HEIGHT PAIRING ON GRADED PIECES OF A BLOCH-BEILINSON FILTRATION

by

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Abstract

For a smooth projective variety X (of dimension d) defined over $\overline{\mathbb{Q}}$, Beilinson (and independently Bloch) constructed a 'height' pairing

$$CH^r_{hom}(X;\mathbb{Q}) \times CH^{d-r+1}_{hom}(X;\mathbb{Q}) \to \mathbb{R},$$

under very reasonable assumptions and with a number of conjectural properties. A folklore conjecture related to this pairing states that the Griffiths Abel-Jacobi map

$$\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J^r(X) \otimes \mathbb{Q}$$

is injective (BBC). But if X is defined over a field of finite transcendence degree over $\overline{\mathbb{Q}}$, then the injectivity of the Abel-Jacobi map doesn't hold any more. Instead we have the concept of a conjectural Bloch-Beilinson filtration, a candidate for which was given by James Lewis. Under some assumptions, specially BBC, the main point of this thesis is to generalize the height pairing to the graded pieces of this candidate Bloch-Beilinson filtration using cohomological machinery.

To Mathematics, pure and impure !

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Contents

Chapter 1

Introduction

Let us fix a subfield $k \subset \mathbb{C}$ (k will be a finitely generated overfield of $\overline{\mathbb{Q}}$ in most of the situations). Henceforth, we will consider smooth projective varieties over k. We will mention the underlying field only if we digress from the above convention. Given a smooth geometrically irreducible projective variety X of dimension d , we can associate its betti and Hodge cohomologies, as well as cycle groups:

- The singular cohomology $H^l(X, \mathbb{Q}) := H^l(X(\mathbb{C}), \mathbb{Q})$, where $X(\mathbb{C})$ denotes the associated complex space. Via the de Rham isomorphism theorem, and the work of Hodge, this singular cohomology comes equipped with a natural Hodge decomposition, as a reflection of the complex structure on $X(\mathbb{C})$:

$$H^l(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^l_{de-Rham}(X, \mathbb{C}) = \oplus_{p+q=l} H^{p,q}(X)$$

  where $H^{p,q}(X)$ is the space of d-closed (p, q)-forms (modulo coboundaries), and $\overline{H^{p,q}(X)} = H^{q,p}(X)$, the complex conjugation induced by conjugation on the second factor of $H^l(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$. Here $H^l_{de-Rham}(X, \mathbb{C})$ denotes the de Rham cohomology of $X(\mathbb{C})$. One can define a Hodge filtration on $H^l(X, \mathbb{C})$ by assigning

$$F^i H^l(X, \mathbb{C}) := \bigoplus_{p+q=l, p \geqq i} H^{p,q}(X) \, ,$$

a situation that holds more generally for compact complex Kähler manifolds.

- The Chow group, which we formally define as follows : For an irreducible subvariety $Y \subset X$, denote by

$$codimension(Y) := d - dim(Y) .$$

Now consider the $\mathbb{Z}$-linear combination of irreducible subvarieties of X of codimension r and denote it by $Z^r(X)$ (we call them *algebraic cycles*). Define the Chow group of codimension r to be

$$CH^r(X) := Z^r(X)/\sim_{rat} ,$$

where $\sim_{rat}$ is an adequate equivalence relation (which, among other things, provides a ring structure on $\bigoplus_{r\geq 0} CH^r(X)$), known as rational equivalence. it is well-known that rational equivalence is the weakest among all other equivalence relations, in the sense that being rationally equivalent implies equivalent under any other adequate equivalence relation.

Given the Chow group of X, there are two cycle class maps associated to the cohomology of X. The first one is known as the fundamental class map into singular cohomology:

$$cl_r : CH^r(X) \to H^{2r}(X, \mathbb{Z}) ,$$

where $H^{2r}(X, \mathbb{Z})$ is the singular cohomology with $\mathbb{Z}$ coefficients (which can have torsion elements). It can be shown that the image of this map lies in

$$H^{r,r}(X, \mathbb{Z}) := H^{2r}(X, \mathbb{Z}) \cap H^{r,r}(X) ,$$

where the latter term is intended to include the torsion classes. One of the most celebrated conjectures in algebraic geometry; known as the Hodge conjecture, states that the (rational) cycle class map

$$cl_r : CH^r(X; \mathbb{Q}) := CH^r(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{r,r}(X, \mathbb{Q}) := H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X) ,$$

is surjective. There is also a generalization of this conjecture, first formulated as an important open question by Hodge, and later amended by Grothendieck, called the general Hodge conjecture (GHC). This will be explained later in the text.

The kernel of the (rational) cycle class map, denoted by $CH^r_{hom}(X;\mathbb{Q})$, is an important algebraic object attached to X. One can define a secondary class map, namely, the Abel-Jacobi map:

$$\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J^r(X) \otimes \mathbb{Q}\,,$$

where $J^r(X)$ is a certain compact complex torus called the Griffiths Jacobian of X.

In the 1970's Bloch conjectured that there should be a 'natural' decreasing filtration on the (rational) Chow groups of smooth projective varieties (resembling the Hodge filtration in cohomology). This was later fortified by Beilinson in terms of motivic extension datum, based on the conjectural existence of the category of mixed motives for varieties over a field. For a smooth projective variety X, if we denote the conjectural filtration by $\{F^iCH^r(X;\mathbb{Q})\}_{i\geq 0}$, then one criteria of it is $F^1CH^r(X;\mathbb{Q}) = CH^r_{hom}(X;\mathbb{Q})$; more precisely, the right hand side (RHS) should be the Chow group of cycles numerically equivalent to zero, but this will be the same as $CH^r_{hom}(X;\mathbb{Q})$ under the Hodge conjecture. So far, there are several candidates for this filtration and one of the most important one was developed by James Lewis, in his paper[38]. It satisfies most of the properties of a Bloch-Beilinson filtration. We will elaborate more on it in the Chapter 3.

One can view the cycle class maps from the Chow groups of smooth projective varieties to the category of "mixed Hodge structures", which will play a role in detecting non-zero cycles. The non-degenerate pairings

$$H^{2r-1}(X,\mathbb{C}) \times H^{2d-2r+1}(X,\mathbb{C}) \to \mathbb{C}\,,\ \text{(Poincaré)}$$

$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}\,,\ \text{(Serre)}$$

induced by

$$(\eta_1, \eta_2) \mapsto \int_X \eta_1 \wedge \eta_2$$

and more importantly (to us), the associate Hodge-Riemann bilinear relations, will be seen to play an important role on the level of Chow groups.

For a smooth projective variety X (of dimension d) defined over a number field k (i.e. $[k : \mathbb{Q}] < \infty$) or more generally over $\overline{\mathbb{Q}}$, Beilinson ([5]) and independently Bloch ([6]) constructed a 'height' pairing (under very reasonable assumptions):

$$CH^r_{hom}(X; \mathbb{Q}) \times CH^{d-r+1}_{hom}(X; \mathbb{Q}) \to \mathbb{R},$$

with a number of conjectural properties. For example, Conjectures 5.4 and 5.5 of [5] seem to mirror the nondegeneracy properties of the pairing stated above. A folklore conjecture, due independently by Bloch and Beilinson, and playing a role in this pairing, states that the (rational) Abel-Jacobi map

$$\Phi_r : CH^r_{hom}(X; \mathbb{Q}) \to J^r(X) \otimes \mathbb{Q}$$

is injective, where the RHS is (again) defined in terms of the associated complex space $X(\mathbb{C})$. This conjecture is referred to as the Bloch-Beilinson conjecture (BBC).

Returning to the conjectural filtration, let

$$Gr^\nu_F CH^r(X; \mathbb{Q}) := F^\nu CH^r(X; \mathbb{Q}) / F^{\nu+1} CH^r(X; \mathbb{Q})$$

denote the graded pieces of the Bloch-Beilinson filtration. It is an important (motivic) invariant of X. We will work in the set-up of the filtration developed by James Lewis ([38]). As with other candidate filtrations, an important feature of which is the fact that $F^2 CH^r(X; \mathbb{Q}) \subset Ker(\Phi_r)$, where

$$Ker(\Phi_r) := \{\eta \in CH^r_{hom}(X; \mathbb{Q})\ ;\ \Phi_r(\eta) = 0\}\ .$$

If X is defined over $\overline{\mathbb{Q}}$ and we assume the BBC about the injectivity of the (rational) Abel-Jacobi map, then

$$Gr_F^1 CH^r(X;\mathbb{Q}) = F^1 CH^r(X;\mathbb{Q}) = CH^r_{hom}(X;\mathbb{Q}),$$

and $Gr_F^{\nu} CH^r(X;\mathbb{Q}) = 0$ for $\nu \geq 2$, since $F^2 CH^r(X;\mathbb{Q}) \subset Ker(\Phi_r) = 0$. The height pairing developed by Beilinson and Bloch can now be viewed as a pairing

$$Gr_F^1 CH^r(X;\mathbb{Q}) \times Gr_F^1 CH^{d-r+1}(X;\mathbb{Q}) \to \mathbb{R}\,.$$

However if X is defined over a field of transcendence degree greater than 0 over $\overline{\mathbb{Q}}$, there are plenty of examples where the Abel-Jacobi map is not injective (see [14], [52], [40] and [45] among others). Hence we (conjecturally) have non zero higher graded pieces. The main purpose of this thesis is to extend the 'height' pairing of Beilinson and Bloch to higher graded pieces of the candidate Bloch-Beilinson filtration developed by James Lewis, in form of the following theorem:

1.0.1 Theorem. *Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety of dimension d and let $K/\overline{\mathbb{Q}}$ be a finitely generated overfield of transcendence degree $\nu - 1$, where $\nu \geq 1$ is an integer. Let us assume Grothendieck amended general Hodge conjecture, together with the BBC, viz., the injectivity of the Abel-Jacobi map for varieties defined over $\overline{\mathbb{Q}}$. Then there exists a pairing*

$$\langle\ ,\ \rangle_{HT} : Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) \times Gr_F^{\nu} CH^{d-r+\nu}(X_K;\mathbb{Q}) \to \mathbb{R}\,,$$

extending the Beilinson height pairing.

We will prove this in Chapter 7.

The set-up of this thesis is as follows: In Chapters 2-6, we develop the background material needed for the main body, including a brief review of arithmetic intersection theory, a pathbreaking area developed by Gillet and Soulé in [18]. Chapter 7 contains the proof of Theorem 1.1. In Chapter 8, we will see some explicit computations of the pairing that we developed. The last chapter, Chapter 9 is more speculative in nature, containing the generalizations of Conjectures 5.3 (a) and 5.5 of [5], for the height pairing on graded pieces.

Chapter 2

Chow group of a smooth projective variety and its connection to cohomology

General references for this chapter are [37], [39] and [42]. We work with the following set up: Unless otherwise stated, by X we will denote a smooth (geometrically irreducible) projective variety of dimension d over a subfield $k \subset \mathbb{C}$. $X(\mathbb{C})$ will denote the complex points of X (which forms a compact complex projective manifold of complex dimension d). We will also denote the singular (or betti) cohomology $H^l_{sing}(X(\mathbb{C}), A)$ by $H^l(X, A)$, where A is one of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$.

2.1 Preview of cohomology theory of X

Let $A^l(X)$ denote the $\mathbb{C}$-valued C^∞ l-forms on $X(\mathbb{C})$. We have the decomposition

$$A^l(X) = \bigoplus_{p+q=l} A^{p,q}(X),\ A^{q,p}(X) = \overline{A^{p,q}(X)}\,, \tag{2.1.0.1}$$

where $A^{p,q}(X)$ are C^∞ (p,q)-forms which in local holomorphic coordinates $z = (z_1, \cdots, z_d) \in X(\mathbb{C})$, are of the form

$$\sum_{|I|=p,|J|=q} f_{IJ} dz_{i_i} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}\ ,$$

where the f_{IJ}'s are complex-valued and C^∞.The differential $d : A^l(X) \to A^{l+1}(X)$ splits into $d = \partial + \overline{\partial}$, where $\partial A^{p,q}(X) \subset A^{p+1,q}(X)$ and $\overline{\partial} A^{p,q}(X) \subset A^{p,q+1}(X)$. Since $d^2 = 0$, we get $0 = \partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial$. The decomposition in (2.1) now descends to the level of cohomology as

2.1.1 Theorem. *(Hodge decomposition)*

$$H^l(X,\mathbb{C}) \cong H^l(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^l_{de-Rham}(X,\mathbb{C}) = \oplus_{p+q=l} H^{p,q}(X)\ , \quad (2.1.1.1)$$

where $H^{p,q}(X)$ are the d-closed (p,q)-forms (modulo coboundaries), and $H^{q,p}(X) = \overline{H^{p,q}(X)}$. All such cohomology groups are finite dimensional and we have the description

$$H^{p,q}(X) \cong \frac{A^{p,q}(X)_{d-closed}}{\partial\overline{\partial}A^{p-1,q-1}(X)}\ .$$

One can define a descending (Hodge) filtration on $H^l(X,\mathbb{C})$ by assigning

$$F^i H^l(X,\mathbb{C}) := \bigoplus_{p+q=l,p\geqq i} H^{p,q}(X)\ .$$

An easy consequence of the theorem is the following

2.1.2 Corollary. *If l is odd, then $H^l(X,\mathbb{Q})$ is even dimensional.*

The following result is well known:

2.1.3 Proposition. *(Poincaré and Serre duality) The pairings*

$$H^{2r-1}(X,\mathbb{C}) \times H^{2d-2r+1}(X,\mathbb{C}) \to \mathbb{C}\ ,\ \text{(Poincaré)}$$

$$H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}\ ,\ \text{(Serre)}$$

induced by

$$(\eta_1,\eta_2) \mapsto \int_X \eta_1 \wedge \eta_2$$

are non-degenerate. Hence one can identify $H^l(X,\mathbb{C}) \cong H^{2d-l}(X,\mathbb{C})^\vee$ and $H^{p,q}(X) \cong H^{d-p,d-q}(X)^\vee$.

2.1.4 Remark. One can also prove Poincaré duality with $\mathbb{Q}$-coefficients and identify $H^l(X,\mathbb{Q}) \cong H^{2d-l}(X,\mathbb{Q})^\vee$.

We also recall

2.1.5 Theorem. *(Künneth decomposition) For varieties X and Y, we have* *the following decomposition for $H^l(X \times_{\mathbb{C}} Y, \mathbb{Q})$:*

$$H^l(X \times_{\mathbb{C}} Y, \mathbb{Q}) \cong \oplus_{p+q=l} H^p(X, \mathbb{Q}) \otimes_{\mathbb{Q}} H^q(Y, \mathbb{Q})$$

which respects the Hodge decomposition of $H^l(X \times_{\mathbb{C}} Y, \mathbb{C})$ in the following manner

$$H^{r,s}(X \times_{\mathbb{C}} Y) \cong \oplus_{r_1+r_2=r, s_1+s_2=s} H^{r_1,s_1}(X) \otimes_{\mathbb{C}} H^{r_2,s_2}(Y) \,.$$

2.1.6 Abstract Hodge theory

The l-th cohomology of a smooth projective variety X is an example of what is known as a pure Hodge structure (of weight l). Formally we define it as follows:

2.1.7 Definition. *Let $\mathbb{A} \subset \mathbb{R}$ be a subring. An $\mathbb{A}$- Hodge structure (HS) of weight $l \in \mathbb{Z}$ is given by the following datum:*

1. *A finitely generated $\mathbb{A}$-module V, and either of the two equivalent statements below:*
2. *A decomposition*
$$V_{\mathbb{C}} = \bigoplus_{p+q=l} V^{p,q}, \ \overline{V^{p,q}} = V^{q,p} \,,$$
where $^-$ is complex conjugation on the second factor of $V_{\mathbb{C}} := V \otimes \mathbb{C}$, or equivalently
3. *A finite descending filtration*
$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\} \,,$$
satisfying
$$V_{\mathbb{C}} = F^r \bigoplus \overline{F^{l-r+1}}, \ \forall r \in \mathbb{Z}.$$

An $\mathbb{A}$-subspace $G \subset V$ is a sub-HS if and only if $G_{\mathbb{C}} = \oplus_{p+q=l} G^{p,q}$ where $G^{p,q} = G_{\mathbb{C}} \cap V^{p,q}$. Also the quotient V/G has a natural HS. The tensor product of two HS, V_1 and V_2, of weights l and m respectively, is a HS, $V_1 \otimes V_2$ of weight $l+m$.

2.1.8 Remark. The equivalence of 2. and 3. can be seen as follows. Given the decomposition in 2., set

$$F^r V_{\mathbb{C}} = \bigoplus_{p+q=l, p\geq r} V^{p,q}.$$

Conversely, given $\{F^r\}$ in 3., we put $V^{p,q} = F^p \cap \overline{F^q}$.

2.1.9 Example. If X/k is smooth projective, then $H^l(X, \mathbb{Z})$ is a $\mathbb{Z}$-Hodge structure of weight l.

2.1.10 Example. $\mathbb{A}(r) := (2\pi i)^r \mathbb{A}$ is an $\mathbb{A}$-Hodge structure of weight $-2r$ and of pure Hodge type $(-r, -r)$, called the Tate twist.

2.1.11 Example. If X/k is smooth projective, then $H^l(X, \mathbb{Q}(r)) := H^l(X, \mathbb{Q}) \otimes \mathbb{Q}(r)$ is a $\mathbb{Q}$-Hodge structure of weight $l - 2r$.

To extend these ideas to a singular varieties, one has the following terminology:

2.1.12 Definition. *An $\mathbb{A}$-mixed Hodge structure ($\mathbb{A}$-MHS) is given by the following datum:*

1. *A finitely generated $\mathbb{A}$-module V,*
2. *A finite descending "Hodge" filtration on $V_{\mathbb{C}} = V \otimes \mathbb{C}$,*

$$V_{\mathbb{C}} \supset \cdots \supset F^r \supset F^{r+1} \supset \cdots \supset \{0\},$$

3. *An increasing weight filtration on $V_{\mathbb{Q}} = V \otimes_{\mathbb{Z}} \mathbb{Q}$,*

$$\{0\} \subset \cdots \subset W_{l-1} \subset W_l \subset \cdots \subset V_{\mathbb{Q}},$$

such that $\{F^r\}$ induces a (pure) HS of weight l on $Gr_l^W := W_l/W_{l-1}$.

2.1.13 Example. (Deligne, [50]) Let $Y/\mathbb{C}$ be an algebraic variety. Then $H^l(Y, \mathbb{Z})$ has a canonical and functorial $\mathbb{Z}$-MHS.

2.1.14 Definition. *A morphism $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ of $\mathbb{A}$-MHS is an $\mathbb{A}$-linear map satisfying*

- *$h(W_l V_{1,\mathbb{Q}}) \subset W_l V_{2,\mathbb{Q}}$, $\forall l$,*
- *$h(F^r V_{1,\mathbb{C}}) \subset F^r V_{2,\mathbb{C}}$, $\forall r$.*

Deligne ([50], Theorem 2.3.5) shows that the category of $\mathbb{A}$-MHS is abelian; in particular if $h : V_{1,\mathbb{A}} \to V_{2,\mathbb{A}}$ is a morphism of $\mathbb{A}$-MHS, then $ker(h)$ and $coker(h)$ are endowed with induced filtration

2.1.15 Example. Let $\overline{U}/\mathbb{C}$ be a compact Riemann surface, $\Xi \subset \overline{U}$ a finite set of points, and $U := \overline{U} - \Xi$. According to the previous example, $H^1(U,\mathbb{Z})(1)) := H^1(U,\mathbb{Z}) \otimes \mathbb{Z}(1)$ carries a $\mathbb{Z}$-MHS. The Hodge filtration on $H^1(U,\mathbb{C})$ is defined in terms of a filtered complex of holomorphic differentials on U with logarithmic poles along Ξ ([50]). We can "observe" the MHS via weights as follows. Poincaré duality gives $H^1_{\Xi}(\overline{U},\mathbb{Z}) \cong H_1(\Xi,\mathbb{Z}) = 0$, and the localization sequence in cohomology below is an exact sequence of MHS:

$$0 \to H^1(\overline{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1)) \to H^0(\Xi,\mathbb{Z}(0))^{\circ} \to 0,$$

where

$$H^0(\Xi,\mathbb{Z}(0))^{\circ} = ker\left(H^2_{\Xi}(\overline{U},\mathbb{Z}(1)) \to H^2(\overline{U},\mathbb{Z}(1))\right) \cong \mathbb{Z}^{|\Xi|-1}.$$

We put $W_0 = H^1(U,\mathbb{Z}(1))$, $W_{-1} = Im\left(H^1(\overline{U},\mathbb{Z}(1)) \to H^1(U,\mathbb{Z}(1))\right)$, $W_{-2} = 0$. Then $Gr^W_{-1}(U,\mathbb{Z}(1)) \cong H^1(\overline{U},\mathbb{Z}(1))$ has pure weight -1 and $Gr^W_0 H^1(U,\mathbb{Z}(1)) \cong \mathbb{Z}^{|\Xi|-1}$ has pure weight 0.

2.1.16 Definition. *Let V be a $\mathbb{A}$-MHS. We put*

$$\Gamma_{\mathbb{A}} V := hom_{\mathbb{A}-MHS}(\mathbb{A}(0),V),$$

and

$$J_{\mathbb{A}}(V) := Ext^1_{\mathbb{A}-MHS}(\mathbb{A}(0),V).$$

In case $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{Q}$, we put $\Gamma = \Gamma_{\mathbb{A}}$ and $J = J_{\mathbb{A}}$.

2.1.17 Example. Suppose $V = V_{\mathbb{Z}}$ is a (pure) HS of weight $2r$. Then $V \otimes \mathbb{Z}(r)$ is of weight 0, and (up to twist) one can identify ΓV with $V_{\mathbb{Z}} \cap F^r V_{\mathbb{C}} = V_{\mathbb{Z}} \cap V^{r,r} := \epsilon^{-1}(V^{r,r})$, where $\epsilon : V \to V_{\mathbb{C}}$

2.1.18 Example. Let V be a $\mathbb{Z}$-MHS. There is the identification due to J.Carlson (see [10] or [27], Lemma 9.2),

$$J(V) \cong \frac{W_0 V_{\mathbb{C}}}{F^0 W_0 V_{\mathbb{C}} + W_0 V},$$

where in the denominator, $V := V_{\mathbb{Z}}$ is identified with its image $V_{\mathbb{Z}} \to V_{\mathbb{C}}$ (quotienting out torsion). For example, if $\{E\} \in Ext^1_{MHS}(\mathbb{Z}(0),V)$ corresponds to the short exact sequence of MHS:

$$0 \to V \to E \xrightarrow{\alpha} \mathbb{Z}(0) \to 0,$$

then one can find $x \in W_0E$ and $y \in F^0W_0E_{\mathbb{C}}$ such that $\alpha(x) = \alpha(y) = 1$. Then $x - y$ descends to a class in $W_0V_{\mathbb{C}}/\{F^0W_0V_{\mathbb{C}} + W_0V\}$, which defines a map from $Ext^1_{MHS}(\mathbb{Z}(0), V)$ to $W_0V_{\mathbb{C}}/\{F^0W_0V_{\mathbb{C}} + W_0V\}$.

2.1.19 A survey of Deligne cohomology

In this subsection, we will consider a smooth projective variety $X/\mathbb{C}$ (of dimension d) and Ω_X will denote the sheaf of holomorphic 1-forms on X. We define $\Omega^l_X := \underbrace{\Omega_X \wedge \cdots \wedge \Omega_X}_{l-times}$. Recall that $\mathbb{A}(r)$ is the Tate-twist, for a subring $\mathbb{A}$ of $\mathbb{R}$. We introduce the Deligne complex $\mathbb{A}_{\mathcal{D}}(r)$:

$$\mathbb{A}(r) \to \underbrace{\Omega_X \to \cdots \to \Omega^{r-1}_X}_{=:\Omega^{\bullet<r}_X}.$$

2.1.20 Definition. *Deligne cohomology is given by the hypercohomology:*

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) := \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r))$$

2.1.21 Remark. We have a product structure on Deligne cohomology

$$H^m_{\mathcal{D}}(X, \mathbb{A}(i)) \otimes H^n_{\mathcal{D}}(X, \mathbb{A}(j)) \to H^{m+n}_{\mathcal{D}}(X, \mathbb{A}(i + j))$$

induced from the multiplication of complexes $\mu : \mathbb{A}_{\mathcal{D}}(i) \otimes \mathbb{A}_{\mathcal{D}}(j) \to \mathbb{A}_{\mathcal{D}}(i + j)$, given in [15], Definition 3.2.

2.1.22 Example. When $\mathbb{A} = \mathbb{Z}$, we have the isomorphism

$$H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong CH^1(X).$$

Alternate take. Let $h : (A^\bullet, d) \to (B^\bullet, d)$ be a morphism of complexes. We define

$$Cone(A^\bullet \xrightarrow{h} B^\bullet)$$

by the formula

$$[Cone(A^\bullet \xrightarrow{h} B^\bullet)]^q := A^{q+1} \oplus B^q, \;\; \delta(a, b) = (-da, h(a) + db).$$

Using the holomorphic Poincaré lemma, one can show that there is a quasi-isomorphism between $\mathbb{A}_{\mathcal{D}}(r)$ and

$$Cone\left(\mathbb{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet)\right)[-1],$$

where ϵ and l are natural maps obtained after a choice of injective resolution of $\mathbb{A}(r)$ and $\Omega^\bullet$. Hence

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) \cong \mathbb{H}^i\left(Cone\left(\mathbb{A}(r) \oplus F^r\Omega_X^\bullet \xrightarrow{\epsilon - l} \Omega_X^\bullet)\right)[-1]\right).$$

From the short exact sequence of sheaves

$$0 \to \Omega_X^{\bullet < r}[-1] \to \mathbb{Z}_{\mathcal{D}}(r) \to \mathbb{Z}(r) \to 0$$

together with Hodge theory, we get the short exact sequence

$$0 \to J(H^{2r-1}(X, \mathbb{Z}(r))) \to H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)) \to \Gamma(H^{2r}(X, \mathbb{Z}(r))) \to 0.$$

Here we note that

$$\Gamma(H^{2r}(X, \mathbb{Z}(r))) = H^{2r}(X, \mathbb{Z}) \cap H^{r,r}(X) = \epsilon^{-1}(H^{r,r}(X)),$$

where $\epsilon : H^{2r}(X, \mathbb{Z}(r)) \to H^{2r}(X, \mathbb{C})$ is induced by the incusion $\mathbb{Z}(r) \hookrightarrow \mathbb{C}$. Further, from the identification of Carlson, (Example 2.1.18 above),

$$J^r(X) := J(H^{2r-1}(X, \mathbb{Z}(r))) = \frac{H^{2r-1}(X, \mathbb{C})}{F^rH^{2r-1}(X, \mathbb{C}) + H^{2r-1}(X, \mathbb{Z}(r))}$$

$$\cong \frac{F^{d-r+1}H^{2d-2r+1}(X, \mathbb{C})^\vee}{H_{2d-2r+1}(X, \mathbb{Z}(d-r))}$$

is a compact complex torus, known as Griffiths jacobian.

2.1.23 Remark. Strictly speaking, $F^rH^{2r-1}(X, \mathbb{C})$ should be replaced by $F^0H^{2r-1}(X, \mathbb{C})$. We resisted that temptation for "obvious" reasons.

Deligne-Beilinson cohomology

The Deligne cohomology described above is not adequate for a smooth quasi-projective variety $U \subset X$. For example, with the above definition we will obtain $H^1_{\mathcal{D}}(U, \mathbb{Z}(1)) = H^0(U, \mathcal{O}_U^*)$, i.e. nowhere zero analytic functions on

U. For obvious reasons, one would accordingly like to recover the nowhere zero algebraic functions, i.e. $H^0_{Zar}(U, \mathcal{O}_U^*)$, where the notation of the Zariski topology Zar, is expected to mean that we now view $\mathcal{O}_U^*$ as the sheaf of nowhere zero regular functions on U. In order to fix this, Beilinson introduced Deligne's logarithmic complex into the picture. We can assume that $j : U = X - Y \hookrightarrow X$, where Y is a Normal Crossing Divisor (NCD) with smooth components. We define $\Omega^\bullet_X\langle Y\rangle$ to be the de Rham complex of meromorphic forms on X, holomorphic on U, with at most logarithmic poles along Y. So for example, in local analytic coordinates $(z_1, \cdots, z_d)$ on X, Y is given by $z_1 \cdots z_l = 0$, and $\Omega^1_X\langle Y\rangle$ has local frame $\{dz_1/z_1, \cdots, dz_l/z_l, dz_{l+1}, \cdots, dz_d\}$. One has a filtered complex

$$F^r\Omega^\bullet_X\langle Y\rangle = \Omega^{\bullet \geq r}_X\langle Y\rangle,$$

with Hodge to de Rham spectral sequence degenerating at E_1. This gives

$$F^r H^i(U, \mathbb{C}) = \mathbb{H}^i(F^r\Omega^\bullet_X\langle Y\rangle) \subset \mathbb{H}^i(\Omega^\bullet_X\langle Y\rangle) = H^i(U, \mathbb{C})$$

as the correct Hodge filtration regarding the MHS $H^i(U, \mathbb{Z})$.

2.1.24 Definition. *([15], Definition 2.6.) The Deligne-Beilinson cohomology $H^i_{\mathcal{D}}(U, \mathbb{A}(r))$ is defined as the hypercohomology of*

$$\mathbb{A}_{\mathcal{D}}(r) := Cone\left(Rj_*\mathbb{A}(r) \bigoplus F^r\Omega^\bullet_X\langle Y\rangle \xrightarrow{\epsilon - l} Rj_*\Omega^\bullet_U\right)[-1],$$

where $Rj_*\mathbb{A}(r)$ (resp. $Rj_*\Omega^\bullet_U$) is the direct image sheaf of $\mathbb{A}(r)$ (resp. of $\Omega^\bullet_U$) and where $Rj_*\Omega^\bullet_U$ is represented in such a way that both ϵ and l exists (for example by the direct image of an injective resolution of $\Omega^\bullet_U$). One can show that this is independent of the good compactification of U.

We get a short exact sequence

$$0 \to \frac{H^{i-1}(U, \mathbb{C})}{F^r H^{i-1}(U, \mathbb{C}) + H^{i-1}(X, \mathbb{A}(r))} \to H^i_{\mathcal{D}}(U, \mathbb{A}(r)) \to F^r \cap H^i(U, \mathbb{A}(r)) \to 0,$$

and (for $\mathbb{A} = \mathbb{Z}$) an isomorphism ([15], Proposition 2.12, iii)

$$H^1_{\mathcal{D}}(U, \mathbb{Z}(1)) \cong H^0_{Zar}(U, \mathcal{O}_U^*) := \mathcal{O}^*_{U,alg}(U)\ .$$

2.2 Chow group of X and its connection to cohomology

In this section we fix an algebraically closed subfield $k \subset \mathbb{C}$ and a smooth projective variety X/k of dimension d.

2.2.1 Adequate equivalence relations

The free abelian group $Z^r(X)$ is too large to work with in a meaningful way. For example, one would like to have a ring structure on $Z^*(X) := \oplus_r Z^r(X)$, for which one can define a ring structure, viz., an intersection theory of algebraic cycles. But one has to quotient out $Z^r(X)$ by an adequate equivalence relation, to accommodate such an intersection theory. This involves a moving lemma to ensure that two cycles meet in the expected dimension, as well as a good notion of intersection multiplicity. The moving lemma involves an adequate equivalence relation. The precise definition of an adequate relation can be found in [37]. We define some of the most studied such equivalence relations, the weakest of which is rational equivalence.

2.2.2 Definition. *Two cycles ξ_1 and ξ_2 in $Z^r(X)$ are **rationally equivalent**, denoted by $\xi_1 \sim_{rat} \xi_2$, if there exists a cycle $w \in Z^r(\mathbb{P}^1_k \times X)$ in sufficiently 'general position' [so that $w_*(t) := Pr_{2,*}((t \times X) \cdot w) \in Z^r(X)$ is defined for all $t \in \mathbb{P}^1_k$] such that $\xi_1 - \xi_2 = w_*(0) - w_*(\infty)$. Equivalently, one can define $\xi_1 \sim_{rat} \xi_2$ if there exists subvarieties W_i of codimension $r-1$ and rational functions $f_i \in k(W_i)^*$ such that $\xi_1 - \xi_2 = \sum_i^N div_{W_i}(f_i)$.*

2.2.3 Definition. *ξ_1 and ξ_2 are **algebraically equivalent**, denoted by $\xi_1 \sim_{alg} \xi_2$, if there exists a smooth connected curve C, a cycle $w \in Z^r(C \times X)$ in sufficiently 'general position' and points $p, q \in C$ such that $\xi_1 - \xi_2 = w_*(p) - w_*(q)$.*

Let $Z^r_{rat}(X) := \{\xi \in Z^k(X); \xi \sim_{rat} 0\}$, $Z^r_{alg}(X) := \{\xi \in Z^k(X); \xi \sim_{alg} 0\}$. We have the following hierarchy relation

$$Z^r_{rat}(X) \subseteq Z^r_{alg}(X) \subseteq Z^r_{\text{hom}}(X) \subset Z^r(X) \,,$$

where $Z^r_{\text{hom}}(X)$ are the null homologous cycles defined in the next section. We define

$$CH^r(X) := Z^r(X)/Z^r_{rat}(X) \ \ (Chow\ group),$$

$$CH^r_{alg}(X) := Z^r_{alg}(X)/Z^r_{rat}(X) \ \ (Chow\ group\ of\ cycles\ \sim_{alg} 0).$$

2.2.4 The cycle class maps

We develop two cycle class maps from the Chow group of a smooth projective variety X to its cohomology.

2.2.5 Definition. *There is a cycle class map*

$$cl_r : CH^r(X) \to H^{2r}_{de-Rham}(X, \mathbb{C}) \cong H^{2d-2r}_{de-Rham}(X, \mathbb{C})^\vee,$$

which can be defined in one of the following two (equivalent) ways:

1. *Let $V \subset X$ be a subvariety of codimension r and $w \in H^{2d-2r}(X, \mathbb{C})$. We define $cl_r(V)(w) = \frac{1}{(2\pi\sqrt{-1})^{d-r}}\delta_V := \frac{1}{(2\pi\sqrt{-1})^{d-r}}\int_{V^*} w$ and extend it linearly to $Z^r(X)$ to obtain*

   $$cl_r : Z^r(X) \to H^{2d-2r}(X, \mathbb{C})^\vee \cong H^{2r}(X, \mathbb{C})\ .$$

   Here $V^* = V - V_{sing}$. It follows from resolution of singularities that the integration is finite. Also, $Z^r_{rat}(X) \subset ker(cl_r)$ and hence one can define $cl_r : CH^r(X) \to H^{2r}(X, \mathbb{C})$.

2. *From twisted Poincare duality, one has the fundamental class generator*

   $$\{V\} \in H_{2d-2r}(V, \mathbb{Z}(d-r)) \cong H^{2r}_V(X, \mathbb{Z}(r)) \to H_{2d-2r}(X, \mathbb{Z}(d-r)) \cong H^{2r}(X, \mathbb{Z}(r)).$$

One can actually show that the image lies in

$$\Gamma(H^{2r}(X, \mathbb{Z}(r))) = H^{r,r}(X) \cap H^{2r}(X, \mathbb{Z}).$$

2.2.6 Example. For $r = d$ the cycle class map $cl_d : CH^d(X) \to \mathbb{Z}$ is the degree map, assigning the integer $\sum_i n_i$ to a zero-cycle $z = \sum_i n_i p_i$. It is obviously surjective.

At this point, we state the famous

2.2.7 Conjecture. (Hodge conjecture)

$$cl_r : CH^r(X;\mathbb{Q}) := CH^r(X) \otimes \mathbb{Q} \to \Gamma(H^{2r}(X,\mathbb{Q}(r))) = H^{2r}(X,\mathbb{Q}) \cap H^{r,r}(X),$$

is surjective.

Here we make the following observation: The original Hodge conjecture was made for smooth projective varieties defined over $\mathbb{C}$. Here our varieties are defined over an algebraically closed subfield k of $\mathbb{C}$, however

2.2.8 Lemma. *Hodge conjecture for smooth projective varieties over* $\mathbb{C} \implies$ *Hodge conjecture for smooth projective varieties over* k.

Proof. Let X/k be a smooth projective variety of dimension d and we denote $X/\mathbb{C} := X \times_k \mathbb{C}$. Let us assume that the cycle class map

$$cl_r : CH^r(X/\mathbb{C};\mathbb{Q}) \to H^{r,r}(X,\mathbb{Q}(r))$$

is surjective and let for $\gamma \in H^{r,r}(X,\mathbb{Q}(r))$, $\xi \in CH^r(X/\mathbb{C};\mathbb{Q})$ be such that $cl_r(\xi) = \gamma$. Now, the defining equations of ξ lies in a field K of finite transcendence degree (say ν) over k. One can find a smooth projective variety S/k such that $k(S) \cong K$ and spread ξ (not uniquely) to $\tilde{\xi} \in CH^r(S \times_k X;\mathbb{Q})$. Let $p \in S(k)$ (which exists by Nullstellensatz, since $k = \bar{k}$). We consider the cycle $p \times X \in CH^\nu(S \times_k X;\mathbb{Q})$ and the morphism $j_p : \underbrace{X \to S \times_k X}_{x \mapsto (p,x)}$. From the commutativity of the cycle class map with morphisms, we have the following chain of commutative diagram

$$\begin{array}{ccc} CH^r(S \times_k X;\mathbb{Q}) & \xrightarrow{j_p^*} & CH^r(X;\mathbb{Q}) \\ \downarrow cl_r & & \downarrow cl_r \\ H^{2r}(S \times_{\mathbb{C}} X,\mathbb{Q}(r)) & \xrightarrow{j_p^*} & H^{2r}(X,\mathbb{Q}(r)). \end{array}$$

Since $H^i(p,\mathbb{Q}) = 0$ for $i > 0$, the map $j_p^* : H^{2r}(S \times_{\mathbb{C}} X,\mathbb{Q}(r)) \to H^{2r}(X,\mathbb{Q}(r))$ factors through $(H^0(S,\mathbb{Q}) \otimes H^{2r}(X,\mathbb{Q}))(r)$. Hence, we get that $j_p^*(cl_r(\tilde{\xi})) = cl_r(\xi) = \gamma$ and the required result. □

2.2.9 Example. Using Lefschetz 1-1 theorem one can show that the cycle class map $cl_1 : CH^1(X) \to \Gamma(H^2(X,\mathbb{Z}(1)))$ is surjective ([37], Chapter 5).

Appendix (General Hodge Conjecture)

Grothendieck was the first to introduce the following notion of coniveau filtration on cohomology ([1]):

2.2.10 Definition. *The (descending) filtration by coniveau*

$$H^l(X,\mathbb{Q}) \supset N_k^1 H^l(X,\mathbb{Q}) \supset N_k^2 H^l(X,\mathbb{Q}) \supset \cdots \supset N_k^l H^l(X,\mathbb{Q}) \supset 0$$

on singular cohomology is defined by any of the following three equivalent definitions

$$N_k^i H^l(X,\mathbb{Q}) := ker\left(H^l(X,\mathbb{Q}) \to \varinjlim_{cd_X Y \geq i} H^l(X - Y, \mathbb{Q}) \right)$$

$$:= Image\left(\sum_{cd_X Y \geq i} H^l_Y(X,\mathbb{Q}) \to H^l(X,\mathbb{Q}) \right)$$

$$:= Gysin\ Images\left(\sum_{cd_X Y \geq i} H^{l-2r}(\tilde{Y},\mathbb{Q}) \to H^l(X,\mathbb{Q}) \right),$$

where $\tilde{Y} \to Y$ is a desingularization.

Note that $N_k^i H^l(X,\mathbb{Q}) \subset F^i H^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q})$ is not an equality, since the coniveau pieces are Hodge substructures of $H^l(X,\mathbb{Q})$ but (as shown by Grothendieck's counterexample in [1]) $F^i H^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q})$ need not be. Let $N_H^i H^l(X,\mathbb{Q})$ be the largest Hodge-substructure contained in $F^i H^l(X,\mathbb{C}) \cap H^l(X,\mathbb{Q})$.

2.2.11 Conjecture. (**Grothendieck Amended General Hodge Conjecture (GHC)**) The inclusion $N_k^i H^l(X,\mathbb{Q}) \subset N_H^i H^l(X,\mathbb{Q})$ is an equality. For $l = 2r$ and $i = r$, we recover the classical Hodge conjecture (Conjecture 2.2.7).

2.2.12 Remark. Grothendieck originally made this conjecture for smooth projective varieties defined over $\mathbb{C}$. But using a "spread" argument similar in spirit to that of Lemma 2.2.8, one can show that $N_k^i H^l(X,\mathbb{Q}) = N_{\mathbb{C}}^i H^l(X,\mathbb{Q})$.

2.2.13 Definition. *(Abel-Jacobi map) Let $CH^r_{hom}(X) = ker(cl_r)$. We define the Abel-Jacobi map*

$$\Phi_r : CH^r_{hom}(X) \to J^r(X) := J(H^{2r-1}(X,\mathbb{Z}(r))),$$

in the following way. Recall that

$$J(H^{2r-1}(X,\mathbb{Z}(r))) = \frac{F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})^\vee}{H_{2d-2r+1}(X,\mathbb{Z}(d-r))}.$$

Let $\xi \in CH^r_{hom}(X)$. *Then* $\xi = \partial\zeta$ *for a real* $2d-2r+1$ *dimensional chain* ζ *in* X. *Let* $\{w\} \in F^{d-r+1}H^{2d-2r+1}(X,\mathbb{C})$. *We define*

$$\Phi_r(\xi)(w) = \frac{1}{(2\pi\sqrt{-1})^{d-r}} \int_\zeta w \Big/ periods\,.$$

It is easy to show that if $\xi = \partial\zeta'$ *for another chain* ζ', *then* $\int_\zeta w = \int_{\zeta'} w$ *modulo periods. Also, from a result of Dolbeault (Lemma 1.7 of [39]), one can show that* Φ_r *is independent of the cohomological representative of* $\{w\}$.

Alternate definition: We observe that

$$H^{2r-1}_{|\xi|}(X,\mathbb{Z}(r)) \cong H_{2d-2r+1}(|\xi|,\mathbb{Z}(d-r)) = 0,$$

as $\dim_{\mathbb{R}}|\xi| = 2d-2r$. Also, there is the cycle class map $cl_r : \xi \mapsto \{\xi\} \in H_{2d-2r}(|\xi|,\mathbb{Z}(d-r)) \cong H^{2r}_{|\xi|}(X,\mathbb{Z}(r))$. Further, since $\xi \in CH^r_{hom}(X)$ (denoted by $\xi \sim_{hom} 0$), we have by duality

$$[\xi] \in H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ := ker\left(H^{2r}_{|\xi|}(X,\mathbb{Z}(r)) \to H^{2r}(X,\mathbb{Z}(r))\right).$$

Hence ξ determines a morphism of MHS, $\mathbb{Z}(0) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ$. From the short exact sequence of MHS

$$0 \to H^{2r-1}(X,\mathbb{Z}(r)) \to H^{2r-1}(X-|\xi|,\mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X,\mathbb{Z}(r))^\circ \to 0,$$

we can pullback via the above morphism to obtain another short exact sequence of MHS,

$$0 \to H^{2r-1}(X,\mathbb{Z}(r)) \to E \to \mathbb{Z}(0) \to 0.$$

Then $\Phi_r(\xi) = \{E\} \in Ext^1_{MHS}(\mathbb{Z}(0), H^{2r-1}(X,\mathbb{Z}(r))) = J(H^{2r-1}(X,\mathbb{Z}(r)))$. It can be shown that this alternate definition of Φ_r agrees with that given in 2.2.13.

2.2.14 Example. It can be shown that the image $\Phi_r(CH^r_{alg}(X)) =: J^r_{alg}(X) \subset J(X)$ is an abelian variety defined over k. Here we recall the following description of $J^r_{alg}(X)_{\mathbb{Q}}$ given in terms of coniveau filtration: Observe that $N^{r-1}_k H^{2r-1}(X,\mathbb{Q})\otimes \mathbb{C} = H^{r,r-1}_a(X)\oplus H^{r-1,r}_a(X)$, where we describe $H^{r,r-1}_a(X)$ as $Pr_{r-1,r}(N^{r-1}_k H^{2r-1}(X,\mathbb{Q}))\otimes \mathbb{C}) \subset H^{r-1,r}(X)$ (similarly for $H^{r-1,r}_a(X)$). Then, $J^r_{alg}(X)_{\mathbb{Q}}$ can be described as

$$J^r_{alg}(X)_{\mathbb{Q}} \simeq J\big(N^{r-1}_k H^{2r-1}(X,\mathbb{Q}(r))\big) \simeq H^{r-1,r}_a(X)/N^{r-1}_k H^{2r-1}(X,\mathbb{Q}(r)) \subset J^r(X)_{\mathbb{Q}}.$$

For details, see Proposition 12.31 of [37]. In general, the following is a deep question: What is the image $\Phi_r(CH^r_{hom}(X))$? We do know that the Griffiths group $Griff^r(X) := CH^r_{hom}(X)/CH^r_{alg}(X)$ is countable (although non-trivial in many cases). From this, and the above description of $J^r_{alg}(X)_{\mathbb{Q}}$ we can conclude that the (rational) Abel-Jacobi map

$$\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J^r(X)_{\mathbb{Q}},$$

is not onto if $N^{r-1}_H H^{2r-1}(X,\mathbb{C}) \neq H^{2r-1}(X,\mathbb{C})$ (see discussions following Proposition 3.2 in [39]).

2.2.15 Example. Recall the isomorphism $\Phi_1 : CH^1_{hom}(X) \cong J^1(X)$, which also shows that $CH^1_{hom}(X) = CH^1_{alg}(X)$. We note that $CH^d_{hom}(X) = CH^d_{alg}(X)$ and the abelian variety $J^d(X)$ (known as the *Albanese* variety of X) is dual to $J^1(X)$. The situation however, is very different for $1 < r < d$, where as seen above, Φ_r is not onto in general and neither can we say $CH^r_{hom}(X) = CH^r_{alg}(X)$. The kernel, $ker(\Phi_r)$ is another important object of study. In [14], Mumford has the following result:

2.2.16 Theorem. *Let X be a smooth projective complex surface (i.e. of dimension 2), with geometric genus $dim_{\mathbb{C}} H^{2,0}(X) \neq 0$. Then*

$$ker\left(\Phi_2 : CH^2_{hom}(X) \to J^2(X)\right),$$

is non-trivial.

Thus, there's no easy answer ! At this point, we recall the folklore conjecture due to Bloch and Beilinson

2.2.17 Conjecture. (**Bloch-Beilinson Conjecture, BBC**) If X is a smooth projective variety defined over $\overline{\mathbb{Q}}$, then the (rational) Abel-Jacobi map

$$\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J(X)_{\mathbb{Q}} = J(H^{2r-1}(X,\mathbb{Q}(r))),$$

is injective.

There are no nontrivial concrete examples of this conjecture, which incidentally is formulated out of exclusion. If $\mathrm{trdeg}_{\overline{\mathbb{Q}}} k = 1$, there are examples by Schoen ([52]), Green-Griffiths-Paranjape ([45]) and James Lewis ([40]) that the kernel of the Abel-Jacobi map is non-zero. The reader is also encouraged to read sections 4 and 5 of [5] (specifically, Lemma 4.0.7, Remark 4.0.8 and the discussion following Lemma 5.6) to get another motivation for this conjecture.

2.2.18 Example. Notice that any $\xi \in CH^r_{alg}(X)$ is in the image of a homomorphism $J^1(\Gamma) \to CH^r_{alg}(X)$ for a smooth projective curve Γ (from definition). Hence we can conclude that $CH^r_{alg}(X)$ is divisible.

2.2.19 Example. We end this section by relating the cycle class maps with Deligne cohomology. One can define a cycle class map into Deligne cohomology

$$cl_{r,\mathcal{D}} : CH^r(X) \to H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)),$$

with the following prescription. Let $\xi \in CH^r(X)$ with support $|\xi|$. One has a long exact sequence of cohomology with support

$$\cdots \to H^{2r-1}_{|\xi|}(X, \mathbb{Z}(r)) \oplus F^r H^{2r-1}_{|\xi|}(X, \mathbb{C}) \to H^{2r-1}_{|\xi|}(X, \mathbb{C})$$

$$\to H^{2r}_{\mathcal{D},|\xi|}(X, \mathbb{Z}(r)) \to H^{2r}_{|\xi|}(X, \mathbb{Z}(r)) \oplus F^r H^{2r}_{|\xi|}(X, \mathbb{C}) \to H^{2r}_{|\xi|}(X, \mathbb{C}) \to \cdots$$

Via Poincaré duality, one has cycle class maps

$$\xi \mapsto [(2\pi i)^{r-d}(\xi, \delta_\xi)] \in ker(H^{2r}_{|\xi|}(X, \mathbb{Z}(r)) \oplus F^r H^{2r}(X, \mathbb{C}) \to H^{2r}_{|\xi|}(X, \mathbb{C})).$$

From the fact $H^{2r-1}_{|\xi|}(X, \mathbb{C}) = 0$, we get an element $[\xi] \in H^{2r}_{\mathcal{D},|\xi|}(X, \mathbb{Z}(r))$ and the cycle class map

$$cl_{r,\mathcal{D}} : CH^r(X) \xrightarrow{\xi \mapsto [\xi]} H^{2r}_{\mathcal{D},|\xi|}(X, \mathbb{Z}(r)) \xrightarrow{\text{``}forgetful\ map} H^{2r}_{\mathcal{D}}(X, \mathbb{Z}(r)).$$

The Deligne cycle class map $cl_{r,\mathcal{D}}$ combines both the classical cycle class and the Abel-Jacobi map in the following (commutative) diagram:

$$\begin{array}{ccccc} CH^r_{hom}(X) & \hookrightarrow & CH^r(X) & \twoheadrightarrow & \dfrac{CH^r(X)}{CH^r_{hom}(X)} \\ \downarrow \Phi_r & & \downarrow cl_{r,\mathcal{D}} & & \downarrow cl_r \\ J(H^{2r-1}(X,\mathbb{Z}(r))) & \hookrightarrow & H^{2r}_{\mathcal{D}}(X,\mathbb{Z}(r)) & \twoheadrightarrow & \Gamma(H^{2r}(X,\mathbb{Z}(r))). \end{array}$$

2.3 Lefschetz theory

Let X/k be a smooth projective variety of dimension d, where k is a subfield of $\mathbb{C}$. We know that $X(\mathbb{C})$ is complex projective algebraic with a choice of ***polarization*** ω_X induced by an algebraic cycle $Y \in CH^1(X)$ (called the hyperplane section of X). Define the morphism (of HS)

$$L_X : A^i(X) \to A^{i+2}(X),\ \eta \mapsto \eta \wedge \omega_X\ ,$$

with an adjoint (with respect to Hodge-inner product)

$$\Lambda_X : A^i(X) \to A^{i-2}(X)\ .$$

From abstract Hodge/Lefschetz theory one gets the following results

2.3.1 Theorem. *(Strong Lefschetz theorem)*

1. *The map* $L^i_X : H^{d-i}(X,\mathbb{Q}(r)) \xrightarrow{\cong} H^{d+i}(X,\mathbb{Q}(r+i))$ *is an isomorphism.*
2. *Moreover, if we define the primitive cohomology*

$$Prim^{d-i}(X,\mathbb{Q}(r)) = Ker\left(L^{i+1}_X : H^{d-i}(X,\mathbb{Q}(r)) \to H^{d+i+2}(X,\mathbb{Q}(r+i+1))\right)\ ,$$

we arrive at the Lefschetz primitive decomposition (for $i = 0,1,2,\cdots$*)*

$$H^i(X,\mathbb{Q}(r)) \cong \oplus_{j\geq(i-d)_+} L^j_X(Prim^{i-2j}(X,\mathbb{Q}(r-j)))\ .$$

The primitive decomposition is compatible with the Hodge decomposition of $H^i(X,\mathbb{C})$*, once we set*

$$Prim^{p,q}(X) := Ker(L^{d-i+1}_X : H^{p,q}(X) \to H^{d-p+1,d-q+1}(X))\ .$$

At this point we would also like to state a weak version of Lefschetz theorem

2.3.2 Theorem. *(Weak Lefschetz Theorem) Let* $Y \xhookrightarrow{j} X$ *be any smooth hyperplane section of* X*. Then the restriction map*

$$j^* : H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$$

is an isomorphism for $i \leq d-2$*, and injective for* $i = d-1$*.*

This theorem is a consequence of the following result by Andreotti and Frankel (using basic Morse theory). We call it the affine version of weak Lefschetz theorem:

2.3.3 Theorem. *Let* U/k *be a smooth affine variety of dimension* d*. Then* $U(\mathbb{C}) \subset \mathbb{C}^r$ *as a closed* d*-dimensional complex submanifold, has the homotopy type of a CW-complex of real dimension* $\leq d$*. As a consequence*

$$H^i(U, \mathbb{Q}) = 0, \; \forall i > d \, .$$

One uses the Lefschetz theory to develop the following bilinear relations on cohomology:

Hodge-Riemann bilinear relations (Untwisted version)

We introduce a real bilinear form on $H^i(X, \mathbb{Q})$ using the following prescription: Given

$$\xi = \oplus_{j \geq (i-d)_+} L_X^j(\xi_j), \;\; \eta = \oplus_{j \geq (i-d)_+} L_X^j(\eta_j) \in H^i(X, \mathbb{Q})$$

with $\xi_j, \eta_j \in Prim^{i-2j}(X, \mathbb{Q})$, set

$$Q(\xi, \eta) = \sum_{j \geq (i-d)_+} (-1)^{(i(i+1)/2) \, +j} \int_X L_X^{d-i+2j}(\xi_j \wedge \eta_j) \, .$$

We also introduce the Weil operator $C = \oplus_{p+q=i}(\sqrt{-1})^{p-q} Pr_{p,q}$ where $Pr_{p,q} : H^i(X, \mathbb{C}) \twoheadrightarrow H^{p,q}(X)$ is the obvious projection. Then , it can be shown that the bilinear form Q has the following property

$$Q(\xi, C(\bar{\xi})) > 0 \; for \; \xi \neq 0 \, . \qquad (2.3.3.1)$$

From equation (2.3), we deduce the

2.3.4 Corollary. *(Hodge-Riemann bilinear relations) (see [23], page 123). The bilinear form Q satisfies the following relations:*

- $Q(Prim^{p,q}(X,\mathbb{Q}), Prim^{s,t}(X,\mathbb{Q})) = 0 \ if \ s \neq q$.
- $(\sqrt{-1})^{-i}(-1)^q Q(\xi,\bar{\xi}) > 0 \ if \ 0 \neq \xi \in Prim^{p,q}(X,\mathbb{Q}) \ (p+q=i)$.

If we set $S = (-1)^i Q$ on $Prim^i(X,\mathbb{Q})$, then from the discussion following Theorem 2.34 it follows that

$$S : Prim^i(X,\mathbb{Q}) \times Prim^i(X,\mathbb{Q}) \to \mathbb{Q} \qquad (2.3.4.1)$$

is bilinear (and non-degenerate) symmetric if i is even, skew if i is odd.

2.3.5 Remark. As we shall see in the next chapter, an analogous Lefschetz theory for Chow groups is largely conjectural, with only a few concrete results. It forms a large part of Grothendieck's collection of standard conjectures in algebraic geometry. Assuming such conjectures, a part of the motivation for this thesis came from the desire to develop 'Hodge-Riemann type bilinear relations' for Chow groups.

We end this chapter by generalizing Corollary 2.37 in case of a pure Hodge structure.

2.3.6 Definition. *A* ***polarization*** *of a (pure) $\mathbb{Q}$-Hodge structure $V_{\mathbb{Q}}$ (of weight i) is a (non-degenerate) bilinear form $S : V_{\mathbb{Q}} \times V_{\mathbb{Q}} \to \mathbb{Q}$, symmetric if i is even, skew if i is odd, and satisfying*

- $S(V^{p,q}, V^{s,t}) = 0$ *unless* $p = t, s = q$.
- $(\sqrt{-1})^{p-q} S(\xi,\bar{\xi}) > 0$ *if* $0 \neq \xi \in V^{p,q}$,

where $p+q=i$. $V_{\mathbb{Q}}$ is called a polarized Hodge structure.

2.3.7 Example. By Corollary 2.37, the cohomology of a smooth projective variety X carries a natural polarization given by the Hodge-Riemann bilinear relations.

2.3.8 Remark. Polarized Hodge structures are semi-simple in the sense that if $V_{\mathbb{Q}}$ is a polarized HS with polarization S and $V_{1,\mathbb{Q}}$ is a a sub-HS, then $V_{1,\mathbb{Q}}$ and $V_{1,\mathbb{Q}}^{\perp} := \{u \in V_{\mathbb{Q}};\ S(u, V_{1,\mathbb{Q}}) = 0\}$ are both polarized HS, with polarization given by restricting S. Moreover, we have

$$V_{\mathbb{Q}} \cong V_{1,\mathbb{Q}} \oplus V_{1,\mathbb{Q}}^{\perp}\,.$$

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Chapter 3

Motives and a conjectural filtration on Chow groups

Unless otherwise stated, k will denote a subfield of $\mathbb{C}$ and X/k will denote a smooth projective variety over k. The category of such varieties will be denoted by $V(k)$.

3.1 Motives

A general reference for this section is Section 4.1 of [48].

3.1.1 Motivation

In the early 1960s Grothendieck, along with Artin and Verdier, developed the l-adic cohomology groups $H^i_{et}(X, \mathbb{Q}_l)$ for every prime $l \neq 0$. Since $k \subset \mathbb{C}$, there is also the classical singular $H^i(X, \mathbb{Q})$ and the de-Rham cohomology groups $H^i_{de-Rham}(X, \mathbb{C})$. This gives us plenty of cohomology theories, each with their own advantages and disadvantages ! There is also the de-Rham isomorphism theorem: $H^i(X, \mathbb{C}) \cong H^i_{de-Rham}(X, \mathbb{C})$ and the comparison isomorphisms:

$$H^i(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong H^i_{sing}(X, \mathbb{Q}_l) \cong H^i_{et}(X, \mathbb{Q}_l) \ ,$$

between the singular and the l-adic cohomology groups. It was Grothendieck's genius that realized the necessity of an underlying category of 'motives' of which all these different cohomology theories share in common as realization functors. Pictorially, one can describe it by the following arrow

$$\mathcal{M}(k) \to (\textit{vector spaces})/F,\ M \mapsto H^*(M, F)\ ,$$

where $\mathcal{M}(k)$ denote the (conjectural) category of motives, F is a field (either $\mathbb{Q}$ or $\mathbb{Q}_l$) and $H^*(\ , F)$ is a cohomology theory (usually l-adic or the singular).

3.1.2 Correspondences and projectors

Before we begin, we define an equivalence relation (given by Grothendieck) known as the *numerical equivalence* which we could have put in Chapter 2. But since it first arose in the theory of motives it is probably apt to define it here !

3.1.3 Definition. *Let X be of dimension d. An algebraic cycle $\xi \in Z^r(X)$ is said to be* ***numerically equivalent*** *to zero, denoted by $\xi \sim_{num} 0$, if the intersection number of $\xi \cdot \xi'$ is zero for all $\xi' \in Z^{d-r}(X)$ (strictly speaking, for all $\xi' \in Z^{d-i}(X)$ for which the intersection number is defined). Let $Z^r_{num}(X) := \{\xi \in Z^r(X); \xi \sim_{num} 0\}$. One has the following inclusions among the different equivalence relations defined so far:*

$$Z^r_{rat}(X) \subset Z^r_{alg}(X) \subset Z^r_{hom}(X) \subset Z^r_{num}(X)\ ,$$

and dividing out by the rational equivalence

$$CH^r_{alg}(X) \subset CH^r_{hom}(X) \subset CH^r_{num}(X).$$

In this context, let us state the following fundamental conjecture, which is an easy consequence of the Hodge conjecture, and indeed a consequence of the weaker hard Lefschetz conjecture, to be discussed later.

3.1.4 Conjecture. $Z^r_{hom}(X) \otimes \mathbb{Q} = Z^r_{num}(X) \otimes \mathbb{Q}$

Notation: From now on, we will avoid torsion and consider Chow groups tensored by $\mathbb{Q}$. We will use the notation $CH^*(X; \mathbb{Q})$ to denote $CH^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

3.1.5 Definition. *Let X and Y be objects in $V(k)$ of dimensions d and e respectively, and we fix an equivalence relation $\sim$ .The* ***group of correspon-***

dences *between* X *and* Y *of* ***degree*** r *with respect to* $\sim$ *is defined by*

$$C^{d+r}_{\sim}(X \times_k Y; \mathbb{Q}) := Z^{d+r}(X \times_k Y; \mathbb{Q})/Z^{d+r}_{\sim}(X \times_k Y; \mathbb{Q}) \ .$$

Let $f \in C^{d+r}_{\sim}(X \times_k Y; \mathbb{Q})$*, then* ${}^t f \in C^{e+(d+r-e)}_{\sim}(Y \times_k X; \mathbb{Q})$ *denotes the* ***transpose*** *of* f*. It is a correspondence from between* Y *and* X *of degree* $(d+r)-e$*.*

3.1.6 Example. Let $\phi : X \to Y$ be the usual morphism of varieties and let Γ_ϕ be the graph. Then $\Gamma_\phi \in C^{d+(e-d)}_{\sim}(X \times_k Y; \mathbb{Q})$ is a correspondence of degree $e-d$ and ${}^t\Gamma_\phi \in C^{e}_{\sim}(Y \times_k X; \mathbb{Q})$ is a correspondence of degree 0.

3.1.7 Remark. As a special case of the above example, consider the identity $Id_X : X \to X$ morphism. Its graph is given by the diagonal correspondence $\Delta_X \in CH^d(X \times_k X; \mathbb{Q})$. Let $[\Delta_X] \in H^{2d}(X \times_{\mathbb{C}} X, \mathbb{Q}(d))$ denote the cycle class image of the diagonal correspondence. By the Künneth decomposition we get

$$[\Delta_X] = \sum_i [\Delta_X]_{2d-i,i}$$

where $[\Delta_X]_{2d-i,i} \in H^{2d-i}(X, \mathbb{Q}) \otimes H^i(X, \mathbb{Q})(d)$ are the Künneth components. They correspond to the identity homomorphism $Id_i : H^i(X, \mathbb{Q}) \to H^i(X, \mathbb{Q})$ through the isomorphism

$$H^{2d-i}(X, \mathbb{Q}) \otimes H^i(X, \mathbb{Q})(d) \cong Hom_{\mathbb{Q}}(H^i(X, \mathbb{Q}), H^i(X, \mathbb{Q})) \ .$$

Composition: Given two correspondences $f \in C^*_{\sim}(X \times_k Y; \mathbb{Q})$ and $g \in C^*_{\sim}(Y \times_k Z; \mathbb{Q})$ the composition $g \bullet f \in C^*_{\sim}(X \times_k Z; \mathbb{Q})$ is defined by

$$g \bullet f := Pr_{X \times_k Z}\left((f \times_k Z) \cdot (X \times_k g)\right)$$

where $\cdot$ is the intersection product of algebraic cycles on $X \times_k Y \times_k Z$.

Operations on algebraic cycles: A correspondence $f \in C^{d+r}(X \times_k Y; \mathbb{Q})$ of degree r operates on $C^*_{\sim}(X; \mathbb{Q})$ by the prescription

$$f_* : C^i_{\sim}(X; \mathbb{Q}) \to C^{i+r}_{\sim}(Y; \mathbb{Q}),\ Z \mapsto f_*(Z) := (Pr_Y)_* \left[f \cdot (Pr_X)^*(Z)\right]$$

for $Z \in C^i_{\sim}(X; \mathbb{Q})$. If $\sim$ is or finer than homological equivalence, then f also operates on cohomology $f_* : H^i(X, \mathbb{Q}) \to H^{i+2r}(X, \mathbb{Q}(r))$.

3.1.8 Remark. 1. Correspondences with respect to rational equivalence operate both on Chow groups and cohomology while those with respect to homological equivalence operate on cohomology but not on Chow groups. Finally, correspondences with respect to numerical equivalence act on the cohomology groups provided Conjecture 3.2 is true.

2. Under the composition of correspondences, $C^*_\sim(X \times_k X; \mathbb{Q})$ becomes a ring with Δ_X as the unity and $C^d_\sim(X \times_k X; \mathbb{Q})$ becomes a subring.

Projectors

3.1.9 Definition. *A correspondence $p \in C^d_\sim(X \times_k X; \mathbb{Q})$ is called a **projector** of X (with respect to $\sim$) if $p^2 := p \bullet p = p$. Two projectors $p, q \in C^d_\sim(X \times_k X; \mathbb{Q})$ are **orthogonal** if $p \bullet q = q \bullet p = 0$*

3.1.10 Example. 1. $p = \Delta_X$ is obviously a projector.

2. For the graph Γ_ϕ of a morphism $\phi : X \to Y$ of finite degree m, $p = \frac{1}{m} {}^t\Gamma_\phi \bullet \Gamma_\phi$ is a projector.

3. For a projector p, $\Delta_X - p$ is a projector orthogonal to p and one has the direct sum decomposition

$$C^*_\sim(X; \mathbb{Q}) \cong p_*(C^*_\sim(X; \mathbb{Q})) \oplus (\Delta_X - p)_*(C^*_\sim(X; \mathbb{Q})) \ .$$

We use the notation $(C^*_\sim(X; \mathbb{Q}))^\perp$ for $(\Delta_X - p)_*(C^*_\sim(X; \mathbb{Q}))$.

3.1.11 Grothendieck's definition of (pure) motives

For an adequate equivalence relation $\sim$, the category $\mathcal{M}_\sim(k)$ of (pure) motives consists of *objects* (X, p, m), where $X \in V(k)$, p is a projector of X and $m \in \mathbb{Z}$ with the following *morphisms*: if $M = (X, p, m)$ and $N = (Y, q, n)$, define

$$Hom_{\mathcal{M}_\sim(k)}(M, N) := \{q \bullet f \bullet p; f \in C^{d+(n-m)}_\sim(X \times_k Y; \mathbb{Q})\}, \ d = dim(X)$$

and the composition of morphisms is defined via the composition of correspondences. The objects $M = (X, p, m)$ are called motives with respect to $\sim$. The full subcategory $\mathcal{M}^+_\sim(k) := \{M' = (X, p, 0)\}$ is usually called the effective (pure) motives.

3.1.12 Example. 1. There exists a functor $h_\sim : V^{opp}(k) \to \mathcal{M}_\sim^+(k)$ defined as $h_\sim(X) = (X, \Delta_X, 0)$.

2. $1_k := (Spec\,k, Id_k, 0)$, is the trivial motive (i.e. the motive of a point).

3. Let $k = \bar{k}$. Fix a point $e \in X(k)$ and consider $\pi_0 := e \times_k X$ and $\pi_{2d} := X \times_k e$ ($d = dim(X)$). They are both projectors orthogonal to each other. Set $h^0_\sim(X) := (X, \pi_0, 0)$ and $h^{2d}_\sim(X) := (X, \pi_{2d}, 0)$. Then we have the following isomorphism in the category of motives: $h^0_\sim(X) \cong 1_k$ and $h^{2d}_\sim(X) \cong (Spec\,k, Id_k, -d)$ where $d = dim(X)$ for any $X \in V(k)$.

4. Set $\mathbb{T} := (Speck, Id_k, 1)$, $\mathbb{L} := (Spec\,k, Id_k, -1)$ and call them **Tate** and **Lefschetz** motive respectively.

So, we have a very concrete definition of motives (or pure motives, but we will just say motives from now on) with examples. What is still conjectural though are some of the properties that a good category of motives should have. For now, let's list some of the known properties:

- It is known that $\mathcal{M}_\sim(k)$ is a pseudo abelian category. It has been proved by Jannsen ([28]) that the category $\mathcal{M}_{num}(k)$ is indeed an abelian, semi-simple category (actually Jannsen proved an if and only if condition).

- $\mathcal{M}_\sim(k)$ has tensor product : for two objects $M = (X, p, m)$, $N = (Y, q, n)$ define $M \otimes N := (X \times_k Y, p \times_k q, m + n)$ and an involution : $M = (X, p, m) \mapsto \hat{M} := (X, {}^t p, d - m)$, $d = dim(X)$.

Relation between various $\mathcal{M}_\sim(k)$: Fundamentally there are (a priori) three different category of motives:

- **Chow motives:** If $\sim$ is rational equivalence, we write $CH\mathcal{M}(k) := \mathcal{M}_{rat}(k)$ and $ch(X) := h_{rat}(X)$

- **Homological motives:** Fixing (since $k \subset \mathbb{C}$) the singular cohomology theory $H^*(X, \mathbb{Q})$, we get $\mathcal{M}_{hom}(k)$ and $h_{hom}(X)$

- **Numerical or Grothendieck motives**: We take $\sim$ to be numerical equivalence and we get $\mathcal{M}_{num}(k)$ and $h_{num}(X)$

We have the following arrows

$$V^{opp}(k) \xrightarrow{ch} CH\mathcal{M}(k) \to \mathcal{M}_{hom}(k) \xrightarrow{\cong ?, Conjecture\ 3.2} \mathcal{M}_{num}(k)\ .$$

Note again that the Hodge-conjecture implies Conjecture 3.1.4 and hence the isomorphism $\mathcal{M}_{hom}(k) \cong \mathcal{M}_{num}(k)$.

3.1.13 Cycle groups and cohomology of motives

For $M = (X, p, m) \in \mathcal{M}_{\sim}(k)$, define

$$C^r_{\sim}(M) := \{Im(p_* : C^{r+m}_{\sim}(X;\mathbb{Q}) \to C^{r+m}_{\sim}(X;\mathbb{Q}))\} .$$

In particular if $M \in CH\mathcal{M}(k)$, then we have the Chow groups/Chow vector spaces of motive $CH^r(M)$. Also, if $\sim$ is equal or finer than homological equivalence, then p acts on cohomology and we define

$$H^i(M) := \{Im(p_* : H^{i+2m}(X, \mathbb{Q}) \to H^{i+2m}(X, \mathbb{Q}))\} ,$$

and get a realization functor

$$real : \mathcal{M}_{\sim}(k) \to (vector\ spaces)/\mathbb{Q} .$$

3.1.14 Remark. The importance of Conjecture 3.1.4 becomes apparent now. The realization functor is from the category of motives with respect to homological equivalence (or finer than homological equivalence). On the other hand, the category $\mathcal{M}_{num}(k)$ is closer to Grothendieck's vision of motives since it does not depend on any cohomology theory (also it is an abelian, semi-simple category by [28]). The truth of Conjecture 3.1.4 will merge these two properties together beautifully.

3.2 Standard conjectures (Section 4.2 of [48])

Let k now denote an algebraically closed subfield of $\mathbb{C}$, as before we fix the category of smooth projective varieties as $V(k)$. All fibre products are taken with respect to the base field k. The first conjecture is an old one, usually called the **Künneth conjecture**:

3.2.1 Conjecture. For $X \in V(k)$ of dimension d, the Künneth components $[\Delta_X]_{2d-i,i}$ of the cohomology of the diagonal class $[\Delta_X] \in H^{2d}(X \times X, \mathbb{Q})$ are algebraic classes, i.e. there exists algebraic cycles $\Delta_X(2d-i,i) \in CH^d(X \times X;\mathbb{Q})$ such that $[\Delta_X(2d-i,i)] = [\Delta_X]_{2d-i,i}$.

This conjecture easily follows from the Hodge conjecture, but can actually be deduced from the hard Lefschetz conjecture stated below. See [33]. For examples where this conjecture holds, see subsection 3.2.2.

3.2.2 Standard conjecture of Lefschetz type

We begin with the following
3.2.3 Proposition. *Let X and Y in $V(k)$ of dimensions d and e respectively and $\xi \in CH^r(X \times Y;\mathbb{Q})$. Let $i = r - d$. Then the Künneth component $[\xi]_{p,q}$ induces $[\xi]_* : H^l(X,\mathbb{Q}(m)) \to H^{l+2i}(Y,\mathbb{Q}(m+i))$, a morphism of Hodge-structure, where $l = 2d - p$.*

Keeping this proposition in mind, we introduce the following
3.2.4 Definition. *Let $p, q \in \mathbb{Z}$ with $p+q$ even. A linear map*

$$\lambda : H^p(X,\mathbb{Q}(m)) \to H^q(Y,\mathbb{Q}((p-q/2)-m))$$

*is said to be **algebraic** if it is induced by $\xi \in CH^{(2d-p+q)/2}(X \times Y;\mathbb{Q})$.*
3.2.5 Remark. By the Hodge conjecture, λ being algebraic is the same thing as saying that λ is a morphism of Hodge structure. Also λ being algebraic does not necessarily mean that the class defined by λ in $H^{2d-p}(X,\mathbb{Q}(d-p+m)) \otimes H^q(Y,\mathbb{Q}((p-q/2)-m))$ is induced by an algebraic cycle (although, Hodge conjecture would imply even that).

Now, let $Y \in CH^1(X;\mathbb{Q})$ be a hyperplane section and $L_X : H^i(X,\mathbb{Q}(r)) \to H^{i+2}(X,\mathbb{Q}(r+1))$ be the operator associated to it. We have seen before the hard and weak versions of Lefschetz theorem in cohomology.

Note that L_X is induced by the algebraic cycle $\Delta_X(Y) := \{(x,x) \in \Delta_X;\, x \in Y\} \in CH^{d+1}(X \times X;\mathbb{Q})$. As seen before, there is an operator

$$\Lambda_X : H^i(X,\mathbb{Q}(r)) \to H^{i-2}(X,\mathbb{Q}(r-1))$$

which serves 'almost as an inverse' of L_X. Now, Λ_X being a linear map in cohomology, using Poincaré duality and Künneth decomposition, it can be seen as a topological correspondence in $H^{2(d-1)}(X \times X, \mathbb{Q}(d-1))$. We have the following conjecture

3.2.6 Conjecture. Λ (and hence Λ^i for any $i \in \mathbb{Z}$) is algebraic.

3.2.7 Remark. As $\Lambda \in H^{2d-i}(X, \mathbb{Q}(d-i+r)) \otimes H^{i-2}(X, \mathbb{Q}(r-1)) \cap H^{d-1,d-1}(X \times X)$, Conjecture 3.11 is implied by Hodge conjecture. As such, this conjecture has the following properties and known cases:

1. If Conjecture 3.2.6 holds for one hyperplane section Y (and the operator L_X), then it holds for any such sections.

2. Conjecture 3.2.6 implies the following conjecture: Let

$$\overline{CH}^r(X; \mathbb{Q}) := CH^r(X; \mathbb{Q}) / CH^r_{hom}(X; \mathbb{Q}) \subset H^{2r}(X, \mathbb{Q}(r)) \, .$$

   Then

   3.2.8 Conjecture. $L^{d-2r} : \overline{CH}^r(X; \mathbb{Q}) \to \overline{CH}^{d-r}(X; \mathbb{Q})$ is an isomorphism

3. Conjecture 3.2.6 implies the Künneth conjecture (Conjecture 3.2.1).

4. Conjecture 3.2.6 is known for projective spaces, Grassmannians, curves (trivial), surfaces (Grothendieck, [33] and abelian varieties (Lieberman, [33]).

3.2.9 Standard conjecture of Hodge-type

Let $X \in V(k)$ of dimension d. Consider

$$\overline{CH}^r(X; \mathbb{Q}) \cap Prim^{2r}(X, \mathbb{Q}(r)) \subset H^{2r}(X, \mathbb{Q}(r)) \, .$$

Let $x, y \in \overline{CH}^r(X; \mathbb{Q}) \cap Prim^{2r}(X, \mathbb{Q}(r))$ for $r \leq d/2$. Then

3.2.10 Conjecture. The pairing

$$x, y \mapsto (-1)^r \langle L_X^{d-2r}(x), y \rangle \in \mathbb{Q}$$

given by the cup product in cohomology, is positive definite.

We state this as a conjecture, although in our situation it is known to be true, first by reducing it to $k = \mathbb{C}$) (Lefschetz principle) and using Hodge-Riemann bilinear relations. But it is still a conjecture if the characteristic of the ground field is nonzero.

3.2.11 Remark. Since Conjecture 3.2.10 is known in our situation, just by assuming Conjecture 3.2.6 we can conclude Conjecture 3.1.4. As an example, in case of abelian varieties we have *numerical equivalence=homological equivalence*, modulo torsion.

3.3 Conjecture of Chow-Künneth type and a filtration (4.2.2 of [48])

We consider a smooth projective variety X over a subfield k of $\mathbb{C}$.

3.3.1 Definition. *Let $X \in V(k)$ of dimension d. We say that X has* ***Chow-Künneth decomposition*** *over k if there exists $\pi_i \in CH^d(X \times X; \mathbb{Q})$, $0 \leqq i \leqq 2d$, such that*

1. *The π_i's are mutually orthogonal projectors, i.e.,*

$$\pi_i \bullet \pi_j = \begin{cases} \pi_i & , \text{ if } i = j \\ 0 & , \text{ otherwise} \end{cases}$$

2. $\sum_i \pi_i = \Delta_X$.
3. $[\pi_i] = [\Delta_X]_{2d-i,i}$, *the usual i-th Künneth components*
4. *Moreover, we expect that $\pi_{2d-i} =^t \pi_i$, $0 \leq i \leq d$.*

If we have such a Chow-Künneth decomposition, then

$$ch(X) := (X, \Delta_X, 0) = \sum_{i=0}^{2d} ch^i(X),\ ch^i(X) := (X, \pi_i, 0)\ . \tag{3.3.1.1}$$

3.3.2 Example. For a smooth projective and irreducible curve C over k and a point $e \in C(k)$, if we choose $\pi_0 = e \times C$, $\pi_2 = C \times e$ and $\pi_1 = \Delta_C - \pi_0 - \pi_1$, then

$$ch(C) = ch^0(C) \oplus ch^1(C) \oplus ch^2(C)\ .$$

Here $ch^0(C)$ and $ch^2(C)$ are the trivial parts of the motive $ch(C)$ and $ch^1(C)$ contains all the 'crucial informations' (see 4.1.8 of [48]).

Now we state the following generalization of Conjecture 3.1.4

3.3.3 Conjecture. (Chow-Künneth conjecture, [48]) Every $X \in V(k)$ has a Chow-Künneth decomposition over $\bar{k}$.

It is evident that Conjecture 3.3.3 implies Conjecture 3.2.1. It actually says that the Künneth components

$$[\Delta_X]_{2d-i,i} \in CH^d_{rat}(X \times X; \mathbb{Q})/CH^d_{hom}(X \times X; \mathbb{Q})$$

of $[\Delta_X]$, can be lifted to $CH^d(X \times X; \mathbb{Q})$.

3.3.4 Example. (Some evidences of Conjecture 3.3.3) The conjecture is known to be true for curves (shown in Example 3.3.2), and if $X, Y \in V(k)$ has the Chow-Künneth decomposition, then so does their product. Hence, it is known for product of curves and surfaces ([48]). It is also known for abelian varieties ([13]), uniruled threefolds ([2]) and elliptic modular varieties ([19]).

3.3.5 Conjectural filtration on Chow groups (4.3.2 of [48])

In the 1970s, Beilinson, based on his (still conjectural) *theory of mixed motives*, conjectured about a possible filtration on the rational Chow groups of a smooth projective variety (it was also independently conjectured by Bloch). We list the conjectural properties of a Bloch-Beilinson filtration below (as formulated by Jannsen in [27])

3.3.6 Definition. *(Conjectural filtration) For $X \in V(k)$ of dimension d, there exists on $CH^r(X; \mathbb{Q})$ a decreasing filtration F^ν, $(\nu \geq 0)$ with the following properties:*

1. *$F^0 = CH^r(X; \mathbb{Q})$, $F^1 = CH^r_{num}(X; \mathbb{Q})$.*
2. *$F^r \cdot F^s \subset F^{r+s}$ under the intersection product.*
3. *$F^\bullet$ is functorial with respect to correspondences.*
4. *Assuming Conjecture 3.2.1 (over $\bar{k}$), the graded pieces $Gr^\nu_F CH^r(X; \mathbb{Q}) := F^\nu / F^{\nu+1}$ depends only on the Grothendieck motive $h^{2r-\nu}_{num}(X) := (X, \Delta_X(2d-$*

$2r+\nu, 2r-\nu), 0)$, *i.e.*,

$$\Delta_X(2d-2r+\ell, 2r-\ell)_*|_{Gr_F^{\nu} CH^r(X;\mathbb{Q})} = \begin{cases} Identity & , \text{ if } \ell = \nu \\ 0 & , \text{ otherwise} \end{cases}$$

5. $F^{r+1} = 0$

3.3.7 Remark. The conjectural filtration is related to Conjecture 3.3.3 in the following way: Suppose $X \in V(k)$ of dimension d satisfies Conjecture 3.3.3 together with

3.3.8 Conjecture. The projectors $\{\pi_{2d}, \pi_{2d-1}, \cdots, \pi_{2r+1}\}$ and $\{\pi_0, \pi_1, \cdots, \pi_{r-1}\}$ operate as zero on $CH^r(X;\mathbb{Q})$.

Then one can define a Bloch-Beilinson type filtration F^{ν} on $CH^r(X;\mathbb{Q})$ with the following characteristics:

1. $Gr_F^{\nu} CH^r(X;\mathbb{Q}) = CH^r(ch^{2r-\nu}(X))$. Hence, one can get the 'Hodge' decomposition at the level of Chow groups

$$CH^r(X;\mathbb{Q}) = \bigoplus_{\nu=0}^{r} Gr_F^{\nu} CH^r(X;\mathbb{Q}).$$

2. (**Conjecture**) The filtration is independent of the ambiguity in the choices of π_i.

3. $F^1 \subset CH^r_{hom}(X;\mathbb{Q})$ and they are conjectured to be equal.

4. $F^2 \subset Ker(\Phi_r)$ and again, they are conjectured to be equal.

Hence, Conjecture 3.3.3 and 3.3.8 defines a filtration with some conjectural properties. It can be shown that this filtration and the one arising from Definition 3.3.6 are equivalent (Theorem 5.2 of [29]).

3.4 A candidate Bloch-Beilinson filtration

The references for this section are [39] (Chapter 9) and [38]. We will consider a smooth projective and irreducible variety X (of dimension d) over $K \subset \mathbb{C}$ which is finitely generated over $\overline{\mathbb{Q}}$. We will discuss a candidate Bloch-Beilinson filtration developed by James Lewis in [38]. Except one, Lewis's filtration has all the desirable properties of the conjectural Bloch-Beilinson filtration.

As seen from Definition 2.21 (2) and the alternate definition following Definition 2.27, we can interpret

$$\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J(X)_{\mathbb{Q}} \cong Ext^1_{MHS}(\mathbb{Q}(0), H^{2r-1}(X,\mathbb{Q}(r)))\ .$$

and

$$cl_r : CH^r(X;\mathbb{Q}) \to H^{r,r}(X,\mathbb{Q}(r)) = Ext^0_{MHS}(\mathbb{Q}(0), H^{2r}(X,\mathbb{Q}(r)))\ .$$

Define $F^0 := CH^r(X;\mathbb{Q})$, $F^1 := Ker(cl_r) = CH^r_{hom}(X;\mathbb{Q})$. In order to get a Bloch-Beilinsion type filtration, the next natural step is to define $F^2 := Ker(\Phi_r)$ and try to find a map

$$F^2 \to Ext^2_{MHS}(\mathbb{Q}(0), H^{2r-2}(X,\mathbb{Q}(r)))\ .$$

Unfortunately, for two MHS's H_1 and H_2, $Ext^{\nu}_{MHS}(H_2,H_1) = 0$ if $\nu \geq 2$, since the functor $Ext^1_{MHS}(H_2,*)$ is right exact. Thus $F^{\nu\geq 2}$ cannot in general be captured by $Ext^{\nu\geq 2}_{MHS}(\mathbb{Q}(0), H^{2r-\nu}(X,\mathbb{Q}(r)))$. Here, note that $Ker(\Phi_r) = 0$ conjecturally (Conjecture 2.31) if $K = \overline{\mathbb{Q}}$ or any number field. Thus, $F^2 = 0$ conjecturally, in case X is defined over $\overline{\mathbb{Q}}$ or a number field k. But if $trdeg_{\mathbb{Q}}k \geq 1$, then there are plenty of examples for which $Ker(\Phi_r) \neq 0$ and hence potentially $F^2 \neq 0$. At this point, we cannot resist the temptation of mentioning Beilinsion's beautiful (conjectural) formula

$$Gr^{\nu}_F CH^r(X;\mathbb{Q}) \cong Ext^{\nu}_{\mathcal{MM}(K)}(\mathbf{1}, h^{2r-\nu}(X)(r))\ ,$$

where $\mathcal{MM}(K)$ is the conjectural category of mixed motives over a given defining field K of smooth projective varieties and $\mathbf{1}$ is the trivial object in the category.

3.4.1 Lewis filtration

Using the cycle class map to absolute Hodge cohomology, James Lewis in [38] developed the following filtration

3.4.2 Theorem. *(Theorem 1.2 of [38]) Assume given a smooth projective* *variety X/K, where $K/\overline{\mathbb{Q}}$ is a finitely generated overfield. Then for all r, there is a filtration*

$$F^0CH^r(X/K;\mathbb{Q}) \supset F^1 \supset F^2 \supset \cdots \supset F^\nu \supset F^{\nu+1} \supset \cdots \supset F^r \supset F^{r+1} = F^{r+2} = \cdots,$$

which satisfies the following

1. *$F^0CH^r(X/K;\mathbb{Q}) := CH^r(X/K;\mathbb{Q})$ and $F^1CH^r(X/K;\mathbb{Q}) = CH^r_{hom}(X/K;\mathbb{Q})$.*
2. *$F^2CH^r(X/K;\mathbb{Q}) \subset Ker(\Phi_r)$.*
3. *$F^l \cdot F^s \subset F^{l+s}$ where $\cdot$ is the intersection product.*
4. *F^ν is preserved under the action of correspondences between smooth projective varieties.*
5. *If we assume that the Künneth components of the diagonal class are algebraic (Conjecture 3.8), then the graded pieces $Gr_F^\nu CH^r(X/K;\mathbb{Q}) := F^\nu/F^{\nu+1}$ depends only on the motive*

   $$h^{2r-\nu}_{hom}(X/K) := (X, \Delta_{X/K}(2d-2r+\nu, 2r-\nu), 0),$$

   i.e.,

   $$\Delta_{X/K}(2d-2r+\ell, 2r-\ell)_*|_{Gr_F^\nu CH^r(X/K;\mathbb{Q})} = \begin{cases} Identity & , \text{ if } \ell = \nu \\ 0 & , \text{ otherwise} \end{cases}$$

6. *Let $D^r(X/K) := \cap_\nu F^\nu$. If we assume that the rational Abel-Jacobi map for smooth quasi projective varieties over $\overline{\mathbb{Q}}$ is injective, then $D^r(X/K) = 0$ and hence $F^{r+1} = 0$.*

3.4.3 Remark. By a conjecture of Jannsen ([27] (5.20)), the above variant of Conjecture 2.2.17 for smooth quasi projective varieties should be true, and indeed can be proven to be the same conjecture under the assumption of the Hodge conjecture.

Although we won't give a complete proof of Theorem 3.4.2, it is instructive to explore the main idea: First we need the formalism of absolute Hodge cohomology

Absolute Hodge cohomology ([4] and Section 3 of [38])

Since we are only interested with the formal properties, we will give a brief definition. Interested readers can find the details mainly in [4].

Let $A \subset \mathbb{R}$ be a subring such that $A \otimes \mathbb{Q}$ is a field.

3.4.4 Definition. *A* ***mixed*** *A****-Hodge complex*** *consists of the following:*

1. *A complex $K_A^\bullet$ of A-modules, that is bounded below, such that $H^p(K_A)$ is an A-module of finite type for all p (technically, we are working in the derived category of complexes).*

2. *A filtered complex $(K_{A\otimes\mathbb{Q}}^\bullet, W)$ of $A \otimes \mathbb{Q}$-vector spaces that is bounded below, and an isomorphism $K_{A\times\mathbb{Q}}^\bullet \xrightarrow{\cong} K_A^\bullet \otimes \mathbb{Q}$ in the derived category.*

3. *A bifiltered complex $(K_\mathbb{C}^\bullet, W, F)$ of $\mathbb{C}$-vector spaces, and a filtered isomorphism $\alpha : (K_\mathbb{C}^\bullet W) \xrightarrow{\cong} (K_{A\otimes\mathbb{Q}}^\bullet, W) \otimes \mathbb{C}$.*

4. *For every $m \in \mathbb{Z}$,*
$$Gr_W^m K_{A\otimes\mathbb{Q}}^\bullet \to (Gr_W^m K_\mathbb{C}^\bullet, F)$$
is a polarizable $A \otimes \mathbb{Q}$-Hodge complex of weight m.

3.4.5 Definition. *A* ***cohomological*** *mixed A-Hodge complex on a space $\mathcal{W}$ is essentially a sheafified version of the definition of a mixed A-Hodge complex. For a precise definition, see [8], Definition 1.8. A cohomological mixed A-Hodge complex naturally gives rise to a mixed Hodge complex by applying the functor $\Gamma(\mathcal{W}, -)$ to a corresponding acyclic resolution of a given complex of sheaves on $\mathcal{W}$.*

We will work under the following set up: X/K is a smooth projective variety (of dimension d), Y/K is a normal crossing divisor, and $j : X - Y \hookrightarrow X$ is an inclusion. The cohomolofical mixed Hodge complex of our interest is

$$(Rj_*\mathbb{Q}, (Rj_*\mathbb{Q}, W), (\Omega_X^\bullet\langle Y\rangle, W, F))\,;$$

and the corresponding mixed Hodge complex will be denoted by

$$\left(K_A^\bullet, (K_{A\otimes\mathbb{Q}}^\bullet, W), (K_\mathbb{C}^\bullet, W, F)\right), \quad A = \mathbb{Q}.$$

Then

3.4.6 Definition. *The* ***absolute Hodge cohomology*** $H^{\bullet}_{\mathcal{H}}((X-Y)_{\mathbb{C}}, \mathbb{Q}(r))$ *is given by the cohomology of the cone complex*

$$\mathcal{M}^{\bullet} := Cone\left(K^{\bullet}_{A} \oplus \hat{W}_0 K^{\bullet}_{A\otimes\mathbb{Q}} \oplus \hat{W}_0 \cap F^0 K^{\bullet}_{\mathbb{C}} \xrightarrow{(\alpha,\beta)} {}'K^{\bullet}_{A\otimes\mathbb{Q}} \oplus \hat{W}_0({}'K^{\bullet}_{\mathbb{C}})\right)[-1],$$

where $\hat{W}_{\bullet} = (Dec\, W)_{\bullet}$ *is the filtration decalée (see [12]) and* α, β *comes from the definition of morphism in a derived category (see Section 3 of [38] for details).*

There is a short exact sequence

$$0 \to J(H^{2r-1}((X-Y)_{\mathbb{C}}, \mathbb{Q}(r))) \to H^{2r}_{\mathcal{H}}((X-Y)_{\mathbb{C}}, \mathbb{Q}(r)) \to \Gamma(H^{2r}((X-Y)_{\mathbb{C}}, \mathbb{Q}(r))) \to 0.$$

Now, we set

$$\underline{H}^{2r}_{\mathcal{H}}((X-Y)_{\mathbb{C}}, \mathbb{Q}(r)) := \Psi(H^{2r}_{\mathcal{D}}(X, \mathbb{Q}(r))),$$

where Ψ is given by restriction (noting that for X/K smooth projective, $H^{2r}_{\mathcal{H}}(X, \mathbb{Q}(r)) = H^{2r}_{\mathcal{D}}(X, \mathbb{Q}(r))$).

Sketch and main ideas for Theorem 3.20

We can find a smooth quasi projective variety $\mathcal{S}/\overline{\mathbb{Q}}$ with generic point

$$\eta_{\mathcal{S}} := \varprojlim_{\mathcal{U}\subset\mathcal{S}/\overline{\mathbb{Q}}} \mathcal{U},$$

(where $\mathcal{U}$ is affine Zariski open subset of $\mathcal{S}$) such that $\overline{\mathbb{Q}}(\mathcal{S}) \cong K$ and ***spread*** out X/K to a family $\rho : \mathcal{X} \to \mathcal{S}$ with $\mathcal{X}_{\eta_{\mathcal{S}}} \cong X$, where $\mathcal{X}$ is smooth and quasi-projective over $\overline{\mathbb{Q}}$ and ρ is smooth and proper (it is called a $\overline{\mathbb{Q}}$-spread). There is a cycle class map

$$CH^r(\mathcal{X}; \mathbb{Q}) \to H^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$$

to absolute Hodge cohomology, which would be injective if we assume the BBC. Further, since $CH^r(\overline{\mathcal{X}}) \to CH^r(\mathcal{X})$ is surjective, the cycle class map takes its image in $\underline{H}^{2r}_{\mathcal{H}}(\mathcal{X}, \mathbb{Q}(r))$. There is a decreasing filtration $\{\mathcal{F}^{\nu} CH^r(\mathcal{X}; \mathbb{Q})\}_{\nu\geq 0}$ with the property that

$$Gr^{\nu}_{\mathcal{F}} CH^r(\mathcal{X}; \mathbb{Q}) \hookrightarrow E^{\nu, 2r-\nu}_{\infty}(\rho)\,,$$

where $E_\infty^{\nu,2r-\nu}(\rho)$ is the ν-th graded piece of a Leray filtration associated to ρ. The term $E_\infty^{\nu,2r-\nu}(\rho)$ fits into the short exact sequence

$$0 \to \underline{E}_\infty^{\nu,2r-\nu}(\rho) \to E_\infty^{\nu,2r-\nu}(\rho) \to \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) \to 0 \,, \tag{3.4.6.1}$$

where

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) = \Gamma\left(H^\nu(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right) \,, \tag{3.4.6.2}$$

and

$$\underline{E}_\infty^{\nu,2r-\nu}(\rho) = \frac{J\left(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)}{\Gamma\left(Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)} \subset J(H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))) \,, \tag{3.4.6.3}$$

(the later inclusion is given by the short exact sequence

$$W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \hookrightarrow W_0H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \,,$$

and the image

$$\Gamma(Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))) \to J\left(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right) \tag{3.4.6.4}$$

can be described in the following way: For $y \in \Gamma(Gr_W^0 H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)))$, we can choose

$$x \in W_0H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))),\ x_{\mathbb{C}} \in F^0W_0H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))) \,,$$

mapping to y under the surjection $W_0 \twoheadrightarrow Gr_W^0$. Then the image of y in 3.5 is given by the image of $x - x_{\mathbb{C}}$ in $J\left(W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)$.) Under the identification $K \cong \overline{\mathbb{Q}}(\eta_{\mathcal{S}})$, we have (by definition)

$$F^\nu CH^r(X/K;\mathbb{Q}) := \varinjlim_{U \subset \mathcal{S}/\overline{\mathbb{Q}}} \mathcal{F}^\nu CH^r(\mathcal{X}_U/\overline{\mathbb{Q}};\mathbb{Q}),\ \mathcal{X}_U := \rho^{-1}(U) \,.$$

We set

$$E_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) := \varinjlim_{U \subset \mathcal{S}/\overline{\mathbb{Q}}} E_\infty^{\nu,2r-\nu}(\rho_U)$$

and same definitions for $\underline{E}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}})$ and $\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}})$. Specifically,

$$\underline{\underline{E}}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = \Gamma(H^\nu(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r))) \,,$$

and

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) = J\left(W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)/\Gamma(Gr_W^0)\,.$$

Similar to 3.4.6.1, we have a short exact sequence

$$0 \to \underline{E}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to \underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to 0\,, \qquad (3.4.6.5)$$

and an injection: $Gr_F^{\nu}CH^r(X/K;\mathbb{Q}) \hookrightarrow E_{\infty}^{\nu,2r-\nu}(\eta_{\mathcal{S}})$.

3.4.7 Remark. Typically in this thesis, we will consider X_K/K, a smooth projective variety which is obtained as a base change from a smooth projective and irreducible X defined over $\overline{\mathbb{Q}}$. Note that, for such a situation, one can choose a product $\overline{\mathbb{Q}}$-bar spread

$$Pr_S : (S \times X)_{\overline{\mathbb{Q}}} \to S\,,$$

where S is a smooth projective variety over $\overline{\mathbb{Q}}$ with generic point η_S, such that $\overline{\mathbb{Q}}(\eta_S) \cong K$ and $\eta_S \times X \cong X_K$.

In later chapters, by a Bloch-Beilinson filtration we will always mean the candidate filtration of Theorem 3.4.2. We note here that assuming a Chow-Künneth decomposition for X and BBC, Lewis filtration is same as the one developed by S. Saito in [51]

As a final remark, we should clarify that for a Chow group $CH^r(X;\mathbb{Q})$ and a candidate Bloch-Beilinson filtration, the condition that $F^{r+1} = 0$ is perhaps the most crucial, as the vector space $D^r(X)$ measures precisely whether we can capture the whole of $CH^r(X;\mathbb{Q})$ using cohomological methods. Further, by the BBC (Conjecture 2.2.17), $D^r(X) = F^{r+1} = 0$.

Chapter 4

Height function and the Néron-Tate pairing

The main references for this short chapter are [53], [7], [49] and [54]. We consider a number field K with its fixed algebraic closure $\overline{K} \subset \mathbb{C}$ and a smooth projective variety X/K. By $X(K)$ we mean the K-rational points of X (similarly $X(\overline{K})$).

4.1 Height Function

As a motivation, one could roughly describe a height function as a function $H : X(\overline{K}) \to \mathbb{R}$ which defines the 'arithmetic complexity' of a point $P \in X(\overline{K})$.

4.1.1 Height of $\overline{K}$-rational points

To start off, suppose $K = \mathbb{Q}$. For $\frac{a}{b} \in \mathbb{Q}$ (written in lowest terms) we can define $H(\frac{a}{b}) := max[|a|, |b|]$ as a height function. More generally, for a non-zero point $P = [x_0; x_1; \cdots; x_N] \in \mathbb{P}^N(\mathbb{Q})$ such that $(x_0, \cdots, x_N) \in \mathbb{Z}$ and $gcd(|x_0|, \cdots, |x_N|) = 1$, we define the **height** of P by

$$H(P) := max[|x_0|, |x_1|, \cdots, |x_N|].$$

It is easy to see that there are only finitely many points of bounded heights in the above case.

Similarly, for a number field K, we define the **height** of a non-zero point $P := [x_0, x_1, \cdots, x_N] \in \mathbb{P}^N_K(K)$ by

$$H_K(P) := \prod_{\nu \in M_K} max[||x_0||_\nu, \cdots, ||x_N||_\nu].$$

Here, M_K denotes the set containing an archimedean prime for each embedding of K in $\mathbb{R}$ or $\mathbb{C}$ and a p-adic absolute value for each prime ideal in O_K, the ring of integers in K.

Sometimes it is more convenient to use the absolute logarithmic height :

$$h(P) := \frac{1}{[K : \mathbb{Q}]} log(H_K(P)).$$

The absolute value is well-defined for $P \in \mathbb{P}^N_K(\overline{K})$.

Heights on Projective varieties

Let X be a smooth projective variety defined over K and $\phi : X \to \mathbb{P}^N_K$ be a morphism. For $x \in X(\overline{K})$, we define :

$$H_\phi(x) := H(\phi(x)),$$
$$h_\phi(x) := log(H_\phi(x)) =: h(\phi(x)).$$

The group $Pic(X)$

Let X be as defined above. We define $Pic(X)$ to be the group of isomorphic classes of algebraic line bundles (locally free sheaves of O_X- modules of rank 1, here O_X is the sheaf of regular functions) on X, with multiplication being the tensor product. One has $Pic(X) = H^1(X, O_X^*) \cong CH^1(X)$. If $f : X \to Y$ is a morphism and c a line bundle on Y, then f^*c defines a line bundle on X and we have a homomorphism

$$f^* : Pic(Y) \to Pic(X).$$

If for any $x \in X$, there is an element of the global section s of a line bundle E such that $s_x \neq 0$, then E is said to be generated by its global sections. For an element $c \in Pic(X)$ generated by its global sections, we get a corresponding morphism $\phi_c : X \to \mathbb{P}^N_K$. We say c is *very ample* if the corresponding morphism ϕ_c is an immersion. We say that c is *ample* if there is an integer $m > 0$ such that mc is very ample.

Heights and line bundles

Let H be the quotient of the vector space of real- valued functions on $X(\overline{K})$ modulo the space of bounded functions. Note that, we can write any $c \in Pic(X)$ as $c = c_\phi - c_\psi$, where $\phi, \psi : X \to \mathbb{P}^N_K$ are immersions and c_ϕ,c_ψ are the corresponding (very ample) line bundles. We thus have the following

4.1.2 Theorem. *There is a unique map $c \mapsto h_c$ of $Pic(X)$ to H such that,*

1. $h_{c+c'} = h_c + h_{c'}$ *for all* $c, c' \in Pic(X)$
2. *If c is very ample then* $h_c = h_{\phi_c}$

The key point in the above theorem is the fact that if $c_{\phi_1} = c_{\phi_2}$, then the corresponding $h_{\phi_1} = h_{\phi_2} + O(1)$, where $O(1)$ denotes 'up to bounded functions'. It uses the fact that the vector space $\Gamma(X, c_\phi)$ of global sections of the line bundle c_ϕ is finite dimensional and change of basis does not change h_ϕ.

4.1.3 Definition. *(Divisors algebraically equivalent to zero in $Pic(X)$)*

For a non-singular variety X, an element $c \in Pic(X)$ is algebraically equivalent to zero if its image is algebraically equivalent to zero in $CH^1(X)$. We denote the subgroup in $Pic(X)$ of elements algebraically equivalent to zero by $Pic^0(X)$. The Neron-Severi group of X is the quotient $NS(X) := Pic(X)/Pic^0(X)$.

4.1.4 Remark. Up to now, we have only defined algebraic equivalence for X defined over an algebraically closed field. More generally, if X is defined over a subfield k of $\mathbb{C}$, we define a cycle $\xi \in CH^r(X)$ to be algebraically equivalent to zero, if its image lies in $CH^r_{alg}(X_{\bar{k}}/\bar{k})$.

4.2 Néron-Tate pairing

4.2.1 Néron-Tate normalization

4.2.2 Proposition (Tate)**.** *Let S be a set and $\pi : S \to S$ a map. Let f be a real-valued function on S such that $f \circ \pi = \lambda f + O(1)$, with $\lambda > 1$. Then there is a unique function $\tilde{f}$ on S such that*

1. $\tilde{f} = f + O(1)$
2. $\tilde{f} \circ \pi = \lambda \tilde{f}$

and we have

$$\tilde{f}(x) = \lim_{n \to \infty} (1/\lambda^n) f(\pi^n x),$$

for every $x \in S$.

The function $\tilde{f}$ satisfies obvious functoriality and commutativity properties.

Suppose for a morphism $\phi : X \to X$ and for $c \in Pic(X)$ that we have $\phi^* c = \lambda c$ with $\lambda (\in \mathbb{Z}) > 1$. Then by the Theorem 4.1 we have $h_c(\phi(x)) = \lambda h_c(x) + O(1)$ on $X(\overline{K})$. By the above proposition of Tate, we get a unique function $\tilde{h}_c$ such that $\tilde{h}_c = h_c + O(1)$ and $\tilde{h}_c(\phi(x)) = \lambda \tilde{h}_c(x)$. This is the normalized logarithmic height.

4.2.3 Height pairing in abelian varieties

4.2.4 Theorem. *Let K be a number field and A be an abelian variety defined over K. There is a unique function $c \mapsto \tilde{h}_c$ on $Pic(A)$ with values in the space of real valued functions on $A(\overline{K})$ such that,*

1. *$\tilde{h}_c(x) = h_c(x) + O(1)$, where h_c is as defined in theorem 4.1.*
2. *Additivity: $\tilde{h}_{c_1 + c_2} = \tilde{h}_{c_1} + \tilde{h}_{c_2}$.*
3. *Functoriality: for all endomorphisms $\phi : A \to A$, we have*

$$\tilde{h}_{\phi^* c} = \tilde{h}_c \circ \phi,$$

for $c \in Pic(A)$. *Further if* B *is another abelian variety and* $\psi : B \to A$ *is a homomorphism, then*

$$\tilde{h}_{\psi^* c} = \tilde{h}_c \circ \psi,$$

for all $c \in Pic(A)$.

Let $c \in Pic^0(A)$, we identify c with a point in $A^\vee(\overline{K})$ where $A^\vee$ is the dual abelian variety. Then, using the Néron-Tate height function, one can define a Néron-Tate pairing

$$(\cdot,\cdot) : A(\overline{K}) \times A^\vee(\overline{K}) \to \mathbb{R},\ (P, c) := \tilde{h}_c(P - 0),$$

where $0 \in A(K)$ is the group identity. In [49], Néron showed that the above pairing could be seen as a sum of local pairings (called Néron's local symbols). Néron-Tate pairing has the property that for every polarization $\lambda : A \to A^\vee$, the bilinear form : $\langle x, y\rangle := (x, \lambda(y))$ is positive definite on $A(\overline{K})_{\mathbb{Q}}$. Also, for any homomorphism $f : A \to B$ of abelian varieties, the pairing satisfies the following projection formula:

$$(x, f^\vee(y))_A = (f(x), y)_B \ \ for\ x \in A(\overline{K}), y \in B^\vee(\overline{K}),$$

where $f^\vee : B^\vee \to A^\vee$ is the dual morphism.

Chapter 5

A brief tour of Arithmetic Intersection Theory

In this chapter we present a brief exposition of arithmetic intersection theory, an area developed by Gillet and Soulé. Interested readers can find the details of arithmetic intersection theory either in [17] or in [9].

5.1 Motivation

For a variety X defined over a number field k, there is a very satisfactory notion of intersection theory on its Chow group $CH^*(X;\mathbb{Q})$ developed by Fulton ([16]), with many desirable properties (actually for any field k, for that matter). Given the successes of such an intersection theory, it is only natural to ask for a similar theory for varieties defined over the ring of algebraic integers O_k of k. Now, O_k has both finite primes and primes at infinity (which corresponds to embeddings of k inside $\mathbb{C}$) and to have a good intersection theory, one has to take into account these infinite primes as well. For example, if one considers the degree map $CH^1(Spec(\mathbb{Z})) \to \mathbb{Z}$ from the usual Chow group, then it is not an invariant under rational equivalence; indeed all such cycles are rationally equivalent to zero, while the definition of the degree of a divisor of a rational number q is $log|q|$. So, we cannot have a good notion of intersection numbers unless we remedy this situation. We can do it by adjoining a point v at infinity to $Spec(\mathbb{Z})$ corresponding to the only real embedding of $\mathbb{Q}$ and define the v-adic valuation of a rational number q to be $-log|q|$. It now follows from the product formula that a principal divisor has degree zero. $Spec(\mathbb{Z})$ is an example of an arithmetic curve (since it has dimension 1), more generally we can consider $X \to Spec(O_k)$, where X is a regular scheme, projective and flat over $Spec(O_k)$. Gillet and Soulé considered a more general version of the usual Chow group for such schemes, by taking into account the 'places at infinity' and systematically developed an intersection theory, which was the correct analog of the one for varieties defined over a number field.

5.2 Green currents

This section is borrowed mainly from Chapter II of [9], including most of the notations. We will state the main results and theorems, the proofs of which could be found in [9] .

5.2.1 Currents on a smooth complex projective variety

Let X be an irreducible smooth complex projective variety of complex dimension d and $A^{p,q}(X)$ denote the vector space of $\mathbb{C}$-valued differential forms of type (p, q). The space $A^n(X)$ of differential forms of degree n is given by

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)\,. \qquad (5.2.1.1)$$

We denote by $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$, $\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$ and $d = \partial + \overline{\partial} : A^n(X) \to A^{n+1}(X)$ the usual differential operators (all of these notions are defined in chapter 2).

Let $D_n(X) := A^n(X)^*$, denoting the space of linear functionals on $A^n(X)$, which are Schwartz continuous: for a sequence $\gamma_r \subset A^n(X)$ with $Supp(\gamma_r)$ contained in some compact set K and $T \in D_n(X)$, we have $T(\gamma_r) \to 0$, if $\gamma_r \to 0$ (which means that all the coefficients in the sequence of forms $\{\gamma_r\}$ together with finitely many of their derivatives tend uniformly to zero on K when $r \to \infty$). By 5.2.1.1 we obtain a similar decomposition

$$D_n(X) = \bigoplus_{p+q=n} D_{p,q}(X), \qquad (5.2.1.2)$$

$D_{p,q}(X)$ being the duals of $A^{p,q}(X)$.

5.2.2 Definition. *We define $D^{p,q}(X) := D_{d-p,d-q}(X)$ to be the space of (p,q)-currents on X.*

The differentials $\partial,\ \overline{\partial},\ d$ induce similar maps $\partial',\ \overline{\partial}',\ d'$ from $D^{p,q}(X)$ to $D^{p+1,q}(X),\ D^{p,q+1}(X)$ and $D^{p+q+1}(X)$ respectively. We have an inclusion map

$$A^{p,q}(X) \hookrightarrow D^{p,q}(X)$$

$$\gamma \mapsto [\gamma]\,,$$

defined by

$$[\gamma](\alpha) := \int_X \gamma \wedge \alpha, \quad \alpha \in A^{d-p,d-q}(X)\,.$$

Here we fix an orientation on X by declaring that

$$\left(\frac{\sqrt{-1}}{2}\right)^n dz_1 \wedge d\bar{z}_1 \cdots dz_n \wedge d\bar{z}_n,$$

has positive orientation on $\mathbb{C}^n$. If $p+q=n$, from Stokes' theorem we get

$$[d\gamma](\alpha) = (-1)^{n+1}(d'[\gamma])(\alpha)\,.$$

Denote $(-1)^{n+1}\partial',\ (-1)^{n+1}\overline{\partial}'\ (-1)^{n+1}d'$ by $\partial,\ \overline{\partial},\ d$ respectively. We have commutative diagrams

$$\begin{array}{ccc}
A^{p,q}(X) & \hookrightarrow & D^{p,q}(X) \\
\partial \big\downarrow & & \big\downarrow \partial \\
A^{p+1,q}(X) & \hookrightarrow & D^{p+1,q}(X)
\end{array}$$

(similarly for $\overline{\partial}$ and d). These diagrams induce isomorphisms on the level of cohomology with respect to $\partial,\ \overline{\partial},\ d$.

For every irreducible analytic subvariety $Y \overset{i}{\hookrightarrow} X$ of codimension p, we can define a current $\delta_Y \in D^{p,p}(X)$ by setting, for all $\alpha \in A^{d-p,d-p}(X)$,

$$\delta_Y(\alpha) := \int_{Y^{ns}} i^*\alpha$$

where Y^{ns} denote the non-singular locus of Y. It follows from Hironaka's theorem on resolution of singularities that δ_Y is well defined and gives a current.

5.2.3 Definition. *Let us define* $d^c := (4\pi i)^{-1}(\partial - \overline{\partial})$ *(so that* $dd^c = -(2\pi i)^{-1}\partial\overline{\partial}$*).*

5.2.4 Definition. *A Green current for a codimension* p *subvariety* Y*, is a current* $g \in D^{p-1,p-1}(X)$ *such that*

$$dd^c g + \delta_Y = [\gamma]$$

for some form $\gamma \in A^{p,p}(X)$*.*

5.2.5 Theorem. *Every subvariety* $Y \subset X$ *has a Green current. If* g_1 *and* g_2 *are two Green currents for Y, then*

$$g_1 - g_2 = [\eta] + \partial S_1 + \overline{\partial} S_2$$

with $\eta \in A^{p-1,p-1}(X)$*,* $S_1 \in D^{p-2,p-1}(X)$*,* $S_2 \in D^{p-1,p-2}(X)$*.*

For subvarieties of codimnsion 1 (divisors) on X, there is a natural choice of Green current given by the following

5.2.6 Theorem. *(The Poincaré-Lelong formula). Let* L *be a holomorphic line bundle on* X *with hermitian metric* $||\cdot||$*,* s *a meromorphic section of* L *and* $c_1(L, ||\cdot||)$ *the first Chern form of* L*. Then* $-log||s||^2 \in L^1(X)$*, hence induces a distribution* $[-log||s||^2] \in D^{0,0}(X)$*. This is a Green current for* $div\, s$ *:*

$$dd^c[-log||s||^2] + \delta_{div\, s} = [c_1(L, ||\cdot||)]\ .$$

5.2.7 Green forms of logarithmic type

As in the previous section, X will denote an irreducible smooth complex projective variety and $Y \subset X$ is an analytic subvariety.

5.2.8 Definition. *A smooth form* α *on* $X - Y$ *is said to be of logarithmic type along* Y*, if there exists a projective map* $\pi : \widetilde{X} \to X$ *such that* $E := \pi^{-1}(Y)$ *is a divisor with normal crossings,* $\pi : \widetilde{X} - E \to X - Y$ *is smooth and* α *is the direct image by* π *of a form* β *on* $\widetilde{X} - E$ *with the following property : Near each* $x \in \widetilde{X}$*, let* $z_1 z_2 \cdots z_k = 0$ *be a local equation of* E*. Then there exists* ∂

and $\overline{\partial}$ closed smooth forms α_i and a smooth form γ such that

$$\beta = \sum_{i=1}^{k} log|z_i|^2 + \gamma \, . \tag{5.2.8.1}$$

If α is of logarithmic type along Y, it is locally integrable on X, hence it defines a current $[\alpha]$, which is the direct image by π of the current $[\beta]$.

5.2.9 Lemma.

- *Let $f : X' \to X$ be a morphism of (irreducible) smooth projective varieties such that $f^{-1}(Y) \neq X'$, and on $X - Y$, let α be a form of logarithmic type along Y. Then the form $f^*(\alpha)$ is of logarithmic type along $f^{-1}(Y)$.*
- *Let $f : X \to X'$ be a projective morphism of (irreducible) smooth projective variety and α be a form on $X - Y$ logarithmic type along Y. Assume that f is smooth outside Y and $f(Y) \neq X'$. Then $f_*(\alpha)$ is of logarithmic type along $f(Y)$ and $f_*([\alpha]) = [f_*(\alpha)]$.*

Now, we state a very important result related to the existence of Green's current of logarithmic type:

5.2.10 Theorem. *For every irreducible subvariety $Y \subset X$ there exists a smooth form g_Y on $X - Y$ of logarithmic type along Y such that $[g_Y]$ is a Green current for Y :*

$$dd^c[g_Y] + \delta_Y = [\omega]$$

where ω is smooth on X.

For the proof, see [9]. After all these set up and results about existence, we give an example of a Green current of logarithmic type :

5.2.11 Example. Let $X = \mathbb{P}^d$, with homogeneous coordinates $X_0, \cdots, X_d$. Y defined by $X_0 = \cdots = X_{p-1}$. Define

$$\theta := log\left(|X_0|^2 + \cdots + |X_d|^2\right), \; \alpha := dd^c\theta \text{ on } X \, ;$$

$$\sigma := log\left(|X_0|^2 + \cdots + |X_{p-1}|^2\right), \; \beta := dd^c\sigma \text{ on } X - Y \, ;$$

$$\Lambda := (\theta - \sigma)\left(\sum_{i=0}^{p-1} \alpha^i \wedge \beta^{p-1-i}\right) \text{ on } X - Y \, .$$

Then, one can show that $[\Lambda]$ defines a Green current of logarithmic type along Y.

Such explicit examples of Green currents are rare.

5.2.12 The $*$-product of Green currents

Let X be as before and $Y, Z \subset X$ be closed irreducible subsets such that $Z \nsubseteq Y$. Denote by g_Y a Green form of logarithmic type for Y. Let $p : \widetilde{Z} \to Z$ be a resolution of singularities of Z and $q : \widetilde{Z} \to X$ its composite with the inclusion $Z \subset X$. Now by Lemma 5.2.9 we know that $q^* g_Y$ is of logarithmic type along $q^{-1}(Y)$. In particular it is integrable and the formula

$$[g_Y] \wedge \delta_Z := q_*[q^* g_Y]$$

defines a current on X. For any Green current g_Z for Z, we define the $*$-product with $[g_Y]$ to be

5.2.13 Definition. *Define* $[g_Y] * g_Z := [g_Y] \wedge \delta_Z + [\omega_Y] \wedge g_Z$. *If* $codim_X Y = n$ *and* $codim_X Z = m$, *then* $[g_Y] * g_Z \in D^{n+m-1,n+m-1}(X)$.

The most important result about the $*$-product of Green currents is the following

5.2.14 Theorem. *If* Y, Z *intersect properly, i.e, if* $Y \cap Z = \cup_i S_i$ *with* $codim_X S_i = codim_X Y + codim_X Z = n + m$, *then*

$$dd^c([g_Y] * g_Z) = [\omega_Y \wedge \omega_Z] - \sum_i \mu_i \delta_{S_i}$$

where the integers $\mu_i = \mu_i(Y, Z)$ *are the intersection multiplicities (see [16], Chapter 7 for intersection multiplicities).*

The essence of this theorem lies in the following observation : Let $Y \bullet Z$ denote the algebraic intersection of the two subvarieties, then $[g_Y] * g_Z$ serves as a Green current for it.

We end this subsection on Green currents by listing down its few properties (from now on, we will write $g_Y * g_Z$ instead of the notation used before):

- Let $Y \subset X$ be a closed irreducible subset and X' be an irreducible smooth projective complex variety and $f : X' \to X$ with $f^{-1}(Y) \neq X'$. If g_Y is a Green current of logarithmic type for Y, then $f^* g_Y$ is a Green current of logarithmic type along $f^{-1}(Y)$ for the cycle $f^*(Y)$:

$$dd^c f^* g_Y = [f^*\omega_Y] - \sum_i \mu_i \delta_{S_i}$$

where $f^{-1}(Y) = \cup_i S_i$ with $codim_{X'} S_i = codim_X Y$ and μ_i are the multiplicities of the cycle $f^*(Y)$.

- Let $Y \subset X$ be a closed irreducible subset and g_Y be a Green current for Y. By Theorem 5.2.10, we get a Green current $\tilde{g}_Y$ of logarithmic type along Y. Also, Theorem 5.2.5 asserts that

$$g_Y = \tilde{g}_Y + [\eta] + \partial S_1 + \overline{\partial} S_2 \,.$$

So, modulo $Im\partial + Im\overline{\partial}$, every Green current can be represented by a Green current of logarithmic type along Y.
Let $Y, Z \subset X$ be closed irreducible subsets and $Z \nsubseteq Y$ and g_Y (resp. g_Z) a Green current for Y (resp. Z). We can define the $*$-product of

$$g_Y * g_Z = \tilde{g}_Y * g_Z \;\; modulo \, (Im\partial + Im\overline{\partial})$$

where $\tilde{g}_Y$ is any Green current of logarithmic type, congruent modulo $Im\partial + Im\overline{\partial}$. One can show that this definition does not depend on the choice of $\tilde{g}_Y$. Furthermore, we have

$$g_Y * g_Z = g_Z * g_Y \;\; modulo \, (Im\partial + Im\overline{\partial}) \;\; (Commutativity)$$

and for closed irreducible subvarieties $Y, Z, W \subset X$ meeting properly and respective choice of Green currents g_Y, g_Z, g_W

$$g_Y * (g_Z * g_W) = (g_Y * g_Z) * g_W \;\; modulo \, (Im\partial + Im\overline{\partial}) \;\; (Associativity) \,.$$

The proof of all these facts can be found in Gillet and Soulé's original paper ([17]). But to get an idea, one can just compute pretending that the above Green currents are forms (see the end of chapter II in [9]).

5.3 Arithmetic Chow groups and the intersection pairing

Although we will only restrict ourselves to arithmetic varieties X over $Spec(O_k)$ (where k is a number field) whose generic fibre X_k is smooth projective, we will give a more general definition (and properties), as in [17].

5.3.1 Definition. *(Arithmetic ring) An **arithmetic ring** is a triple* $(A, \sum, F_\infty)$ *consisting of an excellent noetherian regular integral domain* A*, a finite nonempty set* $\sum$ *of embeddings* $\sigma : A \hookrightarrow \mathbb{C}$ *and a conjugate linear involution of* $\mathbb{C}$*-algebras,* $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$ *such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\quad\delta\quad} & \mathbb{C}^\Sigma \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle F_\infty} \\ A & \xrightarrow{\quad\delta\quad} & \mathbb{C}^\Sigma \end{array}$$

commutes. Here δ *denotes the map induced to the product by the family* $\{\sigma : A \hookrightarrow \mathbb{C}\}_{\sigma \in \sum}$*. We also have the induced commutative diagram*

$$\begin{array}{ccc} \mathbb{C} \otimes_{\mathbb{Z}} A & \xrightarrow{\quad\delta'\quad} & \mathbb{C}^\Sigma \\ {\scriptstyle c \otimes Id}\downarrow & & \downarrow{\scriptstyle F_\infty} \\ \mathbb{C} \otimes_{\mathbb{Z}} A & \xrightarrow{\quad\delta'\quad} & \mathbb{C}^\Sigma \end{array}$$

where $c(z) = \bar{z}$ *and* $\delta' = Id \otimes \sigma_{\sigma \in \sum}$*. We use the notation* $C^\Sigma = \Pi_{\sigma \in \sum} \mathbb{C}_\sigma$*, so that* $\sigma : A \hookrightarrow \mathbb{C}_\sigma$*.*

5.3.2 Example.

- This is the typical example we will consider: Let O_k be the number ring of a number field k, $\sum = Hom(O_k, \mathbb{C})$ and F_∞ be the usual Frobenious on $\mathbb{C}^\Sigma$.
- $A = \mathbb{R}$, $\sum$ is the obvious embedding of $\mathbb{R}$ in $\mathbb{C}$ and F_∞ the complex conjugation
- $A = \mathbb{C}$. Then $(\mathbb{C}, \{Id, c\}, F_\infty)$ is an arithmetic ring, where $c(z) = \bar{z}$ and $F_\infty(a, b) = (\bar{b}, \bar{a})$.

A homomorphism of arithmetic rings $f : (A, \sum, F_\infty) \to (A', \sum', F'_\infty)$ is a pair $f_1 : A \to A'$ and $f_2 : \mathbb{C}^{\Sigma} \to \mathbb{C}^{\Sigma'}$ with f_2 a homomorphism of $\mathbb{C}$ algebras, with some commutativity conditions. Note that, for an extension l/k of number fields, there is an obvious inclusion $O_k \hookrightarrow O_l$ of their number rings. Also, we observe that $\mathbb{Z}$ is an initial object in the category of arithmetic rings.

5.3.3 Arithmetic Chow groups

5.3.4 Definition. *Given an arithmetic ring $(A, \sum, F_\infty)$, an **arithmetic variety** X over A is a scheme which is flat and of finite type over $S = Spec(A)$, $\pi : X \to S$. If F is the field of fractions of A, we write X_F for the generic fibre and suppose that it is smooth. For $s \in S$, we write $X(s) = \pi^{-1}(s)$ and for $\sigma \in \sum$, we denote $X_\sigma = X \times_\sigma \mathbb{C}$ and $X_{\sum} = X \times_A \mathbb{C}^{\Sigma}$. Finally, we denote by $X_\infty = X_{\sum}(\mathbb{C})$, the analytic space associated to the scheme $X_{\sum}$.*

The conjugate-linear automorphism F_∞ induces continuous involution of X_∞. Since X_F is smooth, X_∞ is a complex manifold. We denote by $A^{p,q}(X)$, the space of (p, q)-forms on X_∞, similarly $D^{p,q}(X)$, the space of (p, q)-currents on X_∞. Let $A^{p,q}(X_\mathbb{R})$ (resp. $D^{p,p}(X_\mathbb{R})$) to be the subspace of $A^{p,p}(X)$ (resp. $D^{p,p}(X)$) consisting of real forms (resp. currents) satisfying $F^*_\infty \alpha = (-1)^p \alpha$. Similarly, we define

$$\tilde{A}^{p,p}(X_\mathbb{R}) := A^{p,p}(X_\mathbb{R})/Im\partial + Im\overline{\partial},$$

$$\tilde{A}(X_\mathbb{R}) := \oplus_{p\geqq 0} \tilde{A}^{p,p}(X_\mathbb{R}),$$

and if X_F is projective, then

$$H^{p,p}(X_\mathbb{R}) := \{\alpha \in H^{p,p}(X, \mathbb{R}) = H^{2p}(X, \mathbb{R}) \cap H^{p,p}(X); F^*_\infty \alpha = (-1)^p \alpha\}.$$

For an arithmetic variety X over an arithmetic ring $(A, \sum, F_\infty)$ with smooth and quasi-projective generic fibre X_F , we have the usual notion of (arithmetic) cycles of codimension p, denoted by $Z^p(X)$. Given an integral subscheme $Y \subset X$ of codimension p, $Y_\infty \subset X_\infty$ is an analytic subspace invariant under F_∞. Hence integration over Y_∞ defines a current in $D^{p,p}(X_\mathbb{R})$, which we shall denote by δ_Y. Extending linearly, we get a map

$$Z^p(X) \to D^{p,p}(X_\mathbb{R}).$$

Now, let $\widehat{Z}^p(X)$ be the subgroup of $Z^p(X) \oplus \tilde{D}^{p-1,p-1}(X_{\mathbb{R}})$ consisting of pairs (Z, g_Z) such that g_Z is a Green current for Z.
If $Y \subset X$ is a reduced irreducible subscheme of codimension $p-1$ and $f \in k(Y)^*$, one can define (see [17], 3.3.3 for details)

$$div(f) \in Z^p(X)$$

in the usual way (see for example, the definition of a Chow group of a Noetherian separated scheme in section I.2 of [9]), and an element

$$i_*[log|f|^2] \in D^{p-1,p-1}(X_{\mathbb{R}})$$

where $i : Y \to X$ is the inclusion. Denote by

$$\widehat{div}(f) := (div(f), i_*[log|f|^2]) \in \widehat{Z}^p(X).$$

5.3.5 Definition. *Let $\widehat{R}^p(X)$ be the subgroup by all such pairs $\widehat{div}(f)$ as above. We define*

$$\widehat{CH}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X),\, p \geq 0\,,$$

and call it the ***arithmetic Chow group*** *of X. We use the notation $\widehat{CH}^*(X) := \oplus_{p\geq 0}\widehat{CH}^p(X)$.*

The map

$$\omega : \widehat{CH}^p(X) \to A^{p,p}(X_{\mathbb{R}}),\; \omega(Z, g_Z) = [\omega_Z] = dd^c g_Z + \delta_Z$$

is well defined and helps us to define $\widehat{CH}^p(X)_0 = Ker(\omega)$. Also, let

$$CH^p_{hom}(X) = \{Z \in CH^p(X); Z_F \sim_{hom} 0\}$$

and (assuming X_F to be projective)

$$c : CH^p(X) \to H^{p,p}(X_{\mathbb{R}})$$

is the cycle class map. We have a surjective map $\widehat{CH}^p(X)_0 \twoheadrightarrow CH^p_{hom}(X)$, sending a class $(Z, g_Z) \in \widehat{CH}^p(X)_0$ to $Z \in CH^p_{hom}(X)$ (for the proof, see part (ii) and (iii) of Theorem 3.3.5 of [17]). For computations and examples of arithmetic Chow group, the reader is encouraged to consult either [17] (section 3.4) or [9].

Arithmetic Chow groups behave well under pushforward and pullback of morphisms, as the following theorem states (see 3.6.1 of [17]):

5.3.6 Theorem. *Let $f : X \to Y$ be a morphism between arithmetic varieties over an arithmetic ring $(A, \sum, F_\infty)$. Suppose that f induces a smooth map $X_F \to Y_F$ between generic fibres of X and Y. Then:*

- *If f is flat, for all $p \geq 0$, there is a natural map*

$$f^* : \widehat{CH}^p(Y) \to \widehat{CH}^p(X).$$

- *If f is proper, and X, Y are equidimensional. there is a map*

$$f_* : \widehat{CH}^p(X) \to \widehat{CH}^{p-\delta}(Y)$$

  for $\delta = dim(X) - dim(Y)$. If $f : X \to Y, g : Y \to Z$ are two maps inducing smooth maps on the generic fibres, then $(gf)^* = f^* g^*$ and $(gf)_* = g_* f_*$ when either compositions make sense.

5.3.7 Intersection theory

For a regular, noetherian and separated scheme X (of dimension d) and a closed subscheme $Y \subset X$, define $Z^p_Y(X)$ to be the free abelian group of codimension p integral subschemes supported on Y. One can then accordingly define the Chow group with support in Y, denoted by $CH^p_Y(X)$. Observe that if Y has codimension p, then $CH^p_Y(X) \cong Z^p_Y(X)$. There is an isomorphism

$$CH^p_Y(X; \mathbb{Q}) := CH^p_Y(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong Gr^p_\gamma K^Y_0(X; \mathbb{Q})$$

where $Gr^p_\gamma K^Y_0(X; \mathbb{Q})$ denotes the graded piece associated to the γ-filtration on the K-theory with supports in Y (and $\mathbb{Q}$ coefficients). See [9] for the proof of this result. This isomorphism allows us to define a product

$$CH^p_Y(X; \mathbb{Q}) \otimes CH^q_Z(X; \mathbb{Q}) \to CH^{p+q}_{Y \cap Z}(X; \mathbb{Q})$$

given by the natural (tensor) product in K-theory. This product, together with the $*$-product of Green currents, allows one to define the following intersection product on the arithmetic Chow groups:

5.3.8 Theorem. *For a regular arithmetic variety X, there is a pairing*

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X;\mathbb{Q}),\ (Y, g_Y) \bullet (Z, g_Z) = (Y \cdot Z, g_Y * g_Z),$$

where $\cdot$ denotes the intersection product discussed above and $\widehat{CH}^i(X;\mathbb{Q}) := \widehat{CH}^i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, for any i with $0 \leqq i \leqq dim(X)$. This pairing has the following properties:

1. *$\widehat{CH}^*(X;\mathbb{Q}) := \widehat{CH}^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a commutative graded unitary $\mathbb{Q}$-algebra.*

2. *(Functoriality) Let X, Y be regular arithmetic varieties and let $f : X \to Y$ be a morphism satisfying all the criterions of Theorem 5.3.6. Then one has*
$$f^*(\alpha \bullet \beta) = f^*\alpha \bullet f^*\beta$$
for $\alpha \in \widehat{CH}^p(Y)$ and $\beta \in \widehat{CH}^q(Y)$. $\bullet$ also satisfies the following ***projection formula****:*
$$f_*(f^*\alpha \bullet \beta) = \alpha \bullet f_*\beta \in \widehat{CH}^{p+q-\delta}(Y;\mathbb{Q})$$
where δ is as in Theorem 5.3.6 and $\alpha \in \widehat{CH}^p(Y)$ and $\beta \in \widehat{CH}^q(X)$.

We end this chapter with the definition of arithmetic intersection number for regular and equidimensional (of Krull dimension $d + 1$) arithmetic varieties defined over $S = Spec(O_k)$, where $O_k \subset k$ is the ring of integers of a number field k: To start off, we have the following degree (see [17], section 3.4.3)

$$deg : \widehat{CH}^1(S) \to \mathbb{R}$$

defined by $(Z, g_Z) \mapsto log|Z| + \frac{1}{2}\int_S g$, where $Z = \oplus n_i \wp_i$ for finite primes $\wp_i$ ($|Z| = |O_k/\wp_i|$), $g = \{g_\sigma\}_{\sigma\in\sum}$, $g_\sigma \in \mathbb{R}$ and $\int_S g = \sum_{\sigma\in\sum} g_\sigma$. Now we have the following

5.3.9 Definition. *Let X be a regular, equidimensional (of Krull dimension $d+1$) arithmetic variety over $S = Spec(O_k)$, where $O_k \subset k$ is the ring of integers of a number field k. Given two arithmetic cycles $z_1 \in \widehat{CH}^r(X)$ and $z_2 \in \widehat{CH}^{d-r+1}(X)$, we can define the arithmetic intersection number (or arithmetic degree) $\widehat{deg}(z_1 \bullet z_2)$, through the following sequence of maps*

$$\widehat{CH}^r(X) \otimes \widehat{CH}^{d-r+1}(X) \to \widehat{CH}^{d+1}(X;\mathbb{Q}) \xrightarrow{\pi_*} \widehat{CH}^1(S;\mathbb{Q}) \xrightarrow{deg} \mathbb{R}\,,$$

where $\pi : X \to S$ denotes the structural morphism.

5.3.10 Remark. As pointed out in 4.3.8 (iii) of [17], the above pairing induces a pairing

$$CH^r_{hom}(X) \otimes CH^{d-r+1}_{hom}(X) \to \mathbb{R}\,.$$

In the next chapter, we will see that this pairing induces Beilinson's height pairing (under certain assumptions).

This ends our brief and incomplete survey of this beautiful area. We have only collected the results that we need for the following chapters. There are many details that are being skipped. Interested readers are encouraged to consult [17] and [9] for further reading.

Chapter 6

Beilinson's height pairing via arithmetic intersection theory

In this small chapter, we introduce Beilinson's height pairing on Chow groups of cycles homologous to zero ([5]). We will describe this height pairing in light of arithmetic intersection pairing, as discussed in [34]. First, we have to set up the assumptions and definitions to work with.

6.0.11 Assumption. *Let k, O_k always denote a number field and its ring of integers respectively. In this chapter, an arithmetic variety will denote a scheme which is projective and flat over $S = Spec(O_k)$ and has a smooth generic fibre.*

6.0.12 Definition. *Let X/k be a smooth projective variety over a number field k of dimension d. A **model** of X is an arithmetic variety $\widetilde{X}$ over S such that $\widetilde{X}_k \cong X$. A model $\widetilde{X}$ which is also a regular scheme is called a **regular model**.*

To introduce Beilinson's height pairing through arithmetic intersection theory, we have to assume that a smooth projective variety X/k (of dimension d) has a regular model $\widetilde{X}/S$ (which is equidimensional and of Krull dimension $d+1$).

For a smooth projective variety X/k (with a regular model $\widetilde{X}/S$), define

$$CH^r_{hom}(X;\mathbb{Q}) = Ker(cl_r : CH^r(X;\mathbb{Q}) \to H^{p,p}(\widetilde{X}_{\mathbb{R}}))\ ,$$

where cl_r is the cycle class map. Since for any such model we have an isomorphism $\widetilde{X}_k \cong X$, the above definition is independent of a regular model of X.

We define $CH^r_{fin}(\widetilde{X};\mathbb{Q}) := Ker(CH^r(\widetilde{X};\mathbb{Q}) \to CH^r(X;\mathbb{Q}))$. Note that, $CH^r_{fin}(\widetilde{X};\mathbb{Q}) \subset CH^r_{hom}(\widetilde{X};\mathbb{Q})$. Denote by $CH^r_{fin}(\widetilde{X};\mathbb{Q})^{\perp}$ the orthogonal complement of $CH^r_{fin}(\widetilde{X};\mathbb{Q})$ under the pairing

$$\widehat{deg} : CH^r_{hom}(\widetilde{X};\mathbb{Q}) \otimes CH^{d-r+1}_{hom}(\widetilde{X};\mathbb{Q}) \to \mathbb{R}$$

described in Definition 5.3.9 (and the remark following it). Let $CH^r_{hom}(X;\mathbb{Q})^0$ denote the image of the canonical map

$$\lambda : CH^{d-r+1}_{fin}(\widetilde{X};\mathbb{Q})^{\perp} \to CH^r_{hom}(X;\mathbb{Q}) \ .$$

We now obtain a pairing

$$\langle\ ,\ \rangle_{HT} : CH^r_{hom}(X;\mathbb{Q})^0 \times CH^{d-r+1}_{hom}(X;\mathbb{Q})^0 \to \mathbb{R} \qquad (6.0.12.1)$$

which we can define as follows. Given elements $x = \lambda(x')$ ($x' \in CH^{d-r+1}_{fin}(\widetilde{X};\mathbb{Q})^{\perp}$) and $y = \lambda(y')$ ($y' \in CH^r_{fin}(\widetilde{X};\mathbb{Q})^{\perp}$), we set

$$\langle x\ ,\ y\rangle_{HT} = \widehat{deg}(x' \bullet y') \ .$$

This pairing does not depend on the choice of x' and y', but may depend a priori on the choice of a regular model of X. Now under the
6.0.13 Assumption. $CH^r_{hom}(X;\mathbb{Q}) = CH^r_{hom}(X;\mathbb{Q})^0$,

we define $\langle\ ,\ \rangle_{HT}$ to be the **Beilinson's height pairing**. For a detailed discussion, we refer to section 5 of [34] (the reader should be careful about the change of notations).
6.0.14 Remark. In [5], Beilinson described the height pairing in a different manner, albeit also under the assumption that a smooth projective variety defined over a number field, admits of a regular model.
In brief, his idea is the following: For a number field k, we have the finite primes $\wp$ and the infinite primes $\sigma : k \hookrightarrow \mathbb{C}$. For two cycles $x \in CH^r_{hom}(X;\mathbb{Q})$ and $y \in CH^{d-r+1}_{hom}(X;\mathbb{Q})$, he defined the Archimedean part of the height pairing as

$$\langle x\ ,\ y\rangle_{HT,\infty} := -\int_{X(\mathbb{C})} g_x * g_y$$

where g_x (resp. g_y) is a Green current associated to the complex space related to x (resp. a Green current associated to the complex space related to y). Also, for each finite prime $\wp$ he defined a non-archimedean part of the pairing $\langle x \, , \, y\rangle_{HT,\wp}$, which resembled the local symbol devised by Neron for divisors and zero cycles ([49]). The total height pairing $\langle x, \, y\rangle_{HT}$ is then defined (roughly) as the sum of these local pairings. But under Assumption 6.0.13, the two definitions should agree (section 2 and 4.1 of [5] is very relevant here) ! Except for the trivial case of divisors and zero cycles, the assumption holds if X has a smooth model, and more non-trivially it holds for abelian varieties which has totally degenerate reduction at all places of bad reduction (see [35] for details).

From now till the end of the thesis, we are going to assume that a smooth projective variety X/k admits of a regular model and also that the condition of Assumption 6.0.13 holds.

It follows from the projection formula for the arithmetic intersection pairing, that

6.0.15 Proposition. *(**Projection formula for height pairing**). For X, Y two smooth projective varieties defined over k and a correspondence $\alpha \in CH^r(X \times_k Y; \mathbb{Q})$, we have*

$$\langle x \, , \, \alpha^*(y)\rangle_{HT,X} = \langle \alpha_*(x) \, , \, y\rangle_{HT,Y}$$

for suitable choice of x and y.

6.0.16 Remark. Since regularity doesn't behave well with taking products, in order to be completely rigorous one has to use the cap product construction, as in section 2.3 of [18]. This construction is also relevant when considering base changes (as in Lemma 8.1 of [34]). One can also probably use de-Jong's alteration technique (specifically Theorem 8.2 of [11]) but the author is not sure !

6.1 Height pairing for cycles algebraically equivalent to zero

We begin with a technical definition

6.1.1 Definition. *(Incidence equivalence) For a smooth projective variety X (of dimension d) defined over an algebraically closed field $k \subset \mathbb{C}$, an element $u \in CH^r_{alg}(X;\mathbb{Q})$ is said to be **incidence equivalent** to 0 ($\sim_{inc} 0$) if it satisfies $\alpha_*(u) = 0$ in $CH^1(T)$ for every smooth projective variety T/k and every correspondence $\alpha \in CH^{d-r+1}(X \times T;\mathbb{Q})$. We denote the subgroup of cycles incidence equivalent to zero by $I^r(X;\mathbb{Q})$. We have the isomorphism*

$$CH^r_{alg}(X;\mathbb{Q})/I^r(X;\mathbb{Q}) \cong Pic^r(X)(k),$$

where $Pic^r(X)$ is a certain abelian variety, known as the (higher) r-th Picard variety (see Section 7 of [34] for details).

From now on, k will again denote a number field. For $z \in CH^r_{alg}(X;\mathbb{Q})$, there exists a finite extension K/k, a geometrically irreducible curve C_K over K, an element $z' \in CH^1_{hom}(C_K;\mathbb{Q})$ and a correspondence $\alpha \in CH^r(C_K \times_K X_K;\mathbb{Q})$ such that $\alpha_*(z') = z_K$. Using this, Künnemann ([34], Lemma 8.1) showed that under the assumption of the existence of regular models, we have $CH^r_{alg}(X;\mathbb{Q}) \subset CH^r_{hom}(X;\mathbb{Q})^0$. Hence we have a well-defined Beilinson's height pairing

$$CH^r_{alg}(X;\mathbb{Q}) \times CH^{d-r+1}_{alg}(X;\mathbb{Q}) \to \mathbb{R}\,,$$

which has a description via the Neron-Tate pairing on Picard varieties (Theorem 8.2 of [34]):

6.1.2 Theorem. *For $x \in CH^r_{alg}(X;\mathbb{Q})$ and $y \in CH^{d-r+1}_{alg}(X;\mathbb{Q})$, we get*

$$\frac{1}{[k:\mathbb{Q}]}\langle x, y\rangle_{HT} = \frac{1}{\kappa^r_{X_{\overline{k}}}}\left(\theta^r(x), f^{d-r+1}_{X_{\overline{k}}} \circ \theta^{d-r+1}(y)\right)_{Pic^r(X_{\overline{k}})}\,,$$

where $Pic^r(X_{\overline{k}})$ denotes the r-th Picard variety associated to $X_{\overline{k}}$ (which is an abelian variety defined over k), θ^r is the natural Picard homomorphism, $f^{d-r+1}_{X_{\overline{k}}}$ is the duality homomorphism between $Pic^{d-r+1}(X)$ and $Pic^r(X)^\vee$ (dual of $Pic^r(X)$). The pairing $(\ ,\ )_{Pic^r(X_{\overline{k}})}$ is the Neron-Tate pairing for abelian *varieties.*

To see a detailed proof of this theorem, and to get an idea about Picard varieties, see sections 7 and 8 of [34].

6.1.3 Remark. As mentioned at the end of Section 10 of [34], if we assume that numerical and homological equivalence on $X_{\overline{k}}$ agree up to torsion, then we have an isogeny between $Pic^r(X_{\overline{k}})$ and $J^r_{alg}(X)$ (or cycles incidence equivalent to zero are same as cycles Abel-Jacobi equivalent to zero inside $CH^r_{alg}(X)$). One can get a similar height pairing relation, replacing $Pic^r(X_{\overline{k}})$ by $J^r_{alg}(X)$.

Under the above assumption if one chooses a hyperplane section $L_X \in CH^1(X;\mathbb{Q})$, then $[L_X]^{d-2r+1} : J^r_{alg}(X) \to J^{d-r+1}_{alg}(X)_{\mathbb{Q}} = J^r_{alg}(X)^{\vee}_{\mathbb{Q}}$ is a polarization. We get the following relation:

$$\langle x, L_X^{d-2r+1}(x)\rangle_{HT} \;\equiv\; \left(\Phi_r(x), [L]^{d-2r+1}(\Phi_r(x))\right)_{J^r_{alg}(X)} ,$$

for $x \in CH^r_{alg}(X;\mathbb{Q})$, where $\equiv$ means equality is up to a positive constant.

6.1.4 A Hodge-index result for cycles algebraically equivalent to zero on an abelian variety

While developing the height pairing in [5], Beilinson predicted the following results (conjecture 5.3 and 5.5 of [5]):

6.1.5 Conjecture. Let $L_X : CH^r(X;\mathbb{Q}) \to CH^{r+1}(X;\mathbb{Q})$ be the operator associated to the hyperplane section $L_X \in CH^1(X;\mathbb{Q})$. Then for $2r \leq d+1$ $(d = dim(X))$,

- (i) The operator

  $$L_X^{d-2r+1} : CH^r_{hom}(X;\mathbb{Q}) \to CH^{d-r+1}_{hom}(X;\mathbb{Q})$$

  is an isomorphism

- (ii) If $x \in CH^r_{hom}(X;\mathbb{Q})$, $x \neq 0$ and such that $L_X^{d-2r+2}(x) = 0$ (primitive element), then

  $$(-1)^r\langle x, L^{d-2r+1}(x)\rangle_{HT} > 0 \,.$$

Note that (ii) resembles the Hodge-index conjecture for primitive Chow groups.

We still don't know the status of this conjecture; it seems that (i) should hold only for algebraically closed fields. In any case, Künnemann has the following result for an abelian variety A/k of dimension d, in [34] (Theorem 12.1).

6.1.6 Theorem. *Let $B^r(A;\mathbb{Q}) = CH^r_{alg}(A;\mathbb{Q})/I^r(A;\mathbb{Q})$ where $I^r(A;\mathbb{Q})$ denotes the subgroup of $CH^r_{alg}(A;\mathbb{Q})$ of cycles incidence equivalent to zero in $A_{\overline{k}}$. Let $L_A \in CH^1(A;\mathbb{Q})$ denote a hyperplane section and $2r \leq d+1$.*

- *(i) The operator*

$$L_A^{d-2r+1} : B^r(A;\mathbb{Q}) \to B^{d-r+1}(A;\mathbb{Q})$$

  is an isomorphism

- *(ii) If $x \in B^r(A;\mathbb{Q})$, $x \neq 0$ and such that $L_X^{d-2r+2}(x) = 0$, then*

$$(-1)^r \langle x, L^{d-2r+1}(x)\rangle_{HT} > 0 \; .$$

6.1.7 Remark. Assuming that homological and numerical equivalence agree up to torsion, the subgroup $I^r(A;\mathbb{Q})$ is same as the subgroup

$$CH^r_{alg,AJ}(A;\mathbb{Q}) := Ker\left(\Phi_r : CH^r_{alg}(A;\mathbb{Q}) \to J^r_{alg}(A)_{\mathbb{Q}}\right) \; .$$

Now, if one assumes the Bloch-Beilinson conjecture that the rational Abel-Jacobi map is injective, then $I^r(A;\mathbb{Q}) = CH^r_{alg,AJ}(A;\mathbb{Q}) = 0$. So, it is a reasonable guess that Theorem 6.1.6 should hold for $CH^r_{alg}(A;\mathbb{Q})$ and not just for $B^r(A;\mathbb{Q})$. Also in Remark 12.2 of [34], it is mentioned that the result could be generalized for any smooth projective variety X/k (of dimension d) if one assumes the standard conjectures of Lefschetz type, along with the following:

- (a) The intersection product on $CH^*_{hom}(X;\mathbb{Q})$ is zero (this is actually Conjecture 5.7 of [5]).

- (b) The Künneth components $\Delta_X(2d-i,i)$ induce the zero map on $B^r(X;\mathbb{Q})$ for all $i \neq 2r-1$.

These assumptions all follow from the conjectural Bloch-Beilinson filtration on the Chow group of X and Bloch-Beilinson conjecture mentioned in the previous remark. Specifically since X is defined over a number field ,

$$F^2CH^r(X;\mathbb{Q}) \subset Ker(\Phi_r : CH^r_{hom}(X;\mathbb{Q}) \to J^r(X)) = 0 \; .$$

Hence we can deduce (a) and (b).

This concludes the chapter. Everything in here could be found in [34], in greater details.

Chapter 7

Height pairing between higher graded pieces

This chapter consists of two sections. In the first section, we will develop and discuss our main result (Theorem 7.0.11). In the second section, we will speculate about a possible generalization of Theorem 7.0.11 for a family. Note that from hereon, by a filtration we will mean Lewis Filtration, as discussed in 3.4.1 and k will denote a number field, i.e. a finite extension of $\mathbb{Q}$ with its ring of integers O_k. We begin, and as a reminder, with the following assumption:

7.0.8 Assumption. *(BBC) For a smooth projective variety X defined over a number field (more generally, over $\overline{\mathbb{Q}}$), the rational Abel-Jacobi map*

$$CH^r_{hom}(X;\mathbb{Q}) \to J(H^{2r-1}(X,\mathbb{Q}(r)))$$

is injective.

Since we will be interested in smooth projective varieties $X/\overline{\mathbb{Q}}$, we have to modify the definition of height pairing. We do so through

7.0.9 Proposition. *(Remark 4.0.6 of [5]) Let k'/k be an extension of degree n. Then $CH^r_{hom}(X_k;\mathbb{Q}) \subset CH^r_{hom}(X_{k'};\mathbb{Q})$ (this is immediate by a standard norm argument) and for $a_1 \in CH^r_{hom}(X_k;\mathbb{Q})$ and $a_2 \in CH^{d-r+1}_{hom}(X_k;\mathbb{Q})$ one has*

$$\langle a_1, a_2\rangle_k = \frac{1}{n}\langle a_1, a_2\rangle_{k'}.$$

By means of this formula we can define the height pairing between $CH^r_{hom}(X_{\overline{k}};\mathbb{Q})$ and $CH^{d-r+1}_{hom}(X_{\overline{k}};\mathbb{Q})$; this pairing is Galois-invariant.

For a smooth projective variety X defined over $\overline{\mathbb{Q}}$, we make the following definition:

7.0.10 Definition. *Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety. For algebraic cycles $\alpha \in CH^r_{hom}(X;\mathbb{Q})$ and $\beta \in CH^{d-r+1}_{hom}(X;\mathbb{Q})$, we can find a number field k', a smooth projective variety X'/k' with $X \cong X' \otimes_{k'} \overline{\mathbb{Q}}$ and cycles $\alpha' \in CH^r_{hom}(X';\mathbb{Q})$, $\beta' \in CH^{d-r+1}_{hom}(X';\mathbb{Q})$ such that $\alpha = q^*\alpha'$ and $\beta = q^*\beta'$ for the finite and proper map $q : X \to X'$. We define*

$$\langle \alpha, \beta \rangle_{HT} := \frac{1}{[k' : \mathbb{Q}]} \langle \alpha', \beta' \rangle,$$

Using Proposition 7.0.9, we can see that this pairing is independent of the choice of k', X', α', β'. Note also that the choices of k', X' depend on the cycles α and β.

Now we state and prove the main result of the thesis:

7.0.11 Theorem. *Let $X/\overline{\mathbb{Q}}$ be a smooth projective variety of dimension d and let $K/\overline{\mathbb{Q}}$ be a finitely generated overfield of transcendence degree $\nu - 1$, where $\nu \geq 1$ is an integer. Let us assume Grothendieck amended general Hodge conjecture (GHC) together with the Bloch-Beilinson Conjecture or BBC (Conjecture 2.2.17). Then there exists a pairing*

$$\langle \ , \ \rangle_{HT} : Gr^{\nu}_{F} CH^r(X_K;\mathbb{Q}) \times Gr^{\nu}_{F} CH^{d-r+\nu}(X_K;\mathbb{Q}) \to \mathbb{R}\,,$$

extending Bloch-Beilinson's height pairing.

7.0.12 Remark. Assuming Conjecture 5.3 and 5.5 in [5], we will show later that the above pairing is non-degenerate and induces a 'polarization' on the primitive pieces of $Gr^{\nu}_{F} CH^r(X_K;\mathbb{Q})$, analogous to the situation of Hodge-Riemann bilinear relations on cohomology.

7.1 A key result and the proof of Theorem 7.0.11

We are going to prove a proposition which is essentially the motivation for the heart of the theorem.

Note that $K \cong \overline{\mathbb{Q}}(S)$ where $S/\overline{\mathbb{Q}}$ is a smooth projective variety of dimension $\nu - 1$ and $dim(S \times X) = d + \nu - 1$. Let η_S be the generic point of S.

7.1.1 Proposition. *Assume the general Hodge conjecture. Let us consider the projector*

$$\widetilde{P} := \Delta_S \otimes \Delta_X(2d - 2r + \nu, 2r - \nu) .$$

Then we have a surjection

$$\widetilde{P}_* : CH^r_{hom}(S \times X; \mathbb{Q}) \twoheadrightarrow Gr^{\nu}_F CH^r(X_K; \mathbb{Q}).$$

Proof. First note that we have the surjection

$$CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \xrightarrow{j^*} CH^r((U \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \; ; \; j : U \times X \hookrightarrow S \times X \; ,$$

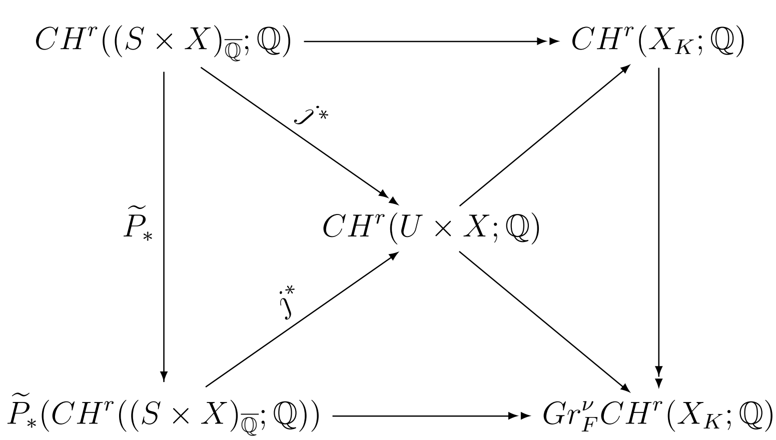
for affine Zariski open subsets $U \subset S/\overline{\mathbb{Q}}$. Now since

$$\varinjlim_{U \subset S/\overline{\mathbb{Q}}} CH^r((U \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) = CH^r(X_K; \mathbb{Q}) \; ,$$

we have the following surjection (using right exactness of $\varinjlim$) :

$$CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \twoheadrightarrow CH^r(X_K; \mathbb{Q}) \cong CH^r((\eta_S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}).$$

Along with the action of $\widetilde{P}_*$ on $CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$, we get the following commutative diagram :



(7.1.1.1)

where, the vertical arrow on the right is given by $\Delta_{X_K}(2d - 2r + \nu, 2r - \nu)_*$.

Thus

$$\widetilde{P}_* : CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \twoheadrightarrow Gr_F^{\nu} CH^r(X_K; \mathbb{Q}).$$

The key issue now is to replace $CH^r((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$ by $CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$ and still get surjectivity. Note that, by the affine Lefschetz theorem, $H^{\nu}(U, \mathbb{Q}) = 0$ for any smooth affine subvariety $U \subset S/\overline{\mathbb{Q}}$. Thus, from the diagram below

$$\begin{array}{ccccc}
CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) & \xrightarrow{\widetilde{P}_*} & \tilde{P}_*(CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) & & \\
\downarrow cl_r & & \downarrow cl_r & & \\
H^{2r}((S \times X)_{\overline{\mathbb{Q}}}, \mathbb{Q}) & \xrightarrow{[\widetilde{P}]_*} & H^{\nu}(S, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q}) & \xrightarrow{j^*} & H^{\nu}(U, \mathbb{Q}) \otimes H^{2r-\nu}(X, \mathbb{Q}) = 0,
\end{array} \tag{7.1.1.2}$$

we conclude that

$$j^*(\widetilde{P}_*(CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) \subset CH^r_{hom}((U \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),$$

for U smooth. If we show

$$j^*(CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) = CH^r_{hom}((U \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),$$

then we can replace $CH^r((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$ from the commutative diagram 7.1 with $CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$ and still get the following surjectivity :

$$\widetilde{P}_* : CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \twoheadrightarrow Gr_F^{\nu} CH^r(X_K; \mathbb{Q}).$$

Hence we conclude the proof by the following lemma:

7.1.2 Lemma. *Let $X/\overline{\mathbb{Q}}$ be smooth projective and $j : U \hookrightarrow X$ be open. Then,*

$$j^* : CH^r_{hom}(X; \mathbb{Q}) \to CH^r_{hom}(U; \mathbb{Q})$$

is surjective.

Proof. Let $Y = X \backslash U$, with desingularization $\widetilde{Y} \to Y$, and the corresponding map:

$$\sigma : \widetilde{Y} \to X.$$

For simplicity, we assume that Y has pure codimension l in X. One has the following exact sequence :

$$H^{2r-2l}(\widetilde{Y},\mathbb{Q}) \xrightarrow{\sigma_*} H^{2r}(X,\mathbb{Q}) \xrightarrow{j^*} H^{2r}(U,\mathbb{Q}).$$

Next, let $\eta \in CH^r_{hom}(U;\mathbb{Q})$. Then there exists $\overline{\eta} \in CH^r(X;\mathbb{Q})$ such that $j^*(\overline{\eta}) = \eta$. Note that $[\eta] = 0 \in H^{2r}(U;\mathbb{Q})$ and thus by Hodge theory, $[\overline{\eta}] = \sigma_*[\gamma]$ for some $[\gamma] \in H^{r-l,r-l}(\widetilde{Y},\mathbb{Q})$. By the Hodge conjecture and a spread argument (since $\overline{\mathbb{Q}}$ is algebraically closed), we can assume that $\gamma \in CH^{r-l}(\widetilde{Y};\mathbb{Q})$. Thus $(\overline{\eta}-\sigma_*\gamma) \in CH^r_{hom}(X;\mathbb{Q})$ and $j^*((\overline{\eta}-\sigma_*\gamma)) = \eta$. □

This indeed shows that $j^*(CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})) = CH^r_{hom}((U\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ and the result follows. □

7.1.3 Proving Theorem 7.0.11

From Lefschetz decomposition and polarization, we obtain the following (perfect) dualities (see [51], Remark 1.3.3, 1.3.4 and Lemma 1.2.4 for details):

$$N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q}) \times N_H^{d-r+1}H^{2d-2r+\nu}(X,\mathbb{Q}) \to \mathbb{Q}\,,$$

$$N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q}) \times N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q}) \to \mathbb{Q}\,.$$

The pairings above induce natural decompositions :

$$H^{2r-\nu}(X,\mathbb{Q}) = N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q}) \bigoplus \{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})\}^{\perp}\,,$$

$$H^{\nu-1}(S,\mathbb{Q}) = N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q}) \bigoplus \{N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})\}^{\perp},$$

and the natural projectors

$$P : H^{2r-\nu}(X,\mathbb{Q}) \twoheadrightarrow N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})\,,$$

$$Q : H^{\nu-1}(S,\mathbb{Q}) \twoheadrightarrow N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})\,.$$

Note that, assuming the Künneth type standard conjectures or more generally the Hodge conjecture, the projectors P and Q are induced by algebraic cycles $P \in CH^d((X\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ and $Q \in CH^{\nu-1}((S\times S)_{\overline{\mathbb{Q}}};\mathbb{Q})$ respectively (we use the same notations for cycles).

Let us revisit Proposition 7.1.1 in light of cohomology. The motivation for this is to initiate the whole idea of an isomorphism between the graded piece $Gr_F^{\nu}CH^r(X_K;\mathbb{Q})$ and a certain subgroup, inside $CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$.

Since S has dimension $\nu-1$, by the affine Lefschetz theorem, we note that

$$\underline{\underline{E}}_{\infty}^{\nu,2r-\nu}(\eta_S)=\Gamma\left(H^{\nu}(\eta_S,R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)=0\,.$$

Hence from the definition of Lewis filtration, it follows that

$$Gr_F^{\nu}CH^r(X_K;\mathbb{Q})\hookrightarrow \underline{E}_{\infty}^{\nu,2r-\nu}(\eta_S)\,.$$

In case of a product spread $Pr_S:(S\times X)_{\overline{\mathbb{Q}}}\to S$, we have the following description of $\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_S)$

7.1.4 Lemma.

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_S)=\frac{J\left(W_{-1}\left(H^{\nu-1}(\eta_S,\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})\right)(r)\right)}{\Gamma\left(Gr_W^0\left(H^{\nu-1}(\eta_S,\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})\right)(r)\right)}\,.$$

Proof. As seen in the description of Lewis filtration, for a general spread $\rho:\mathcal{X}\to\mathcal{S}$ of X_K we have

$$\underline{E}_{\infty}^{\nu,2r-\nu}(\eta_S)=\frac{J\left(W_{-1}H^{\nu-1}(\eta_S,R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)}{\Gamma\left(Gr_W^0\left(H^{\nu-1}(\eta_S,R^{2r-\nu}\rho_*\mathbb{Q}(r))\right)\right)}\,.$$

For the product spread, we have the isomorphism

$$H^{\nu-1}(\eta_S,R^{2r-\nu}Pr_{S,*}\mathbb{Q}(r))\cong\left(H^{\nu-1}(\eta_S,\mathbb{Q})\otimes H^{2r-\nu}(X,\mathbb{Q})\right)(r)\,,$$

and the lemma follows immediately. □

One can actually say even more, through the following

7.1.5 Proposition. *Assume that the General Hodge conjecture holds for smooth projective varieties over $\overline{\mathbb{Q}}$. Then there is an injective map*

$$Gr_F^{\nu}CH^r(X_K;\mathbb{Q})\hookrightarrow J(H_0)$$

Here $J(H_0)$ denotes the Jacobian of the pure Hodge structure H_0 defined by

$$H_0 := \left(\frac{H^{\nu-1}(S,\mathbb{Q})}{N^1_{\overline{\mathbb{Q}}} H^{\nu-1}(S,\mathbb{Q})} \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1} H^{2r-\nu}(X,\mathbb{Q})} \right)(r)\ .$$

7.1.6 Remark. Note that H_0 is actually the lowest weight part of the (mixed) Hodge structure

$$\left(H^{\nu-1}(\eta_S,\mathbb{Q}) \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1} H^{2r-\nu}(X,\mathbb{Q})} \right)(r)\ .$$

Proof. The idea of the proof is essentially in [41], Theorem 4.6. It crucially uses the fact that $\overline{\mathbb{Q}}$ is algebraically closed. We only need S to be smooth and quasi projective for it to work.

Since the projector

$$P : H^{2r-\nu}(X,\mathbb{Q}) \twoheadrightarrow N_H^{r-\nu+1} H^{2r-\nu}(X,\mathbb{Q})$$

is cycle induced, from [41], Section 4 we know that

$$P_*(Gr_F^{\nu} CH^r(X_K;\mathbb{Q})) = 0.$$

(Here we are using the fact that $\overline{\mathbb{Q}}$ is algebraically closed, in order to conclude $N_H^{r-\nu+1} H^{2r-\nu}(X,\mathbb{Q}) = N_{\overline{\mathbb{Q}}}^{r-\nu+1} H^{2r-\nu}(X,\mathbb{Q})$).

Let $U \subset S/\overline{\mathbb{Q}}$ be an affine open subvariety. Let us consider the (mixed) Hodge structures

$$H_1 := \left(H^{\nu-1}(U,\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q}) \right)(r)$$

$$H_2 := \left(H^{\nu-1}(U,\mathbb{Q}) \otimes N_H^{r-\nu-1} H^{2r-\nu}(X,\mathbb{Q}) \right)(r).$$

We have the following short exact sequence on the lowest weight part:

$$0 \to W_{-1}H_2 \to W_{-1}H_1 \to W_{-1}(H_1/H_2) \to 0,$$

and hence at the level of Jacobians, we get

$$J(W_{-1}H_2) \hookrightarrow J(W_{-1}H_1) \twoheadrightarrow J(W_{-1}(H_1/H_2)),$$

since $\Gamma(W_{-1}(H_1/H_2)) = 0$.

From [41] (Lemma 4.5) it follows that $Im(\Gamma Gr^0_{H_2}) = Im(\Gamma Gr^0_{H_1})$ and we obtain the exact sequence

$$J(W_{-1}H_2)/\Gamma Gr^0_{H_2} \hookrightarrow J(W_{-1}H_1)/\Gamma Gr^0_{H_1} \twoheadrightarrow J(W_{-1}(H_1/H_2)) \ .$$

Taking direct limit over all $U \subset S/\overline{\mathbb{Q}}$, we get

$$J\left(W_{-1}\left(H^{\nu-1}(\eta_S,\mathbb{Q}) \otimes N_H^{r-\nu-1}H^{2r-\nu}(X,\mathbb{Q})\right)(r)\right)/\Gamma Gr^0_{H_2}(\eta_S) \hookrightarrow \underline{E}_\infty^{\nu,2r-\nu}(\eta_S)$$
$$\to J(W_{-1}(H_1/H_2)(\eta_S)) \ .$$

Since P_* preserves the intersection

$$Gr^\nu_F CH^r(X_K;\mathbb{Q})$$

$$\bigcap Im\left(J\left(W_{-1}\left[\left(H^{\nu-1}(\eta_S,\mathbb{Q}) \otimes N_H^{r-\nu-1}H^{2r-\nu}(X,\mathbb{Q})\right)(r)\right]\right) \to \underline{E}_\infty^{\nu,2r-\nu}(\eta_S)\right) \ ,$$

it is actually zero (Proposition 4.1 (iii) of [41]). Also, since S is smooth and projective, the lowest weight part

$$W_{-1}(H_1/H_2)(\eta_S) = H_0 \ .$$

Hence, we conclude that

$$Gr^\nu_F CH^r(X_K;\mathbb{Q}) \hookrightarrow J(H_0) \ .$$

□

It is easy to see that this injective map of Proposition 7.1.5 is given by the projector

$$P' := (\Delta_S(\nu-1,\nu-1) - Q) \otimes (\Delta_X(2d-2r+\nu, 2r-\nu) - P) \ .$$

The essence of which could be captured in the following commutative diagram:

$$\begin{array}{ccc} Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) & \xrightarrow{[P']_* = Identity} & Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) \\ \big\downarrow & & \big\downarrow \\ \underline{E}_{\infty}^{\nu,2r-\nu}(\eta_S) & \xrightarrow{[P']_*} & J(H_0) \end{array} \tag{7.1.6.1}$$

7.1.7 Remark. Note that $J(H_0) \hookrightarrow J(H^{2r-1}((S \times X)_{\overline{\mathbb{Q}}}, \mathbb{Q}(r)))$. There is the following decomposition at the level of Jacobians:

$$J(H^{2r-1}((S \times X)_{\mathbb{C}}, \mathbb{Q}(r))) \cong J(H_0) \oplus J(H_0^{\perp}) \, ,$$

where $H_0^{\perp}$ arises due to polarization.

Coming back to the case at hand, let

$$\Phi_r : CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) \hookrightarrow J(H^{2r-1}((S \times X)_{\mathbb{C}}, \mathbb{Q}(r)))$$

be the Abel-Jacobi map and

$$P_1 = \left[\left([(\Delta_S(\nu-1,\nu-1) - Q) \otimes (\Delta_X(2d-2r+\nu, 2r-\nu) - P)] \bullet \widetilde{P} \right) \right]$$

denote the cohomology class of the cycle

$$\left([(\Delta_S(\nu-1,\nu-1) - Q) \otimes (\Delta_X(2d-2r+\nu, 2r-\nu) - P)] \bullet \widetilde{P} \right) \, .$$

Then the surjectivity result of Proposition 7.1.1 has the following cohomology counterpart:

$$\begin{array}{ccc} CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) & \xrightarrow{\widetilde{P}_*} & Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) \\ \Phi_r \big\downarrow & & \big\downarrow \\ J(H^{2r-1}((S \times X)_{\mathbb{C}}, \mathbb{Q}(r))) & \xrightarrow{P_{1,*}} & J(H_0), \end{array} \tag{7.1.7.1}$$

since

$$P_1 : H^{2r-1}((S \times X)_{\mathbb{C}}, \mathbb{Q}(r)) \twoheadrightarrow \left(\frac{H^{\nu-1}(S, \mathbb{Q})}{N^1_{\overline{\mathbb{Q}}} H^{\nu-1}(S, \mathbb{Q})} \otimes \frac{H^{2r-\nu}(X, \mathbb{Q})}{N_H^{r-\nu+1} H^{2r-\nu}(X, \mathbb{Q})} \right)(r)$$

is a projector. Now one can write

$$\Phi_r(CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) \cong Gr_F^{\nu} CH^r(X_K; \mathbb{Q}) \bigoplus (Gr_F^{\nu} CH^r(X_K; \mathbb{Q}))^{\perp} \ , \tag{7.1.7.2}$$

where

$$(Gr_F^{\nu} CH^r(X_K; \mathbb{Q}))^{\perp} = (Id_{H^{2r-1}((S\times X)_{\mathbb{C}}, \mathbb{Q}(r))} - P_1)_*(Gr_F^{\nu} CH^r(X_K; \mathbb{Q})) \, .$$

Via 7.1.7.2, let $\Xi_1 := \Phi_r^{-1}(Gr_F^{\nu} CH^r(X_K; \mathbb{Q})) \subset CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})$. Note that

$$\Xi_1 = \left([(\Delta_S(\nu-1, \nu-1) - Q) \otimes (\Delta_X(2d-2r+\nu, 2r-\nu) - P)] \bullet \widetilde{P} \right)_* (CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) \, .$$

Now the choice of an algebraic cycle (say) w_1, corresponding to P_1 is not unique. But assuming the BBC, we will show that Ξ_1 can indeed be uniquely chosen.

7.1.8 Lemma. *Ξ_1 is independent of the choice of correspondence w_1 modulo*

$$CH^r_{hom,\ AJ}((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) := Ker\left(\Phi_r : CH^r_{hom}(S \times X; \mathbb{Q}) \to J(H^{2r-1}(S \times X, \mathbb{Q}(r))\right) \, .$$

Proof. If $w_1^{'}$ is another such projector, then

$$(w_1 - w_1^{'})_*(CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q})) \subset F^2 CH^r((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \subset CH^r_{hom,\ AJ}((S\times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \, .$$

Hence if we assume the BBC, the choice of Ξ_1 is independent of the projector w_1.

□

We now have a natural isomorphism: $\Xi_1 \cong Gr^{\nu} CH^r(X_K; \mathbb{Q})$, given by the Abel-Jacobi map Φ_r and illustrated more clearly through the commutative diagram:

$$\begin{array}{ccc} CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) & \xhookrightarrow{\Phi_r} & J(H^{2r-1}((S\times X)_{\overline{\mathbb{Q}}},\mathbb{Q}(r))) \\ \downarrow{\scriptstyle w_{1,*}} & & \downarrow{\scriptstyle P_{1,*}} \\ \Xi_1 & \xhookrightarrow{\Phi_r} & J(H_0) \end{array} \tag{7.1.8.1}$$

Following a similar method but for $Gr^{\nu}_F CH^{d-r+\nu}(X_K;\mathbb{Q})$ we get an isomorphism

$$\Xi_2 \cong Gr^{\nu}_F CH^{d-r+\nu}(X_K;\mathbb{Q}).$$

Here $\Xi_2 \subset CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ is obtained as $w_{2,*}(CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}}))$, for an algebraic cycle w_2 corresponding to the projector

$$P_2 : H^{2(d+\nu-r)-1}((S\times X)_{\overline{\mathbb{Q}}},\mathbb{Q}) \twoheadrightarrow \underbrace{\left(\frac{H^{\nu-1}(S,\mathbb{Q})}{N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})} \otimes \frac{H^{2d-2r+\nu}(X,\mathbb{Q})}{N^{d-r+1}_H H^{2d-2r+\nu}(X,\mathbb{Q})}\right)}_{H'_0}.$$

Note that $dim((S\times X)_{\overline{\mathbb{Q}}}) = d+\nu-1$ and $d-r+\nu = (d+\nu-1)-r+1$. Hence, we can use the height pairing introduced by Beilinson (and Bloch)

$$\langle\ ,\ \rangle_{HT} : CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) \times CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) \to \mathbb{R}$$

to get a pairing between Ξ_1 and Ξ_2 and hence (via the natural isomorphisms) between the graded pieces

$$\langle\ ,\ \rangle_{HT} : Gr^{\nu}_F CH^r(X_K;\mathbb{Q}) \times Gr^{\nu}_F CH^{d-r+\nu}(X_K;\mathbb{Q}) \to \mathbb{R}.$$

7.1.9 Remark. For $\nu = 1$ it follows from construction that the above pairing is the one introduced by Beilinson in [5].

Since the choice of S such that $\overline{\mathbb{Q}}(S) \cong K$ is not fixed, apparently the height pairing should a priori vary if we vary S. In our next proposition we show that it does not.

7.1.10 Proposition. *Assuming BBC, the height pairing developed here is independent of the choice of smooth projective variety S with $\overline{\mathbb{Q}}(S) \cong K$.*

Proof. Let S' be another smooth projective variety such that $\overline{\mathbb{Q}}(S') \cong K$. Then S and S' are birational. One can then find a smooth projective variety $S'' \hookrightarrow (S \times S')_{\overline{\mathbb{Q}}}$ and birational morphisms $f_1 : S'' \twoheadrightarrow S$ and $f_2 : S'' \twoheadrightarrow S'$ (see 2.5 of [31]). Hence we have similar birational morphisms $F_1 := f_1 \times Id_X : (S'' \times X)_{\overline{\mathbb{Q}}} \twoheadrightarrow (S \times X)_{\overline{\mathbb{Q}}}$ and $F_2 := f_2 \times Id_X : (S'' \times X)_{\overline{\mathbb{Q}}} \twoheadrightarrow (S' \times X)_{\overline{\mathbb{Q}}}$.

Now, given elements $x \in Gr_F^{\nu} CH^r(X_K; \mathbb{Q})$ and $y \in Gr_F^{\nu} CH^{d-r+\nu}(X_K; \mathbb{Q})$, one can find either

$$x_S \in \Xi_1 \subset CH^r_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),\ y_S \in \Xi_2 \subset CH^{d-r+\nu}_{hom}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),$$

and compute $\langle x_S, y_S \rangle_{HT}$, or

$$x_{S'} \in \Xi_1^{'} \subset CH^r_{hom}((S' \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),\ y_{S'} \in \Xi_2^{'} \subset CH^{d-r+\nu}_{hom}((S' \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}),$$

(where Ξ_1' and Ξ_2' are the counterparts of Ξ_1 and Ξ_2 respectively) and a height pairing $\langle x_{S'}, y_{S'} \rangle_{HT}$. But, assuming BBC, we have isomorphisms: $\Xi_i \underbrace{\cong}_{F_1^*} \Xi_i'' \underbrace{\cong}_{F_{2,*}} \Xi_i'$, $i = 1, 2$ (as before, Ξ_i'' is the counterpart to Ξ_i, for $i = 1, 2$). Moreover $x_{S'} = F_{2,*}(F_1^*(x_S))$ and $y_{S'} = F_{2,*}(F_1^*(y_S))$. Now it follows from the projection formula for height pairing (Proposition 6.0.15) that

$$\langle x_S, y_S \rangle_{HT} = \langle F_1^*(x_S), F_1^*(y_S) \rangle_{HT} = \langle x_{S'}, y_{S'} \rangle_{HT}.$$

□

7.1.11 Height pairing between the algebraic graded pieces.

7.1.12 Definition. *(Algebraic part of the graded piece) : Let*

$$F^{\nu}\underline{CH}^r_{alg}(X_K; \mathbb{Q}) := F^{\nu} CH^r(X_K; \mathbb{Q}) \bigcap \left[Im(CH^r_{alg}((S \times X)_{\overline{\mathbb{Q}}}; \mathbb{Q}) \longrightarrow CH^r(X_K; \mathbb{Q})) \right].$$

Then we can define

$$Gr_F^{\nu}\underline{CH}^r_{alg}(X_K; \mathbb{Q}) := Im\left(F^{\nu}\underline{CH}^r_{alg}(X_K; \mathbb{Q}) \to Gr_F^{\nu} CH^r(X_K; \mathbb{Q})\right).$$

There is one remark in order: If S' is another such variety, then we can dominate both S and S' by a third $S'' \hookrightarrow S \times S'$ (similar to Proposition 7.1.10). From this, and the fact that the rational Chow group of cycles algebraically equivalent to zero being a $\mathbb{Q}$ vector space is divisible, one can show

$$Im\left(CH^r_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) \longrightarrow CH^r(X_K;\mathbb{Q})\right) = Im\left(CH^r_{alg}((S'\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) \longrightarrow CH^r(X_K;\mathbb{Q})\right).$$

Thus the definition of $Gr^{\nu}_F\underline{CH}^r_{alg}(X_K;\mathbb{Q})$ is independent of the choice of S. Now we have the following

7.1.13 Theorem. *Under the same set up as in Theorem 7.0.11 , we have the height pairing*

$$\langle,\rangle_{HT} : Gr^{\nu}_F\underline{CH}^r_{alg}(X_K;\mathbb{Q}) \times Gr^{\nu}_F\underline{CH}^{d-r+\nu}_{alg}(X_K;\mathbb{Q}) \to \mathbb{R},$$

extending the Neron-Tate height pairing.

Proof. Let $J^*_{alg}((S\times X)_{\overline{\mathbb{Q}}})_{\mathbb{Q}} := \Phi_*(CH^*_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}))$. Assuming BBC, we have the following diagram (see [5] for details)

$$\begin{array}{cccccc} \langle,\rangle_{NT} : J^r_{alg}((S\times X)_{\overline{\mathbb{Q}}})_{\mathbb{Q}} & \times & J^{d-r+\nu}_{alg}((S\times X)_{\overline{\mathbb{Q}}})_{\mathbb{Q}} & \longrightarrow & \mathbb{R} \\ \uparrow & & \uparrow & & \\ \langle,\rangle_{HT} : CH^r_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) & \times & CH^{d-r+\nu}_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q}) & \longrightarrow & \mathbb{R}. \end{array} \quad (7.1.13.1)$$

The proof now goes exactly in the same way as Theorem 7.0.11, if we replace $CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ (resp. $CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$) with $CH^r_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ (resp. $CH^{d-r+\nu}_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$). We obtain $\Xi_{1,alg} \subset CH^r_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ (respectively $\Xi_{2,alg} \subset CH^{d-r+\nu}_{alg}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$), such that

$$\Xi_{1,alg} \cong Gr^{\nu}_F\underline{CH}^r_{alg}(X_K;\mathbb{Q})$$

$$\Xi_{2,alg} \cong Gr^{\nu}_F\underline{CH}^{d-r+\nu}_{alg}(X_K;\mathbb{Q})\,.$$

The height pairing is now given as the pairing between $\Xi_{1,alg}$ and $\Xi_{2,alg}$. □

7.1.14 Remark. For a smooth projective variety Z defined over a number field k, let us define

$$CH^r_{alg,AJ}(Z;\mathbb{Q}) := ker\left(\Phi_r : CH^r_{alg}(Z;\mathbb{Q}) \to J(H^{2r-1}(Z,\mathbb{Q}(r)))\right).$$

We have the following lemma ([5], Lemma 4.0.7)

7.1.15 Lemma. *Let Z/k be a smooth projective variety defined over a number field k and $a \in CH^r_{alg,AJ}(Z;\mathbb{Q})$. Then a lies in the kernel of the height pairing.*

In fact, it follows from the remark following Theorem 6.1.2 that the height pairing for cycles algebraically equivalent to zero is given by the Neron-Tate pairing between $J^r_{alg}(Z)_{\mathbb{Q}}$ and $J^{d-r+1}_{alg}(Z)_{\mathbb{Q}}$ (see also [5], Remark 4.0.8).

Thus, one can arrive at the result of Theorem 7.1.13, working modulo the group $CH^*_{alg,AJ}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})$ (instead of assuming it to be zero via BBC).

7.1.16 Motivic viewpoint

This subsection is intended to develop a $\overline{\mathbb{Q}}\otimes_{\mathbb{Q}}\mathbb{R}$ valued height pairing for the graded pieces tensored by $\overline{\mathbb{Q}}$, by reinterpreting it as a pairing between $CH^*_{hom}(\ ;\mathbb{Q})$ of a certain motive and its dual. This beautiful (but conjectural) insight was introduced in section 5 of [5] (see the discussion following Conjecture 5.8). Also [4], section 8.3-8.5 has it in more detail.

For a variety X defined over $\overline{\mathbb{Q}}$, we assume that $F^2CH^r(X;\mathbb{Q}) = 0$. Since $F^2CH^r(X;\mathbb{Q}) \subset ker(\Phi_r)$ this assumption is actually a consequence of the BBC. In particular, it means that the $\mathbb{Q}$ valued intersection pairing for cycles homologous to zero is zero.

For $X/\overline{\mathbb{Q}}$ an irreducible smooth projective variety, we fix the following definitions:

- By motives we will mean motives modulo homological equivalence with coefficients in $\overline{\mathbb{Q}}$.
- For a motive $M := (X,p,l)$, denote by

$$CH^r_{hom}(M;\mathbb{Q}) := Im\left(p_* : CH^{r+l}_{hom}(X;\overline{\mathbb{Q}}) \to CH^{r+l}_{hom}(X;\overline{\mathbb{Q}})\right).$$

- For a motive $M := (X,p,0)$, we have its dual $M^\vee = (X,{}^tp,d-2r+1)$, where d is the dimension of X.

We will need the following observation

7.1.17 Claim. *For the projectors considered in the previous section*

$$P : H^{2r-\nu}(X,\mathbb{Q}) \twoheadrightarrow N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})\ .$$

$$Q : H^{\nu-1}(S,\mathbb{Q}) \twoheadrightarrow N_{\overline{\mathbb{Q}}}^1 H^{\nu-1}(S,\mathbb{Q})$$

and their transpose ${}^tP$ *and* ${}^tQ$*, we have* $Q =^t Q$ *and*

$${}^tP : H^{2d-2r+\nu}(X,\mathbb{Q}) \twoheadrightarrow N_H^{d-r+1}H^{2d-2r+\nu}(X,\mathbb{Q})\ .$$

Proof. Note that since $dim(S) = \nu - 1$, we can think of the map

$${}^tQ : (H^{\nu-1}(S,\mathbb{Q}))^\vee \to (H^{\nu-1}(S,\mathbb{Q}))^\vee$$

as

$${}^tQ : H^{\nu-1}(S,\mathbb{Q}) \to H^{\nu-1}(S,\mathbb{Q}).$$

Also,

$${}^tP : H^{2d-2r+\nu}(X,\mathbb{Q}) \to H^{2d-2r+\nu}(X,\mathbb{Q}).$$

Now from the discussion at the beginning of 7.1.3, we see that

$$(N_{\overline{\mathbb{Q}}}^1 H^{\nu-1}(S,\mathbb{Q}))^\vee \cong \{N_{\overline{\mathbb{Q}}}^1 H^{\nu-1}(S,\mathbb{Q})\}^\perp = N_{\overline{\mathbb{Q}}}^1 H^{\nu-1}(S,\mathbb{Q})$$

and

$$(N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q}))^\vee \cong \{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})\}^\perp = N_H^{d-r+1}H^{2d-2r+\nu}(X,\mathbb{Q})),$$

and the claim follows immediately.

□

As seen in 7.1.3, we can choose

$$w_1 := (\Delta_S(\nu-1,\nu-1) - Q) \otimes (\Delta_X(2d-2r+\nu, 2r-\nu) - P) \bullet \widetilde{P}$$

with the property that

$$[w_1]_* : H^{2r-1}((S\times X)_{\overline{\mathbb{Q}}},\mathbb{Q}) \twoheadrightarrow \{N_{\overline{\mathbb{Q}}}^1 H^{\nu-1}(S,\mathbb{Q})\}^\perp \otimes \{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})\}^\perp$$

and

$$\Xi_1 := w_{1,*}(CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})) \cong Gr^\nu_F CH^r(X_K;\mathbb{Q})\ .$$

The isomorphism carries through if we tensor with $\overline{\mathbb{Q}}$, i.e

$$w_{1,*}(CH^r_{hom}((S\times X)_{\overline{\mathbb{Q}}};\overline{\mathbb{Q}})) \cong Gr^\nu_F CH^r(X_K;\overline{\mathbb{Q}})\ .$$

If we consider

$${}^t w_1 = {}^t \widetilde{P} \bullet \left[\left(\Delta_S(\nu-1,\nu-1) - {}^tQ\right)\otimes\left(\Delta_X(2r-\nu,2d-2r+\nu) - {}^tP\right)\right]$$

then by Claim 7.1.17, we have

$$[{}^t w_1]_* : H^{2(d+\nu-r)-1}((S\times X)_{\overline{\mathbb{Q}}},\mathbb{Q}) \twoheadrightarrow \{N^1_{\overline{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})\}^\perp \otimes \{N^{d-r+1}_H H^{2d-2r+\nu}(X,\mathbb{Q})\}^\perp$$

and

$$\Xi_2 := {}^t w_{1,*}(CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}};\mathbb{Q})) \cong Gr^\nu_F CH^{d-r+\nu}(X_K;\mathbb{Q})\ .$$

which carries through if we tensor with $\overline{\mathbb{Q}}$, i.e

$${}^t w_{1,*}(CH^{d-r+\nu}_{hom}((S\times X)_{\overline{\mathbb{Q}}};\overline{\mathbb{Q}})) \cong Gr^\nu_F CH^{d-r+\nu}(X_K;\overline{\mathbb{Q}})\ .$$

Consider the motive

$$M_{(S\times X)_{\overline{\mathbb{Q}}}} := ((S\times X)_{\overline{\mathbb{Q}}}, w_1, 0)$$

and its dual

$$M^\vee_{(S\times X)_{\overline{\mathbb{Q}}}} = ((S\times X)_{\overline{\mathbb{Q}}},\ {}^t w_1,\ d-2r+\nu).$$

From above, we get

$$CH^r_{hom}(M_{(S\times X)_{\overline{\mathbb{Q}}}};\mathbb{Q}) \cong Gr^\nu_F CH^r(X_K;\overline{\mathbb{Q}})$$

and

$$CH^r_{hom}(M^\vee_{(S\times X)_{\overline{\mathbb{Q}}}};\mathbb{Q}) \cong Gr^\nu_F CH^{d-r+\nu}(X_K;\overline{\mathbb{Q}})\ .$$

In this way, we can develop a $\overline{\mathbb{Q}}\otimes\mathbb{R}$ valued height pairing between $Gr^\nu_F CH^r(X_K;\overline{\mathbb{Q}})$ and $Gr^\nu_F CH^{d-r+\nu}(X_K;\overline{\mathbb{Q}})$ as a height pairing between $CH^r_{hom}(\ ;\ \mathbb{Q})$ of the motive $M_{(S\times X)_{\overline{\mathbb{Q}}}}$ and its dual.

7.2 Speculation about a more general situation

In this incomplete section, we are going to speculate how one can generalize Theorem 7.0.11 in the following situation: we can find a family $\rho : \mathcal{X} \to \mathcal{S}$, where $\mathcal{X}$ and $\mathcal{S}$ smooth and quasiprojective over $\overline{\mathbb{Q}}$ and ρ is smooth and proper. If $\eta_{\mathcal{S}}$ denote the generic point of $\mathcal{S}$, then $\overline{\mathbb{Q}}(\eta_{\mathcal{S}}) \cong K$ and $\mathcal{X}_{\eta_{\mathcal{S}}} \cong X_K$. One can then have the following diagram:

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \overline{\mathcal{X}} \\ \rho \downarrow & & \downarrow \bar{\rho} \\ \mathcal{S} & \hookrightarrow & \overline{\mathcal{S}} \end{array} \tag{7.2.0.1}$$

where $\overline{\mathcal{X}}$ and $\overline{\mathcal{S}}$ are the projective closures of $\mathcal{X}$ and $\mathcal{S}$ respectively. We call this a general $\overline{\mathbb{Q}}$-spread of X_K. In the previous section we worked with the product $Pr_S : (S \times X)_{\overline{\mathbb{Q}}} \to S$.

One has the following (non-canonical) decomposition

$$H^{2r-1}(\mathcal{X}, \mathbb{Q}(r)) \cong \bigoplus_{\nu \geqq 1} H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_* \mathbb{Q}(r)),$$

and (after possibly shrinking $\mathcal{S}$) an inclusion

$$\bigoplus_{\nu \geqq 1} H^{\nu-1}(\mathcal{S}, N_K^{r-\nu+1} R^{2r-\nu}\rho_* \mathbb{Q}(r)) \subset \bigoplus_{\nu \geqq 1} H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_* \mathbb{Q}(r))$$

of MHS. Here, we make the following assumption

7.2.1 Assumption. *Let $\rho : \mathcal{X} \to \mathcal{S}$ be a smooth and proper map of smooth quasiprojective varieties defined over $\overline{\mathbb{Q}}$. Then the images of*

$$\Gamma Gr_0^W H^{\nu-1}(\mathcal{S}, N_K^{r-\nu+1} R^{2r-\nu}\rho_* \mathbb{Q}(r)),$$

and

$$\Gamma Gr_0^W H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r))$$

inside the Jacobian $J(W_{-1}H^{\nu-1}(S, R^{2r-\nu}\rho_*\mathbb{Q}(r)))$*, are same.*

7.2.2 Remark. As seen in the proof of Proposition 7.1.5, this assumption holds in the product situation. It would be interesting to explore further as to which situations it holds.

Now, we hope to generalize Theorem 7.0.11 to

7.2.3 Theorem. *Let* $K/\overline{\mathbb{Q}}$ *be an overfield of transcendence degree* $\nu - 1$ *and* $X/\overline{\mathbb{Q}}$ *be an irreducible smooth projective variety of dimension* d*. Let us assume 7.2.1, together with the Grothendieck amended general Hodge conjecture and BBC. Then using the spread* $\rho : \mathcal{X} \to \mathcal{S}$ *(see discussion at the beginning of the section), one can develop a height pairing*

$$Gr_F^\nu CH^r(X_K;\mathbb{Q}) \times Gr_F^\nu CH^{d-r+\nu}(X_K;\mathbb{Q}) \to \mathbb{R}$$

on the graded pieces of the Lewis filtration ([38]), extending the Beilinson's height pairing.

7.2.4 Remark. We would also want this height pairing to be independent of the spread $\rho : \mathcal{X} \to \mathcal{S}$. In particular we get back the height pairing developed in Theorem 7.0.11, where Assumption 7.2.1 holds.

Proof. Note that $dim(\mathcal{S}) = \nu - 1$ and the relative dimension of $\mathcal{X}$ is d. There is a surjection

$$CH^r(\mathcal{X};\mathbb{Q}) \twoheadrightarrow CH^r(X_K;\mathbb{Q}) \xrightarrow{\Delta_{X_K}(2d-2r+\nu,2r-\nu)_*} Gr_F^\nu CH^r(X_K;\mathbb{Q}).$$

We can choose a lift $\Delta_{\mathcal{X}}(2d-2r+\nu, 2r-\nu) \in \mathcal{X}\times_{\mathcal{S}}\mathcal{X}$ of $\Delta_{X_K}(2d-2r+\nu, 2r-\nu)$, of relative dimension d, and get the following commutative diagram

$$\begin{array}{ccccc} CH^r(\mathcal{X};\mathbb{Q}) & \xrightarrow{\Delta_{\mathcal{X}}(2d-2r+\nu,2r-\nu)_*} & \Delta_{\mathcal{X}}(2d-2r+\nu,2r-\nu)_* CH^r(\mathcal{X};\mathbb{Q}) & & \\ \downarrow{\scriptstyle cl_{\mathcal{H}}} & & \downarrow{\scriptstyle cl_r} & & \\ H^{2r}_{\mathcal{H}}(\mathcal{X},\mathbb{Q}) & \xrightarrow{[\Delta_{\mathcal{X}}(2d-2r+\nu,2r-\nu)]_*} & \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho) & \xrightarrow{j^*} & \underline{\underline{E}}_\infty^{\nu,2r-\nu}(\rho_V) = 0, \end{array} \tag{7.9}$$

where $V \subset \mathcal{S}$ is smooth, affine and open. Further, using the Hodge conjecture we conclude

$$\Delta_{\mathcal{X}}(2d-2r+\nu, 2r-\nu)_* : CH^r_{hom}(\mathcal{X};\mathbb{Q}) \twoheadrightarrow Gr^\nu_F CH^r(X_K;\mathbb{Q}). \quad (7.2.4.1)$$

We also need to use

7.2.5 Proposition. *Let Assumption 7.2.1 hold, then we have an injective map*

$$Gr^\nu_F CH^r(X_K;\mathbb{Q}) \hookrightarrow J\left(W_{-1}H^{\nu-1}\left(\eta_{\mathcal{S}}, \frac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right)\right).$$

The proof of this proposition goes exactly in the same way as that of Proposition 7.1.5, now noting that there is a natural map

$$\underline{E}_\infty^{\nu,2r-\nu}(\eta_{\mathcal{S}}) \to J\left(W_{-1}H^{\nu-1}\left(\eta_{\mathcal{S}}, \frac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right)\right),$$

given by the projector

$$H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow H^{\nu-1}\left(\eta_{\mathcal{S}}, \frac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right).$$

Now, the image $\Phi_r : CH^r_{hom}(\mathcal{X};\mathbb{Q}) \hookrightarrow J(H^{2r-1}(\mathcal{X},\mathbb{Q}(r)))$ actually lands in

$$\frac{J(W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)))}{\Gamma Gr^W_0 H^{2r-1}(\mathcal{X},\mathbb{Q}(r))},$$

via the short exact sequence

$$0 \to W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)) \to W_0H^{2r-1}(\mathcal{X},\mathbb{Q}(r)) \to Gr^W_0 H^{2r-1}(\mathcal{X},\mathbb{Q}(r)) \to 0.$$

But from Assumption 7.2.1, it follows that there is a map

$$\frac{J(W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)))}{\Gamma Gr^W_0 H^{2r-1}(\mathcal{X},\mathbb{Q}(r))} \to J\left(W_{-1}H^{\nu-1}\left(\eta_{\mathcal{S}}, \frac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_K^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right)\right),$$

which is given by the following series of (non-canonical) projections

$$W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)) \twoheadrightarrow W_{-1}H^{\nu-1}(\mathcal{S}, R^{2r-\nu}\rho_*\mathbb{Q}(r)) \twoheadrightarrow W_{-1}H^{\nu-1}(\eta_{\mathcal{S}}, R^{2r-\nu}\rho_*\mathbb{Q}(r))$$

$$\twoheadrightarrow W_{-1}H^{\nu-1}\left(\eta_{\mathcal{S}}, \frac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right).$$

Now we can see the surjection of equation 7.2.4.1 inside the Jacobian, through the following commutative diagram

$$\begin{array}{ccc} CH^r_{hom}(\mathcal{X};\mathbb{Q}) & \hookrightarrow & \dfrac{J(W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)))}{\Gamma Gr_0^W H^{2r-1}(\mathcal{X},\mathbb{Q}(r))} \\ \downarrow & & \downarrow \\ Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) & \hookrightarrow & J\left(\underbrace{W_{-1}H^{\nu-1}\left(\eta_{\mathcal{S}}, \dfrac{R^{2r-\nu}\rho_*\mathbb{Q}(r)}{N_H^{r-\nu+1}R^{2r-\nu}\rho_*\mathbb{Q}(r)}\right)}_{W_{-1}H_0}\right) \end{array} \tag{7.2.5.1}$$

Now, from the (non-canonical) decomposition

$$W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r)) \cong W_{-1}H_0 \bigoplus (W_{-1}H_0)^{\perp},$$

we get a similar decomposition at the level of Jacobians

$$J(W_{-1}H^{2r-1}(\mathcal{X},\mathbb{Q}(r))) = J(W_{-1}H_0) \bigoplus J((W_{-1}H_0)^{\perp}).$$

The idea now is to conclude (viewing everything inside the respective Jacobians)

$$CH^r_{hom}(\mathcal{X};\mathbb{Q}) \cong Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) \bigoplus (Gr_F^{\nu} CH^r(X_K;\mathbb{Q}))^{\perp}. \tag{7.2.5.2}$$

(In progress).

□

We will see some computations of our height pairing in the next chapter.

Chapter 8

Some computations for product of curves

This chapter is motivated towards jump starting calculations.

8.1 Product of general curves

Notation : *Henceforth, for two smooth projective varieties X and Y over any field k, their usual fibre product $X \times_k Y$ will be denoted simply by $X \times Y$. This is done mainly for the ease of writing than for any other reason.*

We will begin with a lemma which will serve as a prototypical example for all the later computations. We thank Dr. José Burgos Gil for providing us with the idea of the proof of this lemma.

8.1.1 Lemma. *Let C be smooth projective curve and X be a smooth projective variety of dimension $d-1$, both defined over a number field k. Let $\alpha_1, \alpha_2 \in CH^1_{alg}(C;\mathbb{Q})$ and $\pi_1 : C \times X \to C$ and $\pi_2 : C \times X \to X$ are the projections. Given $w_1 \in CH^{r-1}(X;\mathbb{Q})$ and $w_2 \in CH^{(d-1)-(r-1)}(X;\mathbb{Q}) = CH^{d-r}(X;\mathbb{Q})$ and the cycles*

$$\xi_1 := \pi_1^*(\alpha_1) \cdot \pi_2^*(w_1) \in CH^r_{alg}(C \times X;\mathbb{Q})$$

$$\xi_2 := \pi_1^*(\alpha_2) \cdot \pi_2^*(w_2) \in CH^{d-r+1}_{alg}(C \times X;\mathbb{Q}).$$

We get the following height pairing relation :

$$\langle \xi_1, \xi_2 \rangle_{HT} = (\deg(w_1 \cdot w_2)_X)\langle \alpha_1, \alpha_2 \rangle_{NT} \; .$$

Here $(w_1 \cdot w_2)_X$ *is the usual intersection pairing on* X*,* $\langle \, , \, \rangle_{HT}$ *and* $\langle \, , \, \rangle_{NT}$ *denotes the Beilinson/arithmetic and the Neron-Tate height pairings respectively.*

Proof. We fix the following notation : For an arithmetic variety Y over $Spec(O_k)$ of a number field k, we will denote the structural morphism $Y \to Spec(O_k) \to Spec(\mathbb{Z})$ by Π_Y.

Let $\widetilde{C}$ be the unique minimal regular model for C over $Spec(O_k)$ (by [44], Proposition 10.1.8 , such a model exists). Choose Z_i, $i = 1, 2$ cycles on $\widetilde{C}$ of codimension 1 such that

- $Z_i|_C = \alpha_i$.
- $Z_i \cdot V = 0$ for any vertical cycle V. We can arrange this, see for example [34], section 6.

Choose g_i, $i = 1, 2$ Green's functions for Z_i such that $dd^c g_i + \delta_{\alpha_i} = 0$. We have

$$[(Z_i, g_i)] \in \widehat{CH}^1(\widetilde{C}), \; i = 1, 2 \; .$$

Then,

$$\langle \alpha_1, \alpha_2 \rangle_{NT} = \Pi_{\widetilde{C},*}([(Z_1, g_1)] \cdot [(Z_2, g_2)]) \in \widehat{CH}^1(Spec(\mathbb{Z})) = \mathbb{R}$$

is independent of the choices of Z_i, g_i.

Now, for any projective and flat model $\widetilde{X}'$ over $Spec(O_k)$ of X, we get, by de Jong's alteration (see [11] for details), a projective, flat and regular scheme $\widetilde{X}$ over $Spec(O_k)$ with a finite and surjective morphism to $\widetilde{X}'$, in particular $dim(\widetilde{X}') = dim(\widetilde{X})$. Let W_i, $i = 1, 2$ be cycles on $\widetilde{X}$ of codimensions $r - 1$ and $d - r$ respectively such that

$$W_i|_X = w_i, \;\; i = 1, 2 \; .$$

Let g_{W_1} (resp. g_{W_2}) be a Green current of logarithmic type for W_1 (resp. W_2). Then

$$[(W_1, g_{W_1})] \in \widehat{CH}^{r-1}(\widetilde{X})$$
$$[(W_2, g_{W_2})] \in \widehat{CH}^{d-r}(\widetilde{X}) .$$

For the scheme $\widetilde{C} \times_{Spec(O_k)} \widetilde{X}$, we can use the alteration trick once again to obtain a regular flat and projective scheme Z over $Spec(O_k)$ and a dominant and finite morphism $f : Z \to \widetilde{C} \times_{Spec(O_k)} \widetilde{X}$. In particular $dim(Z) = dim(\widetilde{C} \times_{Spec(O_k)} \widetilde{X}) = d + 1$. For the projections

$$\pi_{\widetilde{C}} : \widetilde{C} \times_{Spec(O_k)} \widetilde{X} \to \widetilde{C}$$

$$\pi_{\widetilde{X}} : \widetilde{C} \times_{Spec(O_k)} \widetilde{X} \to \widetilde{X} ,$$

consider

$$f_{\widetilde{C}} := \pi_{\widetilde{C}} \circ f$$

$$f_{\widetilde{X}} := \pi_{\widetilde{X}} \circ f \ \ ,$$

and the cycles

$$\tilde{\xi}_1 := f^*_{\widetilde{C}}([(Z_1, g_{Z_1})]) f^*_{\widetilde{X}}([(W_1, g_{W_1})])$$
$$\tilde{\xi}_2 := f^*_{\widetilde{C}}([(Z_2, g_{Z_2})]) f^*_{\widetilde{X}}([(W_2, g_{W_2})]) .$$

Then

$$\langle \xi_1, \xi_2 \rangle_{HT} = \Pi_{Z,*} \left(\tilde{\xi}_1 \cdot \tilde{\xi}_2 \right) \in \widehat{CH}^1(Spec(Z)) = \mathbb{R} .$$

Since $f^*_{\widetilde{C}}$ and $f^*_{\widetilde{X}}$ are morphisms of rings ([17], 4.4.3 (5)),

$$\tilde{\xi}_1 \cdot \tilde{\xi}_2 = f^*_{\widetilde{C}} \left([(Z_1, g_1)] \cdot [(Z_2, g_2)] \right) \cdot f^*_{\widetilde{X}} \left([(W_1, g_{W_1})] \cdot [(W_2, g_{W_2})] \right) \ .$$

By the projection formula for arithmetic intersection pairing ([17], 4.4.3 (7))

$$f_{\widetilde{C},*} \left(\tilde{\xi}_1 \cdot \tilde{\xi}_2 \right) = \underbrace{[(Z_1, g_1)] \cdot [(Z_2, g_2)]}_{\in \widehat{CH}^2(\widetilde{C})} \cdot \underbrace{f_{\widetilde{C},*} \left[f^*_{\widetilde{X}} \left([(W_1, g_{W_1})] \cdot [(W_2, g_{W_2})] \right) \right]}_{\in \widehat{CH}^0(\widetilde{C})} \ .$$

Since

$$\Pi_{Z,*} \left(\tilde{\xi}_1 \cdot \tilde{\xi}_2 \right) = \Pi_{\widetilde{C},*} \left(f_{\widetilde{C},*}(\tilde{\xi}_1 \cdot \tilde{\xi}_2) \right)$$

and

$$f_{\widetilde{C},*} \left[f^*_{\widetilde{X}} \left([(W_1, g_{W_1})] \cdot [(W_2, g_{W_2})] \right) \right] = deg(w_1 \cdot w_2)_X \ ,$$

we obtain our desired result.

□

Now, we state and prove a corollary which will serve as an example for the theory developed through Theorem 7.0.11.

8.1.2 Corollary. *Assume given smooth projective curves* $C_1, \cdots C_d$ *over* $\overline{\mathbb{Q}}$ *and let* $X = C_1 \times \cdots \times C_d$*. For* $\nu \geqq 2$*, we fix an embedding* $K = \overline{\mathbb{Q}}(C_2 \times \cdots \times C_\nu) \hookrightarrow \mathbb{C}$*, and let* $p = (\eta_2, \cdots, \eta_\nu) \in C_2(\mathbb{C}) \times \cdots \times C_\nu(\mathbb{C})$ *be a very general point corresponding to this embedding. We fix* $e_j \in C_j(\overline{\mathbb{Q}})$*,* $j = 2, \cdots, d$*. For distinct points* $p_1, q_1, p_2, q_2 \in C_1(\overline{\mathbb{Q}})$ *and* $\nu \leqq r \leqq d$*, let*

$$\xi_1 := Pr^*_{1,\cdots,\nu}((p_1-q_1)\times(\eta_2-e_2)\times\cdots\times(\eta_\nu-e_\nu)) \bigcap Pr^*_{\nu+1,\cdots,r}(e_{\nu+1}, \cdots, e_r) \in Gr^\nu_F CH^r(X_K; \mathbb{Q}),$$

$$\xi_2 := Pr^*_{1,\cdots,\nu}((p_2-q_2)\times(\eta_2-e_2)\times\cdots\times(\eta_\nu-e_\nu)) \bigcap Pr^*_{r+1,\cdots,d}(e_{r+1}, \cdots, e_d) \in Gr^\nu_F CH^{d-r+\nu}(X_K; \mathbb{Q}).$$

Assume also

$$N^1_H \left(H^1(C_1, \mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu, \mathbb{Q})\right) = N^1_{\overline{\mathbb{Q}}} \left(H^1(C_2, \mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu, \mathbb{Q})\right) = 0.$$

Then, $\langle \xi_1, \xi_2 \rangle_{HT} = \left(\prod_{j=2}^{\nu}[deg(\Delta^2_{C_j}(1,1))_{C_j \times C_j}]\right) \langle p_1 - q_1, p_2 - q_2 \rangle_{NT}$*,*
where $\langle , \rangle_{NT}$ *is the Neron-Tate pairing on* $(J^1(C_1)(\overline{\mathbb{Q}})) \otimes \mathbb{Q}$*.*

We add some comments before we begin the proof :

1. Note that, $X_K = C_{1,K} \times \cdots \times C_{d,K}$ and we view $(\eta_j - e_j) \in CH^1_{hom}(C_{j,K})$ for $2 \leqq j \leqq \nu$.

2. For a smooth projective curve C of genus g, we know that the homology class of $\Delta_C(1,1)$ in $H_1(C, \mathbb{Z})$ is given by

$$\Delta_C(1,1) = \sum_{i=1}^{g} (\alpha_i \otimes \beta_i - \beta_i \otimes \alpha_i),$$

where $\{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}$ are the canonical generators of $H_1(C, \mathbb{Z})$, having the property $\alpha_i \cdot \alpha_j = \beta_i \cdot \beta_j = 0$, $\alpha_i \cdot \beta_j = \delta_{ij}$ and $\alpha_i \cdot \beta_j = -(\beta_j \cdot \alpha_i)$, $\cdot$ denoting the intersection number. Hence, it follows that

$$deg(\Delta^2_C(1,1)) = -2g\ .$$

So, we can rewrite

$$\langle \xi_1, \xi_2 \rangle_{HT} = (-1)^{\nu-1} \cdot 2^{\nu-1} \left(\prod_{j=2}^{\nu} g_j \right) \langle p_1 - q_1, p_2 - q_2 \rangle_{NT} ,$$

where g_j is the genus of the curve C_j.

3. The assumption of this corollary holds for example if we take $X = E_1 \times E_2$, a product of two non-isogenous elliptic curves, and $S = E_2$. Here $N^1_{\mathbb{Q}}(H^1(E_2, \mathbb{Q})) = 0$ is automatic and

$$N^1_H(H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q})) = H^2_{alg}(E_1 \times E_2, \mathbb{Q}) \cap (H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q})) = 0$$

follows from the fact any non-zero element $[\xi] \in H^2_{alg}(E_1 \times E_2, \mathbb{Q}) \cap (H^1(E_1, \mathbb{Q}) \otimes H^1(E_2, \mathbb{Q}))$ will in turn define an isogeny between E_1 and E_2.

Proof. We will closely follow the set up of Theorem 7.0.11 . Set $S = C_2 \times \cdots \times C_\nu$ with projections

$$\pi^S_i : S \to C_i, i = 2, \cdots \nu,$$

$$\pi^X_j : X \to C_j, j = 1, \cdots d,$$

$$\pi_{i,j} := \pi^S_i \times \pi^X_j : S \times X \to C_i \times C_j.$$

We have (from Chow-Kunneth decomposition for smooth curves)

$$\Delta_{C_j}(1,1) = \Delta_{C_j} - e_j \times C_j - C_j \times e_j.$$

We now put

$$\tilde{\xi}_1 := \left(\bigcap_{2}^{\nu} \pi^*_{i,i}(\Delta_{C_i}(1,1)) \right) \bigcap \left(\pi^{X,*}_1(p_1 - q_1) \right) \bigcap \left(\bigcap_{\nu+1}^{r} \pi^{X,*}_j(e_j) \right),$$

$$\tilde{\xi}_2 := \left(\bigcap_{2}^{\nu} \pi^*_{i,i}(\Delta_{C_i}(1,1)) \right) \bigcap \left(\pi^{X,*}_1(p_2 - q_2) \right) \bigcap \left(\bigcap_{r+1}^{d} \pi^{X,*}_j(e_j) \right),$$

and observe that since $(p_j - q_j) \sim_{alg} 0$, it follows from basic intersection theory of algebraic varieties that $\tilde{\xi}_1$ and $\tilde{\xi}_2$ belong to $CH^r_{alg}(S \times X; \mathbb{Q})$ and

$CH^{d-r+\nu}_{alg}(S \times X; \mathbb{Q})$ respectively.

8.1.3 Lemma. *The Abel-Jacobi images of $\tilde{\xi}_1$ and $\tilde{\xi}_2$ lies in*

$$J\left([\otimes_{i=2}^{\nu} H^1(C_i,\mathbb{Q})][\otimes_{j=1}^{\nu} H^1(C_j,\mathbb{Q})][\otimes_{j=\nu+1}^{r} H^2(C_j,\mathbb{Q})][\otimes_{j=r+1}^{d} H^0(C_j,\mathbb{Q})](r)\right)$$

$$\hookrightarrow J(H^{\nu-1}(S,\mathbb{Q}) \otimes H^{2r-\nu}(X,\mathbb{Q})(r)),$$

and

$$J\left([\otimes_{i=2}^{\nu} H^1(C_i,\mathbb{Q})][\otimes_{j=1}^{\nu} H^1(C_j,\mathbb{Q})][\otimes_{j=\nu+1}^{r} H^0(C_j,\mathbb{Q})][\otimes_{j=r+1}^{d} H^2(C_j,\mathbb{Q})](d-r+\nu)\right)$$

$$\hookrightarrow J(H^{\nu-1}(S,\mathbb{Q}) \otimes H^{2d-2r+\nu}(X,\mathbb{Q})(d-r+\nu))$$

respectively.

Proof. We will prove for $\tilde{\xi}_1$ and the argument for $\tilde{\xi}_2$ is exactly similar. Consider the correspondence

$$Z_1 := \pi^*_{C_1\times C_1}\left(\Delta_{C_1}(1,1)\right) \cdot \left(\bigcap_2^{\nu} \pi^*_{i,i}(\Delta_{C_i}(1,1))\right) \bigcap \left(\bigcap_{\nu+1}^{r} \pi_j^{X,*}(e_j)\right)$$

SInce $Z_{1,*}(p_1 - q_1) = \tilde{\xi}_1$, it follows from the commutative diagram

$$\begin{array}{ccc} CH^1_{alg}(C_1;\mathbb{Q}) & \xrightarrow{Z_{1,*}} & CH^r_{alg}(S\times X;\mathbb{Q}) \\ \Phi_1 \downarrow & & \downarrow \Phi_r \\ J(H^1(C_1,\mathbb{Q}(r))) & \xrightarrow{[Z_1]_*} & J(H^{2r-1}(S\times X,\mathbb{Q}(r))) \end{array} \qquad (8.1.3.1)$$

that the Abel-Jacobi image of $\tilde{\xi}_1$ lies in

$$J\left([\otimes_{i=2}^{\nu} H^1(C_i,\mathbb{Q})][\otimes_{j=1}^{\nu} H^1(C_j,\mathbb{Q})][\otimes_{j=\nu+1}^{r} H^2(C_j,\mathbb{Q})][\otimes_{j=r+1}^{d} H^0(C_j,\mathbb{Q})](r)\right).$$

□

For smooth curves we have $H^2(C_j,\mathbb{Q}) = H^{1,1}(C_j,\mathbb{Q})$ from basic Hodge theory. Also from the conditions

$$N^1_{\mathbb{Q}}\left(H^1(C_2,\mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu,\mathbb{Q})\right) = 0$$

and

$$N^1_H\left(H^1(C_1,\mathbb{Q})\otimes\cdots\otimes H^1(C_\nu,\mathbb{Q})\right)=0$$

we obtain that the Abel-Jacobi invariants of $\tilde{\xi}_1$ and $\tilde{\xi}_2$ belongs to

$$P_{1,*}J(H^{2r-1}(S\times X,\mathbb{Q}(r)))$$

and

$$P_{2,*}J(H^{2(d-r+\nu)-1}(S\times X,\mathbb{Q}(d-r+\nu)))$$

respectively. Here, P_1 and P_2 are the projections defined in Chapter 7. Note also that $\tilde{\xi}_1=w_{1,*}(\tilde{\xi}_1)$ and $\tilde{\xi}_2=w_{2,*}(\tilde{\xi}_2)$, if we assume BBC. Thus, $\tilde{\xi}_1\in\Xi_1$ and $\tilde{\xi}_2\in\Xi_2$ where Ξ_1 and Ξ_2 are as defined in Chapter 7.

8.1.4 Lemma. *Under the isomorphism*

$$\Xi_1\cong Gr^\nu_F CH^r(X_K;\mathbb{Q})$$

respectively

$$\Xi_2\cong Gr^\nu_F CH^{d-r+\nu}(X_K;\mathbb{Q})$$

we have that $\tilde{\xi}_1\mapsto\xi_1$ *and* $\tilde{\xi}_2\mapsto\xi_2$.

Proof. Note that, once we fix a general point $p\in S(\mathbb{C})$ and an embedding $ev_p:K\hookrightarrow\mathbb{C}$, the surjection

$$CH^r(S\times X;\mathbb{Q})\twoheadrightarrow CH^r(X_K;\mathbb{Q})$$

is given by $\mathcal{Z}\mapsto\mathcal{Z}\cap(\{p\}\times X)$ where $\mathcal{Z}\in CH^r(S\times X;\mathbb{Q})$. In our situation, $p=(\eta_2,\cdots,\eta_\nu)$ and if we pick up $\pi_1^{X,*}(p_1-q_1)$ as a prototype, we have the following computation :

$$\pi_1^{X,*}(p_1-q_1)=(C_2\times\cdots\times C_\nu)\times\{(p_1-q_1)\}\times(C_2\times\cdots\times C_d)\in CH^1_{alg}(S\times X).$$

Intersecting with $\{(\eta_2,\cdots,\eta_\nu)\}\times X$, we get :

$$\{(\eta_2,\cdots,\eta_\nu)\}\times\{(p_1-q_1)\}\times(C_2\times\cdots\times C_d).$$

Similar calculation shows that the image of $(p_1-q_1)\in CH^1_{hom}(C_1;\mathbb{Q})$ under the map $Pr^*_{1,\cdots,\nu}$ is exactly the same.

Mimicking this computation above for other components and using the fact that $\Delta_{C_j}(1,1) = \Delta_{C_j} - e_j \times C_j - C_j \times e_j$, we get our result. $\square$

By Lemma 8.1.4, it suffices to compute $\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_{HT}$ with respect to the height pairing

$$\langle , \rangle_{HT} : CH^r_{alg}(S \times X; \mathbb{Q}) \times CH^{d-r+\nu}_{alg}(S \times X; \mathbb{Q}) \to \mathbb{R}.$$

Collecting coefficients, we can assume that the curves $C_i,\ i = \overline{1,d}$ are defined over a number field k. We set up the following notations :

- $C := C_1$
- $X := [(C_2 \times C_2) \times \cdots \times (C_\nu \times C_\nu)] \times C_{\nu+1} \times \cdots \times C_d$
- $\alpha_i := p_i - q_i,\ i = 1, 2$
- $w_1 := \left(\bigcap_2^\nu \pi^*_{i,i}(\Delta_{C_i}(1,1))\right) \cdot \left(\bigcap_{\nu+1}^r \pi_j^{X,*}(e_j)\right)$
- $w_2 := \left(\bigcap_2^\nu \pi^*_{i,i}(\Delta_{C_i}(1,1))\right) \cdot \left(\bigcap_{r+1}^d \pi_j^{X,*}(e_j)\right)$,

following Lemma 8.1.1. Then the height pairing is given by

$$\langle \xi_1, \xi_2 \rangle_{HT} = (deg(w_1 \cdot w_2)_X)\, \langle p_1 - q_1, p_2 - q_2 \rangle_{NT}\ .$$

But

$$deg(w_1 \cdot w_2)_X = \prod_{j=2}^{\nu} [deg(\Delta^2_{C_j}(1,1))_{C_j \times C_j}]\ .$$

Thus we obtain the required relation. $\square$

****

8.2 A computation for self product of elliptic curves

The assumption

$$N^1_H\left(H^1(C_1, \mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu, \mathbb{Q})\right) = N^1_{\mathbb{Q}}\left(H^1(C_2, \mathbb{Q}) \otimes \cdots \otimes H^1(C_\nu, \mathbb{Q})\right) = 0$$

of Corollary 8.1.2 is very general condition and was made for the ease of computation. An ideal situation would be to able to compute without this assumption. As an example, if we consider the self product $C \times C$ of an irreducible smooth projective curve C, the aforementioned assumption is no longer valid. Indeed, going further, we will restrict ourselves to the self-product of a CM elliptic curve.

We will stick to the following notation :

Let $X/\overline{\mathbb{Q}} := C_1 \times C_2$ where C_i, $i = 1, 2$ are irreducible smooth projective curves defined over $\overline{\mathbb{Q}}$ and $S := C_2$. Note that

$$\{N^1_{\overline{\mathbb{Q}}} H^1(S, \mathbb{Q})\}^{\perp} = \{0\}^{\perp} = H^1(S, \mathbb{Q}) \ ,$$

and

$$\{N^1_H H^2(X, \mathbb{Q})\}^{\perp} = \{H^2_{alg}(X, \mathbb{Q})\}^{\perp} = H^2_{tr}(X, \mathbb{Q}) \ ,$$

is the transcendental cohomology. We can choose a basis $\{D_1, \cdots, D_N\}$ of $H^2_{alg}(X, \mathbb{Q})$ with the dual basis (with respect to the cup product) $\{D'_1, \cdots, D'_N\}$. The transcendental projector $T : H^2(X, \mathbb{Q}) \twoheadrightarrow H^2_{tr}(X, \mathbb{Q})$ is then given by

$$\Delta_X(2, 2) - A \ .$$

where $A := \sum_1^N D'_j \times D_j$ is the algebraic projector. Since X is defined over $\overline{\mathbb{Q}}$, we can choose D_j, D'_j over $\overline{\mathbb{Q}}$. Let us consider a situation where we can explicitly compute the basis $\{D_1, \cdots, D_N\}$; that of $X/\overline{\mathbb{Q}} := E \times E$ where $E/\overline{\mathbb{Q}}$ is a CM-elliptic curve with complex multiplication given by the lattice $\mathbb{Z}[i]$ (i.e $E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}[i]$), although the method that we are going to adopt should generalize to any CM-elliptic curve.

The basis for $H^2_{alg}(X, \mathbb{Q})$ is given by

$$\{[\Delta_E(2, 0)], [\Delta_E(0, 2)], [\Delta_E(1, 1)], [\Xi_E(1, 1)]\} \ ,$$

where $\Xi_E \in CH^1(E \times E; \mathbb{Q})$ is the graph of the complex multiplication by i.

Note up to a factor of 2, the dual basis is given by

$$\{[\Delta_E(0,2)], [\Delta_E(2,0)], [-\Delta_E(1,1)], [-\Xi_E(1,1)]\} \ .$$

In particular, we have the following intersection numbers (using that the genus g of E is 1)

- $deg(\Delta_E(2,0) \cdot \Delta_E(0,2)) = 1$
- $\deg(\Delta_E(1,1) \cdot \Xi_E(1,1)) = 0$
- $deg(\Delta_E^2(1,1)) = -2$ and $deg(\Xi_E^2(1,1)) = 2$.

Consider the cycle $\xi := (p-q) \times (\eta - o) \in Gr_F^2 CH^2(X_K;\mathbb{Q})$, where $K \cong \overline{\mathbb{Q}}(E)$, $\eta \in E(K)$ is a very general point and $\{p,q,o\} \in E(\overline{\mathbb{Q}})$, with o being the basepoint of E. Our aim is to compute

$$\langle \xi, \xi \rangle_{HT} \ .$$

In this situation $S = E$ and $S \times X = E \times E \times E$, we know from the consideration of Corollary 8.1.2 that

$$\tilde{\xi} := \pi_{13}^*(\Delta_E(1,1)) \cdot \pi_2^*(p-q) \in CH^2_{alg}(E \times E \times E;\mathbb{Q})$$

is a pre image of ξ. To get an unconditional pairing, we need to consider the image of $\tilde{\xi}$ under the projector

$$T := [\Delta_{E,14}(1,1))] \otimes [\Delta_X(2,2) - A] \in CH^3(\underbrace{(E \times E \times E) \times (E \times E \times E)}_{numbered\ 1\cdots 6};\mathbb{Q}) \ ,$$

where A now is given by the projector

$$\Delta_{E,23}(2,0)\times\Delta_{E,56}(0,2)+\Delta_{E,23}(0,2)\times\Delta_{E,56}(2,0)-\Delta_{E,23}(1,1)\times\Delta_{E,56}(1,1)-\Xi_{E,23}(1,1)\times\Xi_{E,56}(1,1) \ .$$

Note that, assuming BBC we get that $[\Delta_{E,14}(1,1) \otimes \Delta_X(2,2)]_*(\tilde{\xi}) = \tilde{\xi}$. Hence we obtain

$$T_*(\tilde{\xi}) = \underbrace{\pi_{13}^*(\Delta_E(1,1)) \cdot \pi_2^*(p-q)}_{\tilde{\xi}_1 = \tilde{\xi}} + \underbrace{\pi_1^*(p-q) \cdot \pi_{23}^*(\Delta_E(1,1))}_{\tilde{\xi}_2} + \underbrace{\pi_1^*(i(p-q)) \cdot \pi_{23}^*(\Xi_E(1,1))}_{\tilde{\xi}_3} \ .$$

It is easy to see, that under the isomorphism defined in Chapter 7,

$$T_*(\tilde{\xi}) \mapsto \xi \ .$$

Hence the height pairing is given by

$$\langle \xi, \xi \rangle_{HT} := \langle T_*(\tilde{\xi}), T_*(\tilde{\xi}) \rangle_{HT} \ .$$

The point now is to get a relation similar to Corollary 8.1.2 for this height pairing. That will be our next

8.2.1 Proposition. *Let $E/\overline{\mathbb{Q}}$ be the CM-elliptic curve such that $E(\mathbb{C}) = \mathbb{C}/\mathbb{Z}[i]$ and $\{p, q\} \in E(\overline{\mathbb{Q}})$. Let Ξ_E be the graph of the morphism of multiplication by i. For the cycle*

$$T_*(\tilde{\xi}) := \underbrace{\pi_2^*(p-q) \cdot \pi_{13}^*(\Delta_E(1,1))}_{\tilde{\xi}_1} + \underbrace{\pi_1^*(p-q) \cdot \pi_{23}^*(\Delta_E(1,1))}_{\tilde{\xi}_2} + \underbrace{\pi_1^*(i(p-q)) \cdot \pi_{23}^*(\Xi_E(1,1))}_{\tilde{\xi}_3} \, ,$$

on $E \times E \times E$ the following height pairing relation holds :

$$\langle T_*(\tilde{\xi}), T_*(\tilde{\xi}) \rangle_{HT} = 2 \langle p-q, p-q \rangle_{NT} \ ,$$

where $\langle \ , \ \rangle_{NT}$ is the Neron-Tate pairing on curves.

Proof. First note that, from linearity of height pairing we obtain

$$\langle T_*(\tilde{\xi}), T_*(\tilde{\xi}) \rangle_{HT} = \sum_1^3 \langle \tilde{\xi}_i, \tilde{\xi}_i \rangle_{HT} \ + \ 2 \left(\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_{HT} \ + \ \langle \tilde{\xi}_1, \tilde{\xi}_3 \rangle_{HT} \ + \langle \tilde{\xi}_2, \tilde{\xi}_3 \rangle_{HT} \right) \ .$$

The computations for each of $\langle \tilde{\xi}_i, \tilde{\xi}_i \rangle_{HT}$ and $\langle \tilde{\xi}_2, \tilde{\xi}_3 \rangle_{HT}$ follows the method used in the proof of Lemma 8.1.1. We get the following relations

- $\langle \tilde{\xi}_1, \tilde{\xi}_1 \rangle_{HT} = [deg(\Delta_E^2(1,1))] \langle p-q, p-q \rangle_{NT} = -2 \langle p-q, p-q \rangle_{NT}$.
- $\langle \tilde{\xi}_2, \tilde{\xi}_2 \rangle_{HT} = -2 \langle p-q, p-q \rangle_{NT}$.
- $\langle \tilde{\xi}_2, \tilde{\xi}_3 \rangle_{HT} = [deg(\Delta_E(1,1) \cdot \Xi_E(1,1))] \langle p-q, i(p-q) \rangle_{NT} = 0$.
- $\langle \tilde{\xi}_3, \tilde{\xi}_3 \rangle_{HT} = 2 \langle p-q, p-q \rangle_{NT}$.

Let's elaborate on the last relation : First note that since $deg(\Xi_E^2(1,1)) = 2$, we get the relation

$$\langle \tilde{\xi}_3, \tilde{\xi}_3 \rangle_{HT} = 2\langle i(p-q), i(p-q) \rangle_{NT} .$$

We compute the height pairing on the right hand side with the following observation

$$\langle i(p-q), i(p-q)\rangle_{NT} = \langle [i]^*(p-q), [i]^*(p-q)\rangle_{NT} = \langle p-q, [i]_*([i]^*(p-q))\rangle_{NT} = \langle p-q, p-q\rangle_{NT} .$$

Hence

$$\langle \tilde{\xi}_3, \tilde{\xi}_3 \rangle_{HT} = 2\langle p-q, p-q \rangle_{NT} .$$

We are left with computing $\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_{HT}$ and $\langle \tilde{\xi}_1, \tilde{\xi}_3 \rangle_{HT}$.

Computation for $\langle \tilde{\xi}_1, \tilde{\xi}_2 \rangle_{HT}$: Note that

$$\Delta_E(1,1) = \Delta_E - \Delta_E(2,0) - \Delta_E(0,2)$$

and

$$\pi_{13}^*(\Delta_E(1,1)) \cdot \pi_{23}^*(\Delta_E(1,1)) = \underbrace{\Delta_{123}}_{\{(x,x,x)\}} - \underbrace{\Delta_{13}}_{\{(x,o,x)\}} - \underbrace{\Delta_{23}}_{\{(o,x,x)\}} + \underbrace{\Delta_{3}}_{\{(o,o,x)\}} \quad \star$$

Here $o \in E(\overline{\mathbb{Q}})$ is the base point. Let $\widetilde{E}/Spec(O_k)$ be the minimal regular model for E and assume (using de-Jong's alteration) that all the self products $\widetilde{E} \times \cdots \times \widetilde{E}$ over $Spec(O_k)$ are regular. Let Z be an arithmetic cycle in $\widetilde{E}$ such that $Z|_E = p - q$ and $Z \cdot v = 0$ for any vertical cycle v. For a choice of Green current g_Z, such that $dd^c g_Z + \delta_{p-q} = 0$, consider

$$\alpha := [(Z, g_Z)] \in \widehat{CH}^1(\widetilde{E}) .$$

Then, as seen before

$$\langle p-q, p-q\rangle_{NT} = \Pi_{\widetilde{E},*}(\alpha \cdot \alpha) \in \widehat{CH}^1(Spec(\mathbb{Z})) \cong \mathbb{R} ,$$

where, as before $\Pi_{\widetilde{E}}$ is the structural morphism. Now, it is evident that the required height pairing is given by

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left[\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot\pi_{13}^*([(\widetilde{\Delta_E(1,1)},g)])\cdot\pi_{23}^*([(\widetilde{\Delta_E(1,1)},g)])\right] .$$

Here, for the cycle $\Delta_E(1,1)\in CH^1(E\times E;\mathbb{Q})$, $\widetilde{\Delta_E(1,1)}\in\widehat{CH}^1(\widetilde{E}\times\widetilde{E})$ denotes an arithmetic cycle with generic fibre $\Delta_E(1,1)$ and g is a suitable Green current. From $\star$ moreover, we can break it up even further : Let us consider the following arithmetic cycles

$$[(\widetilde{\Delta_{123}},g_{123})],[(\widetilde{\Delta_{13}},g_{13})],[(\widetilde{\Delta_{23}},g_{23})],[(\widetilde{\Delta_3},g_3]$$

for suitable choice of Green currents. Then, the height pairing is given by

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{123}},g_{123})]-\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{13}},g_{13})]-\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{23}},g_{23})]$$

$$+\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_3},g_3)]) .$$

We first compute

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{123}},g_{123})]\right) .$$

The idea for the proof was kindly communicated to us by Dr. José Burgos Gill. Let us denote

$$\Delta_{123}:\underbrace{E\to E\times E\times E}_{x\mapsto(x,x,x)},$$

now as an embedding. Note that this has an obvious extension to the regular models. We will use the same notation for it. Since the generic fibre of α is homologically trivial, we get

$$\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{123}},g_{123})]=\Delta_{123,*}\Delta_{123}^*\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\right) .$$

Since $\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}(\Delta_{123,*}())=\Pi_{\widetilde{E},*}(())$, we deduce

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{123}},g_{123})]\right)=\Pi_{\widetilde{E},*}\left(\Delta_{123}^*\pi_1^*(\alpha)\cdot\Delta_{123}^*\pi_2^*(\alpha)\right) .$$

Since each of $\Delta_{123}^*\pi_1^*$ and $\Delta_{123}^*\pi_2^*$ is identity, we get

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{123}},g_{123})]\right)=\Pi_{\widetilde{E},*}\left(\alpha\cdot\alpha\right)=\langle p-q,p-q\rangle_{NT} .$$

This is the only non zero intersection number that we get, as we will see in our next computation. Let

$$\Delta_{13} : \underbrace{E \to E \times E \times E}_{x \mapsto (x,o,x)}$$

denote an embedding. As before, this has an extension to the regular models once we choose and fix a cycle $\tilde{o}$ with generic fibre o. We note here the following observations : $\pi_1 \circ \Delta_{13} = Id_E$ and $\pi_2 \circ \Delta_{13}$ is the constant morphism. Using the same idea as before, we get

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{13}}, g_{13})]\right) = \Pi_{\widetilde{E},*}\left(\Delta_{13}^*\pi_1^*(\alpha)\cdot\Delta_{13}^*\pi_2^*(\alpha)\right) = \Pi_{\widetilde{E},*}\left((\pi_2\circ\Delta_{13})_*\alpha\cdot\alpha\right) \ .$$

The last equality follows from projection formula.
Using the facts that $Z \cdot v = 0$ for all vertical cycles and $(\pi_2 \circ \Delta_{13})_*(p-q) = 0$, we deduce

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{13}}, g_{13})]\right) = 0 \ .$$

Using similar idea, we obtain

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{23}}, g_{23})]\right) = 0$$

$$\Pi_{\widetilde{E}\times\widetilde{E}\times\widetilde{E},*}\left(\pi_1^*(\alpha)\cdot\pi_2^*(\alpha)\cdot[(\widetilde{\Delta_{3}}, g_{3})]\right) = 0 \ .$$

Hence, overall we get

$$\langle\tilde{\xi}_1, \tilde{\xi}_2\rangle_{HT} = \langle p-q, p-q\rangle_{NT} \ .$$

Computation for $\langle\tilde{\xi}_1, \tilde{\xi}_3\rangle_{HT}$: We start with the following observations:

$$\Xi_E(1,1) = \Xi_E - \Delta_E(2,0) - \Delta_E(0,2) \ ,$$

$$\pi_{13}^*(\Delta_E(1,1))\cdot\pi_{23}^*(\Xi_E(1,1)) = \underbrace{\Xi_{123}}_{\{(ix,x,ix)\}} - \Delta_{13} - \underbrace{\Xi_{23}}_{\{(o,x,ix)\}} + \Delta_3 \ .$$

Since we consider the minimal regular model $\widetilde{E}$ of E, the automorphism $\underbrace{[i] : E \to E}_{x\mapsto ix}$ has an extension which we will still denote by $[i] : \widetilde{E} \to \widetilde{E}$. Thus the embedding

$$\underbrace{\Xi_{123} : E \to E \times E \times E}_{x \mapsto (ix, x, ix)}$$

also has an extension to the regular models. Note here that $\pi_1 \circ \Xi_{123} = [i]$ and $\pi_2 \circ \Xi_{123} = Id_E$. Let $[i]_*\alpha$ denote the pushforward of the cycle α with generic fibre $i(p-q)$. Thus $(\pi_1 \circ \Xi_{123})^*[i]_*\alpha = [i]^*[i]_*\alpha = \alpha$. Now, similar computations as before yields

$$\Pi_{\widetilde{E} \times \widetilde{E} \times \widetilde{E}, *} \left(\pi_1^*(\alpha) \cdot \pi_2^*(\alpha) \cdot [(\widetilde{\Xi_{123}}, g'_{123})] \right) = \Pi_{\widetilde{E}, *} (\alpha \cdot \alpha) = \langle p - q, p - q \rangle_{NT}$$

and all other intersection numbers being zero. Overall, we get

$$\langle \tilde{\xi}_1, \tilde{\xi}_3 \rangle_{HT} = \langle p - q, p - q \rangle_{NT} \, .$$

Putting this all together, we get the desired result.

□

Chapter 9

Some Hodge-index type results

In section 5 of [5], Beilinson stated a Hodge-index type conjecture for his height pairing. The idea of this chapter is to extend his conjecture to our situation. We will first see that based on the conjecture, we can actually obtain a Hodge-index type result in our situation. This will be our main goal. Using this and one of Kunnemann's result (see [34], Theorem 12.1), we will obtain a result for the case for abelian varieties and cycles algebraically equivalent to zero. In the second section, we will speculate some results for product of curves, albeit conditionally.

9.1 Hodge-index result for graded pieces

Notation : *The usual fibre product of two (or more) smooth projective varieties X and Y over a field k will be denoted by $X \times Y$.*

Let us make the following assumptions :

9.1.1 Assumption. *For a smooth projective variety X of dimension d defined over a number field k, assume the following (Conjectures 5.3 and 5.5 of [5]):*

- *(hard Lefschetz) : Let $L_X \in CH^1(X;\mathbb{Q})$ be the operation of intersecting with a hyperplane section. Then for $r \leq (d+1)/2$,*

$$L_X^{d-2r+1} : CH^r_{hom}(X;\mathbb{Q}) \to CH^{d-r+1}_{hom}(X;\mathbb{Q})$$

is an isomorphism.

- *(Hodge-index) : Let the hard Lefschetz assumption hold. If* $x \in CH^r_{hom}(X;\mathbb{Q})$*,* $x \neq 0$*, and* $L^{d-2r+2}(x) = 0$ *then*

$$(-1)^r \langle x \, , \, L_X^{d-2r+1}(x) \rangle_{HT} \; > \; 0$$

for $r \leq (d+1)/2$ *.*

If X *is defined over* $\overline{\mathbb{Q}}$ *, we can collect coefficients of the defining polynomials of* X *to get an* X' *defined over a number field* k *(not uniquely) and a finite proper morphism* $X = X' \times_k \overline{\mathbb{Q}} \to X'$*. The assumptions above can now be made for* $X/\overline{\mathbb{Q}}$*.*

Coming back to our situation, if $L_{X_K} \in CH^1(X_K;\mathbb{Q})$ be the operation of intersection hyperplane section in X_K, then $L_{S\times X} := L_S \times X + S \times L_X$ is a natural choice for the same operation in $S \times X$. We also have the following isomorphism for Lewis filtration (see diagram 4.6 in [38])

$$L_{X_K}^{d-er+\nu} : Gr_F^{\nu} CH^r(X_K;\mathbb{Q}) \cong Gr_F^{\nu} CH^{d-r+\nu}(X_K;\mathbb{Q}).$$

Now, under Assumption 9.1.1, together with that made in Theorem 7.0.11, we get the following result :

9.1.2 Proposition. *Let* L_{X_K} *denote the operation of intersecting with a hyperplane section. Then for* $x \neq 0 \in Gr_F^{\nu} CH^r(X_K;\mathbb{Q})$ *such that* $L_{X_K}^{d-2r+\nu+1}(x) = 0$*, the height pairing*

$$(-1)^r \langle x \, , \, L_{X_K}^{d-2r+\nu}(x) \, \rangle_{HT} \; > \; 0 \, ,$$

when $r \leq (d+\nu)/2$*.*

Proof. From the commutativity of the Abel-Jacobi map with correspondences, we get

$$\begin{array}{ccc} \Xi_1 & \xrightarrow{L_{S\times X}^{d-2r+\nu}} & CH_{hom}^{d-r+\nu}(S\times X;\mathbb{Q}) \\ \Big\downarrow \Phi_r & & \Big\downarrow \Phi_{d-r+\nu} \\ J(H_0) & \xrightarrow{[L_{S\times X}]^{d-2r+\nu}} & J(H_0') \end{array} \qquad (9.1.2.1)$$

where H_0 and H_0' are the respective Künneth pieces given by the (cohomology class of) w_1 and w_2 respectively (see Chapter 7 for details). Now, observe the following : For any $x \in \Xi_1$

$$\Phi_{d-r+\nu}\left(L_{S\times X}^{d-2r+\nu}(x) - w_{2,*} \circ L_{S\times X}^{d-2r+\nu}(x)\right)$$

$$= [L_{S\times X}]^{d-2r+\nu}(\Phi_r(x)) - [w_2]_* \circ \underbrace{[L_{S\times X}]^{d-2r+\nu}(\Phi_r(x))}_{\in J(H_0')}$$

$$= [L_{S\times X}]^{d-2r+\nu}(\Phi_r(x)) - [L_{S\times X}]^{d-2r+\nu}(\Phi_r(x)) = 0$$

as $[w_2]_*$ is a projector onto $J(H_0')$. Since we are assuming the BBC, we get

$$L_{S\times X}^{d-2r+\nu}(x) = w_{2,*} \circ L_{S\times X}^{d-2r+\nu}(x)$$

We have shown that $L_{S\times X}^{d-2r+\nu}$ maps Ξ_1 to Ξ_2. Hence, the following diagram commutes:

$$\begin{array}{ccc} \Xi_1 & \xrightarrow{L_{S\times X}^{d-2r+\nu}} & \Xi_2 \\ \Phi_r \downarrow \cong & & \cong \downarrow \Phi_{d-r+\nu} \\ Gr_F^{\nu}CH^r(X_K;\mathbb{Q}) & \xrightarrow[\cong]{L_{X_K}^{d-2r+\nu}} & Gr_F^{\nu}CH^{d-r+\nu}(X_K;\mathbb{Q}). \end{array} \tag{9.1.2.2}$$

It shows that $L_{S\times X}^{d-2r+\nu} : \Xi_1 \cong \Xi_2$. Further, let $\Xi_2' \subset CH_{hom}^{d-r+\nu+1}(S\times X;\mathbb{Q})$ be such that $\Xi_2' \cong Gr_F^{\nu}CH^{d-r+\nu+1}(X_K;\mathbb{Q})$. Now to actually prove Proposition 9.1.2, we note that similar to diagram 9.1.2.2 we can also have the commutative diagram

$$\begin{array}{ccc} \Xi_1 & \xrightarrow{L_{S\times X}^{d-2r+\nu+1}} & \Xi_2' \\ \Phi_r \downarrow \cong & & \cong \downarrow \Phi_{d-r+\nu+1} \\ Gr_F^{\nu}CH^r(X_K;\mathbb{Q}) & \xrightarrow{L_{X_K}^{d-2r+\nu+1}} & Gr_F^{\nu}CH^{d-r+\nu+1}(X_K;\mathbb{Q}). \end{array} \tag{9.1.2.3}$$

Then for $x' \in \Xi_1$

$$\Phi_r(x') = x \in Gr_F^{\nu}CH^r(X_K;\mathbb{Q}) \implies \Phi_{d-r+\nu+1}(L_{S\times X}^{d-2r+\nu+1}(x')) = L_{X_K}^{d-2r+\nu+1}(x)\ .$$

So, $L_{X_K}^{d-2r+\nu+1}(x) = 0 \implies L_{S\times X}^{d-2r+\nu+1}(x') = 0$. We also have

$$(-1)^r \langle x \, , \, L_{X_K}^{d-2r+\nu}(x) \, \rangle_{HT} = (-1)^r \langle x' \, , \, L_{S\times X}^{d-2r+\nu}(x') \, \rangle_{HT} \, .$$

Note that $x' \in \Xi_1 \subset CH^r_{hom}(S \times X; \mathbb{Q})$ and $L_{S\times X}^{d-2r+\nu+1}(x') = 0$. We can apply the Hodge-index assumption (Assumption 9.1.1) to conclude

$$(-1)^r \langle x' \, , \, L_{S\times X}^{d-2r+\nu}(x') \, \rangle_{HT} \; > \; 0 \, ,$$

and Proposition 9.1.2 follows immediately. □

9.1.3 A case for abelian varieties

Here we use Kunnemann's Hodge-index result (see [34], section 12) in the following situation : Consider $X := A$ be an abelian variety of dimension d, and B be another abelian variety of dimension $\nu - 1$, all defined over $\overline{\mathbb{Q}}$ and $K \cong \overline{\mathbb{Q}}(B)$. So, our $S = B$ and $S \times X := B \times A$ is an abelian variety. Since we are assuming the Bloch Beilinson Conjecture, the subgroup of incidence equivalence mentioned in [34] is zero. We have the following result :

9.1.4 Corollary. *Let $X := A$ and B be abelian varieties of respective dimensions d and $\nu - 1$, defined over $\overline{\mathbb{Q}}$ and let $K = \overline{\mathbb{Q}}(B)$. Let L_{A_K} be an ample line bundle on A_K and $2r \leq d + \nu$. Assume the General Hodge Conjecture for $\overline{\mathbb{Q}}$ and the Bloch Beilinson Conjecture. Then for $x \in Gr_F^{\nu}\underline{CH}^r_{alg}(A_K; \mathbb{Q})$, $x \neq 0$, and $L_{A_K}^{d-2r+\nu+1}(x) = 0$, we have*

$$(-1)^r \langle x, L_{A_K}^{d-2r+\nu}(x) \rangle_{HT} \; > \; 0 \, .$$

Proof. From the proof of Proposition 9.2, the height pairing is given by

$$\langle x', L_{B\times A}^{d-2r+\nu}(x') \rangle_{HT} \, ,$$

where $x' \in CH^r_{alg}(B \times A; \mathbb{Q})$ is the unique choice of preimage of x. The corollary now follows from Theorem 12.1 of [34]. □

As a special case of this corollary, if we choose $X := E_1 \times \cdots \times E_d$, to be a product of elliptic curves and $K \cong \overline{\mathbb{Q}}(E_2 \times \cdots \times E_{\nu})$, we obtain a Hodge-index result for $x \in Gr_F^{\nu}\underline{CH}^r_{alg}(X_K; \mathbb{Q})$ and $L_{X_K}^{d-2r+\nu+1}(x) = 0$.

9.1.5 A Non-degeneracy result

Here we present a small result on non-degeneracy of the height pairing on the algebraic graded pieces :

9.1.6 Proposition. *If we assume the Bloch-Beilinson conjecture on the injectivity of Abel-Jacobi map (BBC) and the General Hodge Conjecture for smooth projective varieties over* $\overline{\mathbb{Q}}$*, , then the bilinear pairing*

$$(x,y) := \langle x\ ,\ L_{X_K}^{d-2r+\nu}(y)\rangle_{HT} : Gr_F^{\nu}\underline{CH}^r_{alg}(X_K;\mathbb{Q}) \times Gr_F^{\nu}\underline{CH}^r_{alg}(X_K;\mathbb{Q}) \to \mathbb{R}$$

is non-degenerate.

Proof. As shown in Theorem 7.1.13, the height pairing is given by the height pairing between $\Xi_{1,alg}$ and $\Xi_{2,alg}$ via the following isomorphisms:

$$Gr_F^{\nu}\underline{CH}^r_{alg}(X_K;\mathbb{Q}) \cong \Xi_{1,alg} \subset CH^r_{alg}(S\times X;\mathbb{Q})$$

$$Gr_F^{\nu}\underline{CH}^{d-r+\nu}_{alg}(X_K;\mathbb{Q}) \cong \Xi_{2,alg} \subset CH^{d-r+\nu}_{alg}(S\times X;\mathbb{Q})\ ,$$

where (assuming BBC), we have

$$\Xi_{1,alg} \cong J_{alg}(H_0)$$

$$\Xi_{2,alg} \cong J_{alg}(H_0')\ .$$

Here

$$H_0 := \left(\frac{H^{\nu-1}(S,\mathbb{Q})}{N^{\frac{1}{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})} \otimes \frac{H^{2r-\nu}(X,\mathbb{Q})}{N_H^{r-\nu+1}H^{2r-\nu}(X,\mathbb{Q})}\right)(r)$$

and

$$H_0' := \left(\frac{H^{\nu-1}(S,\mathbb{Q})}{N^{\frac{1}{\mathbb{Q}}}H^{\nu-1}(S,\mathbb{Q})} \otimes \frac{H^{2(d-r+\nu)-\nu}(X,\mathbb{Q})}{N_H^{d-r+1}H^{2(d-r+\nu)-\nu}(X,\mathbb{Q})}\right)(d-r+\nu)\ ,$$

and we define

$$J_{alg}(H_0) := P_{1,*}(J_{alg}(H^{2r-1}(S\times X,\mathbb{Q}(r))))$$

respectively

$$J_{alg}(H_0') := P_{2,*}(J_{alg}(H^{2(d-r+\nu)-1}(S\times X,\mathbb{Q}(d-r+\nu))))$$

for projectors P_1 and P_2 defined in Chapter 7.

Assuming the General Hodge Conjecture over $\overline{\mathbb{Q}}$, there is a natural identification between $J^r_{alg}(S \times X)$ and its dual $J^{d-r+\nu}_{alg}(S \times X)$ via $[L_{S\times X}]^{d-2r+\nu}$. Hence we can identify $J_{alg}(H_0)$ and $J_{alg}(H_0')$ through the commutative diagram:

$$\begin{array}{ccc} J^r_{alg}(S \times X) & \xrightarrow{[L_{S\times X}]^{d-2r+\nu}} & J^{d-r+\nu}_{alg}(S \times X) \\ \downarrow{\scriptstyle P_{1,*}} & & \downarrow{\scriptstyle P_{2,*}} \\ J_{alg}(H_0) & \xrightarrow{[L_{S\times X}]^{d-2r+\nu}} & J_{alg}(H_0') := J^r_{alg}(H_0)^\vee \end{array} \qquad (9.1.6.1)$$

Hence, $[L_{S\times X}]^{d-2r+\nu}$ is a polarization between $J_{alg}(H_0)$ and its dual $J_{alg}(H_0')$. Proposition 9.1.6 now is a consequence of the positivity of the Neron-Tate pairing.

□

9.1.7 Remark. The assumption that the Abel-Jacobi map is injective (BBC) was needed for the ease of writing as much as anything else. We could very well work modulo the kernel of the Abel-Jacobi map and arrive at the same conclusion.

9.2 Hodge-index result for product of curves

In this small section, we try to provide Hodge-index result for a special situation in case of product of curves, modulo assumptions made in Corollary 8.1.2. The bulk of the computations were already done in Chapter 8 and we feed off those results.

We fix $X := C_1 \times \cdots \times C_d$, a product of smooth projective curves defined over $\overline{\mathbb{Q}}$ with $e_j \in C_j(\overline{\mathbb{Q}})$ and $K \cong \overline{\mathbb{Q}}(C_2 \times \cdots \times C_\nu)$. We set $S := C_2 \times \cdots \times C_\nu$ with a very general point $(\eta_2, \cdots, \eta_\nu) \in C_2(\mathbb{C}) \times \cdots \times C_\nu(\mathbb{C})$ and work in the setting of Chapter 8, Corollary 8.1.2.

The result is motivated by Corollary 1.3 of [40].

9.2.1 Proposition. *Let X be as above and consider the situation $\nu = r$. Then for the choice of hyperplane section $L_{X_K} := \sum_{j=1}^{d} \pi_j^*(e_j)$ and*

$$\xi := Pr^*_{1,\cdots,r}((p-q)\times(\eta_2-e_2)\times\cdots\times(\eta_r-e_r)) \in Gr_F^r CH^r(X_K;\mathbb{Q}),\ p,q \in C_1(\overline{\mathbb{Q}}) ,$$

we obtain

$$(-1)^r\langle \xi, L_{X_K}^{d-r}(\xi)\rangle_{HT} \ > \ 0 .$$

Proof. First note that, since $dim(X_K) = d$, $L_{X_K}^{d-r+1}(CH^r(X_K;\mathbb{Q})) = 0$ for any hyperplane section L_{X_K}. In particular the whole of $Gr_F^r CH^r(X_K;\mathbb{Q})$ is primitive. Now, for the hyperplane section

$$L_{X_K} := \sum_{i=1}^{d} \pi_i^*(e_i) ,$$

we have an obvious choice of hyperplane section in $S \times X$, namely

$$L_{S\times X} := S \times L_X + L_S \times X$$

where $L_X = \sum_{i=1}^{d} \pi_i^*(e_i)$ and $L_S = \sum_2^r \pi_j^*(e_j)$. We can be even more explicit to obtain

$$L_{S\times X} = \pi_1^*(e_1) + \left(\sum_2^r \pi^*_{i,i}\left(\Delta_{C_i} - \Delta_{C_i}(1,1)\right)\right) + \sum_{r+1}^{d} \pi_j^*(e_j) .$$

Also, following the assumptions made in Corollary 8.1.2, we see that the unique choice of a preimage for ξ is given by

$$\tilde{\xi} := \left(\pi_1^{X,*}(p-q)\right) \bigcap \left(\bigcap_2^r \pi^*_{i,i}(\Delta_{C_i}(1,1))\right) .$$

Thus, the height pairing is given by

$$\langle \xi, L_{X_K}^{d-r}(\xi)\rangle_{HT} = \langle \tilde{\xi}, L_{S\times X}^{d-r}(\tilde{\xi})\rangle_{HT} .$$

We compute $L^{d-r}_{S\times X}(\tilde{\xi})$ to obtain

$$L^{d-r}_{S\times X}(\tilde{\xi}) = \tilde{\xi}\cdot\left(\sum_{j\geq r+1}^{d}\pi_j^*(e_j)\right)^{d-r}.$$

Using Lemma 8.1.1, we get the following form of height pairing

$$\langle \xi, L^{d-r}_{X_K}(\xi)\rangle_{HT} = (-1)^{r-1}((d-r)!)2^{r-1}\left(\Pi_2^r g_i\right)\langle p-q, p-q\rangle_{NT}.$$

Here g_i is the genus of C_i. From Theorem 6.1 of [34] we know that the Neron-Tate pairing is definite of sign (-1). Our result follows immediately. □

9.2.2 Remark. Since $H^{r,0}(X)\neq 0$, by Corollary 1.3 of [40], the subspace generated by such ξ is of infinite rank inside $Gr^r_F CH^r(X_K;\mathbb{Q})$. Thus, we are able to show that the Hodge-index conjecture holds for this infinite rank subspace, albeit certain assumptions.

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