University of Alberta

MODEL BASED FAULT DETECTION AND DIAGNOSIS OF LTI SYSTEMS USING LIKELIHOOD RATIO TESTS

by

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In loving memory of Dr. Wernher Von Braun who has always been a source of inspiration to me.

Abstract

This study aims to develop a new framework to detect, isolate and estimate the magnitude of additive faults that occur in linear time invariant (LTI) systems. This study starts with introduction of a new framework to deal with additive step type faults to improve the shortcomings attributed to detection of time of occurrence of the fault (TOF) in existing methods. In the next step, a marginalized likelihood ratio based approach is used to decouple detection and isolation phases from the estimation of fault magnitude. The final part of this study is dedicated to investigation of more realistic ramp and truncated ramp type faults.

In chapter 2 an alternative approach to implementation of the generalized likelihood ratio (GLR) test for detection and isolation of the fault in linear systems is proposed. The proposed approach offers the following advantages: 1) It overcomes the shortcomings of the previously suggested methods by accurately detecting the time of occurrence of the fault; 2) It uses statistical fault detection and confirmation tests to obtain a crude estimate of time of occurrence of the fault (TOF) and then refines the estimated TOF using an extended data window and the GLR test; and 3) It avoids performing the isolation in case the number of data points are too few and hence the number of misclassifications is significantly reduced.

Chapter 3 presents a fault detection and isolation (FDI) framework based on the marginalized likelihood ratio (MLR) approach using uniform priors for fault magnitudes in sensors and actuators. The existing methods in the literature use either flat priors with

infinite support or the Gamma distribution as priors for fault magnitudes. In the current study, it is assumed that the fault magnitude is a realization of a uniform prior with known upper and lower limits. The method presented in this study performs detection of time of occurrence of the fault and isolation of the fault type simultaneously while the estimation of the fault magnitude is achieved using a generalized likelihood ratio (GLR) based approach. The third and the final chapter aims to provide a solution for detection and diagnosis of drift type faults in linear time invariant systems using the generalized likelihood ratio (GLR) test. The main goal of this study is to deal with more realistic ramp type faults instead of abrupt jumps. In this regard, several ramp type fault scenarios are considered: pure ramps, truncated ramps and step-type faults which behave as ramp faults during their inter-sample. In addition, a modified GLR-based approach that is capable of accurately detecting the time of occurrence of the fault, is used in this study. The proposed method uses statistical fault detection and confirmation tests to obtain a crude estimate of time of occurrence of the fault (TOF) and then refines the estimated TOF using an extended data window and the GLR test.

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Chapter 1

Introduction

Model based fault detection and diagnosis is one of the methodologies that aims to provide the supervisory control system with sufficient information to take possible corrective actions to compensate for the instrumentation and process related faults (Basseville and Nikiforov, 1993; Gertler, 1998; Narasimhan and Jordache, 2000; Chen and Patton, 1999; Ding, 2008; Patton *et al.*, 1989). The ultimate goal of such methods is to accurately detect the time of occurrence of the fault (TOF), identify its root cause and estimate its magnitude. One of the most comprehensive studies in this area has been proposed by (Basseville and Nikiforov, 1993) which addresses each of the detection, isolation and estimation components using different techniques. In recent years, the concept of fault tolerant control (FTC) and integration of the fault detection and isolation (FDI) module with controller design has generated significant attention (Zhang and Jiang, 2008; Mhaskar *et al.*, 2006; Prakash *et al.*, 2005; Deshpande *et al.*, 2009; Mhaskar, 2006). As a general trend in active fault tolerant control systems, the supervisory control system tries to makes necessary modifications to the controller and thus mitigate the impact of the fault by means of the information provided by the FDI module. It is worth noting that in active fault tolerant control schemes, unlike the passive counterparts, detection of TOF plays a significant role in the overall performance of the control scheme.

The generalized likelihood ratio (GLR) based fault detection and isolation of dynamic systems was first addressed in the study by (Willsky and Jones, 1976) in which the concept of fault signature matrices was introduced to describe the effect of abrupt jumps in the states on the residuals generated by the Kalman filter. The methodology proposed therein used a sliding window and banks of Kalman filters to detect occurrence of the fault. The major drawback behind this approach, which proved to be the main motivation behind the study by (Narasimhan and Mah, 1988), is being “burdensome” from computational aspect. The suggested methodology by (Narasimhan and Mah, 1988) used the statistical time of occurrence detection (TOD) and gross error detection (GED) to detect occurrence of the fault and estimate its occurrence instant. Their proposed method removed the need for continuous operation of the fault detection and isolation (FDI) module which in turn results in significant reduction in computational effort. Nevertheless, using GED and TOD tests is associated with some inaccuracies in detection of time of occurrence of the fault (TOF) (Narasimhan and Mah, 1988; Villez *et al.*, 2011). The proposed method by (Prakash *et al.*, 2002) used similar statistical tests referred to as fault detection test (FDT) and fault confirmation test (FCT) and the the GLR test to detect and isolate the fault and estimate its magnitude and in addition made it possible to detect faults that might sequentially occur in an LTI system. This goal was achieved by maintaining the whiteness of residual through compensations applied to residuals and the Kalman filter. The same FDI framework was used in the study by (Prakash *et al.*, 2005) to form a fault tolerant control (FTC) scheme. However, as stated by (Villez *et al.*, 2011) all the aforementioned FDI approaches suffer from inaccurate detection of TOF which can lead to biased estimation of the fault magnitude and even in some cases misclassification of the fault. The method proposed in the study by (Villez *et al.*, 2011) it is suggested to perform an optimization to find a more accurate estimation of the TOF in an extended data window but no closed form solution is presented for the problem.

The main motivation behind the study which is presented in chapter 2 is to provide a FDI scheme which is capable of accurately detecting the TOF, isolate the fault and estimate its magnitude. It will be assumed that faults occur as abrupt jumps in sensor and actuators of an LTI system. The ultimate goal in this chapter would be providing the GLR test statistic and the closed form solution for the estimated fault magnitude based on a modified GLR-based approach. The proposed approach eliminated the need to solve any optimization problem and hence facilitates the implementation. Moreover, a new strategy would be presented which upon its adoption the FDI avoids performing the isolation and estimation phases when a sufficient number of data points is not available. It is worth noting that performing the isolation and estimation using insufficient data points can lead to biased estimates or even misclassification in the worst case scenario.

Another alternative approach proposed by (Basseville and Nikiforov, 1993), is weighting the likelihood ratio with respect to all possible values of the changing parameter. This concept was further investigated and complemented in the state of the art study by (Gustafsson, 1996). The concept therein, which is referred to as marginalized likelihood ratio (MLR), is based on weighting the likelihood ratio by a non-informative prior distribution and then performing integration in order to marginalize the conditional probability. By doing so, the fault magnitude would be eliminated from the likelihood ratio and therefore the double maximization GLR problem could be reduced to a single maximization over the TOF. However, it should be noted that the method proposed by (Gustafsson, 1996) does not address occurrence of the fault in sensors and actuators and only deals with abrupt jumps in states. Moreover, the isolation issue has not been discussed at all in the study by (Gustafsson, 1996). In another study by (Dos Santos and Yoneyama, 2011) the MLR approach has been used to detect abrupt jumps in sensors and actuators using the Gamma distribution as prior. The major drawback associated with the selection of Gamma distribution as prior can be attributed to the heavy penalization of low and high magnitude faults which could be deemed totally unrealistic. Motivated to overcome these shortcomings, the study presented in chapter 3 aims to use realistic uniform priors to derive a new MLR approach which decouples isolation and estimation phases. Since it is widely known that most process variables are bounded, there arises a need to define realistic and practical priors which are also coherent with the concept of principle of indifference.It is worth noting that similar studies have appeared in the literature addressing change-point detection problem in regression models and time series (Hinkley, 1969; Kalbfleisch and Sprott, 1970; Lee and Heghinian, 1977; Esterby and El-Shaarawi, 1981). Nevertheless, most of these studies address the change-point detection in an offline framework while in the FDI domain it is necessary to find the time of occurrence of the fault (TOF), isolate the fault and estimate its magnitude online. In other words, in the online FDI domain one should deal with dynamic systems where the residuals are generated at each time sample, unlike offline studies mentioned earlier where the whole data set is not available beforehand.

The third and the final chapter aims to provide a solution for detection and diagnosis of incipient faults in LTI systems using GLR-based approach. In order to achieve this goal the new fault signature matrices for ramp and truncated ramp type faults should be first developed. The main goal of this study is to deal with more realistic ramp type faults instead of abrupt jumps. In this regard, both ramp and truncated ramp type faults are considered and the GLR test statistics and the closed form solution for the estimated slope of the ramp are developed. The most critical phase in derivation of the new FDI scheme is detection of the time instant in which the ramp occurs and in case of truncated ramp type faults the time instant at which the fault does the transition to the steady state is also of great importance. The proposed approach can be further extended to deal with step type faults due to the fact that the inter-sample behavior between two consecutive time instants cannot be captured in a discrete system. It is shown that estimation of TOF and the magnitude of ramp type faults using a limited number of data points can cause serious problems as the fault propagates with time even when there exists a small offset between the true and the estimated slope of the fault. Therefore, in the newly proposed FDI scheme a unified framework based on the truncated ramp type faults would be developed to deal with ramp, truncated ramp and step type faults.

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Chapter 2

An Alternative Approach to Implementation of Generalized Likelihood Ratio Test for Fault Detection and Isolation

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2.1 Introduction

Model based fault detection and diagnosis is one of the major approaches that aims to provide the supervisory control system with sufficient information to take possible corrective actions to compensate for the instrumentation related faults (Basseville and Nikiforov, 1993; Gertler, 1998; Narasimhan and Jordache, 1999; Chen and Patton, 1999; Ding, 2008; Patton *et al.*, 1989). The ultimate goal of such methods is to accurately detect the time of occurrence of the fault, identify its root cause and estimate its magnitude. The comprehensive and outstanding study by (Basseville and Nikiforov, 1993) addresses each of the detection, isolation and estimation components using different techniques. In recent years, the concept of fault tolerant control and integration of the fault detection and isolation (FDI) module with controller design has generated significant attention (Zhang and Jiang, 2008; Mhaskar *et al.*, 2006; Prakash *et al.*, 2005; Deshpande *et al.*, 2009; Mhaskar, 2006). As a general trend in active fault tolerant control systems, the supervisory control system tries to makes necessary modifications to the controller and thus mitigate the impact of the fault by means of the information provided by the FDI module.

A seminal study in this field was first proposed by (Willsky and Jones, 1976) which for the first time addressed the online FDI issue using the generalized likelihood ratio (GLR) test. The methodology therein, proposes the computation of signature matrices assuming occurrence of abrupt jumps in states of a linear system, which could be used to solve the composite hypotheses testing problem using the GLR approach. The GLR formulation in thr study by (Willsky and Jones, 1976) tries to find the time of occurrence of the fault using a sliding window and as stated by (Narasimhan and Mah, 1988) the major drawback of the methodology proposed by (Willsky and Jones, 1976) is that it is overly "burdensome" from computational aspects. This is due to the fact that the FDI module solves, in real-time, the composite hypotheses testing problem in a sliding window to detect and isolate the faults. This issue proved to be the motivation in the study by (Narasimhan and Mah, 1988) to develop the fault detection test (FDT) and fault confirmation test (FCT) to detect the time of occurrence of the fault. However, it should be noted that the common issue in both methods is the compromise between type I and type II errors which arises from the thresholds selected for the GLR test or the significance levels of FDT and FCT tests. While the use of FCT and FDT considerably reduces the computational costs associated with detecting time of occurrence of the fault, this approach can result in significant inaccuracy in the estimation of the time of occurrence of the fault. These inaccuracies, in turn, affect the overall performance of the FDI module.

The FDI scheme suggested by Prakash et al. (Prakash *et al.*, 2002) utilized a combination of methodologies proposed in (Willsky and Jones, 1976; Narasimhan and Mah, 1988) to deal with the additive step-type faults in sensors, actuators and process parameters of an LTI system. The scheme proposed therein, took advantage of the FDT and FCT tests to detect time of occurrence of the fault and subsequently, the pre-computed fault signature matrices and the GLR test were used to isolate the fault and estimate its magnitude. The compensation scheme suggested in their study for the states alone, was similar to "Direct State Incrementation" proposed in (Willsky and Jones, 1976) and unlike the approach in (Narasimhan and Mah, 1988), made it possible to deal with sequential faults. The fault tolerant control (FTC) scheme proposed by (Prakash *et al.*, 2005) modified the controller with the help of the data provided by the FDI scheme whilst in the proposed method by (Prakash *et al.*, 2002) the FDI is not integrated with the controller and it is only used to provide supervisory information. However, as stated by (Villez *et al.*, 2011), the detection of time of occurrence of the fault is not properly addressed in the FDI scheme proposed by (Prakash *et al.*, 2002) and this shortcoming makes the performance of the FDI system susceptible to misclassifications.

The main aim of the current study is to develop a FDI scheme for LTI systems to deal with bias type faults which may occur in sensors and actuators separately or consecutively. In particular, it is desired to refine the methodology proposed by (Prakash *et al.*, 2005; Prakash *et al.*, 2002) to accurately detect the time of occurrence of a fault, which in turn, is expected to improve the fault magnitude identification. In this regard a GLR-based approach using an extended data window is used in this study. In the proposed methodology, detection of the time of the occurrence of the fault as well as the isolation of the fault type and estimation of its magnitude would be undertaken by the GLR approach. The proposed approach offers improvement over FDT and FCT tests by applying GLR test over an extended time window, which is constructed with the help of FDT and FCT tests and some extra data points from the historical data. This is a crucial step when FDI is carried out on closed loop systems since any error in estimation of time of occurrence of the fault can lead to misclassification and degradation in the closed loop control performance. The newly proposed FDI scheme is more likely to capture the complete signature of the fault within the batch of data in comparison with the results reported by (Prakash *et al.*, 2002; Prakash *et al.*, 2005). The main contributions of the current study can be summarized as follows:

- Introduction of the concept of an extended data window which is constructed from the FDT/FCT data window and the historical data points;

- Development of a FDI scheme using GLR test and FDT/FCT tests which can accurately detect time of occurrence of the fault (TOF) without the need to use the sliding window. In this scheme the GLR test is used both as a detector and isolator while the FDT/FCT tests provide a crude estimate of TOF;

- Modification of GLR-based FDI scheme in order to prevent it from misclassifying the fault when sufficient number of data points are not available;

- Monte Carlo simulations show reduction in the number of false alarms and misclassifications and improvement in the estimates of the fault magnitudes using the proposed FDI scheme.

- A closed form solution of the test statistic is proposed for additive faults. In other words, for linear systems in the presence of additive faults it is not required to perform cumbersome optimization to gain more accuracy. Moreover, ML-estimate of the fault magnitude would also have a closed form solution and hence extra computational costs related to optimization can be avoided.

It is worth mentioning that, the ultimate goal of the proposed FDI scheme is to provide advisory information to operators so that they can take necessary measures such as re-calibration of the sensors/actuators. This chapter is organized in five sections. In the next section, formulation of change detection problem and a brief review of generalized likelihood ratio test is presented. The newly proposed FDI scheme is discussed in Section 3. Section 4 is dedicated to evaluation of the proposed methodology by application to a CSTR benchmark problem followed by concluding remarks in Section 5.

2.2 Definition of The Problem

2.2.1 Model of the System

Consider the following linear system where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{y} \in R^r$. In this representation $\mathbf{w} \in R^q$ and $\mathbf{v} \in R^r$ are process and measurement noise sequences with known covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ which are mutually uncorrelated and the initial state $x(0)$ follows a Gaussian distribution with known mean and variance. Furthermore, it is assumed that Φ, $\mathbf{C}$, Γ_u and Γ_w are known matrices.

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_u\mathbf{u}(k) + \Gamma_w\mathbf{w}(k) \tag{2.1}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \tag{2.2}$$

In case of occurrence of bias with magnitude $b_{u,j}$ in the jth actuator at time instant t the process would evolve for $k > t$ as follows:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_u\mathbf{u}(k) + \Gamma_w\mathbf{w}(k) + b_{u,j}\Gamma_u\mathbf{e}_{u,j}\sigma(k-t) \tag{2.3}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \tag{2.4}$$

where $\mathbf{e}_{u,j}$ is a unit vector whose jth element is equal to one and all other elements are zero and $\sigma(k-t)$ is a unit step function defined as follows:

$$\sigma(k-t) = \begin{cases} 0 & \text{if } k < t \\ 1 & \text{if } k \geq t \end{cases}$$

Similarly, in case of occurrence of bias with magnitude $b_{y,j}$ in the jth sensor, the measurement equation would be modified as follows while the state equation remains as shown in Eq. 2.1:

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) + b_{y,j}\mathbf{e}_{y,j}\sigma(k-t) \tag{2.5}$$

where $e_{y,j}$ is a unit vector whose jth element is equal to one and all other elements are equal to zero.

2.2.2 Brief Review of GLR

The log-likelihood ratio test assuming occurrence of fault at time instant t with magnitude $b_{f,j}$ where $f \in \{u, y\}$ and j represent the type and location of fault, respectively, can be defined as:

$$\mathbf{T} = \max_{t} \max_{b_{f,j}} 2 \log \frac{p(\Lambda_1^N | t, b_{f,j})}{p(\Lambda_1^N)} \tag{2.6}$$

where, $t \in \{1, \ldots, N\}$ represents the time instant at which the fault has occurred and $\Lambda_1^N = \{\gamma_1, \cdots, \gamma_N\}$ denotes the residuals generated by a Kalman filter using the fault-free model defined in Eqs. 2.1 and 2.2 in the specified window. In this notation the denominator $p(\Lambda_1^N)$ represents the null-hypothesis (fault-free case). It is worth mentioning that the denominator is independent of TOF and fault magnitude due to the fact that it represents the residuals under the fault-free condition and therefore the joint maximum likelihood (ML) estimate of the time of occurrence of the fault (TOF) and fault magnitude, respectively, can be found as follows:

$$\{\hat{b}_{f,j}, \hat{t}\} = arg \max_{t} \max_{b_{f,j}} 2 \log p(\Lambda_1^N | t, b_{f,j}) \tag{2.7}$$

This double maximization problem is further simplified by finding the estimated time of occurrence of the fault $\hat{t}$ by means of FDT and FCT tests (Prakash *et al.*, 2005; Prakash *et al.*, 2002; Narasimhan and Mah, 1988). Thus, the problem reduces to a single maximization over the fault magnitude $b_{f,j}$. The generalized likelihood ratio (GLR) test is as follows:

$$\mathbf{T}^{GLR} = \max_{f,j} \mathbf{T}_{f,j}^{max} \tag{2.8}$$

where

$$\mathbf{T}_{f,j}^{max} = \max_{b_{f,j}} 2\log\frac{p(\Lambda_{\hat{t}}^{N}|b_{f,j})}{p(\Lambda_{\hat{t}}^{N})} \quad (2.9)$$

$$= \max_{b_{f,j}} \mathbf{T}_{f,j} \quad (2.10)$$

The likelihood ratio in Eq.2.9 should be maximized for all hypothesized faults and then the maximum value among all hypothesized faults will determine location and magnitude of the fault. Note that $\hat{t}$ is obtained by application of the FDT/FCT tests and therefore the maximization over TOF (t) is eliminated in Eq.2.9.

The effect of an additive fault on a linear system can be found by modeling the evolution of the innovation sequence after the occurrence of the fault (Willsky and Jones, 1976). However, for a nonlinear system this task should be undertaken by a fault mode observer (Deshpande *et al.*, 2009). The expected value of the innovation sequence at any time $k \geq t$ are given as follows(Prakash *et al.*, 2002; Narasimhan and Mah, 1988):

$$E[\gamma(k)] = b_{f,j}\mathbf{G}_f(k;t)\mathbf{g}_{f,j} \quad ; \quad k \geq t \quad (2.11)$$

where

$$\gamma(k) = \mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k|k-1)$$

Moreover the difference between the true states and the estimated ones ($\delta\hat{\mathbf{x}}$) and its expected value can be defined as follows:

$$\delta\hat{\mathbf{x}}(k) = \hat{\mathbf{x}}(k|k) - \mathbf{x}(k)$$

$$E[\delta\hat{\mathbf{x}}(k)] = b_{f,j}\mathbf{J}_f(k;t)\mathbf{g}_{f,j} \quad ; \quad k \geq t$$

where $\mathbf{g}_{u,j} = \Gamma_u\mathbf{e}_{u,j}$ and $\mathbf{g}_{y,j} = \mathbf{e}_{y,j}$ for actuators and sensors, respectively. In this representation $\mathbf{G}_f(k;t)$ is the signature matrix and it depends on time k at which the innovations are computed and on time t at which the fault has occurred. The signature matrices for the sensor and actuator faults are given as follows(Prakash *et al.*, 2002):

$$\mathbf{G}_y(k;t) = \mathbf{I} - \mathbf{C}\Phi\mathbf{J}_y(k-1;t) \tag{2.12}$$

$$\mathbf{J}_y(k;t) = \Phi\mathbf{J}_y(k-1;t) + \mathbf{K}(k)\mathbf{G}_y(k;t) \tag{2.13}$$

$$\mathbf{G}_u(k;t) = \mathbf{C} - \mathbf{C}\Phi\mathbf{J}_u(k-1;t) \tag{2.14}$$

$$\mathbf{J}_u(k;t) = \Phi\mathbf{J}_u(k-1;t) + \mathbf{K}(k)\mathbf{G}_u(k;t) - \mathbf{I} \tag{2.15}$$

The proof of fault signature matrices is omitted here for the sake of brevity nevertheless the interested reader is referred to Prakash et al. (Prakash *et al.*, 2002) for details. Equations 2.12-2.13 should be initialized by $\mathbf{J}_y(t-1;t) = [0]_{n\times r}$ and $\mathbf{G}_y(t-1;t) = [0]_{r\times r}$ whilst Eqs. 2.14-2.15 should be initialized with $\mathbf{J}_u(t;t) = [0]_{n\times n}$ and $\mathbf{G}_u(t;t) = [0]_{r\times n}$ due to one sample time delay after which the fault in the actuator will affect the system dynamics. In addition, it should be noted that the signature matrices are only function of the system matrices in a linear system.

2.3 Proposed FDI Scheme

2.3.1 Finding the Candidate Data Window for Occurrence of The Fault

The FDI scheme in this study takes advantage of the statistical FDT and FCT tests to find the candidate data window in which the fault has occurred. However, due to the inaccuracies associated with this approach a refining mechanism should be used to find the most probable TOF. The FDT test is based on a quadratic form of the residuals at each time instant normalized using the corresponding covariance matrix while the FCT test is sum of normalized quadratic residuals in a window of data with specific size where the normalization is performed using the corresponding covariance matrices. Assuming rejection of FDT test at time instant t_1 and data window of size N, the FCT test is defined as follows:

$$\epsilon(N; t_1) = \sum_{k=t_1}^{t_1+N} \gamma^T(k)V(k)^{-1}\gamma(k) \tag{2.16}$$

which follows a χ^2 distribution with $r \times (N + 1)$ degrees of freedom assuming the residuals are zero mean white noise with known covariance matrix. The conventional approach in GLR is to compute the FCT test for data window $[t_1, t_1 + N]$ which is obtained after rejection of the FDT test at time instant t_1(Narasimhan and Mah, 1988; Prakash *et al.*, 2005; Prakash *et al.*, 2002). Upon its rejection, the likelihood ratio will be computed for all hypothesized faults in the interval $[t_1, t_1 + N]$ to isolate and estimate the fault magnitude. Now let us consider the following two scenarios as depicted in Figure 2.1, in which in the

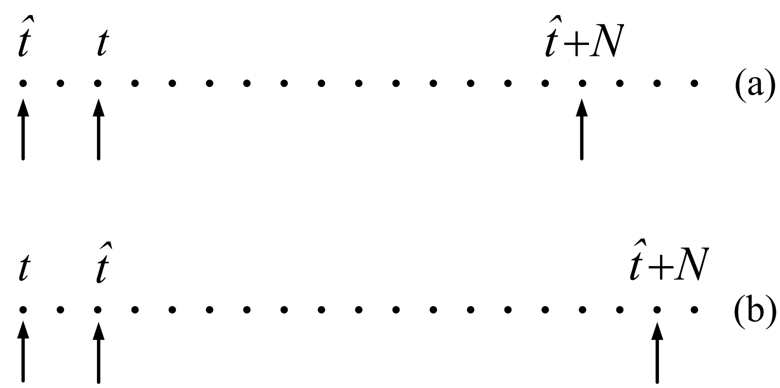

Figure 2.1: Scenarios leading to inaccurate detection of TOF: Figure 2.1a the estimated TOF ($\hat{t}$) precedes the actual TOF (t); Figure 2.1b the estimated TOF ($\hat{t}$) is after the actual TOF (t) (Narasimhan and Mah, 1988).

first case the actual TOF is after the rejection of the FDT test while in second scenario the FDT is rejected after the actual TOF (Narasimhan and Mah, 1988). The second case is most prevalent if the fault magnitude is very small. So in order to accurately estimate TOF, we propose to append the original data window $[t_1, t_1 + N]$ with M extra samples prior to rejection of the FDT test at t_1 and form a new data window as $[t_1 - M, t_1 + N]$. The concept of extended data window is depicted in Figure 2.2.

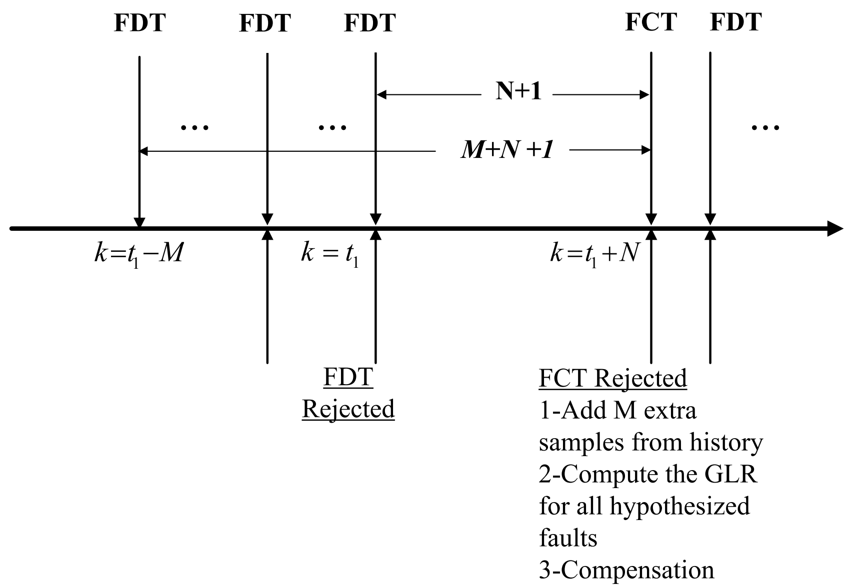

Figure 2.2: Extended FCT data window

2.3.2 An Alternative Approach to GLR test

The main purpose of this section is to propose a FDI scheme which has the following properties:

- Fast and easy detection of TOF using the statistical FDT/FCT tests without using any of sliding window schemes;
- Detecting the most probable TOF considering the scenarios mentioned in the previous section;
- Avoiding any corrective action/fault compensation when only few number of data points is available;

In the wake of the modified data window proposed in the previous section the GLR test for each fault hypothesis assuming occurrence of the fault at time instant $t \in [t_1 - M, t_1 + N]$

is as follows:

$$\mathbf{T}^{GLR} = \max_{f,j,t_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j}} \tag{2.17}$$

where

$$\begin{aligned}\mathbf{T}^{max}_{f,j,t_{f,j}} &= \max_{t} \max_{b_{f,j}} 2\log \frac{p(\Lambda^{t_1+N}_{t_1-M}|t, b_{f,j})}{p(\Lambda^{t_1+N}_{t_1-M})} &(2.18)\\ &= \max_{t} \mathbf{T}_{t,f,j} &(2.19)\end{aligned}$$

it should be noted that:

$$\mathbf{T}_{f,j,t} = \max_{b_{f,j}} 2\log \frac{p(\Lambda^{t_1+N}_{t_1-M}|t, b_{f,j})}{p(\Lambda^{t_1+N}_{t_1-M})}$$

and $\Lambda^{t_1+N}_{t_1-M} = \{\gamma_{t_1-M}, \ldots, \gamma_{t_1+N}\}$. The likelihood ratio on the right hand side of Eq. 2.18 should be maximized for all $t \in [t_1 - M, t_1 + N]$ assuming occurrence of a specific fault type f at location j. Then the same maximization procedure should be repeated for the other hypothesized faults (all combinations of f and j). Finally, the maximum value of $\mathbf{T}^{max}_{f,j,t_{f,j}}$ among all hypothesized faults determines $\hat{t}$, $\hat{f}$ and $\hat{j}$ as follows:

$$\{\hat{f}, \hat{j}, \hat{t}\} = arg \max_{f,j,t_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j}} \tag{2.20}$$

using $\hat{t}$, $\hat{f}$ and $\hat{j}$, the ML-estimate of fault magnitude can be found as:

$$\hat{b}_{\hat{f},\hat{j}} = arg \max_{b_{\hat{f},\hat{j}}} 2\log p(\Lambda^{t_1+N}_{t_1-M}|\hat{t}, b_{\hat{f},\hat{j}}) \tag{2.21}$$

Lemma 2.3.1 *Assuming occurrence of a single fault type f at location j and at time instant $t \in [t_1 - M, t_1 + N]$, the estimated fault magnitude and $\mathbf{T}_{f,j,t}$ can be computed as follows:*

$$\hat{b}_{f,j,t} = \frac{g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_f(k;t) V(k)^{-1} \gamma(k)}{g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_f(k;t) V(k)^{-1} \mathbf{G}_f(k;t) g_{f,j}} \tag{2.22}$$

$$\mathbf{T}_{f,j,t} = \frac{\left(g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_f(k;t) V(k)^{-1} \gamma(k)\right)^2}{g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_f(k;t) V(k)^{-1} \mathbf{G}_f(k;t) g_{f,j}} \tag{2.23}$$

where $\mathbf{G}_f(k;t)$ denotes the fault signature matrix for fault f assuming occurrence of the fault at time instant t.

Proof: Recall the joint maximum likelihood estimate of fault magnitude and TOF:

$$\{\hat{b}_{f,j}, \hat{t}\} = arg \max_{t} \sup_{b_{f,j}} 2 \log p(\Lambda_{t_1-M}^{t_1+N} | t, b_{f,j}) \tag{2.24}$$

Since we have already assumed that true TOF is t, Eq. 2.24 reduces to:

$$\hat{b}_{f,j} = arg \max_{b_{f,j}} 2 \log p(\Lambda_{t}^{t_1+N} | b_{f,j}) = arg \max_{b_{f,j}} \lambda_{f,j}$$

using the definition of the multivariate Gaussian distribution and assuming independence of the residuals, one can write the following:

$$\lambda_{f,j} = 2 \log \left\{ \zeta \exp \left\{ -\frac{1}{2} \sum_{k=t}^{t_1+N} (\gamma(k) - b_{f,j}\mathbf{G}_f(k;t)g_{f,j})^T V(k)^{-1} (\gamma(k) - b_{f,j}\mathbf{G}_f(k;t)g_{f,j}) \right\} \right\} \tag{2.25}$$

where $\zeta = \frac{1}{(2\pi)^{r \times (t_1+N-t+1)/2} \prod_{k=t}^{t_1+N} |V(k)|^{1/2}}$ and r denotes the dimension of $\mathbf{y}$. The ML-estimate of $b_{f,j}$ can be found by solving $\frac{\partial \lambda_{f,j}}{\partial b_{f,j}} = 0$ as follows:

$$-2g_{f,j}^T \sum_{k=t}^{t_1+N} \mathbf{G}_f^T(k;t)V(k)^{-1}\gamma(k) + 2b_{f,j}g_{f,j}^T \sum_{k=t}^{t_1+N} \left\{ \mathbf{G}_f^T(k;t)V(k)^{-1}\mathbf{G}_f(k;t) \right\} g_{f,j} = 0$$

$$\Rightarrow \hat{b}_{f,j,t} = \frac{g_{f,j}^T \sum_{k=t}^{t_1+N} \mathbf{G}_f^T(k;t)V(k)^{-1}\gamma(k)}{g_{f,j}^T \sum_{k=t}^{t_1+N} \mathbf{G}_f^T(k;t)V(k)^{-1}\mathbf{G}_f(k;t)g_{f,j}} \tag{2.26}$$

Note that we have used notation $\hat{b}_{f,j,t}$ and not $\hat{b}_{f,j}$. This is due to the fact that the estimated fault magnitude is based on the assumption of occurrence of the fault at time instant t. Substituting $\hat{b}_{f,j,t}$ into the log-likelihood ratio in the right-hand side (RHS) of Eq.2.18 results in:

$$\mathbf{T}_{f,j,t} = \frac{\left(g_{f,j}^T \sum_{k=t}^{t_1+N} \mathbf{G}_f^T(k;t)V(k)^{-1}\gamma(k) \right)^2}{g_{f,j}^T \sum_{k=t}^{t_1+N} \mathbf{G}_f^T(k;t)V(k)^{-1}\mathbf{G}_f(k;t)g_{f,j}}$$

■

The estimated fault magnitude of the isolated fault can be found by combining Eqs. 2.20 and 2.26 as follows:

$$\hat{b}_{\hat{f},\hat{j},\hat{t}} = \frac{g^T_{\hat{f},\hat{j}} \sum_{k=\hat{t}}^{t_1+N} \mathbf{G}^T_{\hat{f}}(k;\hat{t}) V(k)^{-1} \gamma(k)}{g^T_{\hat{f},\hat{j}} \sum_{k=\hat{t}}^{t_1+N} \mathbf{G}^T_{\hat{f}}(k;\hat{t}) V(k)^{-1} \mathbf{G}_{\hat{f}}(k;\hat{t}) g_{\hat{f},\hat{j}}} \tag{2.27}$$

The procedure which leads to estimation of TOF, isolation of the fault and estimation of its magnitude can be summarized as follows:

1. FDT test is applied at each time instant. Upon rejection of the FDT test, the FCT test is applied.

2. Upon rejection of FCT test, the extended data window is formed by means of adding M extra data points prior to the rejection of FDT test, to original FCT data window.

3. The following test statistic is computed for all $t \in [t_1 - M, t_1 + N]$, for a specific choice of f and j:

$$\mathbf{T}_{f,j,t} = \frac{\left(g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_{f}(k;t) V(k)^{-1} \gamma(k) \right)^2}{g^T_{f,j} \sum_{k=t}^{t_1+N} \mathbf{G}^T_{f}(k;t) V(k)^{-1} \mathbf{G}_{f}(k;t) g_{f,j}}$$

4. $\mathbf{T}^{max}_{f,j,t_{f,j}}$ for the chosen f and j can be computed as follows:

$$\mathbf{T}^{max}_{f,j,t_{f,j}} = \max_t \mathbf{T}_{f,j,t}$$

5. For all other combinations of f and j, steps 3 and 4 are repeated.

6. The maximum test statistic among all hypothesized faults will determine $\hat{t}$, $\hat{f}$ and $\hat{j}$ as follows:

$$\{\hat{f}, \hat{j}, \hat{t}\} = arg \max_{f,j,t_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j}}$$

7. The estimated fault magnitude can be found for the isolated fault using $\hat{t}$, $\hat{f}$ and $\hat{j}$ as follows:

$$\hat{b}_{\hat{f},\hat{j},\hat{t}} = \frac{g^T_{\hat{f},\hat{j}} \sum_{k=\hat{t}}^{t_1+N} \mathbf{G}^T_{\hat{f}}(k;\hat{t}) V(k)^{-1} \gamma(k)}{g^T_{\hat{f},\hat{j}} \sum_{k=\hat{t}}^{t_1+N} \mathbf{G}^T_{\hat{f}}(k;\hat{t}) V(k)^{-1} \mathbf{G}_{\hat{f}}(k;\hat{t}) g_{\hat{f},\hat{j}}}$$

8. Compensation is performed and again the FDT test is applied at the next time instant.

It should be noted that unlike the FDI strategy proposed by (Willsky and Jones, 1976), it is not required that the banks of Kalman filters continuously monitor the residuals for occurrence of the fault. Moreover, by means of adding the extra samples to the data window and searching through the extended data window for the most likely time of occurrence of the fault (TOF), one of the main shortcomings of the proposed approach by (Prakash *et al.*, 2002) is overcome.

Remark 2.3.1 *In order to prevent false alarms and also to accurately estimate the magnitude of the fault, a minimum size of data window* s *is required. In other words, if* $\hat{t}_{\hat{f},\hat{j}} > t_1 + N - s + 1$*, then the number of samples may not be enough to obtain a good estimate of the fault magnitude and hence it seems logical to obtain more data points for the purpose of estimation. Furthermore if* $\hat{t}_{\hat{f},\hat{j}} > t_1 + N - s + 1$ *and the fault has actually occurred, since no corrective action is taken by the FDI methodology, the FDT and FCT tests would be rejected again in the following instants and the fault would be detected in the next or subsequent window due to the fact that the extended data window enables the FDI module to look backwards in time. In choosing the minimum size of data window, the time required for the estimator to converge after a change occurs has to be also considered.*

2.3.3 Online Compensation Schemes

The interaction between the proposed FDI scheme and the process is depicted in Figure 2.3.

The compensation equations should be applied in order to make the residuals white after occurrence of the fault and subsequent detection, isolation and estimation phases. The online corrections, which lead to the whiteness of the residuals and then enable the FDI to detect faults that might occur sequentially, are as follows (Prakash *et al.*, 2002):

$$y_c(k) = y(k) - \hat{b}_{y,\hat{j},\hat{t}}\mathbf{e}_{y,\hat{j}} \quad (\hat{f} = y) \tag{2.28}$$

$$\mathbf{m}_c(k) = \mathbf{m}(k) + \hat{b}_{u,\hat{j},\hat{t}}\mathbf{e}_{u,\hat{j}} \quad (\hat{f} = u) \tag{2.29}$$

$$\hat{\mathbf{x}}_c(t_1 + N|t_1 + N) = \hat{\mathbf{x}}(t_1 + N|t_1 + N) - \hat{b}_{\hat{f},\hat{j},\hat{t}}\mathbf{J}_{\hat{f}}(t_1 + N; \hat{t})\mathbf{g}_{\hat{f},\hat{j}} \tag{2.30}$$

where $k \in [\hat{t}, t_1+N]$ and $f \in \{u, y\}$. In addition, $\mathbf{m}$ denotes the controller output (Prakash *et al.*, 2002). It should be noted that state compensation in Eq. 3.41 is performed only once and after the isolation and estimation phases are carried out. The main purpose of this compensation is to provide the Kalman filter with a bias-free estimated state in the next iteration.

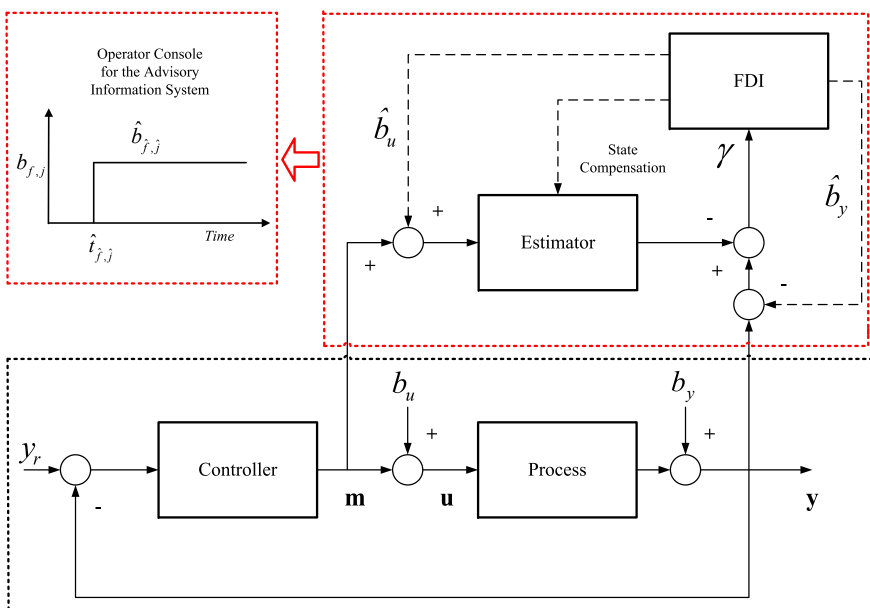

Figure 2.3: The supervisory scheme based on the modified GLR. The advisory information will be provided to the operators/ engineers.

2.4 Simulation Case Study

The nonisothermal continuous stirred tank reactor (CSTR) benchmark example by (Marlin, 1995) which has also been used by (Prakash *et al.*, 2005), is used in this work to evaluate the performance of the proposed methodology. This plant has two states which are the reactor concentration (C_A) and reactor temperature (T) which are both measurable. The reactor feed flow rate (F) and the coolant flow rate (F_c) are selected to be the manipulated variables and the feed concentration (C_{A0}) and feed temperature (T_{cin}) are set as the disturbance variables. Moreover, it should be noted that all the variables are in deviational form.

$$\mathbf{x} = [C_A \quad T]^T; \qquad \mathbf{u} = [F \quad F_c]^T$$

$$\mathbf{w} = [C_{A0} \quad T_{cin}]^T; \qquad \mathbf{y} = [C_A \quad T]^T$$

The stable operating point of the reactor has been used for performing the simulations and the process was linearized around this operating point. The sampling time was selected as $T_{sample} = 0.1\ min$ for discretization. The resulting state space matrices are as follows:

$$\Phi = \begin{bmatrix} 0.1843 & -0.0080 \\ 73.5080 & 1.3330 \end{bmatrix}, \quad \Gamma_u = \begin{bmatrix} 0.1340 & 0.0026 \\ -1.7948 & -.7335 \end{bmatrix}$$

$$\Gamma_w = \begin{bmatrix} 0.0598 & -0.0004 \\ 3.9038 & 0.1208 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The steady state values of inlet and the coolant flow rates are $F_s = 1\ m^3/min$ and $F_{c_s} = 15\ m^3/min$, at this operating point. The MPC weighting matrix, prediction horizon and control horizon were set to the same values as reported in the study by (Prakash *et al.*, 2005) and are reproduced below:

$$N_p = 10, \quad N_c = 1, \quad W_E = \begin{bmatrix} 10^4 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_U = [0]$$

The following constraints were also considered (Prakash *et al.*, 2005):

$$0 \leq F_c \leq 17\ m^3/min \qquad 0 \leq F \leq 2\ m^3/min$$

All the noise parameters are exactly set to values specified in the simulations performed by (Prakash *et al.*, 2005), which are summarized in Table 2.1. The performance of the FDI schemes were tested subject to two different sets of faults. The first set only includes single faults and is summarized in Table 2.2 whilst the sequential faults introduced to the system are included in Table 2.3. In the Monte Carlo simulations (consisting of 100 runs for each case), the performance of the $\chi^2 - GLR$ approach proposed by (Prakash *et al.*, 2002) was compared to the proposed approach in this study which from now on is referred to as $\chi^2 - Modified\,GLR$ or in brief $\chi^2 - MGLR$ method. In the Monte Carlo simulations, the size of FCT data window was set to 21 ($N = 20$) while the number of extra samples appended to the FCT data window to form the extended data window was selected as $M = 21$. The parameter 's' which was discussed in Remark 2.3.1, was set to 15.

Remark 2.4.1 *It is known that a small number of data points in the data window will result in poor estimation of fault magnitude as well as more false alarms or even misclassification of the fault. On the other hand waiting to acquire a large number of data points will result in significant delay in detection, isolation and estimation of the fault magnitude. In the study by (Prakash* et al.*, 2002), based on the simulation results, it is suggested that size of FCT window be chosen approximately half the size required for the Kalman filter to converge after a fault occurs. Considering this argument, the following criterion is suggested for selection of parameter 's':*

$$\tau_K \leq s \leq W_{FCT}$$

where W_{FCT} is the size of FCT window and τ_K is half the size required for Kalman filter to converge if a fault occurs. In our case we have selected $N = 20$, $s = 15$ and $M = 21$ where M is the number of extra points added to the FCT data window. Moreover, based on simulation results it was found that $\tau_K = 10$.

The results of FDI schemes tested subject to the single fault scenarios of Table 2.2 and the sequence of faults shown in Table 2.3, are summarized in Tables 2.4 and 2.5, respectively. As can be seen, the results show that the $\chi^2 - MGLR$ method is capable of estimating the TOF more accurately and with better precision compared to the $\chi^2 - GLR$

Table 2.1: Standard Deviations for Process and Measurement Noise Sequences

Variable	σ**(SD)**
Feed Concentration	0.05 $Kmol/m^3$
Feed Temperature	2.5 $deg.K$
Reactor Concentration	0.01 $Kmol/m^3$
Reactor Temperature	0.5 $deg.K$

Table 2.2: Single Fault Scenarios

Fault Type	**TOF** [1]	**Fault Magnitude**
bias in actuator (F)	100	$\%10F_s = +0.1$
bias in sensor (T)	100	$-3 \times \sigma_T = -1.5$
bias in sensor (C_A)	100	$2 \times \sigma_{C_A} = +0.02$
bias in actuator (F_c)	100	$-\%10F_{c_s} = -1.5$

[1] TOF: Time of Occurrence of the Fault

Table 2.3: Sequence of Faults

Fault Type	**TOF** [1]	**Fault Magnitude**
bias in sensor (C_A)	100	$-3 \times \sigma_{C_A} = -0.03$
bias in actuator (F)	200	$-\%15F_s = -0.15$
bias in sensor (T)	300	$4 \times \sigma_T = +2$
bias in actuator (F_c)	400	$\%13.3F_{c_s} = +2$

[1] TOF: Time of Occurrence of the Fault

Table 2.4: $\chi^2 - GLR$ and $\chi^2 - MGLR$ Results Subject to Single Fault Scenarios Based on 100 Monte Carlo Runs for Each Case

Method	**TOF** [1]	**Fault**	**ETOF** [2] $\hat{t}(\sigma_{\hat{t}})$	**EFM** [3] $\hat{b}(\sigma_{\hat{b}})$	**NMC** [4]
$\chi^2 - GLR$	$t = 100$	$b_F = +0.1$	108.29 (5.979)	0.095 (0.014)	20/100
		$b_T = -1.5$	107.35 (7.706)	-1.562 (0.308)	25/100
		$b_{C_A} = +0.02$	109.53 (8.939)	0.020 (0.003)	6/100
		$b_{F_s} = -1.5$	107.81 (5.324)	-1.531 (0.218)	5/100
$\chi^2 - MGLR$	$t = 100$	$b_F = +0.1$	100.96 (0.559)	0.101 (0.010)	8/100
		$b_T = -1.5$	100.97 (3.232)	-1.527 (0.186)	6/100
		$b_{C_A} = +0.02$	100 (1.231)	0.020 (0.003)	1/100
		$b_{F_s} = -1.5$	101.21 (1.719)	-1.510 (0.102)	1/100

[1] TOF: Time of Occurrence of the Fault
[2] ETOF: Estimated Time of Occurrence of the Fault
[3] EFM: Estimated Fault Magnitude
[4] NMC: Number of Misclassifications

approach used in (Prakash *et al.*, 2005). The low misclassification rates of the proposed method show its superior performance in comparison with the $\chi^2 - GLR$ counterpart. Moreover, the more accurate estimates of the fault magnitudes with smaller standard deviations in comparison with $\chi^2 - GLR$ are clear evidence that precise detection of TOF which is a key factor in the isolation and estimation phases.

The performance of the FDI was also tested subject to the sequential faults. As can be seen in Table 2.5, the performance of the $\chi^2 - GLR$ based FDI is inferior to that of the newly proposed method due to the fact that it cannot accurately identify the time of occurrence of the fault. This problem can critically affect the performance of the FTC scheme if the FDI is integrated with the controller. In order to shed light on the advantages of the $\chi^2 - MGLR$ approach consider the sequence of the faults in Table 2.3. In order to shed light on the advantages of the $\chi^2 - MGLR$ approach consider the sequence of the faults in Table 2.3. The performance of the $\chi^2 - MGLR$ method for a random seed is tabulated in Table 2.6. As it can be seen in this table, the FDT/FCT tests confirm occurrence of the fault at time instant $t = 87$ while we know the true TOF is $t = 100$. The MGLR method now appends the extra samples prior to occurrence of the fault to the FDT/FCT data window which is $[87, 107]$ and forms a new extended data window which would be equal to $[66, 107]$ and tries to isolate the TOF in this newly formed data window. Applying the GLR test to this

Table 2.5: FDI Results Subject to Sequence of Faults Based on 100 Monte Carlo Runs for Each Case

Method	**TOF** [1]	**Fault**	**ETOF** [2] $\hat{\mathfrak{t}}(\sigma_{\hat{\mathfrak{t}}})$	**EFM** [3] $\hat{\mathfrak{b}}(\sigma_{\hat{\mathfrak{b}}})$	**NMC** [4]
$\chi^2 - GLR$	$t = 100$	$b_{C_A} = -0.03$	108.02 (5.721)	-0.029 (0.003)	23/100
	$t = 200$	$b_F = -0.15$	212.09 (9.522)	-0.132 (0.028)	
	$t = 300$	$b_T = 2$	300.55 (8.068)	1.896 (0.322)	
	$t = 400$	$b_{F_c} = 2$	406.37 (9.873)	2.005 (0.279)	
$\chi^2 - MGLR$	$t = 100$	$b_{C_A} = -0.03$	100.02 (0.33)	-0.030 (0.002)	8/100
	$t = 200$	$b_F = -0.15$	201.07 (0.446)	-0.150 (0.009)	
	$t = 300$	$b_T = 2$	300.19 (1.392)	2.053 (0.313)	
	$t = 400$	$b_{F_c} = 2$	401.04 (0.514)	1.999 (0.255)	

[1] TOF: Time of Occurrence of the Fault
[2] ETOF: Estimated Time of Occurrence of the Fault
[3] EFM: Estimated Fault Magnitude
[4] NMC: Number of Misclassifications

extended data window results in $\hat{t} = 102$ and $\hat{b}_{C_A} = -0.018$ which is obviously biased due to lack sufficient samples for estimation of the fault magnitude. Moreover, one may argue that the estimated TOF is also biased. Considering Remark 2.3.1, the MGLR-based FDI avoids taking action in this data window and waits for the subsequent rejection of FDT/FCT tests. Therefore, when the FDT/FCT tests are again rejected due the presence of the fault in the data window $[108, 128]$ the MGLR-based FDI appends the data window $[87, 107]$ from historical data to the recent FDT/FCT data window and forms the extended data window $[87, 128]$ and again repeats the application of the GLR test. There are two main advantages in this approach; first of all since the GLR test is looking for the most probable TOF in an extended data window then it is more likely for it to capture the correct fault signature and secondly enough data points are acquired for estimation of TOF and the fault magnitude. Hence the methodology proposed in this study is totally different from the $\chi^2 - GLR$ approach proposed by (Prakash *et al.*, 2005; Prakash *et al.*, 2002) which assumes $\hat{t} = 87$ and proceeds with isolation. However, assuming $\hat{t} = 87$ as the estimated TOF causes misclassification of the fault by the $\chi^2 - GLR$ due to the fact that fault-free data points in the range $[87, 99]$ cannot be matched with the fault signatures. The true versus the estimated states for both FDI scenarios is depicted in Figures 2.4 and 2.5, respectively. In these graphs the FDI module is integrated with the MPC controller and hence any sort

Table 2.6: The Modified GLR approach performance for a random seed

Estimated TOF by FDT/FCT	87	100	189	201	300	401
Modified GLR FDI Output	No Action	$\hat{b}_{C_A} = -0.029$	No Action	$\hat{b}_F = -0.144$	$\hat{b}_T = 1.85$	$\hat{b}_{F_c} = 2.274$
Interval Used for Estimation of $\hat{b}$	NA	$[100, 128]$	NA	$[201, 230]$	$[300, 318]$	[401,421]

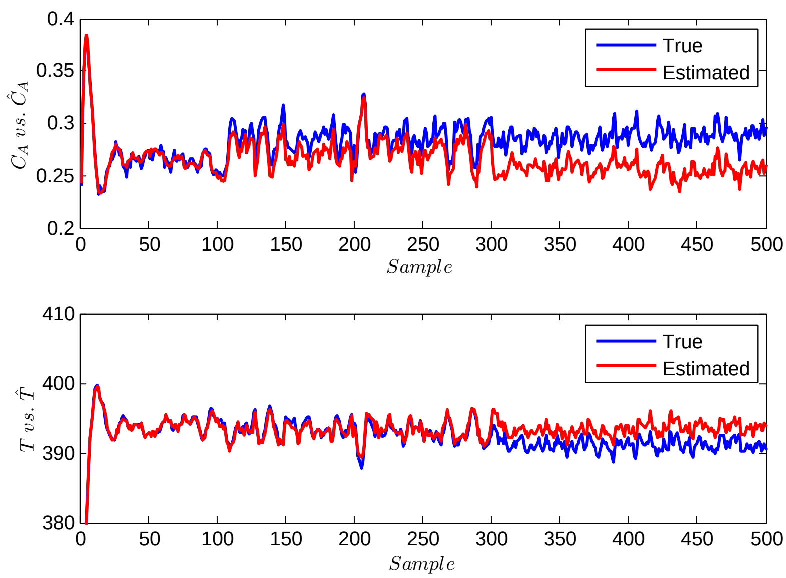

Figure 2.4: Estimated states versus true states for the case of combining the MPC controller with FDI designed based on the $\chi^2 - GLR$ method. The first fault in the sequence is misclassified as a result of incorrect assumption of occurrence of the fault at $t = 87$ and the subsequent diagnoses by the FDI are prone to error and consequently the grade of the final product (specified here by concentration) is likely to be unacceptable.

of incorrect classification will have serious consequences for the process control scheme. It is worth noting that integration of FDI with the controller in this specific case is unlike the general approach suggested in Figure 2.3 where the FDI information is only used for monitoring purposes.

The outputs to the final control elements which are equal to controller outputs plus the corresponding biases are depicted in Figure 2.6 for the proposed FDI scheme subject to the sequence of faults in Table 2.3. It should be noted that the constraints may be violated prior to the isolation of the fault, estimation of the fault magnitude and its compensation.

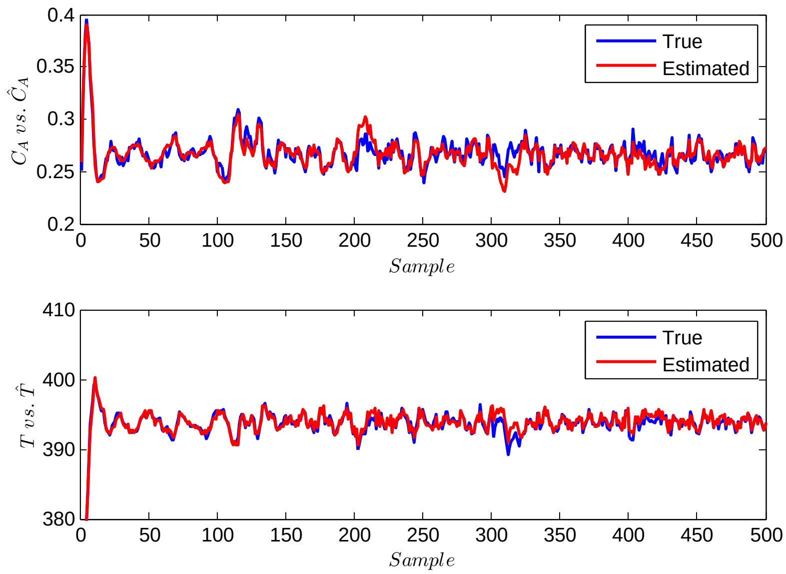

Figure 2.5: Estimated states versus true states for the case of combining the MPC controller with FDI designed based on the $\chi^2 - MGLR$ method. In this case the sensor bias is correctly detected, classified and the compensation applied mitigates the effect of the fault.

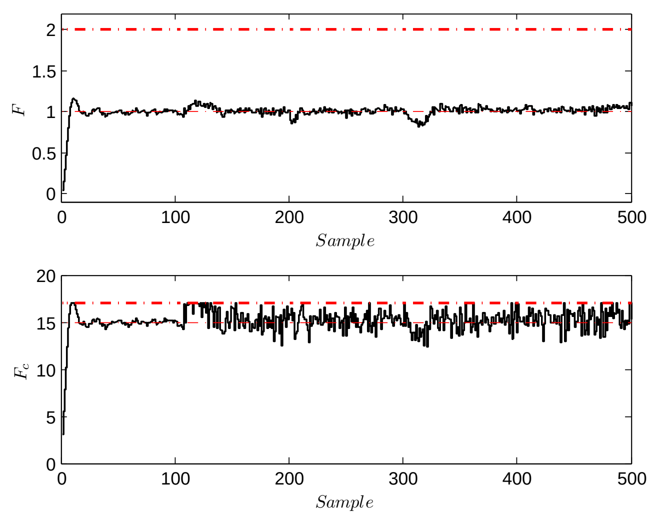

Figure 2.6: Controller outputs plus the corresponding biases for the proposed FDI scheme subject to sequence of faults

2.5 Concluding Remarks

Motivated by the shortcomings of the methods proposed in (Narasimhan and Mah, 1988; Prakash *et al.*, 2002; Prakash *et al.*, 2005), in this study a new FDI scheme has been proposed based on the GLR test and use of statistical FDT and FCT tests proposed by Narasimhan and Mah (Narasimhan and Mah, 1988). The modified GLR-based method overcomes the shortcomings related to the methodology proposed by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault using a modified GLR approach. The proposed method simultaneously performs the detection of TOF, isolation of the fault and estimation of its magnitude.

The new methodology removes the need for continuous operation of banks of Kalman filters over a sliding window as proposed by (Willsky and Jones, 1976) by taking advantage of the FDT and FCT tests. The $\chi^2 - MGLR$ method then tries to refine the estimated TOF by FDT and FCT tests using an extended data window. The method presented in this study is then complemented by a strategy which prevents the FDI from taking any action when sufficient number of data points are not available for estimation of fault magnitude. The method was tested subject to extensive Monte Carlo simulations on a benchmark reactor problem and the results reveal that it outperforms the FDI scheme presented in the study by (Prakash *et al.*, 2005).

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Chapter 3

A Unified Framework for Fault Detection and Isolation of Sensor and Actuator Biases in Linear Time Invariant Systems using Marginalized Likelihood Ratio Test with Uniform Priors

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3.1 Introduction

Model based fault detection and diagnosis is one of the research areas which has attracted significant attention in recent years and aims to provide the supervisory control system with sufficient information to take possible corrective actions to compensate for the instrumentation related faults (Basseville and Nikiforov, 1993; Gertler, 1998; Narasimhan and Jordache, 2000; Chen and Patton, 1999; Ding, 2008; Patton *et al.*, 1989). The critical task of fault detection and isolation (FDI) modules is to accurately detect the time of occurrence of the fault, identify its root cause and estimate its magnitude. Detection, isolation and estimation of fault magnitude techniques have been thoroughly addressed in the comprehensive study by (Basseville and Nikiforov, 1993). In recent years, the concept of fault tolerant control which integrates the FDI with controller design has also generated significant attention (Zhang and Jiang, 2008; Mhaskar *et al.*, 2006; Prakash *et al.*, 2005; Deshpande *et al.*, 2009; Mhaskar, 2006). As a general trend in active fault tolerant control systems, the supervisory control system tries to make necessary modifications/compensations to the controller and thus mitigate the impact of the fault by means of the information provided by the FDI module.

A seminal study in this field was first proposed by (Willsky and Jones, 1976) which addressed online detection and isolation of abrupt jumps using a generalized likelihood ratio (GLR)-based approach. The methodology therein, introduces the concept of fault signature matrices, which could be used to solve the composite hypotheses testing problem using the GLR approach. As stated by (Narasimhan and Mah, 1988) the major drawback of the methodology proposed in the study by (Willsky and Jones, 1976) is that it is overly "burdensome" from computational aspects. This is due to the fact that the FDI module solves, in real-time, the composite hypotheses testing problem in a sliding window to detect and isolate the faults. This issue proved to be the main motivation behind introduction of the fault detection test (FDT) and fault confirmation test (FCT) to estimate the time of occurrence of the fault in the study by (Narasimhan and Mah, 1988)[1]. However, there is always a compromise between type I and type II errors in both methods which arises from the thresholds selected for the GLR test or the significance levels of FDT and FCT tests. It

[1]The FDT test is based on a quadratic form of the residuals at each time instant normalized using the corresponding covariance matrix while the FCT test is the sum of FDT tests in a window of data with specific size. It is worth noting that in the study by (Narasimhan and Mah, 1988), FDT and FCT tests are referred to as "time of occurrence detection (TOD)" and "gross error detection (GED)" tests, respectively.

is worth noting that the use of FDT and FCT tests can result in significant inaccuracies in estimation of the time of occurrence of the fault. These inaccuracies, in turn, degrade the overall performance of the FDI module.

In the study by (Prakash *et al.*, 2002) an online FDI scheme is suggested, based on a combination of methodologies proposed in the studies by (Willsky and Jones, 1976; Narasimhan and Mah, 1988), to deal with the additive step-type faults in sensors, actuators and process parameters of LTI systems. The scheme proposed therein, used the FDT and FCT tests to detect TOF and subsequently, the GLR test were used to isolate the fault and estimate its magnitude using the fault signature matrices. The compensation scheme suggested in their study made it possible to deal with sequential faults. This compensation strategy for the states alone, was similar to "Direct State Incrementation" proposed by (Willsky and Jones, 1976). While in the study by (Prakash *et al.*, 2002) the FDI is used to provide supervisory information, the fault tolerant control (FTC) scheme proposed by (Prakash *et al.*, 2005) modified the controller using the information provided by the FDI scheme. However, as stated by (Villez *et al.*, 2011), the detection of time of occurrence of the fault is not properly addressed in the FDI scheme proposed by (Prakash *et al.*, 2002) and this shortcoming makes the performance of the system susceptible to misclassifications by the FDI module.

As explained in the study by (Wald, 1947; Basseville and Nikiforov, 1993), an alternative solution to the GLR algorithm is weighting the likelihood ratio with respect to the all possible values of the changing parameter. The concept of weighted likelihood ratio and assuming the fault magnitude to be random variable has resulted in emergence of a new generation of change detection algorithms known as the marginalized likelihood ratio (MLR) tests. The MLR-based fault detection method proposed by (Gustafsson, 1996) is a state of the art approach to detect abrupt changes in a dynamic system and incorporates prior knowledge about the faults. It can be shown that GLR test is a specific case of

MLR test and their test statistics are directly related to each other in the asymptotic case (Gustafsson, 1996).

In most cases due to lack of enough prior knowledge of the fault, the distribution of fault magnitude is considered to be noninformative. The main purpose of the marginalized likelihood ratio approach is to detect the occurrence of abrupt jumps in the states while removing the need for having an estimate for the fault magnitude. The two filter implementation of MLR suggested by (Gustafsson, 1996) incorporates a forward-backward filtering method to detect abrupt jumps in the states. However, it should be noted that the methodology proposed by (Gustafsson, 1996) does not address the isolation of faults in sensors and actuators.

It is worth mentioning that similar studies have appeared in the literature addressing change-point detection problem in regression models and time series. In the study by (Hinkley, 1969) in a two-phase linear regression model the maximum likelihood estimator of the change-point is derived. In the study by (Kalbfleisch and Sprott, 1970) the marginal likelihood is used for eliminating the nuisance parameters in order to make inference about the desired parameters of a model. The detection of change-point and estimation of its magnitude is also addressed by (Lee and Heghinian, 1977) using a normally distributed prior for additive fault. In another study by (Esterby and El-Shaarawi, 1981) the relative marginal likelihood function is calculated for change point in a sequence of $(n_1 + n_2)$ independent ordered pairs of observations for which the relationships between variables can be represented by a segmented polynomial regression model. Nevertheless, most of these studies address the change-point detection in an offline framework while in the FDI domain it is necessary to find the time of occurrence of the fault (TOF), isolate the fault and estimate its magnitude online. In other words, in the online FDI domain one should deal with dynamic systems where the residuals are generated at each time sample, unlike offline studies mentioned earlier where the whole data set is not available beforehand.

A preliminary investigation of fault isolation using MLR and uniform priors was proposed in the study by (Kiasi *et al.*, n.d.). In a similar approach,(Dos Santos and Yoneyama, 2011) have proposed a FDI scheme which decouples the isolation and estimation phases by means of marginalization of the likelihood function of the residuals using the Gamma distribution function as the prior. However, as stated by (Dos Santos and Yoneyama, 2011) this choice of prior penalizes low and high magnitude faults, in other words this choice of a prior implicitly and unrealistically assumes that both low and high magnitude faults rarely occur in the system. In addition, one should note that since the Gamma distribution is one-sided, identical positive and negative faults occurring at the same location should be treated as separate faults. Apparently, this will increase the number of hypothesized faults.

It should be noted that as per the principle of indifference (Keynes, 2004), a realistic choice of prior would be the one which assigns equal probability to all fault magnitudes. Isolation of the fault by MLR as well as the fact that most process variables vary within certain limits in the normal operating conditions, motivates one to look for alternative approaches towards online implementation of MLR within a FDI framework.

The main aim of the current study is to develop a FDI scheme for LTI systems to deal with bias type faults which may occur in sensors and actuators separately or consecutively. In particular, it is desired to develop a method to accurately detect the time of occurrence of a fault, which, in turn, is expected to improve the fault magnitude identification. In this regard a MLR-based approach using a uniform prior is proposed in this study. In the proposed methodology, detection of the time of the occurrence of the fault as well as isolation of the fault type and its location would be undertaken by the MLR approach. The remaining task which is the estimation of the isolated fault magnitude would be performed via a GLR-based approach. The proposed approach offers improvement over FDT and FCT tests by applying MLR test over a time window, which is constructed with the help of

FDT and FCT tests. This is a crucial step when FDI is carried out on closed loop systems since any error in estimation of time of occurrence of the fault can lead to misclassification and degradation in the closed loop control performance. The newly proposed FDI scheme is more likely to capture the complete signature of the fault within the batch of data in comparison with the studies by (Prakash *et al.*, 2002; Deshpande *et al.*, 2009). The main contributions of the current study can be summarized as follows:

- Introduction of the concept of realistic and bounded uniform priors for sensors and actuators that is unlike conventionally used priors in the literature such as flat or Gamma distributed priors. The proposed prior is the same for all hypothesized faults.
- Development of a new FDI scheme using MLR and uniform priors for fault magnitudes in which the isolation and fault magnitude estimation phases are decoupled. In this scheme the MLR is used both as a detector and isolator based on the assumption of uniform priors for fault magnitudes.
- Monte Carlo simulations show reduction in the number of false alarms and misclassifications and improvement in the estimates of the fault magnitudes using the proposed FDI scheme.

It is worth mentioning that, the ultimate goal of the proposed FDI scheme is to provide advisory information to operators so that they can take necessary measures such as re-calibration of the sensors/actuators.

This chapter is organized in five sections. In the next section, the modified model of the system/faults and the concept of uniform priors are presented. The newly proposed FDI scheme is discussed in Section 3. Section 4 discusses solution of the proposed FDI scheme and section 5 is dedicated to evaluation of the proposed methodology by application to a CSTR benchmark problem followed by concluding remarks in Section 6.

3.2 Definition of The Problem

Consider the following linear system where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{y} \in R^r$. In this representation $\mathbf{w} \in R^q$ and $\mathbf{v} \in R^r$ are process and measurement noise sequences with known covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ which are mutually uncorrelated and the initial state $x(0)$ follows a Gaussian distribution with known mean and variance. Furthermore, it is assumed that Φ, $\mathbf{C}$, Γ_u and Γ_w are known matrices.

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_u\mathbf{u}(k) + \Gamma_w\mathbf{w}(k) \tag{3.1}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \tag{3.2}$$

It is further assumed that the state space model in Eqs.(3.1-3.2) is derived by linearizing the nonlinear model around the desired operating point $(x_1^{ss}, \cdots, x_n^{ss}, u_1^{ss}, \cdots, u_m^{ss})$ with known parameters.

3.2.1 Modified Model of The System and the Fault Models

To simplify the development of fault diagnosis scheme, in this work, it is desired to work with scaled input and output variables. Let us assume that the calibration range for the ith sensor is (y_i^{min}, y_i^{max}). Similarly, one can consider (u_j^{min}, u_j^{max}) to be the flow range corresponding to the $(0\%, 100\%)$ valve opening for the jth actuator. It is worth mentioning that the lower limit for the valve is set to u_j^{min} and not zero as the valve may not be necessarily shut off completely (for safety reasons as in the case of cooling water to an exothermic reactor). Subsequent to switching to the deviation variables, the ranges can be modified as $(y_i^{min} - y_i^{ss}, y_i^{max} - y_i^{ss})$ and $(u_i^{min} - u_i^{ss}, u_i^{max} - u_i^{ss})$ for transmitters and actuators, respectively. Note that the fact that the steady steady operating point may not be necessarily located in the middle of the original ranges leads to asymmetric intervals around zero after deducting the steady state value. The following change of variables can be used to generate symmetric intervals around zero:

$$\begin{aligned} y_i^{'} &= y_i - \frac{y_i^{max} + y_i^{min} - 2y_i^{ss}}{2} = y_i - \mu_i^y, \quad 1 \le i \le r \\ u_j^{'} &= u_j - \frac{u_j^{max} + u_j^{min} - 2u_j^{ss}}{2} = u_j - \mu_j^u, \quad 1 \le j \le m \end{aligned}$$

after applying the change of variables the corresponding state space representation can be written as:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_u\mathbf{u}'(k) + \Gamma_w\mathbf{w}(k) + \Gamma_u\underline{\mu}^u \tag{3.3}$$

$$\mathbf{y}'(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) - \underline{\mu}^y \tag{3.4}$$

where $\mathbf{y}' \in (-\frac{\Delta y_i}{2}, \frac{\Delta y_i}{2})$ and $\mathbf{u}' \in (-\frac{\Delta u_j}{2} \frac{\Delta u_j}{2})$. In this representation Δy_i and Δu_j are defined as:

$$\Delta y_i = y_i^{max} - y_i^{min}, \quad \Delta u_j = u_j^{max} - u_j^{min}$$

In addition, diagonal $\underline{\mu}^u$ and $\underline{\mu}^y$ matrices are defined as follows:

$$\underline{\mu}^u = \begin{pmatrix} \mu_1^u & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_m^u \end{pmatrix}, \quad \underline{\mu}^y = \begin{pmatrix} \mu_1^y & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mu_r^y \end{pmatrix}$$

As can be seen in Eqs.(3.3-3.4), two constant terms will be added to state and measurement equations to center the transmitter and actuator ranges around zero. Subsequent to centering the input/output ranges, normalization of state space equations can be performed using the following equations:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Gamma_u\Xi_u\mathbf{u}^N + \Gamma_w\mathbf{w}(k) + \Gamma_u\underline{\mu}^u \tag{3.5}$$

$$\mathbf{y}^N(k) = \Xi_y^{-1}\mathbf{C}\mathbf{x}(k) + \Xi_y^{-1}\mathbf{v}(k) - \Xi_y^{-1}\underline{\mu}^y \tag{3.6}$$

where $\mathbf{u}^N = \Xi_u^{-1}\mathbf{u}'$ and $\mathbf{y}^N = \Xi_y^{-1}\mathbf{y}'$ and diagonal normalization matrices Ξ_u and Ξ_y are defined as follows:

$$\Xi_u = \begin{pmatrix} \frac{\Delta u_1}{2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{\Delta u_m}{2} \end{pmatrix}, \quad \Xi_y = \begin{pmatrix} \frac{\Delta y_1}{2} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \frac{\Delta y_r}{2} \end{pmatrix}$$

It is worth mentioning that the evolution of state variable is not affected by means of this transformation. However, all the input/outpt variables would be normalized in the range $[-1, +1]$.

Remark 3.2.1 *The advantages of using normalized state space model will be further clarified in section 3.3.2 where a prior distribution is assigned to each fault. A normalized model enables us to assign identical uniform priors to all sensors and actuators and therefore all the faults can be treated the same from the prior distribution point of view. The fact that sensors may have different calibration ranges and similarly control valves may have different flow ranges for* $[0\%, 100\%]$ *opening, leads to nonidentical uniform priors for hypothesized faults. However, by means of normalizing and centering the state space model, one can reshape the problem so that not only are the prior distributions identical and symmetric but also the test statistics can be easily compared for all hypothesized faults without any worries about the effect of different priors. This procedure lends a great practical utilitarian value to the proposed FDI scheme.*

In the fault free case the Kalman filter can be used to obtain the optimal state estimates of the state variables as follows:

$$\hat{\mathbf{x}}(k|k-1) = \Phi\hat{\mathbf{x}}(k-1|k-1) + \Gamma_u \Xi_u \mathbf{m}^N(k) + \Gamma_u \underline{\mu}^u \tag{3.7}$$

$$\hat{\mathbf{x}}(k|k) = \hat{\mathbf{x}}(k|k-1) + \mathbf{K}(k)\gamma(k) \tag{3.8}$$

$$\gamma(k) = \mathbf{y}^N(k) - \{\Xi_y^{-1}\mathbf{C}\hat{\mathbf{x}}(k|k-1) - \Xi_y^{-1}\underline{\mu}^y\} \tag{3.9}$$

where $\mathbf{m}^N(k)$ denotes the controller output and $\mathbf{K}(k)$ represents the Kalman gain matrix. Note that this notation distinguishes between the manipulated variable $\mathbf{u}^N(k)$ and the controller output $\mathbf{m}^N(k)$ where in the fault-free case $\mathbf{u}^N(k) = \mathbf{m}^N(k)$. The Kalman gain matrix can be calculated from the following set of equations:

$$\mathbf{V}(k) = \Xi_y^{-1}\mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^T\Xi_y^{-1} + \Xi_y^{-1}\mathbf{R}\Xi_y^{-1} \tag{3.10}$$

$$\mathbf{K}(k) = \mathbf{P}(k|k-1)\mathbf{C}^T\Xi_y^{-1}\mathbf{V}(k)^{-1} \tag{3.11}$$

$$\mathbf{P}(k|k) = (\mathbf{I} - \mathbf{K}(k)\Xi_y^{-1}\mathbf{C})\mathbf{P}(k|k-1) \tag{3.12}$$

$$\mathbf{P}(k|k-1) = \Phi\mathbf{P}(k-1|k-1)\Phi^T + \Gamma_w\mathbf{Q}\Gamma_w^T \tag{3.13}$$

In case of occurrence of bias with normalized magnitude $b_{u,j}$ in the jth actuator at time instant t the process would evolve for $k > t$ as follows:

$$\begin{aligned} \mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma_u\Xi_u\mathbf{m}^N(k) + \Gamma_w\mathbf{w}(k) + \Gamma_u\underline{\mu}^u \\ &\quad + \Gamma_u\Xi_u b_{u,j}\mathbf{e}_{u,j}\sigma(k-t) \qquad (3.14) \\ \mathbf{y}^N(k) &= \Xi_y^{-1}\mathbf{C}\mathbf{x}(k) + \Xi_y^{-1}\mathbf{v}(k) - \Xi_y^{-1}\underline{\mu}^y \qquad (3.15) \end{aligned}$$

where $\mathbf{e}_{u,j}$ is a unit vector whose jth element is equal to one and all other elements are zero and $\sigma(k-t)$ is a unit step function which is equal to 1 if $k \geq t$ and equal to zero otherwise. Similarly, in case of occurrence of bias with normalized magnitude $b_{y,j}$ in the jth sensor, the measurement equation would be modified as follows while the state equation remains as shown in Eq.(3.5):

$$\mathbf{y}^N(k) = \Xi_y^{-1}\mathbf{C}\mathbf{x}(k) + \Xi_y^{-1}\mathbf{v}(k) - \Xi_y^{-1}\underline{\mu}^y + b_{y,j}\mathbf{e}_{y,j}\sigma(k-t) \tag{3.16}$$

where e_{y_j} is a unit vector whose jth element is equal to one and all other elements are equal to zero. Moreover, it is worth mentioning that the bias values $b_{f,j}$ $(f \in \{u, y\})$in Eqs.(3.14,3.16) are also normalized using matrices Ξ_u^{-1} and Ξ_y^{-1}.

3.2.2 Justification of Uniform Priors

In many practical situations, it is more appropriate to assume that all possible fault magnitudes have equal probability by invoking the principle of indifference (Keynes, 2004). Although it is not justified to limit the maximum amount of bias in a sensor, in practice if the maximum amount of bias plus the true value of measured variable exceeds the alarm limits then the safety instrumentation system will be activated and shuts down the process. It is also well known that if the measured value is beyond the calibration range then the transmitter will show “Out of Range”. Similarly, manipulated variables are bounded due to actuator constraints, for example a valve can only open from 0% to 100%. Moreover, the transmitters are always calibrated for a certain range dictated by the process requirements.

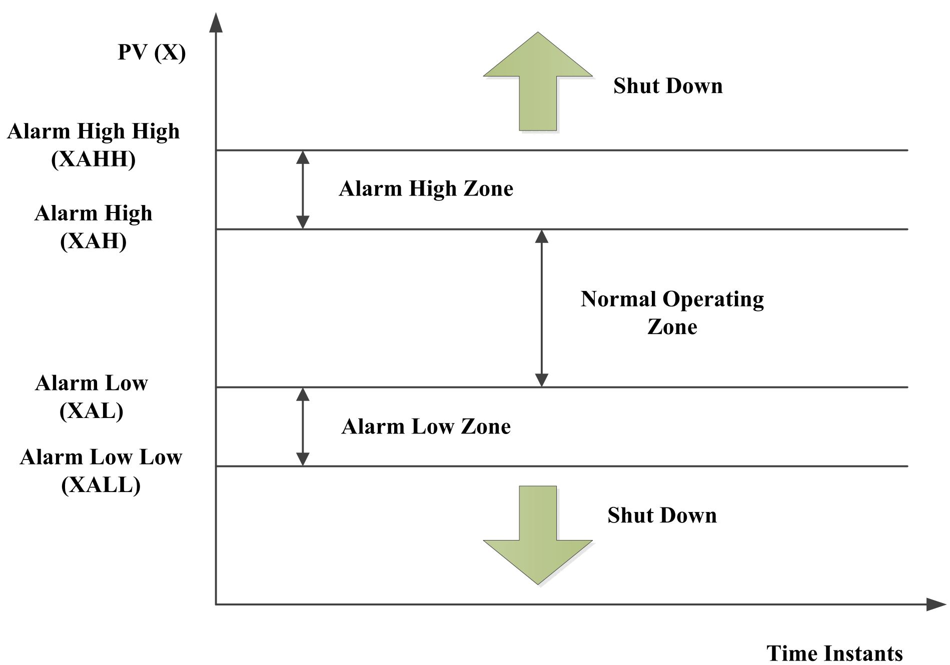

Figure 3.1: Alarm thresholds defined for a specific process variable

The boundedness of process variables can be further clarified using the concept of alarm thresholds depicted in Fig.3.1. It is also known that the calibration range should always encompass the “High/ High High” and “Low/ Low Low” alarm limits and as mentioned earlier the safety systems, for most critical variables, will not allow the operation to be continued upon violation of the “High High (HH)” and “Low Low (LL)” thresholds. Such thresholds are designed to automatically shut down the process and in this case there is no point in performing fault detection and diagnosis. This study is concerned with detection and isolation of soft faults such as biases in actuators and sensors which will not cause shut down of the process.

It is worth noting that when the valve is fully open at 100%, the possible bias size will be in the interval $[-100\%, 0\%]$ and similarly when the valve is fully closed the possible bias size will be $[0\%, 100\%]$. However since we are assuming the system is subject to occurrence of the fault, and we do not know what the actual size of valve opening is, we consider an approximate prior which can handle all the possible cases. Consequently the union of two worst cases i.e. $[-100\%, 0\%]$ and $[0\%, 100\%]$ is considered which results in $[-100\%, 100\%]$. A similar reasoning can be used for describing the choice of prior for all sensors.

3.3 Proposed FDI Scheme

It should be noted that this study is concerned with detection and diagnosis of additive step-type biases in sensors and actuators. In this regard, the following assumptions should be considered:

- Multiple faults occur sequentially in time but not simultaneously.
- In case of occurrence of bias, the controller corrective action will not cause the process variables to violate the shut down limits.
- Occurrence of additive faults in actuators and sensors do not lead to instability of the closed loop system.

It is worth mentioning that in reality the additive faults often appear to have a ramp type characteristic and such faults are not discussed in this study. Although such faults are beyond the scope of this work, an extension to the method proposed in this study could consider ramp type faults instead of the step-type biases.

3.3.1 Finding the Candidate Data Window for Occurrence of The Fault

The FDI scheme in this study takes advantage of the statistical FDT and FCT tests to find the candidate data window in which the fault has occurred. However, due to the inaccuracies associated with this approach a refining mechanism should be used to find the most probable time of occurrence of the fault (TOF). The FDT and FCT tests are defined as follows:

$$\text{FDT:} \qquad \epsilon(k) = \gamma^T(k)V^{-1}(k)\gamma(k) \tag{3.17}$$

$$\text{FCT:} \qquad \epsilon(N;t_1) = \sum_{k=t_1}^{t_1+N} \gamma^T(k)V^{-1}(k)\gamma(k) \tag{3.18}$$

which in the fault-free case follows a central χ^2 distribution with $r \times (N+1)$ degrees of freedom assuming the residuals are zero mean white noise with known covariance matrices. In the GLR approach used in (Prakash *et al.*, 2002; Prakash *et al.*, 2005) the FCT test is computed for data window $[t_1, t_1 + N]$ which is obtained after rejection of the FDT test at time instant t_1. Upon its rejection, the likelihood ratio will be computed for all hypothesized faults in the interval $[t_1, t_1 + N]$ to isolate and estimate the fault magnitude. Now let us consider the following two scenarios as depicted in Fig.3.2, in which in the first case the actual TOF is after the rejection of the FDT test while in second scenario the FDT is rejected after the actual TOF (Narasimhan and Mah, 1988). The second case is most prevalent if the fault magnitude is very small. So in order to accurately estimate the time of occurrence of the fault we propose to append the original data window $[t_1, t_1 + N]$ with M extra samples prior to rejection of the FDT test at t_1 and form a new data window as $[t_1 - M, t_1 + N]$. The concept of extended data window is depicted in Fig.3.3.

3.3.2 Formulation of the Change Detection Problem using MLR

When a fault occurs, if it is desired to estimate the magnitude of the fault, then one of the prime concerns is accurate estimation of the TOF. Let us assume that a fault is suspected

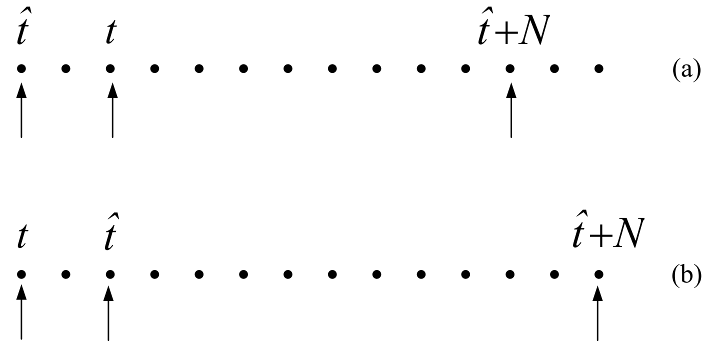

Figure 3.2: Scenarios leading to inaccurate detection of TOF: Figure 3.2a the estimated TOF ($\hat{t}$) precedes the actual TOF (t); Figure 3.2b the estimated TOF ($\hat{t}$) is after the actual TOF (t) (Narasimhan and Mah, 1988).

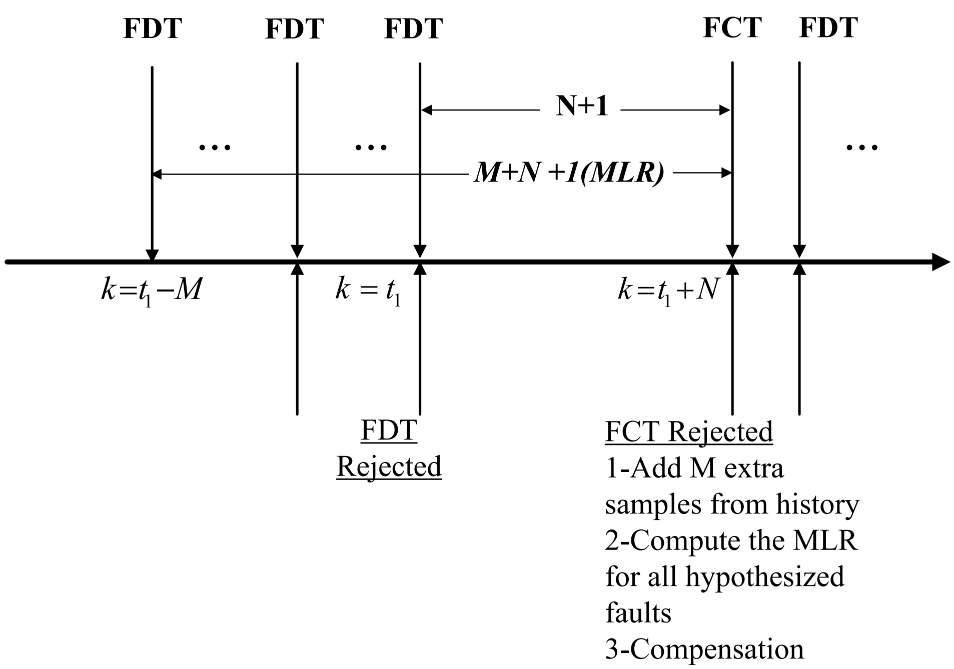

Figure 3.3: Extended FCT data window

to have occurred at some instant t such that t belongs to $[t_1 - M, t_1 + N]$. In contrast with GLR, MLR provides maximum likelihood estimate of the TOF using marginalization of the likelihood function. The main idea in MLR is to assume that the fault magnitude $b_{f,j}$ is a random variable with a certain prior distribution and then perform integration over all possible values of $b_{f,j}$ to accurately estimate the TOF. The maximum likelihood estimate of the time $t_{f,j}$[2], of occurrence of each hypothesized fault using the marginalization of the likelihood function is defined as follows (Gustafsson, 1996):

$$p(\Lambda_{t_1-M}^{t_1+N}|t_{f,j}) = \int p(\Lambda_{t_1-M}^{t_1+N}|t_{f,j}, b_{f,j})p(b_{f,j})db_{f,j} \tag{3.19}$$

$$= \prod_{i=t_1-M}^{t_{f,j}-1} p(\gamma_i) \int \prod_{i=t_{f,j}}^{t_1+N} p(\gamma_i|t_{f,j}, b_{f,j})p(b_{f,j})db_{f,j} \tag{3.20}$$

$$\Rightarrow \hat{t}_{f,j} = \arg\max_{t_{f,j}} p(\Lambda_{t_1-M}^{t_1+N}|t_{f,j}) \tag{3.21}$$

In this notation $\Lambda_{t_1-M}^{t_1+N} = \{\gamma_{t_1-M}, \cdots, \gamma_{t_1+N}\}$ are the residuals generated by a Kalman filter assuming fault free model whilst $p(\Lambda_{t_1-M}^{t_1+N}|t_{f,j}, b_{f,j})$ is the likelihood function assuming occurrence of fault f at time instant t with magnitude $b_{f,j}$ at location j. In this study we assume that the fault magnitude is a realization of a uniform random variable whose domain is known from a priori knowledge of the sensor and actuator faults, defined as:

$$p(b_{f,j}) = \begin{cases} \frac{1}{b_{f,j}^{max} - b_{f,j}^{min}} & b_{f,j}^{min} \leq b_{f,j} \leq b_{f,j}^{max} \\ 0 & \text{elsewehere} \end{cases}$$

Since the normalized representation of the state space equations is used in this study, the prior distribution can be modified accordingly. It should be noted that the input/output variables are all normalized in the range $[-1, 1]$ and consequently all possible values of the bias will be within the range $[-2, 2]$. Using this fact the modified uniform prior can be defined as:

$$p(b_{f,j}) = \begin{cases} \frac{1}{4} & -2 \leq b_{f,j} \leq 2 \\ 0 & \text{elsewehere} \end{cases} \tag{3.22}$$

[2]Time instant at which fault type type f has occurred at location j, e.g. $t_{y,1}$ and $t_{u,2}$ read as TOF at sensor #1 (y,1) and actuator #2 (u,2), respectively.

Another interpretation of this approach as stated by (Wald, 1947) and also section 2.4.1 of the book by (Basseville and Nikiforov, 1993), is weighting the likelihood ratio with respect to all possible values of the changing parameter and then marginalizing it to eliminate the changing parameter. The weighting function can be $dF(b_{f,j})$ where $F(b_{f,j})$ is the cumulative distribution function of a probability measure. Therefore, assuming a uniform prior for fault magnitude and independence of the residuals at each time instant, maximum likelihood (ML) estimate of TOF using marginalization of the the likelihood function assuming occurrence of fault f at location j can be found as:

$$\hat{t}_{f,j} = \arg \max_{\substack{t_{f,j} \\ t_{f,j} \in [t_1, t_1+N]}} \frac{1}{4} \prod_{i=t_1-M}^{t_{f,j}-1} p(\gamma_i) \int_{-2}^{+2} \prod_{i=t_{f,j}}^{t_1+N} p(\gamma_i | t_{f,j}, b_{f,j}) db_{f,j} \tag{3.23}$$

3.3.3 Detection of TOF and Isolation Issue

Using the concept of extended data window, one can estimate the TOF and determine the fault type and its position by maximizing the marginalized likelihood ratio as follows:

$$(\hat{f}, \hat{j}, \hat{t}_{\hat{f},\hat{j}}) = \arg\max_{\substack{f,j,t_{f,j} \\ t_{f,j} \in [t_1-M, t_1+N]}} \left\{ \frac{\frac{1}{4} \int_{-2}^{+2} \prod_{i=t_{f,j}}^{t_1+N} p(\gamma_i | t_{f,j}, b_{f,j}) db_{f,j}}{\prod_{i=t_{f,j}}^{t_1+N} p(\gamma_i)} \right\} \tag{3.24}$$

$$= \arg \max_{\substack{f,j,t_{f,j} \\ t_{f,j} \in [t_1-M, t_1+N]}} T^{MLR}_{f,j,t_{f,j}} \tag{3.25}$$

where $t_{f,j}$ represents the TOF for a specific fault hypothesis assuming occurrence of fault type f at position j. Before presenting the closed form solution of the MLR test, the effect of the additive fault on the innovation sequence should be clearly shown using the concept of fault signature matrices as stated in the following lemma:

Lemma 3.3.1 *Consider the linear system described in Eqs.(3.5-3.6). Upon occurrence of step-type additive fault in the jth sensor at time instant t, the effect of fault on the innovation* *sequence can be computed using the following recurrence equations:*

$$\mathbf{G}_y(k;t) = \mathbf{I} - \Xi_y^{-1}\mathbf{C}\Phi\mathbf{J}_y(k-1;t) \tag{3.26}$$

$$\mathbf{J}_y(k;t) = \Phi\mathbf{J}_y(k-1;t) + \mathbf{K}(k)\mathbf{G}_y(k;t) \tag{3.27}$$

similarly, if the fault occurs in the jth actuator, the recurrence equations would be as follows:

$$\mathbf{G}_u(k;t) = \Xi_y^{-1}\mathbf{C} - \Xi_y^{-1}\mathbf{C}\Phi\mathbf{J}_u(k-1;t) \tag{3.28}$$

$$\mathbf{J}_u(k;t) = \Phi\mathbf{J}_u(k-1;t) + \mathbf{K}(k)\mathbf{G}_u(k;t) - \mathbf{I} \tag{3.29}$$

where

$$\begin{aligned} E[\gamma(k)] &= b_{f,j}\mathbf{G}_f(k;t)\mathbf{g}_{f,j} \quad ; \quad k \geq t \\ E[\delta\hat{\mathbf{x}}(k)] &= b_{f,j}\mathbf{J}_f(k;t)\mathbf{g}_{f,j} \quad ; \quad k \geq t \end{aligned}$$

in the above equations $f \in \mathbf{u}, \mathbf{y}$ and $\mathbf{g}_{u,j} = \Gamma_u \Xi_u \mathbf{e}_{u,j}$ and $\mathbf{g}_{y,j} = \mathbf{e}_{y,j}$ for actuators and sensors, respectively.

Proof: See section 3.7. ■

It is worth noting that in case of occurrence of a fault the multiple hypothesis testing problem will have the following form:

$$\Lambda_{t_{f,j}}^{t_{f,j}+N} = \underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N} +$$

$$\{b_{f,j}\mathbf{G}_f(t_{f,j};t_{f,j})g_{f,j}, b_{f,j}\mathbf{G}_f(t_{f,j}+1;t_{f,j})g_{f,j}, \cdots, b_{f,j}\mathbf{G}_f(t_{f,j}+N;t_{f,j})g_{f,j}\}$$

where $\Lambda_{t_{f,j}}^{t_{f,j}+N} = \{\gamma_{t_{f,j}}, \gamma_{t_{f,j}+1}, \cdots, \gamma_{t_{f,j}+N}\}$ is the set of residuals assuming occurrence of the fault at time instant $t_{f,j}$ and $\mathbf{G}_f$ is the fault signature matrix for fault f. Moreover, realization of the fault magnitude is denoted by $b_{f,j}$. In this notation $\underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N} = \{\underline{\gamma}_{t_{f,j}}, \underline{\gamma}_{t_{f,j}+1}, \cdots, \underline{\gamma}_{t_{f,j}+N}\}$ is the set of fault free residuals in the same data window i.e. $[t_{f,j}, t_{f,j}+N]$.

It is worth mentioning that the elements of the fault free set ($\underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N}$) are Gaussian and uncorrelated with zero mean and known covariance matrices which can be computed using the Kalman filter. Moreover, in this study it is assumed that the residuals at time instant i are independent of residuals at time instant j (with $i \neq j$).

3.4 Solution of the Proposed FDI Scheme

3.4.1 Finding a Refined Estimate of TOF using MLR and uniform priors

The next step is to derive a MLR-based test statistic for a specific fault hypothesis using the fault signature matrices and a uniform prior which could be used for finding the most likely TOF for that specific fault in the data window provided by FDT and FCT tests.

Theorem 3.4.1 *For a linear system with a uniform prior distribution of the fault magnitude as in (3.22) and with fault signature matrices available as in (3.26-3.29) for sensors and actuators, the following conditions apply: upon occurrence of a single fault at sampling instant $t_{f,j}$ ($t_1 - M \leq t_{f,j} \leq t_1 + N$), the test statistic $T_{f,j,t_{f,j}}^{MLR}$ for marginalized likelihood ratio defined in (3.24) can be computed using the following equation:*

$$\begin{aligned} T_{f,j,t_{f,j}}^{MLR} &= \frac{exp(\frac{\beta_{t_{f,j}}^2}{2\eta_{t_{f,j}}})}{4} \sqrt{\frac{\pi}{2\eta_{t_{f,j}}}} \left(erf\left\{ \sqrt{\frac{\eta_{t_{f,j}}}{2}} [2 - \frac{\beta_{t_{f,j}}}{\eta_{t_{f,j}}}] \right\} \right. \\ & \left. - erf\left\{ \sqrt{\frac{\eta_{t_{f,j}}}{2}} [-2 - \frac{\beta_{t_{f,j}}}{\eta_{t_{f,j}}}] \right\} \right) \end{aligned} \tag{3.30}$$

where

$$\beta_{t_{f,j}} \triangleq \mathbf{g}_{f,j}^T \sum_{k=t_{f,j}}^{t_1+N} \mathbf{G}_f^T(k; t_{f,j}) V(k)^{-1} \gamma(k)$$

$$\eta_{t_{f,j}} \triangleq \mathbf{g}_{f,j}^T \sum_{k=t_{f,j}}^{t_1+N} \mathbf{G}_f^T(k; t_{f,j}) V(k)^{-1} \mathbf{G}_f(k; t_{f,j}) \mathbf{g}_{f,j}$$

Proof:

$$T^{MLR}_{f,j,t_{f,j}} = \frac{\frac{1}{4}\int_{-2}^{+2} p(\Lambda^{t_1+N}_{t_1-M}|t_{f,j}, b_{f,j})db_{f,j}}{p(\Lambda^{t_1+N}_{t_1-M})}$$

Since the residuals would follow a normal Gaussian distribution ($\gamma \in R^r$), the above equation can be written as follows:

$$T^{MLR}_{f,j,t_{f,j}} = exp\Big\{-\frac{1}{2}\sum_{k=t_1-M}^{t-1}\gamma^T(k)V(k)^{-1}\gamma(k)\Big\}\frac{1}{4}\times$$
$$\frac{\int_{-2}^{+2} exp\Big\{-\frac{1}{2}\sum_{k=t_{f,j}}^{t_1+N}\gamma^T_{f,j}(k)V(k)^{-1}\gamma_{f,j}(k)\Big\}db_{f,j}}{exp\Big\{-\frac{1}{2}\sum_{k=t_1-M}^{t_1+N}\gamma^T(k)V(k)^{-1}\gamma(k)\Big\}} \quad (3.31)$$

where $\gamma_{f,j}(k) = \gamma(k) - b_{f,j}\mathbf{G}_f(k;t_{f,j})\mathbf{g}_{f,j}$. One can rearrange (3.31) as follows:

$$T^{MLR}_{f,j,t_{f,j}} = \frac{\frac{1}{4}\int_{-2}^{+2} exp\Big\{-\frac{1}{2}\sum_{k=t_{f,j}}^{t_1+N}\gamma^T_{f,j}(k)V(k)^{-1}\gamma_{f,j}(k)\Big\}db_{f,j}}{exp\Big\{-\frac{1}{2}\sum_{k=t_{f,j}}^{t_1+N}\gamma^T(k)V(k)^{-1}\gamma(k)\Big\}} \quad (3.32)$$

defining $T^{MLR}_{f,j,t_{f,j}} \equiv \frac{\Theta}{\Omega}$, the numerator of (3.32) can be written as follows:

$$\begin{aligned}\Theta &= \frac{1}{4}\int_{-2}^{+2} exp\Big\{-\frac{1}{2}\sum_{k=t_{f,j}}^{t_1+N}\big[\gamma^T(k)V(k)^{-1}\gamma(k)\\ &\quad -2b_{f,j}\mathbf{g}^T_{f,j}\mathbf{G}^T_f(k;t_{f,j})V(k)^{-1}\gamma(k)\\ &\quad +b^2_{f,j}\mathbf{g}^T_{f,j}\mathbf{G}^T_f(k;t_{f,j})V(k)^{-1}\mathbf{G}_f(k;t_{f,j})\mathbf{g}_{f,j}\big]\Big\}db_{f,j} \quad (3.33)\end{aligned}$$

similarly, the denominator can be written as:

$$\Omega = exp(-\frac{1}{2}\alpha) = exp\{-\frac{1}{2}\sum_{k=t_{f,j}}^{t_1+N}\gamma^T(k)V(k)^{-1}\gamma(k)\} \quad (3.34)$$

Note that from here onwards for convenience of notation and avoiding unnecessary complexity, the subscript $t_{f,j}$ would be omitted from $\beta_{t_{f,j}}$ and $\eta_{t_{f,j}}$ and they would be shown as β and η, respectively. Using β, η and α, Eq.(3.33) can be rewritten as follows:

$$\begin{aligned}\Theta &= \frac{1}{4}\int_{-2}^{+2} exp\Big\{-\frac{1}{2}\big[\alpha - 2\beta b_{f,j} + \eta b^2_{f,j}\big]\Big\}db_{f,j}\\ &= \frac{exp(-\frac{\alpha}{2}+\frac{\beta^2}{2\eta})}{4}\int_{-2}^{+2} exp\Big\{-\frac{\eta}{2}\big[b_{f,j} - \frac{\beta}{\eta}\big]^2\Big\}db_{f,j} \quad (3.35)\end{aligned}$$

noting that $\int_{-X}^{+X} e^{-c(x-a)^2} dx = \sqrt{\frac{\pi}{4c}} \left\{ erf\left(\sqrt{c}(+X-a)\right) - erf\left(\sqrt{c}(-X-a)\right) \right\}$ where $c > 0$, and after some algebraic manipulations it can be shown that:

$$\Theta = \frac{exp(-\frac{\alpha}{2} + \frac{\beta^2}{2\eta})}{4} \sqrt{\frac{\pi}{2\eta}} \left\{ erf\left\{ \sqrt{\frac{\eta}{2}} [+2 - \frac{\beta}{\eta}] \right\} - erf\left\{ \sqrt{\frac{\eta}{2}} [-2 - \frac{\beta}{\eta}] \right\} \right\} \tag{3.36}$$

using this result and substituting in (3.32) yields Eq.(3.30). ■

As mentioned earlier, Theorem 3.4.1 is only concerned with deriving the MLR test statistic for a single fault hypothesis and the solution for the composite hypothesis testing problem can be stated as:

$$(\hat{f}, \hat{j}, \hat{t}_{\hat{f},\hat{j}}) = \arg\max_{f,j,t_{f,j}} T^{MLR}_{f,j,t_{f,j}} \tag{3.37}$$

The hierarchial procedure used in the proposed MLR-based FDI for isolation of the fault and estimation of its magnitude is depicted in Fig.3.4.

Remark 3.4.1 *Theorem 3.4.1 only addresses the marginalization when the uniform distribution is selected as the prior. Nevertheless, other alternative priors such as flat prior and Gamma distribution can also be used for this purpose. The most general case which is the non-informative flat prior is discussed in (Gustafsson, 1996) and is implemented using a forward-backward filtering method. Use of the Gamma distribution as prior is addressed in (Dos Santos and Yoneyama, 2011). However as mentioned earlier, it is hard to justify the use of Gamma prior for fault magnitude as it heavily penalizes low and high magnitude faults.*

3.4.2 Estimation of the Fault Magnitude

The remaining task of FDI module, which is the estimation of fault magnitude, can be undertaken by a least squares based approach as follows:

$$\hat{b}_{\hat{f},\hat{j}} = \underset{\substack{b_{\hat{f},\hat{j}} \\ s.t. \quad -2 \le b_{\hat{f},\hat{j}} \le +2}}{\arg\min} \sum_{k=\hat{t}_{\hat{f},\hat{j}}}^{t_1+N} \gamma^T_{\hat{f},\hat{j}}(k) V(k)^{-1} \gamma_{\hat{f},\hat{j}}(k) \tag{3.38}$$

where $\gamma_{\hat{f},\hat{j}}(k) = \gamma(k) - b_{\hat{f},\hat{j}} \mathbf{G}_{\hat{f}}(k; \hat{t}_{\hat{f},\hat{j}}) \mathbf{g}_{\hat{f},\hat{j}}$.

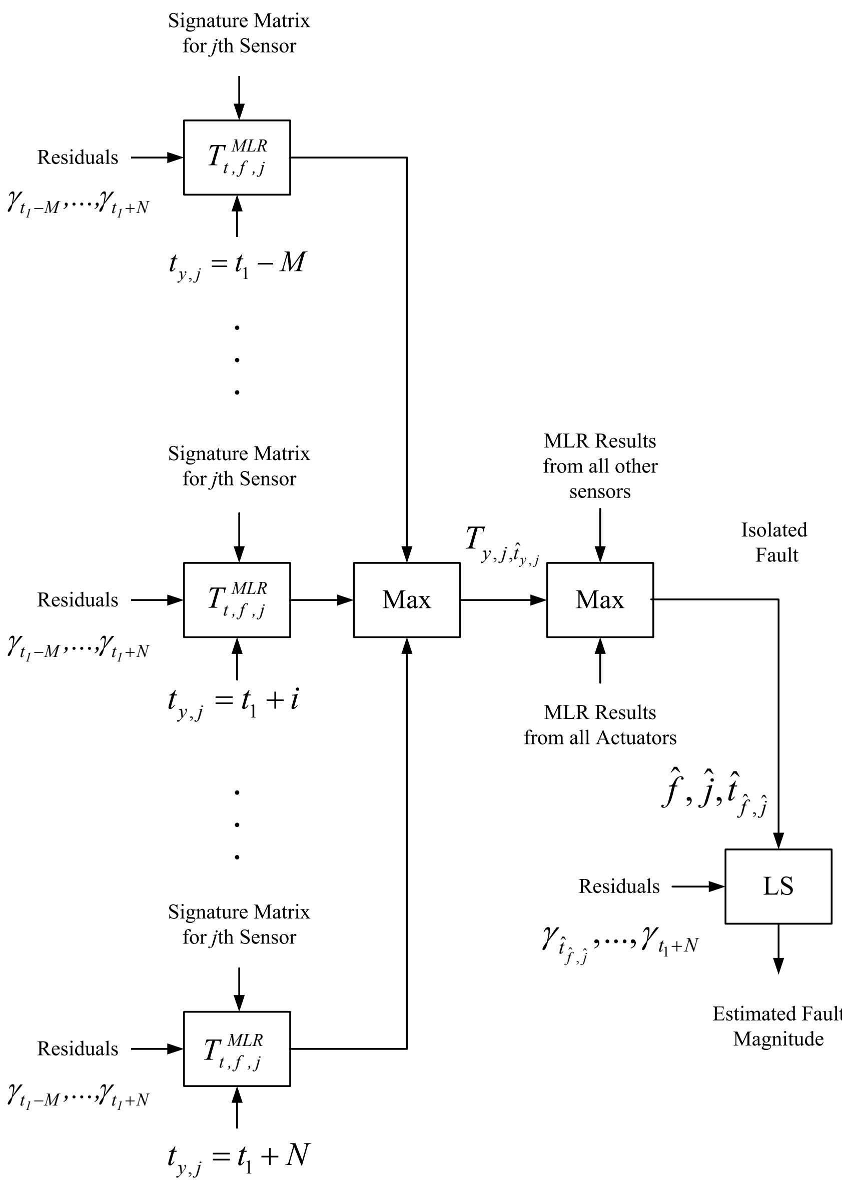

Figure 3.4: Detection of time of occurrence of the fault and its isolation by MLR/ estimation of its magnitude by least squares (LS)

3.4.3 Online Compensation Schemes

The flowchart for online implementation of the proposed FDI scheme is depicted in Fig.(3.5).The interaction between the proposed FDI scheme and the process is depicted in Fig.(3.6). The compensation equations which should be applied to make the residuals white after occurrence of the fault and subsequent detection, isolation and estimation phases which enables the FDI to detect faults that might occur sequentially are as follows (Prakash *et al.*, 2002):

$$y_c(k) = y(k) - \hat{b}_{\hat{y},\hat{j}}\mathbf{e}_{\hat{y},\hat{j}} \tag{3.39}$$

$$\mathbf{m}_c^N(k) = \mathbf{m}^N(k) + \hat{b}_{\hat{u},\hat{j}}\mathbf{e}_{\hat{u},\hat{j}} \tag{3.40}$$

$$\hat{\mathbf{x}}_c(k|k) = \hat{\mathbf{x}}_c(k|k) - \hat{b}_{\hat{f},\hat{j}}\mathbf{J}_{\hat{f}}(k;\hat{t}_{\hat{f},\hat{j}})\mathbf{g}_{\hat{f},\hat{j}} \tag{3.41}$$

where $k \in [\hat{t}_{\hat{f},\hat{j}}, t_1 + N]$ and $f \in \{u, y\}$.

Remark 3.4.2 *In order to prevent false alarms and also to accurately estimate the magnitude of the fault, a minimum size of data window s is required. In other words, if $\hat{t}_{\hat{f},\hat{j}} > t_1 + N - s + 1$, then the number of samples may not be enough to obtain a good estimate of the fault magnitude and hence it seems logical to obtain more data points for the purpose of estimation. Furthermore if $\hat{t}_{\hat{f},\hat{j}} > t_1 + N - s + 1$ and the fault has actually occurred, since no corrective action is taken by the FDI methodology, the FDT and FCT tests would be rejected again in the following instants and the fault would be detected in the next or subsequent window due to the fact that the extended data window enables the FDI module to look backwards in time. In choosing the minimum size of data window, the time required for the estimator to converge after a change occurs has to be also considered.*

Remark 3.4.3 *Let us first consider Eq.(3.35). After dividing Eq.(3.32) by Eq.(3.34), we will have the following test statistic:*

$$T^{MLR}_{f,j,t_{f,j}} = exp(\frac{\beta^2}{2\eta}) \times \frac{1}{4}\int_{-2}^{+2} exp\{-\frac{\eta}{2}[b_{f,j} - \frac{\beta}{\eta}]^2\}db_{f,j} = \Gamma \times \Upsilon \tag{3.42}$$

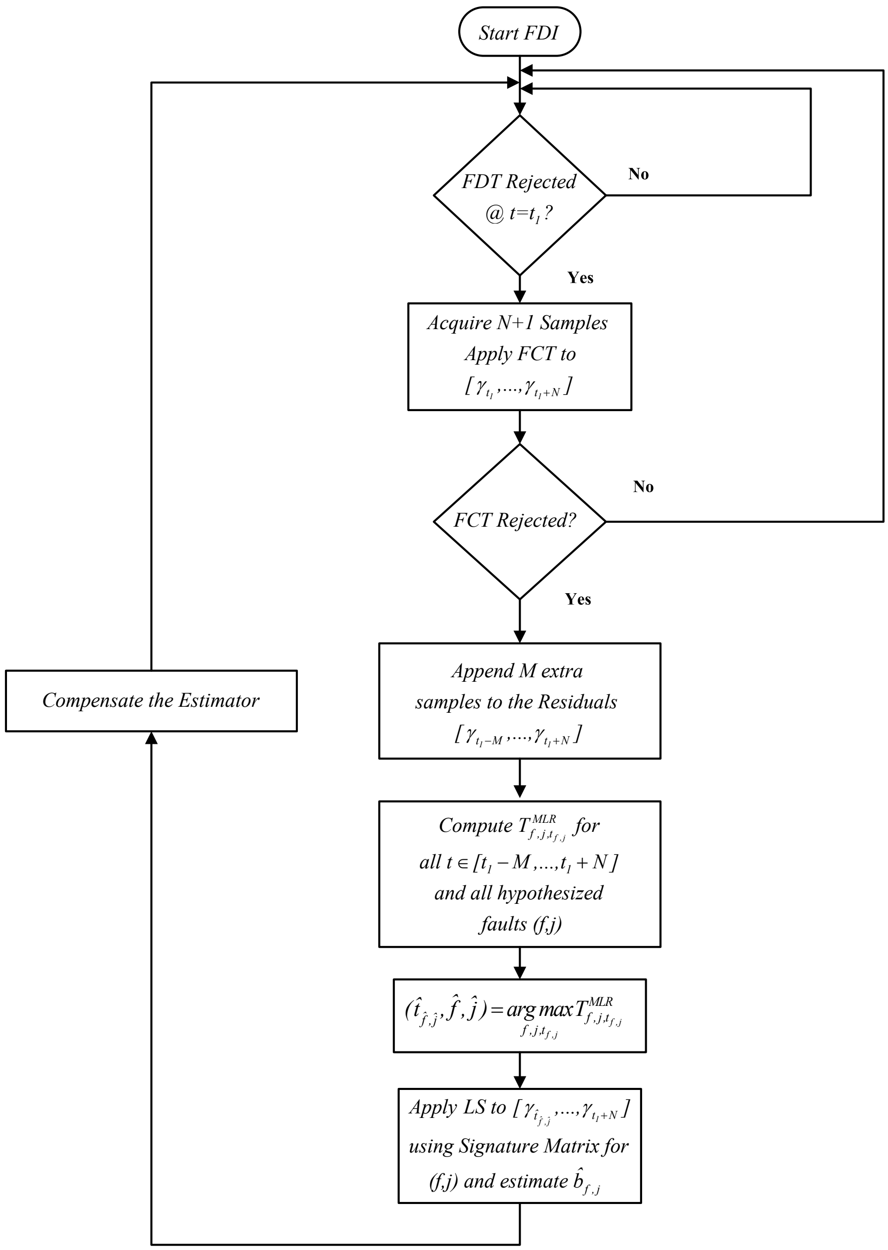

Figure 3.5: Flowchart for the MLR based online FDI scheme where MLR is used as an isolator

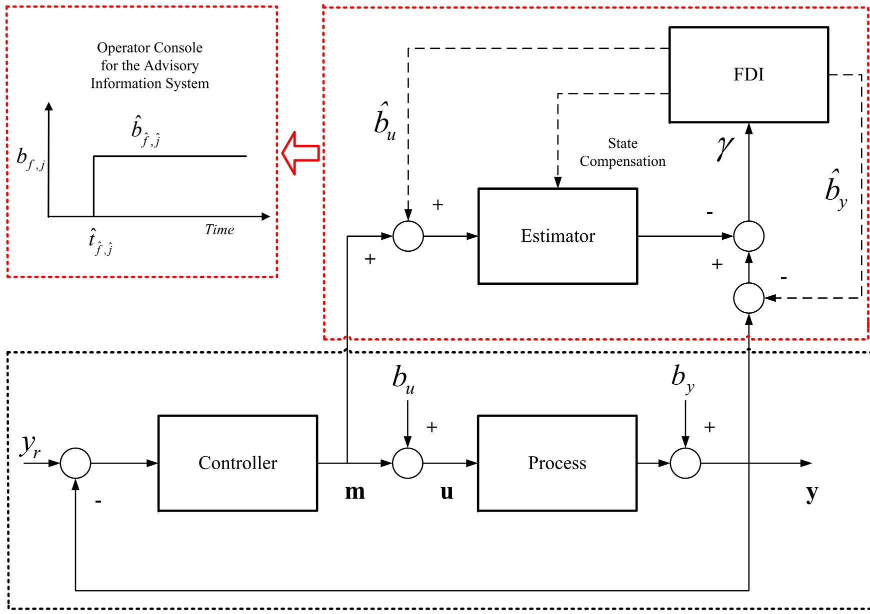

Figure 3.6: The supervisory scheme based on the MLR. The advisory information will be provided to the operators/ engineers.

where $\Gamma = exp(\frac{\beta^2}{2\eta})$ *and* $\Upsilon = \frac{1}{4}\int_{-2}^{+2} exp\{\frac{\eta}{2}[b_{f,j} - \frac{\beta}{\eta}]^2\}db_{f,j}$*. The first term (*$\Gamma$*) is independent of the choice of the prior distribution while the second term (*Υ*) is obviously affected by the prior. In the presence of high magnitude faults near the low and high limits of the prior distribution (i.e.* -2 *and* $+2$*), one may argue that the contribution from the* Υ *term is not completely considered in the test statistic due to the truncation of prior distribution at* -2 *and* $+2$*. One possible solution for such cases is extension of the prior distribution. However, modification of the prior distribution based on the a posteriori observation does not constitute a proper Bayesian approach particularly when no recursive computations of fault magnitude are incurred. It should be noted that such high magnitude faults rarely occur in practice, for example it would be unlikely to see an actuator bias which transforms a* 0% *valve opening signal to* 100%*.*

Table 3.1: Normalization ranges

Sensor	**Meas. Range**	**Actuator**	**Flow Range**
C_A	$[-0.1, +0.1]$	F	$[-1, 1]$
T	$[-25, 25]$	F_c	$[-15, 15]$

3.5 Simulation Case Study

The nonisothermal continuous stirred tank reactor (CSTR) benchmark example by (Marlin, 1995) which has also been used by (Prakash *et al.*, 2005), is used in this work to evaluate the performance of the proposed methodology. This plant has two states which are the reactor concentration (C_A) and reactor temperature (T) which are both measurable. The reactor feed flow rate (F) and the coolant flow rate (F_c) are selected to be the manipulated variables and the feed concentration (C_{A0}) and feed temperature (T_{cin}) are set as the disturbance variables. Moreover, it should be noted that all the variables are in deviational form.

$$\mathbf{x} = [C_A \quad T]^T; \qquad \mathbf{u} = [F \quad F_c]^T$$

$$\mathbf{w} = [C_{A0} \quad T_{cin}]^T; \qquad \mathbf{y} = [C_A \quad T]^T$$

The state space model can be normalized with respect to the ranges specified in Table 3.1. However, it should be noted that all the fault tables and also the simulation results are reported in this section in non-normalized form for easier interpretation and comparison. The stable operating point of the reactor has been used for performing the simulations and the process was linearized around this operating point. The sampling time was selected as $T_{sample} = 0.1\ min$ for discretization. The resulting state space matrices are as follows:

$$\Phi = \begin{bmatrix} 0.1843 & -0.0080 \\ 73.5080 & 1.3330 \end{bmatrix}, \quad \Gamma_u = \begin{bmatrix} 0.1340 & 0.0026 \\ -1.7948 & -0.7335 \end{bmatrix}$$

$$\Gamma_w = \begin{bmatrix} 0.0598 & -0.0004 \\ 3.9038 & 0.1208 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The steady state values of inlet and the coolant flow rates are $F_s = 1\ m^3/min$ and $F_{c_s} = 15\ m^3/min$, at this operating point. The MPC weighting matrix, prediction horizon

Table 3.2: Structure of FDI schemes

Method	**Detection of TOF**	**Isolation of fault**	**Estimation of fault magnitude**
$\chi^2 - GLR$	FDT/FCT test	GLR test	GLR test
$\chi^2 - MLR - LS$	FDT/FCT and MLR test	MLR test	Least Squares (LS)/GLR

and control horizon were set to the same values as reported in the study by (Prakash *et al.*, 2005) and are reproduced below:

$$N_p = 10, \quad N_c = 1, \quad W_E = \begin{bmatrix} 10^4 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_U = [0]$$

The following constraints were also considered:

$$0 \leq F \leq 2\, m^3/min \quad 0 \leq F_c \leq 30\, m^3/min$$

All the noise parameters are exactly set to values specified by (Prakash *et al.*, 2005), which are summarized in Table 3.3. The performance of the FDI schemes were tested subject to two different sets of faults. The first set only includes single faults and is summarized in Table 3.4 whilst the sequential faults introduced to the system are included in Table 3.5. In the Monte Carlo simulations (consisting of 100 runs for each case), the performance of the $\chi^2 - GLR$ approach proposed by (Prakash *et al.*, 2002) was compared to the proposed approach in this study which from now on is referred to as $\chi^2 - MLR - LS$ method. The structures of these FDI schemes are compared in Table 3.2.

In the Monte Carlo simulations, the size of FCT data window was set to 21 ($N = 20$) while the number of extra samples appended to the FCT data window to form the extended data window was selected as $M = 21$. The parameter *'s'* which was discussed in Remark 3.4.2, was set to 15.

The results of FDI schemes tested subject to the single fault scenarios of Table 3.4 and the sequence of faults shown in Table 3.5, are summarized in Tables 3.6 and 3.7, respectively. As can be seen, the results show that the newly proposed method is able to more effectively estimate TOF compared to the $\chi^2 - GLR$ approach used by (Prakash *et al.*, 2002). The low misclassification rates of the proposed method show its superior performance in

Table 3.3: Standard Deviations for Process and Measurement Noise Sequences

Variable	σ**(SD)**
Feed Concentration	0.05 $Kmol/m^3$
Feed Temperature	2.5 $deg.K$
Reactor Concentration	0.01 $Kmol/m^3$
Reactor Temperature	0.5 $deg.K$

Table 3.4: Single Fault Scenarios

Fault Type	**TOF** [1]	**Fault Magnitude**
bias in actuator (F)	100	$\%10F_s = +0.1$
bias in actuator (F)	100	$\%12.5F_s = +0.125$
bias in sensor (T)	100	$3 \times \sigma_T = -1.5$
bias in sensor (T)	100	$3.5 \times \sigma_T = -1.75$
bias in sensor (C_A)	100	$2 \times \sigma_{C_A} = +0.02$
bias in actuator (F_c)	100	$\%10F_{c_s} = -1.5$

[1] TOF: Time of Occurrence of the Fault

Table 3.5: Sequence of Faults

Fault Type	**TOF** [1]	**Fault Magnitude**
bias in sensor (C_A)	100	$-3 \times \sigma_{C_A} = -0.03$
bias in actuator (F)	200	$-15\%F_s = -0.15$
bias in sensor (T)	300	$4 \times \sigma_T = +2$
bias in actuator (F_c)	400	$13.3\%F_{c_s} = +2$

[1] TOF: Time of Occurrence of the Fault

Table 3.6: $\chi^2 - GLR$ and $\chi^2 - MLR - LS$ results based on 100 Monte Carlo runs for each case

Method	**TOF** [1]	**Fault**	**ETOF** [2] $\hat{\mathrm{t}}(\sigma_{\hat{\mathrm{t}}})$	**EFM** [3] $\hat{\mathrm{b}}(\sigma_{\hat{\mathrm{b}}})$	**NMC** [4]
$\chi^2 - GLR$	$t = 100$	$b_F = +0.1$	108.71 (6.406)	0.097 (0.015)	19/100
		$b_F = +0.125$	107.33 (3.369)	0.109 (0.022)	12/100
		$b_T = -1.5$	107.89 (7.967)	-1.574 (0.219)	26/100
		$b_T = -1.75$	107.7 (7.772)	-1.705 (0.281)	18/100
		$b_{C_A} = +0.02$	108.27 (9.139)	0.020 (0.003)	4/100
		$b_{F_c} = -1.5$	108.11 (6.529)	-1.471 (0.229)	6/100
$\chi^2 - MLR - LS$	$t = 100$	$b_F = +0.1$	101.32 (1.483)	0.103 (0.008)	15/100
		$b_F = +0.125$	101 (0.670)	0.123 (0.008)	5/100
		$b_T = -1.5$	101.26 (5.301)	-1.548 (0.174)	4/100
		$b_T = -1.75$	100.28 (1.944)	-1.791 (0.204)	1/100
		$b_{C_A} = +0.02$	100 (1.840)	0.020 (0.002)	1/100
		$b_{F_c} = -1.5$	101.15 (1.855)	-1.510 (0.161)	1/100

[1] TOF: Time of Occurrence of the Fault
[2] ETOF: Estimated Time of Occurrence of the Fault
[3] EFM: Estimated Fault Magnitude
[4] NMC: Number of Misclassifications

comparison with the $\chi^2 - GLR$ counterpart. Moreover, the more accurate estimates of the fault magnitudes with smaller standard deviations in comparison with $\chi^2 - GLR$ are clear evidence that precise detection of TOF plays a critical role in the isolation and estimation phases.

Another important issue which should be taken into consideration is the performance of the FDI subject to the sequential faults. As can be seen in Table 3.7, the performance of the $\chi^2 - GLR$ based FDI is inferior to that of the newly proposed method due to the fact that it cannot accurately identify the time of occurrence of the fault. This problem can critically affect the performance of the FTC scheme if the FDI is integrated with the controller.

In order to show the complexities involved in integration of the FDI with the controller, in the next step we compensate the controller with the information provided by the FDI. The performance of the FDI system in estimation of states subject to the sequence of faults shown in Table 3.5, is depicted in Figs. (3.7-3.8) for a specific random seed. It should be noted that in the generation of these graphs, unlike the procedure used in the Monte Carlo simulations, the controller was integrated with the information provided by the FDI module. As can be seen in Fig.(3.7), the misclassification of fault in the concentration

Table 3.7: FDI results subject to fault sequences based on 100 Monte Carlo runs for each case

Method	**TOF** [1]	**Fault**	**ETOF** [2] $\hat{\mathrm{t}}(\sigma_{\hat{\mathrm{t}}})$	**EFM** [3] $\hat{\mathrm{b}}(\sigma_{\hat{\mathrm{b}}})$	**NMC** [4]
$\chi^2 - GLR$	$t = 100$	$b_{C_A} = -0.03$	107.15 (4.936)	-0.029 (0.002)	26/100
	$t = 200$	$b_F = -0.15$	211.1 (9.636)	-0.122 (0.035)	
	$t = 300$	$b_T = 2$	299.75 (8.841)	1.794 (0.354)	
	$t = 400$	$b_{F_c} = 2$	406.87 (9.993)	2.006 (0.329)	
$\chi^2 - MLR - LS$	$t = 100$	$b_{C_A} = -0.03$	100.15 (0.808)	-0.029 (0.002)	5/100
	$t = 200$	$b_F = -0.15$	200.99 (0.559)	-0.151 (0.011)	
	$t = 300$	$b_T = 2$	300.18 (1.211)	2.048 (0.311)	
	$t = 400$	$b_{F_c} = 2$	401.06 (0.776)	1.964 (0.230)	

[1] TOF: Time of Occurrence of the Fault
[2] ETOF: Estimated Time of Occurrence of the Fault
[3] EFM: Estimated Fault Magnitude
[4] NMC: Number of Misclassifications

sensor (C_A) as bias the inlet flow actuator (F) by $\chi^2 - GLR$ based FDI, results in biased estimation of the states while the $\chi^2 - MLR - LS$ scheme is able to isolate this fault correctly. The major problem in this case is that after the misclassification of the fault at $t = 100$ and the relevant compensation, the residuals would become fairly white[3] from statistical point of view and consequently the FDI assumes that the process is operating under healthy conditions. However in reality this isolation flaw has indeed caused deviation of the state estimates from the true states. The scenario which leads to misclassification is as follows; at $t = 86$, the FDT test is rejected and the occurrence of the fault is confirmed via FCT test at $t = 106$. The $\chi^2 - GLR$ approach assumes, $\hat{t} = 100$ to be the time of occurrence of the fault and performs the isolation. Consequently, due to lack of enough fault signature in the interval $[86, 106]$, the fault in the concentration sensor (C_A) was incorrectly isolated as a fault in inlet flow actuator (F). It is important to remember that the controller believes the sensors and interprets the bias fault as a disturbance and therefore takes the necessary corrective action. In this way the fault propagates to the temperature. On the other hand, in $\chi^2 - MLR - LS$ scheme, after detection of the fault at $t = 86$ and its

[3] Residuals appear white since the fault is considered as a disturbance and its effect is mitigated by the controller action. The 'whiteness' of the residual in this case is an indication that the controller performance is excellent. So unless the fault is detected precisely and acted upon quickly, it can get construed as a step disturbance and this effect can be compensated by a well-tuned controller.

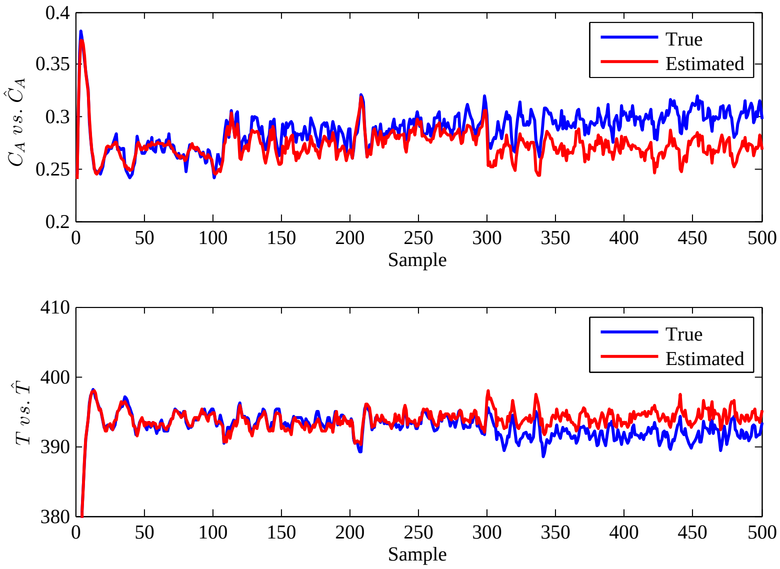

Figure 3.7: Estimated states versus the true states in case of coupling the MPC controller with FDI designed based on the $\chi^2 - GLR$ method. Due to the incorrect isolation at $t = 100$, the subsequent diagnoses by the FDI are prone to error and consequently the grade of the final product (specified here by concentration) is likely to be unacceptable. In this strategy unlike the supervisory scheme depicted in Fig.(3.6) the FDI module is integrated with the controller.

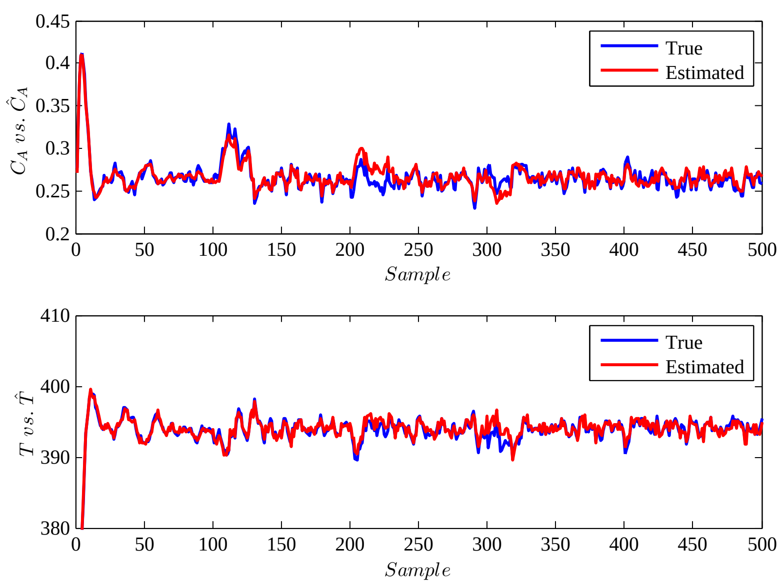

Figure 3.8: Estimated states versus the true states in case of coupling the MPC controller with FDI designed based on the $\chi^2 - MLR - LS$ method. In this strategy unlike the supervisory scheme depicted in Fig.(3.6) the FDI module is integrated with the controller.

confirmation at $t = 106$, the MLR test is used to find the most probable time of occurrence of the fault. The MLR test specifies $\hat{t} = 102$ to be the most likely time of occurrence of the fault. However, since the number of samples which are available for estimation of the fault magnitude is equal to 5 samples (and obviously smaller than $s = 15$), the FDI does not take any corrective action and as described before, the isolation and compensation is carried out in the next FCT window. Using the extended FCT window of $[86, 127]$, MLR then estimates $\hat{t} = 100$.

3.6 Concluding Remarks

In this study a new FDI scheme has been proposed based on the MLR test and use of realistic uniform priors for fault magnitudes to address detection of time of occurrence of the fault, isolation of the fault and estimation of its magnitude. The proposed methods overcome the shortcomings related to the methodology proposed by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault using a MLR-based approach. The proposed method simultaneously performs the detection of time of occurrence and isolation of the fault and the estimation of fault magnitude is undertaken by a least squares approach. The superior performance of the proposed method compared to the FDI schemes proposed in (Prakash *et al.*, 2002; Prakash *et al.*, 2005) can be mainly attributed to accurate detection of time of occurrence of the fault.

In the proposed scheme, statistical FDT and FCT tests are used as the early stage announcers of the time of occurrence of the fault. The method was tested subject to extensive Monte Carlo simulations on a benchmark reactor problem and the results reveal that it outperforms the FDI scheme presented in (Prakash *et al.*, 2002; Prakash *et al.*, 2005).

3.7 Appendix 1: fault signature matrices

First let us consider the case where bias has occurred in sensor j in time instant t. In this case the system evolves for $k \geq t$ as:

$$\begin{aligned}\mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma_u \Xi_u \mathbf{m}^N(k) + \Gamma_w \mathbf{w}(k) + \Gamma_u \underline{\mu}^u \\ \mathbf{y}^N(k) &= \Xi_y^{-1}\mathbf{C}\mathbf{x}(k) + \Xi_y^{-1}\mathbf{v}(k) - \Xi_y^{-1}\underline{\mu}^y + b_{y,j}\mathbf{e}_{y,j}\end{aligned}$$

Defining $\delta\hat{\mathbf{x}}(k) = \hat{\mathbf{x}}(k|k) - \mathbf{x}(k)$ and using Kalman filter equations (3.8-3.9) yields:

$$\begin{aligned}\delta\hat{\mathbf{x}}(k) &= [\mathbf{I} - \mathbf{K}(k)\Xi_y^{-1}\mathbf{C}][\Phi\delta\hat{\mathbf{x}}(k-1) - \Gamma_w \mathbf{w}(k-1)] \\ &\quad + \mathbf{K}(k)[b_{y,j}\mathbf{e}_{y,j} + \Xi_y^{-1}\mathbf{v}(k)] \qquad (3.43)\end{aligned}$$

on the other hand one can rewrite residuals in Eq.(3.9) as:

$$\begin{aligned}\gamma(k) &= \Xi_y^{-1}\mathbf{C}[-\Phi\delta\hat{\mathbf{x}}(k-1) + \Gamma_w \mathbf{w}(k-1)] \\ &\quad + \Xi_y^{-1}\mathbf{v}(k) + b_{y,j}\mathbf{e}_{y,j} \qquad (3.44)\end{aligned}$$

taking expected value from Eqs.(3.43-3.44) results in:

$$\begin{aligned}\mathbf{E}[\delta\hat{\mathbf{x}}(k)] &= [\mathbf{I} - \mathbf{K}(k)\Xi_y^{-1}\mathbf{C}]\Phi\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] \\ &\quad + b_{y,j}\mathbf{K}(k)\mathbf{e}_{y,j} \qquad (3.45) \\ \mathbf{E}[\gamma(k)] &= -\Xi_y^{-1}\mathbf{C}\Phi\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] + b_{y,j}\mathbf{e}_{y,j} \qquad (3.46)\end{aligned}$$

Now let us define the linear dependence of $\mathbf{E}[\delta\hat{\mathbf{x}}(k)]$ on the fault using the following relation:

$$\mathbf{E}[\delta\hat{\mathbf{x}}(k)] = b_{y,j}\mathbf{J}_y(k;t)\mathbf{e}_{y,j}$$

where $b_{y,j}$ is the normalized fault magnitude. Similarly, one can define the expected value of linear dependence of the residuals on the fault using the following:

$$\mathbf{E}[\gamma(k)] = b_{y,j}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}$$

using the above definitions and also substituting Eq.(3.46) and in Eq.(3.45) yields:

$$\mathbf{J}_y(k;t) = \Phi\mathbf{J}_y(k-1;t) + \mathbf{K}(k)\mathbf{G}_y(k;t) \tag{3.47}$$

$$\mathbf{G}_y(k;t) = \mathbf{I} - \Xi_y^{-1}\mathbf{C}\Phi\mathbf{J}_y(k-1;t) \tag{3.48}$$

In case of occurrence of bias in actuator in actuator j at time instant t, for $k \geq t$ the state space model can be represented as:

$$\begin{aligned}\mathbf{x}(k+1) &= \Phi\mathbf{x}(k) + \Gamma_u\Xi_u\mathbf{m}^N(k) + b_{u,j}\Gamma_u\Xi_u\mathbf{e}_{u,j} \\ &\quad +\Gamma_w\mathbf{w}(k) + \Gamma_u\underline{\mu}^u \\ \mathbf{y}^N(k) &= \Xi_y^{-1}\mathbf{C}\mathbf{x}(k) + \Xi_y^{-1}\mathbf{v}(k) - \Xi_y^{-1}\underline{\mu}^y\end{aligned}$$

using a similar approach to the sensor faults, one can find the followings:

$$\begin{aligned}\delta\hat{\mathbf{x}}(k) &= [\mathbf{I} - \mathbf{K}(k)\Xi_y^{-1}\mathbf{C}][\Phi\delta\hat{\mathbf{x}}(k-1) - \Gamma_w\mathbf{w}(k-1)] \\ &\quad +\mathbf{K}(k)[b_{u,j}\Xi_y^{-1}\mathbf{C}\Gamma_u\Xi_u\mathbf{e}_{u,j} + \Xi_y^{-1}\mathbf{v}(k)] \\ &\quad -b_{u,j}\Gamma_u\Xi_u\mathbf{e}_{u,j} \end{aligned} \tag{3.49}$$

$$\begin{aligned}\gamma(k) &= \Xi_y^{-1}\mathbf{C}[b_{u,j}\Gamma_u\Xi_u\mathbf{e}_{u,j} + \Gamma_w\mathbf{w}(k-1)] \\ &\quad -\Xi_y^{-1}\mathbf{C}\Phi\delta\hat{\mathbf{x}}(k-1) + \Xi_y^{-1}\mathbf{v}(k)\end{aligned} \tag{3.50}$$

taking expected value from Eqs.(3.49-3.50) yields:

$$\begin{aligned}\mathbf{E}[\delta\hat{\mathbf{x}}(k)] &= [\mathbf{I} - \mathbf{K}(k)\Xi_y^{-1}\mathbf{C}]\Phi\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] \\ &\quad +b_{u,j}\mathbf{K}(k)\Xi_y^{-1}\mathbf{C}\Gamma_u\Xi_u\mathbf{e}_{u,j} - b_{u,j}\Gamma_u\Xi_u\mathbf{e}_{u,j}\end{aligned} \tag{3.51}$$

$$\mathbf{E}[\gamma(k)] = -\Xi_y^{-1}\mathbf{C}\Phi\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] + b_{u,j}\Xi_y^{-1}\mathbf{C}\Gamma_u\Xi_u\mathbf{e}_{u,j} \tag{3.52}$$

Now let us define the followings:

$$\mathbf{E}[\delta\hat{\mathbf{x}}(k)] = b_{u,j}\mathbf{J}_u(k;t)\Gamma_u\Xi_u\mathbf{e}_{u,j}$$

$$\mathbf{E}[\gamma(k)] = b_{u,j}\mathbf{G}_u(k;t)\Gamma_u\Xi_u\mathbf{e}_{u,j}$$

using the above definitions and also substituting Eq.(3.52) and in Eq.(3.51) yields:

$$\mathbf{J}_u(k;t) = \Phi\mathbf{J}_u(k-1;t) + \mathbf{K}(k)\mathbf{G}_u(k;t) - \mathbf{I} \tag{3.53}$$

$$\mathbf{G}_u(k;t) = \Xi_y^{-1}\mathbf{C} - \Xi_y^{-1}\mathbf{C}\Phi\mathbf{J}_u(k-1;t) \tag{3.54}$$

Regarding initialization of the recursive fault signature equations note that since any type of fault in the sensors immediately affects the residuals then one can deduce that:

$$\mathbf{G}_y(t_{f,j}-1;t_{f,j}) = [0]_{r\times r} \quad \& \quad \mathbf{J}_y(t_{f,j}-1;t_{f,j}) = [0]_{n\times r}$$

where $t_{f,j}$ is the TOF. On the other hand any change in the manipulated variables will take at least one sample time delay to affect the residuals due to the discretization and consequently we have:

$$\mathbf{G}_u(t_{f,j};t_{f,j}) = [0]_{r\times n} \quad \& \quad \mathbf{J}_u(t_{f,j};t_{f,j}) = [0]_{n\times n}$$

3.8 Appendix 2: Relationship between thresholds of GLR and MLR tests

The proposed method in this study does not utilize GLR and MLR test statistic thresholds for detection and isolation of the fault and instead it takes advantage of statistical FDT and FCT tests as early announcers of fault occurrence. Other studies in the literature have also taken advantage of statistical FDT/FCT tests for detection of the fault instead of using thresholds for GLR test (Narasimhan and Mah, 1988; Prakash *et al.*, 2005; Prakash *et al.*, 2002). Please note that in the following proof we assume the observability condition is satisfied i.e. the number of faults present does not exceed the number independent measurements. Moreover, it is assumed that the residuals at time instant i are independent of residuals at time instant j (with $i \neq j$).

As stated in the study by (Willsky and Jones, 1974), the residuals in case of occurrence of the fault in the sensors can be expressed as[4]:

$$\gamma(k) = \mathbf{G}_y(k;t)\bar{\mathbf{b}}_y + \gamma'(k) \tag{3.55}$$

where $\bar{\mathbf{b}}_y = [b_{y1}, b_{y2}, \cdots, b_{yr}]^T$ represents the vector of faults in the sensors and γ' is a zero mean white noise sequence with covariance matrix $V(k)$ and represents the the actual residuals if a fault does not occur. Similarly one can develop the same equation for the fault in the actuators as follows:

$$\gamma(k) = \mathbf{G}_u(k;t)\Gamma_u \Xi_u \bar{\mathbf{b}}_u + \gamma'(k)$$

where $\bar{\mathbf{b}}_u = [b_{u1}, b_{u2}, \cdots, b_{um}]^T$ represents the vector of faults in the actuators. Here we solve the problem for sensors however, the same approach can be used to address the faults in the actuators. Using Eq.(3.55) and assuming occurrence of the fault at time instant t, the following least squares problem can be formed:

$$\begin{bmatrix} \gamma(k) \\ \hdashline \gamma(k+1) \\ \hdashline \vdots \\ \hdashline \gamma(k+N) \end{bmatrix} = \begin{bmatrix} \mathbf{G}_y(k;t) \\ \hdashline \mathbf{G}_y(k+1;t) \\ \hdashline \vdots \\ \hdashline \mathbf{G}_y(k+N;t) \end{bmatrix} \begin{bmatrix} b_{y1} \\ b_{y2} \\ \vdots \\ b_{yr} \end{bmatrix} + \begin{bmatrix} \gamma'(k) \\ \hdashline \gamma'(k+1) \\ \hdashline \vdots \\ \hdashline \gamma'(k+N) \end{bmatrix} \tag{3.56}$$

Equation (3.56) can be written as:

$$\mathbf{Y} = \mathbf{X}\bar{\mathbf{b}}_y + \mathbf{e} \tag{3.57}$$

where

$$\mathbf{Y} = \begin{bmatrix} \gamma(k) \\ \hdashline \gamma(k+1) \\ \hdashline \vdots \\ \hdashline \gamma(k+N) \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \mathbf{G}_y(k;t) \\ \hdashline \mathbf{G}_y(k+1;t) \\ \hdashline \vdots \\ \hdashline \mathbf{G}_y(k+N;t) \end{bmatrix} \quad \mathbf{e} = \begin{bmatrix} \gamma'(k) \\ \hdashline \gamma'(k+1) \\ \hdashline \vdots \\ \hdashline \gamma'(k+N) \end{bmatrix}$$

[4] The notation used in the study by (Willsky and Jones, 1974) is changed to adapt to the notation of the current study.

Let $p_b(\mathbf{Y},\mathbf{X})$ be the joint pdf of $\mathbf{Y}$ and $\mathbf{X}$. The likelihood ratio can be written as:

$$\begin{aligned} l_{\bar{\mathbf{b}}_y/0}(\mathbf{Y},\mathbf{X}) &= 2log\frac{p_{\bar{\mathbf{b}}_y}(\mathbf{Y},\mathbf{X})}{p_0(\mathbf{Y},\mathbf{X})} \\ &= -\frac{1}{2}(\mathbf{Y}-\mathbf{X}\bar{\mathbf{b}}_y)^T\Omega^{-1}(\mathbf{Y}-\mathbf{X}\bar{\mathbf{b}}_y)+\frac{1}{2}\mathbf{Y}^T\Omega^{-1}\mathbf{Y} \end{aligned} \tag{3.58}$$

where the block diagonal matrix Ω can be defined using the covariance matrices as follows:

$$\Omega = \begin{bmatrix} V(k)^{-1} & 0 & \cdots & 0 \\ 0 & V(k+1)^{-1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & V(k+N)^{-1} \end{bmatrix}$$

The GLR test assuming occurrence of the fault at time instant t can be defined as:

$$\mathbf{T}_t^{GLR} = \max_{\bar{\mathbf{b}}_y} l_{\bar{\mathbf{b}}_y/0}(\mathbf{Y},\mathbf{X})$$

As seen in Eq.(3.58), the GLR test coincides with the generalized weighted least squares problem and therefore the estimated fault magnitude can be found as:

$$\hat{\bar{\mathbf{b}}}_y = (\mathbf{X}^T\Omega^{-1}\mathbf{X})^{-1}(\mathbf{X}^T\Omega^{-1}\mathbf{Y})$$

substituting this estimated fault magnitude into the GLR test one can one find:

$$\mathbf{T}_t^{GLR} = (\mathbf{X}^T\Omega^{-1}\mathbf{Y})^T(\mathbf{X}^T\Omega^{-1}\mathbf{X})^{-1}(\mathbf{X}^T\Omega^{-1}\mathbf{Y}) \tag{3.59}$$

One can perform *thin*[5] QR decomposition of $\Omega^{-1/2}\mathbf{X}$ as follows:

$$\Omega^{-\frac{1}{2}}\mathbf{X} = QR; \quad QQ^T = \mathbf{I}$$

Recall that Ω is a positive definite matrix and $V(k) = \Xi_y^{-1}\mathbf{C}\mathbf{P}(k|k-1)\mathbf{C}^T\Xi_y^{-1}+\Xi_y^{-1}\mathbf{R}\Xi_y^{-1}$. Using QR decomposition, one can rewrite GLR test statistic as follows [6]:

$$\begin{aligned} \mathbf{T}_t^{GLR} &= (\mathbf{Y}^T\Omega^{-1/2}QR)(R^TQ^TQR)^{-1}(R^TQ^T\Omega^{-1/2}\mathbf{Y}) \\ &= \mathbf{Y}^T\Omega^{-1/2}QQ^T\Omega^{-1/2}\mathbf{Y} \\ &= \mathbf{Y}^T\Omega^{-1}\mathbf{Y} \end{aligned} \tag{3.60}$$

[5] G. H. Golub, C. F. Van Loan, Matrix computations, Johns Hopkins University Press, 1996.

[6] Note that $\mathbf{X}^T\Omega^{-1}\mathbf{X} = R^TR$ and since we already know that $\mathbf{X}^T\Omega^{-1}\mathbf{X}$ is invertible then it follows that R^TR is also full-rank. Moreover since a matrix cannot gain rank by multiplication it follows that the square R is also full rank.

Substituting Eq.(3.57) into Eq.(3.60) results in:

$$\mathbf{T}_t^{GLR} = (\mathbf{X}\bar{\mathbf{b}}_y + \mathbf{e})^T \Omega^{-1} (\mathbf{X}\bar{\mathbf{b}}_y + \mathbf{e}) \tag{3.61}$$

In the fault-free case (under null-hypothesis):

$$\mathbf{T}_t^{GLR} = \mathbf{e}^T \Omega^{-1} \mathbf{e}$$

Recalling that:

$$\mathbf{e} = \begin{bmatrix} \gamma'(k) \\ \hdashline \gamma'(k+1) \\ \hdashline \vdots \\ \hdashline \gamma'(k+N) \end{bmatrix}$$

One can deduce that the GLR test statistic under the null-hypothesis follows the χ^2 distribution with $r \times (N+1)$ degrees of freedom where r is the dimension of measurement variable $\mathbf{y}$. Moreover, using the properties of the quadratic form one can find that:

$$\begin{aligned} \mathbf{E}[\mathbf{e}^T \Omega^{-1} \mathbf{e}] &= tr[\Omega^{-1} \mathbf{E}[\mathbf{e}\mathbf{e}^T]] + \mathbf{E}[\mathbf{e}^T] \Omega^{-1} \mathbf{E}[\mathbf{e}] \\ &= tr[\Omega^{-1}\Omega] + 0 \\ &= (N+1) \times r \end{aligned}$$

Again using the properties of the quadratic form, in the faulty case one can write the following:

$$\begin{aligned} \mathbf{E}[\mathbf{Y}^T \Omega^{-1} \mathbf{Y}] &= tr[\Omega^{-1} \mathbf{E}[(\mathbf{Y} - \mathbf{X}\bar{\mathbf{b}}_y)(\mathbf{Y} - \mathbf{X}\bar{\mathbf{b}}_y)^T]] + \mathbf{E}[\mathbf{Y}]^T \Omega^{-1} \mathbf{E}[\mathbf{Y}] \\ &= tr[\Omega^{-1}\Omega] + (\mathbf{X}\bar{\mathbf{b}}_y)^T \Omega^{-1} (\mathbf{X}\bar{\mathbf{b}}_y) \\ &= (N+1) \times r + \bar{\mathbf{b}}_y^T \mathbf{X}^T \Omega^{-1} \mathbf{X}\bar{\mathbf{b}}_y \end{aligned} \tag{3.62}$$

Note that $\Omega \in R^{\{r\times(N+1)\}\times\{r\times(N+1)\}}$. In faulty case the test statistic follows a χ^2 distribution with the same degrees of freedom and the non-central parameter can be calculated as:

$$\lambda = \mathbf{E}[\mathbf{T}_t^{GLR}] - \{(N+1) \times r\} = \bar{\mathbf{b}}_y^T \mathbf{X}^T \Omega^{-1} \mathbf{X}\bar{\mathbf{b}}_y$$

It is worth mentioning that:

$$\mathbf{X}^T\Omega^{-1}\mathbf{X} = \sum_{k=t}^{t+N} \mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_(k;t)$$

Regarding the MLR test statistic, in the study by (Gustafsson, 1996) it is shown that the GLR and MLR statistics are related as follows [7]:

$$l_N(k) = l_N(k, \hat{\nu}(k)) - log(det\ R_N(k)) + C_{prior}(k) \tag{3.63}$$

where $R_N(k)$ is the notation used by (Gustafsson, 1996) for representing the following:

$$\eta = \sum_{k=t}^{t+N} \mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y^T(k;t)$$

As stated by Gustafsson (Gustafsson, 1996) "In fact, the likelihood ratios are asymptotically equivalent except for a constant...", considering this, one can deduce that the MLR tests statistic follows the same distribution and the constant only affects selection of the threshold.

A similar reasoning for the GLR test in the univariate case is presented by (Zhang and Basseville, 2003).

3.9 Appendix 3: Uncorrelatedness of residuals

The multiple hypothesis testing problem can be stated as (Willsky and Jones, 1976) [8]:

$$\Lambda_{t_{f,j}}^{t_{f,j}+N} = \underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N} +$$

$$\{b_{f,j}\mathbf{G}_f(t_{f,j};t_{f,j})g_{f,j}, b_{f,j}\mathbf{G}_f(t_{f,j}+1;t_{f,j})g_{f,j}, \cdots, b_{f,j}\mathbf{G}_f(t_{f,j}+N;t_{f,j})g_{f,j}\}$$

[7] Please refer to Theorem 5 in Appendix C of (Gustafsson, 1996)

[8] In the study by(Willsky and Jones, 1976) the notation is slightly different and the residuals are expressed as:

$$\gamma(k) = G(k;\theta)\nu + \gamma_1(k)$$

where γ_1 is a zero mean white noise sequence with known covariance and it represents the "actual measurements residual if a jump does not occur" where $\Lambda_{t_{f,j}}^{t_{f,j}+N} = \{\gamma_{t_{f,j}}, \gamma_{t_{f,j}+1}, \cdots, \gamma_{t_{f,j}+N}\}$ is the set of residuals assuming occurrence of the fault at time instant $t_{f,j}$ and $\mathbf{G}_f$ is the fault signature matrix for fault f which can be pre-computed offline using the system matrices and the steady state Kalman filter equations. Moreover, the realization of a the fault magnitude is denoted by $b_{f,j}$. In this notation $\underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N} = \{\underline{\gamma}_{t_{f,j}}, \underline{\gamma}_{t_{f,j}+1}, \cdots, \underline{\gamma}_{t_{f,j}+N}\}$ is the set of fault free residuals in the same data window i.e. $[t_{f,j}, t_{f,j}+N]$.

It is worth mentioning that the elements of the fault free set ($\underline{\Lambda}_{t_{f,j}}^{t_{f,j}+N}$) are Gaussian and uncorrelated with zero mean and known covariance matrices which can be computed using the Kalman filter. Now let us compute the covariance matrix for two elements of the the faulty set $\Lambda_{t_{f,j}}^{t_{f,j}+N}$:

$$\mathbf{E}\left[(\gamma_k - \mathbf{E}[\gamma_k])(\gamma_{k+m} - \mathbf{E}[\gamma_{k+m}])^T\right]$$

$$= \mathbf{E}\left[(\gamma_k - b_{f,j}\mathbf{G}_f(k;t_{f,j}))(\gamma_{k+m} - b_{f,j}\mathbf{G}_f(k+m;t_{f,j}))^T\right]$$

$$= \mathbf{E}\big[(\underline{\gamma}_k + b_{f,j}\mathbf{G}_f(k;t_{f,j}) - b_{f,j}\mathbf{G}_f(k;t_{f,j}))(\underline{\gamma}_{k+m} + b_{f,j}\mathbf{G}_f(k+m;t_{f,j})$$

$$- b_{f,j}\mathbf{G}_f(k+m;t_{f,j}))^T\big]$$

$$= \mathbf{E}[\underline{\gamma}_k \underline{\gamma}_{k+m}^T] = [0]_{r\times r}$$

In other words, occurrence of the bias type fault does not lead to any change in the correlation between the residuals. In the above notation $k \in \{t_{f,j}, t_{f,j}+N-1\}$ and $1 \leq m \leq N$. Moreover, γ_k and $\underline{\gamma}_k$ denote the faulty and fault-free residuals, respectively.

Next we examine the correlation between the residuals before and after occurrence of the fault. In this case it is assumed that $k-n < t_{f,j}$:

$$\mathbf{E}\left[(\underline{\gamma}_{k-n} - \mathbf{E}[\underline{\gamma}_{k-n}])(\gamma_{k+m} - \mathbf{E}[\gamma_{k+m}])^T\right]$$

$$= \mathbf{E}\left[(\underline{\gamma}_{k-n}])(\gamma_{k+m} - b_{f,j}\mathbf{G}_f(k+m;t_{f,j}))^T\right]$$

$$= \mathbf{E}\left[(\underline{\gamma}_{k-n}])(\underline{\gamma}_{k+m} + b_{f,j}\mathbf{G}_f(k+m;t_{f,j}) - b_{f,j}\mathbf{G}_f(k+m;t_{f,j}))^T\right]$$

$$= \mathbf{E}\left[\underline{\gamma}_{k-n}\underline{\gamma}_{k+m}^T\right] = [0]_{r\times r}$$

It is worth mentioning that a similar reasoning about the uncorrelatedness of residuals is stated in the study by (Dos Santos and Yoneyama, 2011).

However, it is widely known that uncorrelatedness does not necessarily imply independence, the assumption about independence of the residuals is included in section 3.3.3 of manuscript.

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Chapter 4

Detection and Diagnosis of Incipient Faults in Sensors using a Modified GLR-based Approach

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The material of this chapter has been submitted.

Shorter form of this work have also been submitted to IFAC WC (2014).

4.1 Introduction

Model based fault detection and diagnosis has received significant attention from researchers in recent years and aims to provide the supervisory control system with sufficient information to take possible corrective actions to compensate for instrumentation related faults (Basseville and Nikiforov, 1993; Gertler, 1998; Narasimhan and Jordache, 2000; Chen and Patton, 1999; Ding, 2008; Patton *et al.*, 1989). The ultimate goal of such methods is to accurately detect the time of occurrence of the fault, identify its location and estimate its magnitude. More recently, the concept of fault tolerant control and integration of the fault detection and isolation (FDI) module with controller design has been widely studied in the literature (Zhang and Jiang, 2008; Mhaskar *et al.*, 2006; Prakash *et al.*, 2005; Deshpande *et al.*, 2009; Mhaskar, 2006). The general approach in active fault tolerant control systems is to modify the controller according to the supervisory information provided by the FDI module in order to mitigate the impact of the fault on the system performance.

The study proposed by (Willsky and Jones, 1976) for the first time addressed the online fault detection and isolation (FDI) issue using the generalized likelihood ratio (GLR) test framework. The methodology therein, tries to find the time of occurrence of the fault (TOF) using a sliding window assuming occurrence of abrupt jumps in states of a linear system. In the study by (Narasimhan and Mah, 1988) the statistical time of occurrence detection (TOD) and gross error detection (GED) tests were used to overcome the "burdensome" computational aspects of the sliding window approach in the study by (Willsky and Jones, 1976). The FDI scheme suggested by Prakash et al. (Prakash *et al.*, 2002) was based on a combination of methodologies proposed by (Willsky and Jones, 1976; Narasimhan and Mah, 1988) to deal with the additive step-type faults in sensors, actuators and process parameters of an LTI system. The methodology proposed in (Prakash *et al.*, 2002) took advantage of the fault detection (FDT) and fault confirmation (FCT) tests to detect time of occurrence of the fault and subsequently, the pre-computed fault signature matrices and the GLR test were used to isolate the fault and estimate its magnitude. Unlike the approach in (Narasimhan and Mah, 1988), the proposed method in (Prakash *et al.*, 2002) made it possible to deal with sequential faults. The fault tolerant control (FTC) scheme proposed by Prakash et al. (Prakash *et al.*, 2005) made necessary modifications to controller as per information provided by the FDI module whilst in the study by (Prakash *et al.*, 2002) the FDI is only used to provide supervisory information. As stated by (Villez *et al.*, 2011), the major shortcoming in the FDI scheme proposed by (Prakash *et al.*, 2002) is that detection of time of occurrence of the fault is not addressed properly and this can have adverse effects on the performance of the system when the FDI is integrated with the controller.

In a recent study by (Kiasi *et al.*, 2013*b*) this shortcoming is overcome by means of using an extended data window and a modified GLR approach which refines the crude TOF estimate provided by the FDT and FCT test. This alternative approach was shown to have a superior performance compared to the FDI method proposed by (Prakash *et al.*, 2002) in terms of accurate detection of of TOF and estimation of fault magnitude. This superior performance could be mainly attributed to the precise detection of TOF. In another study by (Kiasi *et al.*, 2013*a*) the marginalized likelihood ratio (MLR) approach using uniform priors is utilized to decouple the detection and isolation phases from estimation of the fault magnitude which was also shown to outperform the FDI scheme suggested by (Prakash *et al.*, 2002).

However, the major shortcoming in the aforementioned studies is that they only consider abrupt jumps in the form of step type faults which is not realistic especially in the context of process faults. As it is widely known, most of process faults gradually evolve and in most cases they follow a ramp type pattern before reaching some steady state. The main motivation behind this study is to detect and isolate realistic ramp type faults and truncated ramp type faults which are more common in process industry. The main challenge in dealing with such faults can be attributed to the fact that with a certain data window, one should consider possibility of occurrence of either pure ramp type fault or a combination of step and ramp type faults. For the latter case, in addition to estimation the TOF, the time instant at which the fault reaches steady state should also be estimated. Moreover, the relevant fault signature matrices need to be developed for both cases. The main contributions of the current study can be summarized as follows:

- Developing fault signature matrices for pure ramp type and truncated ramp type faults;
- Computation of estimated fault magnitude and GLR test statistic for both ramp and truncated ramp type faults;
- Accurate estimation of TOF using a modified GLR-based approach and the concept

of extended data window;

- Development of closed form solutions for fault isolation, estimation of TOF and fault magnitude without the need to solve an optimization problem;
- Introduction of a new FDI framework which is capable of dealing with ramp, truncated ramp and step type faults at the same time;
- Reducing the misclassification rate by means of avoiding isolation and estimation when sufficient number of data points are not available

It is worth mentioning that, the ultimate goal of the proposed FDI scheme is to provide advisory information to operators so that they can take necessary measures such as re-calibration of sensors/actuators.

This chapter is organized in five sections. In the next section, formulation of change detection problem using generalized likelihood ratio test is presented. The newly proposed FDI scheme is discussed in Section 3. Section 4 discusses solution of the proposed FDI scheme and section 5 is dedicated to evaluation of the proposed methodology by application to a CSTR benchmark problem followed by concluding remarks in Section 6.

4.2 Definition of The Problem

4.2.1 Model of the system

Consider the following linear system where $\mathbf{x} \in R^n$, $\mathbf{u} \in R^m$ and $\mathbf{y} \in R^r$. In this representation $\mathbf{w} \in R^q$ and $\mathbf{v} \in R^r$ are process and measurement noise sequences with known covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ which are mutually uncorrelated and the initial state $x(0)$ follows a Gaussian distribution with known mean and variance. Furthermore, it is assumed that Φ, $\mathbf{C}$, Γ_u and Γ_w are known matrices.

$$\mathbf{x}(k) = \Phi\mathbf{x}(k-1) + \Gamma_u\mathbf{u}(k-1) + \Gamma_w\mathbf{w}(k-1) \tag{4.1}$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \tag{4.2}$$

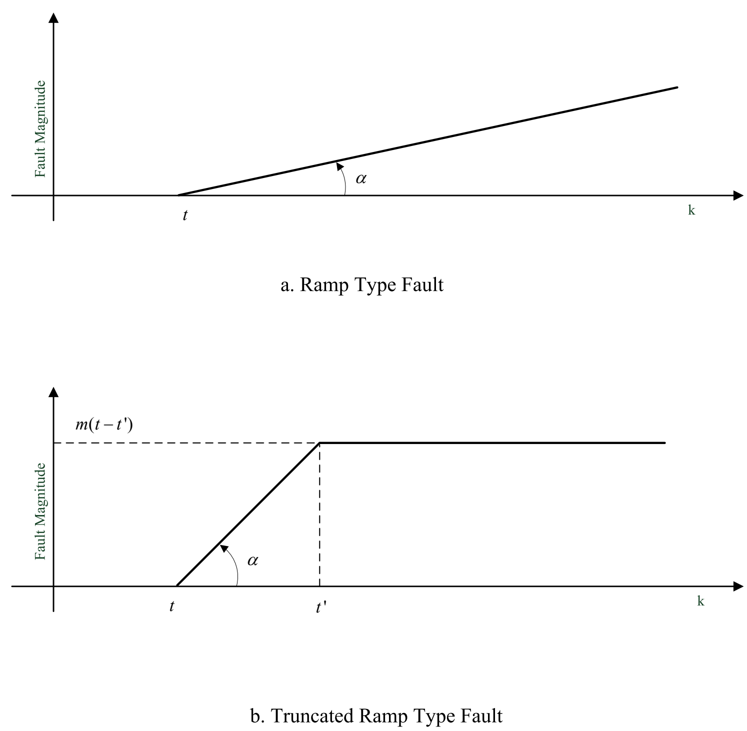


Figure 4.1: Drift Type Faults

In case of occurrence of a ramp type additive fault with slope $m_{u,j}$ as depicted in Fig. 4.1.a , where $m_{u,j} = \tan(\alpha)$ in the jth actuator at time instant t the process would evolve for $k > t$ as follows:

$$\begin{aligned} \mathbf{x}(k) &= \Phi\mathbf{x}(k-1) + \Gamma_u\mathbf{u}(k-1) + \Gamma_w\mathbf{w}(k-1) \\ &\quad + \Gamma_u m_{u,j} \times (k-t-1)\mathbf{e}_{u,j}\sigma(k-t) \qquad (4.3) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \qquad (4.4) \end{aligned}$$

where $\mathbf{e}_{u,j}$ is a unit vector whose jth element is equal to one and all other elements are zero and $\sigma(k-t)$ is a unit step function defined as follows:

$$\sigma(k-t) = \begin{cases} 0 & \text{if } k < t \\ 1 & \text{if } k \geq t \end{cases}$$

Similarly, in case of occurrence of a ramp type additive fault with slope $m_{y,j}$, where $m_{y,j} = \tan(\alpha)$ in the jth sensor, the measurement equation would be modified as follows while the state equation remains as shown in Eq.(4.1):

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) + m_{y,j} \times (k-t)\mathbf{e}_{y,j}\sigma(k-t) \quad (4.5)$$

where $e_{y,j}$ is a unit vector whose jth element is equal to one and all other elements are equal to zero. It is worth noting that the slope, $m_{y,j}$, of the ramp type fault can be defined as follows:

$$m_{y,j} = \frac{b_{y,j}(k) - b_{y,j}(k-1)}{T_s}$$

where T_s is the sampling rate. In case of occurrence of a truncated ramp type fault in $m_{u,j}$ as depicted in Fig. 4.1.b , where $m_{u,j} = \tan(\alpha)$ in the jth actuator at time instant t the process would evolve for $k > t$ as follows:

$$\begin{aligned}
\mathbf{x}(k) &= \Phi\mathbf{x}(k-1) + \Gamma_u\mathbf{u}(k-1) + \Gamma_w\mathbf{w}(k-1) \\
&\quad + m_{u,j} \times (k-t-1)\Gamma_u\mathbf{e}_{u,j}\sigma(k-t) \\
&\quad - m_{u,j} \times (k-t'-1)\Gamma_u\mathbf{e}_{u,j}\sigma(k-t') \quad (4.6) \\
\mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) \quad (4.7)
\end{aligned}$$

Similarly, in case of occurrence of a truncated ramp type additive fault with slope $m_{y,j}$, where $m_{y,j} = \tan(\alpha)$ in the jth sensor, the measurement equation would be modified as follows while the state equation remains as shown in Eq.(4.1):

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k) + m_{y,j} \times (k-t)\mathbf{e}_{y,j}\sigma(k-t') - m_{y,j}\mathbf{e}_{y,j} \times (k-t')\sigma(k-t) \quad (4.8)$$

4.2.2 Fault Signature Matrices of an Additive Ramp Type Fault

In this section the fault signature matrix for a ramp type fault in a sensor is developed. A similar approach can be used to compute the ramp type fault signature in an actuator.

Lemma 4.2.1 *In case of occurrence of a ramp type bias in the* j*th sensor at time instant* t*. The fault signature matrices can be found as follows:*

$$\mathbf{J}_y(k;t) = \frac{k-t-1}{k-t}\Phi\mathbf{J}_y(k-1;t) + \mathbf{K}(k)\mathbf{G}_y(k;t) \tag{4.9}$$

$$\mathbf{G}_y(k;t) = \mathbf{I} - \frac{k-1-t}{k-t}\mathbf{C}\Phi\mathbf{J}_y(k-1;t) \tag{4.10}$$

Proof: Let us consider Eqs.4.1 and 4.5 which describe the evolution of the system after occurrence of the fault. Defining $\delta\hat{\mathbf{x}} = \hat{\mathbf{x}}(k|k) - \mathbf{x}(k)$ and using the Kalman filter equations yields:

$$\begin{aligned} \delta\hat{\mathbf{x}}(k) &= \big[\mathbf{I} - \mathbf{K}(k)\mathbf{C}\big]\big[\Phi\delta\hat{\mathbf{x}}(k-1) - \Gamma_w\mathbf{w}(k-1)\big] \\ &\quad + \mathbf{K}(k)\big[\mathbf{v}(k) + m_{y,j} \times (k-t)\mathbf{e}_{y,j}\big] \end{aligned} \tag{4.11}$$

The corresponding residuals can be expressed as:

$$\gamma(k) = -\mathbf{C}\Phi\delta\hat{\mathbf{x}}(k-1) + \mathbf{v}(k) + \Gamma_w\mathbf{w}(k-1) + m_{y,j}(k-t)\mathbf{e}_{y,j} \tag{4.12}$$

Taking expected value of both sides of Eqs.4.11 and 4.12 yields:

$$\mathbf{E}[\delta\hat{x}(k)] = [\mathbf{I} - \mathbf{KC}]\Phi\mathbf{E}[\delta\hat{x}(k-1)] + m_{y,j} \times (k-t)\mathbf{K}(k)\mathbf{e}_{y,j} \tag{4.13}$$

$$\mathbf{E}[\gamma(k)] = -\mathbf{C}\Phi\mathbf{E}[\delta\hat{x}(k-1)] + m_{y,j} \times (k-t) \tag{4.14}$$

Now let us define the followings:

$$\mathbf{E}[\delta\hat{\mathbf{x}}(k)] = m_{y,j} \times (k-t)\mathbf{J}_y(k;t)\mathbf{e}_{y,j}$$

$$\mathbf{E}[\gamma(k)] = m_{y,j} \times (k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j}$$

using the above definitions and Eqs.4.13 and 4.14 and after some algebraic manipulations, the following recursive equations can be developed:

$$\mathbf{J}_y(k;t) = \frac{k-t-1}{k-t}\Phi\mathbf{J}_y(k-1;t) + \mathbf{K}(k)\mathbf{G}_y(k;t) \tag{4.15}$$

$$\mathbf{G}_y(k;t) = \mathbf{I} - \frac{k-1-t}{k-t}\mathbf{C}\Phi\mathbf{J}_y(k-1;t) \tag{4.16}$$

■

It is worth mentioning that the recursive Eqs.4.9 and 4.10 should be initialized with $\mathbf{J}_y(t;t) = [0]_{n\times r}$ and $\mathbf{G}_y(t;t) = [0]_{r\times r}$. This is due to the fact that at time instant t the fault magnitude is zero. The fault signature matrices for ramp type faults in actuators are similarly developed and the details are given in 4.6.

4.3 Proposed FDI Scheme

4.3.1 GLR approach

The log-likelihood ratio test assuming occurrence of ramp-type fault at time instant t with slope $m_{f,j}$ where $f \in \{u, y\}$ and j represent the type and location of fault, respectively, can be defined as:

$$\mathbf{T} = \max_{t} \max_{m_{f,j}} 2\log \frac{p(\Lambda_1^N|t, m_{f,j})}{p(\Lambda_1^N)} \tag{4.17}$$

where, $t \in [1, N]$ represents the time of occurrence of the fault (TOF) and $\Lambda_1^N = \{\gamma_1, \ldots, \gamma_N\}$ denotes the residuals generated by a fault-free Kalman filter using Eqs. 4.1 and 4.2 in a specified window. In this notation the denominator $p(\Lambda_1^N)$ represents the null-hypothesis (fault-free case). Moreover, the denominator is independent of TOF and the fault magnitude as it represents the residuals under fault-free condition and therefore the joint maximum likelihood (ML) estimate of the time of occurrence of the fault (TOF) and fault magnitude, can be written as follows:

$$\{\hat{m}_{f,j}, \hat{t}\} = arg \max_{t} \max_{m_{f,j}} 2\log p(\Lambda_1^N|t, m_{f,j}) \tag{4.18}$$

This double maximization can be reduced to a single maximization problem by finding the estimated time of occurrence of the fault $\hat{t}$ by means of FDT and FCT tests (Prakash *et al.*, 2005; Prakash *et al.*, 2002; Narasimhan and Mah, 1988). In this case the generalized likelihood ratio (GLR) test is as follows:

$$\mathbf{T}^{GLR} = \max_{f,j} \mathbf{T}_{f,j}^{max} \tag{4.19}$$

where

$$\mathbf{T}_{f,j}^{max} = \max_{m_{f,j}} 2\log\frac{p(\Lambda_{\hat{t}}^{N}|m_{f,j})}{p(\Lambda_{\hat{t}}^{N})} \quad (4.20)$$

$$= \max_{m_{f,j}} \mathbf{T}_{f,j} \quad (4.21)$$

It should be noted that this study is mainly concerned with detection and diagnosis of additive ramp and truncated ramp faults that occur sequentially in sensors and actuators. In this regard, it is assumed that multiple faults occur sequentially in time but not simultaneously and in case of occurrence of a fault, the controller corrective action will not cause the process variables to violate the process safety or alarm shut down limits. Moreover, in this study it is assumed that occurrence of additive faults in actuators and sensors do not lead to instability of the closed loop system.

4.3.2 Finding the candidate data window for occurrence of the fault

This study adopts the approach used in (Kiasi *et al.*, 2013*b*; Kiasi *et al.*, 2013*a*) for constructing a candidate data window. In this approach the FDI takes advantage of the statistical FDT and FCT tests to find the initial data window in which the fault has occurred. However, as mentioned by Villez et al. (Villez *et al.*, 2011) detection of TOF using FDT and FCT tests is associated with inaccuracy and hence in this study a refining mechanism is required to estimate the most probable TOF. The FDT test is based on a the quadratic form of the residuals at each time instant normalized using the corresponding covariance matrix while the FCT test is a sum of FDT tests in a specific data window. Assuming rejection of FDT test at time instant t_1 and data window of size $N + 1$, the FCT test be defined defined as:

$$\epsilon(N;t_1) = \sum_{k=t_1}^{t_1+N} \gamma^T(k)V(k)^{-1}\gamma(k) \quad (4.22)$$

which follows a χ^2 distribution with $r \times (N+1)$ degrees of freedom assuming the residuals are zero mean white noise with known covariance matrix.

In the GLR approach proposed by Prakash et al. (Prakash *et al.*, 2002; Deshpande *et* *al.*, 2009) it is suggested to compute the FCT test for data window $[t_1, t_1 + N]$ which is obtained after rejection of the FDT test at time instant t_1. Upon rejection of FCT test, it is assumed that t_1 is the TOF and subsequently the likelihood ratio is computed for all hypothesized faults in the interval $[t_1, t_1 + N]$ to isolate and estimate the fault magnitude.

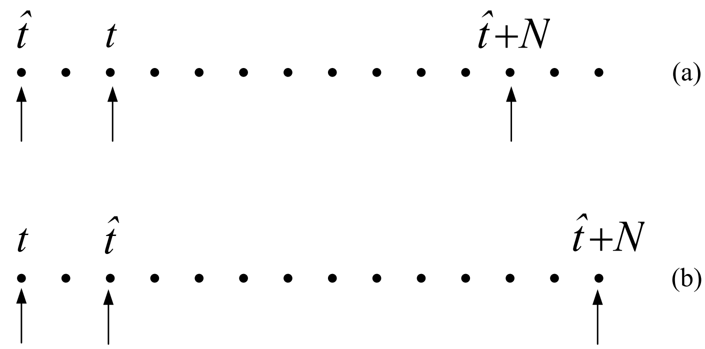


Figure 4.2: Scenarios leading to inaccurate detection of TOF: Fig. (4.2a) the estimated TOF ($\hat{t}$) precedes the actual TOF (t); Fig. (4.2b) the estimated TOF ($\hat{t}$) is after the actual TOF (t) (Narasimhan and Mah, 1988).

Now let us consider the following two scenarios as depicted in Figure 4.2, in which for the first case the actual TOF is after the rejection of the FDT test while in the second scenario the FDT is rejected after the actual TOF (Narasimhan and Mah, 1988). In order to accurately estimate the time of occurrence of the fault we propose to append the original data window $[t_1, t_1 + N]$ with M extra samples prior to rejection of the FDT test at t_1 and form a new data window as $[t_1 - M, t_1 + N]$. The concept of extended data window is depicted in Figure 4.3.

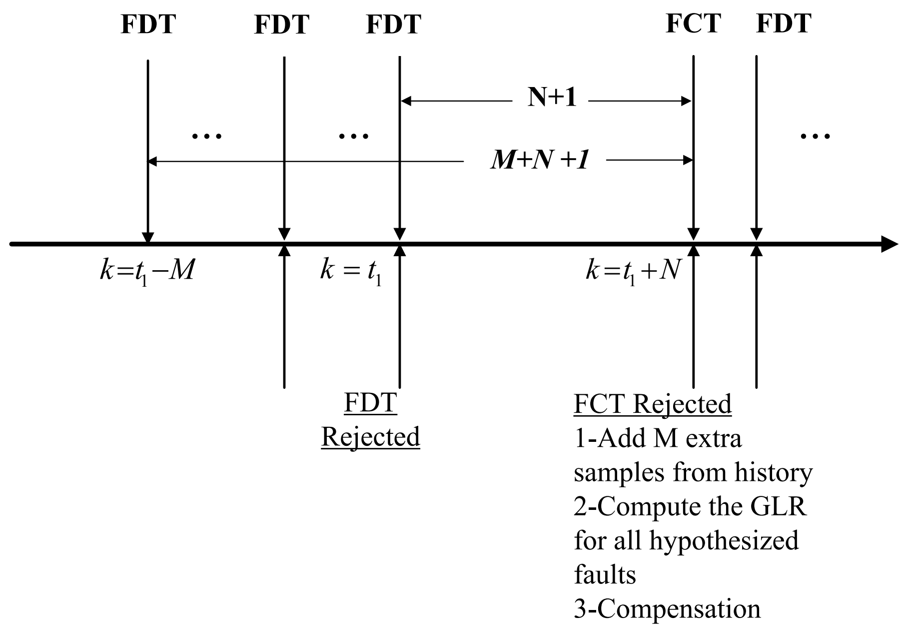


Figure 4.3: Extended FCT data window

4.3.3 GLR test for a ramp type fault

In the wake of the proposed extended data window and assuming occurrence of a ramp-type fault type f ($f \in \{u, y\}$) at time instant $t \in [t_1 - M, t_1 + N]$, the GLR test can be defined as follows:

$$\mathbf{T}^{GLR} = \max_{f,j,t_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j}} \tag{4.23}$$

where

$$\begin{aligned}\mathbf{T}^{max}_{f,j,t_{f,j}} &= \max_{t} \max_{m_{f,j}} 2 \log \frac{p(\Lambda_{t_1-M}^{t_1+N} | t, m_{f,j})}{p(\Lambda_{t_1-M}^{t_1+N})} && (4.24)\\ &= \max_{t} \mathbf{T}_{f,t,j} && (4.25)\end{aligned}$$

it is worth noting that:

$$\mathbf{T}_{f,j,t} = \max_{m_{f,j}} 2 \log \frac{p(\Lambda_{t_1-M}^{t_1+N} | t, m_{f,j})}{p(\Lambda_{t_1-M}^{t_1+N})}$$

and $\Lambda_{t_1-M}^{t_1+N} = \{\gamma_{t_1-M}, \ldots, \gamma_{t_1+N}\}$. The likelihood ratio on the right hand side of Eq. 4.24 should be maximized for all $t \in [t_1 - M, t_1 + N]$ assuming occurrence of a specific fault type f at location j. Then the same maximization procedure should be repeated for the other hypothesized faults (all combinations of f and j). Finally, the maximum value of $\mathbf{T}^{max}_{f,j,t_{f,j}}$ among all hypothesized faults determines $\hat{t}$, $\hat{f}$ and $\hat{j}$ as follows:

$$\{\hat{f},\hat{j},\hat{t}\} = arg \max_{f,j,t_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j}} \tag{4.26}$$

using $\hat{t}$, $\hat{f}$ and $\hat{j}$, the ML-estimate of fault magnitude can be found as:

$$\hat{m}_{\hat{f},\hat{j}} = arg \max_{m_{\hat{f},\hat{j}}} 2\log p(\Lambda^{t_1+N}_{t_1-M}|\hat{t}, b_{\hat{f},\hat{j}}) \tag{4.27}$$

4.3.4 Estimation of ramp type fault magnitude in a sensor

Lemma 4.3.1 *Assuming occurrence of a single ramp type fault in a sensor at location j and at time instant $t \in [t_1 - M, t_1 + N]$, the estimated fault magnitude and $\mathbf{T}_{y,j,t}$ can be computed as follows:*

$$\hat{m}_{y,j,t} = \frac{\sum_{k=t+1}^{t_1+N}\left\{(k-t)\mathbf{e}^T_{y,j}\mathbf{G}^T_y(k;t)V(k)^{-1}\gamma(k)\right\}}{\sum_{k=t+1}^{t_1+N}\left\{(k-t)^2\mathbf{e}^T_{y,j}\mathbf{G}^T_y(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}\right\}} \tag{4.28}$$

$$\mathbf{T}_{y,j,t} = \frac{\left(\sum_{k=t+1}^{t_1+N}\left\{(k-t)\mathbf{e}^T_{y,j}\mathbf{G}^T_y(k;t)V(k)^{-1}\gamma(k)\right\}\right)^2}{\sum_{k=t+1}^{t_1+N}\left\{(k-t)^2\mathbf{e}^T_{y,j}\mathbf{G}^T_y(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}\right\}} \tag{4.29}$$

where $\mathbf{G}_y(k;t)$ denotes the fault signature matrix for a ramp type fault assuming occurrence of the fault at time instant t.

Proof: Recall the joint maximum likelihood (ML) estimate of the ramp slope and TOF:

$$\{\hat{m}_{y,j},\hat{t}\} = \arg\max_{m_{y,j},t} 2\log p(\Lambda^{t_1+N}_{t_1-M}|m_{y,j}, t) \tag{4.30}$$

Assuming t to be the true TOF, the joint ML-estimate reduces to:

$$\hat{m}_{y,j} = \arg\max_{m_{y,j}} 2\log p(\Lambda^{t_1+N}_{t}|m_{y,j}) = \arg\max_{m_{y,j}} \lambda_{y,j} \tag{4.31}$$

invoking the definition of multivariate normal distribution and assuming independence of residuals yields:

$$\lambda_{y,j} = 2\log\left\{\zeta \exp\{-\frac{1}{2}\sum_{k=t}^{t_1+N}\gamma'^T(k)V(k)^{-1}\gamma'(k)\right\} \tag{4.32}$$

where

$$\gamma'(k) = \gamma(k) - m_{y,j} \times (k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j}$$

In the above equation $\zeta = \frac{1}{(2\pi)^{r\times(t_1+N-t+1)/2}\prod_{k=t}^{t_1+N}|V(k)|^{1/2}}$ and r denotes the dimension of $\mathbf{y}$.

$$\begin{aligned}\lambda_{y,j} &= 2\log(\zeta) + 2m_{y,j}\sum_{k=t}^{t_1+N}\{(k-t)\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\gamma(k)\} \\ &\quad -m_{y,j}^2\sum_{k=t}^{t_1+N}\{(k-t)^2\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}\}\end{aligned}$$

The ML-estimate of $m_{f,j}$ can be found by solving the following equation:

$$\begin{aligned}\frac{\partial\lambda_{y,j}}{\partial m_{y,j}} &= 0 \\ \Rightarrow \hat{m}_{y,j,t} &= \frac{\sum_{k=t+1}^{t_1+N}\left\{(k-t)\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\gamma(k)\right\}}{\sum_{k=t+1}^{t_1+N}\left\{(k-t)^2\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}\right\}}\end{aligned} \tag{4.33}$$

Note that the lower limit of the summation is $t+1$ and not t. This is due the fact that $\mathbf{G}_y(t;t) = [0]_{r\times r}$. It is worth mentioning that $\hat{m}_{y,j,t}$ is used to represent the estimated slope and not $\hat{m}_{y,j}$. This is mainly due to the fact that the ML-estimate of the ramp slope is obtained based on the assumption of occurrence of the fault at time instant t. The log-likelihood ratio on the right-hand side (RHS) of Eq. 4.24 can be written as:

$$\mathbf{T} = 2\log\frac{exp\{-\frac{1}{2}\sum_{k=t+1}^{t_1+N}\gamma'(k)^TV(k)^{-1}\gamma'(k)\}}{exp\{-\frac{1}{2}\sum_{k=t+1}^{t_1+N}\gamma(k)^TV(k)^{-1}\gamma(k)\}} \tag{4.34}$$

substituting $\hat{m}_{y,j}$ into log-likelihood ratio in Eq.(4.34) yields:

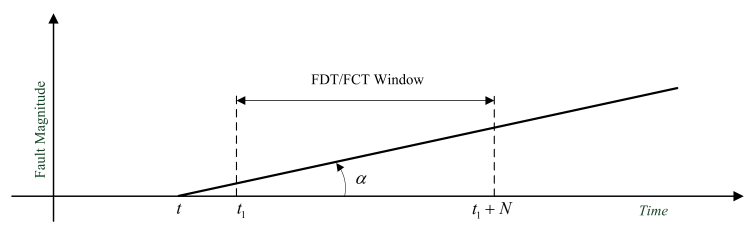
$$\mathbf{T}_{y,j,t} = \frac{\left(\sum_{k=t+1}^{t_1+N}\left\{(k-t)\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\gamma(k)\right\}\right)^2}{\sum_{k=t+1}^{t_1+N}\left\{(k-t)^2\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j}\right\}} \tag{4.35}$$

■

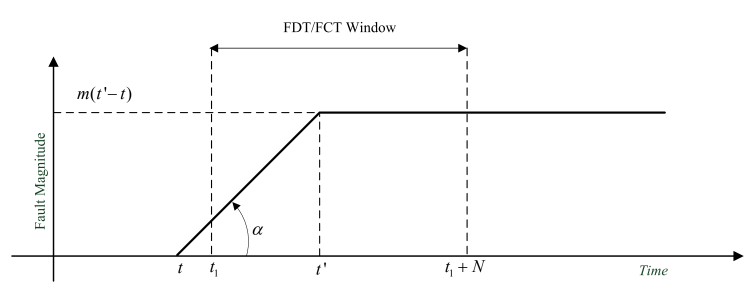
A similar approach can be used to find the estimated slope and $\mathbf{T}_{u,j,t}$ in case of occurrence of a ramp type fault in an actuator. This topic is addressed in 4.7.

Based on the concept of the extended data window four different scenarios can be considered as follows:

1. Occurrence of a ramp type fault where $t_1 - M \leq t < t_1 + N$
2. Occurrence of a truncated ramp type fault where $t_1 - M \leq t < t' < t_1 + N$ and $t' - t > 1$
3. Occurrence of a truncated ramp type fault where $t_1 - M \leq t < t' < t_1 + N$ and $t' - t = 1$
4. Occurrence of a truncated ramp type fault where $t_1 \leq t < t_1 + N$ and $t' \geq t_1 + N$



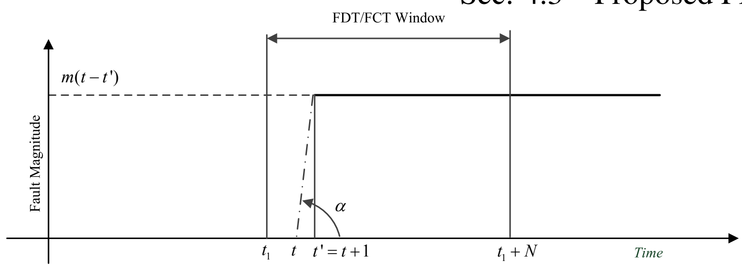
1. Fault Scenario a



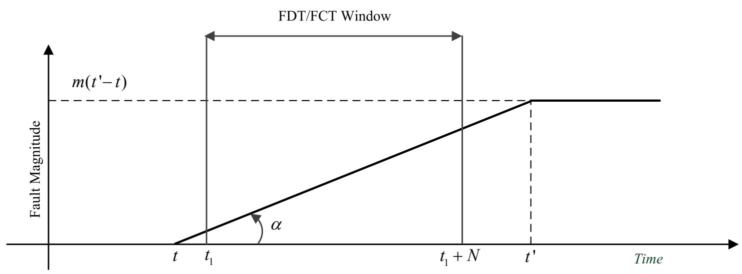
2. Fault Scenario b

Figure 4.4: Fault scenarios a and b

Fault scenarios *a* and *b* are depicted in Fig.4.4 while scenarios *c* and *d* are shown in Fig.4.5. Apparently, the main challenge is to deal with the third and fourth cases. The third scenario



1. Fault Scenario c



2. Fault Scenario d

Figure 4.5: Fault scenarios c and d

in fact represents a step-type bias rather than a truncated ramp. this is mainly due to the fact that in a discrete system, the inter-sample behavior is ignored. Finally, the last case is also likely to occur as the size of extended data window is usually selected with respect to the dynamics of the system and in some cases the dynamics of the fault may not be known beforehand. Therefore, a specific fault might develop very slowly and reach steady state after a long time. In such cases, the concept of truncated ramp can be used again in a more general sense. In other words, the FDI scheme isolates the ramp type fault within the first data window and eventually when the fault reaches steady state the next detection will be performed in another data window.

Remark 4.3.1 *Detection, isolation and estimation of ramp type faults is likely to be non-trivial due to the fact that it might be difficult to obtain an accurate estimate of the ramp slope using some limited data points in the data window. Moreover, as the fault evolves* *through time even a small bias in estimation of the fault magnitude would eventually result in large deviations in the residuals. Considering the limited size of extended data window it would be in some cases difficult to go backwards and correct the errors and hence an alternative solution should be sought. In this study, in order to avoid such critical issues, all faults would be treated as truncated-ramps. This approach has several benefits. First of all, a unified framework is developed to detect ramp and truncated ramp faults. Secondly, in case the fault evolves beyond the data window, the FDI module would have another chance to estimate the slope in the subsequent data windows. It should be noted that a pure ramp type fault within a data window can be modeled using a truncated ramp whose* t' *occurs at the last point of the data window i.e.* $t_1 + N$.

4.3.5 Fault signature matrices for a truncated ramp type fault in a sensor

By invoking the superposition principle in linear systems, the fault signature matrices for a truncated ramp type fault in jth can be expressed as sum of positive and negative ramps occurring at consecutive time instants t and t'. Based on this argument the expected values of $\gamma(k)$ and $\delta\hat{\mathbf{x}}$ in case of occurrence of a truncated ramp type fault, can be defined as follows:

$$\begin{aligned}\mathbf{E}[\delta\hat{\mathbf{x}}] &= m_{y,j} \times (k-t)\mathbf{J}_y(k;t)\mathbf{e}_{y,j} \\ &\quad - m_{y,j} \times (k-t')\mathbf{J}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t') \end{aligned} \tag{4.36}$$

$$\begin{aligned}\mathbf{E}[\gamma(k)] &= m_{f,j}(k-t)\mathbf{G}_y(k-t)\mathbf{e}_{y,j}\sigma(k-t) \\ &\quad - m_{y,j} \times (k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t') \end{aligned} \tag{4.37}$$

Fault signature matrices for a truncated ramp fault in jth actuator are developed in 4.8.

4.3.6 Estimation of truncated ramp type fault magnitude in a sensor

Theorem 4.3.1 *Assuming occurrence of a truncated ramp type fault depicted in Fig. (4.1.b) in a sensor at location j and at time instant $t \in [t_1 - M, t_1 + N]$ and t' to be the time instant at which the fault reaches its steady state value ($t_1 - M \leq t \leq t' \leq t_1 + N$), the estimated fault magnitude and $\mathbf{T}_{y,j,t,t'}$ can be computed as follows:*

$$\hat{m}_{y,j,t,t'} = \frac{\Omega_{y,j}(k,t,t')}{\Upsilon_{y,j}(k,t,t')} \tag{4.38}$$

$$\mathbf{T}_{y,j,t,t'} = \frac{\left[\Omega_{y,j}(k,t,t')\right]^2}{\Upsilon_{y,j}(k,t,t')} \tag{4.39}$$

where

$\Omega_{y,j}(k,t,t') \triangleq$

$$\sum_{k=t+1}^{t_1+N} \left\{ \Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j} - (k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t')\Big]^T V(k)^{-1}\gamma(k) \right\}$$

$\Upsilon_{y,j}(k,t,t') \triangleq$

$$\sum_{k=t+1}^{t_1+N} \left\{ \Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j} - (k-t')\mathbf{G}_y(k;t')\mathbf{e}_{f,j}\sigma(k-t')\Big]^T V(k)^{-1} \right.$$
$$\left. \times\Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j} - (k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t')\Big] \right\}$$

Proof: Similar to ramp type faults discussed earlier, it is assumed that the data window of residuals $\Lambda_{t_1-M}^{t_1+N} = \{\gamma_{t_1-M}, \cdots, \gamma_{t_1+N}\}$ is available. The ML-estimate of the ramp slope can be defined as:

$$\{\hat{m}_{y,j}, \hat{t}, \hat{t'}\} = 2 \log \underset{m_{y,j},t,t'}{\arg\max}\, p(\Lambda_{t_1-M}^{t_1+N} | m_{f,j}, t, t') \tag{4.40}$$

assuming occurrence of the fault at time instant t and reaching the steady state at time instant t' as shown in Fig.(4.1.b) where $t_1 \leq t < t' < t_1 + N$, the above equation can be rewritten as:

$$\hat{m}_{y,j,t,t'} = \arg\max_{m_{y,j}} 2\log p(\Lambda_{t+1}^{t_1+N}|m_{y,j}) = \arg\max_{m_{y,j}} \lambda_{y,j} \tag{4.41}$$

It is worth mentioning that the notation $\hat{m}_{y,j,t,t'}$ is used for the estimated fault slope instead of $\hat{m}_{y,j}$. The main reason behind using this notation is that the estimated slope is based on the assumption of occurrence of the fault at time instant t and reaching steady state at instant t'. Note that the residuals would be considered in a sequence starting at $t+1$ and not t. This is due to the fact that the fault affects the system starting at $t+1$ or in other words $\mathbf{G}_y(t;t) = [0]_{r\times r}$. The log-likelihood function can be defined as Eq.(4.32) and in this case we have:

$$\begin{aligned}\gamma'(k) &= \gamma(k) - m_{y,j}(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j} \\ &\quad + m_{y,j}(k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t')\end{aligned}$$

It is straightforward to find:

$$\begin{aligned}\lambda_{y,j} &= 2\log(\zeta) - \sum_{k=t+1}^{t_1+N}\Big\{\gamma^T(k)V(k)^{-1}\gamma(k) \\ &\quad + m_{y,j}^2(k-t)^2\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y(k;t)\mathbf{e}_{y,j} \\ &\quad + m_{y,j}^2(k-t')^2\mathbf{g}_{y,j}^T\mathbf{G}_y^T(k;t')V(k)^{-1}\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t') \\ &\quad - 2m_{y,j}^2(k-t)(k-t')\mathbf{e}_{y,j}^T\mathbf{G}_y^T(k;t)V(k)^{-1}\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t') \\ &\quad - 2m_{y,j}(k-t)\mathbf{e}_{y,j}\mathbf{G}_y^T(k;t)V(k)^{-1}\gamma(k) \\ &\quad + 2m_{y,j}(k-t')\mathbf{e}_{y,j}\mathbf{G}_y^T(k;t')V(k)^{-1}\gamma(k)\sigma(k-t')\Big\}\end{aligned} \tag{4.42}$$

solving $\frac{\partial\lambda_{y,j}}{\partial m_{y,j}} = 0$ yields:

$$\hat{m}_{y,j,t,t'} = \frac{\Omega_{y,j}(k,t,t')}{\Upsilon_{y,j}(k,t,t')} \tag{4.43}$$

where

$\Omega_{y,j}(k,t,t') \triangleq$

$$\sum_{k=t+1}^{t_1+N}\left\{\Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j}-(k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t')\Big]^T V(k)^{-1}\gamma(k)\right\}$$

$$\begin{aligned}\Upsilon_{y,j}(k,t,t') &\triangleq \\ &\sum_{k=t+1}^{t_1+N}\left\{\Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j}-(k-t')\mathbf{G}_y(k;t')\mathbf{e}_{f,j}\sigma(k-t')\Big]^T V(k)^{-1}\right.\\ &\qquad\left.\times\Big[(k-t)\mathbf{G}_y(k;t)\mathbf{e}_{y,j}-(k-t')\mathbf{G}_y(k;t')\mathbf{e}_{y,j}\sigma(k-t')\Big]\right\}\end{aligned}$$

Assuming the log-likelihood ratio in Eq.(4.34) and substituting $\hat{m}_{y,j,t,t'}$ into this equation results in:

$$\mathbf{T}_{y,j,t,t'} = \frac{\big[\Omega_{y,j}(k,t,t')\big]^2}{\Upsilon_{y,j}(k,t,t')} \tag{4.44}$$

■

A similar approach can be used to find the estimated fault magnitude and $\mathbf{T}_{u,j,t,t'}$ for a truncated ramp fault in an actuator. This topic is discussed in 4.9.

4.3.7 Formulation of the FDI using GLR test and truncated ramp type faults

Let us assume occurrence of a truncated ramp fault type f ($f \in \{u,y\}$) as depicted in Fig.(4.1). In addition, it is assumed that the extended data window of residuals $\Lambda_{t_1-M}^{t_1+N} = \{\gamma_{t_1-M},\cdots,\gamma_{t_1+N}\}$ is available. The GLR test for detection of the TOF, isolation of fault type, its location and estimation of the slope of the truncated ramp type fault can be defined as follows:

$$\mathbf{T}^{GLR} = \max_{f,j,t_{f,j},t'_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}} \tag{4.45}$$

where

$$\begin{aligned}\mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}} &= \max_t \max_{t'} \max_{m_{f,j}} 2\log\frac{p(\Lambda_{t_1-M}^{t_1+N}|t,t',m_{f,j})}{p(\Lambda_{t_1-M}^{t_1+N})} && (4.46)\\ &= \max_t \max_{t'} \mathbf{T}_{f,j,t,t'} && (4.47)\end{aligned}$$

where

$$\mathbf{T}_{f,j,t,t'} = \max_{m_{f,j}} 2\log \frac{p(\Lambda_{t_1-M}^{t_1+N}|t,t',m_{f,j})}{p(\Lambda_{t_1-M}^{t_1+N})}$$

The log-likelihood ratio on the right hand side of Eq.(4.46) should be maximized for all $(t,t') \in [t_1 - M, t_1 + N]$ where $t \leq t'$, assuming occurrence of a specific fault type f at location j. Then the same maximization procedure should be repeated for the other hypothesized faults (all combinations of f and j). Finally, the maximum value of $\mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}}$ among all hypothesized faults determines $\hat{t}$, $\hat{t}'$, $\hat{f}$ and $\hat{j}$ as follows:

$$\{\hat{f},\hat{j},\hat{t},\hat{t}'\} = \arg \max_{f,j,t_{f,j},t'_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}} \tag{4.48}$$

The procedure which leads to estimation of TOF, isolation of the fault and estimation of its magnitude can be summarized as follows. This procedure is discussed for the more general case where faults may be present in both sensors and actuators.

1. FDT test is applied at each time instant. Upon rejection of the FDT test, the FCT test is applied.

2. Upon rejection of FCT test, the extended data window is formed by means of adding M extra data points prior to the rejection of FDT test to the original FCT data window.

3. $\mathbf{T}_{f,j,t,t'}$ is computed for all $t \in [t_1 - M, t_1 + N]$, for a specific choice of f and j:

$$\mathbf{T}_{f,j,t,t'} = \frac{\left[\Omega_{f,j}(k,t,t')\right]^2}{\Upsilon_{f,j}(k,t,t')}$$

   where $f \in \{u, y\}$ and $\mathbf{T}_{f,j,t,t'}$ can be computed as per Eqs.(4.44) and (4.79) for sensors and actuators, respectively.

4. $\mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}}$ for the chosen f and j can be computed as follows:

$$\mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}} = \max_{t} \max_{t'} \max_{m_{f,j}} \mathbf{T}_{f,j,t,t'}$$

5. For all other combinations of f and j, steps 3 and 4 are repeated.

6. The maximum test statistic among all hypothesized faults will determine $\hat{t}$, $\hat{t'}$ $\hat{f}$ and $\hat{j}$ as follows:

$$\{\hat{f}, \hat{j}, \hat{t}, \hat{t'}\} = \arg \max_{f,j,t_{f,j},t'_{f,j}} \mathbf{T}^{max}_{f,j,t_{f,j},t'_{f,j}}$$

7. The estimated fault magnitude can be found for the isolated fault using $\hat{t}$, $\hat{t'}$, $\hat{f}$ and $\hat{j}$ as follows:

$$\hat{m}_{\hat{f},\hat{j},\hat{t},\hat{t'}} = \frac{\Omega_{f,j}(k, \hat{t}, \hat{t'})}{\Upsilon_{f,j}(k, \hat{t}, \hat{t'})}$$

Equations 4.43 and 4.78 should be used for sensors and actuators, respectively.

8. Compensation is performed and again the FDT test is applied at the next time instant.

The compensation mentioned in item 8 is as follows:

$$y_c(k) = y(k) - \hat{m}_{y,\hat{j}} \times (\hat{t}' - \hat{t})\mathbf{e}_{y,\hat{j}} \quad (\hat{f} = y) \tag{4.49}$$

$$\mathbf{m}_c(k) = \mathbf{m}(k) + \hat{m}_{u,\hat{j}} \times (\hat{t}' - \hat{t})\mathbf{e}_{u,\hat{j}} \quad (\hat{f} = u) \tag{4.50}$$

$$\begin{aligned}\hat{\mathbf{x}}_c(t_1+N|t_1+N) &= \hat{\mathbf{x}}(t_1+N|t_1+N) \\ &-\hat{m}_{y,\hat{j},\hat{t},\hat{t}'} \times (t_1+N-\hat{t})\mathbf{J}_y(t_1+N;\hat{t})\mathbf{e}_{y,\hat{j}} \\ &+\hat{m}_{y,\hat{j},\hat{t},\hat{t}'} \times (t_1+N-\hat{t}')\mathbf{J}_y(t_1+N;\hat{t}')\mathbf{e}_{y,\hat{j}} \quad (\hat{f} = y)\end{aligned} \tag{4.51}$$

$$\begin{aligned}\hat{\mathbf{x}}_c(t_1+N|t_1+N) &= \hat{\mathbf{x}}(t_1+N|t_1+N) \\ &-\hat{m}_{u,\hat{j},\hat{t},\hat{t}'} \times (t_1+N-\hat{t}-1)\mathbf{J}_u(t_1+N;\hat{t})\mathbf{g}_{u,\hat{j}} \\ &+\hat{m}_{u,\hat{j},\hat{t},\hat{t}'} \times (t_1+N-\hat{t}'-1)\mathbf{J}_u(t_1+N;\hat{t}')\mathbf{g}_{u,\hat{j}} \quad (\hat{f} = u)\end{aligned} \tag{4.52}$$

$$\begin{aligned}\gamma_c(k) &= \gamma(k) - \hat{m}_{y,\hat{j},\hat{t},\hat{t}'} \times (k-\hat{t})\mathbf{G}_y(k;\hat{t})\mathbf{e}_{y,\hat{j}}\sigma(k-t) \\ &+\hat{m}_{y,\hat{j},\hat{t},\hat{t}'} \times (k-\hat{t}')\mathbf{G}_y(k;\hat{t}')\mathbf{e}_{y,\hat{j}}\sigma(k-t') \quad (\hat{f} = y)\end{aligned} \tag{4.53}$$

$$\begin{aligned}\gamma_c(k) &= \gamma(k) - \hat{m}_{u,\hat{j},\hat{t},\hat{t}'} \times (k-\hat{t}-1)\mathbf{G}_u(k;\hat{t})\mathbf{g}_{u,\hat{j}}\sigma(k-t-1) \\ &+\hat{m}_{u,\hat{j},\hat{t},\hat{t}'} \times (k-\hat{t}'-1)\mathbf{G}_u(k;\hat{t}')\mathbf{g}_{u,\hat{j}}\sigma(k-t'-1) \quad (\hat{f} = u)\end{aligned} \tag{4.54}$$

where $k \in [\hat{t}, t_1+N]$. In addition, $\mathbf{m}$ denotes the controller output (Prakash *et al.*, 2002). It should be noted that state compensation in Eq.(4.51) is performed only once and after the isolation and estimation phases are carried out. The main purpose of this compensation is to provide the Kalman filter with a bias-free estimated state in the next iteration. Moreover, it is necessary to compensate the residuals using Eqs.(4.53) and (4.54) for faults in sensor and actuators, respectively. The main reason for compensation of the residuals is that in case the fault is still existent after the current data window, in the next extended data window the FDI would look backwards into the residuals. Since the FDI has already identified and isolated the fault in the current data window, the compensation of residual would avoid any mistake by the FDI when it look backwards in time. An example of such case can be a ramp type fault which is affecting the system during several consecutive FCT data windows.

Remark 4.3.2 *In order to prevent false alarms and also to accurately estimate the magnitude of the fault, a minimum size of data window 's' is required. In other words, if $\hat{t} > t_1 + N - s + 1$, then the number of samples may not be enough to obtain a good estimate of the fault magnitude and hence it seems logical to obtain more data points for the purpose of estimation. Furthermore if $\hat{t} > t_1 + N - s + 1$ and the fault has actually occurred, since no corrective action is taken by the FDI, the FDT and FCT tests would be rejected again in the following instants and the fault would be detected in the subsequent window. In choosing the minimum size of data window, the time required for the estimator to converge after a change occurs has to be also considered. Note that this constraint is applied to t and not t'. The criteria for selecting parameter 's' is omitted in this study for the sake of brevity and the interested reader is referred to (Kiasi* et al.*, 2013*b*; Kiasi* et al.*, 2013*a*) for details.*

4.4 Simulation Case Study

The following LTI system was used to test the performance of the proposed FDI scheme subject to ramp type faults:

$$\begin{aligned}
\mathbf{x}(k) &= \Phi\mathbf{x}(k-1) + \Gamma_u\mathbf{u}(k-1) + \Gamma_w\mathbf{w}(k-1) \\
\mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k)
\end{aligned}$$

where

$$\Phi = \begin{bmatrix} 0.1843 & -0.0080 \\ 73.5080 & 1.3330 \end{bmatrix}, \quad \Gamma_u = \begin{bmatrix} 0.1340 & 0.0026 \\ -1.7948 & -0.7335 \end{bmatrix}$$

$$\Gamma_w = \begin{bmatrix} 0.0598 & -0.0004 \\ 3.9038 & 0.1208 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Table 4.1: Standard Deviations for Process and Measurement Noise Sequences

Variable	σ**(SD)**
w_1	0.05
w_2	2.5
y_1	0.01
y_2	0.5

Table 4.2: Sequence of Faults

Fault Type	**TOF**	**Fault Magnitude**
Ramp in y_1	$t = 100,\ t' = 110$	$m_{y_1} = \tan(\alpha_1) = -0.01$
Ramp in y_2	$t = 200,\ t' = 210$	$m_{y_2} = \tan(\alpha_2) = 1$
Step in y_2	$t = 300,\ t' = 301$	$m_{y_2} = \tan(\alpha_3) = -2.5$
Ramp in y_1	$t = 400,\ t' = 440$	$m_{y_1} = \tan(\alpha_4) = 0.005$

TOF: Time of Occurrence of the Fault

Moreover $x_{ss} = [0.265, 393.95]^T$ and $u_{ss} = [1, 15]^T$. The MPC weighting matrix, prediction horizon and control horizon were set to following values:

$$N_p = 10, \quad N_c = 1, \quad W_E = \begin{bmatrix} 10^4 & 0 \\ 0 & 1 \end{bmatrix}, \quad W_U = [0]$$

The noise parameters were set to the values in Table 4.1.

In order to evaluate the performance of the proposed FDI scheme, the benchmark system was tested subject to the sequential faults tabulated in Table 4.2. The results for 100 Monte Carlo runs are shown in Table 4.4. As seen in this table, the proposed FDI scheme is capable of detection TOF and the time instant at which the fault reaches steady state (t') fairly accurately. In addition, the results reveal that the proposed method is able to estimate the fault slope precisely. However, it is worth nothing that since the last fault persists for a period of time which is longer than the specified data window, the suggested FDI method tries to isolate and estimate this fault in two subsequent windows. This approach enables the FDI to deal with ramps which might affect the process gradually for a long period of time by means of dividing them into several segments where in each segment the FDI treats the fault as a truncated ramp.

It is worth noting that in the current study the FDI module is not integrated with the

Table 4.3: FDI results subject to fault sequences based on 100 Monte Carlo runs for each case

TOF [1]	**Fault**	**ETOF** [2] $\hat{\mathbf{t}}(\sigma_{\hat{\mathbf{t}}})$	$\hat{\mathbf{t}}'(\sigma_{\hat{\mathbf{t}}'})$	**EFM** [3] $\hat{\mathbf{m}}(\sigma_{\hat{\mathbf{m}}})$	**NMC** [4]
$t = 100, t' = 110$	$m_{y_1} = -0.01$	$t = 101.2$ (0.447)	$t' = 110.8$ (0.836)	-0.0104 (0.0011)	
$t = 200, t' = 210$	$m_{y_2} = 1$	$t = 201$ (0.010)	$t' = 210.6$ (0.547)	1.0552 (0.0597)	
$t = 300, t' = 301$	$m_{y_2} = -2.5$	$t = 300.8$ (0.425)	$t' = 302.2$ (0.425)	-2.5161 (0.2540)	10/100
$t = 400, t' = 440$	$m_{y_1} = 0.005$	$t = 401.6$ (0.894)	$t' = 423.8$ (3.033)	0.0051 (0.0005)	
		$t = 425.4$ (2.792)	$t' = 439.2$ (1.095)	0.0052 (0.0006)	

[1] TOF: Time of Occurrence of the Fault
[2] ETOF: Estimated Time of Occurrence of the Fault
[3] EFM: Estimated Fault Magnitude
[4] NMC: Number of Misclassifications

Table 4.4: FDI results for a specific random seed. Note that indices 1 and 2 correspond to sensor #1 and #2, respectively.

FDT/FCT Rejected	85	106	191	212	296	402	423
$\hat{t}$		101		202	301	402	423
$\hat{t}'$	NA	110	NA	211	302	421	442
Fault Index		1		2	2	1	1
$\hat{m}$		-0.010486		1.0652	-2.3536	0.0053589	0.0052515

controller and the FDI information is only used as an advisory tool. The estimated fault magnitude versus time is plotted in Fig.4.6 for 100 Monte Carlo runs. This concept is depicted in Fig. 4.7. As seen in this figure, the FDI tries to keep the residual white and by doing so would be capable of detecting sequential faults that might occur in the system. However, in order to shed more light on the crucial task of the FDI module in an FTC framework, the FDI was integrated with the controller for a random seed and the results are tabulated in Table 4.4. In addition, true states versus the estimated ones are shown in Fig.4.8. As seen in Table 4.4, by invoking the concept discussed in Remark 4.3.2 the FDI avoids taking any action at times instants 85 and 191 which are both confirmed by FDT/FCT tests as the TOF. Moreover, the proposed method obtains a refined estimate of TOF and consequently an accurate estimate of fault magnitude.

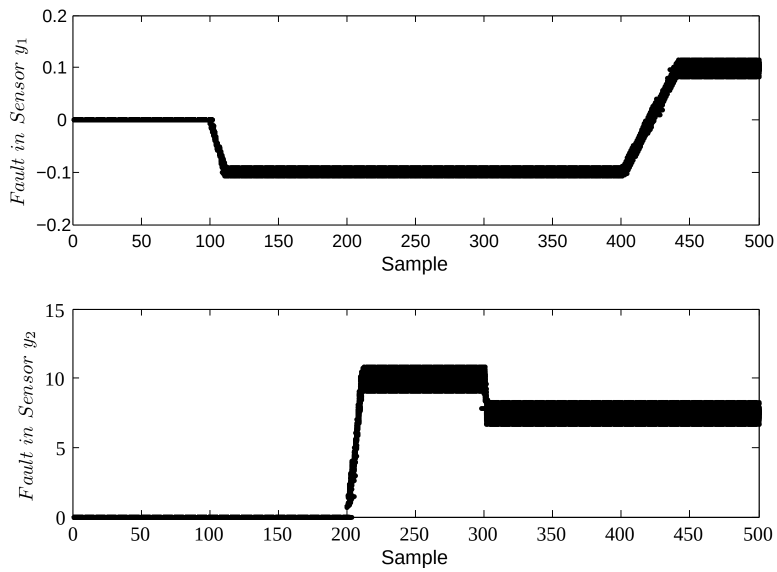


Figure 4.6: Estimated fault evolution based on 100 Monte Carlo runs. In this figure the results from all runs are plotted while excluding the misclassifications.

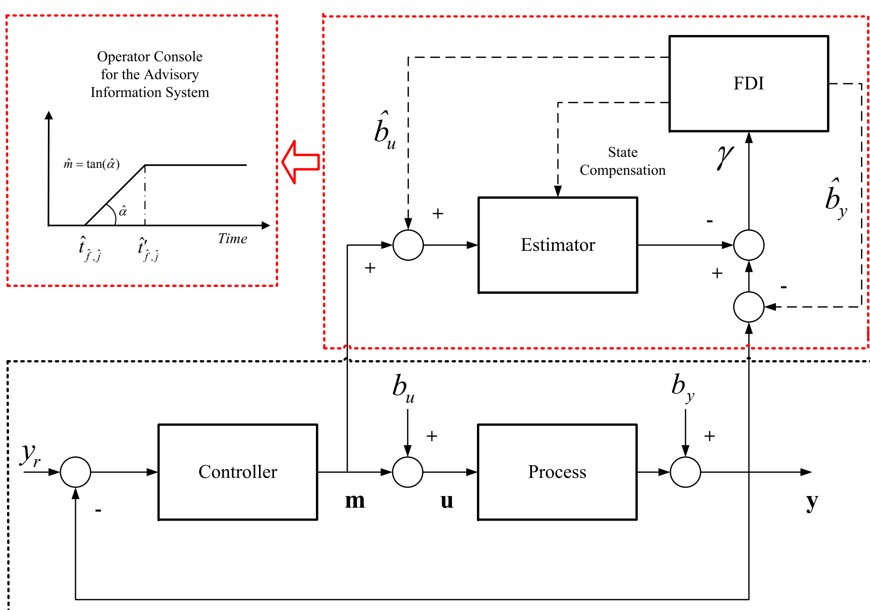


Figure 4.7: The supervisory scheme based on the GLR. The advisory information will be provided to the operators/ engineers.

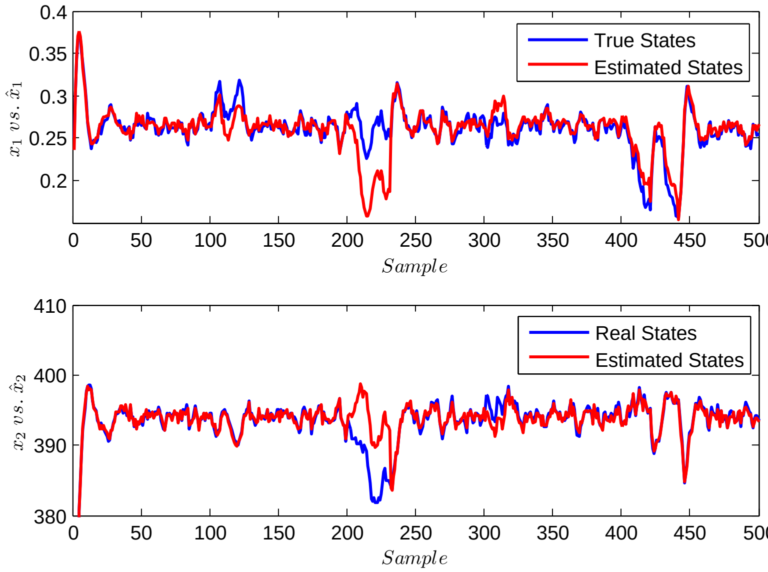


Figure 4.8: True versus estimated states in case of integration of controller with the FDI module where system is subject to sequential faults listed in Table 4.2.

4.5 Concluding Remarks

In this study a new FDI scheme has been proposed based on the GLR test and use of statistical FDT and FCT tests to address detection of time of occurrence, isolation and estimation of the magnitude of more realistic ramp type faults faults. The proposed methods overcome the shortcomings related to the methodology proposed by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault and its limited application to step type faults. In the proposed scheme, statistical FDT and FCT tests are used as the early stage announcers of the time of occurrence of the fault. The suggested scheme is flexible in dealing with ramp, truncated ramp and step type faults and can handle all of them using the same framework. In this study, the fault signature matrices for ramp type faults are developed and a closed form solution is provided to estimate the slope of the ramp and truncated ramp fault.

4.6 Appendix 1: Fault Signature Matrices

Lemma 4.6.1 *In case of occurrence of a ramp type bias in the jth actuator at time instant t. The fault signature matrices can be found as follows:*

$$\mathbf{J}_u(k;t) = (\frac{k-t-2}{k-t-1})\Phi\mathbf{J}_u(k-1;t) + \mathbf{K}(k)\mathbf{G}_u(k;t) - \mathbf{I} \tag{4.55}$$

$$\mathbf{G}_u(k;t) = \mathbf{C} - (\frac{k-t-2}{k-t-1})\mathbf{C}\Phi\mathbf{J}_u(k-1;t) \tag{4.56}$$

Proof: Let us consider Eqs.(4.3-4.4) which describe the evolution of the system after occurrence of the fault. Defining $\delta\hat{\mathbf{x}} = \hat{\mathbf{x}}(k|k) - \mathbf{x}(k)$ and using the Kalman filter equations yields:

$$\begin{aligned}\delta\hat{\mathbf{x}}(k) &= \big[\mathbf{I} - \mathbf{K}(k)\mathbf{C}\big]\big[\Phi\delta\hat{\mathbf{x}}(k-1) - \Gamma_w\mathbf{w}(k-1)\big] \\ &\quad + \mathbf{K}(k)\big[m_{u,j} \times (k-t-1)\mathbf{C}\Gamma_u\mathbf{e}_{u,j} + \mathbf{v}(k)\big] \\ &\quad - m_{u,j} \times (k-t-1)\Gamma_u\mathbf{e}_{u,j}\end{aligned} \tag{4.57}$$

using the definition of the residuals one can write:

$$\gamma(k) = -\mathbf{C}\delta\hat{\mathbf{x}}(k-1) + \mathbf{C}\Gamma_w\mathbf{w}(k-1) + m_{u,j} \times (k-t-1)\mathbf{C}\Gamma_u\mathbf{e}_{u,j} + \mathbf{v}(k) \tag{4.58}$$

Taking expected value from both sides of Eqs.4.57 and 4.58 results in the followings:

$$\begin{aligned}\mathbf{E}[\delta\hat{\mathbf{x}}(k)] &= \big[\mathbf{I} - \mathbf{K}(k)\mathbf{C}\big]\Phi\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] \\ &\quad + m_{u,j} \times (k-t-1)\mathbf{K}(k)\mathbf{C}\Gamma_u\mathbf{e}_{u,j} \\ &\quad - m_{u,j} \times (k-t-1)\Gamma_u\mathbf{e}_{u,j}\end{aligned} \tag{4.59}$$

$$\mathbf{E}[\gamma(k)] = -\mathbf{C}\mathbf{E}[\delta\hat{\mathbf{x}}(k-1)] + m_{u,j} \times (k-t-1)\Gamma_u\mathbf{C}\mathbf{e}_{u,j} \tag{4.60}$$

Now let us define the followings:

$$\begin{aligned}\mathbf{E}[\delta\hat{\mathbf{x}}(k)] &= m_{u,j} \times (k-t-1)\mathbf{J}_u(k;t)\Gamma_u\mathbf{e}_{u,j} \\ \mathbf{E}[\gamma(k)] &= m_{u,j} \times (k-t-1)\mathbf{G}_u(k;t)\Gamma_u\mathbf{e}_{u,j}\end{aligned}$$

Note that the fault would affect the system dynamics after one sample time delay due to the discretization and this fact is considered in the above definitions. Using the above definitions and Eqs.4.59 and 4.60 and after some algebraic manipulations, one can find the followings:

$$\mathbf{J}_u(k;t) = (\frac{k-t-2}{k-t-1})\Phi\mathbf{J}_u(k-1;t) + \mathbf{K}(k)\mathbf{G}_u(k;t) - \mathbf{I} \quad (4.61)$$

$$\mathbf{G}_u(k;t) = \mathbf{C} - (\frac{k-t-2}{k-t-1})\mathbf{C}\Phi\mathbf{J}_u(k-1;t) \quad (4.62)$$

■

It is worth mentioning that the recursive Eqs.4.55 and 4.56 should be initialized with $\mathbf{J}_u(t+1;t) = [0]_{n\times n}$ and $\mathbf{G}_u(t+1;t) = [0]_{r\times n}$. In order to determine the initialization matrices two facts should be considered; first, the fact that at time instant t the fault magnitude is zero and secondly, one inherent sample time delay due to the discretization.

4.7 Appendix 2: Ramp Type Faults in Actuators

Lemma 4.7.1 *Assuming occurrence of a single ramp type fault in an actuator at location j and at time instant $t \in [t_1 - M, t_1 + N]$, the estimated fault magnitude and $\mathbf{T}_{u,j,t}$ can be computed as follows:*

$$\hat{m}_{u,j,t} = \frac{\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\gamma(k)\right\}}{\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t)\mathbf{g}_{u,j}\right\}} \quad (4.63)$$

$$\mathbf{T}_{u,j,t} = \frac{\left(\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\gamma(k)\right\}\right)^2}{\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t)\mathbf{g}_{u,j}\right\}} \quad (4.64)$$

where $\mathbf{G}_y(k;t)$ denotes the fault signature matrix for a ramp type fault assuming occurrence of the fault at time instant t and $\mathbf{g}_{u,j} = \Gamma_u\mathbf{e}_{u,j}$.

Proof: Recall the joint maximum likelihood (ML) estimate of the ramp slope and TOF:

$$\{\hat{m}_{u,j}, \hat{t}\} = \arg\max_{m_{u,j},t} 2\log p(\Lambda_{t_1-M}^{t_1+N}|m_{u,j}, t) \quad (4.65)$$

Assuming t to be the true TOF, the joint ML-estimate reduces to:

$$\hat{m}_{u,j} = \arg\max_{m_{u,j}} 2\log p(\Lambda_t^{t_1+N}|m_{u,j}) = \arg\max_{m_{u,j}} \lambda_{u,j} \tag{4.66}$$

invoking the definition of multivariate normal distribution and assuming independence of residuals yields:

$$\lambda_{u,j} = 2\log\left\{\zeta \exp\{-\frac{1}{2}\sum_{k=t}^{t_1+N} \gamma'^T(k)V(k)^{-1}\gamma'(k)\right\} \tag{4.67}$$

where

$$\gamma'(k) = \gamma(k) - m_{u,j} \times (k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j}$$

In the above equation $\zeta = \frac{1}{(2\pi)^{r\times(t_1+N-t+1)/2}\prod_{k=t}^{t_1+N}|V(k)|^{1/2}}$ and r denotes the dimension of $\mathbf{y}$.

$$\begin{aligned}\lambda_{u,j} &= 2\log(\zeta) + 2m_{u,j}\sum_{k=t}^{t_1+N}\{(k-t-1)\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\gamma(k)\} \\ &\quad -m_{u,j}^2\sum_{k=t}^{t_1+N}\{(k-t-1)^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t)\mathbf{g}_{u,j}\}\end{aligned}$$

The ML-estimate of $m_{u,j}$ can be found by solving the following equation:

$$\begin{aligned}\frac{\partial\lambda_{u,j}}{\partial m_{u,j}} &= 0 \\ \Rightarrow \hat{m}_{u,j,t} &= \frac{\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\gamma(k)\right\}}{\sum_{k=t+2}^{t_1+N}\left\{(k-t-1)^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t)\mathbf{g}_{u,j}\right\}}\end{aligned} \tag{4.68}$$

Note that the lower limit of the summation is $t+2$ and not t. This is due the fact that $\mathbf{G}_u(t+1;t) = [0]_{r\times n}$. It is worth mentioning that $\hat{m}_{u,j,t}$ is used to represent the estimated slope and not $\hat{m}_{u,j}$. This is mainly due to the fact that the ML-estimate of the ramp slope is obtained based on the assumption of occurrence of the fault at time instant t. The log-likelihood ratio on the right-hand side (RHS) of Eq. 4.24 can be written as:

$$\mathbf{T} = 2\log\frac{exp\{-\frac{1}{2}\sum_{k=t+1}^{t_1+N}\gamma'(k)^TV(k)^{-1}\gamma'(k)\}}{exp\{-\frac{1}{2}\sum_{k=t+1}^{t_1+N}\gamma(k)^TV(k)^{-1}\gamma(k)\}} \tag{4.69}$$

substituting $\hat{m}_{f,j}$ into log-likelihood ratio in Eq.(4.69) yields:

$$\mathbf{T}_{u,j,t} = \frac{\left(\sum_{k=t+2}^{t_1+N} \left\{ (k-t-1)\mathbf{g}_{u,j}^T \mathbf{G}_u^T(k;t) V(k)^{-1} \gamma(k) \right\} \right)^2}{\sum_{k=t+2}^{t_1+N} \left\{ (k-t-1)^2 \mathbf{g}_{u,j}^T \mathbf{G}_u^T(k;t) V(k)^{-1} \mathbf{G}_u(k;t) \mathbf{g}_{u,j} \right\}} \tag{4.70}$$

■

4.8 Fault Signature Matrices for Truncated Ramp Fault in an Actuator

The fault signature matrices for a truncated ramp fault in jth actuator can be expressed as sum of positive and negative ramps occurring at consecutive time instants t and t'. Based on this argument the expected values of $\gamma(k)$ and $\delta\hat{\mathbf{x}}$ in case of occurrence of a truncated ramp type fault, can be defined as follows:

$$\begin{aligned} \mathbf{E}[\delta\hat{\mathbf{x}}(k)] &= m_{u,j} \times (k-1-t)\mathbf{J}_u(k;t)\mathbf{g}_{u,j}\sigma(k-t) \\ &\quad - m_{u,j} \times (k-1-t')\mathbf{J}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t') \end{aligned} \tag{4.71}$$

$$\begin{aligned} \mathbf{E}[\gamma(k)] &= m_{u,j} \times (k-1-t)\mathbf{G}_u(k;t)\mathbf{g}_{u,j}\sigma(k-t) \\ &\quad - m_{u,j} \times (k-1-t')\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t') \end{aligned} \tag{4.72}$$

where $\mathbf{g}_{u,j} = \Gamma_u \mathbf{e}_{u,j}$.

4.9 Truncated Ramp Type Faults in an Actuator

Theorem 4.9.1 *Assuming occurrence of a truncated ramp type fault depicted in Fig. (4.1.b) in an actuator at location j and at time instant $t \in [t_1 - M, t_1 + N]$ and t' to be the time instant at which the fault reaches its steady state value ($t_1 - M \le t \le t' \le t_1 + N$), the estimated fault magnitude and $\mathbf{T}_{u,j,t,t'}$ can be computed as follows:*

$$\hat{m}_{u,j,t,t'} = \frac{\Omega_{u,j}(k,t,t')}{\Upsilon_{u,j}(k,t,t')} \tag{4.73}$$

$$\mathbf{T}_{u,j,t,t'} = \frac{\left[\Omega_{u,j}(k,t,t')\right]^2}{\Upsilon_{u,j}(k,t,t')} \tag{4.74}$$

where

$$\Omega_{u,j}(k,t,t') \triangleq$$

$$\sum_{k=t+2}^{t_1+N} \Bigg\{ \Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j} - (k-t'-1)\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big]^T V(k)^{-1}$$

$$\times \gamma(k) \Bigg\}$$

and

$$\Upsilon_{u,j}(k,t,t') \triangleq$$

$$\sum_{k=t+2}^{t_1+N} \Bigg\{ \Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j} - (k-t')\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big]^T V(k)^{-1} \times\Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j} - (k-t')\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big] \Bigg\}$$

Proof: Similar to ramp type faults discussed earlier, it is assumed that the data window of residuals $\Lambda_{t_1-M}^{t_1+N} = \{\gamma_{t_1-M}, \cdots, \gamma_{t_1+N}\}$ is available. The ML-estimate of the ramp slope can be defined as:

$$\{\hat{m}_{u,j}, \hat{t}, \hat{t}'\} = 2\log \arg\max_{m_{u,j},t,t'} p(\Lambda_{t_1-M}^{t_1+N} | m_{u,j}, t, t') \tag{4.75}$$

assuming occurrence of the fault at time instant t and reaching the steady state at time instant t' as shown in Fig.(4.1.b) where $t_1 \leq t < t' < t_1 + N$, the above equation can be rewritten as:

$$\hat{m}_{u,j} = \arg\max_{m_{u,j}} 2\log p(\Lambda_{t+2}^{t_1+N} | m_{u,j}) = \arg\max_{m_{u,j}} \lambda_{u,j} \tag{4.76}$$

Note that the residuals would be considered in a sequence starting at $t+1$ and not t. This is due to the fact that the fault affects the system starting at $t+2$ or in other words

$\mathbf{G}_u(t+1;t) = [0]_{r\times n}$. The log-likelihood function can be defined as Eq.(4.32) and in this case we have:

$$\begin{aligned}\gamma'(k) &= \gamma(k) - m_{u,j}(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j} \\ &\quad + m_{y,j}(k-t'-1)\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\end{aligned}$$

It is straightforward to find:

$$\begin{aligned}\lambda_{u,j} &= 2\log(\zeta) - \sum_{k=t+2}^{t_1+N}\Big\{\gamma^T(k)V(k)^{-1}\gamma(k) \\ &\quad + m_{u,j}^2(k-t-1)^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t)\mathbf{g}_{u,j} \\ &\quad + m_{u,j}^2(k-t')^2\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t')V(k)^{-1}\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1) \\ &\quad - 2m_{u,j}^2(k-t-1)(k-t'-1)\mathbf{g}_{u,j}^T\mathbf{G}_u^T(k;t)V(k)^{-1}\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1) \\ &\quad - 2m_{u,j}(k-t-1)\mathbf{g}_{u,j}\mathbf{G}_u^T(k;t)V(k)^{-1}\gamma(k) \\ &\quad - 2m_{u,j}(k-t'-1)\mathbf{g}_{u,j}\mathbf{G}_u^T(k;t')V(k)^{-1}\gamma(k)\sigma(k-t'-1)\Big\}\end{aligned} \tag{4.77}$$

solving $\frac{\partial\lambda_{u,j}}{\partial m_{u,j}} = 0$ yields:

$$\hat{m}_{u,j,t,t'} = \frac{\Omega_{u,j,t,t'}(k,t)}{\Upsilon_{u,j,t,t'}(k,t)} \tag{4.78}$$

where

$$\Omega_{u,j}(k,t,t') \triangleq$$

$$\sum_{k=t+2}^{t_1+N}\Bigg\{\Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j} - (k-t'-1)\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big]^T V(k)^{-1}$$

$$\times\,\gamma(k)\Bigg\}$$

and

$$\Upsilon_{u,j}(k,t,t') \triangleq$$

$$\sum_{k=t+2}^{t_1+N}\Bigg\{\Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j}-(k-t')\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big]^T V(k)^{-1}$$
$$\times\Big[(k-t-1)\mathbf{G}_u(k;t)\mathbf{g}_{u,j}-(k-t')\mathbf{G}_u(k;t')\mathbf{g}_{u,j}\sigma(k-t'-1)\Big]\Bigg\}$$

Assuming the log-likelihood ratio in Eq.(4.34) and substituting $\hat{m}_{u,j,t,t'}$ into this equation results in:

$$\mathbf{T}_{u,j,t,t'}=\frac{\big[\Omega_{u,j}(k,t,t')\big]^2}{\Upsilon_{u,j}(k,t,t')} \tag{4.79}$$

■

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Chapter 5

Conclusions

The main goal of this study is to propose an FDI scheme which is capable of accurately detection time of occurrence of the fault (TOF), isolate the fault type and estimate its magnitude in LTI systems. The abnormalities addressed in this study are assumed to be additive faults which may occur in sensors or actuators in the form of abrupt jumps or incipient faults. The existing methodologies in the literature are either burdensome from computational point of view (Willsky and Jones, 1976; Gustafsson, 1996) or fail to accurately detect TOF (Prakash *et al.*, 2005; Prakash *et al.*, 2002). Moreover in the study by (Gustafsson, 1996) isolation of the fault and estimation of its magnitude is not addressed at all. It is worth noting that the FDI scheme proposed by (Villez *et al.*, 2011) improves the accuracy of TOF but yet it fails to provide a closed form solution for the GLR based test statistic and the estimated fault magnitude. In another study by (Dos Santos and Yoneyama, 2011), it is suggested to decouple isolation and estimation phases using an MLR-based approach. However, the unrealistic Gamma priors are used to marginalize the fault magnitude which in no way could be justified but mathematical convenience.

In the above studies, the major focus is to detect, isolate and estimate the magnitude of step type faults and there arises a need to provide an FDI scheme which is also capable of dealing with more realistic incipient faults.

5.1 Main Contributions of Chapter 1

Motivated by the shortcomings of the methods proposed in (Narasimhan and Mah, 1988; Prakash *et al.*, 2005; Prakash *et al.*, 2002), in chapter 2 a new FDI scheme has been proposed based on the GLR test and use of statistical FDT and FCT tests proposed by (Narasimhan and Mah, 1988). The modified GLR-based method overcomes the shortcomings related to the methodology proposed in the study by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault using a modified GLR approach. The proposed method simultaneously performs the detection of TOF, isolation of the fault and estimation of its magnitude. The new methodology removes the need for continuous operation of banks of Kalman filters over a sliding window as proposed by (Willsky and Jones, 1976) by taking advantage of the FDT and FCT tests. The $\chi^2 - MGLR$ method then tries to refine the estimated TOF by FDT and FCT tests using an extended data window. The method presented in this study is then complemented by a strategy which prevents the FDI from taking any action when sufficient number of data points are not available for estimation of fault magnitude. The method was tested subject to extensive Monte Carlo simulations on a benchmark reactor problem and the results reveal that it outperforms the FDI scheme presented in the study by (Prakash *et al.*, 2005).

5.2 Main Contributions of Chapter 2

In chapter 3 a new FDI scheme has been proposed based on the MLR test and use of realistic uniform priors for fault magnitudes to address detection of time of occurrence of the fault, isolation of the fault and estimation of its magnitude. The MLR-based approach proposed by (Gustafsson, 1996) deals with detection the abrupt jumps in the states using flat non-informative priors and does not address the isolation of the fault and estimation of its magnitude in sensors and actuators. In another study by (Dos Santos and Yoneyama, 2011) uses the unrealistic Gamma priors for fault magnitudes which heavily penalizes the low and high fault magnitudes. The proposed approach in chapter 3 is based on uniform priors which arise from the concept of bounded process priors and in addition is capable of isolation and estimation of fault magnitude which might occur in both sensors and actuators. The proposed methods overcome the shortcomings related to the methodology proposed by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault using a MLR-based approach. The proposed method simultaneously performs the detection of time of occurrence and isolation of the fault and the estimation of fault magnitude is undertaken by a least squares approach. The superior performance of the proposed method compared to the FDI schemes proposed in (Prakash *et al.*, 2002; Prakash *et al.*, 2005) can be mainly attributed to accurate detection of time of occurrence of the fault.

5.3 Main Contributions of Chapter 3

In the proposed scheme of chapter 4, statistical FDT and FCT tests are used as the early stage announcers of the time of occurrence of the fault. The method was tested subject to extensive Monte Carlo simulations on a benchmark reactor problem and the results reveal that it outperforms the FDI scheme presented in (Prakash *et al.*, 2005; Prakash *et al.*, 2002). In this study a new FDI scheme has been proposed based on the GLR test and use of statistical FDT and FCT tests to address detection of time of occurrence, isolation and estimation of the magnitude of more realistic ramp type faults faults. The proposed methods overcome the shortcomings related to the methodology proposed by (Prakash *et al.*, 2002) in detecting the time of occurrence of the fault and its limited application to step type faults. In the proposed scheme, statistical FDT and FCT tests are used as the early stage announcers of the time of occurrence of the fault. The suggested scheme is flexible in dealing with ramp, truncated ramp and step type faults and can handle all of them using the same framework. In this study, the fault signature matrices for ramp type faults are developed and a closed form solution is provided to estimate the slope of the ramp and truncated ramp fault. The method was evaluated via extensive Monte Carlo simulations on a benchmark reactor problem and the results reveal that it can effectively detect, isolate and estimate the ramp type faults.

5.4 Suggestions for Future Work

This study can be further improved by incorporating the following suggestions:

1. Extending the concept of MLR to multiple simultaneous fault that might occur in a system. In this study it was assumed that single faults might occur sequentially but not simultaneously and by relaxing this assumption, there arises a need to develop a new MLR-based FDI. Moreover, the fault signature matrices should be modified to take into account the evolution of states and the residuals when the system is subject to multiple simultaneous faults.

2. Developing an FDI strategy to deal with additive faults when the system is also subject to unknown inputs. In this case, the typical Kalman filter cannot be used and hence an alternative unknown input observer (UIO) should be used. Nonetheless, the main challenge would be developing the new fault signature matrices and the GLR test statistic. Another improvement to be considered could be replacing the GLR-approach with an MLR-based counterpart.

3. In chapter 4 it is assumed that the faults occur exactly at sampling instants and not in between. This assumption can be relaxed by considering the possibility of occurrence of the fault at inter-sample instants. These faults could be modeled as sum of a step jump and a ramp type fault occurring simultaneously and despite the fact that one

can use the fault signature matrices which are already developed, computation of the GLR test statistic and estimation of the fault magnitudes would be very challenging.

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