Exploring the Implementation of Concept-Rich Instruction (CRI) with University Mathematics Pre-Service Teachers (PSTs): A Tanzanian Case

By

Emmanuel Deogratias

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Department of Secondary Education

University of Alberta

ABSTRACT

This qualitative case study explored the research questions: How does concept-rich instruction (CRI) reveal the way that Tanzanian university mathematics pre-service teachers (PSTs) express their understanding of a mathematical concept? How do PSTs plan to teach a concept given their participation in CRI?

The population for the study comprised undergraduate pre-service mathematics teachers who were in their third year at the University of Dodoma (UDOM) in Tanzania. I selected a sample by employing convenience sampling (Merriam, 1998).The sample comprised nine volunteers for the pilot study that was used to test the concept-rich instructional process, data collection and analysis techniques. There were ten volunteers in the research meetings, the data from which answered the research questions.

This study revealed that by using CRI in the four one-day meetings, PSTs were able to express their understanding of a mathematical concept by defining, decontextualizing, realizing, and re-contextualizing the concept. PSTs also planned to teach the mathematical concept by developing the lesson plans and performing micro-teaching for teaching *pi*.

The analysis of the findings revealed that CRI helped PSTs to develop their understanding of the mathematical concept in the following ways. First, the instructional approach helped PSTs realize a variety of ways to define *pi*. Second, PSTs related the concept with real things available in their local environment. Third, the approach helped PSTs eliminate misconceptions about *pi*. Fourth, the approach helped PSTs realize how to teach the concept to learners in schools using a participatory approach to learning. Fifth, this instructional approach helped PSTs integrate the concept in the Tanzanian K–University mathematics curricula. This study has implications in the development of the PSTs' understandings of the mathematical concepts for teaching.

Key words: Concept-rich instruction, mathematical concepts for teaching, conceptual development and understanding, pi, and pre-service mathematics teachers.

PREFACE

This thesis is the original research by Emmanuel Deogratias. The research project, of which this thesis is a part, received research ethics approval from the University of Alberta Research Ethics Board, Project Name “Tanzanian mathematics pre-service teachers’ expressions of *pi*: A Case Study, No. Pro00066190, Date November 17, 2016.

Chapter 3 of this thesis was the basis for an article that has been published as E. Deogratias, “The efficacy of the concept-rich instruction with university pre-service teachers in a Tanzanian context using Vygotskian perspective,” *World Journal of Educational Research*, vol. 6, issue 3, 373-385. This article is a result of the data that I collected and analysed from the pilot study. I was also responsible for the manuscript composition of this article.

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I am very thankful to everyone who travelled some part of this voyage with me, and without help, the journey would not be completed. To the participants, my supervisors and committee members, government educational agencies in Tanzania, sponsor of this study, my family, colleagues, math meet up group, and friends—I am extremely grateful to you.

First, I must acknowledge the respondents who voluntarily gave their time and provided their open and heartfelt support for this study. Thanks for contributing your expressions of understanding of *pi*, the concerns about teaching and learning *pi*, the way that the concept is integrated into the Tanzanian mathematics curricula, and your suggestions for reform.

My supervisor and members of my supervisory committee, Professors Florence Glanfield, Elaine Simmt, and Mijung Kim were so considerate with their time and added significant suggestions and comments that enriched the quality of this dissertation. Another member of my examining committee, namely, Professor Steven Khan also provided valuable comments and suggestions on the thesis. The comments and suggestions carried new insights and widened my perspectives. My external examiner Professor Jill A. Newton, your annotations were much honoured. My non-examining chair of the committee, Professor Bonita Watt for valuable advice.

I also extend my gratitude to the Tanzanian Ministry of Education, Science, Technology and Vocational Training (MOEST) for giving permission to conduct this study, and Tanzania Institute of Education (TIE) for using Tanzanian mathematics curricula, including textbooks, syllabi, and lesson plan template. I also extend my gratitude and acknowledgement to UDOM for trusting this study to be conducted in the university.

Special thanks and acknowledgments go to Global Affairs Canada for supporting my entire Ed. D. studies through the project for capacity building for primary school mathematics teaching and learning in rural and remote areas in Tanzania. The project was jointly administered by the University of Alberta and Brock University in Canada, and UDOM in Tanzania.

Moreover, special thanks go to my family who encouraged me during this study. I must introduce my wife Lilian Hassan Luhanda, my mother Felista Mukaruka, and my mother-in-law Rogzena Moshi; without their moral comfort I could never have accomplished this journey. Other influential persons in this journey including Laura Paslawski, Professor Margaret Iveson, Dr. Andrew Binde, Professor Joyce Mgombelo, Dr. Athanas Ngalawa, Dr. Perepetua Mlozi, Dr. Leonce Rushubirwa and his wife, Dr. Grace Luiza, Professor Hans Smits and his wife Donna, Diana McIntyre and her family, Dr. Kate Pratt, Dr. Shalini Khan, Debby Waldman, and everyone who added fuelthat boosted my energy.

Most importantly,I must give thanks to God for health, strength, means and chance to accomplish this study. In my journey of Ed. D.studies, I encountered challenging situations, including a loss of my beloved son Joshua, my brother Daniel, my father-in-law Hassan Luhanda, my grandmother Nyabhambu Selemani and my grandfather Mbita Missana. I will cherish your memories throughout my life.

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CHAPTER 1

INTRODUCTION

My doctoral work explores the implementation of concept-rich instruction (CRI) with university mathematics pre-service teachers (PSTs) in Tanzania.

To begin this dissertation, I start by telling about myself and how I came to be interested in this research area.

Context of the Study

This section describes my teaching and learning experiences, and the instructional approaches that have been used in the Tanzanian context to provide a basis for understanding how I position myself as a school mathematics teacher, teachers'-college tutor and university instructor. This positioning has implications for the way in which I view mathematics related to the work of teaching, as well as what PSTs need to learn and practice in university mathematics classes.

Teaching and Learning Experiences

I started primary school at Mahande Primary School in the Ukerewe district in lake zone Tanzania in January 1991 when I was eight years old. I completed Standard VII (equivalent to North American sixth grade) in October 1997. I was awarded a certificate of primary education after passing the Standard VII national examinations. In January 1998, I started secondary school at Bukongo Secondary School, also in the Ukerewe district. Bukongo was the only government-funded public school in the district. I had the opportunity to attend this school because I passed the Standard VII national examinations and I took the advantages of the opportunity because the tuition fees were low. In November 2001, I completed Form IV and was awarded a certificate of completion of secondary education after passing the Form IV national examinations. This level of education qualification is equivalent to the certificate of completion of junior high education in the Canadian system.

In July 2002, I attended Musoma Secondary School, a government-run, boarding high school in the Mara region, where I pursued advanced mathematics, physics, and chemistry as my principal areas of study. In May 2004, I completed my studies and was awarded an advanced certificate of completion of secondary education after passing the Form VI national examinations. This level of education qualification is equivalent to the certificate of completion of high school in the Canadian education system.

After completing high school at Musoma Secondary School, I taught mathematics at Murutunguru Secondary School for a year before enrolling in the University of Dar es Salaam for my undergraduate studies. Murutunguru is a private-run secondary school in the Ukerewe district.

In September 2005, I enrolled in a three-year Bachelor of Education degree program at the University of Dar es Salaam, with a double focus on mathematics and education. The intention of this program was to prepare PSTs to be teacher educators in teacher colleges after they completed their university studies. I took various education courses, including introduction to educational psychology, principles of education, communication skills, principles of curriculum development and teaching, pedagogy of teacher education, teacher education science methods, Information and communications technology (ICT) in science and mathematics education, human development and school learning, teaching practice, pedagogical issues in science and mathematics education, educational measurement and evaluation, professionalism and ethics in education, library education, and management of education and school administration.

I also took various mathematics courses, including ordinary differential equations, rigid body mechanics, algebra, functions, numerical analysis, statistics, and partial differential equations. In September 2008, I completed my undergraduate studies and was awarded a Bachelor of Education degree in science, with honours. While this program provided pedagogical practice and content in the area of mathematics, it was not focused on the mathematics that was being taught in schools. It was, however, beneficial for further studies in the area of mathematics and for my later employment at a university in a department of mathematics.

While pursuing my undergraduate studies, I continued teaching at Murutunguru Secondary School during the holidays. It was while teaching there that I first learned that there was a disconnection between the mathematics taught in schools and the mathematics taught in the university to undergraduate PSTs. For instance, I learned functional analysis, abstract algebra, and rigid body mechanics in mathematics classes for undergraduate PSTs. But these courses were not taught to learners in mathematics classes in primary and secondary schools.

In August 2008, I was employed as a teacher educator by the government of Tanzania to work at one of the diploma-granting teachers' colleges. There, I taught mathematics and education courses. It was while teaching mathematics at this college that I learned that there was a connection between the mathematics that was being taught in schools and in the teacher college, because while at the college, the curriculum for PSTs guided me to teach the mathematics that was actually being taught in schools. This was important to help the PSTs gain knowledge and skills for teaching the mathematics that is taught in schools. However, not all of the topics that were found in school mathematics syllabi were taught to these PSTs at the college.

In February 2009, I was employed by the University of Dodoma (UDOM) as a tutorial assistant in the Department of Mathematics. I was assigned various activities, including conducting tutorial seminars, preparing tutorial questions for PSTs' discussions and presentations in the class, marking tests and university examinations, and compiling course results.

In September 2009, I enrolled in a two-year Master of Science degree program, specializing in mathematical modelling, at the University of Dar es Salaam. In Year 1, I was required to take mathematical courses and to write and defend a thesis proposal. In Year 2, I wrote my dissertation, "Methods for Pricing and Hedging Plain Vanilla Barrier Options." This Master's program aimed to prepare graduate learners to improve their understanding of mathematics and the ways in which mathematics is applied in life. I took various courses, including mathematics of finance, applied functional analysis, insurance mathematics, stochastic differential equations, numerical linear algebra, advanced mathematics statistics, ecological modelling, computer programming, population dynamics, ordinary differential equation, data assimilation, partial differential equation and dissertation-writing. In September 2011, I completed my master's degree program and was awarded a Master of Science degree in mathematical modelling.

In October 2011, I was promoted to assistant lecturer at UDOM. In this position, I worked on various activities, including teaching undergraduate PSTs two mathematics courses per semester, preparing lessons, preparing tests and university examinations, marking tests and university examinations, compiling course results, and attending meetings related to academic issues such as approving the university examination results.

I taught UDOM's undergraduate PSTs courses in linear algebra, ordinary differential equations, rigid body mechanics, complex analysis, and basic functional analysis. I also taught a mathematics project course. Linear algebra and ordinary differential equations were for Year 1, rigid body mechanics and complex analysis were for Year 2, and basic functional analysis and the mathematics project were for Year 3. The table below shows these course details.

Table 1.1

Courses Taught to Undergraduate PSTs at UDOM

Course Code	Course Title	Credits	Year of Study	Semester
MT 112	Linear algebra	10	Year 1	Semester I
MT 122	Ordinary differential equations	7.5	Year 1	Semester II
MT 214	Rigid body mechanics	10	Year 2	Semester I
MT 221	Complex analysis	10	Year 2	Semester II
MT 320	Mathematics project	10	Year 3	Semester II
MT 321	Basic functional analysis	10	Year 3	Semester II

In July 2014, I began a Doctor of Education (Ed. D) program in mathematics education at the University of Alberta in Canada under a project that aims to build capacity in teaching and learning with mathematics teachers in rural and remote areas in Tanzania. Sponsored by Global Affairs Canada, the project is jointly administered by UDOM, Brock University and the University of Alberta. In addition to gaining knowledge and skills in mathematics teaching, learning and researching, I have learned the importance of designing coursework and activities related to the work of teaching mathematics.

Designing coursework and activities has many benefits, including learning how to teach mathematics; using teaching and learning resources that are relevant to the learners, such as local materials, and appropriate teaching strategies and approaches for teaching a mathematical concept; creating a conducive learning environment; and researching mathematics teaching and learning. This opportunity has helped me to position myself well in mathematics education so that I can now separate mathematics-for-teaching from mathematics as a field of study. I feel like these experiences have given me the knowledge and skills I need to improve mathematics teaching and learning with pre-service and in-service teachers in Tanzania.

From these teaching and learning experiences, I have learned that more effort is needed to improve mathematics teaching and learning at universities and schools in Tanzania in order to improve learners' understanding of mathematical concepts. Teachers at schools should have knowledge and skills for teaching mathematics in classes. In doing so, two aspects should be considered. The first aspect is that in-service teachers need ongoing professional learning opportunities to improve their mathematics-for-teaching. The second aspect is that while undergraduate PSTs are at university, they need to gain knowledge and skills for teaching the mathematics that is being taught in schools. I believe that these two aspects can be achieved by using appropriately designed coursework and activities focused on mathematics-for-teaching in the university mathematics programs for PSTs. In doing so, PSTs will be empowered by learning and practicing mathematics related to their work of teaching in their university mathematics classes. In the future, they are likely to become better teachers by engaging their learners with mathematical concepts using participatory approaches in their schools.

The Instructional Approaches in Mathematics Classes with University PSTs

This subsection describes the instructional approaches that have been used in mathematics classes with UDOM's PSTs. The descriptions are important to get a sense of the ways PSTs have been engaged in learning mathematics in classes. It also provides insight into what needs to be done and how to engage these PSTs in mathematics classes so that they will become better teachers in the future.

Lecturing is the main form of instruction in PSTs' mathematics classes at UDOM. This approach is useful for the instructors to deliver course content to PSTs in the class. However, this approach leaves no chance for PSTs to discuss their understanding of mathematics in the class. It also does not engage PSTs in learning mathematics through a participatory approach. Nor does it provide opportunities for the instructors to gain insight during the class into PSTs' ongoing learning about mathematical concepts. There is a move, however, to focus on using participatory approaches with university undergraduate mathematics PSTs. In doing so, various participatory approaches have been used with PSTs. These instructional approaches include project-based learning and group-work assignments.

Project-based learning is currently used with PSTs in Year 3. PSTs in small groups choose a project topic/concept in mathematics and work together by exploring, explaining or describing it using primary or secondary sources of data. Usually, two to four PSTs work on the same mathematical project in the last semester of their undergraduate studies and write a report under one supervisor, who is an instructor from the Department of Mathematics in the College of Natural and Mathematical Sciences. After writing the report, PSTs submit their project work to their supervisor for marking and grading. After grading the work, the supervisor submits it to the Head of the Mathematics Department. The final step in this process is for the examination board to meet to approve the results. This instructional approach is learner-centred: a supervisor facilitates PSTs' learning and provides feedback until the group project is completed. However, the instructional approach is not practiced in other mathematics classes.

Group assignments are an instructional approach that is used during undergraduate PST studies. Instructors provide group assignments to PSTs, who then work together in small groups to do the assignments. The PSTs do this work outside of mathematics classes. After working on the group assignments, PSTs select one representative from the group to present their group work in mathematics class during tutorial sessions. While the overall approach is learner-centred, it does not provide a good opportunity for individual PSTs to discuss their mathematics in the mathematics class because there is only one presenter that presents the group work to the instructor. All of the discussion about the mathematics occurs outside of the class time. The instructor does not have an opportunity to know about the ongoing learning in small group discussions as the work is done outside of class.

Even though there is a move towards using participatory approaches with PSTs, mathematics instructors use a limited participatory approach when delivering lessons to these undergraduate students. As such, there is a need to engage PSTs when teaching them mathematical concepts in class, by using a participatory approach that is concept-oriented.

How I Came to be Interested in CRI

I came to be interested in CRI due to my teaching and learning experiences. As a mathematics teacher educator, I used three different instructional approaches to teach PSTs in the courses I taught at UDOM. The approaches were lecturing, group-work assignments, and project-based learning. I noticed that while using project-based learning and group-work assignments, PSTs were actively engaged in learning a mathematical concept. But when I used a lecturing approach, PSTs were listeners, unless I decided to engage them with a concept by asking questions. Even though I tried both project-based learning and group-work assignments as a teacher educator, I relied heavily on lecturing.

As I described in the previous subsection, I realized and experienced that the Tanzanian university programs for PSTs do not emphasize the use of participatory approaches in university mathematics classes. Also, there is poor performance in mathematics in primary and secondary schools, which could be associated with teachers' lack of understanding about how to engage learners with mathematical concepts in a participatory way of learning. The paragraphs below describe this challenge in detail.

Mathematics is one of the major subjects in Tanzanian primary and secondary school curricula, but learners perform poorly in mathematics in national examinations (Mwananchi, 2016; NECTA, 2015a; 2015b; 2017a; 2017b; 2019). This scenario of poor performance in mathematics has remained constant for the past 17 years in secondary schools: the average failure rate in mathematics national examinations from 1995 to 2002 was 70% (Sugiyama, 2005). From 2003 to 2007 the failure rate increased to 73% (Kafyulilo, 2013). Student success on the Form IV National Examination in mathematics was about 39% in 2011 and dropped to about 18% in 2015 (Mwananchi, 2016). The percentage of learners passing the Form IV National Examination in mathematics increased to about 29% in 2017 (NECTA, 2017b). Still, mathematics was the lowest-ranked subject in terms of performance, while Kiswahili led with a pass rate of about 84% (NECTA, 2017b).

Similarly, in primary education, the scenario of poor performance in mathematics also exists (NECTA, 2014; 2015a; 2017a). For instance, pupils' success in mathematics for the Standard VII National Examination was about 49.5% in 2015 and dropped to about 46.6% in 2016 (Dausen, 2017, para. 4). Uwezo, focusing on literacy and numeracy competency assessments for learners in Standards I to III in primary schools in East African countries—Kenya, Uganda, and Tanzania—reported significantly low performances in numeracy (Uwezo East Africa, 2014).

As stated previously, one possible explanation for poor mathematics performance is the teaching approach. Another is that the teachers may lack a comprehensive understanding of the mathematical concepts at primary and secondary school levels (Chonjo et al., 1996; Kafyulilo, 2010). As such, teachers need to clearly understand the mathematical concepts that are needed in their work of teaching. In addition, these teachers may not have experience with how to engage learners in participatory ways of learning, because the universities that prepare them do not offer coursework and activities to empower them to understand the concepts or teach them how to engage learners this way.

The intention of using suitable coursework and activities is straightforward—PSTs gain knowledge and skills for teaching mathematical concepts in primary and secondary schools and for engaging learners in those schools using participatory approaches while they are enrolled in university. This focus has the potential to enhance learners' understanding of mathematics and achievement in both primary and secondary schools as these instructional approaches focus on participation in learning.

My belief is that engaging PSTs in mathematical concepts for teaching using alternative forms of instruction will contribute to the development of the PSTs' understanding of the concepts themselves. This will enable these teachers to engage learners at schools using participatory approaches, which in turn will improve the learners' performance in mathematics in both primary and secondary schools.

The Problems

Based on my experiences, these two problems emerged:

- University mathematics programs for PSTs do not emphasize the use of participatory approaches in university mathematics classes and the instructional approaches are not concept-oriented. This situation does not empower PSTs to learn and practice teaching in their university mathematics classes.
- Poor performance in mathematics in secondary and primary schools might be connected to school teachers' limited understanding of the mathematical concepts for teaching and how to teach the concepts using participatory approaches. This problem is embedded in the first problem above.

Using CRI to Encourage Participatory Approaches in Tanzanian University Mathematics Classes for PSTs

Based on the two identified problems above, undergraduate mathematics PSTs need to know how to engage learners at schools with mathematical concepts using participatory approaches. This process can be taught by using participatory approaches in university mathematics classes for PSTs, to familiarize them with the concepts taught in schools.

Overall, the current instructional approaches in university mathematics classes do not use participatory approaches for PSTs; CRI, however, can encourage a more participatory approach and better orient PSTs to whatever concept being studied in the mathematics class. CRI is a teaching approach that focuses on actively engaging learners to "learn and think mathematically" (Ben-Hur, 2006, p. vii). This study focuses on exploring the ways that the CRI approach reveals university PSTs' expressions of their understanding of a mathematical concept and the ways PSTs plan to teach a concept. This is the first study to apply this instructional approach to university PSTs in a Tanzanian context.

The examples below exemplify the differences between direct, or teacher-centred, instruction and CRI. Consider a scenario in which two different instructors teach PSTs about a circle. One instructor presents the meaning of circle while PSTs take notes. The second instructor asks, "given a physical circular object, what can you say about a circle? (Think, pair, share)." Here we see two approaches that are different. Each approach leads to different ways of learning and understanding a concept. The first approach presents the meaning of a circle by suggesting that mathematics is abstract and that there is only one definition. The second approach gives the message that mathematics is connected to real things that we see in our daily life, and there are a varieties of ways to understand a concept. As such, it opens up possibilities to the PSTs to explore and understand a concept in small groups and class discussions, including the way in which concepts are connected to each other.

My intention in studying this CRI approach is to improve mathematics teaching, as based on my teaching experience, current mathematics instruction is not concept-oriented and rarely uses participatory approaches in university mathematics classes in Tanzania. As I stated earlier, PSTs have little opportunity to be engaged with a concept using a participatory approach in their university mathematics classes, because most of the instructors use lectures to teach a concept in the classes.

My teaching experience has showed me that university lecturers believe that given time constraints and expectations for higher test results (both on coursework and university examinations), their approaches to mathematics instruction are suitable. PSTs, university lecturers, and learners at schools in Tanzania could benefit from using Ben-Hur's (2006) CRI for participatory learning of mathematical concepts. This study, therefore, is twofold: it aims to explore the ways in which the CRI approach reveals Tanzanian university PSTs' expressions of their understanding of a mathematical concept, and the ways in which PSTs plan to teach a concept, given their participation in CRI.

Research Questions

This study explores two research questions:

- How does CRI reveal the way that Tanzanian university PSTs express their understanding of a mathematical concept?
- How do PSTs plan to teach a concept, given their participation in CRI?

In this study, the term "expressions" refer to the mathematical ideas that are represented in oral or written form (Glanfield, 2003; Pirie & Kieren, 1994). In this context, "expressing is to do with making overt to others or to oneself the nature of those [mental and physical] activities" (Pirie & Kieren, 1994, p.175).

Organization of the Dissertation

Chapter 1 describes the context of the study, problems, and a new approach to teaching mathematics to university PSTs. The chapter begins by describing my teaching and learning experiences. Then, I present the instructional approaches that have been used in mathematics classes with university PSTs in Tanzania. The chapter also shows how I came to be interested in CRI. After that, I describe the identified problems in relation to the purpose of this study. The chapter also provides a new approach to teaching mathematics to university PSTs in Tanzania, and ends by presenting the research questions.

Chapter 2 presents theoretical frameworks and orientations. In particular, it begins with a review of a concept in the Tanzanian mathematics curricula, followed by the meaning of a concept. After that, the chapter presents meaning and forms of understanding. It also describes PSTs' conceptual development and understanding, as set forth in the literature. Then, the chapter describes CRI. After that, the chapter presents instructional approaches and the ways that CRI relates to or differs from other teaching approaches. The chapter ends by presenting the conceptual and theoretical orientations in this study.

Chapter 3 describes the research methodology. The chapter begins by describing a qualitative case study used in this study. The chapter describes the research site for this study, the participants, research design, data collection and analysis techniques. Following this, the chapter addresses the validity and reliability of the data, the ethical considerations of the study and, finally, the study's limitations and delimitations.

Chapter 4 presents the results and analysis of the two research questions.

Chapter 5 presents seven lessons learned through the implementation of CRI in Tanzania. As a teacher educator, the lessons I learned focused on systematic errors, multiple interpretations, open definitions, PSTs getting an opportunity to teach a mathematical concept and seeing how the concept is integrated into Tanzanian mathematics curriculum, community of learners in Tanzanian pre-service teacher education, local resources as a tool to mediate learning, and a better understanding of participatory approaches and social constructivism in university mathematics classrooms.

Chapter 6 presents a summary of results, recommendations, concluding thoughts and implications of the CRI approach in a Tanzanian education context.

CHAPTER 2

THEORETICAL FRAMEWORKS AND ORIENTATIONS

In this chapter, I present a review of *pi* in the Tanzanian mathematics curricula, because *pi* is the example of a mathematical concept that I used to engage pre-service teachers (PSTs) in the research project. Then, I explore the meaning of a concept and understanding, and describe concept-rich instruction (CRI) by focusing on its meaning, components and origin, and how this approach has been used in mathematics classes. To frame this, I present a review of literature on PSTs' understanding of mathematical concepts. I then turn my attention to studies that focus on instructional approaches, and the ways that CRI relates to or differs from other approaches of teaching. Finally, I describe a conceptual orientation followed by the theoretical orientation for this study.

This study is important because it informs the ways that the CRI approach is potentially useful to Tanzanian university mathematics PSTs for strengthening their understanding of the mathematics that they will be teaching. As such, this study provides information about the potential for CRI to strengthen PSTs' understanding of a particular mathematical concept.

By prioritizing the importance of teaching PSTs to engage learners at schools using participatory approaches, this study also gives PSTs an opportunity to engage with the target concept of mathematics through participatory learning.

A Review of the Target Concept in Tanzanian Mathematics Curricula

In this research project, I focused on *pi* as a concept. I reviewed the concept in the Tanzanian mathematics curricula before exploring the implementation of CRI. During the review, I focused on how *pi* is developed in primary and secondary school curricula in Tanzania, as well as how it is articulated in university mathematics curricula at the University of Dodoma

(UDOM). The subsections below present the reasons I chose *pi,* and review it across the Tanzanian mathematics curricula.

Why *Pi*?

There are several reasons to focus on *pi* as an example of a mathematical concept for this study. The first is that *pi* is a concept studied in the Tanzanian school curriculum. This context fits with the idea of Ben-Hur's (2006) work that selecting a concept in a curriculum is one of the aspects of CRI. Ben-Hur proposes choosing a concept in the curriculum before engaging learners with a concept in the class. The second reason is that *pi* appears across Tanzanian schooling and tertiary education, i.e., teacher colleges. The third reason is that it is not presented well in the textbooks used in primary and secondary schools. The fourth reason is that as a concept, it is taught in primary and secondary schools, but it is not in the university PSTs curriculum.

But this is not just about *pi*: if PSTs improve in their understanding of the participatory approach by focusing on *pi,* I believe they will be able to apply CRI teaching and learning strategies to other mathematical concepts and use the participatory learning approach in their future classrooms.

A Review of *Pi* in the Tanzanian Mathematics Curricula

I reviewed various documents related to the Tanzanian mathematics curricula for this study. This review included the Tanzanian primary school, secondary school, and UDOM mathematics curricula as well as syllabi (i.e., TIE, 2005a; 2005b; 2009) and textbooks (e.g., Ben and Company Ltd, 2011a; 2011b; Sichizya & Kwalazi, 2010a; 2010b; TIE, 2009a; 2009b; 2009c; 2009d).Primary and secondary school curricula, namely textbooks and syllabi, are designed and coordinated by the Tanzania Institute of Education (TIE), while the UDOM mathematics curricula are designed by UDOM's lecturers in the Department of Mathematics at the College of Natural and Mathematical Sciences. Table 2.1 below presents these documents.

Table 2.1

Reviewed Tanzanian Mathematics Documents

Reviewed Mathematics Documents
Primary school documents:
Basic Mathematics Syllabus for Primary School (Tanzania Institute of Education, TIE, 2005a).
Primary Basic Mathematics Book 6 (Ben & Company Ltd, 2010; Sichizya & Kwalazi, 2010a).
Primary Basic Mathematics Book 6 (Ben & Company Ltd, 2011a).
Primary Basic Mathematics Book 7 (Ben & Company Ltd, 2011b; Sichizya & Kwalazi, 2010b)
Secondary school documents:
Basic Mathematics Syllabus for Secondary School (TIE, 2005b).
Advanced Mathematics Syllabus for Secondary School (TIE, 2009).
Secondary Basic Mathematics Book 1 (TIE, 2009a).
Secondary Basic Mathematics Book 2 (TIE, 2009b).
Secondary Basic Mathematics Book 3 (TIE, 2009c).
Secondary Basic Mathematics Book 4 (TIE, 2009d).
Advanced Level Pure Mathematics 4th Edition (Tranter, 1975).
University mathematics document:
The UDOM's Revised Programme for the Undergraduate Mathematics PSTs.

Through reading these documents, I learned that *pi* is not well articulated and developed in the Tanzanian mathematics curricula. For instance, *pi* is treated as a value that equals $\frac{22}{7}$ or 3.14 in primary mathematics curricula (Ben & Company Ltd, 2010; Sichizya & Kwalazi, 2010a). In secondary mathematics curricula, *pi* is defined as a ratio of circumference to the diameter of a circle (TIE, 2005b; 2009a). The concept of *pi* is not described in detail in the curriculum. Also, the value of *pi* is used as a radian measure, which is equivalent to 180 degrees in the conversions of an angle measured from degree to a radian measure and vice versa. But, there are no detailed explanations on how to obtain the value (TIE, 2005a; 2009c). In university mathematics curricula, the concept of *pi* is not taught but, rather, university PSTs use *pi* in solving mathematical problems in various courses, including complex analysis, numerical methods, and rigid body mechanics.

I also learned that *pi* in geometry, in the primary mathematics curricula and textbooks, is used to find the circumference and area of a circle, but no clear explanations exist as to what *pi* represents or how to find the value of *pi*. In the same way, one might think that the value of a concept is given as if it is an arbitrary fact (3.14 or $\frac{22}{7}$), rather than a constant value that is derived from the relationships between the circumference of a circle and its diameter (Sichizya & Kwalazi, 2010a). At the same time, learners are told that *pi* is a radian measure (MOEVT, 2011; TIE, 2009a; 2009b) without any relationship developed among these different instantiations of a concept. This shows that the applications of *pi* in mathematics are established in Tanzanian primary and secondary schools and at the university level without any serious development of the meaning of the concept and its value.

University lecturers have assumptions that PSTs in their classes already understand the concept of *pi*, because they studied the concept in primary and secondary schools. This assumption may be misguided, leaving university PSTs little chance to explore such a concept in a way that is critical in order to teach it. This situation highlights the need to integrate the deep study of concepts used in school mathematics courses for PSTs.

What is a Concept?

Sierpinska (1994) defines a concept as "an abstraction and generalization of thoughts about thoughts" (p. 156). Abstraction refers to "a process by which we become aware of regularities in our experience, which we can recognize on future occasions. It is in this way that we are able to make use of our past experience to guide us in the present." (p. 70). She argues that "a concept is not an evolution of the pre-concept; it is a leap to a new and higher level of thought; it is thinking about the thoughts of the previous levels" (p. 156). In this context a pre-concept refers to "an abstraction and generalization of thoughts about things" (p. 156).

According to Tall (2013), abstractions can be grouped into three categories: structural, operational and formal. Structural abstraction focuses on the reasoning of a structure (Tall, 2013). Examples of structural abstractions are the reasoning about why *pi* appears in the volume of a sphere, and why the circumference of a circle is given by $c = 2\pi r$.

Operational abstraction focuses on the operation(s) of an object(s) (Tall, 2013), for instance, counting the number of diameters required to complete the circumference of a circular object. This process involves either adding the diameters or taking a diameter, then multiplying it three times and adding about 0.14 to complete the circumference of a circular object. Another example of this type of abstraction can be found in algebraic expressions in lower grades. For instance, when solving a problem $3x - 4 = x - 12$ involves three operations: addition, subtraction, and division to obtain an object x.

Formal abstraction focuses on a set of theoretic definitions, proof, theorems, and axioms (Tall, 2013). For instance, prove that the area of a circle is given by $A = \pi r^2$. Prove that $(a + b) + c = a + (b + c)$ is another example of formal abstraction.

A concept is an abstraction of thoughts about thoughts, composed coherent generalizations that show the relationship that exists among them (Tall, 2013). For example, *pi* is a ratio of the circumference to the diameter of a circle. This generalization indicates the relationship of the three concepts, namely, a ratio, circumference, and diameter.

The above three categories of abstractions are derived from three worlds of mathematics; conceptual embodiment, operational symbolism, and axiomatic formalism (Tall, 2013). Conceptual embodiment focuses on an object's actions and shapes, and our mental images of the object. Tall (2013) argues that "conceptual embodiment builds on human perceptions and actions developing mental images that are verbalized in increasingly sophisticated ways and become perfect mental entities in our imagination" (p. 16). For instance, using a physical object to teach about *pi* and its value, we need first to think about what kind of a physical object is relevant for teaching about *pi* and its value. Is it circular in shape? If yes, then how can we use it for teaching about *pi* and its value? How can we identify and measure its circumference and diameter? How can we draw its figure?

Operational symbolism focuses on symbols that grow from actions and procedures. Tall (2013) writes that "operational symbolism grows out of physical actions into mathematical procedures. Whereas some learners may conceive the symbols flexibly as operations to perform and also to be operated on through calculation and manipulation" (p. 16). For instance, in the example above (teaching about *pi* and its value), physical actions involve measuring the circumference and diameter of a circular object using measurement tools. Procedures involve taking the circumference of a circular object and dividing by its diameter to obtain the value of *pi.* Operational symbolism involves symbolic representations of the circumference and diameter of a circular object to define *pi* and find its value, *i.e.,* $\pi = \frac{C}{D}$.

Axiomatic formalism focuses on a set of theoretic definitions, which are deduced from proofs. Tall (2013) writes that "axiomatic formalism builds from formal knowledge in axiomatic systems specified by set-theoretic definitions, whose properties are deduced by mathematical proof" (p. 17). For instance, for any periodic function, $sint = \sin(t + 2\pi)$ is an example of a set of theoretic definitions for a periodic function. The properties of this set of theoretic definitions can be deduced by proving whether the first side equals that of the second side.

By prioritizing the above three categories of abstractions, there are two categories of a definition of a concept: structural and procedural (Chesler, 2012; Zaslavsky & Shir, 2005). When a definition of a concept focuses on the way we identify an object's properties, the definition falls into the category of structural (Chesler, 2012). For instance, a period is the time interval needed to complete one cycle in a periodic function. When the definition of a concept focuses on the ways that we construct an object (Chesler, 2012; Zaslavsky & Shir, 2005), the definition falls into the category of procedural. For instance, *pi* is the number of diameters that go around the circumference of a circular object.

Concepts can be grouped into two categories: spontaneous and scientific (Sierpinska, 1994; Harvey, 2011). Spontaneous concepts are "everyday concepts which are developed during day-to-day lived experience whereas scientific concepts are taught most often during formal schooling" (Harvey, 2011, p. 7). In other words, a spontaneous concept is a concept acquired by an individual or group through participation in social practices with other people in everyday life. A spontaneous concept is non-systematic (Sierpinska, 1994; Vygotsky, 1986). Examples of spontaneous concepts in mathematics education are ideas that are not presented and formulated mathematically; they emerge in a social learning context, such as the ideas of circular and periodic. For instance, to teach *pi* as a scientific concept, I introduced local circular materials as teaching aids during research meetings to invite PSTs to share theireveryday spontaneous experiences and concepts.

A scientific concept is a concept that is taught in school, colleges or university. It is systematic because it is "presented and formulated as a body of ideas or principles" (Sierpinska, 1994, p. 156).Examples of scientific concepts in mathematics education are the concepts of *pi* and area of a circle.

Pi is a scientific concept that is taught in Tanzanian primary and secondary schools. It is presented and formulated mathematically in Tanzanian mathematics textbooks and syllabi. It is usually defined as a ratio of the circumference to the diameter of a circle. In this definition, there are underlying concepts related to *pi*, such as a ratio, circumference, diameter and circle, which are critical to know while defining *pi* and finding its value.

In research meetings, I used *pi* as an example of a scientific concept to engage PSTs with a concept using CRI. Also, in a review of the research literature, I focused on the concept(s) that is taught at schools, because it is important for PSTs to learn mathematics-for-teaching (Davis & Renert, 2014).

The table below provides some differences between spontaneous and scientific concepts.

Table 2.2

*The Differences Between Spontaneous and Scientific Concepts (*Tamina-Kingsolver, n.d., para. 6)

Spontaneous Concept	Scientific Concept
"Is not introduced to a child in a systematic fashion or explicitly connected with other	"Is presented as a system of interrelated ideas."

related concepts.”	
“Originates in everyday life [*sic*] i.e., acquired by the child outside of the context of explicit instruction.”	“Originates in classroom instruction [*sic*] i.e., are [*sic*] explicitly introduced by a teacher at school.”
“Develops from “bottom to top” from the child’s experience to generalizations and abstractions.”	“Develops from ‘top to bottom’ from a verbal explanation to concrete everyday phenomena [*sic*].”
“Provides the basis for the development of scientific concepts.”	“Covers the most essential aspects of an area of knowledge and extend[s] the meaning of everyday knowledge.”

The above distinctions about spontaneous and scientific concepts are important because I chose *pi* in the Tanzanian mathematics curriculum as a scientific concept. However, I was able to consider the spontaneous concept about *pi* such as circular through using circular materials as teaching aids for conceptual development and understanding of *pi* in the research meetings. For instance, the spontaneous concept, circular, could be described in different ways such as a circular object, a circular figure and a circular shape. When we think about a circular figure, it might lead to think about a circle. When we think about a circle, it provides an opportunity to develop the meaning of *pi* and its value from a circle by considering measuring its circumference and diameter.

Scientific and spontaneous concepts can be viewed from the perspective of a concept definition and a concept image (Tall & Vinner, 1981; Vinner & Hershkowitz, 1982, 1983). A concept definition is “a form of words used to specify[a] concept” (Tall & Vinner, 1981, p. 152). For instance, the statement “*pi* is a relationship of the circumference and diameter of a circle” is an example of concept definition. Also, a concept definition can be categorized as personal or formal (Tall & Vinner, 1981). A personal concept definition is the form of words that individual learners use to express a concept. A formal concept definition is defined as a form of words established by a large community (NCTM, 2004; Tall & Vinner, 1981).

The notion of a concept image describes the total cognitive structure of a concept—including processes, properties, and mental pictures—which can be in the form of pictorial and symbolic representations (Tall & Vinner, 1981). For instance, for the concept of division of numbers, children in lower grades are taught that dividing a number by zero is impossible; division is only possible when dividing a number by smaller number. When they reach upper grades,children learn that dividing a number by zero is possible, but that the answer in such equations is undefined. This elaboration modifies their concept image. Another example is that children in lower grades learn that the value of *pi* is a number approximate to $\frac{22}{7}$ or 3.14. But when they reach upper grades, they learn that the value of *pi* can also be a radian measure, which is equivalent to 180 degrees.

A concept definition provides its own concept image. The resulting concept image is known as "the concept definition image" (Tall & Vinner, 1981, p. 153).Apart from a concept definition image, other examples of concept images focused on *pi* include formula, symbols, and actions. Below I provide some aspects of concept images with examples focusing on *pi*.

- Symbols: for example, π, which represents *pi* and its value.
- Drawings: for instance, drawings of the graphs of periodic functions, which shows the relationship of *pi* in the periodic functions.
- Formulas: for instance, $A = \pi r^2$, which represents the area of a circle.

- Processes: for instance, dividing a measure of the circumference of a circular object by that of its diameter gives the value of *pi* as a pure number.
- Action: for instance, counting the number of diameters required to complete the circumference of a circle.
- Properties: For instance, the value of *pi* (π)in any periodic functions is half of the period ($\frac{T}{2}$).

We come to recognize concept definitions and concept images through experiences and by using them in appropriate situations. Tall and Vinner (1981) argue that:

> many concepts which we use happily are not formally defined at all; we learn to recognize them by experience and usage in appropriate contexts. Later these concepts may be refined in their meaning and interpreted with increasing subtlety with or without the luxury of a precise definition. Usually in this process, a concept is given a symbol or name which enables it to be communicated and aids in its mental manipulation. But the total cognitive structure which colors the meaning of a concept is far greater than the evocation of a single symbol. It is more than any mental picture, be it pictorial, symbolic or otherwise. During the mental processes of recalling and manipulating a concept, many associated processes are brought into play, consciously and unconsciously affecting the meaning and usage. (p. 151)

For instance, learners are taught that the value of *pi* is an irrational number, which is approximate to 3.14. But later on, they learn that the value of *pi* is a radian measure, which is equivalent to 180 degrees. The first perspective shows that the value of *pi* can be obtained from a relationship between the circumference and diameter of a circle, by taking a ratio of the circumference to diameter of the circle. The second perspective shows that the value of *pi* can be obtained in a unit circle in a relationship of angle measure in degrees and arc length. As a result, there is a dynamic change in the value of *pi* from an irrational number to a radian measure.

Scientific concepts can also be viewed from the perspective of Dubinsky's APOS theory (1991), which I will now explain in detail. The theory postulates that mathematical concepts consist of actions, processes, objects and schema (APOS) which are critical to individual learners when developing an understanding of a concept. To make sense of a concept, learners require these relevant mental structures of actions, processes, objects and schema:

- Actions are mental and physical activities that are expressed by the learners focused on a mathematical concept (Dubinsky, 1991; Maharaj, 2010), for instance, adding $2 + 4 + 6 + 8$.
- Process refers to an internalized action (Dubinsky, 1991), for instance, adding $2 + 4 + 6 + 8$ might be acted on as $2 + (2 + 2) + (2 + 2 + 2) + (2 + 2 + 2 + 2) = 2 \times 10 = 10 \times 2 = 20$.
- Objects can be grouped into two categories: physical and abstract objects (Dubinsky, 1991). An abstract object is an object due to its representations and occurs as a result of encapsulating a process (Dubinsky, 1991; Maharaj, 2010). For instance, in the above example, the number two is added to itself to get the next consecutive term. Then, a series can be extended to the nth term, i.e., $2 + 4 + 6 + 8 + 10 + \cdots + 2n$. In this series, n is an abstract object.
- Schema refers to a fixed unified framework of a mathematical concept that requires multiple actions, objects, and processes. It is unified in that it provides an individual learner with a means of determining whether a schema applies (Dubinsky, 1991;

Maharaj, 2010). For instance, $\sum_{i=1}^{n} 2i$ is a schema for the series $2 + 4 + 6 + 8 + \cdots + 2n$.

Mathematics development begins by mental construction or manipulation of a physical object into action. Then, action is internalized to become a process, and a process is encapsulated to an abstract object (Dubinsky, 1991). Lastly, action, process and abstract object combine together to form a schema (Dubinsky, 1991). For instance, $\sum_{i=1}^{n} 2i$ is a schema formed by combining action, process and an abstract object in the series $2 + 4 + 6 + 8 + \cdots + 2n$.

I believe that Dubinsky, and Tall & Vinner (1981) are describing the same properties of an object. They all view an object as something that is physical (Dubinsky, 1991;Tall& Vinner, 1981). This view refers to an immediate object (Sierpinska, 1994), for instance, using a physical object to find the value of *pi* by measuring its circumference and diameter. In this example, a physical object is an immediate object. Also, they all view an object as an abstract thing, which is a result of encapsulating a process(Dubinsky, 1991; Tall & Vinner, 1981).This view refers to a dynamic object (Sierpinska, 1994). For instance,in the expression$\pi = \frac{C}{D}$, π, C and D are dynamicobjects.

Meaning and Forms of Understanding

Understanding has been defined in different ways by various scholars in mathematics education. Sierpinska (1994) defines understanding as acting on an object in real life. Objects to Sierpinska include mathematical equations, mathematical problems, thoughts, symbols, texts, operations, representations, and generalizations. The process of acting on an object involves explanations and reasonings.

Skemp (1976) defines understanding as knowing something by explaining and reasoning instead of by memorizing facts; and provides a distinction between relational and instrumental understanding. Relational understanding refers to understanding a concept by knowing how and why a concept makes sense. For instance, "how can you find the value of *pi* using a circular object? Please explain for me." This question leads to relational learning because individual learners show how to find the value by explaining and reasoning. Instrumental understanding refers to understanding a concept by focusing on the question "what," for instance, "what are procedures for finding the value of *pi*?" This question leads to rote learning because individual learners might correctly list the procedures but fail to explain how to obtain the value of *pi* from those procedures.

For Kilpatrick and colleagues (2001), understanding is an element of mathematical proficiency. The authors address five strands of mathematical proficiency:

- Conceptual understanding refers to "comprehension of mathematical concepts, operations, and relations." (2001, p. 116)
- Procedural fluency refers "to skill in carrying out procedures flexibly, accurately, efficiently, and appropriately." (2001, p. 116)
- Strategic competence refers to "ability to formulate, represent, and solve mathematical problems."(2001, p. 116)
- Adaptive reasoning refers to "capacity for logical thought, reflection, explanation, and justification." (2001, p. 116)
- Productive disposition refers to "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (2001, p. 131).

There are various forms of understanding described in mathematics education (clearly seen above). When understanding is seen as the acquisition of a body of knowledge and skills (Glanfield, 2003), it is static (for instance, Ball et al., 2008; Shulman, 1986). In this case, knowledge can be viewed as facts and procedures that need “to be memorized and later recalled” (Ben-Hur, 2006, p. 129).

Understanding can be seen as dynamic (Glanfield, 2003; Kilpatrick et al., 2015; Pirie & Kieren, 1994). In this case, learners construct knowledge, and they can use the knowledge in their own way correctly (Pirie & Kieren, 1994).

Understanding can be seen as emergent (Davis & Renert, 2014; Davis & Simmt, 2003, 2006, 2014; Davis & Sumara, 2006). In this case, knowledge is not seen as static but emergent, because learners construct knowledge.

Finally, Ben-Hur suggests that understanding is visible when it is seen as an “outcome of an active process” (2006, p. 3). In this case, knowledge is not static but, rather, it develops through social interactions with knowledgeable others, including peers and teachers in an active learning process. This perspective suggests that learners’ understandings are not pre-determined and that learners’ understandings of the concepts become evident in their reflection(s) on their learning (Ben-Hur, 2006).

The above forms and definitions of understanding are important in this study in the following ways. The forms of understanding are relevant in the section about conceptual orientation: understanding as an outcome of active process. The definitions are relevant in the upcoming section about PSTs’ conceptual development and understanding. The definitions and forms of understanding are also important in this study as they underpin the research questions.

PSTs' Conceptual Development and Understanding

This section describes the research findings from the literature about PSTs' understanding of mathematical concepts. I use the discussion of how "understanding" is addressed in the research studies to define how I used it in this study.

PSTs' Understanding of Mathematical Concepts

"Pre-service teachers' understanding of mathematical concepts" was the key search term used in the literature review. I focused on PSTs for both elementary and secondary schools. From the reviewed literature, it is evident that PSTs' understanding has been researched in relation to a number of mathematical concepts. Studies have examined concepts of fractions (Ball, 1990, 1990a; Castro-Rodriguez, 2016; Kajander & Holm, 2013; Lee & Son, 2016; Nillas, 2003), triangles (Bryan, 1999; Gutierrez & Jaime, 1999), numbers (Crespo & Nicol, 2006; Feldman, 2012; Kaminski, 2002), trigonometry (Akkoc, 2008; Fi, 2003, 2006; Malambo, 2015), functions (Brijlall & Maharaj, 2013; Chesler, 2012; Malambo, 2015; She et al., 2014), and sequences (Yazgan-Sag & Argun, 2012). These are examples of concepts explored in the literature and are summarized in Table 2.3 below.

Table 2.3

Investigated Mathematical Concepts in Relation to PSTs' understanding

Concepts	**Authors**
Number sense	Kaminski, 2002
Number theory	Feldman, 2012
Integer subtraction and fraction division	Ball, 1990, 1990a; Kajander & Holm, 2013; Nillas, 2003
Multiplicative part–whole relationship	Castro-Rodriguez, 2016

Continuity of functions	Brijlall & Maharaj, 2013
Linear functions and slope	She et al., 2014
Definitions of functions	Chesler, 2012
Pythagorean identity and area of a triangle	Bryan, 1999
Altitude of a triangle	Gutierrez& Jaime, 1999
Fraction multiplication	Lee & Son, 2016
Trigonometry	Akkoc, 2008; Fi, 2003, 2006; Malambo, 2015
Sequence	Yazgan-Sag & Argun, 2012

How "Understanding" Has Been Described in the Research

"Understanding" in the reviewed literature was addressed in different ways. I organized these different ways of addressing "understanding" into four categories: meanings of mathematical concept(s) (Akkoc, 2008; Chesler, 2012; Gutierrez & Jaime, 1999; Yazgan-Sag & Argun, 2012); conceptual structure, representations, and contexts of mathematical concepts (Brijlall & Maharaj, 2013; Castro-Rodriguez et al., 2016; Feldman, 2012; Fi, 2003, 2006); concept explanations and justifications (Ball, 1990; Bryan, 1999; Crespo & Nicol, 2006; Kajander & Holm, 2013; Lee & Son, 2016; Malambo, 2015; Nillas, 2003); and relating and applying concepts to real-life experience (Ball, 1990a; Kaminski, 2002). The four categories are important, because they illustrate to what extent PSTs express their understanding of the concepts from defining them to applying the concepts in their daily life practice.

Table 2.4shows the summary of the four categories that I generated from the reviews of the literature about the ways "understanding" has been addressed in prior research. A description of these four categories will follow, after which I connect the way I view these four categories with this current study.

Table 2.4

Emerged Categories of the Reviewed Mathematical Concepts in Relation to PSTs' Understanding

Categories	**Authors**
Meanings of mathematical concepts	Akkoc, 2008; Chesler, 2012; Guetirrez & Jaime, 1999; Yazgan-Sag& Argun, 2012
Conceptual structure, representations, and contexts of mathematical concepts	Brijllal & Maharaj, 2013; Castro-Rodriguez et al., 2016; Feldman, 2012; Fi, 2003; 2006
Concept explanations and justifications	Ball, 1990; Bryan, 1999; Crespo & Nicol, 2006; Kajander & Holm, 2013; Lee & Son, 2016; Malambo, 2015; Nillas, 2003
Relating and applying mathematical concepts to real life	Ball, 1990a; Kaminski, 2002

Meanings of mathematical concepts.

Investigating how the meanings of the concepts have been expressed in the literature about PSTs helped me to realize in general what PSTs know and need to know about the meanings of the concepts. It also informed the CRI approach for conceptual development, including defining the concepts in mathematical words, pictorial and symbolic representations.

Some studies suggest the most foundational task facing PSTs in engaging with mathematical concepts is gaining a solid understanding of the meanings of the concepts. Too often, the meanings of the concepts are taught as if they are facts to be memorized rather than mathematical ideas that are developed through social interactions in an active learning process. A study by Gutierrez and Jaime (1999), examining this process in the European context, found that many PSTs had a limited understanding of the meaning of the altitude of a triangle. The authors explored PSTs' understanding of the altitude of a triangle based on Tall and Vinner's model of concept definition and concept image (Tall & Vinner, 1981).

Gutierrez and Jaime worked with 190 PSTs at the University of Valencia, Spain, administering a test to four groups of the PSTs. The test comprised items related to altitudes of a triangle (drawings) (Gutierrez & Jaime, 1999, p. 261). Group A worked on the test items that included "definition of the altitude of a triangle" (p. 260), and group D worked on test items that did not include the definition. Group B and C were considered the intervention groups. They responded first to the items without the definition and then to the items with the definition.

The authors observed that most of the PSTs in groups B, C, and D had a limited understanding of the concept definition and concept image of the altitude of a triangle while responding to the test items that did not contain a definition of the altitude of a triangle. For example, many PSTs failed to draw an external altitude or an altitude that coincided with the side of a triangle when they were to draw" a side of the triangle or the dotted segment (given in the test to show the side on which the altitude should be drawn)" (p. 270). The authors also found that PSTs in groups B and C performed better on the test with the definition of the altitude than on the test without the definition. Group A performed better on the test with the definition of the altitude of the triangle than Groups B, C and D did on the test that did not have the definition.

A study conducted by Akkoc (2008),using Tall and Vinner's model of concept definition and concept image, found that PSTs at Istanbul University had difficulties with the meaning of the concept definition and concept image of a radian. Akkoc worked with six PSTs who attended a mathematics teacher education program in Turkey. Some of the items that Akkoc designed for the interview and questionnaire included: "[W]hat is a unit circle? In your opinion, what is the place and importance of a unit circle in the teaching of trigonometry?" When interviewing the

PSTs, the author used a “think aloud” interview protocol to observe participants solving problems.

Akkoc noticed that the PSTs with a deep understanding of the concept image of radian—including the meaning, examples, properties and visual representation of a concept—could use the unit circle and relate it to various concepts in trigonometry. Moreover, PSTs who had a strong degree image of a radian in a unit circle used the right triangle to describe concepts and relationships in trigonometry. Akkoc also found PSTs’ concept images of degrees dominated the concept images of radians. She speculated that this might cause problems for understanding trigonometry functions when the functions were defined using real numbers.

One consequence of the struggle to understand the meaning of a particular concept is that the PSTs are unable to distinguish among related concepts. In another study conducted with PSTs in a Turkish university, Yazgan-Sag and Argun (2012) found that most of the 55 participants in the study had some difficulties distinguishing the general concept of sequence from the convergent sequence. The authors explored PSTs’ understanding of sequences from the perspective of Grossman and colleagues (1989). They used two open-ended questions about the concept of sequences to measure the PSTs’ understanding of sequence and to distinguish it from the concept ofa convergent sequence. Yaz-gan-Sag and Argun used Strauss and Corbin’s (1998) constant comparative method to analyse data and found that most of the PSTs were not able to define and differentiate the general concept of sequence from the convergences of a sequence. The authors note that when they asked PSTs to give the definitions of a convergent sequence and the concept of sequence, they found that most of the PSTs expressed that the sequence comprised related sets such as ordered sets, infinite ordered sets and the subject of real numbers; and for the convergent sequence, they expressed a concept as never reaching the boundary limit and every convergent sequence as both monotone and bounded.

PSTs' difficulties understanding the meaning of the concepts and distinguishing among them led to difficulties in their mathematical reasoning (Chesler, 2012). Chesler found that most of the participants in his study had difficulties in reasoning with and about mathematical definitions of functions. Chesler (2012)did a study of 23 pre-service secondary-school teachers enrolled in a capstone course at a large university in the western United States. He investigated their understanding of the definitions of functions and their reasoning. The study took place when the PSTs were analysing a secondary school textbook about the definitions of functions. The task was for the PSTs to choose and apply the definitions of functions, evaluate the equivalence of the definitions of functions, and interpret and critique a secondary-school textbook about the meanings of the particular type of functions.

There were three questions. Two of the questions were given to the PSTs as homework problems and one as a take-home examination. Chesler encouraged the PSTs to work together on the homework problems. However, the PSTs were restricted from collaborating on the take-home examination. The author found that only five out of the 22 PSTs gave correct definitions of functions on the exam. Furthermore, in the analysis of the definitions of a function in secondary-school textbooks, the author found that only six of the 22 PSTs noted that for the rational function, $f(x) = \frac{p(x)}{q(x)}$ exists under the condition that $q(x) \neq 0$ where $p(x)$ and $q(x)$ are polynomial functions. The author found that the PSTs lacked knowledge of school mathematics for the definitions of a function.

Conceptual structure, representations, and contexts of mathematical concepts.

In this study, I investigated the literature about conceptual structure, representations, and contexts of mathematical concepts because I wanted to identify potential problems related to PSTs' understanding of mathematical concepts. I also studied the way research studies were designed to help PSTs learn the conceptual structure, representations, and contexts of a mathematical concept.

Castro-Rodriguez and colleagues(2016) measured 358 PSTs' understanding of a concept through a three-item questionnaire to gather conceptual structure, representations, and contexts of "to fraction" [*sic*], and modes of use of the multiplicative part–whole relationship of the notion of "to fraction." The authors asked PSTs to define the phrase "to fraction" and to draw a picture that demonstrated its meaning. They found that about 16 percent of the PSTs did not mention the whole as a fundamental component when defining "to fraction," and that half of the PSTs did not consider the equal parts in the process of defining the phrase. Furthermore, while about 98 percent of the PSTs were able to pictorially represent the phrase in the form of circles, squares and rectangles, only 2 percent represented it symbolically $(\frac{a}{b})$. Castro-Rodriguez and colleagues (2016) concluded that university courses for the PSTs should deepen the understanding of "to fraction" in the areas of conceptual structure, representations, and contexts and modes of use.

In a study of43 North American PSTs, She, Matterson, Siwatu, and Wilhelm (2014) found that the participants possessed a limited understanding of linear functions and slope and also failed to transfer and apply the concepts appropriately in other situations. She and colleagues (2014) found that many PSTs were unable to find the slope of a line from a sketched graph. The PSTs were unfamiliar with the formula for finding the slope with two points ($m = \frac{y_0 - y_1}{x_o - x_1}$) for the two points (x_0, y_o) and (x_1, y_1). They were not able to apply the formula of a slope of the line ($m = \frac{y_o - y_1}{x_o - x_1}$) to find the y-intercept. They were also unable to find the value of the y-intercept (b) from a sketched graph of the linear functions and the formula $y = mx + b$. Also, they lacked proficiency in calculating.

North American PSTs are not the only ones who struggle to transfer and apply functions. Brijlall and Maharaji (2013)found that African PSTs had a limited understanding of the continuity of functions. Working with PSTs at the University of South Africa, the authors designed written examinations by adapting questions from an earlier study (2008; 2009a). Some of the questions included finding the values of a and b for which the function $f(x) = \begin{cases} ax^3 - bx + 2;\ x \geq 1 \\ bx^2 - a;\ x < 1 \end{cases}$ is differentiable at $x = 1$ and finding the condition for which the function is continuous at $x = b$. The authors found that about 4 percent of the PSTs made calculation errors when solving the function $f(x)$. About 55 percent of the PSTs were able to appropriately evaluate the limit of a split function $f(x)$.

PSTs' difficulties understanding the conceptual structure, representations, and contexts of trigonometry led to difficulty in their explanations and interpretations of trigonometry (Fi, 2003, 2006). Fi (2003) found that PSTs lacked an in-depth understanding of trigonometry in the following areas: radian measure of angles, inverse trigonometric functions, and reciprocal functions. For instance, some PSTs had misconceptions of the properties of the sine and cosine angles in the trigonometry identities in subtraction, inverses, and division (e.g., $cos^{-1}x = \frac{1}{cosx}$, $sin40 - sin10 = sin30$ and $\frac{\cos(60\times x)}{x} = \cos(\frac{60\times x}{x})$). Also, when Fi asked the PSTs, "what does a negative angle measure represent," he found that the PSTs conflated "equal" and "co-terminal" angles. Fi (2006) also found that PSTs had difficulty explaining and interpreting trigonometry in the area of periodicity, one-to-one functions, radian measure, and identities. He noticed that the PSTs "knowledge of school mathematics may not be sufficiently robust to support a meaningful instruction on some key trigonometric ideas" (p. 833).

PSTs can become more proficient at explaining and interpreting concepts by exploring the concepts using an active instructional approach. In a study examining this process with North American PSTs, Feldman (2012) found that the PSTs gained a deep understanding of number theory after a social learning approach on number theory was implemented in their program. He developed PSTs' understanding of number theory based on Dubinsky APOS theory.

Feldman (2012) worked with 59 PSTs in a college mathematics course where he designed and implemented a number theory program. He measured PSTs' understanding of elementary number theory using a number theory knowledge test (NTKT) and individual clinical interviews. He implemented NTKT before and after instruction on number theory. He also interviewed the PSTs before and after the instruction. The instructional sessions occurred during a three-week unit of teaching on number theory. Feldman analysed data obtained through interviewing the PSTs using Dubinsky's (1991) APOS theory to describe how the PSTs' understanding of number theory developed during the instruction. He found that PSTs gained a deep and connected understanding of number theory in the areas of divisibility, prime factorization of the number, greatest and lowest common factors as a combination of prime numbers within the prime factorization of another number and coordinating multiple processes.

Concept explanations and justifications.

My literature search included investigating explanations and justifications of mathematical concepts. For instance, I investigated how PSTs explained and justified their understanding of the division of fractions (Ball,1990; Kajander & Holm, 2013; Nillas, 2003). Several studies found that participants were unable to clearly explain and justify a division of a fraction. Lee and Son (2016) also found that 60 PSTs were unable to clearly explain and justify a fraction multiplication.

PSTs' difficulties with the conceptual understanding of trigonometry lead to difficulty in their explanations and justifications of the subject (Bryan, 1999; Malambo, 2015). Bryan found that participants in his study lacked an in-depth understanding of how to explain the meaning of Pythagorean identity and find the area of a triangle by justification and explanation. Malambo found that most of the participants in his study were proficient in common content knowledge; they were able to apply rules and formulas, but they failed to coherently explain, prove formulas, and translate algebraic-trigonometric functions to the Cartesian plane.

Crespo and Nicol (2006) demonstrated that 32 PSTs had difficulty explaining and justifying concepts. However, using an active instructional approach, they improved the PSTs' understanding of division by zero. The active instructional approach helped the PSTs to learn how to provide arguments, explanations, and justifications. Crespo and Nicol also found that the PSTs improved their pedagogical practice of division by zero. For instance, the PSTs were able to explain a concept of $5 \div 0$ by relating the equation to real life situations such as dividing five cup-cakes among zero children. When one of the PSTs argued that it was not possible to divide cupcakes when there were no children to eat them, a lengthy discussion followed. During that discussion, the researchers were able to observe in real-time how their instructional process helped the PSTs to better understand division by zero, and how to facilitate in-depth explanations and justifications of the concept.

Relating and applying concepts to real-life experience.

I investigated in the literature how PSTs related and applied mathematical concepts in real life, because I wanted to make sense of what PSTs know and need to know about these aspects of the concepts. I also learned how I could engage PSTs with a mathematical concept by relating the concept to real things/objects available in their local environment.

PSTs were unable to clearly relate and apply the concepts to real-life situations (Ball, 1990a; Kaminski, 2002). In a study, Ball (1990a) asked 19 PSTs to solve the problem $1\frac{3}{4} \div \frac{1}{2}$ by relating it to a real-life situation. Most of the participants could not make such a connection. Kaminski (2002) also found that despite the fact that PSTs have difficulties relating and applying concepts to real-life situations. However, using an active instructional approach helped 85 PSTs to learn how to relate and apply number sense to real-life situations. They looked for connections in the following conceptual areas: numeration and number exploration of whole numbers (e.g., being able to write 64 in 10 different ways); mental computation and pattern exploration of whole numbers (e.g., being able to explore the mental computation of 18×7); numeration exploration of rational numbers (e.g., being able to explore the names of the fractions between: $\frac{7}{8}$ and 1, $\frac{1}{4}$ and $\frac{3}{4}$, other than relying on $\frac{1}{2}$); and computational exploration of rational numbers (e.g., being able to find the solution for $[\frac{97\times15}{15}]$).

The instructional approach helped PSTs to develop their understanding of mathematical concepts by relating and applying the concepts to real-life situations. It is important to engage PSTs with a concept by using real objects and situations from daily life, from their local environment, as clearly this helps them to better understand the concepts, which will enable them to be better, more effective teachers.

Each of the four categories (defining a mathematical concept; describing conceptual structure, representations, and contexts of a concept; explaining and justifying a concept; and relating and applying a concept to real life experience) is potentially important to know before I designed and implemented the research activities. I was able to realize how I could design and implement the activities for the concept development of *pi*. This was important to help PSTs learn the concept relationally by building on their understanding of the meaning of *pi* and its value to the applications of the concept in mathematics, other fields, and real-life situation while using CRIand a social constructivist approach. The review of literature of the findings on PSTs' understanding of the mathematical concepts in African and international context is important in this study, because it helped to improve my understanding of how I could integrate the four categories of understanding of a mathematical concept in this study.

Description of Concept-Rich Instruction

As explained earlier, CRI is an instructional approach designed to develop learners' understanding of mathematical concepts. It engages learners with concepts by encouraging them to "learn and think mathematically" in small and large groups (Ben-Hur, 2006, p. vii).

CRI involves five components, all of which contribute to learning mathematical concepts. The five components are meaning, decontextualization, recontextualization, practice and realization.

- Meaning is the focus on helping learners to develop an understanding of a concept by defining it in words, as well as by representing the definition(s) of a concept in mathematical symbols.
- Decontextualization focuses on developing learners' understanding of a concept by experiencing "a variety of its applications."
- Recontextualization focuses on developing learners' understanding of a concept by identifying its applications and using a concept "to connect new experiences with the past experiences."
- Practice focuses on developing learners' understanding of a concept through learning by doing.

- Realization focuses on developing learners' understanding of a concept by integrating a concept into the mathematics curricula and beyond the curricula, such as in other fields and daily life situations (Ben-Hur, 2006, p. 12).

The CRI approach emerged when schools and mathematics teachers in the United States were looking for ways to improve teaching, ways that could be used to actively engage learners in mathematics(Ben-Hur, 2006). The primary motivation was mathematics teachers' desire to achieve first-class status in mathematics education in their schools (Ben-Hur, 2006). National Commission on Mathematics and Science Teaching for 21st Century (2000) (as cited in Ben-Hur, 2006) states: "[learners] are crippled by content limited to 'what?' They get only a little bit about 'how?' (or 'how else?'), and not nearly enough about the 'why?'" (p. 40). Ben-Hur responded to the Glenn Commission by developing CRI training programs for grade 6—8 learners. The instructional approach became popular in mathematics classes in the larger North American context because of its potential to develop learners' understanding of mathematical concepts and facilitate learning achievements (Ben-Hur, 2006).

Ben-Hur and Bellanca (2005) (as cited in Ben-Hur, 2006) used CRI as an instructional approach to guide teaching major concepts for learners in grades 6 to 8 in algebra, geometry, measurement, number sense, probability and proportional reasoning. They also designed instructional sequences and used instructional activities to guide teaching concepts. In the same study, they used CRI to assess ongoing mathematics teaching and learning for grades 6 to 8 through reflections. Teachers have used the Ben-Hur and Bellanca instructional approach to develop learners' understandings of mathematical concepts and to assess ongoing learning of those concepts, including geometry, probability and statistics (Ben-Hur, 2006).

Instructional Approaches and the Ways CRI Relates to or Differs from Other Teaching Approaches

Instructional approaches can be categorized into two forms, namely, direct instruction (DI) and participatory approaches (Kucharcikova & Tokarcikova, 2016). Both approaches are important for mathematics classes, because they can be used by teachers to develop learners' understanding of mathematical concepts.

The DI approach is often thought of as teacher-centred instruction and is premised on learners being receivers of information (Kucharcikova & Tokarcikova, 2016). The lecture method of teaching is an example of a DI approach or teacher-centred instruction. The lecture method of teaching could be an example of Freire's banking concept of education. Freire (2005) defines the banking concept as "An act of depositing, in which the learners are the depositories and the teacher is the depositor. Instead of communicating, the teacher issues communiques and makes deposits which the learners patiently receive, memorize, and repeat" (p. 72).

Learners are treated as empty vessels that need to be filled with knowledge, skills, and values. The teacher is treated as the only source of knowledge, the one who knows everything. Freire (2005) argues that in the banking concept of education: "Knowledge is a gift bestowed by those who consider themselves knowledgeable upon those whom they consider knowing nothing" (p. 72). Freire (2005) provides the characteristics of banking concept in education as described below.

> The teacher thinks and the learners are thought about; the teacher talks and the learners listen—meekly; the teacher disciplines and the learners are disciplined; the teacher chooses and enforces his choice, and the learners comply; the teacher acts and the learners have the illusion of acting through the action of the teacher; the teacher chooses the program content, and the learners (who were not consulted) adapt to it; the teacher

> confuses the authority of knowledge with his or her own professional authority, which she and he sets in opposition to the freedom of the learners; the teacher is the subject of the learning process, while the learners are mere objects (p. 73).

There are challenges associated with using a DI approach such as a lecture. Using that approach to teach mathematics in class, I imagine the classroom arrangement consisting of two faces. The teacher as a depositor of knowledge is seen most of the time in front of the class, close to the chalkboard or white board, facing the learners, the depositories. This kind of seating arrangement is traditionally structured in a way that does not allow a dialogue among learners or between the teacher and learners. There is little chance for learners to discuss among themselves in small groups to learn mathematics during the lesson. In a lecture approach learnersmightnot be provided with an environment that is conducive to creating a community of learners. Giroux (2010) suggests that creating a community of learners "offers learners new ways to think and act independently" (para. 7).

The participatory approach is learner-centred: the teacher acts as a facilitator and learners always construct their own knowledge (Kucharcikova & Tokarcikova, 2016). This approach fosters social learning and interactions among learners in a class (Ben-Hur, 2006; Erickson, Lanning & French, 2017; Wathall, 2016). An example of the participatory approach in mathematics is learning fractions by working on the activities in small group and class discussions.

Participating in activities provides an opportunity for learners to develop their understanding of a concept. Learners work in small groups and discuss their mathematics followed by group presentations to the entire class. In this case, teachers become facilitators of the ongoing learning, including elaborating learners' mathematics in the class. There are many advantages associated with this approach, including that it allows a dialogue among learners, provides the possibility for multiple responses among learners in the class, and transforms the mathematics class from teacher-centred to learner-centred as the community of practice. To create such advantages, mathematics teachers should create an environment of sharing learning, whereby they learn from learners while teaching and the learners learn from them. This kind of instruction includes problem-based learning.

The participatory approach, including activities such as problem-based learning, engages learners in learning mathematics. Freire (2005) argues that in the problem-based learning context, "The learners—no longer docile listeners—are now critical co-investigators in dialogue with the teacher. The teacher presents the material to the learners for their consideration and re-considers her earlier considerations as the learners express their own" (p. 85).

Practicing a learner-centred approach denies that mathematics is abstract, and that mathematics has only one answer; promotes dialogue and class discussions; and stimulates creativity and reflections among learners.

The role that a teacher plays in the DI approach is different from that in the participatory approach. The same goes for the role of the learner in each approach. In the DI approach—for example, a lecture—the teacher presents a lesson and the learner practices with an assignment. But in the participatory approach, a teacher starts by giving activities to the learners to work in small groups and elaborates on their understandings later (Kucharcikova & Tokarcikova, 2016).

The participatory approach encourages learners to work in small groups and think critically in a class. For instance, when encouraging participation in activities, a teacher asks learners to work in small groups on a task."What are the possible values of a and b in a linear equation $ax + b = 4$? Please explain for me" (Think, pair, and share) is an example of a participatory approach, which gives an opportunity for learners to work in small groups and think critically while answering a question. This linear equation encourages learners to think critically to find the values of a and b required when adding the two constant values to get an answer 4, while maintaining a variable x. For instance, when the value of a and b is 2, then the value of a variable x is 1 so that the answer on the left-hand side of the equation becomes equal to that on the right-hand side. There are other values of a and b that give an answer 4 while maintaining a variable x. For instance, the values of a and b are 1 and 3 respectively which give an answer 4, whereby 1 is the value of the variable x. Following these trends, we notice that the equation above has many solutions.

The participatory approach can lead to a relational understanding of a concept among learners in a class. For instance, a teacher asks learners to work in small group activities by explaining and justifying their mathematical ideas. The teacher breaks learners in small groups and asks them to explain and justify their mathematical ideas in those small groups. The teacher might do this by making it clear she/he wants all the learners to get involved by posing a series of questions to the entire group:"Given a circular object, how can you find the value of *pi*? Please explain for me" (Think, pair, and share).

Participatory approaches in mathematics education, include concept-based instruction (CBI) (Erickson, Lanning & French, 2017; Wathall, 2016) and CRI (Ben-Hur, 2006). CBI is addressed in two ways; it can be used to deepen learners' understanding of a concept by focusing on a single subject area such as mathematics or across subjects—interdisciplinary teaching and learning (Erickson, Lanning & French, 2017). For instance, Wathall (2016) used CBI to develop learners' understanding of mathematical concepts. CRI relates to other participatory approaches, such as CBI, in the following ways: both approaches encourage social interaction among learners, focus on developing learners' understanding of the concepts rather than on memorizing facts, give teachers an opportunity to assess ongoing learning of the concepts, focus on developing learners' problem-solving ability, are concept-oriented, and have the potential to improve learning achievement (Ben-Hur, 2006; Wathall, 2016).

CRI and CBI differ with respect to instructional processes. CRI involves five processes: defining, decontextualizing, practicing, re-contextualizing and realizing of a concept (Ben-Hur, 2006). CBI, on the other hand, focuses on six processes: solving a problem, reasoning and proofing, communicating, making connections, creating representations, and investigating (Wathall, 2016).

The guiding principles of CRI are different from those of CBI. CRI involves identifying the concepts in the curriculum, planning the research activities, designing the learning environment, analysing learners' errors in a class, and assessing ongoing learning using a variety of strategies (Ben-Hur, 2006). On the other hand, CBI involves establishing the learning goal(s), implementing, in a class, tasks that foster problem-solving and reasoning, using and connecting representation(s), facilitating meaningful discussions, asking purposeful question(s), assisting struggling learners, and eliciting and using "evidence of student thinking" (Wathall, 2016, p. 217). The guiding principles differ but each instructional approach focuses on helping individual learners understand a mathematical concept relationally rather than instrumentally. The role of a teacher is to facilitate learning of a mathematical concept in a social learning context.

Although CRI and CBI are participatory approaches, this study is focused on engaging PSTs with a concept by encouraging them to participate in learning by doing. The reason is that the participatory approach is rarely used in university mathematics classes for PSTs. I chose CRI because CRI emphasizes practice as one of its instructional components and, in my experience, practice is included in Tanzanian mathematics classrooms. In this study, PSTs practiced a concept through small group and class discussions and group presentations in the meetings. On the last day of implementing CRI, I emphasized practice by having the PSTs plan to teach a concept.

This study aimed to explore the ways CRI revealed PSTs' expressions of their understanding of a mathematical concept and the way PSTs planned to teach a concept, given their participation in CRI. Despite the fact that CRI has the potential to develop learners' understanding of the mathematical concepts, the instructional approach has not yet been researched with respect to university PSTs in their mathematics classes. This study contributes to existing research by broadening the potential of the CRI approach to Tanzanian mathematics PSTs' education.

Conceptual Orientation: Understanding as an Outcome of An Active Process

Understanding as an outcome of an active process is connected to the way understanding in the research meetings with university PSTs was conceptualized, the ways this notion of understanding is compatible with CRI, and the ways this notion of understanding is different from other forms. In the following paragraphs, I will expand on this idea.

As many of the studies cited thus far have shown, PSTs have a limited understanding of various mathematical concepts (Akkoc, 2008; Ball, 1990, 1990a; Brijlall & Maharaj, 2013; Bryan, 1999; Castro-Rodriguez, 2016; Chesler, 2012; Crespo & Nicol, 2006; Feldman, 2012; Fi, 2003, 2006; Gutierrez & Jaime, 1999; Kajander & Holm, 2013; Kaminski, 2002; Lee & Son, 2016; Malambo, 2015; Nillas, 2003; She et al., 2014; Yazgan-Sag & Argun, 2012). However, three investigators (Crespo & Nicol, 2006; Feldman, 2012; Kaminski, 2002) designed studies to help PSTs to improve their understandings of the concepts by using a participatory approach. Figure 2.1 below, gives a summary of the findings from the previous studies about PSTs' understanding of mathematical concepts.

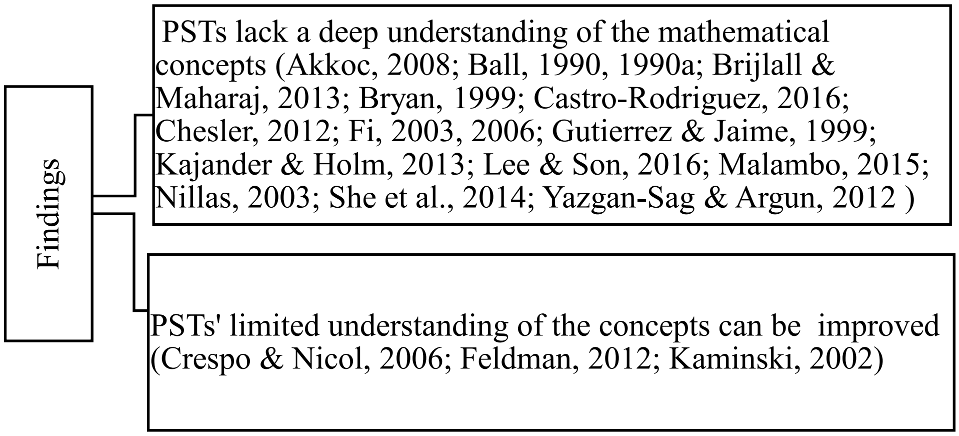

Figure 2.1. The findings from the reviews of literature about PSTs' understanding of the mathematical concepts.

Several researchers have also suggested that PSTs' understanding of mathematical concepts is fundamental to improve learning in schools (Akkoc, 2008; Ball, 1990, 1990a; Brijlall & Maharaj, 2013; Bryan, 1999; Castro-Rodriguez, 2016; Chesler, 2012; Fi, 2003, 2006; Gutierrez & Jaime, 1999; Kajander & Holm, 2013; Lee & Son, 2016; Malambo, 2015; Nillas, 2003; She et al., 2014; Yazgan-Sag & Argun, 2012). Also, PSTs need ongoing learning to improve their understanding of mathematical concepts (e.g., Crespo & Nicol, 2006; Feldman, 2012; Kaminski, 2002).

PSTs' understanding of a concept was regarded as an "outcome of an active process," in my study because I focused on understanding the meanings that PSTs brought forth in the research activities (Ben-Hur, 2006, p. 3). PSTs reflected on their ongoing learning after participating in research activities.

The notion of understanding as an outcome of active processes can be connected with CRI in the research meetings as follows. First, "active processes" refer to CRI processes (defining, decontextualizing, re-contextualizing, realizing, and practicing a mathematical concept). Second, an “outcome” of active processes refers to the ways that CRI processes revealed PSTs’ understanding of the mathematical concepts. This is evidenced through reflections on the ongoing learning, and during class discussions while analysing errors and inconsistencies about the concepts. Ben-Hur (2006) does not give the meaning of the term “processes,” however, the term can be referred to “actions that produce results” (Wathall, 2016, p. 7). Actions include what individual PSTs verbalized about what they did in the discussions in the research meetings. In this study, actions emerged when PSTs interacted with mathematical activities, the mathematics curriculum and CRI activities.

Understanding as an outcome of active processes fits with the reason of exploring the research question: How does CRI reveal the way that PSTs express their understanding of a mathematical concept? I implemented this instructional approach in the research meetings to provide an opportunity for individual PSTs to learn through interacting with peers and the teacher educator in the research meetings during small group and class discussions.

Theoretical Orientation: Social Constructivism

This section starts by presenting types of constructivism, their similarities and differences. This explanation is important in order to frame the study in a social constructivist context based on the ideas of Vygotsky (1978). The section ends by providing the reasons for using the theory in this study.

Types of Constructivism: Similarities and Differences

Constructivism is a learning theory that emphasizes that learners construct knowledge and meanings based on their own experiences (Fosnot, 1996; Steffe & Gale, 1995). Philosophically, the essence of this theory is that reality can be known through individual experience.

There are four epistemological tenets of constructivism:

- Knowledge is a result of an individual's need to be aware of something (von Glasersfeld, 1984; 1996);
- Cognition is an adaptive process that is essential to make an individual's behavior feasible in a certain environment (von Glasersfeld, 1984; 1996);
- Cognition is essential for organizing and making sense of individual's experience. It does not focus on giving an accurate representation of reality (von Glasersfeld, 1984; 1996);
- Knowing is essential in various aspects such as language,social and cultural interactions (Gergen, 1995).

There are three types of constructivism: cognitive, radical, and social. Cognitive constructivism emphasizes the first two philosophical tenets presented above (von Glasersfeld, 1984). Radical constructivism emphasizes the first three epistemological tenets (Larochelle, Bednarz & Garrison, 1998; von Glasersfeld, 1995). Social constructivism emphasizes all four philosophical tenets in the teaching and learning process (Prawat & Floden, 1994).

From the above types of constructivism, I consider social constructivism to be the best in this study because it emphasizes participation in the learning of a mathematical concept. It emphasizes social interactions in a social cultural context for individual and group acquisition and the understanding of knowledge (Gergen, 1995; Vygotsky, 1978). Philosophically, reality is a socially constructed and agreed-upon truth, as a result of "co-participation in cultural practices" (Cobb & Yackel, 1996, p. 37). The meanings that individual learners or groups bring to a social learning environment are essential for socially constructed knowledge within a social activity (Cobb & Yackel, 1996).

Constructivism is important to help teachers understand how learners acquire knowledge. Doolittle and Camp (1999) write that "constructivism posits that knowledge acquisition occurs amid four assumptions:

- Learning involves active cognitive processing.
- Learning is adaptive.
- Learning is subjective, not objective.
- Learning involves both social/cultural and individual processes" (para. 24).

Although the above four assumptions are critical for learners' acquisition of knowledge, an active learning environment is also important. Doolittle and Camp (1999, para. 30—47) emphasize eight recommendations as quoted below:

- Learning should take place in authentic and real-world environments;
- Learning should involve social negotiation and mediation;
- Content and skills should be made relevant to the individual learners;
- Content and skills should be understood within the framework of the individual learners' prior knowledge;
- [Learners] should be assessed formatively, serving to inform future learning experiences;
- [Learners] should be encouraged to become self-regulatory, self-mediated, and self-aware;
- Teachers serve primarily as guides and facilitators of learning, not instructors;
- Teachers should provide for and encourage multiple perspectives and representations of content.

Description of Social Constructivism Based on the Ideas of Vygotsky (1978)

This subsection focuses on social constructivism based on the ideas of Vygotsky (1978) by presenting five concepts: the zone of proximal development (ZPD), more knowledgeable others (MKOs), social interaction (SI), scaffolding, and mediated learning.

Vygotsky argues that higher psychological functions happen on two planes. The first plane is social context. The second plane is internal (Vygotsky, 1978). In other words, these functions happen "first, between people (inter-psychological), and then inside the child (intra-psychological)" (Vygotsky 1978, p. 57). In this study, the PSTs internalized the concept after they worked on the activities or participated in the social learning environment.

A ZPD is important to develop individual learners' understanding of a concept when learners interact with MKOs, such as peers and the teacher in a social learning environment. Vygotsky (1978) argues that "the zone of proximal development is the distance between the actual development level …and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers" (p. 86). This is a zone whereby a learner cannot understand a concept or perform a task on her/his own. But a learner can perform it well with the support of MKO.

A learner needs the guidance of MKOs, such as a teacher or peers, to learn a concept or perform the task within a ZPD. An MKO "is anyone who has higher level of knowledge and skills than a learner on a certain concept, task, event, process or situation" (Cherry, 2019, para. 1). In this case, MKOs can be a teacher, parent, peer or any adult who can provide valuable knowledge and skills to assist the learner (Cherry, 2019). The presence of MKOs within the ZPD is seen as a sensitive strategy for the learners' cognitive development of the learners, as MKOs can provide effective instruction or guidance to the learners, which fosters the development of their skills and knowledge. In doing so, the learners use the developed knowledge and skills in their own lives.

The role of the MKOs such as a teacher is to work with the learners in a ZPD and to present learners with challenges that interrupt their flow of the activities and direct them to the reflections (Ben-Hur, 2006; Verenikina, 2003; Vygotsky, 1978).Within the ZPD, MKOs need to pay attention to individual learners' difficulties to help them develop their understanding of the mathematical concepts in a social learning environment. Ben-Hur (2006) says that,

> [I]n the state of disequilibrium, conceptualization occurs in the zone of proximal development. Individual learners may say the right things, but they may also fail to enact them. Likewise, they may do the right things, but fail to explain them. The learners lose an understanding of the concepts over time if they do not continue to practice and reflect on the concepts. Thus, learners' conceptualization depends upon the amount, nature, and challenge of the practice, which is assigned by the teachers. (p. 14)

Teachers need to encourage learners' interactions with peers during the teaching and learning process to help them observe and gain new knowledge and skills. Such interactions can provide the opportunity for more competent learners to pair and share with less knowledgeable and skilled learners. In doing so, the competent learners also have an opportunity to practice their knowledge and skills.

Social interaction (SI)also plays a major role in the learning process within the ZPD (Vygotsky, 1978; Wertsch, 1979/2008). SI, which is supportive, helps to foster learning in a class. Learners construct knowledge; learning provides the motives and goals for the actions that arise from the activities. However, if the individual learners do not perceive classroom activities as relevant to their goals, or if the activities seem to endanger or contradict their understandings of the context, they will resist participation. If learning support is provided so that it matches the goal of the individual learners, it creates an effective context for learning. When the individual learners lack a motive for participating in the learning activities, they decide whether to continue learning or not. The learners' decision not to participate puts an end to their ongoing learning. They are likely to withdraw from learning, which can result in learning difficulties (Ben-Hur, 2006).

Scaffolding is also an important strategy for helping learners develop their understanding of the concepts within the ZPD. Scaffolding is an instructional strategy whereby a teacher or mentor provides appropriate "activities, tools, instructions, and resources" to guide and help learners develop their understanding of a concept within the ZPD (Cherry, 2019, para.1). Ben-Hur (2006) also suggests that, "teachers must intervene by asking guiding questions, correcting and engaging [learners] with self-evaluation and reflection" (p. 33). In doing so, the learners will be able to work on their own to accomplish the tasks after removing scaffolding, such as "activities, tools, instructions, and resources" (Cherry, 2019, para. 1).

Mediated learning is also important within Vygotsky's ZPD (Ben-Hur, 2006).Mediated learning is a learning process that focuses on developing learners' understanding of a concept through asking questions. Usually the learners are asked how and why questions to facilitate effective thinking. Apart from learning by focusing on questions, learners can also work on a designed task to foster their thinking.

Mediated learning is a process by which learners need to work on a particular task, and discuss and reflect on their learning with MKOs and with one another to develop a thorough understanding of the concepts in the class (Ben-Hur, 2006, p.10).Learning concepts involve reflection. The reflective process is attained through individual learners' interactions with MKOs, such as with a teacher and peers in small group or class discussions. When the teacher mediate learners' interactions with MKOs, learners benefit from improved learning experiences. They are able to communicate their ideas in different ways by describing and reflecting on their learning experiences (Ben-Hur, 2006).

There is “a reciprocal relationship between the teachers and learners,” and between the learners with other learners during mediated learning (Ben-Hur, 2006, p. 55). In this case, a teacher is a mediator of the learning process. The teacher “ensures that mediation is always responsive to emerging learning needs” (p. 56). As such, the teacher must be flexible in the class to create a strong relationship with the learners. In doing so, “the teacher must be ready to adjust their lessons and include objectives that relate to emerging needs,” including giving “more learning experiences” as required, engaging struggling learners “with alternative representations of new concepts,” and facilitating “reflective interactions among learners” (p. 56).

Mediated learning experiences can be attained through interactions among learners and a teacher in the class by asking questions or giving a task. Language plays a role to foster these learning experiences. Through interactions, the teacher can realize what the individual learners know and do not know. Also, the teacher can look for and use the appropriate strategies to help the struggling learners develop their understanding of the concepts. Moreover, the teacher can adjust the instructional activities to meet the needs of struggling learners (Ben-Hur, 2006).

Reasons for Using Social Constructivism as a Theoretical Framework

Social constructivism (Vygotsky, 1978) creates the basis for my interpretation of the world. This is important because there are various forms of constructivism; I needed to understand them and interpret them to underpin the ways I view about knowledge and reality, and assumptions of learning a mathematical concept in this research project. It also gives me a framework for the initial interpretation of the phenomenon of concern, and it provides a framework for understanding a mathematical concept by exploring, noticing, exposing, and interpreting a concept. Social constructivism (Vygotsky (1978) is therefore a useful theoretical framework for me to explore the ways that the CRI approach revealed PSTs’ expressions of their understanding of a concept in the research meetings.

CRI is based on social constructivism to foster the value of thoughtful interactions with and among learners for the improvement of their knowledge and cognitive skills in social learning environments (Ben-Hur, 2006). In social constructivism, human social interactions play an active part in the development of knowledge and skills. The purpose of my study was to explore, during four research days, the implementation of Ben-Hur's CRI approach with university PSTs. The research days provided an opportunity for PSTs and MKOs interact with each other.

Social constructivism (Vygotsky, 1978)was appropriate to use in this studyto explore the implementation of a CRI approach with university PSTs in a Tanzanian context. First, it provided a rationale to have PSTs reflect, during research meetings, on their ongoing learning of a mathematical concept. Second, PSTs could develop their understanding of a concept through interactions with their peers and instructor in the research meetings. Third, it promoted social interactions among the PSTs and the instructor, and among the PSTs with peers (MKOs). Finally,it provided a framework for a researcher like me to design and implement CRI activities in the research meetings. This was important to foster learning of a concept using local materials that were familiar to the PSTs to bring meaningful learning.

Social constructivism based on the ideas of Vygotsky (1978) has been used in education research as an instructional approach (e.g., Bruner, 1997; Daniels, 2001; Harvey & Charnitski, 1998; Nassaji & Cumming, 2000; Siyepu, 2011; Steele, 2001; Yaroshevsky, 1989); for assessment (e.g., Kirschenbaum, 1998; Roth, 1992; Sternberg & Grigorenko, 2002); for cognitive development (e.g., Bruner, 1997; Cole & Cole, 2001; Feuerstein, Rand, Hoffman & Miller, 1980; Rogoff, 1990; Taylor, 1992; Vygotsky, 1978); for communication (e.g.,Gutierrez, Baque-dano-lopez, & Tejeda, 1999; Siyepu, 2013; Wells, 1999); for mediated learning (e.g., Anton, 1999;

Ball, 1993; Cobb, Yackel,& Wood, 1993; Mercer & Littleton, 2007; Radford, 2000; Wertsch, 1991); and for concept formation (e.g., Tall, 1995; Dubinsky, 1991; Czarnocha et al., 1999).

Social constructivism (Vygotsky, 1978) has been presented to frame my exploration of the CRI approach with university PSTs in Tanzania. In doing so, the chapter has provided the theoretical and conceptual orientations of this study, which are important to explore the research questions: How does CRI reveal the way that PSTs express their understanding of a mathematical concept, and how do PSTs plan to teach a concept, given their participation in CRI?

CHAPTER 3

RESEARCH METHODOLOGY

This qualitative case study, carried out in one Tanzanian public university, explored the ways that the concept-rich instruction (CRI)approach revealed: PSTs' expressions of their understanding of a mathematical concept at a Tanzanian university; and the ways PSTs plan to teach a concept, given their participation in CRI. The PSTs participated in four days of CRI activities. Data was collected from those activities. The details of the research design are described in this chapter.

This chapter is organized into eight sections that focus on the following: case study justification, research site description, description of the participants, research design and data collection, data analysis, validity and reliability within the case study, ethical issues, and limitations and delimitations of the study.

Why Use Case Study?

This project focused on one "case" (Yin, 2014, p. 4). Following Stake (1995), this design perspective can help us to "learn about other cases or some general problems" (p. 3). In particular, this design perspective helped me to conduct this study to obtain an in-depth understanding of CRI in a Tanzanian context.

A case study focuses on detailed descriptions and analyses of systems that are bounded by a particular phenomenon (Bell, 2014; Merriam, 1988). Adelman, Jenkins, and Kemmis (1983) describe examples of bounded systems by arguing that "the most straightforward examples of 'bounded systems' are those in which the boundaries have a common-sense obviousness, e.g., an individual teacher, a single school, or perhaps an innovative programme" (p. 3). While CRI in a Tanzanian context is a case about introducing innovative teaching and learning of mathematics in university mathematics classes, third-year pre-service mathematics teachers at the University of Dodoma (UDOM) and the concept of *pi* are the boundaries, and the bounded system includes the study of the ways that the CRI approach revealed PSTs' expressions of their understanding *pi*. This binding is in line with Baxter and Jack (2008), who argue that once the case is determined we need to bind the case to ensure that "the study remained reasonable in scope" (p. 547).

As explained in Chapter 2, this instructional approach had not yet been investigated in Tanzanian university mathematics classes. Having designed and implemented the activities for PSTs based on CRI, I was able, during four one-day research meetings, to observe and study PSTs' expressions of their understanding of one mathematical concept, *pi,* and the ways that they planned to teach the concept.

Baxter and Jack (2008) suggest that "once you have determined that the research question is best answered using a qualitative case study and the case and its boundaries have been determined, then you must consider what type of case study will be conducted" (p. 547).In particular, I used an exploratory case study design to obtain detailed information about CRI in a Tanzanian context. Writing about exploratory case studies, Baxter and Jack (2008) write that "this type of case study is used to explore those situations in which the intervention being evaluated has no clear, single set of outcomes" (p. 548). As such, this type of qualitative case study was appropriate for this study because I wanted to explore a case, namely, CRI in the Tanzanian context.

Research Site

A research site should allow for the collection of data by allowing researchers to interact with participants to investigate the phenomenon under study (Creswell, 2012, 2014). With the intention of using a qualitative case study, I chose the UDOM as a research site as it aligned with my purpose and it was convenient (Nieuwenhuis, 2014; Merriam, 1998).

The motivation for choosing the research site was that I wanted to conduct a study involving undergraduate PSTs in a large, public university in Tanzania. I used convenience sampling and selected UDOM for two reasons: I am an instructor in the department of mathematics there, and consequently I have a personal connection; and UDOM encourages its instructors to conduct research at the university to improve the quality of education.

About the University of Dodoma

UDOM is the largest public (government-run) university in Tanzania. It is located in Dodoma, the capital city. The university began offering academic degree programs in September 2007 through four colleges that focused on social sciences, humanities, informatics, and education. At present, UDOM has seven functioning colleges: College of Humanities and Social Sciences (CHSS), College of Education (CoEd), College of Informatics and Virtual Education (CIVE), College of Earth Sciences (CoES), College of Natural and Mathematical Sciences (CNMS), College of Health and Applied Sciences (CHAS), and College of Business Studies and Law (CBSL). These colleges are physically separate.

The UDOM colleges offer various programs from the certificate to the PhD level. The goal is to increase contributions to higher education through enrolling large numbers of learners who meet entry qualifications. In doing so, the university supports training and education to promote Tanzania’s economic growth, and to help Tanzanians to find solutions to eradicate poverty and improve their social and cultural lives.

UDOM’s objectives, in a broad sense, focus on providing quality knowledge and skills through using appropriate teaching, learning and research techniques/approaches and resources for improving economic growth and prosperity in Tanzania and for improving the quality of life of all Tanzanians.

The Programs at UDOM for Pre-service Mathematics Teachers

There are several three-year (two four-month semesters per year) education degree programs offered at UDOM. A person who successfully completes any one program will be qualified to teach mathematics in Tanzania. A pre-service mathematics teacher could be enrolled in any one of the programs described in Table 3.1.

Table 3.1

Degree Programs Which Are Offered to UDOM's PSTs

Degree Programs	College Responsible
Bachelor of science degree with education (BSc Ed)	College of Natural and Mathematical Sciences
Bachelor of education degree in science with information and communications technology (BED SC ICT)	College of Education
Bachelor of education degree in science (BED SC)	College of Education
Bachelor of education degree in administration and management (BED ADMAN)	College of Education
Bachelor of education degree in policy, planning and management (BED PPM)	College of Education
Bachelor of education degree in information and communications technology (BED ICT)	College of Education
Bachelor of education degree in commerce (BED COM)	College of Education

There are four similar experiences for PSTs enrolled across the seven programs. First, the PSTs are taught mathematics courses by the university instructors from the Department of Mathematics in the College of Natural and Mathematical Sciences in each semester of their program. These PSTs are taught advanced level mathematics content not necessarily related to their work of teaching. Second, the PSTs are taught education courses (such as philosophy of education, methods of teaching, classroom interactions and management, curriculum, and educational psychology)by university instructors from the College of Education in each semester. Third, PSTs engage in an eight-week practicum in either a primary or secondary school at the end of Years 1 and 2. There is no practicum Year 3. Fourth, PSTs are assessed in each course with at least two tests during a semester and with one university examination (UE) at the end of a semester.

Description of the Participants

I used convenience sampling in this study. Convenience sampling, according to Merriam (1998), is when researchers "select a sample based on time, money, location, availability of sites or respondents, and so on" (p. 63). Convenience sampling was convenient for me because I am an employee at UDOM. Both the university and the mathematics undergraduate PSTs were readily available to me, having been an employee at UDOM since 2009 — though I have not taught courses in the last five years due to being on study leave.

The target participant population for the study was undergraduate pre-service mathematics teachers who were in their third year of study. I distributed posters and letters to classes asking PSTs to volunteer for my study. I also put up posters on notice boards in the College of Education Library and in the College of Natural and Mathematical Sciences.

Ten PSTs volunteered to participate in this study. All volunteers had already completed two teaching practicums, and were in the first term of their third and final year of their teacher education programs. Nine participants attended all four days of the CRI. One participant did not take part on the last day, due to family circumstances, and participated in three of the four days.

Three women and seven men participated. All participants were over 20 years old and less than 30 years old. All participants were given a pseudonym expressed as initials to ensure confidentiality. Background information on participants was gathered using a demographic information sheet (see Appendix A).

Table 3.2 provides information on the degree programs of the respondents who participated in this study. Three participants were pursuing a Bachelor of Education in Science with Information Communication and Technology (BED SC ICT), and were majoring in mathematics and information communication and technology (ICT). Another three participants were pursuing a Bachelor of Education in Science (BED SC), with two specializing in mathematics and physics, and one specializing in mathematics and chemistry. One participant was pursuing a Bachelor of Education in Administration and Management (BED ADMAN) specializing in mathematics and physics. Two participants were pursuing a Bachelor of Education in Policy, Planning and Management (BED PPM) specializing in mathematics and economics. One participant was pursuing a Bachelor of Education in Information and Communications Technology (BED ICT) specializing in mathematics and ICT. All these PSTs specialized in mathematics-related areas and were taking mathematics classes in mathematics department.

Table 3.2

Description of Participants

Number	Respondent	AGE (YEARS	DEGREE PROGRAMME	EDUCATIONAL LEVEL	GENDER	NUMBER OF TEACHING EXPERIENCES	SUBJECTS OF SPECIALIZATION	YEAR OF STUDY
1	Cases\\AA	24	BED SC ICT	Undergraduate student	Male	2	MATH AND ICT	3
2	Cases\\DA	24	BED SC	Undergraduate student	Male	2	MATH AND PHYSICS	3
3	Cases\\EB	24	BED SC ICT	Undergraduate student	Female	2	MATHEMATICS AND ICT	3
4	Cases\\KR	29	BED ADMAN	Undergraduate student	Male	2	MATH AND PHYSICS	3
5	Cases\\LD	27	BED PPM	Undergraduate student	Male	2	MATH AND ECONOMICS	3
6	Cases\\LP	29	BED SC ICT	Undergraduate student	Female	2	MATH AND ICT	3
7	Cases\\ML	24	BED SC	Undergraduate student	Male	2	PHYSICS AND MATH	3
8	Cases\\PM	28	BED SC	Undergraduate student	Female	2	MATH AND CHEMISTRY	3
9	Cases\\RL	26	BED ICT	Undergraduate student	Male	2	MATH AND ICT	3
10	Cases\\SI	24	BED PPM	Undergraduate student	Male	2	MATH AND ECONOMICS	3

Research Design and Data Collection

This section starts by presenting what I believe about knowledge, because my assumption about knowledge was important in the design and implementation of the research activities. After that, this section describes the pilot study, which was important in testing the instructional process, data collection and analysis techniques. The section ends by describing the four one-day research meetings, which were important in the process of exploring CRI with university mathematics PSTs in Tanzania.

My Epistemological Stance

I believe that social interactions play a major role in the acquisition and understanding of knowledge. Vygotsky (1978) states that “learning is a necessary and universal aspect of the process of developing culturally organized, specifically human psychological function” (p. 90). I created a learning environment through the use of CRI that focused on social interactions, scaffolding, and mediated learning in the research meetings. This learning environment also provided an opportunity for individual PSTs or groups to interact with more knowledgeable others (MKOs). Also, this learning environment was important to improve PSTs’ understandings of a mathematical concept within the zone of proximal development. Through social interactions during the research meetings, the PSTs worked together in small groups and class discussions to acquire knowledge and develop an understanding of what they were learning.

Research Design

For this case study, I conducted four one-day research meetings with mathematics PSTs. Each meeting was planned based on aspects of CRI as described by Ben Hur (2006). Multiple forms of data were collected throughout each of the four one-day meetings. In preparation for this study, I conducted a one-day pilot study with a group of mathematics PSTs. The purpose of the pilot study was to test CRI in the Tanzanian context. After the results of the pilot study were examined, I proceeded with the plans for the case study. In this section, I first describe the pilot study and then describe the four one-day meetings.

Description of the pilot study.

I conducted a pilot study with nine PSTs at UDOM in October 2016 to test CRI processes, data collection and analysis techniques.

The pilot study consisted of three parts in one day-long meeting. The first part was from 9 to 11 am, the second from 11:30 am to 1:30 pm and the third from 2:30 to 4:30 pm. The three parts were spaced for tea and lunch breaks. The meeting took place on a Saturday in the College of Natural and Mathematical Sciences boardroom. The boardroom provided a comfortable space and a large table around which the participants were able to sit and have easy access to each other.

The first part of the pilot study focused on developing the meaning of *pi.* The second focused on decontextualizing the concept, including finding the value of *pi* using local circular objects and learning how to better approximate the value of *pi* using the Archimedes approach. The third focused on the recontextualization and realization of a concept, including realizing the importance of *pi* in daily life and the meaning of *pi* in the Tanzanian mathematics curricula.

Four important lessons came out of the pilot study. The first lesson, based on feedback, was that the designed and implemented research activities were relevant to PSTs. The activities improved their understanding of *pi:* using the Archimedes approach to approximate the value of *pi*, generating multiple definitions of *pi* and a variety of ways of finding its value by using circular objects, and relating *pi* to local resources available in real life (Deogratias, 2019). The second lesson from the pilot study was that the structured sessions in the day-long meeting were effective for the encouragement of active engagement with a mathematical concept. The third lesson was that the group learning notes, audio and video recordings, and reflections provided relevant data to determine the value of the CRI processes. The fourth lesson was that the data from the pilot study helped to realize the appropriate data analysis techniques to answer the research questions.

Description of the four one-day Research meetings.

The CRI meetings took place in December 2016 in the same boardroom used for the pilot study. To meet the demands of the PSTs' schedules, they were held on four consecutive Saturdays. The space between meetings was designed to allow initial analysis of the collected data and time for planning for the next meetings.

This subsubsection starts by describing the creation of the research activities, which were used to engage PSTs with a concept using a CRI approach during the four one-day meetings. Then, the subsubsection describes the implementation of the research activities, which were important to explore the ways CRI revealed PSTs' expressions of their understanding of a mathematical concept, and the waysPSTs planned to teach a concept. The subsubsection ends by presenting the ways that the four one-day research meetings used social constructivism based on the ideas of Vygotsky (1978).

Creation of the research activities.

I used documents from two sources to create the research activities: Tanzanian mathematics textbooks and syllabi and non-Tanzanian resources. I used Tanzanian mathematics textbooks and syllabi to ensure that the instructional process built on the ideas of *pi* found in Tanzanian primary and secondary schools' mathematics curricula. These documents also helped me to get a sense of how I could help PSTs connect the mathematics about *pi* taught at the university with that taught in primary and secondary schools.

As the Tanzanian mathematics textbooks and syllabi were limited in their expressions of *pi*, I also used non-Tanzanian resources to design the research activities about *pi* as a radian measure and how to obtain the value of *pi* from measuring the circumference and diameter of a circular object/figure (e.g., Evan, n.d., Petr, 1971; Posamentier & Lehman, 2004; Rothman, 2009; Scott, 2008; Tent, 2001).

During the design phase of the research activities, I took into consideration all five components of CRI: meaning, decontextualization, realization, recontextualization, and practice (Ben-Hur, 2006). See Appendix B for the learning activities that were designed. The paragraphs below describe the ways that the five components of CRI were emphasized in the research activities.

In order to address *meaning* of a concept, Ben-Hur (2006) proposes that learners should know to "define concepts" in words and symbols and "elaborate upon them in their general form" (p. 32). For Day One, I designed and used research activities that provided an opportunity for PSTs to express their understanding of the meaning(s) of *pi*, the symbol for *pi,* and connections of *pi* to other mathematical concepts. Using local materials as teaching aids, I designed the research activities to help PSTs connect *pi* with other mathematical concepts. For instance, I asked the PSTs, "When I say the word *pi*, what comes to mind for you? Think, pair, and share."

The intention of this activity was to encourage the PSTs to express their understanding of the meaning of *pi* in words and symbols.

In order to address *decontextualization*, Ben-Hur proposes that learners "must experience a variety of applications to be able to generate a concept" (p. 12). For day two, I designed and used research activities that invited PSTs to generate the concept of *pi* through experiencing its applications, such as measuring the circumference and diameter of a circular object. In doing so, the PSTs expressed their understanding of *pi*: they identified the strategies for finding the value of *pi* from circular objects, and analysed errors and inconsistencies about the value of *pi*. The research activities were designed to provide an opportunity for the PSTs to reflect. For instance, PSTs were asked to reflect on the question, "why does *pi* appear in the volume of a sphere?" and to explain their answers. This activity was intended to help PSTs notice that *pi* is connected to all circular figures.

In order to address *realization*, Ben-Hur (2006) proposes that "teachers must encourage transfer into new experiences across the curriculum" (p. 12). For Days Two and Three, I designed and used research activities to encourage PSTs to experience how *pi* is connected with other mathematical concepts across Tanzanian primary and secondary schools, university mathematics curricula, and beyond the Tanzanian mathematics curricula. Ben-Hur also proposes that learners should understand the applications of a concept "across the curriculum and in everyday life" (p. 39).For instance, I gave the PSTs the following directions for an activity "Given the objects, sort out the objects in which *pi* exists. Draw and name the objects in which *pi* exists. From the drawn and named objects, explain how *pi* is applicable (Think, pair, and share)." The intention of this activity was to help the PSTs notice the circular objects in which *pi* exists and how it is applicable in the Tanzanian mathematics curricula and in their local environment.

In order to address *recontextualization,* Ben-Hur (2006) proposes that learners "must identify new applications for concepts and use concepts to connect new experiences with past or current experiences" (p. 12). In my study, current experiences were the experiences that the PSTs acquired about *pi* by participating in the research meetings. Past experiences were the experiences that they had had prior to attending the research meetings. For Day Three, I designed and used research activities to help the PSTs notice a variety of applications of *pi* in various mathematical concepts. This included asking the PSTs to draw the graphs of sine, cosine and tangent functions. The intention of this activity was to help the PSTs see how *pi* is applicable in a periodic function. In this case, *pi* was used as an interval in a periodic function. The designed research activities also helped the PSTs see the connection of *pi* to other concepts. For instance, I gave the PSTs another activity: "What are the properties of sine, cosine and tangent functions? Please explain your answer(s). Think, pair, and share." The intention of this activity was to help the PSTs learn that *pi* is connected to periodic functions such as sine, cosine and tangent functions. In this case, the connection could have been be period, $T = 2\pi$ in a periodic function.

In order to address *practice*, Ben-Hur (2006) proposes that "learning concepts requires sufficient appropriate practice" (p.12). For Day Four, I designed and used research activities to provide opportunities for PSTs to practice designing a lesson plan to help them gain knowledge and skills about how to teach *pi* as a concept using local materials.

Given the above description of the emphasis of the five components of CRI in the research activities, the designed lessons were built on Ben-Hur's (2006) work (see Table 3.3).

Table 3.3

Description of the Designed Lessons

Day	Ben-Hur's Theme	Activity	Sample Question
	Meaning	Developing the meaning	Watch a video clip about *pi* with

One		of *pi*	the link: https://www.youtube.com/watch?time_continue=452&v=cC0fZ_lkFpQ&feature=emb_logo From the video clip, what can you say about *pi*? Conversations about *pi* after video clip plays.
Day Two	Decontextualization	Developing an understanding of *pi* and its value through experiencing measuring the circumference and diameter of circular objects	Demonstrate how to find the value of *pi* using circular local materials/real objects, a set of Vernier callipers, a string and a ruler.
	Realization	Realizing the meaning of *pi* and its connection with other concepts in the Tanzanian mathematics curricula	Realize the meanings of *pi* across Tanzanian K—University mathematics curricula Discuss its meanings, implications, and look whether there is any connection among the meanings.
Day Three	Recontextualization	Developing an understanding of the applications of *pi* in mathematics	Watch the video clip with the link: https://www.youtube.com/watch?v=593w799sBms From the video clip: What points can you generate on a circle? What can you say about π in a unit circle? Why $\pi\ rad \equiv 180\ degrees$? (Think, pair, and share).
	Realization	Realizing the applications of *pi* in the Tanzanian mathematics curricula	What are the applications of *pi* in sciences and other fields? Please explain your answers.
Day Four	Practice	Developing skills for teaching *pi*	Develop the lesson plan for teaching *pi* using local materials as teaching and learning aids so that learners come to understand *pi* as a concept in your own classroom.

Implementation of the research activities.

In each meeting with PSTs, I implemented the designed research activities (see Appendix B). During the implementation of the research activities, the 10 PSTs worked in three small groups. One group had four members and the other groups each had three. As stated earlier, one participant did not attend the last meeting due to family circumstances. As such, all three groups had three members each that day. No person was assigned to lead the group. The PSTs did not change the groups except on day four: that day, someone from the four-person group joined the group with the absent PST.

Ways that the four one-day research meetings used social constructivism.

The four one-day research meetings illustrated social constructivism (Vygotsky, 1978) in three ways. First, in focusing on the research activities, PSTs worked in small groups. Small group discussions were followed by group presentations and class discussions, all of which required interaction. PSTs in small group and class discussions used two languages, namely, English and Kiswahili to foster interaction. This learning process helped the individual PSTs make sense about *pi*.

The second illustration of social constructivism was the use of local materials to discuss mathematical ideas about *pi*. These materials were available in the local environment and the PSTs used them to develop and demonstrate how to obtain the value of *pi* using circular materials. This instructional strategy of using local resources for conceptual development and understanding in the research meetings relates to the aspects of scaffolding.

The third illustration of using social constructivism was creating opportunities for reflection in each of the meetings. At the end of each meeting, I asked the PSTs to reflect on their ongoing learning of *pi* and its value. These questions required the PSTs to explain and justify a concept throughout the research meetings. The questions were designed as part of the research activities (see Appendix B). The questions also emerged in the research meetings during class discussions such as reflecting on errors and analysing errors and inconsistencies. This instructional strategy of learning a mathematical concept in the research meetings relates to mediated learning, which is fostered through reflection.

Social interactions, MKOs, scaffolding, and mediated learning were important instructional strategies in the research meetings for the individual PSTs' and group understandings of a mathematical concept within the ZPD.As part of the MKOs, I designed and implemented research activities in the research meetings to help PSTs in small groups develop their understanding of *pi* and its value. For instance, on Day Two meeting, the activities focused on finding the value of *pi* by using Archimedes approach. The activities were used as scaffolding to help PSTs learn on how to approximate the value of *pi* using local materials and Archimedes approach. This approach was not in the Tanzanian textbooks and syllabi. Also, as part of the MKOs, I designed research activities for individual PSTs' reflections in each day. Reflections occurred in the research meetings after the group work activities and during class discussions. The reflections were important to help PSTs realize the learned, surprising and understood ideas about *pi* and its value (see Appendix B). The reflections focused on why and how questions. In doing so, I mediated learning for the concept of *pi* and its value in the research meetings. For example, I asked PSTs in small groups to reflect on the questions such as: why *pi* appears in the volume of a sphere? Please explain your answer(s).

Data Collection Methods

Identifying the type of data to be collected on site is essential in qualitative case studies (Creswell, 2012, 2014). According to Stake (1995), the importance of data identification to a researcher looking for evidence at a site is "the most important planning [that] has to do with the matter of the study: What requires to be known?[*sic*] What are some potential relationships that may be discovered?" (p. 54). In line with Stake's (1995) questions, Patton (1990) argues that data collection in a qualitative research study consists "of detailed descriptions of situations, events, people, interactions, and observed behaviours; direct quotes from people about their experiences, opinions, beliefs, and thoughts and excerpts or entire passages from documents, correspondence, records and case histories" (p. 10).

I collected data from the participants based on their experiences, opinions, beliefs, and thoughts about *pi*. The descriptions of the data gathering processes are described below.

Group learning notes.

The participants wrote group learning notes. During the implementation of the CRI activities, the PSTs expressed their understanding of a mathematical concept in small groups in written form on manila sheets. All written notes were gathered as data.

Audio and video recordings.

I used two video cameras to collect data during the four one-day CRI meetings. I used the recorded data to observe and analyse the learning activities related to the development and understanding of *pi* using the CRI approach.

In the meetings, I used the video cameras for two main reasons: "density and permanence" (Bottorff, 1994, p. 245). Density refers to the capacity of video recorded data to capture multiple ongoing behaviours at the same time. It also refers to the ability of the video camera to capture two simultaneous data streams, namely, audio and visual. Permanence refers to the capacity of the researcher to return to the data source for analysis of the moment-to-moment unfolding of expressions that are hard to notice or observe in speech, and in non-verbal mathematical behaviours (Bottorff, 1994; Powell, Francisco, & Maher, 2003).

Video recorded data also has limitations, some of which may include data loss, storage concerns, and user errors (Creswell, 2003). While using video recordings during data collection, Pirie (1996) suggests that we need to realize, "who we are, the type of microphone we use, and where we place the cameras" (p. 3). This suggestion helps us to govern which data we get and lose through video recordings. In line with these concerns, I minimized the loss of data by using two video cameras during data collection: I made sure that the cameras captured the research participants in one small group and overall class discussions and group presentations. I may have lost some data with the small group work of groups one (G1) and three (G3). I also ensured that each part per meeting was recorded.

One camera was placed in front of the room facing the participants to record class discussions about *pi*; the other camera was placed in the room to record activities in a small group (i.e., group two) and during small group presentations and demonstrations throughout each meeting. The cameras helped to capture group work activities, small group presentations and demonstrations, and class discussions about *pi.*

To help collect video data, I recruited someone who was neither a participant nor an MKO. My video requirements included zooming in and out, and shifting the location of one camera from group two to the wall where the materials for group presentations were fixed, and to the presenter of the group during group presentations and demonstrations.

I also used audio tape recorders to record the conversations about *pi* in each of the three small groups. I placed an audio recorder with each group during their discussions. I was expecting that some data might be different from what was written in the group learning notes because the PSTs worked in small groups for the think, pair, and share strategy. After that, they wrote their ideas on a flip chart. I could miss the PSTs' small group discussions while working on the activities before writing their mathematical ideas on the manila sheet for group presentations.

Reflective journals.

Learners' reflective journals were another instrument that I used for data collection. I collected reflections from the individual participants after each meeting. These reflections helped me to gather the individual PSTs' expressions of their understanding of the mathematical concept throughout each meeting.

With regard to the reflections, I included questions in the research meetings for each participant to respond to at the end of the last session each day (see Appendix B).I used the following questions to collect data related to the concept: What surprised you today about *pi* as a concept? What have you understood today that you did not understand before about *pi*? What have you not understood today about *pi* as a concept? What was unfamiliar to you today about *pi* as a concept?

In addition to the individual reflective journals, each group of participants was asked to reflect on their developed lesson plan and its implementation through the construction of group journal entries. This group reflection occurred on Day Four. These group journal entries required participants to review and reflect on what they learned and observed. These occurred through developing the lesson and performing micro-teaching on Day Four.

Data Analysis

There were two phases in the analysis of the collected data. The first phase of analysis focused on the ways that the CRI approach revealed the PSTs' expressions of their understanding of a concept as *individuals,* from the individual reflections, and as *a group,* from group learning notes and audio and video recordings. The second phase of analysis focused on the ways PSTs planned to teach a concept, given their participation in CRI, as a *group.* For this analysis, I used audio and video recordings and group learning notes.

Data from group learning notes, reflections, and audio and video recordings gave me an opportunity to explore the CRI approach as a case in order to respond to the two research questions: How does CRI reveal the way that PSTs express their understanding of a mathematical concept? and How do PSTs plan to teach a concept, given their participation in CRI? Clarke and

Braun (2006) note that the data analysis process involves researchers familiarising themselves with the data, coding, searching for themes, reviewing themes, defining and naming themes, and producing the report. I used Clarke and Braun's (2006) framework to analyze the CRI approach in relation to the two research questions.

I analysed and coded general mathematical ideas or expressions from the participants' responses in the context of Ben-Hur's (2006) work. Each day corresponded to Ben-Hur's themes. The Day One meeting focused on learning the meaning of a concept, the Day Two meeting focused on decontextualizing and realizing a concept, the Day Three meeting focused on re-contextualizing and realizing a concept, and the Day Four meeting focused on practice.

I used six steps proposed by Clarke and Braun (2006) for thematic analysis: becoming familiar with the data; coding; searching for themes; reviewing themes; defining and naming themes; and producing the report. All six steps were followed during the data analysis process, as described below.

In order to address the first step, which is about data familiarisation, I processed the collected data in the following ways. First, I reviewed the raw data. This included reading and re-reading written data and listening and re-listening to recorded data from the research meetings. Then, I did the data transformation. This process refers to transcribing and translating the collected data during the period of study. I transcribed the data using the audio and video in their entirety.

Coding Data Using Ben-Hur's (2006) Work

The second step involved coding. The research activities were designed and implemented based on CRI (Ben-Hur, 2006). As such, I used Ben-Hur's work to code data from the participants' responses. I also used Ben-Hur's work in naming categories or code nodes which represent child nodes and parent nodes in QSR International's NVivo 11 qualitative data analysis software (QSR international Pty Ltd. Version 11.4, 2017). These were important to capture the deep descriptions of the ways that the CRI revealed university PSTs' expressions of their understanding of *pi* and the ways PSTs planned to teach a concept, given their participation in CRI (as Table 3.4).

I started by loading data into the NVivo 11 software. After that, I created a folder in the internal source, "Four-days of Concept-Rich Instruction Data." The folder contained data from the participants' responses, including reflections, audio and video transcribed data, and group learning notes from the research meetings. I used NVivo 11 to categorize the audio and video transcriptions, reflections, and group learning notes into meaningful groups of data.

Table 3.4

Data Analysis Components of CRI and Their Respective Code Nodes Using Ben-Hur's Work

Code Nodes: Parent Nodes in Nvivo	Code Nodes: Child Nodes in Nvivo
Meaning	Defining a concept in words
	Encapsulating conceptual understanding of a concept in symbols
	Elaborating upon a concept in general form
Decontextualization	Reflecting on errors
	Generating a concept with divergent responses
	Analysing errors and inconsistencies about a concept
	Generating a concept through higher order questioning

Recontextualization	Identifying the applications of a concept
	Using a concept to connect new experiences with past or current experiences
Realization	Realizing the importance of a concept beyond school mathematics
	Realizing the connection of a concept with other mathematical concepts across the curriculum
	Realizing the applications of a concept across the mathematics curriculum
	Realizing the applications of a concept in other fields
	Realizing the applications of a concept in everyday life
Practice	Designing the lesson plan
	Performing the microteaching

Descriptions of the Codes

The descriptions of the codes below are presented to make sense of the meanings of each code from the participants' responses in the reflective journals, audio and video recordings, and group learning notes.

Meaning: Defining a concept in words.

According to Ben-Hur (2006), defining a concept refers to understanding the definition of a mathematical concept, which leads to more exploration of a concept. In this case, to increase the possibility of extending their understanding of a concept, when exposed to a variety of mathematical activities learners need to identify and understand the definition(s) of a concept in words

Meaning: Encapsulating their conceptual understanding of a concept in symbols.

Encapsulating a concept in symbols refers to expressing a mathematical concept in a brief summary, which shows understanding rather than rote memorization of facts and formulas. Encapsulation includes defining a concept in both words and symbols (Ben-Hur, 2006).

Meaning: Elaborating upon a concept in general form.

Elaborating upon a concept in general form refers to describing a mathematical concept in detail, including its definition(s), value, and symbols (Ben-Hur, 2006).

Decontextualization: Reflecting on errors.

Reflecting on errors refers to learners thinking deeply and carefully on the mathematical ideas that they bring in the class. This reflection is important to root out misconceptions. When learners are given this opportunity to reflect on their ideas, they can see and identify their errors rooted in misconceptions, and they come to understand the mathematical concept through correcting errors(Ben-Hur, 2006).

Decontextualization: Generating a concept with divergent responses.

Refers to learners expressing their mathematical ideas in a variety of ways, regardless of whether the ideas are correct (Ball & Bass, 2000; Ben-Hur, 2006). Learners know that many approaches may work in solving a problem, and that what is correct or not should be based on the reasoning and roots of mathematics, not on the status of the MKOs. As such, the teacher must promote diverging responses and include all learners as equal members of the learning community (Ben-Hur, 2006).

Encouraging divergent thinking promotes creativity and innovation among learners in mathematics classes. Divergent thinking is not only thinking outside the box, it is also thinking without focusing on what is right or wrong (Ben-Hur, 2006). While teachers and learners should pay attention to the correct answer at the end of the process, divergent thinking facilitates deeper thinking to create divergent responses. As such, the goal of divergent thinking is to create a variety of mathematical ideas about a concept in a short period of time.

Decontextualization: Analysing errors and inconsistencies about a concept.

The analysis of errors and inconsistencies about a concept refers to assessing and justifying the mathematical ideas that emerge among learners in the class. Mathematical ideas that are raised by learners sometimes are not correct, are not described in a meaningful way, or lack consistency. Teachers and peers (MKOs) should be responsible for identifying and analysing the errors and inconsistencies in order to develop learners' understanding of a concept through class discussions (Ben-Hur, 2006).

Decontextualization: Generating a concept through higher order questioning.

This refers to the mathematical ideas that learners express about a concept through higher order questioning. Higher order questioning refers to asking questions that require learners to use higher order thinking skills in order to develop their understanding of a mathematical concept(Ben-Hur, 2006).

Teachers should ask learners questions that require relating experiences, classifying mathematical ideas, developing common ideas from the classifications, and predicting future experiences based on the developed common ideas. Such experiences and ideas occur from learners' thought processes through higher order questioning. In doing so, teachers are responsible for mediated learning, including asking learners to specify their experiences, define their ongoing learning, and reflect on learning processes (Ben-Hur, 2006, p. 26).

Through higher order questioning, learners generate mathematical concepts involving comparisons and generalizations that help them to better understand those concepts (Ben-Hur, 2006).

Recontextualization: Identifying the applications of a concept.

Identifying the applications of a concept refers to identifying the usefulness of a concept in mathematics. Learners need to recognize and understand how a concept is used in other mathematical concepts, as well as the way a concept is useful to various objects/physical objects, and figures (Ben-Hur, 2006).

Recontextualization: Using a concept to connect new experiences with past or current experiences.

Using a concept to connect new experiences with past or current experiences refers to the learners being able to use a new mathematical concept to connect their past experiences with the current experiences after participating in a class (Ben-Hur, 2006). In this context, past experiences refer to the experiences of the individual learners before being taught a concept in the class. Current experiences are those attained by the individual learners after being taught a concept in the class (Ben-Hur, 2006). For instance, PSTs learned, surprising, and observed ideas in the research meetings during group lesson presentations. The learned, surprising, and observed ideas were identified in the reflections.

Realization: Realizing the importance of a concept beyond the school mathematics.

This refers to the learners being aware of the usefulness of a mathematical concept beyond the school mathematics classroom. This realization helps learners to notice the objects or things that are connected to a concept and the importance of a concept in daily life (Ben-Hur, 2006).

Realization: Realizing the connections of a concept with other mathematical concepts across the curricula.

Refers learners understanding the concepts needed to know before and after learning the new concept in the curricula (Ben-Hur, 2006).

Realization: Realizing the applications of a concept across mathematics curricula.

This refers to learners' understanding of the applications of a mathematical concept in the curricula (Ben-Hur, 2006). This realization is important for PSTs, because we expect them to be teachers in the future. They will be, involved in designing lesson plans to teach concepts using textbooks and syllabi in mathematics classes.

Realization: Realizing the applications of a concept in other fields.

Realizing the applications of a concept in other fields refers to the learners' understanding that a mathematical concept can be useful beyond mathematics, including in sciences, communication, photography, engineering and statistics (Ben-Hur, 2006). This realization is important for the PSTs, because they become aware of the applications of the mathematical concepts in other subjects that are taught at schools.

Realization: Realizing the applications of a concept in everyday life.

This realization refers to the learners' understanding of the ways that a mathematical concept is useful in their everyday environment (Ben-Hur, 2006). This realization is important for the PSTs, because they understand the concepts by connecting with the things that they see in local areas. They also understand how to apply the concepts in their daily life practice.

Practice: Designing the lesson plan and performing the micro-teaching.

A lesson plan refers to a teacher's detailed explanation of the course of instruction, in this case, for teaching a particular mathematical concept in a class (Meador, 2017). In brief, a lesson plan is a detailed step-by-step guide for teaching a lesson, which is developed to teach a concept in a single class on a given day (Meador, 2017). The plan describes the objectives of the teaching and learning of a concept for that day. The duration of the single class varies, depending on characteristics of the learners, the learning environment, and the curricula guide.

Micro-teaching refers to the role that the teacher performs in fulfilling the actions inscribed in the lesson plan by teaching a particular mathematical concept to the learners in a single class. In this project, the lesson plans that the PSTs developed and the micro-teaching are part of the PSTs' practices. The lesson plans and micro-teaching for teaching *pi* relate to Ben-Hur's notion of practice. The reason is that practices are all activities related to a concept (Ben-Hur, 2006). Also, in their micro-teaching, the PSTs practiced a concept by demonstrating how to measure the circumference and diameter of a circular object using measurement tools such as a string, ruler and set of Vernier callipers.

Descriptions of the Remaining Four Steps Used in Data Analysis

In order to address the third step, which is about searching for themes, I processed the coded data in several ways. I grouped code nodes according to Ben-Hur's (2006) work: child nodes into five parent nodes. Grouping the child nodes into five parent nodes made it easier as I searched for themes. The seven themes that emerged from the coded child and parent nodes as defined and named in step five below.

Step Four involved reviewing the generated themes that emerged from the participants' responses. I reviewed all seven developed themes by cross checking the code nodes and code references.

Step Five involved defining and naming themes that were developed during the third step and reviewed during the fourth step. For example, the theme "multiple interpretations" was developed by reflecting on the research questions and the conceptual framework with regard to the parent nodes, "meaning" and "decontextualization" (see Table 3.4 above). The same trend was followed with other groups of the parent nodes to define and name the other six themes, namely: “systematic errors,” which resulted from the parent node “recontextualization”; “open definitions,” which resulted from the parent node “decontextualization”; “community of learners in Tanzanian pre-service teacher education,” which resulted from the parent node “practice”; “PSTs getting an opportunity to teach a mathematical concept and seeing how the concept is integrated into Tanzanian mathematics curriculum,” which resulted from the parent node “realization”; and “local resources as a tool to mediate learning,” and “a better understanding of participatory approaches and social constructivism in the university mathematics classrooms” which emerged from all five parent nodes.

Step Six, the last step, involved producing the report as described in Chapters 4 and 5. In particular, Chapter 4 will present the research findings. The chapter will start by presenting the coded categories or parent nodes (meaning, decontextualization, recontextualization, and realization) to answer the first research question, “How does CRI reveal the way that PSTs express their understanding of a mathematical concept?” Then the chapter will present the coded category or parent node, practice, to answer the second research question, “How do PSTs plan to teach a concept, given their participation in CRI?” After that, Chapter 5 will present the seven themes that emerged in the context of a study through implementation of CRI to university PSTs in Tanzania.

Table 3.5 below gives a summary of the steps and procedures used in data analysis.

Table 3.5

Steps and Procedures Used in Data Analysis

Steps	Procedures
Step One	Data familiarisation through reading and re-reading the collected

	data obtained using group learning notes, reflections, and audio and video recordings, and listening and re-listening to the collected data using audio and video recordings. Then, transcribing and translating data collected using audio and video recordings.
Step Two	The generation of codes, including data reduction, organizing the collected data into the NVivo 11 software, and coding the data according to Ben-Hur's (2006) work to generate code nodes or categories.
Step Three	The generation of themes from the categories or code nodes obtained in Step Two.
Step Four	Review of the generated themes obtained in Step Three.
Step Five	Definition and naming of themes
Step Six	Writing the report

Validity and Reliability

Validity and reliability refer to the trustworthiness of data in qualitative case studies (Baxter & Jack, 2008). In this section, validity and reliability are discussed to describe the ways that this study met the criterion of trustworthiness.

The validity of the data was ensured (Baxter &Jack, 2008; Creswell & Miller, 2000) in several ways. The first involved the verification and confirmation of the participants' responses. The PSTs reviewed their responses for trustworthiness and accuracy. This process confirmed the trustworthiness of the data obtained from the transcriptions of the audio and video recordings. When necessary, the PSTs were asked to clarify their responses. I also listened to the audio and video recordings to clarify the transcribed data and ensure the accuracy and trustworthiness of the participants' responses.

The second way that I ensured the validity involved using NVivo 11 software to assist with further analysis. This process ensured accuracy by grouping together common themes, patterns or concepts based on the descriptive framework (Ben-Hur, 2006). The final way that I ensured the validity involved the research participants and the designed research activities and meetings. This aspect provided consistent and relevant information needed to answer the research questions.

The reliability of the data was ensured based on Baxter and Jack (2008) and Creswell and Miller (2000) while focusing on the conducted pilot study and data collection techniques. First, the findings from the pilot study confirmed that the concept-rich instructional processes were effective, because each participant was asked at the end of the research meeting to reflect on the whole day lesson. Second, I designed the four days of CRI activities after the pilot study so that I could include the recommendations from my supervisory committee and pilot study participants. This was important in order to ensure the trustworthiness of the research activities and the reliability of the data.

Ethical Issues in This Study

Before conducting the study, I applied for two letters of approval for data collection from the Research Ethics Board (REB) at the University of Alberta. One letter was for the pilot study meeting and the second was for the four one-day CRI meetings. After that, I asked for permission from the Tanzanian Ministry of Education, Science, Technology and Vocational Training (MOEST) to conduct my research study at UDOM (see Appendix C). After getting the letter of permission from MOEST, I asked the Director of Research Studies at UDOM for permission to conduct this study there (see Appendix D). When I received this permission, I wrote a letter to third-year pre-service mathematics teachers requesting them to volunteer to participate in this study and informing them that they could withdraw at any time from the study (see Appendix E). I also used a poster to recruit PSTs (see Appendix F).

Once I had a group of PSTs, I let them know the purpose of my study before I began collecting data. After that, I requested that each volunteer PST sign the research project consent form. In the consent form, I explained that individual participants could decide not to participate or could withdraw from the study (see Appendix G).

During the data collection process, which involved the use of audio and video recordings, reflections and group learning notes, I asked PSTs to use pseudonyms to maintain confidentiality. I also informed them that the information I collected was for the purpose of this study. Throughout the study, I used unbiased language with these PSTs to respect gender, racial, ethnic, age, and sexual orientation diversities. I also ensured that individual PSTs were actively involved in the research meetings, and that their individual ideas were respected.

After collecting the data, I transcribed it, and sent the transcripts to the individual PSTs to read and ensure that the transcribed data was valid before I used it.

Limitations and Delimitations

Limitations in a study can be defined as "constraints that are largely beyond your control but could affect the study outcome" (Simon & Goes, 2013, para. 5). A qualitative case study reports findings by focusing on a small sample (Dodge, 2011).This study focused only on UDOM's pre-service mathematics teachers from a convenient sampling. The findings are limited to nine PSTs who participated in the four one-day CRI meetings, and one respondent in Days One, Two and Three of the meetings. However, the findings inform PSTs' education about CRI in a Tanzanian context.

The delimitations of a study can be defined as “those characteristics that arise from limitations in the scope of the study (defining the boundaries) and by the conscious exclusionary and inclusionary decisions made during the development of the study plan” (Simon & Goes, 2013, para. 13). By its very nature, a qualitative case study has a narrow scope (Dodge, 2011). The delimitation for this study is that it focused on a small sample because a larger sample would have demanded more time and cost. The second delimitation for the study is the research site, which was restricted to UDOM for convenience. However, this study has implications for UDOM and other universities, as well as for other PSTs and learners at schools in Tanzania (see Chapter 6).

CHAPTER 4

RESEARCH FINDINGS

This chapter presents the findings from the analysis of the data to answer the research questions, How does concept-rich instruction (CRI) reveal the way that Tanzanian university pre-service teachers (PSTs) express their understanding of a mathematical concept? How do PSTs plan to teach a concept, given their participation in CRI?

The analysis conducted using NVivo software generated 323 coding references from the participants' responses: 52 coding references for the parent node meaning, 82 of decontextualization, 99 of recontextualization, 79 of realization, and 11 of practice.

As shown in the previous paragraph, the greatest number of code references pertained to recontextualization.The code references for recontextualization show that PSTs identified a variety of applications of *pi*. They also used *pi* to connect the concept with their past and current experiences. The decontextualization references show that PSTs used a lot of applications of *pi* to generate mathematical ideas about *pi* and its value. The code references for realization show that PSTs realized a lot of ideas about *pi* in mathematics curricula and beyond school mathematics. However, there were few coding references for practice. The participants' responses to the question about how they planned to teach a concept were generated when they worked in small groups (G1, G2, and G3) while designing a lesson plan and performing micro-teaching. There were three lesson plans and micro-teachings performed by each group during the last meeting. Although I still had nine participants, the participants were organized into three groups. Three participants in each group worked together to design a lesson plan and perform a micro-teaching for teaching *pi*.

Research Question 1: How does CRI reveal the way that Tanzanian university PSTs express their understanding of a mathematical concept?

The following sections present a detailed explanation of how the CRI approach revealed PSTs' expressions of their understanding of a mathematical concept while participating in the research meetings. The findings are presented in the following order: from the meaning of a concept to the realization of a concept. In other words, they are presented in the order of the meaning, de-contextualization, re-contextualization, and realization of a concept.

Meaning of a Concept

When responding to the meaning of *pi*, the PSTs expressed their understanding of *pi* in the following ways. First, they defined *pi* in words. Second, they encapsulated their conceptual understanding of *pi* in symbols. Lastly, they elaborated upon *pi* in a general form.

As Ben-Hur (2006) writes, learners "must learn to define concepts and elaborate upon them in their general form. They must learn to encapsulate their conceptual understanding in words and symbols" (p. 32). In other words, learners demonstrate their understanding of the meaning of a concept when they are able to clearly define and describe a concept in greater detail the parameters existing in the definition of a concept.

While Ben-Hur (2006) did not define the terms, "encapsulate," and "elaborate" directly in his work, "encapsulate" is understood as "to express something in a brief summary" (Merriam, n.d., para. 1) while "elaborate" can be defined as "to explain something at greater length or in greater detail" (Merriam, n.d., para. 1).Given these definitions, and in relation to the concept of *pi*, I interpret "encapsulate" in Ben-Hur's work as referring to expressing *pi* in a brief summary, and "elaborate" as referring to describing the concept of *pi* in detail.

Defining a Concept in Words

There were 35 coding references for defining a concept in words: six from group learning notes,13from transcribed data, and 16 from reflections.

I provided an opening activity about *pi* in the Day One meeting. I prompted the PSTs with a question, "what comes in your mind when you hear about *pi*?"and then asked them to work in small groups to discuss their responses. After small group discussions, the PSTs presented their group work. They provided a definition of *pi*. However, there was no clear distinction about the concept of *pi* and its value in group work presentations. The majority of the PSTs defined *pi* as a constant value, either $22/7$ or 3.14. However, G3P (Group Three Presenter) offered the following definition: "*Pi* is the number of diameters required to complete the circumference of a circle, it is about 3.14"[1](Transcribed data). A few of the PSTs defined *pi* as the relationship between the circumference and diameter of a circle. For instance, G2DI (Group Two discussion) defined *pi* as: "a ratio of the circumference to diameter of a circle"[2](Transcribed data).

During instruction, all three of the small groups were asked to complete tasks by using circular-shaped local materials to define *pi* as a concept. After the activities, the PSTs were able to provide a definition of *pi* in words; *pi* is a ratio of the circumference and diameter of a circle. They were also able to present additional definitions of *pi* in words. For instance, all of the PSTs in small groups expressed their understanding of the concept of *pi* by defining it in relation to the circumference and diameter of a circular object/body. This definition is different from the previous definitions because it focuses on the relationship between the diameter and circumference of a circular object. The PSTs used a string and ruler to measure the circumference of a circular object and a set of Vernier callipers to measure its diameter.G1 noted: "*Pi* is the number of diameters to complete one circumference of a circular body adding a part remaining to

[1] Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (3)

[2] Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (1)

complete the circumference"[3] (Group learning notes). This term "adding a part" shows that the PSTs considered the remaining part to cover the circumference of the circular body in terms of a complete diameter. Hence, this remaining part is not part of the circumference but, rather, the part of the diameter that is needed to complete the circumference.

Encapsulating a Concept in Words and Symbols

In the analysis, 11 coding references related to encapsulating the concept in words and symbols: three were from group learning notes, six from transcribed data and two from reflections.

Working on the activities in small groups (G1, G2, and G3), the PSTs focused on the relationship between the circumference and diameter of a circle and defined *pi* in words and symbols. For instance, G3 responded to the meaning of *pi* in words, describing it as a ratio of the circumference to diameter of a circle, and then showed his definition using symbols: "*Pi* refers to constant mathematical symbol (π) which presents the ratio between the circumference of a circle and its diameter i.e., $\pi = \frac{c}{d}$"[4](Group learning notes).

The PSTs in small groups were also asked to complete tasks that involved *pi* as a radian measure. They expressed their understanding of *pi* by presenting their conceptual understanding in symbols in relation to the angle measures in a circle. *Pi* depends on the measurements of angles in radians and in degrees in a circle. For instance, G3 responded to the question about the meaning of *pi* in words as a ratio of two parameters (radian and angle measure) of a circle and then used the mathematical expressions for the definition: "*Pi* is the ratio of radian to angle

[3] Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (1)

[4] Nodes\\Nodes from the Four-day CRI\\Meaning\ Encapsulating their conceptualunderstanding of a concept in symbols (1)

measure and multiply by $180^{\mathbf{0}}$. That is to say, $\pi = \frac{180s}{\theta}$"[5] (Transcribed data).This is an example of how the PSTs were able to coherently express the definition of *pi* in words and symbols well as G3 defined *pi* using the phrase "a ratio of radian to angle measure and by multiplying by 180 degrees."

Elaborating Upon a Concept in General Form

In the analysis, six coding references related to elaboration: one was from group learning notes, four were from transcribed data, and one was from reflections.

Local resources were used as a CRI strategy to help PSTs elaborate *pi* in general form. The PSTs in small groups (G1, G2, and G3) were asked to complete tasks by using circular-shaped local materials to find the value of *pi.* The PSTs engaged in these activities after watching a video clip and reading about the historical value of *pi*. The video clips helped the PSTs to learn about *pi* and its value by considering the diameter and circumference of a circle and the ways in which they can obtain the diameter and circumference of a circle using a circular object. The PSTs expressed the value of *pi* as an irrational number; it cannot be written precisely when approximated to the nearest decimal digits. It varies because we cannot anticipate the next digit of the value of *pi* unless we compute it. G3 put it this way:

> The value of *pi* varies and can be approximated to three, but the value of *pi* is approximated to 3.14, which is the most occurring approximation. Some scholars used π as three, other [scholars] approximated to 3.14 and so on.[6] (Transcribed data)

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[5] Nodes\\Nodes from the Four-day CRI\\Meaning\Encapsulating their conceptual understanding of a concept in symbols (4)

[6] Nodes\\Nodes from the Four-day CRI\\Meaning\Elaborating upon a concept in general form (3)

The above quotation is an example of how the PSTs expressed their understanding of the value of *pi*. They elaborated on the concept that the value can be approximated to one or two decimal places. This is because the value is an irrational number (3.14 ...).

Decontextualization of a Concept

Ben-Hur (2006), writing about decontextualization, notes that learners "must experience a variety of applications to be able to generate a concept" (p. 12). Ben-Hur also states that the decontextualization component of CRI points to the idea that conceptual development is "a reflective process" (p. 18). Learners apply a concept to generate the mathematical ideas. Language facilitates this process either through classroom discussions or written expressions (Asiala *et al.,* 1996; Ben-Hur, 2006; Shepard, 1993). When reflecting on a concept, PSTs "progressively learn to consider, analyse, and compare the procedure without a need to perform it, understand the conditions under which it works, or combine it with other procedures" (Ben-Hur, 2006, p. 18). PSTs worked on a variety of activities to decontextualize a concept in multiple ways, including identifying the procedures for finding the values of *pi* from the circular objects/circular figures.

Reflecting on Errors

In the analysis, there were five coding references related to reflecting on errors: three from reflections, and two from transcribed data.

PSTs reflected on errors about the value of *pi*. This sort of reflection was important to give the PSTS an opportunity to obtain a better understanding of a concept. As evident in past studies (Ben-Hur, 2006; Loughran, 2002; Schon, 1983; 1987), reflection helps learners to think deeply about a concept.

In the research meetings, PSTs reflected on how to obtain the value of *pi* from a circular object. A few of the PSTs were still having difficulty identifying the source of errors while calculating the value of *pi* using a circular object. One participant, EB, tried to measure the circumference and diameter of a circular object and failed to obtain the value of *pi* as approximated equivalent to 3.14.In reflecting about the experience, he said: "I was surprised that there are some calculations in which $\pi \cong 3.14$ was not obtained. So, we failed to show exactly which error occurred."[7] (Reflection).To address this challenge and foster understanding of *pi*, another activity was given to the small groups. They were instructed to take several measurements of the circumference of a small circular object using a string and ruler, and to do the same with the diameter by using a set of Vernier callipers. They found that using the callipers to measure the diameter gave a better approximation of the value of *pi* than using a ruler to measure the circumference. After this activity, EB realized that the errors in the first activity had resulted from using a ruler instead of callipers: the callipers provided a much more accurate measure. GI used EB's experiences:

> … to be more accurate, we require to measure diameter of a sugarcane stem using a set of Vernier calipers and round the thread of the same length to the circumference of the sugarcane stem. We have seen that, when using a ruler and a thread, we got a lot of errors compared when using a set of Vernier calipers because when using a ruler [and a thread], there is enough or a range [value] which brings a lot of errors. But when using a set of Vernier calipers, you reduce errors. It is better to use a set of Vernier calipers than using a ruler [and a thread when measuring the diameter of a circular object] because when using a ruler, you increase the errors.[8] (Transcribed data).

Generating a Concept with Divergent Responses

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[7] Nodes\\Nodes from the Four-day CRI\\De-contextualization\Reflecting over errors (1)

[8] Nodes\\Nodes from the Four-day CRI\\Decontextualization\Reflecting on errors (2)

In the analysis, there were 25 coding references for generating a concept with divergent responses: 13from group learning notes, one from reflections, and 11from transcribed data.

PSTs had very different understanding of the value of *pi*. In the process, the PSTs in small group discussions measured the number of diameters required to go around to complete the circumference of the circular object. After that, they counted the number of diameters required to complete the circumference of the circular object. For instance, a Group one presenter (G1P) explained:

> The idea of *pi,* which represents the number of diameters that goes around the circumference of a circle is unit-less, we are not concerned with the unit (as centimeter, meter, and so on). But we are concerned about finding the number of diameters that goes around the circumference of the circular object. For instance, we have measured the diameter of a circular object as 30cm and circumference as 95 cm by using a string; then the number of diameters to complete the circumference of a circle is three plus a small portion.

$$number\ of\ diameters = 3 + \frac{small\ portion}{diameter}$$

$$= 3 + \frac{5cm}{30cm}$$

$$= 3 + \frac{1}{6}$$

$$= 3 + 0.1666666\ldots$$

$$= 3.1666666\ldots$$

≈ 3.17 is the number of diameters to complete the circumference of a circle. So, 0.1666666 ... is a small portion, which is difficult to measure it [*sic*] using measurement tools, such as a ruler and a set of Vernier calipers.[9] (Transcribed data)

Speaking for Group one, one of the participants expressed an understanding of how to obtain the value of *pi.* He described how to find the value by counting the number of diameters required to complete the circumference of a circle.

PSTs in Group two (G2) illustrated the process for counting the number of diameters needed to go around the circumference of a circular object. They did this by drawing a diagram and performing mathematical computations (see Figure 4.1 below).

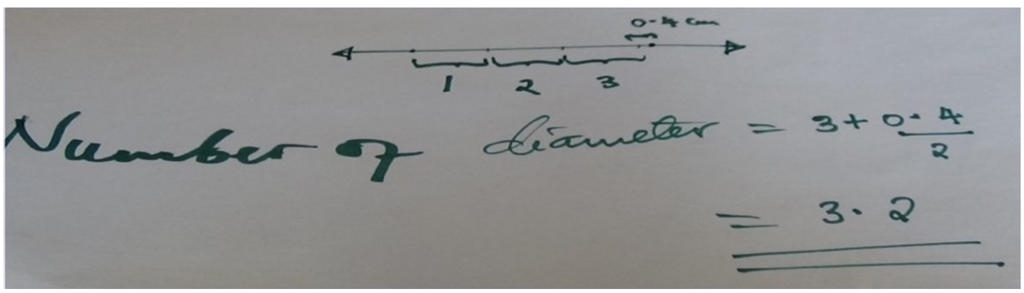

Figure 4.1. Illustrations of the number of diameters to complete the circumference of a circular object.

Figure 4.1 shows how the PSTs expressed their understanding of how to obtain the value of *pi.* They describedcorrectly in a diagram the process for counting the number of diameters required to complete the circumference of a circular figure to give the value of *pi*. From counting, G2 noticed that three diameters plus a remaining small portion were required to complete the circumference.

Analysing Errors and Inconsistencies about a Concept

[9] Nodes\\Nodes from the Four-day CRI\\De-contextualization\Generating the concept with divergent responses (10)

In the analysis, there were 26 coding references related to analysing errors and inconsistencies about the concept: six from group learning notes and 20from transcribed data.

PSTs expressed their understating of *pi* by assessing and justifying their mathematical thinking about a concept and its value. As evident in other studies (Ben-Hur, 2006; Brown & Burton, 1978; Nesher, 1986), analysing errors and inconsistencies helps learners to better understand the concepts.

The PSTs worked on various activities to identify errors and inconsistencies, which were eliminated through group and class discussions. For instance, before instruction, PSTs thought that the value was always given in mathematical problems/questions involving *pi* in textbooks, syllabi, tests, examinations and assignments. However, during instruction, the PSTs learned that the value of *pi* can also be computed.

Researcher: I want to ask you, when you say π is something, which is given. Is that true? Is it true that the value of *pi* is always given?

Participant DA: Yes.

Researcher: Can't we obtain the value of *pi* using mathematical computations? Apart from being given.

Participant EB: When you say it is given [that] means something exists. It is present.

Participant DA: When you are dealing [with] calculating the area of a certain figure/object, π is given as a constant value.

Participant KR: Once you are asked to find π, provided that you are given [the] circumference and radius, will you not calculate the value of *pi*?

Participant EB: Something which you have already given, what is the purpose of finding it?

Participant DA: Take an example, you have given the circumference and π. Can you not get the value of a diameter?

Other participants: You can get [it].

Participant DA: Yes, this is the matter of language.

Participant DA: Let me ask the question: in [the] primary level, when asked to calculate the area of a circle, we are given that $\pi = \frac{22}{7}$. Thus, [that is] why we argued that the value of *pi* is given. What can you comment about this?

Participant DA: So, from now we have to know that π is not $\frac{22}{7}$ and it is not a given value, but it can be given.

Researcher: When I ask you to find the value of *pi* of a sugar cane, will you find it? or you will say it is given.

Other participants: We can find and get the value of *pi*.

Participant DA: But it will depend on each level—in primary, secondary or university. In [the] primary level, you can say "use this as the value of *pi*." But in [the] secondary or university level, it is not necessary to be given. At [the] university or secondary level, you can be given the value of *pi* or you can develop your mind to get the value of *pi*.[10] (Transcribed data).

Before class discussion, most participants, including participant DA, thought that the value of *pi* was always given. However, through class discussions, the participants, including DA improved their understanding of the value of *pi* by realizing that the value can also be computed. The value

[10] Nodes\\Nodes from the Four-day CRI\\De-contextualization\Analysing errors and inconsistencies (2)

can be computed from the diameter and circumference of a circular object/circle by taking the ratio of the circumference of a circle in relation to its diameter.

Generating a Concept Through Higher Order Questioning

In the analysis, there were 26 coding references related to generating a concept through higher order thinking: 18from group learning notes, one from reflections, and seven from transcribed data.

The PSTs generated a variety of ideas about *pi* and its value when asked higher order questions. Ben-Hur argues that "it is teachers who ask [learners] to look back when they are looking forward, to anticipate when they are fixated with the past or present experience, and to compare an individual experience with other experiences" (p. 26).I asked the PSTs several high order questions when they were in small groups so that they could express their understanding of *pi*, its value and symbol, and see the connections to other mathematical concepts. The PSTs were required to answer "why" or "how" questions for each task in the group discussions. This kind of learning process is known as mediated learning (Ben-Hur, 2006; Vygotsky, 1978). The PSTs generated ideas by explaining why *pi* appears in any circular figure, including a sphere. The PSTs were asked, "Why does *pi* appear in this formula $\mathrm{V} = \frac{4}{3}\pi\mathrm{r}^3$ for a sphere? Explain your answer (s)."

> G1: $V = \frac{4}{3}\pi r^3$ for a sphere, *pi* appears due to the fact that it includes the diameter (radius) of the circular object.
>
> G2: Because there is a circular figure.
>
> G3: Because it [i.e., a sphere] consists [of a] circular shape.[11] (Group learning notes)

[11] Nodes\\Nodes from the Four-day CRI\\De-contextualization\Generating concept through higher order questioning (7)

During the small group discussions, the PSTs were able to bring up a number of ideas about *pi*. The PSTs expressed their understanding of the connection of *pi* and its value in a sphere by concluding that the sphere is a circular shape. G1, G2 and G3 focused on the structure of a sphere. This reasoning refers to structural abstractions (Tall, 2013). G1, G2, and G3 connected the sphere with a circular figure or object, because any circular figure/object has a radius and there is a relationship of circumference and diameter. Therefore, the formula for volume of a sphere contains *pi*.

Re-contextualization of a Concept

Ben-Hur (2006) describes "re-contextualization," one of the components of CRI, as the process by which learners "must identify new applications for the [mathematical] concepts and use the concepts to connect new experiences with past or current experiences" (p. 12). In the context of CRI, the re-contextualization is a process by which learners identify new concepts, understand how those concepts apply to "past or current experiences", and connect the new experiences from the new concepts to the "past or current experiences" (Ben-Hur, 2006, p. 36). PSTs identified the applications of *pi* that they knew before participating in the research meetings. Also, PSTs were introduced to new applications of *pi* that they did not know before participating in the meetings. For instance, the PSTs identified applications of *pi* in a variety of ways, including identifying its applications in periodic functions after participating in the research meetings.

The findings are presented in the following order, from identifying the applications of *pi* to connecting the experiences gained by the PSTs in the research meetings.

Identifying the Applications of a Concept

In the analysis, there were 25 coding references identifying applications of the concept: six from group learning notes, 12from reflections, and seven from transcribed data.

A variety of applications of the concept of *pi* were identified based on the use of the value of *pi* (π). For instance, PSTs in each of the small groups identified circular objects. Several objects were placed on the table for each group, including sugar cane stems, cylindrical objects (open two sides, one side open and one closed, and closed two sides), bread (*chapati* in Kiswahili), rectangular table mats and tissues. The PSTs worked in small groups to identify the circular objects. During group discussions (G2D), G2 identified objects in which *pi* exists. G2 reported:

DA: Kuna chapati [There is a circular bread].

EB: How *pi* is applicable to the objects? This is a bread. π is used to find the circumference and area of the bread/circle. $circumference = 2\pi r = \pi d$ and $area = \pi r^2$. This one here (a cylindrical object), π is used to calculate the area of a cylinder, [surface] area= $\pi dh = 2\pi rh$. Also, π is used to find the volume of a cylinder, $V = \pi r^2 h$ for half-closed cylindrical objects.[12] (Transcribed data)

The above quote is an example of how the participants, DA and EB were able to identify the objects in which *pi* exists, including a loaf of bread (when tasked with finding its circumference and area), and the cylindrical objects (when tasked with finding its surface area and volume). *Pi* exists in these objects because they are circular. Each object contains a diameter and circumference. PSTs used the physical objects to identify the applications of *pi* by converting them into abstract objects, which are important in finding circumferences and areas.

PSTs also identified the applications of *pi* as a radian measure. They worked in small groups and drew the diagram of a circular object. After that, they used a protractor to measure

[12]Nodes\\Nodes from the Four-day CRI\\Re-contextualization\Identifying the applications of a concept (2)

and locate angles in the centre of a circular figure. They measured the length of an arc subtended by the angle at the centre of the figure. Through this process, PSTs were able to identify the relationship of the central angle and arc length (i.e., the length of an arc is proportional to the angle at the centre of a circular figure). Also, the PSTs were able to convert an angle measured in degrees to radian measurements by using a protractor. The PSTs were also able to identify the value of *pi* in a radian measure, which is equivalent to 180 degrees i.e., $\pi\ rad \cong 180^0$.

After the research meetings, the PSTs were able to use *pi* as a time interval in the periodic functions. The PSTs worked together in small groups to draw the graphs of the periodic functions of sine, cosine and tangent (see Figures 4.2 and 4.3). Once they had completed the drawings, the PSTs easily identified that these trigonometric functions are periodic because the graphs repeated after a certain interval of time. Furthermore, from the graphs, the PSTs were able to identify that *pi* is connected to the period of any trigonometric function. Period refers to the time interval to complete one circle. The groups identified that a period equals 2π for any sine and cosine functions. For instance, G2 reported the following:

> We are dealing with those functions which tend to repeat their values after a certain interval. Thus why, to show if$sin\ t$ and $cos\ t$ are periodic functions, we need to show as follow[s]: Let $sin\ t = sin\ T$where $T = t + 2\pi$ where 2π is the time interval between one cycle to another cycle. When we are talking about $2\pi,$ we are talking about a complete cycle. So, after completing one cycle, take an example from here to here is a half cycle. After moving up to here, we obtain one cycle. So, from here to here, is about 2π. Thus, why we have said that $sin\ t = sin\ (t + 2\pi)$.[13] (Transcribed data)

[13] Nodes\\Nodes from the Four-day CRI\\Re-contextualization\Identifying the applications ofa concept (5)

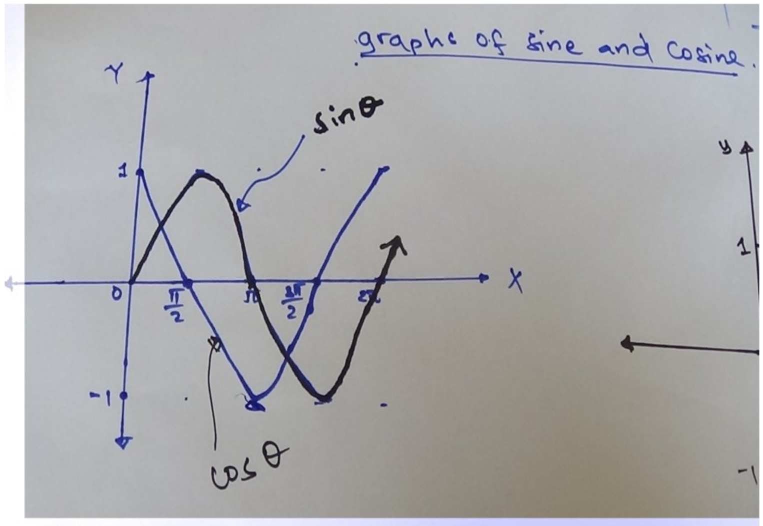

Figure 4.2. The graphs of sine and cosine functions drawn by the group.

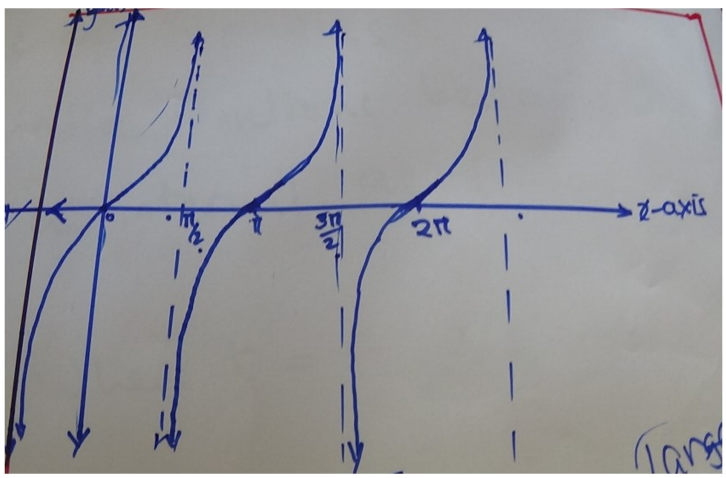

Figure 4.3. The graph of tangent function drawn by the group.

During small group discussions, G2 was able to express their understanding of the application of *pi* in radians. G2 was also able to identify that the value is a radian measure equivalent to 180 degrees. This value is used as the time interval in the periodic functions, including sine, cosine and tangent functions. In any periodic function, the time taken to complete a half cycle gives the value of *pi*. The completion of one cycle doubles the value of *pi*. As such, to complete a half cycle of any periodic functions takes an interval of the value of *pi* in radians, which is equivalent to 180 degrees. In the graphs in Figures 4.2 and 4.3, we can also see that there is connection between the symbol π and shapes of the periodic functions, to show a relationship of *pi* in a periodic function. In the graphs, the symbol π is used as a time interval, and there is a relationship between it and the shapes of the periodic functions. This relationship refers to a blending of operational symbolism and conceptual embodiment (Tall, 2013). Operational symbolism focuses on symbols while conceptual embodiment focuses on shapes (drawings).

Using a Concept to Connect New Experiences with Past or Current Experiences

In the analysis, there were 70 coding references for the new-experiences-with-past-or-current-experiences code node in the NVivo software: 11 were from group learning notes, nine from transcribed data, and 50from reflections.

The PSTs expressed their understanding of *pi* by using it to connect their new experiences with past or current experiences in a variety of ways. They came to understand that learners need to see the connection between mathematical concepts and their daily environment. One strategy they proposed was to ask learners to bring teaching/learning resources to a class. For instance, G1 suggested the following:

> Assign students a homework [assignment] to bring circular materials to school. This strategy will help a teacher to see that students know circular materials, and also [the] teacher can wonder what students will bring at [*sic*] schools and this strategy will help students to understand the lesson.[14] (Transcribed data)

These PSTs expressed their understanding of the teaching and learning strategies for teaching *pi*. *Pi* is embedded in any circular material. As such, bringing local resources into a learning

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[14]Nodes\\Nodes from the Four-day CRI\\Re-contextualization\Using a concept to connect new experiences with past or current experiences (5)

environment gives the PSTs an opportunity to use them for conceptual development and understanding.

Realization of a Concept

Examining the PSTs' responses, I found that the PSTs expressed their understanding of *pi* in the following ways. They realized the importance of *pi* beyond school mathematics. They realized the connections of *pi* with other mathematical concepts across the Tanzanian mathematics curricula. They realized the applications of *pi* across Tanzanian mathematics curricula. They realized the applications of *pi* in other fields beyond school mathematics. They also realized the applications of *pi* in their everyday life.

Ben-Hur (2006) emphasizes that for realization, "teachers must encourage transfer into new experiences across the curriculum" (p.12). To ensure that the research activities would be meaningful and useful across the curriculum for the PSTs, I designed activities for class discussions on Days Two and Three that focused on the following questions:"What are the applications of *pi* in sciences and other fields? Please explain your answers." "Where is *pi* in your daily environment? Explain your answer(s)." "What mathematical concepts do we need to know before and after learning the concept of *pi* across Tanzanian primary, secondary, and undergraduate university mathematics curricula?" and "Suggest how *pi* could be articulated and developed in Tanzanian mathematics curricula for better understanding of a concept." After each question, I encouraged the PSTs to explain their answers.

Realization of mathematical concepts brings meaningful learning by relating local materials in academic settings. This realization means that instead of understanding a concept in the context of classroom lesson, learners connect a concept with what they see in their daily life. In this context, learners need to realize the academic learning of a concept beyond the school mathematics (Ben-Hur, 2006).

Realizing the Importance of a Concept beyond School Mathematics

In the analysis, there were 13 coding references for realizing the importance of a concept beyond school mathematics: three from group learning notes, and ten from transcribed data.

PSTs expressed their understanding of *pi* by realizing its importance beyond school mathematics. Various circular local materials were used pedagogically to help foster the conceptual development and understanding of *pi*. The local materials used as teaching resources included a head of sunflower, a stem of maize, a stem of a sunflower, and a stem of cassava (see Figures 4.4, 4.5, 4.6, and 4.7). The PSTs worked on *pi* and its value in small groups and class discussions to realize the importance of *pi* beyond school mathematics. As G1 noted:

> [*Pi*] is found in all circular (round) objects, such as buckets, round cones, oranges, [and a] sugar cane stem. This is because a circular object can be measured to get circumference and its diameter using a thread and a set of Vernier callipers respectively.[15] (Transcribed data)


Figure 4.4. Sunflowers. *Figure 4.5.* Stems of maize.

[15] Nodes\\Nodes from the Four-day CRI\\Realization\Realizing the importance of a concept beyond the school mathematics (3)

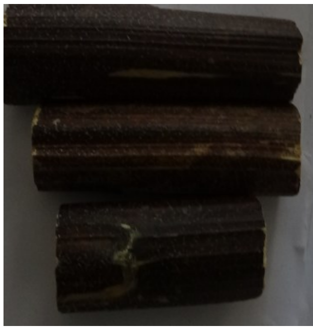


Figure 4.6. Stems of sunflower. *Figure 4.7.* Stems of cassava.

The above example and pictures illustrate that the PSTs came to see the importance of *pi* beyond school mathematics. The PSTs explained that *pi* is found in all circular objects in our daily life, because each of these objects has a diameter and circumference that can be measured using a set of Vernier callipers and string respectively. Based on this experience, the PSTs are likely to use the circular objects available in their local environment to engage learners with the concepts in a mathematics class.

As in past studies about other mathematical concepts, including functions (for instance, Brijlall & Maharaj, 2013; She et al., 2014) and fractions (Castro-Rodriguez et al., 2016), the PSTs in the present study related the concepts in context by connecting the social, physical, cultural and mathematical worlds.

Realizing the Connections of a Concept with Other Mathematical Concepts across Tanzanian Mathematics Curricula

In the analysis, there were 36 coding references to connecting the concept with other mathematical concepts across curricula: 12from group learning notes, seven from transcribed data, and 17from reflections.

The PSTs expressed their understanding of *pi* by connecting it with other mathematical concepts in the curricula. By analysing the content of *pi* in the Tanzanian mathematics curricula, including textbooks (for instance, Ben and Company Ltd, 2011a; 2011b; Sichizya & Kwalazi, 2010a, 2010b; TIE, 2009a; 2009b; 2009c; 2009d) and syllabi (TIE, 2005a; 2005b; 2009), the PSTs were able to realize the concept in the curricula. Consider, for example, G1's notes:

> In primary school, in Standard VI—students should know [what] round objects (circular) look like, the meaning of a diameter, and students should know to measure the diameter and circumference of the round (circular) object after learning [the] measurements topic in Standard IV and V. After learning the above concepts, primary school students should use the value of *pi* to calculate area, perimeter, and volume of 3D objects.
>
> In secondary school, before learning the concept of *pi*, students should know how to measure the central angle and arc length in a circle and sphere topics (Form III). After learning the above concepts, Form III students should know how to use π in calculating [the] area and circumference of 3D objects
>
> [At the] university level (undergraduate), students should know [what] to apply [when] calculating different problems of circular motion by using π in [a] rigid body course (second-year course).[16] (Group learning notes)

G1 realized various mathematical concepts that learners needed to know before and after learning the concept of *pi* in the Tanzanian mathematics curricula. G1 also realized the connections between the concepts in the curricula by aligning them with their respective class levels for teaching to the learners.

PSTs also realized the applications of *pi* in the Tanzanian mathematics textbooks and syllabi. For instance, when they were working together in small groups to identify the objects in which *pi* exists. G3 went further by identifying the circular objects, drawing the circular figures

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[16]Nodes\\Nodes from the Four-day CRI\\Realization\Connecting a concept with other mathematical concepts across the curriculum (1)

of the objects, and connecting the figures to the concepts in the primary school mathematics curricula. G3 described the following:

- A circular bread, $c = \pi d, A = \pi r^2 = \frac{\pi d^2}{4}$ (Standard VI)
- Open pipe (both sides), $A = \pi dh = 2\pi rh$ (Standard VI)
- Closed one side of a pipe, $A_T = \pi r^2 + \pi dh = \pi(r^2 + dh)$ (Standard VI)
- Closed both sides of a pipe, $A = 2\pi r^2 + 2\pi rh$ (Standard VI).[17] (Group learning notes).

The above list is an example of how the PSTs expressed their understanding of the applications of *pi* in the curricula by listing circular objects that can be studied using *pi* and by relating the objects to the 3D figures in which *pi* is applied to find the areas of a circle and cylinder. This is taught in primary school in Standard VI. G3 used drawings of circular objects and introduced formulas to compute their circumferences or areas and made sure that those formulas were consistent with the level of students who would be learning the concepts. For instance, *pi* was useful for finding the circumference and area of the bread, because this particular loaf was shaped in a circle. Another object, the pipe, was cylindrical. To determine its surface area, *pi* was used with other parameters, including diameter, radius, and height. G1 used a pipe that was closed at both sides, which means it contained two circles: one at each end of the pipe. An open-side pipe contains only one circle. But because *pi* exists in any circle, it is applicable to either kind of pipe. Also, G3presented together the shapes of circular objects with their respective names and formulas for finding their areas. This was an example of blending conceptual embodiment and

[17] Nodes\\Nodes from the Four-day CRI\\Realization\Realizing the applications of a concept across the mathematics curriculum (1)

operational symbolism (Tall, 2013). Conceptual embodiment in mathematics emphasizes shapes and figures while operational symbolism focuses on symbols (Tall, 2013).

Realizing the Applications of a Concept in Other Fields

In the analysis, there were 30 coding references for realizing the applications of a concept in other fields: nine from group learning notes,14 from reflections, and seven from transcribed data.

The PSTs expressed their understanding of *pi* by identifying its applications in other fields, including physics and information technology. For instance, the PSTs realized that *pi* is applicable in engineering in multiple ways, including designing and constructing a road. For instance, Group two presenter, G2P expresses that, "in road construction, π can be used in approximation since the length of an arc is proportional to the central angle"[18] (Transcribed data). The group suggested multiple applications of *pi* in relation to the materials that we see in daily environment, including round about. As such, PSTs connected the mathematics that are taught in the class with what they see in their local areas.

Research Question 2: How do PSTs plan to teach a concept, given their participation in CRI?

In the analysis, "practice" was coded as a parent node to answer the second question. The "practice" component of Ben-Hur's (2006) CRI "involves novel activities that are all based on the corresponding concept" (p. 13). Ben-Hur also suggests that "to achieve a high level of student success, teachers must avoid practice tasks that are too easy as well as practice tasks that are too difficult" (2006, p. 16).

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[18] Nodes\\Nodes from the Four-day CRI\\Realization\Realizing the applications of a concept in other fields(11)

To answer the above research question, PSTs' activities of developing lesson plans in small groups and performing micro-teaching are part of their practices with *pi*. On the last day of the CRI meetings, the PSTs designed lesson plans to teach the concept of *pi* in the classroom and then performed micro-teaching.

Designing Lesson Plans for Teaching a Concept

Meaning, decontextualization, recontextualization and realization occurred in a lesson plan and teaching practice. While I emphasized learning by doing from Ben-Hur's practice component, PSTs practiced (meaning, decontextualization, recontextualization and realization) related to the concept of *pi* by planning a lesson and performing micro-teaching on how to teach it in the research meetings. As such, they are not considered separate in the lesson plan and micro-teaching.

The PSTs were given an hour and a half in which to design a lesson including plan notes. They used Tanzanian syllabi (TIE, 2005a; 2005b), textbooks (TIE, 2009a; 2009b; 2009c; 2009d), and lesson plan formats. All the groups planned a lesson focused on understanding *pi* as a concept. For instance, G2 designed the following lesson plan:

Table 4.1

Lesson Development by Group Two

Stages	Time	Teaching Activities	Learning Activities	Assessment
Introduction: Distribute various objects to the students after greetings.	3min	Ask students to identify the circular objects from various objects in small groups. Revise the concepts related to the circle.	Identify the circular objects from the given set of the objects in small groups.	Observe students' responses in identifying the circular objects.

Presentation	10min	Guide students in finding the values of *pi* from the diameter and circumference using the circular object.	Students in their groups of three measure the length of the circumference and the diameter of the circular object provided.	Check students' responses to measuring the diameter and circumference of the circular object.
Reinforcement	10min	Organize students in a group of three students and ask them to once again measure the diameter and circumference of a circle and find the ratio of the circumference and diameter of the circular object.	Students measure the circumference and diameter of the circular object provided, and then find the ratio of the circumference and diameter measured.	Observe the students in their group when measuring the diameter and circumference of a circle and then find the ratio that exists between the circumference and diameter.
Reflection	5min	Provide a question to each individual student and ask the students to define *pi* and find its value when given the circumference and diameter, and to find the value of a diameter when given the circumference.	Individual students attempt the question provided by the teacher.	Mark and observe individual students' attempts on the given question.
Conclusion	2min	Write a question on the chalkboard as individual students' homework.	Individual students to take note of the question provided as home work.	Observe that each student takes note of the question provided.

The G2-designed lesson plan focused on engaging PSTs with the concept of *pi* using participatory approaches. G2 designed the lesson so that the PSTs could understand the meaning of *pi* and find its value from the ratio of two measures (circumference and diameter) of a circular object. The lesson plan had the PSTs being asked how to find the diameter given the circumference after finding the value of *pi* from the ratio of two measures of a circular object. We can see that G2 used circular objects for conceptual development (Dubinsky, 1991; Sierpinska, 1994; Tall, 2013) in the introduction, presentation, and reinforcement stages.

G2 practiced the concept *pi* by designing the activities for teaching *pi* in the following ways. First, in the introduction, PSTs in small groups were asked to identify circular objects from circular and non-circular objects. Second, in the presentation stage, G2 guided PSTs to measure the circumference and diameter of a circular object using tools for measurement. This helped the PSTs to define *pi* and find its value from the ratio of two measures (circumference and diameter) of the object. Third, in the reinforcement stage, PSTs worked to measure the circumference and diameter of the circular object using tools for measurements. Then, they found the ratio of the two measures (circumference and diameter) of the object. Finally, in the reflection stage, the individual PSTs were asked to work on various activities during the research meetings, including defining *pi*, finding its value when given the circumference and diameter, and finding the diameter when given the circumference and the value.

Performing Micro-Teaching for Teaching *Pi* as a Concept

Micro-teaching performed by the PSTs focused on presenting a lesson about how to teach *pi* as a concept. As evident in past studies (for instance, Olusanjo, 2011), micro-teaching is important to help learners understand a concept and is fostered by an effectively designed lesson plan.

After designing the lesson plan, each group was given 30 minutes to present its lesson. During micro-teaching, G1P, G2P, and G3P acted as teachers, and the other PSTs acted as learners working in small groups, A and B. As an example, Appendix H gives the transcript of the micro-teaching performed by G2P in the research meetings.

In the transcript(see Appendix H), G2P practiced teaching *pi* as a concept using circular materials. G2P used the cover of water bucket, a sole tape, a toss cover, a pawpaw stem, a string, and a ruler as teaching and learning resources. These resources were important to help other PSTs understand how to define a concept and find its value from measuring its circumference using a string and a ruler, and its diameter using a set of Vernier callipers. We can also see that circular objects were transformed from immediate to dynamic objects (Dubinsky, 1991; Sierpinska, 1994; Tall, 2013), when PSTs used the physical objects to find the value of *pi* from a ratio of the two measures (circumference and diameter) of a circular object.

PSTs in small groups were asked to measure the diameter and circumference of a circular object. After that, they were asked to find the ratio of two measures (circumference and diameter) of the object. In particular, the quote below was taken from a recording of the small group as its members learned how to measure the circumference and diameter of the circular objects followed by finding the ratios of the two measures in the quote below:

G2P: …Group A, [we will] let you use only this one [a sole tape]. You will get certain measurements. Measure the circumference and then measure the diameter using a string and a ruler to [*sic*] the circumference. Work in a group for about four minutes. (Group work activities continue in the two small groups while teacher is silent observing and waiting for answers).

Group A: The object is sole tape, diameter equals $9.01cm$, andcircumferenceequals $28.4cm$.

Group B: The object is toss top cover, diameter equals $10cm$, and circumference equals $31.5cm$.

G2P: In each group, compute the ratio of circumference to diameter.

Group A: Ratio equals 3.152 ...

Group B: Ratio equals 3.15. (Transcribed data)

In the quote above, G2P was practicing teaching how to obtain the value of pi by measuring the circumference and diameter of a circular object using a participatory approach. To obtain the value of pi, Group A computed the ratio of the two measures (circumference and diameter) of a sole tape while Group B computed the ratio of the two measures (circumference and diameter) of a toss top cover. It is important to find the ratio of the two measures (circumference and diameter) of a circular object, because the ratio of the two measures gives the value of pi.

G2P also guided PSTs about how to measure the circumference and diameter of a circular object using measuring devices through demonstration. In this activity, demonstration was used as a teaching and learning strategy. As evident in past studies (Bruce, et al., 2009; Illine, 2013), demonstration helps learners to develop an understanding of mathematical concepts because the process is visually oriented. When G2P demonstrated the teaching activity, he explained:

> For example, here, this is our circular object (a pipe), okay! So, when you want to measure the circumference of that circular object, use this string. I start first to identify the starting point [of the string] using a pencil. This is my starting point, okay. Then, I take this string. I start measuring from the point identified (This one here, and round the string to complete the circumference of a circular object. This yields you back to the starting point). So, the distance of this string from the starting point around the circle to turn back

to the starting point will make one complete circle which is called the circumference of a circle.[19] (Transcribed data)

From the designed lesson plan and performed micro-teaching for teaching *pi* as a concept, G2 thinks that teaching the concept *pi* involves defining *pi*, finding the value of *pi*, and using the value of *pi* in solving mathematical problems. All these aspects of the concept of *pi* involve the measurements of circumference and diameter of a circular object using tools for measurements, such as a string, a ruler, and a set of Vernier callipers.

Closing Comments on the CRI approach for Empowering PSTs to Practice the Concept

CRI helped the PSTs to gain skills about how to design a lesson plan for teaching a mathematical concept to foster a participatory way of learning. For instance, they learned in the research meetings how to design a lesson for teaching *pi*. In the Day Four meeting, PSTs suggested teaching *pi* using participatory approaches. For instance, G3 suggests: "the best approach that can be used to improve teaching and learning is using learner centred, including questions and answers, demonstrations, and brainstorming." (Group learning notes).

CRI helped the PSTs become teachers in the Day Four meeting by having them do micro-teaching sessions. At that moment, the PSTs were teachers and the teacher educator was a learner. The teacher educator learned from the PSTs how they could teach *pi* so that learners would come to understand a concept. This appeared to be the first time the PSTs had been given the opportunity to act as teachers in the context of the university mathematics class and they were interested in doing so.

CRI helped the PSTs see new ways of teaching a concept and designing the teaching aids. As I suggested, each group used circular objects that they found in their local environment. These teaching and learning resources were different from the ones I used to engage the PSTs during

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[19]Nodes\\Nodes from the Four-day CRI\\Practice\Demonstrating a concept (1)

the first three days of the four one-day meetings (see Figures 4.4, 4.5, 4.6, 4.7). For instance, G1 used stems of pawpaw as one of their teaching and learning aids. G2 used the cover of a water bucket. G3 used plain paper folded on a round sole tape. These teaching and learning aids show that the PSTs used new ways to teach a concept and that they designed teaching aids using resources available in their local environment. These local resources brought by PSTs for the Day Four meeting and used as teaching aids were not pre-determined. As such, CRI empowered the PSTs by facilitating the design of the teaching and learning aids that they used to practice teaching a concept in the research meetings.

CHAPTER 5

LESSONS LEARNED THROUGH THE IMPLEMENTATION OF CRI IN TANZANIA

As a teacher educator, this chapter presents seven lessons that I learned through implementing concept-rich instruction (CRI) with university pre-service teachers (PSTs) in Tanzania. These lessons are related to systematic errors, multiple interpretations, open definitions, pre-service teachers (PSTs) getting an opportunity to teach a mathematical concept and seeing how the concept is integrated into Tanzanian mathematics curriculum, community of learners in Tanzanian pre-service teacher education, local resources as a tool to mediate learning, and a better understanding of participatory approaches and social constructivism in university mathematics classrooms.

Systematic Errors

PSTs' systematic errors about pi were noted as prevalent through analysing data gathered during exploring the implementation of CRI with university PSTs in the research meetings, especially through the component of recontextualization.

My research study in the Tanzanian context revealed systematic errors about the value of *pi* among PSTs. Before the research meetings, the PSTs believed that the value of *pi* was $\frac{22}{7}$. However, this systematic error was eliminated after the PSTs participated in the range of tasks offered in the meetings.

Herholdt and Sapire (2014) define systematic errors as "errors in the declarative or procedural knowledge which occur when someone who makes this type of errors [*sic*] believes that what has been done is correct—thus indicating faulty reasoning" (p. 43). In short, systematic errors arise from misconceptions (Ben-Hur, 2006). Misconceptions are "false ideas that [learners] develop because of careless instruction [*sic*] that are not altered in a subsequent learning" (Ben-

Hur, 2006, p. 43). Effective teachers can see and identify the errors in the work of the learners. According to Ben-Hur (2006), when "teachers examine systematic errors carefully, they can see that the errors are reasoned and not capricious. They are either pre-conceptions or false, naïve intuitions" (p. 44).

When implementing the CRI approach, I noticed systematic errors in the PSTs' work about the value of *pi*. For instance, 3.14 is obtained as a result of taking 22 and dividing by 7. This error resulted from declarative knowledge stated in textbooks (e.g., Sichizya & Kwalazi, 2010a; 2010b). The textbooks use $\frac{22}{7}$. I speculate that this value is used because it makes for easy computation. Learners use this value in a context in which digital devices such as a calculator and computer are not used.

Through the implementation of CRI in the research meetings, the PSTs' systematic errors were used as "instructional tools to alter their misconceptions" (Ben-Hur, 2006, p. 52). I facilitated learning that promoted the conceptual development of the value of *pi* by treating errors as instructional tools to eliminate the PSTs' misconceptions. This treatment of errors as instructional tools can be evidenced in Chapter 4, in the subsection about reflecting on errors and analysing errors. The paragraphs below describe the ways that the PSTs changed their perspective about the value of *pi* from $\frac{22}{7}$ to an irrational number.

In the process of eliminating systematic errors in the research meetings, the PSTs were asked to use circular materials to find the value of *pi*. While using the circular materials, they obtained the value of *pi* for a circular object by measuring the two parameters (circumference and diameter) of the object. However, the PSTs gave different answers for each measured object, and those answers were decimal numbers and not $\frac{22}{7}$.

Systematic errors were also eliminated among PSTs through working on various activities to determine the value of *pi* using an Archimedes approach. Thus, PSTs learned how to use the Archimedes approach for a better and different approximation of the value of *pi* (see Appendix B). This instructional strategy also helped the PSTs to understand the value of *pi* as an irrational number (3.14 ...), a numerical value that can be approached but never reached (Rothman, 2009). The PSTs drew inscribed and circumscribed polygons in a circle to approximatethe value of *pi*. After that, they measured the length of one side of an inscribed or circumscribed polygon and the diameter of a circle. Then, they calculated the perimeter of a polygon by taking the length of one side and multiplied it by the number of sides to help find the number of diameters in the perimeter of the inscribed or circumscribed polygon. Finally, they calculated the number of diameters in the perimeter of the polygon by taking the ratio of the perimeter of the inscribed or circumscribed polygon to the diameter of the circle. In doing so, PSTs improved their understanding of the value of *pi* as an irrational number. For example, participant AA said:

> The one I have been surprised [about] today is that *pi* is not a constant number, but it depends on the certain circumference of a circle and its diameter, but it becomes a constant value when approximated equal to 3.14. Also, π is not the constant fraction $\frac{22}{7}$, because if we say $\pi = \frac{22}{7}$, [that] means each circle has an exact circumference of 22 units and diameter of seven units, which is not true. So, π is not a constant fraction$\frac{22}{7}$. The value of *pi* is an irrational number (i.e., 3.14…). It is not a recurring or terminating decimal, it is an infinity.[20] (Reflection)

[20]Nodes\\Nodes from the Four-day CRI\\Re-contextualization\Using a concept to connect new experiences with past or concurrent experiences (2)

An Archimedes approach helped this PST to eliminate systematic errors from the value of *pi*= $\frac{22}{7}$ to an irrational number that is approximated to 3.14.

As a teacher educator, I learned that systematic errors can occur in the university mathematics classrooms. These errors can be observed in the work of individual PSTs or groups during group presentations and class discussions. After identifying systematic errors in the PSTs' work, a teacher educator is responsible to help them eliminate these errors using various instructional strategies such as scaffolding and mediated learning.

Multiple Interpretations

Multiple interpretations emerged through analysing data gathered during exploring the implementation of CRI with university PSTs in the research meetings. Multiple interpretations as a lesson emerged within the CRI components of meaning and decontextualization.

In this study, multiple interpretations refers to the number of ways that the PSTs interpreted a question during the research meetings. For instance, while working on the designed and implemented research activities, there were multiple interpretations of the definitions of *pi* as well as of finding the value of *pi.*

As stated earlier, one of the overwhelming conclusions of this study is that before the research meetings, the PSTs had few ways to define and find the value of *pi.* They believed that the only definition of *pi* was "the ratio of circumference to diameter of a circle."They also believed that there is only one way to find the value of *pi*: by using the circumference and diameter of a circle. I speculate that these beliefs are due to the fact that Tanzanian textbooks provide only one definition of *pi*. The textbooks also provide only one approach to finding the value of *pi* from circumference and diameter. However, as shown earlier, the CRI approach helped the PSTs to interpret *pi* in multiple ways, including as a ratio of the circumference to diameter of a circular object/figure/circle, and as the number of diameters required to complete the circumference of a circular object/figure/circle. Participating in research meetings also helped the PSTs to develop multiple interpretations of the strategies needed to find the value of *pi*.

Multiple interpretations of the definition of a concept are important in mathematics education. As Chesler (2012) found, "Definitions matter in mathematics. They introduce ideas, they describe objects and concepts, they identify fundamental and essential properties of mathematical objects, they support problem solving and proof, and they facilitate communication of mathematics." (p. 27)

During the research meetings, the PSTs provided multiple interpretations of the definitions of *pi* in a productive way, including using the definitions to further explore a concept. From the two measures (circumference and diameter), they were able to develop multiple interpretations of the definitions of *pi*, such as *pi* is the number of diameters required to complete the circumference of a circular object, and *pi* is a ratio of two measures (circumference and diameter) of a circular object.

According to the PSTs' open interpretations and "shifting meanings did not prove to be obstacles" (Moschkovic, 2008, p. 551) to the discussions in the research meetings. Instead, I used the PSTs' interpretations as resources and built on and connected them and thus strengthened their understanding of *pi*. For instance, the PSTs were exposed to various activities, including measuring the circumference, diameter, arc length and central angle of a circle. Through these activities, the PSTs were able to give multiple interpretations of the definitions of *pi* apart from the ratio of circumference to the diameter of a circle. Here are the PSTs' specific responses about the multiple interpretations of the definitions of *pi*:

G1: *Pi* is the number of diameters to complete one circumference of a circular body adding a part remaining to complete the circumference.[21] (Group learning notes)

G2: *Pi* is the total number of radians occupied in a half circle.[22] (Transcribed data)

G3: *Pi* is the angle measure of a circle in a straight line [passing through the centre of the circle].[23] (Transcribed data)

G3: *Pi* is the ratio of the radian to the angle measure multipl[ied] by 180^0. That is to say, $\pi = \frac{180s}{\theta}$.[24] (Transcribed data)

The above multiple interpretations of the concept definitions of *pi*can be categorized in two different ways. The first way is procedural, because the definitions describe the ways an object is constructed (Chesler, 2012). For instance, consider the definition, *pi* as the number of diameters required to complete the circumference of a circular object. Before counting the number of diameters required, it is important to measure the circumference and diameter of the object using measurement tools such as a string and a set of Vernier calipers. After that, we count the number of diameters around the circumference of the object. These show that there are procedures necessary to know and follow to describe the ways that the value of *pi* can be obtained through this counting process. The second form of categorization, structural, is appropriate because the definitions provide an opportunity to identify the properties of the object (Chesler, 2012; Zaslavsky & Shir, 2005). For example, from the definition, *pi* as the ratio of the

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[21]Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (3)

[22] Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (5)

[23]Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words (6)

[24] Nodes\\Nodes from the Four-day CRI\\Meaning\Defining a concept in words and symbols (7)

two measures (circumference and diameter) of a circular object, we can identify the essential properties of the object, such as circumference and diameter.

The PSTs interacted with the multiple interpretations of the definitions of *pi* in the research meetings in a productive way. For example, G3 described two strategies and the steps to follow in each strategy:

- First strategy:
    - Take a circular object
    - Take instrument/tools for measuring [the] diameter and circumference of the circular objects, including a string, a set of Vernier callipers, and a ruler.
    - Measure the diameter and circumference of the circular object.
    - Record the two measures (the diameter and circumference) [of the circular object]
    - Find the ratio of the circumference of the object and its diameter to obtain the value of *pi.*
- Second strategy:
    - Take a circular object
    - Us[e] a set of Vernier callipers or a string and a ruler
    - Measure the diameter and circumference of the circular object.
    - Finally, measure the number of diameters that goes around the circumference of the circular object. This number of diameters represents the value of *pi*.[25] (From group learning notes)

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[25] Nodes\\Nodes from the Four-day CRI\\Decontextualization\Generating a concept through encouragement of divergent responses (3)

These strategies were important for the PSTs to know how to find the value of *pi* using a circular object and how to define *pi* in relation to the circumference and diameter of a circle/circular figure/circular object. We can also see that in each strategy, a circular object is used as a resource, by converting the circular object into a dynamic object (Dubinsky, 1991; Sierpinska, 1994; Tall, 2013).This occurs when finding the value of *pi* from the relationship between the circumference and diameter of a circular object. The two strategies involve operational abstractions (Tall, 2013). The first strategy involves a ratio of two measures (circumference and diameter) of a circular object to obtain the value of *pi.* The second strategy involves counting the number of diameters required to complete the circumference of a circular object to obtain the value of *pi.*

In the research meetings, the PSTs and I were "responsive to what others said, so that each [contributed mathematical ideas] that buil[t] upon, challenge[d], or extend[ed] a previous one" (Goldenberg, 1991, p. 3). For instance, I placed a sole tape in each group (G1, G2, and G3) with the intention that the PSTs would use it to fix the flip charts on the wall after group work discussions. However, this tape was employed in another way by members of G3, who used it to find the value of *pi* from real objects; they measured its circumference using a string (thread) and a ruler, and its diameter using a set of Vernier callipers. Finally, they obtained the value of *pi* using the ratio of the tape's circumference to its diameter. G3 also drew its figure by indicating the internal and external diameters. The internal diameter of the tape (7 units of measurements) was smaller than the external diameter (10 units of measurements).

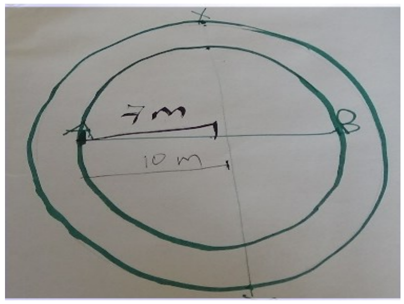

Figure 5.1. The diagram of a sole tape showing the internal and external diameters.

The meaning and decontextualization of the concept were important to develop the multiple interpretations of the definitions of *pi* and finding its value in the research meetings. Engaging PSTs with the activities about meaning of *pi*, helped them to develop multiple interpretations of the definitions of *pi.* Also, engaging PSTs with the activities to use the applications of *pi*, helped them to develop multiple interpretations of finding the value of *pi.*

As a teacher educator, I learned that multiple interpretations can occur in the university mathematics classrooms. These interpretations can be seen in the work of individual PSTs or small groups during group presentations and class discussions. A teacher educator is responsible to promote multiple interpretations of mathematical concepts through using various strategies such as designing open questions that foster PSTs to interpret the concepts in a variety of ways. Also, a teacher educator can ask PSTs a question that yields multiple interpretations in the classrooms, such as "what comes in your mind when you hear about complex numbers? Please explain you answer(s) (Think, pair, and share followed by group presentations and class discussions).

Open Definitions

Open definitions emerged through analysing data gathered during exploring the implementation of CRI with university PSTs in the research meetings. This emerged as a lesson through the component of decontextualization.

Open definitions are meanings that are relevant in a certain situation (Davis & Renert, 2014). But they may infer elaboration to understand a concept (Davis & Renert, 2014). π is $\frac{22}{7}$ is an example of an open definition. This open definition is relevant when working in the context of a fraction. We may also infer other mathematical ideas from this definition, such as the circumference has a length of 22 units and the diameter has a length of seven units when working in the context of a ratio. However, this definition needs elaboration, to explain how it represents the value of *pi*.

The current study showed that before participating in the research meetings, the PSTs had a limited understanding of the open definitions of π; they believed that $\pi = \frac{22}{7}$ or 3.14. I speculate that this belief is due to the fact that Tanzanian textbooks provide only these two values of *pi*. The textbooks also provide only one approach to finding the value of *pi* from circumference and diameter.

In the design and implementation of CRI in the research meetings, I did not focus on giving the PSTs an opportunity to discover the open definitions of π. As such, the PSTs did not realize that they were working on open definitions of π. However, these open definitions emerged during group and class discussions about π and its value. I did not call them open definitions in the research meetings. They emerged for me as I reviewed the collected data.

Understanding an open definition is important in various aspects of mathematics teaching and learning such as in mathematical problems/questions, tests, examinations, and assignments. An open definition is sufficient for certain situations because it is situation-dependent (Davis & Renert, 2014). For instance, when working in the context of a fraction, it is sufficient to use $\pi = \frac{22}{7}$ rather than a decimal value. Also, when working in the context of decimal number, it is sufficient to use a decimal value ($\pi = 3.14$) rather than a fraction). When working in the context of radian measurement, it is sufficient to use the symbol for *pi*, i.e., π, rather than a value,$\frac{22}{7}$, or 3.14.

Davis and Renert (2014) emphasize the importance of structuring learning focused on open definitions. According to them, "All mathematical concepts have 'open definitions.' A mathematical concept must be defined in a manner that is sufficient for the situation at hand, but that permits elaborations as new situations present themselves" (Davis & Renert, 2014, p. 46). During class discussions, the PSTs generated open definitions of π when they were asked to define it in the research meetings. The open definitions of πthat emerged were:

- π is a fractional number, i.e., $\pi \cong \frac{22}{7}$, when the circumference of a circle equals22 and its diameter is seven.
- π is approximated equivalent to 3.14, i.e., $\pi \cong 3.14$, when it is approximated to two decimal places.
- π is an irrational number, i.e., $\pi = 3.14$ ...; it is a decimal number that does not terminate or repeat.
- π is a radian measure equivalent to 180 degrees, i.e., $\pi\ rad \equiv 180^0$, when π involves the measuring process of a central angle in a unit circle.
- π is the number of diameters that goes around to complete the circumference of a circular object when π involves the counting process.

The open definitions emerged in a pedagogical context that is consistent with Bruner's (1966) instruction theory. The theory recognizes the:

- Experiences that most effectively put the individual in a position to learn;
- Ways in which a body of knowledge should be structured so that it can be most readily grasped by the learner;

- Most effective sequences in which to present the materials to be learned;
- Nature and pacing of rewards and punishments in the process of learning and teaching. (Bruner, 1966, pp. 40—41)

In his theory, Bruner addresses three modes of representations about the ways in which knowledge or information is stored and processed in memory. These modes of representations are enactive, iconic and symbolic. In referring to the open definitions above, we can see that the realizations of π could fit into “enactive (action-based), iconic (image-based), and symbolic (language-based) representations” (Davis & Renert, 2014, p. 59), as shown below.

- Enactive (action-based): Counting the number of diameters required to complete the circumference of a circular object (as in Figure 4.1).
- Iconic (image-based): Measuring the central angle in a unit circle (as Figure 5.2).

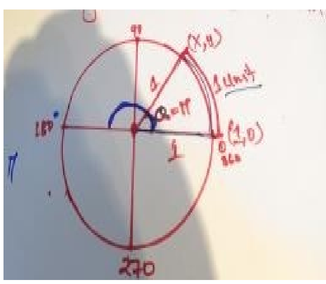

Figure 5.2. Measured central angle of a unit circle equivalent to π radians.

- Symbolic (language-based): “π is an irrational number,” “π is a fractional number, i.e., $\pi \cong \frac{22}{7}$,” and “π is approximated [as being] equivalent to 3.14, i.e., $\pi \cong 3.14$.”

The analysis shows that the CRI approach helped the PSTs to generate open definitions ofπ from measurements of the circumference, diameter, radians, and central angle of a circle or a circular figure/object. The decontextualization of a concept as one of the aspects of CRI helped the PSTs to develop open definitions of π in the research meetings through group and class discussions about π and its value.

As a teacher educator, I learned that it is common that mathematical concepts have open definitions. These definitions can occur in the university mathematics classrooms and can be seen in the work of individual PSTs or small groups during group presentations and class discussions. A teacher educator is responsible to promote open definitions of a mathematical concept through using various strategies such as designing open questions that foster PSTs to provide these definitions. Also, a teacher educator can ask PSTs a question that yields open definitions in the classrooms, such as "what is a function? Please explain you answers (Think, pair, and share followed by group presentations and class discussions).

PSTs Getting an Opportunity to Teach a Mathematical Concept and Seeing how the Concept is Integrated into Tanzanian Mathematics Curriculum

This emergent possibility of CRI in the university mathematics classrooms, occurred through analysing data gathered during exploring the implementation of CRI with university PSTs in the research meetings. This lesson emerged from the CRI component realization.

The CRI approach provided an opportunity for the PSTs to teach *pi* on the fourth (and final) day-long meeting (see Appendix B) and see how the concept is integrated into Tanzanian mathematics curricula such as textbooks and syllabi. It is important for the PSTs to understand the way(s) that a mathematical concept is developed and articulated in the curricula, because teachers use the textbooks and syllabi to design lessons.

The opportunity to see how a mathematical concept is integrated into Tanzanian mathematics curricula is critical for PSTs, as it will help them to know how to teach a concept from the curricula. The curricula provide opportunities for teachers to see mathematics as a subject in which the articulated and integrated concepts have connections. In particular, Mwakapenda (2008) suggests:

> [The curricula] has links within itself and other disciplines. It also allows for the development of teaching strategies that help teachers and teacher educators see teaching in a new perspective. The vision in the curriculum also helps us to conceptualize assessment in ways that recognizes [*sic*] that all learners can do and succeed in mathematics. (p. 189)

So that the learners can and will succeed, the PSTs were asked to discuss how the Tanzanian mathematics curricula could be improved. They identified three concerns, the first of which is that *pi* is taught in standard VI, but with no detailed discussions of the concept.G2 suggested that "The concept of *pi* should be included or introduced in the curriculum in mathematics subjects from [the] primary level to higher school level starting with simple concept[s and moving on] to complex concepts (Bloom's taxonomy).[26] (Reflection).

The second concern that the PSTs identified is that the value of *pi* is always given when it can and probably should be computed. For instance, G2P pointed out that:

> …. in primary level, when asked to calculate the area of a circle, we are given that $\pi = \frac{22}{7}$. Thus, [this is] why we argued that the value of *pi* is given. But it will depend on each level, in primary, secondary or university. [At the] primary level, you can say "use this as the value of *pi*." But [at the] secondary or university level, it is not necessary to be given [the value]. At [the] university or secondary level, you can be given the value of *pi* or you can develop your mind to get the value of *pi*.[27] (Transcribed data).

[26] Nodes\\Nodes from the Four-day CRI\\Recontextualization\Using the concept to connect new experiences with past or concurrent experiences (33)

[27] Nodes\\Nodes from the Four-day CRI\\De-contextualization\Analysing errors and inconsistencies (2)

The third concern was that there is a need to include in the curricula the definition of *pi* as the number of diameters to complete the circumference of a circular object, because this gives the learners an opportunity to practice a concept using circular objects. G1 suggested that:

> In order to articulate the meaning of *pi* in Tanzanian curricula for better understanding, they should use the meaning of *pi* as the number of diameters to complete one circumference of a circular object because they should make the practical sessions by fitting the diameters on a circumference.[28] (Group learning notes).

G1 proposed including in the curriculum the meaning of *pi* as "the number of diameters to complete the circumference of a circular object." The counting the number of diameters around the circumference of a circular object is a practical exercise. This type of counting process helps learners to easily understand the meaning of *pi* and its value. The PSTs identified this issue during the final day-long meeting, when they were asked to use the curricula, including textbooks (e.g., TIE, 2009a; 2009b; 2009c; 2009d) and syllabi (e.g., TIE, 2005a; 2005b) to develop a lesson plan on how to teach *pi* as a concept in small groups.

In the Tanzanian context, *pi* is found across the mathematics curriculum from kindergarten through university. It is used as a constant value and as a radian measure in various mathematical concepts/topics or mathematical contexts including geometry, circles, spheres, complex numbers, calculus, numerical analysis, matrices and trigonometry. Table 5.1 summarizes the applications of *pi* across the Tanzanian mathematics curriculum based on Ben-Hur's notion of realization of a concept. In the Tanzanian education system, Standards I, II, III, IV, V, VI, and VII are primary school levels. Forms I, II, III, and IV are ordinary secondary

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[28] Nodes\\Nodes from the Four-day CRI\\Recontextualization\Using the concept to connect new experiences with past or concurrent experiences (1)

school levels. Forms V and VI are high school levels, and Years I, II, and III are university undergraduate study levels.

Table 5. 1

Realization of Pi Across Tanzanian K—University Mathematics Curriculum

Standard/Form/ Year of Study	**Topic**	**Applications of *Pi***
Standard I		
Standard II		
Standard III		
Standard IV		
Standard V		
Standard VI	Circumference and area of a circle	3.14 or $\frac{22}{7}$is used to find the circumference and area of a circle.
Standard VII	Surface areas and volumes of 3D figures	3.14 or $\frac{22}{7}$is used to find the volume of a sphere and the surface area and volume of a cylinder.
Form I	Circumference, area and volume	3.14 or $\frac{22}{7}$is used to find the circumference and area of a circle and the surface area and volume of a cylinder.
Form II	Trigonometry	A radian measure(π) is used to find the angle measure in trigonometric functions.
Form III	Circle	A radian measure(π) is used to find the angle measure and the arc length of a circle.
	Sphere	3.14 or $\frac{22}{7}$ and a radian measure(π) are used to find the shortest and longest paths of the object along the meridian of a sphere.
Form IV	Matrices & Transformations	A radian measure(π) is used to find the angle measure of an object rotating or reflecting along the xy plane.
Form V	Complex Numbers	A radian measure (π) is used to find the powers of the complex number using de Moivre's identity.
	Calculus	3.14 and a radian measure(π) are used to evaluate the limits of the integral and to integrate the trigonometric function respectively.
Form VI	Numerical methods	3.14…. and radian measure (π) are used in numerical methods to evaluate the roots of the given mathematical expression.

	Coordinate geometry	The value of *pi* is also used in polar coordinates both as 3.14 and a radian measure equivalent to 180 degrees to convert the polar coordinates into Cartesian coordinates and Cartesian coordinates to polar coordinates respectively.
Year 1	Linear algebra	A radian measure(π) is used to find the rotation and reflection of the objects in a two- or three-dimensional plane.
Year 2	Calculus	3.14 or $\frac{22}{7}$and radian measure (π) are used to evaluate the limits of the given integral and integrate the given trigonometric function.
	Rigid body mechanics	A radian measure (π) is used to determine the rotation of a rigid body or particles along a two- or three-dimensional plane.
Year 3	Numerical analysis	3.14 and a radian measure (π) are used to evaluate the roots of the given mathematical expression.

The table above and the syllabi and textbooks show that the value of *pi* is constant and does not change (i.e., 3.14…). Also, while the value of *pi* is an irrational number and $\frac{22}{7}$ and 3.14 are used for the convenience of operating with numbers. For instance, *pi* is taught in Standard VI as an aspect of the geometry of a circle. However, a majority of teachers use 3.14 or $\frac{22}{7}$as the value of *pi*not necessarily because it is correct, but because the mathematics curriculum for primary and secondary schools does not address why the value is approximated to 3.14 or $\frac{22}{7}$before using it to find the circumference and area of a circle (MOEVT, 2011; Sichizya & Kwalazi, 2010a; 2010b; TIE, 2005a; 2005b; 2009).

The curriculum should inform teaching: it is a road map of sorts, and teachers need it to guide their lesson plans. As such, it is critical to integrate the curriculum into practice to meet the learning needs. In addition, Mwakapenda (2008) suggests that, “it needs to be acknowledged that working in integrated ways in the curriculum makes available new visions and realities for schooling" (p. 199). For instance, the PSTs identified that the concept of *pi* is not detailed enough in the Tanzanian K-University mathematics curricula. The PSTs in G2 reported:

> In [the] Tanzanian mathematics curriculum, there is no detailed information for teaching *pi* as a concept. *Pi* is in Standard VI only, and it is about circumference divide[d] by diameter only. The *pi* as a concept [information] is not detailed enough in secondary school and university. We have a challenge in [the] Tanzanian curricula, including [in the] syllabus, that we don't have the detailed explanations/discussions of *pi* as a concept.[29] (Transcribed data)

Furthermore, PSTs identified that in the curricula, including textbooks, when learners at schools are solving mathematical problems that make use of the value of *pi* such as finding the area of a circle, they are usually given $\frac{22}{7}$ as the value. For instance, G2P reported:

> Let me ask the question: in primary school, when asked to calculate the area of a circle, we are given that $\pi = \frac{22}{7}$. Thus, [this is] why we argued that *pi* is given. What can you comment about this? So [after class discussions], from today we have to know that π is approximated to$\frac{22}{7}$. It is an irrational number (3.14…).It can be given and computed.[30] (Transcribed data)

The above quote shows that the PSTs were generalizing that $\pi = \frac{22}{7}$, and it is always given. Before class discussions, PSTs thought that the value of *pi* is$\frac{22}{7}$: still this value was being treated as a

[29]Nodes\\Nodes from the Four-day CRI\\Realization\Realizing the connection of a concept with other mathematical concepts across the curriculum (7)

[30] Nodes\\Nodes from the Four-day CRI\\Realization\Realizing the applications of a concept across the mathematics curriculum (1)

given value. This kind of generalization is called under-generalization (Ben-Hur, 2006), because it is a result of a lack of understanding of the mathematical concept. This under-generalization could also be a result of the Tanzanian mathematics curricula, which presents $\pi = \frac{22}{7}$ without a serious examination of the concept of *pi* and its value.

The CRI approach helped the PSTs to learn how to teach *pi* and realize the concept across Tanzanian mathematics curricula. The PSTs realized how *pi* is articulated and developed in the Tanzanian mathematics curricula. The PSTs engaged in the practice of lesson planning on the fourth day of the research meetings. The realization of a concept gave the PSTs an opportunity to learn the applications of *pi*.

As a teacher educator, I learned I need to provide an opportunity for PSTs to realize school mathematics in the university mathematics classrooms. This realization can occur through designing the research activities that ask PSTs to analyse the mathematical concepts in the mathematics textbooks and syllabi. PSTs realize the concepts in the curricula during small group and class discussions. Also, a teacher educator can ask PSTs in small groups to design a lesson plan using textbooks and syllabi and then perform micro-teaching. After that, class discussions can take place focused on the concepts, such as how the concepts can be articulated in curricula to help learners better understand the concepts.

Community of Learners in Tanzanian Pre-Service Teacher Education

Recognizing the emergence of a community of learners in Tanzanian pre-service teacher education occurred through analysing data gathered around the component of practice during exploring the implementation of CRI with university PSTs in the research meetings.

Through the implementation of the CRI approach, individual PSTs had the opportunity to learn from others in small group and class discussions and during group presentations. In doing so, they become teachers as well as learners in the research meetings. For instance, before participating in this research project, the PSTs had no opportunity to practice teaching in university mathematics classes. One reason is that the mathematics instructors prepare the PSTs to understand the content in their university mathematics class, but the pedagogical practice is usually done during the practicum while the PSTs are at schools. That leaves no chance for the PSTs to practice peer teaching in the university mathematics classes. Another reason is that participatory approaches are rarely practiced in university mathematics classes. The research meetings helped the PSTs to link theory and practice by learning a concept taught in schools and practicing peer-teaching through small group and class discussions and group presentations.

The CRI approach helped the PSTs to think about teaching because during the research meetings the PSTs were not only learners, they were teachers. For instance, the PSTs designed a lesson and performed peer teaching during the last one-day meeting. These activities helped themto belong to a community of learners.

A community of learners can be described as "a group of persons who share values and beliefs and who engage in learning from one another actively, such as educators from learners, learners from learners, and learners from educators" (Blonder, 2014, para. 1). This community of learners "creates a learner-centred environment in which learners and educators or teachers actively and intentionally construct knowledge together" in a social learning setting (Blonder, 2014, para. 1). The PSTs worked together in small groups and class discussions using the think, pair, and share strategy throughout the research meetings. Each PST's ideas were respected during small group discussions, which were followed by group presentations and class discussions.

A community of learners is "connected, collaborative, and supportive" (Blonder, 2014, para. 2). As part of this community during research meetings, a teacher educator and the PSTs shared learning “resources and points of view while maintaining a mutually respectful and shared environment” (Blonder, 2014, para. 2). The research activities also fostered a community of learners.

During the research meetings, all of the PSTs were given an opportunity to participate in the learning process as a community of learners in order to promote diverse abilities. Specifically, a community is identified by feelings of safety among individual learners, as well as a readiness to respond, ask questions, and make mistakes (Ball & Bass, 2000; Ben-Hur, 2006).

A community of learners is supportive because it encourages individual learners so that they are more willing to continue learning when they are challenged or confused (Ball & Bass, 2000; Ben-Hur, 2006). Confusion is a state due to the “shift of operation from the plane of action in which difficulties interfere with the plane of thought” (Ben-Hur, 2006, p. 14). During the research meetings, confusion was evident when the PSTs said the right things but failed to enact them, or did the right things but failed to explain them. For instance, watching a video clip about *pi* and its value in the Day One meeting, the PSTs noticed that the value of *pi* is approximated as being equal to 3.14 and not $\frac{22}{7}$ as they had originally understood. This confusion was treated in the community as an instructional tool. In the small groups I asked the PSTs to find the value of *pi* using a circular object. I took their confusion into consideration as I helped them to improve their understandings of the value of *pi* as an irrational number (3.14 ....), which is approximated as being equal to 3.14. In this way, interactions with more knowledgeable others in the community helped to improve their understanding.

As a community of learners in the research meetings, the PSTs and I were collaborators working to create a conducive learning environment. This was important for the PSTs’ engagement in learning a concept. As such, I encouraged multiple responses from the PSTs, and treated them equally, respecting all of their mathematical ideas. All of the PSTs were part of a community of learners in various activities, including when they were developing their lesson plans, practicing teaching *pi* as a concept, completing group work, and engaging in class discussions.

The CRI approach helped the individual PSTs to belong to a community of learners in the research meetings. The PSTs worked in small group and class discussions and group presentations throughout the research meetings. The CRI activities also helped the PSTs to become a community of learners throughout the meetings. Each activity was designed to give the PSTs an opportunity to connect and collaborate with others while working on the activities using the think, pair, and share strategy.

As a teacher educator, I learned a need to provide an opportunity for PSTs to act as teachers as well as learners in the university mathematics classrooms. This opportunity for PSTs can occur through designing and implementing the research activities, such as that ask PSTs to develop a lesson plan for teaching school mathematics using textbooks and syllabi and then perform micro-teaching. Also, the research activities should promote collaborations among PSTs in the classrooms through using various strategies such as think, pair, and share followed by group presentations and class discussions. In doing so, the university mathematics classrooms can become a community of learners.

Local Resources as a Tool to Mediate Learning

Local resources as a tool to mediate learning occurred through analysing data gathered during exploring the implementation of CRI with university PSTs in the research meetings. This emerged through all five components (meaning, decontextualization, recontextualization, realization, and practice). I used local circular materials as teaching aids in the design and implementation of CRI activities throughout the research meetings.

This research project contributes to the work of Ben-Hur's CRI approach by presenting how local resources were used to engage PSTs with a mathematical concept taught in schools. Vygotsky (1978) suggests that cultural artefacts should be used to accomplish interactions during the teaching and learning process. In keeping with this suggestion, I had the PSTs use circular-shaped local resources as a tool to mediate learning during the research meetings.

Mathematics should not be understood in terms of abstract ideas and concrete images. It is a cultural aspect as suggested by D'Ambrosio's concept of mathematics and culture—ethno-mathematics. We connect formal and informal mathematics.

Ethno-mathematics focuses on teaching and learning mathematics by relating the subject to natural, social and cultural aspects (D' Ambrosio, 2006). Barton (2004) writes that "Ethno-mathematics is a research programme of the way in which cultural groups understand, articulate, and use the concepts and practices which we describe as mathematical. Whether or not the cultural group has a concept of mathematics" (p. 214).

The function of ethno-mathematics as introduced by D' Ambrosio is to raise awareness of how we come to know and do mathematics by contextualizing the local environment (Rosa et al, 2016). This research study was not an ethno-mathematics study, but it used local resources in the design and implementation of research activities to help PSTs improve their understanding of *pi* and its value.

Using local resources as teaching aids can help PSTs achieve how to design and use them after they complete their university degree programs and have their own classrooms. In this case, teachers and local materials are all resources for teaching and learning a mathematical concept (Adler, 2000). Teachers, as a resource, interact with material objects for conceptual development and understanding of the concept (Adler, 2000). The resources extend not only to material objects but also actions that are important for the improvement of mathematics teachers' education

(Adler, 2000). For instance, instead of using local resources only as teaching aids, the resources were also used by PSTs to demonstrate how to measure the circumference and diameter of a circular object using measurement tools, such as a string and a set of Vernier callipers. In this research project, I used local resources while designing and implementing CRI activities based on *pi* as an example of a scientific concept. The local resources were used for PSTs' conceptual development in research meetings. Physical objects were converted from immediate to dynamic objects (Dubinsky, 1991; Sierpinska, 1994; Tall, 2013), while the PSTs were learning a concept of *pi* through CRI. For instance, local resources were used as instructional tools to develop PSTs understanding of how to estimate the value of *pi* using the Archimedes principle.

The CRI activities focused on the meaning, decontextualization, recontextualization, realization, and practice of a concept. These aspects helped the PSTs to relate local resources to teaching situations. The research activities also helped the PSTs to relate a mathematical concept to their local environment. Figure 5.3 shows an instructional design map that addresses the ways in which local resources were used for the conceptual development of *pi* and its value in the research meetings.

Paris (2012) writes about the potential of using local resources to sustain cultural pedagogy for teaching and learning mathematics. He argues that:

> The term culturally sustaining requires that our pedagogies be more than responsive of or relevant to the cultural experiences and practices. It requires that they support [learners] in sustaining the cultural and linguistic competence of their communities while simultaneously offering access to dominant cultural competence. (p. 95).

This is important because mathematics is a cultural phenomenon which consists six mathematical activities, namely, counting, locating, measuring, designing, playing, and explaining (Bishop, 1988). As such, I argue that using local resources as a tool to mediate learning in Tanzanian university mathematics classes contributes to the content and to PSTs' pedagogical practices.

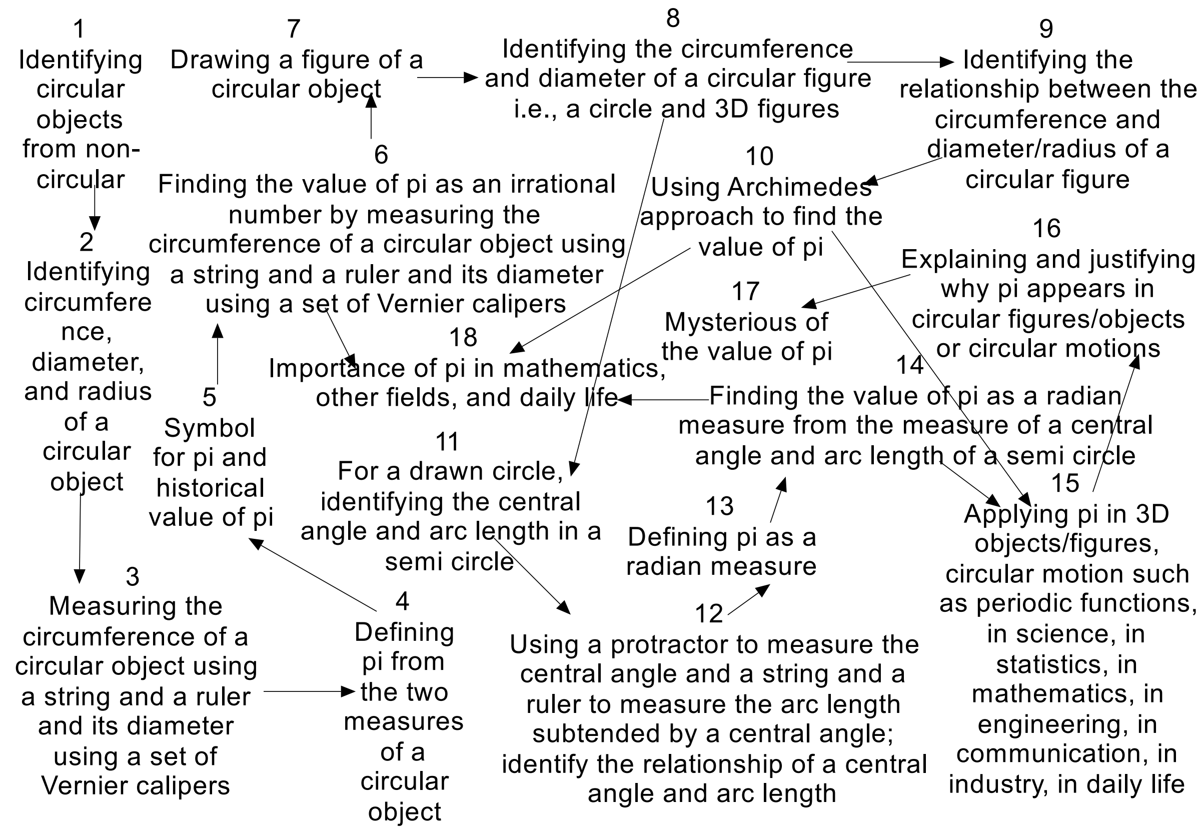

Figure 5.3. Instructional design map that addresses the ways local resources were used for the conceptual development of *pi* and its value in the research meetings.

The numbers in Figure 5.3 represent the stages for conceptual development of *pi* using local resources. We start with number 1 by identifying the circular from non-circular objects to number 18 by understanding the importance of *pi* in mathematics, other fields, and daily life. However, there are stages which give more than one options of what to teach and learn. For instance, in number 6 about finding the value of *pi* as an irrational number by measuring the circumference and diameter of a circular object using measurement tools. We can start either teaching learners by following number 7, which is about drawing a figure of a circular object or by following number 18, which is about importance of *pi* in mathematics, other fields, and daily life.

Designing research activities using local resources involves teachers' ability to focus on the kinds of mathematical concepts that must be taught. In designing the research activities, teacher educators/instructors need to ask themselves, how and to what extent can we improvise the local resources for teaching PSTs? The designed research activities should focus on various processes to mediate learning, including explaining, representing and building connections, relating mathematics to real-life situations, engaging PSTs to practice teaching using local resources, and making explicit connections to the work of teaching and learning mathematics.

Based on the CRI activities, the PSTs responded that local resources are important for teaching and learning mathematics. They suggested using local, circular-shaped materials in class for conceptual development and understandings. For instance, G1 proposed using:

> A piece of watermelon so that you can measure circumference and diameter. Logs—for easy measurement of [their] circumference and diameter to get a constant number π. Sugarcane stem—as you know we have used it in these sessions to find the value of *pi*. Sunflower —as we saw in the previous session, [a] sunflower is in a form of circular. We have to use it so that we can improve teaching and learning.[31] (Transcribed data).

Two aspects of CRI in particular, realization and recontextualization of a concept, helped the PSTs during class discussions to learn the importance of using local resources in mathematics teaching and learning.

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[31] Nodes\\Nodes from the Four-day CRI\\Recontextualization\Using the concept to connect new experiences with past or concurrent experiences (11)

As a teacher educator, I learned a need to use local resources as teaching aids for teaching mathematical concepts in the university mathematics classrooms. This is important to help PSTs relate local materials in the classrooms, as well as to relate the concepts with real objects available in their local contexts. In the future, these PSTs will be aware of the ways that local materials can be used as teaching aids to teach school mathematics in their own classrooms.

A Better Understanding of Participatory Approaches and Social Constructivism in the University Mathematics Classrooms

As a teacher educator and through this study, I gained a better understanding of participatory approaches and social constructivism (Vygotsky, 1978) in teaching mathematics. Before exploring CRI, I used a lecture approach to deliver lessons in the university mathematics classrooms most of the time. I had not thought much about learning theories related to my teaching, I'd largely experienced DI approach, and most often a lecture approach, throughout my school experiences.

Social constructivism (Vygotsky, 1978) was used as a principle in the research meetings to foster PSTs' understanding of *pi* and its value and how to teach it. Throughout the meetings, I used the think, pair, and share strategy followed by group presentations and class discussions. The paragraphs below present the ways that this theory was used in the meetings.

Scaffolding was used to help PSTs develop their understanding of *pi* and its value. For instance, after watching a video clip, in Day One meeting, PSTs realized that the number of diameters that goes around the circumference of a circle is three and remained about 0.14 to complete the circumference of the circle. But, they had not learned this idea before participating in the research meetings. To foster their understanding, I asked PSTs in small groups to take a circular object and measure its circumference and diameter to validate the idea. The PSTs counted the number of diameters that goes around the circumference of a circular object. They were able to validate this argument from the video clip by experimenting themselves through measuring the circumference and diameter of a circular object using a string and a set of Vernier callipers.

There were collaborations among PSTs in small groups during group work activities. For instance, PSTs in small groups (G1, G2 and G3) worked together collaboratively in various group work activities, including measuring the circumferences and diameters of the circular objects, computing the values of *pi*, designing a lesson plan, performing micro-teaching for teaching *pi*, organizing ideas on the flip charts after group work discussions, and fixing their work on a wall for group presentations.

As a part of the more knowledgeable others (MKOs), I facilitated learning on how to obtain the value of *pi* using Archimedes approach. PSTs in small groups (G1, G2, and G3) worked together on the designed research activities. The activities occurred on Day Two meetings. As a researcher of CRI, I needed to have enough mathematical knowledge in order to correct systematic errors that the PSTs had about the value of *pi* before participating in the research meetings. Also, to respond to emerging questions in the research meetings that needed only the attention of the researcher. For instance, I used a scaffolding strategy to help PSTs correct the systematic error of the value of *pi*, by asking them to use various circular objects to find the value.

To foster social interactions (SI) and learning *pi* and its value, two languages, namely, Kiswahili and English were used during group work discussions. These two languages were clearly investigated in Group three (G3) when I was listening to the audio taped data, the group members were using Kiswahili more than English in group work discussions, including designing lesson plans for teaching *pi.* PSTs in Groups one (G1) and two (G2) were using English more than Kiswahili in their group work discussions. However, during group presentations, demonstrations, and class discussions only English language was used.

As a teacher educator, I developed a better understanding of participatory approaches and social constructivism in the university mathematics classrooms. This is important to help PSTs socially construct knowledge through working together in small group and class discussions.

Closing Comments on the CRI Approach for Deepening PSTs' Understanding of a Concept

This chapter presented seven lessons that I learned through exploring the implementation of CRI in Tanzania. These lessons are systematic errors, multiple interpretations, open definitions, PSTs getting an opportunity to teach a mathematical concept and seeing how the concept is integrated into Tanzanian mathematic curriculum, local resources as a tool to mediate learning, community of learners in Tanzanian PSTs' education, and a better understanding of participatory approaches and social constructivism in university mathematics classrooms.

CHAPTER 6

A SUMMARY OF THE RESULTS, POTENTIAL OF CRI, RECOMMENDATIONS, AND CONCLUDING THOUGHTS AND IMPLICATIONS OF THE CRI APPROACH

This chapter provides a summary of the results, potential of concept-rich instruction (CRI), recommendations for practice and future research, and concluding thoughts and implications of the CRI approach in a Tanzanian context. The sections below describe the three aspects of the reflection.

A Summary of the Results

In this study, during four one-day meetings, pre-service teachers (PSTs) used the CRI approach to learn some mathematics related to their work of teaching. I used this instructional approach to engage the PSTs with the mathematical concept of *pi*, which is taught at primary and secondary schools in Tanzania. I used the CRI approach to give the PSTs an opportunity to gain knowledge and skills to teach a concept using participatory learning approaches.

Specific Research Question 1: How does the CRI reveal the way that Tanzanian university PSTs express their understanding of a mathematical concept?

- The PSTs expressed their understanding of the mathematical concept of *pi* by de-contextualizing, realizing, defining, and re-contextualizing it.
- The interpretation of CRI practice gave the PSTs an opportunity to express their understanding of *pi* as a concept to be taught using participatory learning approaches.
- The PSTs learned unfamiliar ideas about *pi* and its value through working on the research activities.
- The CRI approach helped the PSTs to relate *pi* to local resources available in their daily environment.

- The PSTs eliminated systematic errors about *pi* and its value.
- The PSTs developed multiple interpretations of the definitions of *pi* and multiple ways to find its value.
- The PSTs developed open definitions of the value of *pi*, such as it is 3.14 (when approximated to 2 decimal places), and it is an irrational number (3.14 ...).
- The PSTs were able to express their understanding of *pi* and its value in small group and class discussions and group presentations in a collaborative, respective, and supportive way.

Specific Research Question 2: How do PSTs plan to teach a concept, given their participation in CRI?

- The PSTs practiced teaching a concept through developing their lesson plans and performing micro-teaching using participatory learning approaches.
- The PSTs realized how *pi* is articulated and developed across Tanzanian mathematics curricula, including in syllabi and textbooks.
- The PSTs used local materials to design and implement lessons for teaching *pi*.

Potential of the CRI Approach

The CRI was useful for university PSTs in the research meetings, because it fostered a participatory way of learning of a mathematical concept, and it provided conceptual knowledge and skills for PSTs to develop learners' mathematics literacy in Tanzania. The potential of the instructional approach was demonstrated in the following ways. First, it revealed the PSTs' expressions of their understanding of a mathematical concept in a variety of ways both in mathematics and beyond, in the Tanzanian mathematics curricula, and in life outside of school. Second, through the instructional approach, the PSTs were able to learn in research meetings about the mathematical concept of *pi* by exploring, designing lesson plans, and performing micro-teaching a concept using participatory approaches to learning. Third, the instructional approach eliminated PSTs' misconceptions and systematic errors about a concept and its value. Fourth, through the instructional approach, the PSTs were actively engaged. Finally, the instructional approach gave the PSTs an opportunity to develop a community of learners.

Based on the findings, CRI is promising as a potential instructional approach for university PSTs in their mathematics classes in a Tanzanian context. As such, it is suggested that CRI can be useful in Tanzanian primary and secondary schools and teacher colleges in the following ways. First, it can be used to develop learners' understanding of the mathematical concepts. Second, it can be used to foster leaning achievements. Third, it can be used to assess on-going learning of the concepts. Finally, it can be relevant in engaging learners with the concepts in a participatory way of learning.

Recommendations

Recommendations are addressed to the teacher educators at the Tanzania Institute of Education (TIE) and Tanzanian universities. I include myself in the latter cohort, as I am also a Tanzanian university educator. The aforementioned institutions are responsible for designing the curriculum for lower and higher learning institutions in Tanzania. At the same time, teacher educators and instructors are responsible to implement the curriculum for PSTs.

Recommendations for TIE and Tanzanian Universities

TIE and Tanzanian universities could design a curriculum that emphasizes using local resources for the development and understanding of mathematical concepts. This direction is important to help learners at schools and PSTs realize how to relate mathematical concepts to their daily life practice and also to realize how to use local resources in academic settings.

TIE could develop a curriculum to help learners in primary and secondary schools deepen their understanding of mathematical concepts including *pi*. This curriculum could include an approach such as the Archimedes approach for better approximation of the value of *pi* as an irrational number. Also, Tanzanian universities could design the curriculum for PSTs focused on the school mathematics that they will teach. This is important to help university mathematics PSTs gain mathematical knowledge and skills related to their teaching while they are still at universities. This is likely to help them to become better teachers in their own classrooms.

Recommendations for Teacher Educators/Instructors

This study gives recommendations to teacher educators (such as myself) and instructors to learn about CRI and to then, design and implement CRI for mathematics PSTs. This research recommends further studies on CRI with various concepts at other levels of education, such as primary and secondary schools, and teacher colleges in Tanzania. This is important based on the long existing challenges to mathematics teaching and learning in Tanzania (UNESCO, 2018). The current Tanzanian Prime Minister, Kassim Majaliwa, also emphasizes the need to utilize research that has been done in the country and is on-going to address the challenges in the university and college curricula across all fields (Mwananchi, 2019). This study addresses such challenges and suggests what can be done to improve learning achievements in mathematics. The same approach can be applied in other subjects such as chemistry and physics, which involve mathematical concepts.

Recommendations for Grounding Teaching in Social Constructivism for University Mathematics Classes

Instructors have been using social constructivism with university mathematics PSTs outside of classes through group work assignments and project-based learning. The current study emphasizes employing social constructivism to teach mathematics-for-teaching to university

PSTs in the classroom. A possibility for doing this could be to use social constructivism (Vygotsky, 1978) to frame the teaching and learning process in university mathematics classes. This is important to increase PSTs' participation in learning a mathematical concept, give PSTs an opportunity to reflect on ongoing learning of a mathematical concept, foster social interactions among PSTs, use local resources as a change agent for Tanzanian university mathematics classes for PSTs' conceptual development and understanding, and deepen PSTs' understanding of a mathematical concept during small group and class discussions and through collaboration with more knowledgeable others within the zone of proximal development.

Concluding Thoughts about and Implications of the CRI Approach in a Tanzanian Context

Teaching Roles in the University Mathematics Class

Because of a CRI approach, PSTs perceived the opportunity to perform micro- teaching in their university mathematics class as important. PSTs improved their knowledge and skills through performing micro-teaching. While performing micro-teaching, they used participatory approaches, such as small group discussions and demonstrations.

PSTs' Roles in Curriculum Development

After participating in the research meetings and learning about CRI, some PSTs were ready to suggest how *pi* should be articulated and developed across Tanzanian K-University mathematics curricula. For instance, the PSTs requested that *pi* be treated as a topic with detailed information in the curricula to help learners at schools develop a better understanding, and for teachers to understand how to teach it using participatory approaches. TIE might consider studying the needs of PSTs as part of curriculum development.

Implications of the CRI Approach

With the CRI approach, the teacher educator can also be a learner. In the research meetings, I was a learner during small group discussions and presentations. This situation created the learning environment, which was learner-centred, because I focused on what the PSTs discussed in small groups based on the designed research activities. A teacher educator who uses a CRI approach in the class becomes a learner to foster meaningful learning among PSTs.

The CRI approach gave an opportunity for the PSTs' voices to be heard in the research meetings. During four one-day meetings, the PSTs worked together on research activities in small groups. They wrote their mathematical ideas on a flip chart after every small group discussion. After that, each group presented its work followed by a class discussion. In addition, the PSTs designed lesson plans. After that, they practiced teaching a concept. In this case, the PSTs were teachers. Engaging PSTs with a concept to discuss their mathematical ideas and present their thoughts followed by class discussions helped them to elaborate on and ponder about their ideas in the research meetings. In doing so, the PSTs' voices were prevalent in the meetings.

In CRI classes, PSTs should be given an opportunity to reflect on their understanding of a concept. Through reflections, they can realize what they have learned in the research meetings, including mathematical ideas, surprising ideas, and unfamiliar ideas about a concept. These reflections help PSTs to realize the on-going learning of a concept and inform the way that the next sessions should proceed to foster a continuation of this learning experience.

Using a CRI approach, the teacher educator should consider PSTs' "misconceptions as a psychological condition" in order to eliminate them (Ben-Hur, 2006, p. 70). In mathematics classes, a teacher educator should provide an opening activity for small group discussions. The intention is to determine the PSTs' initial understanding of a mathematical concept. A variety of ideas emerge from small group discussions; the mathematical ideas might be incomplete. When these ideas emerge, the teacher educator should consider the PSTs' ideas about a concept to eliminate their misconceptions by using an appropriate instructional strategy.

Although CRI is learner-centred, teacher educators are not powerless in a class, they are actively engaged in the teaching and learning: they design the research activities and sessions before going to deliver the lesson in the class. While in the class, PSTs work on the designed activities in small groups, followed by group presentations and class discussions. This allows the teacher educator to easily observe or investigate how the instructional process is proceeding.

Concluding Thoughts on the CRI Approach in Tanzanian Mathematics Classes

Based on the findings about CRI, it is suggested that there is a need to consider this approach to teach mathematics to university PSTs. This will help in designing the coursework and activities that can empower these PSTs to be able to teach mathematics using participatory approaches after the completion of their university degree programs. For instance, I propose to design the coursework and activities focused on mathematics-for-teaching for PSTs' undergraduate degree programs while emphasizing using local materials as teaching and learning resources.

Designing the coursework and activities focused on mathematics-for-teaching is potentially useful to PSTs in their university mathematics classes. I believe that PSTs should learn mathematics related to their teaching work in a participatory way during each semester of their degree program.

From the findings about CRI, I believe that teaching mathematics for Tanzanian university PSTs should emphasize:

- Designing and implementing classroom activities based on the five components of CRI. This will help to improve learners' development of their understanding of the mathematical concepts and achieve desirable learning outcomes.

- Empowering PSTs by giving them an opportunity to construct their knowledge and elaborate their understandings through interactions with knowledgeable peers and instructors. This brings meaningful learning.
- Improvising teaching and learning resources, such as with local materials that are available in PSTs' daily environment. This can foster learning by relating mathematical ideas in academic settings.
- Implementing this study to teach and learn *pi.* CRI can be used for teaching mathematical concepts within all Tanzanian university mathematics programs.
- Further research should be done about CRI in the university classes for PSTs such as chemistry and physics.

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APPENDIX A

DEMOGRAPHIC INFORMATION SHEET

Attendance sheet for the participants in the four one-day research meetings

S/N	Degree Program	Year of Study	Subject Specialization	Teaching Experience	Gender	Age	Signature

APPENDIX B

FOUR-DAYS OF RESEARCH ACTIVITIES

The Four One-Day Research Meetings with Tanzanian Pre-service Teachers

From December 3, 2016 to December 24, 2016

Table 1

Daily Session Structure

Timeline	Session
9:00 – 11:00	Session one
11:00 – 11:30	Morning tea
11:30 – 1:30	Session two
1:30 – 2:30	Lunch
2:30 – 4:30	Session three

Daily Session Topics

Day One: Developing the meaning of the concept of *pi.*

Day Two: Decontextualization and realization of *pi*.

Day Three: Re-contextualization and realization of *pi.*

Day Four: Practices.

Main Objectives of the Four One-Day Research Meetings

At the end of the four one-day meetings, I expect to understand the ways CRI approach reveals pre-service teachers' expressions of their understanding of a mathematical concept. In particular, the objectives of these research meetings explore:

- The ways CRI approach reveals Tanzanian university pre-service teachers' expressions of their understanding of *pi* when participating in the four one-day research meetings
- The ways pre-service teachers plan to teach *pi,* given their participation in CRI

Summary

Key points

Pi is a concept, which is potentially useful in mathematics and other fields, including sciences. *Pi* distinguishes itself from all other concepts and numbers, because it is connected to cycles. Furthermore, *pi* is important in the areas of mathematics across K—University mathematics curricula. Despite the fact that *pi* is applied in the areas of mathematics, the concept has practical implications in our daily life. In these four one-day meetings, *pi* and its value will be discussed within mathematics classroom; outside mathematics classroom; and other fields, such as sciences, and connect a concept in our daily life practice through CRI approach. The intention is to understand deeply the concept across Tanzanian primary, secondary, and university mathematics curricula, and beyond Tanzanian K—University mathematics curricula.

Background

Considering most of the content taught in mathematics curricula, there has been a debate about a concept of *pi* since 3000 B.C. In these four one-day meetings, *pi* and its value will be discussed within and beyond Tanzanian K—University mathematics curricula through discussion and elaboration of our understandings of a concept in a social learning environment.

Day One – Developing the Meaning of the Concept of *Pi*

Materials

Flip charts, markers

Rulers

Pencils

Mathematical sets

Circular objects

Video clip

Plane papers

Glue paper

A calculator

Thread/string

Rubber bands

Computer and power point projector

Session One Activities

Opening activities

i. What is *pi*? (Think, pair, and share)

ii. When I say the word *pi*, what comes to mind for you? Think, pair, and share.

In a group discuss and come to an agreement, and then write your answer(s) on chart paper. Post the chart paper on the wall for presentation (30 minutes group work, 10 minutes for group presentation followed by 10 minutes class discussions).

Main activity

Watch a video clip (10 minutes watch) about *pi* with the link:

https://www.youtube.com/watch?v=cC0fZ_lkFpQ

From the video clip, what can you say about *pi*? Conversations about *pi* after video clip plays (30 minutes).

Reflection for session One Lesson

What comes in your mind about *pi*? Please explain for me (10 minutes).

Session Two Activities

History of *Pi*:

i. Watch a video clip (10 minutes watch) about *pi* with the link: https://www.youtube.com/watch?v=mZ4CP0vTgEE

From the video clip, what can you say about the history of *pi*? Conversations about *pi* after video clip plays (30 minutes).

ii. The ratio of the circumference of a circle to its diameter, which results to a constant value of *pi* that has been known to humanity since ancient times; yet, even today, despite 2000 years of theories, calculations, thoughts and proofs, the precise value of *pi* remains difficult to find (see http://www-groups.dcs.st-and.ac.uk/history/HistTopics/Pi_through_the_ages.html; http://www.todayifoundout.com/index.php/2014/07/history-pi/; http://www.ualr.edu/lasmoller/pi.html; and https://www.math.rutgers.edu/~cherlin/History/Papers2000/wilson.html). Read a summarized information in Table 2 below about the historical value of *pi.*

Table 2

Chronicle Historical Value of Pi

Year (around)	Name (s) of Scholar(s)	Value of *Pi* approximated	Place
3000 BC		$\frac{22}{7}$	Egypt
1706	William	$\pi \cong \frac{22}{7}$ (Greek	Egypt

BC	Jones	letter ("π") was first used by Jones, W)	
240 BC	Archimedes	$3\frac{10}{71} < \pi < 3\frac{1}{7}$	Greece
213 BC	Chang T'sang	3	China
30 BC	Vitruvius	$3\frac{1}{8}$	Italy
AD 125	Ptolemy	$3\frac{17}{120}$	Greece
AD 480	Tsu Chung-Chih	$\frac{355}{113}$	China
1150	Bhaskara	$\frac{3927}{1250}$	India
1873	William Shanks	π was computed to 707 places	England
1913	Ramanujan	$\pi \approx 3.141592652 \ldots$	India
1949	Eniac	$\pi$was computed to 2035 decimal places using electronic calculator	USA
1961	Wrench	π was computed to	USA

	and Shanks	100265 decimal places using IBM 700	
2005	Dave Anderson	The first 200 million digits of the value of *pi* was calculated	USA

From this traced history, what do you notice about the value of *pi*? Think, pair, and share, followed by presentations (20 minutes group work, 10 minutes presentation, 10 minutes class discussion).

Reflection for Session Two Lesson

How would you describe the value of *pi*? Please explain your answer(s) (10 minutes).

What is the symbol for *pi*? Please explain your answer (5 minutes).

Session Three Activities

a. The Differences between Equal Sign and Approximated Sign

What is the difference(s) between equal sign (=)and approximated sign (≅)? Think, pair, and share.

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (20 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

b. Proper Use of Equal Sign and Approximated Sign

What is the proper use of an equal sign (=) and approximated equivalent sign (≅)? Think, pair, and share. In a group discuss come to an agreement and then write your answer on a chart paper. Post the chart paper on the wall for presentation (15 minutes group work, 5 minutes for group presentation, followed by 5 minutes class discussion).

c. Finding the Value of *Pi* Using Local Materials/Real Objects

How can you find the value of *pi* using real objects/local materials? (Think, pair, and share; and demonstrate)

Anticipated knowledge: Drawing and naming the circular object on chart paper.

In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for group presentation and demonstration (15 minutes for group work, 5 minutes for group presentation, 5 minutes for demonstrations, and 10 minutes for class discussions).

Reflection for Session Three Lesson

What is the difference(s) between an equal sign (=) and approximated equivalent sign (≅)? Please explain your answer(s) (5 minutes).

What is the proper use of an equal sign (=), approximated equivalent sign (≅)? Please explain your answer(s) (5 minutes).

Final Reflection for Day One Meeting

What surprised you today about *pi* as a concept? Please explain for me (5 minutes).

What have you understood today that you did not understand before about *pi*? Please explain for me (5 minutes).

What have you not understood today about *pi* as a concept? Please explain for me (5 minutes).

What was unfamiliar to you today about *pi* as a concept? Please explain for me (5 minutes).

Day Two –Decontextualization and Realization of *Pi*

Materials

Tanzanian primary, secondary school and university mathematics curricula, including textbooks and syllabuses.

Plane papers, flip charts, markers, sticky notes, glue paper, circular objects, a calculator

Rulers

Pencils

Mathematical sets

A set of Vernier callipers

Circular objects

A folded paper

A pen

Thread/string

Session One Activities

Opening Activity

From the given envelope in each group, sort the strategies for finding the value of *pi* using local materials/objects used from Day One activities. Each group decide the strategies for finding the value of *pi* on a chart paper for presentation (Think, pair, and share).In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for group presentation (10 minutes).

Main activities

a. Demonstration how to find the value of *pi* using circular local materials/ real objects, a set of Vernier callipers, a string and a ruler.

   Anticipated knowledge: Measuring and drawing the circular objects.

Group work activities followed by class discussions

i. Measure the diameter of a circular object using a set of Vernier callipers. Record the answer on the manila sheet provided. What is the best way for measuring the diameter of the circular object? Explain your answer.

ii. Measure the circumference of a drawn circular object on a plane paper or manila sheet using a string and a set of Vernier callipers. Record the answer on the sheet provided.

iii. Find the ratio of the circumference with that of a diameter. Record your answer on the sheet provided. What do you notice(s)? What does this ratio imply? Explain your answer (s).

In a group discuss and come to an agreement, and then write your answers on a chart paper. Post the chart paper on the wall for group presentation (13 minutes group work, and 5 minutes presentation, 5 minutes class discussions).

b. Using Archimedes Approach for Better Approximation of the Value of *Pi*

Anticipated knowledge: Measuring the diameter and circumference of a circular object, location of the centre of the circular object, drawing a circle, meaning and drawing of a polygon, namely, inscribed and circumscribed polygons, meaning of a perimeter of the polygon and finding the perimeter of the polygon, and meaning and drawing of a tangential line.

Opening activities: Group work activities

i. Get a circular object, and a length of a string. How many diameters are needed to go around the circumference of the entire object? Describe the process you would use to check your conjectures (Evan, n.d.).

ii. How accurate is your experiment? Describe a process you could use to be as accurate as possible in finding out how many diameters are needed to go around the circumference of the entire circular object (Evan, n.d.).

iii. If you increase the radius of a circular object by one, what will happen to its diameter and circumference, and to the ratio of the circumference of the circular object to its diameter and why? (Evan, n.d.)

iv. If you increase the diameter of a circular object by one, what will happen to the circumference of the object and the ratio of the circumference of the object to its diameter and why? (Evan, n.d.).

v. If you increase the circumference of a circular object by one, what will happen to the diameter and the ratio of circumference of the object to its diameter and why? (Evan, n.d.)

In a group discuss and come to an agreement, and then write your answers on a chart paper. Post the chart paper on the wall for group presentation (14 minutes group work, and 5 minutes presentation, 5 minutes class discussions).

Main activities: Group work activities

i. Get a circular object, measure its diameter using a set of Vernier callipers. Record its diameter on a sheet.

ii. Take a string and mark the measured size of the diameter. After marking the string, lie the string on a folded paper by marking on the folded paper where the size of a diameter ends. Mark the centre on the folded paper. A folded paper should mark two equal halves of the diameter of the circular object in such a way that when you measure the marked centre to either end of the marked paper gives a radius of a circular object.

iii. Take a pencil and a pen, put a pencil at the centre of the folded paper, and a pen at one end of the marked folded paper. Fix part of the folded paper with a pen while drawing a circular diagram with a marked folded paper with a pencil on a manila sheet/plane paper. A drawn diagram yields a circle.

iv. Measure the circumference of the drawn circle using a string and a set of Vernier callipers. Record the value on a sheet.

v. Use similar folded paper, a pencil and a pen you have used in ii above to draw three different inscribed and circumscribed polygons. Record the number of sides, length of one side and perimeter of both inscribed and circumscribed polygons, and the number of diameters in the perimeter of a polygon. Use the tables below to complete the activities. Note that: Each group describe about what you expect to do for the completions of Table 3 and Table 4 below, before you start doing the main activities from iv above.

Table 3

Inscribed Polygons

Type of Polygon	Number of Sides	Length of One Side	Perimeter of Polygon	Diameter of the Circle	Number of Diameters in the Perimeter of the Polygon

Table 4

Circumscribed Polygons

Type of Polygon	Number of Sides	Length of One Side	Perimeter of Polygon	Diameter of the Circle	Number of Diameters in the Perimeter of the Polygon

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for group presentation and demonstration (15 minutes for group work, 5 minutes presentations, 5 minutes for demonstrations, and 5 minutes class discussions).

vi. What do you notice for the number of diameters in the perimeter of the inscribed and circumscribed polygons using Table 3 and Table 4 above? Explain your answer(s). Think, pair, and share.

In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for presentation (5 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

c. Concept(s) That We Need Know in Order to Calculate the Value of *Pi* as Approximated Equivalent to 3.14

What concepts do we need to know in order to calculate the value of *pi* as an approximated equivalent to 3.14? (5 minutes group work followed by 5 minutes presentation).

Step 1 – In your group create a place mat by drawing a large rectangle in the centre of a manila sheet of a paper. Divide the area around the large rectangle into sections, one for each group member.

Step 2 – Individually each group member records a list of concepts that are needed in order to calculate the value of *pi* as approximated equivalent to 3.14 in his or her own section outside the large rectangle.

Step 3 – Individually each person reads their list to the group.

Step 4 – Create themes, as a group, to categorize the different lists of the activities. The themes are written inside of the large rectangle.

Reflection for Session One Lesson

What is the value of *pi*? Please explain your answer (3 minutes).

What concepts do we need to know in order to calculate the value of *pi* as an approximated equivalent to 3.14? (5 minutes).

Session Two Activities

Transition from *pi* as a value to *pi* as an angle measure: We have already discussed *pi* as a value, which is approximated equivalent to 3.14. In this section, we are going to discuss about *pi* in the notion of a radian measure.

Opening Activities

i. What comes in your mind when we write $\pi \cong 3.14$ and $\pi\ rad \equiv 180^0$? Explain your answer(s). Think, pair, and share.

ii. What is a radian? (Think, pair, and share).

iii. What is 1 radian?(Think, pair, and share).

iv. What is the relationship between radian and angle measure?(Think, pair, and share).

v. Convert the following angles in degrees into angle in radians. What do you notice(s) through this process about the value of *pi*? Explain your answer(s) (Think, pair, and share).

Note that in this activity, a turn is defined as a unit of the plan angle measurement that is equal to 360 degrees or 2π radians. Thus, a full circle corresponds to a full turn (one turn), which is equal to 360 degrees or 2π radians. In addition, a turn can be subdivided in many different ways, including quarter turn, half turn, and twelfth turn.

vi. From v. above, what is *pi*? Explain your answer(s) (Think, pair, and share).

In a group discuss and come to an agreement, and then write your answers on a chart paper. Post the chart paper on the wall for presentation (20 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

Main Activities

Group Work Activities

a. Finding the Value of *Pi* as a Radian Measure

Anticipated knowledge: meaning of radians and degrees, and measuring an angle using a protractor.

i. Get a circular object, measure its diameter using a set of Vernier callipers. Record its diameter on a sheet.

ii. Take a string and mark the measured size of the diameter. After marking the string, lie the string on a folded paper by marking on the folded paper where the size of a diameter ends. Mark the centre on the folded paper. A folded paper should mark two equal halves of the diameter of the circular object in such a way that when you measure the marked centre to either end of the marked paper gives a radius of a circular object.

iii. Take a pencil and a pen, put the pencil at the centre of the folded paper, and the pen at one end of the marked folded paper. Fix part of the folded paper with the pen while drawing a circular diagram with a marked folded paper with the pencil on a manila sheet/plane paper. A drawn diagram yields a circle.

iv. Using a protractor measure different angles around the drawn circle in iii above and locate the angles around the circle into both degrees and radians.

v. What do you notice(s) through the process in (iv.) above? Explain your answer(s).

vi. What is the implication(s) of the process in iv above in relation to the value of *pi*? Explain your answer(s).

vii. From vi above, what is the value of *pi*? Explain your answer(s).

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for group presentation and demonstration (20 minutes for group work, 5 minutes presentation, 10 minutes demonstrations, and 5 minutes class discussion).

viii. From the experiment process in a. above, give the meaning(s) of *pi*? Think, pair, and share.

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (10 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

b. Concept(s) that we need to know in order to Calculate the Value of *Pi* as a Radian Measure

What concepts do we need to know in order to find the value of *pi* as a radian measure? (10 minutes group work followed by 5 minutes presentation, and 5 minutes class discussion).

Step One – In your group create a place mat by drawing a large rectangle in the centre of a manila sheet of paper. Divide the area around the large rectangle into sections, one for each group member.

Step Two – Individually each group member records a list of concepts that are needed in order to calculate the value of *pi* as a radian measure equivalent to 180^0 in his or her own section outside the large rectangle.

Step Three – Individually each person reads their list to the group.

Step Four – Create themes, as a group, to categorize the different lists activities. The themes are written inside of the large rectangle.

Reflection for Session Two Lesson

What is the value of *pi*? Please explain your answer (2 minutes).

What concepts do we need to know in order to calculate the value of *pi* as a radian measure? (3 minutes).

How can you find the value(s) of *pi*? Please explain your answer(s) (5 minutes).

Session Three Activities

Group work activities

a. Given the objects. Sort out the objects in which *pi* exists. Draw and name the objects in which *pi* exists. From the drawn and named objects, explain how *pi* is applicable (Think, pair, and share).

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for group presentation (12 minutes for group work, 5 minutes for group presentation, and 5 minutes class discussion).

b. Where is *pi* in your daily environment? Explain your answer(s) (Think, pair, and share).

   In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (10 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

c. Mathematical Concepts That We Need to Know Before and After Learning a Concept of *Pi* Across Tanzanian K—University Mathematics Curricula.

   What mathematical concepts do we need to know before and after learning the concept of *pi*? Across Tanzanian primary, secondary, and undergraduate university curricula. Think, pair, and share; followed by Gallery walk (15 minutes).

   Using the syllabuses and textbooks make list of the mathematical concepts on a chart paper and note the grade level at which the concepts are taught.

d. Meaning(s) of *Pi* Across Tanzanian K—University Mathematics Curricula

   What is *pi*? Write as many different ways as you can. (Discuss the meaning(s) of a concept, its implication(s), and look whether there is any connection among the meanings of a concept). Think, pair, and share.

   Write your ideas on the sticky notes (one idea per sticky note).

   Discuss your ideas with a partner. Sort and classify your ideas into categories: meaning, implication, and connection.

   Table 5

   Meaning(s) of Pi Across the Curricula

Meaning	Implication	Connection

Share your ideas with your group, discuss and come to an agreement of where you will place each sticky note. Post sticky notes on wall in one of the categories (13 minutes group work, 5 minutes presentation, and 5 minutes class discussion).

e. Suggestions of the Ways *Pi* Could be Articulated and Developed in the Tanzanian Mathematics Curricula for Better Understanding of a Concept.

   Suggest the ways *pi* could be articulated and developed in the Tanzanian mathematics curricula for better understanding of a concept. Think, pair, and share.

   Work as a group and then write your answer(s) on a chart paper. Post the chart paper on the wall for group presentation (10 minutes group work, 5 minutes presentation and 5 minutes class discussion).

Reflection for Session Three Lesson

Where is *pi* in your daily environment? Explain your answer(s) (2 minutes).

What mathematical concepts do we need to know before and after learning the concept of *pi?*

Across Tanzanian primary, secondary, and undergraduate university curricula? (4 minutes).

Suggest how *pi* could be articulated and developed in the curricula for better understanding of a concept (4 minutes).

Final Reflection for Day Two Meeting

What surprised you today about *pi* as a concept? Please explain for me (5 minutes).

What have you understood today that you did not understand before about *pi*? Please explain for me (5 minutes).

What have you not understood today about *pi* as a concept? Please explain for me (5 minutes).

What was unfamiliar to you today about *pi* as a concept? Please explain for me (5 minutes).

Day Three – Re-contextualization and Realizations of *Pi*

Materials

Tanzanian primary, secondary, and university mathematics curricula, including textbooks and syllabuses

Rulers

Pencils

Mathematical sets

Manila sheets

Markers

Sticky notes

Glue paper

Real objects

A calculator

Session One Activities

Group work activities

a. Trigonometry: A Unit Circle

Watch the video clip (40 minutes watch) with the link:

https://www.youtube.com/watch?v=593w799sBms

From the video clip:

i. What points can you generate on a circle? (Think, pair, and share).

ii. What can you say about π in a unit circle? (Think, pair, and share).

iii. Why $\pi\ rad \equiv 180\ degrees$? (Think, pair, and share).

iv. If $\pi\ rad \equiv 180\ degrees,$ what is $\frac{\pi}{2}$ rad in degrees? (Think, pair, and share).

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (25 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

b. Draw the graphs of sine, cosine and tangent functions.

i. What do you notice(s) on each of the graphs? (Think, pair, and share).

ii. Why the graph of sine function from a unit circle starts at zero while that of cosine starts at one? (Think, pair, and share).

iii. What can you say about π ? (Think, pair, and share).

iv. What are the properties of sine, cosine and tangent functions? Please explain your answer(s). Think, pair, and share.

In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (30 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

Reflection for Session One Lesson

How isπ connected to a unit circle? Please explain for me (5 minutes).

How is π connected to sine and cosine functions? Please explain for me (5 minutes).

Session Two Activities

Group work activities:

i. Graphs of the sine, cosine, and tangent functions are known as periodic functions. Why do you think they are known as periodic functions? (Think, pair, and share).

ii. What are trigonometric functions? (Think, pair, and share).

iii. Why trigonometric functions are periodic? Explain your answer. Think, pair, and share.

iv. For any angle, t of the periodic function, why $\sin(t)$ is an odd function, but $\cos(t)$ is an even function? Explain your answer(s) (Think, pair, and share).

v. Prove that $\sin(t)\ and\ \cos(t)$ are both periodic functions. What can you say about π ? Explain your answer(s) (Think, pair, and share).

In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for presentation (40 minutes group work, 10 minutes for group presentation followed by 10 minutes class discussions).

i. Define period, T and amplitude, A of a periodic function? Think, pair, and share.

ii. Why the period, T of both cosine and sine functions is 2π? Explain your answer(s). (Think, pair, and share).

iii. What can you say about π in a periodic function? Explain your answer(s). (Think, pair, and share).

Work as a group and then write your answer on a chart paper. Post the chart paper on the wall for group presentation (30 minutes group work, 10 minutes presentation, and 10 minutes class discussion).

Reflection for Session Two Lesson

Why the graphs of the trigonometric functions are called periodic? Explain your answer(s) (10 minutes).

Session Three activities

Group work activities:

a. What are the applications of *pi* in sciences and other fields? Please explain your answers. In a group discuss and come to an agreement, and then write your answers on a chart paper. Post the chart paper on the wall for group presentation (35 minutes group work, 10 minutes presentation and 10 minutes class discussion).

b. Mysterious of the number π

   Why is the number π so mysterious? Think, pair, and share.

   In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for group presentation (25 minutes for group work, 5 minutes for group presentation, and 5 minutes for class discussions).

Reflection for Session Three Lesson

What are the applications of *pi* in sciences? Please explain your answer(s) (5 minutes).

Why is the number π so mysterious? Explain your answer(s) (5 minutes).

Final Reflection for Day Three Meeting

What surprised you today about *pi* as a concept? Please explain for me (5 minutes).

What have you understood today that you did not understand before about *pi*? Please explain for me (5 minutes).

What have you not understood today about *pi* as a concept? Please explain for me (5 minutes).

What was unfamiliar to you today about *pi* as a concept? Please explain for me (5 minutes).

Day Four – Practices

Summary

We have discussed the concept of *pi* within the field of mathematics; other fields, such as sciences; and in our daily life. In this Day Four meeting, we will explore how to develop the lesson plan using local materials and conduct a micro-teaching how to teach *pi* as a concept.

Materials

Tanzanian primary, secondary, and university mathematics curricula, including textbooks and syllabuses

Rulers

Pencils

Mathematical sets

Local materials

Manila sheets

Markers

Session One Activities

Develop a lesson plan on how to teach *pi* as a concept so that learners come to understand it. Use local materials as the teaching and learning resources (group work activities followed by presentation and finally class discussions).

i. Develop the lesson plan for teaching *pi* using local materials as teaching and learning aids so that learners come to understand *pi* as a concept in your own classroom (120 minutes). The lesson plan template for Tanzanian secondary school is adapted and attached with the lesson plan guidelines.

Session Two Activities

ii. Using the lesson plan designed in (i.) above, conduct a micro-teaching how to teach *pi* as concept (total 90 minutes for 3 groups; 30 minutes per group).

iii. From the lesson plan developed and micro-teaching performed by a group

   a. Suggest teaching and learning strategies that could be used to improve their work (Think, pair, and share).

In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for presentation (5 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

b. Suggest teaching and learning aids/resources that could be used to improve their work (Think, pair, and share).

   In a group discuss and come to an agreement, and then write your answer on a chart paper. Post the chart paper on the wall for presentation (5 minutes group work, 5 minutes group presentation followed by 5 minutes class discussions).

Session Three Activities

c. Suggest other local materials that could be used to teach and learn a concept of *pi*. (Think, pair, and share).

   In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for presentation (20 minutes group work, 5 minutes for group presentation followed by 10 minutes class discussions).

d. What else can you put forward? (Think, pair, and share).

   In a group discuss and come to an agreement, and then write your answer(s) on a chart paper. Post the chart paper on the wall for presentation (10 minutes group work, 5 minutes for group presentation followed by 5 minutes class discussions).

e. Write a joint reflection of 3 paragraphs or so describe what was learned from developing the lesson plan, performing the micro-teaching based on the concept of *pi*, and discussing the presented lesson. Append this joint reflection to the lesson plan (20 minutes).

Reflection for Sessions One, Two and Three Lessons

Imagine yourself as a mathematics teacher in your own classroom. How will you introduce *pi* as a concept so that your learners come to understand *pi* as a concept? (10 minutes).

What teaching and learning aids will you use to teach *pi* so that learners come to understand a concept in your own classroom? Please explain for me (7 minutes).

What teaching and learning strategies will you use to help learners understand *pi* as a concept in your own classroom? Please explain for me (8 minutes).

Final Reflection for Day Four Meeting

What surprised you today about *pi* as a concept? Please explain for me (5 minutes).

What have you understood today that you did not understand before about *pi*? Please explain for me (5 minutes).

What have you not understood today about *pi* as a concept? Please explain for me (5 minutes).

What was unfamiliar to you today about *pi* as a concept? Please explain for me (5 minutes).

Appendices

The lesson plan template for Tanzanian secondary school is used.

Lesson plan guidelines

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APPENDIX C

PERMISSION LETTER TO THE DIRECTOR OF TANZANIAN MINISTRY OF EDUCATION, SCIENCE, TECHNOLOGY AND VOCATIONAL TRAINING

University of Alberta

Faculty of Education

Department of Secondary Education

347 Education South, 11210 - 87 Ave

Edmonton, Alberta, Canada T6G 2G5

The Permanent Secretary

Ministry of Education, Science, Technology and Vocational Training

P. O. Box 9121

Dar es Salaam

Tanzania.

Attention: Director, Policy and Planning

Dear Sir/ Madam,

Ref: Research Permit for Emmanuel Deogratias

I write to request the permission from your office to conduct a research study at the University of Dodoma (UDOM), and document analysis of the Tanzanian mathematics curricula with a special focus on the concept of *pi*. In particular, I am interested to conduct the research study with the undergraduate mathematics pre-service teachers in UDOM, and document analysis of a concept in primary, secondary, and university mathematics curricula.

I am a Tanzanian and assistant lecturer at the University of Dodoma, and I am currently enrolled as a PhD Student in Mathematics Education at the University of Alberta in Canada under UDOM/University of Alberta Agreement. The title of my research study is: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case. One of the aspirations of my study is to understand how *pi* is articulated and developed across primary school, secondary school, and teacher's college before implementing concept-rich instruction. In order to meet this aspiration, I need to conduct an in-depth document analysis of the K—University undergraduate mathematics curriculum during my research study before conducting the study at UDOM to explore how concept-rich instruction reveals Tanzanian university pre-service teachers' expressions of their understanding of *pi* when participating in the four one-day meetings.

I wish to assure your office that the findings of this study are purely for my PhD research study and shall not in any way compromise the work of the Tanzanian Ministry of Education, Science, Technology and Vocational Training. For any questions, you are free to contact either me or my supervisor, Professor Florence Glanfield (florence.glanfield@ualberta.ca).

Please find the attached letter from the Project Director at the University of Dodoma of the project about Capacity Development for Mathematics Teachers in Rural and Remote Areas in Tanzania.

Yours Sincerely,

Emmanuel Deogratias, deograti@ualberta.ca

APPENDIX D

PERMISSION LETTER TO THE DIRECTOR OF THE UNIVERSITY OF DODOMA

University of Alberta

Faculty of Education

Department of Secondary Education

347 Education South, 11210 - 87 Ave

Edmonton, Alberta, Canada T6G 2G5

The Director,

University of Dodoma,

P. O. Box 259,

Dodoma,

Tanzania.

Dear Sir/ Madam,

Re: Research Permit for Emmanuel Deogratias

I write to request the permission from your office to conduct a research study at the University of Dodoma (UDOM), and document analysis of UDOM's mathematics curriculum with a special focus on the concept of *pi*. In particular, I am interested to conduct the research study with the undergraduate mathematics pre-service teachers in UDOM, and undergraduate mathematics curriculum.

I am a Tanzanian and assistant lecturer at the University of Dodoma, and I am currently enrolled as a PhD Student in Mathematics Education at the University of Alberta in Canada under UDOM/University of Alberta Agreement. The title of my research study is: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case. One of the aspirations of my study is to understand how *pi is* articulated and developed across primary school, secondary school, and Teacher College before implementing concept-rich instruction. In order to meet this aspiration, I need to conduct an in-depth document analysis of the K—University undergraduate mathematics curriculum during my research study before conducting the study at UDOM to explore how concept-rich instruction reveals Tanzanian university pre-service teachers' expressions of their understanding of *pi* when participating in the four one-day meetings.

I wish to assure your office that the findings of this study are purely for my PhD research study and shall not in any way compromise the work of the University of Dodoma. For any questions, you are free to contact either me or my supervisor, Professor Florence Glanfield (florence.glanfield@ualberta.ca).

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Research Ethics Office at (780) 492-2615.

Yours Sincerely,

Emmanuel Deogratias

deograti@ualberta.ca

APPENDIX E

A LETTER REQUESTING PRE-SERVICE TEACHER'S PARTICIPATION IN FOUR ONE-DAY MEETINGS FOCUSSED ON PI USING CONCEPT-RICH INSTRUCTIONAL APPROACH

Study Title: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case.

Research Investigator:

Emmanuel Deogratias

University of Alberta

deograti@ualberta.ca

You are invited, on a purely voluntary basis, to participate in a research study titled: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case. I am conducting this study for the purpose of my PhD studies in mathematics education from the University of Alberta.

In this study, I request you to participate voluntarily in four one-day case study meetings of the concept-rich instructional approach on Saturday from December 3 to December 24, 2016. The meetings will be audio and video recorded. This participation will include completing reflective journals in each daylong meeting.

The data that will be collected during the four one-day meetings of concept-rich instruction is from a reflective journal, group learning notes, and audio and video recordings. The collected data will be analysed to evaluate the instructional process. I assure you that I will maintain confidentiality and anonymity for your participation in my study. Furthermore, be assured that data that will be collected from this study will not affect your grades in any courses you are enrolled at the University of Dodoma. However, participation in this activity may help you to better understand the ways concept-rich instructional approach can be used to teach mathematics.

Breakfast, lunch and bus fare will be provided to you for each day that you participate in this study.

At the end of these four day-one meetings, you may be asked to verify the data that will be collected from you. I will contact you first via email/phone for the appropriate meeting time. If a person chooses to withdraw from this study is allowed to do so any time, and the data that will be collected will be removed from the study any time within 30 days from which the data was collected.

If you are interested to participate in the four day-one meetings, please you are free to contact Emmanuel Deogratias at the phone number or email address above.

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of the research, contact the Research Ethics Office at +1 (780) 492-2615.

APPENDIX F

A RECRUITMENT POSTER FOR THE FOUR ONE-DAY RESEARCH MEETINGS

RESEARCH PARTICIPANTS NEEDED:

Would you like to learn about a new innovative way to teach mathematics?

I am looking for volunteers to take part in a research case study exploring the ways CRI approach reveals Tanzanian university pre-service teachers' expressions of their understanding of a mathematical concept. Also, the ways pre-service teachers plan to teach a concept, given their participation in CRI.

Volunteers must be:

Third year undergraduate mathematics pre-service teachers for the year of study 2016/2017 at the University of Dodoma.

Your participation will involve four one-day meetings from 9.00 am to 4.30 pm at the University of Dodoma on Saturday starting December 3 to December 24, 2016.

Breakfast, lunch and bus fare will be provided to you for each day that you participate in this study.

For more information about the case study or to volunteer for the study, please contact:

Research Investigator:

Emmanuel Deogratias

University of Alberta

deograti@ualberta.ca

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of the research, contact the Research Ethics Office at (780) 492-2615

APPENDIX G

INFORMATION LETTER AND CONSENT FORM FOR RESEARCH PARTICIPANTS IN THE FOUR ONE-DAY MEETINGS FOCUSED ON PI USING CONCEPT-RICH INSTRUCTIONAL APPROACH

Study Title: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case.

Research Investigator:	Supervisor (if applicable:
Emmanuel Deogratias	Professor Florence Glanfield
University of Alberta	University of Alberta

You are invited, on a purely voluntary basis, to participate in a research study titled: Exploring the implementation of concept-rich instruction with university mathematics pre-service teachers: A Tanzanian Case. I am conducting this study for the purpose of my PhD studies in mathematics education from the University of Alberta.

In this study, I request you to participate voluntarily in four one-day case study meetings of the concept-rich instruction on Saturday from December 1 to December 24, 2016. The meetings will be audio and video recorded. This participation will include completing reflective journal in each daylong meeting.

The data that will be collected during the four one-day meetings from a reflective journal, group learning notes, and audio and video recordings. The collected data will be analysed to evaluate the instructional process. I assure you that I will maintain confidentiality and anonymity for your participation in my study. Furthermore, be assured that data that will be collected from this study will not affect your grades in any courses you are enrolled at the University of Dodoma. However, participation in this activity may help you to better understand the ways concept-rich instructional approach can be used to teach mathematics.

Breakfast, lunch and bus fare will be provided to you for each day that you participate in this study.

At the end of these four day-one meetings, you may be asked to verify the data that will be collected from you. I will contact you first via email/phone for the appropriate meeting time. If a person chooses to withdraw from this study is allowed to do so any time, and the data that will be collected will be removed from the study any time within 30 days from which the data was collected.

If you are interested to participate in the four one-day meetings, please you are free to contact Emmanuel Deogratias at the phone number above.

If you consent to participate in four one-day meetings of the concept-rich instruction, please fill in and sign the consent form attached to this letter. For any questions, you are free to contact Emmanuel Deogratias and Professor Florence Glanfield using the contact details provided above.

The plan for this study has been reviewed for its adherence to ethical guidelines by a Research Ethics Board at the University of Alberta. For questions regarding participant rights and ethical conduct of the research, contact the Research Ethics Office at (780) 492-2615.

Consent Statement

I have read this form and the research study has been explained to me. I have been given the opportunity to ask questions and my questions have been answered. If I have additional questions, I have been told whom to contact. I agree to participate in the research study described above and will receive a copy of this consent form. I will receive a copy of this consent form after I sign it.

____ ________________

Participant's Name (printed) and Signature Date

____ __________________

Name (printed) and Signature of Person Obtaining Consent Date

APPENDIX H

THE TRANSCRIPT OF THE MICRO-TEACHING PERFOMED BY G2P IN THE RESEARCH MEETINGS

G2P: Today we have another session, let us start our session. Our topic today is an introduction to geometry. We are still on the same topic. Our sub-topic is the concept of *pi*, which is the lesson for today. In making revision of what we have learned from the previous lesson, I give you some objects, and I will ask you some questions. Take this one, take this one, and take this one. Let me add this one. Let me add another one.[A teacher distributing objects. In each group, there are some objects [circular and non-circular] which are provided. Let me ask you one question; discuss and identify objects which are circular.

Group A: We discovered a sole tape, top cover of a water bucket. Only two [objects] are circular objects.

Group B: We have toss cover, pawpaw stem.

G2P: For Group A [and Group B], why did you discover that those are circular objects, others not?

Group A: Because when these objects look are round, but this one has four sides, this one is a half. So, they are not circular.

Group B: Because they have a circumference and have no angles.

G2P: So, from there, since we are able to identify circular objects, let us come back to the concept of *pi* as our sub-topic. Okay, before defining anything in my point of view for the concept of *pi*, as we saw from the previous lesson in various terminologies implies a circumference of a circle and diameter of a circle. Let me ask you to measure the circumference of a circular object provided and the diameter. Before measuring in your group, let me demonstrate on how to measure the circumference of that circular object. Look here, all of you. For example, this is a sole tape, okay. The distance from one point to another point around the circle until you come back to the starting point is called the circumference. So, when you measure the distance from a certain point around any circular object or a circle until you come back to that point, the measure or measurement you get will complete only one circle or what we call the circumference of a circle. Do you get a concept?

Group A and B: Yes.

G2P: For example, here, this is our circular object, okay! So, when you want to measure the circumference of that circular object. Assume this or use this string. I will identify the starting point using a pencil. This is my starting point, okay. Then, I will take this string, I will start from the point identified from here, rounding! Worry not, you have to be careful. So, look here. My string started from the point identified which is this one, okay. I started here to that point around the string to turn back to the starting point. So, the distance of that string from the starting point around that circle to turn back to the starting point will make one complete circle which is called the circumference of a circle. So, do you get a concept on how to measure the length of circumference?

Group A and B: Yes.

G2P: This distance obtained from the starting point here to here, this is called the length of the circumference of a circle. Do you get a concept? So, after here you may take this distance you get and then you measure it to the ruler to know the exactly numerical value of that distance surrounding the circumference or surrounding a circle which is called the circumference. So, let say, we get 29 cm as a certain numerical value after placing on a ruler. But, this practice you will do your own in a group. Also, from there you will take a set of Vernier callipers on how to measure the diameter. Okay! So, where is it? (Looking for a set of Vernier callipers). This tool is a set of Vernier callipers which will assist us to measure the diameter of a circle or any circular objects. Okay! So, you may take this circular object, and then you may put like this way. This is called a set of Vernier callipers. Okay. Then, you measure the diameter. Look here. In this case, here there are two measurements. I mean scale, the top one [upper reading], and the bottom one [lower reading]. Okay! So, the top one, this one which reads zero up to 150 mm. Okay! You read it first. After that, you record a certain numerical figure. From there you come to find another figure or measure to the bottom one which reads 0.05mm. After getting the numerical value here, how can you identify this one? This one, it will be identified by looking the line or measure which will correlate with the top one and bottom one. Okay, where they will stand in a straight line or where they will be equal. Okay! Then, if it is, let say 3, you multiply it by 0.05 then you will add to the measurement you got at the top one. For example, here, let say it is 80.5 (the top one). Okay! But when you look clearly, the line which there is an intersection, I mean correlation is 80 and 3, so, you take 3 as a reading here times 0.05 plus 80.5. Okay! I will demonstrate clearly in your group. So, if you get a concept of how to measure this diameter then you record the figure, I mean the measurement. Let us proceed. From the objects, I provided to you for Group B, I would like to assist you to use the toss top cover and the what? For what? (Teacher asked to Group B members, the other circular material to use for measuring its circumference and diameter).

Group B: Toss cover.

G2P: And Group A, let you use only this one [a sole tape]. You will get certain measurements. Measure the circumference and then measure the diameter using a string and a ruler to the circumference. Work in a group for about four minutes. (Group work activities continuing in the two small groups while teacher is silent observing and waiting for the answers).

Group A: The object is sole tape, diameter$= 9.01cm$, and circumference$= 28.4cm$.

Group B: The object is toss top cover, diameter$= 10cm$, and circumference$= 31.5cm$.

G2P: In each group, compute the ratio of circumference to diameter.

Group A: Ratio$= 3.1520$ ...

Group B: Ratio$= 3.15$.

G2P: Okay. This is the ratio. So, in our last session, we learned a concept of the circumference, diameter and so on. So, in today, for the concept of *pi*, as we saw in our groups, we measured the circumference of a circle and diameter of those circles and then we found the ratio existing between circumference and diameter. So, the ratio obtained by taking the circumference dividing by a diameter which is equal to 3.152 ... for Group A, and 3.15 for Group B. This ratio is called the constant π or the value of *pi*. Okay! Do you get a concept? So, the ratio obtained after measuring the circumference and diameter and calculating the ratio of the circumference to the diameter, this constant value is called the value of *pi*. Do you get a concept?

Group A and B: Yes.

G2P: From there, we can define that *pi* is the ratio of the circumference of a circle and the diameter of that circle. Therefore, we can take the notes. *Pi* is the ratio between the measured circumference to the measured diameter of that circle. So, the meaning of *pi* after doing our practices in our groups. We can conclude that or define that *pi* is the ratio of the circumference of a circle to the diameter of that circle. Do you get a concept?

Group A and B: Yes.

G2P: From there, in your group, again, let us find the measurements of another object this one, and you may use another object, this one. You have three minutes. I want to reinforce you, on measurements or how to find the value of *pi*. (Teacher asks learners in small groups to measure the circumference and diameter of another circular object.)

Group A: Circumference= $90.3cm$, diameter= $27cm$.

Group B: Circumference= $6.5cm$, diameter= $2.05cm$.

G2P: So, take ratio. What is the ratio?

Group A: $\pi = ratio = 3.12 \ldots$

Group B: $\pi = ratio = 3.1707 \ldots$

G2P: So, what can I comment for you to know well? Look here, in activity 1 [first activity], Group A got 3.152 ... as a ratio; Group B got 3.15 ... in activity 2 [second activity], Group A, the ratio obtained is 3.12... and Group B, got 3.17 ... So, in your group, what can you say here?

Group B: This means that in order to compute the circumference of a circle, we need 3 diameters and some points to complete the circumference. Thus, why we are going to differ on the [decimal] points. This different in [decimal] points are due to [systematic] errors.

G2P: Another one, Group B? (Silent). Group A, do you have additional point?

Group A: Yes. There is a small difference but the whole numbers on that values start with 3, the difference comes to the decimal places.

G2P: Okay. Thank you for your contribution! Let me comment on that value of *pi*. So that the ratio between circumference and diameter of a circle, the value of *pi* in your measurements, you supposed to get 3.14 ... as a ratio of the circumference to diameter of a circle and its constant to each and every kind of a circle whether its circumference varies or diameter changes. Do you get a concept? But, here, there is certain I mean…I mean what? Can I say, errors in measurement? So, you were supposed to get the value of *pi* equals 3.14 (approximately). Another thing, the symbol used to denote the value of *pi* is this one (π). This is $\pi \cong 3.14$ because there is a certain approximation, this number is still continuing. So, this is the concept of *pi*. I am sure that from there you get a little bit about the concept of *pi* in your group. Okay! So, this is the concept of *pi*. From there, what is my additional [task] for you? I have questions, individually you have to attempt the following questions (a teacher writing the questions to the board). Using a circular object, define *pi* and find its value. Explain your answers. Attempt these two questions. Everyone, you have to attempt(teacher insisting learners after completing writing the questions on the board).

Group A and B: (Learners working individually to the assigned tasks.)

G2P: (The teacher moving around, observing and marking the individual learners' work).

G2P: Okay! Thank you for good attempt of the questions. This is the end of the lesson![32]

(Transcribed data

[32]Nodes\\Nodes from the Four-day CRI\\Practice\Presenting the lesson (2)