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On the Square of AT-Free Graphs

by

Jonathan Roy Backer ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Master of Science**.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On the Square of AT-Free Graphs** submitted by Jonathan Roy Backer in partial fulfillment of the requirements for the degree of **Master of Science**.

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Abstract

An asteroidal triple is an independent set of three vertices such that each pair is joined by some path that does not pass through the neighbourhood of the third. A graph is AT-free if it does not contain any asteroidal triple. The class of AT-free graphs contains various subclasses that each manifest some linearity that has been exploited algorithmically.

Given an AT-free graph, two sweeps of a lexicographic breadth-first search (LBFS) will find a dominating pair and transitively orient the complement of the square. We show how a 2-sweep LBFS orientation of the complement of the square orders every minimal triangulation of an AT-free graph. We also show that if there is a transitive orientation of the complement, then there is such an orientation that agrees with a given 2-sweep LBFS orientation of the complement of the square. Motivated by these results, we show that a graph is cocomparability if and only if there is a particular partial order of its minimal separators.

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Preliminaries

1.1 Introduction

All permutation, interval, trapezoid, and cocomparability graphs are AT-free. These graph classes have models (i.e. partial orders or geometric intersection models or both) with a linearity that is exploited in efficient algorithms to restricted problems that are NP-complete in general. Although AT-free graphs have linear properties, such as dominating pairs, there is no known characterization of this graph class with an explicit linearity.

A recent algorithm [5] which uses 2 sweeps of a lexicographic breadth-first search (LBFS) to find a dominating pair of an AT-free graph also transitively orients the complement of the square [3]. Our motivating question is, "What is ordered by this transitive orientation of the complement of the square?" For AT-free graphs, we show that a given 2-sweep LBFS orientation of the complement of the square orders every minimal triangulation. For cocomparability graphs, we show that some transitive orientation of the complement will agree with a given 2-sweep LBFS orientation of the complement of the square. We develop a new characterization of cocomparability graphs to interpret these results: a graph is cocomparability if and only if it has a particular partial order of its minimal separators. Finally, we conjecture that 2-sweep LBFS orientations of the square generalize transitive orientations of the complement the same way that AT-free graphs generalize cocomparability graphs.

In Chapter 1, we review basic definitions and we survey relevant graph classes. The focus of Chapter 2 is how minimal separators relate to chordal graphs, minimal triangulations, and AT-free graphs. In Chapter 3, we show how transitive orientations of the complement and 2-sweep LBFS orientations of the complement of the square have the same properties with respect to minimal separators. The topic of Chapter 4 is a further examination of the relationship between transitive orientations of the complement and 2-sweep LBFS orientations of the complement of the square. Chapter 5 concludes this thesis with an analysis our main result and a proof of our new characterization of cocomparability graphs.

1.2 Terminology

We shall restrict our attention to simple, undirected, finite graphs. *Simple* graphs are those with no loops or multiple edges. The vertex set and edge set of a graph G are denoted $V(G)$ and $E(G)$ respectively. We shall write $uv \in E(G)$ whenever $u, v \in V(G)$ are adjacent in G. Two sets $S, T \subseteq V(G)$ are *completely adjacent* in G, denoted $S \times T \subseteq E(G)$, if $s \neq t$ implies $st \in E(G)$, for $s \in S$ and $t \in T$. Conversely, S and T are *completely non-adjacent* in G, denoted $S \times T \cap E(G) = \emptyset$, if $st \notin E(G)$, for $s \in S$ and $t \in T$. The set of vertices adjacent to a vertex u, denoted $N(u)$, is the *neighbourhood* of u. We say that a set $S \subseteq V(G)$ is *close to* a vertex u if $S \subseteq N(u)$.

A *u, v-path* in G is an ordered sequence of vertices $\pi = \langle u = x_0, x_1, \ldots, x_k = v \rangle$ such that consecutive vertices are adjacent and no vertex is repeated. The *length* of π is the number of edges connecting consecutive vertices. A *chord* of π is an edge between non-consecutive vertices of π. We say that π is an *induced path* or a *chordless path* if π has no chords in G. Note that every shortest u, v-path is induced. Let π_1 be a u, v-path and π_2 be a v, w-path. Some subsequence of the *concatenation* of π_1 and π_2 is a u, w-path. We denote such a resulting u, w-path as $\pi_1 \cdot \pi_2$. Every path that we refer to is induced unless otherwise noted.

Two vertices u and v are *connected* if there is some u, v-path in G. We say that a subset of $V(G)$ is *connected* if it is pairwise connected. A *component* of G is an inclusion maximal connected subset of $V(G)$, i.e. the subset is not properly contained in any other connected subset of $V(G)$. Whenever $V(G)$ is a component we say that G is *connected.* The *distance* between two vertices u and v, denoted $d_G(u, v)$, is the length of the shortest u, v-path or ∞ if no u, v-path exists. The *diameter* of a connected graph G, denoted $\operatorname{diam}(G)$, is the maximum of all distances between pairs of vertices in G.

A *cycle* of G is a path $C = \langle x_0, x_1, \ldots, x_k \rangle$ such that $x_0x_k \in E(G)$. Alternatively, the vertices of C as ordered form ring such that consecutive vertices are adjacent. With this definition, the *length* of C is the number of edges connecting consecutive vertices, and a *chord* of C is an edge between non-consecutive vertices. A cycle is *long* if it has a length greater than 3, and it is *induced* or *chordless* if it has no chords. C_k is the graph consisting of a chordless cycle of length k. Every cycle that we refer to is chordless unless otherwise noted.

For $S \subseteq V(G)$, the *subgraph* of G *induced* by S, denoted $G[S]$, is the result of restricting G to S. More precisely, $V(G[S]) = S$ and $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. A *graph class* is a set of all graphs with a particular property. We say the that the class $\mathcal{C}$ is *hereditary* if for $G \in \mathcal{C}$, every induced subgraph of G is in $\mathcal{C}$. We shall restrict our attention to hereditary graph classes. If H is an induced subgraph of G then the difference $G \setminus H$ is $G[V(G) \setminus V(H)]$. Analogously, if $S \subseteq V(G)$ then the difference $G \setminus S$ is $G[V(G) \setminus S]$.

Sometimes we shall relax the condition that a subgraph be induced. A graph H is a *subgraph* (resp. *supergraph*) of G whenever $V(H) = V(G)$ and $E(H) \subseteq E(G)$ (resp. $E(H) \supseteq E(G)$). H is a *proper* subgraph or supergraph when the inclusion between the edge sets is strict.

1.3 Graph Classes

What follows is a brief survey of graph classes discussed in this thesis.

1.3.1 Chordal Graphs

Definition. A graph is *chordal* or *triangulated* if every long cycle has a chord.

We say that a graph G is *complete* if its vertices are pairwise adjacent. A *clique* is a subset of vertices that induces a complete subgraph. Every graph has a chordal supergraph, which is called a *triangulation*, because the complete graph with n vertices is chordal. A triangulation H of G is *minimal* if every proper subgraph of H is not a triangulation of G.

Dirac discovered an alternate characterization of chordal graphs in terms of minimal separators that is essential to our understanding of chordal graphs.

Definition. Let G be a connected graph and S be a subset of $V(G)$. We say that S is a *u, v-separator* if u and v are in different components of $G \setminus S$. We call S a *separator* if there are some $u, v \in V$ such that S is a u, v-separator. Trivially, if $uv \notin E(G)$ then $N(u)$ is a u, v-separator close to u. We say that S is a *minimal u, v-separator* if no proper subset of S separates u and v. We call S a *minimal separator* if there are some $u, v \in V$ such that S is a minimal u, v-separator. A minimal separator is close to a vertex u if it is contained in the neighbourhood of u. Finally, we say that minimal separator S is a *$\subseteq$-minimal separator* if every proper subset of S is not a minimal separator.

Observation. *Let G be a connected graph. If $uv \notin E(G)$ then the minimal u, v-separator close to u is unique, for $u, v \in V(G)$.*

Proof. Suppose not and let $S, T \subset N(u)$ be two different minimal u, v-separators close to u. Consider for contradiction $s \in S \setminus T$. There is some u, v-path π such that $\pi \cap S = \{s\}$ because S is minimal. $\pi \cap T \neq \emptyset$ because T separates u and v. Let $t \in \pi \cap T$ be closest to v along π. Now $t \notin S$ because $s \neq t$ and $\pi \cap S = \{s\}$. If s is closer to u along π than t then S is not a

u, v-separator because $tu \in E(G)$. If t is closer to u along π than s then T is not a u, v-separator because $su \in E(G)$. $\square$

Notation. Δ_G is the set of all minimal separators of a connected graph G.

Example. Consider the graph in Figure 1.1. $\{b, c, e\}$ is a d, f-separator but not a minimal d, f-separator because $\{b, e\}$ is also a d, f-separator. Even though $\{b, e\}$ is a minimal separator, it is not $\subseteq$-minimal because $\{b\}$ is a minimal separator as well.

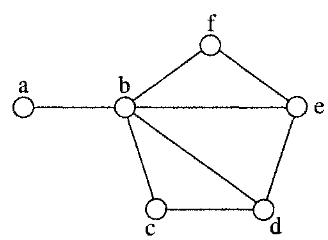

Figure 1.1: A chordal graph with minimal separators $\{b\}$, $\{b, d\}$, and $\{b, e\}$.

Theorem 1.3.1. *[6] A connected graph G is chordal if and only if every minimal separator is a clique.*

1.3.2 Interval Graphs

A graph has an *intersection representation* if we can assign sets to vertices such that two vertices are adjacent if and only their corresponding sets have a non-empty intersection. Every graph has an intersection representation: associate with each vertex the set of all the edges incident with it; then two vertices are adjacent if and only if their corresponding sets have a non-empty intersection. An intersection representation of a graph is also called an *intersection model.*

Definition. A graph is an *interval* graph if it can be represented by the intersection of closed intervals of the real number line.

This graph class has been studied extensively because of its immediate applicability to problems. For example, consider the problem of assigning classrooms to classes such that concurrent classes are assigned different classrooms. The period of time when a class occurs maps directly to a closed interval of real numbers. Two classes are adjacent in the corresponding interval graph whenever their times conflict. If different colours represent different classrooms, the classroom assignment problem reduces to colouring the vertices in the corresponding interval graph such that adjacent vertices have different colours. For any interval graph, there is an efficient algorithm to find such a colouring using the least number of colours. For arbitrary graphs, finding such a colouring using the least number of colours is NP-complete.

Observation 1.3.2. *[10] Every interval graph is chordal.*

Proof. Let G be an interval graph and $\mathcal{I}$ be an intersection model of closed intervals corresponding to G. For $v \in V(G)$, I_v denotes the interval of $\mathcal{I}$ assigned to v. Consider for contradiction a long chordless cycle $\langle v_0, v_1, \ldots, v_{k+2}, v_{k+3} \rangle$ in G. As $v_0 v_{k+2} \notin E(G)$ I_{v_0} is either to the left or to the right of $I_{v_{k+2}}$ on the real number line. We can reflect every interval of $\mathcal{I}$ about a point on the real line and still have a valid intersection representation. So without loss of generality assume that I_{v_0} is to the left of $I_{v_{k+2}}$. Let $I_{v_0} = [a, b]$ and $I_{v_{k+2}} = [x, y]$. If $I_{v_1} \cap [b, x] = \emptyset$ then a straightforward inductive argument shows that $[x, y]$ is left of $[a, b]$, a contradiction. Moreover, $I_{v_{k+3}} \supseteq [b, x]$ because $v_0 v_{k+3} \in E(G)$ and $v_{k+2} v_{k+3} \in E(G)$. Therefore $v_1 v_{k+3} \in E(G)$, which contradicts that the cycle is chordless. $\square$

Some chordal graphs are not interval as demonstrated by the chordal graphs in Figure 1.2.

1.3.3 Comparability and Cocomparability Graphs

An *orientation* of a graph is an assignment of direction to all of its edges. Orientations will always be denoted with an arrow such as $\Rightarrow$, $\rightarrow$, and $\leadsto$.

We only allow edges to be oriented one way: for $uv \in E(G)$, either $u \to v$ or $v \to u$ but not both. An orientation $\to$ is *transitive* if $u \to v$ and $v \to w$ implies $u \to w$, for every $u, v, w \in V(G)$.

Definition. A graph is a *comparability* graph if it has a transitive orientation.

The *complement* of a graph, denoted $\overline{G}$, has the same vertex set as G but $u \neq v$ are adjacent in the complement if and only if they are not adjacent in G. So $E(G) \cap E(\overline{G}) = \emptyset$ and $G \cup \overline{G} = (V(G), E(G) \cup E(\overline{G}))$ is complete.

Definition. A graph is a *cocomparability* graph if it is the complement of a comparability graph.

We are interested in cocomparability graphs because they generalize interval graphs.

Observation 1.3.3. *[8] Every interval graph is a cocomparability graph.*

Proof. Let G be an interval graph. We can create a transitive orientation of $\overline{G}$ by taking an interval model $\mathcal{I}$ and directing $u \to v$ if and only if I_u is to the left of I_v. □

By Observations 1.3.2 and 1.3.3, chordal and cocomparability are necessary conditions for a graph to be interval. Gilmore and Hoffman proved that chordal and cocomparability are also sufficient conditions for a graph to be interval.

Theorem 1.3.4. *[9] A graph is interval if and only if it is chordal and cocomparability.*

Let G be a cocomparability graph and $\to$ be a transitive orientation of $\overline{G}$. A *topological sort* of $\to$ is a total order $\prec$ of $V(G)$ such that $u \to v$ implies $u \prec v$. Consider any topological sort $\prec$ of $\to$. If $u \prec v \prec w$ and $uv, vw \notin E(G)$ then $uw \notin E(G)$ by the transitivity of $\to$. We shall call this type of order a *cocomparability order* because a graph is cocomparability if and only if it has this type of vertex order. A cocomparability order arranges the components of a separator:

Observation 1.3.5. *Let G be a connected graph and $\prec$ be a cocomparability order of G. For every separator S in G, the vertices of a component of $G \setminus S$ occur contiguously in $\prec$ restricted to $V(G) \setminus S$.*

In [7], Gallai provides an alternative characterization of comparability graphs in terms of wreaths.

Definition. If a vertex u is not adjacent to any vertex of a path π then u *misses* π . Alternatively, if u is adjacent to a vertex of π then u *intercepts* π. Two edges $uv, vw \in E(G)$ are *tied* if there is a u,w-path missing v in $\overline{G}$. When uv and vw are tied either $u \to v$ and $w \to v$ or $v \to u$ and $v \to w$, for every transitive orientation $\to$ of G. A *wreath* is a cycle where each pair of consecutive edges along the cycle are tied.

A simple example of a wreath is a long chordless cycle. Consider a graph G and a transitive orientation $\to$ of G. Let $W = \langle v_1, v_2, \ldots v_k \rangle$ be a wreath of G. Label each v_i either *in* (resp. *out*) if the tied consecutive edges of the wreath are directed towards (resp. away) from v_i. No two consecutive vertices of W are both labelled in or both labelled out. Hence, the length of W must be even. Gallai showed that this condition was not only necessary of comparability graphs but also sufficient.

Theorem 1.3.6. *[7] A graph is a comparability graph if and only if it has no odd wreath.*

1.3.4 AT-Free Graphs

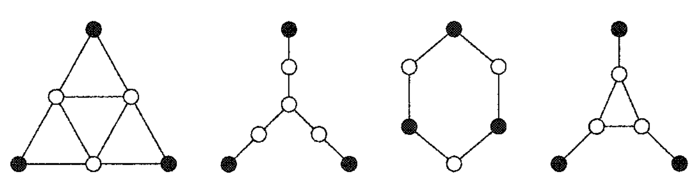

Figure 1.2: Some asteroidal triples. The solid vertices denote the triples.

Definition. A set of pairwise non-adjacent vertices is called an *independent set*. An *asteroidal triple* is a independent set of three vertices such that there is a path between any two missing the third.

Definition. A graph is *AT-free* if it has no asteroidal triple.

Notice that an asteroidal triple is a 3-wreath in the complement and vice versa. So every cocomparability graph is AT-free by Theorem 1.3.6. Thus, chordal and AT-free are necessary conditions for a graph to be interval by Theorem 1.3.4. Lekkerkerker and Boland proved that chordal and AT-free are also sufficient conditions for a graph to be interval.

Theorem 1.3.7. *[11] A graph is interval if and only if it is chordal and AT-free.*

Not only do AT-free graphs characterize interval graphs, but interval graphs also characterize AT-free graphs:

Theorem. *[1] A graph is AT-free if and only if every minimal triangulation is an interval graph.*

We shall now explain our earlier statement that one linear property of AT-free graphs is the existence of dominating pairs.

Definition. Let G be a connected graph. A set $S \subset V(G)$ is called a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S. A pair of vertices (u, v) is a *dominating pair* in G if every u, v-path is a dominating set. Occasionally we will want to talk about the set of vertices *dominated by a pair*, denoted $D_G(u, v)$. By $D_G(u, v)$ we mean the vertices that intercept every u, v-path in G. So (u, v) is a dominating pair whenever $D_G(u, v) = V(G)$.

In Chapter 2, we will prove the following:

Observation. *Let G be a connected graph. The pair (x, y) is a dominating pair in G if and only if (x, y) is a dominating pair in every minimal triangulation.*

If G is a connected cocomparability graph and $\prec$ is a cocomparability order then the first and last vertex of $\prec$ are never in the same component of $G \setminus S$ by Observation 1.3.5. So intuitively dominating pairs correspond to extremities of an order. This next result supports this interpretation.

Theorem 1.3.8. *[4] Every AT-free graph has a dominating pair (u, v) such that the distance between u and v is equal to the diameter of the graph.*

Chapter 2

Minimal Separators

In this chapter, we define the crossing and parallel relations between minimal separators. Then we use these relations to describe chordal graphs, minimal triangulations, and AT-free graphs. Sections 2.1 and 2.2 closely follow Chapter 3 of [1].

2.1 Minimal Separators

Definition. Let S be a separator in a connected graph G and C be a component of $G \setminus S$. Then C is a *full component* of $G \setminus S$ if every vertex of S is adjacent to some vertex of C.

Notation. Let G be a connected graph and S be a separator in G. Then $C_G(S)$ denotes the set of components of $G \setminus S$, $C_G^{\bullet}(S)$ denotes the set of full components of $G \setminus S$, and $C_G^v(S)$ denotes the component of $G \setminus S$ containing the vertex v.

The next property of minimal separators follows directly from the definitions.

Lemma 2.1.1. *Let S be a separator in a connected graph G and C, D be two components of $G \setminus S$. Then the following are equivalent:*

(i) C and D are full components of $G \setminus S$.

(ii) *For every* $c \in C$ *and* $d \in D$, S *is a minimal* c, d*-separator.*

(iii) *There exist* $c \in C$ *and* $d \in D$ *such that* S *is a minimal* c, d*-separator.*

By this next result, if a minimal separator S has a component that is not full, then S is not $\subseteq$-minimal.

Lemma 2.1.2. *[1] Let* S *be a minimal separator in a connected graph* G. *If* C *is a component of* $G \setminus S$ *that is not full, then* $S_C = \{s \in S : s$ *is adjacent to some vertex of* $C\} \in \Delta_G$ *and* C *is a full component of* $G \setminus S_C$.

Example. Consider the connected graph G in Figure 2.1 and the minimal separator $\{b, e\}$. The components of $G \setminus \{b, e\}$ are $\{a\}$, $\{f\}$, and $\{c, d\}$. Clearly $\{c, d\}$ is a full component of $G \setminus \{b, e\}$ but $\{a\}$ is not. By Lemmas 2.1.1 and 2.1.2, $S_{\{a\}} = \{b\}$ is a minimal u, v-separator for $u \in \{a\}$ and $v \in \{c, d\}$.

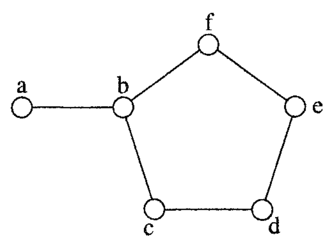

Figure 2.1: $\Delta_G = \{\{b\}, \{b, d\}, \{b, e\}, \{c, e\}, \{c, f\}, \{d, f\}\}$

Now we introduce and examine the crossing and parallel relations.

Definition. Let $S, T \in \Delta_G$. We say that S *crosses* T, denoted $S \# T$, if there are two components C, D of $G \setminus T$ such that S intersects both C and D.

Lemma 2.1.3. *[1] Let* $S, T \in \Delta_G$. *If* $S \# T$ *then* $T \# S$.

A deeper characterization of the crossing relation is:

Lemma 2.1.4. *[1] Let* $S, T \in \Delta_G$. *The following are equivalent:*

(i) $S \# T$.

(ii) $S \cap C \neq \emptyset$ *for every full component* C *of* $G \setminus T$.

(iii) There are some $s, s' \in S$ *such that* T *is a minimal* s, s'*-separator.*

Definition. Let $S, T \in \Delta_G$. We say that S is *parallel* to T, denoted $S \parallel T$, if S does not cross T. By our definition of crossing, this is equivalent to the existence of a component C of $G \setminus T$ such that $S \subseteq T \cup C$.

The next Lemma is an immediate consequence of the previous definition of parallel and Lemma 2.1.3.

Lemma 2.1.5. *[1] Let* $S, T \in \Delta_G$.

(i) If $S \parallel T$ *then* $T \parallel S$.

(ii) If $S \subset T$ *then* $S \parallel T$.

Close separators are always parallel in this sense:

Observation 2.1.6. *Let* $A \in \Delta_G$ *be the minimal* a, b*-separator close to* a *and* $B \in \Delta_G$ *be the minimal* a, b*-separator close to* b. *Then* $A \parallel B$.

Proof. Suppose for contradiction that $A \# B$. By Lemma 2.1.4, A intersects the full component of $G \setminus B$ containing b. So consider $s_a \in A \cap C^b_G(B)$. Note that $as_a \in E(G)$ because A is close to a, and $a \notin B$ because B is close to b. Let π be a s_a, b-path avoiding B. Then π can be extended to make a a, b-path avoiding B, which contradicts that B is a a, b-separator. □

In Figure 2.1 we see that $\{b, e\} \parallel \{b, d\}$ and $\{b, d\} \parallel \{d, f\}$ but $\{b, e\} \# \{d, f\}$. This example shows that $\parallel$ is not necessarily transitive. However, $\parallel$ has the following restricted form of transitivity.

Lemma 2.1.7. *[1] Let* $S, T, U \in \Delta_G$. *If there are two different components* C_S, C_U *of* $G \setminus T$ *such that* $S \subseteq T \cup C_S$ *and* $U \subseteq T \cup C_U$ *then* $S \parallel U$.

When separators are parallel we can say the following about their full components.

Lemma 2.1.8. *Let $S, T \in \Delta_G$ be parallel minimal separators in G. Let C be the component of $G \setminus S$ such that $T \subseteq S \cup C$ and let D be the component of $G \setminus T$ such that $S \subseteq T \cup D$. If $T \not\subseteq S$ then the full components of $G \setminus T$ other than D are contained in C.*

Proof. Let E be a full component of $G \setminus T$ other than D. Consider $t \in T \setminus S$. For every $e \in E$ there is a e, t-path in $G[\{t\} \cup E]$ that does not pass through S. Thus $e \in C$ and consequently $E \subseteq C$. □

Consider two parallel separators S and T. Let C and D be different full components of $G \setminus S$ and $G \setminus T$ such that $T \cap C = \emptyset$ and $S \cap D = \emptyset$. Then S and T are u, v-separators, for each $u \in C$ and $v \in D$. This leads naturally to the next notion of strongly parallel separators.

Definition. Let $S, T \in \Delta_G$ be parallel. We say that S and T are *strongly parallel* if they are both minimal a, b-separators for some $a, b \in V(G)$.

When two separators are strongly parallel a, b-separators the components containing a and b have an inclusion ordering as follows.

Note: The next two Lemmas are a more precise statement of Lemma 3.1.10 in [1]. Although our proof is essentially the same as Parra's proof, we have included it because we changed the statement of the result.

Lemma 2.1.9. *[1] Let $S, T \in \Delta_G$. If S and T are strongly parallel minimal a, b-separators then either $C^a_G(S) \subseteq C^a_G(T)$ and $C^b_G(S) \supseteq C^b_G(T)$ or $C^a_G(S) \supseteq C^a_G(T)$ and $C^b_G(S) \subseteq C^b_G(T)$. The inclusions are strict if $S \neq T$.*

Proof. If $S = T$ then the Lemma is trivially true so suppose that $S \neq T$. Both S and T are minimal a, b-separators so $S \not\subset T$ and $S \not\supset T$. By Lemma 2.1.1, $C^a_G(S)$, $C^a_G(T)$, $C^b_G(S)$, and $C^b_G(T)$ are full components. As $S \parallel T$, either $T \cap C^a_G(S) = \emptyset$ or $T \cap C^b_G(S) = \emptyset$. We will consider the two cases in turn.

1. $T \cap C^a_G(S) = \emptyset$

Clearly $C_G^a(S) \subseteq C_G^a(T)$ because every path in $G \setminus S$ between vertices of $C_G^a(S)$ does not pass through T. Moreover, every vertex $s \in S \setminus T$ is in $C_G^a(T)$ because there is a a, s-path in $G[\{s\} \cup C_G^a(S)]$ avoiding T. Therefore $C_G^a(S) \subset C_G^a(T)$ and $S \cap C_G^a(T) \neq \emptyset$. Consequently $S \cap C_G^b(T) = \emptyset$. By the same arguments we find that $C_G^b(S) \supset C_G^b(T)$.

2. $T \cap C_G^b(S) = \emptyset$

   By the previous lines of reasoning we see that $S \cap C_G^b(T) \neq \emptyset$ and $C_G^b(S) \subset C_G^b(T)$. As a result we find that $T \cap C_G^a(S) \neq \emptyset$ and $C_G^a(S) \supset C_G^a(T)$.

□

Alternatively, if two separators have full components with the same type of inclusion ordering then they are strongly parallel.

Lemma 2.1.10. *[1] Let $S, T \in \Delta_G$. If there are $C_1, C_2 \in C_G^\bullet(S)$ and $D_1, D_2, \in C_G^\bullet(T)$ such that $C_1 \subsetneq D_1$ and $C_2 \supsetneq D_2$ then S and T are strongly parallel minimal a, b-separators for every $a \in C_1$ and $b \in D_2$.*

With respect to close separators we know that:

Lemma 2.1.11. *Let $A, B \in \Delta_G$ be strongly parallel minimal a, b-separators. If A is close to a in G then $C_G^a(A) \subseteq C_G^a(B)$, $C_G^b(A) \supseteq C_G^b(B)$, and $B \subseteq A \cup C_G^b(A)$.*

Proof. If $A = B$ then this Lemma is trivially true. So suppose $A \neq B$ and consider $s_a \in A \setminus B$. By Lemma 2.1.9, either

$$C_H^a(A) \subset C_H^a(B) \text{ and } C_H^b(A) \supset C_H^b(B)$$

or

$$C_H^a(A) \supset C_H^a(B) \text{ and } C_H^b(A) \subset C_H^b(B)$$

Since A is close to a in G, $as_a \in E(G)$. Then $s_a \in B \cup C_G^a(B)$. But $s_a \notin B$ so $s_a \in C_G^a(B) \setminus C_G^a(A)$ and the former must be true. Therefore, $C_G^a(A) \subseteq C_G^a(B)$ and $C_G^b(A) \supseteq C_H^b(B)$.

Consider $s_b \in B \setminus A$. There is a s_b, b-path in $G[\{s_b\} \cup C_G^b(B)]$. This is also a path in $G[\{s_b\} \cup C_G^b(A)]$. So $s_b \in C_G^b(A)$. Therefore, $B \subseteq A \cup C_G^b(A)$. □

2.2 Chordal Graphs and Minimal Triangulations

Now we shall characterize chordal graphs and minimal triangulations in terms of the parallel relation. By definition, a minimal separator S is not a clique if and only if there exist $u, v \in S$ such that u and v are not adjacent. Whenever such a u and v exists, every minimal u, v-separator crosses S by Lemma 2.1.4.

Lemma 2.2.1. *[1] Let $S \in \Delta_G$. Then S is parallel to every minimal separator in G if and only if S is clique in G.*

Theorem 1.3.1, Dirac's characterization of chordal graphs, can now be rewritten as:

Corollary 2.2.2. *[1] A graph G is chordal if and only if the minimal separators of G are pairwise parallel.*

By definition, a triangulation must add chords to every induced long cycle of the original graph. What follows relates this observation to minimal separators.

Observation 2.2.3. *For $S \in \Delta_G$, every non-adjacent pair of vertices in S are end points of a chord of some long induced cycle. Conversely, the end points of every chord of a long induced cycle are contained in some minimal separator.*

Proof. Let $S \in \Delta_G$ and consider $x, y \in S$ such that $xy \notin E(G)$. Let C, D be full components of $G \setminus S$. Let π_C be a shortest x, y-path in $G[\{x, y\} \cup C]$.

Similarly, let π_D be a shortest x, y-path contained in $G[\{x, y\} \cup D]$. Then $\pi_C \cup \pi_D$ induces a $C_{k\geq 4}$ with the chord xy.

Alternatively, let $xy \in E(G)$ be the chord of some induced $C_{k\geq 4}$. Denote the vertices of this cycle C. Now $\{x, y\}$ separates $G[C]$. Let u, v be vertices of different components of $G[C] \setminus \{x, y\}$. Notice that $(V(G) \setminus C) \cup \{x, y\}$ is a u, v-separator. Let $S \subseteq (V(G) \setminus C) \cup \{x, y\}$ be a minimal u, v-separator. Clearly $u, v \in S$. □

Corollary 2.2.2 and Lemma 2.1.4 suggest that the edges of a minimal triangulation either complete the separators or destroy them by joining full components.

Notation. For $S \in \Delta_G$, let G_S denote the graph obtained by joining every non-adjacent pair in S. For $\mathcal{S} \subseteq \Delta_G$, let $G_{\mathcal{S}}$ denote the graph obtained by joining every non-adjacent pair of every separator of $\mathcal{S}$.

We may be able to triangulate a graph by making some of its minimal separators cliques. When we do this, what minimal separators of the original graph are preserved?

Lemma 2.2.4. *[1] Let $\mathcal{S} \subseteq \Delta_G$ and $H = G_{\mathcal{S}}$. Then a minimal a, b-separator T in G that is parallel in G to every $S \in \mathcal{S}$ is also a minimal a, b-separator in the intermediate graph H', for every $G \subseteq H' \subseteq H$.*

The converse question is: what separators of the resulting graph are separators in the original graph?

Lemma 2.2.5. *[1] Let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separators in G and $H = G_{\mathcal{S}}$. Then a minimal a, b-separator T in H is also a minimal a, b-separator in G and $T \parallel S$ in G, for every $S \in \mathcal{S}$.*

Not only is every minimal separator of H a minimal separator of G, but the components of a minimal separator in H are the same as the components of that minimal separator in G.

Corollary 2.2.6. *Let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separators in G and $H = G_{\mathcal{S}}$. For every $T \in \Delta_H$ it holds that $T \in \Delta_G$, $C_H(T) = C_G(T)$, and $C_H^{\bullet}(T) = C_G^{\bullet}(T)$. Moreover, for $C \in C_H(T)$ the vertices of T that are adjacent to some vertex in C are the same in H as in G.*

Proof. This Corollary follows from Lemmas 2.2.5 and 2.1.1 for minimal separators of size 1. So suppose that this Corollary holds for minimal separators of H with size $\leq k$.

Consider $T \in \Delta_H$ such that $|T| = k+1$. Then $T \in \Delta_G$ and $T \parallel S$ in G, for $S \in \mathcal{S}$ by Lemma 2.2.5. Clearly, $C_G^a(T) \subseteq C_H^a(T)$ because any u,v-path in $G[C_G^a(T)]$ connects u and v in H. Suppose that $C_G^a(T) \neq C_H^a(T)$ and consider $z \in C_H^a(T) \setminus C_G^a(T)$. Let π be a a,z-path in $H[C_H^a(T)]$. Let v be the vertex closest to a along π that is not in $C_G^a(T)$. Let u be the vertex closest to a along π adjacent to v. Then $u \in C_G^a(T)$ and $uv \notin E(G)$. So consider $S \in \mathcal{S}$ such that $u,v \in S$. By definition $S \# T$ in G, a contradiction. Hence $C_G(T) = C_H(T)$. Lemmas 2.1.1 and 2.2.5 imply that $C_G^{\bullet}(T) = C_H^{\bullet}(T)$.

Consider a component C of $G \setminus T$ that is not full. By Lemma 2.1.2, $T_C = \{s \in T : s \text{ is adjacent to some vertex of } C \text{ in } H\}$ is a minimal separator of H with C as a full component. Note that $|T_C| < |T|$. By our induction hypothesis, C is a full component of $G \setminus T_C$. □

This next characterization of minimal triangulations follows directly from the other results in this section.

Theorem 2.2.7. *[1] Let $G = (V,E)$ be a graph.*

(i) Let $\mathcal{S}$ be a maximal set of pairwise parallel minimal separators in G. Then $H = G_{\mathcal{S}}$ is a minimal triangulation of G, and $\Delta_H = \mathcal{S}$.

(ii) Let H be a minimal triangulation of G. Then Δ_H is a maximal set of pairwise parallel minimal separators in G, and $H = G_{\Delta_H}$.

2.3 AT-Free Graphs

AT-free graphs can also be characterized by minimal separators and the parallel relation.

Theorem 2.3.1. *[1] Let G be a connected graph. Then G is AT-free if and only if among any three pairwise strongly parallel minimal separators in G, there is a separator S such that the other two intersect different components of $G \setminus S$.*

Corollary 2.3.2. *Let G be a connected AT-free graph and $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel separators of G. Then $G_{\mathcal{S}}$ is AT-free.*

Proof. Follows directly from Theorem 2.3.1 and Corollary 2.2.6. □

Theorem 2.3.3. *[1] Let G be a connected graph. Then G is AT-free if and only if every minimal triangulation H is interval.*

Dominating pairs can be characterized in terms of minimal separators:

Lemma 2.3.4. *Let G be a connected graph. Then (u, v) is a dominating pair in G if and only if u and v are never in the same component of $G \setminus S$, for $S \in \Delta_G$.*

Proof. Suppose that (u, v) is not a dominating pair in G. Then there is some u, v-path π missing a vertex w. Let S be the minimal u, w-separator close to w. Then u and v are in the same component of $G \setminus S$ because π does not pass through S.

Suppose that u and v are in the same component C of $G \setminus S$ for some minimal separator S. Let w be an element of a different component. Then every u, v-path in $G[C]$ misses w. □

Observation. *Let G be a connected graph. The pair (x, y) is a dominating pair in G if and only if (x, y) is a dominating pair in every minimal triangulation.*

Proof. Let G be AT-free and let H be a minimal triangulation of G. Then $H = G_{\Delta_H}$ and $\Delta_H \subseteq \Delta_G$ by Theorem 2.2.7. By Lemma 2.3.4, if (x, y) is a dominating pair in G then x and y are never in the same component of $G \setminus S$, for $S \in \Delta_G$. By Corollary 2.2.6, x and y are never in the same component of $H \setminus S$, for $S \in \Delta_H$. Therefore, (x, y) is a dominating pair in H by Lemma 2.3.4.

Suppose that (x, y) is not a dominating pair in G. There there is some $S \in \Delta_G$ such that x and y are in the same component of $G \setminus S$. Some maximal set $\mathcal{S} \subseteq \Delta_G$ of pairwise parallel separators contains S. By Theorem 2.2.7, $H = G_{\mathcal{S}}$ is a minimal triangulation of G. Corollary 2.2.6 indicates that $S \in \Delta_H$ and that x and y are in the same component of $H \setminus S$. Therefore, (x, y) is not a dominating pair in H by Lemma 2.3.4. □

Later we will need this in a more general setting:

Corollary 2.3.5. *Let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separators. If (x, y) is a dominating pair of G then (x, y) is a dominating pair of $G_{\mathcal{S}}$.*

Proof. Follows immediately from Corollary 2.2.6 and Lemma 2.3.4. □

We also know the following about the structure of a minimal separator separating a dominating pair.

Lemma 2.3.6. *[1] If (x, y) is a dominating pair in a graph G, then every minimal x, y-separator in G is $\subseteq$-minimal.*

Chapter 3

The Square of AT-free Graphs

Corneil et al. published a linear time algorithm to find a dominating pair in an AT-free graph that uses two sweeps of a lexicographic breadth-first search (LBFS) [5]. Chang et al. noticed that the order produced by this algorithm is a cocomparability order of the square of the graph [3]. In this chapter, we start with a survey of these results. Then we show how a transitive orientation of the complement orders a component of a minimal separator relative to that separator and the other components of that separator. We conclude by showing that a 2-sweep LBFS cocomparability order of the complement of the square has the same ordering properties. This similarity is a cornerstone of the next two chapters.

3.1 Lexicographic Breadth-First Search

The lexicographic breadth-first search (LBFS) [13] algorithm is a variant of the standard breadth-first search where ties are broken with a labelling scheme. Given an arbitrary connected G graph and a start vertex x as input, the LBFS algorithm in Figure 3.1 returns a total vertex order σ as output.

LBFS orders are completely characterized as follows:

Lemma 3.1.1. *[2] Every LBFS order σ of graph G has the following property:*

P1: Let $a, b, c \in V(G)$ be vertices such that $\sigma(a) < \sigma(b) < \sigma(c)$. If $ac \in E(G)$

```
PROCEDURE LBFS(G,x).
{ Input: a connected graph G = (V, E) and a distinguished vertex x of G
  Output: a numbering σ of the vertices of G }
begin
    label(x) ← |V|;
    for each vertex v in V \ {x} do
        label(v) ← ∅;
    for i ← |V| downto 1 do
    begin
        pick an unnumbered vertex v with the (lexicographically) largest label;
        σ(v) ← i;
        for each unnumbered vertex u in N(v) do
            append i to label(u);
    end;
end; {LBFS}
```

Figure 3.1: The LBFS algorithm as presented by Corneil et al. [5].

and $bc \notin E(G)$ *then there exists* $d \in V(G)$ *such that* $\sigma(c) < \sigma(d)$, $ad \notin E(G)$, *and* $bd \in E(G)$.

Moreover, every total order σ *with property P1 is the result of some LBFS.*

Definition. In a graph G, vertices u, v are said to be *unrelated with respect to* a vertex x if and only if $u \notin D_G(v, x)$ and $v \notin D_G(u, x)$. If u, v are unrelated with respect to x and $x \notin D_G(u, v)$, then $\{u, v, x\}$ is an asteroidal triple.

In proving the correctness of their 2-sweep LBFS algorithm, Corneil et al. demonstrate two important properties of LBFS orders of connected AT-free graphs.

Theorem 3.1.2. *[5] Let* G *be a connected AT-free graph,* σ *be a LBFS order of* $V(G)$, *and* x *be a vertex of* G. *Then the subgraph of* G *induced by* $\{v \in V(G) : \sigma(v) \geq \sigma(x)\}$ *does not contain any vertices that are unrelated with respect to* x.

So the vertex that is labelled 1 by an LBFS order of a connected AT-free has no unrelated vertices with respect to it by Theorem 3.1.2.

Theorem 3.1.3. *[5] Let G be a connected AT-free graph, and consider $x \in V(G)$ such that G contains no vertices that are unrelated with respect to x. Let σ be the result of LBFS(G, x). Then $\sigma(u) < \sigma(v)$ implies that $v \in D_G(u, x)$, for all $u, v \in V(G)$.*

Let x be the vertex labelled 1 by a LBFS order of a connected AT-free G. If y is the vertex labelled 1 by an order produced by LBFS(G,x) then (x, y) is a dominating pair in G by Theorem 3.1.3.

Definition. We call the vertex order resulting from two such LBFS sweeps of a connected AT-free graph a *2-sweep LBFS order.*

3.2 The Square of AT-Free Graphs

Definition. G^k is the result of taking the k^{th} power of the adjacency matrix of G. Equivalently, $V(G^k) = V(G)$ and $u, v \in E(G^k)$ if and only if $d_G(u, v) \leq k$.

Theorem 3.2.1. *[12] If G^k is AT-free then G^{k+1} is AT-free.*

If G is AT-free then the largest induced cycle of G is a C_5. Even though every long cycle of G has a chord in G^2, G^2 is not necessarily interval because, as illustrated by Figure 3.2, new long cycles may occur.

Example. In Figure 3.2 the graph on the left is AT-free and the graph on the right is its square. The solid vertices on the right are a C_4 of the square.

As we shall soon see, the square of a connected AT-free graph is a cocomparability graph [3]. Before proving this, we shall first prove a more general result.

Lemma 3.2.2. *Let G be a connected AT-free graph and σ be a 2-sweep LBFS order of G. Consider $u, v, w \in V(G)$ such that $\sigma(u) < \sigma(v) < \sigma(w)$. If there is a u, w-path missing v, then v and w have a common neighbour.*

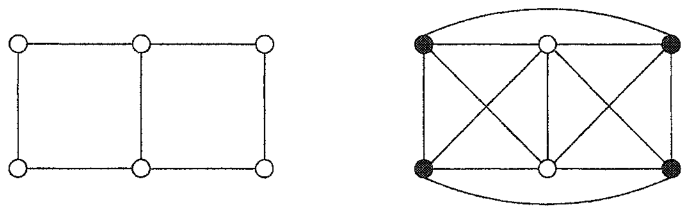

Figure 3.2: Demonstrates that the square of an AT-free graph need not be interval.

Proof. Let z be the last vertex of σ and let π be a u, w-path missing v. By the existence of π, $uv \notin E(G)$ and $vw \notin E(G)$. $v \in D_G(u, z)$ by Theorem 3.1.3. If $w = z$ then π is u, z-path missing v, a contradiction. Note that $v \in D_G(w, z)$ because π misses v.

Let x be rightmost neighbour of w with respect to σ. v is not adjacent to any vertex greater than x in σ by Lemma 3.1.1. Suppose that $vx \notin E(G)$. Then we can extend a x, z-path using vertices greater than x by w to create a w, z-path missing v, a contradiction. Thus $vx \in E(G)$ and $vw \in E(G^2)$. □

Now we are ready to show that the square of an AT-free graph is cocomparability.

Theorem 3.2.3. *[3] Let G be a connected AT-free free graph and σ be a 2-sweep LBFS order of G. Then σ is a cocomparability order of G^2.*

Proof. Consider for contradiction vertices $u, v, w \in V(G)$ such that $\sigma(u) < \sigma(v) < \sigma(w)$, $uv \notin E(G^2)$, $vw \notin E(G^2)$, but $uw \in E(G^2)$. Trivially $uv \notin E(G)$ and $vw \notin E(G)$. If $uw \in E(G)$ then v and w have a common neighbour by Lemma 3.2.2, a contradiction. So let d be a common neighbour of u and w in G. Then $\langle u, d, w \rangle$ is a u, w-path missing v. Again by Lemma 3.2.2, it follows that v and w have a common neighbour, a contradiction. □

Definition. Let G be a connected AT-free graph and σ be a 2-sweep LBFS order of G. Let $\Rightarrow$ be the orientation of $\overline{G^2}$ such that $u \Rightarrow v$ implies $\sigma(u) < \sigma(v)$. By Theorem 3.2.3, $\Rightarrow$ is transitive. We say that $\Rightarrow$ is the orientation of $\overline{G^2}$ *resulting from* σ.

3.3 Minimal Separators and Cocomparability Graphs

A transitive orientation of the complement orders a component of a minimal separator relative to that separator and the other components of that separator in the following sense:

Lemma 3.3.1. *Let G be a connected cocomparability graph and S be a minimal separator of G. Every transitive orientation $\rightarrow$ of $\overline{G}$ has the following properties:*

(1) $\rightarrow$ orients all edges of $\overline{G}$ between every two components of $G \setminus S$ in one direction.

(2) $\rightarrow$ orients all edges of $\overline{G}$ between a component of $G \setminus S$ and S in one direction.

(3) If $\rightarrow$ orients an edge of $\overline{G}$ from a component C of $G \setminus S$ to S then $\rightarrow$ orients all edges of $\overline{G}$ between C and different full components away from C.

(4) If $\rightarrow$ orients an edge of $\overline{G}$ from S to a component C of $G \setminus S$ then $\rightarrow$ orients all edges of $\overline{G}$ between C and different full components towards C.

Proof.

(1) Let C, D be components of $G \setminus S$. Consider for contradiction $c_1, c_2 \in C$ and $d_1, d_2 \in D$ such that $c_1 \rightarrow d_1$ and $d_2 \rightarrow c_2$. Let π_C be a c_1, c_2-path in $G[C]$. Then d_1 misses π_C. Walking from c_1 to c_2 along π_C we see that $\rightarrow$ orients every vertex of π_C towards d_1 in $\overline{G}$ because $c_1 \rightarrow d_1$. Thus $c_2 \rightarrow d_1$. Let π_D be a d_1, d_2 path in $G[D]$. By the same argument we find that $c_2 \rightarrow d_2$, a contradiction.

(2) Let C be a component of $G \setminus S$. Consider for contradiction $s_1, s_2 \in S$ and $c_1, c_2 \in C$ such that $s_1 \to c_1$ and $c_2 \to s_2$. Let D be a full component of $G \setminus S$ such that $D \neq C$. As D is a full component there exist $d_1, d_2 \in D$ such that $d_1 s_1 \in E(G)$ and $d_2 s_2 \in E(G)$.

Clearly $c_1 d_1 \notin E(G)$ and $c_2 d_2 \notin E(G)$. By (1), either $d_1 \to c_1$ and $d_2 \to c_2$ or $c_1 \to d_1$ and $c_2 \to d_2$. Suppose the former. Then $d_2 \to c_2 \to s_2$, contradicting that $d_2 s_2 \in E(G)$. Suppose the latter. Then $s_1 \to c_1 \to d_1$, contradicting that $d_1 s_1 \in E(G)$.

(3) Let C be a component of $G \setminus S$ with $c_s \in C$ and $s \in S$ such that $c_s \to s$. Let D be a full component of $G \setminus S$ such that $D \neq C$. Consider for contradiction $d_c \in D$ and $c_d \in C$ such that $d_c \to c_d$. As D is a full component of $G \setminus S$, there exists $d_s \in D$ such that $d_s s \in E(G)$. So $d_s \to c_s$ because $d_c \to c_d$ by (1). Then $d_s \to c_s \to s$, contradicting that $d_s s \in E(G)$.

(4) Let C be a component of $G \setminus S$ with $c_s \in C$ and $s \in S$ such that $s \to c_s$. Let D be a full component of $G \setminus S$ such that $D \neq C$. Consider for contradiction $c_d \in C$ and $d_c \in D$ such that $c_d \to d_c$. There exists $d_s \in D$ such that $d_s s \in E(G)$ as D is a full component. By (1), $c_s \to d_s$ because $c_d \to d_c$. So $s \to c_s \to d_s$, contradicting that $d_s s \in E(G)$.

□

Example. The graph G on the left in Figure 3.3 demonstrates that the order specified in (3) and (4) of Lemma 3.3.1 must be restricted to full components. To the right is a transitive orientation $\to$ of $\overline{G}$. Now $\{f\}$ is a component of $G \setminus \{a, d\}$ such that $\to$ directs $\{f\}$ to $\{a, d\}$. But $\to$ directs $\{e\}$ to $\{f\}$. So the statement of (3) in Lemma 3.3.1 must be restricted to full components. When $\to$ is reversed, it follows that (4) must be restricted to full components as well.

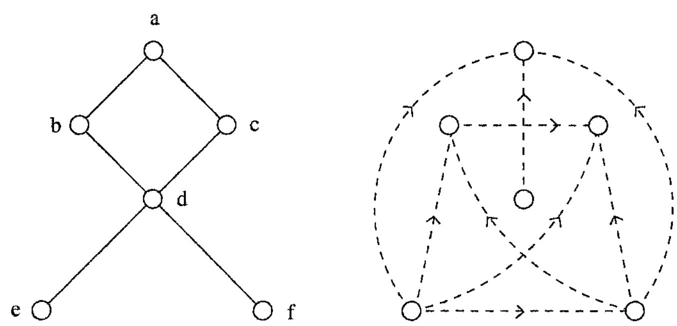

Figure 3.3: Both (3) and (4) of Lemma 3.3.1 are sharp.

Example. The dashed arrows in Figure 3.4 are an orientation of the complement that is not transitive. The solid vertex is the only minimal separator. Clearly, this orientation satisfies Lemma 3.3.1.

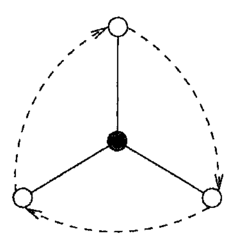

Figure 3.4: An orientation satisfying Lemma 3.3.1 need not be transitive.

Given that the properties of Lemma 3.3.1 are not sufficient for an orientation to be transitive, why is it relevant? If G is a connected AT-free graph, the cocomparability orientation $\Rightarrow$ of G^2 resulting from a 2-sweep LBFS order very has similar properties. But first we need the following Observation.

Observation 3.3.2. *Let G be a connected AT-free graph and S be a minimal separator of G. Then $uv \in E(G^2)$, for every $u, v \in S$.*

Proof. Let C and D be full components of $G \setminus S$. Consider for contradiction $u, v \in S$ such that $uv \notin E(G^2)$. Let π_C be a u,v-path in $G[\{u, v\} \cup C]$ and π_D be a u,v-path in $G[\{u, v\} \cup D]$. As S separates C and D, $\pi_C \cup \pi_D$ induces a chordless cycle in G. However, as $uv \notin E(G^2)$ the length of this cycle is at least 6 which contradicts that G is AT-free. $\square$

Lemma 3.3.3. *Let G be a connected AT-free graph, S be a minimal separator of G, and $\Rightarrow$ be the transitive orientation of $\overline{G^2}$ resulting from a 2-sweep LBFS order $\prec$. Every such $\Rightarrow$ has the following properties:*

(1) $\Rightarrow$ orients all edges of $\overline{G^2}$ between every two components of $G \setminus S$ in one direction.

(2) $\Rightarrow$ orients all edges of $\overline{G^2}$ between a component of $G \setminus S$ and S in one direction.

(3) If $\Rightarrow$ orients an edge of $\overline{G^2}$ from a component C of $G \setminus S$ to S then $\Rightarrow$ orients all edges of $\overline{G^2}$ between C and different full components away from C.

(4) If $\Rightarrow$ orients an edge of $\overline{G^2}$ from S to component C of $G \setminus S$ then $\Rightarrow$ orients all edges of $\overline{G^2}$ between C and different full components towards C.

Proof.

(1) Let C and D be components of $G \setminus S$. Consider for contradiction $c_1, c_2 \in C$ and $d_1, d_2 \in D$ such that $c_1 \Rightarrow d_1$ and $d_2 \Rightarrow c_2$. The remaining argument proceeds by cases:

  (a) $c_1 \prec d_2$

  Then $c_1 \prec d_2 \prec c_2$. There is a c_1, c_2-path through $G[C]$ that misses d_2. By Lemma 3.2.2, $c_2 d_2 \in E(G^2)$, a contradiction.

  (b) $d_2 \prec c_1$

  Then $d_2 \prec c_1 \prec d_1$. There is a d_2, d_1-path through $G[D]$ that misses c_1. By Lemma 3.2.2, $c_1 d_1 \in E(G^2)$, a contradiction.

(2) Let C be a component of $G \setminus S$ and $D \neq C$ be a full component of $G \setminus S$. Consider for contradiction $s_1, s_2 \in S$ and $c_1, c_2 \in C$ such that $s_1 \Rightarrow c_1$ and $c_2 \Rightarrow s_2$.

Suppose that $c_1s_2 \notin E(G^2)$. If $c_1 \Rightarrow s_2$ then $s_1 \Rightarrow c_1 \Rightarrow s_2$, contradicting Observation 3.3.2. If $s_2 \Rightarrow c_1$ then $c_2 \Rightarrow s_2 \Rightarrow c_1$. By Lemma 3.2.2, s_2 must intercept the shortest c_2, c_1 path π in $G[C]$. D is a full component so consider $d \in D$ such that $ds_2 \in E(G)$. Then $\pi \cup \{s_2, d\}$ induces an AT of G, a contradiction. So $c_1s_2 \in E(G^2)$, $c_1 \neq c_2$, and $s_1 \neq s_2$.

Suppose that $c_2s_1 \notin E(G^2)$. If $s_1 \Rightarrow c_2$ then $s_1 \Rightarrow c_2 \Rightarrow s_2$, contradicting Observation 3.3.2. If $c_2 \Rightarrow s_1$ then $c_2 \Rightarrow s_1 \Rightarrow c_1$. By Lemma 3.2.2, s_1 must intercept the shortest c_2, c_1 path π in $G[C]$. D is a full component so consider $d \in D$ adjacent to s_1. Then $\pi \cup \{s_1, d\}$ induces an AT of G, a contradiction. Thus $c_2s_1 \in E(G^2)$.

The remaining proof proceeds by cases:

(a) $c_1c_2 \in E(G)$

Then $c_2s_1 \notin E(G)$ and $c_1s_2 \notin E(G)$. If $s_1 \prec c_2$ then $s_1 \prec c_2 \prec s_2$. There is a s_1, s_2 path through $G[\{s_1, s_2\} \cup D]$ missing c_2. By Lemma 3.2.2 $c_2s_2 \in E(G^2)$, a contradiction. If $c_2 \prec s_1$ then $c_2 \prec s_1 \prec c_1$. There is a c_1, c_2-path missing s_1 because $c_1c_2 \in E(G)$. Again by Lemma 3.2.2, $s_1c_1 \in E(G^2)$, a contradiction.

(b) $c_1c_2 \notin E(G)$

Earlier we observed that $c_2s_1 \in E(G^2)$. Let π_1 be a shortest c_2, s_1-path in G. c_1 misses π_1 because $d(c_2, s_1) \leq 2$, $c_1s_1 \notin E(G^2)$, and $c_1c_2 \notin E(G)$. We also observed that $c_1s_2 \in E(G^2)$. Let π_2 be a shortest c_1, s_2-path in G. Similarly, c_2 misses π_2 because $d(c_1, s_2) \leq 2$, $c_2s_1 \notin E(G^2)$, and $c_2c_1 \notin E(G)$.

Let d be a vertex of D. Any c_1, c_2 path in C misses d. There is a d, s_1-path missing c_1 in $G[\{s_1\} \cup D]$ because D is a full component of $G \setminus S$. This can be extended by π_1 to get a d, c_2-path missing c_1. By symmetry there is a d, c_1-path missing c_2. Therefore $\{d, c_1, c_2\}$ is an AT, a contradiction.

(3) Let C be a component of $G \setminus S$ with $c_s \in C$ and $s \in S$ such that $c_s \Rightarrow s$. Consider for contradiction a full component $D \neq C$ of $G \setminus S$ with $c_d \in C$ and $d_c \in D$ such that $d_c \Rightarrow c_d$. If $c_s \prec d_c$ then $c_s \prec d_c \prec c_d$. There is a c_s, c_d-path through $G[C]$ missing D. Thus $c_d d_c \in E(G^2)$ by Lemma 3.2.2, a contradiction. If $d_c \prec c_s$ then $d_c \prec c_s \prec s$. There is a d_c, s-path missing c_s in $G[\{s\} \cup D]$ because D is a full component of $G \setminus S$. Thus $c_s s \in E(G^2)$ by Lemma 3.2.2, a contradiction.

(4) Let C be a component of $G \setminus S$ with $c_s \in C$ and $s \in S$ such that $s \Rightarrow c_s$. Consider for contradiction a full component $D \neq C$ of $G \setminus S$ with $c_d \in C$ and $d_c \in D$ such that $c_d \Rightarrow d_c$.

Suppose that no vertex of C is adjacent to s in G. If $s \prec c_d$ then $s \prec c_d \prec d_c$. Moreover, D is a full component of $G \setminus S$ so there is some d_c, s-path in $G[\{s\} \cup D]$ missing c_d. So by Lemma 3.2.2 $c_d d_c \in E(G^2)$, a contradiction. If $c_d \prec s$ then $c_d \prec s \prec c_s$. By our assumption, any c_d, c_s-path in $G[C]$ misses s. Again by Lemma 3.2.2 $c_s s \in E(G^2)$, a contradiction. Therefore some vertex of C is adjacent to s in G.

Now we will show that $c_s d_c \in E(G^2)$. If $d_c \prec c_s$ then $c_s d_c \in E(G^2)$ by (1) because $c_d \Rightarrow d_c$. If $c_s \prec d_c$ then $s \prec c_s \prec d_c$. There is a s, d_c-path missing c_s in $G[D \cup \{s\}]$ because D is a full component of $G \setminus S$. Thus $c_s d_c \in E(G^2)$ and $c_d \neq c_d$ by Lemma 3.2.2.

Clearly $c_s d_c \notin E(G)$ because they are in different components of $G \setminus S$. So c_s and d_c have a common neighbour $t \in S$. $c_d t \notin E(G)$ because $c_d d_c \notin E(G^2)$. Similarly, $st \notin E(G)$ because $c_s s \notin E(G^2)$. If $d_c s \notin E(G)$ then $\{c_s, s, d_c\}$ is an AT of G because C and D have a vertex adjacent to s in G. Hence $d_c s \in E(G)$ and consequently $c_d s \notin E(G)$.

If $s \prec c_d$ then $s \prec c_d \prec d_c$. Furthermore, $\langle s, d_c \rangle$ is a s, d_c-path missing c_d. By Lemma 3.2.2, $c_d d_c \in E(G^2)$, a contradiction. So $c_d \prec s$ and $c_d \prec s \prec c_s$. As $sc_s \notin E(G^2)$, s intercepts every c_d, c_s-path in G by

Lemma 3.2.2. Thus $c_dc_s \notin E(G)$. Let π be a shortest c_d, c_s-path in $G[C]$. We claim that $\{c_s, c_d, d_c\}$ is an AT of G because:

- π is a c_s, c_d-path missing d_c
- $\langle c_s, t, d_c\rangle$ is a c_s, d_c-path missing c_d
- Let x be the vertex closest to c_d along π that adjacent to s in G. Since $c_ss \notin E(G^2)$, $c_sx \notin E(G)$. So there is a c_d, s-path missing c_s. There is also a s, d_c-path missing c_s in $G[\{s\} \cup D]$. Hence there is a c_d, d-path missing c_s.

□

Definition. Let G be a connected graph. We will call a transitive orientation $\Rightarrow$ of $\overline{G^2}$ with the properties of Lemma 3.3.3 an *extendable orientation* of $\overline{G^2}$. We call this type of orientation extendable because we shall extend it into other orientations in Chapter 4.

Corollary 3.3.4. *Every connected AT-free graph has an extendable orientation of the complement of its square.*

Observation 3.3.5. *The reversal of every extendable orientation is again an extendable orientation.*

Proof. By the symmetry of the properties of Lemma 3.3.3 with respect to reversal of orientation. □

Example. The graph given by the solid edges in Figure 3.5 is connected and AT-free. The dashed arrows are a transitive orientation of the complement of its square. The solid vertices are a minimal separator that the two arrows cross in opposite directions. Therefore this orientation is not extendable.

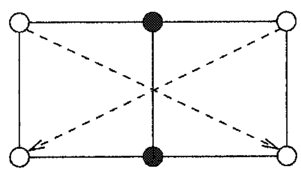

Figure 3.5: Not every transitive orientation of the complement of the square is extendable.

Chapter 4

Extendable Orientations

Lemmas 3.3.1 and 3.3.3 show that an extendable orientation is a weak form of a transitive orientation of the complement with respect to ordering components and minimal separators. In this chapter, we identify instances where an extendable orientation can be extended into a transitive orientation of the complement.

Definition. Let G be a connected graph, $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separators in G, and $H = G_{\mathcal{S}}$. We call H a *partial minimal triangulation* (PMT) of G because there is always some minimal triangulation H' of G such that $E(H) \subseteq E(H')$ by Theorem 2.2.7. As evidenced by Corollary 2.2.6, the separator structure of a PMT is a restriction of the separator structure of the original graph. If a PMT is a cocomparability graph then we say that it is a *cocomparability partial minimal triangulation* (CPMT).

A PMT of a given graph G is a subgraph of the square of G.

Observation 4.0.6. *Let G be a connected AT-free graph and H be a PMT of G. Then $E(H) \subseteq E(G^2)$.*

Proof. Let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separators such that $H = G_{\mathcal{S}}$. Consider $uv \in E(H)$. If $uv \in E(G)$ then $uv \in E(G^2)$. If $uv \notin E(G)$ then there is some $S \in \mathcal{S}$ such that $u, v \in S$. By Corollary 2.2.6, S is a minimal separator of G. It follows that $uv \in E(G^2)$ by Observation 3.3.2. □

If G is a connected AT-free graph, this observation hints that an extendable orientation of $\overline{G^2}$ is orienting something common to every CPMT of G. The main result of this chapter is the Extendable Orientation Theorem:

Theorem (Extendable Orientation Theorem). *Let G be a connected AT-free graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. If H is a CPMT of G then there exists a transitive orientation $\rightarrow$ of $\overline{H}$ such that if $u \Rightarrow v$ then $u \rightarrow v$.*

Observation. *Let G be a connected AT-free graph. Then any minimal triangulation of G is a CPMT of G.*

Proof. Follows immediately from Theorems 1.3.4 and 2.2.7. □

As every minimal triangulation of an AT-free graph is a CPMT, one consequence of the Extendable Orientation Theorem is that an extendable orientation orders every minimal triangulation:

Corollary 4.0.7. *Let G be a connected AT-free graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. If H is a minimal triangulation of G then there exists a transitive orientation $\rightarrow$ of $\overline{H}$ such that if $u \Rightarrow v$ then $u \rightarrow v$.*

When we restrict our attention to cocomparability graphs, every PMT is a CPMT:

Observation 4.0.8. *Let G be a connected cocomparability graph and H be a PMT of G. Then H is a CPMT of G. In particular, any transitive orientation $\rightarrow$ of $\overline{G}$ restricted to $\overline{H}$ is still transitive.*

Proof. Let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel minimal separator in G such that $H = G_{\mathcal{S}}$. Consider for contradiction $u, v, w \in V(H)$ such that $u \rightarrow v$, $v \rightarrow w$, but $uw \in E(H)$. Now $\rightarrow$ is a transitive orientation of $\overline{G}$ so $uw \in E(H) \setminus E(G)$. Thus there is some $S \in \mathcal{S}$ such that $u, w \in S$ but $v \notin S$. This contradicts (2) of Lemma 3.3.1 which states that $\rightarrow$ orients all edges of $E(\overline{G})$ from a component of $G \setminus S$ to S the same way. □

As a result, an extendable orientation orients every PMT of a cocomparability graph:

Corollary 4.0.9. *Let G be a connected cocomparability graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. Then there exists a transitive orientation $\rightarrow$ of $\overline{G}$ such that if $u \Rightarrow v$ then $u \rightarrow v$.*

Example. The opposite of the Extendable Orientation Theorem is not true in general as illustrated by Figure 4.1. The solid edges on the left represent an AT-free graph G. When we add the dashed edge we get minimal triangulation H, which as we observed earlier is a CPMT of G. On the right is a transitive orientation of $\overline{H}$. The solid edges on the right are the edges of $\overline{G^2}$. This orientation of $\overline{G^2}$ is not transitive and therefore not extendable.

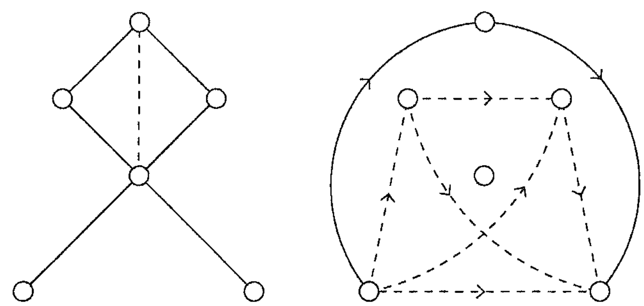

Figure 4.1: The contrary of the Extendable Orientation Theorem is not true in general.

4.1 End Pairs

The proof of the Extendable Orientation Theorem is inductive on the size of the graph. At the beginning of this section, we show that a particular type of dominating pair always exists in a connected AT-free graph. Then we identify how certain parts of the graph which correspond to this dominating pair have the same separator structure and the same complement of the square as the original graph. Finally, in the proof of the Extendable Orientation Theorem, we combine transitive orientations of the complement of various parts of the graph to get a transitive orientation of the complement of the whole.

Lemma 4.1.1. *Let G be a connected AT-free graph and H be a PMT of G. Suppose that (x, y) is a dominating pair in G such that $xy \notin E(H)$. Let S be the minimal x, y-separator close to x in H. Then there exists a vertex $a \in C_H^x(S)$ such that:*

(i) (a, y) *is a dominating pair in* G

(ii) the minimal a, y-separator A close to a in H is completely adjacent to $C_H^a(A)$ in H

(iii) $a = x$ *or* $ax \in E(H)$

Proof. Let $\mathcal{A} = \{a \in C_H^x(S) : (a, y)$ is a dominating pair in $G\}$. Suppose for contradiction that:

A1: for every $a \in \mathcal{A}$ the minimal a, y-separator A close to a in H has a vertex that is not adjacent to some vertex of $C_H^a(A)$.

We will create sequences $\langle a_1, a_2, \ldots, a_k \rangle$ and $\langle A_1, A_2, \ldots, A_k \rangle$ of arbitrary length k such that:

(a) $a_i \in \mathcal{A}$ and A_i is the minimal a_i, y-separator close to a_i in H

(b) if $i < j$ then $C_H^{a_i}(A_i) \supset C_H^{a_j}(A_j)$.

Note that $\langle a_1 = x \rangle$ and $\langle A_1 = S \rangle$ are sequences of length 1 with properties (a) and (b).

Suppose that we have sequences of length $k \geq 1$ with the above properties. Then (a_k, y) is a dominating pair in G and $C_H^{a_k}(A_k) \subseteq C_H^x(S)$. By assumption A1, there is a vertex $u_1 \in C_H^{a_k}(A_k)$ and $s \in A_k$ such that $su_1 \notin E(H)$.

Claim 4.1. *There is a vertex $u \in C_H^{a_k}(A_k)$ such that (u, y) is a dominating pair in G and $su \notin E(H)$.*

Proof. Assume for contradiction that:

A2: for every $u \in C_H^{a_k}(A_k)$, if $su \notin E(H)$ then (u, y) is not a dominating pair in G

We will construct a non-repeating sequence $\langle u_1, u_2, \ldots, u_l \rangle$ of vertices of $C_H^{a_k}(A_k)$ with arbitrary length l having these properties:

(1) $su_i \notin E(H)$

(2) if $i \neq j$ then $u_i u_j \notin E(G)$

(3) if $i < j$ then $u_i \in D_G(u_j, y)$

(4) $u_{i+1} \notin D_G(u_i, y)$.

Note that $\langle u_1 \rangle$ has these properties.

Claim 4.2. *If $\langle u_1, u_2, \ldots, u_l \rangle$ satisfies properties (1) to (4) then there exists $u_{l+1} \in C_H^{a_k}(A_k)$ such that $u_l \neq u_{l+1}$, $su_{l+1} \notin E(H)$, $u_l u_{l+1} \notin E(G)$, $u_l \in D_G(u_{l+1}, y)$, and $u_{l+1} \notin D_G(u_l, y)$.*

Proof. Note that $C_G^{a_k}(A_k) = C_H^{a_k}(A_k)$, $C_G^y(A_k) = C_H^y(A_k)$, and $C_G^{a_k}(A_k), C_G^y(A_k) \in C_G^\bullet(A_k)$ by Lemma 2.1.1 and Corollary 2.2.6.

From property (1) and assumption A2 it follows that (u_l, y) is not a dominating pair in G. So consider $u_{l+1} \notin D_G(u_l, y)$ and let π be a u_l, y-path missing u_{l+1} in G. Suppose that $u_{l+1} \notin C_H^{a_k}(A_k)$. If $u_{l+1} \in A_k$ then $\{u_l, u_{l+1}, y\}$ is an AT in G, a contradiction. Hence, $u_{l+1} \notin A_k$. Since $u_{l+1} \notin C_G^{a_k}(A_k) \cup A_k$, there is some a_k, u_l-path π' missing u_{l+1} in G. Then $\pi' \cdot \pi$ is a a_k, y-path in G missing u_{l+1}, a contradiction because $a_k \in \mathcal{A}$. Therefore $u_{l+1} \in C_H^{a_k}(A_k)$. By choice of u_{l+1}, we know that $u_l \neq u_{l+1}$ and $u_l u_{l+1} \notin E(G)$. Moreover, $u_l \in D_G(u_{l+1}, y)$ because otherwise $\{u_l, u_{l+1}, y\}$ is an AT in G.

Suppose for contradiction that $su_{l+1} \in E(H)$. We will show that there is some u_{l+1}, s-path π in G missing u_l. If $su_{l+1} \in E(G)$ then this is obviously true. Let $\mathcal{S} \subseteq \Delta_G$ be set pairwise parallel minimal separators in G such that $H = G_{\mathcal{S}}$. If $su_{l+1} \notin E(G)$ then there is some minimal separator $T \in \mathcal{S}$ such that $s, u_{l+1} \in T$. As $su_l \notin E(H)$ it follows that $u_l \notin T$. Let C be a full component of $G \setminus T$ that does not contain u_l. There is some u_{l+1}, s-path missing u_l in $G[\{u_{l+1}, s\} \cup C]$. Since $C_G^y(A_k) \in C_G^{\bullet}(A_k)$, there is a s, y-path π' in $G[\{s\} \cup C_G^y(A_k)]$ missing u_l. Thus $\pi \cdot \pi'$ is a u_{l+1}, y-path in G missing u_l, which contradicts that $u_l \in D_G(u_{l+1}, y)$. □

Consider u_2 with the properties of Claim 4.2. Then the sequence $\langle u_1, u_2 \rangle$ satisfies properties (1) to (4). Suppose that we have a sequence $\langle u_1, u_2, \ldots, u_l \rangle$ with properties (1) to (4) where $l \geq 2$. We will show how to extend it to a sequence $\langle u_1, u_2, \ldots, u_{l+1} \rangle$ and maintain these properties.

By Claim 4.2, consider $u_{l+1} \in C_H^{a_k}(A_k)$ such that $u_l \neq u_{l+1}$, $su_{l+1} \notin E(H)$, $u_l u_{l+1} \notin E(G)$, $u_l \in D_G(u_{l+1}, y)$, and $u_{l+1} \notin D_G(u_l, y)$.

Suppose for contradiction that our extended sequence repeats. Let i be the largest $i < l$ such that $u_i = u_{l+1}$. By property (3), we get $u_{l+1} = u_i \in D_G(u_l, y)$, a contradiction.

Suppose for contradiction that our extended sequence violates property (2) or (3). Let $i < l$ be the largest such that $u_i u_{l+1} \in E(G)$ or $u_i \notin D_G(u_{l+1}, y)$. Now $u_{i+1} u_{l+1} \notin E(G)$ and $u_{i+1} \in D_G(u_{l+1}, y)$ because $i < l$. By property (4), $u_{i+1} \notin D_G(u_i, y)$. If $u_i u_{l+1} \in E(G)$ then we can extend any u_i, y-path missing u_{i+1} in G to get a u_{l+1}, y-path missing u_{i+1} in G, a contradiction. So $u_i u_{l+1} \notin E(G)$ and $u_i \notin D_G(u_{l+1}, y)$. Let π be a u_{l+1}, y-path in G missing u_i. As $u_{i+1} \in D_G(u_{l+1}, y)$, u_{i+1} must intercept π in G. Because $u_i u_{i+1} \notin E(G)$ we get an u_{i+1}, y-path that misses u_i in G. But this contradicts property (3).

Therefore $\langle u_1, u_2, \ldots, u_{l+1} \rangle$ maintains properties (1) to (4). By induction, there is a non-repeating sequence of size $|C_H^{a_k}(A_k)| + 1$ contained in $C_H^{a_k}(A_k)$, a contradiction. □

By Claim 4.1, consider $a_{k+1} \in C_H^{a_k}(A_k)$ such that (a_{k+1}, y) is a dominating pair in G and $sa_{k+1} \notin E(H)$. Let A_{k+1} be the minimal a_{k+1}, y-separator close to a_{k+1} in H. Both A_k and A_{k+1} are minimal a_{k+1}, y-separators in H. As $a_{k+1} \in C_H^{a_k}(A_k)$ it follows that $A_{k+1} \subseteq A_k \cup C_H^{a_k}(A_k)$. Hence A_k and A_{k+1} are parallel by Lemma 2.1.4. So A_k and A_{k+1} are strongly parallel a_{k+1}, y-separators. By Lemma 2.1.9, either

$$C_H^{a_{k+1}}(A_k) \subset C_H^{a_{k+1}}(A_{k+1}) \text{ and } C_H^y(A_k) \supset C_H^y(A_{k+1})$$

or

$$C_H^{a_{k+1}}(A_k) \supset C_H^{a_{k+1}}(A_{k+1}) \text{ and } C_H^y(A_k) \subset C_H^y(A_{k+1})$$

There is a s, y-path missing a_{k+1} in $H[\{s\} \cup C_H^y(A_k)]$ because $C_H^y(A_k)$ is a full component of $H \setminus A_k$. So $s \in C_H^y(A_{k+1})$ but $s \notin C_H^y(A_k)$. So the latter must be true and $C_H^{a_{k+1}}(A_k) \supset C_H^{a_{k+1}}(A_{k+1})$. By transitivity, the new sequences $\langle a_1, a_2, \ldots, a_{k+1} \rangle$ and $\langle A_1, A_2, \ldots, A_{k+1} \rangle$ of length $k+1$ satisfy properties (a) and (b).

By induction, we can create sequences of length $|C_H^x(S)| + 1$ where each $C_H^{a_i}(A_i) \subseteq C_H^x(S)$. Thus $a_i \in C_H^x(S)$. Moreover, the a_i are unique by the strict set inclusion of property (b), a contradiction. So our initial assumption A1 was incorrect and there exists $a \in C_H^x(S)$ satisfying conditions (i) and (ii). If $C_H^x(S)$ is completely adjacent to S in H then $a = x$ satisfies this Lemma. So suppose not and consider any $a \in C_H^x(S)$ satisfying properties (i) and (ii). If a is adjacent to every $s \in S$ in H then $A = S$, a contradiction. So a is not adjacent to some $s \in S$ in H. As $C_H^y(S)$ is a full component of $H \setminus S$ there is a s, y-path in $H[\{s\} \cup C_H^y(S)]$ that misses a. Suppose $ax \notin E(H)$. Since $as \notin E(H)$ and $xs \in E(H)$, this s, y-path can be extended to create a x, y-path in H missing a. However, (x, y) is also a dominating pair in H by Corollary 2.3.5, so we have a contradiction. Therefore $ax \in E(H)$. ☐

Example. The shaded box in Figure 4.2 represents a clique. Now $u_3 \prec u_2 \prec u_1 \prec a_1 \prec s \prec s_1 \prec s_2 \prec s_3 \prec y$ is a cocomparability order of this graph G.

Hence, G is AT-free by Theorem 1.3.6. If we let $H = G$ then G as labelled shows the necessity of the induction in Claim 4.1.

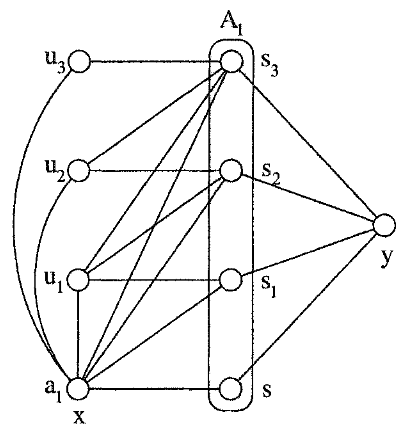

Figure 4.2: The induction in Claim 4.1 is necessary.

Example. The shaded boxes in the graph G of Figure 4.3 represent cliques. Clearly, G can be partitioned into two cliques K and $S \cup \{y\}$. Therefore, the graph is trivially AT-free because there is no independent set of three vertices. If $H = G$ then this graph as labelled shows the necessity of the last induction in Lemma 4.1.1.

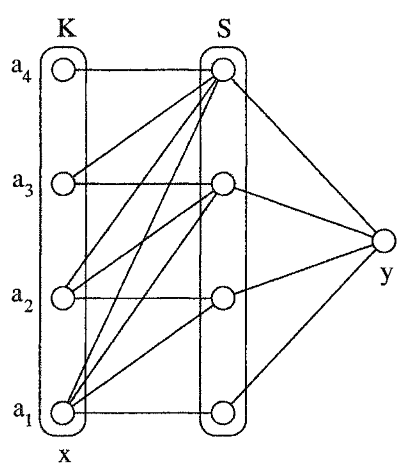

Figure 4.3: The induction at the end of Lemma 4.1.1 is necessary.

Theorem 4.1.2. *Let G be a connected AT-free graph and H be a PMT of G. If* $\operatorname{diam}(G) > 2$ *then there exist vertices a, b and x, y such that:*

(i) $d_G(x,y) > 2$ *and* $ab \notin E(H)$

(ii) (x,y), (a,b), (a,y), *and* (x,b) *are dominating pairs in G*

(iii) $a = x$ *or* $ax \in E(H)$ *and* $b = y$ *or* $by \in E(H)$

(iv) The minimal a, b-separator A in H close to a is completely adjacent to $C_H^a(A)$ in H.

(v) The minimal a, b-separator B in H close to b is completely adjacent to $C_H^b(B)$ in H.

Proof. By Theorem 1.3.8, let (x,y) be a diameter dominating pair in G. So $xy \notin E(H)$ by Observation 4.0.6. Let S be the minimal x, y-separator close to x in H. Let T be the minimal x, y-separator close to y in H. By Lemma 4.1.1 consider $a \in C_H^x(S)$ such that (a,y) is a dominating pair in G, the minimal a, y-separator A close to a in H is completely adjacent to $C_H^a(A)$ in H, and $a = x$ or $ax \in E(H)$. Similarly, consider $b \in C_H^y(T)$ such that (x,b) is a dominating pair of G, the minimal x, b-separator B close to b in H is completely adjacent to $C_H^b(B)$ in H, and $b = y$ or $by \in E(H)$.

Suppose $a \notin C_H^x(B)$. Then $ax \in E(H)$ because $a \neq x$. Thus $a \in B$ which implies that $ab \in E(H)$. But $a \in C_H^x(S)$ so $b \in S \cup C_H^x(S)$. If $b \in C_H^x(S)$ then there is a x, b-path missing y in $G[C_G^x(S)]$ because $C_H^x(S) = C_G^x(S)$ by Corollary 2.2.6, a contradiction. If $b \in S$ then $bx \in E(H)$ which contradicts that x and b are in different components of $H \setminus T$. Therefore, $a \in C_H^x(B)$ and B is the minimal a, b-separator close to b in H. Consequently $ab \notin E(H)$. By symmetry, $b \in C_H^y(A)$ and A is the minimal a, b-separator close to a in H.

All that remains to be shown is that (a,b) is a dominating pair in G. Suppose not and let v be a vertex that misses some a, b-path π in G. Note that $A \in \Delta_G$ and $C_H^\bullet(A) = C_G^\bullet(A)$ by Corollary 2.2.6. Now, if $v \notin A \cup C_G^b(A)$

then we can extend π to get a a, y-path in G missing v because $y \in C_G^b(A)$, a contradiction. So $v \in A \cup C_G^b(A)$. There is a v, b-path missing a in $G[\{v\} \cup C_G^b(A)]$ because $av \notin E(G)$ and $C_G^b(A)$ is a full component. By symmetry there is also a v, a-path missing b in G. Therefore, $\{v, a, b\}$ is an AT in G, a contradiction. □

Example. The dashed lines in Figure 4.4 are edges added by PMTs. On the left and right are connected AT-free graphs that show why the choice of a and b in Theorem 4.1.2 depends on the particular PMT.

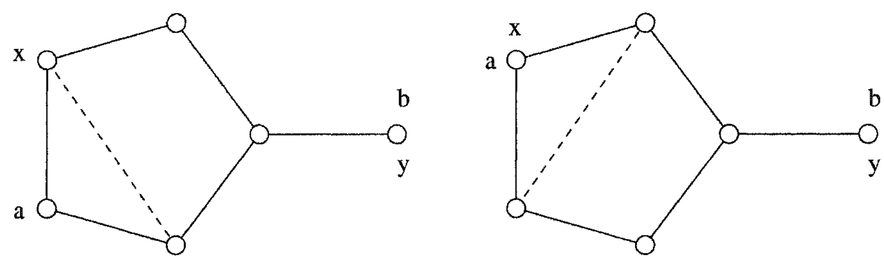

Figure 4.4: a and b in Theorem 4.1.2 depend on the particular PMT.

Observation 4.1.3. *Let G be a connected AT-free graph, T be a minimal separator of G, and C be a full component of $G \setminus T$. Then $G_T[T \cup C]$ is a connected AT-free graph as well.*

Proof. By Corollary 2.3.2, G_T is connected and AT-free. By Lemma 2.2.5, T is a minimal separator of G_T. Corollary 2.2.6 indicates that C is full component of $G_T \setminus T$. Therefore, $G_T[T \cup C]$ is connected and AT-free because the class of AT-free graphs is hereditary. □

By this Observation, the ends $G_A[C_H^a(A) \cup A]$ and $G_B[C_H^b(B) \cup B]$ of Theorem 4.1.2 are candidates for decomposition. The next two Lemmas are properties used in our proof of the Extendable Orientation Theorem.

Lemma 4.1.4. *Let G be a connected graph, H be a CPMT of G, and T be an $\subseteq$-minimal separator in H. If a component C of $H \setminus T$ is completely adjacent to T in H then $H_T[T \cup C]$ is a CPMT of $G_T[T \cup C]$.*

Proof. Since H is a PMT of G, let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel separators in G such that $H = G_{\mathcal{S}}$. As a minimal separator of H, T is parallel in G to every $S \in \mathcal{S}$ by Lemma 2.2.5. It follows immediately that $\mathcal{S}$ is a set of minimal separators of G_T by Lemma 2.2.4. Applying Corollary 2.2.6, $\mathcal{S}$ must be pairwise parallel in G_T because it is pairwise parallel in G. Therefore, H_T is a PMT of G_T because $H_T = (G_T)_{\mathcal{S}}$.

Let $\mathcal{S}' = \{S \in \mathcal{S} : S \subseteq T \cup C \text{ and } S \not\subseteq T\}$. Whenever $S \in \mathcal{S}$ contains a vertex of C, $S \in \mathcal{S}'$ because $S \parallel T$ in G. As a result, $H_T[T \cup C] = (G_T[T \cup C])_{\mathcal{S}'}$.

Let $S \in \mathcal{S}'$. Consider for contradiction $t \in T \setminus S$. By case analysis, we will prove that there is always an u, t-path in $G_T \setminus S$, for $u \notin S$.

1. $u \in C \setminus S$

   Then $ut \in E(H_T)$ because C is completely adjacent to T in H. If $ut \in E(G_T)$ then u to t is such a path. So suppose not and consider $U \in \mathcal{S}$ such that $u, t \in U$. Since $S \parallel U$ in G_T, there is a full component D of $G_T \setminus U$ such that $D \cap S = \emptyset$ by Lemma 2.1.4. Every u, t-path in $G_T[\{u, t\} \cup D]$ is a u, t-path in $G_T \setminus S$.

2. $u \in T \setminus S$

   $\langle u, t \rangle$ is such a path because T is a clique in G_T.

3. $u \in D \setminus S$ where $D \neq C$ is a component of $H \setminus T$

   By Lemma 2.1.2, D is a full component of $H \setminus T$ because T is $\subseteq$-minimal in H. Then $D \in C^{\bullet}_{H_T}(T)$ by Lemmas 2.2.4 and 2.1.1. Applying Corollary 2.2.6 reveals that $D \in C^{\bullet}_{G_T}(T)$ because H_T is a PMT of G_T. So every u, t-path through $G_T[\{t\} \cup D]$ is such a path.

Hence, S is not a separator in G_T, a contradiction. Therefore, $T \subset S$. Consider $s \in S \setminus T$ and a full component D of $G_T \setminus T$. For $d \in D$, there is a s, d-path in $G_T[\{s\} \cup D]$ that avoids T. Consequently, the full components of $G_T \setminus S$ are contained in C. By Lemma 2.1.1, S is a minimal separator of $G_T[T \cup C]$. Moreover, $\mathcal{S}'$ is pairwise parallel in $G_T[T \cup C]$ by Lemma 2.1.4

because $\mathcal{S}'$ is pairwise parallel in G_T. So $H_T[T \cup C]$ is a PMT of $G_T[T \cup C]$. By Observation 4.0.8, H_T is a cocomparability graph. As that is a hereditary graph class, $H_T[T \cup C]$ is a CPMT of $G_T[T \cup C]$. $\square$

Example. The solid edges in Figure 4.5 form a cocomparability graph G as certified by the cocomparability order of the vertex labels. Hence G is AT-free by Theorem 1.3.6. The minimal 2, 3-separator in G is $\{1, 6, 7\}$. By Observation 4.0.8, the graph H formed by adding the dashed edges is a CPMT of G. Let T be the solid vertices and C be the vertices to the left of T. Now $\{1, 6, 7\}$ is the minimal 2, 3-separator in $G_T[T \cup C]$. Hence, $H_T[T \cup C]$ is a PMT of $G_T[T \cup C]$. The orientation $\rightarrow$ of $\overline{H_T[T \cup C]}$ resulting from the vertex labels is transitive. So $H_T[T \cup C]$ is a CPMT of $G_T[T \cup C]$.

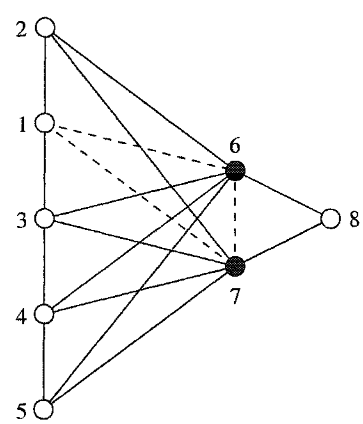

Figure 4.5: Illustrates Lemmas 4.1.4 and 4.1.5.

Lemma 4.1.5. *Let G be a connected AT-free graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. Let H be a PMT of G and T be a minimal separator of H. If a component C of $H \setminus T$ is completely adjacent to T in H then $\Rightarrow$ restricted to $T \cup C$ is an extendable orientation of $\overline{G_T[T \cup C]^2}$*

Proof. First we will show that $d_G(u, v) > 2$ if and only if $d_{G_T[T \cup C]}(u, v) > 2$, for all $u, v \in T \cup C$. Note that if $u, v \in C$ then the neighbourhoods of u and v are exactly the same in G as they are in $G_T[T \cup C]$. Hence, $d_G(u, v) > 2$ if and only if $d_{G_T[T \cup C]}(u, v) > 2$. So without loss of generality suppose that $u \in T$.

1. $v \in C$

   Then $uv \in E(H)$ and $d_G(u,v) \leq 2$ by Observation 4.0.6. If $d_G(u,v) = 1$ then $d_{G_T[T \cup C]}(u,v) = 1$. So suppose not and consider $x \in V(G)$ such that $ux, vx \in E(G)$. As $vx \in E(H)$ it follows that $x \in T \cup C$. Hence, $d_{G_T[T \cup C]}(u,v) = 2$.

2. $v \in T$

   Then $d_G(u,v) \leq 2$ by Observation 3.3.2 and $d_{G_T[T \cup C]}(u,v) = 1$.

Therefore, $\Rightarrow$ restricted to $T \cup C$ is a transitive orientation of $\overline{G_T[T \cup C]^2}$.

Claim 4.3. *Let S be a minimal separator in $G_T[T \cup C]$. Then there is a u,v-path in $G_T[T \cup C] \setminus S$ if and only if there is a u,v-path in $G_T \setminus S$, for $u,v \in T \cup C$.*

Proof. Any u,v-path in $G_T[T \cup C] \setminus S$ is a u,v-path in $G_T \setminus S$.

Let π be a shortest u,v-path in $G_T \setminus S$. Now, $C \in \mathcal{C}^\bullet_G(T)$ by Corollary 2.2.6 because H is a PMT of G. So by Lemmas 2.2.4 and 2.1.1, $C \in \mathcal{C}^\bullet_{G_T}(T)$. If π is not contained in $G_T[T \cup C]$ it must pass through some component $D \neq C$ of $G_T \setminus T$. Let t_u resp. t_v be the vertex of T closest to u resp. v along π. Clearly, t_u and t_v are not consecutive in π because π passes through D. But $t_u t_v \in E(G_T)$ which contradicts that π is a shortest u,v-path in $G_T \setminus S$. So π is a u,v-path in $G_T[T \cup C] \setminus S$. □

From Lemma 2.1.1 and Claim 4.3 we know that $\Delta_{G_T[T \cup C]} \subseteq \Delta_{G_T}$. By Corollary 2.2.6, $\Delta_{G_T} \subseteq \Delta_G$ and $\mathcal{C}_{G_T}(T) = \mathcal{C}_G(T)$. Hence, $\Delta_{G_T[T \cup C]} \subseteq \Delta_G$. Furthermore, Claim 4.3 implies that there is a u,v-path in $G_T[T \cup C] \setminus S$ if and only if there is a u,v-path in $G \setminus S$, for $S \in \Delta_{G_T[T \cup C]}$ and $u,v \in T \cup C$. Therefore, it is routine to verify that if $\Rightarrow$ restricted to $T \cup C$ is not an extendable orientation of $\overline{G_T[T \cup C]^2}$ then $\Rightarrow$ is not an extendable orientation of $\overline{G^2}$. □

Example. As we saw earlier, the solid edges in Figure 4.5 form an AT-free graph G and the graph H that includes the dashed edges is a CPMT of G. Note that $E(\overline{G^2}) = \{1\,5, 1\,8\}$. As such, there is only one transitive orientation of $\overline{G^2}$ up to reversal. So $\{1 \Rightarrow 5, 1 \Rightarrow 8\}$ is an extendable orientation of $\overline{G^2}$ by Corollary 3.3.4 and Observation 3.3.5. If we take T to be the solid vertices and C to be the vertices to the left of T, then $\{1 \Rightarrow 5\}$ is an extendable orientation of $\overline{G_T[T \cup C]^2}$.

4.2 Extendable Orientation Theorem

We are now ready to prove the main result of this chapter:

Theorem 4.2.1 (Extendable Orientation Theorem). *Let G be a connected AT-free graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. If H is a CPMT of G then there exists a transitive orientation $\rightarrow$ of $\overline{H}$ such that if $u \Rightarrow v$ then $u \rightarrow v$.*

Proof. The proof proceeds by induction on the number of vertices n of G. If G has a single vertex then the theorem is trivially true. So suppose that this theorem holds for connected AT-free graphs with fewer than n vertices.

If $\mathrm{diam}(G) \leq 2$ then this theorem is true because any transitive orientation of $\overline{H}$ will suffice. So assume that $\mathrm{diam}(G) > 2$. Consider vertices x, y and a, b of G with the properties described in Theorem 4.1.2. By the symmetry of Theorem 4.1.2, we can assume without loss of generality that $x \Rightarrow y$.

Claim 4.4. *Let S be a minimal a,b-separator in H. For every $u \in C_H^a(S)$ and $v \in C_H^b(S)$ such that $uv \notin E(G^2)$ it holds that $u \Rightarrow v$.*

Proof. Now $x \in S \cup C_H^a(S)$ because $a = x$ or $ax \in E(H)$. Similarly, $y \in S \cup C_H^b(S)$ because $b = y$ or $by \in E(H)$. Since H is a PMT of G, $S \in \Delta_G$ and $C_H^a(S), C_H^b(S) \in C_G^{\bullet}(S)$ by Corollary 2.2.6. By Observation 3.3.2, either $x \notin S$ or $y \notin S$ because $xy \notin E(G^2)$. So consider the following cases:

1. $x \in C_H^a(S)$ and $y \in C_H^b(S)$

   Then $u \Rightarrow v$ because $x \Rightarrow y$ and all edges of $\overline{G^2}$ are oriented the same way between $C_H^a(S)$ and $C_H^b(S)$ by (1) of Lemma 3.3.3.

2. $x \in S$ and $y \in C_H^b(S)$

   Then $u \Rightarrow v$ by (4) of Lemma 3.3.3 because $x \Rightarrow y$ is an edge in $\overline{G^2}$ oriented from S to $C_H^b(S)$.

3. $x \in C_H^a(S)$ and $y \in S$.

   Then $u \Rightarrow v$ by (3) of Lemma 3.3.3 because $x \Rightarrow y$ is an edge in $\overline{G^2}$ oriented from $C_H^a(S)$ to S.

□

Let A be the minimal a, b-separator close to a in H and B be the minimal a, b-separator close to b in H. By Theorem 4.1.2, $A \times C_H^a(A) \subseteq E(H)$ and $B \times C_H^b(B) \subseteq E(H)$. By Observation 2.1.6 and Lemma 2.1.11, $A \parallel B$ in H. and $C_H^a(A) \subseteq C_H^a(B)$ and $C_H^b(A) \supseteq C_H^b(B)$. Consequently, $C_H^a(A) \cap C_H^b(B) = \emptyset$.

Claim 4.5. *$H_{\{A,B\}}$ is a PMT of H. Moreover, it is also a PMT of G.*

Proof. $H_{\{A,B\}}$ is a PMT of H because $A \parallel B$ in H. As H is a PMT of G, let $\mathcal{S} \subseteq \Delta_G$ be a set of pairwise parallel separators in G such that $H = G_{\mathcal{S}}$. By Lemma 2.2.5, both A and B are minimal separators in G and both are parallel to every $S \in \mathcal{S}$ in G. Earlier we proved that $A \parallel B$ in H. Hence $A \parallel B$ in G by Corollary 2.2.6. So $\mathcal{S} \cup \{A, B\}$ is pairwise parallel set of minimal separators in G. Therefore, $H_{\{A,B\}}$ is a PMT of G. □

Let $H'_{\{A,B\}}$ denote $H_{\{A,B\}} \setminus (C_H^a(A) \cup C_H^b(B))$.

Claim 4.6. *Let $S \subseteq V(G)$. There is a u, v-path in $H'_{\{A,B\}} \setminus S$ if and only if there is a u, v-path in $H_{\{A,B\}} \setminus S$, for $u, v \notin C_H^a(A) \cup C_H^b(B)$.*

Proof. Any u, v-path in $H'_{\{A,B\}} \setminus S$ is a u, v-path in $H_{\{A,B\}} \setminus S$.

Consider a shortest u, v-path π in $H_{\{A,B\}} \setminus S$ where $u, v \notin C_H^a(A) \cup C_H^b(B)$. Suppose that π contains some vertex w of $C_H^a(A)$. Let s be the first vertex of A in π and t be the last vertex of A in π. We know that s and t are not consecutive in π because s occurs before w and t occurs after w. But A is a clique in $H_{\{A,B\}}$ which contradicts that π is a shortest u, v-path in $H_{\{A,B\}} \setminus S$. Thus π contains no vertex of $C_H^a(A)$. Analogously, π contains no vertex of $C_H^b(B)$. Hence π is a u, v-path in $H'_{\{A,B\}} \setminus S$. □

As H is a CPMT, let $\rightarrow$ be a transitive orientation of $\overline{H}$. We can assume without loss of generality that $a \rightarrow b$ because the reversal of a transitive orientation is still transitive. The following claim is central to this theorem:

Claim 4.7. *If $u \Rightarrow v$ then $u \rightarrow v$, for all $u, v \notin C_H^a(A) \cup C_H^b(B)$.*

Proof. Consider $u, v \notin C_H^a(A) \cup C_H^b(B)$ such that $u \Rightarrow v$. By Observation 4.0.6 and Claim 4.5, $uv \notin E(H_{\{A,B\}})$ because $uv \notin E(G^2)$ and $H_{\{A,B\}}$ is a PMT of G. By Claim 4.6, $H'_{\{A,B\}}$ is connected because $H_{\{A,B\}}$ is connected. So let S be a minimal u, v-separator in $H'_{\{A,B\}}$. Again by Claim 4.6, S is a minimal u, v-separator in $H_{\{A,B\}}$. Both A and B are parallel to S in $H_{\{A,B\}}$ by Lemma 2.2.1. By Corollary 2.2.6 and Claim 4.5, $S \in \Delta_H$ and S is parallel to both A and B in H because $H_{\{A,B\}}$ is a PMT of H. Clearly, $a, b \notin S$. Note that (a, b) is a dominating pair of H by Corollary 2.3.5 because H is PMT of G. Hence, a and b are in different components of $H \setminus S$ by Observation 2.3.4. Let

$$\begin{aligned} S_a &= \{s \in S : \text{ some vertex of } C_H^a(S) \text{ is adjacent to } s \text{ in } H\} \\ S_b &= \{s \in S : \text{ some vertex of } C_H^b(S) \text{ is adjacent to } s \text{ in } H\} \end{aligned}$$

By Lemma 2.1.2 we know that $S_a, S_b \in \Delta_H$, $C_H^a(S) \in C_H^\bullet(S_a)$, and $C_H^b(S) \in C_H^\bullet(S_b)$. Notice that $C_H^a(S) = C_H^a(S_a)$, $C_H^b(S) \subseteq C_H^b(S_a)$, $C_H^b(S) = C_H^b(S_b)$, and $C_H^a(S) \subseteq C_H^a(S_b)$. To prove that $u \to v$ we consider the following cases:

(1) $u \in C_H^a(S)$

By (1) of Lemma 3.3.1, if $v \in C_H^b(S)$ then $u \to v$ because $a \to b$. So suppose $v \notin C_H^b(S)$. Now $C_H^a(S_b) \in C_H^\bullet(S_b)$ because $C_H^a(S) \subseteq C_H^a(S_b)$ and $C_H^a(S) \in C_H^\bullet(S)$. So by Lemma 2.1.1, S_b is a minimal a,b-separator in H such that $u \in C_H^a(S_b)$ and $v \notin C_H^b(S_b)$.

We know that $S_b \not\subset A$ because A is a minimal a,b-separator in H. Moreover, $S_b \neq A$ because $u \in C_H^a(S_b)$ but $u \notin C_H^a(A)$. So consider $s \in S_b \setminus A$. Suppose for contradiction that $sv \notin E(H)$. There is a a,s-path in $H[\{s\} \cup C_H^a(S)]$ missing v. There is also a s,b-path in $H[\{s\} \cup C_H^b(S_b)]$ missing v. This contradicts that (a,b) is a dominating pair in H.

Now, A and S_b are parallel minimal a,b-separators in H because $A \parallel S$ in H. So either

$$C_H^a(A) \subset C_H^a(S_b) \text{ and } C_H^b(A) \supset C_H^b(S_b)$$

or

$$C_H^a(A) \supset C_H^a(S_b) \text{ and } C_H^b(A) \subset C_H^b(S_b)$$

by Lemma 2.1.9. The former must be true as $u \notin C_H^a(A)$ and $u \in C_H^a(S_b)$.

$C_H^b(S_b)$ is a full component so there is a b,s-path in $H[\{s\} \cup C_H^b(S_b)]$. This is also a b,s-path in $H[\{s\} \cup C_H^b(A)]$. It follows that $s \in C_H^b(A)$ because $s \notin A$. Hence, $v \in A \cup C_H^b(A)$ because $sv \in E(H)$. We know that $av \notin E(H)$ because a and v are in different components of $H \setminus S$. Earlier we chose A close to a in H. So $v \in C_H^b(A)$. By (1) of Lemma 3.3.1, $a \to v$ because $a \to b$. Again by (1) of Lemma 3.3.1, $u \to v$ because $a \to v$, $u \in C^a_H(S)$, and $v \notin C^a_H(S)$.

(2) $v \in C^a_H(S)$

In this case $C^a_H(S_b) \in C^\bullet_H(S_b)$ because $C^a_H(S) \subseteq C^a_H(S_b)$ and $C^a_H(S) \in C^\bullet_H(S)$. By Claim 4.4, $u \notin C^b_H(S)$ because $u \Rightarrow v$. Consequently, S_b is a minimal a, b-separator in H such that $v \in C^a_H(S_b)$ and $u \notin C^b_H(S_b)$.

Consider for contradiction $s \in S_b$ such that $su \notin E(G)$. As S is a minimal a, u-separator in H it is also a minimal a, u-separator in G by Lemma 2.2.5 because H is a PMT of G. There is a a, s-path in $G[\{s\} \cup C^a_G(S)]$ missing u. Note that S_b is a minimal b, u-separator in H because $C^u_H(S) \subseteq C^u_H(S_b)$ and $C^u_H(S) \in C^\bullet_H(S)$. By the preceding argument, there is a s, b-path in $G[\{s\} \cup C^b_G(S_b)]$ missing u. Yet this contradicts that (a, b) is a dominating pair in G.

$C^b_G(S_b) \in C^\bullet_G(S_b)$ so consider $b_s \in C^b_G(S_b)$ and $s \in S_b$ such that $b_s s \in E(G)$. Now, v is completely non-adjacent to S_b in G because u is completely adjacent to S_b in G. Consequently, $b_s v \notin E(G^2)$. So $v \Rightarrow b_s$ by Claim 4.4 because $v \in C^a_H(S_b)$. Thus $u \Rightarrow v \Rightarrow b_s$, contradicting that $su \in E(G)$.

(3) $u \in C^b_H(S)$

Please note that this case is symmetric to case 2. As the symmetry is non-obvious, the same proof is included with the necessary substitutions.

In this case $C^b_H(S_a) \in C^\bullet_H(S_a)$ because $C^b_H(S) \subseteq C^b_H(S_a)$ and $C^b_H(S) \in C^\bullet_H(S)$. By Claim 4.4, $v \notin C^a_H(S)$ because $u \Rightarrow v$. Consequently, S_a is a minimal a, b-separator in H such that $u \in C^b_H(S_a)$ and $v \notin C^a_H(S_a)$.

Consider for contradiction $s \in S_a$ such that $sv \notin E(G)$. As S is a minimal b, v-separator in H it is also a minimal b, v-separator in G by Lemma 2.2.5 because H is a PMT of G. There is a b, s-path in

$G[\{s\} \cup C_G^b(S)]$ missing v. Note that S_a is a minimal a, v-separator in H because $C_H^v(S) \subseteq C_H^v(S_a)$ and $C_H^v(S) \in C_H^\bullet(S)$. By the preceding argument, there is a s, a-path in $G[\{s\} \cup C_G^a(S_a)]$ missing v. This contradicts that (a, b) is a dominating pair in G.

$C_G^a(S_a) \in C_G^\bullet(S_a)$ so consider $a_s \in C_G^a(S_a)$ and $s \in S_a$ such that $a_s s \in E(G)$. Now, u is completely non-adjacent to S_a in G because v is completely adjacent to S_a in G. Consequently, $a_s u \notin E(G^2)$. So $a_s \Rightarrow u$ by Claim 4.4 because $u \in C_H^b(S_a)$. Thus $a_s \Rightarrow u \Rightarrow v$, contradicting that $sv \in E(G)$.

(4) $v \in C_H^b(S)$

This case is symmetric to case 1. The same proof is included with the necessary substitutions because the symmetry is non-trivial.

By (1) of Lemma 3.3.1, if $u \in C_H^a(S)$ then $u \to v$ because $a \to b$. So suppose $u \notin C_H^a(S)$. Now $C_H^b(S_a) \in C_H^\bullet(S_a)$ because $C_H^b(S) \subseteq C_H^b(S_a)$ and $C_H^b(S) \in C_H^\bullet(S)$. So by Lemma 2.1.1, S_a is a minimal a, b-separator in H such that $v \in C_H^b(S_a)$ and $u \notin C_H^a(S_a)$.

We know that $S_a \not\subset B$ because B is a minimal a, b-separator in H. Moreover, $S_a \neq B$ because $v \in C_H^b(S_a)$ but $v \notin C_H^b(B)$. So consider $s \in S_a \setminus B$. Suppose for contradiction that $su \notin E(H)$. There is a b, s-path in $H[\{s\} \cup C_H^b(S)]$ missing u. There is also a s, a-path in $H[\{s\} \cup C_H^a(S_a)]$ missing u. This contradicts that (a, b) is a dominating pair in H.

Now, B and S_a are parallel minimal a, b-separators in H because $B \parallel S$ in H. So either

$$C_H^b(B) \subset C_H^b(S_a) \text{ and } C_H^a(B) \supset C_H^a(S_a)$$

or

$$C_H^b(B) \supset C_H^b(S_a) \text{ and } C_H^a(B) \subset C_H^a(S_a)$$

by Lemma 2.1.9. The former must be true as $v \notin C_H^b(B)$ and $v \in C_H^b(S_a)$.

$C^a_H(S_a)$ is a full component so there is a a,s-path in $H[\{s\} \cup C^a_H(S_a)]$. This is also a a,s-path in $H[\{s\} \cup C^a_H(B)]$. It follows that $s \in C^a_H(B)$ because $s \notin B$. Hence, $u \in B \cup C^a_H(B)$ because $su \in E(H)$. We know that $bu \notin E(H)$ because b and u are in different components of $H \setminus S$. Earlier we chose B close to b in H. So $u \in C^a_H(B)$. By (1) of Lemma 3.3.1, $u \to b$ because $a \to b$. Again by (1) of Lemma 3.3.1, $u \to v$ because $u \to b$, $v \in C^b_H(S)$, and $u \notin C^b_H(S)$.

(5) $u, v \notin C^a_H(S) \cup C^b_H(S)$

Assume for contradiction that $S_a \cap S_b \neq \emptyset$ and consider $s \in S_a \cap S_b$. Suppose that $su \notin E(G)$. Earlier we observed that $S_a \in \Delta_H$ and $C^a_H(S) \in C^\bullet_H(S_a)$. By Corollary 2.2.6, $S_a \in \Delta_G$ and $C^a_G(S) \in C^\bullet_G(S_a)$ because H is a PMT of G. Hence, there is a a,s-path in $G[\{s\} \cup C^a_G(S_a)]$ missing u. Analogously, there is a s,b-path in $G[\{s\} \cup C^b_G(S_b)]$ missing u. This contradicts that (a,b) is a dominating pair in G. Therefore, $su \in E(G)$. By symmetry, $sv \in E(G)$ which contradicts that $uv \notin E(G^2)$. Thus, $S_a \cap S_b = \emptyset$.

Suppose for contradiction that there exist $s_a \in S_a$ and $s_b \in S_b$ such that $s_a u \notin E(G)$ and $s_b u \notin E(G)$. There is a a, s_a-path through $G[\{s_a\} \cup C^a_G(S_a)]$ missing u. By Lemma 2.2.5, S is a minimal u,v-separator in G because H is a PMT of G. So there is a s_a, s_b-path missing u in $G[\{s_a, s_b\} \cup C^v_G(S)]$. Finally, there is a s_b, b-path in $G[\{s_b\} \cup C^b_G(S_b)]$ that misses u. Therefore, there is a a,b-path in G that misses u, a contradiction.

What this means is that u is completely adjacent to S_a or S_b in G. Again by symmetry, v is completely adjacent to S_a or S_b. As $uv \notin E(G^2)$ there are two cases to consider.

(a) $\{u\} \times S_a \subseteq E(G)$, $(\{u\} \times S_b) \cap E(G) = \emptyset$, $(\{v\} \times S_a) \cap E(G) = \emptyset$, and $\{v\} \times S_b \subseteq E(G)$

Consider $s_a \in S_a$ and $s_b \in S_b$. We know that $as_b \notin E(H)$ because

$S_a \cap S_b = \emptyset$. There is a b, s_b-path in $H[S_b \cup C_H^b(S)]$ by definition of S_b. Hence, $s_b \in C_H^b(S_a)$. Now, S_a is a a, b-separator in H because $b \notin C_H^a(S)$ and $C_H^a(S) = C_H^a(S_a)$. So by (1) of Lemma 3.3.1, $a \to s_b$ because $a \to b$. There is a u, s_a-path in $H[S_a \cup C_H^u(S)]$ because $C_H^u(S) \in C_H^\bullet(S)$. Moreover, there is a s_a, a-path in $H[S_a \cup C_H^a(S)]$ by definition of S_a. Thus, $u \in C_H^a(S_b)$. By (2) of Lemma 3.3.1, $u \to s_b$ because $a \to s_b$. Suppose that $v \to u$. Then $v \to s_b$ by transitivity, which contradicts that $vs_b \in E(G)$.

(b) $(\{u\} \times S_a) \cap E(G) = \emptyset$, $\{u\} \times S_b \subseteq E(G)$, $\{v\} \times S_a \subseteq E(G)$, and $(\{v\} \times S_b) \cap E(G) = \emptyset$

$ab \notin E(G^2)$ because $S_a \cap S_b = \emptyset$. By Claim 4.4, $a \Rightarrow b$.

Consider $a_s \in C_G^a(S)$ and $s_a \in S_a$ such that $a_s s_a \in E(G)$ because $C_G^a(S) \in C_G^\bullet(S_a)$. Clearly, $a_s u \notin E(G^2)$ because S_a is a a, u-separator in G and $(\{u\} \times S_a) \cap E(G) = \emptyset$. Consider $s_b \in S_b$. There is a s_b, b-path in $G[S_b \cup C_G^b(S)]$ because $C_G^b(S) \in C_G^\bullet(S_b)$. So $s_b \in C_G^b(S_a)$. Since $us_b \in E(G)$ and $u \notin S$, we know that $u \in C_H^b(S_a)$. By (1) of Lemma 3.3.3, $a_s \Rightarrow u$ because $a \Rightarrow b$. By transitivity, $a_s \Rightarrow v$ which contradicts that $vs_a \in E(G)$.

□

By Observation 4.1.3, $G_a = G_A[A \cup C_H^a(A)]$ is connected and AT-free. By Lemma 4.1.4, $H_a = H_A[A \cup C_H^a(A)]$ is a CPMT of G_a. Lemma 4.1.5 tells us that $\Rightarrow$ restricted to G_a is an extendable orientation of $\overline{G_a^2}$. By our induction hypothesis, let $\xrightarrow{a}$ be a transitive orientation of $\overline{H_a}$ such that for $u, v \in V(G_a)$, if $u \Rightarrow v$ then $u \xrightarrow{a} v$. Define G_b and H_b analogously. Then let $\xrightarrow{b}$ be a transitive orientation of $\overline{H_b}$ such that for $u, v \in V(G_b)$, if $u \Rightarrow v$ then $u \xrightarrow{b} v$.

At this point we are ready to construct an orientation $\rightsquigarrow$ of $\overline{H}$ in a piecewise fashion. Consider the following conditions:

(1) if $u, v \in C_H^a(A)$ then $u \rightsquigarrow v$ if and only if $u \xrightarrow{a} v$

(2) if $u, v \in C^b_H(B)$ then $u \rightsquigarrow v$ if and only if $u \xrightarrow{b} v$

(3) if $u, v \notin C^a_H(A) \cup C^b_H(B)$ then $u \rightsquigarrow v$ if and only if $u \rightarrow v$

(4) if $u \in C^a_H(A)$ and $v \in C^b_H(B)$ then $u \rightsquigarrow v$ if $uv \notin E(H)$

(5) if $u \in C^a_H(A)$ and $v \notin C^a_H(A) \cup C^b_H(B)$ then $u \rightsquigarrow v$ if $uv \notin E(H)$

(6) if $u \notin C^a_H(A) \cup C^b_H(B)$ and $v \in C^b_H(B)$ then $u \rightsquigarrow v$ if $uv \notin E(H)$

Since $C^a_H(A) \cap C^b_H(B) = \emptyset$, $\rightsquigarrow$ is well defined because the cases partition the vertex set. In conditions (1)-(3), if $u \Rightarrow v$ then $u \rightsquigarrow v$.

Suppose that condition (4) holds and $uv \notin E(G^2)$. Just after we proved that $A \parallel B$ in H we demonstrated that $C^a_H(A) \subseteq C^a_H(B)$ and $C^b_H(A) \supseteq C^b_H(B)$. Therefore, $u \Rightarrow v$ by Claim 4.4.

Suppose that condition (5) holds and $uv \notin E(G^2)$. By Observation 4.0.6, $v \notin A$ because A is completely adjacent to $C^a_H(A)$ in H. If $v \in C^b_H(A)$ then $u \Rightarrow v$ by Claim 4.4. So assume that $v \notin A \cup C^b_H(A)$. By Corollary 2.2.6, $A \in \Delta_G$ and $C^a_H(A), C^b_H(A) \in C^\bullet_G(A)$. Consider for contradiction $s_a \in A$ such that $s_a v \notin E(G)$. Then there is an a, s_a-path in $G[\{s_a\} \cup C^a_H(A)]$ missing v. There is also a s_a, b-path in $G[\{s_a\} \cup C^b_H(A)]$ missing v. However, this contradicts that (a, b) is a dominating pair in G. Hence, $\{v\} \times A \subseteq E(G)$ and $(\{u\} \times A) \cap E(G) = \emptyset$. Therefore, $ub \notin E(G^2)$. By Claim 4.4, $u \Rightarrow b$. Let U be the u, v-separator close to u in G. Clearly, $U \subseteq A \cup C^a_G(A)$ and $A \not\subseteq U$. By Lemma 2.1.8, $b \in C^v_G(U)$. By (1) of Lemma 3.3.3, $u \Rightarrow v$ because $u \Rightarrow b$.

This case follows by symmetry as well. It is included with the necessary substitutions for completeness. Suppose that condition (6) holds and $uv \notin E(G^2)$. By Observation 4.0.6, $u \notin B$ because B is completely adjacent to $C^b_H(B)$ in H. If $u \in C^a_H(B)$ then $u \Rightarrow v$ by Claim 4.4. So assume that $u \notin B \cup C^a_H(B)$. By Corollary 2.2.6, $B \in \Delta_G$ and $C^a_H(B), C^b_H(B) \in C^\bullet_G(A)$. Consider for contradiction $s_b \in B$ such that $s_b u \notin E(G)$. Then there is a a, s_b-path in $G[\{s_b\} \cup C^a_H(B)]$ missing u. There is also a s_b, b-path in $G[\{s_b\} \cup C^b_H(B)]$ missing u. However, this contradicts that (a, b) is a dominating pair in G.

Hence, $\{u\} \times B \subseteq E(G)$ and $(\{v\} \times B) \cap E(G) = \emptyset$. Therefore, $av \notin E(G^2)$. By Claim 4.4, $a \Rightarrow v$. Let W be the u, v-separator close to v in G. Clearly, $W \subseteq B \cup C_G^b(B)$ and $B \nsubseteq W$. By Lemma 2.1.8, $a \in C_G^u(W)$. By (1) of Lemma 3.3.3, $u \Rightarrow v$ because $a \Rightarrow v$.

So we have demonstrated that $u \Rightarrow v$ implies $u \rightsquigarrow v$. Hence, all that remains to be shown is that $\rightsquigarrow$ is transitive.

Claim 4.8. *For all $u \notin C_H^a(A)$ and $s_a \in A$ such that $s_a u \notin E(H)$ it holds that $s_a \rightarrow u$.*

Proof. Clearly $u \notin A$ because A is a clique in H. So $u \notin A \cup C_H^a(A)$. Suppose that $u \notin C_H^b(A)$. Then there is a a, s_a-path missing u in $H[\{s_a\} \cup C_H^a(A)]$. There is also a s_a, b-path missing u in $H[\{s_a\} \cup C_H^b(A)]$. This contradicts that (a, b) is a dominating pair in H.

So $u \in C_H^b(A)$. By (1) of Lemma 3.3.1, $a \rightarrow u$ because $a \rightarrow b$. If $u \rightarrow s_a$ then $a \rightarrow s_a$ by transitivity, a contradiction because A is close to a in H. □

Claim 4.9. *For all $u \notin C_H^b(B)$ and $s_b \in B$ such that $s_b u \notin E(H)$ it holds that $u \rightarrow s_b$.*

Proof. By reasoning similar to the previous Claim, $u \in C_H^a(B)$. So by (1) of Lemma 3.3.1, $u \rightarrow b$ because $a \rightarrow b$. If $s_b \rightarrow u$ then $s_b \rightarrow b$ by transitivity, a contradiction because B is close to b in H. □

Consider for contradiction $u, v, w \in V(G)$ such that $u \rightsquigarrow v$, $v \rightsquigarrow w$, and $u \not\rightsquigarrow w$.

One possibility is that $w \rightsquigarrow u$. Perhaps $\{u, v, w\} \cap C_H^a(A) \neq \emptyset$. Then without loss of generality assume that $u \in C_H^a(A)$. Then $w \in C_H^a(A)$ because $w \rightsquigarrow u$. Similarly, $v \in C_H^a(A)$ because $v \rightsquigarrow w$. However, this contradicts the transitivity of $\xrightarrow{a}$. Therefore, $u, v, w \notin C_H^a(A)$. By symmetry, $u, v, w \notin C_H^b(B)$.

So, $u, v, w \notin C_H^a(A) \cup C_H^b(B)$ which contradicts the transitivity of $\rightarrow$. Hence, Therefore, $w \not\rightsquigarrow u$.

The other possibility is that $uw \notin E(H)$. Let us consider when $w \in C_H^a(A)$. Then $v \in C_H^a(A)$ because $v \rightsquigarrow w$. Similarly, $u \in C_H^a(A)$ because $u \rightsquigarrow v$. But this contradicts the transitivity of $\xrightarrow{a}$. Therefore, $w \notin C_H^a(A)$. Again for contradiction, suppose that $u \in C_H^b(B)$. Then $v \in C_H^b(B)$ because $u \rightsquigarrow v$. So $w \in C_H^b(B)$ because $v \rightsquigarrow w$. But this contradicts the transitivity of $\xrightarrow{b}$. Hence, $u \notin C_H^b(B)$.

Earlier we demonstrated that $C_H^a(A) \subseteq C_H^a(B)$ and $C_H^b(A) \supseteq C_H^b(B)$. Suppose $u \in C_H^a(A)$. Then $w \in A \cup C_H^a(A)$ because $uw \in E(H)$. Since $w \notin C_H^a(A)$, $w \in A$. This means that $w \notin C_H^b(B)$ because $C_H^b(A) \supseteq C_H^b(B)$. Thus, $v \notin C_H^b(B)$ because $v \rightsquigarrow w$. Moreover, $v \notin C_H^a(A)$ because $vw \notin E(H)$ and $A \times C_H^a(A) \subseteq E(H)$. So $v \notin C_H^a(A) \cup C_H^b(B)$. By Claim 4.8, $w \rightarrow v$ which contradicts that $v \rightsquigarrow w$. Therefore, $u \notin C_H^a(A)$.

Suppose $w \in C_H^b(B)$. Then $u \in B \cup C_H^b(B)$ because $uw \in E(H)$. Since $u \notin C_H^b(B)$, $u \in B$. If $v \in C_H^a(A)$ then $v \rightsquigarrow u$ because $u \notin C_H^a(B) \cup C_H^b(B)$, a contradiction. If $v \in C_H^b(B)$ the $uv \in E(H)$ because $B \times C_H^b(B) \subseteq E(H)$, a contradiction. So, $v \notin C_H^a(A) \cup C_H^b(B)$. By Claim 4.9, $v \rightarrow u$ which contradicts that $u \rightsquigarrow v$. Therefore, $w \notin C_H^b(B)$.

To recap, $u, w \notin C_H^a(A) \cup C_H^b(B)$. Thus $v \notin C_H^a(A)$ because $u \rightsquigarrow v$. Similarly, $v \notin C_H^b(B)$ because $v \rightsquigarrow w$. But this contradicts the transitivity of $\rightarrow$. So there are not such $u, v, w \in V(G)$. Therefore $\rightsquigarrow$ is transitive orientation of $\overline{H}$ such that $u \Rightarrow v$ implies that $u \rightsquigarrow v$. □

Example. The graph G in Figure 4.6 given by the solid edges illustrates the necessity of the induction step in the proof of Theorem 4.2.1. The vertex labels are a cocomparability order of G. Hence, G is AT-free by Theorem 1.3.6. The minimal $2, 3$-separator close to 2 is $\{1, 6, 7, 9\}$. By Observation 4.0.8, the supergraph H indicated by the dashed edges is a CPMT. In each successive graph we have indicated what x, y, a, and b must be up to reversal. A is indicated by the solid vertices in each graph. The rightmost graph is unlabelled because it is diameter 2.

Notice that $E(\overline{G^2}) = \{1\ 5, 1\ 8, 1\ 10\}$. In any extendable orientation $\Rightarrow$ of $\overline{G^2}$, either 1 is a source or a sink by transitivity. Let us suppose without loss of generality that 1 is a source of $\Rightarrow$. Then the transitive orientation $\rightarrow$ of $\overline{H}$ resulting from the vertex labelling satisfies Theorem 4.2.1.

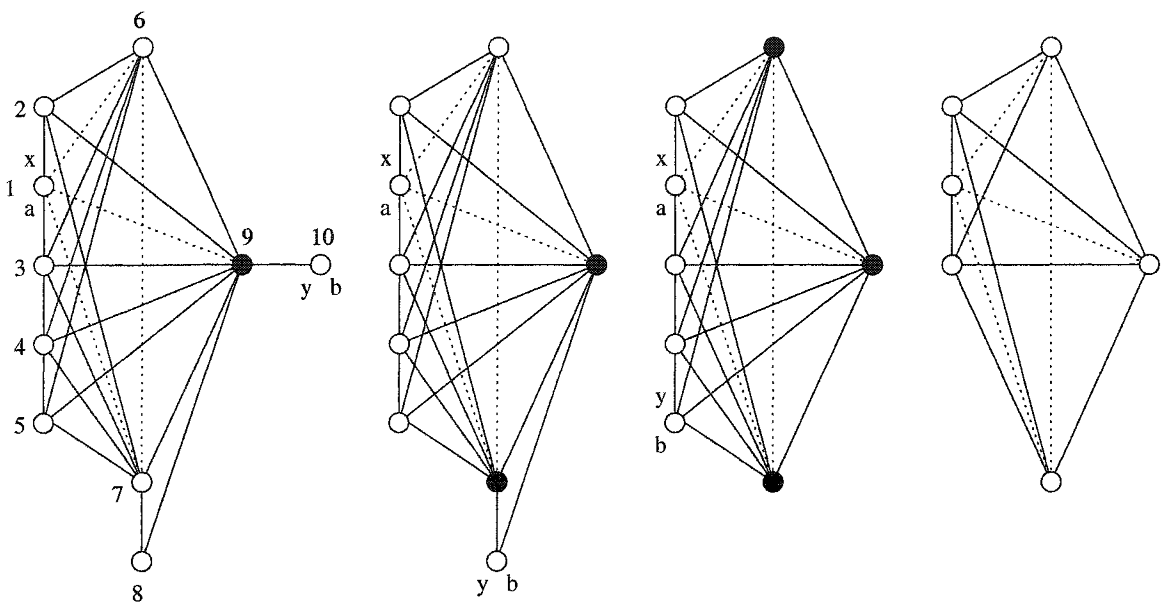

Figure 4.6: Illustrates the necessity of the induction in Theorem 4.2.1. The dashed edges are edges added to form a CPMT.

Example. The graph G in Figure 4.7 demonstrates the necessity of the different cases in Claim 4.7. To the right of it is an interval model proving that G is an interval graph. By Theorem 1.3.7, G is AT-free.

If we take $S = \{s_1, s_3\}$ we get (4) of Claim 4.7. If we take $S = \{s_2, s_4\}$ we get (1) of Claim 4.7. Finally, if we take $S = \{s_1, s_2\}$ we get (5)(a) of Claim 4.7. These are the only cases that can occur because all of the other cases in Claim 4.7 lead to contradiction.

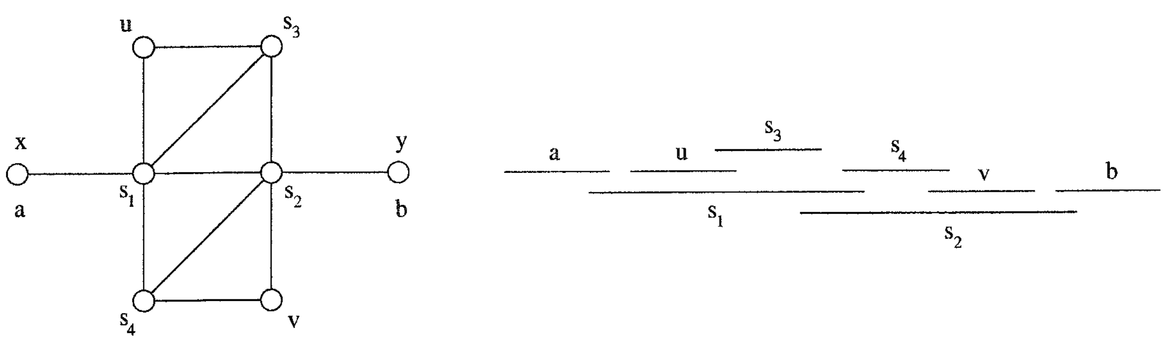

Figure 4.7: Illustrates the various cases in Claim 4.7 of Theorem 4.2.1. To the right is an interval model.

Chapter 5

Cocomparability Graphs

In this chapter, we analyse the Extendable Orientation Theorem, which expresses a relationship between transitive orientations of the complement and extendable orientations of the complement of the square. Motivated by this analysis, we present a new characterization of cocomparability graphs in terms of minimal separators.

With respect to cocomparability graphs, one interpretation of Corollary 4.0.9 is that an extendable orientation of the complement of the square is ordering a subset of what a transitive orientation of the complement orders. This next observation supports this interpretation.

Observation 5.0.2. *Let G be a connected cocomparability graph and $\rightarrow$ be a transitive orientation of $\overline{G}$. Then $\rightarrow$ restricted to $\overline{G^2}$ is an extendable orientation.*

Proof. Let $\Rightarrow$ be $\rightarrow$ restricted to $\overline{G^2}$. By Lemma 3.3.1, $\Rightarrow$ has the necessary properties with respect to Δ_G to be extendable. All that remains to be shown is that $\Rightarrow$ is transitive.

Suppose not and consider $u, v, w \in V(G)$ such that $u \Rightarrow v$, $v \Rightarrow w$, and $u \not\Rightarrow w$. By the transitivity of $\rightarrow$, $u \rightarrow w$. So $uw \in E(G^2)$ but $uw \notin E(G)$. Then consider $x \in V(G)$ such that $ux, wx \in E(G)$. If $vx \in E(G)$ then $uv \in E(G^2)$, a contradiction. So $vx \notin E(G)$. If $v \rightarrow x$ then $u \rightarrow v \rightarrow x$, a contradiction because $ux \in E(G)$. If $x \rightarrow v$ then $x \rightarrow v \rightarrow w$, a contradiction because $xw \in V(G)$. Therefore, $\Rightarrow$ is transitive. $\square$

Observation 5.0.2 and Corollary 4.0.9 raise two important questions: "If G is a cocomparability graph, what does a transitive orientation of the complement order in G?" and "What subset of that does an extendable orientation of $\overline{G^2}$ order?"

5.1 Limitations

In an effort to answer the previous two questions, we shall examine two limitations of the Extendable Orientation Theorem. Each limitation demonstrates an instance where an extendable orientation of the complement of the square does not capture all of the ordering information that it could.

Observation 5.1.1. *Let G be a connected AT-free graph, $\Rightarrow$ be an extendable orientation of $\overline{G^2}$, and H be a PMT of G. If F is a CPMT of H then there exists a transitive orientation of $\rightarrow$ of $\overline{F}$ such that if $u \Rightarrow v$ then $u \rightarrow v$.*

Proof. Repeated application of Corollary 2.2.6 indicates that F is a CPMT of G. The rest follows from Theorem 4.2.1. $\square$

Clearly, $E(\overline{H^2}) \subseteq E(\overline{G^2})$. So there may be less information an extendable orientation of $\overline{H^2}$ than in an extendable orientation of $\overline{G^2}$. However, both extendable orientations can be used to order $\overline{F}$. As in Corollary 2.2.6, the separator structure of H is a restriction of the separator structure of G. This hints that an extendable orientation of $\overline{G^2}$ is orienting something in the separator structure of H because it is still applicable to CPMTs of H. The Extendable Orientation Theorem uses $d_G(u, v) > 2$ as condition for orientation in a connected AT-free graph. We conjecture that this condition for orientation can be replaced with a condition directly related to separator structure which yields more ordering information. This next limitation leads to the same conjecture.

Definition. Let G be a graph. A different graph, denoted $G \times u$, can be derived by adding a new vertex u and making it adjacent to every vertex of G. Observe that $\text{diam}(G \times u) \leq 2$.

Observation 5.1.2. *A graph G is AT-free if and only if $G \times u$ is AT-free.*

Proof. If G has an AT then $G \times u$ also has an AT.

Suppose that $\{x, y, z\}$ is an AT in $G \times u$. Since u is adjacent every vertex of $V(G)$, we know that $u \notin \{x, y, z\}$. Let π be a x, y-path in $G \times u$ missing z. Clearly, u is not a vertex of π because $uz \in E(G \times u)$. So π is a x, y-path in G missing z. By symmetry, $\{x, y, z\}$ is an AT of G. □

Observation 5.1.3. *Let G be a graph. Then $\rightarrow$ is a transitive orientation of $E(\overline{G})$ if and only if $\rightarrow$ is a transitive orientation of $E(\overline{G \times u})$.*

Proof. $E(\overline{G}) = E(\overline{G \times u})$. □

We are interested in $G \times u$ because its separator structure is isomorphic to the separator structure of G:

Lemma 5.1.4. *Let G be a connected graph. Then for every $S \in \Delta_G$:*

1. $S \cup \{u\} \in \Delta_{G \times u}$
2. $C_G(S) = C_{G \times u}(S \cup \{u\})$
3. $\{s \in S : s$ *is adjacent to some vertex of* C *in* $G\} \cup \{u\} = \{s \in S \cup \{u\} : s$ *is adjacent to some vertex of* C *in* $G \times u\}$, *for* $C \in C_G(S)$

Moreover, for $T \in \Delta_{G \times u}$:

1. $T \setminus \{u\} \in \Delta_G$
2. $C_{G \times u}(T) = C_G(T \setminus \{u\})$
3. $\{t \in T : t$ *is adjacent to some vertex of* C *in* $G \times u\} \setminus \{u\} = \{t \in T \setminus \{u\} : t$ *is adjacent to some vertex of* C *in* $G\}$, *for* $C \in C_{G \times u}(T)$

Proof. Suppose $S \in \Delta_G$. Then there is a s,t-path in $G \setminus S$ if and only if there is a s,t-path in $G \times u \setminus (S \cup \{u\})$. So $S \cup \{u\}$ is a separator in $G \times u$ and $C_G(S) = C_{G\times u}(S \cup \{u\})$. The statement about the adjacency of S and $S \cup \{u\}$ to a component C is clearly true. Therefore, $S \in \Delta_{G\times u}$ by Lemma 2.1.1.

Suppose $T \in \Delta_{G\times u}$. Then T is a minimal x,y-separator in $G \times u$, for some $x, y \in V(G \times u)$. Now, $u \neq x$ and $u \neq y$ because u is adjacent to every other vertex of $G \times u$. Moreover, $ux, uy \in E(G \times u)$ so $u \in T$. There is a s,t-path in $(G \times u) \setminus T$ if and only if there is a s,t-path in $G \setminus (T \setminus \{u\})$. By the arguments above, the properties with respect to T and $T \setminus \{u\}$ follow. □

Corollary 5.1.5. *Let G be a connected graph. If H is a PMT of G then $H \times u$ is a PMT of $G \times u$. Similarly, if H is a PMT of $G \times u$ then $H \setminus \{u\}$ is a PMT of G.*

So the Extendable Orientation Theorem implies:

Corollary 5.1.6. *Let G be a connected AT-free graph and $\Rightarrow$ be an extendable orientation of $\overline{G^2}$. Then for every CPMT H of $G \times u$ there is a transitive orientation $\rightarrow$ of $\overline{H}$ such that $u \Rightarrow v$ implies $u \rightarrow v$.*

Proof. An immediate consequence of Observation 5.1.3, Corollary 5.1.5, and Theorem 4.2.1. □

So $G \times u$ preserves the separator structure of G in such a way that $E(\overline{(G \times u)^2}) = \emptyset$. Yet an extendable orientation of $\overline{G^2}$ is directly applicable to $G \times u$.

5.2 Alternate Characterizations

These results motivated a search for characterizations of AT-free and cocomparability graphs in terms of separator structure. As this next example shows, the number of minimal separators in a cocomparability graph may be exponential in the size of the graph.

Example. The shaded boxes in Figure 5.1 represent cliques in a graph G. So $\{s\} \cup K_1$ and $\{t\} \cup K_2$ are two disjoint cliques that partition G. If we direct every edge in $\overline{G}$ from $\{s\} \cup K_1$ to $\{t\} \cup K_2$ we have a transitive orientation. Thus, G is a cocomparability graph and hence is AT-free by Theorem 1.3.6. If we choose a single end point from every edge between K_1 and K_2 in G, we get a minimal s,t-separator. Hence, if $|K_1| = k$ then $|\Delta_G| \geq 2^k$.

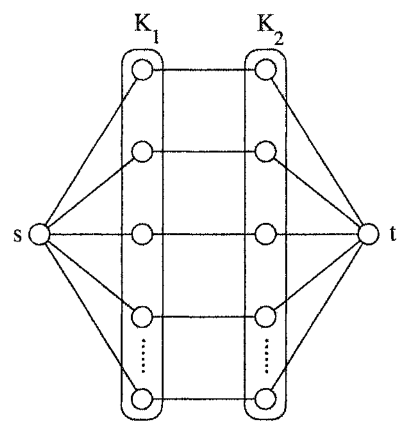

Figure 5.1: The number of minimal separators is not polynomially bounded by the size of the graph.

5.2.1 AT-Free Graphs

Theorem 2.3.1 characterizes AT-free graphs in terms of their separator structure. Since $|\Delta_G|$ can be quite large, Theorem 2.3.1 is not very interesting from an algorithmic perspective. Fortunately, Parra's proof can be readily modified.

Definition. Let G be a connected graph. When $S \in \Delta_G$ is a minimal u,v-separator close to u for some $u, v \in V(G)$, we call S a *close minimal separator.*

Observation. *Let G be a connected graph. The number of close minimal separators in G is $O(n^2)$.*

Proof. The minimal u,v-separator close to u is unique. □

Notation. Let G be a connected graph. If $uv \notin E(G)$ then we denote the minimal u, v-separator close to u as ${}_uS_v$.

Corollary 5.2.1. *Let G be a connected graph. Then G is AT-free if and only if among any three pairwise strongly parallel close minimal separators in G, there is a separator S such that the other two intersect different components of $G \setminus S$.*

Proof. Let $\{x, y, z\}$ be an AT in G. As there is a y, z-path in G missing x we know that $z \in C_G^y({}_xS_y)$. So ${}_xS_y = {}_xS_z$. By symmetry, ${}_yS_x = {}_yS_z$ and ${}_zS_x = {}_zS_y$. By Observation 2.1.6, ${}_xS_y$ is strongly parallel to ${}_yS_x$ and ${}_zS_x$. Moreover, ${}_yS_x, {}_zS_x \subseteq {}_xS_y \cup C_G^y({}_xS_y)$ by Lemma 2.1.11. Therefore by symmetry we have three pairwise strongly parallel close minimal separators ${}_xS_y$, ${}_yS_x$, and ${}_zS_x$ such that for any one the other two intersect the same component. Suppose without loss of generality that ${}_xS_y = {}_yS_x$. Then $z \in C_G^y({}_xS_y)$ and $z \in C_G^x({}_yS_x)$, a contradiction. So the separators are distinct.

Let X, Y, Z be distinct pairwise strongly parallel close minimal separators such that $Y, Z \subseteq X \cup C_X$, $X, Z \subseteq Y \cup C_Y$, and $X, Y \subseteq Z \cup C_Z$, for some $C_X \in \mathcal{C}_G(X)$, $C_Y \in \mathcal{C}_G(Y)$, and $C_Z \in \mathcal{C}_G(Z)$. Let x be a vertex of a full component of $G \setminus X$ other than C_X. Define y and z analogously. Note that $Y \nsubseteq X$ because X and Y are strongly parallel. By Lemma 2.1.8, $y \in C_X$ because $Y \subseteq X \cup C_X$. Similarly, $z \in C_X$. Thus there is a y, z-path in $G[C_X]$ missing x. By symmetry, $\{x, y, z\}$ is an AT in G. □

So AT-free graphs are characterized by the strongly parallel relation between (close) minimal separators. This motivates the following definition:

Definition. Let G be a connected graph. The *separator graph* of G, denoted $\Sigma(G)$, has Δ_G as its vertices and $ST \in E(\Sigma(G))$ if and only if S is strongly parallel to T in G.

Example. As the separator graphs in Figure 5.2 illustrate, we need to keep track of the intersection relationship with respect to components for the separator graph to be useful in terms of Theorem 2.3.1 because the left graph is an asteroidal triple and the right graph is AT-free.

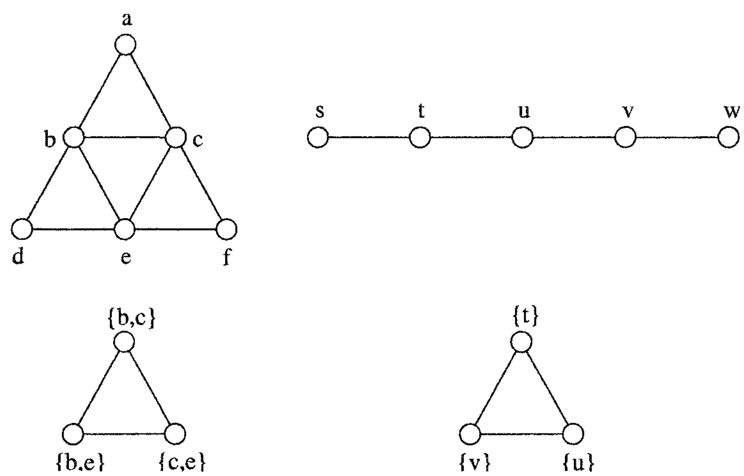

Figure 5.2: Some connected graphs and associated separator graphs.

Rather than introduce a complex labelling scheme, we do the following: Let S be any minimal separator in G. Split the vertex of $V(\Sigma(G))$ corresponding to S into pieces, where each piece represents a component of $G \setminus S$ that has a non-empty intersection with some strongly parallel minimal separator. Suppose that $S, T \in \Delta_G$ are strongly parallel. Then consider $C \in C_G(T)$, $D \in C_G(S)$ such that $S \subseteq T \cup C$ and $T \subseteq S \cup D$. In this case, we connect the piece of S corresponding to D to the piece of T corresponding to C.

Definition. The resulting graph, denoted $\Sigma'(G)$, will be called an *augmented separator graph.*

Example. The graph on the left in Figure 5.3 is a connected AT-free graph. To the right is its augmented separator graph.

Note that by Corollary 2.2.6, if H is a PMT of a connected graph G then $\Sigma'(H) = \Sigma'(G)[\Delta_H]$. Also observe that by Theorem 2.3.1, G is AT-free if and only if no three pieces in $\Sigma'(G)$ form a triangle.

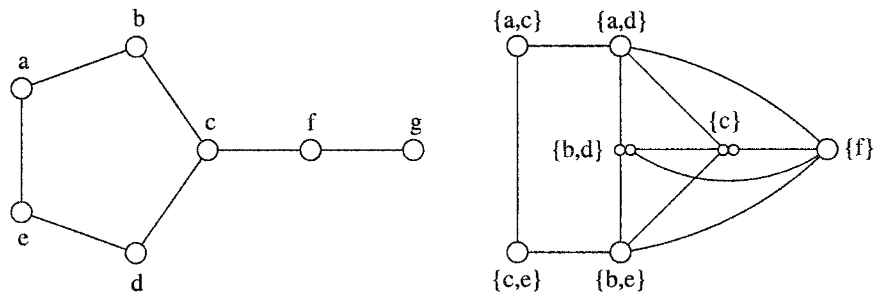

Figure 5.3: A connected AT-free graph and its corresponding augmented separator graph.

Example. The dashed edges in Figure 5.4 are added in a PMT of the graph represented by the solid edges. Observe that $\{b, e\}$ is a close minimal separator in the PMT but not the original graph.

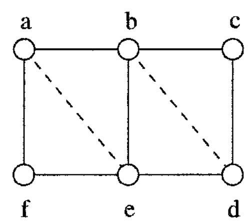

Figure 5.4: A close minimal separator in a PMT is not necessarily a close minimal separator in the original graph.

Observation 5.2.2. *Let G be a connected AT-free graph and S be a minimal separator in G. Then there are at most two full components of $G \setminus S$ with a vertex that is not adjacent to some vertex of S.*

Proof. Suppose not and let $C, D, E \in C^{\bullet}_{G}(S)$ have a vertex that is not adjacent to some vertex of S. Consider $c \in C$ and $s_c \in S$ such that $cs_c \notin E(G)$. Define d, e, s_d, and s_e analogously. Now, there is a d, s_c-path in $G[\{s_c\} \cup D]$ missing c. There is also a s_c, e-path in $G[\{s_c\} \cup E]$ missing c. By symmetry $\{c, d, e\}$ is an AT of G, a contradiction. □

Example. The top left graph in Figure 5.2 shows that this is not a sufficient condition to be AT-free.

Observation 5.2.3. *Let G be a connected graph. If $A, B \in \Delta_G$ are different strongly parallel minimal separators then $B \subseteq A \cup C$ where C is some full component of $G \setminus A$. Moreover, some vertex of A is not adjacent to some vertex of C.*

Proof. Let $A, B \in \Delta_G$ be strongly parallel minimal a, b-separators. Without loss of generality assume that $C^a_G(A) \subset C^a_G(B)$ and $C^b_G(A) \supset C^b_G(B)$ by Lemma 2.1.9. Consider $s_b \in B \setminus A$. There is a b, s_b-path in $G[\{s_b\} \cup C^b_G(B)]$ which avoids A because $C^b_G(A) \supset C^b_G(B)$. So $B \subseteq A \cup C^b_G(A)$. By symmetry $A \subseteq B \cup C^a_G(B)$. Consider $s_a \in A \setminus B$. Then $s_a \in C^a_G(B)$ and $bs_a \notin E(G)$. □

Observations 5.2.3 and 5.2.2 indicate that if G is AT-free then each vertex in $\Sigma'(G)$ will be split into at most 2 pieces.

5.2.2 Cocomparability Graphs

Definition. Let G be a connected graph and $\rightarrow$ be a transitive orientation of $\Sigma'(G)$. We call $\rightarrow$ a *separator orientation* of $\Sigma'(G)$ whenever the following conditions hold:

(1) if $S \rightarrow T \rightarrow U$ then S and U are adjacent to different pieces of T

(2) if either $S \rightarrow U$ and $T \rightarrow U$ or $U \rightarrow S$ and $U \rightarrow T$ then S and T are adjacent to the same piece of U.

Example. The augmented separator graphs in Figure 5.5 correspond to the graphs in Figure 5.2. The transitive orientation on the left is not a separator orientation because $\{b, e\} \rightarrow \{b, c\} \rightarrow \{c, e\}$ violates the first condition.

Theorem 5.2.4. *Let G be a connected graph. Then G is cocomparability if and only if $\Sigma'(G)$ has a separator orientation.*

Proof. Let G be a connected cocomparability graph and $\rightarrow$ be a transitive orientation of $\overline{G}$. Suppose $ST \in E(\Sigma'(G))$ and consider $C_2 \in C_G(S)$ such that

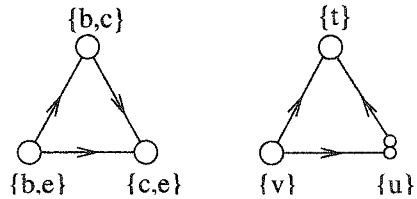

Figure 5.5: A transitive orientation is not necessarily a separator orientation.

$T \subseteq S \cup C_2$. By Lemma 5.2.3, $C_2 \in C^{\bullet}_G(S)$ and some vertex of S is not adjacent to some vertex of C_2. So consider $s \in S$ and $c_2 \in C$ such that $c_2 s \notin E(G)$. Then let $S \rightsquigarrow T$ if $s \to c_2$ and $T \rightsquigarrow S$ if $c_2 \to s$. By (2) of Lemma 3.3.1, this orientation is the same regardless of the particular s and c_2.

To show that $\rightsquigarrow$ is well defined consider $D_1 \in C_G(T)$ such that $S \subseteq T \cup D_1$. By Lemma 5.2.3, $D_1 \in C^{\bullet}_G(T)$ and some vertex of T is not adjacent to a vertex of D_1. So consider $t \in T$ and $d_1 \in D_1$ such that $d_1 t \notin E(G)$. Let C_1 be a full component of $G \setminus S$ other than C_2. Then $C_1 \subseteq D_1$ by Lemma 2.1.8. Similarly, let D_2 be a full component of $G \setminus T$ other than D_1. Then $C_2 \supseteq D_2$ by Lemma 2.1.8. Finally, consider $c_1 \in C_1$ and $d_2 \in D_2$. Suppose that $s \to c_2$. Then $c_1 \to c_2$ by (4) of Lemma 3.3.1. So $c_1 \to d_2$ by (2) of Lemma 3.3.1 because $d_2 \in C_2$. Since $c_1 \in D_1$, $d_1 \to d_2$ by (2) of Lemma 3.3.1. Therefore, $d_1 \to t$ by (4) of Lemma 3.3.1. Suppose that $c_2 \to s$. A similar argument shows that $d_2 \to d_1$. Therefore $t \to d_1$ by (3) of Lemma 3.3.1. Hence, $\rightsquigarrow$ is well defined. Moreover by Lemma 3.3.1, the two additional requirements for $\rightsquigarrow$ to be a separator orientation are true.

All that remains to be shown is that $\rightsquigarrow$ is transitive. Consider $S, T, U \in \Delta_G$ such that $S \rightsquigarrow T$ and $T \rightsquigarrow U$. As every piece is either a source or a sink of $\rightsquigarrow$, S and U must intersect different components of T. Let C_S and C_U be components of $G \setminus T$ such that $S \subseteq T \cup C_S$ and $U \subseteq T \cup C_U$. Clearly, $S \nsubseteq T$, $T \nsubseteq S$, $U \nsubseteq T$, and $T \nsubseteq U$ because S and U are both strongly parallel to T. By Lemma 5.2.3, consider $C_1, C_2 \in C^{\bullet}_G(S)$ of $G \setminus S$ such that $C_1 \neq C_2$ and $T \subseteq S \cup C_2$. Similarly, consider $D_1, D_2 \in C^{\bullet}_G(U)$ such that $D_1 \neq D_2$

and $T \subseteq U \cup D_1$. Now $C_1 \subseteq C_S$ by Lemma 2.1.8 because $S \not\subseteq T$. Similarly $C_S \subseteq D_1$ because $T \not\subseteq U$. So $C_1 \subseteq D_1$. By symmetry, $C_2 \supseteq D_2$. Therefore S and U are strongly parallel by Lemma 2.1.10. Since U intersects the same component of S as T, $S \rightsquigarrow U$.

Now suppose that G is not a cocomparability graph. By Theorem 1.3.6, let $\langle v_1, v_2, \ldots, v_k \rangle$ be a shortest odd wreath in $\overline{G}$. For convenience, the arithmetic with respect to all indices is modulo k. By definition, ${}_{v_{i+1}}S_{v_{i+2}}$ is the minimal v_{i+1}, v_{i+2}-separator close to v_{i+1} in G. The definition of a wreath indicates that there is a v_i, v_{i+2}-path missing v_{i+1} in G. Therefore, $v_i \in C_G^{v_{i+2}}({}_{v_{i+1}}S_{v_{i+2}})$ and ${}_{v_{i+1}}S_{v_{i+2}} = {}_{v_{i+1}}S_{v_i}$.

For contradiction, suppose that ${}_{v_{j+2}}S_{v_{j+3}} = {}_{v_{j+3}}S_{v_{j+4}}$ for some j. We know that $v_{j+1} \in C_G^{v_{j+3}}({}_{v_{j+2}}S_{v_{j+3}})$ and ${}_{v_{j+2}}S_{v_{j+3}} = {}_{v_{j+2}}S_{v_j}$. Similarly, $v_{j+4} \in C_G^{v_{j+2}}({}_{v_{j+3}}S_{v_{j+4}})$ and ${}_{v_{j+3}}S_{v_{j+4}} = {}_{v_{j+3}}S_{v_{j+2}}$. So by assumption, $v_{j+4} \in C_G^{v_{j+2}}({}_{v_{j+2}}S_{v_{j+3}})$ and ${}_{v_{j+2}}S_{v_{j+3}} = {}_{v_{j+3}}S_{v_{j+2}}$. If $k = 3$ then $v_{j+1} = v_{j+4}$ would be in two different components of ${}_{v_{j+2}}S_{v_{j+3}}$, a contradiction. Thus, $k \geq 5$ and $v_{j+1}v_{j+4} \notin E(G)$. Now there is a v_j, v_{j+2}-path in G missing v_{j+1}. There is also a v_{j+2}, v_{j+4}-path in $G[C_G^{v_{j+2}}({}_{v_{j+2}}S_{v_{j+3}})]$ missing v_{j+1}. So there is some v_j, v_{j+4}-path in G missing v_{j+1}. There is a v_{j+1}, v_{j+3}-path in $G[C_G^{v_{j+3}}({}_{v_{j+2}}S_{v_{j+3}})]$ missing v_{j+4}. By the definition of a wreath there is some v_{j+3}, v_{j+5}-path missing v_{j+4} in G. So there is some v_{j+1}, v_{j+5}-path in G missing v_{j+4}. Therefore, $\langle v_{j+1}, v_{j+4}, v_{j+5}, \ldots, v_{j+k} = v_j \rangle$ is a shorter odd wreath of $\overline{G}$, a contradiction.

Since ${}_{v_{i+1}}S_{v_{i+2}} = {}_{v_{i+1}}S_{v_i}$, ${}_{v_i}S_{v_{i+1}}$ is strongly parallel to ${}_{v_{i+1}}S_{v_{i+2}}$ by Lemma 2.1.6. Consequently, ${}_{v_i}S_{v_{i+1}} \subseteq {}_{v_{i+1}}S_{v_{i+2}} \cup C_G^{v_{i+2}}({}_{v_{i+1}}S_{v_{i+2}})$ by Lemma 2.1.11. Similarly, ${}_{v_{i+2}}S_{v_{i+3}} = {}_{v_{i+2}}S_{v_{i+1}}$ implies that ${}_{v_{i+1}}S_{v_{i+2}}$ is strongly parallel to ${}_{v_{i+2}}S_{v_{i+3}}$. Thus, ${}_{v_{i+2}}S_{v_{i+3}} \subseteq {}_{v_{i+1}}S_{v_{i+2}} \cup C_G^{v_{i+2}}({}_{v_{i+1}}S_{v_{i+2}})$ by Lemma 2.1.11. So $\langle {}_{v_1}S_{v_2}, {}_{v_2}S_{v_3}, \ldots, {}_{v_{k-1}}S_{v_k}, {}_{v_k}S_{v_1} \rangle$ is an odd cycle (possibly chorded) in $\Sigma'(G)$. But ${}_{v_i}S_{v_{i+1}}, {}_{v_{i+2}}S_{v_{i+3}} \subseteq {}_{v_{i+1}}S_{v_{i+2}} \cup C_G^{v_{i+2}}({}_{v_{i+1}}S_{v_{i+2}})$. So, the consecutive edges along the cycle are connected to a common piece. Hence, the direction of consecutive edges along the cycle must alternate in any separator orientation. This cannot happen because the cycle is odd. $\square$

Corollary 5.2.5. *Let G be a connected graph. Then G is cocomparability if and only if the augmented separator graph of G restricted to the close minimal separators of G has a separator orientation.*

Proof. Follows from the proof of Theorem 5.2.4. □

Example. $\langle b, d, a, c, e \rangle$ is an odd wreath in the complement of the graph in Figure 5.3. This corresponds to the odd cycle $\langle \{a, c\}, \{c, e\}, \{b, e\}, \{b, d\}, \{a, d\} \rangle$ in the augmented separator graph.

Example. In Figure 5.6, the graphs from left to right are a cocomparability graph, a transitive orientation of the complement, and a corresponding separator orientation.

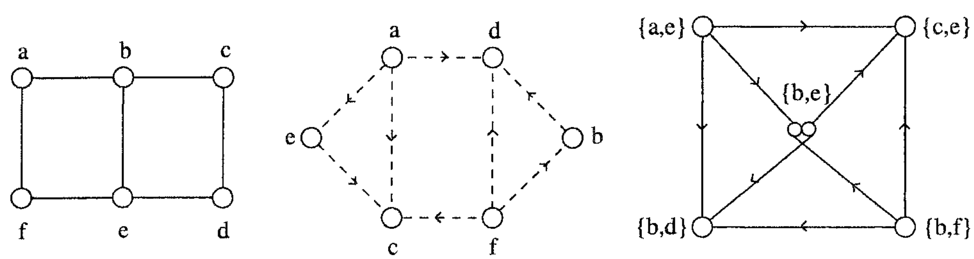

Figure 5.6: An example of a cocomparability orientation and corresponding separator orientation.

5.3 Summary

Earlier we asked two important questions: "If G is a cocomparability graph, what does a transitive orientation of the complement order in G?" and "What subset of that does an extendable orientation of $\overline{G^2}$ order?" Our new characterization shows that a transitive orientation of the complement is ordering strongly parallel minimal separators. It is only a partial order because two minimal separators in a cocomparability graph may not be strongly parallel. By Corollary 2.2.6 and Theorem 4.2.1, an extendable orientation orients the strongly parallel separators in the same way, but it does not order every pair of strongly parallel minimal separators.

In this thesis we started with a review of graph classes. Then we surveyed recent results with respect to minimal separators and the square of AT-free graphs. Our significant contributions are:

1. a characterization of dominating pairs in terms of minimal separators
2. an identification of strong properties of transitive orientations of the complement with respect to minimal separators
3. a demonstration that an extendable orientation is a generalization of a transitive orientation of the complement
4. a proof that an extendable orientation orders every minimal triangulation
5. a new characterization of cocomparability graphs in terms of a partial order of its minimal separators
6. an identification of limitations which suggest areas for further study

In a few words, these results are important because we have an alternative interpretation of the linearity of cocomparability graphs. Moreover, AT-free graphs clearly generalize this linearity. Finally, these results are not sharp and we have identified how they may be strengthened.

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Appendix A

Partial Results

Observation 3.3.2 was used considerably in the results leading up to Theorem 4.2.1; even the statement of Theorem 4.2.1 depends on this observation. One potential generalization of Theorem 4.2.1 is to substitute G_{Δ_G} for G^2 in our definition of extendable orientation. This appendix contains results in this direction.

Definition. Let G be a connected graph. Then G_{Δ_G} is called the *cycle completion* of G because by Observation 2.2.3, G_{Δ_G} is the result of making every long cycle in G a clique.

A.1 Cycle Completions

Lemma A.1.1. *Let G be a connected graph. If there is a x, y-path missing z in G_{Δ_G} then there is a x, y-path missing z in G.*

Proof. Our proof will be inductive on the length k of such a path in G_{Δ_G}. Suppose that $k = 1$. Then there exists some $S \in \Delta_G$ such that $x, y \in S$ but $z \notin S$. Consider $C \in C_G^{\bullet}(S)$ such that $z \notin C$. Then any x, y-path through $G[\{x, y\} \cup C]$ misses z.

Assume that if there is a x, y-path missing z in G_{Δ_G} of length $\leq k$ then there is a x, y-path missing z in G. Let $\langle x = v_0, v_1, \ldots, v_{k+1} = y\rangle$ be a x, y-path missing z in G_{Δ_G}. So $\langle x = v_0, v_1, \ldots, v_k\rangle$ is a x, v_k-path missing z in G_{Δ_G}. By our induction hypothesis there is some x, v_k-path π_1 missing z in G. Similarly $\langle v_k, v_{k+1} = y\rangle$ is a v_k, y-path missing z in G_{Δ_G}. Again, there is some v_k, y-path π_2 missing z in G. Hence, $\pi_1 \cdot \pi_2$ is a x, y-path missing z in G. Therefore, there is always a x, y-path missing z in G by induction. □

Corollary A.1.2. *Let G be a connected AT-free graph. Then G_{Δ_G} is AT-free.*

We have not determined whether being connected and AT-free is sufficient for the cycle completion to be cocomparability. However, this next example shows that connected and AT-free is not sufficient for the cycle completion to be interval.

Example. The independent triples of the graph G in Figure A.1 are $\{a, d, e\}$, $\{a, d, f\}$, $\{c, b, g\}$, and $\{c, b, h\}$. None of these form an AT in G. Since the longest cycle of any AT-free graph is a C_5, cd does not chord any induced cycle because c and d have no common neighbour. Similarly, ab chords no induced cycle. So $\langle a, b, c, d\rangle$ is an induced C_4 of G_{Δ_G}.

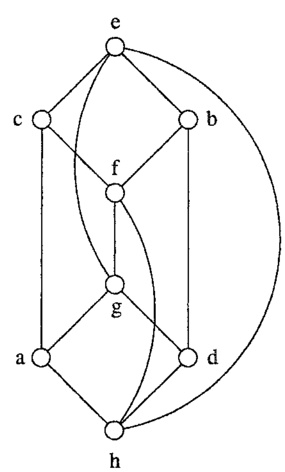

Figure A.1: The cycle completion of an AT-free graph need not be interval.

Observation A.1.3. *Let G be a connected graph and $\rightarrow$ be a transitive orientation of $\overline{G}$. Then $\rightarrow$ restricted to $\overline{G_{\Delta_G}}$ is transitive.*

Proof. Let $\Rightarrow$ denote $\rightarrow$ restricted to $\overline{G_{\Delta_G}}$. Consider for contradiction $u, v, w \in V(G)$ such that $u \Rightarrow v$, $v \Rightarrow w$, and $u \not\Rightarrow w$. Then there is some $S \in \Delta_G$ such that $u, w \in S$ but $v \notin S$. This contradicts (2) of Lemma 3.3.1. $\square$

Corollary A.1.4. *Let G be a connected graph and $\rightsquigarrow$ be a transitive orientation of $\overline{G}$. Let $\Rightarrow$ be $\rightsquigarrow$ restricted to $\overline{G_{\Delta_G}}$. Then for any CPMT H of G there is a transitive orientation $\rightarrow$ of $\overline{H}$ such that $u \Rightarrow v$ implies $u \rightarrow v$.*

Proof. Follows from Observations A.1.3 and 4.0.8. $\square$

The previous result indicates that the equivalent of the Extendable Orientation Theorem holds for cocomparability graphs when we substitute G_{Δ_G} for G^2. This next example shows that this is not true for all connected AT-free graphs.

Example. The solid edges of the graph in the top left of Figure A.2 form a connected AT-free graph G. Now, $\{c, e, f\}$ is the minimal b, g-separator close to b and $\{b, d, g\}$ is the minimal c, f-separator close to c. Hence, the dashed edges of the graphs in the left column are edges added in PMTs of G. To the right of each PMT, we have given the *only* transitive orientations of each complement. Observe that in the top row ad and de must always be oriented in different directions with respect to d. Conversely, in the bottom row ad and de must always be oriented in the same direction with respect to d.

Now, ad and de chord no long cycle of G. So $ad, de \notin E(G_{\Delta_G})$. Let $\Rightarrow$ be any transitive orientation of $\overline{G_{\Delta_G}}$. Suppose that $\Rightarrow$ orients ad and de in the same direction with respect to d. Then the top row is a CPMT H of G such that for every transitive orientation $\rightarrow$ of $\overline{H}$, $\Rightarrow$ disagrees with $\rightarrow$. Suppose that $\Rightarrow$ orients ad and de in different directions with respect to d. Then the bottom row is a CPMT H of G such that for every transitive orientation $\rightarrow$ of $\overline{H}$, $\Rightarrow$ disagrees with $\rightarrow$.

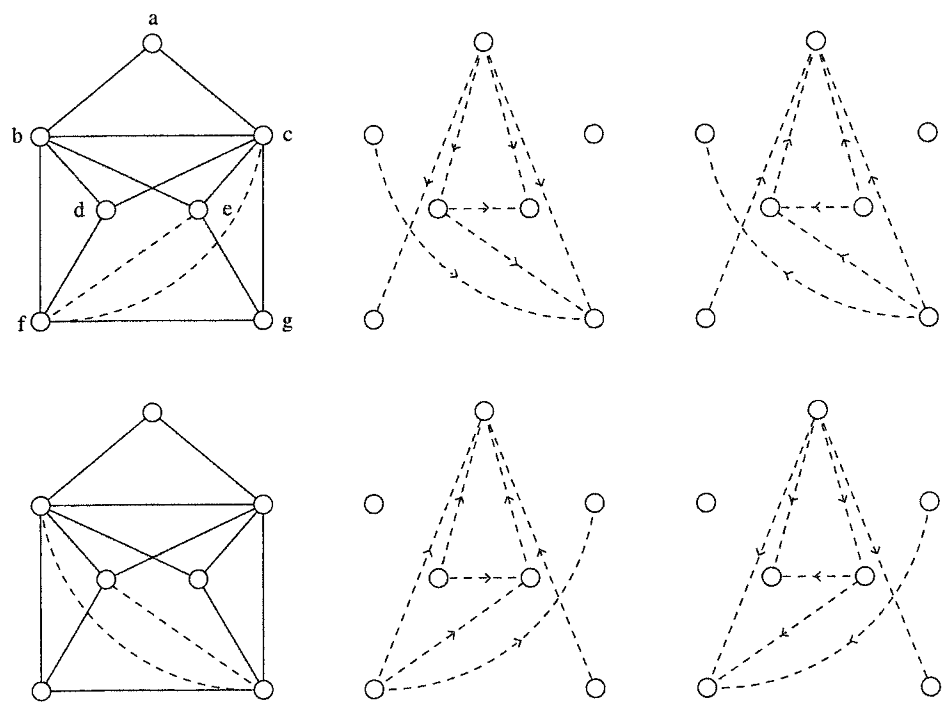

Figure A.2: Theorem 4.2.1 does not hold substituting G_{Δ_G} for G^2.

Definition Index

Notation Index