\rho} \frac{1}{\rho!} F_{g,\rho}[\alpha] x_j \partial_k x_\alpha + \sum_{g\in\mathbb{N}} \hbar^{2g+\rho+2} C^i_{jk} \sum_{\alpha\in I^{\rho+2}} \frac{1}{(\rho+2)!} F_{g,\rho+2}[\alpha]\partial_j\partial_k x_\alpha \\
&\qquad + \sum_{g,g'\in\frac{1}{2}\mathbb{N}} \hbar^{2g+2g'+\rho} C^i_{jk} \sum_{n+n'=\rho+2} \sum_{\alpha'\in I^{n'},\alpha\in I^n} \left(\frac{1}{n!} F_{g,n}[\alpha]\partial_j x_\alpha\right)\left(\frac{1}{n'!} F_{g',n'}[\alpha']\partial_k x_{\alpha'}\right) \\
&\qquad + \hbar^2 D^i \delta_{\rho,0}
\end{aligned}
$$

In which it is understood that $F_{g,n}$ with $n < 1$ should be considered zero. Now we sift by $\hbar$-deg, finding at $\hbar$-deg $= 2\ell + \rho$ the terms:

$$
-\sum_{\alpha\in I^{\rho+1}} \frac{1}{(\rho+1)!} F_{\ell,\rho+1}[\alpha]\partial_i x_\alpha + A^i_{jk} x_j x_k \delta_{\rho,2}\delta_{\ell,0} \tag{3.3}
$$

$$
+ B^i_{jk} \sum_{\alpha\in I^\rho} \frac{1}{\rho!} F_{\ell,\rho}[\alpha] x_j \partial_k x_\alpha + C^i_{jk} \sum_{\alpha\in I^{\rho+2}} \frac{1}{(\rho+2)!} F_{\ell-1,\rho+2}[\alpha]\partial_j\partial_k x_\alpha \tag{3.4}
$$

$$
+ C^i_{jk} \sum_{g,g'=\ell} \sum_{n,n'=\rho+2} \sum_{\alpha'\in I^{n'},\alpha\in I^n} \left(\frac{1}{n!} F_{g,n}[\alpha]\partial_j x_\alpha\right)\left(\frac{1}{n'!} F_{g',n'}[\alpha']\partial_k x_{\alpha'}\right) \tag{3.5}
$$

$$
+ D^i \delta_{\rho,0}\delta_{\ell,1} \tag{3.6}
$$

Finally, we wish to apply $\partial_\gamma \bullet |_0$ to the above, for some multi-index $\gamma \in I^\rho$. The first and fourth terms can be handled immediately by applying Lemma 3 with $\beta = (\gamma, i)$ and $\beta = (\gamma, j, k)$ respectively. The second and sixth terms are trivial. The third term is modified by ∂_γ to:

$$\begin{aligned}
\partial_\gamma \left[B^i_{jk} \sum_\alpha \frac{1}{\rho!} F_{\ell,\rho}[\alpha] x_j \partial_k x_\alpha \right] \Bigg|_0 &= \sum_{(J,K)\models\gamma} \sum_\alpha B^i_{jk} \frac{1}{\rho!} F_{\ell,\rho}[\alpha] (\partial_J x_j)(\partial_{k,K} x_\alpha) \Bigg|_0 \\
&= \sum_{(J,K)\models\gamma} \sum_\alpha B^i_{jk} \frac{1}{\rho!} F_{\ell,\rho}[\alpha] (\delta_{J,j})(\partial_{k,K} x_\alpha)|_0 \\
&= \sum_\alpha n_j(\gamma) B^i_{jk} \frac{1}{\rho!} F_{\ell,\rho}[\alpha] (\partial_{k,\gamma\backslash j} x_\alpha)|_0 \\
&= n_j(\gamma) B^i_{jk} F_{\ell,\rho}[k, \gamma\backslash j]
\end{aligned}$$

which is handled by Lemma 3. As for the remaining term, line (3.5), we write:

$$\partial_\gamma(3.5)|_0 = C^i_{jk} \sum_{(J,K)\models\gamma} \sum_{g+g'=\ell} \sum_{n+n'=\rho+2} \sum_{\alpha' \in I^{n'}, \alpha\in I^n} \left(\frac{1}{n!} F_{g,n}[\alpha] \partial_{j,J} x_\alpha \right) \left(\frac{1}{n'!} F_{g',n'}[\alpha'] \partial_{k,K} x_{\alpha'} \right) \Bigg|_0$$

Using Lemma 3 in each factor gives:

$$\partial_\gamma(3.5)|_0 = C^i_{jk} \sum_{(J,K)\models\gamma} \sum_{g+g'=\ell} F_{g,|J|+1}[j, J] F_{g',|K|+1}[k, K]$$

Altogether, $\partial_\gamma[(3.3) - (3.6)]|_0$ is:

$$\begin{aligned}
&-F_{\ell,\rho+1}[i,\gamma] + 2A^i_{\gamma_1,\gamma_2} \delta_{\rho,2}\delta_{\ell,0} + D^i \delta_{\rho,0}\delta_{\ell,1} \\
&\qquad + n_j(\gamma) B^i_{\gamma_j k} F_{\ell,\rho}[\gamma\backslash j, k] + C^i_{jk} F_{\ell-1,\rho+2}[\gamma, j, k] \\
&\qquad + C^i_{jk} \sum_{(J,K)\models\gamma} \sum_{g+g'=\ell} F_{g,|J|+1}[j, J] F_{g',|K|+1}[k, K]
\end{aligned}$$

Where $\rho = |\gamma|$. The equations $\hat{H}_i Z = 0$ are equivalent to the vanishing of the above expression for all ρ, γ, ℓ, or:

$$F_{\ell,\rho+1}[i,\gamma] = 2A^i_{\gamma_1,\gamma_2}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \tag{3.7}$$

$$+ n_j(\gamma)B^i_{\gamma_j k}F_{\ell,\rho}[\gamma\backslash j,k] + C^i_{jk}F_{\ell-1,\rho+2}[\gamma,j,k] \tag{3.8}$$

$$+ C^i_{jk}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g,|J|+1}[j,J]F_{g',|K|+1}[k,K] \tag{3.9}$$

Since the quantity $2g-2+n$ is strictly larger in the lefthand side than in the right-hand side, we have derived a recursion for the $F_{g,n}[\alpha]$.

The smallest value of the recursion parameter, $2g-2+n$, is 1. The recursion, then, is based on $F_{0,3}$, $F_{\frac{1}{2},2}$, and $F_{1,1}$. We find:

$$F_{0,3}[i,j,k] = 2A^i_{jk} \qquad F_{\frac{1}{2},2} = 0 \qquad F_{1,1} = D^i \tag{3.10}$$

□

Remark 4. *When g is a non-integer, the equation for $F_{g,n}$ only involves $F_{g',n'}$ with g' a non-integer – or, it only involves terms which have at least one $F_{g',n'}$ factor for non-integer g'. Therefore the initial condition $F_{\frac{1}{2},2} = 0$ propagates upward and we find that $F_{g,n} = 0$ whenever $g \notin \mathbb{N}$. This is due to our dismissal of cross-capped QASs. Only in the cross-capped case can one have $F_{g,n} \neq 0$ for $g \notin \mathbb{N}$.*

In our derivation, we simplified matters many times by using the permutation symmetry of the $F_{g,n}[\alpha]$ in their α-argument. If we use the recursion to construct a sequence, not knowing in advance whether they are symmetric, we find that (for example) $F_{g,n}[i,j,\gamma]$ and $F_{g,n}[j,i,\gamma]$ have very different-looking definitions. Their equality is not at all obvious, and in fact not even guaranteed without some crucial relations that constrain A, B, C, D. This is the reason that Proposition 2 does not on its own provide existence.

Put another way: if one constructed a sequence of $F_{g,n}[\alpha]$s using (3.7)-(3.9), placed them into a series $F = \sum_{g,n,\alpha} F_{g,n}[\alpha]x_\alpha$, and then checked the vanishing of $\hat{H}_i e^F$ order-by-order as we've just done, they would end up at equations (3.7)-(3.9) again – involving *not* their $F_{g,n}[\alpha]$s but instead the *symmetrizations*[1] of their $F_{g,n}[\alpha]$s. Unfortunately, (3.7)-(3.9) holding for some $F_{g,n}[\alpha]$s cannot imply that (3.7)-(3.9) holds for their symmetrizations, due (among other things) to non-linearity. Proposition 3, the existence of F, is implied by Proposition 2 along with the following (which is Proposition 2.4, [3] pg. 13):

Proposition 4 (Symmetry)**.** *Each member of the sequence* $\{F_{g,n}[\alpha]\}$ *constructed by the quadratic topological recursion* (3.7)-(3.9) *is completely symmetric in the* α*-argument.*

First, some necessary algebaric manipulations.

Lemma 5. *The subalgebra condition on a quadratic Quantum Airy Structure (in the form of* (3.1)*) is equivalent to the following conditions (for all* $i, p \in I$*):*

$$\begin{aligned}
A^p_{ik} - A^i_{pk} &= 0 && \forall k \\
B^p_{ik} - B^i_{pk} &= f^k_{ip} && \forall k \\
2(A^p_{j'k}B^i_{jj'} - A^i_{j'k}B^p_{jj'}) + 2(A^p_{jj'}B^i_{kk'} - A^i_{jj'}B^p_{kk'}) &= f^a_{ip}A^a_{jk} && \forall j,k \\
2(A^p_{jk}C^i_{jk} - A^i_{jk}C^p_{jk}) &= f^a_{ip}D^a && \\
4(A^p_{j'j}C^i_{kj'} - A^i_{j'j}C^p_{kj'}) + (B^i_{jk'}B^p_{k'k} - B^p_{jk'}B^i_{k'k}) &= f^a_{ip}B^a_{jk} && \forall j,k \\
2(B^p_{j'k}C^i_{j'j} - B^i_{j'k}C^p_{j'j}) + 2(B^p_{j'j}C^i_{j'k} - B^i_{j'j}C^p_{j'k}) &= f^a_{ip}C^a_{jk} && \forall j,k
\end{aligned}$$

Proof. Schematically, a member of a QAS in the quadratic case is of the form

[1] Over α.

$\hat{H}_i = I_i + A_i + B_i + C_i + D_i$. The commutator is:

$$\begin{aligned}
[\hat{H}_i, \hat{H}_p] = \frac{1}{2}[I_i, I_p] + [I_i, A_p] + [I_i, B_p] + [I_i, C_p] + [I_i, D_p] \\
+ \frac{1}{2}[A_i, A_p] + [A_i, B_p] + [A_i, C_p] + [A_i, D_p] \\
+ \frac{1}{2}[B_i, B_p] + [B_i, C_p] + [B_i, D_p] \\
+ \frac{1}{2}[C_i, C_p] + [C_i, D_p] \\
- (i \leftrightarrow p)
\end{aligned}$$

$$\begin{aligned}
&= [I_i, A_p] + [I_i, B_p] + [A_i, B_p] + [A_i, C_p] + \frac{1}{2}[B_i, B_p] + [B_i, C_p] \\
&\qquad - (i \leftrightarrow p)
\end{aligned}$$

$$\begin{aligned}
&= -\hbar^3(2A^p_{ik}x_k) - \hbar^3(B^p_{ik}\partial_k) - \hbar^4(2A^i_{jk}B^p_{j'j}x_{j'}x_k) \\
&\quad - \hbar^4(2A^i_{jk}C^p_{jk} + 4A^i_{jk}C^p_{j'j}x_k\partial_{j'}) \\
&\quad + \hbar^4 B^i_{jk}B^p_{j'k'}(\delta_{kj'}x_j\partial_{k'} + x_j x_{j'}\partial_k\partial_{k'}) \\
&\quad - \hbar^4(2B^i_{jk}C^p_{j'j}\partial_k\partial_{j'}) \qquad - (i \leftrightarrow p)
\end{aligned}$$

The subalgebra condition informs us that this must belong to the ideal $\hbar\mathcal{O}^\hbar \cdot \mathbb{H}$, which is to say that:

$$
\begin{aligned}
&-\hbar(2A^p_{ik}x_k) - \hbar(B^p_{ik}\partial_k) - \hbar^2(2A^i_{jk}B^p_{j'j}x_{j'}x_k)\\
&-\hbar^2(2A^i_{jk}C^p_{jk} + 4A^i_{jk}C^p_{j'j}x_k\partial_{j'})\\
&+\hbar^2(B^i_{jk}B^p_{j'k'}x_jx_{j'}\partial_k\partial_{k'} + B^i_{jk}B^p_{kk'}x_j\partial_{k'})\\
&-\hbar^2(2B^i_{jk}C^p_{j'j}\partial_k\partial_{j'}) \qquad -(i \leftrightarrow p)\\
&\qquad\qquad =\\
&f^a_{ip}(-\hbar\partial_a + A^a_{jk}x_jx_k + B^a_{jk}x_j\partial_k + C^a_{jk}\partial_j\partial_k + D^a)
\end{aligned}
$$

For some elements $f^a_{ip} \in \mathcal{O}^\hbar$. The top expression contains $-\hbar(B^p_{ik} - B^i_{pk})\partial_k$, which is impossible to locate in the bottom expression unless the f^a_{ip} are scalars in $\mathbb{C}$. Therefore, we can match coefficients on the various $\mathcal{O}^\hbar$ elements that we see:

$$A^p_{ik} - A^i_{pk} = 0 \qquad \forall k \tag{3.11}$$

$$B^p_{ik} - B^i_{pk} = f^k_{ip} \qquad \forall k \tag{3.12}$$

$$2(A^p_{j'k}B^i_{jj'} - A^i_{j'k}B^p_{jj'}) + 2(A^p_{jj'}B^i_{kk'} - A^i_{jj'}B^p_{kk'}) = f^a_{ip}A^a_{jk} \qquad \forall j,k \tag{3.13}$$

$$2(A^p_{jk}C^i_{jk} - A^i_{jk}C^p_{jk}) = f^a_{ip}D^a \tag{3.14}$$

$$4(A^p_{j'j}C^i_{kj'} - A^i_{j'j}C^p_{kj'}) + (B^i_{jk'}B^p_{k'k} - B^p_{jk'}B^i_{k'k}) = f^a_{ip}B^a_{jk} \qquad \forall j,k \tag{3.15}$$

$$(B^i_{jk}B^p_{j'k'} - B^p_{jk}B^i_{j'k'}) + (B^i_{j'k'}B^p_{jk} - B^p_{j'k'}B^i_{jk}) = 0 \qquad \forall j,j',k,k'$$

$$2(B^p_{j'k}C^i_{j'j} - B^i_{j'k}C^p_{j'j}) + 2(B^p_{j'j}C^i_{j'k} - B^i_{j'j}C^p_{j'k}) = f^a_{ip}C^a_{jk} \qquad \forall j,k \tag{3.16}$$

In each line, variables unbound by universal quantification are summed (with implied universal quantification over i and p). Note also that in order to match coefficients correctly one must sometimes symmetrize some of the bound variables. $\square$

Corollary 2. *Using* (3.12) *to eliminate* f^a_{ip} *in each of* (3.11)-(3.16)*, we obtain* *the following relations that are satisfied in any quadratic Quantum Airy Structure (written in the form of* (3.1)*), for all* i, p, j, k*:*

$$4A^p_{j'j}C^i_{kj'} + B^i_{jk'}B^p_{k'k} + B^i_{pa}B^a_{jk} = (i \leftrightarrow p) \tag{3.17}$$

$$2B^p_{j'k}C^i_{j'j} + 2B^p_{j'j}C^i_{j'k} + 2B^i_{pa}C^a_{jk} = (i \leftrightarrow p) \tag{3.18}$$

$$B^i_{pk}D^k + C^i_{j'k}A^p_{j'k} = (i \leftrightarrow p) \tag{3.19}$$

$$2(B^i_{p,k'}A^k_{j,k'} + B^i_{k,k'}A^p_{j,k'} + B^i_{j,k'}A^p_{k,k'}) = (i \leftrightarrow p) \tag{3.20}$$

Now on to the proof of Proposition 4.

Proof. The proof is convoluted, but uncomplicated. We will apply the recursion twice to an arbitrary $F_{\ell,\rho+1}[i,p,\gamma]$ (with $\gamma \in I^{\rho-1}$) and argue that all resulting terms are themselves symmetric. This proof is inductive; we take the liberty of assuming that all $F_{g,n}[\alpha]$s with $2g-2+n < 2\ell-2+(\rho+1)$ are already symmetric.

Inductive Step

Recalling (3.7)-(3.9),

$$\begin{aligned}
F_{\ell,\rho+1}[i,p,\gamma] &= 2A^i_{p,\gamma_1}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \\
&\quad + n_j(p,\gamma)B^i_{jk}F_{\ell,\rho}[(p,\gamma)\backslash j,k] + C^i_{jk}F_{\ell-1,\rho+2}[p,\gamma,j,k] \\
&\quad + C^i_{jk}\sum_{(J,K)\models(p,\gamma)}\sum_{g+g'=\ell}F_{g,|J|+1}[j,J]F_{g',|K|+1}[k,K]
\end{aligned}$$

$$\begin{aligned}
&= 2A^i_{p,\gamma_1}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \\
&\quad + B^i_{pk}F_{\ell,\rho}[\gamma,k] + n_j(\gamma)B^i_{jk}F_{\ell,\rho}[p,\gamma\backslash j,k] \\
&\quad + C^i_{jk}F_{\ell-1,\rho+2}[p,\gamma,j,k] \\
&\quad + C^i_{jk}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g,|J|+2}[j,p,J]F_{g',|K|+1}[k,K] \\
&\quad + C^i_{jk}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g,|J|+1}[j,J]F_{g',|K|+2}[p,k,K]
\end{aligned}$$

$$\begin{aligned}
&= 2A^i_{p,\gamma_1}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \\
&\quad + B^i_{pk}{\color{green}F_{\ell,\rho}[\gamma,k]} + n_j(\gamma)B^i_{jk}{\color{red}F_{\ell,\rho}[p,\gamma\backslash j,k]} \\
&\quad + C^i_{jk}{\color{blue}F_{\ell-1,\rho+2}[p,\gamma,j,k]} \\
&\quad + 2C^i_{jk}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}{\color{orange}F_{g,|J|+2}[j,p,J]}F_{g',|K|+1}[k,K]
\end{aligned}$$

At this point, any $F_{\cdot,\cdot}[\bullet]$ with $p \in \bullet$ will be expanded again (and also the green term).

This amounts to making the following substitutions for the colour terms:

$$\begin{aligned}
F_{\ell,\rho}[\gamma,k] = \;& 2A^k_{\gamma_1,\gamma_2}\delta_{\rho,3}\delta_{\ell,0} + D^k\delta_{\rho,1}\delta_{\ell,1} \\
& +n_{j'}(\gamma)B^k_{j'k'}F_{\ell,\rho-1}[\gamma\backslash j',k'] + C^k_{j'k'}F_{\ell-1,\rho+1}[\gamma,j',k'] \\
& +C^k_{j'k'}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g,|J|+1}[j',J]F_{g',|K|+1}[k',K]
\end{aligned}$$

$$\begin{aligned}
F_{\ell,\rho}[p,\gamma\backslash j,k] = \;& 2(A^p_{\gamma_1,k} + A^p_{\gamma_2,k})\delta_{\rho,3}\delta_{\ell,0} + D^p\delta_{\rho,1}\delta_{\ell,1} \\
& +n_{j'}(\gamma\backslash j,k)B^p_{j'k'}F_{\ell,\rho-1}[(\gamma\backslash j,k)\backslash j',k'] + C^p_{j'k'}F_{\ell-1,\rho+1}[\gamma\backslash j,k,j',k'] \\
& +C^p_{j'k'}\sum_{(J,K)\models(\gamma\backslash j,k)}\sum_{m+m'=\ell}F_{m,|J|+1}[j',J]F_{m',|K|+1}[k',K] \\
= \;& 2(A^p_{\gamma_1,k} + A^p_{\gamma_2,k})\delta_{\rho,3}\delta_{\ell,0} + B^p_{kk'}F_{\ell,\rho-1}[\gamma\backslash j,k'] + D^p\delta_{\rho,1}\delta_{\ell,1} \\
& +n_{j'}(\gamma\backslash j)B^p_{j'k'}F_{\ell,\rho-1}[\gamma\backslash\{j,j'\},k,k'] + C^p_{j'k'}F_{\ell-1,\rho+1}[\gamma\backslash j,k,j',k'] \\
& +2C^p_{j'k'}\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell}F_{m,|J|+1}[k,j',J\backslash j]F_{m',|K|+1}[k',K]
\end{aligned}$$

$$F_{\ell-1,\rho+2}[p,\gamma,j,k] = 2A^p_{jk}\delta_{\ell-1,0}\delta_{\rho,1}$$
$$+n_{j'}(\gamma,j,k)B^p_{j'k'}F_{\ell-1,\rho+1}[(\gamma,j,k)\backslash j',k'] + C^p_{j'k'}F_{\ell-2,\rho+3}[\gamma,j,k,j',k']$$
$$+C^p_{j'k'}\sum_{(J,K)\models(\gamma,j,k)}\sum_{m+m'=\ell-1}F_{m,|J|+1}[j',J]F_{m',|K|+1}[k',K]$$

$$= 2A^p_{jk}\delta_{\ell-1,0}\delta_{\rho,1} + B^p_{kk'}F_{\ell-1,\rho+1}[\gamma,j,k'] + B^p_{jk'}F_{\ell-1,\rho+1}[\gamma,k,k']$$
$$+n_{j'}(\gamma)B^p_{j'k'}F_{\ell-1,\rho+1}[\gamma\backslash j',j,k,k'] + C^p_{j'k'}F_{\ell-2,\rho+3}[\gamma,j,k,j',k']$$
$$+C^p_{j'k'}\sum_{(J,K)\vdash\gamma}\sum_{m+m'=\ell-1}2F_{m,|J|+1}[j',j,k,J]F_{m',|K|+1}[k',K]$$
$$+C^p_{j'k'}\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell-1}2F_{m,|J|+1}[j',j,J]F_{m',|K|+1}[k',k,K]$$

$$F_{g,|J|+2}[j,p,J] = 2A^p_{j,J}\delta_{|J|,1}\delta_{g,0}$$
$$+n_{j'}(j,J)B^p_{j'k'}F_{g,|J|+1}[(j,J)\backslash j',k'] + C^p_{j'k'}F_{g-1,|J|+3}[j,J,j',k']$$
$$+C^p_{j'k'}\sum_{(J',K')\models(j,J)}\sum_{m+m'=g}F_{m,|J'|+1}[j',J']F_{m',|K'|+1}[k',K']$$

$$= 2A^p_{j,J}\delta_{|J|,1}\delta_{g,0} + B^p_{jk'}F_{g,|J|+1}[J,k']$$
$$+n_{j'}(J)B^p_{j'k'}F_{g,|J|+1}[J\backslash j',j,k'] + C^p_{j'k'}F_{g-1,|J|+3}[j,J,j',k']$$
$$+C^p_{j'k'}\sum_{(J',K')\models J}\sum_{m+m'=g}2F_{m,|J'|+1}[j,j',J']F_{m',|K'|+1}[k',K']$$

resulting in:

$$
\begin{aligned}
F_{\ell,\rho+1}[i,p,\gamma] = {} & 2A^i_{p,\gamma}\delta_{\rho,2}\delta_{\ell,0} + D^i\delta_{\rho,0}\delta_{\ell,1} \\
& + B^i_{pk}\Big(2A^k_{\gamma_1,\gamma_2}\delta_{\rho,3}\delta_{\ell,0} + D^k\delta_{\rho,1}\delta_{\ell,1} \\
& \qquad + n_{j'}(\gamma)B^k_{j'k'}F_{\ell,\rho-1}[\gamma\backslash j',k'] + C^k_{j'k'}F_{\ell-1,\rho+1}[\gamma,j',k'] \\
& \qquad + C^k_{j'k'}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g,|J|+1}[j',J]F_{g',|K|+1}[k',K]\Big) \\
& + n_j(\gamma)B^i_{jk}\Big(2(A^p_{\gamma_1,k}+A^p_{\gamma_2,k})\delta_{\rho,3}\delta_{\ell,0} + B^p_{kk'}F_{\ell,\rho-1}[\gamma\backslash j,k'] + D^p\delta_{\rho,1}\delta_{\ell,1} \\
& \qquad + n_{j'}(\gamma\backslash j)B^p_{j'k'}F_{\ell,\rho-1}[\gamma\backslash\{j,j'\},k,k'] + C^p_{j'k'}F_{\ell-1,\rho+1}[\gamma\backslash j,k,j',k'] \\
& \qquad + 2C^p_{j'k'}\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell}F_{m,|J|+1}[k,j',J\backslash j]F_{m',|K|+1}[k',K]\Big) \\
& + C^i_{jk}\Big(2A^p_{jk}\delta_{\ell-1,0}\delta_{\rho,1} + B^p_{kk'}F_{\ell-1,\rho+1}[\gamma,j,k'] + B^p_{jk'}F_{\ell-1,\rho+1}[\gamma,k,k'] \\
& \qquad + n_{j'}(\gamma)B^p_{j'k'}F_{\ell-1,\rho+1}[\gamma\backslash j',j,k,k'] + C^p_{j'k'}F_{\ell-2,\rho+3}[\gamma,j,k,j',k'] \\
& \qquad + C^p_{j'k'}\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell-1}2F_{m,|J|+1}[j',j,k,J]F_{m',|K|+1}[k',K] \\
& \qquad + C^p_{j'k'}\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell-1}2F_{m,|J|+1}[j',j,J]F_{m',|K|+1}[k',k,K]\Big) \\
& + 2C^i_{jk}\sum_{(J,K)\models\gamma}\sum_{g+g'=\ell}F_{g',|K|+1}[k,K]\times \\
& \qquad \Big(2A^p_{j,J}\delta_{|J|,1}\delta_{g,0} + B^p_{jk'}F_{g,|J|+1}[J,k'] \\
& \qquad + n_{j'}(J)B^p_{j'k'}F_{g,|J|+1}[J\backslash j',j,k'] + C^p_{j'k'}F_{g-1,|J|+3}[j,J,j',k'] \\
& \qquad + C^p_{j'k'}\sum_{(J',K')\models J}\sum_{m+m'=g}2F_{m,|J'|+1}[j,j',J']F_{m',|K'|+1}[k',K']\Big)
\end{aligned}
$$

This is going to be easier than it looks! Let's re-group:

$$
\begin{aligned}
F_{\ell,\rho+1}[i,p,\gamma] =\ & F_{\ell-1,\rho+1}[\gamma,j',k']\left(B^i_{pk}C^k_{j'k'}+C^i_{j'k}B^p_{kk'}+C^i_{jj'}B^p_{jk'}\right)\\
&+F_{\ell,\rho-1}[\gamma\backslash j',k']n_{j'}(\gamma)\left(B^i_{pk}B^k_{j'k'}+B^i_{j'k}B^p_{kk'}+4C^i_{jk'}A^p_{jj'}\right)\\
&+\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell}F_{m,|J|+1}[j',J]F_{g',|K|+1}[k',K]\left(B^i_{pk}C^k_{j'k'}+2C^i_{jk'}B^p_{jj'}\right)\\
&\\
&+F_{\ell-1,\rho+1}[\gamma\backslash j,k,j',k']n_j(\gamma)\left(B^i_{jk}C^p_{j'k'}+C^i_{j'k}B^p_{jk'}\right)\\
&+\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell-1}2F_{m,|K|+1}[k',K]F_{m',|J|+3}[j,j',k,J]\left(C^i_{jk}C^p_{j'k'}+C^i_{jk'}C^p_{j'k}\right)\\
&+\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell}2F_{m,|J|+1}[J\backslash j,j',k]F_{m',|K|+1}[k',K]\left(n_j(\gamma)B^i_{jk}C^p_{j'k'}+C^i_{j'k'}n_j(J)B^p_{jk}\right)\\
&+\sum_{(J',K',K)\models\gamma}\sum_{g+g'+g''=\ell}4F_{g,|K|+1}[k,K]F_{g',|J'|+1}[j,j',J']F_{g'',|K'|+1}[k',K']\left(C^i_{jk}C^p_{j'k'}\right)\\
&+F_{\ell,\rho-1}[\gamma\backslash\{j,j'\},k,k']n_j(\gamma)n_{j'}(\gamma\backslash j)\left(B^i_{jk}B^p_{j'k'}\right)\\
&+F_{\ell-2,\rho+3}[\gamma,j,k,j',k']\left(C^i_{jk}C^p_{j'k'}\right)\\
&+\sum_{(J,K)\models\gamma}\sum_{m+m'=\ell-1}2F_{m,|J|+1}[j',j,J]F_{m',|K|+1}[k',k,K]\left(C^i_{jk}C^p_{j'k'}\right)\\
&\\
&+2\left(B^i_{pk}A^k_{\gamma_1,\gamma_2}+n_j(\gamma)B^i_{jk}(A^p_{\gamma_1,k}+A^p_{\gamma_2,k})\right)\delta_{\rho,3}\delta_{\ell,0}\\
&+2C^i_{jk}A^p_{jk}\delta_{\ell,1}\delta_{\rho,1}-D\delta_{\rho,0}\delta_{\ell,1}\\
&+n_j(\gamma)B^i_{jk}D^p\delta_{\rho,1}\delta_{\ell,1}+B^i_{pk}D^k\delta_{\rho,1}\delta_{\ell,1}
\end{aligned}
$$

Taking $\ell > 1$ momentarily eliminates the final three lines – we will have to check $F_{0,4}$ and $F_{1,2}$ manually ($F_{1,1}$ is obvious). The middle seven lines are all manifestly symmetric in i and p (assuming, inductively, that the previous $F_{g,n}$ are). We concern ourselves now with the expressions within the round brackets of the first three lines, each of which will turn out to be symmetric:

$$B^i_{pk}C^k_{j'k'} + C^i_{j'k}B^p_{kk'} + C^i_{jj'}B^p_{jk'} \tag{3.21}$$

$$B^i_{pk}B^k_{j'k'} + B^i_{j'k}B^p_{kk'} + 4C^i_{jk'}A^p_{jj'} \tag{3.22}$$

$$B^i_{pk}C^k_{j'k'} + 2C^i_{jk'}B^p_{jj'} \qquad = B^i_{pk}C^k_{j'k'} + C^i_{jk'}B^p_{jj'} + C^i_{jj'}B^p_{jk'} \tag{3.23}$$

We have observed that (3.23) can be re-written, and is equivalent to (3.21). These are the relations (3.17) and (3.18), thus they are symmetric as a result of Corollary 2.

The Base Case

The smallest value of the recursion parameter, $2g-2+n$, is 1. The recursion, then, is based on $F_{0,3}$, $F_{\frac{1}{2},2}$, and $F_{1,1}$. We saw in the proof of Proposition 2 that the second of these was zero, and the third is clearly symmetric. As for $F_{0,3}[i,j,k]$, we saw that it is proportional to A^i_{jk} and so its symmetry results from (3.11) along with the assumed symmetry of A's lower indices.

We must also check $F_{0,4}$ and $F_{1,2}$:

$$\begin{aligned}
F_{0,4}[i,j,k,s] &= n_{j'}(j,k,s)B^i_{j'k'}F_{0,3}[(j,k,s)\backslash j',k'] \\
&= B^i_{j,k'}F_{0,3}[k,s,k'] + B^i_{k,k'}F_{0,3}[j,s,k'] + B^i_{s,k'}F_{0,3}[j,k,k'] \\
&= 2(B^i_{j,k'}A^k_{s,k'} + B^i_{k,k'}A^j_{s,k'} + B^i_{s,k'}A^j_{k,k'})
\end{aligned}$$

$$\begin{aligned}
F_{1,2}[i,j] &= B^i_{jk}F_{0,1}[k] + C^i_{j'k}F_{0,3}[j,j',k] \\
&= B^i_{jk}D^k + C^i_{j'k}A^j_{j'k}
\end{aligned}$$

These are the expressions (3.19) and (3.20), thus they are symmetric by Corollary 2.

□

Chapter 4

Higher Airy Structures

Higher Quantum Airy Structures – or, in our terminology, simply [not-necessarily-quadratic] Quantum Airy Structures – were first defined and studied in [8] by Borot et al. Thinking about enumerative problems characterized by Virasoro constraints leads to the quadratic Airy structures, and in similar fashion higher Airy structures were introduced to handle problems characterized instead by $\mathcal{W}$-constraints. A $\mathcal{W}$-algebra is an extension of the Virasoro algebra, and this additional generality allows the partition functions constructed by higher structures to capture a wider array of geometric specimens. Situations in which $\mathcal{W}$-constraints crop up include Fan-Jarvis-Ruan theories and open intersection theory (see [8]).

We will review here the cornerstone theorem, existence and uniqueness, in the non-quadratic case. The logical sequence of the proofs are similar, although the computations are much more involved.

4.A Conceptual Proof

We now present the proof which we had deferred in Chapter 3. It is a significantly abridged adaption of the proof available in [31]. Our new conventions, which mix up both g and n into the exponent of $\hbar$, has allowed us to approach more straightforwardly, using one induction over the $\hbar$-degree where there had been two inductions over g and n. It manages to avoid all talk of a "Classical Limit" for a Quantum Airy Structure, although that concept remains quite important for Kontsevich and Soibelman's broader point of view.

Theorem 6. *Suppose we have a Quantum Airy Structure $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$. Then, among all $Z \in \mathcal{S}^\hbar$ of the form* (4.1)*, there is a unique solution to the system of equations $\hat{H}_i Z = 0$ for $i \in I$.*

It will be more convenient to prove this statement for Quantum Airy Structures that are possibly cross-capped. We will be making use of notation such as $P[x > 3]$ or $P[\partial = 0]$ which indicate an arbitrary series in $\mathcal{O}^\hbar$ with, respectively, no terms of x-deg ≤ 3 and no terms of ∂-deg > 0. P may refer to a different polynomial in each instance until it gains an index, after which point it is taken to indicate one particular polynomial.

Proof. The proof is by induction on $2g - 1 + n$. We write:

$$Z = \exp(F) = \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha\right] \tag{4.1}$$

for some coefficients $F_{g,n}[\alpha] \in \mathbb{C}$. Secondly, replace $\hat{H}_i e^F = 0$ with the equivalent equations $e^{-F}\hat{H}_i e^F = 0$, which are easier to analyze. Thirdly, we need to answer a question about the expression $e^{-F}\hat{H}_i e^F$: what is the minimum $\hbar$-degree to which a given $F_{g,n}$ contributes any terms? Considering a generic term $\hbar^A x_I \partial_J$ within $\hat{H}_i$ (with $A \geqslant |I| + |J|$[1]), its contribution to $e^{-F}\hat{H}_i e^F$ is:

$$\hbar^A x_I \sum_{\mathbb{K} \vdash J} \prod_{K \in \mathbb{K}} \sum_{g,n,\alpha} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] \partial_K(x_\alpha)$$

Every term in the right-most sum has $\hbar$-degree ≥ 1, so the minimum is obtained when the product has just one factor, or $\mathbb{K} = \{J\}$. We may as well take $I = \varnothing$ and $A = |J| = 1$. The result is that the $\hbar$-minimal term with a particular $F_{g,n}[\alpha]$-factor has, at least, $\hbar$-deg $= 2g - 1 + n$. This is saturated only by $\hat{H}_i$'s linear term.

For any ℓ, the partial sum:

$$S_\ell := \sum_{g,n,\alpha}^{2g-1+n \leq \ell} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha$$

is such that $e^{-F}\hat{H}_i e^F$ and $e^{-S_\ell}\hat{H}_i e^{S_\ell}$ agree in all terms at or below $\hbar$-degree ℓ. The desired equation $e^{-F}\hat{H}_i e^F = 0$ is equivalent to the conditions that, for each ℓ, we have $e^{-S_\ell}\hat{H}_i e^{S_\ell} = P[\hbar > \ell]$.[2]

Inductive Step in ℓ

Fix ℓ. Suppose we have shown, for any (possibly cross-capped) QAS, the existence and uniqueness of a sum:

$$S_\ell = \sum_{g,n,\alpha}^{2g-1+n \leq \ell} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha$$

[1] This condition is, of course, true of any Quantum Airy Structure. But we will see that it is not at all necessary.

[2] This is an existence/uniqueness condition on the S_ℓ, but we also require that each S_ℓ is an initial segment of later $S_{\ell' > \ell}$.

such that:

$$e^{-S_\ell}\hat{H}_i e^{S_\ell} = P[\hbar > \ell] \tag{4.2}$$

The target equation, $e^{-F}\hat{H}_i e^F = 0$, is equivalent to:

$$e^{-F'}\hat{H}'_i e^{F'} := e^{-F+S_\ell}[e^{-S_\ell}\hat{H}_i e^{S_\ell}]e^{F-S_\ell} = 0$$

$$F' := F - S_\ell \qquad\qquad \hat{H}'_i := e^{-S_\ell}\hat{H}_i e^{S_\ell}$$

with the primed operators trivially forming another QAS[3]. Therefore we may search for F' instead of F, knowing that they agree in all $\hbar$-degrees above $\ell - 1$. The benefit is that our induction hypothesis gives us uniqueness of $F'_{g,n}$ whenever $2g-1+n \leq \ell$, which proves that S'_ℓ is necessarily zero. Since we know also that it satisfies (4.2) with primes inserted, we learn the form of the $\hat{H}'$ must be:

$$\begin{aligned}
\hat{H}'_i &= -\hbar\partial_i + P_i[\partial = 0, \hbar > \ell] + Y_i[\partial > 0, \hbar \geq 2] \\
&= -\hbar\partial_i + Q_i[\partial = 0, \hbar = \ell+1] + R_i[\partial = 0, \hbar > \ell+1] + Y_i[\partial > 0, \hbar \geq 2]
\end{aligned}$$

The operator Y_i will only produce terms of $\hbar$-degree $\ell + 2$ or more when we insert

$$S'_{\ell+1} = \sum_{g,n,\alpha}^{2g-1+n=\ell+1} \frac{\hbar^\ell}{n!} F'_{g,n}[\alpha]x_\alpha = \sum_{g,n,\alpha}^{2g-1+n=\ell+1} \frac{\hbar^\ell}{n!} F_{g,n}[\alpha]x_\alpha$$

into $e^{-S'_{\ell+1}}\hat{H}'_i e^{S'_{\ell+1}}$. Thus our present aim (demanding it be equal to $P[\hbar > \ell + 1]$) amounts to $\hbar\partial_i S'_{\ell+1} = Q_i$ for each i.

[3]This is the reason we must include cross-capped QASs; conjugation of a strict QAS does not result necessarily in another QAS.

For any two $\hat{H}'$, their commutator has the form:

$$[\hat{H}'_i, \hat{H}'_j] = -\hbar([\partial_i, Q_j] + [\partial_i, R_j] + [\partial_i, Y_j]) + [Q_i, Y_j] + [R_i, Y_j] + \frac{1}{2}[Y_i, Y_j] \\ - (i \leftrightarrow j)$$

The $\partial = 0$ part of this[4] is:

$$[\partial^0][\hat{H}'_i, \hat{H}'_j] = -\hbar(\partial_i Q_j + \partial_i R_j) - (Y_j \cdot Q_i + Y_j \cdot R_i) \\ - (i \leftrightarrow j)$$

The action of $Y\cdot$ increases $\hbar$-degree by at least two, so the $[\hbar = \ell + 2]$ part of the above is $\hbar(\partial_i Q_j - \partial_j Q_i)$. This commutator must also belong to the ideal $\hbar^2 \mathcal{O}^\hbar \cdot \mathbb{H}$, which requires its ∂-degree zero part to be $P[\hbar > \ell + 2]$, forcing that $\partial_i Q_j = \partial_j Q_i$. Momentarily taking I finite, the Poincare lemma gives existence as well as uniqueness (up to an additive constant $\in \mathbb{C}[[\hbar]]$, which we must choose to be zero) of an $S'_{\ell+1}$ such that $\hbar \partial_i S'_{\ell+1} = Q_i$ for each i. And, of course, it has the appropriate $\hbar$-degree.

By separating the terms of $S'_{\ell+1}$ according to x-monomials and assuming α-symmetry, we get uniqueness for each $F_{g,n}[\alpha]$ whenever $2g - 1 + n \leq \ell + 1$. The inductive hypothesis has been extended to $\ell + 1$.

We must handle the case $I = \mathbb{N}$. Although $\hat{H}_i$ may have infinitely many terms, there are only finitely many at each $\hbar$-degree. Therefore each Q_i is a polynomial.

[4]It may seem this is not well-defined. What is the $\partial = 0$ part of $\partial x = x\partial + 1$? It is well-defined if the operation is "commute all ∂ to the far right and then take the $\partial = 0$ part." In other words, quotient $\mathcal{O}^\hbar$ by the left-ideal $\langle \partial \rangle$, not the right ideal. Another way to define $[\partial^0]F$ would be as [the natural inclusion $\mathcal{S}^\hbar \hookrightarrow \mathcal{O}^\hbar$ of] $F \cdot 1$.

However, there are infinitely many Q_i.

Consider a finite initial segment $\{1, \ldots, J\} \subset I$. Then $\{Q_j : j \leq J\}$ is a finite set, belonging to a finite-dimensional subspace of $\mathbb{C}[[\{x_i\}_{i \in I}]]$, and the Poincare lemma yields an S_ℓ^J such that $\hbar \partial_j S_\ell^J = Q_j$ for all $j \leq J$. We can similarly find S_ℓ^{J+1}. Note that $\partial_j (S_\ell^{J+1} - S_\ell^J) = 0$ for all $j \leq J$, which means that S_ℓ^{J+1} is obtained from S_ℓ^J by the addition of a polynomial involving *only* x_k with $k > J$. The sequence of polynomials S_ℓ^J yields a power series S_ℓ^∞, defined such that its $\{x_j : j \leq J\}$-nonconstant part agrees with S_ℓ^J, which serves in the role of S_ℓ above.

Base Case in ℓ

The smallest value that $\ell = 2g - 1 + n$ can take is 2. We would like to establish the uniqueness and existence of a sum:

$$S_2 = \sum_{2g-1+n=2} \sum_\alpha \frac{1}{n!} \hbar F_{g,n}[\alpha] x_\alpha = \hbar \left(\frac{1}{6} F_{0,3}[i,j,k] x_i x_j x_k + \frac{1}{2} F_{\frac{1}{2},2}[i,j] x_i x_j + F_{1,1}[i] x_i \right)$$

such that $e^{-S_2} \hat{H}_i e^{S_2} = P[\hbar > 2]$. This amounts to (we have provided names for the coefficients of the monomials 1, x_j, and $x_j x_k$ in $\hat{H}_i$):

$$\hbar \partial_i S_2 = \hbar^2 \left(A_i[\varnothing] + A_i[j] x_j + A_i[j,k] x_j x_k \right)$$

Giving, in turn, $F_{1,1}[i] = A_i[\varnothing]$, $F_{\frac{1}{2},1}[i,j] = A_i[j]$, and $F_{0,3}[i,j,k] = 2A_i[j,k]$. Clearly, once the QAS and thus the A_i are chosen, S_2 is uniquely fixed.

$\square$

4.B Computational Proof

Despite having an existence and uniqueness proof already, it does not lay out the recursion formula explicitly. This can be done, and has been in [8]. First, two definitions. We adopt a uniform notation for both x and ∂.

Definition 14. *For any $K = (k_1, k_2, \dots) \in I^m$, set $\overline{K} := (-k_1, -k_2, \dots)$. Define $\mathcal{I} := I \sqcup \overline{I}$, as well as*

$$Q_a := \begin{cases} \partial_a & a > 0 \\ x_{-a} & a < 0 \end{cases}$$

for any $a \in \mathcal{I}$.

Definition 15. *We say that a multi-index $W \in \mathcal{I}^m$ is **normally ordered** if all of its negative values $W_i < 0$ occur before [at smaller i than] any of its positive values. In this case, an operator Q_W has all x-factors to the left of all ∂-factors[5]. If W is a multi-index, let $\mathcal{N}(W)$ refer to any permutation that is normally ordered.*

We can present an expression for the general form of a Higher Quantum Airy Structure:

$$\hat{H}_i = -\hbar Q_i + \sum_{m \geq 2} \sum_{0 \leq \ell \leq m} \frac{\hbar^m}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(m)}[i|\alpha] Q_{\mathcal{N}(\alpha)} \qquad i \in I \tag{4.3}$$

Given a Higher Quantum Airy Structure, we can express the recursion for its partition function in terms of its coefficients, the $C^{(m)}[i|\alpha]$. Recalling Definition 11 for the symbol $\models$, we make one simplification. Instead of writing $(A_k)_{k \in K} \models \beta$ we write $A : K \models \beta$, treating A as a function on K. To refresh, this means that the images $A(k)$ are (possibly empty) tuples $(\beta_j)_{j \in J_\ell}$ which are pair-wise disjoint[6] and whose

[5] At least, it has one or more such representations

[6] Meaning, strictly, that the J_ℓ are pair-wise disjoint.

concatenation is β.[7]

Definition 16. *Given a sequence $\{C^{(m)}[i|\alpha] : m \geqslant 2, 0 \leqslant \ell \leqslant m, \alpha \in \mathcal{I}^\ell, i \in I\}$, the **Topological Recursion** refers to the equations, for each $i \in I$, $\beta \in I^{|\beta|}$, and $G \in \mathbb{N}_0$ such that $2G - 2 + (|\beta| + 1) > 0$,*

$$F_{G,|\beta|+1}[i, \beta] = \sum_{m\geq 2} \sum_{0\leq \ell \leq m} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^\ell} C^{(m)}[i|\alpha] \sum_{\mathbb{J}\vdash\mathcal{N}(\alpha)} \sum''_{\mathbb{A}:\mathbb{J}\models\beta} \sum^{\sim}_{g:\mathbb{J}\to\mathbb{N}_0} \prod_{J\in\mathbb{J}} F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J), J] \tag{4.4}$$

relating the members of a sequence $\{F_{g,n}[\alpha] : g \geq 0, n \geq 1, 2g-2+n > 0, \alpha \in I^n\}$. The g-sum is restricted so that $\sum_{J\in\mathbb{J}}(2g(J) - 2 + |\mathbb{A}(J)| + |J|) = 2G - m + |\beta|$, notated with a $\sim$. The double primes indicate a restriction on the $\mathbb{A}$-sum such that $\mathbb{A}(J) = \varnothing$ does not occur at the same time as $J \cap \mathbb{Z}_- \neq \varnothing$.

What we aim to prove in this section is

Theorem 7. *Let $\mathbb{H} := \{\hat{H}_i\}$ be a Quantum Airy Structure. Among all power series Z of the form:*

$$Z = \exp\left[\sum_{(g,n)\in\frac{1}{2}\mathbb{N}_0\times\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha] x_\alpha\right] \tag{4.5}$$

the solution to the equations $\hat{H}_i Z = 0$ for $i \in I$, if any, is unique. Furthermore, the $F_{g,n}[\alpha] \in \mathbb{C}$ are given by the Topological Recursion.

This will follow easily after a straightforward lemma. First we expand the definition of each $F_{g,n}[\alpha]$ to allow $\alpha \in \mathcal{I}^n$: each $F_{g,n}$[8] vanishes whenever it takes negative arguments. We also introduce $F_{0,2}[a, b] := \delta_{a,-b}$. In the following we

[7] Strictly, the union of the J_ℓ is $\{1, \ldots, |\beta|\}$.
[8] Except $F_{0,2}$, obviously.

briefly suppress the g-summation by writing:

$$F[\alpha] := \sum_{g \in \frac{1}{2}\mathbb{N}_0}^{2g-2+|\alpha|>0} \hbar^{2g-2+|\alpha|} F_{g,|\alpha|}[\alpha]$$

for any $\alpha \in \mathcal{I}^n$.

Lemma 6. *For any n and $W \in \mathcal{I}^n$, we have*

$$\partial_\beta (e^{-F} Q_{\mathcal{N}(W)} \cdot e^F)|_0 = \sum_{\mathbb{J} \vdash W} \sum_{\mathbb{A}:\mathbb{J} \models \beta}'' \prod_{J \in \mathbb{J}} F[\mathbb{A}(J), J]$$

The double prime notation indicates that the $\mathbb{A}$-sum has been restricted in such a way that $\mathbb{A}(A) = \varnothing$ does not occur at the same time as $A \cap \mathbb{Z}_- \neq \varnothing$.

Proof. Let's write $Q_{\mathcal{N}(W)} := x_K \partial_L$ for some K, L and compute:

$$
\begin{aligned}
\partial_\beta(e^{-F} x_K \partial_L \cdot e^F)|_0 &= \sum_{\mathbb{B}:K\sqcup\{L\}\models\beta} \left(\prod_{k\in K} \partial_{\mathbb{B}(k)} x_k\right) \left(\partial_{\mathbb{B}(L)} \left(e^{-F}\partial_L \cdot e^F\right)\right)\Bigg|_0 \\
&= \sum_{\mathbb{B}:K\sqcup\{L\}\models\beta} \left(\prod_{k\in K} \partial_{\mathbb{B}(k)} x_k\right) \left(\partial_{\mathbb{B}(L)} \sum_{\mathbb{J}\vdash L} \prod_{J\in\mathbb{J}} \partial_J F\right)\Bigg|_0 \\
&= \sum_{\mathbb{B}:K\sqcup\{L\}\models\beta} \left(\prod_{k\in K} \partial_{\mathbb{B}(k)} x_k\right) \left(\sum_{\mathbb{J}\vdash L} \sum_{\mathbb{L}:\mathbb{J}\models\mathbb{B}(L)} \prod_{J\in\mathbb{J}} (\partial_{\mathbb{L}(J),J} F)|_0\right) \\
&= \sum_{\mathbb{B}:K\sqcup\{L\}\models\beta} \left(\prod_{k\in K} \delta_{\mathbb{B}(k),k}\right) \left(\sum_{\mathbb{J}\vdash L} \sum_{\mathbb{L}:\mathbb{J}\models\mathbb{B}(L)} \prod_{J\in\mathbb{J}} F[\mathbb{L}(J), J]\right) \\
&= \sum_{\mathbb{B}:\overline{K}\sqcup\{L\}\models\beta} \left(\prod_{k\in \overline{K}} F_{0,2}[k, \mathbb{B}(k)]\right) \left(\sum_{\mathbb{J}\vdash L} \sum_{\mathbb{L}:\mathbb{J}\models\mathbb{B}(L)} \prod_{J\in\mathbb{J}} F[\mathbb{L}(J), J]\right) \\
&= \sum_{\mathbb{J}\vdash L} \sum_{\mathbb{B}:\overline{K}\sqcup\{L\}\models\beta} \sum_{\mathbb{L}:\mathbb{J}\models\mathbb{B}(L)} \left(\prod_{k\in \overline{K}} F_{0,2}[k, \mathbb{B}(k)]\right) \left(\prod_{J\in\mathbb{J}} F[\mathbb{L}(J), J]\right) &\quad (4.6) \\
&\stackrel{?}{=} \sum_{\mathbb{J}\vdash L} \sum_{\mathbb{A}:\overline{K}\sqcup\mathbb{J}\models\beta} \prod_{J\in\overline{K}\sqcup\mathbb{J}} F[\mathbb{A}(J), J] &\quad (4.7)
\end{aligned}
$$

Lets argue on behalf of this final equality. An $\mathbb{A}$ is equivalent to a $(\mathbb{B}, \mathbb{L})$ as follows: for $k \in \overline{K}$ set $\mathbb{B}(k) := \mathbb{A}(k)$, as well as $\mathbb{B}(L) := \{\mathbb{A}(J) : J \in \mathbb{J}\}$; then put $\mathbb{L}(J) := \mathbb{A}(J)$ for all $J \in \mathbb{J}$. However, line (4.6) only includes terms wherein negative indices occur nowhere except $F_{0,2}$, and that each $F_{0,2}$ contains exactly one negative index. Luckily, such are the only non-zero terms of line (4.7), thanks to our extended definitions of the F. We continue just a few more steps:

$$
\begin{aligned}
&= \sum_{\mathbb{J}\vdash L} \sum_{\mathbb{A}:\overline{K}\sqcup\mathbb{J}\models\beta} \prod_{J\in\overline{K}\sqcup\mathbb{J}} F[\mathbb{A}(J), J] &\quad (4.8) \\
&\stackrel{?}{=} \sum_{\mathbb{J}\vdash L\sqcup\overline{K}} \sum_{\mathbb{A}:\mathbb{J}\models\beta} \prod_{J\in\mathbb{J}} F[\mathbb{A}(J), J] &\quad (4.9)
\end{aligned}
$$

We would like to replace the sum over $\mathbb{J} \vdash L$ with a sum over $\mathbb{J} \vdash L \sqcup \overline{K}$, getting line (4.9). This would add additional terms, but almost all of them are zero. The possible cases are these:

- Every time $\mathbb{J}$ partitions $L \sqcup \overline{K}$ in such a way that $\overline{K}$'s elements are sequestered in singletons, we recover a term that was in (4.8) – and each term in (4.8) occurs this way, once.
- All terms vanish in which $\mathbb{J}$ has partitioned more than one element of $\overline{K}$ together.
- All terms vanish in which an element of $\overline{K}$ has been grouped with more than than one element of L.
- Remaining are cases in which some elements of $\overline{K}$ are grouped with exactly one element of L. These pairs (k, ℓ), as they contain a negative index, can only be used as the argument of $F_{0,2}$. This forces that $\mathbb{A}(k, \ell) = \varnothing$.

Cases described in the final bullet are the additional terms we would add if we made our intended replacement. We simply omit them: the $\mathbb{A}$-sum is restricted such that $\mathbb{A}(J) = \varnothing$ does not occur at the same time as $J \cap \mathbb{Z}_- \neq \varnothing$. This restriction is notated by a double-prime over the $\mathbb{A}$-sum. We continue, striking out (4.9) and writing instead:

$$= \sum_{\mathbb{J} \vdash L \sqcup \overline{K}} \sum_{\mathbb{A}: \mathbb{J} \models \beta}'' \prod_{J \in \mathbb{J}} F[\mathbb{A}(J), J]$$

Using the uniform operator notation, we can write

$$\partial_\beta (e^{-F} Q_{\mathcal{N}(W)} e^F)|_0 = \sum_{\mathbb{J} \vdash W} \sum_{\mathbb{A}: \mathbb{J} \models \beta}'' \prod_{J \in \mathbb{J}} F[\mathbb{A}(J), J]$$

for any $n \in \mathbb{N}$ and $W \in \mathcal{I}^n$. $\square$

At this point, it is quite simple to give the proof of our theorem.

Proof. The dirty work has already been done. The equation $\hat{H}_i \cdot e^F$ is equivalent to, for all $N \in \mathbb{N}_0$ and all $\beta \in I^{|\beta|}$, having $[\hbar^N]\partial_\beta(e^{-F}\hat{H}_i \cdot e^F)|_0 = 0$. Now, we know that

$$\begin{aligned}
[\hbar^N]\partial_\beta(e^{-F}\hat{H}_i \cdot e^F)|_0 &= [\hbar^N]\partial_\beta \left[-\hbar e^{-F} Q_i \cdot e^F + \sum_{m\geq 2} \sum_{0\leq \ell \leq m} \frac{\hbar^m}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(m)}[i|\alpha] e^{-F} Q_{\mathcal{N}(\alpha)} \cdot e^F \right]\Bigg|_0 \\
&= -[\hbar^{N-1}]\partial_\beta(e^{-F}Q_i \cdot e^F)|_0 + \sum_{m\geq 2} \sum_{0\leq \ell \leq m} \frac{1}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(m)}[i|\alpha][\hbar^{N-m}]\partial_\beta(e^{-F}Q_{\mathcal{N}(\alpha)} \cdot e^F)|_0 \\
&= -[\hbar^{N-1}]F[i,\beta] + \sum_{m\geq 2} \sum_{0\leq \ell \leq m} \frac{1}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(m)}[i|\alpha][\hbar^{N-m}] \sum_{\mathbb{J} \vdash \mathcal{N}(\alpha)} \sideset{}{''}\sum_{\mathbb{A}:\mathbb{J} \models \beta} \prod_{J \in \mathbb{J}} F[\mathbb{A}(J), J]
\end{aligned} \tag{4.10}$$

We should work out what power of $\hbar$ emerges from $\prod_{J\in\mathbb{J}} F[\mathbb{A}(J), J]$. We will make use of the interchange rule $\prod_{j\in J}\sum_{k\in K} a_{j,k} = \sum_{k:J\to K}\prod_{j\in J} a_{j,k(j)}$, of which one may easily become convinced. Lets see:

$$\begin{aligned}
\prod_{J\in\mathbb{J}} F[\mathbb{A}(J), J] &= \prod_{J\in\mathbb{J}} \sum_{g\in\mathbb{N}_0} \hbar^{2g-2+|\mathbb{A}(J)|+|J|} F_{g,|\mathbb{A}(J)|+|J|}[\mathbb{A}(J), J] \\
&= \sum_{g:\mathbb{J}\to\mathbb{N}_0} \prod_{J\in\mathbb{J}} \hbar^{2g(J)-2+|\mathbb{A}(J)|+|J|} F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J), J] \\
&= \sum_{g:\mathbb{J}\to\mathbb{N}_0} \hbar^{\sum_{J\in\mathbb{J}}(2g(J)-2+|\mathbb{A}(J)|+|J|)} \prod_{J\in\mathbb{J}} F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J), J]
\end{aligned}$$

Extracting the $\hbar^{N-m}$ term, we get

$$[\hbar^{N-m}]\prod_{J\in\mathbb{J}}F[\mathbb{A}(J),J] = \sum_{g:\mathbb{J}\to\mathbb{N}_0}^{\sum_{J\in\mathbb{J}}(2g(J)-2+|\mathbb{A}(J)|+|J|)=N-m}\prod_{J\in\mathbb{J}}F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J),J]$$
$$:= \sum_{g:\mathbb{J}\to\mathbb{N}_0}^{\sim}\prod_{J\in\mathbb{J}}F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J),J]$$

Returning to (4.10), we can see

$$[\hbar^N]\partial_\beta(e^{-F}\hat{H}_i\cdot e^F)|_0$$
$$= -[\hbar^{N-1}]F[i,\beta] + \sum_{m\geq 2}\sum_{0\leq\ell\leq m}\frac{1}{\ell!}\sum_{\alpha\in\mathcal{I}^\ell}C^{(m)}[i|\alpha][\hbar^{N-m}]\sum_{\mathbb{J}\vdash\mathcal{N}(\alpha)}\sum_{\mathbb{A}:\mathbb{J}\models\beta}^{\prime\prime}\prod_{J\in\mathbb{J}}F[\mathbb{A}(J),J]$$
$$= -F_{\frac{1}{2}(N-|\beta|),|\beta|+1}[i,\beta] + \sum_{m\geq 2}\sum_{0\leq\ell\leq m}\frac{1}{\ell!}\sum_{\alpha\in\mathcal{I}^\ell}C^{(m)}[i|\alpha]\sum_{\mathbb{J}\vdash\mathcal{N}(\alpha)}\sum_{\mathbb{A}:\mathbb{J}\models\beta}^{\prime\prime}\sum_{g:\mathbb{J}\to\mathbb{N}_0}^{\sim}\prod_{J\in\mathbb{J}}F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J),J]$$
$$:= -F_{G,|\beta|+1}[i,\beta] + \sum_{m\geq 2}\sum_{0\leq\ell\leq m}\frac{1}{\ell!}\sum_{\alpha\in\mathcal{I}^\ell}C^{(m)}[i|\alpha]\sum_{\mathbb{J}\vdash\mathcal{N}(\alpha)}\sum_{\mathbb{A}:\mathbb{J}\models\beta}^{\prime\prime}\sum_{g:\mathbb{J}\to\mathbb{N}_0}^{\sim}\prod_{J\in\mathbb{J}}F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J),J]$$

We have substituted $G := \frac{1}{2}(N-|\beta|)$, and as before the $\sim$ refers to the condition on g that $\sum_{J\in\mathbb{J}}(2g(J)-2+|\mathbb{A}(J)|+|J|)$. Therefore, the system of equations $\hat{H}_i\cdot e^F = 0$ is equivalent to the system

$$F_{G,|\beta|+1}[i,\beta] = \sum_{m\geq 2}\sum_{0\leq\ell\leq m}\frac{1}{\ell!}\sum_{\alpha\in\mathcal{I}^\ell}C^{(m)}[i|\alpha]\sum_{\mathbb{J}\vdash\mathcal{N}(\alpha)}\sum_{\mathbb{A}:\mathbb{J}\models\beta}^{\prime\prime}\sum_{g:\mathbb{J}\to\mathbb{N}_0}^{\sim}\prod_{J\in\mathbb{J}}F_{g(J),|\mathbb{A}(J)|+|J|}[\mathbb{A}(J),J] \tag{4.11}$$

for all $i\in I$, all $\beta\in I^{|\beta|}$, and all $G\in\mathbb{N}_0$ such that $2G-2+(|\beta|+1)>0$.

This system is recursive. Again, the sum on g is restricted in such a way that $\sum_{J\in\mathbb{J}}(2g(J)-2+|\mathbb{A}(J)|+|J|) = N-m$. Each summand in the lefthand side of that expression is non-negative, since $g(J)\geq 0$ and $|\mathbb{A}(J)|+|J|\geq 2$. Therefore, for each J, we have $2g(J)-2+|\mathbb{A}(J)|+|J| \leq N-m = 2G+|\beta|-m < 2G-2+(|\beta|+1)$ as long as $m > 1$, which is always the case. The familiar expression "$2g-2+n$" is greater on the lefthand side than in any term on the righthand side. $\square$

To complete a treatment of Higher Topological Recursion in parallel to that of Chapter 3, we would now have to give the computational proof of symmetry. And although symmetry is guaranteed, since partition functions exist for Higher Airy Structures as a result of Theorem 6,[9] this explicit calculation remains unperformed. We cannot give that calculation here, but it should proceed in very roughly the same way as in the quadratic case.

It is not hard to twice expand, as we did in Chapter 3:

$$F_{g,n}[k,i,\gamma] = \sum_{\ell,j\geq 0} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^\ell} C^j[k|\alpha] \sum_{\mathbb{J}\vdash\alpha} \sum_{h:\mathbb{J}\to\mathbb{N}}^{\ell+j+\sum_{\mathbb{J}} h(J)=g+|\mathbb{J}|} \times$$
$$\sum_{\mu\vdash_{\mathbb{J}}(i,\beta)}'' \left(\prod_{J\in\mathbb{J}} F_{h(J),|J|+|\mu(J)|}[J,\mu(J)] \right)$$

$$= \sum_{\ell,j\geq 0} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^\ell} C^j[k|\alpha] \sum_{\mathbb{J}\vdash\alpha} \sum_{h:\mathbb{J}\to\mathbb{N}}^{\ell+j+\sum_{\mathbb{J}} h(J)=g+|\mathbb{J}|} \times$$
$$\sum_{\mu\vdash_{\mathbb{J}}\beta}'' \sum_{J'\in\mathbb{J}} \left(F_{h(J')}[J',\mu(J'),i] \prod_{J\in\mathbb{J}\setminus J'} F_{h(J)}[J,\mu(J)] \right)$$

[9]The partition function can clearly be chosen to have symmetric coefficients, which satisfy the recursion, and since the $F_{g,n}$ constructed by the recursion are unique they must be these symmetric coefficients. Of course, the partition function can just as easily be chosen not to have symmetric coefficients, however those would *not* satisfy the recursion – their symmetrizations would.

$$
\begin{aligned}
= \sum_{\ell,j\geq 0} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^\ell} C^j[k|\alpha] \sum_{\mathbb{J}\vdash\alpha} \sum_{h:\mathbb{J}\to\mathbb{N}}^{\ell+j+\sum_{\mathbb{J}} h(J)=g+|\mathbb{J}|} \sum_{\mu\vdash_{\mathbb{J}}\beta}^{\prime\prime} \sum_{J'\in\mathbb{J}} \Bigg[\\
\sum_{\ell',j'\geq 0} \frac{1}{\ell'!} \sum_{\alpha'\in\mathcal{I}^{\ell'}} C^{j'}[i|\alpha'] \sum_{\mathbb{K}\vdash\alpha'} \sum_{h':\mathbb{K}\to\mathbb{N}}^{\ell'+j'+\sum_{\mathbb{K}} h'(K)=h(J')+|\mathbb{K}|} \times \\
\sum_{\mu'\vdash_{\mathbb{K}}\beta}^{\prime\prime} \left(\prod_{K\in\mathbb{K}} F_{h'(K)}[K,\mu'(K)] \right) \prod_{J\in\mathbb{J}\setminus J'} F_{h(J)}[J,\mu(J)] \Bigg]
\end{aligned}
$$

To mimic the earlier proof (of Proposition 4), one would perhaps try to separate out the different J' terms according to the size of J'. This corresponds to the separation of terms on page 45 according to whether nothing was removed from γ, one thing was removed, or γ became partitioned across factors of F. Since the i, p-symmetry worked slightly differently in each of these cases, it is expected that a careful separation like that will need to be done again. As well, in the previous proof we had to dig through the subalgebra condition to find no less than six quadratic relations, each of which was necessary to establish the result. In this case, then, we expect to engage in the analogous but much more difficult process. This is an undertaking for the future.

Chapter 5

Generalization

In both the original quadratic case and the higher-order case, we limited the terms $\hbar^m P_{m,k}$ occurring in the $\hat{H}_k$ in such a way that the degree of $P_{m,k}$ was not more than m. This restriction makes sense in light of a QAS's role as a quantization. However, none of the proofs presented in this paper make essential use of that criterion. We could have defined Higher Airy Structures in the following way:

Definition 17. *A **Quantum Airy Structure in Normal Form** is a collection* $\mathbb{H} = \{\hat{H}_k\}_{k\in I} \subset \mathcal{O}^\hbar$ *of operators such that*

1. $\hat{H}_k = -\hbar\partial_k + \sum_{m\geq 2} \hbar^m P_{m,k}$ *for each* k*, with* $P_{m,k} \in \mathcal{O}$ *arbitrary*[1]

2. *With* $\mathcal{O}^\hbar \cdot \mathbb{H}$ *the left ideal generated by* $\mathbb{H}$ *and* $[\mathcal{O}^\hbar \cdot \mathbb{H}, \mathcal{O}^\hbar \cdot \mathbb{H}]$ *the collection of all* $[s, s']$ *for* $s, s' \in \mathcal{O}^\hbar \cdot \mathbb{H}$*, we have* $[\mathcal{O}^\hbar \cdot \mathbb{H}, \mathcal{O}^\hbar \cdot \mathbb{H}] \subset \hbar^2\mathcal{O}^\hbar \cdot \mathbb{H}$

The same existence and uniqueness theorem could have been stated. This requires a generalization of the form of F: we must allow g to take negative values.

[1]This is tricky, actually, allowing everything to be a series instead of a polynomial. For example, what is the constant term of $(\partial + \partial^2 + \partial^3 + \dots)(x + x^2 + x^3 + \dots)$? One would find it is $1! + 2! + 3! + 4! + \dots$, which is not an element of $\mathbb{C}$. However, if we require that at each $\hbar$-degree and each x-degree there are only finitely many powers of ∂, we will successfully define a function on $\mathcal{S}^\hbar$. That is, we consider series $\sum_{m\in\mathbb{Z}} \sum_{n>0} \sum_{\alpha\in I^n} \hbar^m x_\alpha P_{m,\alpha}(\partial)$ for any polynomials $P_{m,\alpha}$.

The Theorem statement becomes the following:

Theorem 8. *Suppose we have a Quantum Airy Structure $\mathbb{H} = \{\hat{H}_i\} \subset \mathcal{O}^\hbar$. Then, among all*

$$Z = \exp\left[\sum_{g\in\frac{1}{2}\mathbb{Z}} \sum_{n\in\mathbb{N}}^{2g-2+n>0} \sum_{\alpha\in I^n} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}[\alpha]x_\alpha\right]$$

there is a unique solution to the system of equations $\hat{H}_i Z = 0$, $i \in I$.

Proof. We begin with the base case. Since we still require $2g-2+n > 0$, the least value that $2g-1+n$ can take on is 2. We want the existence and uniqueness of a sum (defining $n(g) := 3-2g$):

$$S_2 = \sum_{g\in\frac{1}{2}\mathbb{Z}} \sum_{n\in\mathbb{N}}^{2g-1+n=2} \sum_{\alpha\in I^n} \frac{1}{n!}\hbar F_{g,n}[\alpha]x_\alpha = \hbar\left(\sum_{g<\frac{3}{2}} \sum_{\alpha\in I^{n(g)}} \frac{1}{n(g)!} F_{g,n(g)}[\alpha]x_\alpha\right)$$

such that $e^{-S_2}\hat{H}_i e^{S_2} = P[\hbar > 2]$. If we write $\hat{H}_i = -\hbar\partial_i + \hbar^2 O_i$ then we can tell that no term of O_i which is non-constant in ∂ will contribute to our equation; these will bring down at least one factor of S_2 which, along with the additional $\hbar^2$, will give them $\hbar$-degree 3 or more. For a similar reason, no term in O_i with any $\hbar$ factors can contribute. Everything else (except for the leading linear term) is commutative, and so it is easy to see that:

$$[\hbar^2]e^{-S_2}\hat{H}_i e^{S_2} = -\hbar\partial_i S_2 + \hbar^2\left(\sum_{n\in\mathbb{N}} \sum_{\alpha\in I^n} \frac{1}{n!} C^{(2)}[i|\alpha]x_\alpha\right)$$

Therefore we wish to satisfy:

$$\sum_{g<\frac{3}{2}} \sum_{\alpha\in I^{n(g)}} \frac{1}{n(g)!} F_{g,n(g)}[\alpha]\partial_i x_\alpha = \sum_{n\in\mathbb{N}} \sum_{\alpha\in I^n} \frac{1}{n!} C^{(2)}[i|\alpha]x_\alpha$$

This gives $F_{1-\frac{1}{2}|\alpha|,1+|\alpha|}[\alpha, i] = C^{(2)}[i|\alpha]$ for all $n \geq 0$, $\alpha \in I^n$, and i.

The inductive step is nearly unaffected. The definition of each S_ℓ will change slightly, as they now have additional terms. However, it remains that each term in each S_ℓ has $\hbar$-deg ≥ 1, which is what did most of the lifting.

It also remains that $\partial_i Q_j = \partial_j Q_i$ for all $i, j \in I$. We are still trying to satisfy $\hbar\partial_i S_\ell = Q_i$ for all $i \in I$ and some series S_ℓ with $\hbar$-degree $= \ell - 1$. In this case, however, not only may there be infinitely many Q_i, but each Q_i may also have infinitely many terms.

Fix a finite initial segment $\{1, \ldots, J\} \subset I$. Define Q_i^J by truncating Q_i at degree J, and then setting all x_k to zero for $k > J$. Then $\{Q_j^J : j \leq J\}$ is a finite set of polynomials in finitely many variables, and the Poincare lemma yields a S^J such that $\hbar\partial_j S^J = Q_j^J$ for each $j \leq J$. We can form a power series S^∞, defined such that its $\{x_k : k > J\}$-constant terms of x-degree $\leq J + 1$ agree with those of S^J. This S^∞ can play the role of S_ℓ in the inductive step of the proof.

We need only check consistency. That is, we require that S^J and $S^{K>J}$ agree in their $\{x_k : k > J\}$-constant, x-degree $\leq J + 1$ terms. The S are determined only up to an overall constant, which we are forced to take as zero. So this requirement amounts to, for all $j, \rho < J$, that $[x^\rho]\partial_j(S^K - S^J)$ is equal to (if not zero, then) terms non-constant in $\{x_k : k > J\}$. Since that expression is just $[x^\rho](Q_j^K - Q_j^J)$, we have the result. $\square$

Any additional structures $\mathbb{H}$ permitted by this new definition are *purely* quantum in the sense that they have no well-defined classical limit. The classical limit, as earlier, is an algebra homomorphism that (in our conventions) would send $\hbar x_i \mapsto x_i$

and $\hbar\partial_i \mapsto y_i$[2], interpreted respectively as linear coordinates on a vector space V and the corresponding dual coordinates on V^*. Each $\hat{H}_k$ is taken to a polynomial $H_k(x, y) \in \mathrm{Sym}(V \oplus V^*)$. Their collective zero-locus is a Lagrangian submanifold of $V \oplus V^*$, because the QAS's subalgebra condition ensures that they generate a Poisson ideal. It can further be argued [31] that this submanifold is the image of V under $x \mapsto (x, dS_x)$ for some function $S : V \to \mathbb{C}$. This S is known in the physics setting as the Classical Action. It is a supremely important object classically, but it is also connected to the phase of a quantum-mechanical wavefunction. This connection is explored in depth by the industry of Geometric Quantization [5].

The issue with the new structures is easy to see. If $P_{m,k}$ has a degree higher than m, then we cannot realize it as a polynomial in $\{\hbar x_i, \hbar\partial_i\}$ and we do not have an instruction to map x_i or ∂_i to anything without their $\hbar$-factors. We cannot change the way the classical limit is defined, because the purpose in this point of view is a certain relationship between classical and quantum structures. Specifically, the preimages of x_i and y_i under the classical limit map must have a certain prescribed commutator.

In the older conventions, our generalization amounts to the allowance of (arbitrary, but finitely many) negative $\hbar$-powers within the $\hat{H}_k$. Their classical limit in that notation had been defined as the quotient of $\mathbb{C}[[\hbar]][[\{x_i, \hbar\partial_i\}]]$ by $\langle\hbar\rangle$,[3] which makes clear that introducing an inverse for $\hbar$ would trivialize it. In these same conventions the free energy began with a term $\hbar^{-1}$, which meant that it could be taken as a wavefunction in WKB form. This interpretation, too, would be lost in our generalization, which amounts to filling out F with further negative powers for $\hbar$.

[2]And remaining factors of $\hbar$ to 0.
[3]This is a left and right ideal.

Of course, as the difference is purely aesthetic, neither can our partition function be taken as a WKB wavefunction; it does not behave properly under our classical limit.

The lack of a classical limit does unmoor these structures from some of Topological Recursion's origins and purposes, however it still makes perfect sense to wonder what sort of coefficients may turn up within those generating series. Do they admit a similar connection to Riemann surfaces, to matrix models, Virasoro constraints, or other areas? This is so far not known to us.

Chapter 6

Conclusion

Much well-known and yet fascinating ground has been traversed, its wonders inspected and filed away. Many forms of existence and uniqueness have been displayed for all manner of variations on the Quantum Airy Structure theme, objects that have urgent duties in the realms of enumerative geometry, quantum gravity, string theory, matrix integrals, conformal field theory, and others. Just as much, however, has not been said. Can a direct, computational proof be found for the existence (i.e. symmetry) of partition functions in the general case? If, from an intrinsic perspective, we have no reason not to generalize Quantum Airy Structures, can such a generalization find its own use in enumerative geometry and other fields? Do their partition functions have worthwhile geometric content? Can their accompanying recursion be formulated as proceeding from a spectral curve? Is there any modified notion of a classical limit that could service even them? It is our sincere hope that these questions find satisfying answers which beget more questions.

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