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# SEMIMETRIC, SYMMETRIC, & SEMISTRATIFIABLE SPACES

BY

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SEMIMETRIC, SYMMETRIC, & SEMISTRATIFIABLE SPACES submitted by RICHARD SEIBEL in partial fulfilment of the requirements for the degree of Master of Science.

S. Willard
(Supervisor)

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[illegible]

DATE March 30, 1973

## ABSTRACT

In this thesis, we consider three classes of spaces. The first of these is the class of semimetric spaces, which are generalizations of metric spaces. The next two kinds of spaces we consider are symmetric spaces and semistratifiable spaces, which are non-first countable generalizations of semimetric spaces. Throughout this thesis, all spaces will be assumed to be $T_1$ , unless the contrary is explicitly stated. Also, all terms which are undefined, will be defined as in [W].

Chapter I is devoted to semimetric spaces. Section 1 provides some elementary results. In section 2, we give a useful characterization of semimetric spaces due to R. Heath. In section 3, we examine semimetric spaces to see to what extent various standard properties of metric spaces hold true in semimetric spaces. Chapter I in concluded with some results on countability properties of semimetric spaces.

Symmetrizable and semistratifiable spaces are considered in Chapter II. We give various properties and characterizations of symmetrizable and semistratifiable spaces. Much of our work on symmetrizable spaces relies on the work of A. Arkhangel'skii and much of our work on semistratifiable spaces relies on the work of G.D. Creede. In section 3, we compare symmetric and semistratifiable spaces.

Having introduced semimetric, symmetric, and semistratifiable spaces in the first two chapters, in Chapter III,

we examine under what conditions a topological space is semimetrizable, developable, or metrizable. In particular, in section 1, four theorems are given, each of which give necessary and sufficient conditions for topological space to be semimetrizable, developable, or metrizable. In section 3, we prove that compact symmetrizable spaces are metrizable, a result due to A. Arkhangel'skii. In that section, we also show that compact semistratifiable spaces are metrizable. In section 4, we give necessary and sufficient conditions for semimetrizable, symmetrizable, or semistratifiable spaces to be developable. We conclude this thesis with two results, one on the developability of semimetric spaces, and the other on the metrizability of semimetric spaces.

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## LIST OF SYMBOLS

The following symbols will be used in this thesis:

| SYMBOL | MEANING |
|---|---|
| $S_r(x)$ | sphere of radius $r$ around $x$ |
| $S_r^d(x)$ | sphere of radius $r$ with respect to the distance function $d$ around $x$ |
| $Cl_u A$ | closure of the set $A$ in the usual topology in Euclidean $n$ - space |
| $t$ | the collection of open sets on a set $X$ . |

# CHAPTER I

# SEMIMETRIC SPACES

1. INTRODUCTION. A metric on a set $X$ is a non-negative function $d : X \times X \rightarrow$ satisfying for all $x, y$, and $z \in X$ :

(a) $d(x,y) = 0$ iff $x = y$

(b) $d(x,y) = d(y,x)$

(c) $d(x,z) \leq d(x,y) + d(y,z)$ .

When provided with such a metric, $X$ is called a metric space. Every metric space has, of course, an associated topology, defined by requiring that the open spheres

$$S_r(x) = \{y \in X : d(x,y) < r\} \quad , \quad r > 0$$

form a neighborhood base at each $x \in X$ .

The topological spaces which are metrizable enjoy many interesting properties; to mention but two, every metrizable space is first countable, and in a metrizable space, separability, second countability, and the Lindelof property are equivalent. It is reasonable to ask whether any of these properties will remain true in space admitting a "distance function" which satisfies some, but not all of the properties (a), (b), and (c) of a metric. The boldest attempt to generalize metric geometry in this way eliminates the most powerful axiom, the triangle inequality _ property (c) . The result is the theory of semimetric spaces and their associated topologies.

1.1 *Definition*. Let $X$ be any set. A non-negative function $d : X \times X \to R$ is said to be a *semimetric* iff for all $x, y \in X$,

(a) $d(x,y) = 0$ iff $x = y$ and

(b) $d(x,y) = d(y,x)$ .

The pair $(X,d)$ is then a *semimetric space*, sometimes abbreviated "$X$ is a semimetric space" when usage of convention provide $d$ .

1.2 *Examples*.

(a) Any metric space is, of course, a semimetric space.

(b) We can equip the real line with the distance function

$$d_o(x,y) = \begin{cases} \big||x| - |y|\big| & \text{if } |x| \neq |y| \\ |x - y| & \text{if } |x| - |y| \end{cases} .$$

Easily, $(R,d_o)$ is a semimetric space; of course, $d_o$ is not, in this case, a metric.

(c) Again consider the real line, this time with the distance function

$$d_1(x,y) = \begin{cases} n & \text{if } x = 0 \text{ and } y = \frac{1}{n} \\ & \text{for some } n \in N \text{ or vice versa,} \\ |x-y| & \text{otherwise.} \end{cases}$$

Then $(R,d_1)$ is another example of a semimetric space, and again, $d_1$ is not a metric.

(d) Now consider the real plane, $R^2$, equipped with the distance function

$$d_2(x,y) = |x-y| + a(x,y)$$

where $|x-y|$ is the usual Euclidean distance in the plane and $a(x,y)$ is the smallest non-negative angle (in radians) formed by the line which contains $x$ and $y$ and the line $y = 0$. Hereafter, $a(x,y)$ will refer to the function described in the previous sentence. Clearly $d_2$ is a semimetric which is not a metric.

(e) Again consider the real line, this time equipped with the distance function

$$d_3(x,y) = \begin{cases} |x-y| & \text{if} \quad x \neq y + \frac{1}{m} \quad \text{for any} \quad m \in Z \\ 1 & \text{otherwise.} \end{cases}$$

Then $d_3$ is certainly a non-metric, semimetric function.

1.3 <u>Remark</u>. Just as a metric can be used to define a topology on a set $X$, so can a semimetric. Basically, there are two methods to define a topology using a semimetric $d$, which are equivalent if $d$ is a metric. They are

(a) $x$ is a limit point of $A \subset X$ iff $d(x,A) = 0$, and

(b) $A \subset X$ is closed iff for all $x \notin A$, $d(x,A) > 0$.

1.4 <u>Example</u>. The two ways of assigning a topology using $d$ as given in 1.3 are not necessarily equivalent in general. Define a semimetric $d$ on the real line $R$ as follows:

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in Q \text{ or } x,y \notin Q \\ ||x|-|y|| & \text{otherwise.} \end{cases}$$

Let $A = Q \cap [1,2]$. Then $\bar{A} = A \cup A' = [1,2] \cup [-2,-1] \cap \tilde{Q}$ by 1.3 (a) and $\bar{\bar{A}} = \bar{A} \cup \bar{A}' = [1,2] \cup [-2,-1]$, again by (a). Thus we find that describing a topology by limit points leads to a contradiction, while assigning a topology by a semimetric $d$ using closed sets as in 1.3 (b), clearly always leads to a valid topology. Therefore, 1.3 (a) and 1.3 (b) are not equivalent; in fact 1.3 (a) may not always yield a valid topology.

Developments thus raise the question "when does assignment of a topology using 1.3 (a) yield a valid topology?", to which we now turn our attention.

1.5 Remark. The real line with the topology generated by using the semimetric function $d$ described in 1.4 and assigning closed sets using 1.3 (b) will be referred to throughout this thesis as the Cairns space.

1.6 Lemma [D]. The assignment to each subset $A \subset X$ a set of "limit points" leads to a valid topology iff

(a) $\phi' = \phi$

(b) $(A')' \subseteq A \cup A'$

(c) $(A \cup B)' = A' \cup B'$

(d) for all $x \in X$, $x \notin \{x\}'$ .

1.7 Theorem [Bennet & Hall]. If $d$ is a semimetric satisfying the following condition $*$, then specifications of limit points using 1.3 (a) yields a valid topology.

> $*$: If for all $k$ $N$, there exists a sequence $\{x_n^k\}$ and a point $x^k$ in $X$ such that $\lim_{n\to\infty} d(x_n^k, x^k) = 0$ and furthermore there is a point $p$ in $X$ such that $\lim_{k\to\infty} d(x^k, p) = 0$, then there is a sequence $\{\alpha(k)\}$ in $N$ such that $\lim_{k\to\infty} d(x_{\alpha(k)}^k, p) = 0$ .

Proof: Clearly the assignment of limit points by any semimetric function $d$ satisfies a,c and d of 1.6. Assume that $d$ satisfies $*$ . Let $A \subset X$ and let $p \in (A')' - A$ .

We have to show that $p \in A'$ . Since $p \in (A')'$ , there is a sequence $\{x^k\}$ in $A'$ such that $\lim d(x^k,p) = 0$ . Since each $x^k$ is in $A'$ , there exists a sequence $\{x_n^k\}$ in $A$ such that $\lim_{n\to\infty} d(x_n^k,x^k) = 0$ . Since $x \notin A$ , using * , $x \in A'$ .

1.8 Definition. A semimetric space (symmetric space) is an ordered triple $(X,t,d)$ where $X$ is a set, $t$ a topology on $X$ and $d$ a semimetric, such that limit points (respectively, closed sets) in the topology are given by 1.3 (a) (respectively, 1.3 (b)).

The Cairns space considered in 1.4 shows that a symmetric space is not necessarily a semimetric space. But certainly every semimetric space is a symmetric space. The question then arises, "When is a symmetric space a semimetric space?". One criterion (Bennet & Hall) is given in the next theorem. Another will be given in 2.3.

1.9 Theorem [B&H]. If $(X,t,d)$ is a symmetric space and $d$ satisfies 1.7, then $(X,t,d)$ is a semimetric space.

Proof. We have two topologies on $X$ , the $t'$ topology generated by defining limit points and the $t$ topology generated by defining the closed sets. We must show that they are equal. Let $A \subset X$ be $t$ closed, i.e. $d(x,A) > 0$ for all $x \notin A$ , therefore, if $d(x,A) = 0$ , $x \in A$ , so $A$ contains its $t'$ limit points and so $A$ is $t'$ closed.

Suppose on the other hand that $A$ is $t'$ closed and $d(x,A) = 0$ . Then $x \in A$ , so for all $y \notin A$ , $d(y,A) > 0$ and $A$ is $t$ closed. Therefore, $t = t'$ .

2. SEMIMETRIZABILITY. If we are given any topological space $X$ , a natural question is "Under what conditions does there exist a semimetric $d$ on $X$ such that the topology can be recovered using $d$ ?". In this section, two necessary and sufficient conditions will be given for a topological space $X$ to be a semimetrizable space. The first condition is due to Heath, the second is due to Pareek.

2.1 Lemma. If $X$ is a semimetric space, then the interior of a sphere of radius $r$ around a point $x$ has nonempty interior.

Proof. Consider $S_r(x)$ , the sphere of radius $r$ around $x$ . Then $d(x,X-S_r(x) \geq r$ , so that $x$ is not a limit point of $X-S_r(x)$ , which in turn implies that $x \notin \overline{X-S_r(x)}$ . Therefore, $x \in Int(S_r(x))$.

2.2 Theorem [$Hea_1$]. A $T_1$ space $X$ is a semimetric space iff there exists a function $g : N \times X \to t$ such that the following holds true:

(a) for all $x \in X$, $g_n(x)$ is a nonincreasing local base at $x$

(b) if $y$ is a point in $X$ and $x$ is a point sequence in $X$ such that for all $n \in N$, $y \in g_n(x_n)$, then $x$ converges to $y$.

<u>Proof</u>. Assume $X$ is a semimetric space. Define $g$ as follows:

$$g_n(x) = \mathrm{Int}(S_{1/n}(x)) \quad .$$

Then by 2.1, $g_n(x) \neq \phi$ for any pair $(n,x)$ and clearly $g$ satisfies the conditions (a) and (b) of the theorem.

Assume there exists a $g$ satisfying (a) and (b) of the theorem. Define a function $m : X \times X \to N$ as follows:

$$m(x,y) = \inf\{p : y \notin g_p(x)\} \quad .$$

Define a distance function $d$ for $X$ as follows:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \min\{1/m(x,y)\ ,\ 1/m(y,x)\} & \text{if } x \neq y \end{cases}$$

Then clearly $d$ is a semimetric. It remains to show that the original topology $t$ is the same as the topology $t'$ generated by $d$. It is enough to show that limit points in the $t$ and $t'$ topologies are the same. Since $X$ is first countable with either topology, $x$ is a limit points of $A$ in either sense iff there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$ in the appropriate topology. Assume that $x$ is a limit point of $A$ in the $t$ topology. We may assume that $x \notin A$. Let $\{x_n\}$ be a sequence contained in $A$ which converges to $x$. Then by construction of $d$, $d(x,x_n) \leq \frac{1}{n}$, for all $n$, and therefore $d(x,A) = 0$ and $x$ is a $t'$ limit point of $A$. Conversely, suppose that $x$ is a $t'$ limit point of $A$, $x \notin A$. Choose a sequence $\{x_n\}$ by letting $x_i$ be any element of $A \cap S_{1/i}(x)$. If there exists $i$ such that $A \cap S_{1/i}(x) = \phi$, then $d(x,A) \geq \frac{1}{i}$, a contradiction to the fact that $x$ is a limit point of $A$ iff $d(x,A) = 0$. Clearly $x_n \to x$. Therefore $x$ is a $t$ limit point of $A$ and $t = t'$.

2.3 <u>Corollary</u>. A topological space $X$ is semimetric iff $X$ is first countable and symmetric.

<u>Proof</u>. Clear.

2.4 <u>Corollary</u>. Subspaces of semimetric space are semimetric.

<u>Proof</u>. Clear.

2.5 <u>Definition</u>. Let $\underset{\sim}{P}$ be a collection of ordered pairs of subsets of $X$, $P = (P_1,P_2)$ with $P_1 \subset P_2$ for all $P \in \underset{\sim}{P}$. Then $\underset{\sim}{P}$ is a <u>paired network</u> iff for all $x \in X$ and any arbitrary neighborhood $U$ of $x$, there exists a $P \in \underset{\sim}{P}$ such that $x \in P_1 \subset P_2 \subset U$ . $\underset{\sim}{P}$ is called a <u>cushioned paired network in</u> $X$ iff for each $\underset{\sim}{P}' \subset \underset{\sim}{P}$ , $\overline{\cup(P_1 : P \in \underset{\sim}{P}')} \subset \cup(P_2 : P \in \underset{\sim}{P}')$ . Finally, $\underset{\sim}{P}$ is called a <u>σ-cushioned paired network</u> iff $\underset{\sim}{P}$ is a paired network for $X$ which can be written as a countable union of cushioned paired collections.

2.6 <u>Theorem</u> [P]. If $X$ is a semimetric space with semimetric $d$ , then $X$ has a σ-cushioned paired network.

<u>Proof</u>. For all $x \in X$ and any $n \in N$ , define

$$\underset{\sim}{V}_n = \{S_{1/n}(x) : x \in X\}$$

$$S^{\#}(x;1/n,1/m) = \{z \in X : S_{1/m}(x) \subset S_{1/n}(x)\}$$

$$\underset{\sim}{V}_{n,m} = \{S^{\#}(x;1/n,1/m) , S_{1/n}(x) : x \in X\}$$

for $m \geq n$ . It will now be shown that for fixed $n$ and any $m$ , the collection $\underset{\sim}{V}_{n,m}$ is cushioned; i.e., we want to show that for arbitrary $A \subset X$ ,

$$\overline{\cup(S^{\#}(x;1/n,1/m) : x \in A)} \subset \cup(S_{1/n}(x) : x \in A) .$$

Let $t$ be in the left hand side of above. Then $S_{1/m}(t)$ has nonempty intersection with $M(A;n,m) = \cup(S^{\#}(x;1/n,1/m) : x \in A)$ . If $x' \in M(A;n,m) \cap S_{1/m}(t)$ , then $t \in S_{1/m}(x')$ since $y \in S_r(x)$ implies $x \in S_r(y)$ for any $x,y \in X$ . Since $S_{1/m}(x') \subset S_{1/n}(x')$ , we find that $t \in \cup(S_{1/n}(x) : x \in A)$ .

It remains to show that $\underset{\sim}{V} = \bigcup_{n=1}^{\infty} \bigcup_{m \geq n} \underset{\sim}{V}_{n,m}$ is a paired network for X .

Let U be any open set containing x . Since X is a semimetric space, there exists n(x) such that $x \in S_{n(x)}(x) \subset U$ . Now for all $m \geq n(x)$ , we have $x \in S^{\#}(x;1/n,1/m) \subset S_{1/n(x)}(x) \subset U$ .

Pareek has given an interesting characterization of regular semimetric spaces in terms of σ-cushioned paired networks, as follows.

2.7 <u>Theorem</u> [P]. A regular space X is semimetric iff it is first countable and has a σ-cushioned paired network.

<u>Proof</u>. Necessity is clear. To prove sufficiency, let $V_n(x)$ be a countable decreasing base at each point $x \in X$ . Let $\underset{\sim}{W} = \bigcup_{n=1}^{\infty} \underset{\sim}{W}_n$ be a σ-cushioned paired network for X , where $\underset{\sim}{W}_n = \{(W^n_{\alpha 1}, W^n_{\alpha 2}) : \alpha \in A_n\}$ is a paired cushioned collection for each n . Let us now define

$$M^k_x = \cup(W^n_{\alpha 1} : n \leq k , \alpha \in A_n \text{ and } x \notin W^n_{\alpha 2})$$

$$U_k(x) = V_k(x) - M^k_x$$

$$N(x,y) = \max\{n : U_n(x) \cap U_n(y) \cap (\{x\} \cup \{y\})\} \neq \phi$$

and

$$d(x,y) = \frac{1}{N}(x,y) \quad \text{for all} \quad x,y \in X .$$

Clearly then $d$ is a semimetric function. It remains to show that the topology generated by $d$ is the same as the given topology. Let $M$ be a subset of $X$ and $x_o$ any point of $X$ such that $x \notin \overline{M}$. We wish to show that $d(x_o,M) > 0$. Since $X$ is regular, there exists an $n_o$ such that $V_{n_o} \cap \overline{M} = \phi$. Choose $n_1$ such that $x \in W_{\alpha 1}^{n_1} \subset W_{\alpha 2}^{n_2} \subset X - \overline{M}$. If $y \in \overline{M}$, then $U_{n_o}(x_o) \subset V_{n_1}(y) \subset X - \overline{M} \subset X-\{y\}$ and on the other hand $U_{n_1}(y) \subset V_{n_1}(y) - M_{y_o}^{n_1} \in V_{n_1}(y) - W_1^{n_1} \subset V_{n_1}(y) - \{x_o\}$. Hence, for $n' > \text{Max}\ \{n_o,n_1\}, (U_{n'}(y) \cap U_{n'}(x_o)) \cap (\{x_o\} \cup \{y\}) \subset (U_{n_o}(x_o) \cap U_{n_1}(y)) \cap (\{x_o\} \cup \{y\}) = \phi$. Therefore, $N(x_o,y) \leq \max\{n_o,n_1\}$, which proves that $d(x_o,\overline{M}) > 0$.

On the other hand, assume that $M$ is a subset of $X$ and $x$ is a point of $X$ such that $d(x,M) > 0$. We must show that $x \in \overline{M}$. Let us assume that $d(x,M) = e$ and let us choose $n \in N$ such that $n > \frac{1}{e}$. Then the set $M_x^n$ is a closed set not containing $x$. There exists a neighborhood $V_{n'}(x)$ disjoint from $M_x^n$ since $X$ is regular. Then for $n'' > \max\{n,n'\}$, we have $V_{n''}(x) \subset V_n(x) - M_x^n = U_n(x)$. But if $y \in U_n(x)$, then $N(x,y) \geq n$ and $d(x,y) = \frac{1}{N}(x,y) \leq \frac{1}{n} < e$, so that $M \cap U_{n_o}(x) = \phi$. Therefore, $d$ is a semimetric for the topology on $X$.

3. PROPERTIES OF SEMIMETRIC SPACES. In this section, various properties of metric spaces will be examined to see if they hold true in semimetric spaces. In general, as expected, we find that those properties of metric spaces which depend upon the triangle inequality no longer necessarily hold true in semimetric spaces, while those properties of metric spaces which do not depend upon the triangle inequality do remain true.

3.1 Example. It is well-known that every metric space is normal; unfortunately, the same thing is not true in semimetric spaces. In fact, very little can be said about the separability of semimetric spaces other than every semimetric space if $T_1$ .

(a) A $T_1$ , non - $T_2$ semimetric space. Example 1.2 (b).

(b) A $T_2$ , non - $T_3$ semimetric space. Example 1.2 (c). If $A = \{\frac{1}{n} : n \in N\}$ , then $A$ is clearly closed and cannot be separated from $0$ by disjoint open sets.

(c) A $T_3$ , non - $T_4$ semimetric space. Example 1.2 (d). Regularity is clear. Non-normality follows from Lemma 15.2 in [W].

An interesting open question is "Is every $T_3$ semimetric space a $T_{3\frac{1}{2}}$ semimetric space?".

In a metric space, every sphere is an open set. The question then arises as to whether or not every semimetric space can be resymitrized so that every sphere is open. The following theorm of Heath answers the question in the negative.

3.2 <u>Theorem</u> [$Hea_4$]. There exists a regular semimetrizable space for which there is no compatible semimetric such that all spheres are open.

<u>Proof</u>. Let $X = R \times R$ with a basis made up of

(1) all open discs that do not intersect the x-axis or are centered on rational points of the x-axis

or

(2) all "bow-tie" regions centered on irrational points of the x-axis, i.e. for each irrational point $x$ and each $c > 0$, every set of the form

$$\{y : |x-y| + a(x,y) < c\}\ [1] .$$

Then regularity is clear and the fact that $x$ is a semimetric space follows from the characterization given in 2.2. Assume then that $d$ is any semimetric for the space. If $r$ is any rational point on the axis, then the distance between $r$ and the line $x = r$ is clearly $0$ with respect to the distance function $d$, since if not, then the line $x = r$ is closed, a contradiction. There exists a second category subset $M$ of the irrationals (relative to the topology of the x-axis) and a $c > 0$ such that if $x \in M$, then $S_c^d(x)$ must be contained in a bow-tie region of radius 1 about $x$. Then if $p_n \in M$, $p_n \to r \in Q$, and $q$ has the same abscissa as $r$,

[1] See 1.2 (d) for appropriate interpretation of $a(x,y)$.

then $S^d_{c/2}(q)$ must contain $r$, but no more than a finite number of the $p_n$. There must be such a sequence, therefore by 10.5 in [W], $S^d_{c/2}(q)$ is not open.

If we alter the topology in the space considered above by defining the topology onaall lines parallel to the x-axis and intersecting the y-axis in rational points as was done on the x-axis above and leaving all other neighborhoods fixed, then if $d$ is a compatible semimetric, the set $\{x : S^d_\epsilon(x)$ is not open for some $\epsilon > 0\}$ is dense in the space.

3.3 Theorem. A nonempty product space $\prod_{\alpha\in A} M_\alpha$ is semimetrizable iff each $M_\alpha$ is semimetrizable and $M_\alpha$ is a single point for all but a countable set of indices.

Proof. The proof of this theorem follows essentially the same lines as that of 22.3 in [W].

=> : Each $M_\alpha$ is homeomorphic to a subspace of a semimetrizable space and is therefore semimetrizable. Since a product of first countable spaces if first countable iff each factor space is first countable and there are only countably many factor spaces, then there must be only countably many non-trivial $M_\alpha$.

<= : Let $M_1, M_2, \ldots$ be semimetrizable spaces with semimetric $d_i$, which we can assume is bounded by 1. Define $d$ on $\prod_{i=1}^{\infty} M_i$ as follows for $x = (x_i)$ and $y = (y_i)$,

$$d(x,y) = \sum_{i=1}^{\infty} d_i(x_i, y_i)/2^i \quad .$$

Clearly $d$ is a semimetric.

Let $U$ be a basic neighborhood of $x$ in the product topology. We can assume that $U = S^O_{\epsilon_1}(x_1) \times S^O_{\epsilon_2}(x_2) \times \cdots \times S^O_{\epsilon_n}(x_n) \times \pi\{M_k : k = n+1, n+2, \ldots\}$ . Choose

$$\epsilon = \min \frac{\epsilon_1}{2}, \frac{\epsilon_2}{2^2}, \cdots, \frac{\epsilon_n}{2^n} \quad .$$

Now clearly $d(x,y) < \epsilon$ implies that $d_i(x_i, y_i)$ for each $i = 1, 2, \ldots, n$, so that $S^O_\epsilon(x) \subset U$ .

On the other hand, given $\epsilon > 0$, there exists an $N$ such that $\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}$, so that $S^O_{\frac{\epsilon}{2N}}(x_1) \times S^O_{\frac{\epsilon}{2N}}(x_2) \times \cdots \times S^O_{\frac{\epsilon}{2N}}(x_n) \times \pi\{M_k : k = N+1, N+2, \ldots\} \subset S^d_\epsilon(d)$ . We can conclude therefore, that the topology generated by $d$ is the same as the product topology.

4. THE COUNTABILITY PROPERTIES IN SEMIMETRIC SPACES. In a metric space X , it is well-known that the following are equivalent:

(a) X is second countable

(b) X is Lindelöf

(c) X is separable.

The question then naturally arises as to whether or not this is true in a semimetric space. The answer, unfortunately, is no, as the following two examples show.

4.1 Example. The space X considered in 1.2 (d) is an example of a $T_3$ , separable non-Lindelöf (therefore non-second countable) semimetric space. That X is non-Lindelöf follows since any vertical line has the discrete topology and therefore, by 15.2 in [W], X is non-normal. Then X is non-Lindelöf since any regular, Lindelöf space is normal. The other properties are clear. Therefore, a separable semimetric space is not necessarily either Lindelöf or second countable.

4.2 Example. The following space is an example of a Lindelöf, separable, non-second countable space. Consider the real line equipped with the semimetric d defined as follows:

$$d(x,y) = \begin{cases} |x-y| & \text{if } x \neq y + \frac{1}{n} \text{ for any } n \in Z \\ n & \text{if } x = y + \frac{1}{n} \text{ for some } n \in Z . \end{cases}$$

Then if $A_x = \{y : y = x + \frac{1}{n}$ for some $n \in Z\}$ , a local base at $x \in \chi$ is $g_n(x) = S^u_{1/n}(x) - A_x$ . Then $X$ is clearly Lindelöf and separable. The fact that $x$ is not second countable follows from the fact that if $x \neq y$ , then $A_x \cap A_y$ can contain at most two terms.

We know that in any space, second countable implies Lindelöf and separable. The following theorem gives the equivalence stated in the introduction to this section as it applies to semimetric spaces.

4.3 Theorem. A Lindelöf semimetric space is separable.

Proof. Let $U_i = X_i \bigcup_{X \in X} g_i(x)$ where $g_i(x)$ is the function given in 2.2. Let $U_i^* = \{g_i(x_{i,j})\}$ be the countable subcover of $U_i$ . Then $D = \bigcup_{i,j} \{x_{i,j}\}$ is a countable dense subset of $X$ .

We therefore have the following diagram in any semimetric space:

second countable
↓
Lindelöf
↓
separable

with none of the implications necessarily reversible.

4.4 Definition. A collection B of (not necessarily open) subsets of X is said to be a network for the topology on X iff given any point p and any open set U containing p , then there exists a $b \in B$ such that $p \in b \leq U$ . A network is said to be a countable network iff it has countably many elements. X is quasi second countable iff X has a countable network. A network is said to be σ-discrete iff it is the countable union of discrete collections of subsets of X . (A collection P of sets in X is discrete iff each point of X has a neighborhood meeting at most one element of P .)

4.5 Remark. Certainly the concept of countable network is weaker than that of second countable. The questions that arises as to what relationships exist among the properties:

(a) X has a countable network,

(b) X is Lindelöf, and

(c) X is separable.

The following theorem generalizes the fact that in any space, second countable implies separable and Lindelöf.

4.6 Theorem. If a semimetric space X has a countable network, then X is Lindelöf (and thus X is separable).

Proof. Essentially the same as 16.11 of [W].

4.7 Example. The space considered in 1.2 (d) is an example of a regular, separable, non-Lindelöf (therefore non-quasi second countable) semimetric space. We therefore have the same diagram as was given in 4.3 if we substitute quasi second countable, for second countable.

4.8 Definition. A cosmic space is a $T_3$ space with a countable network. It has been shown [Mi] that X is a cosmic space iff X is the continuous, image of a separable metric space. An interesting question that has been open for several years is "Is every $T_3$, Lindelöf, semimetric space a cosmic space?". It will be the purpose of the next few sections to give a partial solution to this question.

4.9 Remark. One might suspect that a first countable, $T_3$ Lindelöf space is cosmic, but the Sorgenfrey line provides an easy counterexample. For any first countable cosmic space is semimetrizable [$Hea_5$] and the Sorgenfrey line is not semimetrizable.

4.10 Conjecture. The following space, if Lindelöf provides an example of a $T_2$, Lindelöf, semimetric space which does not have a $\sigma$-discrete network.

Let the space X be the real line with a basis at each point x consisting of all sets of the form

$S_{1/m}(x) = S^u_{1/m}(x) - A_x$ where $A_x = \{y : |x-y| \in [\frac{1}{8} 2n+1 , \frac{1}{8} 2n]$ for some $n \in N\}$. Assume that $X$ has a $\sigma$-discrete network, say $\{P_n\}_{n=1}^{\infty}$. Denote by $Cl^u F$ the closure of $F$ in the usual topology on $R$. Define $B_n = \{x : x \in px \in p \subset S_{1/m}(x) \subset S_{1/m}(x)$ for some $p \in P_n$ and $m \in N$. Each $B_n$ is nowhere dense in the usual topology, for if not, then, for some $n$, $[a,b] \subset Cl^u B_n$. Therefore, for every open interval $I \subset [a,b]$, there exists $y_I \in I \cap A_n$. Consider $x \in [a,b]$. Clearly $S_{1/m}(x)$ is open. By the definition of $S_{1/m}(x)$, $S_{1/m}(x)$ contains countably many elements from $B_n$ and must intersect at least two distinct elements $p,p' \in B_n$, a contradiction to the fact that $P_n$ is $\sigma$-discrete. Therefore, each $B_n$ is nowhere dense in the usual topology. But $R$ is not the countable union of nowhere dense subsets, so $X$ is not $\sigma$-discrete.

Clearly $X$ is $T_2$. The fact that $X$ is semimetrizable follows from 2.2.

4.11 <u>Theorem</u> [Ber]. Assuming the continuum hypothesis and the axiom of choice, there exists a $T_3$, Lindelöf semimetric space which does not have a countable network.

<u>Proof</u>. The following lemma will be essential in the proof of the main theorem.

<u>Lemma</u>. There exists a subset $V$ of $I$ such that:

(1) If $U$ is an open subset of $I$, then $\text{card}(U \cap V) = 2^{\aleph_0}$.

(2) There exists no pair $f$ and $D$ such that:

(i) $D \subseteq V$ and $\text{card}\, D = 2^{\aleph_0}$,

(ii) $f : D \to V$,

(iii) either $f$ is strictly increasing and $\{(x, f(x)) : x \in D\}$ is bounded away from $\Delta$, the diagonal in $\{xI\}$, or $f$ is strictly decreasing.

<u>Proof of Lemma</u>. Denote by $\mathcal{C}$ the set of all closed subsets of $I$ and if $A \in \mathcal{C}$, let $\mathcal{F}(A)$ denote the set of all monotone functions from $A$ into $I$ such that point inverses are at most countable. Let

$$\mathcal{F} = \cup\{\mathcal{F}(A) : A \in \mathcal{C}\} .$$

Since $I$ is second countable, there can be at most $2^{\aleph_0}$ open subsets of $I$, and therefore, at most $2^{\aleph_0}$ closed subsets of $I$. Since each closed subset $A$ of $I$ can have at most $2^{\aleph_0}$ monotone functions from $A$ into $I$, there can be at most $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$ elements of $I$. Let $\mathcal{U}$ denote the set of all open subsets of $I$. Well order $\mathcal{U}$ and $\mathcal{F}$ by $\mathcal{U} = \{U_\alpha : \alpha < \Gamma\}$ and $\mathcal{F} = \{F_\alpha : \alpha < \Gamma\}$ respectively, where $\Gamma$ is the first ordinal of cardinal $2^{\aleph_0}$. We will now define the $V$ in the lemma inductively. Pick any point $x_1$ from $U_1$. If $x_\alpha$ has been choosen for all $\alpha < \beta$ , choose $x_\beta$ to be some point in $U_\beta$ such that

$$x_\beta \notin [\cup\{f_\alpha^{-1}(x_\tau) : \alpha \leq \beta, \tau < \beta\} \cup \{x_\tau : \tau < \beta\} \cup \{f_\alpha(x_\tau) : \alpha \leq \beta, \tau < \beta\}] .$$

Then let $V = \{x_\alpha : \alpha < \Gamma\}$ . Then $V$ satisfies condition 1 since each interval $U$ contains an uncountable number of intervals and each interval contained in $U$ contains some element of $V$ . It remains to show that $V$ satisfies condition 2. To show that it does, assume not. Then there exists a pair $f$ and $D$ satisfying 2(i), 2(ii), and 2(iii). Assume that $f$ is strictly decreasing (the argument is similar if $f$ is strictly increasing and $\{(x,f(x)) : x \in D\}$ is bounded away from $\Delta$ ). $f$ can be extended to $\overline{D}$ by $f$ in the following manner:

$$f^*(t) = \begin{cases} f(t) & \text{if } t \in D \\ \text{lub } f(x) \; ; \; x \; t, x \in D & \left\{ \begin{array}{l} \text{if } t \text{ is a limit} \\ \text{point of } D \text{ from} \\ \text{above and } t \notin D \end{array} \right. \\ \text{glb } f(x) \; ; \; x \; t, x \in D & \left\{ \begin{array}{l} \text{if } t \text{ is not a} \\ \text{limit point of } D \\ \text{from above and} \\ t \in D . \end{array} \right. \end{cases}$$

Then clearly $f^* \in \mathcal{F}$ . Consequently, $f^* = f_\alpha$ for some ordinal $\alpha$ . Since $D \subseteq V$ , $D$ is well ordered by the well-ordering of $V$ . Since $f_\alpha$ is strictly decreasing, there exists at most one $x_\theta$ such that $f_\alpha(x_\theta) = x_\theta$ . (If $f_\alpha$ is strictly increasing and $\{(x,f(x)) : x \in D\}$ is bounded away from the diagonal, then for all $x$ , $x \neq f(x)$ . Then there exists $x_\gamma$ , $x_\beta \in D$ such that $\gamma,\beta > \max\{\theta,\alpha\}$ , $\gamma \neq \beta$ , and $f_\alpha(x_\gamma) = x_\beta$ . Since if not, $f_\alpha$ is a one-to-one correspondence between $\{x_\tau : \tau > \max(\theta,\alpha)\}$ and $\{x_\tau : \tau \leq \max(\theta,\alpha)\}$ which is a contradiction since the cardinality of the first set is clearly greater than the cardinality of the second set. But then $\gamma < \beta$ is impossible since $D \subseteq V$ and by the definition of $V$ , $x_\beta \in V$ implies there does not exist $\alpha,\gamma < \beta$ such that $f_\alpha(x_\gamma) = x_\beta$ , since $x_\beta \notin \{f_\alpha(x_\tau) : \alpha \leq \beta , \tau < \beta\}$ . But $\gamma > \beta$ is impossible since $x_\gamma \in f^{-1}(x_\beta)$ , which is impossible since $x_\gamma \notin \cup\{f_\alpha^{-1}(x_\tau) : \alpha \leq \beta , \tau < \beta\}$ by the definition of $V$ . Since $D$ is well-ordered, we must have that $\gamma = \beta$ , and that $f_\alpha(x_\beta) = x_\beta$ which is impossible since $\beta > \theta$ .

Proof of Theorem. Let $V$ be as in the preceding lemma. Choose $M$ to be a subset of $V \times V - \Delta$ such that:

(1) if $U$ is open in $I \times I$ , then $\mathrm{card}(U \cap M) = 2^{\aleph_0}$

and

(2) if $p_1$ and $p_2$ denote the two projection maps of $I \times I$ onto $I$ , $p_i(m) \notin p_i(M - \{m\})$ , $i = 1,2$ and $m \in M$ , i.e. if $m \in M$ , $m = (a,b)$ , then there does not exist $m' \in M$ such that $m' = (a,x)$ or $m' = (y,b)$ . $M$ can be constructed in the following manner. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Gamma\}$ be a well-ordering of all the open subsets of $I \times I$ . Then choose $m_1 \in U_1$ . Assume that $m_\alpha$ has been choosen for all $\alpha < \beta$ . Choose $m_\beta \in U_\beta$ such that

$m_\beta \notin \{p_i(m_\alpha) : \alpha < \beta , i = 1,2\}$ . Then the preceding two properties are clearly satisfied by $M$ .

Let $d$ denote the usual Euclidean metric of $I \times I$ . Define $d^*$ on $M \times M$ as follows: if $(a,b),(c,d) \in M$ , then

$$d^*[(a,b),(c,d)] = \begin{cases} d[(a,b),(c,d)] & \text{if } a \leq c \text{ and } b \leq d \text{ or vice versa} \\ 1 & \text{otherwise.} \end{cases}$$

Then clearly $d^*$ generates a regular semimetric on $M$ , which will be denoted by $M(d^*)$ . We will show that every uncountable subset of $M$ contains an accumulation point of itself, so that assuming the continuum hypothesis, $M$ is Lindelöf. Assume there exists a subset $A$ of $M$ such that card $A = 2^{\aleph_0}$ , and $A$ is discrete in itself. Then there exist $A' \leq A$ and a positive number $\delta$ such that card $A' = 2^{\aleph_0}$ and $m,m' \in A'$ implies $d^*(m,m') > \delta$ . Now let $(p,q) \in A'$ such that for each Euclidean disc $\mathcal{D}$ about $(p,q)$ , card $(S \cap A') = 2^{\aleph_0}$ . In particular, consider the Euclidean disc $S$ of radius $\frac{\delta}{2}$ . Now if $(a,b),(c,d) \in S \cap A'$ , then $a < c$ implies $b > d$ , for otherwise $a < c$ and $b < d$ implies $d^*[(a,b),(c,d)] < \delta$ , a contradiction. If we define a function $f : p_1(S \cap V) \to V$ by $f(a) = b$ , where $(a,b) \in S \cap V$ , then $f$ is strictly decreasing, which is impossible by the lemma. Therefore, assuming the continuum hypothesis, $M$ is Lindelöf. It remains to show that $M$ has no $\sigma$-discrete network. Assume that it does, say

$A = \cap\{B_i : i = 1,2,\ldots\}$ . Then card $B_i < 2^{\aleph_0}$ , since if not choose a $b_\alpha$ from each element of $B_i$ . Then $B = \underset{\alpha}{\cap} \{b_\alpha\}$ is an uncountable set, and thereof has an accumulation point, a contradiction to the assumption that $B_i$ is a discrete collection of subsets of $M$ . Since card $B_i < 2^{\aleph_0}$ for all $i$ , card $A < 2^{\aleph_0}$ . Denote by $N(x,y)$ the $d^*$ sphere of radius $\frac{1}{2}$ about $(x,y)$ . Since $A$ is countable, there exists an $M' \leq M$ of an $A \in A$ such that card $M' = 2^{\aleph_0}$ and $(a,b) \in M'$ implies $(a,b) \in A \leq N(a,b)$ . Let $(p,q) \in M'$ such that, if $S$ is an Euclidean disc about $(p,q)$ , then card$(S\ M') = 2^{\aleph_0}$ . Choose $r$ such that $0 < r < \min \{\frac{1}{4} , \frac{1}{2} d\ [(p,q),\Delta]\}$ and denote by $S$ the $d^*$ sphere of radius $r$ about $(p,q)$ . Let $(a,b),(c,d) \in S\cap M'$ . If $a < c$ , then $b < d$ , for if not, $(c,d) \notin N(a,b)$ , but $(c,d) \in A \leq N(a,b)$ , which is a contradiction. Thus if we define a function $f : P_1(S\cap M') \to V$ by $f(a) = b$ where $(a,b) \in S\cap M'$ we have a strictly increasing function on an uncountable subset of $V$ into $V$ , a contradiction to the definition of $V$ . Therefore, $M(d^*)$ has no $\sigma$-discrete network.

It is well known that the product of two Lindelöf spaces may not be Lindelöf, and that the product of two normal spaces need not be normal. If we put more restrictions on the two spaces by requiring both spaces to be semimetric spaces, does the last statement still hold true? A construction closely related to that found in the proof of 4.11 gives a negative answer.

4.12 <u>Theorem</u>. Assuming the continuum hypothesis, there is a regular, Lindelöf semimetric space $X$ such that $X \times X$ is not normal (therefore not Lindelöf).

<u>Proof</u>. Let $M$ be as in 4.11. Define $d'$ on $M \times M$ by

$$d'[(a,b),(c,d) = \begin{cases} d[(a,b),(c,d)] & \text{if } a \leq c \text{ and } b \geq d \\ & \text{or vice versa} \\ 1 & \text{otherwise,} \end{cases}$$

where $d$ is the usual Euclidean metric in the plane. Then as in 4.11, we have a regular semimetric space which has no countable network. We note that this introduces the continuum hypothesis. Denote by $M(d')$ $M$ with the $d'$ topology. Let $X_1$ denote $M(d^*)$ and let $X_2$ denote $M(d')$. Let $X$ be the free union [D] of $X_1$ and $X_2$, i.e. $X = \{(i,z) : z \in M,\ i = 1,2\}$ with a base for the topology consisting of $\{\{i\} \times U : U$ is open in $X_i$, $i = 1,2\}$. Then clearly $X$ is a regular, Lindelöf space which has no countable network. We will show $X \times X$ is not normal by producing an uncountable closed, relatively discrete set $R$ (by 15.2 in [W], this is sufficient to show that $X \times X$ is not normal

Set $R = \{(1,z),(2,z) : z \in M\}$ is clearly closed in $X \times X$ and $R$ certainly has cardinality $2^{\aleph_0}$. It remains to prove that $R$ is relatively discrete. Let $z \in M$. Let $S_1$ denote the

$d^*$ semimetric sphere of radius $\frac{1}{2}$ about $z$, let $S_2$ denote the $d'$ semimetric sphere of radius $\frac{1}{2}$ about $z$, let $S_1^1 = \{1\} \times S_1$, and $S_2^1 = \{2\} \times S_2$. Assume $z = (x,y)$. Assume that there exists $((1,c),(2,c) \in R \cap (S_1^1 \times S_2^1)$. If $c = (a,b)$ then $a \leq x$ and $y \leq b$ or vice versa since $(1,c) \in S_1^1$. Since $(1,c) \in S_2^1$, we must have that $a \leq x$ and $b \leq y$ or vice versa. Therefore $x = a$ and $y = b$. Since $X \times X$ is regular and non-normal, $X \times X$ cannot be Lindelöf.

# CHAPTER II

## SYMMETRIZABLE AND SEMIMISTRATIFIABLE SPACES

1.0 In the previous chapter, we considered a semimetric function as a weakening of a metric function and examined some of the characteristics of semimetric spaces. In this chapter, we shall examine two kinds of spaces that are weakenings of semimetric spaces. In the first, we will use a semimetric function $d$ to determine a topology on a set $X$ by requiring a set $A$ to be closed iff for all $x \notin A$, $d(x,A) > 0$. This is in contrast to using a semimetric function $d$ to determine a topology be defining a set of limit points. In general, we say in the first chapter that the former always leads to a vlid topology, while the second may not. In the second space we will consider in this chapter, we will examine what happens if we weaker Heath's characterization of semimetric spaces (Chapter I, 2.2) to requiring only the existence of functions $g_n : X \to t$, for $n = 1,2,\ldots$, such that the following holds true:

(1) for all $x \in X$, $\bigcap_n g_n(x) = \{x\}$,

(2) $y \in g_n(x_n)$ for all $n$ implies $\{x_n\} \to \{y\}$.

One obtains Heath's characterization of semimetric spaces from the above by replacing (1) with the condition that $\{g_n(x)\}_{n=1}^{\infty}$ forms a local base at $x$ and $\bigcap_n g_n(x) = \{x\}$.

1.1 Remark. Recall that a symmetric space is an ordered triple $(X,t,d)$ where $X$ is a set, $t$ a topology on $X$, and $d$ is a semimetric function, such that $t$ is obtained from $d$ defining a subset $A$ of $X$ to be closed iff for all $x \notin A$, $d(x,A) > 0$.

1.2 In a metric space, every open sphere is an open set. When we weakened the requirements of the distance function by eliminating the triangle inequality, we found that while the open sphere may not be an open set, it nontheless has nonempty interior. It is natural to ask then, if we weaken a semimetric space to a symmetric space is the interior of an open sphere nonempty. The answer is no, as the following theorem shows.

Theorem. There exists a $T_1$, Lindelöf symmetric space such that the interior of every open sphere around every point is empty.

Proof. Let $X$ be the Cairns space of example 1.4 in Chapter I. Let us denote $(a,b) \cup (-b,-a)$ by $(a,b)^*$ and $[-b,-a] \cup [a,b]$ by $[a,b]^*$. We note that $X$ is separable since the rationals are clearly dense in $X$.

(i) We claim that an open set $A$ in $X$ is the countable union of disjoint open intervals, $I_n$. The proof is analogous to that given in 2.7 [W]. Define $x \sim y$ iff there exists an open interval $(a,b)$ such that $\{x,y\} \subset (a,b) \subset A$. We show that "~" is an equivalence relation on X . If there exists $x \in A$ such that there exist $\{x_n\}_{n=1}^{\infty}$ in $\tilde{A}$ such that $x_n \to x$ in the usual topology (i.e. x is not in any open interval), then $d(x_n,x) \to 0$ which implies $\tilde{A}$ is not closed, which is a contradiction. It is now easily shown that "~" is an equivalence relation on A . The resulting equivalence classes are disjoint, open intervals. The fact that there can be only countably many follows since each interval must contain a rational number. Therefore, since R with the usual topology is Lindelöf, so is X .

(ii) We claim that if F is a closed set with $[a,b] \subset F$, then $[a,b]^* \subset F$ . Let $x \in [-b,-a]$ . Assume that x is rational (the argument is clearly similar if x is irrational). Also assume that $x \notin F$ . Then by (i), there exists $m \in N$, such that $(-x,-1/m\ ,\ -x,+1/m) \subset \tilde{F}$ . But then there exists $\{p_i\}_{i=1}^{\infty} \subset [a,b]$ such that $p_i \notin Q$ for all i . Then $d^u(-x_i,-x) \to 0$ which implies $d(x_i,x) \to 0$ which implies F is not closed.

(iii) X is $T_1$ , but not $T_2$ . X is $T_1$ since each point is a closed set. X is not $T_2$ since by (ii) if $x \neq 0$, x and -x cannot be separated by disjoint open sets.

(iv) For each $x \in X$ , let $B_x$ be a collection of subsets of X , where $B_x$ is defined as follows:

(a) if $X = 0$ , $B_x = \{(-\epsilon,\epsilon) = \epsilon > 0\}$

(b) if $X \neq 0$ , $B_x = \{(x-\epsilon,x+\epsilon)^* - E = \epsilon > 0\}$ and $E$ is any countable subsets of $(-x-\epsilon,-x+\epsilon)$ such that all points of $Cl_\mu E$ are of the same rationality as $x$ . Let $B = \{B_x : x \in X\}$ .

Proof of (iv). Clearly $B$ is a base for some topology on $X$ . It remains to show that $B$ is a base for the given topology on $X$ . At $x = 0$ , it is clear. Let us assume that $x > 0$ ($x < 0$ is similar). Assume that $x$ is of rationality $T$ . We now prove that $\text{Int}\, U_x \neq \phi$ if $U_x \in B$ . Assume $U_x = (-x-\epsilon,-x+\epsilon)^* - E$ . That is, if $y \in U_x$ , we must show that $d(y,\tilde{U}_x) > 0$ . If $y \in (x-\epsilon,x+\epsilon)$ , then since if $y$ is also of rationality $T$ , $d(y,E) = 0$ is impossible, and if $y$ is not of rationality $T$ , then since $y$ is not a limit point of $E$ by construction, $d(y,E) = 0$ is impossible, and thus $d(y,E) \supset 0$ . Similarly $y \in (-x-\epsilon,-x+\epsilon)$ .

We have to prove that if $x \in U'$ for some open $U'$ , then there exists $U \in B$ such that $x \in U \subseteq U'$ . Again, let $x$ be of rationality $T$ . By (i), there exists $r > 0$ such that $(x-r',x+r') \subset U'$ . There exists $r'' > 0$ such that if $y$ is not of rationality $T$ $y \in (-x-r'',-x+r'')$ , then $y \in U'$ , since if not, there exists $\{y_n\}_{n=1}^{\infty}$ contained in $(-x-r',-x+r')$ such that $y_n \to -x$ in the usual topology, which implies that $d(y_n,x) \to 0$ , which implies that $U'$ is not open, a contradiction. Let $r = \min\{r',r''\}$ and let $U = U' \cap (x-r,x+r)^*$ . Then $x \in U \subseteq U'$ . Let $E = (-x-r,-x+r) - U$ . By construction and the definition of closed sets, all limit points of $E$ are of rationality $T$. The cardinality of $E$ is clear.

The proof of the main theorem is now clear. Let $Y = X - \{0\}$. It is easily shown that $Y$ satisfies the conditions of the theorem.

1.3 We have seen that the countable product of semimetric space was semimetric. The question naturally then arises as to whether or not the countable product of symmetric spaces is symmetric? The answer is yes, as the next theorem shows.

<u>Theorem</u>. Let $(X_i, d_i)$ be a symmetric space for all $i \in N$. Then $\prod_i X_i$ is a symmetric space.

<u>Proof</u>. Define $d$ on $X$ by $d(x,y) = \sum_i d_i(x_i,y_i)/2i$. Then $U$ is open in $X$ implies that $U = U_1 \times U_2 \times \cdots \times U_n \times \prod_{k=n+1}^{\infty} X_k$. Where $U_i$ is open in $X_i$, $x \in U$ implies $x_i \in U_i$ for $i = 1,2,\ldots,n$, which implies, since $d(x_i, \tilde{U}_i) > 0$ for $i = 1,2,\ldots,n$ that $d(x,\tilde{U}) > 0$ for $x \notin U$.

Assume on the other hand that $d(x,A) > 0$ for all $x$ not in $A$. If $\tilde{A}$ is not open, then if $\tilde{A} = \prod_i A_i$, some $A_{i_o}$ not open, then there exists $x_{i_o} \in A_{i_o}$ such that $d_{i_o}(x_{i_o}, \tilde{A}_{i_o}) = 0$. Choose $x_i$ to be any element of $A_i$ if $i \neq i_o$. Then $d(x,A) = 0$ where $x = (x_1, x_2, \ldots, x_{i_o}, \ldots)$.

Therefore each $A_i$ is open. The fact that there can be only finitely many $A_i \neq X_i$ follows from the fact that we can assume that each $d_i$ is bounded by 1 . Then given $\epsilon$ , there exists $N$ such that $\sum_{i=N+1}^{\infty} \frac{1}{2} i < \epsilon$ . Then there exists $n$ such tha $n > \frac{1}{\epsilon}$ . Given $x \in \tilde{A}$ , choose $y_n = (x_1, x_2, \ldots)$ where $x_i \in A_i$ for $i = 1,2,\ldots,n$ and $x_i \in \tilde{A}_i$ for $i > n$ (we can assume that $A_i \neq X_i$ for any $i$ ) . Then clearly, $y_n \in A$ for all $n$ and $d(y_n, x) < \frac{1}{n}$ for all $n$ , which implies that $d(x,A) = 0$ , a contradiction.

1.4 In metric and semimetric space, we saw the distance function was invariant with regard to subspaces. In general, in symmetric spaces this will no longer hold true. However, in symmetric spaces, we will find that the distance function is invariant with regard to open or closed subspaces.

<u>Theorem</u>. If $(X,d)$ is a symmetric space, then the distance function is invariant with respect to open or closed subspaces of $X$ , i.e. the restriction of the distance function to the subspace will yield the subspace topology.

<u>Proof</u>. Obvious.

In the past, there has been much work on finding out which spaces have a metric generating their topology. In Chapter I, two characterizations of semimetric spaces were given which depend only upon knowing their topology. The question then arises as to which spaces are symmetrizable, given only their topology. Unfortunately, not too much is known in this area. In 1.7, the major theorem in this area will be given, but first we introduce several concepts.

1.5 Definition. Consider a collection $T_x$ of subsets of $X$ for all $x \in X$ such that $x \in T$ for all $T$ $T_x$ and all finite intersections of elements from $T_x$ are in $T_x$. Define a topology on $X$ using $T_x$ by $P \subset X$ is closed iff for all $x \notin P$, there exists $T \in T_x$ such that $T \cap P = \phi$. $T_c = \bigcup_{x \in X} \{T_x\}$ is called a weak base for the topology and members of $T_c$ are called weak neighborhoods. A topological space $X$ is said to satisfy the weak first axiom of countability (briefly, the gf axiom of countability) iff its topology can be given by a weak base $T_c = \{T_x : x \in X\}$, where each $T_x$ is countable.

1.6 Example. Certainly any base is a weak base, but the converse is not necessarily true. In fact if $T \in T_c$, then $\text{Int}\, T = \phi$ is possible as the following example shows. Let $X = \{(i,j) \mid i,j \in N\}$. That is, $X$ is countably many columns of natural numbers. Define $U \subseteq X$ to be open iff $U$ contains all but a finite number of elements in each column. Then a weak base for the topology at $(m,n) \in X$ can clearly be given by $g_k(m,n) = \{(m,n)\} \cup \{(i,j) : 1 \leq i \leq m+1$, $j > n+k\}$. Then certainly the $g_k(m,n)$ are not open. Also, this space is certainly an example of a space which is gf - countable, but not first countable. The fact that X is gf - countable is clear. To show that X is not first countable, assume that it is, say $b_k(m,n)$ is a base at $(m,n)$ . Then choose U to be a set which does not contain 1 element contained in the kth column of $g_k(m,n)$ . Then U is clearly open and there does not exist k such that $b_k(m,n) \subset U$ .

1.7 Theorem. If X is a $\sigma$ - discrete space, then X is gf - countable iff X is symmetrizable.

Proof. By letting $T_n(x) = S_{1/n}(x)$ if X is symmetrizable, the fact that X is gf - countable is clear.

Assume then that X is gf - countable. We may assume that the members, $Q_k(x)$ , of each system $T_x$ are decreasing. By hypothesis, the network can be put in the form $\gamma = \bigcup_{n\in N} \gamma_n$ , where $\gamma_n = \{S_n^\alpha : \alpha \in L_n\}$ are discrete systems of sets $S_n^\alpha \subseteq X$ ; the last can be regarded as closed. We make the following definitions:

$$M_x^k = \cup\{S_n^\alpha : n \leq k , \alpha \in L_n \text{ and } x \not\in S_n^\alpha\}$$

$$\tilde{Q}_k(x) = Q_k(x) - M_x^k \quad , \quad k = 1,2,\ldots$$

$$N(x,y) = \max\{n : (\tilde{Q}_n(x) \cap \tilde{Q}_n(y)) \cap \{x\}\cap\{y\}) \neq 0\}$$

and

$$d(x,y) = \frac{1}{N(x,y)} \quad \text{for all} \quad x,y \in X .$$

Then clearly $d$ is a semimetric function on $X \times X$ . It remains to show that $d$ is compatible with the topology on $X$ .

Let $x \in X-P$ , where $P$ is a closed subset of $X$ . Then there exists some $n_o$ such that $Q_{n_o} \cap P = \phi$ . Since $X-P$ is open, there exists $S^{\alpha}_{n'} \in \gamma$ such that $x \in S^{\alpha}_{n'} \subseteq X-P$ . Assume that $y \in P$ . Then we have that

$$\tilde{Q}_{n_o}(x) \subseteq Q_{n_o}(x) \subseteq X-P \subseteq X - \{y\} \quad .$$

But on the other hand,

$$Q_{n'}(y) = Q_{n'}(y) - M^{n'}_{y} \subseteq Q_{n'} - S^{\alpha}_{n'} \subseteq Q_{n'}(y) - \{x\} \quad .$$

Therefore, if $n > \mathrm{Max}\{n_o,n'\}$ , then

$$(\tilde{Q}_n(y) \cap \tilde{Q}_n(x)) \cap (\{x\}\ \{y\}) \subseteq (\tilde{Q}_{n_o}(x) \cap \tilde{Q}_{n'}(y)) \cap (\{x\} \cup \{y\}) = \phi \quad ,$$

Therefore, $N(x,y) \leq n$ which implies that $d(x,y) > n = \max\{n_o,n'\}$ for all $y \in P$ . Therefore, $d(x,P) \geq \frac{1}{n} > 0$ .

Assume that $P$ satisfies the condition $d(x,P) > 0$ for all $x \notin P$ . We must show that $P$ is closed in the original topology on $X$ . We will show that for any $\epsilon > 0$ , the set $S_{\epsilon}(x)$ contains a weak neighborhood of $x$ . Choose $n$ such that $n > \frac{1}{\epsilon}$ . Then $M^{n}_{x}$ is closed and does not contain $x$ . Therefore, there exists a weak neighborhood $Q_{n'}(x)$

disjoint from $M_x^n$ . Then if $n'' > \max\{n,n'\}$ , $Q_{n''}(x) \leq Q_\epsilon(x)$ , since the weak neighborhoods decrease monotonically. Therefore

$$Q_n(x) \subseteq Q_{n''}(x) \cap Q_n(x) \subseteq Q_n(x) - M_x^n = \tilde{Q}_n(x) \quad .$$

Then if $y \in \tilde{Q}_n(x)$ , then clearly

$$N(x,y) \geq n \quad \text{and} \quad d(x,y) = \frac{1}{N(x,y)} \leq \frac{1}{n} < \frac{1}{\epsilon} \quad .$$

Thus, a $\sigma$ - discrete space which is gf - countable is symmetrizable.

1.8 <u>Remark</u>. Not every symmetric space has a $\sigma$ - discrete network, as I.4.10 shows. This answers a question of Arkhangel' skii raised in [A].

1.9 In this and the next few sections, several of the properties of symmetric spaces will be examined.

In [A], Arkhangel'skii introduced the definition of the gf - axiom of countability, by asserting, without proof that a $T_{3\frac{1}{2}}$ topological space X satisfies the first axiom of countability iff X is Frechet-Urhyson and satisfies the gf - axiom of countability. The following two examples show that complete regularity is necessary.

(a) The example considered in 1.6.

(b) The space obtained from the real line by identifying the points $\{n\}$ and $\{\frac{1}{n}\}$. This space is clearly $T_2$.

1.10 In 1.4, we saw that the symmetric is invariant with respect to open or closed subspaces. If we consider the example of the space given in 1.6 and construct a symmetric on $X$ by means of the construction of 1.7, then we find that $X$ has a symmetric function $d$ defined by

$$d((m,n),(m',n')) = \begin{cases} 0 & \text{if } m = m' \text{ and } n = n' \\ 1 & \text{if } m<m'+1 \text{ or vice versa} \\ \frac{1}{|n-n'|} & \text{otherwise.} \end{cases}$$

Then if we take the subspace consisting of every fifth column, we arrive at a subspace whose topology is not generated by the above semimetric function.

1.11 Definition. Every map $f : X \to Y$ where $(X,p)$ is a metric space and $Y$ any set induces a quotient distance $d$ defined by

$$d(x,y) = \inf \{p(f^{-1}(x),f^{-1}(y))\} .$$

We write $d = p/f$ .

It is often of interest to know under what circumstances the topology induced by $d$ agrees with the quotient topology. In the next few sections, we answer that question.

$S^d_{1/n}(y) = \{y' : d(y,y') < \frac{1}{n}\} = \{y' : p(f^{-1}(y), f^{-1}(y') < \frac{1}{n}\}$ , then $U = \{x : p(f^{-1}(y), x) < \frac{1}{n}\}$ is a neighborhood of $f^{-1}(y)$ and since $f$ is pseudo-open, $y \in \text{Int } f(U) \subseteq \text{Int } S^d_{1/n}(y)$ .

Conversely, given open $V$ containing $y \in Y$ , there exists $U \subseteq X$ such that $f^{-1}(V) = U$ . Which implies for some $n$ , $S^p_{1/n}(x) \subseteq U$ where $f(x) = y$ . Then $S^d_{1/n}(y) \subseteq V$ .

2.0 In the previous section, we considered what happens if we weaken a semimetric space to symmetric space by using $d$ , the distance function to define closed sets, instead of limit points. In this section we consider what happens if we weaken the requirement that $g_n(x)$ form a base at $x$ in Heath's characterization of semimetric spaces (I,2.2) to merely requiring that the $g_n(x)$ be open.

2.1 Definition. A topological space $X$ is said to be semi-stratifiable iff for all $U \in t$ , there exists a sequence of closed subsets of $X$ such that

(a) $\bigcup_{n \in N} U_n = U$

(b) $U \subset V$ implies that $U_n \subset V_n$ .

$U \to \{U_n\}$ is called a semistratification for $X$ . If instead of condition (a) above, we have

$$\text{(a')} \quad \bigcup_{n \in N} U_n^o = U \text{ and } \overline{U}_n \subset U ,$$

then any space $X$ that fulfills conditions (a') and (b) is called a <u>stratifiable</u> space.

We note that any countable $T_1$ space $X$ is a semi-stratifiable space. For if $X = \{x_1, x_2, \ldots\}$ and if $U$ is open in $X$, then $U = \{X_{k_1}, X_{k_2}, \ldots\}$ where $k_1 \leq k_2 \leq k_3 \leq \cdots$. Let $U_n = \{X_{k_1}, X_{k_2}, \ldots, X_{k_n}\}$. Then since $X$ is $T_1$, $U_n$ is closed.

That $T_1$ is necessary is shown by the space $X = \{a,b\}$ with $\tau = \{\phi, \{a\}, X\}$ .

2.2 The next theorem is a characterization of semistratifiable spaces due to Creede. This characterization is often very useful in deciding whether or not a given space is semi-stratifiable. It also relates the concepts of semimetric space and semistratifiable space.

<u>Theorem</u> [$C_1$]. $X$ is semistratifiable iff there exists $\{g_i\}_1^\infty$ of functions from $X \to t$ such that

(i) $\bigcap_i g_i(x) = Cl\{x\}$ for all $x \in X$

(ii) $y \in g_i(x_i)$ for all $i$ implies $(x_i) \to y$ .

<u>Proof</u>. For each $i \in N$ and $x \in X$ define

$$g_i(x) = X - (X - Cl\{x\})_i \quad .$$

Then clearly $g_i(x)$ satisfies conditions (i) and (ii) of the theorem. Conversely, let $\{g_i(x)\}$ satisfy the conditions of the theorem. For each $n$ and each open set $U$ , define

$$U_n = X - \cup\{g_n(x) : X \quad X-U\} \ .$$

Then the corresponding $U \to \{U_n\}$ is a semistratification for $X$ .

2.3 <u>Corollary</u>. A $T_1$ space $X$ is semimetrizable iff $X$ is semistratifiable and first countable.

In the next few sections, some of the basic properties of semistratifiable spaces will be given. As would be expected, many of the properties enjoyed by semimetrizable space will be enjoyed by semistratifiable spaces.

2.4 <u>Theorem</u> [$C_1$]. The countable product of semistratifiable spaces is semistratifiable.

<u>Proof</u>. Assume that for each $i$ , $X_i$ is a semistratifiable space and $\{g_{i,j}\}_j$ is a sequence of functions satisfying 2.2. Let $X$ be the countable product of the $X_i$ and denote the ith projection map of $X$ onto $X_i$ by $p_i$ . For each pair $i,j$ and each $x \in X$ , define

$$h_{i,j}(x) = \begin{cases} g_{i,j}(p_i(x)) & \text{if} \quad j \leq i \\ \\ X_i & \text{otherwise.} \end{cases}$$

Now define $\{g_j(x)\}_{j=1}^{\infty}$ by $g_j(x) = \prod_{i=1}^{\infty} h_{i,j}(x)$ for each $j$ and $x$. Then clearly the sequence $\{g_j(x)\}_{j=1}^{\infty}$ satisfies the conditions of 2.2 and therefore $X$ is semistratifiable.

2.5 As we saw in 1.4, symmetric spaces are not always "nice" with respect to subspaces. Semistratifiable spaces, however, enjoy pleasent subspace properties.

Theorem $[C_1]$. A semistratifiable space is hereditarily semistratifiable.

Proof. The natural restriction of the semistratification for $X$ to any subspace $Y \subseteq X$ is a semistratification for $Y$.

2.6 Theorem $[C_1]$. If $Y$ is a closed subspace of a semistratifiable space $X$, then there exists a semistratification $V \to V_n$ for $X$ such that $(V \cap Y)_n = (V_n \cap Y)$.

Proof. Assume that $W \to W_n$ is any semistratification for $X$ and that $U \to U_n$ is any semistratification for $Y$. Then let $V_n = (V \cap Y)_n \cup (V - Y)_n$. The correspondence $V \to V_n$ is a semistratification satisfying the conclusion of the theorem.

2.7 Theorem $[C_1]$. The union of two closed (in the union) semistratifiable spaces is semistratifiable.

2.8 Definition. A topological space $X$ is said to be $F_\sigma$ - screenable iff every open cover has a $\sigma$ - discrete closed refinement which covers the space.

2.9 Theorem [$C_1$]. A semistratifiable space if $F_\sigma$ - screenable.

Proof. Let $U \to U_n$ be a semistratification for $X$. Assume that $\{0_\alpha : \alpha \in \Lambda\}$ is an open cover and let $\Lambda$ be well-ordered. Define

$$H_{1,n} = (0_1)_n$$

and

$$H_{\alpha,n} = (0_\alpha)_n - \cup\{0_\beta : \beta\in\Lambda, \beta<\alpha\} \quad \text{for} \quad \alpha > 1 .$$

Then for all $n \in N$, define

$$\mathcal{H}_n = H_{\alpha,n} : \alpha \in \Lambda\} .$$

Then clearly $\mathcal{H}_n$ is a discrete collection of closed sets and by the well-ordering of $\Lambda$, $\mathcal{H} = \bigcup_n \mathcal{H}_n$ covers $X$.

2.10 Theorem [$C_1$]. The closed, continuous image of a semi-stratifiable space is semistratifiable.

Proof. Obvious.

Quite frequently, it is of interest to know whether or not a given topological space is Lindelöf. We will conclude this chapter with an equivalence relationship in semistratifiable spaces between Lindelöf, hereditarily separable, and $\aleph_1$-compact spaces.

2.11 Theorem [$C_1$]. In a semistratifiable $T_1$ space $X$, the following are equivalent:

(1) $X$ is Lindelöf

(2) $X$ is hereditarily separable

(3) $X$ is $\aleph_1$ - compact.

Proof. (1) ⟶ (2). Assume $X$ is a semistratifiable, Lindelöf $T_1$ space. It is sufficient to prove that $X$ is separable since a Lindelöf space in which open sets are $F_\sigma$ is hereditarily Lindelöf. (Since every open set is the countable union of closed sets and closed sets of Lindelöf spaces are Lindelöf, every open subset of a Lindelöf space is Lindelöf, which implies that every subset of a Lindelöf space is Lindelöf.) Assume that $\{g_i(x)\}$ is a sequence of functions satisfying the conditions of 2.2. For each $i$, $\{g_i(x) : x \in X\}$ is an open cover of $X$, and since $X$ is Lindelöf, there exists a countable set $D_i$ such that $\{g_i(x) : x \in D_i\}$ is an open cover. Then $D = \bigcup_i D_i$ is a countable dense subset of $X$.

(2) ⟶ (3) If $E$ is any subset of $X$, card $E > \chi_o$ and $E$ has no accumulation point, then $E$ is discrete in the subset topology, which implies $E$ is not separable (in the subset topology), a contradiction to the fact that $X$ is hereditarily separable.

(3) ⟶ (1) . Let X be an $\aleph_1$ - compact, $T_1$ semistratifiable space. Assume that G is an open cover of X which has no countable subcover. By 2.9, G has a closed refinement $H = \{H_n\}$ , where each $H_n$ is a discrete collection of subsets of X . Since G has no countable subcover, there exists an n such that $H_n$ is countable. Then if X' is a subset of X consisting of exactly one point from each nonempty element of $H_n$ , then X' is uncountable and has no accumulation point.

3.0 In this chapter, we have examined two kinds of spaces that are both weaker than semimetric spaces. Developments thus raise the question, "What is the relationship between semistratifiable spaces and symmetrizable spaces?". If X is first countable, then clearly symmetrizable <=> semistratifiable. We will see that in general, a symmetrizable space need not be a semistratifiable space and that semistratifiable space need not be a symmetrizable space.

3.1 Example. The Cairns space, X , considered in I 1.4 is an example of a symmetrizable space that is not semistratifiable. We need only show $X_1$ is not semistratifiable.

Assume, then, that X is semistratifiable. Consider the open set $(-2,-1) \cup (1,2) - E$ , where E is any set of rational points whose only accumulation point is $-\frac{7}{4}$ . Then there must exist some $U_n$ in the semistratification of X such that $U_n$ contains points arbitrarily close to $-\frac{7}{4}$. Therefore $U_n$ is not closed, a contradiction.

3.2 Example. There exists a countable semistratifiable space $X$ which is not symmetrizable. Let $X$ be a copy of $I \cap Q$ with the usual topology. Let $X$ be the disjoint union of the $X_n$ with the point $\{0\}$ from each $X_n$ identified. Then there does not exist a semimetric function $d$ generating the topology. For if there were one, clearly $d(\{0\},X_n-\{0\}) = 0$ for all $n$, which implies that the set $\bigcup_n [0, \frac{1}{n})$ is not open in $X$, a contradiction.

3.3 The only result I have been able to attain in the direction of examining the relationship between symmetric and semistratifiable spaces is the following theorem.

Theorem. If $X$ is a $T_1$, $\sigma$ - discrete Frechet-Uryhson, semistratifiable space with a function $g : N \times X \to t$ satisfying the conditions of II, 2.2 and the further condition that $y \in g_n(x)$ implies $x \in g_n(y)$, then $X$ is symmetrizable.

Proof. By 1.7, it is sufficient to show that $X$ has a countable weak base for the topology. Let $T_x = \{g_n(x)\}$. We claim that $T_x$ is a countable weak base at $x$. Countability is clear. Remains to show that $T_x$ is a weak base. Let $t'$ be the topology generated using $T_x$. Then clearly $t' \subseteq t$. Assume then that $G$ is $t$ open. To prove that $G$ is $t'$

such that $U_n$ contains points arbitrarily close to $-\frac{7}{4}$. Therefore $U_n$ is not closed, a contradiction.

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open. Assume that $G$ is not. Then there exists $x \in G$ such that $g_n(x) \cap \tilde{G} \neq \phi$ for all $n$. This implies there exists $y_n \in g_n(x) \cap \tilde{G}$ such that $y_n \neq x$. By hypothesis, $x \in g_n(y)$, and therefore $y_n \to x$. Since $X$ is a Frechet-Urhyson space, $x \in \tilde{\tilde{G}}$ in $t$ topology, a contradiction.

3.4 Remark. It is still an open question what the precise relationship between semistratifiable spaces and symmetrizable spaces is.

# CHAPTER III

## DEVELOPABILITY AND METRIZABILITY OF SEMIMETRIC, SYMMETRIC, AND SEMISTRATIFIABLE SPACES

1. INTRODUCTION. In this chapter the concept of developable spaces is introduced. It will be shown that the class of developable spaces lies between the class of semimetric spaces and the class of metric spaces. In section 2, four theorems are given which give necessary and sufficient conditions for a given topological space $X$ to be semimetrizable, developable, or metrizable. In section 3, an interesting theorem of Arkhangel'skii which states that every compact, symmetric space is metrizable is given. In section 4, developability of semimetric symmetric, and semistratifiable spaces is examined. We conclude the thesis with two theorems on the developability and metrizability of semimetric spaces in terms of conditions on the original topology.

1.1 Definition. A topological space $(X,t)$ is developable iff there exists a sequence $\{U_n\}_{n=1}^{\infty}$ of open covers of $X$ such that $U_n$ refines $U_{n-1}$ and for all $x \in X$, $\{st(x,U_n) : n = 1,2,\cdots\}$ is a neighborhood base for $x$. Equivalently, $X$ is developable iff there exists a sequence of open coverings $\{f_n\}$ such that if $x \in U \in t$, then there exists an $n$ such that $st(X,G_n) \subset U$. A Moore Space is a

$T_3$ , developable space.

1.2 <u>Theorem</u>. Every developable space has a semimetric with respect to which all spherical neighborhoods are open.

<u>Proof</u>: Let $\{G_i\}$ be a development for $X$ . Define $d$ as follows:

$$d(x,y) = \inf \{\frac{1}{n} : x \in g \text{ and } y \in g \text{ for some } g \in G_n\} .$$

Then the properties of $d$ are clear.

1.3 <u>Theorem</u>. Every metric space $X$ is developable.

<u>Proof</u>. Let $U_n$ be the cover of $X$ by the $\frac{1}{n}$ spheres around each point $x \in X$ . Then clearly $X$ is developable.

1.4 <u>Example</u> [$Mc_1$]. There exists a regular semimetric space which is not a Moore space. Let $X$ be the space given in the proof of I.3.2. Then $X$ is a regular semimetric space, and by I.3.2 and 1.2, $X$ is not developable.

2. In this section, we consider four theorems, each of which give necessary and sufficient conditions for a given space to be semimetrizable, developable, or metrizable. The first theorem will give the conditions in terms of a distance function $d$ , the second and third in terms of a function $g : N \times N \to t$ (c.f., Heath's characterization of semimetric

spaces, I. 2.2), and the fourth in terms of the topology on the space.

2.1 Theorem. In [$Hea_2$], Heath quoted [A&N]

(i) A $T_3$ space $X$ is developable iff there exists a semimetric function $d$ such that whenever

$$\lim d(x_n,p) = \lim d(y_n,p) = 0 ,$$

then

$$\lim d(x_n,y_n) = 0 .$$

(ii) A $T_3$ space $X$ is metrizable iff there exists a semimetric $d$ for $X$ such that whenever

$$\lim d(x_n,p) = \lim d(x_n,y_n) = 0 ,$$

then

$$\lim d(y_n,p) = 0 .$$

2.2 Theorem [$Hea_1$]. Consider the following three conditions on a fuction $g : N \times X \to t$ :

(i) For each point $x \in X$ , $\{g_n(x)\}$ is a nonincreasing sequence which forms a local base at $x$ such that if $y \in g_m(x_m)$ for all $m \in N$ , then $x_m \to y$ .

(ii) If $y \in X$ and $x$ and $z$ are point sequences such that, for each $m$ , $(y+x_m) \subset g_m(z_m)$ , then $x$ converges to $y$ .

(iii) If $x, y \in X$ and $n \in N$, then $x \in g_n(y)$ implies that $y \in g_n(x)$.

Then the following holds true:

(a) $X$ is semimetric iff there exists a $g$ satisfying condition (i).

(b) $X$ is developable iff there exists a $g$ satisfying conditions (i) and (ii).

(c) $X$ is metrizable iff there exists a function $g$ satisfying conditions (i), (ii), and (iii).

Proof. Part (a). This is just Theorem 2.2 of Chapter I.

Part (b). Sufficiency. Let $G_i = \{g_j(x) : j \geq i$ and $x \in X\}$. We claim that the coverings $G_1, G_2, \ldots$ constitute a development. The fact that $G_{n+1}$ refines $G_n$ is clear. It remains to show that $\{st(x, G_n) : n = 1, 2, \ldots\}$ is a neighborhood base at $x$. Let $x \in U \in t$. Then there exists $n_o$ such that $x \in g_{n_o}(x) \subset U$. Assume that for all $n \geq n_o$, there exists $y_n$ such that $y_n \in st(x, U_n) - g_{n_o}(x)$. Then there exists $z_n$ such that $(x+y_n) \subset g_n(z_n)$ for all $n \geq n_o$. Therefore, by hypothesis, $y_n \to x$, a contradiction, since for all $n \geq n_o$, $y_n \in g_{n_o}(x)$.

Necessity. Assume that $G_1, G_2, \ldots$ is a development for $X$. Then define $g : N \times X \to t$ inductively as follows:

$$g_1(x) = \text{any member of } G_1 \text{ which contains } x$$

$$g_n(x) = \text{any member of } G_n \text{ such that}$$
$$x \in g_n(x) \subset g_{n-1}(x) \text{ , if } n > 1 \text{ .}$$

Then clearly $g$ satisfies conditions (i) and (ii).

Part (c). Necessity is clear. Let $g$ be defined as follows:

$$g_n(x) = S_{1/n}(x) \quad .$$

Sufficiency. Assume that $g$ is a function satisfying conditions (i), (ii), and (iii). For each natural number $n$, define

$$G_n(x) = \{g_m(x) : x \in X \text{ and } m \geq n\} \quad .$$

Assuming that $X$ is not metrizable, there exist two points $p,q \in X$ and $R \in t$ such that for each $n$, there exists $h,k \in G_n$ such that $p \in h$, $h \cap h \neq \phi$, and $k \cap (X-(R-q)) \neq \phi$ (i.e., $k \cap (X-R) \neq \phi$ since $X$ is $T_1$) by Moore's Metrization Theorem. Therefore there exists a point $p$, a region $R$ and point sequences $x,y$, and $z$ such that for all $n$, $y_{m(n)} \in g_n(p)$, so, that, by condition (iii), $p \in g_n(y_{m(n)})$. But $z_{m(n)} \in g_{m(n)}(y_{m(n)})$ since $y_{m(n)} \in g_{m(n)}(z_{m(n)})$ -

therefore $z_{m(n)} \in g_n(y_{m(n)})$ .

Therefore, for each $n$ , $(p+z_{m(n)}) \subset g_n(y_{m(n)})$ , and again by condition (ii) , $\{z_{m(n)}\}$ converges to $p$ . Thus, there exists a subsequence $r$ of $m$ , such that for each natural number $n$ , $z_{r(n)} \in g_n(p)$ , thus $p \in g_n(z_{r(n)})$ . But by assumption, there is a point sequence $u$ such that, for each $n$ , $u_n \in X-R$ and $u_n \in g_n(z_{r(n)})$ , which implies that $u$ converges to $p$ , and therefore, $p \in \overline{(S-R)}$ and $p \in R \in t$ , a contradiction. Therefore $X$ must be metrizable.

2.3 Theorem [$\text{Hea}_1$]. In [$\text{Hea}_1$], a somewhat different version of 2.2, (b) is given. Consider a function $g : N \times X \to t$ and the following condition on $g$ :

(ii') If $y \in R \in t$ and $x$ is a point sequence such that for each $n$ , $y \in g_n(x_n)$ and there is a $k$ such that $\overline{g_{n+k}(x_{n+k})} \in g_n(x_n)$ , then there is a natural number $n$ such that $g_m(x_m) \subset R$ .

Theorem [$\text{Hea}_1$]. If there exists a function $g$ satisfying (i) of 2.2 and (ii') above, and if $X$ is a $T_3$ space, then $X$ is a Moore space.

Proof. Assume that $G = \{g_m(x) : x \in X , m \in N\}$ is a basis for a regular semimetric space satisfying the hypothesis of the theorem to be proved. Well-order $X$ by $\alpha = \{p_1, p_2, \ldots\}$ .

We define, by induction, functions $h : N \times X \to G$, $r : \alpha \times N \to N$, and $n : \alpha \times N \to \alpha$ as follows. If $p_z \in X$, let $h_1(p_z) = g_{r_z}(1) = g_1(p_z)$. If $i > 1$:

(Case 1). If there does not exist $q \in (X-p_z)$ and a $j \in N$ such that $p_z \in h_j(q)$ and $h_j(q) \cap [X-g_{r_z(i-1)+1}(p_z)] \neq \phi$, then let $h_i(p_z) = g_{r_z(i)}(p_z) = g_{r_z(i-1)+1}(p_z)$, i.e., $r_z(i) = r_z(i-1)+1$; or

(Case 2) otherwise. For each such $j < i$, let $p_{n_z}(j)$ be the first member $q$ of $\alpha(q \neq p_z)$ such that $p_z \in h_j(q)$ and $h_j(q) \cap [X-g_{r_z(i-1)+1}(p_z) \neq \phi$; let $r_z(i) = \inf\{m : m \in N$, $m > r_z(i-1)$, and $\overline{g_m(p_z)} \subset \cap [h_j(p_{n_z}(j) : j < i$ and $j$ is not covered by case 1]; and let $h_i(p_z) = g_{r_z(i)}(p_z)$.

Then the basis $H = \{h_i(x) : x \in X$ and $i \in N\}$ satisfies the hypothesis of the theorem since $H$ is a subcollection of $G$ and since, if $x \in g \in G$, then there exists $h \in H$ such that $x \in h < g$.

We claim that the basis $H$ satisfies (ii) of 2.2, for if not, then $X$ is not developable, which implies that there exists $x \in R \in t$ such that, for each $m \in N$, there is a point $q \in X$ such that $x \in h_m(q)$ and $h_m(q) - R \neq \varepsilon$.

Define a sequence $y = \{y_m\}$ requiring $y_m$ to be the first point in $\alpha$ such that $x \in h_m(y_m)$ and $h_m(y_m) - R \neq \phi$.

It is sufficient to prove that for each natural number $i$, there exists an $m > i$ such that $\overline{h_m(y_m)} \subset h_i(y_i)$, for then by (ii'), there is a natural number $N$ such that if $k > N$, then $h_k(y_k) \subset R$, a contradiction to $h_k(y_k) - R \neq \phi$. We now show that for each natural number $i$, there exists an $m > i$, such that $\overline{h_m(y_m)} \subset h_i(y_i)$.

If $i \in N$, then since $y$ converges to $x$ and $x \in h_i(y_i)$, there is a natural number $N_1 > i$ such that, if $m > N_1$, then $y_m \in [h_i(y_i) - y_i]$. There is a natural number $N_2$ such that, if $m > N_2$, then $g_{r_{(y_m)}(m-1)+1}(y_m)$ does not contain $h_i(y_i)$ (otherwise, by condition (i), each point of $h_i(y_i)$ would be a sequential limit point of $y$, and $h_i(y_i)$ contains at least two points, in particular $x$ and a point not in $R$, this is in contradiction to the fact that $X$ is $T_2$); therefore, there exists $m \in N$ such that

$$y_m \in [h_i(y_i)-y_i] \quad \text{and} \quad h_i(y_i) \cap [X-g_{r_{(y_m O}(m-1)+1}(y_m)] \neq \phi .$$

There does not exist a point $q$ in $\alpha$ such that $q$ precedes $y_i$ in $\alpha$ and $h_m(y_m) \subset h_i(q)$ (from the definition of $y_i$ and the fact that $h_m(y_m) \subset h_i(q)$ implies that $h_i(q) \cap [X-R] \neq \phi$); therefore, there is no point $q$ such that $q$ precedes $y_i$ in $\alpha$ and $y_m \in h_i(q)$ and $h_i(q) \cap [X-g_{r_{(y_m)}(m-1)+1}(y_m)] \neq \phi$, since otherwise $h_i(q)$ would contain $h_m(y_m)$ by the definition of $h_m(y_m)$. Therefore, $\overline{h_m(y_m)} \subset h_i(y_i)$ since $y_i \neq y_m$.

2.4 The next theorem is one of the earliest results in distinguishing between semimetrizable, developable, and metrizable spaces. This theorem is somewhat similar to 2.2, particularly part (i).

Consider now the following conditions A, B, and C on a sequence $\{H_i\}$ of collections of subsets of a topological space $X$ :

A. (a) for each $i$ , $H_i$ is a collection of open subsets of $X$ ,

(b) if $p$ is a point and $U$ is an open set containing $p$ , then there exists an integer $n$ such that $H_n$ contains exactly one element, $g(p)$ , associated with $p$ such that $p \in g(p) \subset U$ (cf. the function $g : N \times N \to t$ of I. 2.2),

(c) if $n$ is an integer and $\{g_i(p_i)\}$ is a sequence such that for each $i$ , $g_i(p_i)$ belongs to $H_n$ and is associated with $p_i$ , then $\bigcup_i \{p_i\}$ has no limit point in $X - \bigcup_i \{g_i(p_i)\}$ .

B. If $p \in X$ and $U$ is an open set containing $p$ , then there exists an integer $n$ such that for all $m \geq n$ , each element $g$ of $H_m$ which contains $p$ has the property that $U \subset \overline{g}$ .

C. For each $i$, the sum of closures of any subcollection of $H_i$ is closed.

Theorem [$Mc_1$].

(i) A topological space $X$ is semimetric iff $X$ satisfies condition A.

(ii) A topological space $X$ is developable iff $X$ satisfies conditions A and B.

(iii) A $T_3$ topological space $X$ is metrizable iff $X$ satisfies conditions A, B, and C.

Proof.

Part (i). Proof of this part is very similar to that of I 2.2, but we give it for the sake of completeness. First sufficiency will be shown. Define a semimetric function $d$ for $X$ as follows:

$$d(p,q) = \begin{cases} 0 & \text{if } p = q \\ \dfrac{1}{\min\{i,j\}} & \text{otherwise} \end{cases}$$

where $i$ is the least integer such that $H_i$ contains an element $g(p)$ associated with $p$ but not containing $q$, and similarly $j$ is the least integer such that $H_j$ contains an element $g(q)$ associated with $q$, but not containing $i$.

Then clearly $d$ is a semimetric function for $X$. By (a) and (b) of condition A, $X$ is first countable. It remains to show that limie points are unique. Assume that $p$ is a limit point of $M \leq X$ in the given topology. If $p$ is not a distance limit point of $M$, then there exists a sequence of points $\{p_i\}$ of $M - \{p\}$ converging to $p$ and an $n \in N$ such that $d(p,p_i) > \frac{1}{n}$ for all $i$. Therefore there exists $m \in N$ such that either (i) $p_i \notin g_m(p) \in H_m(p)$ or (ii) $p \notin g_m(p_i) \in H_m(p)$ for infinitely many values of $i$, by the construction of $d$. But since $p_i \to p$, (i) is impossible, and (ii) is impossible by (c) of condition A. By the construction of d and (b) of condition A, it readily follows that if $p$ is a distance limit point of $M$, then $p$ is an open set limit point of $M$. Therefore limit points are invariant with respect to $d$.

Necessity. If $p \in X$ and $h,k \in N$, denote by $R_{h,k}(p)$ and open set, when it exists, such that $S_{1/h}(p) \supset R_{h,k}(p) \supset S_{1/k}(p)$. Let $G_{h,k} = \{R_{h,k}(p) : p \in X\}$. There is a one-one correspondence between $N \times N$ and $N$, say by $f$. Define $H_i = G_{f(h,k)}$. Then clearly $\{H_i\}$ satisfies condition A.

Part (ii). Sufficiency. Let $G_i = \bigcup_{j \geq i} \{g_j(p) : g_j(p) \in H_j\}$. Then $\{G_i\}$ is a development for $X$. Both the refinement property and the base property is clear.

Necessity. Assume that $\{G_i\}$ is a development for $X$. Define a semimetric function on $X$ as was done in 1.2. Define $\{H_i\}$ as was done in part (i) with the additional requirement that $R_{h,k}(p)$ lies in some open set $g \in G_h$. Then it readily follows that $\{H_i\}$ satisfies conditions A and B.

Part (iii). A proof of this readily follows from the results in parts (i) and (ii) and Bing's theorem, [Bi, Theorem 4] that a $T_3$ topological space $X$ is metrizable iff there is a sequence $\{G_i\}$ such that:

(a) $G_i$ is a collection of open subsets such that the sum of the closures of any subcollection of $G_i$ is closed for all $i$, and

(b) if $p \in X$, and $p \in U$, $U$ open, then there exists $n(p,U)$ such that there exists $g \in G_{n(p,U)}$ with $p \in g$, and further, $p \in g' \in G_{n(p,U)}$ implies that $g' \subset U$.

3. In this section we will prove that every $T_2$, compact symmetrizable space or semistratifiable space is metrizable. Throughout this section, every topological space will be assumed to be $T_2$.

3.1 Theorem [N]. A compact semimetrizable space is metrizable.

Proof. By [23.1 of W], it suffices to show that $X$ is second countable, since a compact $T_2$ space is normal [W]. We will show that $X$ is second countable. If $x \in X$ and $n \in N$, there exists $V_n(x)$ open such that $x \in \overline{V_n(x)} \subset S_{1/n}(x)$ by the regularity of $X$. Consider the open cover $\beta_n = \{V_n(x) : x \in X\}$. By compactness there exists a finite subcover $C_n$ where $C_n = V_n(x_n^1), V_n(x_n^2), \cdots, V_n(x_n^{k_n})$. Let $C = \{c : c \in C_n$ for some $n\}$. Denote by $G$ the set of all finite intersections of elements of $C$. Then $C$ is clearly countable. We now show that $G$ is a base.

Assume that $G$ is not a base. Then there exists an $x \in X$ and a $U \in t$ such that there does not exist $g \leq G$ with $x \in g$ and $g-U = \phi$. But there exists $x_1^{i_1}$ such that $x \in V_1(x_1^{i_1}) \in C_1$. Let $G_1 = V_1(x_1^{i_1}) - U$. Then $G_1$ is a closed set.which we can assume is not empty. Assume that $V_j(x_j^{i_j})$ and $G_j$ have been defined for all $j < n$. Define $G_n$ in the following way: there exists $x_n^{i_n}$ such that $x \in V_n(x_n^{i_n}) \in C_n$, set $G_n = [\bigcap_{j=1}^{n} V_j(x_n^{i_n})] - U$. We can again assume that $G_n$ is not empty, and it is clear that $G_n$ is closed. Certainly $\bigcap_{\ell=1}^{k} G_l \neq \phi$, so by compactness, $\bigcap_{\ell=1}^{\infty} G_l \neq \phi$. Assume that $y \in \bigcap_{\ell=1}^{\infty} G_l$. Then $x,y \in V_l(x_l^{i_l}) \subset S_{1/l}(x_l^{i_l})$ for all $\ell$, which implies that $\{x_l^{i_l}\} \to x$ and $y$, and therefore since $X$ is $T_2$, $x = y$, a contradiction.

3.2 <u>Lemma</u>. If $x$ is a $G_\delta$ subset of a compact space $X$, then $x$ has a countable neighborhood base.

<u>Proof</u>. Assume that $x = \bigcap_{n=1}^{\infty} G_n$, where the $G_n$ are decreasing. Then by regularity, there exists, for all $n$, $C_n \in t$ such that $x \in V_n$ and $\overline{V}_n \subset G_n$. Assume that $U \in t$ and that $G_n \not\subset U$ for any $n$. Then by compactness, there exists $y \in \bigcap_{n=1}^{\infty} \overline{V}_n - U$. But $\bigcap_{n=1}^{\infty} \overline{V}_n - u \subset \bigcap_{n=1}^{\infty} \overline{V}_n \subset \bigcap_{n=1}^{\infty} G_n = x$.

3.3 <u>Theorem</u>. A semistratifiable, compact space is metrizable.

<u>Proof</u>. It suffices to show first countability since any first countable, semistratifiable space is semimetrizable. But this is easy. By II. 2.2, any point $x \in X$ is a $G_\delta$ subset of $X$. By the preceding lemma, $X$ is first countable.

3.4 <u>Theorem</u> [A]. A compact symmetrizable space $X$ is metrizable.

<u>Proof</u>. It suffices to show that $X$ is first countable. We will show that any $x \in X$ is a $G_\delta$ subset of $X$. Let $x' \in X$ and let $\gamma = \{V_\alpha\}$ be a well-ordered covering of $X - |x'|$, where $x' \not\in \overline{V}$ ($\gamma$ exists by the regularity of $X$). Define

$$D_\alpha = X - \bigcup_{\beta<\alpha} V_\beta$$

$$J_\alpha^k = \{x \in V_\alpha : d(x, X-V_\alpha) \geq \frac{1}{k}\}$$

$$P_\alpha^k = J_\alpha^k \cap D_\alpha$$

for all $k \in N$ and $\alpha$ . If $\gamma_k = \{P_\alpha^k\}$ , then clearly $\bigcup_{k=1}^{\infty} \gamma_k = X - \{x'\}$ and $\{P_\alpha^k\}$ are a disjoint system of sets. If the set of non-empty members of $\gamma_k$ is countable for all $k$ , then $\cup\{\overline{g} : g \in \gamma_n$ for some $n \in N\} = X - \{x'\}$ , which implies that $x'$ is a $G_\delta$ subset of $X$ . We now show that the set of nonempty members of $\gamma_k$ is countable for any $k$ . Assume the contrary. Choose a point $x_\alpha$ $P_\alpha^k$ and set $Q = \{x_\alpha\}$ . There exists $Q' \subset Q$ such that $Q'$ is uncountable and $d(x',Q') \geq e' > 0$ for some $e'$ . From the definition of $\gamma_k$ , if $x_{\alpha_1}$ , $x_{\alpha_2} \in Q'$ , $d(x_{\alpha_1}, x_{\alpha_2}) \geq \frac{1}{k}$ . Since $X$ is compact, $Q'$ cannot be closed in $X$ since if it were, there would be a point $x_o \in Q'$ such that $d(x_o, Q'-\{x_o\}) = 0$ , a contradiction to the inequality $d(x_o,x) \geq \frac{1}{k}$ if $x \neq x_o$ and $x \in Q'$ . Since $Q'$ is not closed, there exists $x'' \in X$ such that $d(x'',Q') = 0$ . $x'' \neq x'$ since $d(x',Q') \geq e' > 0$ . Let $\alpha'' = \min\{\alpha : x'' \in V \in \gamma\}$ . Choose points $x_{\alpha_i} \in Q'$ $(i-1,2)$ , $\alpha_1 < \alpha_2$ so that $d(x_{\alpha_i}, x'') < \frac{1}{k}$ , and $d(x_{\alpha_i}, x'') < d(x'', X-V_{\alpha''})$. Therefore $x_{\alpha_i} \in V_{\alpha''}$ and $\alpha_i \leq \alpha''$ from the definition of $\gamma_k$ . Thus $\alpha_1 \leq \alpha''$ and $x'' \in V_{\alpha_i}$ . Therefore,

$\frac{1}{k} \leq d(x_{\alpha_1}, X-V_{\alpha_1}) \leq d(x_{\alpha_1}, x'')$ , a contradiction to the choice of $x_{\alpha_1}$ .

4. In this section we will examine necessary and sufficient conditions for a topological space $X$ to be developable. The main theorem of this section is: For a Tychanoff space $X$ , the following are equivalent:

(a) $X$ is developable,

(b) $X$ is a $p$ - space and has a $\sigma$ - discrete network,

(c) $X$ is a semimetrizable $p$ - space,

(d) $X$ is a symmetrizable $p$ - space,

(e) $X$ is a semistratifiable $p$ - space,

(f) $X$ is a $w\Delta$ , semistratifiable space, and

(g) $X$ is a quasicomplete semistratifiable space.

The necessary definitions will be introduced in this section. Unless otherwise stated, in this section, every space will be assumed to be Tychonoff.

4.1 Definition. A topological space $X$ is said to be a p - space iff $X$ is completely regular and if in its Stone-Cech compactification, $\beta X$ , there exists a sequence of families $\{\gamma_n\}$ , where each $\gamma_n$ is a collection of open subsets of $\beta X$ which covers $X$ and satisfies the condition that for all

$x \in X$ , $\bigcap_n St(x,\gamma_n) \subseteq X$ . $\{\gamma_n\}$ is called a <u>pluming for X in $\beta X$</u>[1].

4.2 <u>Definition</u>. A topological space $X$ is said to be <u>$\sigma$-paracompact</u> iff for any open covering $\mathcal{U}$ of $X$ , there exists a sequence $\{\mathcal{U}_n\}$ of open covers of $X$ such that for any $x \in X$ , there is a $n(x) \in N$ and some $U \in \mathcal{U}$ with $St(x,\mathcal{U}_{n(X)}) \subseteq U$ .

4.3 <u>Lemma</u> [B&S]. If $X$ is a topological space with the property that every open cover of $X$ has a $\sigma$ - discrete refinement, then $X$ is $\sigma$ - paracompact.

<u>Proof</u>. Assume that $\mathcal{U}$ is any open cover of $X$ with a $\sigma$ - discrete refinement $\mathcal{P} = \bigcup_n \mathcal{P}_n$ , with each $\mathcal{P}_n$ a discrete collection. We may assume that the sets in $\mathcal{P}$ are closed. For each $P \in \mathcal{P}$ , let $U(P)$ = any set in $\mathcal{U}$ which contains $P$ ; $U(P)$ exists since $\mathcal{P}$ is a refinement of $\mathcal{U}$ . If $x \in P \in \mathcal{P}_n$ , define an open cover $\mathcal{U}_n$ as follows:

$$\mathcal{U}_n(x) = U(P) \cap [X - \cup\{P' : p' \in \mathcal{P}_n , x \notin P'\}] \quad .$$

If $x \in \cup\{P : P \in \mathcal{P}_n\}$ , define $\mathcal{U}_n(X)$ by

[1] In [A], Arkhazelskii defined $X$ to be p - space iff a pluming exists in any one (therefore in all, see [W], 41) of its Hausdorff compactification.

$$\mathcal{U}_n(x) = X - \cup\{P : P \in \mathcal{P}_n\} .$$

Then $\mathcal{U}_n = \{\mathcal{U}_n(x) : x \in X$ is an open cover of $X$ for each $n \in N\}$. Clearly, if $x \in P \in \mathcal{P}_n$, then $St(x,\mathcal{U}_n) \subseteq St(P,\mathcal{U}_n) \subseteq U(P)$. Therefore $X$ is $\sigma$ - paracompact.

4.4 Corollary. Any semistratifiable space (and therefore semimetric space) is $\sigma$ - paracompact.

Proof. Definition II. 2.8 and Theorem II. 2.9.

4.5 Corollary. Any topological space with a $\sigma$ - discrete network is $\sigma$ - paracompact.

Proof. Clear.

4.6 Lemma [B&S]. If $X$ is a $p$ - space and $x \in X$ has the property that $\{x\}$ is a $G_\delta$ set, then $x$ has a countable neighborhood base.

Proof. Assume that $\{\gamma_n\}$ is a pluming for $X$ in $\beta X$ and suppose that $\{x\} = \bigcap_n G_n$, where each $G_n$ is an open subset in $X$. There exists a $G'_n$ open in $\beta X$ for all $n$ such that $G_n = G'_n \cap X$. Therefore $\{x\} = [\bigcap_n G'_n] \cap [\cap St(x,\gamma_n)]$, which implies that $\{x\}$ is a $G_\delta$ set in $\beta X$. Since $\beta X$ is compact and $\{x\}$ is a $G_\delta$ set in $\beta X$, $x$ must have a countable base in $\beta X$, which implies that $x$ has a countable base in $X$ (take the restriction of the base in $\beta X$ to $X$ ).

4.7 Corollary. A semistratifiable $p$ - space is semimetrizaable.

Proof. Clear from II. 2.2 and above.

4.8 Lemma [B&S]. A symmetrizable $p$ - space $X$ is semimetrizable.

Proof. It is enough to show that $X$ is first countable, therefore by 4.6, it is sufficient to show that every one-point set $\{x\}$ is a $G_\delta$ set in $\beta X$. There exists a pluming, say $\{\gamma_n\}$, for $X$ in $\beta X$. If $n$ $N$, there exists an open neighborhood, $U_n(x)$, of $x$ in $\beta X$ such that $\overline{U_n(x)} \subseteq St(x,\gamma_n)$. Let $C = \bigcap_n \overline{U_n(x)} \subseteq \bigcap_n St(x,\gamma_n) \subseteq X$. Then since closed subsets of compact spaces are compact, $C$ is compact in $X$. Since closed subspaces of symmetrizable spaces are symmetrizable and by the preceding lemma, compact symmetrizable spaces are metrizable, $C$ is metrizable. Therefore there exists a sequence $\{N_n(x)\}$ of open subsets of $\beta X$ such that $\{C \cap N_n(x)\}$ is a neighborhood base at $x$, relative to $C$. Then clearly $\{x\} = [\bigcap_n U_n(x)] \cap [\bigcap_n N_n(x)]$.

4.9 Lemma [B&S]. A $p$ - space with a $\sigma$ - discrete network is semimetrizable.

Proof. For all $x \in X$, it is clear that $\{x\}$ is a $G_\delta$ set since $X$ has a $\sigma$ - discrete network. By 4.6 and III. 1.7, $X$ is semimetrizable.

4.10 Lemma [B&S]. A space with a development $\{\mathcal{U}_n\}$ has a $\sigma$ - discrete network.

Proof. As we saw before, a developable space is semimetrizable, therefore each $\mathcal{U}_n$ has a $\sigma$- discrete closed refinement $\beta_n$ by II. 2.10. Then $\bigcup_{n=1}^{\infty} \beta_n$ is clearly a $\sigma$ - discrete network for $X$.

4.11 Lemma [B&S]. A symmetrizable p - space is developable.

Proof. $X$ is semimetrizable by 4.8, so let $d$ be a semimetric for $X$. $X$ is also $\sigma$- paracompact by 2.4. $X$ also has a pluming, say $\{\gamma_n\}$, in $\beta X$. Let $S_n'(x)$ be a neighborhood in $\beta X$ such that $S_n(x) = S_n'(x) \cap X$, where $S_n(x)$ is taken with respect to $d$. For all $x \in X$, let $U_n(x)$ be a neighborhood of $x$ in $\beta X$ such that $\overline{U_n(x)} \subseteq S_n'(x)$ and such that the family $\{\overline{U_n(x)} : x \in X\}$ refines $\gamma_n$. Define $\mathcal{U}(n) = \{U_n(x) \cap X : x \in X\}$. By the $\sigma$ - paracompactness of $X$, there exists for each $n \in N$ a sequence $\{\mathcal{U}_m(n)\}_{m=1}^{\infty}$ of open covers of $X$ such that for all $x \in X$, there is a $m(x) \in N$ and $U \in \mathcal{U}(n)$ with $St(x, \mathcal{U}_{m(x)}(n)) \subseteq U$. We may assume that $\mathcal{U}_{m+1}(n)$ refines $\mathcal{U}_m(n)$ for each $m \in N$. For each $n \in N$, define $\beta_n$ to be an open cover of $X$ such that $\beta_n$ refines each $\mathcal{U}_s(t)$ for $s \leq n$, $t \leq n$ and $\beta_{n+1}$ refines $\beta_n$. We claim that $\{\beta_n\}$ is a development for $X$. Let $x \in X$ and $k \in N$ be fixed. Then there is some $x_k \in X$ and $n_k \in N$ such that $St(x,\mathcal{U}_{n_k}(k)) \subseteq U_k(x_k) \cap X$, which implies that $St(x,\beta_{n_k}) \subseteq St(x,\mathcal{U}_{n_k}(k)) \subseteq U_k(x_k) \cap X$. We may assume that $\{n_k\}_{k=1}^{\infty}$ is an increasing sequence, so that $St(x,\beta_{n_{k+1}}) \subseteq St(x,\beta_{n_k})$. Let $0$ be any open neighborhood of $x$ and let $0'$ be an open set in $\beta X$ such that $0 = 0' \cap X$. It is sufficient to prove that $\bigcap_{k=1}^{m} \overline{U_k(x_k)} \subseteq 0'$ for some $m$, since then $St(x,\beta_{n_m}) \subseteq \bigcap_{k=1}^{m} St(x,\beta_{n_k}) \subseteq [\bigcap_{k=1}^{m} U_k(x_k)] \cap X \leq 0' \cap X = 0$, and therefore, $x \in St(x,\beta_{n_m}) \leq 0$. Assume that $\bigcap_{k=1}^{m} \overline{U_k(x_k)} \not\subseteq 0'$ for any $n \in N$. Then $\{\bigcap_{k=1}^{m} \overline{U_k(x_k)} - 0'\}_{m=1}^{\infty}$ is a decreasing sequence of closed sets in $X$, and hence by the compactness of $\beta X$ has nonempty intersection. But

$$\bigcap_{k=1}^{\infty} \overline{U_k(x_k)} \subseteq [\bigcap_{k=1}^{\infty} St(x,\gamma_k)] \cap [\bigcap_{k=1}^{\infty} S_k'(x_k)]$$

$$\subseteq X \cap [\bigcap_{k=1}^{\infty} S_k'(x_k)] = \bigcap_{k=1}^{\infty} S_k(x_k) ,$$

and therefore $y \in \bigcap_{k=1}^{\infty} \overline{U_k(x_k)} - 0'$ implies that $y \in \bigcap_{k=1}^{\infty} S_k(x_k)$ and so since $X$ is semimetrizable, $x_k \to y$. But also, $x_k \to x$ and so since $X$ is Hausdorff, $x = y$, which is a contradiction.

4.12 Corollary [A]. A collectionwise normal p - space with a symmetric (or with a $\sigma$ - discrete network) is metrizable.

Proof. A collectionwise normal, developable space is metrizable [Bi].

4.13 Definition. A $T_1$ space X is said to be quasi-complete provided that there exists a sequence $\{B_n\}$ of open covers of X such that if $\{A_n\}$ is a decreasing sequence of non-empty closed subsets of X and if there exists an element $x_o$ for which for each n , there is a $b_n \in B_n$ with $A \cup \{x_o\} \subset b_n$ , then $\bigcap_n A_n \neq \phi$ .

4.14 Definition. A $T_1$ space X is said to be a $w\Delta$ - space iff there exists a sequence $\{B_n\}$ of open covers of X such that if $\{A_n\}$ is a decreasing sequence of nonempty closed subsets of X and there exists $x_o \in X$ for which $A_n \subset St(x_o, B_n)$ for all n , then $\bigcap_n A_n \neq \phi$ .

4.15 Remark. Certainly a $w\Delta$ - space is a quasicomplete space. It is still an open question whether or not the reverse implication holds true.

4.16 Lemma [$C_1$]. A topological space X is a Moore space if it is a quasicomplete semistratifiable space.

<u>Proof</u>. Let $\{B_n\}$ be a sequence satisfying the conditions of 2.14 and let $\{h_n\}$ be a sequence of functions satisfying the conditions of II. 2.2. If $x \in X$, let $b_n(x)$ be some member of $B_n(x)$ containing $x$. We define a functions $g : N \times N \to t$ inductively as follows:

$$g_1(x) = \text{some open set containing } x \text{ such that } \overline{g_1(x)} \subset b_1(x) \cap h_1(x)$$

$$g_{n+1}(x) = \text{some open set such that } g_{n+1}(x) \subset g_{n+1}(x) \cap h_{n+1}(x) \cap g_n(x) .$$

Then $g$ fulfills the conditions of 2.3.

4.17 We are now in position to prove the theorem stated in the introduction to this section.

<u>Theorem</u>. If $X$ is a $T_{3\frac{1}{2}}$ topological space, then the following are equivalent:

(a) $X$ is developable,

(b) $X$ is a $p$ - space with a $\sigma$ - discrete network,

(c) $X$ is a semimetrizable $p$ - space,

(d) $X$ is a symmetrizable $p$ - space,

(e) $X$ is a semistratifiable $p$ - space,

(f) $X$ is a $w\Delta$ - semistratifiable space, and

(g) $X$ is a quasicomplete semistratifiable space.

<u>Proof</u>.

(a) ⟶ (b) By 4.10 X has a $\sigma$- discrete network and since every developable space is p - space, we are done.

(b) ⟶ (c) This was shown in 4.9.

(c) ⟶ (d) This is clear.

(d) ⟶ (a) This was shown in 4.11.

(e) ⟶ (c) This was shown in 4.7.

(c) ⟶ (e) This is clear.

(f) ⟶ (g) This is clear.

(g) ⟶ (a) This was shown in 4.16.

(a) ⟶ (e) The fact that X is semistratifiable is clear.

That X is a $w\Delta$ space follows readily from the definition of Moore spaces and $w\Delta$ spaces.

4.18 <u>Corollary</u> [$C_1$]. A locally compact, $T_2$ semistratifiable or symmetrizable space is a Moore space.

<u>Proof</u>. A locally compact, $T_2$ space is a p - space, which follows from 19.2 of [W].

5. In this section, we introduce the concepts of a point-countable base and a strongly complete semimetric space. Developability and metrizability of semimetrizable spaces will be discussed in terms of these concepts.

5.1 Definition. A base $B$ for a topological space $X$ is said to be point countable iff each point of the space belongs to only countably many elements of $B$ .

5.2 In [$\text{Hea}_2$], Heath asserted, without proof, the following theorem:

Theorem. A $T_2$ semimetric space with a point-countable base is developable.

5.3 Definition. A semimetric space $X$ is said to be strongly complete iff there is a semimetric $d$ for $X$ with respect to which every nested sequence $M_1, M_2, \ldots$ of closed sets such that for each $n$ , there is some point $p_n$ for which $M_n \subset \{y : d(y,p_n) < \frac{1}{n}\}$ , has the property that $\bigcap_{n=1}^{\infty} M_n \neq \phi$ .

5.4 Lemma [$\text{Hea}_1$]. Assume that $X$ is a $T_2$ space with a basis $\{g_n(x)\}$ that satisfies condition (i) of 2.2 and the further condition that if $M$ is a nonincreasing sequence of of $X$ such that $M_n \subset g_n(x_n)$ for each $n$ , and there is a natural number $k$ such that $\overline{g_{n+k}(x_{n+k})} \subset g_n(x_n)$ , then $\bigcap_{n=1}^{\infty} M_n \neq \phi$ , then $\{g_n(x)\}$ satisfies (ii') of 2.3.

<u>Proof</u>. If $y \in R \in t$ and $y \in g_n(x_n)$ for all $n$ , and there exists $k$ such that $\overline{g_{n+k}(x_{n+k})} \subset g_n(x_n)$ , we must show there exists $m$ such that $\overline{g_m(x_m)} \subset R$ . Assume the contrary. This means that for all $m$ , $g_m(x_m) - R \neq \phi$ . Therefore, there exists $\chi_o \in \bigcap_{m=k+1}^{\infty} \overline{g_m(x_m)} - R \subset \bigcap_{m=1}^{\infty} g_m(x_m)$ , which implies that $\{x_m\} \rightarrow y, y_o$ . Since $X$ is $T_2$ , $y = y_o$ , but this is impossible since $y \in R$ and $y_o \notin R$ .

5.5 <u>Theorem</u>. [$\text{Hea}_1$]. A strongly complete, regular semimetric space is a Moore space.

<u>Proof</u>. Clearly a strongly complete, regular semimetric space satisfies the hypothesis of the last theorem. Therefore, $X$ has a basis which satisfies the hypothesis of theorem 2.3, and by the conclusion of that theorem, $X$ is a Moore space.

5.6 An even stronger theorem than 5.5 gives

<u>Theorem</u>. [$\text{Hea}_6$]. A strongly complete, seperable, $T_3$ , semimetric space $X$ is metrizable.

<u>Proof</u>. Since a $\chi_1$ - compact Moore space is metrizable [J] and by the last theorem, $X$ is a Moore space, it is sufficient

to prove that $X$ is $\chi_1$ - compact. Assume that $X$ is not $\chi_1$ compact. Then there exists an uncountable subset $M$ without any accumulation points.

Assume that $H$ is a countable dense subset of $X$. We will define inductively a decreasing sequence $\{M_i\}$ as follows:

there exists some $h_1 \in H$ such that $M_1 = \{x : x \in M$ and $d(x,h_1) < 1\}$ is uncountable since $H$ is countable and for every point $m \in M$, $S_1(m)$ must contain a point of $H$. Similarly, for each $n > 1$, there exists some point $h_n \in H$ such that $M_n = \{x : x \in M_{n-1}, d(x,h_n) << \frac{1}{n}\}$ is uncountable. If $\cap M_n$ contains a point $p$ (and $\cap M_i$ can contain at most one point, since $X$ is $T_2$ ), then for each $n$, $d(h_n,p) < \frac{1}{n}$. If $M_n$ not closed for some $n$, then $M_n$ has an accumulaion point, so therefore does $M$, in contradiction to the assumption that $X$ is $\chi_1$ - compact. Thus for each $n$, $M_n - \{p\} = M_n - \bigcap_{i=1}^{\infty} M_n \neq \phi$ and $M_n - \{p\}$ is closed, for if not, then $p$ is an accumulation point of $M_n$, and thus of $M$. Thus, $(M_n - \{p\}) \subset S_{1/n}(h_n)$ and $\bigcap_n (M_n - \{p\}) = \phi$, a contradiction to the assumption that $X$ is strongly complete.

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