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Positive Global Solutions of Nonlinear Elliptic Equations

BY
YIN XI HUANG

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TO MY BELOVED GRANDMOTHER

ABSTRACT

In this thesis we study the existence of global positive solutions of quasilinear elliptic equations and systems. We first introduce some weighted spaces, by means of which we obtain a priori estimates for second order elliptic operators. We then use fixed point theorems to obtain the existence of solutions bounded above and below by positive constants. Together with a spectral procedure and sub-supersolution method, the weighted space–a priori estimate–fixed point theorem approach is further modified to give criteria for the existence of decaying positive solutions. In this way we are able to answer a recent open question for the mixed sublinear-superlinear problem. Appropriate adaptations are then made in order to study degenerate equations and higher order elliptic systems. The existence of bounded positive solutions is shown for both cases. Because of the nonvariational and nonradially symmetric nature of our approach, we can deal with those problems which are not amenable to variational methods nor to procedures involving ordinary differential equations.

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Chapter 1

INTRODUCTION

The study of quasilinear elliptic equations has a long history. At the beginning of the twentieth century Hilbert prophetically raised, along with other questions, his twentieth problem: the solvability of boundary value problems, see Serrin [74]. From then on, the study was carried out in an encyclopedic way and divided into many branches, among which the existence of positive solutions of the quasilinear equation:

$$-\Sigma D_j(a_{ij}(x,u)D_iu) + \Sigma b_i(x)D_iu + cu = f(x,u,\nabla u) \tag{1.1}$$

is of importance. It was observed that problems originating from a variety of practical considerations resulted in equations which were special forms of (1.1). The Emden–Fowler equation: $-\Delta u = f(x)u^\lambda$, arising from the study of the equilibrium configuration of the mass of spherical clouds of gases, received extensive investigation along with its extensions, see Wong [78]. In particular, for $f(x) = (1+|x|^2)^{-1}$, $\lambda > 1$, this is the Matukuma type equation describing the dynamics of a globular cluster of stars, see Ni and Yotsutani [NY]. A similar mathematical problem also arose in studies of the thermal ignition of a chemically active mixture of gases, in considerations of the equilibrium state in a fluid with spherical distribution of density and under mutual attraction of its particles, and in the boundary layer theory of viscous fluids, see Kusano and Swanson [49]. More recently, the Emden–Fowler equation also appears in the theory of nonlinear diffusion generated by nonlinear sources, in membrane buckling, in relativistic mechanics and in nuclear physics.

In the search for some special kinds of solitary waves in nonlinear equations of Klein–Gordon or Schrodinger type, we are also led to some nonlinear second order elliptic equations similar to (1.1). In fact, for the Klein–Gordon equation: $\varphi_{tt} - \Delta\varphi + a^2\varphi = f(\varphi)$, looking for stationary wave of the form $\varphi(x,t) = e^{iwt}u(x)$ leads to the study of

$$-\Delta u + (a^2 - w^2)u = f(u) \quad \text{if} \quad f(e^{i\theta}\varphi) = e^{i\theta}f(\varphi),$$

while the steady wave of the Schrodinger equation:

$$i\varphi_t - \Delta\varphi = f(\varphi) \quad \text{of the form} \quad \varphi = e^{-iwt}u(x)$$

also gives rise to the same type of equation, cf. Berestycki and Lions [11]. One also comes across similar problems in the study of statistical mechanics, constructive field theory, false vacuum in cosmology, nonlinear optics and laser propagation.

In recent years, the study of population dynamics has been widely undertaken. In his book [66], Okubo describes three types of biological diffusion that appear in the theory of population dynamics. In terms of differential equations, all of them are of the type:

$$\frac{\partial u}{\partial t} - \Sigma D_i(a_{ij}(x,u)D_j u) = f(x,t,u),$$

of which the standard heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = f(u)$$

is a particular case. Positive solutions of these equations have a practical meaning and are what we are after for.

From the point of view of mathematics, some geometric problems also give rise to similar problems, see Kenig and Ni [KN1] and Ni [58] for more details.

Because of its importance in the theoretical, as well as in the practical fields, mentioned above, the study of problem (1.1) on both bounded and unbounded domains has attracted much attention. We only mention the articles by Donato and Giachetti [22,23], Noussair and Swanson [61,62,63], Kazdan and Kramer [42], Brezis and Turner [15], Boccardo, Murat and Puel [14], Pohozaev [69], Hess [33,34], Các [16,17], Serrin [73,74], Berestycki and Lions [11], Ni [58,59], Ni and Serrin [60], Gidas and Spruck [28], Joseph and Lundgren [35], Kusano and Swanson [49,50,52], Ding and Ni [DN], Kenig and Ni [KN1, KN2], Li and Ni [LN], Ni and Yotsutani [NY] and Allegretto [3]. See also the expository–survey paper of Ni [N]. More detailed references will be given in each chapter. However, much of the study is either through the "weak" approach, i.e., weak solutions are obtained (we remark that for most cases, $u \equiv 0$ is always a weak solution), through variational techniques, or is restricted to the study of radially symmetric cases, i.e., the coefficients and functions involved are assumed to be radially symmetric, and an ordinary differential equation approach is most often used. For the nonradially symmetric case, we are not aware of such powerful techniques, especially when trying to deal with the problem of the existence of global positive solutions of quasilinear equations.

Over the years, more and more efforts have been put into the study of quasilinear equations. We mention the classical study of Serrin [73] as a major contribution. While considering only bounded domains, Kazdan and Kramer [42], using sub-supersolution technique, proved some existence theorems for general equations of the form $-\Sigma a_{ij}D_iD_ju + \Sigma b_iD_iu = f(x, u, \nabla u)$, while Brezis and Turner [15] obtained existence results through a priori estimates and fixed point theorem arguments. For general unbounded domains, the situation is considerably more complicated. Some of the difficulties arise from the lack of the compact embedding theorems which hold for bounded domains. Another difficulty is due to the failure of Poincare type inequalities in $W_0^{1,p}(\Omega)$ for general unbounded domain Ω. Some authors overcame those difficulties by using weighted Sobolev spaces (see, eg. [10,12,17]) and/or by using the weak solution approach with the help of sub-supersolution techniques (see, eg. [16,22,23,33]). However, in the first place it is often very difficult to construct ordered pairs of sub-supersolutions, and, secondly, we are interested in finding global positive solutions with specific (decaying) behavior at infinity. Observe that positive solutions bounded away from zero (along with some decaying solutions) are not in any of the usual Sobolev spaces defined on R^n.

In this thesis we obtain criteria for the existence of global positive solutions with specific behavior at infinity for quasilinear second order elliptic equations without any assumptions of radial symmetry nor of variational structures. Our main tools will be classical a priori estimates, weighted Sobolev spaces, Schauder fixed point theorem arguments and sub-supersolution methods. Our study differs from previous ones in the following aspects: First, we are mainly interested in those nonsymmetric quasilinear problems containing highly nonradial terms, and hence the usual ordinary differential equation arguments and/or variational methods are not applicable. Secondly we obtain bounded global positive solutions which are bounded away from zero. Thirdly, we also obtain some results about the asymptotic behavior of the solutions, in particular we can employ our results to derive the existence of some decaying positive solutions. Fourthly, the relevant a priori constants can be estimated explicitly. This is crucial in our application to the mixed sublinear-superlinear equation, and it enables us to answer, for the nonradial case, an open question posed in Kusano and Trench [53]. Finally the same methods are extended to obtain global positive solutions for higher order systems and some specific degenerate equations. These degenerate equations arise in applications which include biological models with dispersion to avoid crowding.

This thesis is arranged as follows: In Chapter 2, we introduce some terminology and notations which we need in the sequel. In particular, we introduce some special weighted norms and prove their basic properties. In Chapter 3 we start our estimation of a priori constants, and weighted norms are used to derive a priori estimates for the solutions of some generic second order elliptic equations. The details of the estimation are postponed to Appendix A at the end of this thesis, for they are tedious and lengthy. In Chapter 4 we state and prove our basic existence theorem. In particular, criteria for the existence of positive solutions bounded above and below by positive constants will be obtained by using the Schauder fixed point theorem. Next, in Chapter 5, we adopt the theoretical ideas of Chapter 4, combined with the sub-supersolution method and spectral procedures, to obtain the existence of decaying solutions. We extend our methods to degenerate equations and higher order systems in Chapters 6 and 7 respectively. In each of Chapters 4, 5, 6 and 7, we give applications and make comparison with previously known results. Finally, Chapter 8 concludes the thesis with some remarks and open questions.

Chapter 2

BACKGROUND MATERIAL

The purpose of this chapter is to set up the notational framework for our study. First we will introduce some notations and terminology, in principle following Allegretto [3]. After defining some special weighted norms, we prove that the two norms we introduce in this chapter are not equivalent. Thus appropriate choice of weight functions for actual problems will result in sharper conclusions, as we will see in the sequel.

For a given function $0 < t \in C^\infty(R^n)$, we denote by $L^p_t(D)$ the associated weighted L^p space in a domain D in R^n with norm $\|\varphi\|^p_{L^p_t(D)} = \int_D t \cdot |\varphi|^p dx$. For any $x \in R^n$, we define the ball $B_i(x) = \{y \mid |x-y| < i\}$ and

$$N(\varphi, p, i, D) = \sup_{x \in D} \left[\|\varphi\|_{L^p(B_i(x))}\right].$$

Due to the Hardy inequality in R^n, $n \geq 3$, employed in Chapter 3, we will fix from now on the weight function $t = 1 + |x|^2$. We assume that the other weight function λ we introduce in the sequel, is smooth and $0 < \lambda^{-1} \in L^{n/2}(R^n)$. The explicit choices of λ will depend on the problem under consideration, as we shall see later. Now we can form two weighted spaces $L^2_\lambda(R^n)$ and $L^2_t(R^n)$. Note that $L^2_\lambda \subset L^2_t$ if we choose a λ radial, which increases fast enough at infinity. For nonradially symmetric λ, we have in general the following

THEOREM 2.1. *$L^2_\lambda(R^n)$ and $L^2_t(R^n)$ are two different spaces.*

Proof. Obviously, $t^{-1} = (1 + |x|^2)^{-1} \notin L^{n/2}(R^n)$. Let

$$s(|x|) = (1 + |x|^2)^{-n/2}, \qquad \text{then} \qquad \int_{R^n} s(|x|) = \infty.$$

Define

$$\left(\varphi(r)\right)^{\frac{2n}{n+2}} = s(r)\Big/\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n+2}}.$$

We claim that $\varphi \in L_t^2$, but $\varphi \notin L_\lambda^2$. Observe that

$$\int_0^R \frac{r^{n-1}s(r)dr}{\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n+2}}} = (n+2)\left[\left(1+\int_0^R r^{n-1}s(r)dr\right)^{\frac{1}{n+2}} - 1\right],$$

which tends to infinity as $R \to \infty$, thus

$$\int_{R^n} \varphi^{\frac{2n}{n+2}} = \int_{R^n} \frac{s(r)dx}{\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n+2}}}$$

$$= \omega_n \cdot \int_0^\infty \frac{r^{n-1}s(r)dr}{\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n+2}}} = \infty,$$

where ω_n is the surface area of the unit ball in R^n.

For λ, $0 < \lambda^{-1} \in L^{n/2}(R^n)$, we have

$$\infty = \int_{R^n} \varphi^{\frac{2n}{n+2}} \cdot \lambda^{\frac{n}{n+2}} \cdot \lambda^{-\frac{n}{n+2}}$$

$$\leq \left(\int_{R^n} \left(\varphi^{\frac{2n}{n+2}}\lambda^{\frac{n}{n+2}}\right)^{\frac{n+2}{n}}\right)^{\frac{n}{n+2}} \left(\int_{R^n} \lambda^{-\frac{n}{n+2}\cdot\frac{n+2}{2}}\right)^{\frac{2}{n+2}}$$

$$= \left(\int_{R^n} \varphi^2\lambda\right)^{\frac{n}{n+2}} \left(\int_{R^n} \lambda^{-\frac{n}{2}}\right)^{\frac{2}{n+2}}.$$

Thus we conclude that $\int_{R^n} \varphi^2\lambda = \infty$, i.e., $\varphi \notin L_\lambda^2(R^n)$, φ is independent of λ. On the other hand, observe that

$$\int_{R^n} \varphi^2 t = \int_{R^n} (1+|x|^2)\frac{s(r)^{\frac{n+2}{n}}}{\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n}}}$$

$$= \int_{R^n} \frac{s(r)}{\left(1+\int_0^r \xi^{n-1}s(\xi)d\xi\right)^{\frac{n+1}{n}}} < \infty,$$

that is, $\varphi \in L^2_t(R^n)$. Thus we proved that L^2_λ does not contain L^2_t. To prove the other half of the theorem, let us choose $\lambda(x) = \lambda_1(x_1)\lambda_2(x_2, \ldots, x_n)$ with λ_2 to be chosen below. Choose

$$\lambda_1(x_1) = (1+|x_1|)^{2/n} \ln^{4/n}(2+|x_1|), \quad \text{then} \quad \int \lambda_1^{-n/2} dx_1 < \infty.$$

We also choose a φ such that

$$\varphi(x) = \varphi_1(x_1)\varphi_2(x_2, \ldots, x_n)\,.$$

with φ_2 to be determined along with λ_2 below. We set

$$\varphi_1^2(x_1) = \lambda_1^{-1}(1+|x_1|)^{-1} \ln^{-1-\frac{2}{n}}(2+|x_1|)$$

$$= (1+|x_1|)^{-1-\frac{2}{n}} \ln^{-1-\frac{6}{n}}(2+|x_1|).$$

Observe that

$$\int_0^\infty \varphi_1^2(1+|x_1|^2)dx_1 \geq c \cdot \int_0^\infty (1+|x_1|)^{1-\frac{2}{n}} \ln^{-1-\frac{6}{n}}(2+|x_1|)dx_1 = \infty,$$

$$\int_0^\infty \varphi_1^2 \lambda_1 dx_1 = \int_0^\infty (1+|x_1|)^{-1} \ln^{-1-\frac{2}{n}}(2+|x_1|)dx_1 < \infty.$$

Thus, we conclude that with suitable choices of λ_2 with $\lambda_2^{-1} \in L^{n/2}$ and φ_2 such that $\varphi_2 \in L^2_{\lambda_2} \cap L^2_t$, where the function spaces are defined on R^{n-1}, we have $\varphi \in L^2_\lambda$, but $\varphi \notin L^2_t$.

We have proved that L^2_λ and L^2_t are two different weighted spaces and the proof is complete.

For $q > n$, fixed in the sequel, consider the space $S \subset L^q_{\text{loc}}(R^n)$ with the norm

$$\|s\|_S = N(s, q, 2, R^n),$$

and $s \in S$ if $\|s\|_S < \infty$, then $\{S, \|\cdot\|_S\}$ is a Banach space. We further define $\mathcal{L}_1 = L^2_\xi \cap S$ equipped with norm

$$\|u\|_{\mathcal{L}_1} = \|u\|_{L^2_\xi(R^n)} + \|u\|_S$$

and $\mathcal{L}_2 = L^2_\lambda \cap S$ with norm

$$\|u\|_{\mathcal{L}_2} = e\|u\|_{L^2_\lambda(R^n)} + \|u\|_S,$$

where e is a positive constant, to be chosen explicitly in Theorem 3.4 below. By Theorem 2.1, $\mathcal{L}_1$ and $\mathcal{L}_2$ are two different spaces. Next consider

$$P = \{(u_1, u_2) \mid u_1 \in \mathcal{L}_1,\ u_2 \in \mathcal{L}_2\}$$

and define on P the relation $\sim$ given by: $(u_1, u_2) \sim (u_3, u_4)$ iff $u_1 + u_2 = u_3 + u_4$, a.e. It is easily checked that $\sim$ is an equivalence relation. Let $\mathcal{H}$ be the quotient space $\mathcal{H} = P/\sim$ and define on $\mathcal{H}$ the norm

$$\|(u_1, u_2)\|_{\mathcal{H}} = \inf\{\|v_1\|_{\mathcal{L}_1} + \|v_2\|_{\mathcal{L}_2} \mid (u_1, u_2) \sim (v_1, v_2)\}.$$

We further define a map $J : \mathcal{H} \to L^q_{\text{loc}}(R^n)$ by $J((u_1, u_2)) = u_1 + u_2$. Obviously J is well defined and 1–1 by construction. We have

THEOREM 2.2. *J is linear and has the following order property: if $f \in \text{Range}(J)$ and $|g| \le |f|$, a.e., then $g \in \text{Range}(J)$.*

Proof. Obviously J is linear.

If $f = J((f_1, f_2))$, we define

$$g_1 = g|f_1|/(|f_1| + |f_2|), \quad g_2 = g|f_2|/(|f_1| + |f_2|), \quad \text{if} \quad |f_1| + |f_2| \neq 0,$$

while $g_1 = g_2 = 0$ if $|f_1| + |f_2| = 0$. Then we have $g = J((g_1, g_2))$, hence $g \in$ Range(J). This ends the proof.

On Range(J) we define a norm $M(\cdot)$ by

$$M(f) = \|J^{-1}(f)\|_{\mathcal{H}}.$$

Observe that if $|g| \leq |f|$ then $M(g) \leq M(f)$ while if $f = f_1 + f_2$ with $f_1 \in \mathcal{L}_1$, $f_2 \in \mathcal{L}_2$, then $M(f) \leq \|f_1\|_{\mathcal{L}_1} + \|f_2\|_{\mathcal{L}_2}$.

We note that the norm $M(\cdot)$ and its modifications will be used throughout this thesis, and play an important role in our applications.

Finally, $W_0^{1,p}(\Omega)$ denotes the usual Sobolev space on $\Omega \subset R^n$, bounded or unbounded, and $n \geq 3$.

Chapter 3

A PRIORI ESTIMATES

3.1. Introduction.

Our objective of this chapter is to introduce some a priori estimates, upon which much of the thesis is based. More precisely, for a generic second order elliptic operator

$$\ell u \equiv -\Sigma D_i(a_{ij}(x)D_j u) + \Sigma b_i D_i u + cu$$

with appropriate a_{ij}, b_i and c, we will present some inequalities that the solutions u of the equation

$$\ell u = \Sigma D_i(f_i) + g$$

will satisfy, that is, some inequalities of the form

$$\|u\|_{C^0(B(x_0))} \le C_1 N(f_i, g), \tag{3.0.1}$$

$$\|\nabla u\|_{C^0(B(x_0))} \le C_2 N(f_i, g), \tag{3.0.2}$$

where C_1, C_2 depend on n (dimension of the space) and the operator ℓ, but are independent of u, N is some "norm" of f_i and g, to be explicitly given below.

The existence of inequalities (3.0.1) and (3.0.2) and the estimation for the constants C_1 and C_2 are long known, see Allegretto [3], Ladyzhenskaya and Uraltseva [55] and Gilbarg and Trudinger [29]. The idea of a priori estimates has been proved to be fundamental. Here we are not presenting a "brand new" estimation, but rather we are interested in obtaining some explicit bounds on C_1 and C_2, which as we will see are crucial to us.

Due to the global nature of our problem, the estimates are easier to obtain in the sense that we only need interior estimates. Thus we do not consider the boundary behavior of u and ∇u, as was done in Brezis and Turner [15], de Figueiredo, Lions and Nussbaum [24]. Also, we take a more classical approach, thus the estimation does not rely on the nonlinear structure nor on the symmetry of the elliptic operator, the latter playing an important part in the estimates of [24]. However, because of the global nature of the present problem, we observe that the classical compactness theorems and their extensions are no longer valid for general unbounded domains, and neither is the Poincare inequality. A Poincare-type inequality will however be established by using weighted function spaces. This idea was employed in Berger and Schechter [12], Benci and Fortunato [10], Các [17] and Allegretto [3] to obtain the desired results in different situations. Here we adopt the choice and utilization of weighted inequalities of [3], and our choice of the weighted "norms" on f_i and g will reflect and at the same time justify the above ideas.

In Section 3.2, we will use the classical approach to obtain an L^∞ a priori estimate for u in some norm of f_i and g, as was done in Gilbarg and Trudinger [29] and Ladyzhenskaya and Uraltseva [55]. Our intention is to obtain the explicit bound for C_1, and as we mentioned earlier, only the linear part of the elliptic operator contributes in our estimation, since it can be estimated explicitly with a reasonable amount of effort. Thus the bound is not as sharp, but serves our purpose. We then combine the weighted space idea and the estimate we obtained and eventually reach our goal: an L^∞ a priori bound for ∇u. The explicit bounds for u and ∇u will then be employed in the subsequent development. Since the proofs for these estimates are somewhat lengthy, we postpone them to Appendix A.

3.2. The Basic Estimations.

For our purpose, we find it most beneficial to blend the ideas employed in [3], [29], [55] and elsewhere and to work with vector and matrix functions. In what follows we will assume that the vectors and matrices are normed by the standard Hilbert space norm: for $\vec{v} = (v_0, \ldots, v_m)^T$, $|\vec{v}|^2 = \sum_{i=0}^{m} v_i^2$, etc. Furthermore, we assume that $a_{ij} \in L^\infty(R^n)$ and

$$\lambda_0|\xi|^2 \leq \Sigma a_{ij}\xi_i\xi_j \leq \lambda_1|\xi|^2$$

for some $\lambda_0 > 0$, $\lambda_1 \geq 1$. The main theorem of this chapter is:

THEOREM 3.1. *Let $\vec{u} = (u_0, \ldots, u_m)^T$ be a solution to the system*

$$-\Sigma D_i(a_{ij}(x)D_j\vec{u}) + 2\Sigma B^j(x)D_j\vec{u} + C\vec{u} = -\Sigma D_i(\vec{f}_i) + \vec{g} \tag{3.1}$$

in a ball $B_2(x_0)$. Suppose $\vec{u} \in C^\alpha \cap W^{1,2}(B_2(x_0))$, the vector $\vec{g}$ and the $(m+1) \times (m+1)$ matrix C belong to $L^{q/2}(B_2(x_0))$, the $(m+1) \times (m+1)$ matrices B^j belong to $L^q(B_2(x_0))$, while the vectors $\vec{f}_i$ are in $L^q(B_2(x_0))$ for some $q > n$. Then:

$$\big\||\vec{u}|\big\|_{L^\infty(B_1(x_0))} \leq K_0\left[\big\||\vec{u}|\big\|_{L^2(B_2(x_0))} + \Sigma\big\||\vec{f}_i|^2\big\|^{1/2}_{L^{q/2}(B_2(x_0))} + \big\||\vec{g}|\big\|_{L^{q/2}(B_2(x_0))}\right], \tag{3.2}$$

where $K_0 = K_1(\mu(B_2)^{1/2} + 1)$, *with* $\mu(B_2)$ *the Lebesgue measure of ball* B_2 *in* R^n,

$$K_1 = \left(4H\left(2^{\frac{3}{2}}\ \frac{\lambda_1}{\lambda_0}\right)^{q/(q-n)}\right)^{\frac{n}{2}} \cdot \left(2\left(\frac{2}{n-2}\right)^{\frac{3}{2}\cdot q/(q-n)}\right)^{\frac{n(n-2)}{4}},$$

$$H = T^2(4 + C(\beta_1))$$

$$+ 2\left(T^2C(\beta_1)\left\{\||C|\|_{L^{q/2}(B_2(x_0))} + \|\Sigma|B^i|^2\|_{L^{q/2}(B_2(x_0))} + 2\right\}\right)^{q/(q-n)},$$

$$C(\beta) = \frac{3}{2} + \frac{16}{\beta(\beta+2)}, \qquad \beta_1 = \frac{4}{n-2},$$

$$T = \frac{1}{n\sqrt{\pi}}\left(\frac{n!}{2\Gamma(1+\frac{n}{2})}\right)^{1/n}\left(\frac{n}{n-2}\right)^{1/2}$$ *is the optimum embedding constant from* $W_0^{1,2}(\Omega)$ *to* $L^{\frac{2n}{n-2}}(\Omega)$.

Next we are concerned with a priori estimates of the form (3.0.2), i.e., an estimate for gradient. We can derive this kind of a priori estimate from the previous one. Let $u \in W_0^{1,2}(|x| < t_m)$ be a solution of

$$\ell_0 u \equiv -\Sigma D_i(a_{ij}(x)D_j u) + 2\Sigma b_i(x)D_i u + c(x)u = g, \tag{3.3}$$

where $g \in L^q(|x| < t_m)$, t_m is some positive number. Then $D_k u$ satisfies the following:

$$\ell_0(D_k u) + 2\Sigma D_k b_i D_i u + D_k c u$$

$$- \Sigma D_i(D_k a_{ij})D_j u - \Sigma D_k a_{ij} D_i D_j u = D_k g. \tag{3.4}$$

By introducing the following vector notations:

$\vec{e}_i$: $(n+1)$ vector with 1 in the i-th component and zero in the remaining components;

C: $(n+1)\times(n+1)$ matrix with entry:

$$c_{ij} = 2D_i(b_j) - \sum_k D_k(D_i a_{kj}), \quad i\cdot j > 0,$$

$$c_{00} = c,\ c_{0i} = c,\ c_{i0} = D_i c, \quad \text{for } i \neq 0,$$

B^k: $(n+1)\times(n+1)$ matrix with entry:

$$b^k_{ij} = -\frac{1}{2}D_i(a_{kj}) + b_j, \quad i\cdot j > 0,$$

$$b^k_{00} = b_k, \quad b^k_{ij} = 0 \quad \text{for the other entries,}$$

$\vec{g} = (g,0,\ldots,0)^T$, $\vec{f_i} = g\vec{e}_i, i = 1,\ldots,n$, and $\vec{u} = (u,\nabla u)^T$, we may write (3.4) as

$$-\Sigma D_i(a_{ij}D_j\vec{u}) + 2\Sigma B^k D_k\vec{u} + C\cdot\vec{u} = \vec{g} + \Sigma D_i(\vec{f_i}). \tag{3.5}$$

Using Theorem 3.1, we have the following:

THEOREM 3.2. *Let $u \in W_0^{1,2}(|x| < t_m)$ be a solution to (3.3) and $g \in L^q(|x| < t_m)$ for some $q > n$. Let a_{ij}, B^k, C, $\vec{g}$, and $\vec{f_i}$ be as above and let the conditions of Theorem 3.1 hold. Assume that $C \in L^{q/2}_{\text{loc}}(R^n)$, $B^k \in L^q_{\text{loc}}(R^n)$ and $\sup_{x\in R^n} \||C|\|_{L^{q/2}(B_2(x))} < \infty$, $\sup_{x\in R^n} \||B^k|^2\|_{L^{q/2}(B_2(x))} < \infty$. Assume further that there exists a $\delta > 0$ such that $(\ell_0\varphi,\varphi) \geq \delta(-\Delta\varphi,\varphi)$ for any $\varphi \in C_0^\infty(R^n)$. Then*

$$\max\left\{\sup_{|x|<t_m}|u|,\ \sup_{|x|<t_m-2}|\nabla u|\right\} \leq E_1\left(\|g\|_{L^2_t(R^n)} + \max_{|x|\leq t_m}\|g\|_{L^q(B_2(x))}\right), \tag{3.6}$$

where $E_1 = K_0\max\left(\frac{2}{\delta(n-2)}(1+T\mu(B_2)^{1/n}),\ n+\mu(B_2)^{1/q}\right)$, and K_0 is given in Theorem 3.1, with $\||C|\|_{L^{q/2}(B_2(x_0))}$ and $\|\Sigma|B^k|^2\|_{L^{q/2}(B_2(x_0))}$ replaced by $\sup_{x\in R^n}\||C|\|_{L^{q/2}(B_2(x))}$ and $\sup_{x\in R^n}\|\Sigma|B^k|^2\|_{L^{q/2}(B_2(x))}$ respectively.

We introduce the following Hardy inequality which will be used in the proof of Theorem 3.2 and in Sections 4.3 and 5.3 (compare with Lemma 1 of Allegretto [3]):

Lemma 3.3. *Let $\varphi \in C_0^\infty(R^n)$, $x_0 \in R^n$. Then there exists a constant $c > 0$, independent of x_0 and φ such that*

$$(-\Delta\varphi,\varphi) \geq c\int_{R^n} \frac{\varphi^2}{|x|\cdot|x-x_0|}\,dx, \tag{3.7}$$

and for $x_0 = 0$, $c = \dfrac{(n-2)^2}{4}$.

Now we can state the final estimate of this chapter.

Theorem 3.4. *Let $u \in W_0^{1,2}(|x| < t_m)$, ℓ_0, g, q as given in Theorem 3.2. Assume g has a decomposition $g = g_1 + g_2$ with $g_1 \in L^q \cap L_\ell^2$, $g_2 \in L^q \cap L_\lambda^2$, for some $0 < \lambda^{-1} \in L^{n/2}(R^n)$. Then*

$$\sup_{|x|<t_m} |u| \leq E_1 M(g), \tag{3.8}$$

$$\sup_{|x|<t_m-2} |\nabla u| \leq E_1 M(g), \tag{3.9}$$

where $M(g) \leq \|g_1\|_{L_\ell^2(R^n)} + \sup_{x\in R^n} \|g_1\|_{L^q(B_2(x))} + e\|g_2\|_{L_\lambda^2(R^n)} + \sup_{x\in R^n} \|g_2\|_{L^q(B_2(x))}$ and $e = \frac{n-2}{2}\, T^{1/2}\|\lambda^{-1}\|_{L^{n/2}(R^n)}^{1/2}$, as given in Chapter 2, E_1 is as given in Theorem 3.2.

We explicitly remark that if $a_{ij}(x)$ and $b_i(x)$ are not constants, the expression for E_1 contains some special norms of a_{ij} and b_i, as given in Theorems 3.1 and 3.2. Some explicit estimates for E_1 will be given in Appendix A.

Chapter 4

EXISTENCE THEORY I: BOUNDED SOLUTIONS

4.1. Introduction.

In this chapter we will state and prove our basic existence theorems for bounded positive solutions. Applications of these results will continue in the later chapters. The techniques we will employ use weighted Sobolev spaces and the Schauder fixed point theorem which is applicable, thanks to the L^∞ a priori estimates we obtained in the last chapter. The results we seek use conditions which guarantee the existence of global positive solutions which are bounded above and below by constants and tend to positive constants at infinity. In contrast to previous works, we are mainly interested in those quasilinear problems containing highly nonradial terms and gradient terms. These are cases where ordinary differential equation arguments and/or variational methods do not fit in well. Due to the nonradial and nonlinear nature of our problems, we do not expect to obtain radial solutions, nor do we expect to get sharper results than those known if our problems reduce to those cases upon which these special techniques are applicable.

The typical problem of the form

$$-\Delta u = p(x)u^\gamma$$

was studied by many authors. While variational methods are used, radial arguments (i.e., ordinary differential equation) and the sub-supersolution method play an important role in those investigations and the statements of the theorems obtained. The existence of infinitely many positive solutions was established for

$|p(x)| \leq c/|x|^{\ell}$, $\ell > 2$ by Ni [58], for $|p(x)| \leq \varphi(|x|)$ with $\int_0^\infty r\varphi(r)dr < \infty$ by Kawano [37], see also Kusano and Oharu [46]. There have been many studies to guarantee the existence of positive solutions for semilinear equations, we mention, in addition to [58], [37] and [46], Lions [56], Berestycki and Lions [11], Ding and Ni [DN], Kenig and Ni [KN1] and more recently, Dalmasso [19], Kawano, Satsuma and Yotsutani [41], Li and Ni [LN] and Ni and Yotsutani [NY].

It seems that there are considerably fewer results along these lines for quasilinear equations. If the function f has radial majorants of the form

$$|f(x,u,\xi)| \leq \varphi(|x|)F(u,\xi)$$

with $F(u,\xi)$ either sublinear or superlinear in both u and ξ and $\int_0^\infty r\varphi(r)dr < \infty$, Kusano and Oharu [47] were able to obtain the existence of infinitely many positive solutions which are bounded above and below by positive constants and tend to positive constants at infinity. Usami [76] relaxed the restriction on $\varphi(r)$ somewhat. More details will be given in Section 4.4. For the mixed sublinear-superlinear equation

$$-\Delta u + \varphi(|x|)u^{\lambda} + \psi(|x|)u^{\mu} = 0,$$

Kusano and Trench [53] studied the radial case while Furusho [27] obtained criteria for positive solutions under restrictions on the radial majorants of φ and ψ. If no radial majorants are allowable, the problem was given as an open question.

In this chapter we establish a general existence theorem for positive entire solutions which are bounded above and below by positive constants and tend to positive constants at infinity. We have no restrictions on the growth of f as a function of u and ∇u in the formulation of the existence theorem. The applications of this existence theorem thus cover the cases we mentioned above, especially for the cases where there are no radial majorants which satisfy the integral criteria presented by the previous authors. We also obtain solutions for mixed sublinear-superlinear quasilinear equations, and thus answer the open question mentioned above.

The material in this chapter is arranged as follows: in Section 4.2 we give the existence theorem and its proof. This is the core of this chapter. In Section 4.3, we study the asymptotic behavior of solutions given by the methods of Section 4.2. Then in Section 4.4 several applications are considered and explicit comparisons are made between our results and what was previously known.

4.2. Existence Theorem.

We consider the following problem

$$\ell_0 u \equiv -\Sigma D_i(a_{ij}(x)D_j u) + 2\Sigma b_i(x)D_i u = f(x, u, \nabla u) \tag{4.1}$$

in R^n. We assume that $a_{ij} \in L^\infty(R^n)$ and $D^2 a_{ij} \in L^q_{\text{loc}}(R^n)$ with $N(D^2 a_{ij}, q, 2, R^n) < \infty$ and (a_{ij}) is symmetric,

$$\lambda_0|\xi|^2 \leq \Sigma a_{ij}\xi_i\xi_j \leq \lambda_1|\xi|^2 \tag{4.2}$$

for some $\lambda_0,\ \ \lambda_1 > 0$, and $b_i,\ \nabla b_i \in L^q_{\text{loc}}(R^n)$ such that $N(\nabla b_i, q, 2, R^n) < \infty$. We also need the following structure assumption in order to employ the a priori estimates obtained in Chapter 3:

$$\exists\delta > 0, \quad (\ell_0\varphi,\ \varphi) \geq \delta(-\Delta\varphi,\ \varphi), \quad \forall\varphi \in C_0^\infty(R^n). \tag{4.3}$$

Furthermore, we assume that f satisfies Caratheodory's conditions, that is: f is measurable with respect to x for all $(u,\xi) \in R \times R^n$, and continuous with respect to (u,ξ) for almost all $x \in R^n$. Note that we do not impose any growth restriction on f and the above conditions on a_{ij} and b_i are required in Chapter 3.

We state the main theorem of this chapter.

THEOREM 4.1. *Let*

$$F(x,a,b) = \sup_{\substack{0 \le u \le a \\ -b\vec{1} \le \xi \le b\vec{1}}} |f(x,u,\xi)| \tag{4.4}$$

satisfy $M(F(x,a,b)) < \infty$ *for any positive constants* a, b. *Then for any positive constant triple* (a,b,σ), *with* $\sigma < 1$, *satisfying*

$$E_1 M(F(x,a,b)) \le \min\left(b, \frac{1-\sigma}{2}\, a\right), \tag{4.5}$$

there exists a solution $u \in C^1(R^n)$ *to (4.1) such that*

$$\sigma a \le u \le a, \qquad |\nabla u| \le b, \tag{4.6}$$

where M *and* E_1 *are given in Chapter 2 and Chapter 3 respectively.*

Proof. We will first prove the existence locally and use standard diagonal arguments to extend the local solution to all of R^n. Let $\{t_m\}$, $\{\varphi_m\}$ denote a sequence of positive numbers and $C_0^\infty(R^n)$ functions respectively such that

$$t_m \uparrow +\infty, \qquad t_1 > 4, \qquad 0 \le \varphi_m \le 1,$$

$$\varphi_m \in C_0^\infty(|x| < t_m - 3), \qquad \varphi_m \equiv 1 \text{ in } (|x| < t_m - 4).$$

For any chosen m, set

$$\mathcal{B} = C^0(|x| \leq t_m) \cap C^1(|x| \leq t_m - 3)$$

and norm $\mathcal{B}$ with

$$\|u\|_{\mathcal{B}} = \max\left(\|u\|_{C^0(|x|\leq t_m)}, \ \frac{1-\sigma}{2}\ \frac{a}{b}\ \|\nabla u\|_{C^0(|x|\leq t_m - 3)}\right), \tag{4.7}$$

where a, b, σ satisfy (4.5).

We claim that

(i) $\{\mathcal{B}, \|\cdot\|_{\mathcal{B}}\}$ is a Banach space and

(ii) $\mathcal{B} \hookrightarrow \text{Range}(J)$ is continuous, where J is given in Chapter 2.

Obviously, if $\{u_n\}$ is a Cauchy sequence in $\mathcal{B}$, then by the standard Arzela–Ascoli theorem, there is a $u_0 \in \mathcal{B}$ such that $u_n \to u_0$ in $\mathcal{B}$. Secondly, for any $u \in \mathcal{B}$ define $u \equiv 0$ outside $(|x| < t_m)$, then we have

$$N(u, q, 2, R^n) \leq \|u\|_{\mathcal{B}} \cdot \mu(B_2)^{1/q},$$

$$\|u\|^2_{L^2_t(R^n)} = \int_{|x|<t_m} (1+|x|^2)u^2 \leq \|u\|^2_{\mathcal{B}} \cdot \int_{|x|<t_m} (1+|x|^2)dx,$$

$$\|u\|^2_{L^2_\lambda(R^n)} \leq \|u\|^2_{\mathcal{B}} \cdot \int_{|x|<t_m} \lambda dx \qquad (\text{note: } \ 0 < \lambda^{-1} \in L^{n/2}(R^n)).$$

Recall that $\text{Range}(J) = \{u_1 + u_2 \mid u_1, u_2 \in L^q_{\text{loc}}(R^n), \ \|u_1\|_{\mathcal{L}_1} = N(u_1, q, 2, R^n) + \|u_1\|_{L^2_t(R^n)} < \infty$, and $\|u_2\|_{\mathcal{L}_2} = N(u_2, q, 2, R^n) + \|u_2\|_{L^2_\lambda(R^n)} < \infty\}$ and the norm in $\text{Range}(J)$ is given by $\inf\{\|v_1\|_{\mathcal{L}_1} + \|v_2\|_{\mathcal{L}_2} \mid v_1 + v_2 = u_1 + u_2\}$, we conclude that the embedding is continuous.

Let ℓ_0^{-1} denote the Dirichlet inverse of ℓ_0 in $(|x| < t_m)$, more precisely, if u solves $\ell_0 u = g$ in $(|x| < t_m)$, $u = 0$ on $|x| = t_m$, then $\ell_0^{-1}(g) = u$. Then, by Theorem 3.4 of Chapter 3, $\ell_0^{-1} : \text{Range}(J) \to \mathcal{B}$ and for $g \in \text{Range}(J)$:

$$\begin{aligned} \|\ell_0^{-1}(g)\|_{C^0(|x|\le t_m)} &\le E_1 M(g), \\ \|\nabla(\ell_0^{-1}(g))\|_{C^0(|x|\le t_m-3)} &\le E_1 M(g), \end{aligned} \tag{4.8}$$

where E_1 is independent of m. Now let K denote the ball in $\mathcal{B}$ centered at $a(1+\sigma)/2$ with radius $a(1-\sigma)/2$ and define an operator P on K by:

$$P(u) = \frac{a(1+\sigma)}{2} + \ell_0^{-1}(f(x, u, \varphi_m \nabla u)).$$

Since $u \in K$, then $|f(x, u, \varphi_m \nabla u)| \le F(x, a, b)$, while $M(F(x, a, b)) < \infty$, whence $f(x, u, \varphi_m \nabla u) \in \text{Range}(J)$ and

$$M(f(x, u, \varphi_m \nabla u)) \le M(F(x, a, b)).$$

We claim the following:

(i) $P : K \to K$.

In fact, for $u \in K$,

$$\begin{aligned} \Big\|P(u) - \frac{a(1+\sigma)}{2}\Big\|_{C^0(|x|\le t_m)} &= \|\ell_0^{-1}(f(x, u, \varphi_m \nabla u))\|_{C^0(|x|\le t_m)} \\ &\le E_1 M(f(x, u, \varphi_m \nabla u)) \end{aligned}$$

by (4.8) and (4.5), we have

$$E_1 M(f(x, u, \varphi_m \nabla u)) \le E_1 M(F(x, a, b)) \le \frac{1-\sigma}{2}\, a.$$

Similarly,

$$\left\|\nabla\Big(P(u) - \frac{a(1+\sigma)}{2}\Big)\right\|_{C^0(|x|\le t_m-3)} \le E_1 M(F(x,a,b)) \le b.$$

Hence $P(u) \in K$.

(ii) P is a compact continuous operator.

By Theorem 15.1 of [55, p. 203], $\ell_0^{-1}(f(x,u,\varphi_m \nabla u)) \in C^{1,\alpha}(|x| \le t_m)$ where $\alpha = 1 - n/q$, and Theorem 1.31 of [1] ensures that $C^{1,\alpha}(|x| \le t_m) \hookrightarrow C^1(|x| \le t_m)$ is compact. Thus $P : K \to K$ is compact, and for $u_n \to u$ in K, by Caratheodory's conditions we see $f(x, u_n, \varphi_m \nabla u_n) \to f(x, u, \varphi_m \nabla u)$ in $L^q(|x| < t_m)$ by the Lebesgue Convergence Theorem, hence P is a compact continuous map.

From the Schauder fixed point theorem, we conclude that there exists a $u_m \in K$ such that $P(u_m) = u_m$. Equivalently, in $(|x| < t_m)$,

$$\ell_0\Big(u_m - \frac{a(1+\sigma)}{2}\Big) = f(x, u_m, \varphi_m \nabla u_m),$$

with $a\sigma \le u_m \le a, \ |\nabla u_m| \le b$.

Next we will show using diagonal arguments that a subsequence of the sequence $\{u_m\}$ we just generated will converge in $W^{1,2}$ locally to a solution u.

Let $\varphi \in C_0^\infty(R^n), \ 0 \le \varphi \le 1, \ \varphi \equiv 1$ in $(|x| < t_m)$ for some $m > 0$. We observe that

$$(\ell_0(\varphi v), \varphi v) = \int_{R^n} |v|^2 (\Sigma a_{ij} D_i\varphi \cdot D_j\varphi + 2b_i D_i\varphi \cdot \varphi) + (\ell_0 v, \varphi^2 v) \tag{4.9}$$

holds for $v \in W^{1,2}_{\text{loc}}(R^n)$. Setting $v = u_m$ and $\varphi = \varphi_m$ in (4.9) and combining with

(4.3) we obtain

$$\delta\|\nabla(\varphi_m u_m)\|^2_{L^2} \le (\ell_0(\varphi_m u_m), \varphi_m u_m)$$

$$= \int |u_m|^2(\Sigma a_{ij} D_i\varphi_m D_j\varphi_m + 2b_j D_j\varphi_m \cdot \varphi_m)$$

$$+ \int \varphi_m^2(u_m \cdot f(x, u_m, \varphi_m \nabla u_m)). \qquad (4.10)$$

Taking into consideration that $\sigma a \le u_m \le a$, $|\nabla u_m| \le b$, $|f| \le F$ and $M(F) < \infty$, we conclude from (4.10) that

$$\|\nabla(\varphi_m u_m)\|^2_{L^2} \le c,$$

where c depends on a_{ij}, b_i, f, but not on m. Hence $\{\varphi_m u_m\}$ is bounded in $W^{1,2}$ and has a subsequence which converges in L^2. Letting $m \to \infty$ and using the standard diagonal argument, we conclude that there are a $u \in L^2_{\text{loc}}$, and a subsequence $\{u_{m'}\}$ such that $\|u_{m'} - u\|_{L^2(B_R)} \to 0$, for any $R > 0$. Setting $v = u_{m'} - u_{n'}$ in (4.9) yields

$$\delta\|\nabla(\varphi(u_{n'} - u_{m'}))\|^2_{L^2} \le \int |u_{n'} - u_{m'}|^2(\Sigma a_{ij} D_i\varphi D_j\varphi + 2b_i D_i\varphi\varphi)$$

$$+ \int \varphi^2(u_{m'} - u_{n'})(f(x, u_{m'}, \varphi_{m'}\nabla u_{m'}) - f(x, u_{n'}, \varphi_{n'}\nabla u_{n'}))$$

$$\le c'\|u_{n'} - u_{m'}\|_{L^2(\text{supp}\varphi)}$$

for some $c' > 0$ depending on φ, a_{ij}, b_i, f, but not on m', n'. Thus we conclude that $\{u_{m'}\}$ converges in $W^{1,2}$ locally. Then u is the desired solution and by the regularity theorem, cf. Theorem 15.1 of [55, p. 203] $u \in C^1(R^n)$ and the proof is complete.

We have the following immediate consequence:

COROLLARY 4.2. *Under the assumptions of Theorem 4.1, with (4.5) replaced by*

$$\lim_{a\to\beta} \frac{M(F(x,a,a))}{a} < \frac{1}{2}\, E_1^{-1}, \qquad \beta = 0^+, \text{ or } +\infty. \tag{4.5$'$}$$

Then (4.1) has infinitely many positive solutions which are bounded above and below by positive constants.

Proof. Set $\beta = 0^+$. Since (4.5$'$) implies that (4.5) holds for small $a = b$ with some $\sigma < 1$, we find a solution u such that $\sigma a \le u \le a$, $|\nabla u| \le a$. Replacing a by σa and σa by $\sigma^2 a$ respectively, we find another solution $\tilde{u}$ with $\sigma^2 a \le \tilde{u} \le \sigma a$, $|\nabla \tilde{u}| \le \sigma a$, and so on. Repeating this procedure generates infinitely many solutions.

For $\beta = +\infty$, (4.5$'$) implies that (4.5) holds for large $a = b$ with some $\sigma < 1$. This gives a solution u with $\sigma a \le u \le a$, $|\nabla u| \le a$. Replacing σa by a and repeating the procedure give another solution $\tilde{u}$ with $a \le \tilde{u} \le a/\sigma$, $|\nabla \tilde{u}| \le a/\sigma$ and so on. The corollary is proved.

From the above results we can see that the explicit bound on E_1 is important, and we will return to this in the applications. However, for two special cases we do not need any knowledge of E_1, that is,

$$\lim_{a\to\beta} \frac{M(F(x,a,a))}{a} = 0, \quad \text{for} \quad \beta = 0^+ \text{ or } +\infty. \tag{4.5$''$}$$

COROLLARY 4.3. *If (4.5$''$) holds, then the conclusion of Corollary 4.2 holds.*

Remark. (4.5$''$) essentially says that if f is either sublinear or superlinear in u as well as in ∇u, then we have infinitely many positive solutions bounded above and below by positive constants.

From (4.5′) we can also see that some balancing between the powers of u and ∇u will result in the existence of a solution, and it is this observation that enables us to deal with mixed sublinear-superlinear cases. The details will be discussed in Section 4.4.

4.3. Asymptotic Behavior.

We have seen in the last section that under the assumptions of Theorem 4.1, solutions to (4.1) are found in the ordered interval $[\sigma a, a]$. It is natural to ask if u will eventually approach some constant. This is the problem we are going to study in this section. Lemma 3.3 will be used.

We state the following:

THEOREM 4.4. *Under the same assumptions of Theorem 4.1, if further* $|b_i| \le c_0/(1+|x|)$ *and* $F = J([F_1, F_2])$ *with*

$$\lim_{|x|\to\infty} \|F_1\|_{L^q(B_2(x))} = \lim_{|x|\to\infty} \|F_2\|_{L^q(B_2(x))} = 0,$$

then the solution u *given by Theorem 4.1 approaches some positive constant* c *as* $|x| \to \infty$.

Proof. Choose $\alpha > 0$ and a function $h \in C^1$ such that $h(x) = |x|^{-\alpha}$ for $|x| > 2$, $|h(x)| < D(\alpha)$ (constant) for $|x| < 2$, and $|D_i h/h| < c(\alpha)/(1+|x|)$, with $c(\alpha) \to 0$ as $\alpha \to 0$, $h(0) = 1$. For a fixed generic point $x_0 \in R^n$, let $h_0(x) = h(x - x_0)$,

$\beta_i = D_i h_0/h_0$, $w = \bar{u}h_0$, $\bar{u} = u - \frac{a(1+\sigma)}{2}$. For $\varphi \in C_0^\infty(R^n)$, consider

$$(\ell_0(\bar{u}h_0), \varphi) = (\Sigma a_{ij} D_i(\bar{u}h_0), D_j\varphi) + 2(\Sigma b_i D_i(\bar{u}h_0), \varphi)$$

$$= (\Sigma a_{ij} D_i\bar{u}, h_0 D_j\varphi) + (\Sigma a_{ij}(D_i h_0)\bar{u}, D_j\varphi)$$

$$+ 2(\Sigma b_i D_i\bar{u}, h_0\varphi) + 2(\Sigma b_i \bar{u} D_i h_0, \varphi)$$

$$= (\ell_0\bar{u}, h_0\varphi) - (\Sigma a_{ij} D_i\bar{u}, \varphi D_j h_0) + \Sigma a_{ij}(D_i h_0)\bar{u}, D_j\varphi)$$

$$+ 2(\Sigma b_i \bar{u} D_i h_0, \varphi).$$

We observe that

$$(a_{ij} D_i h_0 \bar{u}, D_j\varphi) = (a_{ij}\beta_i w, D_j\varphi),$$

$$(a_{ij} D_i\bar{u} D_j h_0, \varphi) = (a_{ij} D_i\bar{u} h_0 \beta_j, \varphi)$$

$$= (a_{ij} D_i(\bar{u}h_0)\beta_j, \varphi) - \left(a_{ij}\bar{u}h_0 \cdot \frac{D_i h_0}{h_0}\beta_j, \varphi\right)$$

$$= (a_{ij} D_i w \beta_j, \varphi) - (a_{ij} w \beta_i \beta_j, \varphi),$$

thus we have

$$(\ell_0(\bar{u}h_0), \varphi) = (f h_0, \varphi) - (\Sigma D_j(a_{ij}\beta_i w), \varphi) + (a_{ij} w \beta_i \beta_j, \varphi)$$

$$- (a_{ij} D_i w \beta_j, \varphi) + 2(\Sigma b_i \beta_i w, \varphi).$$

Now, define a new operator

$$\tilde{\ell} w = \ell_0 w + \Sigma D_j(a_{ij}\beta_i w) - \Sigma a_{ij}\beta_i\beta_j w + \Sigma a_{ij}\beta_j D_i w - 2\Sigma b_i \beta_i w,$$

and we have, for $w = \bar{u}h_0$

$$(\tilde{\ell}w, w) = (\ell_0 w, w) - (\Sigma a_{ij}\beta_i\beta_j w, w) - 2(\Sigma b_i\beta_i w, w),$$

and $\tilde{\ell}w = fh_0$.

Note that since $|\beta_i| \le \frac{c(\alpha)}{1+|x-x_0|}$ and $|b_i| \le \frac{c_0}{1+|x|}$, Lemma 3.3 implies that

$$|(\Sigma a_{ij}\beta_i\beta_j w, w)| + 2|(\Sigma b_i\beta_i w, w)| \le \frac{\delta}{2}(-\Delta w, w)$$

by proper choice of α small enough, and since $(\ell_0 w, w) \ge \delta(-\Delta w, w)$, we thus conclude that

$$(\tilde{\ell}w, w) \ge \frac{\delta}{2}(-\Delta w, w),$$

i.e., $\tilde{\ell}$ satisfies the same conditions as ℓ_0 and hence the proof of Theorem 3.2 implies that there is a constant K' independent of x_0 such that

$$\left|u(x_0) - \frac{a(1+\sigma)}{2}\right| \le K'[\|F_1h_0\|_{\mathcal{L}_1} + \|F_2h_0\|_{\mathcal{L}_2}].$$

By our assumption

$$\|F_ih_0\|_{L^q(B_2(x_0))} \le D(\alpha)\|F_i\|_{L^q(B_2(x_0))} \to 0 \quad \text{as} \quad |x_0| \to \infty,$$

for $i = 1, 2$. On the other hand,

$$\|F_1h_0\|^2_{L^2_\ell(R^n)} \le \|F_1\|^2_{L^2_\ell(|x|>\frac{|x_0|}{2})} + \frac{c}{|x_0|^{2\alpha}}\|F_1\|^2_{L^2_\ell(R^n)}.$$

From this we conclude $\|F_1h_0\|_{\mathcal{L}_1} \to 0$ as $|x_0| \to \infty$. Similarly we conclude $\|F_2h_0\|_{\mathcal{L}_2} \to 0$ as $|x_0| \to \infty$. This ends the proof.

Observe that the convergence of u to a constant at infinity can be estimated in terms of the properties of F_1, F_2.

4.4. Applications.

In this section we discuss the applications of our results and compare them with results previously obtained by other authors using radial arguments and weighted spaces.

The existence theorems in Benci and Fortunato [10] are quite general. In fact, in an unbounded domain Ω, they proved that, for weight function $\rho(x) > 0$ satisfying $\rho(x) \to \infty$ as $|x| \to \infty$, and $\forall r \in R$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in N^n$, $\exists c \in R_+$, $|D^\alpha \rho^r(x)| \leq c(\rho(x))^r$, if

$$|f(x,y,z)| \leq h(x) + (\rho(x))^\alpha |y|^\beta + (\rho(x))^\gamma |z|^\delta,$$

where $h \in L^{q'}_{\rho^\epsilon(\Omega)}$, $q \in \left[2, \frac{2n}{n-2}\right]$, $\varepsilon > 0$, $\alpha < 0$, $1 \leq \beta < \frac{n+2}{n-2}$, $\gamma < 0$, $1 \leq \delta < \frac{n+2}{n}$, then $-\Delta u + \lambda u = f(x, u, \nabla u)$ always has solutions in $W_0^{1,2}(\Omega)$.

However, if we consider our problem in R^n and seek positive solutions, these conditions seem not to suffice for our purpose, and the conditions we give will be more restrictive.

We will assume the positivity of the functions involved in the following examples.

Example 1. Consider the following typical problem

$$-\Sigma D_i(a_{ij} D_j u) + 2k\Sigma \frac{x_i}{1+|x|^2} D_i u = q(x) u^\delta + p(x) |\nabla u|^\gamma \tag{4.10}$$

in R^n, $n \geq 3$, with either $0 < \delta,\ \gamma < 1$, or $1 < \delta, \gamma$, and $k > 0$, where (a_{ij}) is a symmetric constant matrix and satisfies (4.2). It is easy to check that (4.3) holds provided $\frac{\lambda_0}{k} > \frac{4n}{(n-2)^2}$. Then by Corollary 4.3, the problem has infinitely many positive solutions bounded above and below by positive constants provided $M(q(x)), M(p(x)) < \infty$. In a simplified version, we need only to require that:

$$\int_{R^n} (1+|x|^2)p^2(x) < \infty, \qquad \int_{R^n} (1+|x|^2)q^2(x) < \infty. \tag{4.11}$$

Kenig and Ni [KN1] treated the similar problem

$$-\sum D_i(a_{ij}D_j u) \ + k(x)u = K(x)u^p$$

with $p > 1$ in R^n. Similar existence results were obtained for certain classes of k and K. The same conclusion was obtained by Kusano and Oharu [46,47] under the condition that for $\varphi(r) = \max_{|x|=r}(p(x), q(x))$, $\int_0^\infty r\varphi(r)dr < \infty$ with the left hand side of (4.10) replaced by $-\Delta u$. Usami [76] was able to weaken the condition to the following:

$$\text{let} \qquad \tilde{p}(r) = \max_{|x|\leq r} p(x), \qquad \tilde{q}(r) = \max_{|x|\leq r} q(x),$$

$$\int_0^\infty r\tilde{p}(r)dr < \infty, \qquad \int_0^\infty r^{1-\gamma}\tilde{q}(r)dr < \infty.$$

Thus, it is clear that if one could find radial majorants $\tilde{p}$ and $\tilde{q}$ for p and q respectively, this includes the case that p and q are both radially symmetric, our result is not as sharp. However, there exist functions φ with $M(\varphi) < \infty$ which do not satisfy the radial integral test, that is, under some circumstances our result is applicable but the radial argument test fails. We will illustrate this with the following example.

Example 2. Define

$$\varphi_1(x_1,x_2)=\begin{cases}|x_1|^2\cdot c\cdot e^{\frac{e^{-|x_1|}}{|x_2|^2-e^{-|x_1|}}}, & |x_2|^2<e^{-|x_1|}\\ 0, & |x_2|^2\geq e^{-|x_1|}\end{cases}$$

where c is chosen such that $\int_{-1}^{1} c(1+|x|^2)e^{\frac{1}{x^2-1}}dx=1$.

We further define $\varphi^2(x_1,x_2,x_3)=\varphi_1(x_1,x_2)\cdot e^{-|x_3|}$. It is easy to see that $\tilde\varphi(r)=\max\limits_{|x|=r}\varphi(x_1,x_2,x_3)=k\cdot r^1$, k some positive constant, and hence $\int_0^\infty r\tilde\varphi(r)dr=\infty$.

However, we claim that $\int_{R^3}(1+|x|^2)\varphi^2(x)dx<\infty$. Obviously,

$$\int_{R^3}(1+|x|^2)\varphi^2(x)dx=\int_{R^3}(1+|x_1|^2+|x_2|^2+|x_3|^2)\varphi_1(x_1,x_2)e^{-|x_3|}dx$$

$$\leq\int_{-\infty}^{\infty}(1+|x_3|^2)e^{-|x_3|}dx_3\cdot\int_{-\infty}^{\infty}(1+|x_1|^2)\int_{-\infty}^{+\infty}(1+|x_2|^2)\varphi_1(x_1,x_2)dx_2dx_1.$$

Now, we observe that for $0<a\leq 1$

$$\int_{-a}^{a}c(1+|x|^2)e^{\frac{a^2}{x^2-a^2}}dx\leq\int_{-a}^{a}c\left(1+\left(\frac{x}{a}\right)^2\right)e^{\frac{1}{(\frac{x}{a})^2-1}}d\left(\frac{x}{a}\right)\cdot a=a.$$

Thus

$$\int_{-\infty}^{+\infty}(1+|x_2|^2)\varphi_1(x_1,x_2)dx_2\leq|x_1|^2\cdot e^{-\frac{1}{2}|x_1|},$$

and since

$$\int_{-\infty}^{+\infty}(1+|x_1|^2)|x_1|^2e^{-\frac{1}{2}|x_1|}dx_1<\infty,$$

$$\int_{-\infty}^{+\infty}(1+|x_3|^2)e^{-|x_3|}dx_3<\infty,$$

we conclude that $\int_{R^3}(1+|x|^2)\varphi^2(x)dx<\infty$.

Example 3. Again consider problem (4.10). If $p(x) \equiv 0$, $\delta \neq 0$, and $\delta < 1$, under the condition (4.11), the same conclusion holds. In this case we allow δ to be negative and hence we include the singular case. We note that Dalmasso [19] studied the singular equations, $-\Delta u = f(x)u^{-\lambda}$, $\lambda > 0$, under the condition that

$$\int |x|^{\lambda(n-2)} f(x) < \infty.$$

His result guarantees that the solution $u \in M^{n/n-2}(R^n)$ and $\Delta u \in L^1(R^n)$, where $M^{n/n-2}(R^n)$ is the Marcinkiewicz space with norm

$$\|u\|_{M^{n/n-2}} = \inf\left\{c \in [0, +\infty) \mid \int_K |u| dx \leq c \cdot (\mu(K))^{n/2}, \quad K \text{ measurable}\right\}.$$

Our criterion does not involve the singular index λ but we do not get the solution u with $\Delta u \in L^1(R^n)$, though the bounded solutions are always in the Marcinkiewicz space. We remark that Kusano and Swanson [49] studied the existence of decaying solutions for singular equations.

For computational simplicity, we will only consider other examples for $a_{ij} = \delta_{ij}$ and $b_i = 0$.

Example 4. Consider the mixed sublinear-superlinear problem

$$-\Delta u = p(x)u^{\delta} + q(x)u^{\gamma} \tag{4.12}$$

in R^n, $n \geq 3$, $0 < \delta < 1 < \gamma$. Let $P = M(p(x))$, $Q = M(q(x))$. Obviously, $M(p(x)a^{\delta} + q(x)a^{\gamma}) \leq M(p(x))a^{\delta} + M(q(x))a^{\gamma} = Pa^{\delta} + Qa^{\gamma}$. We look for the minimum of the following function

$$f(x) = Px^{\delta-1} + Qx^{\gamma-1}.$$

We observe that

$$f'(x) = (\delta - 1)Px^{\delta-2} + (\gamma - 1)Qx^{\gamma-2},$$

thus $x_0 = \left(\frac{1-\delta}{\gamma-1}\frac{P}{Q}\right)^{1/\gamma-\delta}$ is a critical point and

$$f(x_0) = P^{\frac{\gamma-1}{\gamma-\delta}}Q^{\frac{1-\delta}{\gamma-\delta}}\left[\left(\frac{\gamma-1}{1-\delta}\right)^{\frac{1-\delta}{\gamma-\delta}} + \left(\frac{1-\delta}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma-\delta}}\right] \eta$$

is the minimum.

Hence from Theorem 4.1, if $\eta < \frac{1}{2}E_1^{-1}$, there exists a positive solution u to (4.12) such that $0 < (1 - 2\eta E_1)\left(\frac{1-\delta}{\gamma-1}\frac{P}{Q}\right)^{\frac{1}{\gamma-\delta}} \leq u \leq \left(\frac{1-\delta}{\gamma-1}\frac{P}{Q}\right)^{\frac{1}{\gamma-\delta}}$.

We note that Kusano and Trench [53] also studied the mixed sublinear-superlinear problem and obtained some results about decaying solutions.

Example 5. Consider

$$-\Delta u = p(x)u^{\delta} + q(x)u^{\gamma} + h(x)|\nabla u|^{\delta} + g(x)|\nabla u|^{\gamma} \tag{4.13}$$

in R^n, $n \geq 3$, with $0 < \delta < 1 < \gamma$. Let $H = M(h(x))$, $G = M(g(x))$, then if

$$\pi \ (P+H)^{\frac{\gamma-1}{\gamma-\delta}}(Q+G)^{\frac{1-\delta}{\gamma-\delta}}\left[\left(\frac{\gamma-1}{1-\delta}\right)^{\frac{1-\delta}{\gamma-\delta}} + \left(\frac{1-\delta}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma-\delta}}\right] < \frac{1}{2E_1},$$

there is a bounded positive solution u such that

$$0 < (1 - 2\pi E_1)\left(\frac{1-\delta}{\gamma-1}\ \frac{P+H}{Q+G}\right)^{\frac{1}{\gamma-\delta}} \leq u \leq \left(\frac{1-\delta}{\gamma-1}\cdot\frac{P+H}{Q+G}\right)^{\frac{1}{\gamma-\delta}}.$$

Consider also

$$-\Delta u = p(x)u^{\delta} + q(x)u^{\gamma}|\nabla u|^{\mu} \tag{4.14}$$

in R^n, $n \geq 3$. Kusano and Oharu [47] proved the existence of infinitely many positive solutions which are bounded above and below by constants for the following cases:

$0 \le \mu \le 2$, (i) $\delta > 1$, $\gamma > 1$; (ii) $0 \le \delta < 1$, $\gamma \ge 0$, $\gamma+\mu < 1$; (iii) $\delta \le 0$, $\gamma+\mu \le 0$. We could recover these results here without any restriction on μ. Moreover, our existence theorem also enables us to establish the following result, which is new. Let $0 < \delta < 1 < \gamma + \mu$, let

$$\sigma = P^{\frac{\gamma+\mu-1}{\gamma+\mu-\delta}} Q^{\frac{1-\delta}{\gamma+\mu-\delta}} \left[\left(\frac{\gamma+\mu-1}{1-\delta} \right)^{\frac{1-\delta}{\gamma+\mu-\delta}} + \left(\frac{1-\delta}{\gamma+\mu-1} \right)^{\frac{\gamma+\mu-1}{\gamma+\mu-\delta}} \right],$$

then if $\sigma < \frac{1}{2E_1}$, this problem has a positive solution which is bounded above and below by constants and approaches some positive constant as $|x| \to \infty$.

We note that the results we presented here are extensions of the results reported in Kusano and Oharu [46,47], Kusano and Swanson [49], Usami [76], Kusano and Trench [53] and Dalmasso [19]. Example 4 of Section 5.3 studies the existence of decaying solutions.

Example 6. We consider

$$-\Delta u = K(x)e^{2u} \tag{4.15}$$

and

$$-\Delta u = K(x)e^{-2u} \tag{4.16}$$

in R^n, $n \ge 3$, $K(x) > 0$. Applying Theorem 4.1, we conclude that if $M(K(x)) \cdot \frac{e^{2a}}{a} < \frac{1}{2E_1}$ for some positive constant a, then (4.15) has a bounded positive solution bounded away from zero. We note that (4.15) was studied by Ni [59], Kusano and Oharu [46], and Cheng and Lin [18]. Ni [59] proved that if $|K(x)| \le c/|x|^\delta$, for $\delta > 2$, then (4.15) has infinitely many bounded solutions in R^n. Cheng and Lin [18] studied the case $\delta = 2$. In [46], assuming $|K(x)| \leq \varphi(|x|)$ and $\int_0^\infty t\varphi(t)dt < \infty$, Kusano and Oharu obtained a bounded positive solution bounded away from zero if further for some positive constants a, b, $\int_0^\infty t\varphi(t)dt \leq \frac{n-2}{2}ae^{-a}$ and $\int_0^\infty t\varphi(t)dt \leq (n-2)(e^{-b} - e^{2b})$ hold. So, essentially we obtain the same type of criteria. See also Kusano and Oharu [47, Example 5], Kawano [37] and Ni and Yotsutani [NY]. For $n = 2$, entire solutions were also considered by McOwen [M2].

Applying Corollary 4.2, we conclude that (4.16) has infinitely many bounded positive solutions bounded away from zero, provided $M(K) < \infty$. By assuming $K \in L^{n/2}$, we require essentially $\int_{R^n} K(x)dx < \infty$. See also Example 7 below.

We remark that the more general problems

$$-\Delta u = K(x)e^{u^\gamma}, \qquad \text{or} \qquad -\Delta u = K(x)e^{-u^\gamma}$$

with $\gamma > 0$ can be considered accordingly, and similar results follow.

Finally we conclude this chapter by illustrating the advantage of introducing a different weight $\lambda > 0$ with $\lambda^{-1} \in L^{n/2}(R^n)$, though we did not exploit this feature fully in our examples.

Example 7. Consider the simple superlinear problem

$$-\Delta u = q(x)u^\gamma$$

in R^3 with $1 < \gamma$. Assume that $q(x) \in L^{3/2} \cap L^1(R^3)$. We select $\lambda^{-1} = q(x)$, to be the weight function, then for $F(x,a) = q(x)a^\gamma = J([(v_1, v_2)])$ with $v_1 = 0$, $v_2 = q(x)$, $\int \lambda q^2(x) = \int q(x) < \infty$ hence $M(F) < \infty$. Because $\gamma > 1$, we conclude that $-\Delta u = q(x)u^\gamma$ has infinitely many positive solutions which tend to positive constants as $|x| \to \infty$ provided $q(x) \in L^{3/2} \cap L^1$. The same result, without the asymptotic behavior part, was given in Allegretto [3] under the condition that $|x|q(x) \in L^\infty \cap L^1$. This is an improvement since it is easy to see, for example $q(x) = \frac{1}{1+|x|^4} \in L^{3/2} \cap L^1$, but $|x|q(x) \notin L^\infty \cap L^1$.

We remark that, with the presence of radial majorants, slower decaying rate for $p(r)$ is allowed, see, e.g. Ni [58] and Naito [NA]. For $\gamma = (n+2)/(n-2)$, some results were given by Ding and Ni [DN].

Now, for Example 3, if we choose $\delta < 0$ and $q(x) \in L^{n/2}$, then we need only to require $\int_{R^n} q(x) < \infty$. This is an improvement over the criterion of Dalmasso [19].

Chapter 5

EXISTENCE THEORY II: DECAYING SOLUTIONS

5.1. Introduction.

We have studied the following equation

$$-\Sigma D_i(a_{ij}(x)D_ju) + \Sigma b_i(x)D_iu = f(x, u, \nabla u) \tag{5.0.1}$$

in the previous chapter and have obtained existence results for bounded positive entire solutions. In this chapter we focus our attention on the study of the existence of decaying entire solutions. This task is carried out by using the existence theorem of the last chapter and sub-supersolution method. However, in order to construct an ordered pair of sub- and supersolutions, Caratheodory's conditions no longer suffice and new structure restraints on f have to be imposed. Thus, the results we present in this chapter are not as general as those of the last chapter. Nevertheless we again expect to cover cases where the usual radial arguments are violated because of the highly nonradial symmetry of the coefficients.

Under the premise of the existence of ordered pairs of sub-supersolutions, Amann and Crandall [6] gave some multiplesolution results. Hess [34], Deuel and Hess [20] and Boccardo, Murat and Puel [13,14] presented the existence of weak solutions under different growth conditions for f upon ∇u. Pohozaev [69] gave an unimprovable growth condition on ∇u for very general f. While dealing with the unbounded domain case, Hess [33], Các [16] obtained the existence of weak solutions for f depending on ∇u subquadratically, and Donato and Giachetti [22,23] successfully raised the dependence of f on ∇u to square. However, this weak solution plus sub-supersolution approach can hardly supply any information about the positivity of solutions since for most cases $u \equiv 0$ is always a weak solution. Further, decaying solutions are not usually found in the standard Sobolev spaces on R^n. On the other hand, Berger and Schechter [12] studied quasilinear problems in divergence form within the framework of weighted spaces. Benci and Fortunato [10] studied the boundary value problem in weighted spaces with f depending on ∇u subquadratically, and Các [16] considered the weak solutions for a different type of weighted spaces for exterior domains, through the assumption of the existence of sub-supersolutions.

For semilinear equations of the form

$$-\Delta u = f(x, u),$$

the existence of decaying positive solutions was obtained by many authors, see, e.g. Gidas and Spruck [28], Berestycki and Lions [11], Ni [58] and others. More recently, Noussair and Swanson [62,63,64] and Kawano, Satsuma and Yotsutani [41], proved the existence of decaying solutions for f depending on u sublinearly or superlinearly. Kusano and Swanson [49] studied the existence of decaying solutions for the singular case. Kusano and Trench [53] and Furusho [27] considered decaying solutions for the mixed sublinear-superlinear cases. Recently, Li and Ni [LN], Ni and Yotsutani [NY] studied the Matukuma type equation and presented precise asymptotes of decaying solutions. A delicate nonexistence theorem was proved in [LN]. However, as far as quasilinear problems are concerned, few results have been obtained in this direction.

We mention the work of Kusano and Swanson [51], in which the decaying solutions were obtained for f depending on u and ∇u sublinearly. Again, radial ideas and strong sub-supersolution methods were extensively used.

Our method is based on the following simple observation: if we can find $0 < z \in C^\infty$ such that u solves (5.0.1) if and only if $v = \frac{u}{z}$ solves a problem to which Theorem 4.1 can be applied, then as v tends to some positive constant as indicated by Theorem 4.4, u tends to a constant multiple of z. However, instead of trying to find a function z such that u solves (5.0.1), it is easier to find z such that u is a supersoluton of (5.0.1). We then employ a spectral procedure to construct a suitable subsolution and the existence theorem will follow. In Section 5.2 we present the arguments. Applications to various situations and comparison will be presented in Section 5.3.

5.2. Existence Theorems.

We consider the existence of decaying solutions of

$$\ell_0 u = -\Sigma D_i(a_{ij}(x)D_j u) + 2\Sigma b_i D_i u = f(x, u, \nabla u) \tag{5.1}$$

in R^n, $n \geq 3$.

A function $v \in W^{1,2}_{\text{loc}}(R^n)$ is a weak supersolution of (5.1) if

$$\int_{R^n} \Sigma a_{ij}(x) D_j v D_i \varphi + 2\Sigma b_i D_i v \varphi \geq \int_{R^n} f(x, v, \nabla v)\varphi$$

for any $\varphi \in C_0^\infty(R^n)$, $\varphi \geq 0$. A weak subsolution is defined accordingly. We assume $a_{ij} = a_{ji} \in C^{1+\alpha}_{\text{loc}}(R^n) \cap W^{2,q}_{\text{loc}}(R^n)$ with $N(D^2 a_{ij}, q, 2, R^n) < \infty$, $\lambda_0|\xi|^2 \leq \Sigma a_{ij}\xi_i\xi_j \leq \lambda_1|\xi|^2$, for some $\lambda_0, \lambda_1 > 0$, $b_i \in C^\alpha_{\text{loc}}(R^n) \cap W^{1,q}_{\text{loc}}(R^n)$ with $N(\nabla b_i, q, 2, R^n) <$

∞. We note that we can relax the smoothness assumption somewhat if either we consider only weak solutions or the function f in (5.1) does not depend on ∇u. Assume also that f satisfies the following Nagumo type condition: for $(x, u, \xi) \in R^n \times R \times R^n$,

$$|f(x,u,\xi)| \leq b(|u|)(h(x) + k(x)|\xi|^2) \tag{5.2}$$

with $b : R^+ \to R^+$ nondecreasing, $b \in L^\infty(R^+)$, and $k, h \in L^\infty(R^n)$. We remark that a Nagumo type condition is needed here because sub-supersolution methods are used. However, it is reasonable since, by the nonexistence results of Serrin [73] (see also [74, pp. 512–513]), if a Nagumo type condition does not hold, then for any given smooth domain, there are smooth data for which the Dirichlet problem has no solutions. Concerning the restrictions on b, h, and k, Pohozaev [69] proved the following result: if

$$|f(x,u,\xi)| \leq b(x,u)(1 + |\xi|^\mu)$$

with f satisfying Caratheodory's conditions and for any given $\ell > 0$

$$\sup_{|u| \leq \ell} b(\cdot, |u|) \in L^p(\Omega), \qquad p > n,$$

where $\Omega \subset R^n$ with C^2 boundary, then $\mu = 2 - \frac{n}{p}$ cannot be improved, i.e., generic a priori estimates will not hold for $2 \geq \mu > 2 - \frac{n}{p}$. Thus, taking into account the above results, our conditions on f, b, h and k are rather reasonable.

We first state an existence theorem under the assumption of the existence of sub-supersolutions.

THEOREM 5.1. *Let f satisfy (5.2) and be locally Hölder continuous with exponent $\mu \in (0,1)$. Let a_{ij}, b_i be as stated above. Assume that $w, v \in W^{1,\infty}_{\mathrm{loc}} \cap L^\infty(R^n)$ are an ordered pair of weak sub-supersolutions with $w \leq v$ and $w \leq 0 \leq v$ near ∞. Then (5.1) has a solution $u \in C^2(R^n)$ with $w \leq u \leq v$.*

Proof. The existence part of this theorem is contained in Theorem 5.3 of [23]. We will use standard bootstrap arguments to establish the regularity. Let B_r be a ball in R^n centered at 0 with radius r. Then w, v are an ordered pair of weak sub-supersolutions for (5.1) on B_r. Denote the solution by u_r. By the regularity theory, see, eg. Theorem 12.1 of [55, p. 195], or Theorem 3.1 of [55, p. 266] we have

$$\|u_r\|_{C^1(B_{r-1})} \leq K[\|u_r\|_{C^0(B_r)} + \|f\|_{C^0(B_r)}],$$

where K is some constant independent of r. Since $w, v \in L^\infty(R^n)$, $\|u_r\|_{C^0(R^n)} \leq C_1$, C_1 independent of r. Further, by the conditions on f, we obtain $\|f\|_{C^0(R^n)} \leq C_2$. Then $f(\cdot, u, \nabla u) \in L^\infty$, thus $u \in C^{1+\alpha}$ and then, since $f \in C^\mu$ and $u \in C^1$, we have $f(\cdot, u, \nabla u) \in C^{\mu'}$, thus $\|u_r\|_{C^2(B_r)} \leq C_3$ independent of r. Hence $u \in C^2(R^n)$. The proof is complete.

From now on we assume that f satisfies the conditions of Theorem 5.1. Further restrictions on f will be introduced below.

As we mentioned in the introduction, we need to find a function z such that $v = \frac{u}{z}$ solves a problem to which Theorem 4.1 can be applied. We introduce a function z such that

(i) $z \in C^1 \cap W^{2,2}_{\mathrm{loc}}(R^n)$,

(ii) $\ell_0 z \geq 0$,

(iii) $z, \nabla z \in L^\infty(R^n)$,

(iv) $z \to 0$ as $|x| \to \infty$,

(v) $\left(\ell_0\varphi - 2\Sigma\beta_i \frac{\partial\varphi}{\partial x_i}, \varphi\right) \geq \delta(-\Delta\varphi, \varphi)$ for some $\delta > 0$ and any $\varphi \in C_0^\infty(R^n)$, where $\beta_i = \sum_j a_{ij} \frac{\partial}{\partial x_j}(\ln z)$ is assumed in $L^\infty(R^n)$.

THEOREM 5.2. *Let*

$$F(x,a,b) = \sup_{\substack{0 \leq \xi \leq a \\ -b\vec{1} \leq \vec{r} \leq b\vec{1}}} \frac{|f(x, \xi z, \xi \nabla z + z\vec{r})|}{z} \tag{5.3}$$

satisfy $M(F(x,a,b)) < \infty$ for any positive a, b. If for some positive constants a, b, σ with $\sigma < 1$, we have

$$E_1 M(F(x,a,b)) \leq \min\left(b, \frac{1-\sigma}{2}\, a\right), \tag{5.4}$$

where E_1 is given by Theorem 3.2, then (5.1) has a positive weak supersolution $v \in W^{1,\infty}_{\text{loc}}(R^n)$ such that $v \sim z$ at ∞. Furthermore, if $|b_i|$, $|\beta_i| \leq \frac{c}{1+|x|}$ and $F = J([F_1, F_2])$ with

$$\lim_{|x|\to\infty} \|F_1\|_{L^q(B_2(x))} = \lim_{|x|\to\infty} \|F_2\|_{L^q(B_2(x))} = 0,$$

then $v/z \to c$, some positive constant, as $|x| \to \infty$.

Proof. Letting $u = \hat{u}z$, we have

$$\begin{aligned}
\ell_0(\hat{u}z) &= -\Sigma D_i(a_{ij}D_j(\hat{u}z)) + 2\Sigma b_i D_i(\hat{u}z) \\
&= -\Sigma D_i(a_{ij}(D_j\hat{u})z) - \Sigma D_i(a_{ij}(D_jz)\hat{u}) + 2\Sigma b_i(D_i\hat{u})z + 2\Sigma b_i(D_iz)\hat{u} \\
&= -\Sigma D_i(a_{ij}D_j\hat{u})z + 2\Sigma b_i(D_i\hat{u})z - \Sigma a_{ij}D_j\hat{u}D_iz \\
&\quad - \Sigma D_i(a_{ij}D_jz)\hat{u} + 2\Sigma b_i(D_iz)\hat{u} - \Sigma a_{ij}D_i\hat{u}D_jz \\
&= \ell_0(\hat{u})z + \ell_0(z)\hat{u} - 2\Sigma a_{ij}D_j\hat{u}D_iz.
\end{aligned}$$

Hence (5.1) is reduced to

$$\ell_0(\hat{u}) - 2\Sigma\beta_i D_i \hat{u} + \frac{\ell_0(z)}{z}\,\hat{u} = f(x, \hat{u}z,\ \hat{u}\nabla z + z\nabla\hat{u})/z. \tag{5.1'}$$

Since $\ell_0(z) \geq 0$, the existence of a positive solution to

$$\ell_1(\hat{u}) = \ell_0(\hat{u}) - 2\Sigma\beta_i D_i \hat{u} = f(x, \hat{u}z, z\nabla\hat{u} + \hat{u}\nabla z)/z \tag{5.5}$$

implies the existence of a positive supersolution to (5.1′). It is readily seen that Theorems 4.1 and 4.4 imply the desired result.

We go on to construct a subsolution to (5.1). We assume that

$$\lim_{u\to 0+} \frac{f(x, u, \xi)}{u} = +\infty \tag{5.6}$$

uniformly in a neighborhood Q of zero in $R^n \times R^+ \times R^n$.

We remark that (5.6) holds if in Q, $f(x, 0, \xi) > 0$ or $f(x, u, \xi) \geq p(x)u^\gamma$, with $\gamma < 1$ and $p(x) \geq \varepsilon_0 > 0$. We also note that (5.6) contains the condition (6) given in [4]. Observe that (5.6) ensures that f is not globally nonpositive — a situation explicitly forbidden by the maximum principle, since we cannot have a solution with an interior maximum if $f \leq 0$. Finally, we assume globally that

$$f(x, u, \xi) \geq g(x, u, \xi) \tag{5.7}$$

for some $g \in C^1(R^n \times R \times R^n)$, $g(x, 0, 0) \geq 0$.

We have the following existence result:

Theorem 5.3. *Assume that f satisfies the conditions of Theorems 5.1 and 5.2, and that (5.6), (5.7) hold. Then (5.1) has a classical positive solution u such that $0 < u \leq cz$ at ∞.*

Proof. Let us first construct a subsolution. Let $B_\epsilon(0) \subset\subset Q^1$ where Q^1 is the projection of Q onto its first n components and let u_1 be a positive eigenfunction of the Dirichlet problem: $\ell_0(u_1) = \lambda u_1$ in $B_\epsilon(0)$. By (5.6) we could choose ε_1 small enough such that $\ell_0(\varepsilon_1 u_1) \leq f(x, \varepsilon_1 u_1, \nabla(\varepsilon_1 u_1))$ and $\varepsilon_1 u_1 \leq v$ in $B_\epsilon(0)$, where v is the supersolution given by Theorem 5.2. Now, we extend u_1 to R^n by setting $u_1 = 0$ in $R^n - B_\epsilon(0)$ and observe $w = \varepsilon_1 u_1 \leq v$ globally. Integrating by parts shows that w is a weak subsolution of (5.1) in R^n. Then Theorem 5.1 implies the existence of a nonnegative solution $u \in C^2$. We have to show u is positive. If not, assume $u(x_0) = 0$ at some $x_0 \in R^n$. Then x_0 is a minimum of u and hence $\nabla u(x_0) = 0$. Since $f(x, u, \nabla u) \geq g(x, u, \nabla u)$ and $g \in C^1$, $g(x, 0, 0) \geq 0$, we also observe that for x near x_0,

$$
\begin{aligned}
g(x, u, \nabla u) &\geq \int_0^1 \frac{d}{dt}\,[g(x, tu, t\nabla u)]dt \\
&= \sum_{i=1}^{n} \psi_i(x) D_i u + \psi_0(x) u
\end{aligned}
$$

for some $\psi_i \in L^\infty$, $i = 0, 1, \ldots, n$. We conclude that

$$
\ell_0(u) - \sum_{i=1}^{n} \psi_i(x) D_i u - \psi_0(x) u \geq 0
$$

and $u \geq 0$ near x_0. But by e.g. [29, p. 194], $\|u\|_{L^p(B_{2R}(x_0))} \leq c \inf_{B_R(x_0)} u = 0$ for some R, p. Thus we conclude that $u \equiv 0$ near x_0 and that the set $S = \{x \mid u(x) = 0\}$

is open. On the other hand S is closed since u is continuous. Then $S = R^n$ or $\emptyset$. But since $u \geq w$ and $w \not\equiv 0$, we must have $S = \emptyset$. This completes the proof.

Remark. The assumption $z \to 0$ in condition (iv) is not necessary if we do not want decaying solutions. In fact, instead of (iv), we can put different asymptotic behavior condition there and obtain solutions with different behaviors at infinity. We will see this in the next section.

5.3. Applications.

From the results of the last section we can see that the behavior of the (decaying) solution is determined by that of z. One may generate a very large number of results by different choices of z. We first consider an example which will give us very nonradially symmetric solutions by special choice of z.

Example 1. Consider the problem

$$-\Delta u + m^2 u = g(x, u, \nabla u) \tag{5.8}$$

in R^n, $n \geq 3$ and $m > 0$. We have the following

THEOREM 5.4. *Let $\rho_i \in R^n$ with $|\rho_i| = m$, $i = 1, \ldots, k$, and let $z = \sum_{i=1}^{k} c_i e^{\rho_i \cdot x}$ with $c_i \geq 0$, $\sum_{i=1}^{k} c_i > 0$. Let*

$$G(x, a) = \sup_{\substack{0 \leq \xi \leq a \\ -a\vec{1} \leq \vec{r} \leq a\vec{1}}} |g(x, \xi z, \xi \nabla z + z\vec{r})/z|$$

satisfy $G(x, a) < \infty$ for any positive a. If for some $a > 0$, $0 < \sigma < 1$, $E_1 G(x, a) \leq \frac{1-\sigma}{2}\, a$, then (5.8) has a solution u such that $a\sigma z \leq u \leq za$. In particular,

$u \to \infty$ *in* $\bigcup_i \{x \mid x \cdot \rho_i > \varepsilon|x|\}$, $u \to 0$ *in* $\bigcap_i \{x \mid x \cdot \rho_i < -\varepsilon|x|\}$, *and* $u \in L^\infty$ *in* $\bigcap_i \{x \mid x \cdot \rho_i = 0\}$.

We only note that by our choice z satisfies

$$-\Delta z + m^2 z = 0. \tag{5.8'}$$

Since $\left|\frac{\nabla z}{z}\right| \leq m$, for $\varphi \in C_0^\infty(R^n)$,

$$\begin{aligned}\left(2\,\frac{\nabla z}{z}\cdot\nabla\varphi,\ \varphi\right) &= \left(\frac{\nabla z}{z}, \nabla\varphi^2\right)\\ &= \left(-\ \frac{\Delta z}{z}, \varphi^2\right) + \left((\frac{\nabla z}{z})^2, \varphi^2\right)\\ &\leq (-m^2\varphi,\varphi) + (m^2\varphi,\varphi) \leq 0.\end{aligned}$$

We conclude that $((-\Delta + m^2)\varphi,\varphi) - \left(2\frac{\nabla z}{z}\ \nabla\varphi,\varphi\right) \geq ((-\Delta + m^2)\varphi,\varphi)$. Thus Theorem 5.3 can be applied to deduce the above result.

We note that the same semilinear problem was studied by Kusano and Swanson [50] and Fukagai [25] for f depending on u sublinearly or superlinearly, and unbounded positive solutions were obtained. We point out that our result is an immediate extension, since we cover the quasilinear case and we obtain unbounded as well as decaying positive solutions. We also remark that this example reveals the perturbation nature of our method: if the perturbation term $g(x,u,\nabla u)$ is "small" enough the perturbed solutions are "close" to the unperturbed ones.

As we have seen, by choosing different z, we could obtain entirely "wild" solutions.

To be specific, in the following examples, we always assume the functions $p(x)$, $q(x)$, $h(x)$, $g(x) \geq 0$ are nontrivial and in $C^\infty(R^n)$, and $a_{ij} = \delta_{ij}$, $b_i = 0$. We note that even though we could choose z from a large class of functions, radial or not, we select a very simple one in order to illustrate the idea. Specifically, we set

$$z = \begin{cases} |x|^{-\alpha}, & |x| > 1 \\ 1 + \frac{\alpha}{2} - \frac{\alpha}{2}|x|^2, & 0 \le |x| \le 1, \end{cases} \tag{5.9}$$

$\alpha \ge 0$ to be determined. We observe that

$$\nabla z = \begin{cases} -\alpha|x|^{-\alpha-2} \cdot x, & |x| > 1 \\ -\alpha x, & 0 \le |x| \le 1 \end{cases}$$

and

$$\Delta z = \begin{cases} -\alpha(n-\alpha-2)|x|^{-\alpha-2}, & |x| > 1 \\ -\alpha n, & 0 \le |x| \le 1. \end{cases}$$

Set $\alpha \le n-2$, then we have $z \in C^1(R^n) \cap W^{2,2}_{\text{loc}}(R^n)$, $-\Delta z \ge 0$, $z, \nabla z \in L^\infty(R^n)$, $z \to 0$ as $|x| \to \infty$.

Next, we choose α such that (v) is satisfied as follows: for $\varphi \in C_0^\infty(R^n)$, since $\left(-2\,\frac{\nabla z}{z} \cdot \nabla\varphi, \varphi\right) = \left(\text{div}\,\frac{\nabla z}{z},\, \varphi^2\right)$,

$$\left|\left(-2\frac{\nabla z}{z} \cdot \nabla\varphi, \varphi\right)\right| = \left|\left(\text{div}\,\frac{\nabla z}{z}, \varphi^2\right)\right| \le \int_{R^n} \left|\text{div}\,\frac{\nabla z}{z}\right| \cdot \varphi^2.$$

We observe that

$$\text{div}\,\frac{\nabla z}{z} = \begin{cases} \alpha(2-n)|x|^{-2}, & |x| > 1 \\ \dfrac{-\alpha n}{(1+\alpha/2-\alpha/2|x|^2)} - \dfrac{\alpha^2|x|^2}{(1+\alpha/2-\alpha/2|x|^2)^2}, & 0 \le |x| \le 1, \end{cases}$$

$$\left|\text{div}\,\frac{\nabla z}{z}\right| \le \begin{cases} \alpha(n-2)|x|^{-2}, & |x| > 1 \\ |-\alpha^2 - \alpha n| \le \dfrac{\alpha(n+\alpha)}{|x|^2}, & 0 \le |x| \le 1. \end{cases}$$

By Lemma 3.3, $\left(\frac{n-2}{2}\right)^2 \int_{R^n} \frac{1}{|x|^2}\, \varphi^2 \le \int_{R^n} |\nabla\varphi|^2$, hence by setting $0 \le \alpha < \frac{1}{2}(\sqrt{(n-2)^2+n^2}-n)$ we conclude that (v) holds:

$$\left(-\Delta\varphi - 2\,\frac{\nabla z}{z}\cdot\nabla\varphi,\ \varphi\right) > \left(\frac{(n-2)^2}{4} - \alpha^2 - \alpha n\right)(-\Delta\varphi,\ \varphi)$$

for $\varphi \in C_0^\infty(R^n)$.

Now, with z chosen, we will go on to illustrate our results by examining some examples.

Example 2. Consider

$$-\Delta u = p(x)u^\delta + q(x)|\nabla u|^\gamma \tag{5.10}$$

in R^n, $n \ge 3$, with $0 \le \delta, \gamma < 1$. We observe first that for $a, b \ge 0$, $(a+b)^\gamma \le 2^\gamma(a^\gamma + b^\gamma)$ for $\gamma \ge 0$. Note that since

$$\begin{aligned} &M\left(p(x)\,\frac{z^\delta a^\delta}{z} + q(x)a^\gamma|z+\nabla z|^\gamma/z\right) \\ &\quad\le M(p(x)z^{\delta-1})a^\delta + a^\gamma M\left(q(x)\left(z^{\gamma-1} + \frac{|\nabla z|^\gamma}{z}\right)2^\gamma\right) \\ &\quad\le M(p(x)z^{\delta-1})a^\delta + M(q(x)z^{\gamma-1})2^\gamma a^\gamma + (2a)^\gamma M(q(x)|\nabla z|^\gamma z^{-1}), \end{aligned}$$

we conclude that if $M(p(x)z^{\delta-1}) = P < \infty$, $M(q(x)z^{\gamma-1}) = Q < \infty$, and $M(q(x)|\nabla z|^\gamma z^{-1}) = Q_1 < \infty$, (5.10) has a positive solution u such that $0 < u \le c\cdot z$. The conditions we give essentially require the following:

(i) $\displaystyle\int_{R^n} p^2(x)(1+|x|^2)(1+|x|)^{-2(\delta-1)\alpha} < \infty$,

(ii) $\displaystyle\int_{R^n} q^2(x)(1+|x|^2)(1+|x|)^{-2(\gamma-1)\alpha} < \infty$,

(iii) $\displaystyle\int_{R^n} q^2(x)(1+|x|^2)(1+|x|)^{-2(\alpha+1)\gamma+2\alpha} < \infty.$

If $q \in L^{n/2}$, we need only to require (i) and

(ii)′ $\displaystyle\int_{R^n} q(x)(1+|x|)^{-2(\gamma-1)\alpha} < \infty,$

(iii)′ $\displaystyle\int_{R^n} q(x)(1+|x|)^{-2(\alpha+1)\gamma+2\alpha} < \infty.$

See Example 7 in Section 4.4 for comparison.

The decaying solutions for (5.10) were studied by many authors in the semilinear case using radial arguments and/or variational methods. For example, Noussair and Swanson [62,63,64] studied the problem $-\Delta u = p(|x|)u^\gamma$. They obtained decaying solutions for

(1) $p(r) \le (1+r^2)^{-a/2}$, $a \in (0,2]$, $\frac{n+2-2a}{n-2} < \gamma < \frac{n+2}{n-2}$ ([63], [64]),

(2) $\gamma < 1$, $p(r)$ bounded ([62]).

On the other hand, Li and Ni [LN, Theorem 1.4] proved that, if $K(x) = O(|x|^\delta)$ for $\delta < -2$, $K \in C^1(R^n)$ and $(n-(n-2)(p+1)/2)K(x) + x\cdot\nabla K(x)$ does not change sign in R^n, then $-\Delta u = K(x)u^p$ has no bounded decaying positive solutions. They also pointed out that the above hypotheses are not satisfied if $p < \frac{n+2}{n-2}$ unless $K \equiv 0$. We note that, since $p(x) = (1+|x|^2)^{-1}$ does not satisfy our criterion, the results we present here do not apply to the Matukuma equation, thus our study does not include those of Li and Ni [LN] and Ni and Yotsutani [NY]. Kawano, Satsuma and Yotsutani [41] studied the case $-\Delta u = \varphi(|x|)u^m$, $m > 1$ or $m < 1$, $\int_0^\infty r\varphi(r) < \infty$. Kusano and Swanson [51] consider exactly (5.10) for $0 < \delta$, $\gamma < 1$ or $1 < \delta$, $1 < \gamma < 2$ and radial p, q. Their criteria are:

$$\int_0^\infty tp(t) < \infty, \qquad \int_0^\infty t^{1-\gamma}q(t) < \infty.$$

We explicitly remark that even with (ii)′ and (iii)′, our criteria are not as good.

Example 3. We consider the following mixed sublinear–superlinear problem

$$-\Delta u = p(x)u^{\delta} + q(x)u^{\gamma} \tag{5.11}$$

in R^n, $n \geq 3$ with $0 < \delta < 1 < \gamma$. Let $P = M(pz^{\delta-1})$, $Q = M(qz^{\gamma-1})$. If $\eta\ P^{\frac{\gamma-1}{\gamma-\delta}} Q^{\frac{1-\delta}{\gamma-\delta}} \left[\left(\frac{\gamma-1}{1-\delta}\right)^{\frac{1-\delta}{\gamma-\delta}} + \left(\frac{1-\delta}{\gamma-1}\right)^{\frac{\gamma-1}{\gamma-\delta}} \right] < \frac{1}{2E_1}$, then (5.11) has a solution u such that $0 < u \leq cz$, c is a constant (see Example 4 of Section 4.4 for the calculation). The radially symmetric case was studied in Kusano and Trench [53]. They required, in particular, that $\int_0^\infty t^{n-1-\delta(n-2)}p(t) < \infty$ and $\int_0^\infty t^{n-1-\gamma(n-2)}q(t) < \infty$. Here we require $p, q \in L^{n/2}$, $\int_{R^n} p(x)(1+|x|)^{-2(\delta-1)\alpha} < \infty$ and $\int_{R^n} q(x)(1+|x|)^{-2(\gamma-1)\alpha} < \infty$. We remark that Furusho [27] also studied the mixed sublinear–superlinear problem for the nonradially symmetric case but with radial majorants. His statements and results are expressed in a different manner. We also note that this example is an open question in [53].

Example 4. Consider

$$-\Delta u = p(x)u^{\delta}(1 + |\nabla u|^{\lambda}) + q(x)u^{\gamma}(1 + |\nabla u|^{\mu}) \tag{5.12}$$

in R^n, $n \geq 3$, $0 \leq \lambda$, $\mu \leq 2$, $0 < \delta, \gamma$ and $\delta < 1$. Let $P_1 = M(pz^{\delta-1})$, $P_2 = 2^{\lambda}M(pz^{\lambda+\delta-1})$, $P_3 = 2^{\lambda}M(pz^{\delta-1}|\nabla z|^{\lambda})$, $Q_1 = M(qz^{\gamma-1})$, $Q_2 = 2^{\mu}M(qz^{\mu+\gamma-1})$, $Q_3 = 2^{\mu}M(qz^{\gamma-1}|\nabla z|^{\mu})$. Direct calculation shows that if there is a positive constant β such that

$$P_1\beta^{\delta-1} + (P_2 + P_3)\beta^{\delta+\lambda-1} + Q_1\beta^{\gamma-1} + (Q_2 + Q_3)\beta^{\mu+\gamma-1} < \frac{1}{2E_1},$$

then (5.12) has a decaying solution. We note that Kusano and Oharu [47] considered the bounded solutions of $-\Delta u = p(x)u^{\delta}+q(x)u^{\gamma}|\nabla u|^{\mu}$, see Example 5 of Section 4.4.

Example 5. Consider

$$-\Delta u = K(x)e^{u} \tag{5.13}$$

and

$$-\Delta u = K(x)e^{-u} \tag{5.14}$$

in R^n, $n \geq 3$, $K(x) > 0$. (5.13) is related to the Eddington equation in astrophysics (see e.g. Ni and Yotsutani [NY]) and we have studied the existence of bounded solutions to both cases in Example 6 of Section 4.4. Observe that condition (5.6) is satisfied for both problems. By Theorem 5.3, if there exists a positive constant a such that $M\left(\frac{K}{z}\right) < \frac{1}{2E_1}\, ae^{-a}$, then (5.13) has a decaying solution u such that $0 < u \leq az$. The criterion requires that

(i) $\displaystyle\int_{R^n} K^2(x)(1+|x|^2)(1+|x|)^{2\alpha} \leq \frac{1}{2E_1}\, ae^{-a}$, or

(i)′ $\displaystyle K(x) \in L^{n/2}, \quad \int_{R^n} K(x)(1+|x|)^{2\alpha} \leq \frac{1}{2E_1}\, ae^{-a}$.

For radially symmetric K, the existence of decaying solutions was proved by Ni and Yotsutani [NY] under the condition that

$$\int_0^{\infty} rK(r) \; < \frac{n-2}{2e},$$

while nonexistence of global positive solutions was given for $\int rK(r) \; = \infty$. We note that we obtain similar type of criterion here for nonradially symmetric K.

For (5.14), since e^{-z} is bounded, it is easy to see that for $a > 0$,

$$M\left(\frac{K(x)}{z}(e^{-z})^{a}\right) \leq M\left(\frac{K(x)}{z}\right) \cdot c,$$

where c is a constant independent of a. Hence by Theorem 5.3 and the sublinear nature of the problem, we conclude that if $M(K/z) < \infty$, i.e.,

(i) $$\int_{R^n} K^2(x)(1+|x|^2)(1+|x|)^{2\alpha} < \infty, \quad \text{or}$$

(i)′ $$K(x) \in L^{n/2}, \quad \int_{R^n} K(x)(1+|x|)^{2\alpha} < \infty,$$

then (5.14) has a decaying solution.

Chapter 6

DEGENERATE EQUATIONS

6.1. Introduction.

This chapter deals with the following problem:

$$-\Sigma D_j(a_{ij}(x,u)D_i u) = f(x,u,\nabla u) \tag{6.1}$$

in R^n, $n \geq 3$, with $\Sigma a_{ij}(x,u)\xi_i\xi_j > 0$ for $u \neq 0$, $a_{ij}(x,0) = 0$. The intention here is to obtain global existence results for positive solutions of (6.1), by extending the results of previous chapters.

The prototype of (6.1)

$$-\Delta\varphi(u) = f(x,u) \tag{6.2}$$

arises from a variety of physical and biological problems. Indeed, it is the stationary case for some nonhomogeneous reaction–diffusion equations. For example, if $\varphi'(u) > 0$ for $u \neq 0$, $\varphi'(0) = 0$, as in the case $\varphi(u) = |u|^{m-1}u$ with $m > 1$, (6.2) reduces to the porous media equation, while if $\varphi'(u) > 0$ for $u \neq 0$, $\varphi'(0) = +\infty$, as in the case $\varphi(u) = |u|^{m-1}u$ with $0 < m < 1$, we obtain a model for a heated plasma. We refer to Aronson [7], Diaz [21] and Peletier [67] for more physics background, references and related questions. Gurtin and MacCamy [30,31] introduced the same mathematical model to describe the dynamics of population with dispersion to avoid crowding. Okubo summarized all biological diffusions into three types: Fick, repulsive and attractive, all of which can be described by equations of the form (6.2), see [66].

The existence and uniqueness of nonnegative solutions to (6.2) have been investigated by several authors. Schatzman [72] proved the existence of nonnegative solutions for $\varphi(u) = u^m$ with $m > 1$ and $f(x,u) = f(x)$ while Spruck [75] proved the uniqueness of solutions for $\varphi(u) = u^m$ $(m > 1)$ and $f(x,u) = f(x)\cdot u$. By using a shooting method, Peletier and Tesei [68] studied the bifurcation and attractivity of 1-dimensional problems for $f(x,u) = a(x)u^p$ $(p \geq 1)$ and $\varphi(u) = u^m$ $(m > p)$. In [8], Aronson, Crandall and Peletier studied the 1-dimensional equilibria stability for $f(x,u) = u(1-u)(u-a)$ and $\varphi(u) = u^m$ $(m > 1)$. See also de Mottoni, Schiaffino and Tesei [57]. For the case of a more general φ which satisfies $\varphi(0) = \varphi'_+(0) = 0$, Pozio and Tesei [70] and Bandle, Pozio and Tesei [9] investigated the support properties of solutions for (6.2) and obtained conditions for the existence of nonnegative solutions with compact support. Some generalizations were then obtained by Rakotoson [71], who studied the problem $Au + F(x,u,\nabla u) = 0$ in a bounded domain, where A is an operator of Leray–Lions type. The Cauchy problem for $u_t = \Delta u^m$ with $0 < m < 1$ was considered by Herrero and Pierre [32] in $(0,\infty) \times R^n$, who also proved the existence and uniqueness of nonnegative solutions. In [36], a system of degenerate equations, originating from biological genetics, was studied by Kalashnikov.

While considering the more general equation (6.1), which embraces the porous media equations, the plasma physics equations as well as the biological equations, we treat the problem from a different point of view and obtain results in different directions. Instead of studying $\varphi(u) = u^m$ for $m > 1$ and $0 < m < 1$ separately, as has been done in most of the relevant literature, we present a unified approach which enables us to obtain global positive solutions. This treatment relies on the theory and methods we established and utilized in previous chapters.

At the beginning of Section 6.2, we will obtain the existence of solutions bounded above and below by positive constants. In order to obtain decaying solutions, we have to restrict ourselves to some specific cases, on which we can apply a variable change technique. Finally we will give three examples in Section 6.3 to illustrate our approach.

6.2. Existence Results.

In this section we will prove the existence of positive global bounded solutions for the following degenerate equation

$$\ell_1 u = -\Sigma D_j(a_{ij}(x,u)D_i u) + \Sigma b_i(x)D_i u = f(x,u) \tag{6.3}$$

in R^n, $n \geq 3$, with appropriate a_{ij} and b_i.

We define the following differential operator

$$\ell_1(v)u = -\Sigma D_j(a_{ij}(x,v)D_i u) + \Sigma b_i(x)D_i u \tag{6.4}$$

with $0 < \sigma a \leq v \leq a$, a and $\sigma < 1$ to be determined a posteriori. We assume that for such a v, $a_{ij} \in C^1(R^{n+1})$ and that (a_{ij}) is symmetric. We also assume that, for $0 < \sigma a \leq v \leq a$, either

$$\text{for} \quad m > 0, \qquad \lambda_0 \sigma^m a^m |\xi|^2 \leq \Sigma a_{ij}(x,v)\xi_i\xi_j \leq \lambda_1 a^m |\xi|^2 \tag{6.5}$$

or

$$\text{for} \quad m < 0, \qquad \lambda_0 a^m |\xi|^2 \leq \Sigma a_{ij}(x,v)\xi_i\xi_j \leq \lambda_1 \sigma^m a^m |\xi|^2, \tag{6.5'}$$

where λ_0, λ_1 are positive constants independent of σ, a and x. To use the a priori estimates of Chapter 3, we require the following structure condition

$$\exists \delta > 0, \qquad (\ell_1(v)\varphi, \varphi) \geq \delta(-\Delta\varphi, \varphi), \qquad \forall \varphi \in C_0^\infty(R^n), \tag{6.6}$$

where δ depends on σ and a. Furthermore, we assume that f satisfies Caratheodory's conditions and $b_i(x) \in L^q_{\text{loc}}(R^n)$ such that $N(b_i, q, 2, R^n) < \infty$ (see also Chapter 4). Now we can formulate the existence theorem:

THEOREM 6.1. *Let*

$$F(x,a) = \sup_{0\le u\le a} |f(x,u)| \tag{6.7}$$

satisfy $M(F(x,a)) < \infty$ *for any positive constant* a. *If for some positive constant pair* (a,σ) *with* $\sigma < 1$,

$$E_1 M(F(x,a)) \le \frac{1-\sigma}{2}\, a \tag{6.8}$$

holds, where E_1 *is given in Chapter 3, depending on* σ *and* a *in this case, then there exists a solution* $u \in C^1(R^n)$ *to (6.3) such that*

$$\sigma a \le u \le a.$$

The proof of this theorem follows along the same lines as those of the proof of Theorem 4.1, thus we will only sketch it.

Proof. Let K denote the ball in $B = C^0(|x| \le t_m)$ with norm $\|u\|_B = \|u\|_{C^0(|x|\le t_m)}$, centered at $a(1+\sigma)/2$ with radius $a(1-\sigma)/2$. Define an operator P on K by

$$P(u) = \frac{a(1+\sigma)}{2} + \ell_1^{-1}(u)(f(x,u)), \qquad u \in K,$$

where $\ell_1^{-1}(u)$ is the Dirichlet inverse of $\ell_1(u)$ in $(|x| < t_m)$. Then condition (6.8) and the same arguments as in the proof of Theorem 4.1 yield the existence of a fixed point of P, i.e., there exists a $u_m \in K$,

$$\ell_1\left(u_m - \frac{a(1+\sigma)}{2}\right) = f(x, u_m),$$

in $(|x| < t_m)$ with $a\sigma \leq u_m \leq a$.

Iterating the same process produces a sequence of $\{u_m\}$ and the convergence of $\{u_m\}$ to the solution u of (6.3) with desired properties follows from the same arguments as in the proof of Theorem 4.1. This ends the proof.

We note that in this particular case, condition (6.8) is very complicated to verify, since σ and a are intrinsically involved in the calculation of E_1. However the following corollary provides criteria which are easy to verify:

COROLLARY 6.2. *Assume for simplicity that* $-\Sigma D_i(b_i) \geq 0$. *Assume that for* $m > 0$

$$\text{either} \qquad \lim_{a \to 0^+} M(F(x,a))a^{-m-1} = 0,$$

$$\text{or} \qquad \lim_{a \to +\infty} M(F(x,a))a^{-1} = 0,$$

and for $m < 0$

$$\text{either} \qquad \lim_{a \to +\infty} M(F(x,a))a^{-m-1} = 0,$$

$$\text{or} \qquad \lim_{a \to 0^+} M(F(x,a))a^{-1} = 0.$$

Then (6.3) has infinitely many positive solutions bounded above and below by positive constants.

Proof. We only give the proof for $m > 0$, since the $m < 0$ case is analogous. From the proof of Theorem 3.2, we conclude that, since

$$(\ell_1(v)\varphi, \varphi) \geq a^{-m}\sigma^m \lambda_0(-\Delta\varphi, \varphi) \qquad \text{for} \quad \varphi \in C_0^\infty(R^n),$$

$$E_1 = c \cdot \sigma^{-mq/(q-n)\cdot\frac{n}{2}} \max(a^{-m}\sigma^{-m}c_1, c_2),$$

where c_1, c_2 and c are positive constants independent of a and σ. Observe that for constant a large enough, $\max(a^{-m}\sigma^{-m}c_1, c_2) = c_2$, thus if $\lim_{a\to\infty} M(F(x,a))/a = 0$, there exist infinitely many a's such that $E_1 M(F(x,a))/a < 1/2$. This implies the existence of infinitely many bounded solutions. Similarly for constant a small enough, $a^{-m}\sigma^{-m}c_1$ majorizes $\max(a^{-m}\sigma^{-m}c_1, c_2)$ and $E_1 M(F(x,a))/a < 1/2$ if $M(F(x,a))/a^{m+1}$ is small enough. Thus the proof is complete.

Next we concentrate on the existence of decaying solutions for (6.3). We observe that previous methods used in Chapter 5 are not directly applicable in the present case since it is difficult to find a decaying subsolution or a decaying supersolution for the general equation $\ell_1 u = 0$. We can only deal with some special cases which, however, lead to new results.

Instead of studying (6.3), we consider the following more special equation

$$-\Sigma D_i(\varphi(x)\psi(u)D_i u) = f(x,u) \tag{6.9}$$

with $c_2 \geq \varphi(x) > c_1 > 0$, $u \cdot \psi(u) > 0$ for $u \neq 0$, $\psi(0) = 0$. The technique we employ is a combination of variable change and the method we used in Chapter 5.

We first state an auxiliary lemma.

LEMMA 6.3. *Let $u \in W_0^{1,2}(|x| < r)$ be a solution to*

$$\ell u = f + D_i(g_i),$$

where ℓ satisfies the conditions of Theorem 3.2, $M(f) < \infty$ and $g_i \in L^2(R^n) \cap L^q_{\text{loc}}(R^n)$, $q > n$. Then

$$\sup_{|x|<r} |u| \leq E_1(M(f) + \|g_i\|_{L^2} + N(g_i, q, 2, R^n)), \tag{6.10}$$

where E_1 is given in Theorem 3.2.

Proof. From (A.20) in the proof of Theorem 3.2, we have

$$|u(x_0)| \leq K_0(\|u\|_{L^2(B_2)} + \|g_i\|_{L^q(B_2)} + \|f\|_{L^q(B_2)} \cdot \mu(B_2)^{1/q}). \tag{6.11}$$

By the Sobolev embedding theorem, we obtain

$$\|u\|_{L^2(B_2)} \leq \mu(B_2)^{1/n} T \|\nabla u\|_{L^2(B_r)}, \tag{6.12}$$

where T is the optimum embedding constant.

Observe that

$$\delta \|\nabla u\|^2_{L^2(|x|<r)} \leq (\ell u, u) = (f, u) + (D_i(g_i), u),$$

and

$$|(D_i(g_i), u)| = |(-g_i, D_i u)| \leq \|g_i\|_{L^2} \cdot \|\nabla u\|,$$

$$|(f, u)| \leq \frac{2}{n-2} \|f\|_{L^2_t} \cdot \|\nabla u\|.$$

Thus we obtain

$$\|\nabla u\|_{L^2(|x|<r)} \leq \frac{1}{\delta}(\|g_i\|_{L^2(|x|<r)} + \frac{2}{n-2}\|f\|_{L^2_t}). \tag{6.13}$$

Combining (6.11), (6.12) and (6.13) yields the lemma.

In the following we assume that ψ is a continuous function in R^1 and $\int_0^u \psi(s)ds$ is well defined. Note that in the case $\psi(u) = u^m$, we require $m > -1$. We introduce a new variable as follows:

Define

$$\pi(u) = \int_0^u \psi(s)ds \qquad \text{and} \qquad H = \varphi(x)\pi(u).$$

Since $u \cdot \psi(u) > 0$ for $u \neq 0$, $\pi^{-1}(u)$ exists and $u = \pi^{-1}(H/\varphi(x))$. Denote $f_1(x, H) = f\left(x, \pi^{-1}(H/\varphi(x))\right)$, then it is easily checked that

$$\nabla H = \varphi(x)\psi(u)\nabla u + (\nabla \ln \varphi)H,$$

and

$$-\Delta H = f_1(x, H) - \nabla((\nabla \ln \varphi)H). \tag{6.14}$$

We note that a decaying solution H of (6.14) will give a decaying solution u of (6.9) if $H/\varphi(x)$ also decays. Thus it suffices to study (6.14). We assume that f_1 and φ satisfy the following conditions:

(i) $\nabla \ln \varphi \in L^2 \cap L^\infty_{\text{loc}} \cap L^q_{\text{loc}}(R^n)$, $|\Delta \ln \varphi| < \dfrac{(n-2)^2}{4|x|^2}$ and $M\left(\nabla \ln \varphi \ \frac{\nabla z}{z}\right) < \infty$,

(ii) In a neighborhood Q of zero in $R^n \times R^+$,

$$\lim_{a \to 0+} \frac{f_1(x, a)}{a} = +\infty$$

uniformly, and

$$f_1(x, a) \geq g(x, u)$$

for some $g \in C^1$, $g(x, 0) \geq 0$.

Then we have

THEOREM 6.4. *Let z be the function given in Chapter 5 and let*

$$F(x, a) = \max_{0 \leq \xi \leq a} |f_1(x, \xi z)|/z$$

satisfy $M(F(x,a)) < \infty$ for any positive a. If there is a positive constant a such that

$$E_1\Big(M(F(x,a))/a+M\Big(\nabla\ln\varphi\,\frac{\nabla z}{z}\Big)+\|\nabla\ln\varphi\|_{L^2}+N(\nabla\ln\varphi,q,2,R^n)\Big) < \frac{1}{2}, \quad (6.15)$$

then there exists a solution H to (6.14) such that $0 < H \leq cz$, where c is a positive constant.

Proof. By substituting $H = vz$ into (6.14) we obtain

$$\begin{aligned}-\Delta v - 2\,\frac{\nabla z}{z}\,\nabla v - \frac{\nabla z}{z}\,v + \Big(-\frac{\Delta z}{z}\Big)v &= \frac{f_1(x,vz)}{z} - \frac{1}{z}\nabla((\nabla\ln\varphi)vz)\\ &= f_1(x,vz)/z - \nabla((\nabla\ln\varphi)v) - (\ln\varphi)v\,\frac{\nabla z}{z}.\end{aligned}$$

Thus we conclude, by Theorem ь.2, the existence of a supersolution cz to (6.14).

On the other hand, (6.14) is equivalent to

$$-\Delta H + (\Delta\ln\varphi)H = f_1(x,H) - \nabla\ln\varphi\cdot\nabla H, \quad (6.14')$$

and $|\Delta\ln\varphi| < \dfrac{(n-2)^2}{4|x|^2}$ implies that the eigenvalue problem

$$\begin{cases} -\Delta H + (\Delta\ln\varphi)H = \lambda H, & \text{in } \Omega\\ H = 0, & \text{on } \partial\Omega\end{cases}$$

has a positive solution in Ω, where Ω is the projection of Q onto its first n components. Observe that $\lim_{a\to 0+} \dfrac{f_1(x,a)}{a} = +\infty$ in Q ensures that the spectral procedure we employed in Chapter 5 is applicable here and hence there exists a nonnegative subsolution to (6.14). The rest of the proof follows along the same lines as those of the proof of Theorem 5.3. This concludes the proof.

Remark. We note that $\frac{\nabla z}{z}$ is of the order $|x|^{-1}$. Thus, if we ignore the weight function λ, then $M\left(\nabla \ln \varphi \ \frac{\nabla z}{z}\right) < \infty$ can be replaced by $\nabla \ln \varphi \in L^2(R^n)$. In particular if $\varphi(x)$ is a constant, then (6.15) becomes $E_1 M(F(x,a)) < a/2$. We further observe that in general, $\varphi = c + |x|^{-k}$ with $c > 0$, $k + 1 > n$ satisfies the required condition (i) preceding Theorem 6.4.

THEOREM 6.5. *Suppose that the conditions of Theorem 6.4 hold. Assume that* $0 < c_1 \leq \varphi(x) \leq c_2$. *Then (6.9) has a positive decaying solution* u *such that* $0 < u \leq c\pi^{-1}(H/\varphi(x))$, *where* H *is given by Theorem 6.4.*

Remark. The right hand side of (6.9) can depend on ∇u also. We choose not to pursue this particular case in order to avoid the complicated notations which ensue but we will illustrate this in the next section with an example. However, we explicitly point out that the dependence of f on ∇u in (6.3) is excluded. In order to estimate $||\nabla u||_\infty$, we have to know $||D^2 a_{ij}||_q$ locally (see Theorems 3.1 and 3.2), where the partial derivative is taken on all variables. The implicit expression of a_{ij} makes the estimation of $||D^2 a_{ij}||_q$ extremely difficult.

6.3. Examples.

Example 1. Consider the following problem

$$-\nabla(\varphi(x)|u|^\gamma \nabla u) = p(x)u^\lambda \tag{6.16}$$

in R^n, $n \geq 3$, with $\lambda \geq 0$. For simplicity we assume $p(x) > 0$. Assume that $c_1 \leq \varphi(x) \leq c_2$ for two positive constants c_1 and c_2 and that $M(p(x)) < \infty$. Then by Corollary 6.2, (6.16) has infinitely many positive solutions which are bounded above and below by positive constants provided that, for $\gamma \geq 0$, either (i) $\lambda > \gamma+1$ or (ii) $\lambda < 1$, and for $\gamma < 0$, either (i) $\lambda < \gamma+1$ or (ii) $\lambda > 1$.

Example 2. We consider the existence of decaying solution of

$$-\nabla(|u|^\gamma \nabla u) = p(x)u^\sigma + q(x)u^\delta \tag{6.17}$$

in R^n, $n \geq 3$, with $p(x), q(x) \geq 0$, $\gamma > -1$, and σ and δ to be specified later. We introduce the new variable

$$H = u^{\gamma+1}/(\gamma+1).$$

Then

$$u = ((\gamma+1)H)^{\frac{1}{\gamma+1}}.$$

Thus we obtain the following new equation

$$-\Delta H = p(x)(\gamma+1)^{\frac{\sigma}{\gamma+1}} H^{\frac{\sigma}{\gamma+1}} + q(x)(\gamma+1)^{\frac{\delta}{\gamma+1}} H^{\frac{\delta}{\gamma+1}}. \tag{6.17'}$$

By Example 3 of Chapter 5, let $P = M\Big((\gamma+1)^{\frac{\sigma}{\gamma+1}} p \cdot z^{\frac{\sigma}{\gamma+1}-1}\Big)$, $Q = M\Big((\gamma+1)^{\frac{\delta}{\gamma+1}} q \cdot z^{\frac{\delta}{\gamma+1}-1}\Big)$, if

$$\eta = P^{\frac{\delta-\gamma-1}{\delta-\sigma}} Q^{\frac{\gamma+1-\sigma}{\delta-\sigma}} \left[\left(\frac{\delta-\gamma-1}{\gamma+1-\sigma}\right)^{\frac{\gamma+1-\sigma}{\delta-\sigma}} + \left(\frac{\gamma+1-\sigma}{\delta-\gamma-1}\right)^{\frac{\delta-\gamma-1}{\delta-\sigma}}\right] < \frac{1}{2E_1}$$

where E_1 and z are given in Chapters 3 and 5 respectively, and $0 < \sigma < \gamma+1 < \delta$, then (6.17′) has a solution H such that $0 < H \leq cz$. Hence (6.17) has a solution u such that $0 < u \leq c \cdot z^{\frac{1}{\gamma+1}}$. This is the mixed sublinear–superlinear case. For the case $0 < \sigma,\ \delta < \gamma+1$, the estimate about η is not needed. Note that for the following cases: (i) $0 < \sigma,\ \delta < \gamma+1$, (ii) $0 < \gamma+1 < \sigma, \delta$, (iii) $\sigma, \delta < 1$ and

(iv) $\sigma, \delta > 1$, we also obtain infinitely many bounded positive solutions bounded away from zero.

Example 3. Consider

$$-\nabla(\varphi(x)|u|^{\gamma}\nabla u) = p(x)u^{\sigma} + q(x)|\nabla u|^{\delta}u^{\lambda} \tag{6.18}$$

in R^n, $n \geq$ [illegible], with $p(x) > 0$, $q(x) \geq 0$, $0 < \sigma < 1+\gamma$, and $0 \leq \delta \leq 2$. Following the construction before Theorem 6.4, we define

$$H = \varphi(x)u^{\gamma+1}/(\gamma+1).$$

Now we have

$$u = \left(\frac{(\gamma+1)H}{\varphi}\right)^{\frac{1}{\gamma+1}},$$

$$\nabla u = (\gamma+1)^{-\frac{\gamma}{\gamma+1}}\left(\frac{H}{\varphi}\right)^{\frac{1}{\gamma+1}}(\nabla \ln H - \nabla \ln \varphi).$$

Thus we obtain

$$u^{\lambda}|\nabla u|^{\delta} \leq (\gamma+1)^{\frac{\lambda-\gamma}{\gamma+1}}\, 2^{\delta}\left(\frac{H}{\varphi}\right)^{\frac{\delta+\lambda}{\gamma+1}}(|\nabla \ln H|^{\delta} + |\nabla \ln \varphi|^{\delta}),$$

whence

$$F(x,a,a) \leq p(x)(\gamma+1)^{\frac{\sigma}{\gamma+1}}\left(\frac{z}{\varphi}\right)^{\frac{\sigma}{\gamma+1}} \cdot a^{\frac{\sigma}{\gamma+1}}$$

$$+ 2^{2\delta}(\gamma+1)^{\frac{\lambda-\gamma}{\gamma+1}}(1+|\nabla \ln \varphi|^{\delta})(1+|\nabla \ln z|^{\delta})\varphi^{-\frac{\lambda+\delta}{\gamma+1}}q(x) \cdot a^{\frac{\delta+\lambda}{1+\gamma}}$$

$$G(x,a).$$

Theorem 6.5 implies that if for some $a > 0$,

$$E_1(M(G(x,a))/a + M(\nabla \ln \varphi \cdot \nabla \ln z) + \|\nabla \ln \varphi\|_{L^2} + N(\nabla \ln \varphi, q, 2, R^n)) < 1/2,$$

then (6.18) has a solution u such that

$$0 < u \le c \cdot (z/\varphi)^{\frac{1}{\gamma+1}}.$$

Hence we obtain a decaying solution. If, in particular, $\varphi(x) \equiv 1$, then

$$G(x,a) = (\gamma+1)^{\frac{\sigma}{\gamma+1}}\ a^{\frac{\sigma}{\gamma+1}}\ z^{\frac{\sigma}{\gamma+1}}\ p(x) + 2^{\delta}(\gamma+1)^{\frac{\lambda-\delta}{\gamma+1}}(1 + |\nabla \ln z|^{\delta})a^{\frac{\delta+\lambda}{\gamma+1}}q(x),$$

and we only require

$$E_1 M(G(x,a)) < \frac{a}{2}$$

for the existence of decaying solution. If we further have $\gamma + 1 > \max(\delta + \lambda, \sigma)$, σ, $\delta + \lambda > 0$, then the following conditions will suffice for our purpose:

(i) $$\int_{R^n} (1+|x|^2)(1+|x|)^{\frac{2\sigma\alpha}{1+\gamma}} p^2(x) < \infty,$$

(ii) $$\int_{R^n} (1+|x|^2) q^2(x) < \infty,$$

(iii) $$\int_{R^n} (1+|x|^2)(1+|x|)^{-2\delta} q^2(x) < \infty,$$

where α is given in Chapter 5.

Chapter 7

HIGHER ORDER ELLIPTIC SYSTEMS

7.1. Introduction.

In this chapter, we are concerned with the extension of the results of previous chapters to higher order systems. We first note that the extension to second order systems is straightforward, see Noussair and Swanson [65] for example, and we choose not to do so. In extending to higher order systems, we rely on the theory we established in Chapters 3 and 4. In a general sense, this chapter can be regarded as an application of the theory in Chapters 3 and 4. For simplicity, we focus on fourth order systems. The same methods apply to higher even order systems, but the calculations become more awkward as the order gets higher.

Higher order elliptic equations have received extensive attention recently. Kusano and Swanson [52] studied the biharmonic problem $\Delta^2 u = f(|x|, u)$ in R^n, $n \geq 3$, and obtain unbounded as well as decaying solutions ($n \geq 5$ for decaying case). Fukagai [26] studied the semilinear equations of the form $\Delta^N u + \sum_{i=1}^{N-1} a_i \Delta^i u = f(|x|, u)$ in R^n, $n \geq 3$ and $N \geq 1$, a_i constants, and solutions with specific behavior were obtained. For higher even order semilinear equations with f depending on u superlinearly, Usami [77] proved the existence of infinitely many positive solutions which behave like $|x|^{2m-2}$ ($2m =$ the order of the equation). Kusano, Naito and Swanson [45] proved the existence of infinitely many positive and eventually negative solutions in R^2. Higher even order quasilinear problems were treated by Kusano, Naito and Swanson [44].

This chapter represents an extension of the results in Allegretto and Huang [5], where the system $\ell_1\ell_0\vec{u} = \vec{F}(x,\vec{u})$ was under consideration, with ℓ_1, ℓ_0 being second order elliptic operators. In [5], we obtained the existence of infinitely many positive solutions bounded above and below by positive constants with $\vec{F}$ depending on $\vec{u}$ either sublinearly or superlinearly. We will study here a more general problem

$$\ell_1\ell_0\vec{u} = \vec{F}(x,\vec{u},\nabla\vec{u}) \tag{7.0.1}$$

in R^n, $n \geq 3$, and we intend to obtain the existence of bounded positive solutions bounded away from zero. Since one may reduce (7.0.1) to the following system

$$\begin{cases} \ell_0\vec{u} = \vec{v}, \\ \ell_1\vec{v} = \vec{F}(x,\vec{u},\nabla\vec{u}), \end{cases} \tag{7.0.1$'$}$$

it might at first sight appear that this problem is contained in the framework of previous chapters. However, due to the presence of the coefficient 1 on the right hand side, which does not belong to any of the weighted spaces we introduced, this system does not fulfill the requirements we postulated. Thus, the study of (7.0.1) is meaningful. We remark that the reduction of (7.0.1) into (7.0.1$'$) is however the starting point of our process, which is basically an iteration procedure. More precisely, we use $\ell_0\vec{u} = \vec{v}$ to obtain a priori bound on $\vec{u}$ by $\vec{v}$ and then use $\ell_1\vec{v} = \vec{F}(x,\vec{u},\nabla\vec{u})$ to obtain another a priori bound on $\vec{v}$. We combine those two steps to obtain the required a priori bound. We note that each step is an application of our previous theory and the final utilization of the Schauder fixed point theorem is standard. These procedures are the main content of Section 7.2. As before, in Section 7.3 we consider some concrete examples and present some comparisons with previously known results.

However, we remark that, due to the structure of fourth (or higher) order problems, we are unable to adopt the techniques of Chapter 5 to obtain the existence of decaying solutions, and because of the implicit formulation of weight function λ, we also leave this part aside when we define a new norm. These technical reasons will be made clearer in the sequel.

7.2. Main Theorem.

In this section we will use the techniques of Chapters 3 and 4 to derive our existence theorem. We will employ the same notation used in previous chapters.

Let $\ell_k = -\Sigma D_i(a_{ij}^k(x)D_j u) + \Sigma b_i^k D_i u,\ k = 0, 1$, be two elliptic operators in R^n, with $a_{ij}^k = a_{ji}^k,\ \lambda_0^k|\xi|^2 \le \Sigma a_{ij}^k \xi_i \xi_j \le \lambda_1^k |\xi|^2$, for some positive constants $\lambda_0^k, \lambda_1^k,\ k = 0, 1$. We assume that $a_{ij}^k \in L^\infty(R^n),\ D^2 a_{ij}^k \in L_{\text{loc}}^q(R^n)$ with $N(D^2 a_{ij}^k, q, 2, R^n) < \infty$ and $b_i^k \in W_{\text{loc}}^{1,q}(R^n)$ such that $N(\nabla b_i^k, q, 2, R^n) < \infty,\ q > n,\ k = 0, 1$. To apply the a priori estimates of Chapter 3, we assume that

$$(\ell_k \varphi, \varphi) \ge \delta^k(-\Delta\varphi, \varphi)$$

for some $\delta^k > 0$ and any $\varphi \in C_0^\infty(R^n),\ k = 0, 1$. We further assume that

$$\sum_i \Big(\sum_j |D_i a_{ij}^1|\Big)^2 \le |x|^{-2}, \qquad \Sigma|b_i^1|^2 \le |x|^{-2}, \qquad -\Sigma D_i(b_i^1) \ge 0. \tag{7.1}$$

Let $\vec{F}$ satisfy Caratheodory's conditions and since we can multiply $\vec{F}$ by a positive constant, we may assume, without loss of generality that $\lambda_1^1 \le 1$. We consider the following problem

$$\ell_1 \ell_0 \vec{u} = \vec{F}(x, \vec{u}, \nabla \vec{u}) \tag{7.2}$$

in $R^n,\ n \ge 3$.

We begin with some technical lemmas.

Lemma 7.1. *Let $\varphi \in C_0^\infty(R^n)$, then for any real number $\alpha > 2-n$,*

$$\int_{R^n} |x|^\alpha |\nabla\varphi|^2 \geq \frac{1}{4}(2-n-\alpha)^2 \int_{R^n} |x|^{\alpha-2}\varphi^2. \tag{7.3}$$

Proof. Let $\Omega = R^n - B_\varepsilon(0)$, for $\varepsilon > 0$. Let $\varphi = |x|^\beta \psi$ with $\beta = (2-n-\alpha)/2$, $\psi \in C_0^\infty(R^n)$. Then we have

$$\int_\Omega |x|^\alpha |\nabla\varphi|^2 = \int_\Omega |x|^{2\beta+\alpha}|\nabla\psi|^2 + \int_\Omega \beta^2 |x|^{\alpha+2\beta-2}\psi^2 + \int_\Omega \beta |x|^{\alpha+2\beta-2} \cdot x \cdot 2\psi \cdot \nabla\psi.$$

Using Green's formula on the last term on the right hand side, we obtain

$$\begin{aligned}\int_\Omega \beta|x|^{\alpha+2\beta-2}x\cdot\nabla\psi^2 &= -\int_\Omega \beta|x|^{\alpha+2\beta-2}n\psi^2 - \int_\Omega \beta(\alpha+2\beta-2)|x|^{\alpha+2\beta-2}\psi^2 \\ &\quad + \int_{\partial\Omega} \beta|x|^{\alpha+2\beta-2}\psi^2 x\cdot\vec{n} \\ &= -\beta(n+\alpha+2\beta-2)\int_\Omega |x|^{\alpha+2\beta-2}\psi^2 + \int_{\partial\Omega}\beta|x|^{\alpha+2\beta-2}\psi^2 x\cdot\vec{n},\end{aligned}$$

where $\vec{n}$ is the inward unit normal of $\partial B_\varepsilon(0)$.

Combining these equations we obtain

$$\begin{aligned}\int_\Omega |x|^\alpha|\nabla\varphi|^2 = \int_\Omega |x|^{\alpha+2\beta}|\nabla\psi|^2 \\ + (-\beta^2 + \beta(2-n-\alpha))\int_\Omega |x|^{\alpha+2\beta-2}\psi^2 \\ + \int_{\partial\Omega}\beta|x|^{\alpha+2\beta-2}\psi^2 x\cdot\vec{n}.\end{aligned}$$

We observe that

$$\int_\Omega |x|^{\alpha+2\beta}|\nabla\psi|^2 \geq 0$$

and

$$\left|\int_{\partial\Omega}\beta|x|^{\alpha+2\beta-2}\psi^2 x\vec{n}\right| \leq K\cdot\|\varphi\|_\infty^2\cdot\varepsilon^{n+\alpha-2} \to 0 \quad \text{as} \quad \varepsilon\to 0.$$

Thus, letting $\varepsilon \to 0$ we obtain the desired inequality.

Remark. This is Lemma 3 of Allegretto [2], where the proof is given for $\varphi \in C_0^\infty(\Omega)$, $0 \notin \Omega$. We essentially adopt the proof there.

Now we introduce another weight function $s = (1+|x|^2)(1+|x|^4)$ and state the following

LEMMA 7.2. *Let $\psi \in C_0^\infty(R^n)$. Assume that $\lambda_1^1 \le 1$, $\frac{8(n+4)}{n^2} < \lambda_0^1$, then*

$$\|\varphi\|_{L_t^2} \le \frac{4}{n^2\lambda_0^1 - 8(n+4)} \|\ell_1\varphi\|_{L_s^2}, \tag{7.4}$$

where $t = 1+|x|^2$, $s = (1+|x|^2)(1+|x|^4)$.

Proof. For any $\varphi \in C_0^\infty(R^n)$, by Lemma 7.1, we have

$$\int_{R^n} |x|^2\varphi^2 \le \frac{4}{(n+2)^2}\int_{R^n} |x|^4|\nabla\varphi|^2,$$

$$\int_{R^n} \varphi^2 \le \frac{4}{n^2}\int_{R^n} |x|^2|\nabla\varphi|^2.$$

We sum these and use the ellipticity of (a_{ij}^1) to obtain

$$\begin{aligned}\int_{R^n}(1+|x|^2)\varphi^2 &\le \frac{4}{n^2}\int_{R^n}(|x|^2+|x|^4)|\nabla\varphi|^2 \\ &\le \frac{4}{n^2\lambda_0^1}\int_{R^n}(|x|^2+|x|^4)\Sigma a_{ij}D_i\varphi D_j\varphi \qquad \text{(we drop the superscript)} \\ &= \frac{4}{n^2\lambda_0^1}\int_{R^n}(|x|^2+|x|^4)(\Sigma a_{ij}D_i\varphi D_j\varphi + \Sigma b_iD_i\varphi\cdot\varphi - \Sigma b_iD_i\varphi\cdot\varphi).\end{aligned}$$

By using the Divergence Theorem we have

$$\begin{aligned}\int_{R^n}(1+|x|^2)\varphi^2 \le{}& \frac{4}{n^2\lambda_0^1}\int_{R^n}[-\Sigma D_j(|x|^2+|x|^4)a_{ij}D_i\varphi\cdot\varphi - \frac{1}{2}\Sigma b_i(|x|^2+|x|^4)D_i\varphi^2] \\ &+ \frac{4}{n^2\lambda_0^1}\int_{R^n}(-\Sigma D_j(a_{ij}D_i\varphi) + \Sigma b_iD_i\varphi)\varphi\cdot(|x|^2+|x|^4),\end{aligned}$$

whence

$$\int_{R^n}(1+|x|^2)\varphi^2 \leq \frac{4}{n^2\lambda_0^1}\int_{R^n}\frac{1}{2}[\Sigma D_i(a_{ij}D_j(|x|^2+|x|^4))+\Sigma D_i(b_i(|x|^2+|x|^4))]\varphi^2$$

$$+\frac{4}{n^2\lambda_0^1}\int_{R^n}\ell_1\varphi\cdot\varphi\cdot|x|^2(1+|x|^2).$$

We observe the following

$$\Sigma D_i(a_{ij}D_j(|x|^2+|x|^4))+\Sigma D_i(b_i(|x|^2+|x|^4))$$

$$=\Sigma 8x_ix_ja_{ij}+\Sigma 2(1+2|x|^2)x_jD_i(a_{ij})+\Sigma a_{ii}2(1+2|x|^2)$$

$$+\Sigma 2(1+2|x|^2)x_ib_i+|x|^2(1+|x|^2)\Sigma D_ib_i$$

$$\leq 8\lambda_1^1|x|^2+2(1+2|x|^2)n+2(1+2|x|^2)|x|\cdot\Big(\sum_i\Big(\sum_j D_i(a_{ij})\Big)^2\Big)^{1/2}$$

$$+2(1+2|x|^2)|x|\cdot(\Sigma b_i^2)^{1/2}$$

$$\leq(8\lambda_1^1+4n+8)|x|^2+2n+4$$

$$\leq(16+4n)|x|^2+2n+4\leq 4(n+4)(1+|x|^2).$$

Note that we used (7.1) in the above process. We conclude

$$\int_{R^n}\Big(1-\frac{8(n+4)}{n^2\lambda_0^1}\Big)(1+|x|^2)\varphi^2\leq\frac{4}{n^2\lambda_0^1}\Big(\int_{R^n}\varphi^2(1+|x|^2)\int_{R^n}(1+|x|^2)|x|^4|\ell_1\varphi|^2\Big)^{1/2}.$$

Hence if $8(n+4)/n^2<\lambda_0^1$, and since $(1+|x|^2)|x|^4\leq(1+|x|^2)(1+|x|^4)$, we obtain

$$\int_{R^n}(1+|x|^2)\varphi^2\leq\Big(\frac{4}{n^2\lambda_0^1-8(n+4)}\Big)^2\int_{R^n}(1+|x|^2)(1+|x|^4)|\ell_1\varphi|^2.$$

This completes the proof of Lemma 7.2.

Observe that the proof of Lemma 7.2 depends on the estimate

$$\frac{-2}{n^2\lambda_0^{\frac{1}{2}}}\int_{R^n}\ell_1^*(|x|^2+|x|^4)\varphi^2 \leq K\int_{R^n}(|x|^2+|x|^4)\ell_1\varphi\cdot\varphi+\varepsilon\int_{R^n}\varphi^2(1+|x|^2)$$

for some $K>0$ and $0\leq\varepsilon<1$, where ℓ_1^* denotes the formal adjoint of ℓ_1. Our technical explicit conditions on the coefficients and n are merely criteria which ensure the validity of this inequality for ℓ_1 "near" $-\Delta$ and $n\geq 11$. Since the explicit forms of the weight functions are also used in the calculation, we are unable to derive a similar inequality for an implicit weight function λ. We remark that a limit argument shows that (7.4) is valid for any function $g\in W_0^{1,2}(|x|<r)$ with $\ell_1 g=f\in L^q(|x|<r)$, $q>n/2$. For the special case $\ell_1=-\Delta$, we have, by the same estimate,

$$\|\varphi\|_{L_t^2}\leq\frac{4}{n^2-8(n+2)}\|\Delta\varphi\|_{L_s^2}$$

for any $\varphi\in C_0^\infty(R^n)$. Observe that now $n\geq 10$ is required.

We define a new norm $M_1(\cdot)$ in this section by

$$M_1(f)=\|f\|_{L_s^2(R^n)}+N(f,q,2,R^n).$$

We note that $M_1(f)$ is parallel to $\|f\|_{\mathcal{C}_1}$ in previous chapters.

We are ready to state the following lemma, which is a new formulation of Theorem 3.2 for the present situation.

LEMMA 7.3. *Let f be such that $M_1(f)<\infty$, for $q>n$. Then for ξ_r with ξ_r, $\ell_0\xi_r\in W_0^{1,2}(|x|<r)$ such that $\ell_1\ell_0\xi_r=f$, we have*

$$\max\{\|\xi_r\|_{C^0(|x|<r)},\|\nabla\xi_r\|_{C^0(|x|<r-2)}\}$$

$$\leq M_1(f)\cdot E_1\mu(B_2)^{1/q}\cdot\max\left\{1,\frac{4}{n^2\lambda_0^{\frac{1}{2}}-8(n+4)}\right\}, \quad (7.5)$$

where E_1 is given by Theorem 3.2 with $\delta = \min\{\delta^1, \delta^2\}$ and is independent of f, r, ξ_r.

Proof. We set $\ell_0\xi_r = g$ and $\ell_1 g = f$. For any x_0 such that $B_2(x_0) \subset (|x| < r)$, we have

$$|\xi_r(x_0)| \leq E_1(\|g\|_{L^2_t} + \|g\|_{L^q(B_2(x_0))}),$$

and

$$|\nabla\xi_r(x_0)| \leq E_1(\|g\|_{L^2_t} + \|g\|_{L^q(B_2(x_0))}).$$

Since $\ell_1 g = f$, we also have

$$|g(y)| \leq E_1(\|f\|_{L^2_t} + \|f\|_{L^q(B_2(y))}).$$

Hence we conclude

$$\Big(\int_{B_2(x_0)} g^q\Big)^{1/q} = \|g\|_{L^q(B_2(x_0))} \leq E_1(\|f\|_{L^2_t} + N(f, q, 2, R^n)) \cdot \Big(\int_{B_2(x_0)} 1\Big)^{1/q}.$$

We observe that

$$\|f\|_{L^2_t} \leq \|f\|_{L^2_s},$$

and by Lemma 7.2,

$$\|g\|_{L^2_t} \leq \frac{4}{n^2\lambda_0^1 - 8(n+4)}\|\ell_1 g\|_{L^2_s} = \frac{4}{n^2\lambda_0^1 - 8(n+4)} \cdot \|f\|_{L^2_s}.$$

Thus we conclude

$$\|g\|_{L^2_t} + \|g\|_{L^q(B_2(x_0))} \leq \frac{4}{n^2\lambda_0^1 - 8(n+4)}\|f\|_{L^2_s} + N(f, q, 2, R^n).$$

This proves the lemma.

We are ready to state our main theorem of this chapter.

Theorem 7.4. *Let*

$$\vec{F}(x,a,b) = \sup_{\substack{0\le\xi\le a\vec{1}\\ -b\vec{1}\le\vec{r}\le b\vec{1}}} |\vec{F}|(x,\xi,\vec{r})$$

satisfy $M_1(\vec{F}(x,a,b)) < \infty$, *for any positive* a, b, *where* $|\vec{f}| = (|f_1|,\dots,|f_k|)^T$ *if* $\vec{f} = (f_1,\dots,f_k)^T$. *Assume that there exist three positive constants* a, b, σ *with* $\sigma < 1$ *such that*

$$EM_1(\vec{F}(x,a,b)) < \min\left(b,\ \frac{1-\sigma}{2}\ a\right), \tag{7.6}$$

where $E = E_1 \cdot \mu(B_2)^{1/q} \cdot \max\left(1,\ \frac{4}{n^2\lambda_0^{\frac{1}{2}} - 8(n+4)}\right)$. *Then (7.2) has a positive solution* $\vec{u}$ *such that* $\sigma a\vec{1} \le \vec{u} \le a\vec{1}$, $|\nabla\vec{u}| \le b$.

We omit the proof since it is identical to that of Theorem 4.1, by defining $P(\vec{u}) = \frac{1+\sigma}{2}\ a\vec{1} + \ell_0^{-1}\ell_1^{-1}(\vec{F}(x,\vec{u},\varphi_m\nabla\vec{u}))$.

We can also state the following corollary:

Corollary 7.5. *Assume that the conditions of Theorem 7.4 are satisfied, with (7.6) replaced by*

$$\lim_{a\to\beta} \frac{M_1(\vec{F}(x,a,a))}{a} < \frac{1}{2E}, \qquad \beta = 0^+ \quad or \quad +\infty. \tag{7.6'}$$

Then problem (7.2) has infinitely many positive solutions which are bounded above and below by positive constants.

We further have

Theorem 7.6. *Assume that conditions of Theorem 7.4 hold. If further*

$$\lim_{|x|\to\infty} \|\vec{F}\|_{L^q(B_2(x))} = 0, \qquad |b_i^0| \le c_0/(1+|x|),$$

then the soluton $\vec{u}$ given by Theorem 7.4 tends to a positive constant vector.

We only remark here that the other condition required in Theorem 4.4: $|b_i^1| \leq c_0/(1+|x|)$, is valid automatically by (7.1). For completeness, we include a brief sketch of the proof which follows the ideas of the proof of Lemma 7.3.

Proof. Choose $\alpha > 0$ and $h \in C^1$ such that $h(x) = |x|^{-\alpha}$, for $|x| > 2$, $|h(x)| \leq D(\alpha)$ (constant) for $|x| < 2$, $\left|\frac{D_i h}{h}\right| \leq \frac{c(\alpha)}{1+|x|}$, with $c(\alpha) \to 0$ as $\alpha \to 0$, $h(0) = 1$, and for any fixed $x_0 \in R^n$, $h_0(x) = h(x - x_0)$. As in the proof of Lemma 7.3, denote $\ell_0\vec{u} = g$, $\bar{u} = \vec{u} - \frac{a(1+\sigma)}{2}\vec{1}$, where a and σ are given in (7.6). Then by the proof of Theorem 4.4, we have

$$|\bar{u}(x_0)| = \left|\vec{u}(x_0) - \frac{a(1+\sigma)}{2}\,\vec{1}\right| \leq K(\|gh_0\|_{L_t^2} + \|gh_0\|_{L^q(B_2(x_0))}).$$

Since also $\ell_1 g = f$, in the same manner as the proof of Theorem 4.4, we define an operator $\tilde{\ell}_1$ such that $\tilde{\ell}_1(gh_0) = fh_0$, with $(\tilde{\ell}_1\varphi, \varphi) \geq \frac{\delta^1}{2}(-\Delta\varphi, \varphi)$ for any $\varphi \in C_0^\infty(R^n)$. Then we obtain

$$|gh_0(y)| \leq E_1(\|fh_0\|_{L_t^2} + \|fh_0\|_{L^q(B_2(y))})$$

and Lemma 7.2 yields

$$\|gh_0\|_{L_t^2} \leq c \cdot \|\tilde{\ell}_1(gh_0)\|_{L_s^2} = c \cdot \|fh_0\|_{L_s^2}.$$

We observe that

$$\|fh_0\|_{L_t^2} \leq \|fh_0\|_{L_s^2}.$$

Thus we conclude that

$$\left|\vec{u}(x_0) - \frac{a(1+\sigma)}{2}\,\vec{1}\right| \leq K \cdot (\|fh_0\|_{L_s^2} + \|fh_0\|_{L^q(B_2(x_0))}).$$

We have

$$\|fh_0\|_{L^q(B_2(x_0))} \le D(\alpha)\cdot\|f\|_{L^q(B_2(x_0))} \to 0 \qquad \text{as} \quad |x_0|\to\infty,$$

and

$$\|fh_0\|^2_{L^2_s} \le \|f\|^2_{L^2_s(|x|>\frac{|x_0|}{2})} + \frac{c}{|x_0|^{2\alpha}}\|f\|_{L^2_s}$$

also tends to zero as $|x_0|\to\infty$.

Thus we conclude $\vec{u}(x_0)\to \frac{a(1+\sigma)}{2}\,\vec{1}$. This completes the proof.

7.3. Examples.

We will briefly discuss some examples, noting that the comments made in Chapter 4 are still true here.

Example 1. Consider the following system

$$\begin{aligned} \Delta^2 u &= p(x)u^{\alpha_1}[\ln(1+u)]^{\beta_1}v^{\alpha_2}[\ln(1+v)]^{\beta_2},\\ \Delta^2 v &= q(x)u^{\alpha_3}[\ln(1+u)]^{\beta_3}v^{\alpha_4}[\ln(1+v)]^{\beta_4} \end{aligned} \tag{7.7}$$

in R^n, $n\ge 10$. Assume the following: p,q are bounded functions and

(i) $\int_{R^n}(1+|x|^2)(1+|x|^4)p^2(x)dx<\infty$, or $|x|^3p(x)\in L^\infty\cap L^1(R^n)$,

(ii) $\int_{R^n}(1+|x|^2)(1+|x|^4)q^2(x)dx<\infty$, or $|x|^3q(x)\in L^\infty\cap L^1(R^n)$,

(iii) $\alpha_1+\alpha_2>1,\quad \alpha_3+\alpha_4>1,\quad \beta_1,\beta_2,\beta_3,\beta_4$ arbitrary.

Then (7.7) has infinitely many positive solutions which are bounded and bounded away from zero.

If we replace (iii) by

(iii)′ $\alpha_1+\alpha_2<1,\quad \alpha_3+\alpha_4<1,$

then the same conclusion holds.

For the single equation $\Delta^2 u = p(x)u^\gamma$, $\gamma \neq 1$, Kusano and Swanson [52] obtained:

(i) For $n \geq 3$, if $\int_0^\infty t^{2\gamma+1}p(t) < \infty$, then there are infinitely many unbounded solutions;

(ii) For $n \geq 5$, $\int_0^\infty t^3 p < \infty$, there exist infinitely many positive solutions which are bounded above and below by positive constants;

(iii) $\int_0^\infty t^{n-1-\gamma(n-4)}p < \infty$, $n \geq 5$, there exist decay solutions behaving as $|x|^{4-n}$.

We note that (ii) is very close to our result, but we require a higher space dimension. Usami [77] also obtained the existence of infinitely many positive solutions behaving as $|x|^{2m-2}$ for $\Delta^m u = p \cdot u^\gamma$, $\gamma > 1$. In [44], this result was extended to the case $\gamma \neq 1$.

Example 2. Consider the following system

$$\begin{aligned} \Delta^2 u &= p(x)e^{\alpha_1 u + \alpha_2 v}, \\ \Delta^2 v &= q(x)e^{\alpha_3 u + \alpha_4 v}, \end{aligned} \tag{7.8}$$

in R^n, $n \geq 10$. If there is a positive constant a such that

(i) $\displaystyle\int_{R^n} (1+|x|^2)(1+|x|^4)p^2(x)dx < \frac{1}{2E}a \cdot e^{-(\alpha_1+\alpha_2)a}$,

(ii) $\displaystyle\int_{R^n} (1+|x|^2)(1+|x|^4)q^2(x)dx < \frac{1}{2E}a \cdot e^{-(\alpha_3+\alpha_4)a}$,

then (7.8) has a solution (u, v) such that $0 < u, v \leq a$. Note that if we write $p(x) = \delta p_1(x)$, $q(x) = \gamma q_1(x)$ and assume $|x|^3 p_1(x)$, $|x|^3 q_1(x) \in L^\infty \cap L^1(R^n)$, then we can always choose δ, γ small such that (i), (ii) hold for some $a > 0$.

We remark that this example was also considered in Kawano and Kusano [39], where existence criteria were given in the following form

$$\int_0^\infty tp(t)dt \leq \frac{n-2}{2}\, ae^{-(\alpha_1+\alpha_2)a},$$

$$\int_0^\infty tq(t)dt \leq \frac{n-2}{2}\, ae^{-(\alpha_3+\alpha_4)a}.$$

If we assume that

(i) $\alpha_1 + \alpha_2 < 0, \qquad \alpha_3 + \alpha_4 < 0,$

(ii) $\int_{R^n} (1+|x|^2)(1+|x|^4)p^2(x)dx < \infty, \quad \int_{R^n} (1+|x|^2)(1+|x|^4)q^2(x)dx < \infty,$

then (7.8) has infinitely many positive solutions which are bounded above and below by positive constants.

For the single equation $\Delta^m u = p(x)e^u$, Usami [77] obtained the existence of infinitely many solutions with the behavior of $|x|^{2m-2}$ if $\int_0^\infty tp \cdot e^{ct^{2m-2}} < \infty$ for some $c > 0$, and Kusano, Naito and Swanson [44] gave existence results for infinitely many positive as well as eventually negative solutions with $\Delta^i u \sim |x|^{2m-2i-2}$, under the condition $\int tp \cdot e^{-c^\gamma t^{2\gamma(m-1)}} dt < \infty$ for $\Delta^m u = p(x)e^{|u|^{\gamma-1}\cdot u}$.

As a conclusion, we remark that, by the same methods, we could also consider higher order systems. For example, for the equation

$$\ell\vec{u} = (-\Delta)^3\vec{u} = \vec{F}(x, \vec{u}, \nabla\vec{u})$$

in R^n, we can establish a weighted inequality similar to Lemma 7.2 of the form:

$$\|\nabla\varphi\|_{L^2_s} \leq c \cdot \|\Delta^2\varphi\|_{L^2_p},$$

where $p = (1+|x|^2)(1+|x|^4)^2$.

We can then adopt the proof of Lemma 7.3 to establish a similar inequality and finally prove a corresponding theorem. However, in doing so, we have to restrict ourselves to higher dimension, i.e., we require $n \geq 21$.

Chapter 8

CONCLUSION

This last chapter briefly reviews the thesis and presents some remarks and open problems which are related to the thesis and are of practical and theoretical interests.

As we mentioned before, the whole thesis is based on the successful acquisition and application of a priori estimates for general second order elliptic operators. The procedure of estimating the global a priori constants basically follows the classical approach employed by Ladyzhenskaya and Uraltseva [55] and Gilbarg and Trudinger [29], and is reduced to a reasonable length. The explicit value of the a priori constant E_1 has been estimated and plays an essential role in some applications, as we showed in Chapter 5. In establishing such a priori estimates, we make use of Hardy's inequality and weighted Sobolev spaces. These techniques enable us to overcome the difficulties due to the lack of compactness of embedding between Sobolev spaces and the failure of Poincare type inequalities for general unbounded domains. We leave one of the weight functions λ free, so that by choosing λ properly we can derive sharper results. We then apply the Schauder fixed point theorem to obtain the existence of global positive bounded solutions which are bounded away from zero for quasilinear equations of the form $-\Sigma D_i(a_{ij}D_ju) + 2\Sigma b_iD_iu = f(x, u, \nabla u)$. Under moderate conditions, we can describe the asymptotic behavior for the above solutions. Other authors obtain similar conclusions for equations with symmetric coefficients through standard variational and/or ordinary differential equation arguments. Our results extend those previously known and a number of examples in this thesis offer significant improvements over criteria given by other authors, see, e.g. Example 7 in Chapter 4.

In Chapter 5 we obtain the existence of decaying positive global solutions. This task is more complex as it involves sub-supersolution methods as well as spectral procedures. We first obtain a decaying supersolution v through a variable change technique. Then, by postulating some structure conditions on the coefficients and using a spectral procedure, we are able to construct a subsolution w such that $w \not\equiv 0$, $w \leq v$ globally. The existence of a decaying positive solution immediately follows. Applications of the above abstract existence theorem yield interesting consequences. In particular, Example 3 in Chapter 5 answers the open question of Kusano and Trench [53] for mixed sublinear-superlinear equations.

We further modify these basic ideas to deal with special problems. First we study degenerate equations, and we discuss the existence of global positive solutions bounded above and below by positive constants. In addition, the existence of decaying positive global solutions is obtained for a more restricted class of such equations. In contrast to most of the literature, the typical problem $-\Delta(|u|^{m-1}u) = f(x, u)$, which includes the porous media equation ($m > 1$, slow diffusion) and the plasma physics equation ($m < 1$, fast diffusion), is treated in a unified manner and the existence results obtained are global. As a second application we investigate higher order elliptic systems through a two step iteration process and obtain the existence of global positive solutions.

It is clear that since we do not use variational nor ordinary differential equation arguments, the methods developed in this thesis are applicable to cases which are not amenable to conventional methods. However, it is not realistic to expect that our methods provide comparable results to those obtained by variational or radial techniques, if such techniques are applicable. Moreover, the application is restricted to R^n with $n \geq 3$ since Hardy's inequality is not valid in R^2. We mention that, however, 2-dimensional problem is also treated by Kenig and Ni [KN2] and McOwen [M1, M2]. The restriction on the dimension becomes more stringent when higher order elliptic equations are considered. Furthermore, in some particular cases our methods do not yield as much qualitative information for solutions as other methods do. For example, for decaying solutions we prove that $0 < u(x) \leq C|x|^{-\alpha}$, but the exact asymptotic behavior of the solution is not known. In particular it is not clear whether or not $u(x) \sim |x|^{-\alpha}$. On the other hand, the maximum allowable decaying rate α is less than $n-2$, which is the rate closely associated with the radial arguments. We observe that since the choice of α depends on the behavior of z, a "better" choice of z could result in a larger (better) value of α. However, how to choose a "better" z is not obvious to us. We further note that, for a given z, the optimum value of E_1 is unknown.

Next we offer some considerations about the possible extension of the methods in this thesis. A direct application of our methods covers the p-Laplacian equations. More specifically, by using the weighted spaces-Schauder fixed point theorem approach, we can obtain at least the existence of global positive solutions bounded above and below by positive constants for equations of the form $-\nabla(|\nabla u|^{p-2}\nabla u) = f(x, u, \nabla u)$ in R^n with $1 < p < n$. We also observe that, in our discussion, R^n can be replaced by a domain Ω. In fact, the extension of the content of Chapter 4 in this direction is obvious. For the existence of decaying solutions, we need only modify f by continuation at the boundary of Ω such that f is continuous in Ω_ϵ $\{ x \in R^n | \ d(x, \Omega) < \epsilon \}$ and $f \equiv 0$ in $R^n \setminus \Omega_\epsilon$. We then construct a supersolution in R^n, which is obviously also a supersolution in Ω, while the subsolution is constructed locally. Then we can repeat exactly the arguments of Chapter 5 and derive a solution u in $W^{1,2}_{\mathrm{loc}}(\Omega)$, such that $u > 0$ in Ω, $u = 0$ on $\partial\Omega$. We note that in this procedure the estimates of E_1 may be different and $M(f)$ is likely to be replaced by other norms. A special example is the case where Ω is an exterior domain, which we can also study for $n = 2$.

To conclude this chapter and this thesis, we indicate some open questions which are closely related to our study. Due to the lack of Hardy's inequality in R^2, we are unable to invoke our methods for global problems in R^2. Nevertheless, it would be of interest to obtain global existence results in R^2 for equations with nonradial coefficients. Problems of the form $-\sum D_i(a_{ij}D_iu) + k(x)u = K(x)u^p$ with $k(x) \geq 0$, $p > 1$ were studied in $R^n (n > 2)$ by Kenig and Ni [KN1], it is natural to ask whether a similar existence result still holds for the following equation $-\sum D_i(a_{ij}D_iu) + \sum b_iD_iu + k(x)u = K(x)u^p$. Considerable attention has been given to the existence of radial global decaying solutions for higher order elliptic equations, and some interesting results have recently been given in a paper by Kusano, Naito and Swanson (Can. J. Math. 40 (1988), 1281–1300). However, the existence of nonradial decaying positive solutions remains unclear and the resolution for this problem is of importance. Finally we point out that the existence results for degenerate equations obtained in this thesis are restricted to some specific cases, and the method for dealing with more general equations is unknown.

APPENDIX A

Here we present the proofs for all the theorems and lemma in Chapter 3.

THEOREM 3.1. *Let $\vec{u} = (u_0, \ldots, u_m)^T$ be a solution to the system*

$$-\Sigma D_i(a_{ij}(x) D_j \vec{u}) + 2\Sigma B^j(x) D_j \vec{u} + C\vec{u} = -\Sigma D_i(\vec{f}_i) + \vec{g} \tag{A.1}$$

in a ball $B_2(x_0)$. Suppose $\vec{u} \in C^\alpha \cap W^{1,2}(B_2(x_0))$, the vector $\vec{g}$ and the $(m+1) \times (m+1)$ matrix C belong to $L^{q/2}(B_2(x_0))$, the $(m+1) \times (m+1)$ matrices B^j belong to $L^q(B_2(x_0))$, while the vectors $\vec{f}_i$ are in $L^q(B_2(x_0))$ for some $q > n$. Then:

$$\left\| |\vec{u}| \right\|_{L^\infty(B_1(x_0))} \leq K_0 \left[\left\| |\vec{u}| \right\|_{L^2(B_2(x_0))} + \Sigma \left\| |\vec{f}_i|^2 \right\|^{1/2}_{L^{q/2}(B_2(x_0))} + \left\| |\vec{g}| \right\|_{L^{q/2}(B_2(x_0))} \right], \tag{A.2}$$

where $K_0 = K_1(\mu(B_2)^{1/2} + 1)$, with $\mu(B_2)$ the Lebesgue measure of ball B_2 in R^n,

$$K_1 = \left(4H\left(2^{\frac{3}{2}} \frac{\lambda_1}{\lambda_0}\right)^{q/(q-n)}\right)^{\frac{n}{2}} \cdot \left(2\left(\frac{2}{n-2}\right)^{\frac{3}{2} \cdot q/(q-n)}\right)^{\frac{n(n-2)}{4}},$$

$$H = T^2(4 + C(\beta_1))$$
$$+ 2\left(T^2 C(\beta_1)\left\{ \left\| |C| \right\|_{L^{q/2}(B_2(x_0))} + \left\| \Sigma |B^i|^2 \right\|_{L^{q/2}(B_2(x_0))} + 2 \right\}\right)^{q/(q-n)},$$

$$C(\beta) = \frac{3}{2} + \frac{16}{\beta(\beta+2)}, \qquad \beta_1 = \frac{4}{n-2},$$

$$T = \frac{1}{n\sqrt{\pi}} \left(\frac{n!}{2\Gamma(1+\frac{n}{2})}\right)^{1/n} \left(\frac{n}{n-2}\right)^{1/2}$$ *is the optimum embedding constant from $W_0^{1,2}(\Omega)$ to $L^{\frac{2n}{n-2}}(\Omega)$.*

Proof. We adopt the proof of [29] with a test function motivated by arguments in [55]. We set: for $k \geq 0$, $\beta > 0$ and $\eta \in C_0^\infty(B_2(x_0))$ to be chosen below,

$$v = |\vec{u}| + k$$

and

$$\vec{\varphi} = \vec{u}v^{\beta}\eta^2.$$

Then $\vec{\varphi}$ is a proper test function by the chain rule of differentiation (cf. [29, p. 151]) and

$$D_i\vec{\varphi} = D_i\vec{u}v^{\beta}\eta^2 + \vec{u}\beta v^{\beta-1}(D_iv)\eta^2 + 2\vec{u}v^{\beta}\eta D_i\eta, \tag{A.3}$$

·here

$$D_iv = \begin{cases} \langle D_i\vec{u}, \vec{u}\rangle/|\vec{u}|, & |\vec{u}| > 0, \\ 0, & |\vec{u}| = 0. \end{cases}$$

Multiplying $\vec{\varphi}$ on both sides of (A.1) and integrating over $B_2(x_0)$ (for notational convenience we drop the integration domain and denote $B_2(x_0)$ by B_2 in the following) yield:

$$\int [\Sigma a_{ij}(x)\langle D_j\vec{u}, D_i\vec{\varphi}\rangle + 2\Sigma\langle B^j \cdot D_j\vec{u}, \vec{\varphi}\rangle + \langle C\vec{u}, \vec{\varphi}\rangle] = \int [\Sigma\langle \vec{f}_i, D_i\vec{\varphi}\rangle + \langle \vec{g}, \vec{\varphi}\rangle],$$

where $\langle\ ,\ \rangle$ denotes the R^n-inner product. Using (A.3) we have

$$\begin{aligned} \int [\Sigma a_{ij}(x)\langle D_j\vec{u}, D_i\vec{u}\rangle v^{\beta}\eta^2 &+ \Sigma a_{ij}\langle D_j\vec{u}, \vec{u}\rangle v^{\beta-1}\beta(D_iv)\eta^2 \\ &+ 2\Sigma a_{ij}\langle D_j\vec{u}, \vec{u}\rangle v^{\beta}\eta D_i\eta \\ &+ 2\Sigma\langle B^j \cdot D_j\vec{u}, \vec{u}\rangle v^{\beta}\eta^2 + \langle C\vec{u}, \vec{u}\rangle v^{\beta}\eta^2] \\ &= \int \langle \vec{g}, \vec{u}\rangle v^{\beta}\eta^2 + \int [\Sigma\langle \vec{f}_i, D_i\vec{u}\rangle v^{\beta}\eta^2 \\ &+ \Sigma\langle \vec{f}_i, \vec{u}\rangle v^{\beta-1}\beta(D_iv)\eta^2 + 2\Sigma\langle \vec{f}_i, \vec{u}\rangle v^{\beta}\eta D_i\eta]. \end{aligned} \tag{A.4}$$

Now, using the expression of $D_i v$, we obtain the following, by the ellipticity of the operator and Hölder's inequality:

$$\int \Sigma a_{ij}(x)\langle D_j\vec{u}, D_i\vec{u}\rangle v^\beta\eta^2 \geq \lambda_0 \int |\nabla\vec{u}|^2 v^\beta\eta^2,$$

$$\int \Sigma a_{ij}\langle D_j\vec{u}, \vec{u}\rangle v^{\beta-1}\beta(D_i v)\eta^2 = \int \Sigma a_{ij}|\vec{u}|v^{\beta-1}\beta D_j v(D_i v)\eta^2$$

$$\geq \lambda_0\beta \int |\vec{u}|v^{\beta-1}|\nabla v|^2\eta^2,$$

$$2\int \Sigma a_{ij}\langle D_j\vec{u}, \vec{u}\rangle v^\beta\eta D_i\eta = 2\int \Sigma a_{ij}|\vec{u}|v^\beta\eta D_j v D_i\eta,$$

$$2\int \Sigma\langle B^j D_j\vec{u}, \vec{u}\rangle v^\beta\eta^2 \leq 2\int \Sigma|B^j|\ |\vec{u}|\ |D_j v|v^\beta\eta^2,$$

$$\int \Sigma\langle \vec{f}_i, D_i\vec{u}\rangle v^\beta\eta^2 \leq \frac{1}{2}\int \left(\lambda_0|\nabla\vec{u}|^2 v^\beta\eta^2 + \frac{1}{\lambda_0}\ \Sigma|\vec{f}_i|^2 v^\beta\eta^2\right),$$

$$\int \Sigma\langle \vec{f}_i, \vec{u}\rangle v^{\beta-1}\beta(D_i v)\eta^2 \leq \frac{1}{2}\int \left(\lambda_0\beta|\vec{u}|^2|\nabla v|^2 v^{\beta-2}\eta^2 + \frac{1}{\lambda_0}\ \beta\Sigma|\vec{f}_i|^2 v^\beta\eta^2\right)$$

$$\leq \frac{1}{2}\int \lambda_0\beta|\nabla v|^2|\vec{u}|v^{\beta-1}\eta^2 + \frac{1}{2}\int \frac{\beta}{\lambda_0}\ \Sigma\ \frac{|\vec{f}_i|^2}{v} v^{\beta+1}\eta^2,$$

since $|v| \geq |\vec{u}|$.

Substituting the above back into (A.4) yields:

$$\int [\frac{\lambda_0}{2}\ |\nabla\vec{u}|^2 v^\beta\eta^2 + \frac{\lambda_0\beta}{2}\ |\vec{u}|v^{\beta-1}\eta^2|\nabla v|^2 + 2\Sigma a_{ij}|\vec{u}|v^\beta\eta D_j v D_i\eta$$

$$- 2\Sigma|B^j|\ |\vec{u}|\ |D_j v|v^\beta\eta^2 + \langle C\vec{u}, \vec{u}\rangle v^\beta\eta^2]$$

$$\leq \int [\frac{1}{2\lambda_0}\Sigma|\vec{f}_i|^2 v^\beta\eta^2 + \frac{\beta}{2\lambda_0}\ \frac{\Sigma|\vec{f}_i|^2}{v}\ v^{\beta+1}\eta^2 + 2\Sigma\langle \vec{f}_i, \vec{u}\rangle v^\beta\eta D_i\eta]$$

$$+ \int \langle \vec{g}, \vec{u}\rangle v^\beta\eta^2. \qquad \text{(A.5)}$$

Using Hölder's inequality again we have

$$\langle \vec{g}, \vec{u}\rangle v^\beta\eta^2 \leq |\vec{g}|\cdot|\vec{u}|v^\beta\eta^2 \leq \frac{|\vec{g}|}{v}\cdot v^{\beta+2}\eta^2,$$

$$|2\langle \vec{f}_i, \vec{u}\rangle v^\beta \eta D_i\eta| \le 2|\vec{f}_i|\cdot|\vec{u}|v^\beta\eta|\nabla\eta| \le 2\,\frac{|\vec{f}_i|}{v}\cdot v^{\beta+2}\eta|\nabla\eta|$$

$$\le \Big(\frac{|\vec{f}_i|^2}{v^2}\,\eta^2 + |\nabla\eta|^2\Big)v^{\beta+2},$$

$$|\langle C\vec{u},\vec{u}\rangle|v^\beta\eta^2 \le |C|\cdot|\vec{u}|^2v^\beta\eta^2 \le |C|v^{\beta+2}\eta^2,$$

$$\Big|2\Sigma a_{ij}|\vec{u}|v^\beta\eta D_iv D_j\eta\Big| \le |\vec{u}|\Big(|\nabla v|^2v^{\beta-1}\eta^2\varepsilon + \frac{\lambda_1^2}{\varepsilon}\,v^{\beta+1}|\nabla\eta|^2\Big),$$

$$2\Big|\Sigma|B^j|\ |\vec{u}|\ |D_jv|v^\beta\eta^2\Big| \le |\vec{u}|\Big(|\nabla v|^2\cdot v^{\beta-1}\eta^2\varepsilon + \frac{1}{\varepsilon}\eta^2v^{\beta+1}\Sigma|B^j|^2\Big).$$

From these estimates and (A.5) we conclude

$$\int[\frac{\lambda_0}{2}|\nabla\vec{u}|^2v^\beta\eta^2 + \frac{\lambda_0\beta}{2}\,|\vec{u}|v^{\beta-1}|\nabla v|^2\eta^2]$$

$$\le \int[\Sigma\,\frac{|\vec{f}_i|^2}{v^2}\,\eta^2v^{\beta+2}\Big(\frac{1+\beta}{2\lambda_0}\Big) + \Big(\frac{\Sigma|\vec{f}_i|^2}{v^2}\,\eta^2 + |\nabla\eta|^2\Big)v^{\beta+2} + \eta^2v^{\beta+2}\Big(\frac{|\vec{g}|}{v} + |C|\Big)$$

$$+ |\vec{u}|\ |\nabla v|^2v^{\beta-1}\eta^2\cdot 2\varepsilon + |\vec{u}|\cdot v^{\beta+1}(|\nabla\eta|^2 + \eta^2\cdot\Sigma|B^j|^2)\cdot\frac{\lambda_1^2}{\varepsilon}].$$

By choosing $\varepsilon = \frac{\beta\lambda_0}{8}$ we obtain:

$$\int[\frac{\lambda_0}{2}|\nabla\vec{u}|^2v^\beta\eta^2 + \frac{\lambda_0\beta}{4}\,|\vec{u}|\cdot|\nabla v|^2v^{\beta-1}\eta^2]$$

$$\le \int v^{\beta+2}\Big\{\eta^2\,\frac{\Sigma|\vec{f}_i|^2}{v^2}\Big(\frac{1+\beta}{2\lambda_0}+1\Big) + \eta^2\Big[\frac{|\vec{g}|}{v} + |C| + \Sigma|B^j|^2\cdot\frac{8}{\quad}\,\lambda_1^2\Big]$$

$$+ \Big(\frac{8}{\lambda_0\beta}\,\lambda_1^2 + 1\Big)|\nabla\eta|^2\Big\}. \quad \text{(A.6)}$$

Since $|\nabla\vec{u}|^2 \ge |\nabla v|^2$ we find

$$\int v^\beta|\nabla v|^2\eta^2 \le \int v^{\beta+2}\Big\{\eta^2\cdot\Big[\frac{|\vec{g}|}{v} + |C| + \Sigma|B^i|^2 + \frac{\Sigma|\vec{f}_i|^2}{v^2}\Big] + |\nabla\eta|^2\Big\}$$

$$\cdot\frac{2}{\lambda_0}\Big(\frac{1+\beta}{2\lambda_0} + 1 + \frac{8}{\lambda_0\beta}\,\lambda_1^2\Big). \quad \text{(A.7)}$$

Set

$$\bar{b} = \Sigma \frac{|\vec{f}_i|^2}{v^2} + \frac{|\vec{g}|}{v} + |C| + \Sigma |B^i|^2,$$

$$w = v^{(\beta+2)/2},$$

then $D_i w = \frac{\beta+2}{2} v^{\beta/2} \cdot D_i v$. Hence we have

$$\int |\nabla w|^2 \eta^2 \leq \frac{(\beta+2)^2}{4} \cdot \frac{2}{\lambda_0} \Big(\frac{1+\beta}{2\lambda_0} + \frac{8\lambda_1^2}{\lambda_0 \beta} + 1 \Big) \int w^2 (\eta^2 \bar{b} + |\nabla \eta|^2).$$

Now

$$\frac{1}{\beta+2} \Big(\frac{1+\beta}{2\lambda_0} + \frac{8\lambda_1^2}{\lambda_0 \beta} + 1 \Big) \leq \frac{\lambda_1^2}{\lambda_0} \Big(\frac{3+\beta}{2} + \frac{8}{\beta} \Big) \cdot \frac{1}{\beta+2},$$

and $\frac{3+\beta}{2+\beta} \leq \frac{3}{2}$. Thus we have

$$\int |\nabla w|^2 \eta^2 \leq \frac{(\beta+2)^3}{4} \cdot \frac{\lambda_1^2}{\lambda_0^2} C(\beta) \cdot \int w^2 (\eta^2 \bar{b} + |\nabla \eta|^2), \tag{A.8}$$

where $\frac{C(\beta)}{2} = \frac{3}{4} + \frac{8}{\beta(\beta+2)}$ and $C(\beta)$ is a decreasing function of β.

From Sobolev's embedding theorem (cf. [29, p. 155]) we have

$$\|\eta w\|_{2n/(n-2)}^2 \leq T^2 \int |\nabla(\eta w)|^2 \leq 2T^2 \int [\eta^2 |\nabla w|^2 + w^2 |\nabla \eta|^2].$$

Using (A.8) we obtain

$$\begin{aligned} \|\eta w\|_{2n/(n-2)}^2 &\leq 2T^2 \int [w^2 |\nabla \eta|^2 + \frac{(\beta+2)^3}{4} \cdot \frac{\lambda_1^2}{\lambda_0^2} \cdot C(\beta) w^2 (\eta^2 \bar{b} + |\nabla \eta|^2)] \\ &= 2T^2 \int \Big[w^2 |\nabla \eta|^2 \Big(1 + \frac{(\beta+2)^3}{4} \frac{\lambda_1^2}{\lambda_0^2} C(\beta) \Big) \\ &\quad + \frac{(\beta+2)^3}{4} \frac{\lambda_1^2}{\lambda_0^2} C(\beta) w^2 \eta^2 \bar{b} \Big], \end{aligned} \tag{A.9}$$

while for any $\varepsilon > 0$, by the interpolation inequality (7.10) of [29, p. 146],

$$\int w^2\eta^2\bar{b} \leq \|\bar{b}\|_{q/2}\|\eta w\|^2_{(2q)/(q-2)}$$
$$\leq \|\bar{b}\|_{q/2} \cdot (\varepsilon\|\eta w\|_{(2n)/(n-2)} + \varepsilon^{-\sigma}\|\eta w\|_2)^2,$$

where

$$\sigma = \Big(\frac{1}{2} - \frac{1}{\frac{2q}{q-2}}\Big)\Big/\Big(\frac{1}{\frac{2q}{q-2}} - \frac{1}{\frac{2n}{n-2}}\Big)$$
$$= \Big(\frac{1}{2} - \frac{q-2}{2q}\Big)\Big/\Big(\frac{q-2}{2q} - \frac{n-2}{2n}\Big) = \frac{2}{2q}\Big/\frac{2(q-n)}{2qn} = \frac{n}{q-n}.$$

By substituting the above back into (A.9) we conclude

$$\|\eta w\|^2_{2n/(n-2)} \leq \frac{T^2}{2}\Big(4 + (\beta+2)^3\ \frac{\lambda_1^2}{\lambda_0^2}\ C(\beta)\Big)\int w^2|\nabla\eta|^2$$
$$+ \frac{T^2}{2}(\beta+2)^3\ \frac{\lambda_1^2}{\lambda_0^2}C(\beta)\|\bar{b}\|_{q/2}\cdot 2\Big(\varepsilon^2\|\eta w\|^2_{2n/(n-2)} + \varepsilon^{-2\sigma}\|\eta w\|_2^2\Big).$$

Choosing ε^2 such that $T^2(\beta+2)^3\ \frac{\lambda_1^2}{\lambda_0^2}\ C(\beta)\varepsilon^2 = \frac{1}{2}$ leads to

$$\|\eta w\|^2_{2n/(n-2)} \leq T^2\Big[4 + (\beta+2)^3\ \frac{\lambda_1^2}{\lambda_0^2}\ C(\beta)\Big]\int w^2|\nabla\eta|^2$$
$$+ \Big(2T^2(\beta+2)^3\ \frac{\lambda_1^2}{\lambda_0^2}\ C(\beta)\|\bar{b}\|_{q/2}\Big)^{1+\sigma}\|\eta w\|_2^2.$$

Thus we have

$$\|\eta w\|^2_{2n/(n-2)} \leq \|w(\eta + |\nabla\eta|)\|_2^2\cdot$$
$$\cdot\Big[T^2(4 + C(\beta)) + (2T^2C(\beta)\|\bar{b}\|_{q/2})^{1+\sigma}\Big]\cdot(\beta+2)^{3(1+\sigma)}\cdot\Big(\frac{\lambda_1^2}{\lambda_0^2}\Big)^{1+\sigma}.$$

Setting $\gamma = \beta + 2$, and

$$H(\beta) = \{T^2(4 + C(\beta)) + (2T^2C(\beta)\|\bar{b}\|_{q/2})^{1+\sigma}\}^{1/2},$$

we obtain

$$\|\eta w\|_{2n/(n-2)} \leq H(\beta) \cdot \gamma^{3(1+\sigma)/2} \cdot \left(\frac{\lambda_1^2}{\lambda_0^2}\right)^{\frac{1+\sigma}{2}} \|w(\eta + |\nabla\eta|)\|_2. \tag{A.10}$$

We now choose the cut-off function η more specifically. Let r_1, r_2 satisfy $1 \leq r_1 < r_2 \leq 3$ and set $\eta = 1$ in B_{r_1}, $\eta = 0$ outside B_{r_2}, and $|\nabla\eta| \leq \frac{2}{r_2 - r_1}$, thus $\eta + |\nabla\eta| \leq \frac{4}{r_2 - r_1}$ as $\eta \leq 1 \leq \frac{2}{r_2 - r_1}$. Setting $x = \frac{n}{n-2}$, from (A.10) we obtain

$$\|w\|_{L^{2x}(B_{r_1})} \leq \frac{4H(\beta)}{r_2 - r_1}\, \gamma^{\frac{3}{2}(1+\sigma)} \left(\frac{\lambda_1^2}{\lambda_1^2}\right)^{\frac{1+\sigma}{2}} \cdot \|w\|_{L^2(B_{r_2})}. \tag{A.11}$$

Set $r_m = 1 + \frac{1}{2^m}$, then $r_m - r_{m+1} = \frac{1}{2^m} - \frac{1}{2^{m+1}} = \frac{1}{2^{m+1}}$, and we have

$$\|w\|_{L^{2x}(B_{r_m})} \leq \frac{4H(\beta)}{(\frac{1}{2})^m}\, \gamma^{\frac{3}{2}(1+\sigma)} \left(\frac{\lambda_1^2}{\lambda_0^2}\right)^{\frac{1}{2}(1+\sigma)} \cdot \|w\|_{L^2(B_{r_{m-1}})}. \tag{A.12}$$

Recalling that $w = v^{(\beta+2)/2} = v^{\gamma/2}$, then we have

$$\left(\int_{B_{r_m}} v^{\gamma x}\right)^{\frac{1}{x\gamma}} \leq \left[\frac{4H(\beta)}{(\frac{1}{2})^m}\, \gamma^{\frac{3}{2}(\sigma+1)} \left(\frac{\lambda_1^2}{\lambda_0^2}\right)^{\frac{1}{2}(1+\sigma)}\right]^{2/\gamma} \left(\int_{B_{r_{m-1}}} v^{\gamma}\right)^{1/\gamma},$$

i.e.,

$$\|v\|_{L^{\gamma x}(B_{r_m})} \leq \left[4H(\beta)2^m \gamma^{\frac{3}{2}(\sigma+1)} \left(\frac{\lambda_1^2}{\lambda_0^2}\right)^{\frac{1}{2}(1+\sigma)}\right]^{2/\gamma} \cdot \|v\|_{L^{\gamma}(B_{r_{m-1}})}. \tag{A.13}$$

By setting $\gamma_m = x^m 2 = \beta_m + 2$, then $\beta_1 = x \cdot 2 - 2 = \frac{4}{n-2}$. We use (A.13) as iteration formula to get:

$$\begin{aligned}
\|v\|_{L^{\gamma_{m+1}}(B_{r_{m+1}})} &\leq \left[4H(\beta_m)2^m (x^m 2)^{\frac{3}{2}(1+\sigma)} \left(\frac{\lambda_1}{\lambda_0}\right)^{1+\sigma}\right]^{2/x^m 2} \cdot \|v\|_{L^{\gamma_m}(B_{r_m})} \\
&\leq \prod_{\ell=1}^{m} \left(4H(\beta_1)2^{\frac{3}{2}(1+\sigma)} \cdot \left(\frac{\lambda_1}{\lambda_0}\right)^{1+\sigma}\right)^{1/x^\ell} \cdot \prod_{\ell=1}^{m} \left(x^{\frac{3}{2}(1+\sigma)} 2\right)^{\ell/x^\ell} \cdot \|v\|_{L^{\gamma_1}(B_{r_1})} \\
&= \left(4H(\beta_1)2^{\frac{3}{2}(1+\sigma)} \left(\frac{\lambda_1}{\lambda_0}\right)^{1+\sigma}\right)^{\sum_{\ell=1}^{m} \frac{1}{x^\ell}} \left(x^{\frac{3}{2}(1+\sigma)} 2\right)^{\sum_{\ell=1}^{m} \frac{\ell}{x^\ell}} \cdot \|v\|_{L^{\gamma}(B_{r_1})},
\end{aligned} \tag{A.14}$$

since $H(\beta)$ is a decreasing function of β. By letting $\beta = 0$ in the expression for $\vec{\varphi}$ and repeating the calculation, we obtain

$$\|v\|_{L^\gamma(B_{r_1})} \leq \frac{4\widetilde{H}}{\frac{1}{2}} 2^{\frac{3}{2}(1+\sigma)}\left(\frac{\lambda_1}{\lambda_0}\right)^{1+\sigma}\|v\|_{L^2(B_2)}, \qquad \text{(A.12')}$$

where $\widetilde{H} = \left\{T^2\left(4+\frac{3}{2}\right)+\left(2T^2\ \frac{3}{2}\ \|\bar{b}\|_{q/2}\right)^{1+\sigma}\right\}^{1/2} \leq H(\beta_1)$. Combining (A.12′) and (A.14) and letting $m \to \infty$ we conclude

$$\sup_{B_1} v \leq \left[4H(\beta_1)2^{\frac{3}{2}(1+\sigma)}\left(\frac{\lambda_1}{\lambda_0}\right)^{1+\sigma}\right]^{\sum_{m=0}^{\infty}\frac{1}{x^m}} \cdot \left(2x^{\frac{3}{2}(1+\sigma)}\right)^{\sum_{m=0}^{\infty}\frac{m}{x^m}} \cdot \|v\|_{L^2(B_2)}. \qquad \text{(A.15)}$$

Now, since $x = \frac{n}{n-2} > 1$, we have

$$\sum_{m=0}^{\infty}\frac{1}{x^m} = \frac{1}{1-x^{-1}} = \frac{n}{2},$$

$$\sum_{m=0}^{\infty}\frac{m}{x^m} = \frac{x}{(x-1)^2} = \frac{n(n-2)}{4},$$

and recalling that $\sigma = \frac{n}{q-n}$, we have

$$\sup_{B_1} v \leq \left[4H(\beta_1)\left(2^{\frac{3}{2}}\frac{\lambda_1}{\lambda_0}\right)^{q/(q-n)}\right]^{n/2}\left(2\cdot\left(\frac{n}{n-2}\right)^{\frac{3}{2}q/(q-n)}\right)^{\frac{n(n-2)}{4}}\|v\|_{L^2(B_2)}, \qquad \text{(A.16)}$$

i.e.,

$$\sup_{B_1} v \leq K_1 \cdot \|v\|_{L^2(B_2)}. \qquad \text{(A.16')}$$

But $\bar{b} = \frac{\Sigma|\vec{f}_i|^2}{v^2}+\frac{|\vec{g}|}{v}+|C|+\Sigma|B^i|^2$, and $v = |\vec{u}|+k$. We choose $k = \left(\Sigma\||\vec{f}_i|^2\|^{1/2}_{L^{q/2}(B_2)}+\||\vec{g}|\|_{L^{q/2}(B_2)}\right)$, then

$$\|\bar{b}\|_{L^{q/2}(B_2)} \leq \left\||C|+\Sigma|B^i|^2\right\|_{L^{q/2}(B_2)}+2,$$

and from (A.16′), we obtain

$$\sup_{B_1}(|\vec{u}|+k) \leq K_1(\|\vec{u}\|_{L^2(B_2)}+k\cdot\mu(B_2)^{1/2}).$$

Thus

$$\sup_{B_1} |\vec{u}| \le K_1(\mu(B_2)^{1/2}+1)(\|\vec{u}\|_{L^2(B_2)}+k)$$

$$= K_1(\mu(B_2)^{1/2}+1)\Big(\|\vec{u}\|_{L^2(B_2)}+\Sigma\big\||\vec{f}_i|^2\big\|^{1/2}_{L^{q/2}(B_2)}+\big\||\vec{g}|\big\|_{L^{q/2}(B_2)}\Big).$$

This concludes the proof of Theorem 3.1.

We remark that for $\ell u = -\Delta u + \Sigma b_i D_i u + cu$, $\frac{\lambda_1}{\lambda_0} = 1$, the proof is given in Allegretto and Huang [4].

Let $u \in W_0^{1,2}(|x| < t_m)$ be a solution of

$$\ell_0 u = -\Sigma D_i(a_{ij}(x)D_j u) + 2\Sigma b_i(x)D_i u + cu = g \tag{A.17}$$

where $g \in L^q(|x| < t_m)$, t_m some positive number. Denote $\vec{u} = (u, \nabla u)^T$, then $\vec{u}$ satisfies the following equation

$$-\Sigma D_i(a_{ij}(x)D_j\vec{u}) + 2\Sigma B^k \cdot D_k\vec{u} + C\vec{u} = \vec{g} + \Sigma D_i(\vec{f}_i) \tag{A.18}$$

where $\vec{g} = (g, 0, \ldots, 0)^T$, $\vec{f}_i = g\vec{e}_i$,

$\vec{e}_i$: $(n+1)$ vector with 1 in the i-th component and zero in the remaining components,

C: $(n+1)\times(n+1)$ matrix with entry:

$$c_{ij} = 2D_i(b_j) - \sum_k D_k(D_i a_{kj}), \qquad i\cdot j > 0,$$

$$c_{00} = c,\ c_{oi} = c,\ c_{i0} = D_i c, \quad \text{for } i \ne 0,$$

B^k: $(n+1)\times(n+1)$ matrix with entry:

$$b^k_{ij} = -\frac{1}{2}D_i(a_{kj}) + b_j, \qquad i\cdot j > 0,$$

$$b^k_{00} = b_k, \qquad b^k_{ij} = 0 \quad \text{for the other entries.}$$

THEOREM 3.2. *Let $u \in W_0^{1,2}(|x| < t_m)$ be a solution to (A.17) and $g \in L^q(|x| < t_m)$ for some $q > n$. Let a_{ij}, B^k, C, $\vec{g}$, and $\vec{f_i}$ be as above and let the conditions of Theorem 3.1 hold. Assume that $C \in L^{q/2}_{\text{loc}}(R^n)$, $B^k \in L^q_{\text{loc}}(R^n)$ and $\sup_{x\in R^n} \||C|\|_{L^{q/2}(B_2(x))} < \infty$, $\sup_{x \in R^n} \|\Sigma|B^k|^2\|_{L^{q/2}(B_2(x))} < \infty$. Assume further that there exists a $\delta > 0$ such that $(\ell_0\varphi, \varphi) \geq \delta(-\Delta\varphi, \varphi)$ for any $\varphi \in C_0^\infty(R^n)$. Then*

$$\max\left\{\sup_{|x|<t_m} |u|, \sup_{|x|<t_m-2} |\nabla u|\right\} \leq E_1\left(\|g\|_{L^2_t(R^n)} + \max_{|x|\leq t_m} \|g\|_{L^q(B_2(x))}\right), \quad \text{(A.19)}$$

where $E_1 = K_0 \max\left(\frac{2}{\delta(n-2)}(1 + T\mu(B_2)^{1/n}),\ n + \mu(B_2)^{1/q}\right)$, and K_0 is given in Theorem 3.1, with $\||C|\|_{L^{q/2}(B_2(x_0))}$ and $\|\Sigma|B^k|^2\|_{L^{q/2}(B_2(x_0))}$ replaced by $\sup_{x\in R^n} \||C|\|_{L^{q/2}(B_2(x))}$ and $\sup_{x\in R^n} \|\Sigma|B^k|^2\|_{L^{q/2}(B_2(x))}$ respectively.

Proof. As we pointed out, $\vec{u} = (u, \nabla u)^T$ satisfies (A.18) and Theorem 3.1 implies that, for $B_2(x_0) \subset (|x| < t_m)$,

$$\begin{aligned} |\vec{u}(x_0)| &\leq K_0\left[\||\vec{u}|\|_{L^2(B_2(x_0))} + \Sigma\||\vec{f_i}|^2\|^{1/2}_{L^{q/2}(B_2)} + \||\vec{g}|\|_{L^{q/2}(B_2)}\right] \\ &\leq K_0\left[\|\vec{u}\|_{L^2(B_2)} + (n + \mu(B_2)^{1/q})\|g\|_{L^q(B_2)}\right]. \end{aligned} \quad \text{(A.20)}$$

Now we investigate $\|\vec{u}\|_{L^2(B_2)}$. We observe that

$$\begin{aligned} \|u\|_{L^2(B_2)} &= \left(\int_{B_2} |u|^2\right)^{1/2} \\ &\leq \mu(B_2)^{1/n} \cdot \|u\|_{L^{(2n)/(n-2)}(B_2)} \\ &\leq \mu(B_2)^{1/n} \cdot \|u\|_{L^{(2n)/(n-2)}(|x|<t_m)} \\ &\leq \mu(B_2)^{1/n} \cdot T \cdot \||\nabla u|\|_{L^2(|x|<t_m)}, \end{aligned} \quad \text{(A.21)}$$

by applying Hölder's inequality and the Sobolev embedding theorem, where T is the optimum embedding constant we used in Theorem 3.1. Since $\ell_0 u = -\Sigma D_i(a_{ij}D_j u) + 2\Sigma b_i D_i u + cu = g$ and $(\ell_0\varphi,\varphi) \ge \delta(-\Delta\varphi,\varphi)$ for all $\varphi \in C_0^\infty(R^n)$, we have

$$\begin{aligned}\delta\|\nabla u\|^2_{L^2(|x|<t_m)} &= \delta(\nabla u,\nabla u) \le (\ell_0 u, u)\\ &= (g,u) = \int g\cdot u\\ &= \int g(1+|x|^2)^{1/2}u(1+|x|^2)^{-1/2}\\ &\le \Big(\int g^2(1+|x|^2)\Big)^{1/2}\cdot\Big(\int u^2(1+|x|^2)^{-1}\Big)^{1/2}.\end{aligned}$$

Note that $1+|x|^2$ is exactly the weight function t we introduced in Chapter 2. We have

$$\delta\|\nabla u\|^2_{L^2(|x|<t_m)} \le \|g\|_{L^2_t(|x|<t_m)}\cdot\|u\|_{L^2_{t^{-1}}}(|x|<t_m).$$

Before we proceed, we prove the following technical lemma which will also be used in Sections 4.3 and 5.3.

LEMMA 3.3. *Let $\varphi \in C_0^\infty(R^n)$, $x_0 \in R^n$. Then there exists a constant $c > 0$, independent of x_0 such that*

$$(-\Delta\varphi,\varphi) \ge c\int_{R^n}\frac{\varphi^2}{|x|\cdot|x-x_0|}\,dx, \tag{A.22}$$

and for $x_0 = 0$, we have $c = \left(\frac{n-2}{2}\right)^2$.

Proof. Let $\Omega = R^n - B_{\varepsilon^*}(x_0) - B_{\varepsilon^*}(0)$ for $\varepsilon^* > 0$ small. Note that for $v > 0$,

$$\begin{aligned}0 &\le \int_\Omega v^2\cdot\Sigma\Big(D_i\Big(\frac{\varphi}{v}\Big)\Big)^2 = \int_\Omega\frac{1}{v^2}\Sigma(vD_i\varphi - \varphi D_i v)^2\\ &= \int_\Omega \Sigma(D_i\varphi)^2 + \Sigma\Big(\frac{\varphi^2(D_iv)^2}{v^2} - \frac{1}{v}\,D_ivD_i(\varphi^2)\Big)\\ &= \int_\Omega\Sigma(D_i\varphi)^2 - \int_\Omega\Sigma D_i\Big(\frac{\varphi^2}{v}\Big)D_iv.\end{aligned}$$

Hence we obtain

$$\int_\Omega \Sigma(D_i\varphi)^2 \geq \int_\Omega \Sigma D_i\Big(\frac{\varphi^2}{v}\Big)D_i v$$
$$= \int_\Omega \frac{\varphi^2}{v}(-\Delta v) + \int_{\partial\Omega} \frac{\varphi^2}{v}\,\frac{\partial v}{\partial n}\,ds. \qquad \textbf{(A.23)}$$

Now choose $v = |x|^\alpha|x-x_0|^\beta$ with $\alpha = -\frac{(n-2)}{2}$, $\beta = -\varepsilon$, $\varepsilon > 0$ to be determined later, then

$$\frac{\partial v}{\partial x_i} = \alpha|x|^{\alpha-2}x_i|x-x_0|^\beta + \beta|x-x_0|^{\beta-2}(x_i - x_{0i})|x|^\alpha,$$

$$\frac{\partial^2 v}{\partial x_i^2} = \alpha(\alpha-2)|x|^{\alpha-4}x_i^2|x-x_0|^\beta + \alpha|x|^{\alpha-2}|x-x_0|^\beta$$
$$+ 2\alpha\beta|x|^{\alpha-2}|x-x_0|^{\beta-2}x_i(x_i-x_{0i})$$
$$+ \beta(\beta-2)|x-x_0|^{\beta-4}(x_i-x_{0i})^2|x|^\alpha + \beta|x-x_0|^{\beta-2}|x|^\alpha.$$

Hence

$$\frac{-\Delta v}{v} = -\alpha(\alpha+(n-2))\frac{1}{|x|^2} - \beta(\beta+(n-2))\frac{1}{|x-x_0|^2} -$$
$$2\,\frac{\alpha\beta}{|x||x-x_0|}\,\Sigma\,\frac{x_i(x_i-x_{0i})}{|x||x-x_0|}$$
$$= \Big(\frac{n-2}{2}\Big)^2\frac{1}{|x|^2} + \frac{\varepsilon(n-2-\varepsilon)}{|x-x_0|^2} - \frac{(n-2)\varepsilon}{|x||x-x_0|}\,\Sigma\,\frac{x_i}{|x|}\cdot\frac{x_i-x_{0i}}{|x_0-x_0|}.$$

Since

$$\frac{(n-2)^2}{4}\,\frac{1}{|x|^2} + \frac{\varepsilon(n-2-\varepsilon)}{|x-x_0|^2} \geq 2\,\frac{n-2}{2}\,\frac{1}{|x|}\cdot\frac{\sqrt{\varepsilon(n-2-\varepsilon)}}{|x-x_0|},$$

we conclude in Ω:

$$\frac{-\Delta v}{v} \geq (n-2)|x|^{-1}|x-x_0|^{-1}[\sqrt{\varepsilon(n-2-\varepsilon)} - \varepsilon].$$

Choosing ε such that $0 < \varepsilon < \frac{n-2}{2}$, thus $\sqrt{\varepsilon(n-2-\varepsilon)} - \varepsilon > 0$, we conclude, for some $c > 0$,

$$-\frac{\Delta v}{v} \geq c \cdot \frac{1}{|x||x-x_0|}. \tag{A.24}$$

Observe also that

$$\left| \int_{\partial\Omega} \frac{\varphi^2}{v} \frac{\partial v}{\partial n}\, ds \right| \leq K_1 \cdot \frac{\|\varphi\|_\infty^2}{\varepsilon^*} (\varepsilon^*)^{n-1} \to 0 \quad \text{as} \quad \varepsilon^* \to 0.$$

Thus letting $\varepsilon^* \to 0$ and substituting (A.24) back into (A.23) gives rise to (A.22).

By setting $x_0 = 0$ in the above calculation, we obtain the following inequality

$$\int_{R^n} \frac{\varphi^2}{|x|^2}\, dx \leq \left(\frac{2}{n-2}\right)^2 (-\Delta\varphi, \varphi), \tag{A.25}$$

and hence

$$\|\varphi\|_{L^2_{t-1}(R^n)} \leq \frac{2}{n-2} \|\nabla\varphi\|_{L^2(R^n)}. \tag{A.26}$$

From (A.26) we conclude that

$$\begin{aligned}\delta\|\nabla u\|^2_{L^2(|x|<t_m)} &\leq \|g\|_{L^2_t(R^n)} \cdot \|u\|_{L^2_{t-1}(|x|<t_m)} \\ &\leq \|g\|_{L^2_t(R^n)} \cdot \frac{2}{n-2} \|\nabla u\|_{L^2(|x|<t_m)}.\end{aligned}$$

Thus we obtain

$$\|\nabla u\|_{L^2(|x|<t_m)} \leq \frac{2}{\delta(n-2)} \|g\|_{L^2_t(R^n)}. \tag{A.27}$$

Combining (A.20), (A.21) and (A.27) gives

$$|\vec{u}(x_0)| \leq K_0 \left(\frac{2}{\delta(n-2)} (T\mu(B_2)^{\frac{1}{n}} + 1) \|g\|_{L^2_t(R^n)} + (n + \mu(B_2)^{\frac{1}{q}}) \|g\|_{L^q(B_2)} \right). \tag{A.28}$$

This is the conclusion of Theorem 3.2 and hence we complete the proof.

We remark that the original proof of Lemma 3.3 and the idea of Theorem 3.2 are from Allegretto [3] and were used in Allegretto and Huang [4].

We conclude the appendix with the proof of Theorem 3.4.

THEOREM 3.4. *Let* $u \in W_0^{1,2}(|x| < t_m)$, ℓ_0, g, q *as given in Theorem 3.2. Assume* g *has a decomposition* $g = g_1 + g_2$ *with* $g_1 \in L^q \cap L_t^2$, $g_2 \in L^q \cap L_\lambda^2$, *for some* $0 < \lambda^{-1} \in L^{n/2}(R^n)$. *Then*

$$\sup_{|x|<t_m} |u| \le E_1 M(g), \tag{A.29}$$

$$\sup_{|x|<t_m-2} |\nabla u| \le E_1 M(g), \tag{A.30}$$

where $M(g) \le \|g_1\|_{L_t^2(R^n)} + \sup_{x\in R^n} \|g_1\|_{L^q(B_2(x))} + e\|g_2\|_{L_\lambda^2(R^n)} + \sup_{x\in R^n} \|g_2\|_{L^q(B_2(x))}$ *and* $e = \frac{n-2}{2} T^{1/2} \|\lambda^{-1}\|_{L^{n/2}(R^n)}^{1/2}$, *as given in Chapter 2,* E_1 *as in Theorem 3.2.*

Proof. Let $\ell_0(u_i) = g_i$, $i = 1,2$, then obviously $|u_1(x)| \le E_1 M(g_1)$ by Theorem 3.2. Now for $\ell_0(u_2) = g_2$, we note that (A.20) still holds. In order to obtain an inequality parallel to (A.27), we observe that, for $g_2 \in L_\lambda^2(R^n)$, $0 < \lambda^{-1} \in L^{n/2}(R^n)$, Sobolev's embedding theorem yields:

$$\begin{aligned}\|u_2\|^2_{L^2_{\lambda^{-1}}(R^n)} = \int \frac{u_2^2}{\lambda} &\le \Big(\int \lambda^{n/2}\Big)^{2/n} \Big(\int u_2^{(2n)/(n-2)}\Big)^{(n-2)/(2n)} \\ &\le \|\lambda^{-1}\|_{L^{n/2}(R^n)} \cdot T \cdot \|\nabla u_2\|^2_{L^2(R^n)},\end{aligned}$$

and

$$\begin{aligned}\|\nabla u_2\|^2_{L^2(R^n)} = (\nabla u_2, \nabla u_2) &\le \frac{1}{\delta}(\ell_0 u_2, u_2) \\ &= \frac{1}{\delta}(g_2, u_2) = \frac{1}{\delta}\int \frac{u_2}{\lambda^{1/2}} \cdot g_2 \cdot \lambda^{1/2} \\ &\le \frac{1}{\delta} \cdot \|u_2\|_{L^2_{\lambda^{-1}}(R^n)} \cdot \|g_2\|_{L^2_\lambda(R^n)}.\end{aligned}$$

Then we have

$$\|u_2\|_{L^2_{\lambda^{-1}}(R^n)} \le \frac{T}{\delta} \cdot \|\lambda^{-1}\|_{L^{n/2}(R^n)} \|g_2\|_{L^2_\lambda(R^n)},$$

whence

$$\|\nabla u_2\|^2_{L^2(R^n)} \leq \frac{T}{\delta^2}\|\lambda^{-1}\|_{L^{n/2}(R^n)} \cdot \|g_2\|^2_{L^2_\lambda(R^n)}. \tag{A.27'}$$

Substituting (A.27′) into (A.20) gives

$$|u_2| \leq \frac{K_0}{\delta}(\mu(B_2)^{1/n}T + 1)T^{1/2}\|\lambda^{-1}\|^{1/2}_{L^{n/2}(R^n)}\|g_2\|_{L^2_\lambda(R^n)}$$

$$+ K_0(n + \mu(B_2)^{1/q})\|g_2\|_{L^q(B_2)}. \tag{A.31}$$

Combining (A.28) and (A.31) and noting that the constant E_1 is independent of the decomposition $g = g_1 + g_2$, we complete the proof.

As mentioned above, we found it convenient to estimate E_1 by means of a computer programme. We obtained, in this way, for $a_{ij} = \delta_{ij}$ and $b_i = 0$, if $\alpha = 0$, then $1/E_1 \approx 1.625 \times 10^{-5}$, while if $\alpha = 0.5, 1/E_1 \approx 2.069 \times 10^{-7}$.

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