All lobsters with perfect matchings are graceful

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Abstract

Given a tree T consider one of its longest paths P_T. We define T to be m-distant if all of its vertices are a distance at most m from P_T. We will show that any 3-distant tree satisfying both of the following properties is graceful.

1. The tree has a perfect matching.
2. The tree can be constructed by attaching paths of length two to the vertices of a 1-distant tree (caterpillar); these attachments are made by identifying an end vertex of each path of length two with a vertex of the 1-distant tree.

Consequently, all 2-distant trees (lobsters) with perfect matchings are graceful.

1 Introduction

All graphs considered herein will be finite with no loops or multiple edges. Moreover, the vertex set and edge set of a graph G will be denoted by V_G and E_G, respectively. We begin by defining a graceful labelling of a graph [12].

Definition 1 *A graceful labelling of a graph G is an injective function $l : V_G \longrightarrow \{0, 1, \dots, |E_G|\}$ for which the associated function $g : E_G \longrightarrow \{1, \dots, |E_G|\}$ defined by $g(\{u, v\}) = |l(u) - l(v)|$ is bijective. A graph which exhibits a graceful labelling is said to be graceful.*

An example of a gracefully labelled tree is shown in Figure 1. The notion of a graceful labelling was introduced by Rosa [12] who proved that if all trees are graceful then the Ringel-Kotzig Conjecture [11] holds. This

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conjecture states that, for any n, K_{2n+1} can be cyclically decomposed into $2n+1$ copies of any tree with n edges. In the same work, Rosa showed that all caterpillars (trees for which the deletion of the pendant vertices gives a path) are graceful. Bermond [1] conjectured that all lobsters (trees for which the deletion of the pendant vertices gives a caterpillar) are graceful, where most advancements towards verifying this conjecture consider only very specific cases ([10],[4],[2]). In Section 2 we prove that all lobsters with perfect matchings are graceful; however, before doing so we introduce the definition of an m-distant tree.

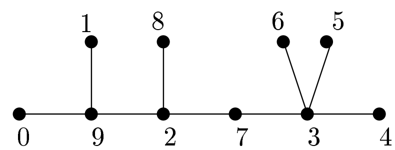

Figure 1: A graceful labelling of a caterpillar.

Definition 2 *For any tree T let P_T be one of its longest paths. We define T to be m-distant if all of its vertices are a distance at most m from P_T.*

Given this definition, we observe equivalences between paths and 0-distant trees, caterpillars and 1-distant trees, and lobsters and 2-distant trees. In [6], Ling defined a big lobster as a tree for which the deletion of the pendant vertices gives a lobster. This progression of animal related nomenclature motivates the definition of m-distant trees as a simplification of the terminology.

2 Gracefulness of certain 3-distant trees

The following is our main result.

Theorem 3 *Any 3-distant tree satisfying both of the following properties is graceful.*

1. *The tree has a perfect matching.*

2. *The tree can be constructed by attaching paths of length two to the vertices of a 1-distant tree (caterpillar); these attachments are made by identifying an end vertex of each path of length two with a vertex of the 1-distant tree.*

Before proving Theorem 3, we would like to clarify the restriction placed on 3-distant trees by the second condition of this theorem. To this effect, an example of a 3-distant tree which has a perfect matching but does not satisfy this condition is given in Figure 2; in this tree one might consider the path of length two being attached to a 2-distant tree.

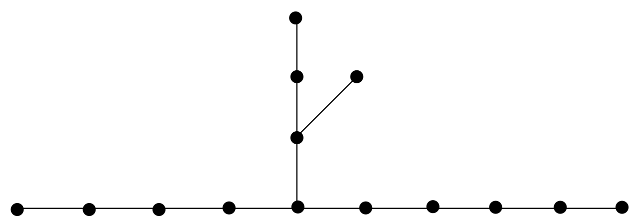

Figure 2: An example of a 3-distant tree with a perfect matching that does not satisfy the second condition of Theorem 3. Note that this tree has the minimum number of vertices of all such trees with perfect matchings.

Proof. Consider a 3-distant tree T on $n = 2m$ vertices which satisfies the conditions listed in Theorem 3. Let C_T be a maximum vertex subcaterpillar to which the paths of length two are attached. Given that T has a perfect matching, so does C_T; moreover, the perfect matching on C_T is a subset of the perfect matching on T. We denote the edge sets that constitute these perfect matchings by M_T and M_{C_T}, respectively.

Let P_{C_T} be the maximal path in C_T and let $e_{0,1}$ be an edge containing a pendant vertex of P_{C_T}. Note that our use of maximal subgraphs guarantees that $e_{0,1}$ also contains a pendant vertex of T, thereby, $e_{0,1}$ is an edge of M_T. For $0 < i < |M_{C_T}|$, let $e_{i,1}$ be the unique edge of M_{C_T} which is at distance 1 from $e_{i-1,1}$ and furthest away from $e_{0,1}$. Now let the other edges in M_T which are at distance 1 from $e_{i,1}$ be $e_{i,2}, \ldots, e_{i,f(i)}$; if there are no such edges let $f(i) = 1$. Note that $f(0) = 1$, and that T consists exclusively of the $\frac{n}{2}$ edges of the form $e_{i,j}$ and the $\frac{n}{2} - 1$ edges that connect them. An example of the edge denotation described above is given in Figure 3.

For each edge $e_{i,j}$ of M_T we associate a unique value $s_{i,j}$. Let $s_{0,1} = 0$ and define $s_{i,1}$, $0 < i < |M_{C_T}|$, by

$$s_{i,1} = 2\sum_{k=0}^{i-1} f(k) - s_{i-1,1} - 1. \tag{1}$$

Knowing that $s_{0,1} = 0$ and $s_{1,1} = 1$, we can solve this recursive definition to

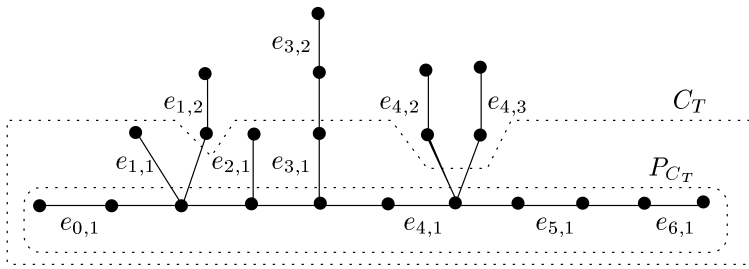

Figure 3: The edges of the perfect matching in a 3-distant tree satisfying the conditions of Theorem 3.

obtain

$$s_{i,1} = \begin{cases} 2\sum_{k=0}^{\frac{i}{2}-1} f(2k+1) & \text{if } i \text{ is even,} \\ 1 + 2\sum_{k=0}^{\frac{i-1}{2}-1} f(2k+2) & \text{if } i \text{ is odd.} \end{cases} \tag{2}$$

Similarly for $0 < i < |M_{C_T}|$ and $1 < j \leq f(i)$, we define $s_{i,j}$ by

$$s_{i,j} = 2j - 2 + 2\sum_{k=0}^{i-1} f(k) - s_{i,1} - 1, \tag{3}$$

which, using equation (2), can be solved to obtain

$$s_{i,j} = \begin{cases} 2j - 1 + 2\sum_{k=0}^{\frac{i}{2}-2} f(2k+2) & \text{if } i \text{ is even,} \\ 2j - 2 + 2\sum_{k=0}^{\frac{i-1}{2}-1} f(2k+1) & \text{if } i \text{ is odd.} \end{cases} \tag{4}$$

A graceful labelling of T is achieved by labelling the endpoints of each edge $e_{i,j}$ as $s_{i,j}$ and $n-1-s_{i,j}$ such that all labels of the form $s(\alpha)$ are adjacent only to labels of the form $n-1-s(\beta)$, and vice versa. In order to prove that such a labelling is graceful we must show that it satisfies the following properties.

i. The vertex labels are distinct and between 0 and $n-1$.

ii. The edge labels are distinct.

i. Since $f(i) > 0$ for all i, $0 \leq i < |M_{C_T}|$, equations (2) and (4) give that

$$0 \leq s_{i,j} \tag{5}$$

for all i, j. Using equations (1), (3), and (5), we observe that

$$s_{i,j} < 2j + 2\sum_{k=0}^{i-1} f(k) \leq 2f(i) + 2\sum_{k=0}^{i-1} f(k) \leq 2\sum_{k=0}^{i} f(k) \leq \sum_{k=0}^{|M_{C_T}|-1} f(k) = n,$$

which gives $0 \leq s_{i,j} \leq n-1$ and $0 \leq n-1-s_{i,j} \leq n-1$. That is, all the vertex labels are between 0 and $n-1$.

Assume, however, that the labels are not distinct. That is, there exists i_1, i_2, j_1, j_2, $(i_1, j_1) \neq (i_2, j_2)$, such that $s_{i_1,j_1} = s_{i_2,j_2}$ or $s_{i_1,j_1} = n - 1 - s_{i_2,j_2}$. Let us first consider when there exists i_1, i_2, j_1, j_2, $(i_1, j_1) \neq (i_2, j_2)$, such that $s_{i_1,j_1} = s_{i_2,j_2}$, which requires that $s_{i_1,j_1} \equiv s_{i_2,j_2} \pmod 2$. There are ten cases to consider, however, we will show contradictions for only four of them as the remaining six cases use similar arguments. In many of the arguments we will use the fact that $f(i) > 0$ for all i, $0 \leq i < |M_{C_T}|$.

(a) $\mathbf{j_1 = j_2 = 1, i_1 \equiv i_2 \equiv 0 (mod\ 2)}$. Without loss of generality, assume that $i_2 \geq i_1$. In this case

$$\begin{aligned} & s_{i_1,j_1} = s_{i_2,j_2} \\ \implies\ & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) = 2\sum_{k=0}^{\frac{i_2}{2}-1} f(2k+1) \\ \implies\ & i_1 = i_2, \end{aligned}$$

which contradicts $(i_1, j_1) \neq (i_2, j_2)$.

(b) $\mathbf{j_1 = 1 < j_2, i_1 \equiv 0 (mod\ 2), i_2 \equiv 1 (mod\ 2), i_2 > i_1}$. In this case

$$\begin{aligned} & s_{i_1,j_1} = s_{i_2,j_2} \\ \implies\ & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) = 2j_2 - 2 + 2\sum_{k=0}^{\frac{i_2-1}{2}-1} f(2k+1) \\ \implies\ & j_2 - 1 + \sum_{k=\frac{i_1}{2}}^{\frac{i_2-1}{2}-1} f(2k+1) = 0, \end{aligned}$$

which is a contradiction as $j_2 - 1 > 0$ and $\displaystyle\sum_{k=\frac{i_1}{2}}^{\frac{i_2-1}{2}-1} f(2k+1) \geq 0$.

(c) $\mathbf{j_1 = 1 < j_2, i_1 \equiv 0 (mod\ 2), i_2 \equiv 1 (mod\ 2), i_2 < i_1}$. In this case

$$
\begin{aligned}
& s_{i_1,j_1} = s_{i_2,j_2} \\
\implies\ & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) = 2j_2 - 2 + 2\sum_{k=0}^{\frac{i_2-1}{2}-1} f(2k+1) \\
\implies\ & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) < 2f(i_2) + 2\sum_{k=0}^{\frac{i_2-1}{2}-1} f(2k+1) \\
\implies\ & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) < 2\sum_{k=0}^{\frac{i_2-1}{2}} f(2k+1) \\
\implies\ & \sum_{k=\frac{i_2-1}{2}+1}^{\frac{i_1}{2}-1} f(2k+1) < 0,
\end{aligned}
$$

which is a contradiction as $\displaystyle\sum_{k=\frac{i_2-1}{2}+1}^{\frac{i_1}{2}-1} f(2k+1) \geq 0$.

(d) $\mathbf{j_1, j_2 > 1, i_1 = i_2 \equiv 0 (mod\ 2)}$. In this case

$$
\begin{aligned}
& s_{i_1,j_1} = s_{i_2,j_2} \\
\implies\ & 2j_1 - 1 + 2\sum_{k=0}^{\frac{i_1}{2}-2} f(2k+2) = 2j_2 - 1 + 2\sum_{k=0}^{\frac{i_2}{2}-2} f(2k+2) \\
\implies\ & j_1 = j_2,
\end{aligned}
$$

which contradicts $(i_1, j_1) \neq (i_2, j_2)$.

(e) $\mathbf{j_1 = j_2 = 1, i_1 \equiv i_2 \equiv 1 (mod\ 2)}$.

(f) $\mathbf{j_1 = 1 < j_2, i_1 \equiv 1 (mod\ 2), i_2 \equiv 0 (mod\ 2), i_2 > i_1}$.

(g) $\mathbf{j_1 = 1 < j_2, i_1 \equiv 1 (mod\ 2), i_2 \equiv 0 (mod\ 2), i_2 < i_1}$.

(h) $\mathbf{j_1, j_2 > 1, i_1 = i_2 \equiv 1 (mod\ 2)}$.

(i) $\mathbf{j_1, j_2 > 1, i_1 \equiv i_2 \equiv 0 (mod\ 2)}$.

(j) $\mathbf{j_1, j_2 > 1, i_1 \equiv i_2 \equiv 1 (mod\ 2)}$.

Let us now consider when there exists i_1, i_2, j_1, j_2, $(i_1, j_1) \neq (i_2, j_2)$, such that $s_{i_1,j_1} = n - 1 - s_{i_2,j_2}$, which requires that $s_{i_1,j_1} \not\equiv s_{i_2,j_2}$ (mod 2). There are six cases to consider; however, we will show contradictions for only two of them as the remaining four cases use similar arguments to reach a contradiction.

(a) $\mathbf{j_1 = j_2 = 1, i_1 \equiv 0 (mod\ 2), i_2 \equiv 1 (mod\ 2)}$. In this case

$$
\begin{aligned}
& s_{i_1,j_1} = n - 1 - s_{i_2,j_2} \\
\implies \quad & 2 \sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) = n - 1 - 1 - 2 \sum_{k=0}^{\frac{i_2-1}{2}-1} f(2k+2) \\
\implies \quad & 2 + 2 \sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) + 2 \sum_{k=0}^{\frac{i_2-1}{2}-1} f(2k+2) = n \\
\implies \quad & 2f(0) + 2 \sum_{k=1}^{|M_{C_T}|-2} f(k) \geq n \\
\implies \quad & 2 \sum_{k=0}^{|M_{C_T}|-1} f(k) > n,
\end{aligned}
$$

which is a contradiction as $2 \sum_{k=0}^{|M_{C_T}|-1} f(k) = n$.

(b) $\mathbf{j_1 = 1 < j_2, i_1 \equiv i_2 \equiv 0 (mod\ 2)}$. In this case

$$\begin{aligned}
& s_{i_1,j_1} = n - 1 - s_{i_2,j_2} \\
\implies & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) = n - 1 - 2j_2 + 1 - 2\sum_{k=0}^{\frac{i_2}{2}-2} f(2k+2) \\
\implies & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) + 2j_2 + 2\sum_{k=0}^{\frac{i_2}{2}-2} f(2k+2) = n \\
\implies & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) + 2f(i_2) + 2\sum_{k=0}^{\frac{i_2}{2}-2} f(2k+2) \geq n \\
\implies & 2\sum_{k=0}^{\frac{i_1}{2}-1} f(2k+1) + 2\sum_{k=0}^{\frac{i_2}{2}-1} f(2k+2) \geq n \\
\implies & 2\sum_{k=1}^{|M_{C_T}|-1} f(k) \geq n \\
\implies & 2\sum_{k=0}^{|M_{C_T}|} f(k) > n,
\end{aligned}$$

which is a contradiction as $2\sum_{k=0}^{|M_{C_T}|-1} f(k) = n$.

(c) $\mathbf{j_1 = j_2 = 1, i_1 \equiv 1 (mod\ 2), i_2 \equiv 0 (mod\ 2)}$.

(d) $\mathbf{j_1 = 1 < j_2, i_1 \equiv i_2 \equiv 1 (mod\ 2)}$.

(e) $\mathbf{j_1, j_2 > 1, i_1 \equiv 0 (mod\ 2), i_2 \equiv 1 (mod\ 2)}$.

(f) $\mathbf{j_1, j_2 > 1, i_1 \equiv 1 (mod\ 2), i_2 \equiv 0 (mod\ 2)}$.

Having shown the vertex labels distinct, it remains to show the edge labels distinct.

ii. First consider edges of the form $e_{i,j}$. The value of such an edge is $|n - 1 - s_{i,j} - s_{i,j}| = |n - 1 - 2s_{i,j}|$, which is odd. Since the $s_{i,j}$ values are distinct,

$$\begin{aligned} & |n-1-2s_{i_1,j_1}| = |n-1-2s_{i_2,j_2}| \\ \Longrightarrow\quad & n-1-2s_{i_1,j_1} = 1+2s_{i_2,j_2}-n \\ \Longrightarrow\quad & n-1-s_{i_1,j_1} = s_{i_2,j_2} \\ \Longrightarrow\quad & (i_1,j_1) = (i_2,j_2). \end{aligned}$$

Thereby, the edges of the form $e_{i,j}$ have distinct labels, utilizing all the odd labels from 1 to $n-1$.

Now consider the edges formed in connecting $e_{i,1}$ with $e_{i-1,1}$, $1 \le i < |M_{C_T}|$. Such an edge has value

$$\begin{aligned} & |n-1-s_{i-1,1}-s_{i,1}| \\ = \quad & \left| n-1-s_{i-1,1}-2\sum_{k=0}^{i-1} f(k)+1+s_{i-1,1} \right| \\ = \quad & n-2\sum_{k=0}^{i-1} f(k), \end{aligned}$$

which is even. Moreover, no two such edges have the same value because $f(i) > 0$ for all i, $0 \le i < |M_{C_T}|$.

Finally consider the edges formed in connecting $e_{i,1}$ with $e_{i,j}$, $1 \le i < |M_{C_T}|$, $1 < j \le f(i)$. Such an edge has value

$$\begin{aligned} & |n-1-s_{i,j}-s_{i,1}| \\ = \quad & |n-1-2j+2-2\sum_{k=0}^{i-1} f(k)+s_{i,1}+1-s_{i,1}| \\ = \quad & n-2j+2-2\sum_{k=0}^{i-1} f(k), \end{aligned}$$

which is even. No two such edges have the same value, otherwise there would be some $e_{i,j}$ for which $j > f(i)$. Using the same argument, no such edge can have the same value as an edge connecting $e_{i,1}$ with $e_{i-1,1}$, $1 \le i \le |M_{C_T}|$.

Having now shown the edge labels distinct, T is graceful. □

□

The graceful labelling of an 3-distant tree satisfying both of the conditions of Theorem 3 is shown in Figure 4. Given that all 2-distant trees are 3-distant, and that all 2-distant trees with perfect matchings satisfy the second condition of Theorem 3, we obtain the following corollary which partially addresses the conjecture of Bermond that all 2-distant trees (lobsters) are graceful [1].

Corollary 4 *All 2-distant trees (lobsters) with perfect matchings are graceful.*

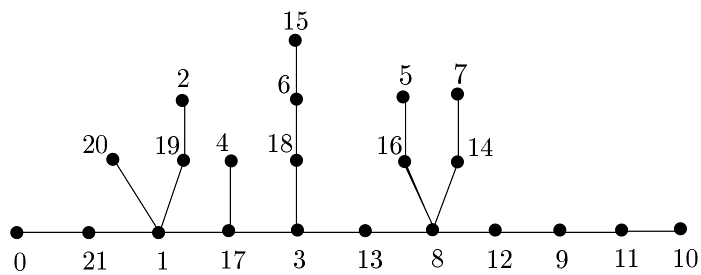

Figure 4: The graceful labelling of a 3-distant tree satisfying the conditions of Theorem 3.

References

[1] J.-C. Bermond, *Graceful graphs, radio antennae and french windmills*, Graph Theory and Combinatorics, Pittman, London, 1979, pp. 18–37.

[2] V. N Bhat-Nayak and U. N Deshmukh, *New families of graceful banana trees*, Proceedings of the Indian Academy of Science **106** (1996), no. 2, 201–216.

[3] G.S. Bloom, *A chronology of the Ringel-Kotzig conjecture and the continuing quest to call all trees graceful*, Annals New York Academy of Sciences **326** (1979), 32–51.

[4] W.C. Chen, H.I Lu, and Y.N. Yeh, *Operations of interlaced trees and graceful trees*, Southeast Asian Bulletin of Mathematics **4** (1997), 337–348.

[5] J.A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics **DS6** (2002), 1–106.

[6] J. Ling, *Graceful labellings of two classes of big lobster trees*, Neimenggu Shida Xuebao Ziran Kexue Ban (1988), no. 4, 8–15 (Chinese).

[7] D. Morgan, *Gracefully labeled trees from Skolem sequences*, Congressus Numerantium **42** (2000), 41–48.

[8] ———, *Gracefully labelled trees from Skolem and related sequences*, Master's thesis, Memorial University of Newfoundland, 2001.

[9] D. Morgan and R. Rees, *Using Skolem and hooked-Skolem sequences to generate graceful trees*, Journal of Combinatorial Mathematics and Combinatorial Computing **44** (2003), 47–63.

[10] H.K. Ng, *Gracefulness of a class of lobsters*, AMS Abstracts **7** (1986), abstract no. 825–05–294.

[11] G. Ringel, *Problem 25 in theory of graphs and its applications*, Proceedings of the Symposium in Smolenice 1963, Academic Press, New York, 1964, p. 162.

[12] A. Rosa, *On certain valuations of the vertices of a graph*, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York and Dunod, Paris, 1967, pp. 349–355.