University of Alberta

WAVELETS AND APPLICATIONS

by

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in partial fulfillment of the requirements for the degree of
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in

Mathematics

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Chapter 1

Wavelet Bases of Hermite Cubic Splines on an Interval

1.1 Introduction

In this chapter we shall construct wavelet bases of Hermite cubic splines on the interval. These wavelet bases are suitable for numerical solutions of differential equations.

By $L_2(\mathbb{R})$ we denote the linear space of all square-integrable real-valued functions on $\mathbb{R}$. The inner product in $L_2(\mathbb{R})$ is defined as

$$\langle u, v \rangle := \int_{\mathbb{R}} u(x)v(x)\,dx, \quad u, v \in L_2(\mathbb{R}).$$

If $\langle u, v \rangle = 0$, then we say that u and v are orthogonal. The norm of a function f in $L_2(\mathbb{R})$ is given by $\|f\|_2 := \sqrt{\langle f, f \rangle}$.

Smooth orthogonal wavelets with compact support were constructed by Daubechies (see [22]). The Daubechies orthogonal wavelets were adapted to the interval $[0, 1]$ by Cohen, Daubechies, and Vial ([17]). Semi-orthogonal spline wavelets were constructed by Chui and Wang [16]. These spline wavelets were adapted to the interval [0,1] by Chui and Quak [15]. In [50] Wang constructed cubic spline wavelet bases for Sobolev spaces.

Orthogonal multi-wavelets were constructed by Donovan, Geronimo, Hardin,

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and Massopust [23]. In [30], Heil, Strang, and Strela considered the possibility of construction of wavelets on the basis of Hermite cubic splines.

Let ϕ_1 and ϕ_2 be the cubic splines given by

$$\phi_1(x) := \begin{cases} (x+1)^2(1-2x) & \text{for} \quad x \in [-1,0], \\ (1-x)^2(2x+1) & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [-1,1], \end{cases}$$

and

$$\phi_2(x) := \begin{cases} x(x+1)^2 & \text{for} \quad x \in [-1,0], \\ x(x-1)^2 & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [-1,1]. \end{cases}$$

In [20], Dahmen, Han, Jia, and Kunoth constructed biorthogonal multiwavelets on the basis of the Hermite cubic splines ϕ_1 and ϕ_2. These wavelets were adapted to the interval $[0,1]$. However, their construction for the wavelet basis on the interval $[0,1]$ was quite complicated.

In this chapter we take a new approach to the construction of wavelet bases of Hermite cubic splines. In contrast to the semi-orthogonal wavelets of Chui and Wang, the wavelets at different levels are orthogonal with respect to the inner product $\langle u', v' \rangle$, rather than $\langle u, v \rangle$. This requirement of orthogonality is more pertinent to applications of wavelets to numerical solutions of differential equations.

As is well-known, the semi norm is a norm in the underlying Sobolev space for the second order elliptic problems with zero Dirichlet boundary condition. Hence, the wavelets with the above orthogonality form a Riesz basis in Sobolev space and thus stiffness matrices arising from the discretization of the problems by the wavelets have the uniformly bounded condition numbers. this, in turn, ensures the efficiency of iterative methods applied to solving the discretized linear system.

On the other hand, Hermite cubic splines, unlike Daubechie's scaling functions, have explicit expressions with short supports, which are favorite in numerical solutions of partial differential equations. Furthermore, our wavelets

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have the same short supports as those of Hermite cubic splines, and this guarantees the efficiency in algorithm. The potential use of such wavelets maybe the numerical solutions of differential equations, and the tensor-product counterparts of our wavelets may serve well for solving partial differential equations in multidimensional spaces. Moreover, changing the orthogonality property with different inner products results in wavelets suitable for numerical solutions of higher order differential equations or integral equations. This is also the motivation of constructing such wavelets.

In Section 1.2 we will give two wavelets ψ_1 and ψ_2 as follows:

$$\psi_1(x) = -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) - 21\phi_2(2x+1) + 21\phi_2(2x-1),$$
$$\psi_2(x) = \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(2x+1) + 12\phi_2(2x) + 9\phi_2(2x-1).$$

Clearly, ψ_1 and ψ_2 are supported on $[-1,1]$; ψ_1 is symmetric and ψ_2 is antisymmetric. Moreover,

$$\langle \psi_1', \phi_m'(\cdot - j)\rangle = \langle \psi_2', \phi_m'(\cdot - j)\rangle = 0, \quad m = 1, 2, \quad \forall \quad j \in \mathbb{Z}.$$

These wavelets can be easily adapted to the interval $[0,1]$.

By $L_2(0,1)$ we denote the space of all square-integrable real-valued functions on $(0,1)$. The inner product in $L_2(0,1)$ is defined as

$$\langle u, v\rangle := \int_0^1 u(x)v(x)\,dx, \quad u, v \in L_2(0,1).$$

Let $H^1(0,1)$ be the space of all functions u in $L_2(0,1)$ for which (the distributional derivative) $u' \in L_2(0,1)$. Let $H_0^1(0,1)$ be the closure of the set

$$\{u \in C[0,1] \cap C^1(0,1) : u(0) = u(1) = 0\}$$

in the space $H^1(0,1)$, where $C[0,1]$ denotes the space of all continuous functions on $[0,1]$, and $C^1(0,1)$ denotes the space of those continuous functions on $(0,1)$ whose derivatives are also continuous.

For a nonnegative integer k, we denote by Π_k the set of all polynomials of degree at most k. For $n \geq 1$, let V_n be the space of those cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$ and

$$v|_{(j/2^n,(j+1)/2^n)} \in \Pi_3|_{(j/2^n,(j+1)/2^n)} \quad \text{for } j = 0, \ldots, 2^n - 1.$$

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The dimension of V_n is 2^{n+1}. It is easily seen that the set

$$(1.1)\quad \Phi_n := \{\phi_1(2^n\cdot - j) : j = 1, \ldots, 2^n - 1\} \cup \{\phi_2(2^n\cdot - j)|_{(0,1)} : j = 0, \ldots, 2^n\}$$

is a basis for V_n. We label the elements in Φ_n as $\{v_1, v_2, \ldots, v_{2^{n+1}}\}$.

Let Ψ_n be the set of wavelets given by

(1.2)

$$\Psi_n := \{\psi_1(2^n\cdot - j) : j = 1, \ldots, 2^n - 1\} \cup \{\psi_2(2^n\cdot - j)|_{(0,1)} : j = 0, \ldots, 2^n\}.$$

Let W_n be the linear span of Ψ_n. It is easily seen that Ψ_n is a basis for W_n. Consequently, the dimension of W_n is 2^{n+1}. In Section 1.3 we shall show that

$$\int_0^1 w'(x)v'(x)\,dx = 0 \quad \forall\, w \in \Psi_n \text{ and } v \in \Phi_n.$$

It follows that $V_n \cap W_n = \{0\}$. Moreover, we have $V_{n+1} \supseteq V_n + W_n$ and

$$\dim(V_{n+1}) = \dim(V_n) + \dim(W_n).$$

This shows that V_{n+1} is the direct sum of V_n and W_n. Therefore, we have the following decomposition of $H_0^1(0,1)$:

$$H_0^1(0,1) = V_1 + W_1 + W_2 + \cdots.$$

Recall that $\Phi_1 = \{v_1, v_2, v_3, v_4\}$. For $n = 1, 2, \ldots$, we label the elements in Ψ_n as follows:

$$\Psi_n = \{w_{2^{n+1}+1}, \cdots, w_{2^{n+2}}\}.$$

Let $g_k := v_k/\|v_k'\|_2$ for $k = 1, 2, 3, 4$ and $g_k := w_k/\|w_k'\|_2$ for $k > 4$. Thus, $\|g_k'\|_2 = 1$ for $k = 1, 2, \ldots$. In Section 1.3 we will show that $(g_k')_{k=1,2\ldots}$ is a Riesz sequence in $L_2(0,1)$.

In Section 1.4 we shall apply the wavelets constructed in Section 1.3 to numerical solutions of the Sturm-Liouville equation of the form

$$(1.3)\qquad -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) = f(x), \quad x \in (0,1),$$

with the Dirichlet boundary condition $u(0) = u(1) = 0$. We assume that p and q are continuous functions on $[0,1]$ and

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$p(x) > 0$, $q(x) \geq 0$ for all $x \in [0,1]$. Let

$$(1.4) \quad a(u,v) := \int_0^1 p(x)u'(x)v'(x)\,dx + \int_0^1 q(x)u(x)v(x)\,dx, \quad u,v \in H_0^1(0,1).$$

Then the variational form of the above equation with the Dirichlet boundary condition is

$$a(u,v) = \langle f, v\rangle \quad \forall\, v \in H_0^1(0,1).$$

Wavelets have been used to discretize differential equations. In particular, Xu and Shann [52] successfully applied the wavelet method to numerical solutions of the Sturm-Liouville equation (1.3). The wavelet bases in their paper are anti-derivatives of the Daubechies orthogonal wavelets. Consequently, their basis functions are not locally supported and, in general, the corresponding stiffness matrix is full (not sparse). Furthermore, the condition number of the stiffness matrix is not uniformly bounded.

In application of the wavelet method one often encounters the difficulty that the boundary conditions are hard to impose on wavelets. In our construction, only two wavelets in Ψ_n, $\psi_2(2^n\cdot)$ and $\psi_2(2^n\cdot - 2^n)$, needed to be adapted to the interval $(0,1)$ by means of restriction. This is in sharp contrast to the complexity of the construction of boundary wavelets given in [20].

Recall that $\{g_k : k = 1,\ldots,2^{n+1}\}$ is a wavelet basis for V_n. Let A_n denote the stiffness matrix $\big(a(g_j,g_k)\big)_{j,k=1,\ldots,2^{n+1}}$. In Section 1.4 we will prove that the condition number of A_n is uniformly bounded (independent of n). In particular, for the case $p = 1$ and $q = 1$, numerical computation suggests that the condition number of A_n be less than 3.75 for all n. By comparison, the condition number of the stiffness matrix with respect to the wavelet basis constructed in [20] is very large.

At the end of this chapter, we shall provide two numerical examples using the above wavelet basis. The computational results demonstrate the advantage of our wavelet basis.

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1.2 Spline Wavelets

In this section we construct wavelets on the basis of Hermite cubic splines.

Let ϕ_1 and ϕ_2 be the cubic splines given in Section 1.1. The graphs of ϕ_1 and ϕ_2 are depicted in Figure 1.1. Clearly, both ϕ_1 and ϕ_2 belong to $C^1(\mathbb{R})$. Moreover, we have

$$\phi_1(0) = 1, \quad \phi_1'(0) = 0, \quad \phi_2(0) = 0, \quad \phi_2'(0) = 1.$$

Hence, for a function $f \in C^1(\mathbb{R})$,

$$u = \sum_{j\in\mathbb{Z}} f(j)\phi_1(\cdot - j) + \sum_{j\in\mathbb{Z}} f'(j)\phi_2(\cdot - j)$$

is a Hermite interpolant to f on $\mathbb{Z}$, that is, $u(j) = f(j)$ and $u'(j) = f'(j)$ for all $j \in \mathbb{Z}$.

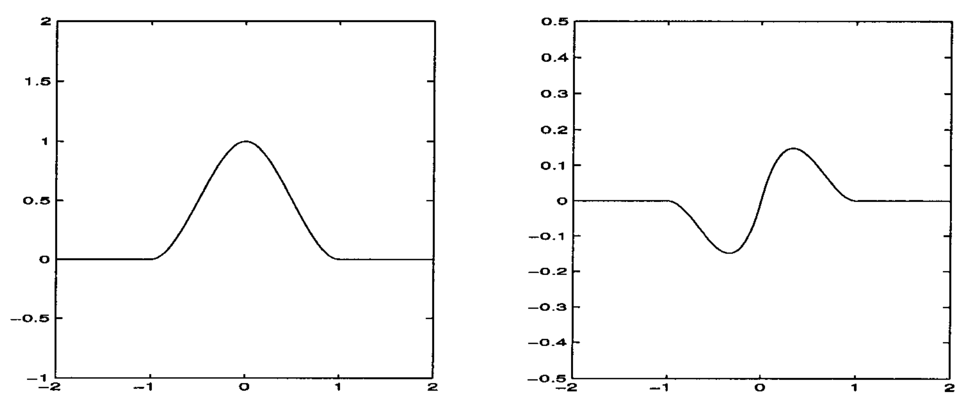

Figure 1.1: Hermit cubic splines on $\mathbb{R}$

Let $\Phi := (\phi_1, \phi_2)^T$, the transpose of the 1×2 vector (ϕ_1, ϕ_2). Then Φ satisfies the following vector refinement equation (see [30]):

$$\Phi(x) = \sum_{j=-1}^{1} a(j)\Phi(2x - j), \quad x \in \mathbb{R},$$

where

$$a(-1) = \begin{bmatrix} 1/2 & 3/4 \\ -1/8 & -1/8 \end{bmatrix}, \ a(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \text{ and } a(1) = \begin{bmatrix} 1/2 & -3/4 \\ 1/8 & -1/8 \end{bmatrix}.$$

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Let S be the shift invariant space generated by ϕ_1 and ϕ_2. A function g belongs to S if and only if there are two sequences b_1 and b_2 on $\mathbb{Z}$ such that

$$g = \sum_{j\in\mathbb{Z}} \big[b_1(j)\phi_1(\cdot - j) + b_2(j)\phi_2(\cdot - j)\big].$$

Let $S_1 := \{g(2\cdot) : g \in S\}$. Then $S \subset S_1$, since Φ is refinable. We look for a wavelet space W such that S_1 is the direct sum of S and W. We wish to find two wavelets ψ_1 and ψ_2 such that their shifts generate W. Moreover, we require

$$(1.5) \qquad \langle \psi_1', \phi_m'(\cdot - j)\rangle = \langle \psi_2', \phi_m'(\cdot - j)\rangle = 0, \quad m = 1, 2, \quad \forall\, j \in \mathbb{Z}.$$

For this purpose we need to calculate the inner product of the derivatives of shifts of ϕ_1 and ϕ_2. Note that

$$\phi_1'(x) := \begin{cases} -6x^2 - 6x & \text{for} \quad x \in [-1, 0], \\ 6x^2 - 6x & \text{for} \quad x \in [0, 1], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_2'(x) := \begin{cases} 3x^2 + 4x + 1 & \text{for} \quad x \in [-1, 0], \\ 3x^2 - 4x + 1 & \text{for} \quad x \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

$$\psi(x) = \sum_{k\in\mathbb{Z}} \big[b_1(k)\phi_1(2x - k) + b_2(k)\phi_2(2x - k)\big], \quad x \in \mathbb{R}.$$

Then for $j \in \mathbb{Z}$ we have

$$\begin{aligned} \langle \psi', \phi_1'(\cdot - j)\rangle \;=\; & \frac{1}{20}\big[-21b_1(2j-2) + 42b_1(2j) - 21b_1(2j+2) \\ & -3b_2(2j-2) + 4b_2(2j-1) - 4b_2(2j+1) + 3b_2(2j+2)\big] \end{aligned}$$

and

$$\begin{aligned} \langle \psi', \phi_2'(\cdot - j)\rangle \;=\; & \frac{1}{120}\big[33b_1(2j-2) - 60b_1(2j-1) + 60b_1(2j+1) \\ & -33b_1(2j+2) + 4b_2(2j-2) - 12b_2(2j-1) \\ & +28b_2(2j) - 12b_2(2j+1) + 4b_2(2j+2)\big]. \end{aligned}$$

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For $z \in \mathbb{C} \setminus \{0\}$, let

$$q_{11}(z) := \sum_{j\in\mathbb{Z}} b_1(2j+1)z^{2j+1}, \quad q_{12}(z) := \sum_{j\in\mathbb{Z}} b_1(2j)z^{2j},$$

$$q_{21}(z) := \sum_{j\in\mathbb{Z}} b_2(2j+1)z^{2j+1}, \quad q_{22}(z) := \sum_{j\in\mathbb{Z}} b_2(2j)z^{2j}.$$

Then $\langle \psi', \phi'_m(\cdot - j)\rangle = 0$ for $m = 1, 2$ and all $j \in \mathbb{Z}$ if and only if

$$B(z)\,(q_{11}(z), q_{12}(z), q_{21}(z), q_{22}(z))^T = 0 \quad \forall\, z \in \mathbb{C} \setminus \{0\},$$

where

$$B(z) := \begin{bmatrix} 0 & -21z^2+42-21z^{-2} & 4z-4z^{-1} & -3z^2+3z^{-2} \\ -60z+60z^{-1} & 33z^2-33z^{-2} & -12z-12z^{-1} & 4z^2+28+4z^{-2} \end{bmatrix}.$$

We find two independent solutions as follows:

$$\begin{bmatrix} q_{11}(z) \\ q_{12}(z) \\ q_{21}(z) \\ q_{22}(z) \end{bmatrix} = \begin{bmatrix} -2z^{-1}-2z \\ 4 \\ -21z^{-1}+21z \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} q_{11}(z) \\ q_{12}(z) \\ q_{21}(z) \\ q_{22}(z) \end{bmatrix} = \begin{bmatrix} z^{-1}-z \\ 0 \\ 9z^{-1}+9z \\ 12 \end{bmatrix}.$$

These two solutions induce two wavelets ψ_1 and ψ_2 given by

$$\psi_1(x) = -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) - 21\phi_2(2x+1) + 21\phi_2(2x-1),$$
$$\psi_2(x) = \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(2x+1) + 12\phi_2(2x) + 9\phi_2(2x-1).$$

By our construction, ψ_1 and ψ_2 are supported on $[-1, 1]$, they satisfy the conditions in (1.5), and their shifts generate the wavelet space W such that S_1 is the direct sum of S and W. Moreover, ψ_1 is symmetric and ψ_2 is anti-symmetric (see Figure 1.2).

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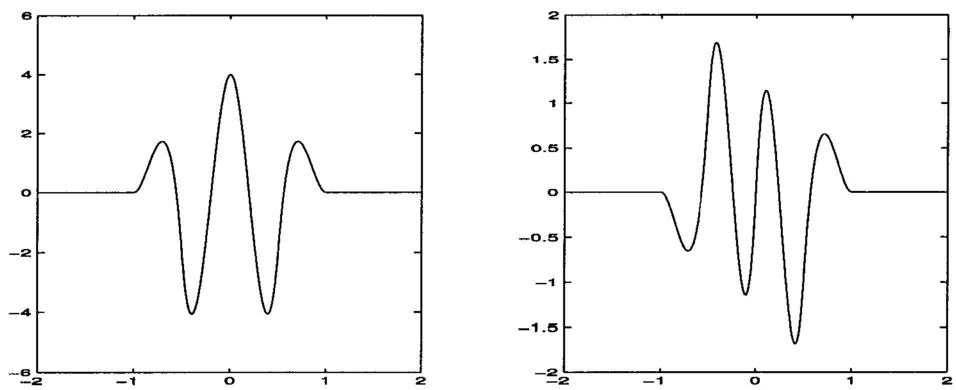

Figure 1.2: Wavelets ψ_1 and ψ_2

Let us take a look at ψ_1' and ψ_2'. For $0 \le x \le 1/2$ we have

$$\begin{aligned} \psi_1'(x) &= 792x^2 - 312x, \quad \psi_1'(x-1) = -408x^2 + 120x, \\ \psi_2'(x) &= 552x^2 - 288x + 24, \quad \psi_2'(x-1) = 168x^2 - 48x. \end{aligned}$$

For $1/2 \le x \le 1$ we have

$$\begin{aligned} \psi_1'(x) &= 408x^2 - 696x + 288, \quad \psi_1'(x-1) = -792x^2 + 1272x - 480, \\ \psi_2'(x) &= 168x^2 - 288x + 120, \quad \psi_2'(x-1) = 552x^2 - 816x + 288. \end{aligned}$$

Hence, the shifts of ψ_1' and ψ_2' are linearly independent on the interval $(0, 1)$. Because of shift invariance, the shifts of ψ_1' and ψ_2' are linear independent on the interval $(k, k+1)$ for every $k \in \mathbb{Z}$. Suppose b_1 and b_2 are two square summable sequences on $\mathbb{Z}$. Let

$$u := \sum_{j\in\mathbb{Z}} \big[b_1(j)\psi_1'(\cdot - j) + b_2(j)\psi_2'(\cdot - j)\big].$$

For $j < k$ or $j > k+1$, $\psi_1'(\cdot - j)$ and $\psi_2'(\cdot - j)$ vanish on $(k, k+1)$. Since the shifts of ψ_1' and ψ_2' are linearly independent on $(k, k+1)$, there exist two positive constants C_1 and C_2 independent of k, b_1, and b_2 such that

$$C_1^2 \sum_{j=k}^{k+1} \big[|b_1(j)|^2 + |b_2(j)|^2\big] \le \int_k^{k+1} |u(x)|^2\,dx \le C_2^2 \sum_{j=k}^{k+1} \big[|b_1(j)|^2 + |b_2(j)|^2\big].$$

It follows that

$$2C_1^2 \sum_{j\in\mathbb{Z}} \big[|b_1(j)|^2 + |b_2(j)|^2\big] \le \int_{\mathbb{R}} |u(x)|^2\,dx \le 2C_2^2 \sum_{j\in\mathbb{Z}} \big[|b_1(j)|^2 + |b_2(j)|^2\big].$$

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In other words, the shifts of ψ_1' and ψ_2' are stable. See [35] for a study of stability of shifts of several functions.

1.3 Wavelets on the Interval

In this section we use the spline wavelets in the previous section to construct a wavelet basis for the space $H_0^1(0,1)$.

Recall that V_n is the linear space of those cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$ and

$$v|_{(j/2^n,(j+1)/2^n)} \in \Pi_3|_{(j/2^n,(j+1)/2^n)} \quad \text{for } j = 0, \ldots, 2^n - 1.$$

The dimension of V_n is 2^{n+1}. Moreover,

(a) $V_1 \subset V_2 \subset \cdots \subset H_0^1(0,1)$;

(b) $\cup_{n=1}^{\infty} V_n$ is dense in $H_0^1(0,1)$.

Let Φ_n and Ψ_n be the sets defined in (1.1) and (1.2), respectively. Then Φ_n is a basis for V_n. Let W_n be the linear span of Ψ_n. Clearly, Ψ_n is a basis for W_n. Consequently, the dimension of W_n is 2^{n+1}.

We claim that

$$\text{(1.6)} \qquad \int_0^1 w'(x)v'(x)\,dx = 0 \quad \forall\, w \in \Psi_n \text{ and } v \in \Phi_n.$$

Suppose $w = \psi_r(2^n \cdot - j)$ for some $r \in \{1,2\}$ and $j \in \{1, \ldots, 2^n - 1\}$. Then $\psi_r'(2^n \cdot - j)$ is supported in the interval $[0,1]$. Hence, for $s = 1,2$ and $k \in \mathbb{Z}$, we have

$$\int_0^1 \psi_r'(2^n x - j)\phi_s'(2^n x - k)\,dx = \int_{\mathbb{R}} \psi_r'(2^n x - j)\phi_s'(2^n x - k)\,dx = 0,$$

where (1.5) has been used to derive the second equality. For the same reason, (1.6) is valid if $v = \phi_s(2^n \cdot - k)$ for some $s \in \{1,2\}$ and $k \in \{1, \ldots, 2^n - 1\}$. Thus, in order to complete the proof of (1.6), it remains to deal with the case

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$w = \psi_2(2^n\cdot - j)|_{(0,1)}$ and $v = \phi_2(2^n\cdot - k)|_{(0,1)}$ for $j, k \in \{0, 2^n\}$. We have $v'(x)w'(x) = 0$ for $x \in (0,1)$ if $j = 0$ and $k = 2^n$, or if $j = 2^n$ and $k = 0$. Hence (1.6) is valid in this case. Suppose $j = k = 0$. Since ψ_2 and ϕ_2 are anti-symmetric, ψ_2' and ϕ_2' are symmetric. It follows that

$$\int_{-1}^{0} \psi_2'(x)\phi_2'(x)\,dx = \int_0^1 \psi_2'(x)\phi_2'(x)\,dx.$$

But (1.5) tells us that

$$\int_{-1}^{1} \psi_2'(x)\phi_2'(x)\,dx = 0.$$

Therefore,

$$\int_0^1 \psi_2'(x)\phi_2'(x)\,dx = 0.$$

Consequently,

$$\int_0^1 \psi_2'(2^n x)\phi_2'(2^n x)\,dx = 2^{-n}\int_0^{2^n} \psi_2'(x)\phi_2'(x)\,dx = 0.$$

This verifies (1.6) for $w = \psi_2(2^n\cdot)|_{(0,1)}$ and $v = \phi_2(2^n\cdot)|_{(0,1)}$. An analogous argument shows that (1.6) is valid for $w = \psi_2(2^n\cdot - 2^n)|_{(0,1)}$ and $v = \phi_2(2^n\cdot - 2^n)|_{(0,1)}$. The proof of (1.6) is complete.

It follows from (1.6) that

$$\int_0^1 w'(x)v'(x)\,dx = 0 \quad \forall\, w \in W_n \text{ and } v \in V_n.$$

In particular, $V_n \cap W_n = \{0\}$. We have $V_{n+1} \supseteq V_n + W_n$ and

$$\dim(V_n + W_n) = \dim(V_n) + \dim(W_n) = 2^{n+1} + 2^{n+1} = \dim(V_{n+1}).$$

This shows that V_{n+1} is the direct sum of V_n and W_n. Consequently,

$$V_{n+1} = V_1 + W_1 + \cdots + W_n.$$

Therefore, we have the following decomposition of $H_0^1(0,1)$:

$$H_0^1(0,1) = V_1 + W_1 + W_2 + \cdots.$$

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Suppose $v \in V_1$ and $w_n \in W_n$ for $n = 1, 2, \ldots$. The preceding discussion tells us that $\langle v', w'_n \rangle = 0$ for all n and $\langle w'_m, w'_n \rangle = 0$ for $m \neq n$. Hence,

$$\left\| v' + \sum_{n=1}^{\infty} w'_n \right\|_{L_2(0,1)}^2 = \|v'\|_{L_2(0,1)}^2 + \sum_{n=1}^{\infty} \|w'_n\|_{L_2(0,1)}^2. \tag{1.7}$$

For $n = 1, 2, \ldots$ and $x \in (0, 1)$, let

$$\psi_{n,j}(x) := (2^{-n/2}/\sqrt{729.6})\, \psi_1(2^n x - j/2) \quad \text{for } j = 2, 4, \ldots, 2^{n+1} - 2,$$

$$\psi_{n,j}(x) := (2^{-n/2}/\sqrt{153.6})\, \psi_2(2^n x - (j-1)/2) \quad \text{for } j = 3, 5, \ldots, 2^{n+1} - 1,$$

and

$$\psi_{n,1}(x) := (2^{-n/2}/\sqrt{76.8})\, \psi_2(2^n x), \quad \psi_{n,2^{n+1}}(x) := (2^{-n/2}/\sqrt{76.8})\, \psi_2(2^n x - 2^n).$$

Note that $\psi_{n,j}$ are so normalized that $\|\psi'_{n,j}\|_{L_2(0,1)} = 1$ for $j = 1, \ldots, 2^{n+1}$.

Theorem 1.1. *The sequence $(\psi'_{n,j})_{n=1,2,\ldots,1 \le j \le 2^{n+1}}$ is a Riesz sequence in $L_2(0,1)$. In other words, there exist two positive constants A and B such that*

$$A \left(\sum_{n=1}^{\infty} \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2 \right)^{1/2} \le \left\| \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)} \le B \left(\sum_{n=1}^{\infty} \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2 \right)^{1/2}$$

for every sequence $(b_{n,j})_{n=1,2,\ldots,1 \le j \le 2^{n+1}}$.

Proof. By (1.7) we have

$$\left\| \sum_{n=1}^{\infty} \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2 = \sum_{n=1}^{\infty} \left\| \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2.$$

In light of the discussion at the end of Section 1.3, we assert that the shifts of ψ'_1 and ψ'_2 are linearly independent on $(k, k+1)$ for every $k \in \mathbb{Z}$. Hence, there exist two positive constants A and B (independent of n) such that

$$A^2 \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2 \le \left\| \sum_{j=1}^{2^{n+1}} b_{n,j} \psi'_{n,j} \right\|_{L_2(0,1)}^2 \le B^2 \sum_{j=1}^{2^{n+1}} |b_{n,j}|^2.$$

This completes the proof of the theorem. □

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For $x \in (0,1)$, let

$$\begin{aligned}
\phi_{1,1}(x) &:= \sqrt{5/24}\,\phi_1(2x-1), \\
\phi_{1,2}(x) &:= \sqrt{15/4}\,\phi_2(2x), \\
\phi_{1,3}(x) &:= \sqrt{15/8}\,\phi_2(2x-1), \\
\phi_{1,4}(x) &:= \sqrt{15/4}\,\phi_2(2x-2).
\end{aligned}$$

Note that each $\phi_{1,j}$ is so normalized that $\|\phi'_{1,j}\|_{L_2(0,1)} = 1$, $j = 1, \ldots, 4$. Clearly, V_1 is spanned by $\phi_{1,j}$, $j = 1, \ldots, 4$. Consequently, $H_0^1(0,1)$ is spanned by $\phi_{1,j}$, $j = 1, \ldots, 4$, together with $\psi_{n,j}$, $n = 1, 2, \ldots$, $j = 1, \ldots, 2^{n+1}$. We relabel these functions as follows. Let $g_j := \phi_{1,j}$ for $j = 1, \ldots, 4$, and let $g_{2^{n+1}+j} := \psi_{n,j}$ for $n = 1, 2, \ldots$ and $j = 1, \ldots, 2^{n+1}$. With the same reasoning as in the proof of Theorem 1.1, we see that the sequence $(g'_k)_{k=1,2,\ldots}$ is a a Riesz sequence in $L_2(0,1)$. In other words, there exist two positive constants A and B such that

$$A\left(\sum_{k=1}^{\infty} |b_k|^2\right)^{1/2} \le \left\|\sum_{k=1}^{\infty} b_k g'_k\right\|_{L_2(0,1)} \le B\left(\sum_{k=1}^{\infty} |b_k|^2\right)^{1/2} \tag{1.8}$$

for every square summable sequence $(b_k)_{k=1,2,\ldots}$.

1.4 Applications

In this section the wavelets constructed in the previous section are used to solve differential equations. We shall confine ourselves to the Sturm-Liouville equation of the form (1.3) with the Dirichlet boundary condition $u(0) = u(1) = 0$. We assume that p and q are continuous functions on $[0,1]$ and $p(x) > 0$, $q(x) \ge 0$ for all $x \in [0,1]$.

For $u, v \in H_0^1(0,1)$, let $a(u,v)$ be the bilinear form given in (1.4). Then the variational form of equation (1.3) with the Dirichlet boundary condition is

$$a(u,v) = \langle f, v\rangle \quad \forall\, v \in H_0^1(0,1). \tag{1.9}$$

The corresponding Galerkin approximation problem is the following: find $u_n \in V_n$ such that

$$a(u_n,v) = \langle f, v\rangle \quad \forall\, v \in V_n. \tag{1.10}$$

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By the Lax-Milgram lemma (see, *e.g.*, [7, p. 60]), the approximation problem (1.10) has a unique solution.

We propose to use the wavelet set $G_n := \{g_1, \dots, g_{2^{n+1}}\}$ as a basis for V_n. Recall that $g_j := \phi_{1,j}$ for $j = 1, \dots, 4$, and $g_{2^{n+1}+j} := \psi_{n,j}$ for $n = 1, 2, \dots$ and $j = 1, \dots, 2^{n+1}$, where $\phi_{1,j}$ $(j = 1, \dots, 4)$ and $\psi_{n,j}$ $(j = 1, \dots, 2^{n+1})$ are the functions constructed in the previous section. With this basis for V_n, the Galerkin approximation problem (1.10) can be discretized as follows:

$$\sum_{k=1}^{2^{n+1}} a(g_j, g_k)\eta_k = \langle g_j, f \rangle, \quad j = 1, \dots, 2^{n+1}.$$

The stiffness matrix

$$\big(a(g_j, g_k)\big)_{1 \le j,k \le 2^{n+1}}$$

is denoted by A_n. We will prove that the condition number of A_n is uniformly bounded (independent of n). Therefore, the wavelet basis G_n is a good tool for preconditioning.

Let us recall that the condition number of an invertible square matrix A is defined by

$$\text{cond}\,(A) := \|A\|\,\|A^{-1}\|,$$

where $\|\cdot\|$ is a matrix norm. The spectral condition number of A is defined as

$$\frac{\max_i |\lambda_i(A)|}{\min_i |\lambda_i(A)|},$$

where the numbers $\lambda_i(A)$ are eigenvalues of A. If A is a (real) symmetric matrix, then its condition number with respect to the 2-norm is equal to its spectral condition number (see [10, p. 51]).

Theorem 1.2. *The condition number of the stiffness matrix A_n is uniformly bounded (independent of n).*

Proof. It suffices to show that there exist two positive constants B and C independent of n such that

(1.11)
$$B\left(\sum_{j=1}^{4}|c_j|^2 + \sum_{m=1}^{n-1}\sum_{j=1}^{2^{m+1}}|b_{m,j}|^2\right) \le a(u,u) \le C\left(\sum_{j=1}^{4}|c_j|^2 + \sum_{m=1}^{n-1}\sum_{j=1}^{2^{m+1}}|b_{m,j}|^2\right)$$

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for any

$$u = \sum_{j=1}^{4} c_j \phi_{1,j} + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} b_{m,j} \psi_{m,j}.$$

By (1.8) there exists a positive constant C_1 independent of n such that

$$\begin{aligned} \|u'\|_{L_2(0,1)} &= \left\| \sum_{j=1}^{4} c_j \phi'_{1,j} + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} b_{m,j} \psi'_{m,j} \right\|_{L_2(0,1)} \\ &\geq C_1 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{1/2}. \end{aligned}$$

But

$$a(u,u) \geq \langle pu', u' \rangle \geq \mu \langle u', u' \rangle = \mu \|u'\|^2_{L_2(0,1)},$$

where $\mu := \min_{x \in [0,1]} p(x) > 0$. This establishes the first inequality in (1.11). Furthermore, we observe that

$$a(u,u) \leq \nu (\|u\|^2_{L_2(0,1)} + \|u'\|^2_{L_2(0,1)}),$$

where $\nu := \max_{0 \leq x \leq 1} \{p(x), q(x)\} < \infty$. By (1.8) there exists a positive constant C_2 independent of n such that

$$\|u'\|_{L_2(0,1)} \leq C_2 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{\frac{1}{2}}.$$

Moreover,

$$\|u\|_{L_2(0,1)} \leq \left\| \sum_{j=1}^{4} c_j \phi_{1,j} \right\|_{L_2(0,1)} + \sum_{m=1}^{n-1} \left\| \sum_{j=1}^{2^{m+1}} b_{m,j} \psi_{m,j} \right\|_{L_2(0,1)}.$$

Note that $\|\psi_{m,j}\|_{L_2(0,1)} = O(2^{-m})$ as $m \to \infty$. Hence, there exists a positive constant C_3 independent of n such that

$$\|u\|_{L_2(0,1)} \leq C_3 \left[\left(\sum_{j=1}^{4} |c_j|^2 \right)^{1/2} + \sum_{m=1}^{n-1} 2^{-m} \left(\sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right)^{1/2} \right].$$

With the help of the Schwarz inequality we see that there exists a positive constant C_4 independent of n such that

$$\|u\|^2_{L_2(0,1)} \leq C_4 \left(\sum_{j=1}^{4} |c_j|^2 + \sum_{m=1}^{n-1} \sum_{j=1}^{2^{m+1}} |b_{m,j}|^2 \right).$$

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The second inequality in (1.11) follows. The proof of the theorem is complete. $\square$

In what follows we apply the wavelet basis G_n to two numerical examples.

Example 1. Consider the Dirichlet problem:

$$\begin{cases} -u'' = f & \text{on} \quad (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where f is given by

$$f(x) = (53.7\pi)^2 \sin(53.7\pi x) + (2.3\pi)^2 \sin(2.3\pi x), \quad x \in (0,1).$$

The exact solution of the problem is

$$u(x) = \sin(53.7\pi x) + \sin(2.3\pi x), \quad x \in (0,1), \tag{1.12}$$

which could be regarded as the sum of a high-frequency component and a low-frequency component. Let us use the wavelet basis $G_n := \{g_1, \dots, g_{2^{n+1}}\}$ to solve the Dirichlet problem. With $u_n = \sum_{k=1}^{2^{n+1}} \eta_k g_k$, the Galerkin approximation problem (1.10) is discretized as

$$\sum_{k=1}^{2^{n+1}} \langle g_j', g_k' \rangle \eta_k = \langle g_j, f \rangle, \quad j = 1, \dots, 2^{n+1}. \tag{1.13}$$

The stiffness matrix $A_n := \big(\langle g_j', g_k' \rangle\big)_{1 \le j,k \le 2^{n+1}}$ is block diagonal. Moreover, each block is a banded matrix. By Theorem 1.2, the condition number of the matrix A_n is uniformly bounded (independent of n). This assertion is confirmed by numerical computation of the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the condition number $\kappa = \lambda_{\max}/\lambda_{\min}$ of the matrix A_n for $n = 6, \dots, 12$ (see Table 1.1).

We use the CG (conjugate gradient) method to solve the above system (1.13) of linear equations. Since the stiffness matrix A_n is well conditioned, the CG method converges very fast. Up to $n = 12$, only 21 iterations are needed for convergence to the solution of the system of linear equations. Here and in what follows, we take 10^{-10} as the threshold to stop the iterations. For

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n	6	7	8	9	10	11	12
$\lambda_{\max}$	1.5780	1.5787	1.5789	1.5789	1.5789	1.5789	1.5789
$\lambda_{\min}$	0.4220	0.4213	0.4211	0.4211	0.4211	0.4211	0.4211
κ	3.7397	3.7474	3.7494	3.7498	3.7498	3.7498	3.7498

Table 1.1: Condition number of the matrix A_n

$n = 1, 2, \ldots$, let $e_n := \|u_n - u\|_{L_2(0,1)}$, where u is the exact solution given in (1.12). For $n = 6, \ldots, 12$, the following table lists the error e_n and the rate of convergence $\log_2 e_{n-1}/e_n$.

n	6	7	8	9	10	11	12
e_n	1.21-2	1.33-3	1.08-4	7.36-6	4.71-7	2.96-8	1.85-9
$\log_2(\frac{e_{n-1}}{e_n})$	4.10	3.19	3.62	3.88	3.97	3.99	4.00

Table 1.2: Error e_n and its convergence rate

It is well known from approximation theory that the Hermite cubic splines provide approximation of order 4. The preceding computation confirms this assertion.

If we use the finite elements in Φ_n given in (1.1) to discretize the equation (1.10), then the resulting stiffness matrix is ill conditioned. For $n = 12$, the system of linear equations has 8192 unknowns. Without preconditioning, it takes more than 2000 iterations for the CG method to converge. The following graph depicts the error against the number of iterations.

In [6], Bramble, Pasciak, and Xu proposed the so-called BPX method for preconditioning. This method was developed on the nodal basis (piecewise linear functions). The corresponding *spectral* condition number (not necessarily the condition number) was shown to be uniformly bounded. For $n = 6, \ldots, 12$, the following table gives the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the spectral condition number of the corresponding matrix after preconditioning:

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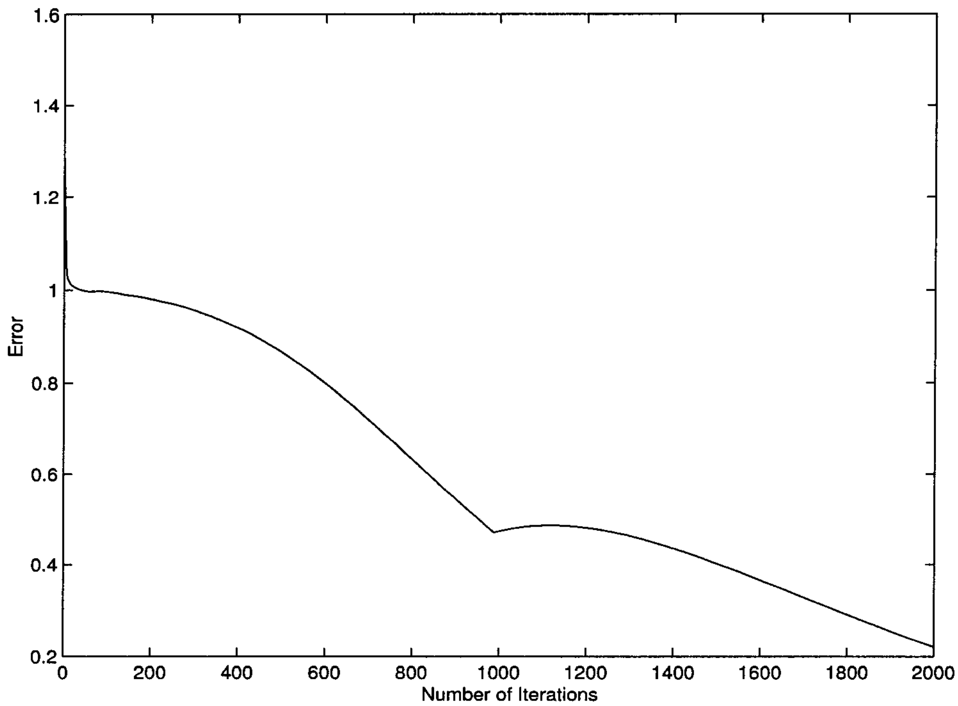

Figure 1.3: The error against the number of iterations without preconditioning

n	6	7	8	9	10	11	12
λ_{max}	4.390	4.725	5.004	5.238	5.437	5.607	5.753
λ_{min}	0.9311	0.9297	0.9291	0.9323	0.9316	0.9311	0.9308
κ	4.715	5.082	5.385	5.619	5.836	6.021	6.180

Table 1.3: BPX preconditioning for nodal basis

We observe that piecewise linear functions only provide approximation of order 2. In order to achieve convergence of order 4, one may extend the BPX method (or additive Schwarz method) to Hermite cubic splines. We will prove that BPX method is still valid for Hermite cubic splines in **Appendix A** [37]. For $n = 6, \ldots, 12$, the following table gives the maximal eigenvalue λ_{max}, the minimal eigenvalue λ_{min}, and the spectral condition number of the corresponding matrix after preconditioning:

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n	6	7	8	9	10	11	12
$\lambda_{\max}$	3.562	3.632	3.682	3.718	3.743	3.763	3.777
$\lambda_{\min}$	0.7693	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696
κ	4.630	4.719	4.784	4.831	4.864	4.890	4.907

Table 1.4: BPX preconditioning for Hermite cubic splines

We see that the condition number induced by our wavelet basis is smaller than that given by the BPX method. For $n = 12$, after preconditioning by the BPX method, it takes 26 iterations for the PCG (preconditioned conjugate gradient) method to converge. Hence, the preconditioning method induced by our wavelet basis is competitive.

Example 2. Consider the Dirichlet problem

$$\begin{cases} -u'' + u = f & \text{on} \quad (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where

$$f(x) = \left[(53.7\pi)^2 + 1\right] \sin(53.7\pi x) + \left[(2.3\pi)^2 + 1\right] \sin(2.3\pi x), \quad x \in (0,1).$$

The function u given in (1.12) is the exact solution of the problem.

In this case, the bilinear form $a(u,v)$ is given by

$$a(u,v) = \langle u', v' \rangle + \langle u, v \rangle, \quad u, v \in H_0^1(0,1).$$

With the wavelet basis G_n the Galerkin approximation problem (1.10) is discretized as

$$\text{(1.14)} \qquad \sum_{k=1}^{2^{n+1}} \big(\langle g_j', g_k' \rangle + \langle g_j, g_k \rangle\big) \eta_k = \langle g_j, f \rangle, \quad j = 1, \ldots, 2^{n+1}.$$

The stiffness matrix

$$A_n := \big(\langle g_j', g_k' \rangle + \langle g_j, g_k \rangle\big)_{1 \le j,k \le 2^{n+1}}$$

is still a sparse matrix. By Theorem 1.2, the condition number of the matrix A_n is uniformly bounded (independent of n). This assertion is confirmed by

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n	6	7	8	9	10	11	12
$\lambda_{\max}$	1.5780	1.5787	1.5789	1.5789	1.5789	1.5789	1.5789
$\lambda_{\min}$	0.4220	0.4213	0.4211	0.4211	0.4211	0.4211	0.4211
κ	3.7396	3.7474	3.7494	3.7498	3.7498	3.7498	3.7498

Table 1.5: Condition number of the matrix A_n

numerical computation of the maximal eigenvalue $\lambda_{\max}$, the minimal eigenvalue $\lambda_{\min}$, and the condition number κ of A_n for $n = 6, \ldots, 12$ (see Table 1.5).

We use the CG method to solve the above system (1.14) of linear equations. The computational results are similar to those in Example 1. Up to $n = 12$, only 19 iterations are needed for convergence to the solution of the system of linear equations. For $n = 6, \ldots, 12$, the following table lists the error e_n and the rate of convergence $\log_2 e_{n-1}/e_n$.

n	6	7	8	9	10	11	12
e_n	1.21-2	1.33-3	1.08-4	7.36-6	4.71-7	2.97-8	1.92-9
$\log_2(\frac{e_{n-1}}{e_n})$	4.10	3.19	3.62	3.88	3.97	3.99	3.95

Table 1.6: Error e_n and its convergence rate

BPX method is applied for the comparison. Up to $n = 12$, 21 iterations are needed for convergence to the solution of the system of linear equations. The maximal and minimal eigenvalues of the preconditioned system, as well as spectral condition numbers, are listed in Table 1.7.

n	6	7	8	9	10	11	12
$\lambda_{\max}$	3.562	3.632	3.682	3.718	3.743	3.763	3.777
$\lambda_{\min}$	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696	0.7696
κ	4.628	4.719	4.784	4.831	4.864	4.890	4.907

Table 1.7: BPX preconditioning for Hermite cubic splines

Finally, we remark that our wavelet basis can also be used to solve inte-

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gral equations numerically. A discrete wavelet Petrov-Galerkin method was developed by Chen, Micchelli, and Xu [12] for numerical solutions of integral equations of the second kind with weakly singular kernels. Recently, Shen and Lin [45] used the wavelet basis G_n constructed in this chapter to find numerical solutions of integral equations on the upper half plane.

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Chapter 2

Modified Hierarchy Basis For Solving Singular Boundary Value Problems

2.1 Background

Our investigations in this chapter is concerned with the preconditioning method on the basis of the modified hierarchy basis for the numerical solution of the singular boundary value problem arising from the radically symmetric elliptic partial differential equations, a problem with numerous applications (see, e.g., [44]).

When the Dirichlet problem

$$\begin{aligned} -\triangle u(\mathbf{x}) + q(\mathbf{x})u(\mathbf{x}) &= f(\mathbf{x}), \quad in \quad \mathbf{B}, \\ u &= 0, \quad on \quad \partial\mathbf{B}, \end{aligned}$$

is defined on a unit ball $\mathbf{B} := B_1(0)$ in $\mathbb{R}^d$ and the data depend only on the radical coordinate, then after a change of variables, the problem will reduce to a one dimensional singular boundary value problem,

$$\begin{aligned} &-u''(x) - \frac{d-1}{x}u'(x) + q(x)u(x) = f(x), \quad x \in (0,1) \\ &u'(0) = u(1) = 0, \end{aligned}$$

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where $q(x) \geq 0$ and $q(x) \in L_\infty(0,1)$.

For the smooth data, it has been proven (see, e.g., [24, 32, 43, 44]) that the (smooth) solution can be approximated with high order accuracy by the Galerkin method with a piecewise polynomial subspace. Therefore, no special functions are required in the basis.

Convergence results of the Galerkin method for the singular boundary values problems have been studied for the case $q(x) = 0$ in detail in [32]. In [24], Eriksson and Thomee established the general optimal order error estimates and even generalized their results to the corresponding time dependent problems. It shows that the Galerkin method would give the same convergence results for the singular problems as for the nonsingular problems.

For the solution with certain smoothness (such as in H^2), the simple piecewise linear nodal basis shall satisfy the approximation requirement. By the error estimates provided in Section 2.3, we show that a slightly modified piecewise linear nodal basis provides the suitable approximation order.

However, it is still a challenging problem to efficiently solve the large system of linear equations arising from the Galerkin method for the singular boundary value problems. Like its counterpart for the regular elliptic problems, the linear system arising from the Galerkin method for the singular boundary value problems is also ill conditioned. For the regular elliptic boundary value problem, multigrid methods (see, e.g., [5, 4]), and numerous other preconditioning methods (see, e.g., [6]), were successfully developed. Nevertheless, to our best knowledge, presently there are few references about preconditioning methods of the Galerkin method for the singular problems. To design an easily implemented preconditioning method through the construction of the modified hierarchy basis shall be the principle goal of this chapter.

The hierarchy basis has been discussed extensively in [55, 56], and has been proven to be an efficient preconditioning method for low dimensional regular elliptic problems. In this chapter, we construct a modified hierarchy basis based on the concept of "stability" (see, e.g., [33, 35, 41]), and the "norm equivalence" for the Sobolev space (see, e.g., [2, 34, 41, 29]). Such basis is

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then adapted to the nodal basis introduced in section 2.2 for the singular boundary value problem, and thus the preconditioning can be achieved. It will be shown later that after applying the preconditioning method based on the modified hierarchy basis, the condition number of the stiffness matrix arising from the Galerkin method will be uniformly bounded. In particular, the condition number is nicely bounded by 2 for the case $q(x) = 0$.

This chapter is divided into three parts. In section 2.2, we propose the preconditioning method on the basis of the modified hierarchy basis for the singular boundary value problem, and show the connection between the concept of norm equivalence and stability of the modified hierarchy basis. The condition number of the preconditioned stiffness matrix is proven to be uniformly bounded. In section 2.3, we provide basic error estimates for the Galerkin approximation from the piecewise linear nodal basis subspace V_h with its element v satisfying the boundary conditions $v'(0) = v(1) = 0$. We will show such subspace provides the same approximation order as the linear nodal basis subspace without the condition $v'(0) = 0$. Numerical examples are computed to confirm our results in section 2.4.

2.2 The Galerkin method and the modified hierarchy basis

We consider the boundary value problem of the form

$$-(x^\alpha u'(x))' + x^\alpha q(x)u(x) = x^\alpha f(x), \quad x \in (0,1), \tag{2.1}$$

$$u'(0) = u(1) = 0, \tag{2.2}$$

where $\alpha = d - 1$.

Let v be a real-valued Lebesgue measurable function on $\mathbb{R}$. We define the $L_2(0,1)$ inner product by

$$\langle u, v \rangle := \int_0^1 u(x)v(x)dx,$$

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and $L_2(0,1)$ space by

$$L_2(0,1) := \{v : \quad ||v||_{L_2(0,1)} < \infty\}.$$

The weighted L_2 space $\dot{L}_2(0,1)$ is defined by

$$\dot{L}_2(0,1) := \Big\{v : \int_0^1 |x^{\frac{\alpha}{2}} v(x)|^2 dx < \infty\Big\}.$$

The weighted Sobolev space $\dot{H}_0^1(0,1)$ is the closure of the set $\{v : v \in C([0,1]) \cap C^1(0,1), v(1) = 0\}$ in the sense of the weighted Sobolev norm

$$||v||_{\dot{H}^1} := \Big(\int_0^1 x^\alpha (|v(x)|^2 + |v'(x)|^2) dx\Big)^{1/2}.$$

Define the symmetric bilinear form $a(\cdot,\cdot)$ as follows: for $u, v \in \dot{H}_0^1(0,1)$,

$$a(u,v) := \int_0^1 x^\alpha u'(x) v'(x) dx + \int_0^1 q(x) x^\alpha u(x) v(x) dx. \tag{2.3}$$

Then the solution u of the singular boundary value problem also solves the variational problem

$$a(u,v) = \langle x^\alpha f(x), v(x)\rangle, \quad \forall v \in \dot{H}_0^1(0,1). \tag{2.4}$$

Here, with some ambiguity, we also use x^α to denote function $x \mapsto x^\alpha$, $x \in (0,1)$, and we assume that $f \in \dot{L}_2(0,1)$ $(x^{\frac{\alpha}{2}} f(x) \in L_2(0,1))$.

We have the following Poincare-type inequality ([32]).

Lemma 2.1.

$$||x^{\alpha/2} v||_{L_2} \le \frac{2}{\alpha+1} ||x^{\alpha/2} v'||_{L_2}, \quad v \in \dot{H}_0^1.$$

Proof. We have

$$\begin{aligned}
\int_0^1 x^\alpha v^2(x) dx &= \int_0^1 \Big(\frac{x^{\alpha+1}}{\alpha+1}\Big)' v^2(x) dx \\
&= -\int_0^1 \Big(\frac{x^{\alpha+1}}{\alpha+1}\Big) 2v(x) v'(x) dx + \Big(\frac{x^{\alpha+1}}{\alpha+1}\Big) v^2(x)|_0^1 \\
&\le \frac{2}{\alpha+1} ||x^{\frac{\alpha}{2}} v||_{L_2(0,1)} ||x^{\frac{\alpha+2}{2}} v'||_{L_2(0,1)} \\
&\le \frac{2}{\alpha+1} ||x^{\frac{\alpha}{2}} v||_{L_2(0,1)} ||x^{\frac{\alpha}{2}} v'||_{L_2(0,1)} ||x||_{L_\infty(0,1)}.
\end{aligned}$$

Since $||x||_{L_\infty(0,1)} \le 1$, this completes the lemma. □

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Now we define another inner product for $\dot{H}_0^1(0,1)$ by

$$\langle u, v\rangle_E := \int_0^1 x^\alpha u'(x)v'(x)dx, \quad u, v \in \dot{H}_0^1. \tag{2.5}$$

By Lemma 2.1, we have the following inequalities:

$$\langle v, v\rangle_E \le a(v,v) \le \Big(1 + (\frac{2}{\alpha+1})^2 ||q||_{L_\infty(0,1)}\Big)\langle v, v\rangle_E, \quad v \in \dot{H}_0^1. \tag{2.6}$$

Hereafter, we fix $\alpha = 1$ for simplicity. The case $\alpha > 1$ can be handled in the same way without any extra difficulty.

For the uniform partition of $[0,1]$, $0 = x_0 < x_1 < ... < x_{2^n} = 1$, $x_j = 2^{-n}j$, $j = 0, ..., 2^n$, let ϕ be the hat function $\phi(x) := \max\{0, 1 - |x|\}$, and

$$\phi_{n,1} := (\phi(2^n\cdot) + \phi(2^n \cdot -1))\chi_{[0,1]}, \tag{2.7}$$

$$\phi_{n,j} := \phi(2^n \cdot -j), j = 2, ..., 2^n - 1, \tag{2.8}$$

where $\chi_{[a,b]}$, $a < b$, is the characteristic function on the interval $[a, b]$. Let

$$V_n := \text{span}\{\phi_{n,j} : j = 1, ..., 2^n - 1\}.$$

It is easily seen that $V_n \subset V_{n+1}$ for $n = 1, 2, \ldots$.

The Galerkin method is defined as seeking the element $u_n \in V_n$ such that

$$a(u_n, v) = \langle xf, v\rangle, \quad v \in V_n. \tag{2.9}$$

Lemma 2.1 shows that $a(\cdot,\cdot)$ is elliptic, and by the Lax-Milgram theorem, existence and uniqueness of the solution are guaranteed for both (2.4) and (2.9).

Taking

$$u_n = \sum_{j=1}^{2^n-1} c_{n,j}\phi_{n,j},$$

we can rewrite (2.9) as

$$\sum_{j=1}^{2^n-1} a(\phi_{n,j}, \phi_{n,l})c_{n,j} = \langle xf, \phi_{n,l}\rangle, \quad l = 1, \ldots, 2^n - 1, \tag{2.10}$$

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or more briefly,

$$A_n C_n = F_n, \tag{2.11}$$

where (j, l) entry of the $2^n - 1$ by $2^n - 1$ stiffness matrix A_n is $a(\phi_{n,j}, \phi_{n,l})$, $C_n := (c_{n,1}, \ldots, c_{n,2^n-1})^T$, and $F_n := (\langle xf, \phi_{n,1}\rangle, \ldots, \langle f, \phi_{n,2^n-1}\rangle)^T$. Here, the superscript T denotes the transpose of a vector or a matrix.

The condition number of a nonsingular M by M matrix A is defined by

$$\kappa(A) := ||A|| ||A^{-1}||,$$

where $||A|| := \sup_{\mathbf{x}\in\mathbb{R}^M} \frac{||A\mathbf{x}||}{||\mathbf{x}||}$, $\mathbf{x} := (x_1, \ldots, x_M)^T$, and $||\mathbf{x}|| := (\sum_{i=1}^M x_i^2)^{\frac{1}{2}}$.

When A is positive definite and symmetric, we have

$$\kappa(A) = \frac{\lambda_{\max,A}}{\lambda_{\min,A}},$$

where $\lambda_{\max,A}, \lambda_{\min,A}$ are the maximum and the minimum eigenvalues of the matrix A, respectively.

The following error estimate will be established in the next section:

$$||x^{1/2}(u - u_n)||_{L_2} \leq C(2^{-n})^2 ||x^{1/2}u''||_{L_2}.$$

Consequently, the subspace V_n has to be large enough to guarantee that the error $u - u_n$ is sufficiently small. However, increasing the number n will dramatically increase the condition number of the associated stiffness matrix A_n (see, e.g., [5]), which makes solving u_n numerically difficult. It is well-known in the literature that for an ill-conditioned large linear system, without any preconditioning, it's impossible to find an efficient solver. Therefore, seeking a suitable preconditioning method will be important for solving the discretized system numerically. There is an abundance of literature contributed to this topic for the regular elliptic boundary problems, such as [6, 29, 55, 56]. Recently, wavelet methods have been introduced to serve as new preconditioning methods (see, e.g., [18, 28, 41, 51, 52]). Stability plays the key role in the

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wavelet preconditioning method. In other words, if one is able to find a basis which is stable in the corresponding Sobolev space, then the condition number of the associated stiffness matrix is uniformly bounded. A basis, say $\{\psi_i\}_{i=1}^{\infty}$, is stable if it satisfies,

$$C_0\Big(\sum_{i=1}^{\infty} c_i^2\Big) \le \Big\|\sum_{i=1}^{\infty} c_i\psi_i\Big\|^2 \le C_1\Big(\sum_{i=1}^{\infty} c_i^2\Big),$$

where C_0, C_1 are two positive constants independent of $\{c_i\}_{i=1}^{\infty}$, and $||\cdot||$ refers to the norm for the space in which we are interested. Stability of the shift invariant space has been studied extensively in [33, 35].

To find a proper preconditioning matrix for A_n in (2.11), we introduce the following lemma.

Lemma 2.2. *If two positive definite symmetric $M \times M$ matrices A, B satisfy the following condition*

$$C_0\mathbf{x}^T B\mathbf{x} \le \mathbf{x}^T A\mathbf{x} \le C_1\mathbf{x}^T B\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^M.$$

Then for any $M \times M$ matrix S,

$$\kappa(SAS^T) \le \frac{C_1}{C_0}\kappa(SBS^T).$$

Proof. Since

$$\begin{aligned}\lambda_{\max,SAS^T} &= \sup_{\mathbf{x}\in\mathbb{R}^M} \frac{\mathbf{x}^T SAS^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \sup_{\mathbf{x}\in\mathbb{R}^M} \frac{(S^T\mathbf{x})^T A(S^T\mathbf{x})}{\mathbf{x}^T\mathbf{x}} \\ &\le C_1 \sup_{\mathbf{x}\in\mathbb{R}^M} \frac{\mathbf{x}^T SBS^T\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = C_1\lambda_{\max,SBS^T}.\end{aligned}$$

Likewise we obtain

$$\lambda_{\min,SAS^T} \ge C_0\lambda_{\min,SBS^T},$$

and hence

$$\kappa(SAS^T) \le \frac{C_1}{C_0}\kappa(SBS^T).$$

□

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Lemma 2.2 tells that once one finds a good preconditioning matrix for B, then it is also a good preconditioning matrix for A provided that the ratio C_1/C_0 is not large. Basic properties of positive definite matrices and their condition numbers maybe found in ([31], chapter 7).

Lemma 2.3. *For $n = 1, 2, ...,$ let $\chi_n := \sum_{k=1}^{n} 2^{-k/2}\chi_{[2^{-k}, 2^{-k+1}]}$, and $g_{n,j} := \chi_n\phi'_{n,j}$, $j = 1, ..., 2^n - 1$. Let $u = \sum_{j=1}^{2^n-1} c_{n,j}\phi_{n,j}$. Then*

$$\int_0^1 \Big|\sum_{j=1}^{2^n-1} c_{n,j}g_{n,j}(x)\Big|^2 dx \le \langle u, u\rangle_E \le 2\int_0^1 \Big|\sum_{j=1}^{2^n-1} c_{n,j}g_{n,j}(x)\Big|^2 dx, \tag{2.12}$$

where $\langle\cdot,\cdot\rangle_E$ is defined in (2.5) with $\alpha = 1$.

Note: We may think of $g_{n,j}$ as the weighted derivative of $\phi_{n,j}$, and the weights are $2^{-k}, k = 1, ..., n$, on the subintervals $(2^{-k}, 2^{-k+1})$, $k = n, ..., 1$. In other words, we discretize the weight x in the inner product form $\langle\cdot,\cdot\rangle_E$ through χ_n.

Proof. Noting that $g_{n,j} = 0$, $j = 1, ..., 2^n - 1$ on $(0, 2^{-n})$, we get

$$\langle u, u\rangle_E = \sum_{k=1}^{n}\int_{2^{-k}}^{2^{-k+1}} x|u'(x)|^2 dx.$$

Accordingly,

$$\langle u, u\rangle_E = \sum_{k=1}^{n}\int_{2^{-k}}^{2^{-k+1}} x\Big|\sum_{j\in I_k} c_{n,j}\phi'_{n,j}(x)\Big|^2 dx,$$

where $I_k := \{2^{n-k}, \ldots, 2^{n-k+1}\}$.

Now we have

$$\langle u, u\rangle_E \ge \sum_{k=1}^{n}\int_{2^{-k}}^{2^{-k+1}} 2^{-k}\Big|\sum_{j\in I_k} c_{n,j}\phi'_{n,j}(x)\Big|^2 dx \tag{2.13}$$

$$= \sum_{k=1}^{n}\int_{2^{-k}}^{2^{-k+1}} \Big|\sum_{j\in I_k}(c_{n,j}2^{-\frac{k}{2}}\phi'_{n,j}(x))\Big|^2 dx. \tag{2.14}$$

By the definition of $\phi_{n,j}$ in (2.7,2.8), we have

$$\phi'_{n,j}(x) = \begin{cases} 2^n, & (j-1)2^{-n} < x < j2^{-n}, \\ -2^n, & j2^{-n} < x < (j+1)2^{-n}, \\ 0, & otherwise. \end{cases}$$

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Then, on each subinterval $(2^{-k}, 2^{-k+1})$, $k = n, n-1, \ldots, 1$, it follows that

$$\sum_{j \in I_k} c_{n,j} 2^{-\frac{k}{2}} \phi'_{n,j} = \sum_{j \in I_k} c_{n,j} \chi_n \phi'_{n,j} = \sum_{j \in I_k} c_{n,j} g_{n,j}.$$

This together with (2.14) yields

$$\langle u, u \rangle_E \geq \int_0^1 \left| \sum_{j=1}^{2^n-1} (c_{n,j} g_{n,j}(x)) \right|^2 dx.$$

The proof of the right inequality of (2.12) is similar and is omitted. □

Combining Lemma 2.3 with inequality (2.6), we have

Lemma 2.4. *Denote by* A_n *the matrix* $(a(\phi_{n,j}, \phi_{n,l}))_{j,l=1,\ldots,n}$, $A_{E,n}$ *the matrix* $(\langle \phi_{n,j}, \phi_{n,l} \rangle_E)_{j,l=1,\ldots,n}$ *and by* $\tilde{A}_n$ *the matrix* $(\langle g_{n,j}, g_{n,l} \rangle)_{j,l=1,\ldots,n}$. *Then the inequalities*

$$\mathbf{x}^T A_{E,n} \mathbf{x} \leq \mathbf{x}^T A_n \mathbf{x} \leq (1 + ||q||_{L_\infty(0,1)}) \mathbf{x}^T A_{E,n} \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{2^n-1}, \tag{2.15}$$

and

$$\mathbf{x}^T \tilde{A}_n \mathbf{x} \leq \mathbf{x}^T A_{E,n} \mathbf{x} \leq 2 \mathbf{x}^T \tilde{A}_n \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^{2^n-1} \tag{2.16}$$

hold.

The following theorem is a simple consequence of Lemma 2.2 and Lemma 2.4.

Theorem 2.1. *For any matrix* S *of the same size as* A_n ,

$$\kappa(S A_n S^T) \leq 2(1 + ||q||_{L_\infty(0,1)}) \kappa(S \tilde{A}_n S^T).$$

By Theorem 2.1, we reduce the problem to preconditioning the much simpler matrix $\tilde{A}_n$ instead of A_n. Due to the similarity between the basis $\{g_{l,j}\}$ and the derivative of the basis $\{\phi_{n,i}\}$, it's natural to construct another orthogonal basis similar to the hierarchy basis to preconditioning $\tilde{A}_n$ (see, e.g., [55,56]). We will construct such a basis in the rest of this section.

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Proposition 2.1. *Let $\tilde{V}_n$ be the linear span of $g_{n,j}$, $j = 1, ..., 2^{n-1}$. The sequence $\{\tilde{V}_n\}_{n=1,2,...}$ of subspaces is nested, that is, $\tilde{V}_n \subset \tilde{V}_{n+1}$ for all n.*

Proof. We shall show that the following relation is valid almost everywhere:

$$g_{n-1,j} = \begin{cases} g_{n,1} + g_{n,2} + \frac{1}{2}g_{n,3}, & j = 1, \\ \frac{1}{2}g_{n,2j-1} + g_{n,2j} + \frac{1}{2}g_{n,2j+1}, & j = 2, ..., 2^{n-1} - 1. \end{cases} \tag{2.17}$$

For brevity, we define $\eta := 2^{-n}$. From the definition of $g_{n-1,1}$, we have

$$g_{n-1,1} = \begin{cases} -2^{\frac{n-1}{2}}, & 2\eta < x < 4\eta, \\ 0, & otherwise. \end{cases}$$

Note that

$$g_{n,1}(x) = \begin{cases} -2^{\frac{n}{2}}, & \eta < x < 2\eta, \\ 0, & otherwise, \end{cases}$$

$$g_{n,2}(x) = \begin{cases} 2^{\frac{n}{2}}, & \eta < x < 2\eta, \\ -2^{\frac{n+1}{2}}, & 2\eta < x < 3\eta, \\ 0, & otherwise, \end{cases}$$

and

$$g_{n,3}(x) = \begin{cases} 2^{\frac{n+1}{2}}, & 2\eta < x < 3\eta, \\ -2^{\frac{n+1}{2}}, & 3\eta < x < 4\eta, \\ 0, & otherwise. \end{cases}$$

Hence, for $x \in (0, 4\eta) \setminus \{\eta, 2\eta, 3\eta\}$, we have

$$g_{n-1,1}(x) = g_{n,1}(x) + g_{n,2}(x) + \frac{1}{2}g_{n,3}(x).$$

To verify the second equation in (2.17), first we recall that

$$\phi'_{n-1,j} = \frac{1}{2}\phi'_{n,2j-1} + \phi'_{n,2j} + \frac{1}{2}\phi'_{n,2j+1}, \ a.e. \ j = 2, ..., 2^{n-1} - 1.$$

Moreover, χ_{n-1} and χ_n agree on the interval $[2^{-n+1}, 1]$ and, for $j = 2, ..., 2^{n-1} - 1$, $\phi_{n-1,j}$ is supported in $[2^{-n+1}, 1]$. Therefore,

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$$
\begin{aligned}
g_{n-1,j} &= \chi_{n-1}\phi'_{n-1,j} = \chi_n\phi'_{n-1,j} \\
&= \chi_n(\frac{1}{2}\phi'_{n,2j-1} + \phi'_{n,2j} + \frac{1}{2}\phi'_{n,2j+1}) \\
&= \frac{1}{2}g_{n,2j-1} + g_{n,2j} + \frac{1}{2}g_{n,2j+1}.
\end{aligned}
$$

This proves the proposition. □

Similar to the construction of the hierarchy basis, let

$$
\tilde{\psi}_{l-1,j} := g_{l,2j-1}, \quad j = 1, ..., 2^{l-1}, l = n, n-1, ..., 1, \tag{2.18}
$$

and

$$
\tilde{W}_{l-1} := \text{span}\{\tilde{\psi}_{l-1,j} : j = 1, ..., 2^{l-1}\}.
$$

Then we have

Proposition 2.2. $\{\tilde{\psi}_{l,j} : l = 1, ..., n, j = 1, ..., 2^{l-1}\}$ *is an orthogonal basis for* $\tilde{V}_n$.

Proof. We shall verify the following properties:
i) $\langle\tilde{\psi}_{l-1,j}, g_{l-1,j'}\rangle = 0, \quad j = 1, ..., 2^{l-1}, j' = 1, ..., 2^{l-1} - 1,$
ii) $\langle\tilde{\psi}_{l-1,j}, \tilde{\psi}_{l-1,j'}\rangle = 0, \quad j \neq j',$
iii) $\tilde{V}_n = \tilde{W}_0 + \tilde{W}_1 + \cdots + \tilde{W}_{n-1}$.

Considering i), for $j \neq 1$, there exists k such that $2j - 1 \in \{2^{l-k} + 1, ..., 2^{l-k+1} - 1\}$, and

$$
\tilde{\psi}_{l-1,j} = g_{l,2j-1} = 2^{l-\frac{k}{2}} \begin{cases} 1, & (2j-2)2^{-l} < x < (2j-1)2^{-l}, \\ -1, & (2j-1)2^{-l} < x < (2j)2^{-l}, \\ 0, & \textit{otherwise}. \end{cases}
$$

Since $g_{l-1,j'}$ is a constant on $\text{supp}\{\tilde{\psi}_{l-1,j}\} = [(2j-2)2^{-l}, (2j)2^{-l}]$ for $j' = 1, ..., 2^{l-1} - 1$, i) is true. For the case $j = 1$, we obtain that $\tilde{\psi}_{l-1,1}(= g_{l,1})$ is orthogonal to $\tilde{V}_{l-1}$ because $g_{l-1,j'}, j' = 1, ..., 2^{l-1} - 1$, have no overlapped support with $\tilde{\psi}_{l-1,1}$.

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ii) follows from

$$\text{supp}\{\tilde{\psi}_{l-1,j}\}\bigcap\text{supp}\{\tilde{\psi}_{l-1,j'}\}=\emptyset.$$

Finally, we turn to iii). First, $\{g_{l,j}\}_{j=1}^{2^l-1}$ is defined to be a basis for $\tilde{V}_l$. Second, by i) and ii), we have

$$\tilde{V}_n=\tilde{V}_1+\tilde{W}_1+\tilde{W}_2+\cdots+\tilde{W}_{n-1}. \tag{2.19}$$

According to definitions, $\tilde{\psi}_{0,1}=g_{1,1}$ by (3.21), $\tilde{W}_0=\text{span}\{\tilde{\psi}_{0,1}\}$, and $\tilde{V}_1=\text{span}\{g_{1,1}\}$. Therefore, $\tilde{V}_1$ can be replaced by $\tilde{W}_0$ in (2.19).

This completes the proof. □

In what follows we shall provide the preconditioning method for $\tilde{A}_n$ in (2.16). More precisely, we can find two sparse matrices P and H based on the change of bases from $\{g_{n,j}\}_j$ to $\{\tilde{\psi}_{l,j}\}_{l,j}$ such that $(PH)\tilde{A}_n(PH)^T$ is an identity matrix. By Theorem 2.1, it is clear that (PH) is also a good preconditioner for the stiffness matrix A_n. To find the matrices P and H, we shall write (2.17,3.21) into the matrix form for the convenience of explanation.

Denote by G_l, $\tilde{\Psi}_l$ the vectors of functions $(g_{l,1},...,g_{l,2^l-1})^T$, $(\tilde{\psi}_{l,1},...,\tilde{\psi}_{l,2^l})^T$, respectively. Let $\tilde{\Psi}:=(\tilde{\Psi}_0^T,...,\tilde{\Psi}_{n-1}^T)^T$, and denote by $\tilde{A}_{\tilde{\Psi},n}$ the matrix $\langle\tilde{\Psi},(\tilde{\Psi})^T\rangle$. Then $\tilde{A}_{\tilde{\Psi},n}$ is a diagonal matrix by Proposition 2.2. Furthermore, one can find a diagonal matrix P such that

$$I_{2^n-1}\equiv P\tilde{A}_{\tilde{\Psi},n}P^T. \tag{2.20}$$

where (l,l) entry of the matrix P is $||\tilde{\Psi}(l)||_{L_2}^{-1}$, and $\tilde{\Psi}(l)$ denotes the $l-$th entry of the vector $\tilde{\Psi}$.

Clearly, $P\tilde{\Psi}$ is a stable (orthonormal) basis for $\tilde{V}_n$, and due to the simple transformation from the basis G_n to the basis $P\tilde{\Psi}$ (see (2.17),(3.21)), $\tilde{A}_n$ can be preconditioned through a basis transformation from G_n to $P\tilde{\Psi}$.

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By (2.17), we have

$$G_{l-1} = B_{g,l-1} G_l, \tag{2.21}$$

where $B_{g,l-1}$ is a $2^{l-1}-1$ by 2^l-1 matrix (only nonzero entries are listed.)

$$\begin{pmatrix} 1 & 1 & \frac{1}{2} & & & & & \\ & & \frac{1}{2} & 1 & \frac{1}{2} & & & \\ & & & & & \ddots & & \\ & & & & & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix}.$$

Denote by $B_{\tilde{\psi},l-1}$ the 2^{l-1} by 2^l-1 matrix

$$\begin{pmatrix} 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Then, (3.21) becomes

$$\tilde{\Psi}_{l-1} = B_{\tilde{\psi},l-1} G_l. \tag{2.22}$$

Thus, (2.21) and (2.22) together yield

$$\begin{pmatrix} G_{l-1} \\ \tilde{\Psi}_{l-1} \end{pmatrix} = \begin{pmatrix} B_{g,l-1} \\ B_{\tilde{\psi},l-1} \end{pmatrix} G_l.$$

By H_{l-1} we denote the 2^n-1 by 2^n-1 transformation matrix

$$H_{l-1} := \begin{pmatrix} B_{g,l-1} & 0 \\ B_{\tilde{\psi},l-1} & 0 \\ 0 & I_{2^n-2^l} \end{pmatrix}.$$

Then we have

$$\tilde{\Psi} = H_1 \cdots H_{n-1} G_n.$$

For brevity, let

$$H := H_1 \cdots H_{n-1},$$

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and thus we have the transformation between two bases

$$\tilde{\Psi} = HG_n.$$

Note that $\tilde{A}_{\tilde{\Psi},n} = H\tilde{A}_n H^T$. By (2.20), we have

$$I_{2^n-1} \equiv (PH)\tilde{A}_n(PH)^T. \tag{2.23}$$

Let S in Theorem 2.1 be PH in (2.23). Then

$$\kappa(SA_nS^T) \leq 2(1 + ||q||_{L_\infty(0,1)}).$$

Consequently, (PH) is a suitable preconditioner for A_n. Furthermore, it's easily seen that (PH) has $O(N)$ nonzero entries, where $N = 2^n - 1$ is the size of the basis functions for V_n. Therefore, implementation of the preconditioning shall be efficient. Detail discussion may be found in [52, Prop. 4.6].

Corollary 2.1. *For the case $q(x) = 0$, the condition number of the matrix $(PH)A_n(PH)^T$ is bounded by 2 for all n.*

Now we provide a preconditioning algorithm for solving (2.11). Notice that

$$A_nC_n = F_n \Leftrightarrow (PH)A_n(PH)^T((PH)^T)^{-1}C_n = (PH)F_n.$$

Then (2.11) is equivalent to the following linear equations with $\mathbf{x} = ((PH)^T)^{-1}C_n$,

$$[(PH)A_n(PH)^T]\mathbf{x} = (PH)F_n. \tag{2.24}$$

To solve (2.11) for C_n, we first solve (2.24) for $\mathbf{x}$, and the solution of (2.11) is

$$C_n = (PH)^T\mathbf{x}.$$

Note that the matrix $[(PH)A_n(PH)^T]$ is well conditioned. Therefore it's efficient to solve (2.24) for $\mathbf{x}$ numerically.

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2.3 Error Estimates

We provide basic error estimates in this section and show that finite dimensional subspaces used in section 2.2 do provide the suitable approximation order.

To keep the practical applicability, and for the convenience of stating the results, we restrict ourselves to the uniform partition case in the previous section. Under such setting, it's easier to describe the preconditioning method based on the multi-level nested subspaces.

However, error estimates stated in this section hold for the general non-uniform partition case. Furthermore, the preconditioning method developed in the previous section is readily generalized to the non-uniform partition case as long as the sequence of subspaces are nested.

For the general non-uniform partition defined by $0 = x_0 < x_1 < ... < x_M = 1$, let

$$\phi_j := \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \frac{x_{j+1}-x}{x_{j+1}-x_j}, & x \in [x_j, x_{j+1}], \\ 0, & otherwise. \end{cases}$$

We also let $h_j := x_j - x_{j-1}$, and $h := \max_{1\leq j\leq M}\{h_j\}$, where the later quantity measures the mesh size. The finite dimensional space is spanned by the nodal basis functions $\{\phi_j\}$,

$$V_h := \text{span}\{\phi_0 + \phi_1, \phi_2, ..., \phi_{M-1}\}.$$

Then the Galerkin method is to find $u_h \in V_h$ such that

$$a(u_h, v) = \langle xf, v\rangle, \quad \forall v \in V_h.$$

We will follow several lemmas to obtain error estimates in this section.

In the following, the solution u is assumed to be smooth ($u \in H^2$, where H^2 denotes the usual Sobolev space of functions with the second weak derivative on $(0,1)$) with the boundary conditions $u'(0) = u(1) = 0$. We let the same letter C which is independent of h denote the different constants in the different inequalities.

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Lemma 2.5. *There exists a constant h_0 such that for all $h < h_0$,*

$$||x^{1/2}(u' - u'_I)||_{L_2} \le Ch||x^{1/2}u''||_{L_2},$$

where C is a constant depending on $\max_{i>1}\{x_i/x_{i-1}\}$ *and that $u_I \in V_h$ is the interpolant of u defined by*

$$u_I(x) := u(x_1)(\phi_0(x) + \phi_1(x)) + \sum_{i=2}^{M-1} u(x_j)\phi_j(x).$$

Proof. On the interval $I_i := (x_{i-1}, x_i)$, $i > 1$, similar to the proof of Lemma 2 in [24], we have

$$\int_{x_{i-1}}^{x_i} x(u' - u'_I)^2 dx \le x_i \int_{x_{i-1}}^{x_i} (u' - u'_I)^2 dx.$$

By the well-known result (see, e.g., [7], p. 7)

$$||u' - u'_I||_{L_2(I_i)} \le Ch||u''||_{L_2(I_i)},$$

it follows that

$$\begin{aligned}\int_{x_{i-1}}^{x_i} x(u' - u'_I)^2 dx &\le x_i Ch^2 \int_{x_{i-1}}^{x_i} (u'')^2 dx \le Ch^2 \frac{x_i}{x_{i-1}} \int_{x_{i-1}}^{x_i} x(u'')^2 dx \\ &\le Ch^2 ||x^{1/2}u''||^2_{L_2(I_i)},\end{aligned}$$

which implies,

$$||x^{1/2}(u' - u'_I)||_{L_2(I_i)} \le Ch||x^{1/2}u''||_{L_2(I_i)}. \tag{2.25}$$

On the interval $I_1 = (0, x_1)$, let $e(x) := u'(x) - u'(x_1)$. Then $e(x_1) = 0$ and $e'(x) = u''(x)$. Following the idea in Lemma 2.1, we have

$$\begin{aligned}\int_0^{x_1} x|e(x)|^2 &= \int_0^{x_1} x\left|\int_x^{x_1} e'(t)dt\right|^2 dx \le \int_0^{x_1} x\left|\int_x^{x_1} \left(\frac{t}{x}\right)^{1/2} |e'(t)|\right|^2 dt \\ &\le ||t^{1/2}e'||^2_{L_2(I_1)}h^2,\end{aligned}$$

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and hence

$$(2.26) \qquad \int_0^{x_1} x|u'(x) - u'(x_1)|^2 dx \le h^2 \int_0^{x_1} x|u''(x)|^2 dx.$$

Since

$$\int_0^{x_1} x|u'(x) - u'(x_1)|^2 dx = \Gamma_1 + \Gamma_2 - 2\int_0^{x_1} xu'(x)u'(x_1)dx,$$

where $\Gamma_1 := \int_0^{x_1} x|u'(x)|^2 dx$, and $\Gamma_2 := |u'(x_1)|^2 \int_0^{x_1} xdx$, we have

$$\int_0^{x_1} x|u'(x) - u'(x_1)|^2 dx \ge \Gamma_1 - \alpha\Gamma_1 - \frac{1}{\alpha}\Gamma_2 + \Gamma_2.$$

Let $\alpha = 1/2$. Note that $u_I'(x) = 0,\ x \in I_1$. Then together with (2.26), we have

$$(2.27) \quad ||x^{1/2}(u' - u_I')||_{L_2(I_1)} = ||x^{1/2}u'||^2_{L_2(I_1)} = \Gamma_1 \le 2h^2||x^{1/2}u''||^2_{L_2(I_1)} + 2\Gamma_2.$$

Combing (2.25) with (2.27) yields

$$||x^{1/2}(u' - u_I')||^2_{L_2(0,1)} \le Ch^2||x^{1/2}u''||^2_{L_2(0,1)} + 2\Gamma_2.$$

The proof will be completed by estimating Γ_2:

$$2\Gamma_2 = |u'(x_1)|^2(x_1)^2 \le h^2|u'(x_1)|^2 = h^2\left|\int_0^{x_1} u''(t)dt\right|^2 \le \frac{h^3}{2}||u''||^2_{L_2(I_1)}.$$

Hence there exists a constant h_0 such that

$$h||u''||^2_{L_2(0,x_1)} \le ||x^{1/2}u''||^2_{L_2(0,1)}, \quad \forall h < h_0,$$

and thus completes the proof. □

Theorem 2.2. *There exists a constant h_0 such that for any $h < h_0$,*

$$(2.28) \qquad ||x^{1/2}(u' - u_h')||_{L_2(0,1)} \le Ch||x^{1/2}u''||_{L_2(0,1)},$$

and

$$(2.29) \qquad ||x^{1/2}(u - u_h)||_{L_2(0,1)} \le Ch^2||x^{1/2}u''||_{L_2(0,1)}$$

hold.

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Proof. (2.28) is a standard error estimate.

$$\begin{aligned}||x^{1/2}(u'-u_h')||_{L_2}^2 &\le Ca(u-u_h,u-u_h)\le Ca(u-u_h,u-u_I)\\ &\le C||x^{1/2}(u'-u_h')||_{L^2}||x^{1/2}(u'-u_I')||_{L_2},\end{aligned}$$

and hence,

$$||x^{1/2}(u'-u_h')||_{L_2}\le C||x^{1/2}(u'-u_I')||_{L_2}\le Ch||x^{1/2}u''||_{L_2},$$

where the last step used Lemma 2.5.

(2.29) can be obtained through a duality argument.

Let w solves $a(v,w)=\langle x(u-u_h),v\rangle,\quad \forall v\in \dot{H}_0^1$. Then we have

$$a(u-u_h,w)=\langle x(u-u_h),u-u_h\rangle=||x^{1/2}(u-u_h)||_{L_2}^2,$$

and

$$\begin{aligned}a(u-u_h,w) &= a(u-u_h,w-w_I)\le C||x^{1/2}(u'-u_h')||_{L_2}||x^{1/2}(w'-w_I')||_{L_2}\\ &\le Ch^2||x^{1/2}u''||_{L_2}||x^{1/2}w''||_{L_2}.\end{aligned}$$

Once we prove the regularity of w, i.e.,

$$||x^{1/2}w''||_{L_2}\le C||x^{1/2}(u-u_h)||_{L_2},$$

(2.29) holds.

w satisfies the following equation, (from $-(xw')'=x(u-u_h-qw)$)

$$w(x)=\int_x^1\frac{1}{t}\int_0^t(u(s)-u_h(s)-q(s)w(s))sdsdt.$$

Differentiating both sides of the above expression twice, we have

$$w''(x)=\frac{1}{x^2}\int_0^x(u(s)-u_h(s)-q(s)w(s))sds-(u(x)-u_h(x))+q(x)w(x),$$

and thus

$$\begin{aligned}||x^{1/2}w''||_{L_2} &\le \left\|\frac{1}{x}\int_0^x(\frac{s}{x})^{1/2}s^{1/2}(u-u_h-qw)ds\right\|_{L_2}\\ &+ ||x^{1/2}(u-u_h)||_{L_2}+C||x^{1/2}w||_{L_2}.\end{aligned}$$

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By the Hardy's inequality

$$\left\|\frac{1}{x}\int_0^x f(t)dt\right\|_{L_2} \leq 2||f||_{L_2},$$

we get

$$\begin{aligned}||x^{1/2}w''||_{L_2} &\leq \left\|\frac{1}{x}\int_0^x (s^{1/2}(u-u_h) - s^{1/2}qw)ds\right\|_{L_2} + ||x^{1/2}(u-u_h)||_{L_2} \\ &+ C||x^{1/2}w||_{L_2} \leq C[||x^{1/2}(u-u_h)||_{L_2} + ||x^{1/2}w||_{L_2}].\end{aligned}$$

It remains to prove $||x^{1/2}w||_{L_2} \leq C||x^{1/2}(u-u_h)||_{L_2}$:

$$\begin{aligned}||x^{1/2}w||_{L_2}||x^{1/2}(u-u_h)||_{L_2} &\geq \langle x(u-u_h), w\rangle = a(w,w) \\ &\geq C||x^{1/2}w'||^2_{L_2} \geq C||x^{1/2}w||^2_{L_2}. \quad \blacksquare\end{aligned}$$

Finally, for the case $q(x) = 0$, we provide error estimates for $||u' - u_h'||_{L_2}$ and $||u - u_h||_{L_2}$.

Theorem 2.3. *If $q(x) = 0$, then*

$$||u' - u_h'||_{L_2} \leq Ch||u''||_{L_2}.$$

Proof. u_h satisfies the following equation,

(2.30) $$\langle x(u' - u_h'), v'\rangle = 0, \quad \forall v \in V_h.$$

In fact, $V_h' := \{v' : v \in V_h\}$ is $\{w = \sum_{j=2}^M c_j\chi_{I_j} : \{c_j\} \in \mathbb{R}^{M-1}\}$. In other words, V_h' is the linear span of the piecewise constants on each interval $I_j, j > 1$ with 0 on the interval I_1.

Let $u_h' = \sum_{j=2}^M c_j\chi_{I_j}$. Then (2.30) is equivalent to

$$\int_{x_{i-1}}^{x_i} (u'(x) - c_i)xdx = 0, \quad i > 1.$$

Therefore,

$$c_i = \int_{x_{i-1}}^{x_i} u'(t)tdt \Big/ \int_{x_{i-1}}^{x_i} tdt.$$

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Setting $\Gamma := \int_{x_{i-1}}^{x_i} tdt$, we obtain

$$(2.31)\ ||u' - u_h'||^2_{L_2(I_i)} = \int_{x_{i-1}}^{x_i} (u'(x) - u_h'(x))^2 dx = \int_{x_{i-1}}^{x_i} \Big(\int_{x_{i-1}}^{x_i} (u'(x) - u'(t)) tdt \Big)^2 dx / \Gamma^2, \quad i > 1.$$

Estimating the last term in the above equation gives

$$(2.32)\ \int_{x_{i-1}}^{x_i} \Big(\int_{x_{i-1}}^{x_i} (u'(x) - u'(t)) tdt \Big)^2 dx \le \int_{x_{i-1}}^{x_i} \Big(\int_{x_{i-1}}^{x_i} |u'(x) - u'(t)|^2 dt \int_{x_{i-1}}^{x_i} t^2 dt \Big) dx.$$

Now we have

$$(2.33)\quad \int_{x_{i-1}}^{x_i} (u'(x) - u'(t))^2 dt = \int_{x_{i-1}}^{x_i} \Big| \int_x^t u''(s) ds \Big|^2 dt \le \int_{x_{i-1}}^{x_i} \Big| \int_x^t |u''(s)|^2 ds \Big| \Big| \int_x^t 1^2 ds \Big| dt \le ||u''||^2_{L_2(I_i)} h_i^2.$$

Hence, after plugging (2.34) into (2.32), we estimate the term $||u' - u_h'||^2_{L_2(I_i)}$ in (2.32) by

$$\int_{x_{i-1}}^{x_i} |u'(x) - u_h'(x)|^2 dx \le h_i^3 ||u''||^2_{L_2(I_i)} \frac{\int_{x_{i-1}}^{x_i} t^2 dt}{(\int_{x_{i-1}}^{x_i} tdt)^2} \le h_i^3 ||u''||^2_{L_2(I_i)} \frac{x_i^2 \int_{x_{i-1}}^{x_i} dt}{(x_{i-1} \int_{x_{i-1}}^{x_i} dt)^2} \le (\frac{x_i}{x_{i-1}})^2 h^2 ||u''||^2_{L_2(I_i)}.$$

On the interval I_1,

$$\int_0^{x_1} |u'(x)|^2 dx = \int_0^{x_1} \Big| \int_0^x u''(s) ds \Big|^2 dx \le \int_0^{x_1} \Big| \int_0^x |u''(s)|^2 ds \int_0^x 1^2 ds \Big| dx \le ||u''||^2_{L_2(0,x_1)} h^2.$$

Thus the proof is completed. □

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Theorem 2.4. *If $q(x) = 0$, there exists a constant h_0 such that for any $h < h_0$,*

$$||u - u_h||_{L_2(0,1)} \leq Ch||u' - u_h'||_{L_2(0,1)}.$$

Proof. Let $e := u - u_h$, and let w solve the problem,

$$-(xw')' = e, \quad \text{with} \quad w(1) = w'(0) = 0.$$

Then

$$w(x) = \int_x^1 \frac{1}{t} \int_0^t e(s)dsdt. \tag{2.34}$$

Differentiating (2.34) twice, it follows that

$$w''(x) = \frac{1}{x^2} \int_0^x e(s)ds - \frac{1}{x} e(x).$$

By using Hardy's inequality, we have

$$||xw''||_{L_2} \leq 2||e||_{L_2} + ||e||_{L_2} = 3||e||_{L_2}. \tag{2.35}$$

On the other hand,

$$||e||_{L_2}^2 = \langle e, e \rangle = -\int_0^1 (xw')'e.$$

Integrating it by parts yields

$$||e||_{L_2}^2 = \int_0^1 xw'e'dx = \int_0^1 x(w' - v')e'dx, \quad \forall v \in V_h,$$

which implies

$$||e||_{L_2}^2 \leq ||e'||_{L_2} \inf_{v \in V_h} ||x(w' - v')||_{L_2}.$$

Suppose we have the result $\inf_{v \in V_h} ||x(w' - v')||_{L_2} \leq Ch||xw''||_{L_2}$ for the moment. Then together with (2.35), we have

$$||e||_{L_2}^2 \leq Ch||e||_{L_2}||e'||_{L_2}.$$

Thus completes the proof. □

It remains to prove

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Lemma 2.6. *For an element $w \in \dot{H}_0^1 \cap H^2$, $w'(0) = 0$, there exists a constant h_0 such that for any $h < h_0$,*

$$\inf_{v \in V_h} ||x(w' - v')||_{L_2} \leq Ch||xw''||_{L_2}.$$

Proof. Let v be an element in V_h. Then there exist $c_2, ..., c_M$ such that $v = \sum_{j=2}^{M} c_j \chi_{I_j}$. Let

$$c_j = \int_{I_j} xw'(x)dx / \int_{I_j} xdx.$$

Then a similar approach in Theorem 2.3 can be taken to obtain

$$||x(w' - v')||_{L_2}^2 \leq Ch||xw''||_{L_2}^2.$$

Detail steps are as follows:

On I_j, $j > 1$, let $\Gamma := \int_{I_j} xdx$. It follows that

$$\begin{aligned}
\int_{I_j} |x(w' - v')|^2 dx &= \int_{I_j} \left| x \int_{x_{j-1}}^{x_j} (w'(x) - w'(t))tdt \right|^2 dx / (\Gamma^2) \\
&\leq \frac{1}{\Gamma^2} \int_{I_j} \left| x \int_{x_{j-1}}^{x_j} t \int_t^x w'' dsdt \right|^2 dx \\
&\leq C_1 \frac{1}{\Gamma^2} \int_{I_j} \left| x \int_{x_{j-1}}^{x_j} \left| \int_t^x |w''(s)s| ds \right| dt \right|^2 dx \\
&\leq C_1 \frac{1}{\Gamma^2} \int_{I_j} |x(||sw''||_{L_2} \sqrt{h_j} h_j|^2 dx \\
&= C_1 h_j^3 ||sw''||_{L_2}^2 \frac{\int_{I_j} x^2 dx}{\Gamma^2} \\
&\leq Ch^2 ||sw''||_{L_2}^2,
\end{aligned}$$

where $C_1 = x_j / x_{j-1}$ (since $\frac{s}{t} \leq \frac{x_j}{x_{j-1}}$).

On the interval I_1, we use the trick in Lemma 2.5 again, and we obtain

$$\begin{aligned}
\int_0^{x_1} |x(w'(x) - w'(x_1))|^2 dx &= \int_0^{x_1} \left| x \int_x^{x_1} w''(t)dt \right|^2 dx \\
&\leq \int_0^{x_1} \left| x \int_x^{x_1} |w''(t)(t/x)| dt \right|^2 dx \\
&\leq \int_0^{x_1} \left| \int_x^{x_1} |w''(t)t| dt \right|^2 dx \\
&\leq h^2 ||w''(t)t||_{L_2(I_1)}^2.
\end{aligned}$$

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Let $\Gamma_1 := \int_0^{x_1} |xw'(x)|^2 dx$, and $\Gamma_2 := \int_0^{x_1} |xw'(x_1)|^2 dx = \frac{x_1^3}{3}|w'(x_1)|^2$. Then we have

$$\Gamma_1 \leq 2h^2||tw''(t)||^2_{L_2(I_1)} + 2\Gamma_2.$$

Estimating $2\Gamma_2$ gives

$$2\Gamma_2 = \frac{2}{3}x_1^3|w'(x_1)|^2 \leq \frac{2}{3}h^3\Big(\int_0^{x_1} w''(t)dt\Big)^2 \leq \frac{2}{3}h^2(||w''||^2_{L_2(I_1)}h^2).$$

Hence there exists h_0 depending on w, such that for any $h < h_0$,

$$||w''||^2_{L_2(I_1)}h^2 \leq ||xw''||^2_{L_2(0,1)}.$$

This completes the proof. □

2.4 Examples

Now we shall provide the results of numerical examples to illustrate the theory developed earlier.

For the first example, let $q(x) = 0$ and

$$f(x) = -\frac{\pi}{2}x^{-1}(\sin\frac{\pi}{2}x + x\frac{\pi}{2}\cos\frac{\pi}{2}x).$$

Then the solution is

$$u = -\cos\frac{\pi}{2}x.$$

To define the coarsest grid, we split the domain $(0,1)$ into $2^1 = 2$ pieces, and then divide each piece into 2 pieces of equal length. We keep on splitting until there are 2^n pieces. Therefore, we have n levels of nested subspaces.

Concerning the singular boundary value problem (2.4), once the finite dimensional subspace V_n is fixed, the stiffness matrix A_n is also fixed, as well as the preconditioner PH. Therefore, we will demonstrate the performance of the preconditioner first by comparing the condition numbers of A_n with those of $(PH)A_n(PH)^T$ with different n. In Table 2.1, we display the maximum eigenvalues, minimum eigenvalues and the condition numbers of the two matrices A_n and $(PH)A_n(PH)^T$ for the different n.

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n	5	6	7	8	9	10
$\lambda_{\max,A_n}$	1.10+2	2.33+2	4.83+2	9.87+2	2.00+3	4.04+3
$\lambda_{\min,A_n}$	0.0477	0.0232	0.0115	0.0057	0.0028	0.0014
$\kappa(A_n)$	2.30+3	1.00+4	4.21+4	1.74+5	7.06+5	2.85+6
$\lambda_{\max,PHA_n(PH)^T}$	1.9688	1.984	1.992	1.996	1.998	1.999
$\lambda_{\min,PHA_n(PH)^T}$	1.031	1.016	1.008	1.004	1.003	1.001
$\kappa(PHA_n(PH)^T)$	1.909	1.954	1.977	1.988	1.994	1.997

Table 2.1: Condition numbers of the matrix A_n and $(PH)A_n(PH)^T$

Computing results in Table 2.1 show that condition numbers of the preconditioned stiffness matrices are uniformly bounded by 2, which verifies the result of the Corollary in section 2.2.

Now, we are ready to implement our preconditioning method to solve the first example. We use the Galerkin method to solve the problem with mesh size $1/2^n$ and let u and u_h denote the solution and the Galerkin solution of the singular problem, respectively. $|(u-u_h)|_{H^1}, ||(u-u_h)||_{L_2}$ with different n are listed in Table 2.2. As predicted by Theorem 2.3, 2.4, the Galerkin method with the piecewise linear nodal basis preserves $O(h)$, $O(h^2)$ convergence rates for $|(u-u_h)|_{H^1}, ||(u-u_h)||_{L_2}$, respectively.

n	$\|u-u_h\|_{H^1}$	$\|\|u-u_h\|\|_{L_2}$
5	0.0307	1.73×10^{-4}
6	0.0134	4.53×10^{-5}
7	0.0062	1.20×10^{-5}
8	0.0029	3.16×10^{-6}
9	0.0014	8.17×10^{-7}
10	7.05×10^{-4}	2.12×10^{-7}

Table 2.2: Estimates of $|u-u_h|_{H^1}, ||u-u_h||_{L_2}$

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After we apply the weighted Jacob iterative method with $\omega = 2/3$ (see, e.g., [8]), and an initial guess of 0, to the preconditioned linear system for 10,15,20 iterative times, we obtain the numerical solutions u_{it}. The errors between the numerical solution and the Galerkin solution, estimated in H^1 semi-norm and L_2 norm, are given in Table 2.3.

	$\|u_{it} - u_h\|_{H^1}$			$\|\|u_{it} - u_h\|\|_{L_2}$		
n	itno=10	itno=15	itno=20	itno=10	itno=15	itno=20
5	4.00-4	1.02-5	2.60-7	1.31-4	3.28-6	8.34-8
6	4.05-4	1.04-5	2.67-7	1.38-4	3.48-6	8.91-8
7	4.06-4	1.05-5	2.69-7	1.41-4	3.57-6	9.14-8
8	4.07-4	1.05-5	2.70-7	1.43-4	3.61-6	9.25-8
9	4.07-4	1.05-5	2.70-7	1.43-4	3.63-6	9.30-8
10	4.07-4	1.05-5	2.70-7	1.43-4	3.64-6	9.32-8

Table 2.3: Estimates of $|u_{it} - u_h|_{H^1}, ||u_{it} - u_h||_{L_2}$ with the different iterative numbers

Since the condition number is strictly bounded by 2, it can be expected that the convergence rate of $u_{it} - u_h$ shall be like $O(\rho^m)$, where iterative number is denoted by m, and $\rho < 1$. Numerical results in Table 2.3 indicate that $\rho \approx \frac{1}{2}$. In fact, a careful analysis of iterative methods with a given condition number bound may give an estimate of the convergence rate, which shall not be discussed in detail here.

Similar to the example shown in [24], let $q(x) = 1 - x^2, f(x) = (1 - x^2)^2 + 4$ in our second example. Subspace level is up to $n = 10$. In this situation, $u(x) = 1 - x^2$, and $|(u - u_h)|_{H^1}, ||(u - u_h)||_{L_2}$ are computed in Table 2.4 for different n. Condition numbers of the preconditioned system are shown in Table 2.5. Similar computing results to example one are obtained. These numerical results confirm the performance of our preconditioning method.

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n	$\|u - u_h\|_{H^1}$	$\|\|u - u_h\|\|_{L_2}$
5	0.0306	2.07×10^{-4}
6	0.0141	5.49×10^{-5}
7	0.0067	1.44×10^{-5}
8	0.0033	3.70×10^{-6}
9	0.0016	9.45×10^{-7}
10	8.03×10^{-4}	2.39×10^{-7}

Table 2.4: Estimates of $|u - u_h|_{H^1}, ||u - u_h||_{L_2}$

n	5	6	7	8	9	10
$\lambda_{\max,A_n}$	1.10+2	2.33+2	4.83+2	9.87+2	2.00+3	4.04+3
$\lambda_{\min,A_n}$	5.38-2	2.61-2	1.29-2	6.4-3	3.2-3	1.6-3
$\kappa(A_n)$	2.04+3	8.91+3	3.75+4	1.54+5	6.28+5	2.54+6
$\lambda_{\max,PHA_n(PH)^T}$	2.016	2.021	2.023	2.024	2.024	2.024
$\lambda_{\min,PHA_n(PH)^T}$	1.036	1.018	1.009	1.004	1.002	1.001
$\kappa(PHA_n(PH)^T)$	1.946	1.986	2.005	2.015	2.019	2.022

Table 2.5: Condition numbers of the matrix A_n and $(PH)A_n(PH)^T$

	$\|u_{it} - u_h\|_{H^1}$			$\|\|u_{it} - u_h\|\|_{L_2}$		
n	itno=10	itno=15	itno=20	itno=10	itno=15	itno=20
5	3.89-4	1.04-5	2.81-7	1.10-4	2.78-6	7.19-8
6	3.92 -4	1.05-5	2.86-7	1.14-4	2.89-6	7.48-8
7	3.93 -4	1.05-5	2.87-7	1.16-4	2.93-6	7.59-8
8	3.93-4	1.06-5	2.87-7	1.16-4	2.95-6	7.64-8
9	3.93 -4	1.06-5	2.87-7	1.17-4	2.96-6	7.66-8
10	3.93-4	1.06-5	2.87-7	1.17-4	2.97-6	7.67-8

Table 2.6: Estimates of $|u_{it} - u_h|_{H^1}, ||u_{it} - u_h||_{L_2}$ with the different iterative numbers

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Chapter 3

C^1 wavelets on two dimensional triangular meshes

3.1 Introduction

We are interested in the multilevel analysis on a sequence of nested subspaces

$$V_0 \subset V_1 \subset V_2 \cdots$$

on triangulations $\{\mathcal{T}_k\}_{k=0}^{\infty}$ of a polygonal domain Ω in $\mathbb{R}^2$. The multilevel decomposition of $\{V_k\}_{k=0}^{\infty}$ is to seek the proper subspaces that

$$V_J = V_0 + \widetilde{W}_1 + \widetilde{W}_2 + \cdots + \widetilde{W}_{J-1} + \widetilde{W}_J.$$

$\widetilde{W}_k \subset V_k$ is chosen to be orthogonal to V_{k-1} with respect to some kind of inner product. The basis functions for $\widetilde{W}_k$ are generally called wavelets. If the usual L_2 inner product $\langle \cdot, \cdot \rangle_{L_2}$ on Ω is applied, then basis functions, say $\{\psi_j^k\}_{j \in I_{\psi,k}}$ for $\widetilde{W}_k$, are traditionally called semi-wavelets [14, 16], where $I_{\psi,k}$ denotes the index set of wavelets. Let $\{\phi_j^k\}_{j \in I_k}$ be basis functions for V_k. Then

$$\langle \psi_j^k, \phi_{j'}^{k'} \rangle_{L_2} = 0, \quad k' < k,\ j \in I_k,\ j' \in I_{\psi,k'}.$$

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A basis $\{\phi_j\}_{j=1}^{\infty}$ is stable if we have

$$||\sum_{j=1}^{\infty} \alpha_j \phi_j||^2 \simeq \sum_{j=1}^{\infty} \alpha_j^2,$$

where $||\cdot||$ is the norm of interest. Here, $\simeq$ refers to that two terms can be bounded by some constant multiple of each other, with the constant independent of the parameters on which these two terms may depend. Similarly, We let $\lesssim$ ($\gtrsim$) denote that the first (second) term can be bounded by a constant multiple of the second (first) term.

Moreover, $\{\phi_j\}_{j=1}^{\infty}$ is a Bessel sequence if and only if [16]

$$||\sum_{j=1}^{\infty} \alpha_j \phi_j||^2 \lesssim \sum_{j=1}^{\infty} \alpha_j^2.$$

A well-known norm equivalence theory in literature (for example, see [21, 54]) reads

$$||u||_{H^s}^2 \simeq \sum_{k=0}^{J} 2^{2ks} ||u_k||_{L_2}^2, \tag{3.1}$$

where

$$u = \sum_{k=0}^{J} u_k, \ u_k \in \widetilde{W}_k, \ \widetilde{W}_0 \equiv V_0.$$

Consequently, if the (semi-)wavelet basis $\{\psi_j^k\}_{j\in I_{\psi,k}}$ for $\widetilde{W}_k$ is stable in the L_2 space, then wavelet system $\{2^{-ks}\psi_j^k\}_{k=0,1,\dots,\, j\in I_{\psi,k}}$ forms a stable basis in the Sobolev space H^s [1, 10], where s is some positive real number.

Stable bases in Sobolev spaces have broad applications in numerically solving partial differential equations as well as integral equations [12, 13, 11, 19, 21, 34, 38, 45, 52]. For higher order problems, wavelets with higher order smoothness, such as C^1 wavelets, are required. This motivates people to construct the stable wavelet systems in Sobolev spaces. In particular, some stable wavelets in Sobolev spaces have been constructed and discussed on the uniform meshes in [39, 41, 50, 51].

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Validation of (3.1) requires some mild conditions on the subspaces $\{V_k\}_{k=0}^{\infty}$, i.e., the Jackson inequality and the Bernstein inequality. In general, the Jackson inequality implies the suitable approximation capability provided by the subspaces $\{V_k\}_{k=0}^{\infty}$, and the Bernstein inequality, known as the inverse inequality for a long time in finite element literature, usually holds true for the underlying subspaces $\{V_k\}_{k=0}^{\infty}$.

Floater and Quak have constructed the semi-wavelets from the piecewise linear nodal basis with small supports on the irregular meshes in [25, 26, 27]. Their wavelets are orthogonal among different levels with respect to the L_2 inner product. By (3.1), it is clear that their wavelets also form a stable basis in the Sobolev space after properly scaled. To reduce the support of wavelet, Stevenson introduced the discrete L_2 inner product in [46, 47] for each subspace V_k, which is also generated from the piecewise linear nodal basis. $\widetilde{W}_k$ is then the orthogonal complement of V_{k-1} in V_k with respect to the discrete inner product for V_k. His wavelet for the linear nodal basis for two dimensional case is simple and has three coefficients. Numerical results on the regular mesh (see [39]) show the potential for numerically solving partial differential equations. The idea of constructing wavelets which are not L_2 orthogonal among levels can also be found in several other papers (see [48, 49]).

C^1 wavelets on general meshes are of particular importance for fulfilling the smoothness condition required by numerically solving high order problems. However, the difficulty in choosing the proper C^1 scaling functions on general meshes as well as the lack of general theory to verify the stability of wavelets make it intensely difficult to construct stable C^1 wavelets in Sobolev spaces. Main goal of this chapter is to investigate the general theory on the construction of C^1 continuous wavelets on the general meshes. We wish to develop the theory independent on the Fourier analysis, which is considered as the fundamental theoretical tool for the wavelet analysis on uniform meshes. On the other hand, to further generalize Stevenson's idea to more complicated finite element spaces may not be suitable because one key estimate of the projection operator (Theorem 4.3 in [47]) involves three levels of subspaces. In

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this chapter, we shall overcome this weakness by estimating the same operator only involving two levels. Our method greatly reduces the complexity in the estimate and thus makes the estimate for more complicated basis functions possible. Moreover, our wavelets are extremely short supported, and this leads to the fast algorithm for applications.

We will take Powell-Sabin elements (PS element, see [42]) as basis functions for subspaces $\{V_k\}_{k=0}^{\infty}$. On the three-direction meshes, where grid lines run only in three directions (see [36] and [3], p.294), PS element is C^1 continuous on the meshes and the subspaces $\{V_k\}_{k=0}^{\infty}$ are nested. We design the discrete L_2 inner product $\langle \cdot, \cdot \rangle_{D_2(k)}$ for each subspace V_k and our wavelet subspace $\widetilde{W}_k$ is orthogonal to V_{k-1} with respect to the discrete inner product $\langle \cdot, \cdot \rangle_{D_2(k)}$. One improvement in our estimate is that the proof is proceeded on the standard equilateral triangle in stead of an arbitrary triangle due to the affine invariance property of the basis functions. The stability of the wavelet basis in the Sobolev space H^2 is verified.

Although the three-direction meshes are a type of regular triangular meshes, our theory does not employ any property of the regular structure of meshes. The obstacle preventing us from applying our theory to irregular meshes is the difficulty in seeking the nested subspaces from suitable C^1 continuous refinable functions. However, our theory is sufficiently flexible to be extendable to irregular meshes once proper C^1 continuous refinable basis functions on irregular meshes are obtained.

This chapter is divided into four parts. In Section 3.2, we introduce the knowledge of Powell-Sabin C^1 element which is employed later as scaling functions to produce the sequence of the nested subspaces. We construct the wavelets in Section 3.3, and some basic properties of the wavelets are presented. More precisely, we define one discrete L_2 inner product for each subspace, and on the basis of such inner products, we construct the wavelets to form a Riesz basis in Sobolev spaces. In Section 3.4, we focus on the general theory of Riesz basis in the Sobolev space. We shall verify that our wavelets satisfy all required conditions to form a Riesz basis in the Sobolev space H^2.

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The last section is devoted to the examples of the computation for wavelets on the regular triangular meshes.

3.2 PS element and C^1 basis functions

Powell-Sabin (PS) element has been traced back to 1977 and was widely used in CAGD because of its simple structure (compared with other C^1 elements). On a three-direction mesh, it is refinable and thus the sequence of subspaces from PS element is nested. On a general mesh, PS element is no longer guaranteed to be C^1 continuous globally. Applications utilizing these nonconforming PS element were discussed in [40]. In this chapter, we use PS element as the scaling functions to generate the sequence of nested subspaces on two dimensional three-direction meshes.

The structure of this section is organized as follows. In Section 3.2.1, we give the definition of PS element and study the local basis functions on an arbitrary triangle. In connection with the local basis functions on the standard equilateral triangle of edge length one, we discuss the affine map and affine invariance properties of the local basis in Section 3.2.2. By (3.10), we shows that affine invariance property allows us computing independently on the shapes of the triangles and thus it provides an efficient way to carry out computing and constructing wavelets. In Section 3.2.3, we combine local basis functions together to get C^1 continuous basis functions on a three-direction mesh. Associated with such basis, discrete L_2 inner products and norms are introduced.

3.2.1 Powell-Sabin element

PS element is composed of piecewise quadratic polynomials on an arbitrary triangle $\triangle P_1P_2P_3$ with 9 degrees of freedom [42]. We concern with one type of subdivision of a triangle shown in Figure 3.1. Let O be the centroid of the triangle $T := \triangle P_1P_2P_3$, and **PS element** used throughout this chapter

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is defined to be a piecewise quadratic function on 6 small sub-triangles and C^1 continuous on T. O is chosen to be the centroid because we wish the multilevel subspaces from PS element are nested. For the PS element on a non-degenerated triangle, it has 9 degrees of freedom. In other words, we have the following property for PS element.

Proposition 3.1. *For a given piecewise quadratic function on the splitting sub-triangles as shown in Figure 3.1, if it is C^1 on the triangle $\triangle P_1P_2P_3$ with given values, as well as derivatives at P_1, P_2, and P_3, then this function is uniquely determined.*

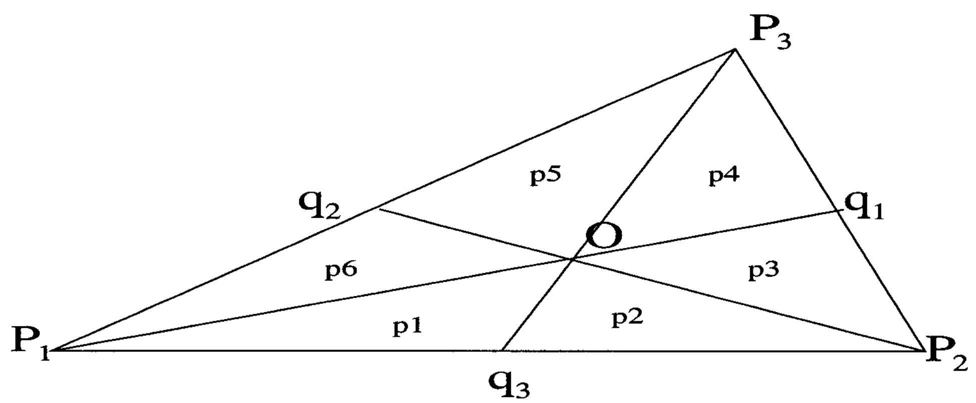

Figure 3.1: PS element is composed of 6 piecewise quadratic polynomials p_1, p_2, ..., p_6, respectively on a triangle

Existence of such PS element is a consequence of the affine invariance property of PS element, and this will be shown in Section 3.2.2. Uniqueness has been proved in [42]. The interpolation data we discussed here have 9 degrees of freedom, i.e., function values and the derivatives at 3 vertices. For derivatives at a vertex, there are 2 degrees of freedom if we notice that any two different directional derivatives of a C^1 continuous function at a point determine its tangent plane at that point from fundamental calculus.

We may construct 9 basis functions associated with these 9 degrees of freedom for PS elements on a triangle. Such basis functions are so called **local**

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basis functions, compared to the basis functions defined on the whole mesh. We may drop **local** if the context is clear. For a given triangle $\triangle P_1P_2P_3$ in Figure 3.2, we may define 9 basis functions $\phi_{T,P,0}$, $\phi_{T,P,1}$ and $\phi_{T,P,2}$ centered at the vertex $P \in \{P_1, P_2, P_3\}$. For $\phi_{T,P_1,0}$, we have

$$(3.2) \qquad \phi_{T,P_1,0}\big|_{P_1} = 1, \quad \frac{\partial \phi_{T,P_1,0}}{\partial d_1(P_1)}\Big|_{P_1} = 0, \quad \frac{\partial \phi_{T,P_1,0}}{\partial d_2(P_1)}\Big|_{P_1} = 0,$$

and

$$\phi_{T,P_1,0}\big|_{P_j} = 0, \quad \frac{\partial \phi_{T,P_1,0}}{\partial d_1(P_j)}\Big|_{P_j} = 0, \quad \frac{\partial \phi_{T,P_1,0}}{\partial d_2(P_j)}\Big|_{P_j} = 0, \quad j = 2, 3.$$

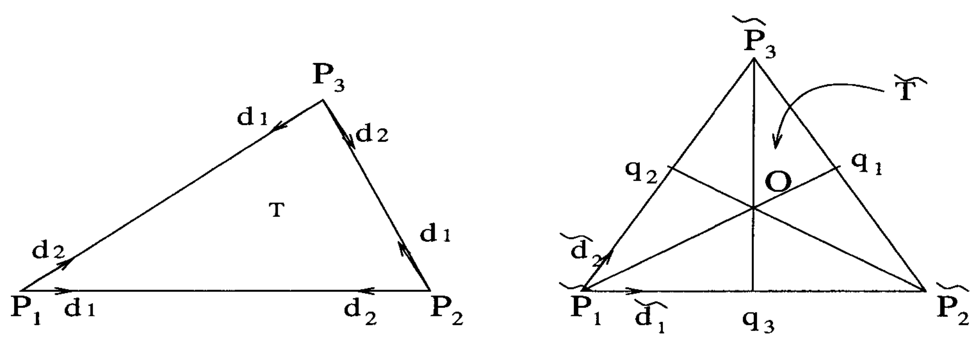

Figure 2a

Figure 2b

Figure 3.2: Notation for the triangle T

Recall that directional derivative of a function f in $\vec{d}$ direction is $\frac{\partial f}{\partial \vec{d}} = \cos(\theta)\frac{\partial f}{\partial \vec{x}} + \sin(\theta)\frac{\partial f}{\partial \vec{y}}$, where $\vec{d}$ is a direction with the unit length and the angle between $\vec{d}$ and x axis is θ. Here, by $d_1(P), d_2(P)$ we denote the directions of the unit length along the edges starting from the vertex P following the right hand rule. For example, $d_1(P_1)$ is the direction in $\overrightarrow{P_1P_2}$, and $d_2(P_1)$ is in $\overrightarrow{P_1P_3}$. We drop P from $d_1(P)$, $d_2(P)$ if the context is clear. Let $|d_j(P)|$, $j = 1, 2$ be the length of the edge starting from the vertex P and in the direction $d_j(P)$. For instance, $|d_1(P_1)| = |P_1P_2|$. T may come into the notation to suggest that the relevant symbols be based on the triangle T, such as $d_j(T, P)$, $|d_j(T, P)|$ in Section 3.2.3. Subscript T is used to imply that $\phi_{T,P_1,0}$ is defined on the

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triangle T locally. If the context is clear, the subscript T in $\phi_{T,P_1,0}$ is omitted. Clearly, $\phi_{P_1,0}$ interpolates the value at the vertex P_1 while keeps all other interpolation data such as derivatives at P_1, P_2, P_3 and values at P_2, P_3 zeros. This is similar to the Hermite interpolants in one dimensional case [34]. $\phi_{P_1,1}$ interpolates the derivative at P_1 in d_1 direction, i.e.,

$$\frac{\partial \phi_{P_1,1}}{\partial d_1}\Big|_{P_1} = 1/|P_1P_2|, \quad \phi_{P_1,1}\big|_{P_1} = 0, \quad \frac{\partial \phi_{P_1,1}}{\partial d_2}\Big|_{P_1} = 0, \tag{3.3}$$

and

$$\phi_{P_1,1}\big|_{P_j} = 0, \quad \frac{\partial \phi_{P_1,1}}{\partial d_1}\Big|_{P_j} = 0, \quad \frac{\partial \phi_{P_1,1}}{\partial d_2}\Big|_{P_j} = 0 \quad j = 2, 3,$$

where scale parameter $1/|P_1P_2|$ is used for the affine invariance purpose. In particular, if $|P_1P_2| = 1$, then $\phi_{P_1,1}$ has the unit derivative in $\overrightarrow{P_1P_2}$ direction at P_1. Function $\phi_{P_1,2}$ which interpolates the directional derivative at P_1 in the direction $\overrightarrow{P_1P_3}$ (i.e., $d_2(P_1)$) is defined similarly. Likewise, we may define the basis functions centered at P_2, P_3.

These 9 basis functions are PS elements with prescribed interpolation data. By Proposition 3.1, we claim such basis functions are uniquely determined. They are linear independent. If not, then we have a PS element which is a linear combination of them such that this PS element is identically zero on the triangle. If we notice that each basis function interpolates the different data of values and derivatives at 3 vertices, then this PS element must have non zero value or directional derivative in some direction at one vertex. This contradicts the fact that this PS element is identically zero on the triangle. Thus we have

Proposition 3.2. $\{\phi_{T,P,j}\}_{P=P_1,P_2,P_3,j=0,1,2}$ *is a local basis for PS elements on the triangle* $\triangle P_1P_2P_3$.

Now we consider when two PS elements on two neighboring triangles join C^1 continuously. First, we give a proposition on the property of PS element (see Figure 3.1).

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Proposition 3.3. *If quadratic functions $p1, p2$ are C^1 continuous across the common boundary Oq_3* (see Figure 3.1)*, then*

$$p1 = p2 + \lambda l^2, \tag{3.4}$$

where λ is an arbitrary constant, and $l = 0$ refers to the function of the line through O and $q3$.

Let's consider the directional derivative of $p1$, $p2$ in Figure 3.1 in $\overrightarrow{Oq_3}$ direction. By (3.4), we have

$$\frac{\partial p1}{\partial \overrightarrow{Oq_3}} = \frac{\partial p2}{\partial \overrightarrow{Oq_3}}, \quad (\frac{\partial l}{\partial \overrightarrow{Oq_3}} = 0).$$

Then directional derivative of the PS element in $\overrightarrow{Oq_3}$ direction on P_1P_2 is linear. Moreover, it interpolates two corresponding directional derivatives at P_1 P_2. However, the directional derivative in $\overrightarrow{P_1P_2}$ direction of the PS element are respectively two linear functions on P_1q_3 and q_3P_2, and they join C^1 continuously. The same analysis is applied to $T' := \triangle P_1P_3P_4$ in Figure 3.3. Let q be the midpoint of P_1P_3, and O, O' be the centroids of $\triangle P_1P_2P_3$, $\triangle P_1P_3P_4$, respectively. Then we may anticipate that two PS elements on two neighboring triangles sharing the same values and derivatives at vertices P_1 and P_3 could fail to join C^1 continuously across the common boundary P_1P_3 unless O, q, O' are co-linear.

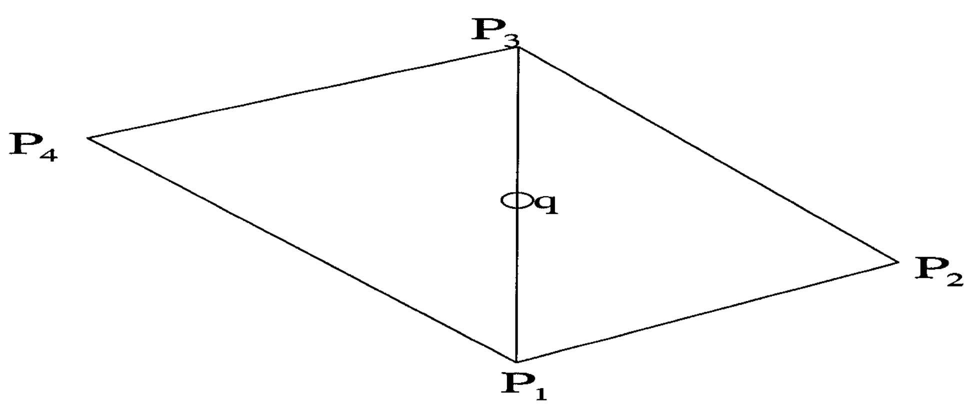

Figure 3.3: Two neighboring triangles sharing the common edge P_1P_3

However, on a three-direction mesh, $P_1P_2P_3P_4$ forms a parallelogram, and then O, q, O' fall on the line through P_2 and P_4 automatically. Hence, two PS

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elements join C^1 continuously in this case. This is the reason why we restrict ourselves to three-direction meshes.

3.2.2 Affine map and the local basis on the standard equilateral triangle

In this section, we illustrate how to compute basis functions efficiently in a uniform way. Affine map plays an important role in the connection of a PS element on an arbitrary triangle with a PS element on the standard triangle. The following proposition is a basic property of the affine map.

Proposition 3.4. *There is a unique affine map mapping from T to T_1 or from T_1 to T, where T, T_1 are two non-degenerated triangles.*

Proof. First we prove there is a unique affine map which maps three vertices of T to the corresponding vertices of T_1. Let $P_1(x_1, y_1), P_2(x_2, y_2), P_3(x_3, y_3)$ be three vertices of T and $P_1'(x_1', y_1'), P_2'(x_2', y_2'), P_3'(x_3', y_3')$ be the vertices of T_1, respectively. The affine map $\mathcal{A} : T \to T_1$ satisfies

$$\begin{pmatrix} x_i' \\ y_i' \end{pmatrix} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad i = 1, 2, 3.$$

Rearranging the above equations yields

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{12} \\ b_1 \\ a_{21} \\ a_{22} \\ b_2 \end{pmatrix}.$$

Obviously, unique existence of the affine map is equivalent to the non-singularity of the matrix

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix},$$

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which implies that P_1, P_2, P_3 are non-collinear. This uniquely fixed a_{ij}, b_j, $i = 1, 2$, $j = 1, 2$ and gives the affine map. Moreover, the map is injective. If not, then there exist two different points x_1, x_2 in T with the expression

$$x_1 = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3, \quad x_2 = \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3$$

such that $\mathcal{A}x_1 = \mathcal{A}x_2$, that is

$$(\alpha_1 - \beta_1)P_1' + (\alpha_2 - \beta_2)P_2' + (\alpha_3 - \beta_3)P_3' = 0.$$

This shows that $\alpha_j = \beta_j$, $j = 1, 2, 3$ and thus $x_1 = x_2$. This is a contradiction. The map is surjective. For any point $x' = \sum_{i=1}^{3} \alpha_i P_i' \in T_1$, it is clear that $\mathcal{A}x = x'$, with $x = \sum_{i=1}^{3} \alpha_i P_i \in T$.

It shows that this affine map is a one to one correspondence from T to T_1. Existence and uniqueness of the affine map from T_1 to T can be verified in an analogue manner. This completes the proof. □

Proposition 3.5. *Let the affine map $\mathcal{A}$: $T \to T_1$, with $\mathcal{A}(P_1) = P_1'$, $\mathcal{A}(P_2) = P_2'$. If $x = P_1 + t(P_2 - P_1)$, then $x' = \mathcal{A}(x) = P_1' + t(P_2' - P_1')$ is on the line $P_1'P_2'$.*

Proof.

$$\begin{aligned} x' &= \mathcal{A}(x) = \mathcal{A}(P_1) + t\mathcal{A}(P_2 - P_1) = t\mathcal{A}(P_2) + (1-t)\mathcal{A}(P_1) \\ &= tP_2' + (1-t)P_1' = P_1' + t(P_2' - P_1'). \end{aligned}$$

This completes the proof. □

Proposition 3.5 tells that if a function $f(x')$ is defined on the line segment $P_1'P_2'$, and the affine map satisfies $\mathcal{A}(P_1) = P_1'$, $\mathcal{A}(P_2) = P_2'$, then $f(\mathcal{A}(x))$ is defined on the line segment P_1P_2. Based on Proposition 3.5, next proposition explores the relationship of the derivatives of the two functions $f(x')$ and $g(x) = f(\mathcal{A}(x))$.

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Proposition 3.6. *If the affine map $\mathcal{A}$ satisfies $\mathcal{A}(P_1) = P_1'$, $\mathcal{A}(P_2) = P_2'$, and $g(x) = f(\mathcal{A}(x))$, then*

$$|P_1P_2|\frac{\partial g}{\partial(\overrightarrow{P_1P_2})}|_{P_1} = |P_1'P_2'|\frac{\partial f}{\partial(\overrightarrow{P_1'P_2'})}|_{P_1'},$$

where $\frac{\partial g}{\partial(\overrightarrow{P_1P_2})}|_{P_1} = \nabla g \cdot \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|}$ *is the directional derivative of g at point P_1.*

By Proposition 3.6 we have

$$\phi_{T,P_i,j}(x) = \phi_{\tilde{T},\tilde{P}_i,j}(\mathcal{A}_T(x))/|d_j(T,P_i)|, \quad i = 1,2,3, j = 1,2, \tag{3.5}$$

and

$$\phi_{T,P_i,0}(x) = \phi_{\tilde{T},\tilde{P}_i,0}(\mathcal{A}_T(x)), \quad i = 1,2,3. \tag{3.6}$$

Here, $\tilde{T} := \triangle\tilde{P}_1\tilde{P}_2\tilde{P}_3$ is the standard equilateral triangle with unit edge lengths and affine map $\mathcal{A}_T : T \to \tilde{T}$. By $|d_j(T, P_i)|$ we denote the length of the edge of the triangle T starting from P_i in d_j direction.

Proposition 3.6 provide a general way to construct the PS element on an arbitrary triangle. More precisely, We construct the basis functions on the standard triangle $\tilde{T}$ first, and find the affine map mapping a given general triangle T to $\tilde{T}$ next. Finally, (3.5, 3.6) determines the basis functions on T.

If $\mathcal{A}(T) = \tilde{T}$, $g(x) = f(\mathcal{A}(x)), x \in T, \tilde{x} \in \tilde{T}$, and $f(\tilde{x})$ is a PS element on $\tilde{T}$, then $g(x)$ is also a PS element on T. Furthermore, $\{\phi_{\tilde{T},\tilde{P},j}(\mathcal{A}(x))\}_{\tilde{P}=\tilde{P}_1,\tilde{P}_2,\tilde{P}_3,j=0,1,2}$ is a basis of PS elements on T. $\{\phi_{\tilde{T},\tilde{P},j}(\tilde{x})\}_{\tilde{P}=\tilde{P}_1,\tilde{P}_2,\tilde{P}_3,j=0,1,2}$ are given explicitly on $\tilde{T}$ in the end of this section, and this, on the other hand, verifies the existence of the PS element in Proposition 3.1.

The representation of a PS element on T and its correspondence after being affinely mapped onto $\tilde{T}$ have the following affine invariance property.

Proposition 3.7. *Let the PS element f be defined on T with the expression*

$$f(x) = \sum_{P_i \in \mathcal{N}(T), j=0,1,2} \alpha_{T,P_i,j}\phi_{P,j}(x), \tag{3.7}$$

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where $\mathcal{N}(T) := \{P_1, P_2, P_3\}$ is the set of three vertices of triangle T. By the affine map, we map the function f on the triangle T to its correspondence $\tilde{f}$ on the standard triangle $\tilde{T}$ by

$$\tilde{f}(\tilde{x}) := f(\tilde{A}\tilde{x} + \tilde{b}), \tag{3.8}$$

where $P_j = \tilde{A}\tilde{P}_j + \tilde{b}$, $j = 1, 2, 3$ and $\{\tilde{P}_j,\ j = 1, 2, 3\}$ are three vertices of $\tilde{T}$. $\tilde{f}$ is also a PS element on $\tilde{T}$, and it has the expression

$$\tilde{f}(\tilde{x}) = \sum_{i=1,2,3,j=0,1,2} \alpha_{\tilde{T},\tilde{P}_i,j}\phi_{\tilde{T},\tilde{P}_i,j}(\tilde{x}). \tag{3.9}$$

Then, we have

$$\alpha_{T,P_i,j} = \alpha_{\tilde{T},\tilde{P}_i,j}, \quad i = 1, 2, 3, j = 0, 1, 2. \tag{3.10}$$

Proof. Taking directional derivative of (3.7), we have

$$\frac{\partial f}{\partial d_j}|_{P_i} = \alpha_{T,P_i,j}\frac{\partial \phi_{T,P_i,j}}{\partial d_j}|_{P_i},$$

which implies

$$\alpha_{T,P_i,j} = \frac{\partial f}{\partial d_j(T, P_i)}|_{P_i}|d_j(T, P_i)|, \quad j = 1, 2.$$

Likewise, we have

$$\tilde{\alpha}_{\tilde{T},\tilde{P}_i,j} = \frac{\partial \tilde{f}}{\partial d_j(\tilde{T}, \tilde{P}_i)}|_{\tilde{P}_i}|d_j(\tilde{T}, \tilde{P}_i)|, \quad j = 1, 2.$$

Using the property of affine map in Proposition 3.6 and (3.8), we obtain

$$\frac{\partial \tilde{f}}{\partial d_j(\tilde{T}, \tilde{P}_i)}|_{\tilde{P}_i}|d_j(\tilde{T}, \tilde{P}_i)| = \frac{\partial f}{\partial d_j(T, P_i)}|_{P_i}|d_j(T, P_i)|.$$

This implies (3.10) and thus completes the proof. □

Equation (3.10) motivates us to introduce the scale parameter $1/|P_1P_2|$ in the definition of PS local basis in (3.3). The representations of the PS element therfore are invariant on different triangles. This enables us to carry

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out computing on the standard reference triangle instead of a general triangle. Furthermore, (3.10) provide a quick way to map a PS element on a general triangle to a PS element on $\tilde{T}$.

To end this subsection, we shall construct the basis functions on $\tilde{T}$ with explicit expressions due to their importance. As shown in Figure 3.2b, $\tilde{T}$ is divided into 6 sub-triangles. We set up the coordinate system as follows. Let $\tilde{P}_1$ be the origin, and $\overrightarrow{\tilde{P}_1\tilde{P}_2}$ be in the positive x-axis direction. On each sub-triangle, a basis function is a quadratic function. Let $p1, ..., p6$ be polynomials on sub-triangles $\triangle O\tilde{P}_1 q3$, $\triangle Oq3\tilde{P}2$,..., $\triangle Oq2\tilde{P}_1$, respectively.

Using Proposition 3.3, together with the prescribed values and derivatives at three vertices of $\tilde{T}$, we compute the basis functions on $\tilde{T}$. For the basis function $\phi_{P_1,1}$, we have

$$\begin{aligned} (3.11) \qquad p1 = p5 = p6 &= x - \frac{1}{\sqrt{3}}y - \frac{3}{2}x^2 + \frac{1}{\sqrt{3}}xy + \frac{1}{6}y^2, \\ p2 = p3 = p4 &= p1 + 2(x - 1/2)^2. \end{aligned}$$

$\phi_{P_1,2}$ shall be symmetric to $\phi_{P_1,1}$ about the line $y = (\tan 30^o)x$, and $\phi_{P_1,0}$ has the expression

$$\begin{aligned} (3.12)\ p1 &= 1 - 2x^2 - 2y^2, \\ p2 &= 1 - 2x^2 - 2y^2 + 4(x - 1/2)^2, \\ p3 &= p4 = 1 - 2x^2 - 2y^2 + 4(x - 1/2)^2 + 3(y + \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}})^2, \\ p5 &= 1 - 2x^2 - 2y^2 + 3(y + \frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}})^2, \\ p6 &= p1. \end{aligned}$$

Other six basis functions on the vertices $\tilde{P}_2, \tilde{P}_3$ can be described in an analogue manner, or be obtained by using the symmetric property of the basis functions.

As a conclusion of this sub-section, we derive the basic properties of the affine map, and compute the basis functions on the standard triangle. Combining the affine map with the basis functions on $\tilde{T}$, we get the basis functions on a general triangle. Moreover, we also give an explicit method to transform a PS element on a general triangle to its correspondence on $\tilde{T}$ by (3.10). In

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general, we provide the necessary basic knowledge of the local basis functions, which are the fundamental bricks to build the basis functions on three-direction meshes.

3.2.3 Basis functions on the meshes

Once we have the basis functions on an arbitrary triangle $\triangle P_1P_2P_3$, we shall define the basis functions on three-direction meshes. Let $\mathcal{T}$ be the triangulation, and $\mathcal{N}$ be the vertex set. There are 3 basis functions denoted by $\{\phi_{P,j}\}_{j=0,x,y}$ associated with each vertex P. Here, in order to define the basis function uniformly, let $\phi_{P,x}, \phi_{P,y}$ be the functions interpolating the derivatives at the node P in the directions of the positive x-axis, y-axis, respectively. The supports of $\phi_{P,x}, \phi_{P,y}$ consist of the neighboring triangles of the vertex P. On each neighboring triangle T, $\phi_{P,x}, \phi_{P,y}$ can be written as the linear combination of the local basis functions $\phi_{T,P,1}, \phi_{T,P,2}$, if we notice that the tangent plane at a vertex of a C^1 function is uniquely determined by any two different directional derivatives. Let the set S be $S := \{\phi_{P,j},\ P \in \mathcal{N},\ j = 0, x, y\}$. Then the space V is the linear span of the set S. V reproduces the polynomial of degree 2 on each triangle in the given mesh.

We assume that the refinement procedure is to subdivide each triangle by connecting the middle point of each edge. After each refinement, one triangle is subdivided into 4 similar sub-triangles. Furthermore, we assume that for a triangle $T \in \mathcal{T}$, three interior angles are bounded below and above by some constants. This assumption implies that the ratio l_i/l_j is bounded up and below for $i, j = 1, 2, 3$ by some constants, where l_1, l_2, l_3 are the lengthes of a given triangle T. The mesh size thus can be measured by any one of l_1, l_2, and l_3. If the mesh size is h, which is defined by $h := \max_{T\in\mathcal{T}}\{\text{diameter of } T\}$, then $||\phi_{P,j}||_{L_2} \simeq h$.

On a triangle, say $\triangle P_1P_2P_3$, we have two representations of a function in V, i.e., in terms of local basis functions $\{\phi_{T,P_i,j}\}_{i=1,2,3,j=0,1,2}$ or $\{\phi_{P_i,j}\}_{i=1,2,3,j=0,x,y}$. In the following, we wish to discuss the basis $\{\phi_{P,j}\}_{P\in\mathcal{N},j=0,x,y}$ in detail. $\phi_{P,j}$

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is defined on the six neighboring triangles (denoted by $\mathcal{T}(P)$) of the vertex P in $\mathcal{T}$. On its supported triangle, $\phi_{P,j}$ is a PS element, and

$$\phi_{P,0}\big|_P = 1,\ \phi_{P,0}\big|_q = 0,\ \mathcal{N} \ni q \neq P,\ \frac{\partial \phi_{P,0}}{\partial j}\Big|_q = 0,\ q \in \mathcal{N},\ j = x, y,$$

and

$$\frac{\partial \phi_{P,x}}{\partial x}\Big|_P = 1/h,\ \frac{\partial \phi_{P,x}}{\partial y}\Big|_q = 0,\ \phi_{P,x}\big|_q = 0,\ q \in \mathcal{N},\ \frac{\partial \phi_{P,x}}{\partial x}\Big|_q = 0,\ \mathcal{N} \ni q \neq P.$$

$\phi_{P,y}$ is defined exactly in the same way. For any $T \in \mathcal{T}(P_1)$, say T in Figure 3.1, $\phi_{P_1,x}$ (or $\phi_{P_1,y}$) on T can be represented in terms of local basis $\phi_{T,P_1,1}$ and $\phi_{T,P_1,2}$. If the angle between P_1P_2 and x axis is θ_1 and the angle between P_1P_3 and x axis is θ_2, then

$$\frac{\partial \phi_{P_1,x}}{\partial d_1(T,P_1)}\Big|_{P_1} = \cos(\theta_1)\frac{\partial \phi_{P_1,x}}{\partial x}\Big|_{P_1},\quad \frac{\partial \phi_{P_1,x}}{\partial d_2(T,P_1)}\Big|_{P_1} = \cos(\theta_2)\frac{\partial \phi_{P_1,x}}{\partial x}\Big|_{P_1}.$$

Recall that $d_i(T,P_1), i = 1,2$, are two directions along two edges of T starting from the vertex P_1. $\phi_{P_1,x}$ restricted on T can be written as

$$\phi_{P_1,x}\big|_T = \cos(\theta_1)\frac{|d_1(T,P_1)|}{h}\phi_{T,P_1,1} + \cos(\theta_2)\frac{|d_2(T,P_1)|}{h}\phi_{T,P_1,2}. \tag{3.13}$$

$\phi_{P_1,0}$ is simply the same as $\phi_{T,P_1,0}$ on T, i.e.,

$$\phi_{P_1,0}|_T = \phi_{T,P_1,0}. \tag{3.14}$$

(3.13-3.14) gives the local representations of the (global) basis functions. It is clear that $\phi_{P,j},\ j = 0, x, y$ can be locally represented by the local basis function $\phi_{T,P,j}$ $T \in \mathcal{T}(P), P \in \mathcal{N}(T), j = 0,1,2$. In other words, basis function $\phi_{P,j},\ j = 0, x, y$ are PS elements on each neighboring triangle T of P. Now we propose the definition of discrete L_2 inner product and its associated norm. For any function $f \in V$, it is a PS element on a triangle $T \in \mathcal{T}$. f can be represented in terms of the (global) basis functions or in terms of local basis

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functions triangle by triangle. We use the latter representation. Suppose f is represented as (3.7) on a triangle $T \in \mathcal{T}$. By (3.8), we obtain

$$||f||^2_{L_2(T)} = ||\tilde{f}||^2_{L_2(\tilde{T})}\text{vol}(T). \tag{3.15}$$

Note that

$$||\phi_{\tilde{T},\tilde{P},j}||^2_{L_2} \simeq 1, \quad \tilde{P} \in \mathcal{N}(\tilde{T}), j = 0, 1, 2.$$

Then

$$||\tilde{f}||^2_{L_2(\tilde{T})} \simeq \sum_{i=1,2,3,j=0,1,2} \tilde{\alpha}^2_{\tilde{T},\tilde{P}_i,j}.$$

Combining with (3.15), we have

$$||f||^2_{L_2(T)} \simeq \sum_{i=1,2,3,j=0,1,2} \alpha^2_{T,P_i,j}\text{vol}(T),$$

where $\text{vol}(T) \simeq h^2$ if the mesh size is h.

For two given functions $f(x), g(x) \in V$ which have the representations like (3.7) with the coefficients $\{\alpha_{T,P_i,j}\}$ and $\{\beta_{T,P_i,j}\}$, respectively, their discrete L_2 inner product can be defined as follows,

$$\langle f, g\rangle_{D_2} := \sum_{T\in\mathcal{T}} \text{vol}(T) \sum_{P\in\mathcal{N}(T),j=0,1,2} \alpha_{T,P,j}\beta_{T,P,j}. \tag{3.16}$$

Obviously, the discrete L_2 norm induced by the discrete L_2 inner product is equivalent to the L_2 norm for any function in V, i.e.,

Proposition 3.8.

$$\langle f, f\rangle_{D_2} \simeq ||f||^2_{L_2}, \quad f \in V.$$

By (3.10), we may replace $\alpha_{P_i,j}, \beta_{P_i,j}$ with $\widetilde{\alpha}_{\tilde{P}_i,j}, \widetilde{\beta}_{\tilde{P}_i,j}$ in the definition of discrete inner product (3.16). It suggests that we use affine map to map the function on T to the function on the standard triangle $\tilde{T}$ first with the representation (3.9) on $\tilde{T}$. The discrete L_2 inner product or norm then can be computed on $\tilde{T}$, respectively. Let T run over all triangles in $\mathcal{T}$ and sum the discrete inner products or norms up, then we get the corresponding discrete

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L_2 inner product or norm. Next, we introduce the new notation for multilevel analysis purpose. Recall that the uniform subdivision procedure is utilized to refine the mesh. We have 3 basis functions sitting on each vertex in the mesh for each level. We add the superscript or the subscript k to denote the triangulations, set of nodes, discrete inner products, etc, in the corresponding level k. For instance, for the k-th level mesh, we may use the following notation: $\mathcal{T}_k$, $\mathcal{N}_k$, $\langle\cdot,\cdot\rangle_{D_2(k)}$, $\phi^k_{P,j}$, $\alpha^k_{P,j}$, S_k, V_k etc.

Since (global) basis functions are PS elements on each triangle and C^1 continuous, we may claim the refinability of the basis functions.

Proposition 3.9. *Basis functions* $\{\phi^k_{P,j}\}_{P\in\mathcal{N},j=0,x,y}$ *are refinable.*

Proof. We prove that $\phi^0_{P_1,x}$ can be represented as the linear combination of the higher level basis functions. Refinability of other basis functions shall be proved similarly. We take the triangle T in Figure 3.1. Other neighboring triangles of P_1 on which $\phi^0_{P_1,x}$ is supported can be dealt with in the same way.

It is easily seen that $\phi^0_{P_1,x}$ determines the values and derivatives in x, y directions at P_1, q_2, q_3. Let these data be $(\alpha_{P_1,0} = 0, \alpha_{P_1,x} = 1, \alpha_{P_1,y} = 0)$, $(\alpha_{q_2,0}, \alpha_{q_2,x}, \alpha_{q_2,y})$, and $(\alpha_{q_3,0}, \alpha_{q_3,x}, \alpha_{q_3,y})$. Then we prove that

$$f = \sum_{P\in\{P_1,q_2,q_3\}} \alpha_{P,0}\phi^1_{P,0} + \frac{1}{2} \sum_{P\in\{P_1,q_2,q_3\},j=x,y} \alpha_{P,j}\phi^1_{P,j}$$

is identical to $\phi^0_{P_1,x}$ on $\triangle P_1q2q3$.

On $\triangle P_1q_2q_3$, f shares the same values and derivatives at P_1, q_2, q_3 with $\phi^0_{P_1,x}$. Furthermore, f and $\phi^0_{P_1,x}$ are both PS elements on $\triangle P_1q_2q_3$ in V_1. By Proposition 3.1, they are identical. $f \equiv \phi^0_{P_1,x}$ can be proved on the rest subtriangles of T similarly. This completes the proof. □

Refinability of the basis functions implies that the subspaces $\{V_k\}_{k=0}^{\infty}$ are nested, i.e.,

$$V_0 \subset V_1 \cdots .$$

Here, $S_k := \{\phi^k_{P,j}\}_{P\in\mathcal{N}_k;\, j=0,x,y}$ and $V_k := \text{span}S_k$.

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The wavelet subspace $\widetilde{W}_k$ is defined to be the orthogonal complement of V_{k-1} in V_k with respect to the discrete inner product $\langle \cdot, \cdot \rangle_{D_2(k)}$ for V_k. More precisely, for any $w \in \widetilde{W}_k$, we have

$$\langle w, v \rangle_{D_2(k)} = 0, \quad \forall v \in V_{k-1}.$$

$\widetilde{W}_k$ is the counterpart of W_k, which denotes the L_2 orthogonal complement of V_{k-1} in V_k. Here, we define the L_2 orthogonal projection to be $Q_k : L_2 \to V_k$ with $\langle Q_k u, v \rangle_{L_2} = \langle u, v \rangle_{L_2}$, $u \in L_2$, $v \in V_k$. Then we have

$$W_k := P_k \left(\bigcup_{l=0}^{\infty} V_l \right),$$

where $P_k := Q_k - Q_{k-1}$, $k = 0, 1, ...,$ $(Q_{-1} := 0)$.

For $\{V_k\}_{k=0}^{\infty}$, it is easily seen that the following Jackson inequality

$$||v_l - Q_k v_l||_{L_2} \lesssim 2^{-2(l-k)} |v_l|_{H^2}, \ v_k \in V_k, \ 0 \leq k < l \tag{3.17}$$

and the Bernstein inequality

$$||v_k||_{H^2} \lesssim 4^k ||v_k||_{L_2}, \ v_k \in V_k \tag{3.18}$$

hold true.

In the next section, the basis functions for $\widetilde{W}_k$ shall be constructed, and we call these functions wavelets.

3.3 Construction of wavelets

One favorite feature for the wavelets is the simple structure, which means the local (or small) supports of the wavelets. For each refinement, we get more newly created vertices. These vertices are middle points of edges in the old mesh. There are wavelet functions associated with these new vertices in the finer level mesh. We shall construct the wavelets for $\widetilde{W}_1$. Wavelet functions for other wavelet subspaces can be constructed similarly.

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Let two triangles $\triangle P_1P_2P_3$ and $\triangle P_1P_3P_4$ in the mesh $\mathcal{T}_0$ share the common edge P_1P_3 (see Figure 3.3). After the refinement, new vertex q, the middle point of the edge P_1P_3, is created. We define 3 wavelets associated with q in the following form,

$$\psi^1_{q,j'} := \beta^1_{q,j'}\phi^1_{q,j'}(x) + \sum_{i=1,3,j=0,x,y} \beta^1_{P_i,j}\phi^1_{P_i,j}(x), \quad j' = 0, x, y. \tag{3.19}$$

$\psi^1_{q,j'}$ is automatically orthogonal to all basis functions of V_0, except $\{\phi^0_{P,j},\ P = P_1, P_3, j = 0, x, y\}$ with respect to the discrete L_2 inner product $\langle\cdot,\cdot\rangle_{D_2(1)}$. Since we have 7 coefficients in the expression of the wavelet with only 6 constraints (which are the orthogonality conditions between the wavelet and 6 basis functions sitting at P_1, P_3), there is at least one nontrivial solution for the coefficients in (3.19). We want further that $\beta^1_{q,j'} \neq 0$, and thus we may normalize the coefficients to make $\beta^1_{q,j'} = 1$. Suppose that there is only one nontrivial solution with $\beta^1_{q,j'} = 0$. Then, we prove that the rest coefficients must be zeros, too. For a fixed j', suppose we have

$$\psi^1_{q,j'} = \sum_{i=1,3,j=0,x,y} \beta^1_{P_i,j}\phi^1_{P_i,j}(x). \tag{3.20}$$

Considering the support of $\phi^1_{P_i,j}(x),\ i = 1,3,\ j = 0,x,y$, we find that $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ shall be orthogonal to $\phi^0_{P_1,j}(x),\ j = 0,x,y$, because $\phi^1_{P_3,j}(x),\ j = 0,x,y$ are orthogonal to $\phi^0_{P_1,j}(x),\ j = 0,x,y$ automatically. Here, orthogonality is in the sense of discrete L_2 inner product.

First, we shall note that $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ has zero values and zero derivatives at vertices other than P_1 in $\mathcal{N}_1$. $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ orthogonal to $\phi^0_{P_1,j}(x),\ j = 0,x,y$ implies that $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ must have zero value, as well as zero derivatives at vertex P_1. Thus $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ has zero values as well as zero derivatives at all vertices in $\mathcal{N}_1$. Since on each triangle in $\mathcal{T}_1$, $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ is a PS element, $\sum_{j=0,x,y}\beta^1_{P_1,j}\phi^1_{P_1,j}(x)$ is then identically zero on all triangles in $\mathcal{T}_1$ by Proposition 3.1. Hence $\beta^1_{P_1,j} = 0,\ j = 0,x,y$. Likewise, we obtain $\beta^1_{P_3,j} = 0,\ j = 0,x,y$.

This contradicts the fact that there is at least one non-trivial solution for

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coefficients and thus excludes the solution with $\beta^1_{q,j'} = 0$. Consequently, we may write the wavelet in the form,

$$\psi^1_{q,j'} := \phi^1_{q,j'} + \sum_{i=1,3,j=0,x,y} \beta^1_{P_i,j}\phi^1_{P_i,j}(x), \quad j' = 0, x, y. \tag{3.21}$$

Clearly, our short supported wavelet has at most 7 non-zero coefficients. For convenience of stating the next lemma, we write $\psi^1_{q,j'}$ in a clearer expression:

$$\psi^1_{q,j'} := \phi^1_{q,j'} + \sum_{P\in\mathcal{N}_q,j=0,x,y} \beta^1_{q,j',P,j}\phi^1_{P,j}(x), \quad j' = 0, x, y. \tag{3.22}$$

Here, $\mathcal{N}_q$ is defined to be the set of two vertices in $\mathcal{N}_0$, whose midpoint is q. Accordingly, $\mathcal{N}^k_q$, $k = 0, 1, 2, ...$ is defined for the meshes in different levels.

Note that for two different wavelets $\psi^1_{q,j}$ and $\psi^1_{q',j'}$ with $\{q, j\} \neq \{q', j'\}$, they have the different components $\phi^1_{q,j}$ and $\phi^1_{q',j'}$. This leads to the following lemma.

Lemma 3.1. *$\{\psi^1_{q,j}\}_{q\in\mathcal{N}_1\backslash\mathcal{N}_0,j=0,x,y}$ is a stable basis for $\widetilde{W}_1$ in L_2 space.*

Proof. Let

$$g = \sum_{q\in\mathcal{N}_1\backslash\mathcal{N}_0} \sum_{j=0,x,y} \gamma_{q,j}\psi^1_{q,j}.$$

Recalling that

$$\psi^1_{q,j} = \phi^1_{q,j} + \sum_{P\in\mathcal{N}_q} \sum_{j_1=0,x,y} \beta^1_{q,j,P,j_1}\phi^1_{P,j_1},$$

we have

$$\begin{aligned} g &= \sum_{q\in\mathcal{N}_1\backslash\mathcal{N}_0,j=0,x,y} [\gamma_{q,j}\phi^1_{q,j} + \gamma_{q,j} \sum_{P\in\mathcal{N}_q,j_1=0,x,y} \beta^1_{q,j,P,j_1}\phi^1_{P,j_1}] \\ &= \sum_{q\in\mathcal{N}_1\backslash\mathcal{N}_0,j=0,x,y} \gamma_{q,j}\phi^1_{q,j} + \sum_{P\in\mathcal{N}_0,j=0,x,y} \alpha_{P,j}\phi^1_{P,j}, \end{aligned} \tag{3.23}$$

where

$$\alpha_{P,j} = \sum_{P\in\mathcal{N}_q,j_1=0,x,y} \gamma_{q,j_1}\beta^1_{q,j_1,P,j}.$$

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Since there are six neighboring triangles around the vertex $P \in \mathcal{N}_0$ in a three-direction mesh, we have

$$\alpha_{P,j}^2 \le C \sum_{P \in \mathcal{N}_q, j_1=0,x,y} \gamma_{q,j_1}^2, \tag{3.24}$$

where C is a fixed constant only dependent on the number of the neighboring triangles and coefficients $\{\beta_{q,j_1,P,j}^1\}$.

Note that $\{\phi_{P,j}^1\}_{P \in \mathcal{N}_1, j=0,x,y}$ is a Riesz basis of V_1 in L_2 space. By (3.23), we have

$$||g||_{L_2}^2 \simeq \sum_{q \in \mathcal{N}_1 \setminus \mathcal{N}_0, j=0,x,y} \gamma_{q,j}^2 ||\phi_{q,j}^1||_{L_2}^2 + \sum_{P \in \mathcal{N}_0, j=0,x,y} \alpha_{P,j}^2 ||\phi_{P,j}^1||_{L_2}^2.$$

Assume the mesh size of $\mathcal{T}_1$ is h. Then

$$||\phi_{P,j}^1||_{L_2}^2 \simeq h^2, \quad P \in \mathcal{N}_1, j = 0, x, y.$$

By the estimate of (3.24), we have

$$||g||_{L_2}^2 h^{-2} \simeq \sum_{q \in \mathcal{N}_1 \setminus \mathcal{N}_0, j=0,x,y} \gamma_{q,j}^2.$$

This completes the proof. □

As a direct consequence of Lemma 3.1, we have

Corollary 3.1.

$$|| \sum_{P \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}, j=0,x,y} \alpha_{P,j}^k \psi_{P,j}^k ||_{L_2}^2 \simeq (2^{-k})^2 \sum_{P \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}, j=0,x,y} (\alpha_{P,j}^k)^2, k = 0, 1, ...,$$

provided that the initial mesh size is $\simeq 1$.

Counting the number of basis functions, we claim that $\{\psi_{q,j}^k\}_{q \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}}$ is a L_2 stable basis for $\widetilde{W}_k$. To keep the consistence of the notation, we let $\psi_{q,j}^0 = \phi_{q,j}^0$ and $\mathcal{N}_{-1}$ be the empty set in what follows. Then we have $\widetilde{W}_0 = V_0$.

Let Y_k be the orthogonal projection from V_{k+1} to V_k with respect to $\langle \cdot, \cdot \rangle_{D_2(k+1)}$, i.e., $\langle Y_k v_{k+1}, v' \rangle_{D_2(k+1)} = \langle v_{k+1}, v' \rangle_{D_2(k+1)}$, $v_k \in V_{k+1}$, $v' \in V_k$. The following proposition exhibits the stability of the wavelet basis for $\widetilde{W}_k$ in the Sobolev space H^2.

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Proposition 3.10.

$$4^k||u_k||_{L_2} \simeq |u_k|_{H^2}, \quad u_k \in \widetilde{W}_k, \;\; k = 1, 2, \cdots .$$

Proof. We have

$$\begin{aligned} ||u_k||_{L_2} &\simeq ||u_k||_{D_2(k)} = ||(I - Y_{k-1})u_k||_{D_2(k)} \\ &\le ||(I - Q_{k-1})u_k||_{D_2(k)} \simeq ||(I - Q_{k-1})u_k||_{L_2} \lesssim 2^{-2k}|u_k|_{H^2}. \end{aligned}$$

In the last inequality, we use the approximation property (the Jackson inequality (3.17)) of the subspace V_k. Hence,

$$||u_k||_{L_2} \lesssim 4^{-k}|u_k|_{H^2}.$$

By the inverse inequality, we obtain

$$|u_k|_{H^2} \lesssim 4^k||u_k||_{L_2}.$$

Combining the previous two inequalities together completes the proof. □

3.4 Stability of the wavelets in Sobolev space H^2

In this section, we consider the stability of the wavelets in Sobolev space H^2. We go through several necessary lemmas before we finally reach the result. The first lemma is a type of strengthened Cauchy-Schwarz inequality, which is of particular importance in ensuring that the wavelets form a Bessel sequence in H^2. Similar lemmas appeared in [2, 47]. In [47], a particular short proof is presented on the basis of the advanced knowledge of spaces interpolation. For self-contain purpose, we shall give here a quite basic proof based on some cancellation properties employed in [2].

Lemma 3.2. (Strengthened Cauchy-Schwarz inequality)
(3.25)

$$\langle \triangle u_k, \triangle v_l \rangle_{L_2} \le C 2^{-(l-k)} (2^{2k}||u_k||_{L_2})(2^{2l}||v_l||_{L_2}), \quad u_k \in V_k, \; v_l \in V_l, \; k \le l,$$

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where C is a constant independent of k, l. Here, $\triangle$ is the Laplacian operator defined by $\triangle u := u_{xx} + u_{yy}$.

Proof. Without loss of generality, we take $k = 0$. We consider the case $l > k+2$, and (3.25) holds true for the case $l \le k+2$ automatically by Cauchy-Schwarz inequality and inverse inequality.

We concern with L_2 inner product of $\triangle u_0$ and $\triangle v_l$ on a given triangle $T \in \mathcal{T}_0$, as shown in Figure 3.1. By the rule of the refinement procedure, the sub-triangles in the mesh with level number l are all similar to T. Moreover, v_l can be written in terms of basis functions in V_l sitting on the vertices in $\mathcal{N}_l \cap \overline{T}$.

Note that $\triangle u_0$ are piecewise constants on 6 sub-triangles of T, and $\triangle v_l$ are piecewise constants on all sub-triangles of $T' \in \mathcal{T}_l$. We focus on the basis functions in level l with their support in one of 6 sub-triangles of T in level 0, say $\triangle OP_3q_2$. The shaded area in Figure 3.4 is defined in such a way that every basis function $v_l \in V_l$ centered in the shaded area has the support totally within $\triangle OP_3q_2$. We choose the largest possible triangle in $\triangle OP_3q_2$ as the shaded area. For any basis function $v_l \in V_l$ sitting on the vertex within the shaded area in Figure 3.4, because its first derivatives vanish on the boundary of $\triangle OP_3q_2$, it is orthogonal to $\triangle u_0$ because

$$\int_{\triangle OP_3q_2} (\triangle u_0)(\nabla\cdot\nabla v_l) = (\triangle u_0)\int_{\triangle OP_3q_2} (\nabla\cdot\nabla v_l) = (\triangle u_0)\int_{\partial\triangle OP_3q_2} \nabla v_l\cdot\vec{n}ds = 0,$$

where $\partial\triangle OP_3q_2$ is the boundary of the triangle $\triangle OP_3q_2$.

Roughly speaking, the distance between the shaded area and the boundary $\partial\triangle OP_3q_2$ is about the same size of the support of basis function in V_l, i.e., 2^{-l} multiple of the length of the edge of $\triangle OP_3q_2$. 2^{-l} comes from the uniform refinement procedure. This leads to the estimate

$$\frac{\text{area}(\triangle OP_3q_2) - \text{shaded area}}{\text{area}(\triangle OP_3q_2)} \le C2^{-l}, \quad (k=0). \tag{3.26}$$

Note that the basis functions centered on the boundary of $\triangle OP_3q_2$ has the support outside the shaded area and $\triangle u_0$ is a constant on $\triangle OP_3q_2$. Then, for

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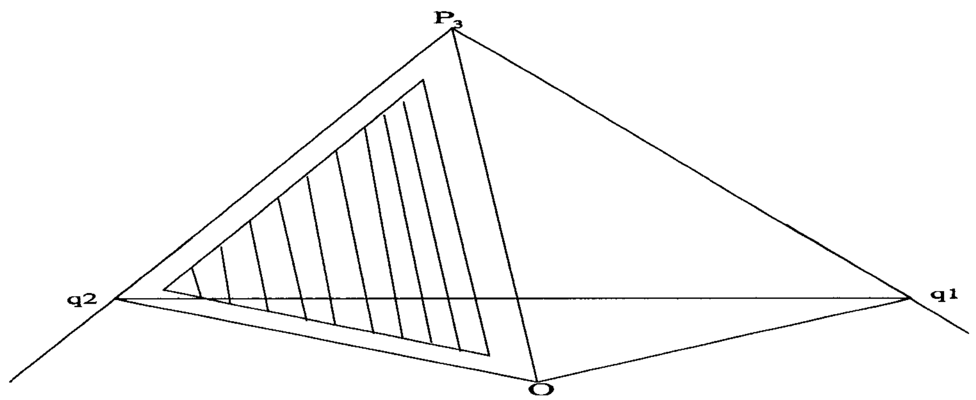

Figure 3.4: Supports of two PS elements in two different levels

a function $v_l \in V_l$, we have

$$\begin{aligned}\int_{\triangle OP_3q_2}(\Delta u_0)(\Delta v_l) &= \int_{\triangle OP_3q_2\backslash \text{ShadedArea}}(\Delta u_0)(\Delta v_l)\\ &\le ||\Delta u_0||_{L_2(\triangle OP_3q_2\backslash\text{ShadedArea})}||\Delta v_l||_{L_2(\triangle OP_3q_2\backslash\text{ShadedArea})}\\ &\le ||\Delta u_0||_{L_2(\triangle OP_3q_2\backslash\text{ShadedArea})}||\Delta v_l||_{L_2(\triangle OP_3q_2)}\\ &\le C2^{-l}||\Delta u_k||_{L_2(\triangle OP_3q_2)}||\Delta v_l||_{L_2(\triangle OP_3q_2)},\end{aligned}$$

where 2^{-l} comes from (3.26).

Inequality (3.25) on the sub-triangle $\triangle OP_3q_2$ can be obtained by the use of inverse inequality. After we sum the estimate on each sub-triangle up, the result follows. □

Lemma 3.2, together with Corollary 3.1 shows that the wavelet system $\{2^{-k}\psi_{q,j}^k\}_{k,q,j}$ is a Bessel sequence in H^2.

Theorem 3.1. *Let* $u = \sum_{k=0}^J u_k$ *with* $u_k = \sum_{q\in\mathcal{N}_k\backslash\mathcal{N}_{k-1}}\sum_{j=0,x,y}\beta_{q,j}^k(2^{-k}\psi_{q,j}^k) \in \widetilde{W}_k$, $k = 0, 1, ..., J$. *Then*

(3.27) $$||u||_{H^2}^2 \lesssim \sum_{k=0}^J \sum_{q\in\mathcal{N}_k\backslash\mathcal{N}_{k-1}} \sum_{j=0,x,y}(\beta_{q,j}^k)^2.$$

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Proof. It follows that

$$(3.28) \qquad |u|_{H^2}^2 \simeq \sum_{l,m=0}^{J} \langle \Delta u_l, \Delta u_k \rangle_{L_2}$$
$$\lesssim \sum_{l,m=0}^{J} 2^{-|l-m|}(4^l ||u_l||_{L_2})(4^m ||u_m||_{L_2}) \text{ (Lemma 3.2)}$$
$$\lesssim \sum_{l=0}^{J} 4^{2l} ||u_l||_{L_2}^2.$$

Moreover,

$$||u||_{L_2}^2 = ||\sum_{l=0}^{J} u_l||^2 \leq \sum_{l,m=0}^{J} 4^{-(l+m)}(4^l ||u_l||_{L_2}^2)(4^m ||u_k||_{L_2}^2)$$
$$\lesssim \sum_{l=0}^{J} 4^{2l} ||u_l||_{L_2}^2.$$

Combining above two inequalities, we have

$$||u||_{H^2}^2 \lesssim \sum_{k=0}^{J} 4^{2k} ||u_k||_{L_2}^2 \lesssim \sum_{k=0}^{J} 4^{2k} (2^{-2k} \sum_{q \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}} \sum_{j=0,x,y} (2^{-k} \beta_{q,j}^k)^2),$$

where we used Corollary 3.1 in the last inequality.

This completes the proof.

□

Bessel sequence property of the wavelet basis guarantees the upper bound for $||u||_{H^2}$ in (3.27). If we can prove the lower bound for $||u||_{H^2}$, then the wavelet basis form a Riesz basis in H^2. This is the task of the remaining section.

Let the projection operator Z_k^l be defined by,

$$Z_k^l := (1 - Y_{k-1}) Y_k \cdots Y_{l-1}, \quad k = 1, 2, ..., l-1; \quad Z_0^l := Y_0 Y_1 \cdots Y_{l-1}.$$

Z_k^l projects a function in V_l into $\widetilde{W}_k (k < l)$ and the estimate of this operator is the key in verifying the stability of the wavelet system in Sobolev space.

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Lemma 3.3.

$$||Z_k^l||_{L_2\to L_2} \le C2^{\lambda(l-k)}, \quad k<l,$$

where $\lambda<2$*, and* C *is a constant independent of* l,k.

Suppose Lemma 3.3 holds true for a moment, then we claim

Theorem 3.2. $\{2^{-k}\psi_{q,j}^k\}_{k,q,j}$ *is a Riesz basis for* H^2.

Proof. Let $u=\sum_{k=0}^J u_k$ with $u_k=\sum_{q\in\mathcal{N}_k\setminus\mathcal{N}_{k-1}}\sum_{j=0,x,y}\beta_{q,j}^k(2^{-k}\psi_{q,j}^k)\in\widetilde{W}_k$, $k=0,...,J$. First, we verify that

$$||u||_{H^2}^2 \gtrsim \sum_{k=0}^J 4^{2k}||u_k||_{L_2}^2.$$

For simplicity, by Z_k we denote Z_k^J in the proof. Note that $u_k=Z_ku,\ k=0,...,J$. Let $s=2$. Then we have

$$\begin{aligned}
\text{RHS} &\equiv \sum_{k=0}^J 4^{ks}||Z_ku||_{L_2}^2 \\
&= \sum_{k=0}^J 4^{ks}\langle Z_ku, Z_ku\rangle_{L_2} = \sum_{k=0}^J 4^{ks}\langle Z_k\sum_{l=0}^J P_lu, Z_k\sum_{m=0}^J P_mu\rangle_{L_2} \\
&= \sum_{l,m=0}^J\sum_{k=0}^{\min(l,m)} 4^{ks}\langle Z_kP_lu, Z_kP_mu\rangle_{L_2} \\
&\le \sum_{l,m=0}^J\sum_{k=0}^{\min(l,m)} 4^{ks}||Z_kP_lu||_{L_2}||Z_kP_mu||_{L_2} \quad (||Z_k^l||_{L_2\to L_2}\lesssim 2^{\lambda(l-k)},\ k<l) \\
&\lesssim \sum_{l,m=0}^J\sum_{k=0}^{\min(l,m)} 4^{ks}||P_lu||_{L_2}2^{\lambda(l-k)}||P_mu||_{L_2}2^{\lambda(m-k)} \\
&= \sum_{l,m=0}^J\sum_{k=0}^{\min(l,m)} 2^{-(l+m-2k)(s-\lambda)}(2^{ls}||P_lu||_{L_2})(2^{ms}||P_mu||_{L_2}) \\
&\lesssim \sum_{l,m=0}^J 2^{-(l+m-2\min(l,m))(s-\lambda)}(2^{ls}||P_lu||_{L_2})(2^{ms}||P_mu||_{L_2}) \\
&\lesssim \sum_{l=0}^J (2^{ls}||P_lu||_{L_2})^2.
\end{aligned}$$

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Therefore, we obtain

$$\text{RHS} \lesssim ||u||_{H^2}^2.$$

By Corollary 3.1,

$$||u||_{H^2}^2 \gtrsim \text{RHS} \simeq \sum_{k=0}^{J} \sum_{q\in\mathcal{N}_k\setminus\mathcal{N}_{k-1}} \sum_{j=0,x,y} (\beta_{q,j}^k)^2.$$

Note that $\{2^{-k}\psi_{q,j}^k\}_{k,q,j}$ is a Bessel sequence in H^2 by Theorem 3.1 and $\cup_{k=0}^{\infty} V_k$ is dense in H^2. It follows that $\{2^{-k}\psi_{q,j}^k\}_{k,q,j}$ is a Riesz basis in H^2. This completes the proof.

□

It remains to prove Lemma 3.3.

Proof.

$$\begin{aligned} ||Z_k^l||_{L_2\leftarrow L_2} &:= \sup_{u_l\in V_l} \frac{||Z_k^l u_l||_{L_2}}{||u_l||_{L_2}} \\ &= \sup_{u_l\in V_l} \frac{||Z_k^l u_l||_{L_2}}{||v_k||_{D_2(k)}} \frac{||v_k||_{D_2(k)}}{||v_{k+1}||_{D_2(k+1)}} \cdots \frac{||v_{l-1}||_{D_2(l-1)}}{||v_l||_{D_2(l)}} \frac{||v_l||_{D_2(l)}}{||v_l||_{L_2}}, \end{aligned}$$

where $v_j := Y_j^l u_l$, $j < l$, $v_l := u_l$ and $Y_j^l := Y_j \cdots Y_{l-1}$.

By Proposition 3.8, we have for $k = 1, 2, ..., l-1$,

$$\frac{||Z_k^l u_l||_{L_2}}{||v_k||_{D_2(k)}} \simeq \frac{||(1-Y_{k-1})v_k||_{D_2(k)}}{||v_k||_{D_2(k)}} \lesssim 1, \quad \frac{||v_l||_{D_2(l)}}{||v_l||_{L_2}} \simeq 1.$$

For $k = 0$, $Z_0^l u_l = v_0$, and thus

$$\frac{||Z_0^l u_l||_{L_2}}{||v_0||_{D_2(0)}} \simeq 1.$$

Now we consider $\frac{||v_j||_{D_2(j)}}{||v_{j+1}||_{D_2(j+1)}}$, $k \le j < l$. Note that $v_j = Y_j v_{j+1}$. Then

$$\frac{||v_j||_{D_2(j)}}{||v_{j+1}||_{D_2(j+1)}} = \frac{||v_j||_{D_2(j)}}{||v_j||_{D_2(j+1)}} \frac{||Y_j v_{j+1}||_{D_2(j+1)}}{||v_{j+1}||_{D_2(j+1)}} \le \frac{||v_j||_{D_2(j)}}{||v_j||_{D_2(j+1)}}.$$

$\frac{||Y_j v_{j+1}||_{D_2(j+1)}}{||v_{j+1}||_{D_2(j+1)}} \le 1$ because Y_j is the orthogonal projection from V_{j+1} to V_j with respect to the discrete L_2 inner product $\langle\cdot,\cdot\rangle_{D_2(j+1)}$.

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If we assume $\frac{||v_j||_{D_2(j)}}{||v_j||_{D_2(j+1)}} \leq 2^\lambda$ $(\lambda < 2)$, $k \leq j < l$ for a moment and postpone its lengthy and tedious proof to the next lemma, then we have $\frac{||v_j||_{D_2(j)}}{||v_{j+1}||_{D_2(j+1)}} \leq 2^\lambda$. Hence,

$$||Z_k^l||_{L_2 \to L_2} \leq C2^{\lambda(l-k)}.$$

This completes the proof. □

As pointed out in the previous lemma, the key step to verify $||Z_k^l||_{L_2 \to L_2} \leq C2^{\lambda(l-k)}$ is that the ratio $||f||_{D_2(j)}/||f||_{D_2(j+1)} \leq 2^\lambda$ for any $f \in V_j$.

Lemma 3.4. *There exists a positive number* $\lambda < 2$ *that*

$$||f||^2_{D_2(k)}/||f||^2_{D_2(k+1)} \leq 2^{2\lambda}, \quad f \in V_k.$$

Proof. Without loss of generality, we take $k = 0$ and verify that for some $\lambda < 2$,

$$||f|_T||^2_{D_2(0)}/||f|_T||^2_{D_2(1)} \leq 2^{2\lambda}$$

is true for any $f \in V_0$ on any triangle T in $\mathcal{T}_0$.

For a function $f(x) \in V_0$ on T as shown in Figure 3.1, let

$$f(x) = \sum_{i=1,2,3,j=0,1,2} \alpha_{P_i,j}\phi_{P_i,j}(x), \quad x \in T. \tag{3.29}$$

By the unique affine map $\mathcal{A} : \tilde{T} \to T$, we map $f(x)$ to $\tilde{f}(\tilde{x})$ by letting $\tilde{f}(\tilde{x}) := f(\mathcal{A}(\tilde{x}))$, where $\tilde{T}$ is a standard equilateral triangle. By (3.10) ($\alpha_{P_i,j} = \widetilde{\alpha}_{\tilde{P}_i,j}$), we have

$$\tilde{f}(\tilde{x}) = \sum_{i=1,2,3,j=0,1,2} \alpha_{P_i,j}\phi_{\tilde{P}_i,j}(\tilde{x}), \quad \tilde{x} \in \tilde{T}. \tag{3.30}$$

This indicates that we may compute the discrete L_2 norm of a given function after it is affine mapped onto the standard triangle. Therefore, without loss of generality, we assume that f is a function defined on the standard equilateral triangle T with unit edge lengths in the following proof.

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Let $f(x)$ be the PS element on T in the form of (3.29). To compute $||f|_T||_{D_2(0)}$, we simply have

$$||f|_T||^2_{D_2(0)} = \text{vol}(T) \sum_{i=1,2,3,j=0,1,2} \alpha^2_{P_i,j}.$$

It is clear that $||f|_T||^2_{D_2(0)}$ is the sum of squares of 9 variables $\{\alpha_{P_i,j}\}_{i=1,2,3,j=0,1,2}$. On the other hand, these variables uniquely determine the PS element on T. To compute $||f|_T||_{D_2(1)}$ of the discrete L_2 norm in higher level, we need values and derivatives of f at the midpoints of three edges of T. In other words, we shall write $f(x)$ in terms of higher level PS elements with local basis on each sub-triangle $T_1 := \triangle P_1q3q2$, $T_2 := \triangle q2q3q1$, $T_3 := \triangle q3P_2q1$, $T_4 := \triangle q2q1P_3$, respectively. Since T is symmetric, the data (values and derivatives) of two basis functions $\{\phi_{P_1,0}, \phi_{P_1,1}\}$ at q_1, q_2, and q_3 are sufficient for the computation. Data of other 7 basis functions at q_1, q_2, and q_3 can be obtained similarly due to the symmetry of the basis functions.

We concern with the values and derivatives of $f(x)$ at the midpoint $q3$. Note that $\phi_{P_3,j}$, $j = 0, 1, 2$ has zeros values and derivatives at any point on the line P_1P_2. Thus, $\alpha_{P_3,j}$, $j = 0, 1, 2$ have no contribution to the value or derivatives at $q3$. Only $\{\alpha_{P,j}, \ P = P_1, P_2, j = 0, 1, 2\}$ have contribution to $q3$. Such contribution can be obtained by studying the values and derivatives of $\{\phi_{P_1,0}, \phi_{P_1,1}\}$ at $q3, q2$. First, we derive the value and derivatives of $\phi_{P_1,0}$ at $q3$. $\phi_{P_1,0}$ is a piecewise quadratic polynomials on T, and its expression is given explicitly in (3.12). By simple computing, we have

$$\begin{aligned} \phi_{P_1,0}(q3) &= 1/2, \\ \frac{\partial \phi_{P_1,0}}{\partial d_2(T_1, q3)} &= 2 \\ \frac{\partial \phi_{P_1,0}}{\partial d_1(T_1, q3)} &= 1 \\ \frac{\partial \phi_{P_1,0}}{\partial d_2(T_3, q3)} &= -1 \\ \frac{\partial \phi_{P_1,0}}{\partial d_1(T_3, q3)} &= -2, \end{aligned} \tag{3.31}$$

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where $d_j(T, P)$ type definition of direction is given in Section 3.2.1. Since $\phi_{P_1,0}$ is symmetric about the line $P_1 q1$, its value and derivatives at $q2$ can be obtained by (3.31) (see Figure 3.5). Consequently, contribution of $\alpha_{P_1,0}$ to $q3, q2$ can be computed from (3.31). Because of the symmetry, contribution of $\alpha_{P_2,0}$ to $q3$ can be computed from (3.31), too.

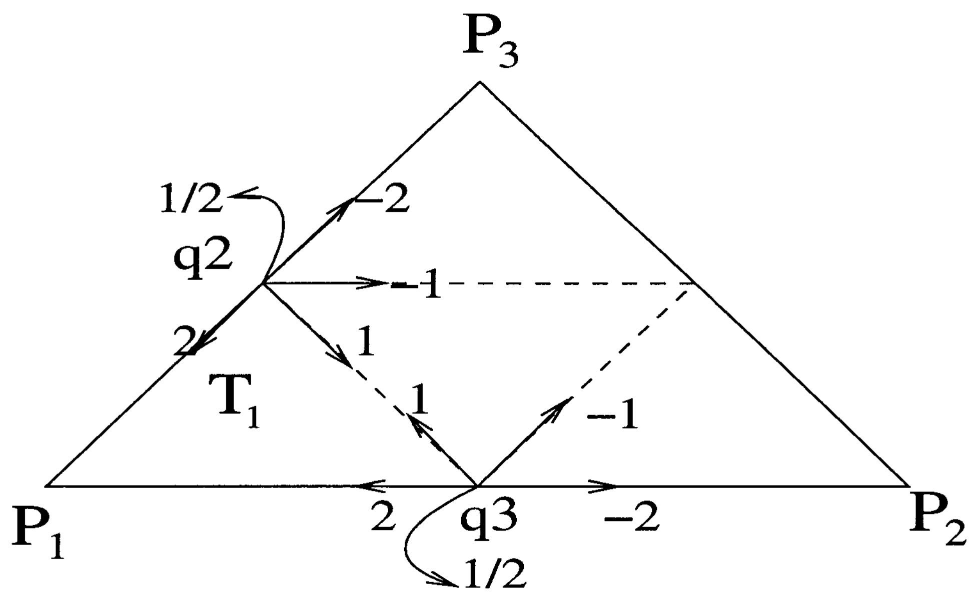

Figure 3.5: Value and derivatives of $\phi_{P_1,0}$ on T

Second, we study the contribution of $\phi_{P_1,1}$ at $q3, q2$. By the expression of $\phi_{P_1,1}$ as in (3.11), we have (see Figure 3.6)

$$
\begin{aligned}
\phi_{P_1,1}(q3) &= 1/8, \\
\frac{\partial \phi_{P_1,1}}{\partial d_2(T_1, q3)} &= 1/2 \\
\frac{\partial \phi_{P_1,1}}{\partial d_1(T_1, q3)} &= 0 \\
\frac{\partial \phi_{P_1,0}}{\partial d_2(T_3, q3)} &= -1/2 \\
\frac{\partial \phi_{P_1,1}}{\partial d_1(T_3, q3)} &= -1/2.
\end{aligned}
\tag{3.32}
$$

With (3.31-3.32) in hand, we compute the value and the derivatives of $f(x)$ at $q3$ as follows,

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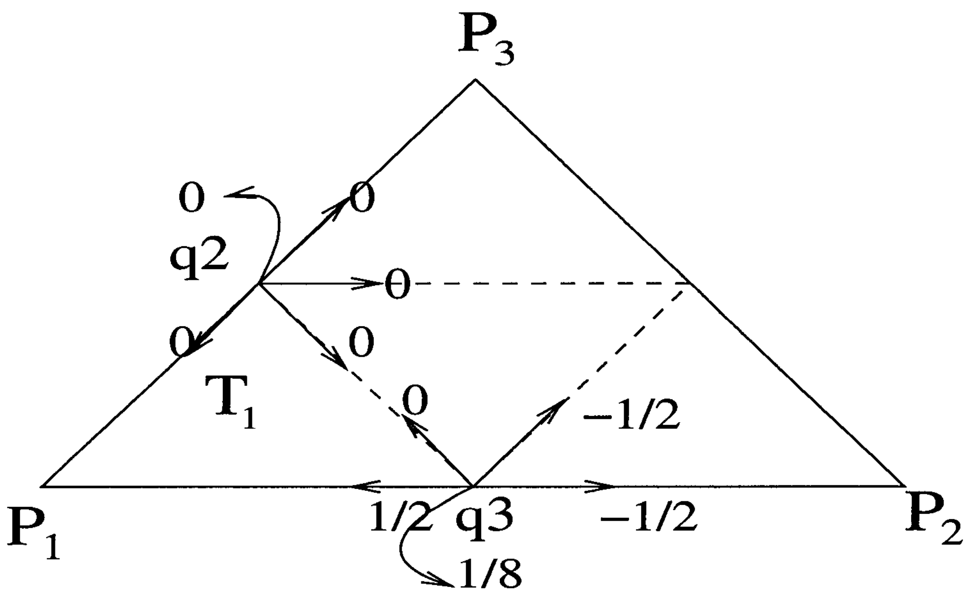

Figure 3.6: Value and derivatives of $\phi_{P_1,1}$ on T

$$\begin{aligned}
f(q3) &= \sum_{j=0,1,2,P=P_1,P_2} \alpha_{P,j}\phi_{P,j}(q3) \\
\frac{\partial f}{\partial d}(q3) &= \sum_{j=0,1,2,P=P_1,P_2} \alpha_{P,j}\frac{\partial \phi_{P,j}(q3)}{\partial d}(q3),
\end{aligned}$$

where d refers to one of four directions $\{d_2(T_1,q3),\ d_1(T_1,q3),\ d_2(T_3,q3),\ d_1(T_3,q3)\}$ at $q3$ shown in Figure 3.5 or Figure 3.6. Hence, the value and the derivatives of $f(x)$ at $q3$ is a linear combination of $\{\alpha_{P_1,0},\alpha_{P_1,1},\alpha_{P_1,2},\alpha_{P_2,0},\alpha_{P_2,1}\alpha_{P_2,2}\}$. We give the details in the following,

$$\begin{aligned}
f(q3) &= \frac{1}{2}(\alpha_{P_1,0}+\alpha_{P_2,0})+\frac{1}{8}(\alpha_{P_1,1}+\alpha_{P_2,2}), \\
\frac{\partial f}{\partial d_2(T_1,q3)}(q3) &= 2(\alpha_{P_1,0}-\alpha_{P_2,0})+\frac{1}{2}(\alpha_{P_1,1}-\alpha_{P_2,2}), \\
\frac{\partial f}{\partial d_1(T_1,q3)}(q3) &= (\alpha_{P_1,0}-\alpha_{P_2,0})-\frac{1}{2}\alpha_{P_2,2}+\frac{1}{2}(\alpha_{P_1,2}+\alpha_{P_2,1}), \\
\frac{\partial f}{\partial d_2(T_3,q3)}(q3) &= (\alpha_{P_2,0}-\alpha_{P_1,0})-\frac{1}{2}\alpha_{P_1,1}+\frac{1}{2}(\alpha_{P_1,2}+\alpha_{P_2,1}), \\
\frac{\partial f}{\partial d_1(T_3,q3)}(q3) &= 2(\alpha_{P_2,0}-\alpha_{P_1,0})+\frac{1}{2}(\alpha_{P_2,2}-\alpha_{P_1,1}).
\end{aligned}$$

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Using the symmetric property of T, we get other values and derivatives at $q2, q1$.

To compute $||f|_{T_1}||_{D_2(1)}$, we have

$$\alpha^1_{T_1,q,0} = f(q), \quad q = P_1, q2, q3,$$

and

$$\alpha^1_{T_1,q,j} = \frac{1}{2}\frac{\partial f}{\partial d_j(T_1,q)}(q), \quad q = P_1, q2, q3, \ j = 1,2,$$

where $\frac{1}{2}$ comes from the definition of $\frac{\partial \phi^1_{T_1,P,j}}{\partial d_j(T_1,P)} = 2, \quad j = 1,2$. It follows that

$$||f|_{T_1}||^2_{D_2(1)} = \frac{\mathrm{vol}(T)}{4} \sum_{P=P_1,q2,q3} \sum_{j=0,1,2} (\alpha^1_{T_1,P,j})^2.$$

It is easily seen that $||f|_{T_1}||^2_{D_2(1)}$ is a quadratic form of the variables $\{\alpha_{P,j}\}_{P=P_1,P_2,P_3,j=0,1,2}$. Let α be the column vector form of these variables. Then in a similar way, we may compute $||f|_{T_j}||^2_{D_2(1)}$, $j = 2,3,4$, which are all quadratic forms of α. Hence, we may write

$$||f|_T||^2_{D_2(1)} = \sum_{i=1}^{4} ||f|_{T_i}||^2_{D_2(1)} = \mathrm{vol}(T)\alpha^T D\alpha,$$

where D is a 9×9 symmetric positive definite matrix, and α^T is the transpose of the vector α. It follows that

$$1/\lambda_{D,\max} \le \frac{||f|_T||^2_{D_2(0)}}{||f|_T||^2_{D_2(1)}} = \frac{\alpha^T\alpha}{\alpha^T D\alpha} \le 1/\lambda_{D,\min},$$

where $\lambda_{D,\min}, \lambda_{D,\max}$ are respectively the minimal, maximal eigenvalues of the matrix D. From the computation, we have

$$\lambda_{D,\min} > 0.0679 > 4^{(-2)}.$$

Let $\lambda = \log_4(1/0.0679)$. Then we have

$$\frac{||f|_T||^2_{D_2(0)}}{||f|_T||^2_{D_2(1)}} < 2^{2\lambda}.$$

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It is clear that on an arbitrary triangle T, we have the same estimate. After we sum up the above inequality over all triangles in $\mathcal{T}_0$, we have

$$\frac{||f||^2_{D_2(0)}}{||f||^2_{D_2(1)}} < 2^{2\lambda}.$$

This completes the proof. □

3.5 Examples

In this section, we shall demonstrate the computation of basis functions, as well as wavelet functions, on the regular triangular mesh. The mesh is constructed by connecting lower left vertex with upper right vertex of each square in the uniform tensor product mesh. The support of each basis function contains 6 triangles, as shown in Figure 3.7. On each triangle, the basis function is a PS element which is a piecewise continuous quadratic function on its six pieces of sub-triangles. Therefore, we have to compute each PS element one by one, and put them together to obtain the basis functions.

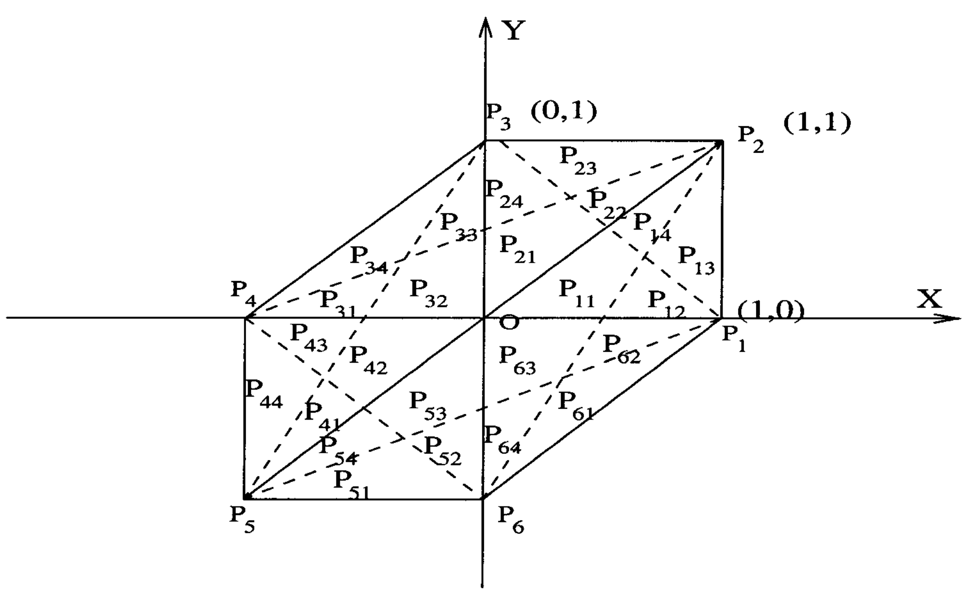

Figure 3.7: Supports of the basis functions $\phi^0_{O,j}$, $j = 0, x, y$

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For the basis function $\phi^0_{O,0}$, it satisfies the following conditions,

$$\begin{aligned} \phi^0_{O,0}(O) &= 1, \quad \phi^0_{O,0}(P) = 0,\ P \in \mathcal{N}e(O) \setminus \{O\} \\ \frac{\partial \phi^0_{O,0}}{\partial x}(P) &= 0, \quad \frac{\partial \phi^0_{O,0}}{\partial y}(P) = 0,\ P \in \mathcal{N}e(O), \end{aligned}$$

where $\mathcal{N}e(O) := \{O, P_1, P_2, ..., P_6\}$ is the set of vertices neighboring O plus O itself in Figure 3.7. $\phi^0_{O,x}, \phi^0_{O,y}$ satisfy the similar conditions at the vertex O in $\mathcal{N}e(O)$.

For one triangle in the support of $\phi^0_{O,0}$, say $\triangle OP_1P_2$, the PS element of the basis function $\phi^0_{O,0}$ satisfies the above conditions at three vertices O, P_1, P_2, too. By the properties of PS element, unique PS element is determined.

There are two ways to carry out the computation. The first method is based on the affine transformation. Because we have the basis functions on a standard equilateral triangle, we may compute the basis functions on an arbitrary triangle by finding the affine map between these two triangles. The interpolation data on the vertices, such as directional derivatives, shall be computed accordingly. The second method uses the properties of PS element and computes the quadratic functions on each sub-triangles directly with the given conditions on three vertices.

In the following, we give the basis functions for $\{\phi^0_{O,j}\}_{j=0,x,y}$(see Figure 3.7).

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$$\phi^{\mathbf{0}}_{\mathbf{O,0}}$$

$$
\begin{aligned}
P_{11} &= P_{21} = P_{32} = P_{42} = P_{53} = P_{63} = 1 - 2x^2 + 2xy - 2y^2;\\
P_{12} &= (1 - 2x^2 + 2xy - 2y^2) + (y - 2x + 1)^2;\\
P_{13} &= (1 - 2x^2 + 2xy - 2y^2) + (y - 2x + 1)^2 + (x + y - 1)^2;\\
P_{14} &= (1 - 2x^2 + 2xy - 2y^2) + (x + y - 1)^2;\\
P_{22} &= P_{14};\\
P_{23} &= (1 - 2x^2 + 2xy - 2y^2) + (x + y - 1)^2 + 4(y - x/2 - 1/2)^2;\\
P_{24} &= (1 - 2x^2 + 2xy - 2y^2) + 4(y - x/2 - 1/2)^2;\\
P_{31} &= (1 - 2x^2 + 2xy - 2y^2) + (y - 2x - 1)^2;\\
P_{33} &= P_{24};\\
P_{34} &= (1 - 2x^2 + 2xy - 2y^2) + (y - 2x - 1)^2 + 4(y - x/2 - 1/2)^2;\\
P_{41} &= (1 - 2x^2 + 2xy - 2y^2) + (x + y + 1)^2;\\
P_{43} &= P_{31};\\
P_{44} &= (1 - 2x^2 + 2xy - 2y^2) + (x + y + 1)^2 + (y - 2x - 1)^2;\\
P_{51} &= (1 - 2x^2 + 2xy - 2y^2) + (x + y + 1)^2 + 4(y - x/2 + 1/2)^2;\\
P_{52} &= (1 - 2x^2 + 2xy - 2y^2) + 4(y - x/2 + 1/2)^2;\\
P_{54} &= P_{41}\\
P_{61} &= (1 - 2x^2 + 2xy - 2y^2) + 4(y - x/2 + 1/2)^2 + (y - 2x + 1)^2;\\
P_{62} &= P_{12};\\
P_{64} &= P_{52};
\end{aligned}
$$

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$$\phi^0_{\mathbf{O,x}}$$

$$\begin{aligned}
P_{11} &= -\frac{3}{2}x^2 + xy - y^2 + x; \\
P_{12} &= P_{11} + \frac{1}{2}(y - 2x + 1)^2; \\
P_{13} &= P_{11} + \frac{1}{2}(y - 2x + 1)^2 + \frac{1}{2}(x + y - 1)^2; \\
P_{14} &= P_{11} + \frac{1}{2}(x + y - 1)^2; \\
P_{21} &= P_{11} + (x - y)^2; \\
P_{22} &= P_{21} + \frac{1}{2}(x + y - 1)^2; \\
P_{23} &= P_{22}; \\
P_{24} &= P_{21}; \\
P_{31} &= P_{21} + 2x^2 - \frac{1}{2}(y - 2x - 1)^2; \\
P_{32} &= P_{21} + 2x^2; \\
P_{33} &= P_{32}; \\
P_{34} &= P_{31}; \\
P_{41} &= (-\frac{3}{2}x^2 - xy + y^2 + x) - \frac{1}{2}(x + y + 1)^2; \\
P_{42} &= (-\frac{3}{2}x^2 - xy + y^2 + x); \\
P_{43} &= (-\frac{3}{2}x^2 - xy + y^2 + x) - \frac{1}{2}(y - 2x - 1)^2; \\
P_{44} &= (-\frac{3}{2}x^2 - xy + y^2 + x) - \frac{1}{2}(y - 2x - 1)^2 - \frac{1}{2}(x + y + 1)^2; \\
P_{51} &= (\frac{1}{2}x^2 + xy + x) - \frac{1}{2}(x + y + 1)^2; \\
P_{52} &= (\frac{1}{2}x^2 + xy + x); \\
P_{53} &= P_{52}; \\
P_{54} &= P_{51}; \\
P_{61} &= (-\frac{3}{2}x^2 + xy + x) + \frac{1}{2}(y - 2x + 1)^2; \\
P_{62} &= P_{61}; \\
P_{63} &= (-\frac{3}{2}x^2 + xy + x); \\
P_{64} &= P_{63};
\end{aligned}$$

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and

$$
\begin{aligned}
& \phi^0_{O,y} \\
P_{11} &= -xy - \frac{1}{2}y^2 + y; \\
P_{12} &= P_{11}; \\
P_{13} &= (-xy - \frac{1}{2}y^2 + y) + \frac{1}{2}(x + y - 1)^2; \\
P_{14} &= P_{13}; \\
P_{21} &= -x^2 + xy - \frac{3}{2}y^2 + y; \\
P_{22} &= (-x^2 + xy - \frac{3}{2}y^2 + y) + \frac{1}{2}(x + y - 1)^2; \\
P_{23} &= (-x^2 + xy - \frac{3}{2}y^2 + y) + \frac{1}{2}(x + y - 1)^2 + 2(y - x/2 - 1/2)^2; \\
P_{24} &= (-x^2 + xy - \frac{3}{2}y^2 + y) + 2(y - x/2 - 1/2)^2; \\
P_{31} &= xy - \frac{3}{2}y^2 + y; \\
P_{32} &= P_{31}; \\
P_{33} &= (xy - \frac{3}{2}y^2 + y) + 2(y - x/2 - 1/2)^2; \\
P_{34} &= P_{33}; \\
P_{41} &= (xy + \frac{1}{2}y^2 + y) - \frac{1}{2}(x + y + 1)^2; \\
P_{42} &= (xy + \frac{1}{2}y^2 + y); \\
P_{43} &= P_{42}; \\
P_{44} &= P_{41}; \\
P_{51} &= (x^2 - xy + \frac{3}{2}y^2 + y) - \frac{1}{2}(x + y + 1)^2 - 2(y - x/2 + 1/2)^2; \\
P_{52} &= (x^2 - xy + \frac{3}{2}y^2 + y) - 2(y - x/2 + 1/2)^2; \\
P_{53} &= (x^2 - xy + \frac{3}{2}y^2 + y); \\
P_{54} &= (x^2 - xy + \frac{3}{2}y^2 + y) - \frac{1}{2}(x + y + 1)^2; \\
P_{61} &= (-xy + \frac{3}{2}y^2 + y) - 2(y - x/2 + 1/2)^2; \\
P_{62} &= (-xy + \frac{3}{2}y^2 + y); \\
P_{63} &= P_{62}; \\
P_{64} &= P_{61}.
\end{aligned}
$$

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To compute the wavelets, we shall use the discrete L_2 inner product defined by (3.16) to determine the coefficients in (3.19). Let's recall the definition of the wavelet in (3.22)

$$\psi^1_{q,j'} = \beta^1_{q,j'}\phi^1_{q,j'}(x) + \sum_{P\in\mathcal{N}_q, j=0,x,y} \beta^1_{q,j',P,j}\phi^1_{P,j}(x),\ j'=0,x,y, \tag{3.33}$$

where vertex q is the mid-point of two vertices in $\mathcal{N}_q$ in level 0 mesh. Recall that for a regular triangular mesh, level 0 mesh has the mesh size 1, and after the refinement, the level 0 mesh becomes level 1 mesh with the mesh size $h_1 = 1/2$.

$\psi^1_{q,j'}$ is required to be orthogonal to $\{\phi^0_{P,j}\}_{\{P\in\mathcal{N}_q, j=0,x,y\}}$ with respect to the discrete inner product $\langle\cdot,\cdot\rangle_{D_2(1)}$ for V_1. To find the suitable coefficients in (3.33) for wavelets, we need the following inner products,

$$\langle\phi^1_{P',j'}, \phi^0_{P,j}\rangle_{D_2(1)}, \quad P' \in \{q, \mathcal{N}_q\}, P \in \mathcal{N}_q,\ j', j = 0, x, y.$$

Let's recall the definition of the discrete L_2 inner product $\langle\cdot,\cdot\rangle_{D_2(1)}$ in level 1 before we carry out the computation.

$$\langle f, g\rangle_{D_2(1)} := h_1^2 \sum_{T\in\mathcal{T}_1} \sum_{P\in\mathcal{N}(T), j=0,1,2} \alpha_{T,P,j}\beta_{T,P,j}, \tag{3.34}$$

where

$$f(x) = \sum_{T\in\mathcal{T}_1} \sum_{P\in\mathcal{N}(T), j=0,1,2} \alpha_{T,P,j}\phi^1_{T,P,j}(x)$$

and

$$g(x) = \sum_{T\in\mathcal{T}_1} \sum_{P\in\mathcal{N}(T), j=0,1,2} \beta_{T,P,j}\phi^1_{T,P,j}(x).$$

Here, $\mathcal{T}_1$ is the triangulation of level 1, and $\mathcal{N}(T)$ is the set of three vertices of the triangle T. Therefore, we shall write the basis functions in (3.33) and $\{\phi^0_{P,j}\}_{P\in\mathcal{N}_q, j=0,x,y}$ in terms of the local basis functions in V_1, and use their coefficients to compute the discrete inner product $\langle\cdot,\cdot\rangle_{D_2(1)}$.

In Figure 3.8, we shall compute 9 wavelets sitting on $q1, q2, q3$, and their dilations and shifts form the wavelet basis. First, we give the explicit forms of nine wavelets in the following,

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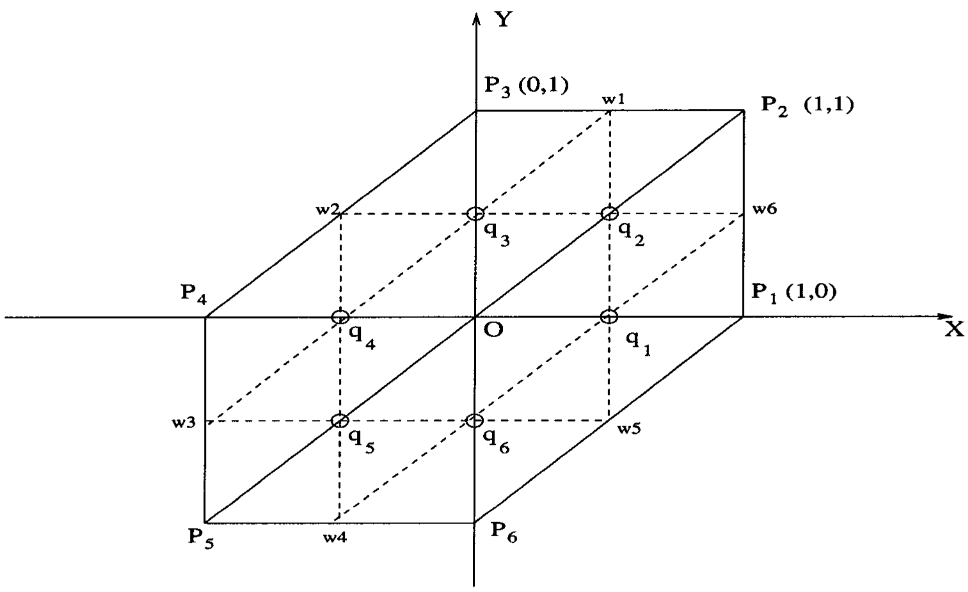

Figure 3.8: Computation of the wavelets

$$
\begin{aligned}
\psi^1_{q3,0} &= \frac{-1}{8}\left(-8\phi^1_{q3,0} + 4\phi^1_{O,0} - \phi^1_{O,x} + 2\phi^1_{O,y} + 4\phi^1_{P_3,0} + \phi^1_{P_3,x} - 2\phi^1_{P_3,y}\right),\\
\psi^1_{q3,x} &= \frac{-1}{2}\left(-2\phi^1_{q3,x} + \phi^1_{O,x} + \phi^1_{P_3,x}\right),\\
\psi^1_{q3,y} &= \frac{1}{2}\left(2\phi^1_{q3,y} + 2\phi^1_{O,0} - \phi^1_{O,x} + \phi^1_{O,y} - 2\phi^1_{P_3,0} - \phi^1_{P_3,x} + \phi^1_{P_3,y}\right),\\
\psi^1_{q2,0} &= \frac{-1}{8}\left(-8\phi^1_{q2,0} + 4\phi^1_{O,0} + \phi^1_{O,x} + \phi^1_{O,y} + 4\phi^1_{P_2,0} - \phi^1_{P_2,x} - \phi^1_{P_2,y}\right),\\
\psi^1_{q2,x} &= \frac{1}{2}\left(2\phi^1_{q2,x} + 2\phi^1_{O,0} + \phi^1_{O,y} - 2\phi^1_{P_2,0} + \phi^1_{P_2,y}\right),\\
\psi^1_{q2,y} &= \frac{1}{2}\left(2\phi^1_{q2,y} + 2\phi^1_{O,0} + \phi^1_{O,x} - 2\phi^1_{P_2,0} + \phi^1_{P_2,x}\right),\\
\psi^1_{q1,0} &= \frac{-1}{8}\left(-8\phi^1_{q1,0} + 4\phi^1_{O,0} + 2\phi^1_{O,x} - \phi^1_{O,y} + 4\phi^1_{P_1,0} - 2\phi^1_{P_1,x} + \phi^1_{P_1,y}\right),\\
\psi^1_{q1,x} &= \frac{-1}{2}\left(-2\phi^1_{q1,x} - 2\phi^1_{O,0} - \phi^1_{O,x} + \phi^1_{O,y} + 2\phi^1_{P_1,0} - \phi^1_{P_1,x} + \phi^1_{P_1,y}\right),\\
\psi^1_{q1,y} &= \frac{-1}{2}\left(-2\phi^1_{q1,y} + \phi^1_{O,y} + \phi^1_{P_1,y}\right).
\end{aligned}
$$

Next, we illustrate how to obtain the wavelets by using an example for computing $\psi^1_{q3,x}$ in Figure 3.8.

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Let

$$\psi^1_{q3,x}(x) = \beta^1_{q3,x}\phi^1_{q3,x} + \sum_{P\in\{O,P_3\},j=0,x,y} \beta^1_{P,j}\phi^1_{P,j}(x).$$

Then $\psi^1_{q3,x}$ shall be orthogonal to six basis functions in level 0, i.e.,

$$V := \{\phi^0_{O,0}, \phi^0_{O,x}, \phi^0_{O,y}, \phi^0_{P_3,0}, \phi^0_{P_3,x}, \phi^0_{P_3,y}\}.$$

Let $V^1 := \{\phi^1_{q3,x}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_3,0}, \phi^1_{P_3,x}, \phi^1_{P_3,y}\}$, then the orthogonality between $\psi^1_{q3,x}$ and V can be written in the matrix form

$$\left(\langle V^T, V^1\rangle_{D_2(1)}\right)\beta^T = 0,$$

Where vector $\beta := \{\beta^1_{q3,x}, \beta^1_{O,0}, \beta^1_{O,x}, \beta^1_{O,y}, \beta^1_{P_3,0}, \beta^1_{P_3,x}, \beta^1_{P_3,y}\}$.

$\left(\langle V^T, V^1\rangle_{D_2(1)}\right)$ is a 6 by 7 matrix with (i, j) element defined by $< f, g >_{D_2(1)}$, where f, g are i-th and j-th elements of the vectors V, V^1, respectively.

By the definition of discrete inner product, we shall write all involved functions in terms of local basis functions to obtain the matrix $\left(\langle V^T, V^1\rangle_{D_2(1)}\right)$. We shall give an example to compute $\langle\phi^1_{q3,x}, \phi^0_{O,0}\rangle_{D_2(1)}$ to illustrate how to get the required discrete inner products.

First, the support of $\phi^1_{q3,x}$ is composed of six small triangles around the vertex $q3$, such as $\triangle q3P_3w1$. $\phi^0_{O,0}$'s support are six triangles around O with $P_1, P_2, ..., P_6$ as vertices. It is clear that the overlap of the supports of two functions is the support of $\psi^1_{q3,x}$, i.e., six small triangles around vertex $q3$.

Second, we shall represent two functions $\phi^1_{q3,x}, \phi^0_{O,0}$ in terms of local basis functions. For the derivatives, we focus on the six directional derivatives in six directions, i.e., $0, \pi/4, \pi/2$ and their opposite directions. It's easy to find that

$$\frac{\partial\phi^1_{q3,x}}{\partial x}(q3) = 1/h_1 = 2, \ \frac{\partial\phi^1_{q3,x}}{\partial y}(q3) = 0,$$

and

$$\frac{\partial\phi^1_{q3,x}}{\partial d_{\pi/4}} = \frac{\partial\phi^1_{q3,x}}{\partial x}(q3)/\sqrt{2} = 2/\sqrt{2},$$

where $d_{\pi/4}$ denotes the direction with an anti-clockwise angle $\pi/4$ and $h_1 = 1/2$. Note that $\phi^1_{q3,x}$ has all zeros data for its values and derivatives at vertices

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other than $q3$ and

$$\frac{\partial \phi^1_{\triangle q2q3w1,q3,2}}{\partial d_{\triangle q2q3w1,q3,2}} = 1/|d_{\triangle q2q3w1,q3,2}| = 1/|q3w1| = 2/\sqrt{2}.$$

Thus we have the local representation for $\phi^1_{q3,x}$ on its support. We list the representation for $\phi^1_{q3,x}$ on $\triangle q2q3w1$ as follows,

$$\phi^1_{q3,x}|_{\triangle q2q3w1} = \phi^1_{\triangle q2q3w1,q3,1} + \phi^1_{\triangle q2q3w1,q3,2}. \tag{3.35}$$

Next, we shall compute the local representation for $\phi^0_{O,0}$. Note that $\phi^1_{q3,x}$ has all zeros data for its values and derivatives at all vertices other than $q3$. We thus only interested in the directional derivatives of $\phi^0_{O,0}$ at $q3$ in six directions.

Since we have the explicit expression of $\phi^0_{O,0}$, we may calculate the value and directional derivatives of $\phi^0_{O,0}$ at $q3$ by

$$\phi^0_{O,0}(q3) = 1/2,\ \frac{\partial \phi^0_{O,0}}{\partial x}(q3) = 1,\ \frac{\partial \phi^0_{O,0}}{\partial y}(q3) = -2,$$

and

$$\frac{\partial \phi^0_{O,0}}{\partial d_{\pi/4}} = (1-2)/\sqrt{2} = -1/\sqrt{2}.$$

Therefore, we write the local representation of $\phi^0_{O,0}$ on $\triangle q2q3w1$ by

$$\phi^0_{O,0}|_{\triangle q2q3w1} = \frac{1}{2}\phi^1_{\triangle q2q3w1,q3,0} + \frac{1}{2}\phi^1_{\triangle q2q3w1,q3,1} - \frac{1}{2}\phi^1_{\triangle q2q3w1,q3,2} + I, \tag{3.36}$$

where I includes the basis functions centered not at $q3$.

By (3.35) and (3.36), we get the discrete inner product of $\phi^1_{q3,x}$ and $\phi^0_{O,0}$ on $\triangle q2q3w1$ as follows

$$\langle \phi^1_{q3,x}, \phi^0_{O,0}\rangle_{D_2(1)}|_{\triangle q2q3w1} = h_1^2(0 \times \frac{1}{2} + 1 \times \frac{1}{2} + 1 \times \frac{-1}{2}) = 0.$$

Likewise, discrete inner products on other 5 sub-triangles in the support of $\phi^1_{q3,x}$ can be done. In fact, we only need the information of directional derivatives of $\phi^1_{q3,x}$ and $\phi^0_{O,0}$ at $q3$ in the computing. Taking the summation of inner products on all sub-triangles, we have

$$\langle \phi^1_{q3,x}, \phi^0_{O,0}\rangle_{D_2(1)} = 0.$$

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In a similar way, we get the matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ and list it in the following

$$h_1^2 \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2/6 & 0 & 4/6 & 2/6 & 0 & 0 & 0 \\ 1/6 & 0 & 2/6 & 4/6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/6 & 0 & 0 & 0 & 0 & 4/6 & 2/6 \\ 1/6 & 0 & 0 & 0 & 0 & 2/6 & 4/6 \end{pmatrix}.$$

Null space of the column vectors is the solution for the wavelet $\psi^1_{q3,x}$, and existence of the solution is proved previously in Section 3.3. From computation, we have

$$\beta = \frac{-1}{2}\{-2, 0, 1, 0, 0, 1, 0\}^T.$$

If we change V^1 to be $\{\phi^1_{q3,0}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_3,0}, \phi^1_{P_3,x}, \phi^1_{P_3,y}\}$, then the associated matrix $\left(\langle V^T, V^1\rangle_{D_2(1)}\right)$ becomes

$$h_1^2 \begin{pmatrix} 1/2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4/6 & 2/6 & 0 & 0 & 0 \\ 1/8 & 0 & 2/6 & 4/6 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4/6 & 2/6 \\ -1/8 & 0 & 0 & 0 & 0 & 2/6 & 4/6 \end{pmatrix}.$$

For this case, $\psi^1_{q3,0} = V^1\beta^T$, where $\beta = \frac{-1}{8}\{-8, 4, -1, 2, 4, 1, -2\}$.

It's worth to point out that the above two matrices for $\left(\langle V^T, V^1\rangle_{D_2(1)}\right)$ are the same except for the first column. Therefore, in the following, we shall only list the first column of the associated matrix $\left(\langle V^T, V^1\rangle_{D_2(1)}\right)$ to compute other wavelets.

Let $V^1 := \{\phi^1_{q3,y}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_3,0}, \phi^1_{P_3,x}, \phi^1_{P_3,y}\}$, and the first column of the matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{-1, 1/6, -1/6, 1, 1/6, -1/6\}^T$. Thus, $\psi^1_{q3,y} = V^1\beta^T$, with $\beta = \frac{1}{2}\{2, 2, -1, 1, -2, -1, 1\}$.

In the following, we give $\{\psi^1_{q2,j}, \psi^1_{q1,j}\}_{j=0,x,y}$.

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1.) $\psi^1_{q2,0}$:
Let $V^1 = \{\phi^1_{q2,0}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_2,0}, \phi^1_{P_2,x}, \phi^1_{P_2,y}, \}$, and
$V = \{\phi^0_{O,0}, \phi^0_{O,x}, \phi^0_{O,y}, \phi^0_{P_2,0}, \phi^0_{P_2,x}, \phi^0_{P_2,y}, \}$. The first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{1/2, 1/8, 1/8, 1/2, -1/8, -1/8\}^T$. Then

$$\psi^1_{q2,0} = V^1\beta^T,$$

where

$$\beta = \frac{-1}{8}\{-8, 4, 1, 1, 4, -1, -1\}.$$

2.) $\psi^1_{q2,x}$:
Let $V^1 = \{\phi^1_{q2,x}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_2,0}, \phi^1_{P_2,x}, \phi^1_{P_2,y}, \}$, and the first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{-1, -1/6, -2/6, 1, -1/6, -2/6\}^T$. Then

$$\psi^1_{q2,x} = V^1\beta^T,$$

where

$$\beta = \frac{1}{2}\{2, 2, 0, 1, -2, 0, 1\}.$$

3.) $\psi^1_{q2,y}$:
Let $V^1 = \{\phi^1_{q2,y}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_2,0}, \phi^1_{P_2,x}, \phi^1_{P_2,y}, \}$, and the first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{-1, -2/6, -1/6, 1, -2/6, -1/6\}^T$. Then

$$\psi^1_{q2,y} = V^1\beta^T,$$

where

$$\beta = \frac{1}{2}\{2, 2, 1, 0, -2, 1, 0\}.$$

4.) $\psi^1_{q1,0}$:
Let $V^1 = \{\phi^1_{q1,0}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_1,0}, \phi^1_{P_1,x}, \phi^1_{P_1,y}, \}$, and
$V = \{\phi^0_{O,0}, \phi^0_{O,x}, \phi^0_{O,y}, \phi^0_{P_1,0}, \phi^0_{P_1,x}, \phi^0_{P_1,y}, \}$. The first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{1/2, 1/8, 0, 1/2, -1/8, 0\}^T$. Then

$$\psi^1_{q1,0} = V^1\beta^T,$$

where

$$\beta = \frac{-1}{8}\{-8, 4, 2, -1, 4, -2, 1\}.$$

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5.) $\psi^1_{q1,x}$:

Let $V^1 = \{\phi^1_{q1,x}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_1,0}, \phi^1_{P_1,x}, \phi^1_{P_1,y},\}$, and the first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{-1,-1/6,1/6,1,-1/6,1/6\}^T$. Then

$$\psi^1_{q1,x} = V^1\beta^T,$$

where

$$\beta = \frac{-1}{2}\{-2,-2,-1,1,2,-1,1\}.$$

6.) $\psi^1_{q1,y}$:

Let $V^1 = \{\phi^1_{q1,y}, \phi^1_{O,0}, \phi^1_{O,x}, \phi^1_{O,y}, \phi^1_{P_1,0}, \phi^1_{P_1,x}, \phi^1_{P_1,y},\}$, and the first column of the associated matrix $\left(\langle V^T, V1\rangle_{D_2(1)}\right)$ is $\{0,1/6,2/6,0,1/6,2/6\}^T$. Then

$$\psi^1_{q1,y} = V^1\beta^T,$$

where

$$\beta = \frac{-1}{2}\{-2,0,0,1,0,0,1\}.$$

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Appendix A

Additive Schwarz-Type Preconditioner for Hermite Cubic Splines

Due to its built in parallelism as well as simple implementation, additive Schwarz type preconditioner has been received more and more attention recently [6, 29, 53, 57]. In Appendix A, we shall construct the additive Schwarz preconditioner for the Hermite cubic splines and prove that the preconditioned system has the uniformly bounded condition number. Hermite cubic splines are well known in the field of the approximation [20, 34], and their C^1 continuity and high order approximation property make them attractive in practice.

This Appendix is divided into three parts. In section A.2, we sketch the basic framework of the additive Schwarz preconditioner. Hermite cubic splines and their properties shall be briefly reviewed in section A.3. Finally, we construct the nested finite element spaces with Hermite cubic splines, and show that the condition number of the preconditioned system by the additive Schwarz preconditioner for the Hermite cubic splines is uniformly bounded in section A.4.

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A.1 Abstract additive Schwarz preconditioner

In this section, we introduce the notation, and the basic concepts of the additive Schwarz precndtioner we may use later. We are following the setting introduced in [7, 29].

Let $S_0 \subset S_1 \subset \cdots \subset S_n = S$ be a nested sequence of finite dimensional Hilbert spaces and

$$S = \sum_{j=0}^{n} S_j$$

where n is a positive integer.

Let $a(\cdot,\cdot) : S \times S \to \mathbb{R}$ be a positive definite and symmetric bilinear form with the properties

$$a(v,w) = a(w,v) \quad \forall v,w \in S,$$

and

$$a(v,v) > 0.$$

Define $A: \ S \to S$, and

$$a(v,w) = (Av,w) \quad \forall v,w \in S,$$

where $(\cdot,\cdot)$ is the scalar product in S.

Let each subspace S_j, $j = 1,...,n$, equipped with a positive definite and symmetric form $b_j(v,w) = (B_j v, w), \quad v,w \in S_j$ with $B_j : \ S_j \to S_j$. Finally, we define the operator $I_j : S_j \to S$ to be the nature injection operator, and its transpose is denoted by I_j^t.

The abstract additive Schwarz preconditioner can be written as

$$B = \sum_{j=0}^{n} I_j B_j^{-1} I_j^t. \tag{A.1}$$

Remark 1. In practice, $a(\cdot,\cdot)$ usually is the bilinear form introduced from the given second order elliptic problem, and A corresponds to the stiffness matrix. I_j is referred to as the transformation matrix for the basis in S_j and the basis in S. I_j^t is the transpose of I_j. $b_j(\cdot,\cdot)$ is closely related to the scalar product

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in S_j, and B_j usually can be represented as a diagonal matrix. Therefore, the additive Schwarz preconditioner (A.1) can be easily implemented.

Once we have a sequence of nested subspaces, we may estimate the maximum and minimum eigenvalues by the following theorem

Theorem A.1. *The maximum and minimum eigenvalues of BA can be characterized by*

$$\lambda_{\max}(BA) := \max_{0 \neq v \in S} \frac{a(v,v)}{\min_{v=\sum_{j=0}^{n} I_j v_j,\, v_j \in S_j} \sum_{j=0}^{n} b_j(v_j, v_j)},$$

and

$$\lambda_{\min}(BA) := \min_{0 \neq v \in S} \frac{a(v,v)}{\min_{v=\sum_{j=0}^{n} I_j v_j,\, v_j \in S_j} \sum_{j=0}^{n} b_j(v_j, v_j)}.$$

Proofs of the theorem can be found in several sources [7, 29, 41].

We may find that whether the additive Schwarz preconditioner works well or not is depending on the ratio (the condition number) of $\lambda_{\max}$ to $\lambda_{\min}$ in the Theorem A.1.

A.2 Hermite cubic splines

Recall that the Hermite cubic splines ϕ_1 and ϕ_2 be given by

$$\phi_1(x) := \begin{cases} \phi_{10} := (x+1)^2(1-2x) & \text{for} \quad x \in [-1,0], \\ \phi_{11} := (1-x)^2(2x+1) & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [-1,1], \end{cases}$$

and

$$\phi_2(x) := \begin{cases} \phi_{20} := x(x+1)^2 & \text{for} \quad x \in [-1,0], \\ \phi_{21} := x(x-1)^2 & \text{for} \quad x \in [0,1], \\ 0 & \text{for} \quad x \in \mathbb{R} \setminus [-1,1]. \end{cases}$$

Then

$$\phi_1(j) = \delta(j), \quad \phi_1'(j) = 0, \quad \phi_2(j) = 0, \quad \phi_1'(j) = \delta(j), \; j \in \mathbb{Z},$$

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where

$$\delta(j) := \begin{cases} 0, & 0 \neq j \in \mathbb{Z} \\ 1, & j = 0 \end{cases}.$$

Applications of the Hermite cubic splines in solving ODE numerically can be found in [34].

On a mesh with $0 < x_0 < x_1 < \cdots < x_N = 1$, we may scale and shift ϕ_1, ϕ_2 to construct the basis functions for the finite dimensional space as follows

$$\phi_{1,i}(x) := \begin{cases} \phi_{10}(\frac{x-x_i}{x_i-x_{i-1}}) & for \quad x \in [x_{i-1}, x_i], \\ \phi_{11}(\frac{x-x_i}{x_{i+1}-x_i}) & for \quad x \in [x_i, x_{i+1}], \\ 0 & for \quad x \in \mathbb{R} \setminus [x_{i-1}, x_{i+1}], \end{cases}$$

and

$$\phi_{2,i}(x) := \begin{cases} \phi_{20}(\frac{x-x_i}{x_i-x_{i-1}}) & for \quad x \in [x_{i-1}, x_i], \\ \phi_{21}(\frac{x-x_i}{x_{i+1}-x_i})\frac{(x_{i+1}-x_i)}{(x_i-x_{i-1})} & for \quad x \in [x_i, x_{i+1}], \\ 0 & for \quad x \in \mathbb{R} \setminus [x_{i-1}, x_{i+1}], \end{cases}$$

where $i = 0, ..., N$.

We may verify that $\phi_{1,i}(x_j) = \delta(i-j), \phi'_{1,i}(x_j) = 0, \phi_{2,i}(x_j) = 0,\ \phi'_{2,i}(x_j) = \delta(i-j)/(x_i - x_{i-1}), \quad i, j = 0, ..., N.$

Now we introduce one lemma on the Hermite cubic splines.

Lemma A.1. *Let* $v = \alpha_0\phi_{11}(x) + \alpha_1\phi_{10}(x-1) + \beta_0\phi_{20}(x) + \beta_1\phi_{21}(x-1)$, *then we have*

$$C_1||v||^2_{L_2(0,1)} \leq \alpha_0^2 + \alpha_1^2 + \beta_0^2 + \beta_1^2 \leq C_2||v||^2_{L_2(0,1)}, \tag{A.2}$$

$$C_3(\beta_0^2 + \beta_1^2) \leq ||v'||^2_{L_2(0,1)}, \tag{A.3}$$

and

$$||v'||_{L_2(0,1)} \leq C_4||v||_{L_2(0,1)} \tag{A.4}$$

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where C_1, C_2, C_3 and C_4 are four constants independent of v, and L_2 norm on the interval $I \subset [0,1]$ is defined to be $||v||_{L_2(I)} := (\int_I |v(x)|^2 dx)^{1/2}$.

Proof. Let the vector $\alpha := (\alpha_0, \alpha_1, \beta_0, \beta_1)^t$, then

$$||v||^2_{L_2(0,1)} = \alpha^t D\alpha,$$

where

$$D = \begin{pmatrix} \frac{13}{35} & \frac{9}{70} & \frac{-11}{210} & \frac{13}{420} \\ \frac{9}{70} & \frac{13}{35} & \frac{-13}{420} & \frac{11}{210} \\ \frac{-11}{210} & \frac{-13}{420} & \frac{1}{105} & \frac{-1}{140} \\ \frac{13}{420} & \frac{11}{210} & \frac{-1}{140} & \frac{1}{105} \end{pmatrix}.$$

Note that D is symmetric and positive definite (i.e., its eigenvalues are strictly positive and bounded). Then (A.2) holds true.

Likewise, we have

$$||v'||^2_{L_2(0,1)} = \alpha^t D_1\alpha,$$

where

$$D_1 = \begin{pmatrix} \frac{6}{5} & \frac{-6}{5} & \frac{-1}{10} & \frac{-1}{10} \\ \frac{-6}{5} & \frac{6}{5} & \frac{1}{10} & \frac{1}{10} \\ \frac{-1}{10} & \frac{1}{10} & \frac{2}{15} & \frac{-1}{30} \\ \frac{-1}{10} & \frac{1}{10} & \frac{-1}{30} & \frac{2}{15} \end{pmatrix}.$$

Note that $D_1 - 0.082D_2$ is symmetric and with the eigenvalues nonnegative, where D_2 is a diagonal matrix with the diagonal entries $(0,0,1,1)$. Then (A.3) follows if we set $C_3 = 0.082$.

Last inequality (A.4) is true because

$$||v'||_{L_2(0,1)} \leq C_4(\alpha_0^2 + \alpha_1^2 + \beta_0^2 + \beta_1^2).$$

If we note that D_1's maximum eigenvalue is bounded, then, by (A.2), (A.4) holds.

This completes the proof. □

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A.3 Additive Schwarz preconditioner for the Hermite cubic splines

For the given second order elliptic model two points boundary value problem

$$-(p(x)u(x)')' + q(x)u(x) = f(x), \quad x \in (0,1),$$

with the boundary conditions

$$u(0) = u(1) = 0,$$

we may define the bilinear form $a(\cdot,\cdot)$ to be

$$a(v,w) = \int_0^1 p(x)v'(x)w'(x)dx + \int_0^1 q(x)v(x)w(x)dx, \; v,w \in H_0^1(0,1),$$

where $p(x) > 0, q(x) \geq 0$ for $x \in (0,1)$, and $H_0^1(0,1)$ is the usual Sobolev space with the norm, semi-norm denoted by $||\cdot||_1$, $|\cdot|_1$, respectively. It's well known that

$$a(v,v) \simeq ||v||_1^2.$$

Here and in what follows, we use $X \simeq Y$ to denote the equivalence of the two terms X and Y (X, Y can be bounded each other by multiply by some constants independent of the mesh.), and let $C, C_i (i = 1, 2...)$ denote the generic constants independent of the mesh.

For $H_0^1(0,1)$, H^1 semi-norm is an equivalent norm to H^1 norm.

The Galerkin method is to seek an element u in $H_0^1(0,1)$, such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(0,1), \tag{A.5}$$

where $(v,w) := \int_0^1 v(x)w(x)dx$ is the traditional inner product for the real-valued function space $L_2(0,1)$.

If we have a finite dimensional subspace $S \subset H_0^1(0,1)$, then the finite element method is to seek an element $u_n \in S$ such that

$$a(u_n,v) = (f,v) \quad \forall v \in S. \tag{A.6}$$

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Now we construct the finite element space based on the Hermite cubic splines. For the convenience of statement, we focus on the uniform mesh, although quasi-uniform mesh also admits the additive Schwarz preconditioner for the Hermite cubic splines.

Let $\phi^e_{j,k}(x) := \phi_e(2^j x - k), e = 1, 2$, where j, k, known as scales and shifts, are both nonnegative integers. Then we may check that $\mathrm{supp}(\phi^e_{j,k}) = [k-1, k+1]/2^j$.

Let the finite dimensional space S_j be the linear span of the basis functions $\{\phi^e_{j,k}\}$, $k = 1, ..., 2^j - 1$ for $e = 1$, and $k = 0, ..., 2^k$ for $e = 2$. Then

$$S = S_n = \sum_{j=0}^{n} S_j, \quad S \subset H^1_0(0,1).$$

We may write the basis function in a vector by $\Phi_j := \{\varphi_{j,l}\}$, where $\varphi_{j,2k} = \phi^2_{j,k}$, $k = 0, ..., 2^j - 1$, $\varphi_{j,2k-1} = \phi^1_{j,k}$, $k = 1, ..., 2^j - 1$, and $\varphi_{j,2^{j+1}-1} = \phi^2_{j,2^j}$.

The corresponding stiffness matrix is generated by $A := (a(\varphi_{n,k1}, \varphi_{n,k2}))$, and the transformation matrix I_j can be obtained from the so called refinement equations

$$\Phi(x) = \sum_{k=-1}^{1} R_k \Phi(2x - k),$$

where the vector of functions Φ is defined to be $(\phi^0(x), \phi^1(x))^T$ and the matrices

$$R_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{-1}{8} & \frac{-1}{8} \end{bmatrix}, R_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, R_1 = \begin{bmatrix} \frac{1}{2} & \frac{-3}{4} \\ \frac{1}{8} & \frac{-1}{8} \end{bmatrix}.$$

In other words, every basis function $\varphi_{j,k}$ can be written as a linear combination of no more than 6 basis functions in $\{\varphi_{j+1,k}\}$. Therefore, $S_j \subset S_{j+1}$. Moreover, we have the 2^{j+2} by 2^{j+1} transformation matrix T_j, such that

$$\Phi^t_j = \Phi^t_{j+1} T_j.$$

Thus, I_j can be written as

$$I_j := T_{n-1} T_{n-2} \cdots T_j, \quad j = 0, ..., n-1,$$

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and I_n is just the 2^{n+1} by 2^{n+1} identity matrix. Furthermore

$$\Phi_j^t = \Phi_n^t I_j \quad j = 0, ..., n. \tag{A.7}$$

For any element $v_j \in S_j$, we may write it as a linear combination of the basis functions, i.e.

$$v_j = \sum_{k=0}^{2^{j+1}-1} v_{j,k}\varphi_{j,k}.$$

Denote by $\mathbf{v_j}$ the vector of the coefficients $\{v_{j,k}\}$ associated to $v_j \in S_j$. Then by Lemma A.1, we have

$$h_j \mathbf{v_j}^T \mathbf{v_j} \simeq ||v||^2_{L_2(0,1)},$$

where $\mathbf{v_{j_1}}^T \mathbf{w_{j_2}}$ is the usual vector product and h_j, known as the mesh size of S_j, is 2^{-j}.

Let's take a look at I_j again. For a given $v_j \in S_j$, we associate it with its coefficient vector $\mathbf{v_j}$, and $I_j v_j \in S_n$. Here I_j is an injection mapping $v_j \in S_j$ naturally into S_n. With some ambiguity, we use the same notation I_j in (A.7) as a transformation matrix mapping a vector in $\mathbb{R}^{2^{j+1}}$ to $\mathbb{R}^{2^{n+1}}$. More precisely, if $\mathbf{v_n}$ is the vector of coefficients of v_j with the basis functions in S_n (i.e., $v_j = \Phi_n^t \mathbf{v_n}$), then

$$\mathbf{v_n} = I_j \mathbf{v_j}.$$

Thus, vectors $\mathbf{v_j}, \mathbf{v_n}$ can be think of as two representations of the same function in S in terms of bases in S_j, S_n, respectively. I_j, as a transformation matrix, connects such basis change. It's important in the numerical implementation.

Let the bilinear form $b_j(\cdot, \cdot)$ on S_j be

$$b_j(v_j, w_j) = h_j^{-1} \mathbf{v_j}^T \mathbf{w_j}, \quad v_j, w_j \in S_j.$$

Then B_j^{-1}, written in the matrix form, is a 2^{j+1} by 2^{j+1} identity matrix multiply by $h_j^{-1} = 2^j$.

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Now the additive Schwarz preconditioner can be defined in the matrix form by

$$B = \sum_{j=0}^{n} 2^{-j} I_j I_j^T.$$

Remark 2. The computational work for $B\mathbf{v_n}$ can be calculated as $O(2^{n+1})$ if we note that the computational work for $T_j\mathbf{v_j}$ is $O(2^{j+1})$, and 2^{j+1} is the dimension of the space S_j.

After we introduce the additive Schwarz preconditioner, we may estimate the maximum and the minimum eigenvalues of the product of the matrices B and A. By Theorem A.1, we need to estimate the ratio of $a(v,v)$ to $\min_{v=\sum_{j=0}^{n} I_j v_j,\, v_j \in S_j} \sum_{j=0}^{n} 2^j \mathbf{v_j}^T \mathbf{v_j}$.

For Hermite cubic splines, we have the following theorem,

Theorem A.2. *For the finite dimensional space $\{S_j\}$ generated by the Hermite cubic splines, we have*

$$\lambda_{\max} := \max_{0\neq v \in S} \frac{a(v,v)}{\min_{v=\sum_{j=0}^{n} I_j v_j,\, v_j \in S_j} \sum_{j=0}^{n} 2^j \mathbf{v_j}^T \mathbf{v_j}} = O(1),$$

and

$$\lambda_{\min} := \min_{0\neq v \in S} \frac{a(v,v)}{\min_{v=\sum_{j=0}^{n} I_j v_j,\, v_j \in S_j} \sum_{j=0}^{n} 2^j \mathbf{v_j}^T \mathbf{v_j}} = O(1).$$

Before we prove the theorem, we need three Lemmas. Let $Q_j : S_n \to S_j, j \geq 0$ $(Q_{-1} = 0)$ be the orthogonal projection operator, i.e.,

$$(Q_j v, w_j) = (v, w_j), \quad \forall w_j \in S_j,$$

where v is an element in S_n. Then for the sequence of subspaces $\{S_j\}$, we have

Lemma A.2. *For $v \in S$, we have*

$$||v||_1^2 \simeq \sum_{j=0}^{n} 4^j ||(Q_j - Q_{j-1})v||_{L_2}^2.$$

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As a result of norm equivalence, it's well known in the literature. It's proofs may be found in [21, 41].

Lemma A.3. *For any $v_j \in S_j$, $j = 0, 1, ..., n$, we have*

$$\text{(A.8)} \qquad ||v_j||_{L_2}^2 \simeq \sum_{k=0}^{2^{j+1}} v_{j,k}^2 h_j = 2^{-j} \mathbf{v_j}^t \mathbf{v_j},$$

and

$$\text{(A.9)} \qquad ||v_j'||_{L_2}^2 \leq C \sum_{k=0}^{2^{j+1}} (v_{j,k}/h_j)^2 h_j = C 2^j \mathbf{v_j}^t \mathbf{v_j},$$

Proof. If we note that $||\phi_{j,k}^e||_{L_2(0,1)} \simeq h_j = 2^{-j}$, and $||(\phi_{j,k}^e)'||_{L_2(0,1)} \leq C 2^j$, then (A.8, A.9) can be obtained directly by Lemma A.1. (A.8) usually is referred to as the stability of the Hermite cubic splines in the sense of L_2, and (A.9) is the inverse inequality. □

The next Lemma is a Cauchy-Schwarz type inequality,

Lemma A.4.

$$\int_0^1 v_j'(x) w_k'(x) dx \leq C 2^{-|j-k|/2} (h_j^{-1} ||v_j||_{L_2})(h_k^{-1} ||v_k||_{L_2}), \quad \forall v_j \in S_j, w_k \in S_k.$$

Proof. For the case $k = j$, by Cauchy-Schwarz inequality

$$\int_0^1 v_j'(x) w_k'(x) dx \leq ||v_j'||_{L_2} ||w_k'||_{L_2} \leq C (h_j^{-1} ||v_j||_{L_2})(h_k^{-1} ||v_k||_{L_2}),$$

where we use the inverse inequality in the last step.

For the case $j < k$, we consider one sub-interval, say $(m/2^j, (m+1)/2^j)$ in the mesh for S_j. Furthermore, let $\alpha_1 = v_j(m/2^j), \alpha_2 = v_j((m+1)/2^j), \beta_1 = h_j v_j'(m/2^j), \beta_2 = h_j v_j'((m+1)/2^j)$. Then, on the interval $(m/2^j, (m+1)/2^j)$, v_j can be written as

$$v_j = \alpha_1 \phi_{j,m}^1 + \alpha_2 \phi_{j,m+1}^1 + \beta_1 \phi_{j,m+1}^2 + \beta_2 \phi_{j,m+1}^2.$$

Let $a = m/2^j, b = (m+1)/2^j$. Then

$$\int_a^b v_j'(x) w_k'(x) dx = v_j' w_k |_a^b - \int_a^b v_j''(x) w_k(x) dx.$$

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First, we estimate the term $v_j' w_k|_a^b$ by

$$\begin{aligned} v_j' w_k|_a^b &\leq \frac{1}{2}((v_j'(a))^2 + (v_j'(b))^2)^{1/2}((w_k(a))^2 + (w_k(b))^2)^{1/2} \\ &\leq C(||v_j'||_{L_2(a,b)} h_j^{-1/2})(||w_k||_{L_2(a,b)} h_k^{-1/2}). \end{aligned}$$

Since by the Lemma A.1 (after a scale), we have

$$((v_j'(a))^2 + (v_j'(b))^2) h_j \leq C||v_j'||^2_{L_2(a,b)},$$

and

$$((w_k(a))^2 + (w_k(b))^2) h_k \leq C||w_k||^2_{L_2(a,b)}.$$

Hence, by inverse inequality,

$$\begin{aligned} (||v_j'||_{L_2(a,b)} h_j^{-1/2})(||w_k||_{L_2(a,b)} h_k^{-1/2}) &\leq C(h_j^{-3/2}||v_j||_{L_2(a,b)})(h_k^{-1/2}||w_k||_{L_2(a,b)}) \\ &\leq C(h_k/h_j)^{1/2}(h_j^{-1}||v_j||_{L_2(a,b)})(h_k^{-1}||v_k||_{L_2(a,b)}). \end{aligned}$$

Note that $h_j = 2^{-j}, h_k = 2^{-k}$. Then

$$v_j' w_k|_a^b \leq C2^{-|j-k|/2}(h_j^{-1}||v_j||_{L_2})(h_k^{-1}||v_k||_{L_2}).$$

Second, we estimate the term $-\int_a^b v_j''(x) w_k(x)$.

$$\begin{aligned} |-\int_a^b v_j''(x) w_k(x)| &\leq C||v_j''||_{L_2(a,b)}||w_k||_{L_2(a,b)} \\ &\leq Ch_j^{-2}||v_j||_{L_2(a,b)}||w_k||_{L_2(a,b)} \\ &\leq C2^{-|j-k|/2}(h_j^{-1}||v_j||_{L_2(a,b)})(h_k^{-1}||v_k||_{L_2(a,b)}). \end{aligned}$$

If we add up the estimates on all intervals and apply the Cauchy-Schwarz inequality, then Lemma A.4 follows. Thus completes the proof. □

Now we give the proof of Theorem

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[theoremA.2]

Proof of Theorem A.2.

Let $v = \sum_{j=0}^{n} I_j v_j$, then we have

$$\begin{aligned} a(v,v) &\simeq C\|v'\|^2_{L_2(0,1)} \\ &= C(\sum_{j=0}^{n} v_j', \sum_{k=0}^{n} v_k') \\ &\leq C\sum_{j,k=0}^{n} 2^{-|j-k|/2}(h_j^{-1}\|v_j\|_{L_2})(h_k^{-1}\|v_k\|_{L_2}) \\ &\leq \sum_{j=0}^{n}(h_j^{-1}\|v_j\|_{L_2})^2 \\ &\simeq \sum_{j=0}^{n} 2^j \mathbf{v_j}^T\mathbf{v_j}. \qquad \text{(by lemma A.3)} \end{aligned}$$

Since the splitting of v is arbitrary, this implies that

$$a(v,v) \leq C \min_{v=\sum_{j=0}^{n} I_j v_j, v_j \in S_j} \sum_{j=0}^{n} 2^j \mathbf{v_j}^T\mathbf{v_j}. \tag{A.10}$$

Now let $v_j = (Q_j - Q_{j-1})v$, $j = 0, ..., n$, then by Lemma A.2, we have

$$a(v,v) \simeq \sum_{j=0}^{n} 4^j \|v_j\|^2_{L_2} \simeq \sum_{j=0}^{n} 2^j \mathbf{v_j}^T\mathbf{v_j}. \tag{A.11}$$

Combing (A.10) with (A.11) yields that

$$a(v,v) \simeq \min_{v=\sum_{j=0}^{n} I_j v_j, v_j \in S_j} \sum_{j=0}^{n} 2^j \mathbf{v_j}^T\mathbf{v_j}.$$

Thus completes the proof. □

Remark 3. For quasi-uniform mesh, note that $h_j \simeq 2^{-j}$, $j = 0, ..., n$. Then we can obtain the same result on the additive Schwarz preconditioner for the Hermite cubic splines using the same proof in the note.

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Finally, we show the numerical results of the additive Schwarz preconditioner for the model problem

$$u''(x) = f(x) \quad x \in (0,1),$$

with the boundary conditions

$$u(0) = u(1) = 0.$$

The bilinear form arising from the elliptic problem is $a(v,w) = \int_0^1 v'(x)w'(x)dx$, and thus A is the stiffness matrix with (k_1,k_2) entry $\int_0^1 \varphi'_{n,k_1}(x)\varphi'_{n,k_2}(x)dx$. To obtain better results on the condition number, we normalize $\varphi_{j,k}, j = 0,...,n,\ k = 0,...,2^{j+1}-1$, such that

$$\int_0^1 |\varphi'_{j,k}(x)|^2 dx = 2^{-j}.$$

The additive Schwarz preconditioner is given in (A.1). The condition numbers with respect to different n are listed in the following table. The numerical results confirm the claims in theorem A.2.

n	6	7	8	9	10	11	12
κ	4.62	4.71	4.78	4.83	4.86	4.89	4.89

Table A.1: Condition numbers(κ) of BA with different n

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