University of Alberta

THE MODULAR INVARIANTS OF $A_{2,p'} \oplus A_{2,p}$ WHEN $\mathrm{GCD}(p', p) = 1$

by

Elaine Myra Beltaos ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Master of Science

in

Mathematics

Department of Mathematical and Statistical Sciences
Edmonton, Alberta
Spring, 2004

Your file Votre référence
ISBN: 0-612-96450-7
Our file Notre référence
ISBN: 0-612-96450-7

ABSTRACT

In this thesis, we classify the modular invariants of the affine algebra $(A_{2,p'} \oplus A_{2,p})^{(1)}$ where p' and p are coprime. The importance to conformal field theory of classifying modular invariants for affine algebras goes back to Witten. The first modular invariant classification for an affine algebra was done by Cappelli-Itzykson-Zuber in 1986 for $A_1^{(1)}$ in [4]. An almost identical problem to the $(A_2 \oplus A_2)^{(1)}$ classification, and the motivation for the work done in this thesis, is the classification of the (nonunitary) minimal W_3 models. To date, only one nonunitary conformal field theory classification exists; namely, for W_2 (the Virasoro minimal models). We include a review of Gannon's $A_1^{(1)}$ classification [11] as a demonstration of our approach.

Key Words: modular invariants, affine algebras, conformal field theory

Dedicated to my mother, Lillian Beltaos.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Dr. Terry Gannon, for his advice and guidance during the period of this work, and for his helpful comments and suggestions regarding this thesis.

I would also like to thank my brother, Andrew Beltaos for LaTeX help and lessons.

The style file mathesis.sty, created by Hongtao Yang, was used in the preparation of this thesis.

I would also like to acknowledge the financial support from the Department of Mathematical and Statistical Sciences at the University of Alberta.

Table of Contents

List of Tables

Chapter 1

Introduction

The impact of rational conformal field theory on mathematics has been profound (e.g. knot invariants, the definition of vertex operator algebras and quantum groups). Conformal field theory arises naturally in physics, most notably in the area of string theory, which attempts to describe all forces of nature within a single theory and thus resolve the conflict between general relativity and quantum mechanics [17]. The relationship between conformal field theory and string theory is described for instance in [20]. Another relationship between conformal field theory and physics involves statistical systems at criticality.

Roughly speaking, a conformal field theory is a quantum field theory in 2-dimensional space-time, whose symmetries include the conformal transformations. The rational theories obey in addition a certain finiteness condition. In the case of string theory, this space-time is the surface ("world sheet") traced out as the strings collide and separate through time.

A special class of rational conformal field theories, namely the Wess-Zumino Witten models [16], has symmetries given by affine Kac-Moody algebras. Their physical importance lies primarily in the fact that large classes of other models can be constructed from them by the Goddard-Kent-Olive coset construction. For example, the so-called W_N models are constructed using $A_{N-1}^{(1)}$ models.

Classifying rational conformal field theories is essentially the same as classifying their modular invariant partition functions. In particular, much work has been done on the classification of the modular invariant partition functions of affine algebras. The classifications of the W_N minimal models are very similar to the modular invariant classifications of the affine $A_{N-1} \oplus A_{N-1}$ algebras.

The ultimate purpose of the research done in this thesis is to obtain the classification of the nonunitary W_3 minimal models. The unitary ones were classified in [14]. A simpler but almost identical problem is the classification of the modular invariants of $(A_2 \oplus A_2)^{(1)}$, which is the problem solved in this thesis. The difference between the two classifications will be discussed more in the Concluding Remarks (§7.1). The author plans to complete and publish the W_3 classification in the near future. One reason the nonunitary classification is interesting, is that there is only one classification of nonunitary models that has ever been done, namely the "minimal Virasoro models" = "W_2 minimal models". It is known that typically [12], nonunitary classifications will look very different than unitary ones, so this W_3 minimal nonunitary classification should generate interest for that reason alone.

The history of the problem of classifying modular invariant partition functions of an affine algebra began when A. Cappelli, C. Itzykson, and J.B. Zuber achieved the first modular invariant classification, for the affine algebra $A_1^{(1)}$ [4]. This problem is simple to state: for any affine algebra $X_r^{(1)}$ of rank r, we can write its modular invariant partition function as

(1.1) $$\mathcal{Z} = \sum_{\lambda,\mu \in P^n_{++}(X_r)} M_{\lambda\mu} \chi_\lambda \chi_\mu^*,$$

where $P^n_{++}(X_r)$ is the set of highest weights of $X_r^{(1)}$ of height n; the χ's are the characters associated to the corresponding representations of $X_r^{(1)}$, and $*$ denotes the complex conjugate. Equation (1.1) defines the one-to-one correspondence between a modular invariant partition function $\mathcal{Z}$ and its coefficient matrix M, and we do not distinguish between them. M is called a *modular invariant* if the following three conditions hold:

(1.2a) $$M_{\rho\rho} = 1 \quad (\textit{uniqueness of vacuum}),$$

(1.2b) $$M_{\lambda\mu} \in \mathbb{Z}_{\geq 0}\ \forall \lambda, \mu \in P^n_{++}(X_r) \quad (\textit{integrality and positivity}),$$

(1.2c) $$SM = MS,\ TM = MT \quad (\textit{modular invariance}),$$

where ρ is the vacuum, and S and T are the $X_r^{(1)}$ modular data.

Remark: In many articles, M is called a modular invariant if only (1.2c) holds, and if all of equations (1.2) hold, then M is called a *physical invariant* or a *positive physical invariant*. However, in this thesis, we refer to any M satisfying all of equations (1.2) as a modular invariant.

Cappelli-Itzykson-Zuber's result [4] led to the problem of trying to find a complete classification for all affine algebras $X_r^{(1)}$. Much work has been done since that first classification[1]. For example, Gannon followed up with the $A_2^{(1)}$ classification for any level, and the classification for $A_r^{(1)}$ at level 2 and 3 and any rank r, and the work of Degiovanni and Gannon yielded the classification for all simple affine algebras at level 1. Gannon also worked on the first semi-simple classification, for $(A_1 \oplus \cdots \oplus A_1)^{(1)}$; he found a solution for any level $k = (k_1, \ldots, k_s)$, such that $\gcd(k_i, k_j) \leq 3$ whenever $i \neq j$, and of $(A_1 \oplus A_1)^{(1)}$ at any level $k = (k_1, k_2)$. A main feature of the $(A_1 \oplus \cdots \oplus A_1)^{(1)}$ classification was that its methods could be (and were intended to be) generalized to other affine algebras, something which was not found to be true of [4]. In [11], Gannon found a new proof of Cappelli-Itzykson-Zuber's result, applying the "generalizable" method to the affine $A_1^{(1)}$ algebra. We include a review of this proof in Chapter 2. The method can be outlined as follows: for a given affine algebra $X_r^{(1)}$, we first find the automorphism invariants (these correspond to those M whose vacuum column is 0 except at $M_{\rho\rho} = 1$); next, find the simple-current extensions, which are built up in a natural way from symmetries of the extended $X_r^{(1)}$ Dynkin diagram; and finally, to find all exceptional invariants - those modular invariants that are not of the first two types. The completed classifications seemed to suggest that the exceptional invariants occur only for "small" levels, and in fact, the following was found by Gannon and Ocneanu [13, 19]:

Theorem 1.1. *All possible modular invariants appearing in RCFT (or the subfactor interpretation), corresponding to any fixed choice of simple affine algebra $X_r^{(1)}$, and all sufficiently high levels, are known.*

The "known" modular invariants referred to in Theorem 1.1 are of the "extended Dynkin diagram"-type, plus some exceptionals that have already been found. By "simple affine algebra", we mean the affinization of a simple Lie algebra.

[1] For more on the classification of modular invariant partition functions, and references, see for example [10].

The result of this thesis is the classification of the modular invariants of the semi-simple affine algebra $(A_2 \oplus A_2)^{(1)}$, given in Theorem 2.1. Our proof follows as closely as possible the work done in [9] and [14]. In Chapter 2, we set up our problem specifically for $(A_2 \oplus A_2)^{(1)}$. We also include, as an illustration, a section on Gannon's $A_1^{(1)}$ classification, which is the most concise modular invariant classification, due to the fact that $A_1^{(1)}$ is the least complicated affine algebra. In Chapter 3, we find the automorphism invariants[2]. In Chapter 4, we use T-invariance and a Galois symmetry to find out where a nonzero entry on the vacuum row or column could appear. The possibilities for these happen to be very limited, and for all but a few exceptional levels, they turn out to be just a simple current orbit of the vacuum. In Chapter 5, we find the modular invariants at the non-exceptional heights, and in Chapter 6, we find the exceptional invariants by considering each exceptional height separately.

We include one table: Table 3.1 lists all of the $(A_2 \oplus A_2)^{(1)}$ simple-current invariants.

[2]these were found by Gannon for any $A_{r_1} \oplus \cdots \oplus A_{r_s}$ in [8]

Chapter 2

The Problem

The classification of the modular invariants of $(A_2 \oplus A_2)^{(1)}$ will follow closely that of A_2 in [9]. In §2.1, we define the problem of the classification for $A_2 \oplus A_2$, and in §2.2, we review the "modern" classification for A_1 [11]: this is a model for our approach.

2.1 Basic Definitions and Calculations

The $(A_2 \oplus A_2)^{(1)}$ data is built up from the $A_2^{(1)}$ data in the natural way, so in this section, we will generally introduce a concept for $A_2^{(1)}$ first and then write down the $(A_2 \oplus A_2)^{(1)}$ version. The nontwisted affine X_r algebra is denoted by $X_r^{(1)}$; however, in this thesis, we will usually leave off the superscript (1), since we are dealing only with nontwisted affine algebras.

We associate to the affine A_2 algebra a *level* k; however, in many instances it will be more useful to work with the *height* $n := k+3$. We denote A_2 at height n by $A_{2,n}$. Let $\Lambda_0, \Lambda_1, \Lambda_2$ be the $A_2^{(1)}$ fundamental weights. We translate all $A_{2,n}$ highest weights by the Weyl vector $\rho = \Lambda_0 + \Lambda_1 + \Lambda_2$ in order to make our equations easier to use.

We identify a highest weight $\lambda = \lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \lambda_2\Lambda_2$ with its Dynkin labels: we say $\lambda = (\lambda_0, \lambda_1, \lambda_2)$. We can, and generally will, omit λ_0 since $\lambda_0 = n - \lambda_1 - \lambda_2$, and so λ is completely determined by λ_1 and λ_2. The set of "shifted" highest weights is

$$P_{++}^n(A_2) = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : 0 < \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < n\},$$

as opposed to the set $P_+^n(A_2) = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : 0 \le \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < n - 2\}$ of (unshifted) highest weights. We use P_{++}^n instead of P_+^n due to the translation of all weights by $\Lambda_0 + \Lambda_1 + \Lambda_2$. For the direct sum $A_{2,p'} \oplus A_{2,p}$, we call (p', p) the height and $(k, l) := (p' - 3, p - 3)$ the level. For our classification, we will always assume $\gcd(p', p) = 1$[1]. The set of (shifted) highest weights for $A_{2,p'} \oplus A_{2,p}$ is

$$P_{++}^{p',p} := \{(\lambda, \mu) \in \mathbb{Z}^4 : 0 < \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < p' \text{ and } 0 < \mu_1, \mu_2, \mu_1 + \mu_2 < p\},$$

so $(\lambda, \mu) \in P_{++}^{p',p}$ iff $\lambda \in P_{++}^{p'}(A_2)$ and $\mu \in P_{++}^{p}(A_2)$. The highest weight $(\rho, \rho) := ((1,1),(1,1))$ is called the *vacuum.* We will often abbreviate a highest weight (λ, μ) by $\lambda\mu$. Let $\chi^{(p')}$, $\chi^{(p)}$ be the characters corresponding to the height p' and p representations of $A_2^{(1)}$ respectively, and let $\chi^{(p',p)}$ be the $A_{2,p'} \oplus A_{2,p}$ character. Then

$$\chi_{\lambda\mu}^{(p',p)} = \chi_\lambda^{(p')} \chi_\mu^{(p)}. \tag{2.1}$$

Let M be the coefficient matrix for the partition function

$$\mathcal{Z} = \sum_{\lambda\mu, \kappa\nu \in P_{++}^{p',p}} M_{\lambda\mu,\kappa\nu} \chi_{\lambda\mu}^{(p',p)} \chi_{\kappa\nu}^{(p',p)*} \tag{2.2}$$

of a WZW rational conformal field theory with chiral algebra $A_{2,p'} \oplus A_{2,p}$. The characters $\chi_{\lambda\mu}^{(p',p)}$ are functions of a complex number τ. For $A_2 \oplus A_2$, Equations (1.2) become

$$M_{\rho\rho,\rho\rho} = 1, \tag{2.3a}$$

$$M_{\lambda\mu,\kappa\nu} \in \mathbb{Z}_{\ge 0} \text{ for all } \lambda\mu, \kappa\nu \in P_{++}^{p',p}, \tag{2.3b}$$

$$MS^{(p',p)} = S^{(p',p)}M; \; MT^{(p',p)} = T^{(p',p)}M, \tag{2.3c}$$

where $S^{(p',p)}$ and $T^{(p',p)}$ are given in (2.6). Any M satisfying (2.3) is an $A_{2,p'} \oplus A_{2,p}$ modular invariant: the goal of this thesis is to find all such M. The S and T matrices that M commute with are called *modular data*, and Property (2.3c) is called *modular invariance.* A partition function corresponding to a modular invariant is called a

[1] One reason for this is that the classification for arbitrary (p', p) would be very difficult. Another is that the W_N minimal model classifications have p' and p coprime, so removing the coprime condition would not contribute to the W_3 case.

modular invariant partition function. We will switch back and forth between using a modular invariant or its partition function, depending on which is more convenient to use at a given time.

Up to conformal equivalence, we can identify a torus with $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, for some $\tau \in \mathbb{C}$ with $Im(\tau) > 0$. Moreover, the tori corresponding to τ and $(a\tau+b)/(c\tau+d)$, for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, are also conformally equivalent. $SL_2(\mathbb{Z})$ is the set of 2×2 matrices with integer entries and determinant 1. This is the final redundancy, as far as conformal equivalence is concerned. For this reason, $SL_2(\mathbb{Z})$ is called the modular group of the torus.

The partition function $\mathcal{Z}$ in (2.2) should be well-defined on each conformal equivalence class of tori. This means that $\mathcal{Z}$ is a function of τ, and must satisfy

$$\mathcal{Z}(\tau) = \mathcal{Z}\left(\frac{a\tau+b}{c\tau+d}\right), \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}),$$

where $SL_2(\mathbb{Z}) = \{2 \times 2 \text{ matrices with integer entries and determinant } 1\}$. For this to happen, it is enough to have

$$\mathcal{Z}(\tau) = \mathcal{Z}(\tau+1) \text{ and} \mathcal{Z}(\tau) = \mathcal{Z}\left(\frac{-1}{\tau}\right),$$

because the actions $\tau \mapsto 1+\tau$ and $\tau \mapsto -1/\tau$ generate all of $SL_2(\mathbb{Z})$. This is what we mean by modular invariance of the partition function.

The characters χ_λ of integrable representations $\lambda \in P^n_{++}$ of affine Kac-Moody algebras $X^{(1)}_r$ have the remarkable property that they are also functions of τ, and satisfy

$$\chi_\lambda\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_{\mu \in P^n_{++}} \rho\begin{pmatrix} a & b \\ c & d \end{pmatrix} \chi_\mu(\tau)$$

for some unitary representation ρ of $SL_2(\mathbb{Z})$ (see Chapter 13 of [18]). We are especially interested in the two generators

$$S = \rho\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \rho\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where ρ is a representation. For physical reasons, we know the partition function has the form (1.1), and so modular invariance reduces to (1.2c).

The S and T matrices for A_2 at height n, denoted $S^{(n)}$ and $T^{(n)}$, are

(2.4a) $$S^{(n)}_{\lambda\mu} = \frac{-i}{\sqrt{3}n} \sum_{\omega \in \mathcal{W}} det\omega \exp[-2\pi i \frac{\omega(\lambda)\cdot\mu}{n}],$$

(2.4b) $$T^{(n)}_{\lambda\mu} = \exp[2\pi i \frac{\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 - n}{3n}]\delta_{\lambda,\mu},$$

where $\mathcal{W}$ is the Weyl group for A_2. $S^{(n)}$ and $T^{(n)}$ are unitary and symmetric (see Chapter 13 of [18]). Notice that $T^{(n)}$ is diagonal. Generally, we will not need to calculate individual entries of the S-matrix; however, in Chapter 6, we will need to use the explicit formula for $S^{(n)}_{\lambda\mu}$, so it is worthwhile to write it here. Let $(t, r, s) := (\lambda_0, \lambda_1, \lambda_2)$ and $(t', r', s') := (\mu_0, \mu_1, \mu_2)$ (recall that $\lambda_0 = n - \lambda_1 - \lambda_2$). Then the $\lambda\mu$-th entry of $S^{(n)}$ is given by:

(2.5) $$S^{(n)}_{\lambda\mu} = \frac{-i}{n\sqrt{3}}\xi^{-(2rr'+2ss'+rs'+r's)}\left(1 + \zeta^{tt'-rs'} + \zeta^{tt'-r's} - \zeta^{rr'} - \zeta^{ss'} - \zeta^{tt'}\right),$$

where $\xi = e^{\frac{\pi i}{3n}}$ and $\zeta = e^{\frac{\pi i}{n}}$ (see for instance [2]).

The S and T matrices for $(A_2 \oplus A_2)^{(1)}$ at height (p', p), are

(2.6a) $$S^{(p',p)}_{\lambda\mu,\kappa\nu} = S^{(p')}_{\lambda\kappa} \cdot S^{(p)}_{\mu\nu},$$

(2.6b) $$T^{(p',p)}_{\lambda\mu,\kappa\nu} = T^{(p')}_{\lambda\kappa} \cdot T^{(p)}_{\mu\nu}.$$

It follows from (2.4) that $S^{(p',p)}$ and $T^{(p',p)}$ are also unitary and symmetric, and $T^{(p',p)}$ is diagonal.

The Dynkin diagram for $A_2^{(1)}$ is an equilateral triangle. *Charge conjugation* is the reflection of the triangle through the 0th node that exchanges the other two nodes, and a *simple current* is a rotation of the triangle through $2\pi/3$ radians. We denote a charge conjugation for $A_2^{(1)}$ at height n by C_n and a simple current by A_n. C_n and A_n act on a weight λ as follows: $C_n(\lambda_0, \lambda_1, \lambda_2) = (\lambda_0, \lambda_2, \lambda_1)$, and $A_n(\lambda_0, \lambda_1, \lambda_2) = (\lambda_2, \lambda_0, \lambda_1)$. C_n has order 2 and A_n has order 3, and together they generate the group of order 6 of all symmetries of the $A_2^{(1)}$ Dynkin diagram. This is the group of outer automorphisms of $A_2^{(1)}$, which we denote by $\mathcal{O}$. Keeping in mind that $\lambda_0 = n - \lambda_1 - \lambda_2$, we can write the action of C_n and A_n on a weight $\lambda = (\lambda_1, \lambda_2)$ as

(2.7a) $$C_n(\lambda_1, \lambda_2) = (\lambda_2, \lambda_1),$$

(2.7b) $$A_n(\lambda_1, \lambda_2) = (n - \lambda_1 - \lambda_2, \lambda_1).$$

Let $\mathcal{O}\lambda := \{C_n^i A_n^j \lambda : i = 0,1, j = 0,1,2\}$. Notice that if $\lambda = (\lambda_1, \lambda_2)$ has $\lambda_1 = \lambda_2$, then $\mathcal{O}\lambda = \{A^j\lambda : j = 0,1,2\}$.

For $A_2 \oplus A_2$, we define the charge conjugations to be

(2.8) $$C^{(i,j)}(\lambda, \mu) := C_{p'}^i C_p^j(\lambda, \mu) := (C_{p'}^i \lambda, C_p^j \mu),$$

and the simple currents

(2.9) $$A^{(i,j)}(\lambda, \mu) := A_{p'}^i A_p^j(\lambda, \mu) := (A_{p'}^i \lambda, A_p^j \mu).$$

Each $C^{(i,j)}$ and $A^{(i,j)}$ has order 2 and 3 respectively. The charge conjugations and simple currents generate the group of outer automorphisms of $(A_2 \oplus A_2)^{(1)}$ of order 36. A special subgroup of these is the simple currents $\{A^{(i,j)}\}$, which we denote by **A**. The charge conjugations and simple currents for $A_2^{(1)}$ satisfy the relations (2.10) below, which we will use throughout Chapters 3, 4, 5 and 6. Define $t(\lambda) = \lambda_1 - \lambda_2$, called the *triality* of λ. A simple calculation shows that

$$t(A^a\lambda) \equiv na + t(\lambda) \pmod 3.$$

We then get

(2.10a) $$T^{(n)}_{C\lambda, C\mu} = T^{(n)}_{\lambda\mu},$$

(2.10b) $$S^{(n)}_{C\lambda,\mu} = S^{(n)}_{\lambda,C\mu} = S^{(n)*}_{\lambda\mu},$$

(2.10c) $$T^{(n)}_{A^a\lambda, A^a\mu} = \exp[\frac{2\pi i}{3}(a^2 n - at(\lambda))]T^{(n)}_{\lambda\mu}$$

(2.10d) $$S^{(n)}_{A^a\lambda, A^b\mu} = \exp[\frac{2\pi i}{3}(bt(\lambda) + at(\mu) + nab)]S^{(n)}_{\lambda\mu}.$$

Equations (2.10) are equations (1.6) of [9]. Another important property of the S-matrix, which we will use frequently, is the value of $S^{(n)}_{\lambda\mu}$ when μ has $\mu_1 = \mu_2$, and especially when $\mu = \rho = (1,1)$

(2.11) $$S^{(n)}_{\lambda,(a,a)} = \frac{8}{\sqrt{3}n} \sin\left(\pi\frac{a\lambda_1}{n}\right) \sin\left(\pi\frac{a\lambda_2}{n}\right) \sin\left(\pi\frac{a(\lambda_1 + \lambda_2)}{n}\right),$$

for all $1 \leq a \leq \frac{n-1}{2}$.

Putting $(a,a) = \rho$ into (2.11), we get the useful fact

$$S^{(n)}_{\lambda\rho} \geq S^{(n)}_{\rho\rho} > 0, \tag{2.12}$$

with equality iff $\lambda \in \mathcal{O}\rho$. Equations (2.11) and (2.12) are (2.1) of [9].

We are now ready to state the result of this thesis as the following theorem.

Theorem 2.1. *Let p' and p be positive coprime integers. The modular invariants for $(A_2 \oplus A_2)^{(1)}$ at height (p',p) are*

(a) the automorphism invariants, listed in Theorem 3.1,

(b) the simple-current invariants, listed in Theorem 5.1

(c) the exceptional invariants given in equations (6.12), (6.14), (6.25), (6.27), (6.28), (6.29).

Remark: In this section, we dealt exclusively with A_2 and $A_2 \oplus A_2$; however, apart from the specifics, such as entries of the S and T matrices and the symmetries of the extended A_2 Dynkin diagram, all of the concepts from this section hold for general X_r and their modular data.

2.2 The $A_1^{(1)}$ classification

For clarity, we will demonstrate Gannon's classification of the A_1 modular invariants [11]. As our $A_2 \oplus A_2$ classification follows this method, we will point out the main ideas of the proof so that the reader can relate the corresponding steps in Chapters 3, 4, 5 and 6 to the ones done here. The aim of this section is to prove Theorem 2.2.

2.2.1 The Problem for $A_1^{(1)}$

For the affine algebra $A_1^{(1)}$ at height n, $P^n_{++}(A_1) = \{a \in \mathbb{Z} : 0 < a < n-1\} = \{1, 2, ..., n-1\}$; we write P_{++} for short. The vacuum is 1, and there is only one outer automorphism, which we call J. J acts on P_{++} by $Ja = n - a$ (ie, $\lambda_1 \longmapsto \lambda_0 = n - \lambda_1$). The S and T matrices are

$$S_{ab} = \sqrt{\frac{2}{n}} \sin\left(\pi \frac{ab}{n}\right), \tag{2.13a}$$

(2.13b) $$T_{ab} = \exp[-\frac{\pi i}{4}] \exp[\pi i \frac{a^2}{2n}] \delta_{a,b}.$$

S is orthogonal and symmetric and obeys the relation

(2.14) $$S_{Ja,b} = (-1)^{b+1} S_{ab},$$

which follows directly from the definitions of S and J. Putting $(a,1)$ and $(1,1)$ into (2.13a), we have

(2.15) $$S_{a1} \geq S_{11} > 0,$$

with equality iff $a \in \{1, J1\}$. Let

(2.16) $$\mathcal{Z} = \sum_{a,b=1}^{n-1} M_{ab} \chi_a \chi_b^*$$

be a partition function for $A_1^{(1)}$ with coefficient matrix M. We call M a *modular invariant* if

(2.17a) $$M_{11} = 1,$$

(2.17b) $$M_{ab} \in \mathbb{Z}_{\geq 0},\ \forall a, b \in P_{++},$$

(2.17c) $$MS = SM, MT = TM,$$

and as usual we identify M with its partition function $\mathcal{Z}$.

Theorem 2.2. *The complete list of modular invariants for $A_1^{(1)}$ at height n is*

(2.18a) $$\mathcal{A}_{n-1} = \sum_{a=1}^{n-1} |\chi_a|^2,\ \forall\, n \geq 3,$$

(2.18b) $$\mathcal{D}_{\frac{n}{2}+1} = \sum_{a=1}^{n-1} \chi_a \chi_{J^{a+1}a}^*,\ \textit{whenever } \tfrac{n}{2} \textit{ is even},$$

(2.18c) $$\mathcal{D}_{\frac{n}{2}+1} = |\chi_1 + \chi_{J1}|^2 + |\chi_3 + \chi_{J3}|^2 + \cdots + 2|\chi_{\frac{n}{2}}|^2,\ \textit{whenever } \tfrac{n}{2} \textit{ is odd},$$

(2.18d) $$\mathcal{E}_6 = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2,\ \textit{for } n = 12,$$

(2.18e)
$$\mathcal{E}_7 = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2 + \chi_9(\chi_3 + \chi_{15})^* + (\chi_3 + \chi_{15})\chi_9^* + |\chi_9|^2,\ \textit{for } n = 18,$$

(2.18f) $$\mathcal{E}_8 = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2,\ \textit{for } n = 30.$$

2.2.2 T-invariance and the Galois selection rule

In this subsection we find that S satisfies a Galois symmetry (2.23), and we derive a simple formula (2.20) which comes from T-invariance. These are the two tools which will later give us the "1-couplings'; ie, those $a \in P_{++}$ such that $M_{a1} \neq 0$ or $M_{1a} \neq 0$. This subsection is the analogue to §4.1.

Since M commutes with T (ie, M is *T-invariant*), we have

$$\sum_{c=1}^{n-1} M_{ac}T_{cb} = \sum_{c=1}^{n-1} T_{ac}M_{cb}, \tag{2.19}$$

for any $a, b \in P_{++}$. But T is diagonal, so this gives us $M_{ab}T_{bb} = T_{aa}M_{ab}$. If $M_{ab} \neq 0$, we can cancel out the M_{ab}, so $\exp[\pi i \frac{a^2}{2n}] = \exp[\pi i \frac{b^2}{2n}]$, by (2.13b). This gives us the selection rule (what we will call the norm condition in Chapter 4)

$$M_{ab} \neq 0 \implies a^2 \equiv b^2 \pmod{4n}. \tag{2.20}$$

Our next tool is the *parity rule*, or *Galois selection rule* (2.25), which comes from a symmetry obeyed by the S-matrix. Let $\mathcal{L} := \{\ell \in \mathbb{Z} : 0 < \ell < 2n, \text{and } \gcd(\ell, 2n) = 1\}$. For each $\ell \in \mathcal{L}$, we will find a permutation $a \longmapsto [\ell a]$ of P_{++} and a choice of signs $\epsilon_\ell : P_{++} \longrightarrow \{\pm 1\}$ such that

$$M_{ab} = \epsilon_\ell(a)\epsilon_\ell(b)M_{[\ell a],[\ell b]} \text{ for all } a, b \in P_{++},\ \ell \in \mathcal{L}. \tag{2.21}$$

Let $\{x\}$ be the unique integer congruent to $x \pmod{2n}$ satisfying $0 \leq \{x\} < 2n$. Notice that for all $\ell \in \mathcal{L}$ and $a \in P_{++}$, $\{\ell a\} \neq n$, so either $\{\ell a\} < n$ or $\{\ell a\} > n$. We will define our permutation and choice of signs as follows: If $\{\ell a\} < n$, put $[\ell a] = \{\ell a\}$ and $\epsilon_\ell(a) = +1$. If $\{\ell a\} > n$, put $[\ell a] = 2n - \{\ell a\}$ and $\epsilon_\ell(a) = -1$. Then this permutation and choice of signs will satisfy (2.21), as we will show.

Let ξ be a primitive $2n$th root of unity, and denote by φ the Euler totient function $\varphi(m) = \|\{m' \in \mathbb{Z}_{>0} : m' < m \text{ and } \gcd(m, m') = 1\}\|$. We know that $[\mathbb{Q}(\xi) : \mathbb{Q}] = \varphi(2n)$; ie, $\mathbb{Q}(\xi)$ is a $\varphi(2n)$ dimensional vector space over $\mathbb{Q}$ with basis the primitive roots $\{\xi^i : \gcd(i, 2n) = 1\}$ [6]. For any $\ell \in \mathcal{L}$, define $\sigma_\ell(\xi) = \xi^\ell$. Then ξ^ℓ is another primitive root, so $\sigma_\ell \in Gal(\mathbb{Q}(\xi)/\mathbb{Q})$. For all $a, b, c, d \in P_{++}$, $M_{ab} \in \mathbb{Z} \subset \mathbb{Q}(\xi)$, and $S_{ab}S_{cd} \in \mathbb{Q}(\xi)$ (using the formula $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$), so σ_ℓ can be applied to them.[2] By S-invariance and orthogonality of S, $M = SMS$, so

[2] We consider the product $S_{ab}S_{cd}$ to avoid the $\sqrt{\frac{2}{n}}$ in the definition of the S-matrix. This way, we get a factor of $\frac{2}{n}$, which is rational, so is sent to itself by σ_ℓ.

$\sigma_\ell(M_{ab}) = \sigma_\ell((SMS)_{ab})$ for all $a, b \in P_{++}$. But $M_{ab} \in \mathbb{Z} \subset \mathbb{Q}$, so σ_ℓ fixes all M_{ab}. Therefore,

$$M_{ab} = \sum_{k,j=1}^{n-1} \sigma_\ell(S_{ak}S_{jb})M_{kj}. \tag{2.22}$$

To find out what $\sigma_\ell(S_{ak}S_{jb})$ is, we write

$$S_{ak}S_{jb} = \frac{2}{n}\sin\left(\pi\frac{ak}{n}\right)\sin\left(\pi\frac{jb}{n}\right) = -\frac{1}{2n}\left(e^{i\frac{\pi ak}{n}} - e^{-i\frac{\pi ak}{n}}\right)\left(e^{i\frac{\pi jb}{n}} - e^{-i\frac{\pi jb}{n}}\right),$$

so

$$\begin{aligned}\sigma_\ell(S_{ak}S_{jb}) &= -\frac{1}{2n}\left(e^{i\frac{\pi \ell ak}{n}} - e^{-i\frac{\pi \ell ak}{n}}\right)\left(e^{i\frac{\pi \ell jb}{n}} - e^{-i\frac{\pi \ell jb}{n}}\right) \\ &= \sqrt{\frac{2}{n}}\sin\left(\pi\frac{\ell ak}{n}\right)\sqrt{\frac{2}{n}}\sin\left(\pi\frac{\ell jb}{n}\right).\end{aligned}$$

If $\{\ell a\} < n$, then $[\ell a] = \{\ell a\} = \ell a + 2nm$, for some $m \in \mathbb{Z}$, so $\sin\left(\pi\frac{\ell ak}{n}\right) = \sin\left(\pi\frac{[\ell a]k}{n}\right)$. If $\{\ell a\} > n$, then $[\ell a] = -\ell a + 2nm$, for some $m \in \mathbb{Z}$, so $\sin\left(\pi\frac{\ell ak}{n}\right) = -\sin\left(\pi\frac{[\ell a]k}{n}\right)$. Either way,

$$\sin\left(\pi\frac{\ell ak}{n}\right) = \epsilon_\ell \sin\left(\pi\frac{[\ell a]k}{n}\right), \tag{2.23}$$

for our definition of ϵ_ℓ, and the same holds for $\sin\left(\pi\frac{\ell jb}{n}\right)$. Therefore, we have shown that

$$\sigma_\ell(S_{ak}S_{jb}) = \epsilon_\ell(a)\epsilon_\ell(b)S_{[\ell a],k}S_{j,[\ell b]}. \tag{2.24}$$

In general, the S-matrix of an affine algebra obeys such a symmetry for some permutation of the highest weights and choice of signs ϵ.

With (2.24), the right-hand side of equation (2.22) now becomes

$$\epsilon_\ell(a)\epsilon_\ell(b)\sum_{k,j=1}^{n-1} S_{[\ell a],k}M_{kj}S_{j,[\ell b]},$$

which is $\epsilon_\ell(a)\epsilon_\ell(b)M_{[\ell a],[\ell b]}$. We thus have (2.21). Since every entry of M is nonnegative, it follows from (2.21) and the fact that $\epsilon_\ell(a)\epsilon_\ell(b) = \pm 1$, that $M_{[\ell a],[\ell b]} > 0$ whenever $M_{ab} > 0$. But then $\epsilon_\ell(a)\epsilon_\ell(b) = 1$, so we get the *Galois selection rule*

$$M_{ab} \neq 0 \implies \epsilon_\ell(a) = \epsilon_\ell(b), \tag{2.25}$$

for all $\ell \in \mathcal{L}$.

2.2.3 The $A_1^{(1)}$ Permutation Matrices

This subsection is analogous to Chapter 3: the permutation matrices here are what we call the automorphism invariants in Chapter 3. Define M to be a *permutation matrix* if $M_{ab} = \delta_{b,\pi a}$ for some permutation π of P_{++}; ie, there is only one nonzero entry in each row or column of M, and that entry is 1. The following lemma tells us that all modular invariants of the form

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix} \tag{2.26}$$

are of this type.

Lemma 2.1. *Let M be a modular invariant, and suppose $M_{a1} = \delta_{a,1}$. Then M is a permutation matrix for some permutation π of P_{++}, and $S_{\pi a, \pi b} = S_{ab}$ for all $a, b \in P_{++}$.*

Proof. We will first show that the entries of any modular invariant are bounded above. For any $a, b \in P_{++}$,

$$1 = M_{11} = (SMS)_{11} = \sum_{i,j=1}^{n-1} S_{1i} M_{ij} S_{j1} \geq S_{11}^2 \sum_{i,j=1}^{n-1} M_{ij} \geq S_{11}^2 M_{ab}, \tag{2.27}$$

where the first inequality comes from (2.15) and the second from the fact that all M_{ij} are nonnegative. This tells us that for any $a, b \in P_{++}$, $M_{ab} \leq \frac{1}{S_{11}^2}$. Notice that multiplying modular invariants gives us at least a matrix commuting with S and T, as does taking transpose. Therefore, defining $N := M^T M$, N^L (N to the power of L) commutes with S and T for any positive integer L. The diagonal entries of N are

$$N_{aa} = (M^T M)_{aa} = \sum_{i=1}^{n-1} M_{ai}^T M_{ia} = \sum_{i=1}^{n-1} (M_{ia})^2.$$

We will show that, unless there is at most one nonzero entry in each column of M, and it equals 1, some $(N^L)_{aa}$ will be unbounded as L goes to infinity.

Suppose that there is an entry in the ath column of M that is greater than 1. Then, it must be at least 2, since all the entries of M are integers. So $N_{aa} = \sum_{i=1}^{n-1} (M_{ia})^2 \geq 2^2 = 4$. On the other hand, suppose there are two entries in the

ath column of M that equal 1 (or more). Then $N_{aa} = \sum_{i=1}^{n-1}(M_{ia})^2 \geq 1^2 + 1^2 = 2$. Either way, $N_{aa} \geq 2 > 1$.

Next we will use induction to show that $(N^L)_{aa} \geq (N_{aa})^L$. Since $N_{aa} \geq 2$, then this will imply $N_{aa}^L \geq 2^L$. For $L = 1$, we have $(N^L)_{aa} = (N^1)_{aa} = N_{aa} = (N_{aa})^1 = (N_{aa})^L$. Now suppose that $(N^L)_{aa} \geq (N_{aa})^L$, and consider $(N^{L+1})_{aa}$:

$$\begin{aligned}
(N^{L+1})_{aa} &= (N \times N^L)_{aa} \\
&= \sum_{i=1}^{n-1} N_{ai} N_{ia}^L \\
&= N_{aa} N_{aa}^L + \sum_{i=1, i \neq a}^{n-1} N_{ai} N_{ia}^L \\
&\geq N_{aa}(N_{aa})^L + \sum_{i=1, i \neq a}^{n-1} N_{ai} N_{ia}^L
\end{aligned}$$

Because all of the entries of M are nonnegative, all of the entries of N must be nonnegative as well, so the last summation is nonnegative. Therefore, $(N^{L+1})_{aa} \geq N_{aa}(N_{aa})^L + 0 = (N_{aa})^{L+1}$, and so we have shown that $(N^L)_{aa} \geq 2^L$ for that $a \in P_{++}$. But now, $N_{11} = \sum_{i=1}^{n-1} M_{i1}^2 = M_{11}^2$ by hypothesis, so $N_{11} = 1$. A simple induction argument shows that $N_{11}^L = 1$ for all L, so by (2.27), the entries of N^L are bounded above. In particular, $N_{aa}^L \leq \frac{1}{S_{11}^2}$ for all L. This contradicts $(N^L)_{aa} \geq 2^L$. Therefore each column can have at most one nonzero entry, and that entry is 1.

By S-invariance, $(MS)_{11} = (SM)_{11}$, so $\sum_{i=1}^{n-1} M_{1i}S_{i1} = \sum_{i=1}^{n-1} S_{1i}M_{i1} = S_{11}$, since $M_{i1} = \delta_{1,i}$. This gives us $S_{11} + \sum_{i=2}^{n-1} M_{1i}S_{i1} = S_{11}$, so $\sum_{i=2}^{n-1} M_{1i}S_{i1} = 0$. By (2.15), $S_{i1} > 0$ for all i, so we must have $M_{1i} = 0$ for all $i \geq 2$; ie, $M_{1i} = \delta_{i,1}$.

Now letting $N' := MM^T$, a similar argument shows that $N'^L \to \infty$ as $L \to \infty$ unless there is at most one nonzero entry in each row of M, and that entry is 1. To show that there is a 1 in each row of M, we calculate $(MS)_{a1} = (SM)_{a1}$, so

$$\sum_{i=1}^{n-1} M_{ai}S_{i1} = \sum_{i=1}^{n-1} S_{ai}M_{i1} = S_{a1} > 0.$$

The left-hand side is positive iff $M_{aa'} \neq 0$ for some $a' \in P_{++}$. Similarly, evaluating $(MS)_{1b} = (SM)_{1b}$, we must have $M_{b'b} \neq 0$ for some $b' \in P_{++}$. Therefore, there is at least one nonzero entry in each row and column of M, so we have shown that

$M_{ab} = \delta_{b,\pi a}$ for some permutation π of P_{++}. $S_{\pi a,\pi b} = S_{ab}$ comes from S-invariance; ie,

$$\sum_{i=1}^{n-1} M_{ai}S_{ib} = \sum_{i=1}^{n-1} S_{ai}M_{ib} \iff M_{a,\pi a}S_{\pi a,b} = S_{a,\pi^{-1}b}M_{\pi^{-1}b,b}$$
$$\iff S_{\pi a,b} = S_{a,\pi^{-1}b},\ \forall a,b \in P_{++}. \quad \square$$

With this lemma, we can now find all modular invariants M such that the entries in the first row and column are all zero except for $M_{11} = 1$.

Suppose that $M_{a1} \neq 0$ or $M_{1a} \neq 0 \implies a = 1$. By Lemma 2.1, $M_{ab} = \delta_{b,\pi a}$, so we need to find which permutations π define a modular invariant. Since $M_{11} = 1$, we already know that $\pi 1 = 1$. To see what $\pi 2$ is, the last part of Lemma 2.1 gives us $S_{12} = S_{\pi 1,\pi 2} = S_{1,\pi 2}$, so

$$\sin\left(\frac{\pi}{n}\right) = \sin\left(\frac{m\pi}{n}\right),$$

where $m := \pi 2$. The only $m \in P_{++}$ that can satisfy this are $m = 2$ and $m = J2$. If $m = J2$, then by (2.20), $2^2 \equiv (n-2)^2 \pmod{4n}$, which implies $4 \mid n$. Therefore, $n/2$ is even, so in this case $\mathcal{D}_{\frac{n}{2}+1}$ is the permutation matrix defined by $\pi' a = a$ when a is odd and $\pi' a = n - a$ when a is even. Inverses and products of permutation matrices are permutation matrices, so if $\pi 2 = J2$, we can let $M' := \mathcal{D}_{\frac{n}{2}+1}^{-1}M$, and then M' will be a permutation matrix satisfying $\pi 2 = 2$. Therefore, we may assume that $\pi 2 = 2$, replacing M with M' when necessary.

Now let $a \in P_{++}$ and let $b = \pi a$. Then $S_{1a} = S_{1a} = S_{1b}$, and $S_{2a} = S_{2b}$, so we have $\sin\left(\frac{\pi a}{n}\right) = \sin\left(\frac{\pi b}{n}\right)$ and $\sin\left(\frac{2\pi a}{n}\right) = \sin\left(\frac{2\pi b}{n}\right)$. Therefore,

$$1 = \frac{\sin(\frac{2a\pi}{n})}{\sin(\frac{2b\pi}{n})} = \frac{2\sin(\frac{a\pi}{n})\cos(\frac{a\pi}{n})}{2\sin(\frac{b\pi}{n})\cos(\frac{b\pi}{n})} = \frac{\cos(\frac{a\pi}{n})}{\cos(\frac{b\pi}{n})},$$

so $\cos\left(\frac{a\pi}{n}\right) = \cos\left(\frac{b\pi}{n}\right)$. But $0 < a, b \leq n-1$, and this implies $b = a$; ie $\pi a = a$ for all $a \in P_{++}$. Therefore, either M or M' is the identity, so M can be the identity $\mathcal{A}_{n-1}$, or the permutation matrix $\mathcal{D}_{\frac{n}{2}+1}$ when $4 \mid n$.

2.2.4 The 1-couplings

The goal of this subsection is to find all places on the first row and column of M that could contain a nonzero entry; ie, those a for which $M_{a1} \neq 0$ or $M_{1a} \neq 0$. It turns out that there is usually only one possibility for such an a, other than $a = 1$.

Proposition 2.1. *Suppose $M_{a1} \neq 0$ or $M_{1a} \neq 0$. Then $a \in \{1, J1\}$ for all $n \neq 12, 18, 30$.*

Proof. Suppose that $M_{a1} \neq 0$ or $M_{1a} \neq 0$. Putting $(a,1)$ into (2.20) gives us

$$(a+1)(a-1) \equiv 0 \pmod{4n}. \tag{2.28}$$

By (2.25), $\epsilon_\ell(a) = \epsilon_\ell(1)$, so by definition of ϵ_ℓ, $\{\ell a\} < n \iff \{\ell\} < n$. Therefore, $sgn\left(\sin(\frac{\ell a\pi}{n})\right) = sgn\left(\sin(\frac{\ell\pi}{n})\right)$, which is equivalent to $\sin(\frac{\ell a\pi}{n})\sin(\frac{\ell\pi}{n}) > 0$. Using the formula $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$, we have

$$\cos\left(\frac{\ell(a-1)\pi}{n}\right) > \cos\left(\frac{\ell(a+1)\pi}{n}\right). \tag{2.29}$$

Our strategy will be to show that there are no solutions $a \notin \{1, J1\}$ to (2.28) and (2.29), other than at the *exceptional* heights $n = 12, 18, 30$.

Notice that $a^2 \equiv 1 \pmod{4n}$ implies a is odd, so $a+1$ and $a-1$ are even. Putting $\ell + n$ into (2.29), we get:

$$\cos\left(\pi(\ell+n)\frac{a-1}{n}\right) > \cos\left(\pi(\ell+n)\frac{a+1}{n}\right)$$

iff

$$\cos\left(\pi\ell\frac{a-1}{n} + \pi(a-1)\right) > \cos\left(\pi\ell\frac{a+1}{n} + \pi(a+1)\right).$$

But $\pi(a-1)$ and $\pi(a+1)$ are multiples of 2π, since $a-1$, $a+1$ are even, so we have the above inequality iff $\cos(\pi\ell\frac{a-1}{n}) > \cos(\pi\ell\frac{a+1}{n})$. Therefore, ℓ obeys (2.29) iff $\ell + n$ does, so that we can take ℓ in (2.29) to be coprime to n, rather than to $2n$.

Let $\mathcal{L}' := \{\ell : \gcd(\ell, n) = 1\}$. Define $d := \gcd(a-1, 2n)$, and $d' := \gcd(a+1, 2n)$. Since they are both even, $a-1$ and $a+1$ have a factor of 2 in common. However, they cannot have any other factor in common because their difference is 2; ie, $\gcd(d, d') = 2$. Next we try to find out what dd' is. Let $n = r_1^{\gamma_1} r_2^{\gamma_2} \dots r_l^{\gamma_l}$, where the r_i's are distinct primes. By (2.28), $(a-1)(a+1) = 4nk$, for some $k \in \mathbb{Z}_+$, so $(a-1)(a+1) = 4r_1^{\gamma_1} r_2^{\gamma_2} \dots r_\ell^{\gamma_\ell} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where $\prod_{i=1}^m p_i^{\alpha_i}$ is the prime power decomposition for k. Therefore, $a - 1 = 2r_1^{\gamma_1'} r_2^{\gamma_2'} \dots r_{\ell'}^{\gamma_{\ell'}'} p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{m'}^{\alpha_{m'}}$, and $a+1 = 2r_{\ell'+1}^{\gamma_{1'+1}'} \dots r_l^{\gamma_l'} p_{m'+1}^{\alpha_{m'+1}} \dots p_m^{\alpha_m}$ (renaming the primes if necessary, and to keep the p_i's and r_i's distinct, $\gamma_i' \geq \gamma_i$ for all $i = 1, \ldots, \ell$). Now,

$$\begin{aligned} d &= \gcd(a-1, 2n) \\ &= \gcd(2r_1^{\gamma_1'} r_2^{\gamma_2'} \ldots r_{\ell'}^{\gamma_{\ell'}'} p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{m'}^{\alpha_{m'}}, 2r_1^{\gamma_1} r_2^{\gamma_2} \ldots r_\ell^{\gamma_\ell}) \\ &= 2r_1^{\gamma_1} \ldots r_{\ell'}^{\gamma_{\ell'}}, \end{aligned}$$

and

$$d' = \gcd(2r_{\ell'+1}^{\gamma_{\ell'+1}'} \ldots r_\ell^{\gamma_\ell'} p_{m'+1}^{\alpha_{m'+1}} \ldots p_m^{\alpha_m}, 2r_1^{\gamma_1} r_2^{\gamma_2} \ldots r_\ell^{\gamma_\ell}) = 2r_{\ell'+1}^{\gamma_{\ell'+1}} \ldots r_\ell^{\gamma_\ell},$$

and now it is easy to see that $dd' = 4n$. $\gcd(d, d') = 2$ implies $d = 2s$, and $d' = 2s'$, for some s, s' with $\gcd(s, s') = 1$.

We now look at what happens if one of d and d' is less than 6. Say $d = 2$ or 4 (d is even). If $d = 2$, then $dd' = 2d' = 4s' = 4n$, so $s' = n$. So $2n = \gcd(a+1, 2n)$, which means that $a+1$ is a multiple of $2n$, a contradiction since $a \in P_{++}$. If $d = 4$, then $dd' = 4d' = 4n$, so $d' = n$. Therefore $n = \gcd(a+1, 2n)$, which implies that $a + 1 = mn$ for some $m \in \mathbb{Z}_{\geq 0}$. But the only m that could possibly work here is $m = 1$. In that case, $a = n - 1 = J1$, which is a contradiction because we assumed $a \neq 1, J1$. Therefore, $d \geq 6$, and a similar argument shows that $d' \geq 6$ as well (we would get in one case $a = 1$, and in the other case, $a \notin P_{++}$). So $d, d' \geq 6$.

Since $\gcd(a+1, 2n) = d'$, and $a+1$ is even, $a+1 = a'd'$, for some a' with $\gcd(a', 2n) = 1$. Therefore, a' has an inverse mod $2n$. Let $\ell \in L$ be such that $\ell a' \equiv 1 \pmod{2n}$. If $\ell < n$, then let $\ell' := \ell$, and we have $\ell'(a+1) = \ell' a' d' \equiv d' \pmod{2n}$. If $\ell > n$, then let $\ell' = \ell - n$, and we have: $\ell'(a+1) = (\ell - n)a'd' = \ell a' d' - n a' d' = \ell a' d' - n(a+1) \equiv \ell a' d' \equiv d' \pmod{2n}$. Therefore, we can choose $\ell' \in \mathcal{L}'$ so that $\ell'(a+1) \equiv d' \pmod{2n}$. Next choose $\ell_0 \in \mathcal{L}'$ so that

$$\ell_0(a-1) \equiv \begin{cases} n - d, \text{ if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is even,} \\ n - 2d, \text{ if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is odd,} \\ n - \frac{d}{2}, \text{ otherwise, ie if } \frac{d'}{2} \text{ is odd} \end{cases} \pmod{2n}, \tag{2.30}$$

and define $\ell_i = \frac{2ni}{d} + \ell_0$. Then $\ell_i(a-1) = (\frac{2ni}{d} + \ell_0)(a-1) = \frac{2ni}{d}(a-1) + \ell_0(a-1)$, but $d \mid a-1$, so $\frac{a-1}{d} \in \mathbb{Z}$. Therefore, $2ni\frac{a-1}{d}$ is a multiple of $2n$, so $\frac{2ni}{d}(a-1) + \ell_0(a-1) \equiv \ell_0(a-1) \pmod{2n}$.

Now we will show that for all i such that $0 \leq i < \frac{d}{2}$, the numbers $\ell_i(a+1)$ will all be distinct. Let $\frac{2n}{d} =: m$. Then $\ell_i = m_i + l_0$. Suppose that $\ell_k(a+1) \equiv \ell_{k'}(a+1)$

(mod $2n$), so $(km - k'm)(a+1) \equiv 0 \pmod{2n}$, or $m(k-k')(a+1) \equiv 0 \pmod{2n}$. Therefore, $m(k-k')(a+1) = 2ns = mds$, for some $s \in \mathbb{Z}$, and so $(k-k')(a+1) = ds$, ie, $(k-k')(a+1)$ is a multiple of d. But $a+1 = d'm'$, for some $m' \in \mathbb{Z}_{\geq 0}$, and $\gcd(d, d') = 2$. We also know that $d, d' \geq 6$, so $a+1$ cannot be a multiple of d. However, $a+1$ is even, so $k-k'$ must be a multiple of $\frac{d}{2}$. This is a contradiction: if $0 \leq k, k' < \frac{d}{2}$, their difference cannot be $\frac{d}{2}$ or greater. The only multiple of $\frac{d}{2}$ that works is 0. But then $k - k' = 0$, and so $\ell_k = \ell_{k'}$.

Now we must show that there are precisely $\varphi(\frac{d}{2})$ numbers i with $0 \leq i < \frac{d}{2}$, such that $\ell_i \in L'$, where φ is the Euler totient function.

$$\left(\varphi\left(\frac{d}{2}\right) - 1\right) d' < \begin{cases} 2d, & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is even,} \\ 4d, & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is odd,} \\ d, & \text{otherwise} \end{cases} \tag{2.31}$$

Putting $\ell = \ell'$ into (2.29), we get $\cos(\pi\ell'\frac{a-1}{n}) > \cos(\pi\ell'\frac{a+1}{n})$, so $\cos(\pi\ell'\frac{md}{n}) > \cos(\pi\frac{d'+2nk}{n})$, for some integers m and k, and $\gcd(\ell', n) = 1$. Therefore $\cos(\pi\ell'\frac{md}{n}) > \cos(\frac{\pi d'}{n})$. Since $d' > d$, $\frac{d}{d'} < 1$, so dividing both sides of (2.29) by d', we obtain:

$$\varphi\left(\frac{d}{2}\right) - 1 < \begin{cases} \frac{2d}{d'} < 2, & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is even,} \\ \frac{4d}{d'} < 4, & \text{if } \frac{d}{2} \text{ is odd and } \frac{n}{2} \text{ is odd,} \\ \frac{d}{d'} < 1, & \text{otherwise} \end{cases} . \tag{2.32}$$

We can use (2.32) to solve for d in cases:

Case 1: $\varphi(d/2) < 3$. In this case, $\varphi(d/2) = 2$. If $d/2$ is a prime p, then $\varphi(d/2) = \varphi(p) = p - 1 = 2$, so $p = 3$, which implies $d = 6$.

If $d/2$ is a product of two primes p and q, then $\varphi(d/2) = \varphi(pq) = \varphi(p)\varphi(q) = (p-1)(q-1) = 2$, and so $p = 3$, $q = 2$ (or vice-versa). Therefore $d/2 = 6$. But we cannot have this because $d/2$ should be odd. $d/2$ cannot be a product of 3 or more distinct primes, so we try a prime power. Let $d/2 = p^2$. Then $\varphi(d/2) = \varphi(p^2) = p(p-1) = 2$. The only choice for p here is $p = 2$. But then $d/2 = 2^2 = 4$, which is even. There are no other possibilities for $\frac{d}{2}$ (they would give too many factors for 2), so in this case, $d = 6$.

We can now solve for d' and n. We know that $dd' = 4n$, so $6d' = 4n$. But n is a multiple of 4, so $n = 4m$ for some $m \in \mathbb{Z}_{\geq 0}$. Therefore $6d' = 16m$, or $d' = 16m/6$.

So $16m/6$ must be an integer greater than 6. For this to happen, m must be a multiple of 3. If $m = 3$, we get $d' = \frac{8\times3\times2}{6} = 8$, and $n = 4m = 12$. But now, notice that we need $(\varphi(d/2) - 1)d' < 2d$, ie, $(2-1)d' < 2 \times 6 = 12$, so $d' < 12$. If we take m to be larger than 3, then d' will be too big (the next lowest choice for m is 6, which is already too large). Therefore we get $d = 6$, $d' = 8$, and $n = 12$.

Case 2: $\varphi(d/2) < 5$. In this case, $\varphi(d/2)$ can equal 2, 3, or 4. If $\varphi(d/2) = 2$, we get the same possibilities as above, except that now $n/2$ is odd, so n is not a multiple of 4, and $(\varphi(d/2) - 1)d' < 4d$. As above, $d = 6$, so now $dd' = 6d' = 4n = 8m$, for some odd integer m. $6d' = 8m$ implies $d' = \frac{8m}{6}$, so again m will have to be a multiple of 3. $m = 3$ gives us $d' = 4$ which is too small. The next odd multiple of 3 is 9, which gives us $d' = 12$, and $n = 2m = 18$. Here, $(\varphi(d/2) - 1)d' = d' = 12 < 24 = 4d$. The next odd multiple of 3, 15, gives us $d = 20 < 24$, and $n = 2m = 30$. $\varphi(d/2) = 3$ is not a possibility since 3 is odd, so let us consider the case $\varphi(d/2) = 4$. If $d/2 = p$, a prime, then $\varphi(d/2) = \varphi(p) = p - 1 = 4$ implies $p = 5$, so $d = 10$. Now, $d' = 8m/10$, where m is odd. If $m = 5$, then $d' = 4$, which is too small. If $m = 15$, then $d' = 12$, and $n = 2m = 30$. In this case, $(\varphi(d/2) - 1)d' = 3d' = 36 < 40 = 4d$. If $m = 25$, then $d' = 20$, and $3d' = 60 \not< 40$. So the only possibility we get here is $d = 10$, $d' = 12$, and $n = 30$.

If $d/2 = pq$, where p and q are distinct primes, then $\varphi(d/2) = \varphi(pq) = \varphi(p)\varphi(q) = (p-1)(q-1) = 4$. There are 2 choices here: $p = q = 3$ (but then they are not distinct), and $p = 5$, $q = 2$ (but then $d/2$ is even), so there are actually no possibilities here. $d/2$ cannot be the product of three distinct primes either, so we try $d/2 = p^2$. Then $\varphi(d/2) = p(p-1) = 4$, which has no solutions. Therefore, we have only the following four possibilities for (d, d', n):

$(d, d', n) \in \{(6, 8, 12),\ (6, 12, 18),\ (6, 20, 30),\ (10, 12, 30)\}$ These three n, $n = 12, 18, 30$ will be our exceptional heights.

We now know all possible first rows and columns of M at the non-exceptional heights. In the next subsection, we extend this to all rows and columns of M.

2.2.5 The J-extensions

In §§2.2.3, we found all M such that $M_{a1} = M_{1a} = \delta_{1,a}$. Therefore, for this subsection, we will assume $M_{a1} \neq 0$ or $M_{1a} \neq 0 \implies a = 1$ and $a = J1$. We begin with some simple calculations that will give us important information about M.

By S-invariance,

$$M_{J^i1,J^j1} = \sum_{a,b=1}^{n-1} S_{J^i1,a} M_{ab} S_{b,J^j1} = \sum_{a,b=1}^{n-1} (-1)^{(a+1)i} S_{1a} M_{ab} (-1)^{(b+1)j} S_{1b},$$

so applying the Triangle Inequality, we get

$$|M_{J^i1,J^j1}| \leq \sum_{a,b=1}^{n-1} |(-1)^{(a+1)i}||S_{1a}||M_{ab}||(-1)^{(b+1)j}||S_{1b}|,$$

so $M_{J^i1,J^j1} \leq \sum_{a,b=1}^{n-1} S_{1a} M_{ab} S_{1b} = M_{11} = 1$. Again, using the fact that $M_{ab} \in \{0, 1, 2, \dots\}$, this implies that M_{J^i1,J^j1} must be 0 or 1.

Suppose $M_{J^i1,J^j1} = 1$. Then $\sum_{a,b=1}^{n-1} (-1)^{(a+1)i} S_{1a} M_{ab} (-1)^{(b+1)j} S_{1b} = 1 = M_{11} = \sum_{a,b=1}^{n-1} S_{1a} M_{ab} S_{1b}$, so $(-1)^{(a+1)i}(-1)^{(b+1)j} = 1$, (if $M_{ab} \neq 0$); ie, $(-1)^{(a+1)i+(b+1)j} = 1$. Therefore, $(a+1)i + (b+1)j$ is even, so we get the selection rule:

$$(a+1)i \equiv (b+1)j \pmod 2 \text{ whenever } M_{ab} \neq 0. \tag{2.33}$$

Applying a similar calculation to any $a, b \in P_{++}$, we have

$$\begin{aligned} M_{J^ia,J^jb} &= \sum_{k,l=1}^{n-1} S_{J^ia,k} M_{kl} S_{l,J^jb} \\ &= \sum_{k,l=1}^{n-1} (-1)^{(k+1)i} S_{ak} M_{kl} (-1)^{(l+1)j} S_{lb} \\ &= \sum_{k,l=1}^{n-1} (-1)^{(k+1)i+(l+1)j} S_{ak} M_{kl} S_{lb}. \end{aligned}$$

But whenever $M_{kl} \neq 0$, $(k+1)i + (l+1)j \equiv 0 \pmod 2$ by the selection rule. Therefore, the above sum is just $\sum_{k,l=1}^{n-1} S_{ak} M_{kl} S_{lb} = M_{ab}$, so

$$M_{J^ia,J^jb} = M_{ab} \; \forall a, b \in P_{++}. \tag{2.34}$$

Equation(2.34) is the analogue of Lemma 4.3(c) in Chapter 4.

When n is even, there is an $a \in P_{++}$ such that $Ja = a$, namely $a = n/2$. We call such an a a *fixed point* of J. The following lemma determines all M_{ab} with $a, b \neq n/2$.

Lemma 2.2. *Let M be a modular invariant, and suppose $M_{a1} \neq 0$ only for $a = 1$ and $a = J1$, and similarly for M_{1a}, ie, the first row and column of M are all zeroes except for $M_{J^i 1 J^j 1} = 1$. Then the ath row (or column) of M will be identically 0 iff a is even. Moreover, let a, $b \in P_{++}$, both different from $n/2$, and suppose $M_{ab} \neq 0$. Then,*

$$M_{ac} = \begin{cases} 1, & \text{if } c = b \text{ or } c = Jb \\ 0, & \text{otherwise} \end{cases},$$

and a similar formula holds for M_{cb}.

Proof. The proof is similar to the proof of Lemma 2.1. We know that $M_{a1} = M_{a,J1} = M_{1a} = 1$, and by the selection rule (2.33), $M_{J1,J1} = M_{11} = 1$ also. All other entries in the first and last rows and columns of M are zeroes, so M looks like

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 1 \\ 0 & * & \dots & * & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & * & \dots & * & 0 \\ 1 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

We first need to show that the even rows and columns of M are identically 0. Suppose $M_{ab} \neq 0$. Then $M_{Ja,b} \neq 0$ either, because $M_{ab} = M_{Ja,b}$ by the selection rule (2.33). Letting $i = 1$ and $j = 0$, we get the congruence $a + 1 \equiv 0 \pmod 2$, or $a \equiv -1 \equiv 1 \pmod 2$, which means a must be odd. But if a is odd, then $(a+1)i \equiv (b+1)j$ implies $0 \equiv (b+1)j \pmod 2$. But this must be true whether $j = 0$ or 1, so $b + 1 \equiv 0 \pmod 2$, or $b \equiv 1 \pmod 2$, so b is odd as well. Therefore, if $M_{ab} \neq 0$, then both a and b are odd, so the even rows and columns must be identically 0.

For the second part of the lemma, write

$$M \sim \oplus_{i=1}^{n} B_i = \begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_n \end{pmatrix},$$

where the B_i's are the indecomposable submatrices of M as in equation (4.13), and all other enties of M are 0. Put $B_1 = B(1,2)$ (see (4.17)); ie, B_1 is the block containing $M_{11}, M_{J1,1}, M_{1,J1}$, and $M_{J1,J1}$. We will show that each non-trivial B_i that does not involve $n/2$ is of the form $B(1,2)$. Let B be some $B_i \neq (0)$ which does not contain any entries of M involving $n/2$, and write

$$B = \begin{pmatrix} x_{11} & \cdots & x_{1m} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mm} \end{pmatrix}.$$

Since the even rows and columns of M are identically 0, the M_{ab}'s contained in B must have both a and b odd. Also, each odd row of M has at least one nonzero entry, and since $M_{J^i a J^j b} = M_{ab}$ for all $i, j = 0, 1$, each row of B must have at least two nonzero entries (otherwise, we would get a block of zeros inside B, and so B would not be indecomposable). Now consider $N := M^T M$. Then the ith block of $M^T M$ not involving $n/2$ is

$$B^T B = \begin{pmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & & \vdots \\ y_{m1} & \cdots & y_{mm} \end{pmatrix},$$

where $y_{ab} = \sum_{c=1}^{m} x_{ca} x_{cb}$. In particular, the diagonal entries of $B^T B$ are given by

$$y_{aa} = \sum_{c=1}^{m} x_{ca}^2. \tag{2.35}$$

But we know that there at least two nonzero entries in the cth row of B; ie, $x_{ca} > 0$ for at least two choices of c. Therefore, $y_{aa} \geq 2$.

As in the proof of Lemma 2.1, $N_{aa}^L \geq (N_{aa})^L$, so let N_L be the matrix defined by $(N_L)_{ab} = (N_{ab})^L$. Then

$$N_L = \begin{pmatrix} 2^L & 2^L & & & & \\ 2^L & 2^L & & & & \\ & & y_{11}^L & \cdots & y_{1m}^L & \\ & & \vdots & \ddots & \vdots & \\ & & y_{m1}^L & \cdots & y_{mm}^L & \\ & & & & & \ddots \end{pmatrix}.$$

Again, as in the proof of Lemma 2.1, the entries of $\frac{N_L}{2L}$ must be bounded above. But this means $\frac{y^L_{aa}}{2L} \leq 1$, so $y_{aa} \in \{0,1,2\}$. But $y_{aa} \geq 2$. Now going back to (2.35), we see that we must have $x_{ca} = 1$ for $c \in \{c', Jc'\}$ for some c', and $x_{ca} = 0$ for all other c. Therefore, $B = B(1,2)$. □

Now suppose that we have the hypothesis of Lemma 2.2; ie, $M_{J^i1,J^j1} \neq 0$ for all $i,j = 0,1$. By (2.20), $1^2 \equiv (J1)^2 \pmod{4n}$, ie $1 \equiv (n-1)^2 \pmod{4n}$. If $n/2$ is even, then $n/2 = 2k$, for some $k \in \mathbb{Z}$, so $n = 4k$. But then $1 \equiv (4k-1)^2 = 16k^2 - 8k + 1 = 4(4k)k - 2(4k) + 1 = 4nk - 2n + 1 \equiv -2n + 1 \pmod{4n}$. This implies that $0 \equiv -2n \equiv 2n \pmod{4n}$, which cannot be true. So $n/2$ must be odd. The first such n is $n = 6$. Then, by Lemma 2.2,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so the only unknown entry is M_{33}.

Evaluating MS at $(1,3)$ gives:

$$\begin{aligned} (MS)_{13} &= M_{11}S_{13} + M_{12}S_{23} + M_{13}S_{33} + M_{14}S_{43} + M_{15}S_{53} \\ &= S_{13} + S_{53}, \end{aligned}$$

and $(SM)_{13} = S_{13}M_{33}$ so, by S-invariance, $S_{13} + S_{53} = S_{13}M_{33}$, which implies $S_{53} = S_{13}(M_{33} - 1)$, and so $\sin(\frac{15\pi}{6}) = \sin(\frac{3\pi}{6})(M_{33} - 1)$. Solving this for M_{33}, we see that $M_{33} = 2$. Therefore, we have M if $n = 6$, so we now look at $n \geq 10$ ($n = 8$ gives us $n/2$ even). We know that the even rows and columns are identically 0, and there are two 1's in each odd row and column, except we do not know what happens in the $\frac{n}{2}$th position. Consider the third row and column of M. Just as we assumed $M_{22} = 1$ above, we will assume $M_{33} = M_{J^i3,J^j3} = 1$ here. Otherwise, since there is a 1 in the third and $J3$rd rows and columns, we can permute the rows and columns of M so that this is the case and replace M with this matrix.

Now we look at what happens at $M_{3,\frac{n}{2}}$. If $M_{3,\frac{n}{2}} \neq 0$, then $M_{3,a} = 0$ unless $a = \frac{n}{2}$ or $J\frac{n}{2} = \frac{n}{2}$, and $M_{33} = 1 \neq 0$, so $M_{a,3} = 0$ unless $a = 3$ or $a = J3$. Evaluating $MS = SM$ at $(3,1)$, we get $M_{3,\frac{n}{2}}S_{\frac{n}{2},3} = S_{33} + S_{3,n-3}$. Therefore we have $M_{3,\frac{n}{2}}\sin(\pi\frac{3n}{2n}) = \sin(\frac{9\pi}{n})+\sin(\frac{(3n-9)\pi}{n})$, which implies $\sin(\frac{3\pi}{2})M_{3,\frac{n}{2}} = \sin(\frac{9\pi}{n})+\sin(\frac{9\pi}{n}) = 2\sin(\frac{9\pi}{n})$, so $M_{3,\frac{n}{2}} = 2\sin(-\frac{9\pi}{n}) = 2\sin(-\frac{9\pi}{n}+\frac{12\pi}{n}) = 2\sin(\frac{3\pi}{n})$. We need $2\sin(\frac{3\pi}{n})$ to be a positive integer, so $\sin(\frac{3\pi}{n}) = \frac{k}{2}$, for some $k \in \mathbb{Z}_+$. But because we need k to be positive, k must equal 1. So $\sin(\frac{3\pi}{n}) = \frac{1}{2}$, which implies that $\frac{3\pi}{n} = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. $\frac{3}{n} = \frac{1}{6}$ implies $n = 18$ (which is an exceptional level), and $\frac{3}{n} = \frac{5}{6}$ implies $n = \frac{18}{5} \notin \mathbb{Z}_{\geq 0}$. Therefore, $n = 18$ is the only possibility for $M_{3,\frac{n}{2}} \neq 0$.

Suppose that $M_{3,\frac{n}{2}} = M_{\frac{n}{2}} = 0$. By Lemma 2.2, there is one 1 in the third row to the left of the $\frac{n}{2}$th column, say at M_{3m}. Then also $M_{3,Jm} = M_{J3,m} = M_{J3,Jm} = 1$. Evaluating $MS = SM$ at $(3,1)$ gives $2\sin(\pi\frac{m}{n}) = 2\sin(\pi\frac{3}{n})$, so $m = 3$ or $J3$. But $m < \frac{n}{2}$, so $m = 3$.

Let a be any odd element of P_{++} such that $a \neq n/2$, and suppose $M_{\frac{n}{2},a} \neq 0$. Then evaluating $MS = SM$ at $(1,a)$ and $(3,a)$ gives us

$$2\sin\left(\pi\frac{a}{n}\right) = M_{\frac{n}{2},a}, \text{ and } 2\sin\left(\pi\frac{3a}{n}\right) = -M_{\frac{n}{2},a}$$

respectively. This implies that $\sin(\pi\frac{a}{n}) = -\sin(\pi\frac{3a}{n}) = \sin(-\pi\frac{3a}{n})$, which cannot happen, so $M_{\frac{n}{2},a} = 0$. Then, by Lemma 2.2, we have a unique $b < \frac{n}{2}$ such that $M_{ba} \neq 0$ (because a is odd, we get one 1 above the $\frac{n}{2}$th row). Evaluating $SM = MS$ at (b,a), we get

$$\sin\left(\pi\frac{a^2}{n}\right) = \sin\left(\pi\frac{b^2}{n}\right),$$

so $a^2 = b^2$, and so $a = b$. Therefore we have a matrix whose diagonal odd entries are 1, and the corresponding M_{J^ia,J^ja} entries are 1. As in the case $n = 6$, we can evaluate $SM = MS$ at $(1,\frac{n}{2})$ to obtain $M_{\frac{n}{2},\frac{n}{2}} = 2$. Therefore,

$$M = |\chi_1 + \chi_{J1}|^2 + |\chi_3 + \chi_{J3}|^2 + \cdots + 2|\chi_{\frac{n}{2}}|^2 = \mathcal{D}_{\frac{n}{2}+1},$$

whenever $\frac{n}{2}$ is odd.

This completes the proof of Theorem 1 when $n \neq 12, 18, 30$. These three n – the exceptional heights – must each be evaluated separately, and they will give us the remaining modular invariants of Theorem 1.

2.2.6 The Exceptionals

By the proof of Proposition 2.1, we already have some information about M at the exceptional heights. S-invariance and the Galois selection rule were used here to complete the A_1 classification: $n = 12$ corresponds to $\mathcal{E}_6$; $n = 18$ corresponds to $\mathcal{E}_7$, and $n = 30$ gives us the exceptional $\mathcal{E}_8$.

Chapter 3

The Automorphism Invariants of $A_2 \oplus A_2$ at Height (p', p)

In this chapter, we classify the automorphism invariants of $A_{2,p'} \oplus A_{2,p}$, which is the first step of the classification of all modular invariants. We call M an *automorphism invariant* if M is a modular invariant and $M_{\lambda\mu,\kappa\nu} = \delta_{\kappa\nu,\sigma(\lambda\mu)}$, for some permutation σ of $P_{++}^{p',p}$. In other words, M has only one nonzero entry in each row or column, and that entry is 1. The condition that $M_{\rho\rho,\rho\rho} = 1$ ensures $\sigma(\rho\rho) = \rho\rho$ for any permutation σ that defines an automorphism invariant.

Remark: We will often refer to the permutation σ that defines an automorphism invariant as an automorphism invariant.

3.1 Preliminary Calculations

Let σ be a permutation of $P_{++}^{p',p}$ such that σ defines an automorphism invariant. Then

$$\begin{aligned}
(MS^{(p',p)})_{\lambda\mu,\kappa\nu} &= (S^{(p',p)}M)_{\lambda\mu,\kappa\nu} \\
\iff \sum_{\alpha\beta} M_{\lambda\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\kappa\nu} &= \sum_{\alpha\beta} S^{(p',p)}_{\lambda\mu,\alpha\beta} M_{\alpha\beta,\kappa\nu} \\
\iff M_{\lambda\mu,\sigma(\lambda\mu)} S^{(p',p)}_{\sigma(\lambda\mu),\kappa\nu} &= S^{(p',p)}_{\lambda\mu,\sigma^{-1}(\kappa\nu)} M_{\sigma^{-1}(\kappa\nu),\kappa\nu} \\
\iff S^{(p',p)}_{\sigma(\lambda\mu),(\kappa\nu)} &= S^{(p',p)}_{\lambda\mu,\sigma^{-1}(\kappa\nu)},
\end{aligned}$$

and a similar calculation holds for $T^{(p',p)}$. Thus a permutation σ of $P_{++}^{p',p}$ is an automorphism invariant iff σ satisfies the conditions

(3.1a) $$S^{(p',p)}_{\sigma(\lambda\mu),\sigma(\kappa\nu)} = S^{(p',p)}_{\lambda\mu,\kappa\nu};$$

(3.1b) $$T^{(p',p)}_{\sigma(\lambda\mu),\sigma(\lambda\mu)} = T^{(p',p)}_{\lambda\mu,\lambda\mu}.$$

Equations (3.1) tell us that inverses and compositions of automorphism invariants are also automorphism invariants.

The charge conjugations are automorphism invariants. Let $u = (u_1, u_2)$, where $u_1, u_2 \in \{0, 1\}$, and let $C^u = C_{p'}^{u_1} C_p^{u_2}$ (see (2.8)). Then each C^u is a charge conjugation (and hence a permutation) and has order 2. By (2.10d),

$$S^{(p',p)}_{C^u(\lambda\mu),C^u(\kappa\nu)} = S^{(p')}_{C^{u_1}\lambda,C^{u_1}\kappa} S^{(p)}_{C^{u_2}\mu,C^{u_2}\nu} = S^{(p')}_{\lambda(C^{u_1})^2\kappa} S^{(p)}_{\mu(C^{u_2})^2\nu} = S^{(p',p)}_{\lambda\mu,\kappa\nu},$$

and by (2.10c),

$$T^{(p',p)}_{C^u(\lambda\mu),C^u(\lambda\mu)} = T^{(p')}_{C^{u_1}\lambda,C^{u_1}\lambda} T^{(p)}_{C^{u_2}\mu,C^{u_2}\mu} = T^{(p')}_{\lambda\lambda} T^{(p)}_{\mu\mu} = T^{(p',p)}_{\lambda\mu,\lambda\mu}.$$

We call the automorphism invariants defined by the charge conjugations ${}^C I$, I^C, and ${}^C I^C$, corresponding to $u = (1,0), (0,1)$ and $(1,1)$ respectively. A useful fact about ${}^C I^C$ is that it commutes with any modular invariant M. This is because M commutes with $S^{(p',p)}$, and hence with $(S^{(p',p)})^2$. Then, for $u = (1,1)$, (2.10d) tells us that

$$\begin{aligned}
(S^{(p',p)})^2_{\lambda\mu,\kappa\nu} &= \sum_{\alpha\beta} S^{(p',p)}_{\lambda\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\kappa\nu} \\
&= \sum_{\alpha\beta} S^{(p',p)}_{\lambda\mu,\alpha\beta} S^{(p',p)*}_{\alpha\beta,C^u(\kappa\nu)} \\
&= {}^C I^C \sum_{\alpha\beta} S^{(p',p)}_{\lambda\mu,\alpha\beta} S^{(p',p)*}_{\alpha\beta,\kappa\nu} \\
&= {}^C I^C \delta_{\kappa\nu,\lambda\mu} \\
&= \delta_{\kappa\nu,C^{(1,1)}(\lambda\mu)},
\end{aligned}$$

by unitarity of $S^{(p',p)}$, so we have shown that $(S^{(p',p)})^2 = {}^C I^C$.

We can define another automorphism invariant out of the simple currents, as in [8]. Let $a := (a_{11}, a_{21}, a_{12}, a_{22})$ be a quadruple of integers such that

(3.2a) $$2a_{ii} + ka_{i1}^2 + la_{i2}^2 \equiv 3ka_{i1} + 3la_{i2} \pmod 6,$$

(3.2b) $$a_{ij} + a_{ji} + ka_{i1}a_{j1} + la_{i2}a_{j2} \equiv 0 \pmod 3,$$

for all $i, j \in \{1,2\}$, and where $k = p' - 3, l = p - 3$. Define

(3.3) $$\sigma_a : P_{++}^{p',p} \longrightarrow P_{++}^{p',p},\ (\lambda,\mu) \mapsto (A_{p'}^{a_{11}t(\lambda)+a_{21}t(\mu)}\lambda, A_p^{a_{12}t(\lambda)+a_{22}t(\mu)}\mu).$$

We can take the integers $\{a_{ij}\}$ such that $a_{ij} \in \{0, \pm 1\}$ since the simple currents have order 3. Taking the composition $\sigma_b \circ \sigma_a$ (where a and b satisfy (3.2)), we see that $\sigma_b \circ \sigma_a$ has the form σ_c with $c_{ij} = a_{ij} + b_{ij} + ka_{i1}b_{1j} + la_{i2}b_{2j}$ as in (3.3). If we let $b_{ij} = a_{ji}$, we then get $c_{ij} = a_{ij} + a_{ji} + ka_{i1}a_{j1} + la_{i2}a_{j2}$, which is 0 (mod 3) by (3.2b). Therefore σ_a is a permutation, and $\sigma_a^{-1} = \sigma_{a'}$, where $a' = (a_{11}, a_{12}, a_{21}, a_{22})$. Now using (3.2), straightforward calculations show that σ_a satisfies (3.1), and so is an automorphism invariant. We will denote the automorphism invariant defined by σ_a as I^A.

We will use the following lemma in the next section to find out what happens at the small weights $(\rho,(1,2)), ((1,2),\rho), (\rho,(2,1))$ and $((2,1),\rho)$ under σ. The information about the pairs $(2,2)$, $(1,4)$ and $(4,1)$ will be needed in Chapter 5.

Lemma 3.1. *Let $\lambda \in P_{++}(A_2, n)$, for $n \neq 12$. Then (a) $S^{(n)}_{\lambda\rho} = S^{(n)}_{(1,2),\rho}$ or $S^{(n)}_{\lambda\rho} = S^{(n)}_{(2,1),\rho} \implies \lambda \in \mathcal{O}(1,2)$, (b) For $(a,b) \in \{(2,2),(1,4),(4,1)\}$, $S^{(n)}_{\lambda\rho} = S^{(n)}_{(a,b),\rho} \implies \lambda \in \mathcal{O}(a,b)$.*

Proof. (a) was done in [9]. The proof of (b) will be along the same lines. Since $S^{(n)}_{C^aA^b\lambda,\rho} = S^{(n)}_{A^b\lambda,C^a\rho} = S^{(n)}_{A^b\lambda,\rho} = e^{\frac{2\pi i}{3}bt(\rho)}S^{(n)}_{\lambda\rho} = S^{(n)}_{\lambda\rho}$, we see that for any $\lambda' \in \mathcal{O}\lambda = \{(\lambda_1,\lambda_2),(\lambda_2,\lambda_1),(\lambda_1,n-\lambda_1-\lambda_2),(\lambda_2,n-\lambda_1-\lambda_2),(n-\lambda_1-\lambda_2,\lambda_1),(n-\lambda_1-\lambda_2,\lambda_2)\}$, $S^{(n)}_{\lambda'\rho} = S^{(n)}_{\lambda\rho}$. Therefore, choosing the proper representative of $\mathcal{O}\lambda$, we may suppose without loss of generality, that $\lambda_1 \leq \lambda_2 \leq n - \lambda_1 - \lambda_2$. We cannot have more than two of λ_1, λ_2, or $n - \lambda_1 - \lambda_2$ greater than or equal to $n/2$, because then at least one of the weights $\lambda' \in \mathcal{O}\lambda$ would have $\lambda_1' + \lambda_2' \geq n$. Therefore, $\lambda_1 \leq \lambda_2 < n/2$. Define $s(w) := \sin\left(\frac{w\pi}{n}\right)$. Then $S^{(n)}_{\lambda\rho} = S^{(n)}_{(1,4),\rho}$ gives us

(3.4) $$s(x)s(y)s(z) = s(1)s(4)s(5),$$

where $x = \lambda_1$, $y = \lambda_2$, and $z = \lambda_1 + \lambda_2$ or $n - \lambda_1 - \lambda_2$, by (2.11) and the fact that $s(n-z) = s(z)$ for $0 \leq z \leq n$. Now, one of $\lambda_1+\lambda_2$ and $n-\lambda_1-\lambda_2$ is less than or equal to $n/2$, so choosing that one to be z, we may assume that $0 < x \leq y \leq z \leq n/2$, and either $z = x + y$, or $x + y > n/2$. Choosing $n \geq 10$ for now (we will check the cases $n = 4, 5, 6, 7, 8, 9$ later), we then have all arguments $\frac{x\pi}{n}, \frac{y\pi}{n}, \frac{z\pi}{n}, \frac{\pi}{n}, \frac{4\pi}{n}$ and $\frac{5\pi}{n}$ in the first quadrant of the unit circle, and so $s(w) \geq s(w') \iff w \geq w'$ with equality iff $w = w'$.

A direct comparison of (x, y, z) with $(1, 4, 5)$ immediately gives us the left-hand side of (3.4) greater than or less than the right-hand side for all but the following triples: (i)(1, 4, 5), (ii)(2, 2, z), (iii)(2, 3, z), (iv)(2, 4, 4), (v)(3, 3, z), or (vi)(4, 4, 4). Notice that (ii) has $z = 4$ for $n \geq 8$, (iii) has $z = 5$ for $n \geq 10$, and (v) has $z = 6$ unless $n = 11$ (in which case $z = 5$), or $n = 10$ (in which case $z = 4$). We would like to eliminate all of the above cases except for (i). For (ii), (iv), (v) and (vi), we will show that

$$s(a)s(b) < s(a')s(b'), \tag{3.5}$$

whenever $a + b = a' + b'$ and $a < a' \leq b' < b$.

Consider the function $f(t)$ defined by $t \mapsto s(a+t)s(b-t)$. Then $f'(t) = \frac{\pi}{n} s(b - a - 2t) > 0$ iff $x < (b-a)/2 < n/2 + x$. But this is true whenever $t \geq 0$, since then $0 < b - a < b + a < n$, so $0 < (b-a)/2 < n/2 \leq n/2 + t$. Therefore, $f'(t) > 0$ when $0 \leq t < (b-a)/2$, so f is increasing there. Notice that we have $0 < a' - a = b - b' < (b-a)/2$, so $f(0) < f(a'-a) = f(b-b')$. Evaluating this immediately gives us (3.5).

Now comparing $s(2)s(3)s(5)$, $s(2)s(4)s(4)$, and $s(3)s(3)s(4)$ with the right-hand side of (3.4), we see that we cannot have (ii) with $z = 4$ (unless $n = 12$), (iii) with $z = 5$, or (v) with $z = 4$, because in each case the left-hand side is greater than the right-hand side. This also eliminates (v) when $z = 6$ and (vi), because now comparing $s(3)s(3)s(6)$ and $s(4)s(4)s(4)$ with $s(3)s(3)s(5)$, we see that these triples will also give us the left-hand side of (3.4) greater than the right-hand side.

At $n = 10$ and 11, $(2, 2, z)$ does not solve (3.4); however, it does at $n = 12$. At $n = 4$ and 5, $(1, 4) \notin P^n_{++}(A_2)$, so there is nothing to check, and at the heights $n = 6, 7, 8$ and 9, we find that $S^{(n)}_{\lambda\rho} = S^{(n)}_{(1,4),\rho} \implies \lambda \in \mathcal{O}\rho$. The above argument also applies to $(a, b) = (4, 1)$ as $S^{(n)}_{(4,1),\rho} = S^{(n)}_{C(1,4),\rho} = S^{(n)}_{(1,4),C\rho} = S^{(n)}_{(1,4),\rho}$ by (2.10b).

For $(a, b) = (2, 2)$, we need to solve the equation

$$s(x)s(y)s(z) = s^2(2)s(4). \tag{3.6}$$

In this case we let $n \geq 8$ so that all arguments are in the first quadrant, and we can assume $x \leq y \leq z \leq n/2$, for $z = x + y$ or $z = n - x - y$. We get the following choices for (x, y, z) : (i)(2,2,4), (ii)$(1, y, z)$, (iii)(2,3,3), and (iv)(3,3,3).

(iii) and (iv) imply $n = 8$ and $n = 9$ respectively, and evaluating (3.6) at both of these, we find that $S^{(n)}_{\lambda\rho} \neq S^{(n)}_{(2,2),\rho}$. (ii) is the most difficult case. Suppose $n > 12$. Then from the argument for (3.4), we know that $s(1)s(5) > s^2(2)$. Choosing any $(1, y, z)$ with $4 < y \leq z \leq n/2$ then gives us $s(1)s(y)s(z) > s(1)s(4)s(5) > s^2(2)s(4)$, so for $n > 12$ and $y \geq 4$, $(1, y, z)$ is not a solution to (3.6). For $n = 8, 9, 10, 11$, there are no solutions of the form $(1, y, z)$, but as in (ii) above, (3.6) has the solution $(1, 4, 5)$ if $n = 12$. If $y < 4$, then the only possible $(1, y, z)$ has $y = 3$, and since $n \geq 8$, $z = 4$. But $s(1)s(3)s(4) = s^2(2)s(4)$ iff $\cos(\frac{2\pi}{n}) = 1$, which cannot happen since $n \geq 4$. Therefore, for $n \geq 8$, the only solution to (3.6) is (i)(2,2,4), so $\lambda \in \mathcal{O}(2, 2)$.

It remains to check the heights $n = 4, 5, 6$ and 7. If $n = 4$, $(2, 2) \notin P^n_{++}(A_2)$, and if $n = 5, 6$ or 7, we have $\lambda \in \mathcal{O}(2, 2)$ whenever $S^{(n)}_{\lambda\rho} = S^{(n)}_{(2,2),\rho}$. $\square$

By the Weyl-Kac character formula [18], we can write ratios of the $A_{2,n}$ S-matrix in terms of the Weyl characters as

$$\frac{S^{(n)}_{\lambda\kappa}}{S^{(n)}_{\rho\kappa}} = ch_{\lambda-\rho}(-2\pi i \frac{\kappa}{n}), \tag{3.7}$$

where ch_β is the Weyl character of the irreducible A_2-module $L(\beta)$ with highest weight β. The "$\lambda - \rho$" is used instead of λ because the Weyl characters are written in terms of the unshifted highest weights in P^n_+. Formula (3.7) will be used frequently throughout this thesis. When $\kappa = \rho$, we refer to the S-ratio in (3.7) as the *q-dimension* (or *quantum dimension*) of λ; we write

$$Q^{(n)}(\lambda) = \frac{S^{(n)}_{\lambda\rho}}{S^{(n)}_{\rho\rho}}. \tag{3.8}$$

The q-dimensions for $A_2 \oplus A_2$ are

$$Q^{(p',p)}(\lambda\mu) = Q^{(p')}(\lambda)Q^{(p)}(\mu) = S^{(p',p)}_{\lambda\mu,\rho\rho}/S^{(p',p)}_{\rho\rho,\rho\rho}. \tag{3.9}$$

By (2.12), we see that $Q^{(p',p)}(\lambda\mu) \in \mathbb{R}\ \forall \lambda\mu \in P_{++}^{p',p}$, and $Q^{(p',p)}(\lambda\mu) \geq 1$ with equality iff $\lambda \in \mathcal{O}\rho$, $\mu \in \mathcal{O}\rho$. We also have $Q^{(p',p)}(\lambda'\mu') = Q^{(p',p)}(\lambda\mu)$ whenever $\lambda' \in \mathcal{O}\lambda$ and $\mu' \in \mathcal{O}\mu$. We have defined q-dimensions in terms of weights $\lambda = (\lambda_1, \lambda_2)$ where λ_1 and λ_2 are positive integers; however, in [7], q-dimensions were extended to have domain $\{(\lambda_1, \lambda_2) : \lambda_1, \lambda_2 \in \mathbb{R}\}$. The proof of Lemma 3.2 below treats q-dimensions as functions of real vectors.

The following lemma will be used in the next section to find out what happens to the small weights $(\rho,(1,2)), ((1,2),\rho), (\rho,(2,1))$ and $((2,1),\rho)$ under an automorphism invariant σ. The information about the pairs $(2,2), (1,4)$ and $(4,1)$ will be needed in Chapter 5.

Lemma 3.2. *(a) The smallest q-dimension $Q^{(n)}(\lambda)$ such that $\lambda \neq \rho$ is $Q^{(n)}(1,2)$, and $Q^{(n)}(\rho) < Q^{(n)}(1,2)$;*
(b) Let $n \geq 12$. The smallest q-dimensions $Q^{(n)}(\lambda)$ such that $\lambda \neq \rho$ and $t(\lambda) \equiv 0 \pmod 3$, are $Q^{(n)}(2,2)$ and $Q^{(n)}(1,4)$, and $Q^{(n)}(\rho) < Q^{(n)}(2,2) \leq Q^{(n)}(1,4)$ with equality iff $n = 12$.

Proof. (a) was done for all $A_r^{(1)}$ in [8]. We will use the same idea to prove (b). Let $\lambda = (\lambda_1, \lambda_2)$, and suppose that both $\lambda_1, \lambda_2 \geq 4$. We will first show that $Q^{(n)}(3,3) < Q^{(n)}(\lambda_1, \lambda_2)$. Define $\mu(t) := (\lambda_1, \lambda_2) + (-t, t) = (\lambda_1 - t, \lambda_2 + t)$, where $t \in [3 - \lambda_2, \lambda_1 - 3]$. Notice that since $\lambda_1, \lambda_2 \geq 4$, $0 \in [3 - \lambda_2, \lambda_1 - 3]$, and $\mu(0) = (\lambda_1, \lambda_2)$. Define a function $f(t) := Q^{(n)}(\mu(t))$. Our strategy will be to show that f has no minimum value on $(3 - \lambda_2, \lambda_1 - 3)$, and so will take its minimum at one of the endpoints of the interval (a minimum must exist because f is differentiable everywhere and is non-constant). Either of the endpoints gives us a weight where one of the Dynkin labels is 3, so $f(\text{endpoint}) < f(0)$ tells us $Q^{(n)}(3,b) < Q^{(n)}(\lambda)$, for some b. By the definition of $Q^{(n)}$, we have

$$
\begin{aligned}
f(t) &= \frac{8\sin\left(\frac{(\lambda_1 - t)\pi}{n}\right)\sin\left(\frac{(\lambda_2 + t)\pi}{n}\right)\sin\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)}{\sqrt{3}n\sin^2\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)} \\
&= \frac{4\sin\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)}{\sqrt{3}n\sin^2\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)}\left(\cos\left(\frac{(\lambda_1 - \lambda_2 - 2t)\pi}{n}\right) - \cos\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)\right), \qquad (3.10)
\end{aligned}
$$

$$f'(t) = \frac{8\pi \sin\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)}{\sqrt{3}n^2 \sin^2\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right)} \sin\left(\frac{(\lambda_1+\lambda_2-2t)\pi}{n}\right), \tag{3.11}$$

$$f''(t) = -\frac{16\pi^2 \sin\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)}{\sqrt{3}n^3 \sin^2\left(\frac{\pi}{n}\right) \sin\left(\frac{2\pi}{n}\right)} \cos\left(\frac{(\lambda_1+\lambda_2-2t)\pi}{n}\right). \tag{3.12}$$

Suppose that f has a minimum at t_0, for some $t_o \in (3-\lambda_2, \lambda_1-3)$. Then, since f is differentiable everywhere, we must have $f'(t_0) = 0$ and $f''(t_0) > 0$. But putting t_0 into (3.11) and (3.12), we see that these two conditions can be satisfied iff $\cos\left(\frac{(\lambda_1+\lambda_2-2t)\pi}{n}\right) = -1$. Putting this into (3.10), we get

$$\begin{aligned} f(t_0) &= Q^{(n)}(\mu(t_0)) \\ &= \frac{4\sin\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)}{\sqrt{3}n \sin^2\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)}\left(-1-\cos\left(\frac{(\lambda_1+\lambda_2)\pi}{n}\right)\right) \\ &< 0, \end{aligned}$$

since $\lambda_1+\lambda_2 < n$. But this is a contradiction, because $S^{(n)}_{\kappa\rho} \geq S^{(n)}_{\rho\rho} > 0$ for all $\kappa \in P^n_{++}(A_2)$, so we see by (3.10) that $f(t)$ must be positive for our choice of λ. Therefore, the minimum value of f occurs at $3-\lambda_2$ or λ_1-3, either of which gives us a weight with one Dynkin label equalling 3. Since $Q^{(n)}$ is constant along simple current orbits; ie, $Q^{(n)}(\kappa') = Q^{(n)}(\kappa)$ for all $\kappa' \in \mathcal{O}\kappa$, we can assume the first Dynkin label is 3. Therefore, for any $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1, \lambda_2 \geq 4$, there exists a b such that $Q^{(n)}(3,b) < Q^{(n)}(\lambda)$.

A similar argument with $f(t) = Q^{(n)}(3,t)$ shows that among those $(3,b)$ with $b \geq 3$, the minimum value of $Q^{(n)}(3,b)$ occurs at $b=3$ (or $b=n-6$, but $(3,n-6) \in \mathcal{O}(3,3)$, so $Q^{(n)}$ has the same value either way). Therefore $Q^{(n)}(3,3) \leq Q^{(n)}(\lambda)$ for all λ with $\lambda_1, \lambda_2 \geq 3$. This tells us that the weights κ with smallest q-dimension must have one of their Dynkin labels equal to 1 or 2. Among those κ of the form $(1,b)$, $b=4$ gives the smallest $Q^{(n)}(1,b)$ such that $t(\kappa) \equiv 0 \pmod 3$, and since $Q^{(n)}(1,b) = Q^{(n)}(1,n-b-1)$, we can define $Q^{(n)}$ on the interval to be $[4, n-5]$ without losing any information. Setting $f'(t_0) = 0$ and $f''(t_0) > 0$ gives us a contradiction as before, so $Q^{(n)}(1,b)$ is smallest at one of $b=4$ or $b=n-5$. But $Q^{(n)}(1,4) = Q^{(n)}(1,n-5)$, so we can take $b=4$, and this will also ensure

$t(1,b) \equiv 0 \pmod 3$. In this way, we also get $(2,2)$ as the smallest weight of the form $(2,b)$ with $t(2,b) \equiv 0 \pmod 3$. By the proof of Lemma 3.1, we know that $Q^{(n)}(2,2) < Q^{(n)}(1,4)$ for $n > 12$, and $Q^{(12)}(2,2) = Q^{(12)}(1,4)$ We will now show that

$$Q^{(n)}(\rho) < Q^{(n)}(2,2) < Q^{(n)}(1,4) < Q^{(n)}(2,3) < Q^{(n)}(3,3). \tag{3.13}$$

This will be enough, because then $Q^{(n)}(1,4)$ is smaller than the q-dimension of all $t(\lambda) \equiv_3 0$ weights with one Dynkin label equal to 1 or 2, other than $\lambda = (2,2)$, and $Q^{(n)}(2,2)$ is smaller than the q-dimension of all $t(\lambda) \equiv_3 0$ weights with one Dynkin label equalling 1 or 2.

If $n \geq 12$, then $\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{5\pi}{n}\right) < \sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{6\pi}{n}\right)$, so this gives us $Q^{(n)}(2,3) < Q^{(n)}(3,3)$. To show that $Q^{(n)}(1,4) < Q^{(n)}(2,3)$, we need $\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{4\pi}{n}\right)\sin\left(\frac{5\pi}{n}\right) < \sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\sin\left(\frac{5\pi}{n}\right)$. But this holds iff $\cos\left(\frac{3\pi}{n}\right) - \cos\left(\frac{5\pi}{n}\right) < \cos\left(\frac{\pi}{n}\right) - \cos\left(\frac{5\pi}{n}\right)$, iff $\cos\left(\frac{3\pi}{n}\right) < \cos\left(\frac{\pi}{n}\right)$, which is true for all $n \geq 6$, so (3.13) holds. The fact that $Q^{(n)}(\rho)$ is the smallest q-dimension follows from (2.12).
□

3.2 The Automorphism Invariant Classification

The goal of this section is to show that all automorphism invariants are defined by compositions of charge conjugations and σ_a's as in the following theorem:

Theorem 3.1. *Let M be an automorphism invariant of $A_{2,p'} \oplus A_{2,p}$, where p' and p are coprime. Then M is one of*

$$\mathcal{A}_{p',p} = \sum_{\lambda\mu \in P_{++}^{p',p}} \chi_{\lambda\mu}^{(p',p)} \chi_{\lambda\mu}^{(p',p)*}, \tag{3.14a}$$

$$\mathcal{D}_{p',p}^{(1)} = \sum_{\lambda\mu \in P_{++}^{p',p}} \chi_{\lambda\mu}^{(p',p)} \chi_{\sigma_a(\lambda\mu)}^{(p',p)*}, \quad \text{where } \sigma_a \text{ is given in Table 3.1}, \tag{3.14b}$$

or their conjugations.

We begin by seeing how σ acts on the weights $(\rho, (1,2))$ and $((1,2), \rho)$, and then in Proposition 3.1, we will extend σ to all weights $\kappa\nu \in P_{++}^{p',p}$.

Claim 3.1. *Let σ be an automorphism invariant. Then*

$$\sigma(\rho,(1,2)) = (C_{p'}^a A_{p'}^w \rho, C_p^b A_p^x (1,2)),$$

and

$$\sigma((1,2),\rho) = (C_{p'}^c A_{p'}^y (1,2), C_p^d A_p^z \rho),$$

for some $a,b,c,d \in \{0,1\}$, $w,x,y,z \in \{0,1,2\}$.

Proof. By (3.1a), σ is a symmetry of $S^{(p',p)}$, and since $\sigma(\rho\rho) = \rho\rho$, we have

$$Q^{(p',p)}(\lambda,\mu) = \frac{S^{(p',p)}_{\lambda\mu,\rho\rho}}{S^{(p',p)}_{\rho\rho,\rho\rho}} = \frac{S^{(p',p)}_{\sigma(\lambda\mu),\rho\rho}}{S^{(p',p)}_{\rho\rho,\rho\rho}} = Q^{(p',p)}(\sigma(\lambda,\mu))$$

for all $\lambda\mu \in P_{++}^{p',p}$. Therefore, letting $(\rho',(1,2)') := \sigma(\rho,(1,2))$, we get

$$\frac{Q^{(p')}(\rho)}{Q^{(p')}(\rho')} = \frac{Q^{(p)}(1,2)'}{Q^{(p)}(1,2)} \tag{3.15}$$

Since $S^{(p')}_{\lambda\rho} \geq S^{(p')}_{\rho\rho} > 0$, $\forall \lambda \in P_{++}^{p'}(A_2)$, the left-hand side of (3.15) is less than or equal to 1, and so $Q^{(p)}(1,2)' \leq Q^{(p)}(1,2)$. By Lemma 3.2, we then have $Q^{(p)}(1,2)' = Q^{(p)}(\rho)$ or $Q^{(p)}(1,2)' = Q^{(p)}(1,2)$.

Suppose for a contradiction that $Q^{(p)}(1,2)' = Q^{(p)}(\rho)$. Then $S^{(p)}_{(1,2)',\rho} = S^{(p)}_{\rho\rho}$, and by (2.12), this is true iff $(1,2)' \in \mathcal{O}\rho = \{(1,1),(p-2,1),(1,p-2)\}$. But the decoupled norm condition (4.3) tells us this can happen only if $p \mid 4$, so since $p \geq 4$, $p = 4$ is the only possibility for $(1,2)' \in \mathcal{O}\rho$. Therefore, for $p \neq 4$, $Q^{(p')}(1,2)' = Q^{(p')}(1,2)$. Now, Lemma 3.1a gives us $(1,2)' \in \mathcal{O}(1,2)$. If $p = 4$, $\mathcal{O}\rho = \mathcal{O}(1,2)$, so in any case $(1,2)' \in \mathcal{O}(1,2)$. Going back to (3.15), we get $Q^{(p')}(\rho) = Q^{(p')}(\rho')$, and by (2.12), $\rho' \in \mathcal{O}\rho$.

Therefore,

$$\sigma(\rho,(1,2)) = (C_{p'}^a A_{p'}^w \rho, C_p^b A_p^x (1,2))$$

for some $a,b \in \{0,1\}, w,x \in \{0,1,2\}$, and by the same argument, we get that $\sigma((1,2),\rho) = (C_{p'}^c A_{p'}^y (1,2), C_p^d A_p^z \rho)$, for some $c,d \in \{0,1\}$ and $y,z \in \{0,1,2\}$. □

Since $C\rho = \rho$, we may assume that $a = c$ and $b = d$ in Claim 3.1, and now, letting $u = (c,b)$, so that $C^u = C_{p'}^c C_p^b$, we see that $C^u \circ \sigma(\rho,(1,2)) = (A_{p'}^w \rho, A_p^x (1,2))$ and $C^u \circ \sigma((1,2),\rho) = (A_{p'}^y (1,2), A_p^z \rho)$. Putting $\sigma' := C^u \circ \sigma$, σ' is an automorphism invariant that fixes $(\rho,(1,2))$ and $((1,2),\rho)$ up to simple current orbits.

Our next step is to define $\sigma'' := \sigma_a^{-1} \circ \sigma'$, for some σ_a as in (3.3) that will be found later, and show that σ'' acts as the identity on the weights $(\rho, (1,2))$ and $((1,2), \rho)$. Since σ_a and σ' are automorphism invariants, we will then have an automorphism invariant σ'' fixing the small weights, and the final step is to show that any such σ'' is in fact the identity on $P_{++}^{p',p}$.

Since $\sigma'(\rho, (1,2)) = (A_{p'}^w \rho, A_p^x(1,2))$ and $\sigma'((1,2), \rho) = (A_{p'}^y(1,2), A_p^z \rho)$, we can evaluate

$$\begin{aligned} S^{(p',p)}_{(\rho,(1,2)),(\rho,(1,2))} &= S^{(p',p)}_{\sigma'(\rho,(1,2)),\sigma'(\rho,(1,2))}, \\ S^{(p',p)}_{((1,2),\rho),((1,2),\rho)} &= S^{(p',p)}_{\sigma'((1,2),\rho),\sigma'((1,2),\rho)}, \\ S^{(p',p)}_{(\rho,(1,2)),((1,2),\rho)} &= S^{(p',p)}_{\sigma'(\rho,(1,2)),\sigma'((1,2),\rho)}, \end{aligned}$$

and using (2.10d), we get the following relations that (w, x, y, z) must satisfy.

$$\begin{aligned} kw^2 + lx^2 + x &\equiv 0 \pmod 3; \\ ky^2 + lz^2 + y &\equiv 0 \pmod 3; \\ kwy + lxz - w - z &\equiv 0 \pmod 3, \end{aligned} \tag{3.16}$$

where (k, l) is the level. All solutions to (3.16) are listed in Table 3.1.

Claim 3.2. *Suppose that $\sigma(\rho, (1,2)) = (A_{p'}^w \rho, A_p^x(1,2))$ and $\sigma = ((1,2), \rho) = (A_{p'}^y(1,2), A_p^z \rho)$ for some automorphism invariant σ. Then there is a quadruple $a = (a_{11}, a_{21}, a_{12}, a_{22})$ of integers satisfying (3.2) such that $\sigma(\rho, (1,2)) = \sigma_a(\rho, (1,2))$ and $\sigma((1,2), \rho) = \sigma_a((1,2), \rho)$, where σ_a is defined in (3.3).*

Proof. Let $a = (a_{11}, a_{21}, a_{12}, a_{22})$. Then

$$\sigma_a(\rho, (1,2)) = (A_{p'}^{a_{11}t(\rho)+a_{21}t(1,2)} \rho, A_p^{a_{12}t(\rho)+a_{22}t(1,2)}(1,2)) = (A_{p'}^{-a_{21}} \rho, A_p^{-a_{22}}(1,2)),$$

and $\sigma_a((1,2), \rho) = (A_{p'}^{-a_{11}}(1,2), A_p^{-a_{12}} \rho)$. Therefore, we can put $a = (-y, -w, -z, -x)$ to get $\sigma = \sigma_a$ at $(\rho, (1,2))$ and $((1,2), \rho)$. To show that this σ_a is an automorphism invariant, we must show that the a_{ij}'s satisfy (3.2), where $a_{11} = -y$, $a_{21} = -w$, $a_{12} = -z$, and $a_{22} = -x$. But (w, x, y, z) must satisfy Equations (3.16), which gives us the relations $ka_{21}^2 + la_{22}^2 - a_{22} \equiv 0_3$, $ka_{11}^2 + la_{12}^2 - a_{11} \equiv_3 0$ and $ka_{21}a_{11} + la_{22}a_{12} + a_{21} + a_{12} \equiv_3 0$, and these in turn give us (3.2). Therefore, $a = (-y, -w, -z, -x)$ is our quadruple and it gives an automorphism invariant. □

Earlier in this section, we defined $\sigma' = C^u \circ \sigma$, for any automorphism invariant σ. Now define $\sigma'' := \sigma_a^{-1} \circ \sigma'$. Then σ'' is an automorphism invariant and so commutes with C^u for $u = (1,1)$. Since σ' satisfies the hypothesis of Claim 3.2, we then have $\sigma''(\rho,(2,1)) = \sigma''(C_{p'}\rho, C_p(1,2)) = \sigma'' \circ C^{(1,1)}(\rho,(1,2)) = C^{(1,1)} \circ \sigma''(\rho,(1,2)) = C^{(1,1)}(\rho,(1,2)) = (\rho,(2,1))$, and similarly, $\sigma''((2,1),\rho) = ((2,1),\rho)$, so σ'' fixes all of the small weights. It now remains to show that σ'' is the identity.

Proposition 3.1. *Let σ be an automorphism invariant such that σ fixes the weights $(\rho,(1,2)), ((1,2),\rho), (\rho,(2,1))$ and $((2,1),\rho)$. Then $\sigma(\lambda\mu) = \lambda\mu$ for all $\lambda\mu \in P_{++}^{p',p}$.*

Proof. Let $\sigma(\kappa\nu) = (\kappa'\nu')$. Then by (3.1a) and the fact that σ sends (ρ,ρ) and $(\rho,(1,2))$ to themselves, we have for any $\kappa\nu \in P_{++}^{p',p}$,

$$\frac{S^{(p)}_{(1,2),\nu}}{S^{(p)}_{\rho\nu}} = \frac{S^{(p')}_{\rho\kappa}}{S^{(p')}_{\rho\kappa}} \cdot \frac{S^{(p)}_{(1,2),\nu}}{S^{(p)}_{\rho\nu}} = \frac{S^{(p',p)}_{(\rho,(1,2)),\kappa\nu}}{S^{(p',p)}_{\rho,\rho,\kappa\nu}} = \frac{S^{(p',p)}_{(\rho,(1,2)),\kappa'\nu'}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}} = \frac{S^{(p')}_{\rho\kappa'}}{S^{(p')}_{\rho\kappa'}} \cdot \frac{S^{(p)}_{(1,2),\nu'}}{S^{(p)}_{\rho\nu'}}.$$

A similar calculation carries through for all of the small weights, so we have

$$\begin{aligned} \frac{S^{(p)}_{(1,2),\nu}}{S^{(p)}_{\rho\nu}} &= \frac{S^{(p)}_{(1,2),\nu'}}{S^{(p)}_{\rho\nu'}}, \quad \frac{S^{(p)}_{(2,1),\nu}}{S^{(p)}_{\rho\nu}} = \frac{S^{(p)}_{(2,1),\nu'}}{S^{(p)}_{\rho\nu'}}; \\ \frac{S^{(p')}_{(1,2),\kappa}}{S^{(p')}_{\rho\kappa}} &= \frac{S^{(p')}_{(1,2),\kappa'}}{S^{(p')}_{\rho\kappa'}}, \quad \frac{S^{(p')}_{(2,1),\kappa}}{S^{(p')}_{\rho\kappa}} = \frac{S^{(p')}_{(2,1),\kappa'}}{S^{(p')}_{\rho\kappa'}}, \end{aligned} \tag{3.17}$$

where $(\kappa'\nu') = \sigma(\kappa\nu)$.

By Theorem 1 of Chapter VI of [3], we can write the Weyl character ch_β for a highest weight $\beta \in P_+^n$[1] as a polynomial in $ch_{(1,0)}$ and $ch_{(0,1)}$. Therefore, for any weight $\lambda \in P_{++}^n$,

$$ch_{\lambda-\rho}(-2\pi i\frac{\kappa}{n}) = P_\lambda\left(ch_{(1,0)}(-2\pi i\frac{\kappa}{n}), ch_{(0,1)}(-2\pi i\frac{\kappa}{n})\right). \tag{3.18}$$

Now by (3.7), we can write

$$\frac{S^{(n)}_{(1,2)\mu}}{S^{(n)}_{\rho\mu}} = ch_{(0,1)}(-2\pi i\frac{\mu}{n}), \text{ and } \frac{S^{(n)}_{(2,1)\mu}}{S^{(n)}_{\rho\mu}} = ch_{(1,0)}(-2\pi i\frac{\mu}{n}),$$

so we have

$$\frac{S^{(p')}_{\lambda,\kappa}}{S^{(p')}_{\rho\kappa}} = P_\lambda\left(\frac{S^{(p')}_{(1,2),\kappa}}{S^{(p')}_{\rho\kappa}}, \frac{S^{(p')}_{(2,1),\kappa}}{S^{(p')}_{\rho\kappa}}\right) = P_\lambda\left(\frac{S^{(p')}_{(1,2),\kappa'}}{S^{(p')}_{\rho\kappa'}}, \frac{S^{(p')}_{(2,1),\kappa'}}{S^{(p')}_{\rho\kappa'}}\right) = \frac{S^{(p')}_{\lambda\kappa'}}{S^{(p')}_{\rho\kappa'}},$$

[1] P_+^n is the set of unshifted highest weights $\{\lambda = (\lambda_1,\lambda_2) \in \mathbb{Z}^2 : 0 \le \lambda_1, \lambda_2, \lambda_1+\lambda_2 < n-2\}$

by (3.17), and a similar calculation applies to $S^{(p)}_{\mu\nu}/S^{(p)}_{\rho\nu}$. Therefore, for any $\lambda\mu, \kappa\nu \in P^{p',p}_{++}$, we have

$$\frac{S^{(p',p)}_{\lambda\mu,\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa\nu}} = \frac{S^{(p')}_{\lambda\kappa} S^{(p)}_{\mu\nu}}{S^{(p')}_{\rho\kappa} S^{(p)}_{\rho\nu}} = \frac{S^{(p')}_{\lambda\kappa'}}{S^{(p')}_{\rho\kappa'}} \cdot \frac{S^{(p)}_{\mu\nu'}}{S^{(p)}_{\rho\nu'}} = \frac{S^{(p',p)}_{\lambda\mu,\kappa'\nu'}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}},$$

where $\kappa'\nu' = \sigma(\kappa\nu)$. Multiplying both sides by $S^{(p',p)*}_{\lambda\mu,\kappa\nu}$ and summing over all $\lambda\mu \in P^{p',p}_{++}$, we get, by unitarity of $S^{(p',p)}$,

$$\frac{1}{S^{(p',p)}_{\rho\rho,\kappa\nu}} = \frac{\delta_{\kappa'\nu',\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}}.$$

Therefore, we must have $\kappa'\nu' = \kappa\nu$; ie, $\sigma(\kappa\nu) = \kappa\nu$. □

Since we defined $\sigma'' = \sigma_a^{-1} \circ C^u \circ \sigma$ (where σ is the original automorphism invariant we started with), and σ'' satisfies the hypothesis of Proposition 3.1, we have shown that any automorphism invariant σ has the form $\sigma = C^u \circ \sigma_a$, where C^u is a charge conjugation and σ_a has $a = (a_{11}, a_{21}, a_{12}, a_{22})$, for some quadruple a listed in Column B of Table 3.1. Therefore, the modular invariant partition function associated to M is

$$\mathcal{Z} = \sum_{\lambda\mu} ch_{\lambda\mu} ch^*_{C^u\sigma_a(\lambda\mu)},$$

for the above a and $u \in \{(0,0), (0,1), (1,0), (1,1)\}$. In terms of matrices, this means that M is the matrix product of one of I, ${}^CI$, I^C, or ${}^CI^C$ with some I^A defined by a σ_a from Table 3.1.

Solutions to (3.16) and their associated simple current invariants σ_a					
A	B	A	B	A	B
$k \equiv 0, l \equiv 1$		$k \equiv l \equiv 1$		$k \equiv 1, l \equiv 0$	
(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)	(0,0,0,0)
(-1,0,-1,1)	(1,1,-1,0)	(0,0,-1,0)	(1,0,0,0)	(0,0,-1,0)	(1,0,0,0)
(1,0,-1,-1)	(1,-1,1,0)	(0,-1,0,0)	(0,0,0,1)	(1,-1,0,-1)	(0,1,1,1)
(1,-1,-1,1)	(1,-1,-1,1)	(0,-1,-1,0)	(1,0,0,1)	(1,-1,-1,1)	(1,-1,-1,1)
(-1,-1,-1,-1)	(1,1,1,1)	(1,1,1,1)	(-1,-1,-1,-1)	(-1,-1,0,1)	(0,1,-1,1)
(0,-1,0,0)	(0,0,0,1)	(1,1,1,-1)	(-1,-1,1,-1)	(-1,-1,-1,-1)	(1,1,1,1)
		(-1,1,1,1)	(-1,1,-1,-1)		
$k \equiv 0, l \equiv -1$		(-1,1,1,-1)	(-1,1,1,-1)	$k \equiv -1, l \equiv 0$	
(0,0,0,0)	(0,0,0,0)			(0,0,0,0)	(0,0,0,0)
(-1,0,1,-1)	(-1,1,-1,0)	$k \equiv l \equiv -1$		(0,0,1,0)	(-1,0,0,0)
(1,0,1,-1)	(-1,-1,1,0)	(0,0,0,0)	(0,0,0,0)	(1,1,1,1)	(-1,-1,-1,-1)
(0,1,0,0)	(0,0,0,-1)	(0,0,1,0)	(-1,0,0,0)	(-1,1,1,-1)	(-1,1,1,-1)
(1,1,1,1)	(-1,-1,-1,-1)	(0,1,0,0)	(0,0,0,-1)	(1,0,0,-1)	(0,-1,1,-1)
(-1,1,1,-1)	(-1,1,1,-1)	(0,1,1,0)	(-1,0,0,-1)	(-1,1,0,1)	(0,1,-1,-1)
		(1,-1,-1,1)	(1,-1,-1,1)		
$k \equiv 1, l \equiv -1$		(1,-1,-1,-1)	(1,-1,1,1)	$k \equiv -1, l \equiv 1$	
(0,0,0,0)	(0,0,0,0)	(-1,-1,-1,1)	(1,1,-1,1)	(0,0,0,0)	(0,0,0,0)
(0,0,-1,0)	(1,0,0,0)	(-1,-1,-1,-1)	(1,1,1,1)	(0,0,1,0)	(-1,0,0,0)
(0,1,0,0)	(0,0,0,-1)			(0,-1,0,0)	(0,0,0,1)
(0,1,-1,0)	(1,0,0,-1)			(0,-1,1,0)	(-1,0,0,1)
Column A: (w, x, y, z) satisfying (3.16)					
Column B: $(a_{11}, a_{21}, a_{12}, a_{22}) = (-y, -w, -z, -x)$ as in Claim 3.2					

Table 3.1: Solutions to (3.16) and their corresponding simple current invariants σ_a, where all equivalences are taken modulo 3

Chapter 4

The $\rho\rho$-Couplings

The purpose of this chapter is to find the possible $\rho\rho$-couplings for $A_{2,p'} \oplus A_{2,p}$; that is, those $\lambda\mu \in P_{++}^{p',p}$ such that $M_{\rho\rho,\lambda\mu} \neq 0$ or $M_{\lambda\mu,\rho\rho} \neq 0$. The two main tools we will need for this are the norm condition (4.1) (T-invariance) and the parity rule (4.5), both of which we will "decouple" so that we can use the results from the A_2 classification as much as possible (by "decoupling", we mean to take the result for $A_{2,p'} \oplus A_{2,p}$ and get the corresponding result for each of $A_{2,p'}$ and $A_{2,p}$).

The decoupled versions of T-invariance and the parity rule; ie, (4.9b) and (4.10), are almost the single A_2 versions of these, and they will give us already a very small set of possibilities for the $\rho\rho$-couplings. Putting these back into (4.9a), we regain some of the information we lost in the decoupling and narrow down the choices even further. Finally, in §4.3, we eliminate all but the $\rho\rho$-orbits as possible $\rho\rho$-couplings, other than at the exceptional levels.

4.1 The Norm Condition and the Parity Rule

By T-invariance, we have that $(MT^{(p',p)})_{\lambda\mu,\lambda'\mu'} = (T^{(p',p)}M)_{\lambda\mu,\lambda'\mu'}$ for any $\lambda\mu, \lambda'\mu' \in P_{++}^{p',p}$, so $\sum_{\kappa\nu \in P_{++}^{p',p}} M_{\lambda\mu,\kappa\nu} T^{(p',p)}_{\kappa\nu,\lambda'\mu'} = \sum_{\kappa\nu \in P_{++}^{p',p}} T^{(p',p)}_{\lambda\mu,\kappa\nu} M_{\kappa\nu,\lambda'\mu'}$. Since T is diagonal, the above sums have only one nonzero term each: at $\kappa\nu = \lambda'\mu'$ and $\kappa\nu = \lambda\mu$ respectively, so we get $M_{\lambda\mu,\lambda'\mu'} T^{(p',p)}_{\lambda'\mu',\lambda'\mu'} = T^{(p',p)}_{\lambda\mu,\lambda\mu} M_{\lambda\mu,\lambda'\mu'}$. If $M_{\lambda\mu,\lambda'\mu'} \neq 0$, we can cancel the M terms to get

$$T^{(p')}_{\lambda'\lambda'} T^{(p)}_{\mu'\mu'} = T^{(p')}_{\lambda\lambda} T^{(p)}_{\mu\mu},$$

where $T^{(p')}_{\lambda'\lambda'}$ and $T^{(p')}_{\lambda\lambda}$ are $A_{2,p'}$ T-matrices, and $T^{(p)}_{\mu'\mu'}$ and $T^{(p)}_{\mu\mu}$ are $A_{2,p}$ T- matrices.

Define $\langle\lambda\rangle := \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = 3\frac{\lambda^2}{2}$. Then, putting in the definitions for T, we have

$$\exp[2\pi i\{\frac{\langle\lambda'\rangle}{3p'} + \frac{\langle\mu'\rangle}{3p} - \frac{\langle\lambda\rangle}{3p'} - \frac{\langle\mu\rangle}{3p}\}] = 1,$$

which tells us that the sum inside the braces must be an integer. Therefore, $\frac{\langle\lambda'\rangle}{3p'} + \frac{\langle\mu'\rangle}{3p} \equiv \frac{\langle\lambda\rangle}{3p'} + \frac{\langle\mu\rangle}{3p}$ (mod 1), so we get the *norm condition*

(4.1)

$M_{\lambda\mu,\lambda'\mu'} \neq 0 \implies$

$$\frac{\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2}{p'} + \frac{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}{p} \equiv \frac{{\lambda'_1}^2 + \lambda'_1\lambda'_2 + {\lambda'_2}^2}{p'} + \frac{{\mu'_1}^2 + \mu'_1\mu'_2 + {\mu'_2}^2}{p} \pmod 3.$$

We can now decouple (4.1) so that we can use the results for one copy of A_2 from [9]. Multiplying (4.1) by p', we have

$$\langle\lambda\rangle + \frac{p'\langle\mu\rangle}{p} \equiv \langle\lambda'\rangle + \frac{p'\langle\mu'\rangle}{p} \pmod{3p'}, \tag{4.2}$$

so

$$\langle\lambda\rangle - \langle\lambda'\rangle \equiv p'\frac{\langle\mu'\rangle - \langle\mu\rangle}{p} \pmod{3p'}.$$

But $\langle\lambda\rangle - \langle\lambda'\rangle \in \mathbb{Z}$, so since gcd$(p',p)$=1, we must have $\langle\mu'\rangle - \langle\mu\rangle \equiv 0 \pmod p$. Dividing (4.1) by p', we get $\langle\lambda\rangle - \langle\lambda'\rangle \equiv 0 \pmod{p'}$, so we have the decoupled norm condition

(4.3)

$\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2 \equiv \lambda_1'^2+\lambda_1'\lambda_2'+\lambda_2'^2 \pmod{p'}$ $\mu_1^2+\mu_1\mu_2+\mu_2^2 \equiv \mu_1'^2+\mu_1'\mu_2'+\mu_2'^2 \pmod p$.

Notice that while (4.3) is a simplification of (4.1), they are not equivalent: (4.3), while easier for us to use, is weaker than (4.1). Also, the equations (4.3) are not exactly T-invariance for A_2 at height p' and p respectively: T-invariance for A_2 at height n is taken mod $3n$ (so the difference is a factor of 3 in the modulus).

We now introduce the Galois selection rule, or parity rule, for $A_2 \oplus A_2$. As for A_1, the parity rule for $A_2 \oplus A_2$ comes from a Galois symmetry obeyed by the S-matrix; however, the A_2 Galois permutation is not as simple to find as for A_1, and we will not need to use it here. (see [5] for more about Galois symmetry in rational conformal field theory). Unlike T-invariance, the $A_2 \oplus A_2$ parity rule decouples completely for any choice of (p', p), so it is exactly two copies of the A_2 parity rule.

Define, for any ℓ coprime to $3p'p$, a function $\epsilon_\ell : P_{++}^{p',p} \to \{\pm 1\}$, as follows:

$$\epsilon_\ell(\lambda\mu) = \epsilon^{(p'p)}(\ell)\epsilon_\ell^{(p')}(\lambda)\epsilon_\ell^{(p)}(\mu),$$

where $\epsilon^{(p'p)}(\ell)$ is a consta$\epsilon_\ell^{(n)}(\lambda) = \Big\{$uals ± 1, and

$$\epsilon_\ell^{(n)}(\lambda) = \begin{cases} +1, & \text{if } \{\ell\lambda_1\}_n + \{\ell\lambda_2\}_n < n, \\ -1, & \text{if } \{\ell\lambda_1\}_n + \{\ell\lambda_2\}_n > n \end{cases},$$

where $\{x\}_n$ is the unique integer $0 \leq \{x\}_n < n$, that is congruent to $x \pmod{n}$. $\epsilon_\ell(\lambda\mu)$ is called the *parity* of $\lambda\mu$ (see [9]). The parity rule gives us, for any ℓ coprime to $3p'p$:

$$M_{\lambda\mu,\kappa\nu} = \epsilon_\ell(\lambda\mu)\epsilon_\ell(\kappa\nu)M_{[\ell\lambda][\ell\mu],[\ell\kappa][\ell\nu]}, \text{ where } [\ell\lambda][\ell\mu],\ [\ell\kappa][\ell\nu] \in P_{++}^{p',p}. \tag{4.4}$$

Equation (4.4) tells us that $M_{\lambda\mu,\kappa\nu} = \pm M_{[\ell\lambda][\ell\mu],[\ell\kappa][\ell\nu]}$. If $M_{\lambda\mu,\kappa\nu} \neq 0$, then, since every entry of M is nonnegative, we cannot have $M_{\lambda\mu,\kappa\nu} = -M_{[\ell\lambda][\ell\mu],[\ell\kappa][\ell\nu]}$. Therefore, if $M_{\lambda\mu,\kappa\nu} \neq 0$, we must have $M_{\lambda\mu,\kappa\nu} = +M_{[\ell\lambda][\ell\mu],[\ell\kappa][\ell\nu]}$, and so we get the *Galois selection rule*

$$M_{\lambda\mu,\kappa\nu} \neq 0 \implies \epsilon_\ell(\lambda\mu) = \epsilon_\ell(\kappa\nu), \tag{4.5}$$

for any ℓ with $\gcd(\ell,\ 3p'p) = 1$. To decouple (4.5) like we decoupled the norm condition, we set $\epsilon_\ell(\lambda\mu) = \epsilon_\ell(\kappa\nu)$, and we want to show that for any ℓ coprime to $3p'$, $\epsilon_\ell^{(p')}(\lambda) = \epsilon_\ell^{(p')}(\kappa)$, and for any ℓ coprime to $3p$, $\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\nu)$. We have

$$\epsilon^{p'p}(\ell)\epsilon_\ell^{(p')}(\lambda)\epsilon_\ell^{(p)}(\mu) = \epsilon^{pp'}(\ell)\epsilon_\ell^{(p')}(\kappa)\epsilon_\ell^{(p)}(\nu)$$

by (4.5), so

$$\epsilon_\ell^{(p')}(\lambda)\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p')}(\kappa)\epsilon_\ell^{(p)}(\nu), \text{ for any } \ell \text{ coprime to } 3p'p. \tag{4.6}$$

Let ℓ_1 and ℓ_2 be any integers such that $\ell_1 \equiv \ell_2 \pmod 3$. Then, by the Chinese Remainder Theorem, we can find an ℓ such that

$$\ell \equiv \ell_1 \pmod{3p'}, \text{ and } \ell \equiv \ell_2 \pmod{3p}. \tag{4.7}$$

We will take ℓ_1 to be an integer coprime to $3p$. Then $\ell_1 \not\equiv 0 \pmod 3$, so $\ell_1 \equiv \pm 1 \pmod 3$. If $\ell_1 \equiv 1 \pmod 3$, then let $\ell_2 = 1$, and if $\ell_1 \equiv -1 \pmod 3$, then let

$\ell_2 = -1$. Putting these ℓ_1 and ℓ_2 into (4.7), we see that the ℓ we get is coprime to $3p'p$, and so we can put it into (4.6).

Consider $\epsilon_\ell^{(p)}(\mu)$ and $\epsilon_\ell^{(p)}(\nu)$. If $\ell_2 = 1$, then, since $\ell \equiv \ell_2 = 1 \pmod p$, $\{\ell\mu_1\}_p + \{\ell\mu_2\}_p = \{\ell_2\mu_1\}_p + \{\ell_2\mu_2\}_p = \{\mu_1\}_p + \{\mu_2\}_p = \mu_1 + \mu_2 < p$, since $\mu_1, \mu_2, \mu_1+\mu_2 < p$, and the same calculation holds for ν. Therefore, in this case, we have $\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\nu) = 1$.

If $\ell_2 = -1$, then $\{\ell\mu_1\}_p + \{\ell\mu_2\}_p = \{-\mu_1\}_p + \{-\mu_2\}_p$. To evaluate these, we see that $0 < \mu_1, \mu_2 < p$, so $-p < -\mu_1, -\mu_2 < 0$, and so $0 < -\mu_1 + p, -\mu_2 + p < p$. Then, $\{-\mu_1\}_p + \{-\mu_2\}_p = (-\mu_1 + p) + (-\mu_2 + p) = 2p - (\mu_1 + \mu_2) > p$. Therefore, in this case, $\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\nu) = -1$. Putting the fact that $\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\nu)$ into (4.6), we see that, for any ℓ_1 coprime to $3p'$, $\epsilon_{\ell_1}^{(p')}(\lambda) = \epsilon_{\ell_1}^{(p')}(\kappa)$. This argument is symmetric with respect to ℓ_1 and ℓ_2, so reversing them, we also have $\epsilon_{\ell_2}^{(p)}(\mu) = \epsilon_{\ell_2}^{(p)}(\nu)$, for any ℓ_2 coprime to $3p$. We therefore have the decoupled Galois selection rule,

$$M_{\lambda\mu,\kappa\nu} \neq 0 \implies \epsilon_\ell^{(p')}(\lambda) = \epsilon_\ell^{(p')}(\kappa) \text{ and } \epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\nu), \tag{4.8}$$

for any ℓ with $\gcd(\ell, 3p'p) = 1$. Equation (4.8) is equivalent to (4.5).

Now suppose that $\lambda\mu$ is a (left) $\rho\rho$-coupling; ie, $M_{\lambda\mu,\rho\rho} \neq 0$. We can put $\lambda'\mu' = \rho\rho$ into (4.1) and (4.3) to get

$$M_{\lambda\mu,\rho\rho} \neq 0 \implies$$

$$\frac{\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2}{3p'} + \frac{\mu_1^2 + \mu_1\mu_2 + \mu_2^2}{3p} \equiv \frac{1}{p'} + \frac{1}{p} \pmod 1 \tag{4.9a}$$

$$\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 \equiv 3 \pmod{p'}; \quad \mu_1^2 + \mu_1\mu_2 + \mu_2^2 \equiv 3 \pmod p. \tag{4.9b}$$

(4.9b) is Equation (4.3) of [9], and is weaker than (4.9a).

We can put $\lambda\mu$ and $\rho\rho$ into (4.8), so $\epsilon_\ell^{(p')}(\lambda) = \epsilon_\ell^{(p')}(\rho)$, and $\epsilon_\ell^{(p)}(\mu) = \epsilon_\ell^{(p)}(\rho)$ for any ℓ coprime to $3p'p$. Now, for $n = p'$ or p, $\epsilon_\ell^{(n)}(\rho) = \epsilon_\ell^{(n)}(1,1) = \{\ell\}_n + \{\ell\}_n = 2\{\ell\}_n$. Therefore,

$$\epsilon_\ell^{(n)}(\rho) = \begin{cases} +1, & \text{if } 0 < \{\ell\}_n < \frac{n}{2} \\ -1, & \text{if } \frac{n}{2} < \{\ell\}_n < n \end{cases}.$$

But $\epsilon_\ell^{(p')}(\lambda_1,\lambda_2)=\epsilon_\ell^{(p')}(\rho)$ and $\epsilon_\ell^{(p)}(\mu_1,\mu_2)=\epsilon_\ell^{(p)}(\rho)$ by (4.8), so we have

$$
\begin{aligned}
&0<\{\ell\}_{p'}<\frac{p'}{2} \implies \{\ell\lambda_1\}_{p'}+\{\ell\lambda_2\}_{p'}<p' \\
&\frac{p'}{2}<\{\ell\}_{p'} \implies \{\ell\lambda_1\}_{p'}+\{\ell\lambda_2\}_{p'}>p'; \\
&0<\{\ell\}_{p}<\frac{p}{2} \implies \{\ell\mu_1\}_{p}+\{\ell\mu_2\}_{p}<p \\
&\frac{p}{2}<\{\ell\}_{p} \implies \{\ell\mu_1\}_{p}+\{\ell\mu_2\}_{p}>p.
\end{aligned}
\tag{4.10}
$$

Equation (4.10) is (4.1b) of [9]. Equations (4.9b) and (4.10) are what we need to get Proposition 1 of [9], which will give us Lemma 4.2, our first list of possible $\rho\rho$-couplings.

4.2 Searching for the Possible $\rho\rho$-couplings

We now have (4.9b) and (4.10) as in (4.3) and (4.1b) of [9], so we are in a position to write down some possible $\rho\rho$-couplings. We will need the following lemma, which is Proposition 1 of [9].

Lemma 4.1. *The set of all solutions λ to (4.9b) and (4.10), is:*
(a) for $p'\equiv_4 1,2,3$, $p'\neq 18$: $\lambda\in\mathcal{O}\rho$;
(b) for $p'\equiv_4 0$, $p'\neq 12,24,60$: $\lambda\in\mathcal{O}\rho\cup\mathcal{O}(\rho'')$, where $\rho''=(\frac{p'-2}{2},\frac{p'-2}{2})$.
(c) for $p'=12,18,24,60$, respectively, λ lies in

$$
\begin{aligned}
&\mathcal{O}\rho \cup \mathcal{O}(3,3)\cup\mathcal{O}(5,5)\\
&\mathcal{O}\rho \cup \mathcal{O}(1,4)\\
&\mathcal{O}\rho \cup \mathcal{O}(5,5)\cup\mathcal{O}(7,7)\cup\mathcal{O}(11,11)\\
&\mathcal{O}\rho \cup \mathcal{O}(11,11)\cup\mathcal{O}(19,19)\cup\mathcal{O}(29,29),
\end{aligned}
$$

and the set of all solutions μ is the same with μ instead of λ and p instead of p'.

We can put these choices into (4.9a) to further narrow down the possibilities. This is done on a case-by-case basis, but is made much simpler by a few observations. First, we can discard many cases due to the condition that p' and p be coprime. Then, the symmetry of the norm condition itself means we can cut the cases in half.

Also, for those λ such that $\lambda_1 = \lambda_2$, we have that $\langle A\lambda \rangle = \langle A^2\lambda \rangle$. Since the only weight appearing in Lemma 4.1 such that $\lambda_1 \neq \lambda_2$ is $(4,1)$, this greatly reduces the number of cases we have to check.

The result of putting all of the pairs into (4.9a) is the following lemma.

Lemma 4.2. *Let p' and p be coprime. Then the only pairs $\lambda\mu$ such that λ and μ each satisfy (4.9b) and (4.10), and $\lambda\mu$ satisfies (4.9a) are:*

(a) (i) $p' \equiv_{12} 1, 7, 10$ and $p \equiv_{12} 2, 5, 11$: $\lambda\mu = (A^i_{p'}\rho, A^j_p\rho)$, $i = \pm j = 0, 1, 2$;

(ii) $p' \equiv_{12} 3, 6, 9$, and $p \equiv_{12} \pm 1, \pm 2, \pm 5$; p', $p \neq 18$: $\lambda\mu = (A^i_{p'}\rho, \rho)$, $i = 0, 1, 2$;

(b) (i) $p' \equiv_{12} 1, 5, 7, 11$ and $p \equiv_{12} 0$; $p \neq 12, 24, 60$: $\lambda\mu = (\rho, A^j_p\rho)$, or $\lambda\mu = (\rho, A^j_p\rho'')$, $j = 0, 1, 2$;

(ii) $p' \equiv_{12} 1, 7$ and $p \equiv_{12} 8$, or $p' \equiv_{12} 5, 11$ and $p \equiv_{12} 4$: $\lambda\mu = (A^i_{p'}\rho, A^i_p\rho)$, or $\lambda\mu = (A^i_{p'}\rho, A^i_p\rho'')$, $i = 0, 1, 2$;

(iii) $p' \equiv_{12} 3, 9$ and $p \equiv_{12} 4, 8$: $\lambda\mu = (A^i_{p'}\rho, \rho)$, or $\lambda\mu = (A^i_{p'}\rho, \rho'')$, $i = 0, 1, 2$;

(c) (i) $p' = 18$ and $p \equiv_{12} 1, 7$: $\lambda \in \mathcal{O}\rho$, $\mu = \rho$;

(ii) $p' = 18$ and $p \equiv_{12} 5, 11$: $\lambda \in \mathcal{O}\rho$ and $\mu = \rho$, or $\lambda \in \mathcal{O}(1, 4)$ and $\mu = A^i_p\rho$, for $i = 1, 2$;

(d) (i) $p' = 12$ and $p \equiv_{12} 1, 7$: $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(5, 5)$ and $\mu = \rho$, or $\lambda \in \mathcal{O}(3, 3)$ and $\mu = A^i_p\rho$, for $i = 1, 2$;

(ii) $p' = 12$ and $p \equiv_{12} 5, 11$: $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(5, 5)$, $\mu = \rho$;

(e) $p' = 24$ and $p \equiv_{12} 1, 5, 7, 11$: $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(5, 5) \cup \mathcal{O}(7, 7) \cup \mathcal{O}(11, 11)$, $\mu = \rho$);

(f) $p' = 60$ and $p \equiv_{12} 1, 5, 7, 11$: $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(11, 11) \cup \mathcal{O}(19, 19) \cup \mathcal{O}(29, 29)$, $\mu = \rho$), plus a symmetric list, where p' and p are reversed, λ and μ are reversed, and $\rho'' = (\frac{p'-2}{2}, \frac{p'-2}{2})$.

Proof. Because p' and p are coprime, Lemma 4.1 gives us the following list of choices for $\lambda\mu$

$$p' \equiv_4 1, 2, 3,\ p \equiv_4 1, 2, 3,\ p', p \neq 18 : \lambda \in \mathcal{O}\rho, \mu \in \mathcal{O}\rho; \tag{4.11a}$$

$$p' \equiv_4 1, 3,\ p \equiv_4 0,\ p' \neq 18,\ p \neq 12, 24, 60 : \lambda \in \mathcal{O}\rho,\ \mu \in \mathcal{O}\rho \cup \mathcal{O}\rho''; \tag{4.11b}$$

$$p' = 18,\ p \equiv_4 1, 3 : \lambda \in \mathcal{O}\rho \cup \mathcal{O}(1, 4),\ \mu \in \mathcal{O}\rho; \tag{4.11c}$$

$$p' = 12,\ p \equiv_4 1, 3 : \lambda \in \mathcal{O}\rho \cup \mathcal{O}(3, 3) \cup \mathcal{O}(5, 5),\ \mu \in \mathcal{O}\rho; \tag{4.11d}$$

(4.11e) $p' = 24,\ p \equiv_4 1,3 : \lambda \in \mathcal{O}\rho \cup \mathcal{O}(5,5) \cup \mathcal{O}(7,7) \cup \mathcal{O}(11,11),\ \mu \in \mathcal{O}\rho;$

(4.11f) $p' = 60,\ p \equiv_4 1,3 : \lambda \in \mathcal{O}\rho \cup \mathcal{O}(11,11) \cup \mathcal{O}(19,19) \cup \mathcal{O}(29,29),\ \mu \in \mathcal{O}\rho.$

We first test the possible $\lambda\mu$ from (4.11a): suppose $p' \equiv_4 1,2,3$, $p \equiv_4 1,2,3$, and p', $p \neq 18$. Then (λ, μ) can be one of nine pairs, $(\lambda,\mu) \in \{(\rho,\rho), (\rho, A_p\rho), (\rho, A_p^2\rho), (A_{p'}\rho, \rho), (A_{p'}\rho, A_p\rho), (A_{p'}\rho, A_p^2\rho), (A_{p'}^2\rho, \rho), (A_{p'}^2\rho, A_p\rho), (A_{p'}^2\rho, A_p^2\rho)\}$. Notice that if $\langle\alpha\rangle = \langle\beta\rangle$, then putting α or β into (4.9a) will give us the same information. Calculating $\langle\rho\rangle$, $\langle A\rho\rangle$, and $\langle A^2\rho\rangle$, we see that $\langle\rho\rangle = 3$, $\langle A\rho\rangle = \langle A^2\rho\rangle$. Therefore, we can test in classes of pairs, as follows: (1) test (ρ,ρ); (2) test $(\rho, A_p\rho)$ to get the cases $(\rho, A_p\rho)$ and $(\rho, A_p^2\rho)$; (3) test $(A_{p'}\rho, A_p\rho)$ to get the cases $(A_{p'}\rho, A_p\rho)$, $(A_{p'}\rho, A_p^2\rho)$, $(A_{p'}^2\rho, A_p\rho)$, $(A_{p'}^2\rho, A_p^2\rho)$, and (4) to get $(A_{p'}\rho, \rho)$ and $(A_{p'}^2\rho, \rho)$, notice that this case is symmetric to (2).

(1) is trivial. For (2), we get $\frac{\langle\rho\rangle}{p'} + \frac{\langle A_p\rho\rangle}{p} \equiv \frac{3}{p'} + \frac{3}{p} \pmod 3$, or $\frac{3}{p'} + p + \frac{3}{p} \equiv \frac{3}{p'} + \frac{3}{p} \pmod 3$, so $(\rho, A_p\rho)$ satisfies (4.9a) iff $p \equiv 0 \pmod 3$. Therefore, if $p' \equiv_4 1,2,3$, $p \equiv_4 1,2,3$, and $p \equiv_3 0$, $\lambda = \rho$, $\mu \in \{A_{p'}\rho, A_p^2\rho\}$. But we also have $(\lambda,\mu) = (\rho,\rho)$ from (1), so $(\lambda,\mu) \in (\rho, \mathcal{O}\rho)$. From this, we also automatically have (4), which tells us if $p \equiv_4 1,2,3$, $p' \equiv_4 1,2,3$ and $p' \equiv_3 0$, then $\lambda \in \mathcal{O}\rho$, $\mu = \rho$. Testing $(A_{p'}\rho, A_p\rho)$, we see that the pairs in (3) will satisfy (4.9a) iff $p' + p \equiv 0 \pmod 3$. This means that either $p' \equiv 1 \pmod 3$ and $p \equiv 2 \pmod 3$, or vice-versa, since $p' \equiv p \equiv 0 \pmod 3$ violates the coprime condition. Again, we add $(\lambda,\mu) = (\rho,\rho)$ because this also satisfies (4.9a) for such p' and p, and so we have $(\lambda,\mu) = (\rho,\rho)$, or $\lambda, \mu \in \{A_{p'}\rho, A_p^2\rho\}$. Putting these congruence conditions together with p', $p \equiv_4 1,2,3$, we have Lemma 4.2(a).

Lemma 4.2(b) is the longest case, so we will do it in the most detail. Suppose that $p' \equiv_4, 1,2,3$, as in (4.11b), so $\lambda \in \mathcal{O}\rho$, $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$. Suppose first that both λ and μ are in $\mathcal{O}\rho$. From the proof of (a), we get $\lambda\mu = \rho\rho$ with no further conditions, $\lambda = \rho$, $\mu \in \{A_{p'}\rho, A_p^2\rho\}$ with the added condition that $p \equiv_3 0$, and λ, $\mu \in \{A_{p'}\rho, A_p^2\rho\}$ with the added condition that $p' + p \equiv_3 0$.

If $\lambda \in \mathcal{O}\rho$ and $\mu \in \mathcal{O}\rho'$, we need to test the following pairs: (ρ, ρ''), $(\rho, A_p\rho'')$, $(A_{p'}\rho, \rho'')$, and $(A_{p'}\rho, A_p\rho'')$. For $(\lambda,\mu) = (\rho,\rho'')$, we need no further conditions on p' and p. For $(\lambda,\mu) = (\rho, A_p\rho'')$, we need also $p \equiv_3 0$, and for $(\lambda,\mu) = (A_{p'}\rho, \rho'')$, we must have $p' \equiv_3 0$. Finally, $(\lambda,\mu) = (A_{p'}\rho, A_p\rho'')$ satisfies (4.9a) iff $p' + p \equiv 0 \pmod 3$.

Putting together the congruence we found with $p' \equiv_4 1, 2, 3$, and $p \equiv_4 0$, we get the following list of choices mod 12:

(1) $p' \equiv_{12} \pm 1, \pm 3, \pm 5$ and $p \equiv_{12} 0, 4, 8$, $p \neq 12, 24, 60$: $(\lambda, \mu) \in \{(\rho, \rho), \rho, \rho''\})$;

(2) $p' \equiv_{12} \pm 1, \pm 3, \pm 5$ and $p \equiv_{12} 0$, $p \neq 12, 24, 60$: $\lambda = \rho$, $\mu \in \{A_p\rho, A_p^2\rho, A_p\rho'', A_p^2\rho''\}$;

(3) $p' \equiv_{12} 1, 7, 10$ and $p \equiv_{12} 8$: $\lambda \in \{A_{p'}\rho, A_{p'}^2\rho\}$, $\mu \in \{A_p\rho, A_p^2\rho, A_p\rho'', A_p^2\rho''\})$;

(4) $p' \equiv_{12} 5, 11$ and $p \equiv_{12} 4$: same as (3);

(5) $p' \equiv_{12} 3, 9$ and $p \equiv_{12} 0, 4, 8$, $p \neq 12, 24, 60$: $\lambda \in \{A_{p'}\rho, A_{p'}^2\rho\}$, $\mu = \rho''$.

Eliminating cases which violate $\gcd(p', p)=1$, and putting together the overlapping cases gives us Lemma 4.2(b).

In Lemma 4.2(c),(d),(e),(f), $p' \equiv 0 \pmod 3$, so by the coprime condition, we can eliminate any cases which require $p \equiv 0 \pmod 3$ (which is about half of the remaining cases). From the proof of (a), we know that this would eliminate any (λ, μ) with $\lambda \in \mathcal{O}\rho$ and $\mu \in \{A\rho, A^2\rho\}$. Therefore, for all of (c)-(f), we have only the possibility that $\lambda \in \mathcal{O}\rho$, $\mu = \rho$, for $\lambda, \mu \in \mathcal{O}\rho$. All that remains now is to check the special cases not involving $\mathcal{O}\rho$.

Putting (4.11c) into (4.9a), we find that the additional condition $p \equiv 2 \pmod 3$ gives us the choices $\lambda \in \mathcal{O}(1,4)$ and $\mu \in \{A_p\rho, A_p^2\rho\}$; (4.11d) gives us $\lambda \in \mathcal{O}(5,5)$ and $\mu = \rho$, and if $p \equiv 1 \pmod 3$, $\lambda \in \mathcal{O}(3,3)$ and $\mu \in \{A_p\rho, A_p^2\rho\}$; (4.11e) gives us $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(5,5) \cup \mathcal{O}(7,7) \cup \mathcal{O}(11,11)$ and $\mu = \rho$ with no further restrictions on p, and (4.11f) gives $\lambda \in \mathcal{O}\rho \cup \mathcal{O}(11,11) \cup \mathcal{O}(19,19) \cup \mathcal{O}(29,29)$ and $\mu = \rho$ with no further restrictions on p' and p. □

4.3 The $\rho\rho$-couplings

We now use Lemma 4.2 to further narrow down the possible $\rho\rho$-couplings. Throughout this section (and also in the next chapter), we will use Lemma 4.3 below, which is Lemma 4 of [14].

Define $\mathcal{P}_R(p', p) = \{\lambda\mu \in P_{++}^{p',p} : M_{\kappa\nu,\lambda\mu} \neq 0,$ for some $\kappa\nu \in P_{++}^{p',p}\}$, and $\mathcal{J}_R = \{A_{p'}^i A_p^j : M_{A^i\rho A^j\rho,\rho\rho} \neq 0\}$. $\mathcal{P}_L(p', p)$ and $\mathcal{J}_L$ are defined similarly. We will often write $\mathcal{P}_R, \mathcal{P}_L, \mathcal{J}_R, \mathcal{J}_L$ for short.

Lemma 4.3. *(a) For each $\lambda\mu \in P_{++}^{p',p}$, define $s_L(\lambda, \mu) = \sum_{\kappa\nu} M_{\kappa\nu,\rho\rho} S_{\lambda\mu,\kappa\nu}^{(p',p)}$; $s_R(\lambda, \mu) = \sum_{\kappa\nu} M_{\rho\rho,\kappa\nu} S_{\lambda\mu,\kappa\nu}^{(p',p)}$. Then $s_L(\lambda, \mu), s_R(\lambda, \mu) \geq 0$, and $s_L(\lambda, \mu) > 0$ iff $\lambda\mu \in \mathcal{P}_L$ and $s_R(\lambda\mu) > 0$ iff $\lambda\mu \in \mathcal{P}_R$;*

(b) $M_{A^a\rho A^b\rho,\rho\rho} = 1$*, for all* $A^a_{p'}A^b_p \in \mathcal{J}_L$*, and* $M_{\rho\rho,A^c\rho A^d\rho} = 1$*, for all* $A^c_{p'}A^d_p \in \mathcal{J}_R$*;*

(c) For all a, b, c, d *such that* $M_{A^a\rho A^b\rho,A^c\rho A^d\rho} = 1$*,* $M_{A^a\lambda A^b\mu,A^c\kappa A^d\nu} = M_{\lambda\mu,\kappa\nu}$*, for all* $\lambda\mu, \kappa\nu \in P^{p',p}_{++}$*. In particular,* $M_{A^a\lambda A^b\mu,\kappa\nu} = M_{\lambda\mu,\kappa\nu} = M_{\lambda\mu,A^c\kappa A^d\nu}$*, forall* $A^a_{p'}A^b_p \in \mathcal{J}_L, A^c_{p'}A^d_p \in \mathcal{J}_R$*.*

(d) $A^a_{p'}A^b_p \in \mathcal{J}_L \iff at(\lambda) + bt(\mu) \equiv 0 \pmod 3$*, for all* $\lambda\mu \in \mathcal{P}_L$*, and* $A^c_{p'}A^d_p \in \mathcal{J}_R \iff ct(\kappa) + dt(\nu) \equiv 0 \pmod 3$*, for all* $\kappa\nu \in \mathcal{P}_R$*;*

(e) Suppose that $M_{\lambda\mu,\rho\rho} \neq 0 \iff \lambda\mu \in \mathcal{J}_L(\rho\rho)$ *and* $M_{\rho\rho,\kappa\nu} \neq 0 \iff \kappa\nu \in \mathcal{J}_R(\rho\rho)$*. Then* $\mathcal{P}_L = \{\lambda\mu \in P^{p',p}_{++} : at(\lambda) + bt(\mu) \equiv_3 0,\ \forall A^a_{p'}A^b_p \in \mathcal{J}_L\}$ *and* $\mathcal{P}_R = \{\kappa\nu \in P^{p',p}_{++} : ct(\kappa) + dt(\nu) \equiv_3 0,\ \forall A^c_{p'}A^d_p \in \mathcal{J}_L\}$*.*

Proof. (a) We evaluate $MS^{(p',p)} = S^{(p',p)}M$ at $(\lambda\mu, \rho\rho)$, so

$$\sum_{\kappa\nu} M_{\lambda\mu,\kappa\nu} S^{(p',p)}_{\kappa\nu,\rho\rho} = \sum_{\kappa\nu} S^{(p',p)}_{\lambda\mu,\kappa\nu} M_{\kappa\nu,\rho\rho}.$$

The left-hand side is nonnegative since $S^{(p',p)}_{\kappa\nu,\rho\rho} > 0$ for all $\kappa\nu \in P^{p',p}_{++}$ and $M_{\lambda\mu,\kappa\nu} \in \mathbb{Z}_{\geq 0}$, and the right-hand side is $s_L(\lambda, \mu)$. $M_{\lambda\mu,\kappa\nu} > 0$ for some $\kappa\nu \in P^{p',p}_{++}$ iff $\lambda\mu \in \mathcal{P}_L$, so we get $s_L(\lambda, \mu) \geq 0$ with strict inequality iff $\lambda\mu \in \mathcal{P}_L$. Evaluating $MS^{(p',p)} = S^{(p',p)}M$ at $(\rho\rho, \lambda\mu)$ gives the result for $s_R(\lambda, \mu)$.

(b) By unitarity of $S^{(p',p)}$, S-invariance is equivalent to $M = S^{(p',p)*}MS^{(p',p)}$. By the positivity of M, Equations (2.10d), and the triangle inequality,

$$\begin{aligned}
M_{A^a\rho A^b\rho,\rho\rho} &= |M_{A^a\rho A^b\rho,\rho\rho}| \\
&= |\sum_{\lambda\mu,\kappa\nu} S^{(p',p)*}_{A^a\rho A^b\rho,\lambda\mu} M_{\lambda\mu,\kappa\nu} S^{(p',p)}_{\kappa\nu,\rho\rho}| \\
&= |\sum_{\lambda\mu,\kappa\nu} e^{-\frac{2\pi i}{3}(at(\lambda)+bt(\mu))} S^{(p',p)*}_{\rho\rho,\lambda\mu} M_{\lambda\mu,\kappa\nu} S^{(p',p)}_{\kappa\nu,\rho\rho}| \\
&\leq \sum_{\lambda\mu,\kappa\nu} |e^{-\frac{2\pi i}{3}(at(\lambda)+bt(\mu))}|\, |S^{(p',p)*}_{\rho\rho,\lambda\mu} M_{\lambda\mu,\kappa\nu} S^{(p',p)}_{\kappa\nu,\rho\rho}| \\
&= \sum_{\lambda\mu,\kappa\nu} S^{(p',p)*}_{\rho\rho,\lambda\mu} M_{\lambda\mu,\kappa\nu} S^{(p',p)}_{\kappa\nu,\rho\rho} \\
&= (S^{(p',p)*}MS^{(p',p)})_{\rho\rho,\rho\rho} = M_{\rho\rho,\rho\rho} = 1.
\end{aligned}$$

Therefore, $M_{A^a\rho A^b\rho,\rho\rho} \in \{0, 1\}$. But $M_{A^a\rho A^b\rho,\rho\rho} \neq 0 \iff A^a_{p'}A^b_p \in \mathcal{J}_L$, so $M_{A^a\rho A^b\rho,\rho\rho} = 1 \iff A^a_{p'}A^b_p \in \mathcal{J}_L$. But $M_{A^a\rho A^b\rho,\rho\rho} = 1 \iff M_{A^a\rho A^b\rho,\rho\rho} = M_{\rho\rho,\rho\rho}$, so we must have $e^{-\frac{2\pi i}{3}(at(\lambda)+bt(\mu))} = 1$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$ for some $\kappa\nu \in P^{p',p}_{++}$. This gives us (*b*), and that $A^a_{p'}A^b_p \in \mathcal{J}_L$ iff $at(\lambda) + bt(\mu) \equiv 0 \pmod 3$ $\forall \lambda\mu \in \mathcal{P}_L$, which is (d).

(c) Let $A^a_{p'}A^b_p \in \mathcal{J}_L$. Then

$$\begin{aligned} M_{A^a\lambda A^b\mu,\kappa\nu} &= \sum_{\alpha\beta,\gamma\delta} S^{(p',p)*}_{A^a\lambda A^b\mu,\alpha\beta} M_{\alpha\beta,\gamma\delta} S^{(p',p)}_{\gamma\delta,\kappa\nu} \\ &= \sum_{\alpha\beta,\gamma\delta} e^{-\frac{2\pi i}{3}(at(\alpha)+bt(\beta))} S^{(p',p)*}_{\lambda\mu,\alpha\beta} M_{\alpha\beta,\gamma\delta} S^{(p',p)}_{\gamma\delta,\kappa\nu}. \end{aligned}$$

But by the proof of (b), $at(\alpha)+bt(\beta) \equiv 0 \pmod 3$ whenever $M_{\alpha\beta,\gamma\delta} \neq 0$, so the exponential term is just a "1" whenever there is a nonzero term in the sum, and so we have (c).

(d) This was done in the proof of (b).

(e) By S-invariance, $(MS^{(p',p)})_{\lambda\mu,\rho\rho} = (S^{(p',p)}M)_{\lambda\mu,\rho\rho}$, for all $\lambda\mu \in P^{p',p}_{++}$, so

$$\begin{aligned} \sum_{\alpha\beta} M_{\lambda\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\rho\rho} &= \sum_{\alpha\beta} S^{(p',p)}_{\lambda\mu,\alpha\beta} M_{\alpha\beta,\rho\rho} \\ &= \sum_{a,b:A^aA^b\in\mathcal{J}_L} S^{(p',p)}_{\lambda\mu,A^a\rho A^b\rho} M_{A^a\rho A^b\rho,\rho\rho} \\ &= \sum_{a,b:A^aA^b\in\mathcal{J}_L} S^{(p',p)}_{\lambda\mu,A^a\rho A^b\rho}, \end{aligned}$$

by (b). By (2.10d), this is equal to $\sum_{a,b:A^aA^b\in\mathcal{J}_L} e^{\frac{2\pi}{3}(at(\lambda)+bt(\mu))} S^{(p',p)}_{\lambda\mu,\rho\rho}$, so we have

$$\sum_{\alpha\beta} M_{\lambda\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\rho\rho} = \sum_{a,b:A^aA^b\in\mathcal{J}_L} e^{\frac{2\pi i}{3}(at(\lambda)+bt(\mu))} S^{(p',p)}_{\lambda\mu,\rho\rho}. \tag{4.12}$$

If $\lambda\mu \in \mathcal{P}_L$, then since $S^{(p',p)}_{\alpha\beta,\rho\rho} > 0$ for all $\alpha\beta$, the left-hand side of (4.12) is nonzero iff $at(\lambda)+bt(\mu) \equiv 0 \pmod 3$ $\forall A^a_{p'}A^b_p \in \mathcal{J}_L$ (otherwise, we get the sum of the third roots of unity, which is 0). Therefore, $\lambda\mu \in \mathcal{P}_L \iff at(\lambda)+bt(\mu) \equiv 0 \pmod 3$. □

Corollary 4.1. *$\mathcal{J}_R$ and $\mathcal{J}_L$ are abelian groups.*

Proof. The set of all simple currents is a finite group **A** under composition, with identity $A^0_{p'}A^0_p$. Suppose that $A^a_{p'}A^b_p$, $A^c_{p'}A^d_p \in \mathcal{J}_L$. Then $(A^a_{p'}A^b_p)(A^c_{p'}A^d_p) \in \mathcal{J}_L$, because $A^a_{p'}A^b_pA^c_{p'}A^d_p = A^{a+c}_{p'}A^{b+d}_p$, so $M_{A^{a+c}\rho A^{b+d}\rho,\rho\rho} = M_{A^a(A^c\rho)A^b(A^d)\rho,\rho\rho} = M_{A^c\rho A^d\rho,\rho\rho} = M_{\rho\rho,\rho\rho}$, by 4.3(c), and the same argument applies to $\mathcal{J}_R$. Therefore, $\mathcal{J}_L$ and $\mathcal{J}_R$ are closed under composition, and hence are subgroups of **A**. They are abelian because $A^{a+c}_{p'}A^{b+d}_p = A^{c+a}_{p'}A^{d+b}_p$. □

We will need to use some properties of the Perron-Frobenius eigenvalue, which is described in the following theorem.

Theorem 4.1. (Perron-Frobenius Theorem) *[15] Let B be a square matrix with non-negative real entries. Then there is an eigenvalue $r(B)$, called the* Perron-Frobenius eigenvalue, *such that $r(B)$ is real and nonnegative, and for any other eigenvalue s of B, $r(B) \geq |s|$.*

For any square matrix M, we can simultaneously permute the rows and columns to write M as a direct sum of indecomposable submatrices as

$$M \sim \bigoplus_{i=1}^{\ell} B_i = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & B_\ell \end{pmatrix}, \tag{4.13}$$

where by $A \sim B$, we mean that A is some permutation of the rows and columns of B. By an indecomposable submatrix B, we mean that we cannot further write $B = B_1 \oplus B_2$. As an example, take

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 3 & 0 \\ 1 & 0 & 6 & 0 & 1 \\ 0 & 4 & 0 & 5 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}. \tag{4.14}$$

We can then write

$$M \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 6 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \end{pmatrix}, \tag{4.15}$$

so $M \sim B_1 \oplus B_2$, where $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, and $B_2 = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Of course, the order in which the B_i's appear in (4.13) is not unique. For M a modular invariant, we will always take B_1 to be the block containing $M_{\rho\rho,\rho\rho}$. Since each B_i is a submatrix of M, each B_i has nonnegative real entries, and so by Theorem 4.1 has a Perron-Frobenius eigenvalue $r(B_i)$. Let B be any B_i as in (4.13), and suppose B is also symmetric (which we will find later is always the case for an indecomposable submatrix of a modular invariant). Then we further have the properties:

(4.16a)
$$\min_a \sum_b B_{ab} \leq r(B) \leq \max_a \sum_b B_{ab};$$

(4.16b)
$$\max_a B_{aa} \leq r(B).$$

Define $B(m,\ell)$ to be the $\ell \times \ell$ matrix

(4.17)
$$B(m,\ell) = \begin{pmatrix} m & \cdots & m \\ \vdots & & \vdots \\ m & \cdots & m \end{pmatrix}.$$

We will find later that for any modular invariant M, most of its submatrices B_i will of the form $B(1,\ell)$. With these properties of the Perron-Frobenius eigenvalue, we get the following two lemmas:

Lemma 4.4. *[9] Let M be a modular invariant. Then $\mathcal{R}_R = \{(\rho,\rho)\}$ iff $\mathcal{R}_L = \{(\rho,\rho)\}$ iff M is an automorphism invariant.*

A consequence of Lemma 4.4 is that it means we have already found all modular invariants M having $\rho\rho$ as the only the $\rho\rho$-coupling, and so we do not need to consider that case in Chapter 5. The following lemma is the basis for much of Chapter 5.

Lemma 4.5. *[9] (a) Suppose M has $M_1 = B(1,m)$ for some m. Then for each i, either $M_i = (0)$ or $r(M_i) = m$. Also, for each $(\lambda,\mu) \in P_{++}^{p',p}$, $\sum_{\kappa\nu} M_{\lambda\mu,\kappa\nu}^2 \leq m^2/\|\mathcal{J}_L(\lambda,\mu)\|$.*

(b) Now suppose $\mathcal{R}_L = \mathcal{J}_L(\rho\rho)$ and $\mathcal{R}_R = \mathcal{J}_R(\rho\rho)$, and suppose that $M_{\lambda\mu,\kappa\nu} \neq 0$. Then
$$M_{\lambda\mu,\kappa\nu} \leq \frac{\|\mathcal{J}_L\|}{\sqrt{\|\mathcal{J}_L(\lambda\mu)\| \; \|\mathcal{J}_R(\kappa\nu)\|}}.$$

If, in addition, (λ,μ) *is not a fixed point of* $\mathcal{J}_L$ *(ie,* $J \in \mathcal{J}_L, J \neq A^0_{p'}A^0_p$ *implies* $J(\lambda\mu) \neq (\lambda\mu)$*), and also* (κ,ν) *is not a fixed point of* $\mathcal{J}_R$*, then* $M_{\lambda\mu,\kappa\nu} = 1$*. Moreover,* $M_{\lambda\mu,\alpha\beta} \neq 0$ *iff* $(\alpha\beta) \in \mathcal{J}_R(\kappa\nu)$*, and* $M_{\alpha\beta,\kappa\nu} \neq 0$ *iff* $(\alpha\beta) \in \mathcal{J}_L(\lambda\mu)$*.*

Lemmas 4.4 and 4.5 are the $A_2 \oplus A_2$ versions of Lemmas 2.1 and 2.2, respectively. We are now ready to state the $\rho\rho$-couplings as the following theorem.

Theorem 4.2. *Let* $\mathcal{R}_R = \{\lambda\mu \in P^{p',p}_{++} : M_{\rho\rho,\lambda\mu} \neq 0\}$ *and* $\mathcal{R}_L = \{\lambda\mu \in P^{p',p}_{++} : M_{\lambda\mu,\rho\rho} \neq 0\}$*, and define* $\rho'' := (\frac{p-2}{2}, \frac{p-2}{2})$*. Then* $M_{\lambda\mu,\rho\rho} \in \{0,1\}$*, and the choices for* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *are*

(a)(i) $p' \equiv_{12} 1,7,10$ *and* $p \equiv_{12} 2,5,11$*:*

$\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho)\}$,

$\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho),(A_{p'}\rho, A_p\rho),(A^2_{p'}\rho, A^2_p\rho)\}$,

$\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho),(A_{p'}\rho, A^2_p\rho),(A^2_{p'}\rho, A_p\rho)\}$,

$\mathcal{R}_R = \{(\rho,\rho),(A_{p'}\rho A_p\rho),(A^2_{p'}\rho, A^2_p\rho)\}$ and $\mathcal{R}_L = \{(\rho,\rho),(A_{p'}\rho, A^2_p\rho),(A^2_{p'}\rho, A_p\rho)\}$,

$\mathcal{R}_R = \{(\rho\rho),(A_{p'}\rho, A^2_p\rho),(A^2_{p'}\rho, A_p\rho)\}$ and $\mathcal{R}_L = \{(\rho,\rho),(A_{p'}\rho, A_p\rho),(A^2_{p'}\rho, A^2_p\rho)\}$;

(ii) $p' \equiv_{12} 3,6,9$ *and* $p \equiv_{12} \pm1,\pm2,\pm5, p' \neq 18$*:* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho)\}$*, or* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho),(A_{p'}\rho,\rho),(A^2_{p'}\rho,\rho)\}$*;*

(b) (i) $p' \equiv_{12} 1,5,7,11$ *and* $p \equiv_{12} 0, p \neq 12,24,60$*:* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho)\}$*, or* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho),(\rho,A_p\rho),(\rho,A^2_p\rho)\}$*;*

(ii) $p' \equiv_{12} 1,7$ *and* $p \equiv_{12} 8$*,* $p \neq 8$*, or* $p' \equiv_{12} 5,7,11$ *and* $p \equiv_{12} 4$*: the same choices for* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *as in* (a)*(i);*

(iii) $p' \equiv_{12} 3,9$ *and* $p \equiv_{12} 4,8$*,* $p \neq 8$*: the same choices for* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *as in* (a)*(ii);*

(c) $p' = 12$ *and* $p \equiv_{12} 1,7$*:* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho)\}$*,* $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{p'}\rho,\rho) : i = 0,1,2\}$*, or* $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{p'}\rho,\rho),(A^i_{p'}\rho'',\rho) : i = 0,1,2\}$

(d) $p' = 24$ *and* $p \equiv_{12} 1,5,7,11$*:* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho)\}$*, or* $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{p'}\rho,\rho),(A^i_{p'}\rho',\rho),(A^i_{p'}\rho'',\rho),(A^i_{p'}\rho''',\rho) : i = 0,1,2\}$*, where* $\rho' = (5,5)$*,* $\rho''' = (7,7)$*.*

(e) (i)$p' \equiv_{12} 1,7$ *and* $p = 8$*: the same choices for* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *as in (b)(ii), or* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho),(\rho,\rho'')\}$*, or any combination of* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *such that :*

$$\mathcal{R}_{R,L} = \{(\rho,\rho), (A_{p'}\rho, A_p\rho), (A_{p'}^2\rho, A_p^2\rho), (\rho,\rho''), (A_{p'}\rho, A_p\rho''), (A_{p'}^2\rho, A_p^2\rho'')\},$$

or

$$\mathcal{R}_{R,L} = \{(\rho,\rho), (A_{p'}\rho, A_p^2\rho), (A_{p'}^2\rho, A_p\rho), (\rho,\rho''), (A_{p'}\rho, A_p^2\rho''), (A_{p'}^2\rho, A_p\rho'')\}.$$

(ii) $p' \equiv_{12} 3, 9$ *and* $p = 8$*: the same choices for* $\mathcal{R}_R$ *and* $\mathcal{R}_L$ *as in (a)(ii), or* $\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho), (\rho,\rho'')\}$*, or*

$$\mathcal{R}_R = \mathcal{R}_L = \{(\rho,\rho), (A_{p'}\rho,\rho), (A_{p'}^2\rho,\rho), (\rho,\rho''), (A_{p'}\rho,\rho''), (A_{p'}^2\rho,\rho'')\},$$

plus a symmetric list as in Lemma 4.2.

We will prove (a), (b), and (e) here, and prove the remainder in Chapter 6.

Proof of (a), (b) and (e). First, let us suppose that $p' \equiv 3, 9 \pmod{12}$, and $p \equiv 4, 8 \pmod{12}$, as in Lemma 4.2(b)(*iii*). Then $\mathcal{J}_L$, $\mathcal{J}_R = \{A_{p'}^0 A_p^0\}$, or $\{A_{p'}^0 A_p^0, A_{p'}^1 A_p^0, A_{p'}^2 A_p^0\}$, by Corollary 4.1.

Define

$$m_L := \sum_{a=0}^{2} M_{A^a\rho\rho,\rho\rho}, \text{ and } m_L' := \sum_{a=0}^{2} M_{A^a\rho\rho'',\rho\rho}.$$

(m_R, m_R' are defined similarly). Suppose that $m_L' \neq 0$ (so $m_L' \geq 1$), and put $\lambda\mu = (\rho,(2,2))$ and $\lambda\mu = (\rho,(1,4))$ into Lemma 4.3(a). Since $S^{(p')}_{\rho,A^a\rho} = S^{(p')}_{\rho\rho}$ for all $a = 0,1,2$, we get that

$$\begin{aligned} s_L(\rho,(2,2)) &= \sum_{\kappa\nu} M_{\kappa\nu,\rho\rho} S^{(p')}_{\rho\kappa} S^{(p)}_{(2,2)\nu} \\ &= \left(\sum_{a=0}^{2} M_{A^a\rho\rho,\rho\rho}\right) S^{(p')}_{\rho\rho} S^{(p)}_{(2,2)\rho} + \left(\sum_{a=0}^{2} M_{A^a\rho\rho'',\rho\rho}\right) S^{(p')}_{\rho\rho} S^{(p)}_{(2,2)\rho''} \\ &= S^{(p')}_{\rho\rho} S^{(p)}_{(2,2)\rho} (m_L - m_L'), \end{aligned}$$

since $S^{(p)}_{(2,2)\rho''} = -S^{(p)}_{(2,2)\rho}$. But $S^{(p')}_{\lambda\rho}$, $S^{(p)}_{\mu\rho} > 0$, $\forall\lambda\mu$, so we must have $m_L - m_L' \geq 0$, or $m_L \geq m_L'$. But by Lemma 4.3(b), $M_{A^a\rho\rho'',\rho\rho} = M_{\rho\rho'',\rho\rho}$ for all a such that $A_{p'}^a A_p^0 \in \mathcal{J}_L$, so this means (since $M_{\rho\rho'',\rho\rho} \geq 1$), that if $m_L' \neq 0$, then $m_L' \geq m_L$. Therefore, we must have $m_L = m_L'$.

Putting $\lambda\mu = (\rho, (1,4))$ into Lemma 4.3(a) gives us

$$s_L(\rho(1,4)) = (S^{(p)}_{(1,4)\rho} + S^{(p)}_{(1,4)\rho''})S^{(p')}_{\rho\rho}m'_L \geq 0. \tag{4.18}$$

But $S^{(p)}_{(1,4)\rho} = \frac{8}{\sqrt{3}p}\sin\left(\frac{\pi}{p}\right)\sin(\frac{4\pi}{p})\sin(\frac{5\pi}{p}) = \frac{2}{\sqrt{3}p}\left(\sin(\frac{8\pi}{p}) + \sin(\frac{2\pi}{p}) - \sin(\frac{10\pi}{p})\right)$,

and $S^{(p)}_{(1,4)\rho''} = -\frac{8}{\sqrt{3}p}\cos(\frac{\pi}{p})\sin(\frac{4\pi}{p})\cos(\frac{5\pi}{p}) = \frac{2}{\sqrt{3}p}\left(-\sin(\frac{8\pi}{p}) + \sin(\frac{2\pi}{p}) - \sin(\frac{10\pi}{p})\right)$,

so (4.18) is true iff $\sin(\frac{2\pi}{p}) - \sin(\frac{10\pi}{p}) \geq 0$. This is a contradiction for $p \geq 20$, so for those p, $m'_L = 0$, and so $\mathcal{R}_R = \{(\rho,\rho)\}$ or $\{(A^i_{p'}\rho, \rho) : i = 0,1,2\}$.

We check the cases $p = 4, 8, 16$ separately. If $p = 4$, then $\rho'' = (\frac{4-2}{2}, \frac{4-2}{2}) = (1,1) = \rho$, so there is nothing to check, and $p = 16$ implies that $\sin(\frac{\pi}{8}) \geq \sin(\frac{5\pi}{8})$, which is false. However, $p = 8$ gives us $\sin(\frac{\pi}{4}) \geq \sin(\frac{5\pi}{4})$, which is true. Therefore, we have that for $p \neq 8$, $p' \equiv 3, 9 \pmod{12}$, and $p \equiv 4, 8 \pmod{12}$, $\mathcal{R}_R = \{(\rho,\rho)\}$ or $\mathcal{R}_R = \{(A^i_{p'}\rho, \rho) : i = 0,1,2\}$.

Since M commutes with $S^{(p',p)}$, we evaluate $(MS^{(p',p)})_{\rho\rho,\rho\rho} = (S^{(p',p)}M)_{\rho\rho,\rho\rho}$ to get $\sum_{\kappa\nu} M_{\rho\rho,\kappa\nu}S^{(p',p)}_{\kappa\nu,\rho\rho} = \sum_{\kappa\nu} S^{(p',p)}_{\rho\rho,\kappa\nu}M_{\kappa\nu,\rho\rho}$. Since $S^{(p',p)}_{A^a\rho A^b\rho,\rho\rho} = S^{(p',p)}_{\rho\rho,A^c\rho A^d\rho}$ for all $a,b,c,d \in \mathbb{Z}$, we get $S^{(p',p)}_{\rho\rho,\rho\rho}\sum_{a=0}^2 M_{\rho\rho,A^a\rho\rho} = S^{(p',p)}_{\rho\rho,\rho\rho}\sum_{a=0}^2 M_{A^a\rho\rho,\rho\rho}$, or $m_R = m_L$. Lemma 4.3(b) tells us that $M_{A^a\rho A^b\rho,\rho\rho} = M_{\rho\rho,A^c\rho A^d\rho} = M_{\rho\rho,\rho\rho} = 1$, so for all $A^a_{p'}A^b_p \in \mathcal{J}_L$, $A^c_{p'}A^d_p \in \mathcal{J}_R$, $\|\mathcal{R}_R\| = \|\mathcal{R}_L\|$, and we have Theorem 4.2(b)(iii) and (g)(ii).

Next we consider Lemma 4.2(b)(i). Corollary 4.1 tells us that $\mathcal{J}_L = \{A^0_{p'}A^0_p\}$ or $\{A^0_{p'}A^0_p, A^0_{p'}A^1_p, A^0_{p'}A^2_p\}$. Here, we define $m_L := \sum_{a=0}^2 M_{\rho A^a\rho,\rho\rho}$, and $m'_L := \sum_{a=0}^2 M_{\rho A^a\rho'',\rho\rho}$. $s_L(\rho,(2,2)) \geq 0$ and Lemma 4.3(b) again tell us that if $m'_L \neq 0$, then $m_L = m'_L$, and as above, $s_L(\rho,(1,4)) \geq 0$ iff $\sin(\frac{2\pi}{p}) \geq \sin(\frac{10\pi}{p})$. This is clearly true for $p \geq 20$, and since $p \neq 12$, we get Theorem 4.2(b)(i).

For (b) (ii), we again apply Corollary 4.1 to Lemma 4.2(b)(ii) to get $\mathcal{J}_L = \{A^0_{p'}A^0_p\}$, $\{A^0_{p'}A^0_p, A^1_{p'}A^1_p, A^2_{p'}A^2_p\}$, or $\{A^0_{p'}A^0_p, A^1_{p'}A^2_p, A^2_{p'}A^1_p\}$. Therefore, if $\lambda\mu \in \mathcal{R}_L$, then $\lambda\mu \in \mathcal{J}_L(\rho\rho)$, for one of the above choices of $\mathcal{J}_L$, and $\lambda\mu$ could also be in the set $\{(\rho,\rho''), (A_{p'}\rho, A_p\rho''), (A^2_{p'}\rho, A^2_p\rho''), (A_{p'}\rho, A^2_p\rho''), (A^2_{p'}\rho, A_p\rho'')\}$.

We define $m_L := \sum_{a=0}^2 M_{A^a\rho A^a\rho,\rho\rho}$, $m_\ell := \sum_{a=0}^2 M_{A^a\rho A^{-a}\rho,\rho\rho}$, $m'_L := \sum_{a=0}^2 M_{A^a\rho A^a\rho'',\rho\rho}$, and $m'_\ell := \sum_{a=0}^2 M_{A^a\rho A^{-a}\rho'',\rho\rho}$. First suppose that $\mathcal{J}_L = \{A^0_{p'}A^0_p\}$ or $\{A^0_{p'}A^0_p, A^1_{p'}A^1_p, A^2_{p'}A^2_p\}$. Then, as usual, if $m'_L \neq 0$, $m'_L \geq m_L$ by Lemma 4.3(b). If $\mathcal{J}_L = \{A^0_{p'}A^0_p, A^1_{p'}A^1_p, A^2_{p'}A^2_p\}$, $(A_{p'}\rho, A^2_p\rho'')$ and $(A^2_{p'}\rho, A_p\rho'')$ cannot be $\rho\rho$-couplings.

For example, if $M_{A\rho A^2\rho'',\rho\rho} \neq 0$, then Lemma 4.3(b) tells us that $M_{A^2\rho\rho'',\rho\rho} \neq 0$ either, which is false Lemma (4.2(b)(ii)), so we do not need to consider m'_ℓ.

We calculate $s_L(\rho,(2,2))$ and $s_L(\rho,(1,4))$ as above, which gives us that $m'_L = 0$ unless $p = 8$. For $\mathcal{J}_L = \{A^0_{p'}A^0_p, A^1_{p'}A^2_p, A^2_{p'}A^1_p\}$, we use m_ℓ and m'_ℓ and find that $m'_\ell = 0$ unless $p = 8$.

If $\mathcal{J}_L = \{A^0_{p'}A^0_p\}$, then the usual argument gives us contradictions for all $p \neq 8$. When $p = 8$, we still must have $m_L = m'_L$ (or $m_\ell = m'_\ell$), so we cannot have $\mathcal{R}_L = \{(\rho,\rho),(\rho,\rho''),(A_{p'}\rho, A_p\rho''),(A^2_{p'}\rho, A^2_p\rho'')\}$ or $\{(\rho,\rho),(\rho,\rho''),(A_{p'}\rho A^2_p\rho''),$ $(A^2_{p'}\rho, A_p\rho'')\}$. This gives us Theorem 4.2(b)(ii) and 4.2(g)(i). The fact that $M_{\lambda\mu,\rho\rho} \in \{0,1\}$ follows from Lemma 4.5(b) for (a) and (b). For (e), we have $m_L = m'_L$ (or $m_\ell = m'_\ell$) whenever $m'_L \neq 0$. If $m_L = 1$, then $m'_L = 1$ gives us $M_{\rho\rho'',\rho\rho} = 1$, and if $m'_L = m_L = 3$, then at least for one $J \in \mathcal{J}_L$, $M_{J(\rho\rho''),\rho\rho} \neq 0$. But now by Lemma 4.3(c), $M_{J(\rho\rho''),\rho\rho} \neq 0$ for all $J \in \mathcal{J}_L$, so they must all be 1. □

Chapter 5

The Simple-Current Extensions

In this chapter we complete the $A_{2,p'} \oplus A_{2,p}$ classification for the non-exceptional heights; that is, where neither of the heights p' or p is 8, 12, 18, 24, or 60. By the previous chapter, we know the $\rho\rho$ row and column of M. We will use Lemma 4.3 to extend this knowledge to all weights $\lambda\mu \in \mathcal{P}_L(p',p)$, $\kappa\nu \in \mathcal{P}_R(p'p)$. The strategy is the same in all cases: we see what happens to M at the "small" weights and build up to all weights $\lambda\mu$ from there. The weights we will use as the small weights differ from those in Chapter 3. In §5.2, they are $(\rho,(2,2))$, $(\rho,(1,4))$, $(\rho,(4,1))$, $((2,2),\rho)$, $((1,4),\rho)$, $((4,1),\rho)$, and in §5.3, they are $((2,2),\rho)$, $((1,4),\rho)$ and $((4,1),\rho)$.

In each case of Theorem 4.2(a) and (b), $M_{\lambda\mu,\rho\rho} \neq 0 \iff \lambda\mu = A^a_{p'}\rho A^b_p\rho$ for some $A^a_{p'} A^b_p \in \mathcal{J}_L$, and $M_{\rho\rho,\kappa\nu} \neq 0 \iff \kappa\nu = A^c_{p'}\rho A^d_p\rho$ for some $A^c_{p'} A^d_p \in \mathcal{J}_R$, so by Lemma 4.3(e), we have $\mathcal{P}_L = \{\lambda\mu \in P^{p',p}_{++} : at(\lambda) + bt(\mu) \equiv 0 \pmod 3\ \forall A^a_{p'} A^b_p \in \mathcal{J}_L\}$ and $\mathcal{P}_R = \{\kappa\nu \in P^{p',p}_{++} : ct(\kappa) + dt(\nu) \equiv 0 \pmod 3\ \forall A^c_{p'} A^d_p \in \mathcal{J}_R\}$. We also have $\|\mathcal{J}_L\| = \|\mathcal{J}_R\| = 1$ or 3. If $\|\mathcal{J}_L\| = \|\mathcal{J}_R\| = 1$, then M is an automorphism invariant by Lemma 4.4, and so has already been done in Chapter 3. Therefore, for all of this chapter, we will assume that $\|\mathcal{J}_L\| = \|\mathcal{J}_R\| = 3$. The goal of this chapter is the following theorem:

Theorem 5.1. *Let M be a modular invariant, and suppose that M has $\mathcal{R}_R = \mathcal{J}_R(\rho\rho)$ and $\mathcal{R}_L = \mathcal{J}_L(\rho\rho)$ for some $\mathcal{J}_R$ and $\mathcal{J}_L$; ie, the $\rho\rho$-couplings of M are just simple* *currents of $\rho\rho$. Then M is one of the following:*

(5.1a)
$$\mathcal{D}^{(2)}_{p',p} = \frac{1}{3} \sum_{\substack{\lambda\mu\in P^{p',p}_{++}\\ t(\lambda)+t(\mu)\equiv_3 0}} (\chi^{(p',p)}_{\lambda\mu} + \chi^{(p',p)}_{A\lambda A^{\pm1}\mu} + \chi^{(p',p)}_{A^{-1}\lambda A^{\mp1}\mu})(\chi^{(p',p)*}_{\lambda\mu} + \chi^{(p',p)*}_{A\lambda A^{\pm1}\mu} + \chi^{(p',p)*}_{A^{-1}\lambda A^{\mp1}\mu}) \text{ (for } p',p \not\equiv_3 0),$$

(5.1b)
$$\mathcal{D}^{(3)}_{p',p} = \frac{1}{3} \sum_{\substack{\lambda\mu: t(\lambda)\equiv_3 0\\ \lambda\neq\phi}} |\chi^{(p',p)}_{\lambda\mu} + \chi^{(p',p)}_{A\lambda\mu} + \chi^{(p',p)}_{A^2\lambda\mu}|^2 + 3 \sum_{\mu\in P^p_{++}} \chi^{(p',p)}_{\phi\mu}\chi^{(p',p)*}_{\phi\mu}$$
$$\text{where } \phi = (p'/3, p'/3) \text{ (for } p' \equiv_3 0, p \not\equiv_3 0),$$

up to multiplication by an automorphism invariant.

5.1 $\mathcal{J}$−orbits

In light of Lemma 4.3, it will be useful in this chapter to work with $\mathcal{J}$−orbits of weights, rather than the weights themselves. For $\lambda\mu \in \mathcal{P}_L$, we define $\langle\lambda\mu\rangle_L$ to be the $\mathcal{J}_L$−orbit $\{J(\lambda\mu) : J \in \mathcal{J}_L\}$, and for $\kappa\nu \in \mathcal{P}_R$, $\langle\kappa\nu\rangle_R := \{J'(\kappa\nu) : J' \in \mathcal{J}_R\}$. We will usually drop the subscripts L and R when it is clear to which we are referring, or our comments can apply to either. We denote the set of all $\mathcal{J}_L$−orbits $\langle\lambda\mu\rangle$ by $P_L/\langle\,\rangle$, and the set of all $\mathcal{J}_R$−orbits by $\mathcal{P}_R/\langle\,\rangle$. By Lemma 4.3(c), $M_{\lambda'\mu',\kappa'\nu'} = M_{\lambda\mu,\kappa\nu}$ for all $\lambda'\mu' \in \mathcal{J}_L$, $\kappa'\nu' \in \mathcal{J}_R$, so we write $M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle}$ to mean any representative $M_{\lambda'\mu',\kappa'\nu'}$ such that $\lambda'\mu' \in \mathcal{J}_L(\lambda\mu)$, $\kappa'\nu' \in \mathcal{J}_R(\kappa\nu)$. We also define

$$ch_{\langle\lambda\mu\rangle} := \sum_{\lambda'\mu'\in\langle\lambda\mu\rangle} \chi^{(p',p)}_{\lambda'\mu'}.$$

With this definition, we see that

$$\begin{aligned}
\mathcal{Z} &= \sum_{\lambda\mu,\kappa\nu} M_{\lambda\mu,\kappa\nu}\chi^{(p',p)}_{\lambda\mu}\chi^{(p',p)*}_{\kappa\nu} \\
&= \sum_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \sum_{\substack{\lambda'\mu'\in\langle\lambda\mu\rangle\\ \kappa'\nu'\in\langle\kappa\nu\rangle}} \chi^{(p',p)}_{\lambda'\mu'}\chi^{(p',p)*}_{\kappa'\nu'} \\
&= \sum_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \left(\sum_{\lambda'\mu'\in\langle\lambda\mu\rangle} \chi^{(p',p)}_{\lambda'\mu'}\right)\left(\sum_{\kappa'\nu'\in\langle\kappa\nu\rangle} \chi^{(p',p)*}_{\kappa'\nu'}\right),
\end{aligned}$$

so the partition function associated to M can be written as

(5.2)
$$\mathcal{Z} = \sum_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} ch_{\langle\lambda\mu\rangle} ch^*_{\langle\kappa\nu\rangle}.$$

5.2 The classification when $\mathcal{J}$ has no fixed points

We begin with Theorem 4.2(a)(i). Here, $\mathcal{J}_L$ and $\mathcal{J}_R$ are either $\{A^0_{p'}A^0_p,\ A^1_{p'}A^1_p,\ A^2_{p'}A^2_p\}$ or $\{A^0_{p'}A^0_p,\ A^1_{p'}A^2_p,\ A^2_{p'}A^1_p\}$, and $\mathcal{P}_L, \mathcal{P}_R = \{\lambda\mu \in P^{p',p}_{++} : t(\lambda) \pm t(\mu) \equiv 0 \pmod 3\}$, where the plus or minus sign depends on which $\mathcal{J}_L$ we choose. When we deal with $M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle}$ however, we may suppose that $t(\lambda) \equiv 0 \pmod 3$, so in either case we can assume that $t(\lambda) \equiv t(\mu) \equiv 0 \pmod 3$, whenever $M^e_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0$ for some $\langle\kappa\nu\rangle$. We can do this because $t(A^i_{p'}\lambda) = ip' + t(\lambda)$, and since $3 \nmid p'$, $p' \equiv \pm 1 \pmod 3$. This means that exactly one of the representatives $A^i_{p'}\lambda$ of the $\mathcal{J}_L$−orbit $\langle\lambda\mu\rangle$ will have $t(A^i_{p'}\lambda) \equiv 0 \pmod 3$, and so $\|\mathcal{P}_L/\langle\,\rangle\| = \|\mathcal{P}_R/\langle\,\rangle\| = \|\{\lambda\mu \in P^{p',p}_{++} : t(\lambda) \equiv t(\mu) \equiv 0 (\mathrm{mod}\ 3)\}\|$.

Both of the above possibilities for $\mathcal{J}_L$ and $\mathcal{J}_R$ have no fixed points: suppose $A^i_{p'}A^j_p(\lambda\mu) = (\lambda\mu)$, where $i = \pm j \neq 0$. Then, in particular, $A^i_{p'}\lambda = \lambda$. But this implies $t(\lambda) = t(A^i_{p'}\lambda)$; ie, $ip' + t(\lambda) \equiv t(\lambda) \pmod 3$. Since $i \neq 0$, this means that $p' \equiv 0 \pmod 3$, which is a contradiction (for Theorem 4.9b(a)(i)). Therefore, we may use Lemma 4.5(b), and it applies to every weight in $P^{p',p}_{++}$. Suppose $M_{\lambda\mu,\kappa\nu} \neq 0$. By Lemma 4.3(c), $M_{\lambda'\mu',\kappa'\nu'} = M_{\lambda\mu,\kappa\nu}\ \forall \lambda'\mu' \in \mathcal{J}_L(\lambda\mu)$, $\kappa'\nu' \in \mathcal{J}_R(\kappa\nu)$ and by Lemma 4.5(b), these equal 1, and $\lambda'\mu' \in \mathcal{J}_L(\lambda\mu)$ are the only weights that can couple to $\kappa'\nu' \in \mathcal{J}_R(\kappa\nu)$ (and vice-versa).

Therefore, we can define a permutation $\sigma : \{\mathcal{J}_L - \text{orbits}\} \longrightarrow \{\mathcal{J}_R - \text{orbits}\}$ such that $M^e_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} \neq 0 \iff \langle\kappa\nu\rangle_R = \sigma\langle\lambda\mu\rangle_L$, and $M^e_{\langle\lambda\mu\rangle,\sigma\langle\lambda\mu\rangle} = 1$. Notice that, since σ takes an orbit to an orbit, if $\sigma\langle\lambda\mu\rangle = \langle\kappa\nu\rangle$, we may take $\kappa\nu$ to be any representative of $\langle\kappa\nu\rangle_R = \{J'(\kappa\nu) : J' \in \mathcal{J}_R(\kappa\nu)\}$. Equation (5.2) tells us that the partition function associated to M in this case is

$$\mathcal{Z} = \sum_{\langle\lambda\mu\rangle} ch_{\langle\lambda\mu\rangle} ch^*_{\sigma\langle\lambda\mu\rangle}, \tag{5.3}$$

where $\lambda\mu \in \mathcal{P}_L$. To describe M, we must now describe σ, and find which permutations σ will give us a modular invariant. We already know what happens to $\rho\rho$ under σ. By Lemma 4.3(b), $M_{J(\rho\rho),J'(\rho\rho)} = M_{\rho\rho,\rho\rho}\ \forall J \in \mathcal{J}_L, J' \in \mathcal{J}_R$, so $\sigma\langle\rho\rho\rangle = \langle\rho\rho\rangle$.

Let $\lambda\mu \in \mathcal{P}_L$, $\kappa\nu \in \mathcal{P}_R$. Then by Lemma 4.5,

$$(MS^{(p',p)})_{\lambda\mu,\kappa\nu} = \sum_{\alpha\beta} M_{\lambda\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\kappa\nu} = \sum_{\lambda'\mu' \in \sigma\langle\lambda\mu\rangle} S^{(p',p)}_{\lambda'\mu',\kappa\nu}.$$

But $\sigma\langle\lambda\mu\rangle$ is a $\mathcal{J}_R$–orbit, so the above sum is equal to

$$\sum_{c,d:A^cA^d\in\mathcal{J}_R} S^{(p',p)}_{A^c\lambda' A^d\mu',\kappa\nu} = \sum_{c,d} e^{\frac{2\pi i}{3}(ct(\kappa)+dt(\nu))} S^{(p',p)}_{\lambda'\mu',\kappa\nu} = 3S^{(p',p)}_{\lambda'\mu',\kappa\nu},$$

because $A^cA^d \in \mathcal{J}_R$ and $\kappa\nu \in \mathcal{P}_R$, so by Lemma 4.3(d), $ct(\kappa) + dt(\nu) \equiv 0 \pmod 3$. Similarly, $(S^{(p',p)}M)_{\lambda\mu,\kappa\nu} = 3S^{(p',p)}_{\lambda\mu,\kappa' nu'}$, for any $\kappa'\nu' \in \sigma^{-1}\langle\kappa\nu\rangle$. S-invariance then gives us $S^{(p',p)}_{\lambda'\mu',\kappa\nu} = S^{(p',p)}_{\lambda\mu,\kappa'\nu'}$; or equivalently,

$$S^{(p',p)}_{\lambda\mu,\kappa\nu} = S^{(p',p)}_{\lambda'\mu',\kappa'\nu'}, \tag{5.4}$$

for $\lambda\mu$ any representative of $\langle\lambda\mu\rangle \in \mathcal{P}_L/\langle\,\rangle$, $\kappa\nu$ any representative of $\langle\kappa\nu\rangle \in \mathcal{P}_L/\langle\,\rangle$, any $\lambda'\mu' \in \sigma\langle\lambda\mu\rangle$ and any $\kappa'\nu' \in \sigma\langle\kappa\nu\rangle$.

We are now ready to check what happens to the small weights under σ, using the following claim.

Claim 5.1. *$\sigma\langle\rho,(2,2)\rangle = \langle\rho,(2,2)\rangle$ and $\sigma\langle\rho,(1,4)\rangle = \langle\rho, C_p^a(1,4)\rangle$, for some $a \in \{0,1\}$.*

Proof. Let $\sigma\langle\rho,(2,2)\rangle = \langle\rho',(2,2)\rangle$ (and similarly for $\langle\rho,(1,4)\rangle$. By (5.4), $Q^{(p',p)}(\rho,(2,2)) = Q^{(p',p)}(\rho',(2,2)')$, so since $Q^{(p',p)} = Q^{(p')}Q^{(p)}$,

$$\frac{Q^{(p')}(\rho)}{Q^{(p')}(\rho')} = \frac{Q^{(p)}(2,2)'}{Q^{(p)}(2,2)}. \tag{5.5}$$

Since $S^{(p')}_{\rho'\rho} \geq S^{(p')}_{\rho\rho}$,

$$1 \geq \frac{S^{(p')}_{\rho\rho}}{S^{(p')}_{\rho'\rho}} = \frac{Q^{(p')}(\rho)}{Q^{(p')}(\rho')} = \frac{Q^{(p)}(2,2)'}{Q^{(p)}(2,2)},$$

so $Q^{(p)}(2,2)' \leq Q^{(p)}(2,2)$. By Lemma 3.2, we then have $Q^{(p)}(2,2)' = Q^{(p)}(2,2)$ or $Q^{(p)}(\rho)$ when $p \geq 12$.

Suppose that $Q^{(p)}(2,2)' = Q^{(p)}(\rho)$. Then, by Lemma 3.1, $(2,2)' \in \mathcal{O}\rho$. Because $\sigma\langle\rho,(2,2)\rangle = \langle\rho',(2,2)'\rangle$, we are assuming $M_{\rho(2,2),\rho'(2,2)'} \neq 0$, so $(2,2)$ and $(2,2)'$ must satisfy the decoupled norm (4.3). This implies $3 \equiv 12 \pmod p$. But then $p \mid 9$ which is a contradiction since $3 \nmid p$ and $p \geq 4$, so $Q^{(p)}(2,2)' = Q^{(p)}(2,2)$. Now by Lemma 3.1, $(2,2)' \in \mathcal{O}(2,2)$.

Equation (5.5) now tells us that

$$\frac{Q^{(p')}(\rho)}{Q^{(p')}(\rho')} = 1; \text{ ie, } S^{(p')}_{\rho\rho} = S^{(p')}_{\rho\rho'},$$

and this can happen iff $\rho' \in \mathcal{O}\rho$. Therefore, $\sigma\langle\rho,(2,2)\rangle = \langle A^i_{p'}\rho, A^j_p(2,2)\rangle$ for some $i,j \in \{0,1,2\}$. But we can assume $t(A^i_{p'}\rho) \equiv t(A^j_p(2,2)) \equiv 0 \pmod 3$, so we put $i = j = 0$. In this case, we did not need to consider the charge conjugations, because $C_{p'}\rho = \rho$ and $C_p(2,2) = (2,2)$.

Replacing $(2,2)$ in (5.5) with $(1,4)$, we get $Q^{(p)}(1,4)' \leq Q^{(p)}(1,4)$, and by Lemma 3.2, $Q^{(p)}(1,4)' \in \{Q^{(p)}(\rho), Q^{(p)}(2,2), Q^{(p)}(1,4) = Q^{(p)}(4,1)\}$ for $p \geq 12$. By Lemma 3.1, $(1,4)' \in \{\mathcal{O}\rho, \mathcal{O}(2,2), \mathcal{O}(1,4)\}$. If $(1,4)' \in \mathcal{O}\rho$, then (4.3) tells us that $p \mid 18$, which implies $p = 1$ or 2, contradicting $p \geq 4$. If $(1,4)' \in \mathcal{O}(2,2)$, then we again have $3 \mid p$. Therefore, $(1,4)' \in \mathcal{O}(1,4)$, and choosing $t((1,4)') \equiv 0 \pmod 3$, we have that $(1,4)' = (1,4)$ or $(4,1)$, and so $\sigma\langle\rho,(1,4)\rangle = \langle\rho, C^a_p(1,4)\rangle$.

Checking each of $p = 4,\ldots,11$ separately, we find that we also have $(2,2)' \in \mathcal{O}(2,2)$ and $(1,4)' \in \mathcal{O}(1,4)$ at those heights. □

The proof of Claim 5.1 also applies to the weights $((2,2),\rho)$ and $((1,4),\rho)$, so we have $\sigma\langle(2,2),\rho\rangle = \langle(2,2),\rho\rangle$ and $\sigma\langle(1,4),\rho\rangle = \langle C^b_{p'}(1,4),\rho\rangle$ for some $b \in \{0,1\}$. As in Chapter 3, letting $\sigma' := C^{(b,a)} \circ \sigma$, we have $\sigma'\langle\alpha\beta\rangle = \langle\alpha\beta\rangle$ for $\alpha\beta \in \{(\rho,(2,2)), (\rho,(1,4)), ((2,2),\rho), ((1,4),\rho)\}$. Since $(1,4) = C(4,1)$ and σ' commutes with $C^{(1,1)}$, we also have $\sigma'\langle\rho,(4,1)\rangle = \langle\rho,(4,1)\rangle$ and $\sigma'\langle(4,1),\rho\rangle = \langle(4,1),\rho\rangle$. In matrix terms, this amounts to multiplying our modular invariant M by one of the charge conjugations ${}^CI$, ${}^CI^C$, I^C. Multiplying a modular invariant by an automorphism invariant gives another modular invariant, so the product M' defined by σ' is a modular invariant. As in Chapter 3, we will show that any σ fixing the small weights is the identity.

Let $\lambda\mu \in \mathcal{P}_L$. Then by (5.4) and Claim 5.1,

$$\frac{S^{(p)}_{(2,2)\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p')}_{\rho\lambda}}{S^{(p')}_{\rho\lambda}} \cdot \frac{S^{(p)}_{(2,2)\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p',p)}_{\rho(2,2),\lambda\mu}}{S^{(p',p)}_{\rho\rho,\lambda\mu}} = \frac{S^{(p',p)}_{\rho(2,2),\lambda'\mu'}}{S^{(p',p)}_{\rho\rho,\lambda'\mu'}} = \frac{S^{(p')}_{\rho\lambda'}}{S^{(p')}_{\rho\lambda'}} \cdot \frac{S^{(p)}_{(2,2)\mu'}}{S^{(p)}_{\rho\mu'}} = \frac{S^{(p)}_{(2,2)\mu'}}{S^{(p)}_{\rho\mu'}},$$

and a similar calculation holds for all of the other small weights. We therefore have equations (5.6) below.

Choose any $\lambda\mu, \kappa\nu \in \mathcal{P}_L$. Let $\lambda'\mu' \in \sigma\langle\lambda\mu\rangle$ and $\kappa'\nu' \in \sigma\langle\kappa\nu\rangle$. Then

$$\text{(5.6a)} \qquad \frac{S^{(p')}_{(1,4)\lambda}}{S^{(p')}_{\rho\lambda}} = \frac{S^{(p')}_{(1,4)\lambda'}}{S^{(p')}_{\rho\lambda'}}; \quad \frac{S^{(p')}_{(4,1)\lambda}}{S^{(p')}_{\rho\lambda}} = \frac{S^{(p')}_{(1,4)\lambda'}}{S^{(p')}_{\rho\lambda'}}; \quad \frac{S^{(p')}_{(2,2)\lambda}}{S^{(p')}_{\rho\lambda}} = \frac{S^{(p')}_{(2,2)\lambda'}}{S^{(p')}_{\rho\lambda'}};$$

$$(5.6b)\qquad \frac{S^{(p)}_{(1,4)\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p)}_{(1,4)\mu'}}{S^{(p)}_{\rho\mu'}};\ \frac{S^{(p)}_{(4,1)\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p)}_{(4,1)\mu'}}{S^{(p)}_{\rho\mu'}};\ \frac{S^{(p)}_{(2,2)\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p)}_{(2,2)\mu'}}{S^{(p)}_{\rho\mu'}}.$$

Recall that in the proof of Proposition 3.1, we wrote a Weyl character ch_β as a polynomial in $ch_{(1,0)}$ and $ch_{(0,1)}$. When β has $t(\beta) \equiv_3 0$, we do this with $ch_{(1,1)}$, $ch_{(0,3)}$, and $ch_{(3,0)}$ as well; ie,

$$ch_\beta = P_\beta(ch_{(1,1)}, ch_{(3,0)}, ch_{(0,3)}),$$

provided $t(\beta) \equiv 0 \pmod 3$ [3]. Therefore, (3.7) gives us

$$(5.7)\qquad \frac{S^{(n)}_{\beta\alpha}}{S^{(n)}_{\rho\alpha}} = P_\beta\left(\frac{S^{(n)}_{(2,2)\,\alpha}}{S^{(n)}_{\rho\alpha}}, \frac{S^{(n)}_{(1,4)\,\alpha}}{S^{(n)}_{\rho\alpha}}, \frac{S^{(n)}_{(4,1)\,\alpha}}{S^{(n)}_{\rho\alpha}}\right).$$

Now let $\lambda\mu,\ \kappa\nu \in \mathcal{P}_L$, where $\kappa\nu$ is any representative of $\langle\kappa\nu\rangle$ and $t(\lambda) \equiv t(\mu) \equiv 0 \pmod 3$. Then for any $\kappa'\nu' \in \sigma\langle\kappa\nu\rangle$, equations (5.7) and (5.6) give us

$$\begin{aligned}
\frac{S^{(p',p)}_{\lambda\mu\,\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa\nu}} &= \frac{S^{(p')}_{\lambda\kappa}}{S^{(p')}_{\rho\kappa}}\cdot\frac{S^{(p)}_{\mu\nu}}{S^{(p)}_{\rho\nu}}\\
(5.8)\qquad &= P_\lambda\left(\frac{S^{(p')}_{(2,2)\,\kappa}}{S^{(p')}_{\rho\kappa}}, \frac{S^{(p')}_{(1,4)\kappa}}{S^{(p')}_{\rho\kappa}}, \frac{S^{(p')}_{(4,1)\kappa}}{S^{(p')}_{\rho\kappa}}\right)\cdot P_\mu\left(\frac{S^{(p)}_{(2,2)\nu}}{S^{(p)}_{\rho\nu}}, \frac{S^{(p)}_{(1,4)\nu}}{S^{(p)}_{\rho\nu}}, \frac{S^{(p)}_{(4,1)\nu}}{S^{(p)}_{\rho\nu}}\right)\\
&= P_\lambda\left(\frac{S^{(p')}_{(2,2)\,\kappa'}}{S^{(p')}_{\rho\kappa'}}, \frac{S^{(p')}_{(1,4)\kappa'}}{S^{(p')}_{\rho\kappa'}}, \frac{S^{(p')}_{(4,1)\kappa'}}{S^{(p')}_{\rho\kappa'}}\right)\cdot P_\mu\left(\frac{S^{(p)}_{(2,2)\nu'}}{S^{(p)}_{\rho\nu'}}, \frac{S^{(p)}_{(1,4)\nu'}}{S^{(p)}_{\rho\nu'}}, \frac{S^{(p)}_{(4,1)\nu'}}{S^{(p)}_{\rho\nu'}}\right)\\
&= \frac{S^{(p')}_{\lambda\kappa'}}{S^{(p')}_{\rho\kappa'}}\cdot\frac{S^{(p)}_{\mu\nu'}}{S^{(p)}_{\rho\nu'}} = \frac{S^{(p')}_{\lambda\kappa'}S^{(p)}_{\mu\nu'}}{S^{(p')}_{\rho\kappa'}S^{(p)}_{\rho\nu'}} = \frac{S^{(p',p)}_{\lambda\mu,\kappa'\nu'}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}}.
\end{aligned}$$

Now we can apply the fact that $S^{(p',p)}$ is unitary by multiplying (5.8) by $\sum_{a=0}^{2} S^{(p',p)*}_{\lambda\mu,A^a\kappa A^{\pm a}\nu}$ and summing over all $\lambda\mu \in \mathcal{P}_L$, so

$$(5.9)\qquad \sum_{\lambda\mu\in\mathcal{P}_L}\frac{S^{(p',p)}_{\lambda\mu,\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa\nu}}\left(\sum_{a=0}^{2} S^{(p',p)*}_{\lambda\mu,A^a\kappa A^{\pm a}\nu}\right) = \sum_{\lambda\mu\in\mathcal{P}_L}\frac{S^{(p',p)}_{\lambda\mu,\kappa'\nu'}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}}\left(\sum_{a=0}^{2} S^{(p',p)*}_{\lambda\mu,A^a\kappa A^{\pm a}\nu}\right),$$

where by Formulas (2.10d),

$$\begin{aligned}
&\textstyle\sum_{a=0}^{2}\ S^{(p',p)*}_{\lambda\mu,A^a\kappa A^{\pm a}\nu}\\
&= S^{(p')*}_{\lambda\kappa}S^{(p)*}_{\mu\nu} + e^{\frac{2\pi i}{3}-t(\lambda)}S^{(p')*}_{\lambda\kappa}e^{\frac{2\pi i}{3}(\mp t(\mu))}S^{(p)*}_{\mu\nu} + e^{\frac{2\pi i}{3}(t(\lambda))}S^{(p')*}_{\lambda\kappa}e^{\frac{2\pi i}{3}(\pm t(\mu))}S^{(p)*}_{\mu\nu}\\
&= \{1 + e^{-\frac{2\pi i}{3}(t(\lambda)\pm t(\mu))} + e^{\frac{2\pi i}{3}(t(\lambda)\pm t(\mu))}\}S^{(p',p)*}_{\lambda\mu,\kappa\nu}.
\end{aligned}$$

If $t(\lambda) \pm t(\mu) \equiv 0 \pmod 3$ (the plus or minus sign depends on whether we are taking $\mathcal{J}_L$ to be $\{A^0_{p'}A^0_p, A^1_{p'}A^1_p, A^2_{p'}A^2_p\}$ or $\{A^0_{p'}A^0_p, A^1_{p'}A^2_p, A^2_{p'}A^1_p\}$), then the sum in the brackets is 3; otherwise we get the sum of the third roots of unity, which is 0. Therefore, $\sum_{a=0}^{2} S^{(p',p)*}_{\lambda\mu, A^a\kappa A^{\pm a}\nu} = 0$ unless $\lambda\mu \in \mathcal{P}_L$, so (5.9) becomes

$$\frac{3}{S^{(p',p)}_{\rho\rho,\kappa\nu}} \sum_{\lambda\mu} S^{(p',p)}_{\kappa\nu,\lambda\mu} S^{(p',p)*}_{\lambda\mu,\kappa\nu} = \frac{3}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}} \sum_{\lambda\mu} S^{(p',p)}_{\kappa'\nu',\lambda\mu} S^{(p',p)*}_{\lambda\mu,\kappa\nu},$$

where the sum is taken over all $\lambda\mu \in P^{p',p}_{++}$.

Because $S^{(p',p)}$ is unitary, the left-hand side of this equation is just $3/S^{(p',p)}_{\rho\rho,\kappa\nu}$, whereas the right-hand side is $3\delta_{\kappa\nu,\kappa'\nu'}/S^{(p',p)}_{\rho\rho,\kappa'\nu'}$. As the left-hand side is nonzero, we must have $\kappa'\nu' = \kappa\nu$. But we chose $\kappa'\nu'$ to be any weight in $\sigma\langle\kappa\nu\rangle$, and $\kappa\nu$ any weight in $\langle\kappa\nu\rangle$, so this means $\langle\kappa'\nu'\rangle = \langle\kappa\nu\rangle$; ie, $\sigma\langle\kappa\nu\rangle = \langle\kappa\nu\rangle$.

What we have just shown is that $\sigma' = C^{(a,b)} \circ \sigma$ is the identity on $\mathcal{P}_L/\langle\ \rangle$; ie, $\sigma'\langle\lambda\mu\rangle_L = \langle\lambda\mu\rangle_R$. Putting back the charge conjugations, we see that the partition function associated to M is

$$\mathcal{Z} = \sum_{\langle\lambda\mu\rangle} ch_{\langle\lambda\mu\rangle} ch^*_{C^u\langle\lambda\mu\rangle}, \tag{5.10}$$

where $u \in \{(0,0),(0,1),(1,0),(1,1)\}$. The above argument also carries through for Theorem 4.2(b)(ii), because $3 \nmid p', p$ there either.

5.3 The classification when $\mathcal{J}$ has a fixed point

In this section, we consider $M_{\lambda\mu,\kappa\nu}$ when one of the weights $\lambda\mu, \kappa\nu$ may be a fixed point of $\mathcal{J}_L$ or $\mathcal{J}_R$. This occurs in Theorem 4.2 (a)(ii), (iii), and (b)(i),(iii). Notice that in all of these cases, $\mathcal{J}_L = \mathcal{J}_R$. We will do Theorem 4.2(a)(ii). Here we have

$$\mathcal{J}_L = \mathcal{J}_R = \{A^0_{p'}A^0_p, A^1_{p'}A^0_p, A^2_{p'}A^0_p\},$$

and by Lemma 4.3(e),

$$\mathcal{P}_L = \mathcal{P}_R = \{\lambda\mu \in P^{p',p}_{++} : t(\lambda) \equiv_3 0\},$$

so we put $\mathcal{J} := \mathcal{J}_L = \mathcal{J}_R$ and $\mathcal{P} := \mathcal{P}_L = \mathcal{P}_R$. Let $\phi := (\frac{p'}{3}, \frac{p'}{3})$. Then we have a fixed point of $\mathcal{J}$ at $(\phi\mu)$, for any $\mu \in P_{++}(A_2, p)$. Let $\mathcal{K}^\phi := \{\alpha\beta \in \mathcal{P} : \alpha \neq \phi \text{ and } M_{\alpha\beta,\phi\mu} \neq 0\}$, so $\mathcal{K}^\phi$ is the set of nonfixed points that couple to a fixed point.

We can define a permutation σ, as before, on all the $\mathcal{J}$-orbits of nonfixed points that are not in $\mathcal{K}^\phi$, so for all $\lambda\mu \notin \mathcal{K}^\phi$, $\lambda \neq \phi$, $M^e_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} \neq 0 \iff M^e_{\langle\lambda\mu\rangle,\langle\kappa\nu\rangle} = 1$ and $\langle\kappa\nu\rangle = \sigma\langle\lambda\mu\rangle$.

We would like to apply Claim 5.1 to the weights $((2,2),\rho)$, $((1,4),\rho)$ and $((4,1),\rho)$; however, we must first show that these weights are neither fixed points nor couple to a fixed point so that we can apply σ to them. We will need a few results before we can do this. First we show that the $\mathcal{J}$-orbit $\langle\rho\mu\rangle$ is more or less mapped to itself by σ.

Claim 5.2. *The weights $\rho\mu$ are not fixed points of $\mathcal{J} = \{A^0_{p'}A^0_p, A^1_{p'}A^0_p, A^0_{p'}A^2_p\}$ and $\rho\mu \notin \mathcal{K}^\phi$. Define $M^{(p)}_{\mu\mu'} = 1$ if $M_{\rho\mu,\rho\mu'} = 1$ and 0 for all other entries of $M^{(p)}$. Then $M^{(p)}$ is an automorphism invariant of $A_{2,p}$.*

Proof. The weight $\rho\mu$ is a fixed point of $\mathcal{J}$ iff $p'-2 = 1$. But $p' \geq 4$, so this cannot happen. Now, suppose that $M_{\rho\mu,\phi\nu} \neq 0$. Then, by Lemma 4.1, $\phi \in \mathcal{O}\rho \cup \mathcal{O}\rho''$[1]. But $\phi \notin \mathcal{O}\rho$ since $p' \geq 4$, $\phi \in \mathcal{O}\rho'' \iff \frac{p'}{3} = \frac{p'-2}{2} \iff p' = 6$, in which case $\phi = (2,2)$. Putting ρ and $(2,2)$ into the decoupled norm condition (4.9b) gives us a contradiction, so $M_{\rho\mu,\phi\nu} = 0$ at every non-exceptional height (p',p), and so we are done the first part.

We may now apply σ to $\rho\mu$. Let $\sigma\langle\rho\mu\rangle = \langle\rho\mu\rangle = \langle\rho'\mu'\rangle = \{(\rho'\mu'), (A_{p'}\rho', \mu'),$ $(A^2_{p'}\rho', \mu')\}$, so that we have $M_{\rho\mu,\rho'\mu'} \neq 0$. Again by Lemma 4.1, we have $\rho' \in \mathcal{O}\rho \cup \mathcal{O}\rho''$. Suppose that $\rho' \in \mathcal{O}\rho''$. We may assume that $\rho' = \rho''$. Putting $\rho\mu = \lambda\mu$ and $\rho\rho = \kappa\nu$ into (5.4) and using $\sigma\langle\rho\rho\rangle = \langle\rho\rho\rangle$ and $\sigma\langle\rho\mu\rangle = \langle\rho''\mu'\rangle$, we have $S^{(p')}_{\rho\rho}/S^{(p')}_{\rho''\rho} = S^{(p)}_{\mu'\rho}/S^{(p)}_{\mu\rho}$. By the definition of $S^{(p')}$ and using the identity $\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$, we have

$$\frac{S^{(p')}_{\rho\rho}}{S^{(p')}_{\rho''\rho}} = \frac{[e^{i\frac{2\pi}{p'}} - e^{-i\frac{2\pi}{p'}}][e^{i\frac{\pi}{p'}} - e^{-i\frac{\pi}{p'}}]^2}{[e^{i\frac{\rho''_1\pi}{p'}} - e^{-i\frac{\rho''_1\pi}{p'}}][e^{i\frac{\rho''_2\pi}{p'}} - e^{-i\frac{\rho''_2\pi}{p'}}][e^{i\frac{(\rho''_1+\rho''_2)\pi}{p'}} - e^{-i\frac{(\rho''_1+\rho''_2)\pi}{p'}}]},$$

which is a polynomial over $\mathbb{Q}$ in $\xi_{p'} = e^{\frac{2\pi i}{p'}}$. Similarly, $S^{(p)}_{\rho\mu'}/S^{(p)}_{\rho\mu}$ is a polynomial over $\mathbb{Q}$ in ξ_p. But $\gcd(p',p) = 1 \implies \mathbb{Q}(\xi_{p'}) \cap \mathbb{Q}(\xi_p) = \mathbb{Q}$ [21], so

$$\frac{S^{(p')}_{\rho\rho}}{S^{(p')}_{\rho''\rho}} = \frac{S^{(p)}_{\rho\mu'}}{S^{(p)}_{\rho\mu}} \in \mathbb{Q}(\xi_{p'}) \cap \mathbb{Q}(\xi_p) = \mathbb{Q}, \tag{5.11}$$

[1] Since we are actually doing Theorem 4.2(a)(ii), Lemma 4.1(a) gives us just $\phi \in \mathcal{O}\rho$, but we want the proof to apply to all the non-exceptional levels with fixed points, so we may as well consider the case $\phi \in \mathcal{O}\rho''$ now.

but by the proof of Claim 3 in [14], $S^{(p')}_{\rho\rho}/S^{(p')}_{\rho''\rho} \in \mathbb{Q}$ iff $p' \leq 6$. Now $3 \mid p'$ and $p' \geq 4$ leave only the possibility $p' = 6$, so we must have $\rho' \in \mathcal{O}\rho$. Putting $\langle\lambda\mu\rangle = \langle\rho\mu\rangle, \langle\kappa\nu\rangle = \langle\rho\nu\rangle$ into (5.4), we then have

$$S^{(p)}_{\mu\nu} = S^{(p)}_{\mu'\nu'}, \text{ where } \sigma\langle\rho\mu\rangle = \langle\rho\mu'\rangle,\ \sigma\langle\rho\nu\rangle = \langle\rho\nu'\rangle. \tag{5.12}$$

Let $\pi : P^p_{++}(A_2) \to P^p_{++}(A_2)$ be defined by $\pi\mu = \mu'$, where $\sigma\langle\rho\mu\rangle = \langle\rho\mu'\rangle$. Notice that μ' does not change with the $\mathcal{J}$−orbit $\langle\rho\mu'\rangle = \{(A^i_{p'}\rho, \mu') : i = 0, 1, 2\}$, so π is well-defined. We know that $\rho\mu \in \mathcal{P}$ for all $\mu \in P^p_{++}(A_2)$, since $t(\rho) = 0$, so for any $\mu \in P^p_{++}(A_2)$, $\mu = \pi\mu'$, where $\langle\rho\mu\rangle = \sigma\langle\rho\mu'\rangle$, and π is one-to-one since σ is. Therefore π is a permutation on $P_{++}(A_2, p)$, and $M^{(p)}_{\mu\alpha} = \delta_{\alpha,\pi\mu}$. Since $\sigma\langle\rho\rho\rangle = \langle\rho\rho\rangle$, we know that $\pi\rho = \rho$, so $M^{(p)}_{\rho\rho} = 1$. By (5.12), $S^{(p)}_{\mu\nu} = S^{(p)}_{\pi\mu,\pi\nu}$. But this is true iff $S^{(p)}$ commutes with $M^{(p)}$ (this is similar to the derivation of (5.4) in §5.1). We also know that $M^{(p)}$ commutes with T since $M_{\rho\mu,\rho\mu'} \neq 0 \implies \mu$ and μ' satisfy T-invariance for $\mathrm{A}_{2,p}$. □

It is known that the only automorphism invariants of $A_{2,p}$ are $\mathcal{A}_p$, $\mathcal{D}_p$[2], or their conjugations [9], so $M^{(p)}$ must be one of these. Therefore $\mathcal{A}_{p'} \otimes M^{(p)}$ is an automorphism invariant of $A_{2,p'} \oplus A_{2,p}$[3]. Let $M' := M(\mathcal{A}_{p'} \otimes M^{(p)})^{-1}$, for any modular invariant M. Then M' is a modular invariant and has $M'_{\langle\rho\mu\rangle,\langle\kappa\nu\rangle} \neq 0 \iff \langle\kappa\nu\rangle = \langle\rho\mu\rangle$. Therefore, replacing M with M', we may assume that $\sigma\langle\rho\mu\rangle = \langle\rho\mu\rangle$.

Claim 5.3. *Let $\lambda\mu, \kappa\nu \in \mathcal{P}$ and suppose that $M_{\lambda\mu,\kappa\nu} \neq 0$. Then $\mu = \nu$.*

Proof. Let $\lambda\mu \in \mathcal{P}$. By S-invariance, $(SM)_{\rho\kappa,\lambda\mu} = (MS)_{\rho\kappa,\lambda\mu}$ for any weight $\kappa \in \mathrm{P}^p_{++}(A_2)$, so

$$\sum_{\alpha\beta} S^{(p',p)}_{\rho\kappa,\alpha\beta} M_{\alpha\beta,\lambda\mu} = \sum_{\alpha\beta} M_{\rho\kappa,\alpha\beta} S^{(p',p)}_{\alpha\beta,\lambda\mu}. \tag{5.13}$$

To evaluate the right-hand sum, notice that by the Claim 5.2, $M_{\rho\kappa,\alpha\beta} \neq 0 \implies \alpha\beta \in \langle\rho\kappa\rangle = \{(\rho\kappa), (A_{p'}\rho, \kappa), (A^2_{p'}\rho, \kappa)\}$, so the right-hand side of (5.13) is equal to $M_{\rho\kappa,\rho\kappa} S^{(p',p)}_{\rho\kappa,\lambda\mu} + M_{\rho\kappa,A\rho\kappa} S^{(p',p)}_{A\rho\kappa,\lambda\mu} + M_{\rho\kappa,A^2\rho\kappa} S^{(p',p)}_{A^2\rho\kappa,\lambda\mu}$. But $S^{(p',p)}_{A^i\rho\kappa,\lambda\mu} = e^{\frac{2\pi i}{3}t(\lambda)} S^{(p',p)}_{\rho\kappa,\lambda\mu} = S^{(p',p)}_{\rho\kappa,\lambda\mu}$ because $t(\lambda) \equiv 0 \pmod 3$. Since both (ρ, κ) and $(A^i_{p'}\rho, \kappa)$ are not fixed points, $M_{\rho\kappa,A^i\rho\kappa} = 1$, so the right-hand side is just $3S^{(p',p)}_{\rho\kappa,\lambda\mu}$.

---

[2] $\mathcal{A}_p = \sum_\lambda \chi^p_\lambda \chi^{p^*}_\lambda$ and $\mathcal{D}_p = \sum_\lambda \chi^p_\lambda \chi^{p^*}_{A^{pt(\lambda)}\lambda}$, for p not divisible by 3

[3] $\mathcal{A}_{p'} \otimes \mathcal{D}_p$ corresponds to $(0, 0, 0, b)$ in Column B of Table 3.1, and these appear for all $b \in \{0, 1, 2\}$. $\mathcal{A}_{p'} \otimes \mathcal{A}_p$ is the identity.

Multiplying both sides of (5.13) by $S^{(p)*}_{\kappa\nu}$ and summing over κ gives

$$\begin{aligned}
\sum_{\kappa}\sum_{\alpha\beta} S^{(p')}_{\rho\alpha} S^{(p)}_{\kappa\beta} S^{(p)*}_{\kappa\nu} M_{\alpha\beta,\lambda\mu} &= 3S^{(p')}_{\rho\lambda} \sum_{\kappa} S^{(p)}_{\kappa\mu} S^{(p)*}_{\kappa\nu} \\
&= \sum_{\alpha\beta} S^{(p')}_{\rho\alpha} M_{\alpha\beta,\lambda\mu} \sum_{\kappa} S^{(p)}_{\beta\kappa} S^{(p)*}_{\kappa\nu} \\
&= 3S^{(p')}_{\rho\lambda} \sum_{\kappa} S^{(p)}_{\mu\kappa} S^{(p)*}_{\kappa\nu} \\
&= 3S^{(p')} \delta_{\mu\nu},
\end{aligned}$$

by unitarity of $S^{(p')}$. Switching the summation signs on the left-hand side, we have

$$\begin{aligned}
LHS &= \sum_{\alpha\beta}\sum_{\kappa} S^{(p')}_{\rho\alpha} S^{(p)}_{\kappa\beta} S^{(p)*}_{\kappa\nu} M_{\alpha\beta,\lambda\mu} \\
&= \sum_{\alpha\beta} S^{(p')}_{\rho\alpha} M_{\alpha\beta,\lambda\mu} \delta_{\beta\nu} \\
&= \sum_{\alpha} S^{(p')}_{\rho\alpha} M_{\alpha\nu,\lambda\mu}
\end{aligned}$$

by unitarity of $S^{(p)}$. Therefore, equating $LHS = RHS$, we have

$$\sum_{\alpha} S^{(p')}_{\rho\alpha} M_{\alpha\nu,\lambda\mu} = 3S^{(p')}_{\rho\lambda} \delta_{\mu\nu}.$$

Since every entry of M is nonnegative, and $S^{(p')}_{\rho\alpha} > 0 \ \forall\alpha$, the left-hand sum can be zero iff $M_{\alpha\nu,\lambda\mu} = 0$ for all α. Therefore, if there is at least one α for which $M_{\alpha\nu,\lambda\mu} \neq 0$, we must have $3S^{(p')}_{\rho\lambda}\delta_{\mu\nu} \neq 0$; ie, $\mu = \nu$. □

Remark: Because of Claim 5.3, we will write $M_{\lambda\mu,\kappa\nu}$ as $M_{\lambda\mu,\kappa\mu}$ for the rest of this chapter.

Claim 5.4. *Recall that $\phi = (\frac{p'}{3}, \frac{p'}{3})$. Suppose $M_{\phi'\mu,\phi\mu} \neq 0$ for some nonfixed point ϕ' (ie, $\phi' \neq \phi$) and any μ. Then $M_{\phi'\mu,\lambda\mu} = \delta_{\lambda\phi}$ for any λ.*

Proof. Suppose $M_{\phi'\mu,\phi\mu} \neq 0$. Then by Lemma 4.5,

$$M_{\phi'\mu,\phi\mu} \leq \frac{\|\mathcal{J}_L\|}{\sqrt{\|\mathcal{J}_L(\phi'\mu)\| \ \|\mathcal{J}_R(\phi\mu)\|}}. \tag{5.14}$$

We have $\mathcal{J}_L = \mathcal{J}_R = \mathcal{J}$, so $\|\mathcal{J}_L\| = \|\mathcal{J}\| = 3$, and $(\phi\mu)$ is fixed by $\mathcal{J}$, so $\|\mathcal{J}_R(\phi\mu)\| = \|\mathcal{J}\| = 1$. Now $\|\mathcal{J}(\phi'\mu)\| = 1$ or 3 by Lemma 4.3(c). But by hypothesis, $\phi'\mu$ is not fixed by $\mathcal{J}$, so $\|\mathcal{J}(\phi'\mu)\| = 3$. Therefore (5.14) gives us $M_{\phi'\mu,\phi\mu} \leq \frac{3}{\sqrt{3 \cdot 1}} = \sqrt{3} < 2$, and since $M_{\phi'\mu,\phi\mu} \neq 0$, $M_{\phi'\mu,\phi\mu} = 1$.

Now suppose that $M_{\phi'\mu,\kappa\mu} \neq 0$ for some $\kappa \neq \phi$. Then the facts that $\phi'\mu$ and $\kappa\mu$ are both not fixed points of $\mathcal{J}$ and $M_{\phi'\mu,\phi\mu} \neq 0$ imply $\kappa\mu \in \mathcal{J}(\phi\mu)$, by Lemma 4.5. But this implies $\kappa = \phi$. □

Claim 5.4 says that if $\phi'\mu$ is any weight that can couple to a fixed point $\phi\mu$, then $M_{\phi'\mu,\phi\mu} = 1$. It also says that if a weight $\phi'\mu$ couples to a fixed point, it cannot couple to any other weight. With this, we are now in a position to show that $((2,2),\rho)$, $((1,4),\rho)$ and $((4,1),\rho)$ are not in $\mathcal{K}^\phi$; ie, they do not couple to a fixed point.

Let $\kappa\nu \in \mathcal{P}$. Then by Claim 5.4 and S-invariance,

$$\begin{aligned} S^{(p',p)}_{\phi\mu,\kappa\nu} &= 1 \cdot S^{(p',p)}_{\phi\mu,\kappa\nu} = M_{\phi'\mu,\phi\mu} S^{(p',p)}_{\phi\mu,\kappa\nu} = \sum_{\alpha\beta} M_{\phi'\mu,\alpha\beta} S^{(p',p)}_{\alpha\beta,\kappa\nu} \\ &= (MS)_{\phi'\mu,\kappa\nu} = (SM)_{\phi'\mu,\kappa\nu} = \sum_{\alpha\beta} S^{(p',p)}_{\phi'\mu,\alpha\beta} M_{\alpha\beta,\kappa\nu}. \end{aligned}$$

If $\kappa\nu$ is not a fixed point and $\kappa\nu \notin \mathcal{K}^\phi$, we know that $M_{\alpha\beta,\kappa\nu} \neq 0$ iff $M_{\alpha\beta,\kappa\nu} = 1$ and $\alpha\beta \in \sigma^{-1}\langle\kappa\nu\rangle$. Let $\kappa'\nu'$ be any representative of $\sigma^{-1}\langle\kappa\nu\rangle$. Then the last sum is equal to

$$S^{(p',p)}_{\phi'\mu,\kappa'\nu'} M_{\kappa'\nu',\kappa\nu} + S^{(p',p)}_{\phi'\mu,A\kappa'\nu'} M_{A\kappa'\nu',\kappa\nu} + S^{(p',p)}_{\phi'\mu,A^2\kappa'\nu'} M_{A^2\kappa'\nu',\kappa\nu} = \sum_{a=0}^{2} S^{(p',p)}_{\phi'\mu,A^a\kappa'\nu'},$$

which is $3S^{(p',p)}_{\phi'\mu,\kappa'\nu'}$ when $t(\nu') \equiv 0 \pmod 3$. Therefore we have

$$S^{(p',p)}_{\phi\mu,\kappa\nu} = 3S^{(p',p)}_{\phi'\mu,\kappa'\nu'}, \tag{5.15}$$

whenever $\kappa \neq \phi$, $\kappa\nu \notin \mathcal{K}^\phi$, $\kappa'\nu' \in \sigma^{-1}\langle\kappa\nu\rangle$, and $t(\nu') \equiv_3 0$.

Suppose for a contradiction that $((2,2),\rho) \in \mathcal{K}^\phi$. We know that $(\rho,\rho) \notin \mathcal{K}^\phi$, so putting $((2,2),\rho) = \phi'\mu$ and $\rho\rho = \kappa\nu$ into (5.15), and choosing $\kappa'\nu' = \rho\rho$, we get $S^{(p',p)}_{\phi\rho,\rho\rho} = 3S^{(p',p)}_{(2,2)\rho,\rho\rho}$. Factoring out the $S^{(p)}_{\rho\rho}$, we have $S^{(p')}_{\phi\rho} = 3S^{(p')}_{(2,2)\rho}$, so by (2.11),

$$\frac{8}{\sqrt{3}p'} \sin^2\left(\frac{\pi}{3}\right) \sin\left(\frac{2\pi}{3}\right) = 3\frac{8}{\sqrt{3}p'} \sin^2\left(\frac{2\pi}{p'}\right) \sin\left(\frac{4\pi}{p'}\right),$$

which is true iff

$$\sin^2\left(\frac{2\pi}{p'}\right) \sin\left(\frac{4\pi}{p'}\right) = \frac{\sqrt{3}}{8}. \tag{5.16}$$

For $p' \geq 8$, $\frac{2\pi}{p'}$ and $\frac{4\pi}{p'}$ are in the first quadrant (so $\sin\theta$ increases with θ), and we see that the left-hand side of (5.16) decreases as p' increases. Therefore, there can be only one value of $p' \geq 8$ so that equality holds in (5.16). At $p' = 6$ we do not get equality, and writing $\frac{\sqrt{3}}{8} = (\frac{1}{2})^2 \cdot \frac{\sqrt{3}}{2} = \sin^2(\frac{\pi}{6})\sin(\frac{\pi}{3})$, we see by inspection that $p' = 12$ is this value, which is an exceptional case. Putting $((1,4),\rho)$ and $((4,1),\rho)$ into (5.15) gives us

$$\sin\left(\frac{\pi}{p'}\right)\sin\left(\frac{4\pi}{p'}\right)\sin\left(\frac{5\pi}{p'}\right) = \frac{\sqrt{3}}{8},$$

which is true iff

$$\sin\left(\frac{8\pi}{p'}\right) + \sin\left(\frac{2\pi}{p'}\right) - \sin\left(\frac{10\pi}{p'}\right) = \frac{\sqrt{3}}{2}. \tag{5.17}$$

This has only one solution $p' \geq 10$, which we can see by inspection to be $p' = 12$. For $p' < 10$, we find that $p' = 6$ is also a solution to (5.17), but if $p' = 6$, then $\phi = (2,2)$, and $(1,4)$ and $(2,2)$ do not satisfy decoupled T-invariance (4.3). Therefore, for $p' \neq 12$, $((2,2),\rho)$, $((1,4),\rho)$, $((4,1),\rho) \notin \mathcal{K}^{\phi}$.

Now that we know that these small weights do not couple to a fixed point, we can put them and (ρ,ρ) into (5.15) (as $\kappa\nu$) to obtain

$$S^{(p')}_{\phi\rho} = 3S^{(p')}_{\phi'\rho};\ S^{(p')}_{\phi(2,2)} = 3S^{(p')}_{\phi'(2,2)};\ S^{(p')}_{\phi(1,4)} = 3S^{(p')}_{\phi'(1,4)};\ S^{(p')}_{\phi(4,1)} = 3S^{(p')}_{\phi'(4,1)},$$

so

$$\frac{S^{(p')}_{(2,2)\phi}}{S^{(p')}_{\rho\phi}} = \frac{S^{(p')}_{(2,2)\phi'}}{S^{(p')}_{\rho\phi'}};\ \frac{S^{(p')}_{(1,4)\phi}}{S^{(p')}_{\rho\phi}} = \frac{S^{(p')}_{(1,4)\phi'}}{S^{(p')}_{\rho\phi'}};\ \frac{S^{(p')}_{(4,1)\phi}}{S^{(p')}_{\rho\phi}} = \frac{S^{(p')}_{(4,1)\phi'}}{S^{(p')}_{\rho\phi'}}.$$

Therefore, for any $\phi' \in \mathcal{K}^{\phi}$, $\phi' \neq \phi$, and any λ with $t(\lambda) \equiv_3 0$, we have

$$\begin{aligned}\frac{S^{(p')}_{\lambda\phi}}{S^{(p')}_{\rho\phi}} &= P_\lambda\left(\frac{S^{(p')}_{(2,2)\phi}}{S^{(p')}_{\rho\phi}}, \frac{S^{(p')}_{(1,4)\phi}}{S^{(p')}_{\rho\phi}}, \frac{S^{(p')}_{(4,1)\phi}}{S^{(p')}_{\rho\phi}}\right)\\ &= P_\lambda\left(\frac{S^{(p')}_{(2,2)\phi'}}{S^{(p')}_{\rho\phi'}}, \frac{S^{(p')}_{(1,4)\phi'}}{S^{(p')}_{\rho\phi'}}, \frac{S^{(p')}_{(4,1)\phi'}}{S^{(p')}_{\rho\phi'}}\right) = \frac{S^{(p')}_{\lambda\phi'}}{S^{(p')}_{\rho\phi'}}.\end{aligned} \tag{5.18}$$

Equation (5.18) then gives us

$$\frac{S^{(p',p)}_{\kappa\nu,\phi\mu}}{S^{(p',p)}_{\rho\rho,\phi\mu}} = \frac{S^{(p')}_{\kappa\phi}}{S^{(p')}_{\rho\phi}} \cdot \frac{S^{(p)}_{\nu\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p')}_{\kappa\phi'}}{S^{(p')}_{\rho\phi'}} \cdot \frac{S^{(p)}_{\nu\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p',p)}_{\kappa\nu,\phi'\mu}}{S^{(p',p)}_{\rho\rho,\phi'\mu}}, \tag{5.19}$$

for any $\kappa\nu \in \mathcal{P}$ and $\phi' \in \mathcal{K}^{\phi}$, $\phi' \neq \phi$.

Let $\kappa = (\kappa_1, \kappa_2)$. Then

$$S^{(p')^*}_{\kappa\phi} = \frac{8}{\sqrt{3}p'} \sin\left(\frac{\kappa_1 \pi}{3}\right) \sin\left(\frac{\kappa_2 \pi}{3}\right) \sin\left(\frac{(\kappa_1 + \kappa_2)\pi}{3}\right),$$

which is 0 if $\kappa_1 \equiv -\kappa_2 \pmod 3$. In other words, if $t(\kappa) \equiv 0 \pmod 3$, then $S^{(p',p)^*}_{\kappa\nu,\phi\mu} = 0$, which tells us $\kappa\nu \notin \mathcal{P} \implies S^{(p',p)}_{\kappa\nu,\phi\mu} = 0$. Therefore, multiplying both sides of (5.19) by $S^{(p',p)^*}_{\kappa\nu,\phi\mu}$ and summing over $\kappa\nu \in \mathcal{P}$ is equivalent to multiplying by $S^{(p',p)^*}_{\kappa\nu,\phi\mu}$ and summing over all $\kappa\nu \in P^{p',p}_{++}$. Unitarity of $S^{(p',p)}$ then gives us

$$\frac{1}{S^{(p',p)}_{\rho\rho,\phi\mu}} = \frac{\delta_{\phi'\mu,\phi\mu}}{S^{(p',p)}_{\rho\rho,\phi'\mu}} = 0,$$

since $\phi' \neq \phi$. But this implies $0 \neq LHS = 0$. Therefore, we must have $M_{\phi'\mu,\phi\mu} = 0$ for all $\phi' \neq \phi$.

We have now shown that $M_{\lambda\mu,\phi\mu} = 0$ for all $\lambda\mu \neq \phi\mu$, so to determine the fixed point behaviour of M, we must find the value of $M_{\phi\mu,\phi\mu}$. We will use Lemma 4.5(a). Since $\|\mathcal{J}_L\| = \|\mathcal{J}_R\| = 3$, and by Lemma 4.3(b), $M_{A^i\rho\rho,A^j\rho\rho} = 1$, $\forall i,j \in \{0,1,2\}$, $B_1 = B(1,3)$. Let B_ϕ be the block containing $M_{\phi\mu,\phi\mu}$. Then B_ϕ is the 1×1 matrix $(M_{\phi\mu,\phi\mu})$. Lemma 4.5(a) now tells us that, since $B_\phi \neq (0)$, $r(B_\phi) = 3$, and so $M_{\phi\mu,\phi\mu} = 3$. Therefore $M_{\lambda\mu,\phi\mu} = 3\delta_{\lambda\mu,\phi\mu}$.

It now remains to determine the values of M at the nonfixed points of $\mathcal{J}$. This will be similar to what we have already done in §5.2: the proof carries through with some minor adjustments. The difference in this case, is that $3 \mid p$, and $t(\lambda)$ and $t(\mu)$ are not necessarily congruent to 0 (mod 3) for some $\lambda\mu \in \langle\lambda\mu\rangle$.

For $(m,n) \in \{(2,2),(1,4),(4,1)\}$, a similar argument to that in Claim 5.1 now tells us only that $\sigma\langle(m,n),\rho\rangle = \langle C^a_{p'} A^b_{p'}(m,n), C^c_p A^d_p \rho\rangle$ where $a, c \in \{0,1\}$, $b \in \{0,1,2\}$. By Claim 5.3, we know that $c = d = 0$, and multiplying our modular invariant by one of the charge conjugations (see §3.1), we may assume that $a = 0$. Since $\langle(m,n),\rho\rangle$ runs through all $(A^i(m,n),\rho)$, $i = 0,1,2$, we can put $b = 0$. Therefore, letting $M'' := M'C$, where C is one of the charge conjugations ${}^C I^C$, ${}^C I$ or I^C, M'' satisfies $\sigma\langle(2,2),\rho\rangle = \langle(2,2),\rho\rangle$ and $\sigma\langle(1,4),\rho\rangle = \langle(1,4),\rho\rangle$. Because σ commutes with $C^{(1,1)}$, we also have $\sigma\langle(4,1),\rho\rangle = \langle(4,1),\rho\rangle$.

For $(m,n) \in \{(2,2),(1,4),(4,1)\}$, (5.4) gives us

$$\frac{S^{(p')}_{(m,n)\lambda}}{S^{(p')}_{\rho\lambda}} = \frac{S^{(p')}_{(m,n)\lambda}}{S^{(p')}_{\rho\lambda}} \cdot \frac{S^{(p)}_{\rho\mu}}{S^{(p)}_{\rho\mu}} = \frac{S^{(p',p)}_{(m,n)\rho,\lambda\mu}}{S^{(p',p)}_{\rho\rho,\lambda\mu}} = \frac{S^{(p',p)}_{(m,n)\rho,\lambda'\mu'}}{S^{(p',p)}_{\rho\rho,\lambda'\mu'}} = \frac{S^{(p')}_{(m,n)\lambda'}}{S^{(p')}_{\rho\lambda'}} \cdot \frac{S^{(p)}_{\rho\mu'}}{S^{(p)}_{\rho\mu'}} = \frac{S^{(p')}_{(m,n)\lambda'}}{S^{(p')}_{\rho\lambda'}},$$

so we have equations (5.6a).

Now let $\lambda\mu, \kappa\nu \in \mathcal{P}$. Then $t(\lambda) \equiv_3 t(\kappa) \equiv_3 0$, so by (5.6a) and (5.7),

$$\begin{aligned}
\frac{S^{(p',p)}_{\lambda\mu,\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa\nu}} &= \frac{S^{(p')}_{\lambda\kappa}}{S^{(p')}_{\rho\kappa}} \cdot \frac{S^{(p)}_{\mu\nu}}{S^{(p)}_{\rho\nu}} \\
&= P_\lambda\left(\frac{S^{(p')}_{(2,2)\kappa}}{S^{(p')}_{\rho\kappa}}, \frac{S^{(p')}_{(1,4)\kappa}}{S^{(p')}_{\rho\kappa}}, \frac{S^{(p')}_{(4,1)\kappa}}{S^{(p')}_{\rho\kappa}}\right) \cdot \frac{S^{(p)}_{\mu\nu}}{S^{(p)}_{\rho\nu}} \\
&= P_\lambda\left(\frac{S^{(p')}_{(2,2)\kappa'}}{S^{(p')}_{\rho\kappa'}}, \frac{S^{(p')}_{(1,4)\kappa'}}{S^{(p')}_{\rho\kappa'}}, \frac{S^{(p')}_{(4,1)\kappa'}}{S^{(p')}_{\rho\kappa'}}\right) \cdot \frac{S^{(p)}_{\mu\nu}}{S^{(p)}_{\rho\nu}} \\
&= \frac{S^{(p')}_{\lambda\kappa'}}{S^{(p')}_{\rho\kappa'}} \cdot \frac{S^{(p)}_{\mu\nu}}{S^{(p)}_{\rho\nu}} = \frac{S^{(p',p)}_{\lambda\mu,\kappa'\nu}}{S^{(p',p)}_{\rho\rho,\kappa'\nu}},
\end{aligned}$$

and so

$$\text{(5.20)} \qquad \sum_{\lambda\mu:t(\lambda)\equiv_3 0} \frac{S^{(p',p)}_{\lambda\mu,\kappa\nu}}{S^{(p',p)}_{\rho\rho,\kappa\nu}}\left(\sum_{a=0}^{2} S^{(p',p)*}_{A^a\lambda\mu,\kappa\nu}\right) = \sum_{\lambda\mu:t(\lambda)\equiv_3 0} \frac{S^{(p',p)}_{\lambda\mu,\kappa'\nu'}}{S^{(p',p)}_{\rho\rho,\kappa'\nu'}}\left(\sum_{a=0}^{2} S^{(p',p)*}_{A^a\lambda\mu,\kappa\nu}\right),$$

for any $\lambda\mu \in \langle\lambda\mu\rangle$, and $\kappa\nu \in \langle\kappa\nu\rangle$. But by (2.10d),

$$\sum_{a=0}^{2} S^{(p',p)*}_{A^a\lambda\mu,\kappa\nu} = \left(\sum_{a=0}^{2} e^{\frac{2\pi i}{3}at(\lambda)}\right) S^{(p',p)*}_{\lambda\mu,\kappa\nu},$$

which is 0 if $t(\lambda) \not\equiv 0 \pmod 3$, because in that case, we get the sum of the third roots of unity. If $t(\lambda) \equiv 0 \pmod 3$, we get $3S^{(p',p)*}_{\lambda\mu,\kappa\nu}$, so summing over all $\lambda\mu$ with $t(\lambda) \equiv 0 \pmod 3$ will give us the same result as summing over all $\lambda\mu \in \mathcal{P}^{p'p}_{++}$. Now as usual, summing over all $\lambda\mu$ gives us $\sigma\langle\kappa\nu\rangle = \langle\kappa'\nu\rangle = \langle\kappa\nu\rangle$, by unitarity of $S^{(p',p)}$.

We have shown that when $\mathcal{J}$ has a fixed point $\phi\mu$, $M_{\phi\mu,\phi\mu} = 3$, and $M_{\phi\mu,\kappa\mu} = M_{\kappa\mu,\phi\mu} = 0$, for all $\kappa\mu \in \mathcal{P}$ with $\kappa \neq \phi$. We have also shown that at the nonfixed points $\lambda\mu$ of $\mathcal{J}$, $\sigma\langle\lambda\mu\rangle = \langle\lambda\mu\rangle$ up to multiplication by an automorphism invariant. We can define σ on all of $\mathcal{P}$ by letting $\sigma\langle\phi\mu\rangle = \langle\phi\mu\rangle$ (keeping in mind that the $\mathcal{J}$−orbit is used here only for consistency of notation, since $\langle\phi\mu\rangle = \{\phi\mu\}$), so that the partition function for a modular invariant with $\sigma\langle\lambda\mu\rangle = \langle\lambda\mu\rangle$, such as our M'', can be written as

$$\text{(5.21)} \qquad \mathcal{Z} = \sum_{\substack{\langle\lambda\mu\rangle:t(\lambda)\equiv_3 0\\ \lambda\neq\phi}} ch_{\langle\lambda\mu\rangle} ch^*_{\langle\lambda\mu\rangle} + 3\sum_{\mu\in P^p_{++}(A_2)} ch_{\langle\phi\mu\rangle} ch^*_{\langle\phi\mu\rangle}.$$

Our M'' has this form. But we defined $M'' = M'C = M(\mathcal{A}_k \otimes M^{(p)})^{-1}C$, where $M^{(p)}$ is an automorphism invariant of $A_{2,p}$. Therefore, for any modular invariant M, $M = M''C(\mathcal{A}_{p'} \otimes M^{(p)})$, where M'' is defined by (5.21).

All of §5.3 applies to Theorem 4.2(b)(iii) and, reversing p' and p, we also have the modular invariants for Theorem 4.9b(b)(i), so other than at the exceptional levels, we know the modular invariants for $A_{2,p'} \oplus A_{2,p}$.

Chapter 6

The Exceptional Heights

In this chapter, we consider the exceptional heights $(p', p) = (12, p), (24, p)$ and $(p', 8)$ (it turns out that $(18, p)$ and $(60, p)$ have already been done in Chapter 5). We have Lemma 4.3, Lemma 4.5(a) and (b), S-invariance and T-invariance (the norm condition), and the Galois condition. Also, throughout this chapter, we use Maple[1] to check which weights satisfy certain inequalities or congruences. Lemma 6.1 below is a special case of the A_2 Galois condition, and will apply to all of our exceptional heights other than when $p = 8$ and $3 \mid p'$. Because of this, the height $(p', 8)$ where $3 \mid p'$ will be the most difficult case, and we will use the general Galois condition to solve it. The A_2 Galois condition was completely solved by Aoki for all but 33 relatively small heights [1].

Lemma 6.1. *[9] Suppose p' is coprime to 6. Then λ and κ satisfy the decoupled parity rule (4.8); ie, $M_{\lambda\mu,\kappa\nu} \neq 0 \implies \epsilon_\ell^{(p')}(\lambda) = \epsilon_\ell^{(p')}(\kappa)$ for all ℓ with $gcd(\ell, 3p') = 1$ iff $\kappa \in \mathcal{O}\lambda$, and a similar statement holds for p, μ, ν when p is coprime to 6.*

6.1 The Exceptional $\rho\rho$-couplings

We will first finish the proof of Theorem 4.2.

Proof of Theorem 4.2(c), (d), (e). First suppose that $p' = 18$ and $p \equiv_{12} 5, 11$, as in Lemma 4.2(c)(ii). If $\lambda \in \mathcal{O}(1,4)$ and $\mu = A^i\rho$, $i = 1, 2$, then evaluat-

[1] Maple is accurate to nine significant figures, which is acceptable enough for these calculations. A potential inaccuracy can occur when Maple returns a 0 value; for example, it could be possible that an expression could have a value of -10^{-20} but Maple has rounded it up to 0.

ing $s_L((8,8),\rho) \geq 0$ leads to a contradiction whether $m = 1$ or 3 (recall $m = \sum_{i=0}^{2} M_{A^i\rho\rho,\rho\rho}$). Therefore, if $p' = 18$ and $p \equiv_{12} 1,5,7,11$, $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{p'}\rho,\rho) : i = 0,1,2\}$ and so is covered by Theorem 4.2(a)(ii).

Next suppose $p' = 12$ and $p \equiv_{12} 1,5,7,11$, as in Lemma 4.2(d). Let $m := \sum_i M_{A^i\rho\rho,\rho\rho}$ and $m'' := \sum_i M_{A^i\rho'',\rho\rho}$. Applying the parity rule with $\ell \equiv -1 \pmod{3p}$ and $\ell \equiv \pmod{3p'}$, we get $M_{(3,3)A\rho,\rho\rho} = M_{(3,3)A^2\rho,\rho\rho}$, and $M_{A(3,3)A\rho} = M_{A(3,3)A^2\rho} = M_{A^2(3,3)A\rho} = M_{A^2(3,3)A^2\rho}$. As in Chapter 4, Lemma 4.3(c) tells us either $m'' \geq m$, or $m'' = 0$. Let $b := M_{(3,3)A\rho,\rho\rho}$, $b' := M_{A(3,3)A\rho,\rho\rho}$, and $B := \sum_{\substack{i=0 \\ j=1}}^{2} M_{A^i(3,3)A^j\rho,\rho\rho}$, so $B = 2b + 4b'$. Then $b = 0$ or $2b \geq 2$, and $b' = 0$ or $4b' \geq 4$. We will show that $B = 0$.

If $m'' = 0$, then $s_L((3,3),\rho) \geq 0$ implies $B \leq m$, so if $m = 3$, then $B \leq 3$ implies $b' = 0$. Therefore the only choice with $m = 3$ has $b \leq 3$, so $b = 0$ or 1. If $m = 1$, then $B \leq 1$, a contradiction unless $B = 0$. We thus have the possibility $m = 3$, $m'' = 0$, and $B = 2b = 2$. But this cannot happen because $m = 3$ implies $M_{A^i(3,3)A^j\rho,\rho\rho} = M_{(3,3)A^j\rho,\rho\rho}$ for all $i \in \{0,1,2\}$, which implies $b' \neq 0$, a contradiction.

Therefore, suppose $m'' \neq 0$. Then $s_L((2,2),\rho) \geq 0$ implies $m'' \leq m$. But $m'' \geq m$ as usual, so $m'' = m$. Now evaluating $s_L((3,3),\rho) \geq 0$ gives

$$B \leq 2m. \tag{6.1}$$

If $m = 3$, then $s_L((2,1),\rho) \geq 0$ implies $b' \geq b$, and $s_L((3,2),\rho) \geq 0$ implies $b' \geq b$ so $b = b'$. Together with (6.1), this gives us the possibility

$$m = m'' = 3,\ b = b' = 1,\ B = 2b + 4b' = 6. \tag{6.2}$$

If $m = 1$, then by (6.1), $B \leq 2$, which rules out $b' \neq 0$. Therefore, for $B \neq 0$, we must have $b \neq 0$, so $b = 1$ and $B = 2$. We then get the possibility

$$m = m'' = 1,\ b = 1,\ B = 2b = 2. \tag{6.3}$$

The next step is to eliminate the choices in (6.2) and (6.3). Suppose we have $m = m'' = 3$ and $B = 2b + 4b' = 6$, as in (6.2). Then $s_L((3,3),\rho) > 0$, so $((3,3),\rho) \in \mathcal{P}_L$. But then by T-invariance, $M_{(3,3)\rho,\kappa\nu} \neq 0 \implies \kappa\nu = (A^i_{12}(3,3),\rho)$, for some $i \in \{0,1,2\}$. But now, evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $(\rho\rho,(3,3)\rho)$ implies $M_{(3,3)\rho,A^i(3,3)\rho} = 0$ for all $i = 0,1,2$, contradicting $((3,3),\rho) \in \mathcal{P}_L$.

To eliminate the possibility in (6.3), we find that in that case, $s_L((2,3),\rho)$ is negative, so we cannot have (6.3). Therefore $B=0$, so now we just have to consider $\lambda \in \mathcal{O}\rho \cup \mathcal{O}\rho''$.

If $m''=0$, then $\mathcal{R}_R = \mathcal{R}_L = \{(A_{12}^i\rho,\rho) : i=0,1,2\}$. By Lemma 4.3(a), we must have $s_L(\lambda,\mu) = \sum_{\kappa\nu} M_{\kappa\nu,\rho\rho} S^{(12,p)}_{\lambda\mu,\kappa\nu} \geq 0$. Letting $\kappa=(2,2)$ and $\nu=\rho$, we have

$$\begin{aligned} s_L((2,2),\rho) &= \sum_{i=0}^{2} M_{A^i\rho\rho,\rho\rho} S^{(12,p)}_{(2,2)\rho,A^i\rho\rho} + \sum_{i=0}^{2} M_{A^i\rho''\rho,\rho\rho} S^{(12,p)}_{(2,2)\rho,A^i\rho''\rho} \\ &= S^{(p)}_{\rho\rho}\{mS^{(12)}_{(2,2)\rho} + m''S^{(12)}_{(2,2)\rho''}\} \\ &\geq 0. \end{aligned}$$

But by (2.11),

$$S^{(12)}_{(2,2)\rho''} = \frac{-8}{\sqrt{3}n}\sin^2\left(\frac{10\pi}{12}\right)\sin\left(\frac{8\pi}{12}\right) = \frac{-8}{\sqrt{3}n}\sin^2\left(\frac{2\pi}{12}\right)\sin\left(\frac{4\pi}{12}\right) = -S^{(12)}_{(2,2)\rho},$$

so factoring out $S^{(12)}_{(2,2)\rho}$, we get that the above inequality iff $S^{(p)}_{\rho\rho}S^{(12)}_{(2,2)\rho}(m-m'')\geq 0$ iff $m \geq m''$ (since $S^{(p)}_{\rho\rho}S^{(12)}_{(2,2)\rho} > 0$). Therefore, either $m''=0$, or $m''=m$. By Lemma 4.3(b), either $M_{\rho\rho,\rho\rho}=1$ and $M_{\lambda\mu,\rho\rho}=0\ \forall\lambda\mu\neq\rho\rho$, or $M_{A^i\rho\rho,\rho\rho}=M_{\rho\rho,\rho\rho}=1\ \forall i=0,1,2$, so $m=1$ or 3. Therefore, if $m''\neq 0$, then $m''=m=1$ or 3.

Suppose $m''=m=1$. Then there exists exactly one ℓ for which $M_{A^i\rho''\rho,\rho\rho}=\delta_{i,\ell}$. Consider $s_L((1,2),\rho)$. For $s_L((1,2),\rho)\geq 0$, we must have $S^{(12)}_{(1,2)\rho}+S^{(12)}_{(1,2)A^\ell\rho''}\geq 0$. But if $\ell=0$, we can calculate the left-hand side using (2.11), and we find that it is negative. If $\ell\neq 0$, then the left-hand side is non-real by (2.5). Therefore, we cannot have $m''=m=1$, so we have Theorem 4.2(c).

Now suppose we have $p'=24$ and $p\equiv_{12} 1,5,7,11$ as in Lemma 4.2(e). Let $m'=\sum_{i=0}^2 M_{A^i\rho'\rho,\rho\rho}$, $m''':=\sum_{i=0}^2 M_{A^i\rho'''\rho,\rho\rho}$, where $\rho'=(5,5)$ and $\rho'''=(7,7)$ ($\rho''=(11,11)$ here). Lemma 4.3(c) again tells us $m'=0$ or $m'\geq m$; $m''=0$ or $m''\geq m$, and $m'''=0$ or $m'''\geq m$. The case $m'=m''=m'''=0$ was covered by Theorem 4.2(a)(ii), so we will assume m',m'', or $m'''>0$. Evaluating $s_L((2,2),\rho), s_L((3,3),\rho)$, and $s_L((4,4),\rho)$ gives us the following equations

(6.4a) $$m-m''-m'+m'''\geq 0$$

(6.4b) $$m-m''+m'-m'''\geq 0$$

(6.4c) $$m+m''-m'-m'''\geq 0.$$

Adding (6.4a) and (6.4b) gives us $m \geq m''$; (6.4b) + (6.4c) gives us $m \geq m'''$, and (6.4a) + (6.4c) gives us $m \geq m'$. Therefore, whenever one of m', m'', or m''' is nonzero, that one must equal m, which is 1 or 3. Putting all possibilities into Maple, we find that every choice of m, m', m'' and m''' violates one of equations (6.4), or one of $s_L((2,2),\rho), s_L((3,3),\rho) \geq 0$ except for $m = m'' = 3$ and $m' = m''' = 0$; $m = m' = m'' = m''' = 1$, or $m = m' = m'' = m''' = 3$. But $m = m'' = 3$ and $m' = m''' = 0$ violates $s_L((1,4),\rho) \geq 0$. If $m = m' = m'' = m''' = 1$, then there exist $j, k, \ell \in \{0,1,2\}$ such that $M_{A^i\rho'\rho,\rho\rho} = \delta_{j,i}$; $M_{A^i\rho''\rho,\rho\rho} = \delta_{k,i}$, and $M_{A^i\rho'''\rho,\rho\rho} = \delta_{\ell,i}$. But any choice of j, k and ℓ gives either $s_L((3,2),\rho)$ negative or non-real. Therefore, we have only the choices $m = m' = m'' = m''' = 3$, or $m = 1$ or 3 and $m' = m'' = m''' = 0$, which is covered by Theorem 4.2(a)(ii).

Finally, suppose $p' = 60$ and $p \equiv_{12} 1, 5, 7, 11$ (Lemma 4.2(f)), and put $\rho' = (11,11), \rho''' = (19,19)$. The details are similar to the $p' = 24$ argument: here $s_L((3,3),\rho), s_L((6,6),\rho)$ and $s_L((10,10),\rho) \geq 0$ yield equations (6.4) which imply $m = m' = m'' = m''' = 1$ or 3, unless $m' = m'' = m''' = 0$. But both $m = m' = m'' = m''' = 1$ and 3 imply $s_L((2,5),\rho) < 0$. Therefore, the case $p' = 60$ is covered by Theorem 4.2(a)(ii). □

6.2 The Exceptional Invariants at $(p', 8)$

6.2.1 The Exceptionals at $p = 8$ when $p' \equiv_{12} 1, 7$ and $\mathcal{R}_R = \mathcal{R}_L = \{\rho\rho'', \rho\rho\}$

We begin with $(p', p) = (p', 8)$ as in Theorem 4.2(e)(i), so $p' \equiv_{12} 1, 7$. We will do this case in the most detail. Suppose first that $\mathcal{R}_R = \mathcal{R}_L = \{\rho\rho, \rho\rho''\}$ ($\rho'' = (3,3)$ here). Then $M_{\rho\rho,\rho\rho''} = M_{\rho\rho'',\rho\rho} = M_{\rho\rho,\rho\rho} = 1$, and $M_{\rho\rho,\lambda\mu} = M_{\lambda\mu,\rho\rho} = 0 \; \forall \lambda\mu \notin \{\rho\rho, \rho\rho''\}$.

Suppose $M_{\lambda\mu,\kappa\nu} \neq 0$. Then λ and κ must satisfy the decoupled parity rule, so by Lemma 6.1, $\kappa \in \mathcal{O}\lambda$, and since $p = 8$, there are very few possibilities for μ. Putting these possibilities into 4.3(a) (using Maple), we get $s_L(\lambda, \mu) > 0$ for any $\lambda \in P_{++}^{p'}$, $\mu \in \mathcal{O}\rho \cup \mathcal{O}(3,3) \cup \mathcal{O}(1,3)$, so $\mathcal{P} = P_{++}^{p'} \times \mathcal{O}\rho \cup \mathcal{O}(3,3) \cup \mathcal{O}(1,3)$. Now putting all possible μ and ν into the decoupled norm condition (4.3), we have $M_{\lambda\mu,\kappa\nu} \neq 0 \implies \mu, \nu \in \mathcal{O}\rho \cup \mathcal{O}(3,3)$ or $\mu, \nu \in \mathcal{O}(1,3)$.

Consider $M_{A\rho A\rho, A\rho A\rho}$. Evaluating $S^{(p',8)}M = MS^{(p',8)}$ at $(A_{p'}\rho A_8\rho, \rho\rho)$, we get

$$S^{(p')}_{\rho\rho}\{\sum_{a,b=0}^{2} M_{A\rho A\rho, A^a\rho A^b\rho} S^{(8)}_{\rho\rho} + \sum_{c,d=0}^{2} M_{A\rho A\rho, A^c\rho A^d\rho''} S^{(8)}_{\rho''\rho}\} = S^{(p')}_{\rho\rho}(S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''}).$$

But $S^{(8)}_{\rho\rho}/S^{(8)}_{\rho\rho''} = 3 - 2\sqrt{2}$, so $S^{(8)}_{\rho\rho}$ and $S^{(8)}_{\rho\rho''}$ are linearly independent over $\mathbb{Q}$. Therefore, equating coefficients, we find that $M_{A\rho A\rho, A^a\rho A^b\rho} = 1$ for exactly one choice of a and b. By the norm condition (4.1), we must have either $a = b = 0$, or both a and b nonzero; however, $(A_{p'}\rho, A_8\rho)$ is not a $\rho\rho$-coupling, so we cannot have $a = b = 0$. Therefore $(A^a_{p'}\rho, A^b_8\rho) \in \{(A_{p'}\rho, A_8\rho), (A^2_{p'}\rho, A_8\rho), (A_{p'}\rho, A^2_8\rho), (A^2_{p'}\rho, A^2_8\rho)\}$. Notice that these are all conjugations of each other, so multiplying by the appropriate conjugation matrix if necessary, we may assume that $a = 2$ and $b = 1$; ie, $M_{A\rho A\rho, A^2\rho A\rho} = 1$. Then by Lemma 4.3 (c), $M_{A\lambda A\mu, A^2\kappa A\nu} = M_{\lambda\mu,\kappa\nu}$ for all $\lambda\mu, \kappa\nu \in P^{p',8}_{++}$, and

$$t(\lambda) + t(\mu) \equiv -t(\kappa) + t(\nu) \pmod 3 \tag{6.5}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$.

Evaluating $S^{(p',8)}M = MS^{(p',8)}$ at $(\rho\rho, \rho A_8\rho)$, we get that $M_{\rho A\rho, A^a\rho A^b\rho} = 1$ for exactly one choice of a and b. By (6.5), we get $a + 2b \equiv 2 \pmod 3$, and this together with the norm condition (4.1) tells us $a = 0$ and $b = 1$. By Lemma 4.3(c), we therefore have

$$t(\mu) \equiv t(\nu) \pmod 3 \tag{6.6}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$. Now by (6.5) and (6.6), we also have

$$t(\lambda) \equiv -t(\kappa) \pmod 3 \tag{6.7}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$.

Suppose $M_{\lambda\mu,\kappa\nu} \neq 0$, where $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$. Then $\nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$ as well, and by Lemma 6.1, $\kappa \in \mathcal{O}\lambda$. For now take $\mu = A^a\rho$ for some fixed $a \in \{0, 1, 2\}$ (the argument for $\mu \in \mathcal{O}\rho''$ will be similar). Then $M_{\lambda A^a\rho,\kappa\nu} \neq 0 \implies (\kappa, \nu) = (A^b_{p'}C^c_{p'}\lambda, A^a_8\rho)$ or $(A^{b'}_{p'}C^{c'}_{p'}\lambda, A^a_8\rho'')$, since $\kappa \in \mathcal{O}\lambda$ and $t(\nu) \equiv -t(A^a_8\rho) \pmod 3$ by (6.7). Evaluating $S^{(p',8)}M = MS^{(p',8)}$ at $(\lambda A^a\rho, \rho\rho)$, we get

$$M_{\lambda A^a\rho, A^bC^c\lambda A^a\rho} = M_{\lambda A^a\rho, A^{b'}C^{c'}\lambda A^a\rho''} = 1$$

for exactly one choice of b, c and b', c'. Putting $((\lambda, A_8^a\rho), (A_{p'}^b C_{p'}^c \lambda, A_8^a \rho))$ into (6.5), we get $b \equiv_3 b' \equiv_3 t(\lambda)$ and $c \equiv_3 c' \equiv_3 0$. Therefore, $M_{\lambda A^a\rho,\kappa\nu} = 1$ for $(\kappa, \nu) = (\lambda', A_8^a\rho)$ and $(\kappa, \nu) = (\lambda'', A_8^a\rho'')$, where $\lambda', \lambda'' \in \{\lambda, A_{p'}^{t(\lambda)}\lambda\}$.

Now evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $((2,1)\mu, \kappa\nu)$ for any $\kappa \in P_{++}^{p'}$, and any $\mu, \nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, we get

$$S^{(p')}_{(2,1)\kappa}(S^{(8)}_{\rho\rho} + S^{(8)}_{\rho''\rho}) = S^{(p')}_{(2,1)\kappa'}S^{(8)}_{\rho\rho} + S^{(p')}_{(2,1)\kappa''}S^{(8)}_{\rho''\rho}, \tag{6.8}$$

where $\kappa', \kappa'' \in \{\kappa, A_{p'}^{t(\kappa)}\kappa\}$. The reason for finding equation (6.8) is to get an S symmetry, analogous to (3.17). If $\kappa' = \kappa''$, then (6.8) becomes $S^{(p')}_{(2,1)\kappa} = S^{(p')}_{(2,1)\kappa'}$. Now suppose they are not necessarily equal: put $\kappa' = \kappa$ and $\kappa'' = A_{p'}^{t(\kappa)}\kappa$. The right-hand side of (6.8) then becomes $S^{(p')}_{(2,1)\kappa}S^{(8)}_{\rho\rho} + e^{\frac{2\pi i}{3}t(\kappa)}S^{(p')}_{(2,1)\kappa}S^{(8)}_{\rho\rho''}$, so equating $LHS = RHS$, we must have $e^{\frac{2\pi i}{3}t(\kappa)} = 0$. But this can be true iff $t(\kappa) \equiv 0 \pmod 3$, in which case, $\kappa'' = A_{p'}^0\kappa = \kappa = \kappa'$ anyway. Choosing $\kappa' = A_{p'}^{t(\kappa)}\kappa$ and $\kappa'' = \kappa$ also gives us $\kappa'' = \kappa' = \kappa$, so in any case, we at least have $\kappa' = \kappa''$ and $S^{(p')}_{(2,1)\kappa} = S^{(p')}_{(2,1)\kappa'}$. Since $\kappa' \in \mathcal{O}\kappa$, we also know that $S^{(p')}_{\rho\kappa} = S^{(p')}_{\rho\kappa'}$, so we have

$$\frac{S^{(p')}_{(2,1)\kappa}}{S^{(p')}_{\rho\kappa}} = \frac{S^{(p')}_{(2,1)\kappa'}}{S^{(p')}_{\rho\kappa'}}; \quad \frac{S^{(p')}_{(1,2)\kappa}}{S^{(p')}_{\rho\kappa}} = \frac{S^{(p')}_{(1,2)\kappa'}}{S^{(p')}_{\rho\kappa'}}, \tag{6.9}$$

where the second equation uses $C_{p'}(2,1) = (1,2)$ and the fact that $\{C_{p'}\kappa'\} = \{(C_{p'}\kappa)'\}$ (where by $\{\lambda'\}$, we mean the set $\{\lambda, A_{p'}^{t(\lambda)}\lambda\}$). The fact that the sets are equal follows from calculating each set. By (3.18), we therefore have

$$\frac{S^{(p')}_{\lambda\kappa}}{S^{(p')}_{\rho\kappa}} = \frac{S^{(p')}_{\lambda\kappa'}}{S^{(p')}_{\rho\kappa'}}$$

for all $\lambda \in P_{++}^{p'}$, and where $\kappa' \in \{\kappa, A_{p'}^{t(\kappa)}\kappa\}$. As usual, unitarity of $S^{(p')}$ now gives us $\kappa' = \kappa$ (see the proof of Proposition 3.1). Therefore, what we have shown is that whenever $\mu, \nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$,

$$M_{\lambda\mu,\kappa\nu} \neq 0 \iff M_{\lambda\mu,\kappa\nu} = 1 \text{ and } \kappa\nu \in \{\lambda\rho, \lambda\rho''\},$$

If $M_{\lambda A^a\rho'',\kappa\nu} \neq 0$ for some fixed a, the same steps apply, so we have the following when $\mu, \nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$:

$$M_{\lambda\mu,\kappa\nu} \neq 0 \iff M_{\lambda\mu,\kappa\nu} = 1 \text{ and } \kappa\nu \in \{\lambda\rho, \lambda\rho''\}. \tag{6.10}$$

Finally, suppose $M_{\lambda(1,3),\kappa\nu} \neq 0$. Then $\nu \in \mathcal{O}(1,3)$, and by (6.6), $\nu = (1,3)$ or $(4,3)$. As above, T-invariance and (6.5) give us $\kappa \in \{\lambda, A_{p'}^{t(\lambda)}\lambda\}$. Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\lambda(1,3), \rho\rho)$ yields

$$M_{\lambda(1,3),\lambda(1,3)} + M_{\lambda(1,3),\lambda(4,3)} = 2, \tag{6.11}$$

so at most two and at least one of the above M terms is nonzero. Without loss of generality, suppose $M_{\lambda(1,3),\lambda(1,3)} \neq 0$. Then evaluating $MS^{(p',8)} = S^{(p',8)}$ at $(\lambda(1,3), \rho(1,2))$ implies

$$M_{\lambda(1,3),\lambda(1,3)} S^{(p',8)}_{\lambda(1,3),\rho(1,2)} + M_{\lambda(1,3),\lambda(4,3)} S^{(p',8)}_{\lambda(4,3),\rho(1,2)} = 0,$$

because $(\rho,(1,2)) \notin \mathcal{P}$, so $M_{\alpha\beta,\rho(1,2)} = 0$ for all $\alpha\beta \in P_{++}^{p',8}$. Therefore

$$S^{(p')}_{\lambda\rho}\{M_{\lambda(1,3),\lambda(1,3)} S^{(8)}_{(1,3)(1,2)} + M_{\lambda(1,3),\lambda(4,3)} S^{(8)}_{(4,3),(1,2)}\} = 0.$$

But using (2.5), we see that

$$S^{(8)}_{(1,3)(1,2)} = \frac{-i}{8\sqrt{3}}(2e^{-\frac{7\pi i}{12}} - 2e^{\frac{11\pi i}{12}}) = -S^{(8)}_{(4,3)(1,2)},$$

so $M_{\lambda(1,3),\lambda(1,3)} = M_{\lambda(1,3),\lambda(4,3)}$. Together with (6.11), we get $M_{\lambda(1,3),\lambda(1,3)} = M_{\lambda(1,3),\lambda(4,3)}$ $= 1$. A similar argument holds for any $\mu \in \mathcal{O}(1,3)$, so we actually have the following:

$$M_{\lambda\mu,\lambda\mu} = M_{\lambda\mu,\lambda\nu} = 1,$$

where (μ,ν) can be one of the pairs $((1,3),(4,3))$, $((3,1),(3,4))$, or $((4,1),(1,4))$, or vice-versa.

We therefore get the exceptional invariants

$$\begin{aligned}\mathcal{E}^{(1)}_{p',8} = \sum_{\lambda\in P^{p'}_{++}} & |\chi^{(p',8)}_{\lambda\rho} + \chi^{(p',8)}_{\lambda(3,3)}|^2 + |\chi^{(p',8)}_{\lambda(1,3)} + \chi^{(p',8)}_{\lambda(4,3)}|^2 + |\chi^{(p',8)}_{\lambda(3,1)} + \chi^{(p',8)}_{\lambda(3,4)}|^2 \\ & + |\chi^{(p',8)}_{\lambda(4,1)} + \chi^{(p',8)}_{\lambda(1,4)}|^2 + |\chi^{(p',8)}_{\lambda(6,1)} + \chi^{(p',8)}_{\lambda(2,3)}|^2 + |\chi^{(p',8)}_{\lambda(1,6)} + \chi^{(p',8)}_{\lambda(3,2)}|^2 \\ & (\text{for } p' \not\equiv_3 0, p = 8),\end{aligned} \tag{6.12}$$

up to multiplication by an automorphism invariant.

6.2.2 The Exceptionals at $p = 8$ when $p' \equiv_{12} 1, 7$ and $\mathcal{R}_R = \mathcal{J}_R(\rho\rho) \cup \mathcal{J}_R(\rho\rho'')$ and $\mathcal{R}_L = \mathcal{J}_L(\rho\rho) \cup \mathcal{J}_L(\rho\rho'')$

The second type of exceptional when $p = 8$ is given in Theorem 4.2(e)(i). There are several choices, all of which are done similarly. We will do the case $\mathcal{R} := \mathcal{R}_L = \mathcal{R}_R =$

$\{(A^i_{p'}\rho, A^i_8\rho), (A^i_{p'}\rho, A^i_8\rho'') : i = 0,1,2\}$. Put $\mathcal{J} := \{A^0_{p'}A^0_8, A^1_{p'}A^1_8, A^2_{p'}A^2_8\}$. Then $M_{\rho\rho,J(\rho\rho)} = M_{J(\rho\rho),\rho\rho} = 1\ \forall J \in \mathcal{J}$, and by Lemma 4.3(c), (d), $M_{A^i\lambda A^i\mu, A^j\kappa A^j\nu} = M_{\lambda\mu,\kappa\nu}\ \forall \lambda\mu,\kappa\nu \in P^{p',8}_{++}$, and $t(\lambda)+t(\mu) \equiv t(\kappa)+t(\nu) \pmod 3$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$.

Evaluating $s_L(\rho,(2,2)) \geq 0$, we find that $M_{A^i\rho A^i\rho'',\rho\rho} \leq 1$. But $M_{A^i\rho A^i\rho'',\rho\rho} > 0$ since $A^i_{p'}\rho A^i_8\rho''$ is a $\rho\rho$-coupling, so $M_{A^i\rho A^i\rho'',\rho\rho} = 1\ \forall i = 0,1,2$. Similarly, evaluating $s_R(\rho,(2,2)) \geq 0$, we get $M_{\rho\rho,A^i\rho A^i\rho''} = 1$. By Lemma 4.3(a), we can find $\mathcal{P}$ by evaluating $s_L(\lambda,\mu) > 0$, which reduces to finding all μ with $S^{(8)}_{\mu\rho} + S^{(8)}_{\mu\rho''} > 0$ ($S^{(p')}_{\lambda\rho}$ is factored out, which is always positive, so λ can be anything in $P^{p'}_{++}$). We find that $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho'' \cup \mathcal{O}(1,3)$. Therefore, $\mathcal{P} := \mathcal{P}_R = \mathcal{P}_L = \{\lambda\mu \in P^{p',8}_{++} : t(\lambda)+t(\mu) \equiv_3 0$ and $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho'' \cup \mathcal{O}(1,3)\}$. We can use $\mathcal{J}$-orbits here as we did in Chapter 5 (ie, let $\langle\lambda\mu\rangle = \{(A^i_{p'}\lambda, A^i_8\mu) : i = 0,1,2\}$). Since $3 \nmid p'$, can always choose a representative of $\langle\lambda\mu\rangle$, such that $t(\lambda) \equiv_3 t(\mu) \equiv_3 t(\kappa) \equiv_3 t(\nu)$ whenever $M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0$. Putting all weights $\lambda\mu \in \mathcal{P}$ into T-invariance, we find that $M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0 \implies \mu,\nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, or $\mu,\nu \in \mathcal{O}(1,3)$.

Suppose $M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0$, where $\mu,\nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$ Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\lambda\mu,\rho\rho)$ and choosing $t(\lambda) \equiv_3 t(\mu)$, we get

$$S^{(p')}_{\lambda\rho}\{(\sum_{i,j,k} M_{\lambda\rho,A^iC^j\lambda A^k\rho})S^{(8)}_{\rho\rho} + (\sum_{i,j,k} M_{\lambda\rho,A^iC^j\lambda A^k\rho''})S^{(8)}_{\rho\rho''}\} = 3S^{(p')}_{\lambda\rho}(S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''}),$$

where $i,k = 0,\ldots,2$ and $j = 0,\ldots,1$. But since we can choose $t(A^i_{p'}C^j_{p'}\lambda) \equiv_3 t(A^k_8\rho) \equiv_3 0$, we can assume $i = k = 0$. Therefore, by the linear independence of $S^{(8)}_{\rho\rho}$ and $S^{(8)}_{\rho\rho''}$ over $\mathbb{Q}$, we have

$$3\sum_{j=0}^{1} M_{\lambda\rho,C^j\lambda\rho} = 3\sum_{j=0}^{1} M_{\lambda\rho,C^j\lambda\rho''} = 3,$$

so $M_{\langle\lambda\rho\rangle\langle C^a\lambda\rho\rangle} = M_{\langle\lambda\rho\rangle\langle C^b\lambda\rho''\rangle} = 1$ for some choice of a and b. Multiplying by a charge conjugation if necessary, we may assume $a = b$.

Notice that $S^{(8)}_{\rho''\rho} + S^{(8)}_{\rho''\rho''} = S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''}$, so for any choice of $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$ we have

$$M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0 \iff M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} = 1 \text{ and } \langle\kappa\nu\rangle \in \{\langle\lambda\rho\rangle, \langle\lambda\rho''\rangle\},$$

whenever $\mu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$.

Now suppose $M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0$, where $\mu,\nu \in \mathcal{O}(1,3)$. First let $\mu = (1,4)$ so that we have $t(\mu) \equiv_3 0$. Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\lambda(1,3), \rho\rho)$, we get

$$\sum_{i,j=0}^{1} M_{\langle\lambda(1,4)\rangle,\langle C^i\lambda C^j(1,4)\rangle} = 2. \tag{6.13}$$

Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\lambda(1,4), \rho(5,2))$, we get

$$\sum_{i=0}^{1} M_{\langle\lambda(1,4)\rangle\langle C^i\lambda(1,4)\rangle} S^{(8)}_{(1,4)(5,2)} + \sum_{i=0}^{1} M_{\langle\lambda(1,4)\rangle\langle C^i\lambda(4,1)\rangle} S^{(8)}_{(4,1)(5,2)} = 0,$$

since $(\rho, (5,2)) \notin \mathcal{P}$. Using (2.5) to calculate the $S^{(8)}$ values and equating the real and imaginary parts of this equation, we have

$$M_{\langle\lambda(1,4)\rangle\langle C^a\lambda(1,4)\rangle} = M_{\langle\lambda(1,4)\rangle\langle C^a\lambda(4,1)\rangle}.$$

Therefore, together with (6.13), we have

$$M_{\langle\lambda(1,4)\rangle\langle C^a\lambda(1,4)\rangle} = M_{\langle\lambda(1,4)\rangle\langle C^a\lambda(4,1)\rangle} = 1,$$

giving us the invariants

$$\begin{aligned} \mathcal{E}^{(2)}_{p',8} = {} & + \sum(\chi^{(p',8)}_{A^i\lambda A^{\pm i}(1,4)} + \chi^{(p',8)}_{A^i\lambda A^{\pm i}(4,1)}) \sum(\chi^{(p',8)*}_{A^i\lambda A^{\pm i}(1,4)} + \chi^{(p',8)*}_{A^i\lambda A^{\pm i}(4,1)}), \\ & + \sum_{i=0}^{2}(\chi^{(p',8)}_{A^i\lambda A^{\pm i}(1,4)} + \chi^{(p',8)}_{A^i\lambda A^{\pm i}(4,1)}) \sum_{i=0}^{2}(\chi^{(p',8)*}_{A^i\lambda A^{\pm i}(1,4)} + \chi^{(p',8)*}_{A^i\lambda A^{\pm i}(4,1)}), \\ & (\text{for } p' \not\equiv_3 0, p = 8), \end{aligned} \tag{6.14}$$

up to multiplication by an automorphism invariant.

6.2.3 The Exceptionals at $(p', 8)$ when $3 \mid p'$

In the case of Theorem 4.2(e)(ii), we do not have p' coprime to 6, but we can still use Galois [1]. There are 33 heights at which the Galois condition has not been solved; of those, the ones that affect us are $p' = 9, 15, 21, 39$. Therefore, we will have to consider the heights $(9,8)$, $(15,8)$, $(21,8)$, and $(39,8)$ separately. The exceptionals at $(9,8)$ were found in [14]. For the remaining three heights, we do not work them out here, but we expect to also find $\mathcal{E}^{(1)}_{p',8}$ (6.12) and $\mathcal{E}^{(3)}_{p',8}$ (6.25) as the only exceptional invariants. They are finite and can be found by a computer search, and by some of the methods used in this chapter.

For general p', the Galois condition tells us that there are two choices of orbits for λ when $M_{\lambda\mu,\kappa\nu} \neq 0$, namely $\lambda \in \mathcal{O}\kappa \cup \mathcal{O}\kappa'$ for some κ'. Our strategy is to show that we cannot have any of the $\mathcal{O}\kappa'$ cases, so λ must be in $\mathcal{O}\kappa$. At that point, the argument can be worked out as for the other cases.

We have here $\mathcal{R}_R = \mathcal{R}_L = \{\rho\rho, \rho\rho''\}$ or $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{p'}\rho, \rho), (A^i_{p'}\rho, \rho'') : i = 0, 1, 2\}$. The arguments for them are similar; we will do the first one. As before, $\mathcal{P} = P^{p'}_{++} \times \mathcal{O}\rho \cup \mathcal{O}\rho'' \cup \mathcal{O}(1,3)$, and $M_{\lambda\mu,\kappa\nu} \neq 0$ implies either $\mu, \nu \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, or $\mu, \nu \in \mathcal{O}(1,3)$. Similar calculations as in the previous cases give us

$$t(\lambda) \equiv t(\kappa) \pmod 3, \tag{6.15}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$.

First, consider κ with $t(\kappa) \not\equiv 0 (\mathrm{mod}\ 3)$. Due to the choices of weights μ, ν whenever $M_{\lambda\mu,\kappa\nu} \neq 0$, μ and ν cancel out of full T-invariance (4.1), leaving

$$\frac{\lambda}{p'} \equiv \frac{\kappa}{p'} \pmod 3, \tag{6.16}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$. Then of the 12 possible weights in $\mathcal{O}(\kappa) \cup \mathcal{O}(\kappa')$ at most 2 will satisfy (6.15) and (6.16). By (2.10c) and T-invariance, the weight in the κ orbit will be κ, and we will call the weight in the other orbit κ'.

Applying $MS^{(p',8)} = S^{(p',8)}M$ at $(\rho(2,5), \kappa\rho)$ and $((2,1)(2,5), \kappa\rho)$, we get

$$0 = S^{(p')}_{\rho\kappa}(M_{\kappa\rho,\kappa\rho} - M_{\kappa\rho'',\kappa\rho}) + S^{(p')}_{\rho\kappa'}(M_{\kappa'\rho,\kappa\rho} - M_{\kappa'\rho'',\kappa\rho}), \tag{6.17a}$$

$$0 = S^{(p')}_{(2,1)\kappa}(M_{\kappa\rho,\kappa\rho} - M_{\kappa\rho'',\kappa\rho}) + S^{(p')}_{(2,1)\kappa'}(M_{\kappa'\rho,\kappa\rho} - M_{\kappa'\rho'',\kappa\rho}), \tag{6.17b}$$

where the left-hand sides are 0 because $(\lambda, (2,5)) \notin \mathcal{P}$ for any λ.

Let $\Delta_\kappa := M_{\kappa\rho,\kappa\rho} - M_{\kappa\rho'',\kappa\rho}$ and $\Delta_{\kappa'} := M_{\kappa'\rho,\kappa\rho} - M_{\kappa'\rho'',\kappa\rho}$. Then equations (6.17) become

$$S^{(p')}_{\rho\kappa}\Delta_\kappa = -S^{(p')}_{\rho\kappa'}\Delta_{\kappa'} \tag{6.18a}$$

$$S^{(p')}_{(2,1)\kappa}\Delta_\kappa = -S^{(p')}_{(2,1)\kappa'}\Delta_{\kappa'} \tag{6.18b}$$

We see from (6.18a) that Δ_κ and $\Delta_{\kappa'}$ must be both 0 or both nonzero. If $\Delta_\kappa, \Delta_{\kappa'} \neq 0$, then dividing equations (6.18), we get

$$\frac{S^{(p')}_{(2,1)\kappa}}{S^{(p')}_{\rho\kappa}} = \frac{S^{(p')}_{(2,1)\kappa'}}{S^{(p')}_{\rho\kappa'}}, \tag{6.19}$$

and the usual argument implies $\kappa = \kappa'$. Therefore we can take $\Delta_\kappa = \Delta_{\kappa'} = 0$. This means

$$M_{\kappa\rho,\kappa\rho} = M_{\kappa\rho'',\kappa\rho} \text{ and } M_{\kappa'\rho,\kappa\rho} = M_{\kappa'\rho'',\kappa\rho}.$$

Then evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\rho\rho, \kappa\rho)$, we get

$$S^{(p')}_{\rho\kappa}\{S^{(8)}_{\rho\rho} + S^{(8)}_{\rho''\rho}\} = S^{(p')}_{\rho\kappa}(S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''})M_{\kappa\rho,\kappa\rho} + S^{(p')}_{\rho\kappa'}(S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''})M_{\kappa'\rho,\kappa\rho},$$

and a similar equation from $MS^{(p',8)} = S^{(p',8)}M$ at $((2,1)\rho, \kappa\rho)$. Dividing both sides by $S^{(8)}_{\rho\rho}$ then gives us

(6.20a) $$(4-2\sqrt{2})S^{(p')}_{\rho\kappa} = S^{(p')}_{\rho\kappa}M_{\kappa\rho,\kappa\rho}(4-2\sqrt{2}) + S^{(p')}_{\rho\kappa'}M_{\kappa'\rho,\kappa\rho}(4-2\sqrt{2}),$$

(6.20b) $$(4-2\sqrt{2})S^{(p')}_{(2,1)\kappa} = S^{(p')}_{(2,1)\kappa}M_{\kappa\rho,\kappa\rho}(4-2\sqrt{2}) + S^{(p')}_{(2,1)\kappa'}M_{\kappa'\rho,\kappa\rho}(4-2\sqrt{2}),$$

because $S^{(8)}_{\rho\rho}/S^{(8)}_{\rho\rho''} = 3-2\sqrt{2}$. Equations (6.20) reduce to

(6.21a) $$S^{(p')}_{\rho\kappa}(1 - M_{\kappa\rho,\kappa\rho}) = S^{(p')}_{\rho\kappa'}M_{\kappa'\rho,\kappa\rho}$$

(6.21b) $$S^{(p')}_{(2,1)\kappa}(1 - M_{\kappa\rho,\kappa\rho}) = S^{(p')}_{(2,1)\kappa'}M_{\kappa'\rho,\kappa\rho}.$$

Therefore, if $M_{\kappa'\rho,\kappa\rho} \neq 0$, $M_{\kappa\rho,\kappa\rho} = 0$. Dividing these, we get $S^{(p')}_{(2,1)\kappa}/S^{(p')}_{\rho\kappa} = S^{(p')}_{(2,1)\kappa'}/S^{(p')}_{\rho\kappa'}$, again implying $\kappa' = \kappa$. Therefore, what we have shown is that if $t(\kappa) \not\equiv_3 0$, $M_{\lambda\mu,\kappa\nu} \neq 0$ iff $\lambda = \kappa$. The analysis for the weights μ and ν is the same as in §6.2.1.

Now suppose $t(\kappa) \equiv 0 \pmod 3$. This is more difficult, because then any weight in $\mathcal{O}\kappa \cup \mathcal{O}\kappa'$ can potentially couple to κ. We will proceed as above.

Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\rho\rho, \kappa\rho)$, we get

$$\begin{aligned} S^{(p')}_{\rho\kappa}\{S^{(8)}_{\rho\rho} + S^{(8)}_{\rho\rho''}\} =& S^{(p')}_{\rho\kappa}\{\sum_{\alpha\in\mathcal{O}\kappa} S^{(8)}_{\rho\rho}M_{\alpha\rho,\kappa\rho} + \sum_{\alpha\in\mathcal{O}\kappa} S^{(8)}_{\rho\rho''}M_{\alpha\rho'',\kappa\rho}\} \\ &+ S^{(p')}_{\rho\kappa'}\{\sum_{\alpha\in\mathcal{O}\kappa'} S^{(8)}_{\rho\rho}M_{\alpha\rho,\kappa'\rho} + \sum_{\alpha\in\mathcal{O}\kappa'} S^{(8)}_{\rho\rho''}M_{\alpha\rho'',\kappa\rho}\}, \end{aligned}$$

and dividing by $S^{(8)}_{\rho\rho''}$, we have

(6.22) $$S^{(p')}_{\rho\kappa}(4-2\sqrt{2}) = S^{(p')}_{\rho\kappa}(\Sigma_\rho + (3-2\sqrt{2})\Sigma_{\rho''}) + S^{(p')}_{\rho\kappa'}(\Sigma'_\rho + (3-2\sqrt{2})\Sigma'_{\rho''}),$$

where $\Sigma_\rho := \sum_{\alpha\in\mathcal{O}\kappa} M_{\alpha\rho,\kappa\rho}$, $\Sigma_{\rho''} := \sum_{\alpha\in\mathcal{O}\kappa} M_{\alpha\rho'',\kappa\rho}$, $\Sigma'_\rho := \sum_{\alpha\in\mathcal{O}\kappa'} M_{\alpha\rho,\kappa'\rho}$, and $\Sigma'_{\rho''} := \sum_{\alpha\in\mathcal{O}\kappa'} M_{\alpha\rho'',\kappa'\rho}$. Evaluating $MS^{(p',8)} = S^{(p',8)}M$ at $(\rho(2,2), \kappa\rho)$, we get

$$0 = (\Sigma_\rho - \Sigma_{\rho''}) + (\Sigma'_\rho - \Sigma'_{\rho''}). \tag{6.23}$$

Subtracting (6.22) and (6.23) gives us

$$S^{(p')}_{\rho\kappa}(1 - \Sigma_{\rho''}) = S^{(p')}_{\rho\kappa'}\Sigma'_{\rho''}. \tag{6.24}$$

If we knew that $S^{(p')}_{\rho\kappa} = S^{(p')}_{\rho\kappa'}$, then we would be done, because then we would be in the situation of the $t(\kappa) \equiv_3 0$ case. Without loss of generality, suppose $S^{(p')}_{\rho,\kappa} \leq S^{(p')}_{\rho,\kappa'}$. If $\Sigma'_{\rho''} \neq 0$, then by (6.24), $\Sigma_{\rho''} = 0$ and $\Sigma'_{\rho''} = 1$, which implies $S^{(p')}_{\rho,\kappa} = S^{(p')}_{\rho\kappa'}$. So take $\Sigma_{\rho''} \neq 0$. Then $\Sigma_\rho = 1$ and $\Sigma'_{\rho''} = 0$, which implies $\Sigma_\rho = 1, \Sigma'_\rho = 0$, and again $S^{(p')}_{\rho\kappa} = S^{(p')}_{\rho\kappa'}$, which is what we needed to show, because now the argument reduces to the $t(\kappa) \not\equiv_3 0$ case, and we get $\lambda \in \mathcal{O}\kappa$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$. We get the exceptionals $\mathcal{E}^{(1)}_{p',8}$, and

$$\begin{aligned}\mathcal{E}^{(3)}_{p',8} =& \frac{1}{3}\sum_{\lambda\in P^{p'}_{++}}\sum_{i=0}^{2}(\chi^{(p',8)}_{A^i\lambda\rho} + \chi^{(p',p)}_{A^i\lambda(3,3)})\sum_{i=0}^{2}(\chi^{(p',8)*}_{A^i\lambda\rho} + \chi^{(p',p)*}_{A^i\lambda(3,3)}) \\ &+ \sum_{i=0}^{2}(\chi^{(p',8)}_{A^i\lambda(1,4)} + \chi^{(p',8)}_{A^i\lambda(4,1)})\sum_{i=0}^{2}(\chi^{(p',8)*}_{A^i\lambda(1,4)} + \chi^{(p',8)*}_{A^i\lambda(4,1)}), \\ &(\text{for } p' \equiv_3 0, p = 8),\end{aligned} \tag{6.25}$$

up to multiplication by an automorphism invariant.

6.3 The Exceptionals at $(12, p)$

6.3.1 The Exceptionals at $p' = 12$ when $\mathcal{R}_R = \mathcal{R}_L = \mathcal{J}(\rho\rho)$

We will do the case $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{12}\rho, \rho) : i = 0, 1, 2\}$. Here, $\mathcal{J} := \mathcal{J}_R = \mathcal{J}_L = \{A^0_{12}A^0_p, A^1_{12}A^2_p, A^0_{12}A^0_p\}$, and we have a fixed point of $\mathcal{J}$ at $((4,4), \mu) =: \phi\mu$, for any $\mu \in P^p_{++}$. We have $M_{J(\rho\rho),\rho\rho} = M_{\rho\rho,J(\rho\rho)} = 1$ for all $J \in \mathcal{J}$, and $M_{\lambda\mu,\rho\rho} = M_{\rho\rho,\lambda\mu} = 0$ for all $\lambda\mu \notin \mathcal{J}(\rho\rho)$. Therefore, $B_1 = B(1,3)$, and by Lemma 4.3 (d), (c),

$$t(\lambda) \equiv t(\kappa) \equiv 0 \pmod 3, \tag{6.26}$$

whenever $M_{\lambda\mu,\kappa\nu} \neq 0$ and $M_{A^i\lambda\mu,A^j\kappa\nu} = M_{\lambda\mu,\kappa\nu}$ $\forall\lambda\mu, \kappa\nu \in P^{12,p}_{++}$. By Lemma 4.3(a), $\mathcal{P} = \{\lambda\mu \in P^{12,p}_{++} : s_L(\lambda,\mu) > 0\}$, so using Maple to calculate $s_L(\lambda,\mu)$ for all $\lambda\mu$

with $t(\lambda) \equiv_3 0$, we get that $\mathcal{P} = \{\lambda\mu \in P_{++}^{12,p} : t(\lambda) \equiv_3 0 \text{ and } S_{\lambda\rho}^{(12)} S_{\mu\rho}^{(p)} > 0\}$. But $S_{\lambda\rho}^{(12)} S_{\mu\rho}^{(p)} > 0$ for all λ, μ, so we have $\mathcal{P} = \mathcal{O}\rho \cup \mathcal{O}(2,2) \cup \mathcal{O}(3,3) \cup \mathcal{O}(4,4) \cup \mathcal{O}(5,5) \cup \mathcal{O}(1,4) \times P_{++}^{p}$. Putting these possibilities into T-invariance, we get that $M_{\lambda\mu,\kappa\nu} \neq 0$ implies either $\lambda, \kappa \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, $\lambda, \kappa \in \mathcal{O}(2,2) \cup \{(4,4)\}$, $\lambda, \kappa \in \mathcal{O}(3,3)$, or $\lambda, \kappa \in \mathcal{O}(1,4)$. Therefore, by Lemma 4.5(b), we have a permutation σ of J−orbits $\langle \lambda\mu \rangle$ whenever $\lambda \notin \mathcal{O}(2,2) \cup \{\phi\}$, as in Chapter 5, where $\phi = (4,4)$.

Evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $(\rho A_p\rho, \rho\rho)$ gives us $\sum_{i=0}^{2} M_{\rho A\rho, \rho A^i\rho} = 1$, so $M_{\rho A\rho, \rho A^a\rho} = 1$ for exactly one choice of a. By T-invariance (4.1), we cannot have $a = 0$, so $a = 1$ or 2. Therefore, $\sigma\langle \rho, A_p\rho \rangle = \langle \rho, C_p^b(A^2\rho) \rangle$ for some $b \in \{0,1\}$.

Now consider $\sigma\langle (1,4), \mu \rangle$, and let $\sigma\langle (1,4), \mu \rangle = \langle (1,4)', \mu' \rangle$. By Lemma 6.1, $\mu' \in \mathcal{O}\mu$, so evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $((1,4)\mu, \rho\rho)$, we get

$$\sum_{j,k,\ell} M_{(1,4)\mu, C^j(1,4)A^kC^\ell\mu} = 1,$$

and so $M_{(1,4)\mu, C^a(1,4)A^cC^d\mu} = 1$ for exactly one choice of a, c, d. Therefore we have $\sigma\langle (1,4), \mu \rangle = \sigma\langle C_{12}^a(1,4), A_p^c C_p^d \mu \rangle$, as well as $\sigma\langle \rho A_p \rho \rangle = \langle \rho, C_p^b(A^2\rho) \rangle$ from before. Multiplying by the appropriate conjugation matrix (and adjusting c and d accordingly), we may suppose $a = b = 0$; ie, $\sigma\langle \rho, A_p\rho \rangle = \langle \rho, A_p^2\rho \rangle$ and $\sigma\langle (1,4), \mu \rangle = \langle (1,4), A_p^c C_p^d \mu \rangle$, for some $c \in \{0,1,2\}, d \in \{0,1\}$, and similarly, $\sigma\langle (4,1), \mu \rangle = \langle (4,1), \mu' \rangle$. By Lemma 4.3(b), $M_{\rho A\rho, \rho A\rho} = 1 \implies t(\mu) \equiv_3 2t(\nu)$ whenever $M_{\lambda\mu,\kappa\nu}$, so evaluating $t(\mu) \equiv_3 2t(C_p^k A_p^\ell \mu)$, we find that $k = 0$ and $\ell = pt(\mu)$.

Now suppose $M_{\phi\mu,\kappa\nu} \neq 0$. Then $\kappa \in \mathcal{O}(2,2) \cup \{\phi\}$. For $\kappa \in \mathcal{O}(2,2)$, we can choose $\kappa = (2,2)$ without loss of generality. Notice that $M_{\phi\mu,(2,2)\nu} \neq 0$ iff $M_{(2,2)\nu,\phi\mu} \neq 0$, by Lemma 4.5(b). As in the previous case, we know that $\nu \in \{\mu, A_p^{pt(\mu)}\mu\}$, and multiplying by a simple current invariant if necessary, we can take $\nu = \mu$. Then evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $((2,2)\mu, \rho\rho)$ gives us

$$M_{(2,2)\mu,(2,2)\mu} + M_{(2,2)\mu,\phi\mu} = 1,$$

so one of the above terms is 1, and the other one is zero. If $M_{(2,2)\mu,(2,2)\mu} = 1$ and $M_{(2,2)\mu,\phi\mu} = 0$, then $((2,2), \mu)$ does not couple to a fixed point, so $\sigma\langle (2,2), \mu \rangle = \langle (2,2)\mu \rangle$. We then find that $M_{\phi\mu,\phi\mu} = 3$ (see the proof that $M_{\phi\mu,\phi\mu} = 3$ in Chapter 5). This gives us the modular invariants $M = \mathcal{D}_{12} \otimes \mathcal{A}_p$ and $\mathcal{D}_{12} \otimes \mathcal{D}_p$, both of which have already been found in Chapter 5. Therefore we can assume that $M_{(2,2)\mu,(2,2)\mu} = 0$ and $M_{(2,2)\mu,\phi\mu} = 1$.

We now need to determine the value of $M_{\phi\mu,\phi\mu}$. As usual, we use S-invariance, at $(\phi\mu, \rho\rho)$, which gives us $M_{\phi\mu,\phi\mu} + M_{\phi\mu,(2,2)\mu} = 3$. But we just showed that $M_{\phi\mu,(2,2)\mu} = 1$, so $M_{\phi\mu,\phi\mu} = 2$.

It remains to find $\sigma\langle\rho,\mu\rangle$, $\sigma\langle(3,3),\mu\rangle$, and $\sigma\langle(5,5),\mu\rangle$. But S-invariance at $(\rho\mu,\rho\rho)$, $((3,3)\mu,\rho\rho)$, and $((5,5)\mu,\rho\rho)$ give us $\sigma\langle\rho,\mu\rangle = \langle\rho,\mu\rangle$, $\sigma\langle(3,3),\mu\rangle = \langle(3,3),\mu\rangle$, and $\sigma\langle(5,5),\mu\rangle = \langle(5,5),\mu\rangle$. Therefore, M is one of the invariants

$$\begin{aligned}\mathcal{E}^{(1)}_{12,p} = \sum_{\mu\in P^p_{++}} &|\chi^{(12,p)}_{\rho\mu} + \chi^{(12,p)}_{(10,1)\mu} + \chi^{(12,p)}_{(1,10)\mu} + \chi^{(12,p)}_{(5,5)\mu} + \chi^{(12,p)}_{(2,5)\mu} + \chi^{(12,p)}_{A^2(5,2)\mu}|^2 \\ &+ 2|\chi^{(12,p)}_{(3,3)\mu} + \chi^{(12,p)}_{(6,3)\mu} + \chi^{(12,p)}_{(3,6)\mu}|^2, \text{ (for } p' = 12, p \not\equiv_3 0),\end{aligned} \tag{6.27}$$

up to multiplication by an automorphism invariant.

6.3.2 The Exceptionals at $p' = 12$ when $\mathcal{R}_R = \mathcal{R}_L = \mathcal{J}(\rho\rho) \cup \mathcal{J}(\rho''\rho)$

The second exceptional case when $p' = 12$ is $\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{12}\rho,\rho).(A^i\rho''_{12},\rho) : i = 0,1,2\}$. Let $\mathcal{J} := \{A^0_{12}A^0_p, A^1_{12}A^0_p, A^2_{12}A^0_p\}$. Here, we have $M_{J(\rho\rho),\rho\rho} = M_{\rho\rho,J(\rho\rho)} = M_{\rho\rho,J(\rho''\rho)} = M_{J(\rho''\rho),\rho\rho} = 1\ \forall J \in \mathcal{J}$, and by Lemma 4.3(d), (c), $t(\lambda) \equiv_3 t(\kappa)$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$, and $M_{A^i\lambda\mu,A^j\kappa\nu} = M_{\lambda\mu,\kappa\nu}\ \forall i,j \in \{0,1,2\}$. We can use Lemma 6.1 again since p is coprime to 6, so $M_{\lambda\mu,\kappa\nu} \neq 0 \implies \nu \in \mathcal{O}\mu$.

To find $\mathcal{P}$, we check for which $\lambda\mu \in P^{12,p}_{++}$ is $s_L(\lambda,\mu) > 0$. We find these to be all $\lambda\mu$ with $\lambda \in \mathcal{O}\rho \cup \mathcal{O}\rho'' \cup \mathcal{O}(3,3)$; ie, $\mathcal{P} = \mathcal{O}\rho \cup \mathcal{O}\rho'' \cup \mathcal{O}(3,3) \times P^p_{++}$. Putting these into T-invariance gives $M_{\lambda\mu,\kappa\nu} \neq 0 \iff \lambda,\kappa \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, or $\lambda,\kappa \in \mathcal{O}(3,3)$. As in the previous cases, we also have $t(\mu) \equiv_3 2t(\nu)$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$ (conjugating if necessary), so $\nu \in \{\mu, A^{pt(\mu)}_p\mu\}$. As usual, multiplying by a simple current invariant, we can take $\nu = \mu$. Suppose $M_{\langle\rho\mu\rangle\langle\kappa\nu\rangle} \neq 0$. Evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $(\rho\mu,\rho\rho)$, we get $\langle\kappa\nu\rangle = \langle\rho\mu\rangle$ and $\langle\kappa\nu\rangle = \langle\rho''\mu\rangle$ (and the value of M at these weights is 1), and similarly, $M_{\langle\rho''\mu\rangle\langle\kappa\nu\rangle} \neq 0 \implies \langle\kappa\nu\rangle = \langle\rho\mu\rangle$ and $\langle\kappa\nu\rangle = \langle\rho''\mu\rangle$. Therefore, $\langle\rho\mu\rangle$ and $\langle\rho''\mu\rangle$ each have two couplings, both of which give an M value of 1, so we have the following:

$$M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} \neq 0 \iff M_{\langle\lambda\mu\rangle\langle\kappa\nu\rangle} = 1 \text{ and } \langle\kappa\nu\rangle \in \{\langle\rho\mu\rangle, \langle\rho''\mu\rangle\},$$

for any $\lambda\mu$ with $\lambda \in \mathcal{O}\rho \cup \mathcal{O}\rho''$, and where $\mu' \in \{\mu, A^{pt(\mu)}_p\mu\}$.

Finally, suppose $M_{\langle(3,3)\mu\rangle\langle\kappa\nu\rangle} \neq 0$. Evaluating $MS^{(12,p)} = S^{(12,p)}M$ at $((3,3)\mu,\rho\rho)$

gives us $M_{\langle(3,3)\mu\rangle\langle(3,3)\mu\rangle} = 2$. Therefore, M is given by

(6.28)

$$\mathcal{E}^{(2)}_{12,p} = \sum_{\mu\in P^p_{++}} |\chi^{(12,p)}_{\rho\mu} + \chi^{(12,p)}_{(10,1)\mu} + \chi^{(12,p)}_{(1,10)\mu}|^2 + |\chi^{(12,p)}_{(3,3)\mu} + \chi^{(12,p)}_{(6,3)\mu} + \chi^{(12,p)}_{(3,6)\mu}|^2 +$$
$$|\chi^{(12,p)}_{(5,5)\mu} + \chi^{(12,p)}_{(2,5)\mu} + \chi^{(12,p)}_{(5,2)\mu}|^2 + |\chi^{(12,p)}_{(1,4)\mu} + \chi^{(12,p)}_{(7,1)\mu} + \chi^{(12,p)}_{(4,7)\mu}|^2$$
$$|\chi^{(12,p)}_{(4,1)\mu} + \chi^{(12,p)}_{(7,4)\mu} + \chi^{(12,p)}_{(1,7)\mu}|^2 + 2|\chi^{(12,p)}_{(4,4)\mu}|^2$$
$$+ (\chi^{(12,p)}_{(2,2)\mu} + \chi^{(12,p)}_{(8,2)\mu} + \chi^{(12,p)}_{(2,8)\mu})\chi^{(12,p)*}_{(4,4)\mu} + \chi^{(12,p)}_{(4,4)\mu}(\chi^{(12,p)*}_{(2,2)\mu} + \chi^{(12,p)*}_{(8,2)\mu} + \chi^{(12,p)*}_{(2,8)\mu}),$$
$$(\text{for } p' = 12, p \not\equiv_3 0),$$

up to multiplication by an automorphism invariant.

6.4 The Exceptional Invariants at $(24, p)$

There is only one case to consider here; we have

$$\mathcal{R}_R = \mathcal{R}_L = \{(A^i_{24}\rho, \rho), (A^i_{24}\rho'', \rho), (A^i_{24}(5,5), \rho), (A^i_{24}(7,7), \rho) : i = 0, 1, 2\}.$$

Let $\mathcal{R} := \mathcal{R}_R = \mathcal{R}_L$. Then $M_{\lambda\mu,\rho\rho} = M_{\rho\rho,\lambda\mu} = 1\ \forall\lambda\mu \in \mathcal{R}$, and $M_{\lambda\mu,\rho\rho} = M_{\rho\rho,\lambda\mu} = 0\ \forall\lambda\mu \notin \mathcal{R}$. Lemma 4.3(b) gives us $t(\lambda) \equiv_3 t(\kappa) \equiv_3 0$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$. Also, as usual, evaluating $MS^{(24,p)} = S^{(24,p)}M$ at $(\rho A_p\rho, \rho\rho)$ (and conjugating if necessary), we get that $t(\mu) \equiv 2t(\nu) \pmod 3$ whenever $M_{\lambda\mu,\kappa\nu} \neq 0$.

Using Maple to calculate $s_L(\lambda, \mu)$ for all $\lambda\mu \in P^{24,p}_{++}$, we find that $\mathcal{P} = (\mathcal{O}\rho \cup \mathcal{O}(5,5) \cup \mathcal{O}(7,7) \cup \mathcal{O}(11,11) \cup \mathcal{O}(1,7) \cup \mathcal{O}(5,8)) \times P^p_{++}$. S-invariance at $((5,5)\rho, \rho\rho)$ gives us $M_{(5,5)\rho,(5,5)A^i_p\rho} = M_{(5,5)\rho,(7,7)A^j_p\rho} = M_{(5,5)\rho,(11,11)A^j_p\rho} = 1$ for exactly one choice of i, j, k. But now $-t(\rho) \equiv t(A^i_p\rho) \equiv t(A^j_p\rho) \equiv t(A^k_p\rho) \pmod 3$ implies $i = j = k = 0$. Similarly, we can evaluate $S^{(24,p)}M = MS^{(24,p)}$ at $((7,7)\rho, \rho\rho)$ and $((11,11)\rho, \rho\rho)$ to get $M_{\lambda\mu,\kappa\nu} \neq 0 \iff M_{\lambda\mu,\kappa\nu} = 1$ and $\lambda\mu, \kappa\nu \in \mathcal{R}$; ie, $M_1 = B(1,12)$.

Multiplying by a simple current invariant if necessary, S-invariance at $(\lambda\mu, \rho\rho)$ gives us $M_{\langle\lambda\mu\rangle\langle\lambda'\mu\rangle} = 1$ for all $\lambda, \lambda' \in \mathcal{O}\rho \cup \mathcal{O}(5,5) \cup \mathcal{O}(7,7) \cup \mathcal{O}(11,11)$.

Finally, suppose $M_{\lambda\mu,\kappa\nu} \neq 0$, where $\lambda \in \mathcal{O}(1,7) \cup \mathcal{O}(5,8)$. S-invariance at $(\lambda\mu, \rho\rho)$ gives us

$$M_{\langle\lambda\mu\rangle\langle(1,7)\mu\rangle} + M_{\langle\lambda\mu\rangle\langle(7,1)\mu\rangle} + M_{\langle\lambda\mu\rangle\langle(5,8)\mu\rangle} + M_{\langle\lambda\mu\rangle\langle(8,5)\mu\rangle} = 4,$$

for any $\lambda \in \mathcal{O}(1,7) \cup \mathcal{O}(5,8)$. In particular, put $\lambda = (5,8)$. Now evaluating $MS^{(24,p)} = S^{(24,p)}M$ at $((5,8)\mu, (1,4)\rho)$ and $((5,8)\mu, (2,5)\rho)$, we have $M_{\langle(5,8)\mu\rangle\langle(5,8)\mu\rangle} = M_{\langle(5,8)\mu\rangle\langle(8,5)\mu\rangle} = M_{\langle(5,8)\mu\rangle\langle(1,7)\mu\rangle} = M_{\langle(5,8)\mu\rangle\langle(7,1)\mu\rangle}$, and similarly for $\langle\lambda\mu\rangle = \langle(1,7)\mu\rangle$. Therefore, we have found the exceptional invariants

(6.29)

$$\begin{aligned}\mathcal{E}_{24,p} = \sum_{\mu \in P^p_{++}} & |\chi^{(24,p)}_{\rho\mu} + \chi^{(24,p)}_{(5,5)\mu} + \chi^{(24,p)}_{(7,7)\mu} + \chi^{(24,p)}_{(11,11)\mu} + \chi^{(24,p)}_{(22,1)\mu} + \chi^{(24,p)}_{(1,22)\mu} + \chi^{(24,p)}_{(14,5)\mu} \\ & + \chi^{(24,p)}_{(5,14)\mu} + \chi^{(24,p)}_{(11,2)\mu} + \chi^{(24,p)}_{(2,11)\mu} + \chi^{(24,p)}_{(10,7)\mu} + \chi^{(24,p)}_{(7,10)\mu}|^2 + |\chi^{(24,p)}_{(16,7)\mu} + \chi^{(24,p)}_{(7,16)\mu} \\ & + \chi^{(24,p)}_{(1,16)\mu} + \chi^{(24,p)}_{(16,1)\mu} + \chi^{(24,p)}_{(8,11)\mu} + \chi^{(24,p)}_{(11,8)\mu} + \chi^{(24,p)}_{(5,11)\mu} + \chi^{(24,p)}_{(11,5)\mu} \\ & + \chi^{(24,p)}_{(5,8)\mu} + \chi^{(24,p)}_{(8,5)\mu} + \chi^{(24,p)}_{(1,7)\mu} + \chi^{(24,p)}_{(7,1)\mu}|^2 \text{ (for } p' = 24, p \not\equiv_3 0),\end{aligned}$$

up to multiplication by an automorphism invariant.

Chapter 7

Concluding Remarks

In this thesis, we found the $A_2 \oplus A_2$ modular invariants[1]. We used some powerful tools, among them the Galois symmetry for the S-matrix, and the Weyl character formula. These are not unique to $A_2 \oplus A_2$; they hold for any affine algebra X_r. In future work we will apply these methods to the W_3 classification. We include in this chapter a brief discussion of the minimal W_3 model classification and some suggestions of further work in the area.

7.1 The Nonunitary W_3 Minimal Models

This thesis sets the stage for the classification of the nonunitary W_3 minimal models. As with $A_2 \oplus A_2$, we associate to W_3 a pair (p', p) with $\gcd(p', p) = 1$. In the case of all minimal W_N models, the difference between a unitary theory and a nonunitary one is that a unitary theory has $p = p' + 1$. The unitary W_3 minimal models were classified in [14]. The W_3 minimal model problem can be stated in much the same way as for the $A_2 \oplus A_2$ classification: find all M satisfying (2.3). The difference in this case is that the S and T matrices are given in (7.1). As in the $A_2 \oplus A_2$ classification, we have a set

$$P_{++}^{p',p} = \{(\lambda, \mu) \in \mathbb{Z}^4 : 0 < \lambda_1, \lambda_2, \lambda_1 + \lambda_2 < p' \text{ and } \ 0 < \mu_1, \mu_2, \mu_1 + \mu_2 < p\},$$

that will index our modular invariant M. An element $\lambda\mu \in P_{++}^{p',p}$ is called a *primary*.

[1]Although not explored in this thesis, it is interesting to note here that there is a mysterious connection between the A_1 modular invariant classification and the ADE pattern [11], and between the A_2 classification and Fermat curves [9]

The S and T matrices for the W_3 classification are slightly different than for $A_2 \oplus A_2$. They are given by

$$S_{\lambda\mu,\kappa\nu} = \alpha_{p',p} \exp[-2\pi i \frac{t(\lambda)t(\nu) + t(\mu)t(\kappa)}{3}] S^{(p'/p)}_{\lambda\kappa} S^{(p/p')}_{\mu\nu}, \tag{7.1a}$$

$$T_{\lambda\mu,\kappa\nu} = \beta_{p',p} \exp[\pi i \frac{(p'\lambda - p\mu)^2}{p'p}], \tag{7.1b}$$

where $\alpha_{p',p}$ and $\beta_{p',p}$ are constants, and $S^{(p'/p)}$ is just the usual $S^{(n)}$ matrix evaluated at the fractional height $n = p'/p$.

Recall that $\langle\lambda\rangle = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = \frac{3}{2}\lambda^2$. The norm condition for W_3 is given by

$$M_{\lambda\mu,\lambda'\mu'} \neq 0 \implies \frac{p'}{p}\langle\lambda\rangle + \frac{p}{p'}\langle\mu\rangle + t(\lambda)t(\mu) \equiv_3 \frac{p'}{p}\langle\lambda'\rangle + \frac{p}{p'}\langle\mu'\rangle + t(\lambda')t(\mu'). \tag{7.2}$$

The most significant difference between the $A_2 \oplus A_2$ classification and the nonunitary minimal W_3 model classification (and in general between unitary and nonunitary theories), is that *we do not have the vacuum column of S strictly positive* (see (2.12)). However, there is a unique primary called the *minimal primary* and denoted o, such that $S_{\lambda\mu,o} > 0$ for all $\lambda\mu \in P^{p',p}_{++}$. The $S_{\updownarrow o}$ column will play the role in the nonunitary W_3 classification that the vacuum column $S^{(p',p)}_{\updownarrow,\rho\rho}$ did for $A_2 \oplus A_2$. In the case of any W_N, the minimal primary also obeys the property that $M_{oo} = 1$. This is important because it bounds the entries of M (the proof is similar to the proof of (2.27)) and thus proves that for any W_N minimal model, and in particular for our classification, there are finitely many choices for M.

In some rational conformal field theories, we can relate the minimal primary and the vacuum in a relatively simple way, via the *Galois shuffle*, which is a composition of a simple current[2] and a Galois automorphism (see §§2.2.2).

We say that a rational conformal field theory has the *GS property* if there is a simple current J_o and a Galois automorphism σ_o (these are not necessarily unique) such that

$$o = J_o\sigma_o(\rho, \rho).$$

Not all rational conformal field theories possess the GS property; however all W_N minimal models do. In fact, in the case of W_3, J_o is trivial [12].

[2] A simple current is defined as a primary $\lambda\mu$ such that $S_{\lambda\mu,o} = S_{\rho\rho,o}$

Our approach to the W_3 minimal models will follow the same three steps as we did for $A_2 \oplus A_2$ in this thesis; namely, find the automorphism invariants, then the simple-current extensions, and finally, find the exceptional invariants.

7.2 Reflections and Further Work

Working on this thesis was most interesting and enjoyable, and in particular, we plan to continue with the W_3 classification. Understanding the proofs of the previous classifications presented the biggest challenge; however, once the problem was well-understood, most of the proof went as expected.

In addition to the classification of the minimal W_3 models described above, some further work on this problem could be to remove the $\gcd(p', p) = 1$ condition. For example, we should be able to assume that $\gcd(p', p) \leq 3$, as was done for $A_1 \oplus \cdots \oplus A_1$.

Bibliography

[1] N. Aoki. Simple factors of the Jacobian of a Fermat curve and the Picard number of a product of Fermat curves. *Amer. J. Math*, 113:779–833, 1991.

[2] M. Bauer, A. Coste, C. Itzykson, and P. Ruelle. Comments on the links between su(3) modular invariants, simple factors in the jacobian of Fermat curves, and rational triangular billiards. *J. Geom. Phys.*, 22:134–189, 1997.

[3] N. Bourbaki. *Groupes et Algèbres de Lie*, volume IV-VI. Hermann, 1968.

[4] A. Cappelli, C. Itzykson, and J.B. Zuber. The A-D-E classification of $A_1^{(1)}$ and minimal conformal field theories. *Commun. Math. Phys.*, 113:1–26, 1987.

[5] A. Coste and T. Gannon. Remarks on Galois symmetry in RCFT. *Phys. Lett.*, B323:316–321, 1994.

[6] D.S. Dummit and R.M. Foote. *Abstract Algebra.* John Wiley and Sons, 2nd edition, 1999.

[7] J. Fuchs. Simple WZW currents. *Commun. Math. Phys.*, 136:345–356, 1991.

[8] T. Gannon. Symmetries of the Kac-Peterson modular matrices of affine algebras. *Invent. math.*, 122:341–357, 1995.

[9] T. Gannon. The classification of SU(3) modular invariants revisited. *Ann. Inst. Henri Poincaré:Phys. Théor.*, 65(1):15–55, 1996.

[10] T. Gannon. The level two and three modular invariants of SU(n). *Letters in Mathematical Physics*, 39:289–298, 1997.

[11] T. Gannon. The Cappelli-Itzykson-Zuber A-D-E classification. *Reviews in Mathematical Physics*, 12(5):739–748, 2000.

[12] T. Gannon. Comments on nonunitary conformal field theories. *Nucl. Phys.*, B670:335–358, 2003.

[13] T. Gannon. The ADE_7-type modular invariants of the affine algebras. (in preparation).

[14] T. Gannon and M. Walton. On the classification of diagonal coset modular invariants. *Commun. Math Phys.*, 173:175–197, 1995.

[15] R. Gantmacher. *The Theory of Matrices*, volume II. Chesea Publishing Co., 1964.

[16] D. Gepner and E. Witten. String theory on group manifolds. *Nucl. Phys.*, B278(3):493–549, 1986.

[17] B. Greene. *The Elegant Universe.* Vintage, 2000.

[18] V. C. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, 1990.

[19] A. Ocneanu. The classification of subgroups of quantum SU(N). In R. Coquereaux et al, editor, *Quantum Symmetries in Theoretical Physics and Mathematics*, Providence, 2002. Amer. Math. Soc.

[20] J. Polchinski. *String Theory*, volume I, II. Cambridge University Press, 1999.

[21] L.C. Washington. *Introduction to Cyclotomic Fields.* Springer, 1982.