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Resilient Estimation and Control on k-Nearest Neighbor Platoons: A Network-Theoretic Approach

Mohammad Pirani * **Ehsan Hashemi** ** **Baris Fidan** **
John W. Simpson-Porco *** **Henrik Sandberg** *
Karl Henrik Johansson *

* *Department of Automatic Control, KTH Royal Institute of Technology.*
** *Department of Mechanical and Mechatronics Engineering, University of Waterloo.*
*** *Department of Electrical and Computer Engineering, University of Waterloo.*

Abstract: This paper is concerned with the network-theoretic properties of so-called k-nearest neighbor intelligent vehicular platoons, where each vehicle communicates with k vehicles, both in front and behind. The network-theoretic properties analyzed in this paper play major roles in quantifying the resilience and robustness of three generic distributed estimation and control algorithms against communication failures and disturbances, namely resilient distributed estimation, resilient distributed consensus, and robust network formation. Based on the results for the connectivity measures of the k-nearest neighbor platoon, we show that extending the traditional platooning topologies (which were only based on interacting with nearest neighbors) to k-nearest neighbor platoons increases the resilience of distributed estimation and control algorithms to both communication failures and disturbances.

Keywords: Network connectivity, Network robustness, resilient estimation and control

1. INTRODUCTION

Intelligent transportation systems are an important real-world instance of a multi-disciplinary cyber-physical system Lu et al. (2014). In addition to classical electromechanical engineering, designing intelligent transportation systems requires synergy with and between outside disciplines, including communications, control, and network theory. In this direction, estimation and control theory are pivotal parts in designing algorithms for the active safety of automotive and intelligent transportation systems Pirani et al. (2017a); Turri et al. (2017). From another perspective, networks of connected vehicles are quite naturally mathematically modeled using tools from networks and graph theory, with associated notions such as degree, connectivity and expansion. While these modeling tools are in general distinct, the primary goal of this paper is to investigate connections between the control-theoretic and network-theoretic approaches to intelligent platoons.

The interplay between the network and system-theoretic concepts in network control systems has attracted much attention in recent years Fitch and Leonard (2013); Pirani et al. (2017b). There is a vast literature on revisiting the system-theoretic notions from the network's perspective. In this direction, some new notions have emerged such as *network coherence* Bamieh et al. (2012); Pirani et al. (2017b) which is interpreted as the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms of a network dynamical system showing the ability of the network in mitigating the effect of disturbances. The advantage of this approach is in large-scale networks for which working with systemic notions is a burdensome task and tuning network properties is more implementable.

The above-mentioned reciprocity between the system and network-theoretic concepts finds many applications in mobile networks and in particular in networks of connected vehicles. There is much research on designing distributed estimation and control algorithms for traffic networks to ensure the safety or optimality of the energy consumption Liang et al. (2016); Turri et al. (2017). In all of those settings, there exist system-theoretic conditions which ensure the effectiveness of the proposed algorithms. However, as the scale of the network increases and the interactions become more sophisticated, e.g., from simple platooning to more complex topologies, testing those system-theoretic conditions becomes harder and the need to redefine those conditions in terms of network-theoretic properties is seriously felt. To this end, our approach is to reinterpret the performance of distributed estimation and control algorithms in terms of graph-theoretic properties of k-nearest neighbor platoons. We first quantify how densely connected this network is, as there are many non-equivalent metrics used in the literature to quantify the network connectivity. Then we make a connection between each connectivity measure with its corresponding system performance metric. From this view, the contributions of this paper are as follows. We first discuss some network connectivity measures for a generalized form of vehicle platoons (called k-nearest neighbor platoon) and show that this particular network topology provides high levels of connectivity for most of the connectivity measures. Interestingly, most of these measures depend only on the number of local interactions of each vehicle in the platoon. Then, we apply the connectivity measures of k-nearest neighbor platoon to provide network-theoretic conditions for the performance of three well-known distributed estimation and control algorithms and show the positive effect of such network topology in enhancing the resilience of those algorithms. The overall structure of the paper in a glance is schematically shown in Fig. 1.

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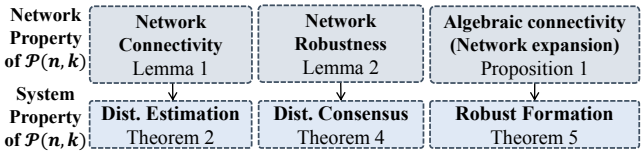

Fig. 1. Network-theoretic approaches to the performance of distributed estimation and control algorithms on k-nearest neighbor platoons, $\mathcal{P}(n,k)$.

2. NOTATIONS AND DEFINITIONS

In this paper, an undirected network (graph) is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes (or vertices) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges. Neighbors of node $v_i \in \mathcal{V}$ are given by the set $\mathcal{N}_i = \{v_j \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E}\}$. The degree of each node v_i is denoted by $d_i = |\mathcal{N}_i|$ and the minimum and maximum degrees in graph $\mathcal{G}$ are shown by $d_{\min}$ and $d_{\max}$, respectively. The adjacency matrix of the graph is a symmetric and binary $n \times n$ matrix A, where element $A_{ij} = 1$ if $(v_i, v_j) \in \mathcal{E}$ and zero otherwise. For a given set of nodes $X \subset \mathcal{V}$, the *edge-boundary* (or just boundary) of the set is defined as $\partial X \triangleq \{(v_i, v_j) \in \mathcal{E} \mid v_i \in X, v_j \in \mathcal{V} \backslash X\}$. The *isoperimetric constant* of $\mathcal{G}$ is defined as Chung (1997)

$$i(\mathcal{G}) \triangleq \min_{S \subset V, |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}. \tag{1}$$

where ∂S is the edge-boundary of a set of nodes $S \subset V$. The Laplacian matrix of the graph is $L \triangleq D - A$, where $D = \text{diag}(d_1, d_2, \ldots, d_n)$. The eigenvalues of the Laplacian are real and nonnegative, and are denoted by $0 = \lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_n(L)$ and $\lambda_2(L)$ is called the algebraic connectivity of the network Godsil and Royle (2001). Given a connected graph $\mathcal{G}$, an orientation of the graph $\mathcal{G}$ is defined by assigning a direction (arbitrarily) to each edge in $\mathcal{E}$. For graph $\mathcal{G}$ with m edges, numbered as $e_1, e_2, ..., e_m$, its node-edge incidence matrix $\mathcal{B}(\mathcal{G}) \in \mathbb{R}^{n \times m}$ is defined as Godsil and Royle (2001)

$$[\mathcal{B}(\mathcal{G})]_{kl} = \begin{cases} 1 & \text{if node } k \text{ is the head of edge } l, \\ -1 & \text{if node } k \text{ is the tail of edge } l, \\ 0 & \text{otherwise.} \end{cases}$$

The graph Laplacian satisfies $L = \mathcal{B}(\mathcal{G})\mathcal{B}(\mathcal{G})^{\mathsf{T}}$ Godsil and Royle (2001).

For positive integers $n, k \geq 1$ such that $n > k$, a k-Nearest Neighbor platoon containing n vehicles, which we denote as $\mathcal{P}(n,k)$, is a specific class of networks which captures the physical properties of wireless sensor networks in vehicular platoons. It is a network comprised of n nodes (or vehicles), where each node can communicate with its k nearest neighbors from its back and k nearest neighbors from its front, for some $k \in \mathbb{N}$. This definition is compatible with wireless sensor networks, due to the limited sensing and communication range for each vehicle and the distance between the consecutive vehicles Pirani et al. (2017a). An example of such network topology for $n = 5$ and $k = 2$ is shown in Fig. 3.

3. NETWORK-THEORETIC PROPERTIES

In this section, we examine four network connectivity measures which, as we will see, each play a fundamental role in understanding the system-theoretic performance of different algorithms on k-nearest neighbor platoons. These properties, as mentioned in the previous sections, are *network connectivity, network robustness*, and *network expansion* and *algebraic connectivity*. Fig. 2 (b) provides a visual sense of the strength of each of these connectivity measures in general graphs Shahrivar et al. (2013). Fig 2 (c) shows the values of each connectivity measure in k-nearest neighbor platoons which are discussed in detail in the subsequent subsections. The main insight is that, while these connectivity notions are distinct in general networks, they collapse to one equivalent notion of connectivity for k-nearest neighbor platoons.

3.1 Vertex and Edge Connectivity

First, we have the following definitions of graph vertex and edge connectivities.

Definition 1. **(Cuts in Graphs):** A *vertex-cut* in a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ is a subset $\mathcal{S} \subset \mathcal{V}$ of vertices such that removing the vertices in $\mathcal{S}$ (and any resulting dangling edges) from the graph causes the remaining graph to be disconnected. A (j,i)*-cut* in a graph is a subset $\mathcal{S}_{ij} \subset \mathcal{V}$ such that if the nodes $\mathcal{S}_{ij}$ are removed, the resulting graph contains no path from vertex v_j to vertex v_i. Let κ_{ij} denote the size of the smallest (j,i)-cut between any two vertices v_j and v_i. The graph $\mathcal{G}$ is said to have *vertex connectivity* $\kappa(\mathcal{G}) = \kappa$ (or κ*-vertex-connected*) if $\kappa_{ij} = \kappa$ for all $i, j \in \mathcal{V}$. The *edge connectivity* $e(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum number of edges whose deletion disconnects the graph.

For the vertex and edge connectivity and graph's minimum degree the following inequalities hold

$$\kappa(\mathcal{G}) \leq e(\mathcal{G}) \leq d_{\min}. \tag{2}$$

The following lemma discusses the connectivity of k-nearest neighbor platoons.

Lemma 1. A k-nearest neighbor platoon $\mathcal{P}(n,k)$ is a k-vertex and a k-edge connected graph, i.e., $\kappa(\mathcal{G}) = e(\mathcal{G}) = k$.

Proof. The proof is via contradiction. Suppose $\mathcal{P}(n,k)$ is a ℓ-connected graph, with $\ell < k$. Thus, there exists a minimum vertex cut $\mathcal{S}_{ij}$ between two vertices v_i and v_j where $|\mathcal{S}_{ij}| = \ell$. Without loss of generality, label the vertices from v_i to v_j as $v_i, v_{i+1}, ..., v_j$. Since $\ell < k$, there is a vertex $\bar{v}$ among $v_{i+1}, ..., v_{i+k}$ (which are directly connected to v_i) which does not belong to $\mathcal{S}_{ij}$. By replacing v_i with $\bar{v}$ in the above discussion, we will find a path from v_i to v_j which does not include vertices in $\mathcal{S}_{ij}$ and this contradicts the claim that $\mathcal{S}_{ij}$ is a vertex cut. Hence $\mathcal{P}(n,k)$ is a k-vertex connected graph. For the edge connectivity, observe that for graphs $\mathcal{P}(n,k)$ we have $d_{\min} = k$. The result then follows immediately from (2). □

3.2 Network Robustness

The notion of network robustness is another network connectivity measure, which finds application in the study of distributed consensus algorithms LeBlanc et al. (2013).

Definition 2. **(r-Reachable/Robust Graphs):** Let $r \in \mathbb{N}$. A subset $S \subset \mathcal{V}$ of nodes in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be r*-reachable* if there exists a node $v_j \in S$ such that $|\mathcal{N}_j \setminus S| \geq r$. A graph $G = (\mathcal{V}, \mathcal{E})$ is said to be r*-robust* if for every pair of nonempty, disjoint subsets of $\mathcal{V}$, at least one of them is r-reachable.

Generally speaking, r-robustness is a stronger notion than r-connectivity LeBlanc et al. (2013), as shown in the following example.

Example 1. The graph shown in Fig. 2 (a) is comprised of two complete graphs on n nodes (S_1 and S_2) and each

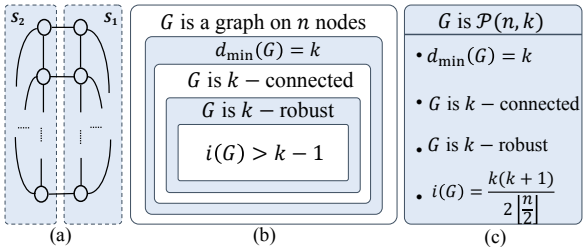

Fig. 2. (a) A graph with a large connectivity and small robustness, (b) Venn diagram of network connectivity measures for general graphs, (c) Connectivity measures for k-nearest neighbor platoons.

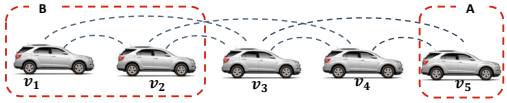

Fig. 3. Node Selection for calculating robustness (set A) and isoperimetric constant (set B) in $\mathcal{P}(5,2)$.

node in S_1 has exactly one neighbor in S_2 and vice-versa. The minimum degree and the vertex connectivity are both n; however, the network is only 1-robust.

As discussed in Example 1 and schematically shown in the Venn diagram in Fig. 2 (b), the network minimum degree, network connectivity and network robustness have different strength in general graphs. However, our next result shows that these notions coincide for k-nearest neighbour platoons.

Based on the above definition of network robustness, we have the following lemma for the robustness of k-nearest neighbor platoons.

Lemma 2. A k-nearest neighbor platoon $\mathcal{P}(n,k)$ with $k \leq \lfloor \frac{n}{2} \rfloor$ is a k-robust network.

Proof. The sketch of the proof is that, from Definition 2, by choosing every two disjoint sets of vertices in $\mathcal{P}(n,k)$, we see that the minimum number r for which a subset is r-reachable is $r = k$ and one of its corresponding subset is the ending node in the platoon, as shown in subset A in Fig. 3. Here k should satisfy $k \leq \lfloor \frac{n}{2} \rfloor$ as it is proven that no graph is $\lfloor \frac{n}{2} \rfloor + 1$ robust LeBlanc et al. (2013).□

3.3 Algebraic Connectivity and Network Expansion

One can further extend inequalities (2) to include the algebraic connectivity and obtain Godsil and Royle (2001)

$$\lambda_2(L) \leq \kappa(\mathcal{G}) \leq e(\mathcal{G}) \leq d_{\min}(\mathcal{G}).$$

Based on the above inequalities, we conclude that the algebraic connectivity of $\mathcal{P}(n,k)$ is less than k. However, by using the notion of *network expansion*, some tighter bounds on the algebraic connectivity can be obtained.

Definition 3. (**Expander Graph**). Expander graphs are graph sequences for which each graph in the sequence has an expansion property, meaning that there exists $\gamma > 0$ (independent of n) such that each subset S of nodes with size $|S| \leq \frac{n}{2}$ has at least $\gamma|S|$ edges to the rest of the network. In particular, we say that the graph $\mathcal{G}$ is a γ-expander network if $i(\mathcal{G}) = \gamma$ for some $\gamma > 0$, where $i(\mathcal{G})$ is the isoperimetric constant defined in Section 2.

The algebraic connectivity of the graph is related to the network expansion (or the isoperimetric constant) by the following bounds Chung (1997)

$$\frac{i(\mathcal{G})^2}{2d_{\max}} \leq \lambda_2(L) \leq 2i(\mathcal{G}). \tag{3}$$

Using these bounds, we present the following proposition.

Proposition 1. Given a k-nearest neighbor platoon $\mathcal{P}(n,k)$ its algebraic connectivity is bounded by

$$\max\left\{2k - n + 2, \frac{k(k+1)^2}{16\bar{n}^2}\right\} \leq \lambda_2(L) \leq \frac{k(k+1)}{\bar{n}}, \tag{4}$$

where $\bar{n} = \lfloor \frac{n}{2} \rfloor$.

Proof. First we use bounds given in (3). For this, we should calculate the isoperimetric constant in $\mathcal{P}(n,k)$ by finding a set in $\mathcal{P}(n,k)$ which minimizes $\frac{|\partial S|}{|S|}$ with $|S| \leq \frac{n}{2}$. A set which contains $\lfloor \frac{n}{2} \rfloor$ nodes, minimizes this function (Fig. 3, set B). Hence, the isoperimetric constant will be $i(\mathcal{G}) = \frac{1+2+\ldots+k}{\lfloor \frac{n}{2} \rfloor} = \frac{k(k+1)}{2\lfloor \frac{n}{2} \rfloor}$. Substituting this value into (3) and considering the fact that $d_{\max} \leq 2k$ provides the upper bound and the lower bound $\frac{k(k+1)^2}{16\bar{n}^2}$. The second lower bound comes from bound $2d_{\min} - n + 2 \leq \lambda_2(L)$ proposed in Fiedler (1973) and considering the fact that $d_{\min} = k$. □

The maximum over two lower bounds in (4) is due to the fact that for certain values of k one of the lower bounds is tighter than the other. For instance, for $k \leq \frac{n-2}{2}$ the left lower bound is zero or negative and the right lower bound is tighter. However, for $k = n - 1$ the left lower bound is tighter.

4. DISTRIBUTED ESTIMATION AND CONTROL ALGORITHMS

In this section, three estimation and control policies for vehicle platoons will be studied, and we will show how the connectivity measures introduced in Section 3 can be directly applied to quantify the performance of these algorithms.

4.1 Distributed Estimation, Robust to Communication Faults

Distributed estimation (or calculation) is a procedure by which vehicles in a network may estimate unavailable quantities based on incomplete localized measurements and cooperation with nearby vehicles. Distributed estimation can potentially have diverse applications in vehicle networks, such as fault detection or prediction, as schematically shown in the upper box in Fig. 4.

The state of vehicle v_j, which can be its kinematic state, e.g., velocity, or some spatial parameter, e.g., road condition, is denoted simply by the scalar $x_j[0]$. The objective is to enable vehicle v_i in the network (which is not in the communication range of vehicle v_j) to calculate this value. To yield this, vehicle v_i performs a linear iterative policy using the following time invariant updating rule

$$x_i[k+1] = w_{ii}x_i[k] + \sum_{j \in \mathcal{N}_i} w_{ij}x_j[k]\,, \tag{5}$$

where $w_{ii}, w_{ij} > 0$ are predefined weights. In addition to dynamics (5), at each time step, vehicle v_i has access to its own value (state) and the values of its neighbors. Hence, the vector of measurements for v_i is defined as

$$y_i[k] = \mathcal{C}_i \boldsymbol{x}[k], \tag{6}$$

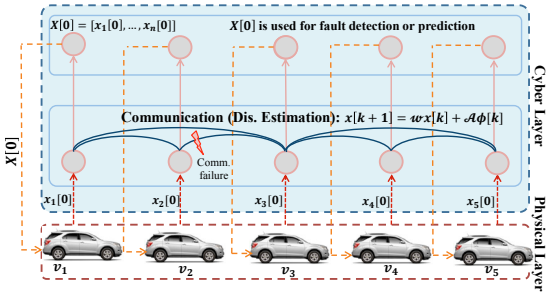

Fig. 4. Cyber-physical representation of the distributed estimation algorithm.

where $\mathcal{C}_i$ is a $(d_i+1)\times n$ matrix with a single 1 in each row that denotes the positions of the state-vector $\boldsymbol{x}[k]$ available to vehicle v_i (i.e., these positions correspond to vehicles that are neighbors of v_i, along with vehicle v_i itself).

Remark 1. **(Cyber-Physical Representation)**: Fig. 4 provides a cyber-physical interpretation of the distributed estimation algorithm. According to this figure, algorithm (5) is developed in the cyber layer, which receives the physical states of vehicles from the physical layer as initial conditions for its algorithm (red dashed lines), perform the distributed estimation to obtain the initial states of all vehicles in the network, and finally returns those initial states back to the physical layer (orange dashed lines). It should be noted that state $x_i[k]$ in (5) evolves in the cyber layer and it does not represent the evolution of vehicle's physical state based on the communication; the dynamics (5) is only used for implementing a distributed calculation algorithm. Here, it is only $\mathbf{x}[0] = [x_1[0], x_2[0], ..., x_n[0]]^T$ that reflects the physical states of the vehicles.

For such distributed estimation algorithms, we consider the possibility that there may exist some vehicles which fail to disseminate their information in a correct way, and some robust distributed estimation algorithms have been proposed to overcome such communication failures Sundaram and Hadjicostis (2011). More formally, suppose that some vehicles do not precisely follow (5) to update their value. In particular, at time step k, suppose vehicle v_i's update rule deviates from the predefined policy (5) and (likely, unintentionally) adds an arbitrary value $\phi_i[k]$ to its updating policy. [1] In this case, the updating rule (5) will become

$$x_i[k+1] = w_{ii}x_i[k] + \sum_{j\in\mathcal{N}_i} w_{ij}x_j[k] + \phi_i[k], \qquad (7)$$

and if there are $f > 0$ of these faulty vehicles, (7) in vector form becomes

$$\boldsymbol{x}[k+1] = \mathcal{W}\boldsymbol{x}[k] + \underbrace{[\mathbf{e}_1 \quad \mathbf{e}_2 \quad ... \quad \mathbf{e}_f]}_{\mathcal{A}}\boldsymbol{\phi}[k], \qquad (8)$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)^\mathsf{T}$, $\mathcal{W} \in \mathbb{R}^{n\times n}$ is the matrix of communication weights w_{ij}, $\boldsymbol{\phi}[k] = [\phi_1[k], \phi_2[k], ..., \phi_f[k]]^\mathsf{T}$ and $\mathbf{e}_i$ denotes the ith unit vector of $\mathbb{R}^n$. The set of faulty vehicles in (8) is unknown and consequently the matrix $\mathcal{A}$ is unknown. However, each vehicle knows an upper bound for the number of faulty vehicles.

The following theorem provides a condition which ensures that each vehicle is able to determine the (initial) states of all other vehicles in the network, despite of the action of some faulty vehicles. The details of the estimator design

[1] In the literature such agents are called *adversarial* or *malicious*.

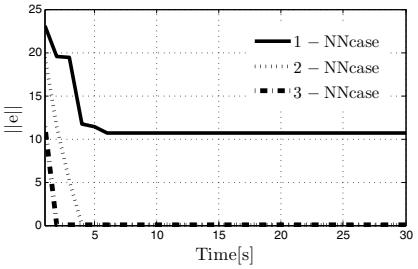

Fig. 5. Distributed estimation error for a vehicle in 1, 2 and 3 nearest-neighbour platoons of 10 vehicles.

(which is in the form of an unknown input observer) is not discussed in this paper and we refer the reader to Sundaram and Hadjicostis (2011).

Theorem 1. (Sundaram and Hadjicostis (2011)). Let $\mathcal{G}$ be a fixed graph and let f denote the maximum number of faulty vehicles that are to be tolerated in the network. Then, regardless of the actions of the faulty vehicles, v_i can uniquely determine all of the initial values of linear iterative strategy (8) for almost [2] any choice of weights in the matrix $\mathcal{W}$ if $\mathcal{G}$ is at least $(2f+1)$-vertex connected.

Theorem 2 together with Lemma 1 yield the following theorem which shows the ability of $\mathcal{P}(n,k)$ in performing distributed estimation algorithms.

Theorem 2. For a k-nearest neighbor platoon $\mathcal{P}(n,k)$, regardless of the actions of up to $\lfloor\frac{k-1}{2}\rfloor$ faulty vehicles, each vehicle can uniquely determine all of the initial values in the network via linear iterative strategy (8) for almost any choice of weights in the matrix $\mathcal{W}$.

Fig. 5 illustrates via an example how Theorem 2 provides network-theoretic sufficient condition for distributed estimation on $\mathcal{P}(n,k)$. In this example, there exists a single faulty vehicle in a network of 10 vehicles. Based on Theorem 2, it is sufficient to have $\mathcal{P}(10,3)$ to overcome the action of the faulty vehicle; the corresponding trace in Fig 5 shows that the Euclidean norm of the error of the estimated initial states of the vehicles in the network observed by a single vehicle goes to zero. More formally, if the true initial values are denoted by vector $\mathbf{x}[0]$ and the estimation (calculation) of these initial values by each vehicle at time step k is $\hat{\mathbf{x}}[k]$, then Fig. 5 shows the Euclidean norm of the error vector $\mathbf{e}[k] = \hat{\mathbf{x}}[k] - \mathbf{x}[0]$. However, the faulty vehicle in this example is not optimally malicious, so the distributed estimation algorithm works here for $\mathcal{P}(10,2)$.

4.2 Distributed Consensus, Robust to Communication Faults

In the distributed consensus scenario, the network of connected vehicles tries to reach to a consensus value, e.g., velocity or road condition, despite the existence of some faults, biases, or signal drops in inter-vehicle communications. In order to overcome the actions of faulty vehicles, the following iteration policy, called Weighted-Mean-Subsequence-Reduced (W-MSR) Zhang et al. (2015), is proposed to overcome their actions.

Definition 4. **(W-MSR Algorithm):** For some non-negative integer f, at each time-step, each node knows the number of faulty vehicles (or at least an upper bound

[2] The *almost* in Theorem 2 is due to the fact that the set of parameters for which the system is not observable has Lebesgue measure zero Reinschke (1987).

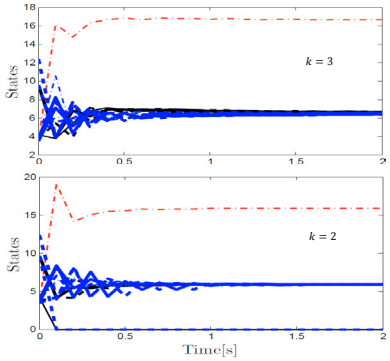

Fig. 6. Distributed consensus in the presence of a single faulty vehicle (red dashed line) for $\mathcal{P}(10,2)$ (bottom) and $\mathcal{P}(10,3)$ (top).

of that) and disregards the largest and smallest f values in its neighborhood ($2f$ in total) and updates its state to be a weighted average of the remaining values. More formally, this yields

$$x_j[k+1] = w_{jj}x_j[k] + \sum_{p\in\mathcal{N}_j[k]} w_{jp}x_p[k]. \tag{9}$$

where $\mathcal{N}_j[k]$ is the set of vehicles which are the neighbors of vehicle j and are not ignored.

In particular, if there exist f faulty vehicles, the dynamics is similar to (5), except the following two additional restrictions on matrix $\mathcal{W}$: (i) $w_{jp} > 0$, $\forall p \in \mathcal{N}_j[k] \cup \{v_j\}, v_j \in \mathcal{V}$, and (ii) $\sum_{p\in\mathcal{N}_j[k]\cup\{v_j\}} w_{jp} = 1, \forall v_j \in \mathcal{V}$. Similar to the case of distributed estimation mentioned in subsection A, the underlying network has to satisfy a certain level of connectivity to ensure that consensus can be achieved despite the actions of malicious or faulty vehicles. However, compared to the distributed estimation, distributed consensus requires *r-robustness* which a stronger notion of network connectivity as discussed in the previous section. The following theorem provides a sufficient condition for the iteration (9) to reach to a consensus despite of the actions of faulty vehicles in the network.

Theorem 3. (LeBlanc et al. (2013)). Suppose there exist at most f faulty vehicles in the network. Then the resilient asymptotic consensus is reached under the W-MSR iteration if the network is $(2f+1)$-robust.

Lemma 2 and Theorem 3 present the following theorem to show the ability of $\mathcal{P}(n,k)$ to perform robust distributed consensus.

Theorem 4. Suppose there are at most $\lfloor\frac{k-1}{2}\rfloor$ faulty vehicles in a k-nearest neighbor platoon. Then resilient asymptotic consensus on $\mathcal{P}(n,k)$ is reached under W-MSR dynamics, despite the action of faulty vehicles.

Fig. 6 confirms the connectivity condition proposed by Theorem 4 for distributed consensus in the presence of faulty vehicles. Here, there exists a single faulty vehicle in the network (whose state is shown with red dashed line) and it is shown that $\mathcal{P}(10,3)$ is robust enough to overcome the action of the faulty vehicle.

4.3 Network Formation in the Presence of Communication Disturbances

The vehicle network formation is the third problem analyzed in this paper. Let p_i and u_i denote the position and longitudinal velocity of vehicle v_i. The objective is for each vehicle to maintain specific distances from its neighbors. The desired vehicle formation will be formed by a specific constant distance Δ_{ij} between vehicles v_i and v_j, which should satisfy $\Delta_{ij} = \Delta_{ik} + \Delta_{kj}$ for every triple $\{v_i, v_j, v_k\} \subset \mathcal{V}$. Considering the fact that each vehicle v_i has access to its own position, the positions of its neighbors, and the desired inter-vehicular distances Δ_{ij}, the control law for vehicle v_i is Hao et al. (2010)

$$\begin{aligned}\ddot{p}_i(t) = &\sum_{j\in\mathcal{N}_i} k_p\left(p_j(t) - p_i(t) + \Delta_{ij}\right)\\ &+k_u\left(u_j(t) - u_i(t)\right) + w_i(t),\end{aligned} \tag{10}$$

where $k_p, k_u > 0$ are control gains and $w_i(t)$ models communication disturbances. Dynamics (10) in matrix form become

$$\dot{\boldsymbol{x}}(t) = \underbrace{\begin{bmatrix}\mathbf{0}_n & I_n\\ -k_pL & -k_uL\end{bmatrix}}_{A}\boldsymbol{x}(t) + \underbrace{\begin{bmatrix}\mathbf{0}_{n\times 1}\\ k_p\boldsymbol{\Delta}\end{bmatrix}}_{B} + \underbrace{\begin{bmatrix}\mathbf{0}_n\\ I\end{bmatrix}}_{F}\mathbf{w}(t), \tag{11}$$

where $\boldsymbol{x} = [\mathbf{P}\ \dot{\mathbf{P}}]^\mathsf{T} = [p_1, p_2, ..., p_n, \dot{p}_1, \dot{p}_2, ..., \dot{p}_n]^\mathsf{T}$, $\boldsymbol{\Delta} = [\Delta_1, \Delta_2, ..., \Delta_n]^\mathsf{T}$ in which $\Delta_i = \sum_{j\in\mathcal{N}_i}\Delta_{ij}$. Here $\mathbf{w}(t)$ is the vector of disturbances. We want to quantify the effect of the communication disturbances on the inter-vehicular distances. For this, we need to define an appropriate performance measurement. One such choice is $\mathbf{y} = \mathcal{B}^\mathsf{T}\mathbf{P}$, where $\mathcal{B} \in \mathbb{R}^{n\times|\mathcal{E}|}$ is the incidence matrix associated with the network and $\mathbf{P} = [p_1, p_2, ..., p_n]^\mathsf{T}$ is the vector of positions. In this case we have an output associated with each connection, i.e., $y_{ij} = p_i - p_j$ which is the distance between v_i and v_j at each time. With such performance output, we can quantify the sensitivity of inter-vehicular distances to communication disturbances. This sensitivity can be captured by an appropriate system norm from the disturbance signal to the desired output measurement. Here the system $\mathcal{H}_\infty$ norm is used which represents the worst case amplification of the disturbances over all frequencies and is widely used in the robustness analysis of vehicle platoons Herman et al. (2015). Such effect is discussed more formally in the following theorem.

Theorem 5. The system $\mathcal{H}_\infty$ norm of (11) from the external disturbances $\mathbf{w}(t)$ to $\mathbf{y} = \mathcal{B}^\mathsf{T}\mathbf{P}$ is

$$||G||_\infty = \begin{cases}\dfrac{2}{k_u\lambda_2\sqrt{4k_p - k_u^2\lambda_2}}, & \text{if } \dfrac{\lambda_2k_u^2}{2k_p} \le 1,\\ \dfrac{1}{k_p\lambda_2^{\frac{1}{2}}} & \text{otherwise.}\end{cases} \tag{12}$$

Proof. First we show that the system $\mathcal{H}_\infty$ norms of (11) from disturbance signals $\mathbf{w}(t)$ to performance outputs $\mathbf{y} = \mathcal{B}^\mathsf{T}\mathbf{P}$ and $\mathbf{y} = L^{\frac{1}{2}}\mathbf{P}$ are the same. For the output measurement $\mathbf{y} = \mathcal{B}^\mathsf{T}\mathbf{P}$ we have $G^*G = F^\mathsf{T}(s^*I - A)^{-\mathsf{T}}\mathcal{B}\mathcal{B}^\mathsf{T}(sI-A)^{-1}F = F^\mathsf{T}(s^*I-A)^{-\mathsf{T}}L(sI-A)^{-1}F$ and as system $\mathcal{H}_\infty$ norm is a function of the spectrum of G^*G, identical results will be obtained as if one used $\mathbf{y} = L^{\frac{1}{2}}\mathbf{P}$ instead of $\mathbf{y} = \mathcal{B}^\mathsf{T}\mathbf{P}$. Hence, it is sufficient to find the system $\mathcal{H}_\infty$ norm of (11) from disturbances to $\mathbf{y} = L^{\frac{1}{2}}\mathbf{P}$. Let $\Lambda = V^\mathsf{T}LV$ be the eigendecomposition of L, where V may be taken to be orthogonal. Consider the invertible change of states $\tilde{\boldsymbol{x}} = (V^\mathsf{T}\boldsymbol{x}, V^\mathsf{T}\dot{\boldsymbol{x}})$. Then a straightforward computation shows that

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}} &= \begin{bmatrix}0 & I_n\\ -k_p\Lambda & -k_u\Lambda\end{bmatrix}\tilde{\boldsymbol{x}} + \begin{bmatrix}0\\ V^\mathsf{T}\end{bmatrix}w\\ y &= \begin{bmatrix}L^{\frac{1}{2}}V & 0\end{bmatrix}\tilde{\boldsymbol{x}}.\end{aligned} \tag{13}$$

The model (13) has the same transfer function as (11), and hence the same system norm. Now consider an input/output transformation on (13), where $\bar{y} = V^{\mathsf{T}} y$ and $\bar{w} = V^{\mathsf{T}} w$, knowing the fact that such input/output transformation preserves the system $\mathcal{H}_\infty$ norm Pirani et al. (2017c). Hence, the transformed system

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}} &= \begin{bmatrix} 0 & I_n \\ -k_p\Lambda & -k_u\Lambda \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} 0 \\ \underbrace{V^{\mathsf{T}}V}_{=I_n} \end{bmatrix} \bar{w} \\ \bar{y} &= \underbrace{\begin{bmatrix} V^{\mathsf{T}} L^{\frac{1}{2}} V & 0 \end{bmatrix}}_{=\begin{bmatrix} \Lambda^{\frac{1}{2}} & 0 \end{bmatrix}} \tilde{\boldsymbol{x}} . \end{aligned} \tag{14}$$

has the same system norm as (13). The system (14) is comprised of n decoupled subsystems, each of the form

$$\begin{aligned} \dot{\tilde{\boldsymbol{x}}}_i &= \begin{bmatrix} 0 & 1 \\ -k_p\lambda_i & -k_u\lambda_i \end{bmatrix} \tilde{\boldsymbol{x}}_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \bar{w}_i \\ \bar{y}_i &= \begin{bmatrix} \lambda_i^{\frac{1}{2}} & 0 \end{bmatrix} \tilde{\boldsymbol{x}}_i . \end{aligned} \tag{15}$$

with transfer functions

$$\tilde{G}_i(s) = \frac{\lambda_i^{\frac{1}{2}}}{s^2 + k_u\lambda_i s + k_p\lambda_i}, \qquad i \in \{1, \dots, n\} .$$

which gives $\tilde{G}_1(s) = 0$. For $i \in \{2, \dots, n\}$, we have

$$|\tilde{G}_i(j\omega)|^2 = \tilde{G}_i(-j\omega)\tilde{G}_i(j\omega) = \underbrace{\frac{\lambda_i}{(k_p\lambda_i - \omega^2)^2 + k_u^2\lambda_i^2\omega^2}}_{f(\omega)} .$$

Maximizing $|\tilde{G}_i(j\omega)|^2$ with respect to ω is equivalent to minimizing $f(\omega)$. By setting $\frac{df(\omega)}{d\omega} = 0$ we get $\bar{\omega}_1 = 0$ and $\bar{\omega}_2 = (k_p\lambda_i - \frac{1}{2}k_u^2\lambda_i^2)^{\frac{1}{2}}$ as critical points. Here $\bar{\omega}_2$ is the global minimizer of $f(\omega)$, unless $\frac{k_u^2\lambda_i}{2k_p} > 1$. Substituting these critical values back into the formula for $|\tilde{G}_i(j\omega)|^2$, we find for $i \in \{2, \dots, n\}$ that

$$||\tilde{G}_i||_\infty = \begin{cases} \dfrac{2}{k_u\lambda_i\sqrt{4k_p - k_u^2\lambda_i}}, & \text{if } \dfrac{\lambda_i k_u^2}{2k_p} \le 1, \\ \dfrac{1}{k_p\lambda_i^{\frac{1}{2}}} & \text{otherwise.} \end{cases} \tag{16}$$

Since $0 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n$ and $||\tilde{G}_i||_\infty$ is a monotonically decreasing function of λ_i, the result follows. □

Based on the above theorem, the algebraic connectivity of the network, λ_2, plays a major role in the $\mathcal{H}_\infty$ performance of the system. Figure 7 shows the considerable effect of increasing the connectivity index k on the $\mathcal{H}_\infty$ performance of dynamics (11) with parameters $k_p = 5$ and $k_u = 10$. According to this figure, for the case of 1-nearest neighbor platoon with size $n = 20$, the $\mathcal{H}_\infty$ norm is about 2, while it drops below 1 for $k = 2$ and below 0.5 for $k = 4$. This shows the effect of increasing the connectivity index k on the system $\mathcal{H}_\infty$ performance, as predicted by bounds on the algebraic connectivity in (4).

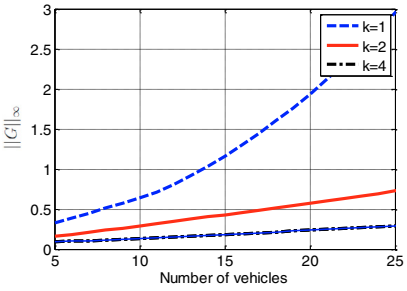

Fig. 7. The effect of increasing connectivity k on the $\mathcal{H}_\infty$ performance of dynamics (11).

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