UNIVERSITY OF ALBERTA

BOUNDARY VALUE PROBLEMS OF ANTI-PLANE COSSERAT ELASTICITY

©

by

STANISLAV POTAPENKO

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy

Department of Mechanical Engineering

Edmonton, Alberta

Spring 2003

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

National Library of Canada
Bibliothèque nationale du Canada

Acquisitions and Bibliographic Services
Acquisisitons et services bibliographiques

395 Wellington Street
Ottawa ON K1A 0N4
Canada

395, rue Wellington
Ottawa ON K1A 0N4
Canada

Your file Votre référence
ISBN: 0-612-82157-9
Our file Notre référence
ISBN: 0-612-82157-9

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou aturement reproduits sans son autorisation.

Canada

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

UNIVERSITY OF ALBERTA

LIBRARY RELEASE FORM

Name of Author:	**Stanislav Potapenko**
Title of Thesis:	**Boundary value problems of anti-plane Cosserat elasticity**
Degree:	**Doctor of Philosophy**
Year this Degree Granted:	**2003**

Permission is hereby granted to the University of Alberta Library to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as herein before provided, neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

S. Potapenko

Stanislav Potapenko
10154-87 Avenue,
Edmonton, AB, T6E 2P2

Date: March 05, 2003

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Boundary Value Problems of Anti-Plane Cosserat Elasticity** submitted by Stanislav Potapenko in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Peter Schiavone

Dr. Peter Schiavone, Supervisor

Dr. Andrew Mioduchowski, Co-Supervisor

C. Constanda

Dr. Christian Constanda

Dr. Jozef Szymanski

Ru Chongqing

Dr. Chongqing Ru

Date: February 28, 2003

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

For my mother

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

ABSTRACT

Problems involving mechanical behavior of materials with microstructure are receiving an increasing amount of attention in the literature. First of all, it can be attributed to the fact that a number of recent experiments shows a significant discrepancy between results of the classical theory of elasticity and the actual behavior of materials for which microstructure is known to be significant (e.g. synthetic polymers, human bones). Second, materials, for which microstructure contributes significantly in the overall deformation of a whole body, are becoming more and more important for applications in different areas of modern day mechanics, physics and engineering.

Since the classical theory is not adequate for modeling the elastic behavior of such materials, a new theory, which allows us to incorporate microstructure into a classical model, should be used.

The foundations of a theory allowing to account for the effect of material microstructure were developed in the middle of the twentieth century and is known now as the theory of Cosserat elasticity. For the last forty years significant results have been accomplished leading to a better understanding of processes occurring in a Cosserat continuum. In particular, progress has been achieved in the area of investigation of

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

three-dimensional, plane-strain problems of Cosserat elasticity and also some problems related to the theory of Cosserat plates and shells.

However, some certain problems of Cosserat elasticity have remained untouched until today. Among them is the anti-plane problem of Cosserat elasticity. Meanwhile, the anti-plane problem is regarded as very important for applications in mechanics, since from the point of view of mechanics, the anti-plane problem with Neumann boundary conditions is the problem of torsion of a beam with significant microstructure.

The objective of this work is to formulate and solve rigorously basic boundary value problems of anti-plane Cosserat elasticity. To achieve this goal we use the boundary integral equation method in order to derive the exact analytical solutions for the corresponding boundary value problems in terms of integral potentials. The exact solutions are then approximated numerically using the method of generalized Fourier series in order to obtain quantitative characteristics of the solutions to the corresponding boundary value problems. In particular, it has been found that in the case of torsion of a circular Cosserat beam, microstructure does have a significant effect on the warping function, provided that the cross-section is elliptic.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

ACKNOWLEDGEMENTS

I would like to thank professors Peter Schiavone and Andrew Mioduchowski for their invaluable guidance, tremendous support, confidence and patience for the time of my study at the University of Alberta and for a wide range of useful advise and critical comments during my work on this dissertation.

Stanislav Potapenko

Edmonton, AB

Winter 2003

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

TABLE OF CONTENTS

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

1 Introduction

The classical theory of elasticity is based on an ideal model of the elastic continuum in which the transfer of loading through any interior surface element occurs only by means of the (force) stress vector. This assumption leads to a description of the strain of the body in terms of symmetric strain and stress tensors.

Results from analytical models derived on the basis of classical elasticity are in good agreement with experiments performed within the elastic range on numerous structural materials for example such as concrete, steel or aluminium.

The classical theory of elasticity, however, fails to produce acceptable results when the microstructure of the material contributes significantly to the overall deformation of the body, for example, in the case of granular bodies with large molecules (e.g. polymers) or human bones (see, for example, [48]-[52]). These cases are becoming increasingly important in the design and manufacture of modern day advanced materials as small-scale effects become paramount in the prediction of the overall mechanical behavior of these materials.

An attempt to eliminate these shortcomings was first made by Voigt

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

[103] in 1886 who assumed that the transfer of the interaction between two elements of the body through a surface element occurs not only by means of a force (stress) vector but also by means of an independent moment (couple-stress) vector.

However, the first more or less harmonious theory was introduced only in 1909 by the brothers E. and F. Cosserat [12]-[14]. In their theory, the Cosserat brothers made further developments of the Voigt's theory. They suggested that deformation of the body should be described by the displacement vector $u(x,t)$ and independent microrotation vector $\Phi(x,t)$. The assumption that a material element has six degrees of freedom leads to the description of deformation of the body in terms of asymmetric strain and stress tensors unlike the classical theory of elasticity in which deformations can be described by only one symmetric stress tensor.

In spite of these new ideas Cosserats' work remained unnoticed for a significant period of time. The major drawbacks of the theory could have been that first, the theory had already been non-linear, secondly it was formulated in a very unclear manner and thirdly and probably the most important reason, the theory contained many problems lying very far from the framework of elasticity theory. In addition to problems related to the theory of elasticity

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

the authors considered problems of non-ideal fluids, the quasi-elastic continuum model of the "MacCullagh and Kelvin ether", some problems related to electrodynamics and magnetism, in other words, they made an attempt to create a unified theory containing mechanics, optics and electrodynamics. No wonder that the theory was found to be very complicated and was not heeded.

However, the investigations in the area of solid mechanics and mechanics of fluids in the middle of the twentieth century demonstrated that the behavior of certain classes of materials and fluids cannot be described in terms of classical theory, hence the Cosserat theory was rediscovered and drew attention of many workers.

The investigations first have been concentrated on the simplified Cosserat theory, i.e. on the asymmetric elasticity in so-called Cosserat pseudocontinuum, sometimes this theory is also called couple-stress elasticity. In a Cosserat pseudocontinuum there is still a possibility of the generation of asymmetric stresses and couple stresses during deformation of the body, but at the same time the whole deformation of the body is described only by the displacement field. In other words, for reasons of simplicity it is assumed that the microrotation vector Φ and the displacement vector u are dependent

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

as in the classical theory of elasticity (see for example [55]) by means of the following relation

$$\Phi = \frac{1}{2}\text{curl } u.$$

Among the papers on the couple-stress theory of elasticity, first of all it is necessary to notice the works of Toupin [99], [100] and Truesdell and Toupin [101], on the linear and non-linear elasticity of Cosserat pseudocontinuum. This work was further developed by Grioli [32], Mindlin [58], [59] and Mindlin and Thiersten [60]. We must also mention the whole series of papers on the Cosserat pseudocontinuum that appeared after the above-mentioned works devoted to both the investigation of general problems of the couple stress theory related to derivation and methods of solutions of governing equations [7], [44], [75] and to specific applications of the theory, for instance, to the problems of determination of the effect of couple stresses on stress concentration factor around holes and rigid inclusions or to the investigation of bending of plates in pseudo-Cosserat media [3]-[6], [34], [54], [57], [77], [94], [105].

However, like almost any simplified theory, the couple-stress theory of elasticity could not entirely and precisely describe the deformations of granular bodies. The series of more recent experiments [22], [40], [48]-[52], [95],

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

[96] clearly confirmed this fact once again. Mostly, the theory was represented only for the reason of simplicity, for example, the governing equations of the theory are just Navier's equations with respect to three unknown displacements - exactly as in the classical theory of elasticity. No wonder, that soon after appearance of the first papers on couple-stress theory, the general stipulations of the mathematically rigorous more general theory of Cosserat elasticity were introduced.

The foundations of the theory of a Cosserat continuum, when the microrotations and displacements are no longer dependent, were formulated by Gunther [33] and Schaefer [78], [79] in the late fifties and early sixties of the twentieth century. The first author examined the three-dimensional model of the Cosserat continuum and emphasized the importance of the Cosserat continuum for dislocation theory. The second author rediscovered the foundations of Cosserat theory for the plane state of strain. Then, several years later Aero and Kuvshinsky [2] and Palmov [67], [68] presented constitutive relations and governing equations of the general theory of Cosserat elasticity.

The interesting exposition of the theory of Cosserat elasticity was given by Eringen and his co-workers [23]-[27] who introduced the new name for the theory - the theory of micropolar or asymmetric elasticity. Eringen has

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

also formulated the general provisions of the theory of micropolar plates [25]. Also, there must be mentioned the works on approximate methods of solutions of the theory of micropolar plates and shells [20], [104] and several works devoted to the problem of crack propagations in micropolar media [18], [107]. An extensive description of the theory can be found in the paper by Schaeffer [78] and, in particular, including the extensive bibliography, in the book by Nowacki [64].

Parallel to the works in the area of Cosserat elasticity, investigations have also been conducted in the area of Cosserat fluids. A relatively complete bibliography in this field can be found, for example, in [69] and [70].

However, in spite of the importance of all afore-mentioned work, none of these papers or monographs dealt with both the mathematically rigorous formulation of the boundary value problems arising in the theory of micropolar elasticity and the methods of their solutions. Mostly it can be explained by the fact that methodology, methods and approaches of the classical theory of elasticity (for example, theory of analytical functions, Fredholm's theory of integral equations, theory of one-dimensional singular integral equations) are inadequate for the rigorous mathematical analysis of the governing equations and boundary conditions of such a complicated structure. Fortunately, this

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

situation is now changing mostly due to the important work in the area of three-dimensional classical elasticity carried out in the last 40 years.

The theory of three-dimensional problems of classical elasticity can be worked out by a variety of means. Some of these approaches may be further successfully applied to the analysis of the boundary value problems of micropolar elasticity. The first possibility is the modern theory of generalized solutions of differential equations (the method of Hilbert spaces, variational methods). The second one is the theory of multidimensional singular potentials and singular integral equations.

The first direction - based on the ideas of the modern functional analysis which are novel to the classical mechanics - is characterized by great generality involving the case of variable coefficients and boundary manifolds of the general type. Owing to such generality, it may be employed in the first place for proving theorems on the existence of non-classical solutions, requiring additional, sometimes essential, restrictions when used for classical solutions.

A fine, concise treatment of these topics may be found, for example, in [29], [30], [17] or in the book by Chudinovich and Constanda [9].

The second direction based on the rapidly developing theory of singular

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

integrals and integral equations is a direct extension of the concepts of the theory of potentials and Fredholm equations which are, as known, the prevailing concepts of the classical mechanics. This approach, being not so general as the first one, allows to investigate in detail cases most important for the theory and applications, retaining the efficiency of the methods of the classical mechanics of continua. The breakthrough in this direction occurred after the pioneering work of Muskhelishvili on singular integral equations [62]. Further this approach has been extensively developed and applied to the rigorous investigation of the boundary value problems of three-dimensional theory of elasticity in the works by Kupradze and his co-workers [46], [47] and to the analysis of the bending of plates with transverse shear deformation in the work by Constanda [10].

The work of Kupradze has provided researchers with effective tools for investigations in the micropolar theory of elasticity. Iesan [36], [37], using the approach proposed by Kupradze for the treatment of three-dimensional problems of micropolar elasticity, formulated uniqueness and existences theorems for the boundary-value problems of a micropolar state of plain strain. However, the analysis presented in [36], [37] overlooks certain differentiability requirements to establish the rigorous solution to the problem. In a series

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

of works by Schiavone [80]-[85], and Schiavone and Constanda [86], [87], the framework of singular integral equations has been successfully adapted for establishing analytical solutions and analysis of boundary value problems of the theory of micropolar plates. In addition, in [85] the boundary integral equation method was extended for the rigorous treatment of plane strain problems of micropolar elasticity allowing to overcome deficiency represented in [36], [37]. However, to the author's knowledge, a rigorous treatment of the boundary value problems of the anti-plane theory of micropolar elasticity has remained absent from the literature.

Anti-plane shear deformations are one of the simplest classes of deformations that solids can undergo. In classical anti-plane shear (or longitudinal shear, generalized shear) of a cylindrical body, the (single) displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. Thus, classical anti-plane shear, with just a single scalar axial displacement field, may be viewed as complementary to the more complicated (yet perhaps more familiar) classical plane strain deformations, with its two in-plane displacements. In recent years, considerable attention has been paid to the analysis of anti-plane shear deformations within the context of various constitutive theories (linear and nonlinear) of solid mechanics

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

[35]. Such studies were largely motivated by the promise of relative analytic simplicity compared with plane problems since the governing equations are a single second-order linear or quasi-linear partial differential equation rather than higher-order or coupled-systems of partial differential equations. Thus the antiplane shear problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics may be examined in a particularly simple setting. Perhaps the most widely studied problem is the antiplane shear crack problem which has been the subject of numerous investigations (for overview of classical antiplane problems and extensive bibliography see [35]).

Unlike its classical counterpart, however, the theory describing antiplane deformations of a linearly elastic Cosserat solid is not marked by its relative analytic simplicity. The governing equations and fundamental boundary value problems describing the anti-plane deformations of a linearly elastic, homogeneous and isotropic Cosserat elastic solid have been formulated by Nowacki [65] and later applied to the analysis of the torsion of micropolar beam by Iesan [38] and Smith [91]. In fact, in the case of a Cosserat solid, the theory reduces to a coupled system of three partial differential equations for three unknowns: one describing the single antiplane displacement

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

and two more representing the "out-of-plane" microrotations. It is not surprising, therefore, that a rigorous analysis and solution of the corresponding fundamental boundary value problems remains absent from the literature.

The objective of this dissertation is to formulate and solve rigorously the fundamental Dirichlet, Neumann, Robin, mixed interior and exterior boundary value problems for the antiplane deformations of a Cosserat homogeneous linearly elastic solid by means of the boundary integral equation method. The solutions to these problems have numerous applications in areas where classical models of antiplane deformations have been applied [35] but for advanced materials where material microstructure plays a significant role. The importance of the presented theory for applications is demonstrated on the example of torsion of a cylindrical micropolar beam. As a result of investigations performed in this thesis, the following works [71]-[74] of the author have been recently published.

The thesis is organized as follows.

In Chapter 2, we provide a brief overview of the three-dimensional theory of micropolar elasticity, presented in detail in [64] . The purpose of this chapter is to introduce the governing equations describing three-dimensional deformations of a linearly elastic Cosserat solid and to formulate the ba-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

sic constitutive and kinematic relations that will be used for derivation of corresponding relations of the theory of antiplane Cosserat elasticity in the subsequent chapters.

Chapter 3 is devoted entirely to the antiplane problems of micropolar elasticity. On the basis of the governing equations and constitutive relations of the three-dimensional Cosserat theory, we derive the governing equations and formulate the fundamental boundary value problems of antiplane micropolar elasticity. Using the boundary integral equation method we prove uniqueness and existence theorems and obtain exact solutions to these problems in the form of integral potentials.

Since the solutions in the form of integral potentials may not be convenient for applications, in Chapter 4 we introduce the modification of generalized Fourier series method by means of which we can approximate numerically the solutions obtained in Chapter 3. We show that the generalized Fourier method can be successfully used for construction of approximate solutions of Dirichlet, Neumann interior and exterior problems in the form of series. We provide the evidence that the generalized Fourier series converge rapidly to the exact solutions of the corresponding boundary-value problems.

In Chapter 5 we demonstrate the importance of the presented theory for

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

applications. As an example, we consider the problem of torsion of a cylindrical beam with microstructure. Using the theoretical background of the preceding chapters we show that the torsion problem can be formulated as an interior Neumann boundary value problem of antiplane micropolar elasticity and as a result of that the exact analytical solution may be represented in the form of a single layer integral potential. Then, using the Fourier method developed in Chapter 4, we construct the approximate solution in the form of generalized Fourier series. Finally, we consider two practical examples related to torsion of micropolar beams of circular and elliptic cross-sections. These examples confirm the importance of the effect of material microstructure on the warping function measuring anti-plane displacement of an elliptic micropolar beam.

Finally, in Chapter 6 we make several important conclusions and recommendations for future work.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

2 The Basic Foundations of the Three-Dimensional Theory of Cosserat Elasticity

The purpose of this chapter is to present a brief overview of the general provisions of the three-dimensional theory of Cosserat elasticity. Since, this chapter summarizes only what has been done before, we skip certain details related to the derivation of constitutive equations and methods of solutions of the system of governing equations. Detailed description of the three-dimensional theory of Cosserat elasticity can be found in [64].

Throughout what follows, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, the convention of summation over repeated indices is understood, $\mathcal{M}_{m\times n}$ is the space of $(m \times n)$- matrices, E_n is the identity element in $\mathcal{M}_{n\times n}$, a superscript T indicates matrix transposition and $(...)_{,\alpha} \equiv \partial(...)/\partial x_\alpha$. Also, if X is a space of scalar functions and v a matrix, $v \in X$ means that every component of v belongs to X.

Let an elastic isotropic Cosserat body occupy a domain V in $\mathbb{R}^3$ and be bounded by surface S. Assume that the body undergoes deformation due to the action of external forces $\mathbf{X} = (X_1, X_2, X_3)^T$ and external moments $\mathbf{Y} = (Y_1, Y_2, Y_3)^T$. The elastic properties of the body can be characterized by

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

elastic constants $\lambda, \mu, \gamma, \beta, \alpha, \kappa$, where λ and μ are usual Lame coefficients as in the classical theory of elasticity and γ, β, α and κ are micropolar elastic constants, representing the contribution of material microstructure to the elastic properties of the body. The state of deformation is characterized by a displacement field

$$u(x) = (u_1(x), u_2(x), u_3(x))^T$$

and a microrotation field

$$\Phi(x) = (\phi_1(x), \phi_2(x), \phi_3(x))^T,$$

where $x = (x_1, x_2, x_3)$ is a generic point in $\mathbb{R}^3$. This leads to the description of deformation of the body in terms of asymmetric strain , torsion, stress and couple-stress tensors [64], [65] of the form

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}, \quad \varkappa = \begin{bmatrix} \varkappa_{11} & \varkappa_{12} & \varkappa_{13} \\ \varkappa_{21} & \varkappa_{22} & \varkappa_{23} \\ \varkappa_{31} & \varkappa_{32} & \varkappa_{33} \end{bmatrix}, \tag{2.1}$$

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \varrho = \begin{bmatrix} \varrho_{11} & \varrho_{12} & \varrho_{13} \\ \varrho_{21} & \varrho_{22} & \varrho_{23} \\ \varrho_{31} & \varrho_{32} & \varrho_{33} \end{bmatrix}, \tag{2.2}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where ε is the strain tensor, $\varkappa$ the torsion tensor, σ the stress tensor, ϱ the couple-stress tensor.

Following the procedure given in detail in [64], [65] and [24] we can derive the equations of equilibrium in terms of stresses and couple stresses of the form

$$\begin{aligned} \sigma_{ji,j} + X_j &= 0, \\ \epsilon_{ijk}\sigma_{jk} + \varrho_{ji,j} + Y_i &= 0. \end{aligned} \tag{2.3}$$

where $\epsilon_{ijk}-$ alternating symbol.

Note, that in case of micropolar media, the equilibrium equations are more complicated than in the classical case because of the appearance of the extra system of equations due to the presence of couple stresses. It leads us to a description of the elastic behavior of a Cosserat solid in terms of asymmetric stress and couple-stress tensors. It can be easily shown that if we set all couple stresses equal to zero we again obtain a symmetric stress tensor as in the classical case. Consequently, the presence of couple stresses prevents the symmetry of the stress tensor.

Using the constitutive relations [64]

$$\sigma_{ji} = (\mu + \alpha)\varepsilon_{ji} + (\mu - \alpha)\varepsilon_{ij} + \lambda_{kk}\delta_{ji} \tag{2.4}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$\varrho_{ji} = (\gamma + \kappa)\varkappa_{ji} + (\gamma - \kappa)\varkappa_{ij} + \beta\varkappa_{kk}\delta_{ji}, \tag{2.5}$$

where δ_{ji} is the Kronecker symbol,

and the kinematic relations

$$\varepsilon_{ji} = u_{i,j} - \epsilon_{kij}\phi_k, \quad \varkappa_{ji} = \phi_{i,j}, \tag{2.6}$$

we can formulate the equilibrium equations in terms of displacements and microrotations in the following vector form

$$\begin{aligned}(\mu + \alpha)\Delta u + (\lambda + \mu - \alpha)\mathrm{graddiv}u + 2\alpha\mathrm{curl}\Phi + X &= 0, \\ [(\gamma + \kappa)\Delta - 4\alpha]\,\Phi + (\beta + \gamma - \kappa)\mathrm{graddiv}\Phi + 2\alpha\mathrm{curl}u + Y &= 0, \quad (2.7)\end{aligned}$$

where Δ is the Laplace operator.

As can be seen from (2.7), the governing equations of micropolar elasticity have much more complicated structure than those of classical elasticity. In the case of a Cosserat solid, the system of governing equations is a system of coupled partial differential equations with six unknowns: three usual displacements as in the classical theory of elasticity and three more representing independent microrotations. We can conclude that the theory of Cosserat elasticity is much more general in comparison with the classical one. If we assume, that micropolar elastic constants $\alpha, \beta, \gamma, \kappa$ are equal to zero, we can

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

easily see that the governing equations are reduced to the well-known Navier's equations : the governing equations of classical theory of elasticity.

In [46] the boundary value problems corresponding to (2.3) and (2.7) were shown to be well-posed and solved rigorously by means of the boundary integral equation method. In [64], [42], [43], [15], [16] system (2.7) was integrated by means of the method of potentials under different sets of boundary conditions. Since our goal is to investigate antiplane problems of Cosserat elasticity we will not pay significant attention to the integration of equations (2.7) but we will use them in order to derive the governing equations for anti-plane shear deformations.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

3 Anti-Plane Problems of Cosserat Elasticity

3.1 Preliminaries

In this chapter we first derive the governing equations of anti-plane micropolar elasticity on the basis of the general three-dimensional equations of Cosserat theory presented in Chapter 2. After that we apply the boundary integral equation method for the analysis of the corresponding boundary value problems.

Let S be a domain in $\mathbb{R}^2$ bounded by a closed C^2-curve ∂S and occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants $\lambda, \mu, \alpha, \beta, \gamma$ and κ. The state of micropolar anti-plane shear is characterized by a displacement field $u(x') = (u_1(x'), u_2(x'), u_3(x'))^T$ and a microrotation field $\Phi(x') = (\phi_1(x'), \phi_2(x'), \phi_3(x'))^T$ of the form

$$\begin{aligned} u_\alpha(x') &= 0, \qquad u_3(x') = u_3(x), \\ \phi_3(x') &= 0, \qquad \phi_\alpha(x') = \phi_\alpha(x), \end{aligned} \tag{3.1}$$

where $x' = (x_1, x_2, x_3)$ and $x = (x_1, x_2)$ are generic points in $\mathbb{R}^3$ and $\mathbb{R}^2$,respectively. From (3.1) we find that the equilibrium equations of micropolar anti-plane

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

shear written in terms of displacements and microrotations are given by [65]:

$$L(\partial_x)u(x) = F(x), \qquad x \in S, \tag{3.2}$$

in which now, denoting ϕ_α by u_α,we have $u(x) = (u_1, u_2, u_3)^T$. The matrix partial differential operator $L(\partial x) = L(\partial/\partial x_\alpha)$ is defined by [71]

$$L(\xi) = L(\xi_\alpha)$$
$$= \begin{pmatrix} (\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_1^2 & (\beta+\gamma-\kappa)\xi_1\xi_2 & 2\alpha\xi_2 \\ (\beta+\gamma-\kappa)\xi_1\xi_2 & (\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_2^2 & -2\alpha\xi_1 \\ -2\alpha\xi_2 & 2\alpha\xi_1 & (\mu+\alpha)\Delta \end{pmatrix}, \tag{3.3}$$

where $\Delta = \xi_\alpha\xi_\alpha$ and $F = (F_1, F_2, F_3)^T$ represent body forces and couples.

Together with L we consider the boundary stress operator $T(\partial x) = T(\partial/\partial x_\alpha)$ defined by [71]

$$T(\xi) = T(\xi_\alpha)$$
$$= \begin{pmatrix} (2\gamma+\beta)\xi_1 n_1 + (\gamma+\kappa)\xi_2 n_2 & (\gamma-\kappa)\xi_2 n_1 + \beta\xi_1 n_2 & -2\alpha n_2 \\ (\gamma-\kappa)\xi_1 n_2 + \beta\xi_2 n_1 & (\gamma+\kappa)\xi_1 n_1 + (2\gamma+\beta)\xi_2 n_2 & 2\alpha n_1 \\ 0 & 0 & (\mu+\alpha)\xi_\alpha n_\alpha \end{pmatrix}^T, \tag{3.4}$$

where $n = (n_1, n_2)^T$ is the unit outward normal to ∂S.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

The internal energy density is given by [71]

$$E(u,u) = E_1(u,u) + E_2(u,u) \qquad (3.5)$$
$$+\frac{1}{4}[2\mu u_{3,\alpha} u_{3,\alpha} + 2\alpha\left((2u_2 + u_{3,1})^2 + (2u_1 - u_{3,2})^2\right)]$$

where

$$E_1(u,u) = \frac{\gamma+\kappa}{2}u_{1,2}^2 + \frac{\gamma+\kappa}{2}u_{2,1}^2 + (\gamma-\kappa)u_{1,2}u_{2,1} ,$$
$$E_2(u,u) = (\gamma+\frac{\beta}{2})u_{1,1}^2 + (\gamma+\frac{\beta}{2})u_{2,2}^2 + \beta u_{1,1}u_{2,2} .$$

Throughout what follows we assume that

$$2\gamma+\beta>0, \quad \kappa,\alpha,\gamma,\mu>0. \qquad (3.6)$$

Noting that the matrix $L_0(\xi)$ corresponding to the second order derivatives in the system (3.2) is invertible for all $\xi \neq 0$ since

$$\det L_0(\xi) = (\mu+\alpha)(\gamma+\kappa)(2\gamma+\beta)(\xi_1^2+\xi_2^2)^3,$$

it is clear [71] that (3.2) is an elliptic system and that $E(u,u)$ is a positive quadratic form. In fact, $E(u,u) = 0$ if and only if

$$u(x) = (0, \quad 0, \quad c)^T, \qquad (3.7)$$

where c is an arbitrary constant. This is the most general rigid displacement and microrotation associated with (3.1). Clearly, the space of such rigid

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

displacements and microrotations is spanned by the single vector $(0,\ 0,\ 1)$. Accordingly, we denote by $\mathcal{F}$, the matrix

$$\mathcal{F} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.8}$$

from which it can be seen that, $L\mathcal{F} = 0$ in $\mathbb{R}^2$, $T\mathcal{F} = 0$ on ∂S and a generic vector of the form (3.7) can be written as $\mathcal{F}k$, where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary.

Let S^+ be the bounded domain enclosed by ∂S and $S^- = \mathbb{R}^2 \setminus (S^+ \cup \partial S)$.

3.1. Remark. *The following two assertions can be proved without difficulty as in [71] using classical techniques from [46]:*

(i) (Betti Formula) If $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ is a solution of the homogeneous system (3.2) in S^+, then

$$2\int_{S^+} E(u,u)\ d\sigma = \int_{\partial S} u^T T u\ ds$$

(ii) (Reciprocity relation). If $u, v \in C^2(S^+) \cap C^1(\overline{S}^+)$, then

$$\int_{S^+} (v^T L u - u^T L v) d\sigma = \int_{\partial S} (v^T T u - u^T T v) ds$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

3.2 Fundamental Solutions

We seek a Galerkin representation for the solution of (3.2). Following the method described in [10], [11] and [76], denoting by $L^*(\xi)$ the adjoint of $L(\xi)$ (transposed matrix of cofactors of $L(\xi)$), taking in turn each component of F equal to $-\delta(|x-y|)$, where δ is the Dirac distribution and setting the other two equal to zero,we obtain a matrix of fundamental solutions for (3.2) of the form [38], [71]

$$D(x,y) = L^*(\partial x)\Psi(x,y), \tag{3.9}$$

where

$$\Psi(x,y) = -\frac{d^2}{2\pi}\{\frac{1}{c_1^2c_2^2}\ln|x-y| + \frac{1}{(c_1^2-c_2^2)}[\frac{1}{c_1^2}K_0(c_1|x-y|) + \frac{1}{c_2^2}K_0(c_2|x-y|)]\}, \tag{3.10}$$

$d^2 = (\mu+\alpha)(\gamma+\kappa)(2\gamma+\beta)$,the constants c_1^2 , c_2^2 are defined by

$$c_1^2 = \frac{8\alpha}{(2\gamma+\beta)}, \qquad c_2^2 = \frac{8\alpha\mu}{(\mu+\alpha)(\gamma+\kappa)}$$

and K_0 is the modified Bessel function of order zero. Here and in what follows we assume that $c_1 \neq c_2$.

In view of (3.9)-(3.10) we conclude that

$$D(x,y) = (D(y,x))^T. \tag{3.11}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Along with $D(x,y)$ we consider the matrix of singular solutions

$$P(x,y;\ n) = (T(\partial_y; n)D(y,x))^T, \tag{3.12}$$

writing for simplicity $P(x,y;\ n) \equiv P(x,y)$.

To determine the behavior of $D(x,y)$ and $P(x,y)$ in the neighborhood of $x = y$ we note [1] that, as $\xi \to 0$,

$$K_0(\xi) = -(1 + \frac{1}{4}\xi^2 + \frac{1}{64}\xi^4 + ...) \ln \xi,$$

so that from (3.10) we deduce that in the neighborhood of $x = y$,

$$\Psi(x,y) = -\frac{d^2}{128\pi} |x-y|^4 \ln |x-y| + ..., \tag{3.13}$$

We denote by $\{E_{ij}\}$ the standard ordered basis for the vector space of (3×3)-matrices. From (3.9)-(3.12), we find that for y close to x,

$$D(x,y) = -\frac{1}{2\pi}\left(bE_{\gamma\gamma} + \frac{1}{\mu+\alpha}E_{33}\right) \ln |x-y| + \Omega(x,y) \tag{3.14}$$

and

$$P(x,y) = -\frac{1}{2\pi}\left(\frac{\partial}{\partial n(y)} \ln |x-y|\, E_{kk} + p\frac{\partial}{\partial s(y)} \ln |x-y|\, \epsilon_{\alpha\beta}E_{\alpha\beta}\right) + \Pi(x,y). \tag{3.15}$$

Here, $\dfrac{\partial}{\partial n\,(y)}$ and $\dfrac{\partial}{\partial s_y}$ represent derivatives in each of the normal and tangential directions respectively, that is, $\dfrac{\partial}{\partial n\,(y)} \equiv n.\nabla_y$, $\dfrac{\partial}{\partial s_y} \equiv \tau.\nabla_y$ where

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$\tau = (-n_2, n_1)$ is the tangential direction at y, chosen so that $\{n(y), \tau(y)\}$ is right-handed.,

$$b = \frac{(3\gamma + \beta + \kappa)}{2(\gamma + \kappa)(2\gamma + \beta)}, \qquad p = \frac{(\gamma + \kappa)(\gamma - \kappa) - \beta(2\gamma + \beta)}{2(\gamma + \kappa)(2\gamma + \beta)},$$

and $\Omega(x, y)$, $\Pi(x, y)$ are weakly singular in the sense of [10].

It can be verified that the columns of $D(x, y)$ and $P(x, y)$ are solutions of the homogeneous system (3.2) at all $x \in \mathbb{R}^2$, $x \neq y$, and for any direction n independent of x .

3.2. Remark. *(Somigliana formulae). Using classical techniques [10], [46], we can prove that if* $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ *is a solution of the homogeneous system (3.2) in* S^+, *then*

$$\int_{\partial S} [D(x, y)T(\partial_y)u(y) - P(x, y)u(y)] \, ds_y = \begin{cases} u(x), & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ 0, & x \in S^-. \end{cases}$$

3.3 Exterior Domain

The analogues of the Betti and Somigliana formulae in the exterior domain S^- require that we restrict the behavior of u at infinity. To this end, we consider the class $\mathcal{A}$ of vectors $u \in \mathcal{M}_{3\times1}$ whose components in terms of polar coordinates, admit an asymptotic expansion (as $r = |x| \to \infty$), of the

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

form

$$
\begin{aligned}
u_1(r,\theta) &= r^{-2}\left[m_0 \sin 2\theta + m_1(1-\cos 2\theta) + m_2\right] + O(r^{-3}), \qquad (3.16)\\
u_2(r,\theta) &= r^{-2}\left[-m_0 \sin 2\theta - m_1(1-\cos 2\theta) + m_3\right] + O(r^{-3}),\\
u_3(r,\theta) &= r^{-1}\left[(m_3 - m_0)\cos\theta - (m_2 - m_1)\sin\theta\right] + O(r^{-2}),
\end{aligned}
$$

where $m_0, ..., m_3$ are arbitrary constants.

We introduce also the set

$$
\mathcal{A}^* = \left\{u : u = \mathcal{F}k + s^{\mathcal{A}}\right\},
$$

where $k \in \mathcal{M}_{3\times 1}$ is constant and arbitrary and $s^{\mathcal{A}} \in \mathcal{M}_{3\times 1} \cap \mathcal{A}$. In view of (3.5), $\mathcal{A}$ and $\mathcal{A}^*$ are classes of finite energy functions.

For simplicity, and without loss of generality (see, for example, [46] where it is shown that boundary value problems for the non-homogeneous system (3.2) can be reduced to those studied below for the homogeneous system by means of a suitable constructed particular solution of the non-homogeneous system), throughout what follows, we consider only the homogeneous system (3.2), that is

$$
Lu = 0. \qquad (3.17)
$$

3.3. Remark. *The following assertions are proved using classical techniques [10], [46].*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

(i) (Betti Formula in Exterior Domain) If $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ *is a solution of (3.17) in* S^-*,then*

$$2\int_{S^-} E(u,u)\ d\sigma = -\int_{\partial S} u^T T u\ ds$$

(ii) (Somigliana Formula in Exterior Domain) If $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$ *is a solution of (3.17) in* S^-*,then*

$$-\int_{\partial S} [D(x,y)T(\partial_y)u(y) - P(x,y)u(y)]\ ds_y = \begin{cases} 0, & x \in S^+, \\ \frac{1}{2}u(x), & x \in \partial S, \\ u(x), & x \in S^-. \end{cases}$$

3.4 Boundary-value Problems

Let $f, g, h, q, r, l \in C(\partial S) \cap \mathcal{M}_{3\times 1}$ be prescribed on ∂S and let $\iota \in \mathcal{M}_{3\times 3}$ be a given positive definite matrix. We consider the following interior and exterior Dirichlet, Neumann and Robin problems:

Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (3.17) in S^+ such that

$$u\,|_{\partial S} = f. \tag{D^+}$$

Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (3.17) in S^+ such that

$$Tu\,|_{\partial S} = g. \tag{N^+}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Find $u \in C^2(S^+) \cap C^1(\overline{S}^+)$ satisfying (3.17) in S^+ such that

$$(Tu + \iota u)\,|_{\partial S} = h \tag{R^+}$$

Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (3.17) in S^- such that

$$u\,|_{\partial S} = q \tag{D^-}$$

Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}$ satisfying (3.17) in S^- such that

$$Tu\,|_{\partial S} = r \tag{N^-}$$

Find $u \in C^2(S^-) \cap C^1(\overline{S}^-) \cap \mathcal{A}^*$ satisfying (3.17) in S^- such that

$$(Tu - \iota u)\,|_{\partial S} = l \tag{R^-}$$

3.4. Theorem.

(i) Each of the problems $(D^+), (R^+), (D^-), (N^-)$ and (R^-) has at most one solution.

(ii) Any two solutions of (N^+) differ only by rigid displacement and microrotation of the form (3.8).

Proof: The proof of the theorem is given in detail in [71] and performed by means of the classical techniques from [46], [10] that make use of the Betti formulae established above.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

3.5 Elastic Potentials

We introduce the single layer potential

$$(V\varphi)(x) = \int_{\partial S} D(x,y)\varphi(y)\, ds_y,$$

and the double layer potential

$$(W\varphi)(x) = \int_{\partial S} P(x,y)\varphi(y)\, ds_y,$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is an unknown density matrix. We define also an operator $\mathcal{P}$ on continuous functions $\psi \in \mathcal{M}_{3\times 1}$ on ∂S by

$$\mathcal{P}\psi = \int_{\partial S} \mathcal{F}^T \psi\, ds.$$

3.5. Theorem. *If* $\varphi \in C(\partial S)$ *then*

(i) $W\varphi \in \mathcal{A}$

(ii) $V\varphi \in \mathcal{A}$ *if and only if* $\mathcal{P}\varphi = 0$.

Proof: The first part of the assertion is obtained by direct verification. The second part follows from the fact that, from (3.9) – (3.10), as $r = |x| \to \infty$, the components of $(V\varphi)(x)$ may be represented in the following form

$$\begin{aligned}
(V\varphi)_1(r,\theta) &= \widetilde{(V\varphi)}_1(r,\theta),\\
(V\varphi)_2(r,\theta) &= \widetilde{(V\varphi)}_2(r,\theta),\\
(V\varphi)_3(r,\theta) &= \widetilde{(V\varphi)}_3(r,\theta) + A\ln|x| \int_{\partial S} \varphi_3\, ds,
\end{aligned}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where $\widetilde{(V\varphi)}_1(r,\theta)$, $\widetilde{(V\varphi)}_2(r,\theta)$, $\widetilde{(V\varphi)}_3(r,\theta) \in \mathcal{A}$, and

$$A = -\frac{1}{2\pi}\left[\frac{[(2\gamma+\beta)(\gamma+\kappa)(\mu+\alpha)]^2}{4\mu}\right].$$

3.6. Theorem.

(i) *If* $\varphi \in C(\partial S)$, *then* $V\varphi$, $W\varphi$ *are analytic and satisfy* $L(V\varphi) = L(W\varphi) = 0$ *in* $S^+ \cup S^-$.

(ii) *If* $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then the direct values* $V_0\varphi, W_0\varphi$ *of* $V\varphi$, $W\varphi$ *on* ∂S *exist (the latter as principal value), the functions*

$V^+(\varphi) = (V\varphi)|_{\overline{S}^+}$, $V^-(\varphi) = (V\varphi)|_{\overline{S}^-}$ *are of class* $C^{1,\alpha}(\overline{S}^+)$ *and* $C^{1,\alpha}(\overline{S}^-)$, *respectively and*

$$TV^+(\varphi) = (W_0^* + \tfrac{1}{2}I)\varphi, \qquad TV^-(\varphi) = (W_0^* - \tfrac{1}{2}I)\varphi \quad \text{on } \partial S,$$

where W_0^* *is the adjoint of* W_0 *and* I *- the identity operator.*

(iii) *If* $\varphi \in C^{1,\alpha}(\partial S)$, $\alpha \in (0,1)$, *then the functions*

$$W^+(\varphi) = \begin{cases} (W\varphi)|_{S^+}, & \text{in } S^+, \\ (W_0 - \frac{1}{2}I)\varphi, & \text{on } \partial S, \end{cases} \qquad W^-(\varphi) = \begin{cases} (W\varphi)|_{S^-}, & \text{in } S^+, \\ (W_0 + \frac{1}{2}I)\varphi, & \text{on } \partial S, \end{cases}$$

are of class $C^{1,\alpha}(\overline{S}^+)$ *and* $C^{1,\alpha}(\overline{S}^-)$, *respectively, and* $TW^+(\varphi) = TW^-(\varphi)$ *on* ∂S.

Proof: The proof of part *(i)* follows from classical arguments for systems of partial differential equations (see, for example, [61]) and direct verification.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

For parts *(ii)* and *(iii)* we use the following expressions for $V\varphi, W\varphi$ when $x \to y$

$$\begin{aligned} V\varphi &= \frac{1}{2\pi}\left[bE_{\gamma\gamma} + \frac{1}{\mu+\alpha}E_{33}\right]v(\varphi) + \widetilde{(V\varphi)}, \\ W\varphi &= \frac{1}{2\pi}\left[w(\varphi)E + w^f(\varphi)p\epsilon_{\alpha\beta}E_{\alpha\beta}\right] + \widetilde{(W\varphi)}, \end{aligned}$$

where

$$\begin{aligned} v(\varphi)(x) &= -\int_{\partial S} \ln|x-y|\,\varphi(y)\,ds_y, \\ w(\varphi)(x) &= -\int_{\partial S}\left(\frac{\partial}{\partial n(y)}\ln|x-y|\right)\varphi(y)\,ds_y, \\ w^f(\varphi)(x) &= \int_{\partial S}\left(\frac{\partial}{\partial s(y)}\ln|x-y|\right)\varphi(y)\,ds_y. \end{aligned}$$

The result now follows from established results concerning the behavior of these functions in the neighborhood of ∂S [10].

3.6 Solution of the Boundary Value Problems

We consider first the Dirichlet and Neumann problems. If we seek the solution of $(D^+), (D^-), (N^+), (N^-)$ in the form $W^+(\varphi)$, $W^-(\varphi) + \mathcal{F}k$ ($k \in \mathcal{M}_{3\times 1}$ is constant), $V^+(\varphi)$ and $V^-(\varphi)$, respectively, then, by Theorem 3.6, each of the corresponding problems can be reduced to the following systems of singular

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

boundary integral equations

$$\begin{aligned}
(W_0 - \frac{1}{2}I)\varphi &= f && (\mathcal{D}^+)\\
(W_0^* + \frac{1}{2}I)\varphi &= g && (\mathcal{N}^+)\\
(W_0 + \frac{1}{2}I)\varphi &= q - Fk && (\mathcal{D}^-)\\
(W_0^* - \frac{1}{2}I)\varphi &= r && (\mathcal{N}^-)
\end{aligned}$$

The corresponding homogeneous equations are denoted by $(\mathcal{D}_0^+), (\mathcal{D}_0^-), (\mathcal{N}_0^+), (\mathcal{N}_0^-)$

3.7 Theorem. *If $f \in C^{1,\alpha}(\partial S), \alpha \in (0,1)$, then any solution $\varphi \in C^{0,\alpha}(\partial S)$ of (D^+) is of class $C^{1,\alpha}(\partial S)$. A similar result holds for (D^-) if $q \in C^{1,\alpha}(\partial S)$.*

Proof: The proof of this assertion is based on the use of the relation

$$\frac{d\varsigma}{\varsigma - z} = \left[\frac{\partial}{\partial n\,(y)} \ln|x-y| + i\frac{\partial}{\partial s\,(y)} \ln|x-y|\right] ds_y$$

where $z = x_1 + ix_2,\ \ \varsigma = y_1 + iy_2$. This allows us to express the operator W_0 as a sum of a singular part and a weakly singular part. That is:

$$W_0 = (W_0^s\varphi)(z) + \widetilde{W_0}(z)$$

where the singular part is given by

$$(W_0^s\varphi)(z) = \frac{1}{2\pi} p\epsilon_{\alpha\beta}E_{\alpha\beta}\int_{\partial S}\frac{\varphi(\varsigma)\,d\varsigma}{\varsigma - z}.$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

and $\widetilde{W}_0(z)$ is the weakly singular part. This allows us to write $(\mathcal{D}^{\pm})$ as systems of Cauchy singular integral equations. Using regularization procedures [62] and the mapping properties of the relevant operators [10], we obtain the required result.

3.8 Theorem. *The Fredholm Alternative holds for* (D^{+}) , (N^{-}) *and for* (N^{+}), (D^{-}) *in the real dual system* $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$, $\alpha \in (0,1)$, *with the bilinear form*

$$(\varphi,\psi) = \int_{\partial S} \varphi^T(y)\psi(y)\; ds_y \tag{3.18}$$

Proof: Let $\mathcal{D}$, $\mathcal{N}$ be the integral operators occurring in $(\mathcal{D}^{\pm})$ and $(\mathcal{N}^{\pm})$, respectively. Using the definition of $P(x,y) = (T(\partial_y)D(y,x))^T$ we can see that for any $\varphi, \psi \in C^{0,\alpha}(\partial S)$

$$\begin{aligned}
(\mathcal{D}\varphi,\psi) &= \int_{\partial S}\left[\int_{\partial S} P(x,y)\varphi(y)\; ds_y\right]^T \psi(x)ds_x \\
&= \int_{\partial S}\left[\int_{\partial S} T(\partial_y)D(y,x)^T\varphi(y)ds_y\right]^T \psi(x)ds_x \\
&= \int_{\partial S}\varphi^T(y)\left[\int_{\partial S} T(\partial_y)D(y,x)\psi(x)ds_x\right] ds_y \\
&= (\varphi,\mathcal{N}\psi).
\end{aligned}$$

Due to the symmetry of the bilinear form (3.18), we have also that $(\mathcal{N}\varphi,\psi) = (\varphi,\mathcal{D}\psi)$, which means that $\mathcal{D}$ and $\mathcal{N}$ are mutually adjoint in the given dual system. The next step of the proof is to show that the index

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

ϱ of the complex version of $(\mathcal{D}^+)$ is zero. From [62], the index of the complex version of the system $(\mathcal{D}^+)$ is given by:

$$\varrho = \frac{1}{2\pi}\left[\arg\frac{\det(-\frac{1}{2}E - \pi i\widehat{k}(z,z))}{\det(-\frac{1}{2}E + \pi i\widehat{k}(z,z))}\right]_{\partial S}$$

where

$$\widehat{k}(z,z) = -\frac{1}{2\pi}p\epsilon_{\alpha\beta}E_{\alpha\beta}, \quad z \in \partial S.$$

Consequently

$$\det(-\frac{1}{2}E \pm \pi i\widehat{k}(z,z)) = -\frac{1}{8}(1-p^2) < 0,$$

from which we can immediately deduce that $\varrho = 0$. Consequently, the Fredholm Alternative holds for the operator $\mathcal{D}$ in the (complex) dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$ with the bilinear form (3.18) and hence also for $\mathcal{D}$ in the real dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$ with the same bilinear form. The argument is similar for the pair $(\mathcal{D}^-)$, $(\mathcal{N}^+)$.

3.9 Theorem. *(i)* (D^+) *has a unique solution for any* $f \in C^{1,\alpha}(\partial S), \alpha \in (0,1)$. *This solution can be represented as* $W^+(\varphi)$ *with* $\varphi \in C^{1,\alpha}(\partial S)$.

(ii). (N^-) *has a unique solution for any* $r \in C^{0,\alpha}(\partial S), \alpha \in (0,1)$, *if and only if* $\mathcal{P}r = 0$. *The solution can be represented as* $V^-(\varphi)$ *with* $\varphi \in C^{0,\alpha}(\partial S)$.

(iii) (N^+) *is soluble for any* $g \in C^{0,\alpha}(\partial S)$, $\alpha \in (0,1)$ *,if and only if* $\mathcal{P}g = 0$. *The solution is unique up to a matrix of the form* $\mathcal{F}k$, *where* $k \in M_{3\times 1}$ *is*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

constant and arbitrary, and can be represented as $V^+(\varphi)$ *with* $\varphi \in C^{0,\alpha}(\partial S)$.

(iv) (D^-) *has a unique solution for any* $q \in C^{1,\alpha}(\partial S), \alpha \in (0,1)$. *This solution can be represented as the sum of* $W^-(\varphi)$ *with* $\varphi \in C^{1,\alpha}(\partial S)$ *and a specific matrix* $\mathcal{F}k$.

Proof: The proof of *(i)*, *(ii)* and *(iii)* follows as in [10]. To prove assertion *(iv)* we can find $G \in \mathcal{M}_{3\times 3}$ such that

$$\mathcal{P}G = E_3. \tag{3.19}$$

Then, in order to satisfy the solubility condition, we choose

$$k = \int_{\partial S} G^T q \; ds$$

With this choice of k, by (3.19), we obtain

$$\int_{\partial S} G^T(q - \mathcal{F}k) \; ds = \int_{\partial S} G^T q \; ds - \left(\int_{\partial S} G^T \mathcal{F} \; ds\right) k = k - k = 0,$$

and the solvability condition for (D^-) is satisfied for any $q \in C^{1,\alpha}(\partial S)$.

3.10. Remark. *The conditions* $\int_{\partial S} \mathcal{F}^T r \; ds = 0$, $\int_{\partial S} \mathcal{F}^T g \; ds = 0$ *represent zero resultant force and moment acting on* ∂S.

Let us now consider the problems (R^+) and (R^-). The methods used in [10] and [46] to investigate the analogous problems in classical elasticity do not accommodate (R^+) and (R^-). This is attributed to the asymptotic

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

behavior of $V(\varphi)$ and, in particular, to the additional conditions imposed on the density φ for regularity at infinity. To overcome this difficulty we seek the solutions of (R^+) and (R^-) in the form of $V^+(\varphi - \mathcal{F}k\varphi) + \mathcal{F}k\varphi$ and $V^-(\varphi - \mathcal{F}k\varphi) + \mathcal{F}k\varphi$, respectively, where $k\varphi \in \mathcal{M}_{3\times1}$ is given by

$$k\varphi = \left(\int_{\partial S} \mathcal{F}^T\mathcal{F}\, ds\right)^{-1} \int_{\partial S} \mathcal{F}^T\varphi\, ds, \tag{3.20}$$

which ensures that $V(\varphi - \mathcal{F}k\varphi) \in \mathcal{A}$ by Theorem.3.5 (ii).

It is clear that

$$\int_{\partial S} \mathcal{F}^T\mathcal{F}\, ds \in \mathcal{M}_{3\times3},$$

is positive definite and hence invertible for any closed C^2 -curve ∂S. By Theorem 3.6, these problems can be reduced to the following systems of singular boundary integral equations

$$\begin{aligned} (W_0^* + \frac{1}{2}I + mV_0Jk)\varphi &= h, && (\mathcal{R}^+) \\ (W_0^* - \frac{1}{2}I - mV_0Qk)\varphi &= l, && (\mathcal{R}^-) \end{aligned}$$

where $J, Q \in \mathcal{M}_{3\times3}$ are given by

$$J = \left(m - \frac{1}{2}I\right)\mathcal{F} - W_0^*\mathcal{F} - mV_0\mathcal{F}, \qquad Q = -J - 2W_0^*\mathcal{F}.$$

3.11. Theorem.

(i) The Fredholm Alternative holds for $(R^+)((R^-))$ and its adjoint in the real

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

dual system $(C^{0,\alpha}(\partial S), C^{0,\alpha}(\partial S))$, $\alpha \in (0,1)$, *with the bilinear form*

$$(\varphi, \psi) = \int_{\partial S} \varphi^T(y)\psi(y)\, ds_y.$$

(ii) (R_0^+), *and* (R_0^-) *have only the trivial solution.*

Proof: (i) We follow the procedures used in the proof of Theorem 3.8, and show that the index of each of the singular operators in $(\mathcal{R}^+)$ and $(\mathcal{R}^-)$ is zero

(ii) First, let us consider $(\mathcal{R}_0^+)$. Let $\varphi_0 \in C^{0,\alpha}(\partial S), \alpha \in (0,1)$, be a solution of $(\mathcal{R}_0^+)$. Then $u = V^+(\varphi - \mathcal{F}k\varphi) + \mathcal{F}k\varphi$ satisfies the homogeneous problem (R_0^+). By the uniqueness result for (R^+), (Theorem 3.4 *(i)*), we now have that $u = 0$ in S^+. From Theorem 3.6 *(i)* we have that $u^+ = u = u^- = 0$ on ∂S. Also, since $k\varphi$ is given by (3.20), $u \in \mathcal{A}^*$ by Theorem 3.5 *(ii)*. Thus, u satisfies (D_0^-) and $u = 0$ in S^- by the uniqueness result for (D^-) (Theorem 3.4 *(i)*) Hence, $k\varphi = 0$ and $(Tu)^+ - (Tu)^- = \varphi_0 = 0$. The same procedure is applicable to prove the assertion for $(\mathcal{R}_0^-)$.

3.12. Theorem. *The problem* $(R^+)((R^-))$ *has a unique solution for any* h $(l) \in C^{0,\alpha}(\partial S), \alpha \in (0,1)$. *The solution can be represented in the form* $V^+(\varphi - Fk\varphi) + Fk\varphi$ $(V^-(\varphi - Fk\varphi) + Fk\varphi)$ *with* $\varphi \in C^{0,\alpha}(\partial S)$ *from* $(R^+)((R^-))$ *and* $k\varphi$ *given by (3.20)*

Proof: From Theorem 3.11 and standard results on the mapping proper-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

ties of the corresponding integral operators [10], we deduce that (R^+) and (R^-) are always uniquely soluble in the space $C^{0,\alpha}(\partial S)$ for prescribed boundary data h and l, respectively. To complete the proof, we remark that from Theorems 3.8 and 3.9, with $\varphi \in C^{0,\alpha}(\partial S), \alpha \in (0,1)$, from (R^+) and $k\varphi$ given by (3.20), $V^+(\varphi - Fk\varphi) + Fk\varphi$ satisfies all requirements of (R^+). A similar statement can be made for the problem (R^-).

The treatment of mixed boundary value problems is a little different from that of Dirichlet, Neumann and Robin and at the same time - more difficult. This can perhaps be attributed to the fact that solutions of mixed boundary value problems generally suffer from a discontinuity in the first derivative across the points on the boundary where the data change from Dirichlet-type to Neumann-type. This makes it extremely difficult to construct classical solutions (see, for example, [28] and [106]).

Because of this reason, we find it necessary to consider mixed boundary value problems separately.

3.7 Solution of Mixed Problems

Let S^+ be the bounded domain enclosed by ∂S and $S^- = \mathbb{R}^2 \backslash (S^+ \cup \partial S)$. Further, divide ∂S into two (for simplicity) arcs ∂S_1 and ∂S_2 with common

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

endpoints a and b. As in [46], the set $\gamma = \{a, b\}$ is included in ∂S_1 so that ∂S_2 is taken as an open set and ∂S_1 as a closed one. It is worth noting that the generalization of our method to the case where ∂S is divided into more than two parts is relatively straightforward and proceeds as in [47].

We consider the following interior mixed boundary value problem for (3.17).

Find $v \in C^2(S^+) \cap C^1(\bar{S}^+ \backslash \gamma)$ such that

$$\begin{aligned} L(\partial x)\, v(x) &= 0, & x &\in S^+ \\ v(x) &= B(x), & x &\in \partial S_1 \\ T(\partial x)\, v(x) &= C(x), & x &\in \partial S_2, \end{aligned}$$

where $B, C \in \mathcal{M}_{3\times 1}$ are prescribed on ∂S_1 and ∂S_2, respectively. Proceeding as in [47], this problem can be reduced to the simpler problem:

$(\mathcal{M}^+)$ Find $u \in C^2(S^+) \cap C^1(\bar{S}^+ \backslash \gamma)$ satisfying (3.17) in S^+ and bounded on $\overline{S}^{\,+}$ such that

$$u(x) = 0, \quad x \in \partial S_1 \tag{3.21}$$

$$T(\partial x)\, u(x) = f(x), \quad x \in \partial S_2, \tag{3.22}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where $f = C - T\Phi \in \mathcal{M}_{3\times 1}$, $u = v - \Phi$ and Φ is the (known) solution of a related Dirichlet problem for (3.17).

Similarly, we consider the exterior problem:

$(\mathcal{M}^-)$ Find $u \in C^2(S^-) \cap C^1(\bar{S}^-\backslash\gamma) \cap \mathcal{A}^*$ satisfying (3.2) in S^- and

$$u(x) = 0, \quad x \in \partial S_1 \tag{3.23}$$

$$T(\partial x)\, u(x) = q(x), \quad x \in \partial S_2, \tag{3.24}$$

where $q \in \mathcal{M}_{3\times 1}$, is prescribed on ∂S_2. In view of (3.7), we pose the exterior problem $(\mathcal{M}^-)$ in $\mathcal{A}^*$ to allow as large a set of admissible matrix functions as possible.

3.13. Remark. *(Betti Formulae).*

The following subsidiary formulae are proved using classical results as in [46].

(i) If $u \in C^2(S^+) \cap C^1(\bar{S}^+\backslash\gamma)$ is a solution of (3.17) in S^+,

$$2\int_{S^+} E(u,u)d\sigma = \int_{\partial S} u^T T u ds. \tag{3.25}$$

(ii) If $u \in C^2(S^-) \cap C^1(\bar{S}^-\backslash\gamma) \cap \mathcal{A}^*$ is a solution of (3.17) in S^-,

$$2\int_{S^-} E(u,u)d\sigma = -\int_{\partial S} u^T T u ds. \tag{3.26}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Using (3.25) in the case of the interior problem $(\mathcal{M}^{+})$ and (3.26) in the case of the exterior problem $(\mathcal{M}^{-})$, classical arguments [46] lead to the following uniqueness result for problems $(\mathcal{M}^{+})$ and $(\mathcal{M}^{-})$.

3.14. Theorem. *$(\mathcal{M}^{+})$ and $(\mathcal{M}^{-})$ have at most one solution.*

Proof: The proof of the theorem is given in detail in [72].

3.8 Solvability of the Mixed Boundary Value Problems

3.8.1 Interior Mixed Problem

Consider first the interior mixed problem $(\mathcal{M}^{+})$. We seek the solution in the form

$$u(x) = (V\varphi)(x) = \int_{\partial S_2} [D(x,y) - H(x,y)]\,\varphi(y)\,ds(y), \tag{3.27}$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is some unknown matrix-density and the matrix $H(x,y) \in \mathcal{M}_{3\times 3}$ is constructed as follows.

Let Ω_1 be a bounded domain with C^2-boundary $\partial\Omega_1$ such that:

$$(i)\ S^{+} \subset \Omega_1 \qquad (ii)\ \partial S_2 \subset \Omega_1 \qquad (iii)\ \partial S_1 \subset \partial\Omega_1.$$

We denote the columns of the matrix H by $H^{(j)}$ each of which satisfies the

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

boundary value problem

$$L(\partial x)H^{(j)}(x,y)=0, \qquad x\in\Omega_1,$$

$$H^{(j)}(x,y)=D^{(j)}(x,y), \qquad x\in\partial\Omega_1.$$

From the existence result for the interior Dirichlet problem of anti-plane micropolar elasticity [71], it is clear that $H^{(j)}(x,y)$ exists uniquely for each $y\in\partial S_2$ in the class $C^2(\Omega_1)\cap C^1(\bar{\Omega}_1)$. In fact, for each y, $H^{(j)}(x,y)$ takes the form of an elastic double layer potential [72].

Suppose that the unknown density φ of (3.27) is of the class $H^*(\partial S_2)$ [102, p.80] i.e. φ is Hölder-continuous on ∂S_2 but may admit 'weak singularity' at the endpoints γ. Proceeding as in [47], using the properties of the single layer potential of anti-plane micropolar elasticity [71] and the properties of the matrix H, u from (3.27) satisfies the continuity conditions of the problem $(\mathcal{M}^+)$, (3.17) in S^+ and the displacement condition (3.21) on ∂S_1. The remaining traction condition (3.22) leads to the following system of singular integral equations over the open arc ∂S_2:

$$\frac{1}{2}\varphi(x)+\int_{\partial S_2}T(\partial x)\,D(x,y)\,\varphi(y)\,ds(y)-\int_{\partial S_2}T(\partial x)\,H(x,y)\,\varphi(y)\,ds(y)=f(x). \tag{3.28}$$

Using the properties of anti-plane micropolar elastic potentials [71], it is clear

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

that the first integral on the left-hand side of (3.28) must be interpreted in the sense of principal value while the second is a Fredholm integral.

It is clear that $u(x)$ from (3.27) will be the unique solution of $(\mathcal{M}^+)$ provided (3.28) yields a solution $\varphi \in H^*(\partial S_2)$ for sufficiently smooth boundary data f.

3.15. Lemma. *The homogeneous system $(3.28)^0$ from (3.28) has only the trivial solution in the space $H^*(\partial S_2)$.*

Proof: Let $\varphi_0 \in H^*(\partial S_2)$ be a solution of $(3.28)^0$. Then $(V\varphi_0)(x)$ from (3.27) solves the homogeneous problem $(\mathcal{M}^+)^0$. Theorem 3.14 now yields $(V\varphi_0)(x) = 0$, $x \in S^+$. The continuity of a single layer potential [72] now gives that $(V\varphi_0)(x) = 0$, $x \in \partial S$. Further, using the boundary value of $H(x,y)$, $(V\varphi_0)(x) = 0$, $x \in \partial\Omega_1$. Hence $(V\varphi_0)(x) = 0$ on the boundary of the bounded domain $\Omega_1 \backslash S^+$. By the uniqueness result for the interior Dirichlet problem of anti-plane micropolar elasticity [71], $(V\varphi_0)(x) = 0$, $x \in \Omega_1 \backslash S^+$so that the jump relations arising from the application of the T-operator to a single layer potential [72] yield

$$(TV)^+(\varphi_0) - (TV)^-(\varphi_0) = \varphi_0 = 0 \quad \text{on} \quad \partial S_2,$$

which completes the proof.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

In [102, §23], Vekua developed a theory of solvability for systems of singular integral equations with discontinuous coefficients. To see that this theory applies to our system, we rewrite (3.28) as a system with discontinuous coefficients over the closed curve ∂S. In fact, noting that for $x \in \partial S_2$,

$$\int_{\partial S_2} T(\partial x) D(x,y)\varphi(y)ds(y) = \frac{a}{\pi i} E^* \int_{\partial S_2} \frac{\varphi(\xi)}{\xi - z} d\xi + \int_{\partial S_2} m(z,\xi)\varphi(\xi)d\xi,$$

where

$$a = \pi i \alpha_1, \quad \alpha_1 = -\frac{2\mu^2 + \mu\kappa - \lambda\kappa}{4\pi(\lambda + 2\mu + \kappa)(\mu + \kappa)}, \quad E^* = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\xi = y_1 + iy_2 \in \partial S_2$, $z = x_1 + ix_2 \in \partial S_2$ and m is a weakly singular (Fredholm) kernel [28], we can rewrite (3.28) in the form

$$A(z)\varphi(z) + \frac{B(z)}{\pi i} \int_{\partial S} \frac{\varphi(\xi)}{\xi - z} d\xi + \int_{\partial S} C(z,\xi)\varphi(\xi)d\xi = g(z), \quad z \in \partial S, \tag{3.29}$$

or

$$A(z)\varphi(z) + \frac{1}{\pi i} \int_{\partial S} \frac{K_1(z,\xi)}{\xi - z} \varphi(\xi)d\xi = g(z), \quad z \in \partial S,$$

where, $K_1(z,\xi) = B(z) + \pi i(\xi - z)C(z,\xi)$, $z \in \partial S$,

$$A(z) = \begin{cases} E_3, & z \in \partial S_1, \\ -\frac{1}{2}E_3, & z \in \partial S_2; \end{cases} \qquad B(z) = \begin{cases} 0, & z \in \partial S_1, \\ aE^*, & z \in \partial S_2; \end{cases}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$C(z,\xi)=\begin{cases}0, & z \quad \text{or} \quad \xi\in\partial S_1,\\ \\ m(z,\xi)-[T(\partial x)H(x,y)](z,\xi), & z \quad \text{and} \quad \xi\in\partial S_2\end{cases}$$

and

$$g(z)=\begin{cases}0, & z\in\partial S_1,\\ \\ f(z), & z\in\partial S_2.\end{cases}$$

It is clear that A and K_1 satisfy the Hölder condition everywhere on ∂S except perhaps at the points of γ where they have discontinuity of the first kind [106].

3.16. Lemma. *The Fredholm Alternative holds for the system (3.29) and its adjoint or associated system in the space $H^*(\partial S)$.*

Proof: According to Vekua [102, §23] Noether's theorems are valid for the system (3.29). Further, proceeding as in [28], a routine calculation shows that the (total) index [106] of the singular integral operator from (3.29) is zero so that Noether's theorems reduce to Fredholm's theorems. Finally, the endpoints a and b are easily shown to be 'special' [102, §13] so that any solution of (3.29) with $g\in H^*(\partial S)$ is necessarily of the same class [106].

3.17. Theorem. *The mixed problem $(\mathcal{M}^+)$ is uniquely solvable for any $f\in H^*(\partial S_2)$. The solution is given by (3.27) with $\varphi\in H^*(\partial S_2)$ obtained from the system (3.28).*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Proof: Using Lemma 3.15, we see that the homogeneous system $(3.29)^0$ has only the trivial solution in $H^*(\partial S)$. Hence, since Fredholm's theorems apply (by Lemma 3.16), the associated homogeneous system has also only the trivial solution in $H^*(\partial S)$ and (3.29) is uniquely solvable in $H^*(\partial S)$ for any $g \in H^*(\partial S)$. This means that the system (3.28) is uniquely solvable in $H^*(\partial S_2)$ whenever the boundary data $f \in H^*(\partial S_2)$. Consequently, the unique (by Theorem 3.14) solution of $(\mathcal{M}^+)$ with $f \in H^*(\partial S_2)$, is given by (3.27) with $\varphi \in H^*(\partial S_2)$ obtained from the system (3.28).

3.8.2 Exterior Mixed Problem

In the case of the exterior mixed problem, the asymptotic behavior (3.9) of the matrix $D(x,y)$ requires that we seek the solution in the form

$$u(x) = (V_E\varphi)(x) = \int_{\partial S_2} [D(x,y) - M^\infty(x) F^T(y) - \Psi(x,y)]\varphi(y)\, ds(y), \tag{3.30}$$

where $\varphi \in \mathcal{M}_{3\times 1}$ is again an unknown matrix-density and $M^\infty \in \mathcal{M}_{3\times 3}$ is given by

$$\frac{4\alpha^2}{\pi}(\mu+\alpha)(\gamma+\varepsilon)(2\gamma+\beta)M^\infty(r,\theta) =$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$\begin{pmatrix} 0 & 0 & r^{-1}\sin\theta \\ 0 & 0 & -r\sin\theta \\ 0 & 0 & 2\ln r \end{pmatrix}.$$

Here, polar coordinates (r,θ) are given by $r=|x|$ and $\theta=\tan^{-1}(x_2/x_1)$. It is easily verified that $LM^{\infty}=0$ in $\mathbb{R}^2\backslash\{0\}$. The matrix Ψ is constructed using a procedure similar to that used to construct the matrix H for $(\mathcal{M}^{+})$. That is, let Ω_2 be an infinite domain with closed C^2- boundary $\partial\Omega_2$ such that

$$(i)S^{-}\subset\Omega_2 \quad (ii)\partial S_2\subset\Omega_2 \quad (iii)\partial S_1\subset\partial\Omega_2 \quad (iv)\{0\}\notin\bar{\Omega}_2.$$

The columns $\Psi^{(j)}(x,y)$, $y\in\partial S_2$ satisfy

$$\begin{aligned} L(\partial x)\Psi^{(j)}(x,y) &= 0, && x\in\Omega_2, \\ \Psi^{(j)}(x,y) &= \mathcal{G}^{(j)}(x,y), && x\in\partial\Omega_2, \end{aligned}$$

where $\mathcal{G}\in\mathcal{M}_{3\times 3}$ is given by $\mathcal{G}(x,y)=D(x,y)-M^{\infty}(x)F^{T}(y)$. The existence result for the exterior Dirichlet problem of anti-plane micropolar elasticity [71], guarantees that $\Psi^{(j)}(x,y)$ exists uniquely for each $y\in\partial S_2$ in the class $C^2(\Omega_2)\cap C^1(\bar{\Omega}_2)\cap\mathcal{A}^{*}$. In fact, for each y, $\Psi^{(j)}(x,y)$ takes the form of the sum of an elastic double layer potential and a matrix of the form (3.8).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

The fact that with $\varphi \in H^*(\partial S_2)$, u from (3.30) satisfies the continuity conditions of the problem $(\mathcal{M}^-)$, (3.17) in S^- and the displacement condition (3.23) on ∂S_1 follows as in the case of $(\mathcal{M}^+)$ using the properties of M^∞ and Ψ described above and the smoothness properties of the elastic single-layer potential from anti-plane micropolar elasticity [71]. The fact that $u(x) = (V_E\varphi)(x) \in \mathcal{A}^*$ follows from the fact that, as $|x| \to \infty$ [71],

$$\begin{aligned}
&\int_{\partial S_2} [D(x,y) - M^\infty(x) F^T(y)]\varphi(y)\, ds(y) \\
=\;& M^\infty(x) \int_{\partial S_2} F^T(y)\varphi(y)\, ds(y) + u_0 - M^\infty(x) \int_{\partial S_2} F^T(y)\varphi(y)\, ds(y) \\
=\;& u_0 \in \mathcal{A}.
\end{aligned}$$

Also,

$$\int_{\partial S_2} \Psi(x,y)\varphi(y)\, ds(y) = \int_{\partial S_2} \Psi^{(j)}(x,y)\varphi_j(y)\, ds(y) \in \mathcal{A}^*,$$

since, as noted above, $\Psi^{(j)}(x,y) \in \mathcal{A}^*$ for each y. Hence, $u(x) = (V_E\varphi)(x) \in \mathcal{A}^*$. As in the case of problem $(\mathcal{M}^+)$, the remaining traction condition (3.24) leads to the following system of singular integral equations over the open arc ∂S_2.

$$\begin{aligned}
&-\frac{1}{2}\varphi(x) + \int_{\partial S_2} T(\partial x)\, D(x,y)\varphi(y)\, ds(y) \\
&- \int_{\partial S_2} T(\partial x)\ [\Psi(x,y) + M^\infty(x) F^T(y)]\varphi(y)\, ds(y) \\
=\;& q(x), \quad x \in \partial S_2 \qquad (3.31)
\end{aligned}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Consequently, u from (3.20) will be the unique solution of $(\mathcal{M}^-)$ provided (3.31) yields a solution $\varphi \in H^*(\partial S_2)$ whenever $q \in H^*(\partial S_2)$.

3.18. Lemma. *The homogeneous system* $(3.31)^0$ *from (3.31) has only the trivial solution in the space* $H^*(\partial S_2)$.

Proof: Let $\varphi_0 \in H^*(\partial S_2)$ be a solution of $(3.31)^0$. Then $(V_E\varphi_0)(x)$ from (3.30) solves the homogeneous problem $(\mathcal{M}^-)^0$. Theorem 3.14 now yields $(V_E\varphi_0)(x) = 0$, $x \in S^-$. Proceeding as in the proof of Lemma 3.15, we obtain that $(V_E\varphi_0)(x) = 0$, $x \in \Omega_2 \backslash S^-$so that

$$(TV_E)^+(\varphi_0) - (TV_E)^-(\varphi_0) = \varphi_0 = 0 \text{ on } \partial S_2,$$

which completes the proof.

The solvability result for the exterior problem $(\mathcal{M}^-)$ now follows from Lemma 3.16:

3.19. Theorem. *The mixed problem* $(\mathcal{M}^-)$ *is uniquely solvable for any* $q \in H^*(\partial S_2)$. *The solution is given by (3.30) with* $\varphi \in H^*(\partial S_2)$ *obtained from the system (3.31).*

Proof: The system (3.31) is similar in nature to the system (3.28). Following the steps leading to system (3.29), we can again rewrite (3.31) as a system with discontinuous coefficients over the closed curve ∂S. As in the proof of

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Lemma 3.16, the index of the resulting system over ∂S is shown to be zero and the endpoints a and b to be 'special' [106]. Vekua's theory again shows that Noether's theorems reduce to the Fredholm Alternative in the space $H^*(\partial S)$. Using Lemma 3.18 and the Fredholm Alternative, the associated homogeneous system has also only the trivial solution in $H^*(\partial S)$. Arguing as in the proof of Theorem 3.19, we conclude that (3.31) is uniquely solvable in $H^*(\partial S_2)$ for any $q \in H^*(\partial S_2)$. Consequently, the unique (by Theorem 3.14) solution of $(\mathcal{M}^-)$ with $q \in H^*(\partial S_2)$, is given by (3.30) with $\varphi \in H^*(\partial S_2)$ obtained from the system (3.31).

3.9 Summary

In this chapter we formulated Dirichlet, Neumann, Robin, mixed interior and exterior boundary value problems of antiplane micropolar elasticity. The existence and uniqueness results for these problems have been established and analytical solutions have been obtained in the form single and double-layer integral potentials.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

4 Generalized Fourier Series

After carrying out a detailed mathematical analysis of boundary value problems of anti-plane micropolar elasticity the solutions should be expressed in a form allowing us to employ them for practical purposes. The reason is, that after deriving analytical solutions in the form of integral potentials we can only assert that these solutions exist and that they are unique, but we still cannot find quantitative characteristics for the unknown density φ. Unfortunately, it is extremely hard, if not impossible, to find φ using analytical procedures, therefore numerical techniques for obtaining the function φ should be used. One of the common ways to attain this end is the approximation of the required function by means of a set of known functions with well-studied and sufficiently simple properties. This idea underlies, in particular, the Fourier method. The present section deals with modification of the method of generalized Fourier series proposed by Kupradze [46] for the treatment of three dimensional boundary value problems of the theory of elasticity. This method, based on the same idea as the standard Fourier method, has certain advantages from the computational point of view. For example, if we exclude a few particular cases when its application is quite efficient, the importance of the classical Fourier method for computations in

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

this sense is limited. The application of the method involves the expansion of the required function in a series of the functions which are themselves solutions of no less complicated boundary value problems and the numerical computation is possible only if we know eigenvalues and eigenfunctions of the latter problems.

The modification of the generalized Fourier method to be described below and to be applied for the construction of approximate solutions of boundary value problems of antiplane Cosserat elasticity lacks this inefficiency and affords the possibility to construct the required system of functions directly from the problem data.

Thus the generalized Fourier method may be used for obtaining approximate numerical solutions. In this dissertation we will consider only the application of generalized Fourier method to Dirichlet and Neumann interior and exterior boundary value problems. The corresponding treatment of mixed boundary value problems lies beyond the scope of the present investigation and is the objective of future work.

In this chapter we suspend the convention of summation over repeated indices, as well as that regarding the values taken by Latin subscripts. Greek subscripts and superscripts continue to take values 1, 2.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

4.1 The Interior Neumann Problem

4.1. Definition. *Let X be a normed space. A subset $X \subset \mathcal{X}$ is called a fundamental set in X if span $\mathcal{X}$ is dense in X.*

The following assertion is a well-known result of functional analysis (see for example [45]).

4.2. Theorem. *If X is a Hilbert space, then $\mathcal{X} \subset X$ is a fundamental set in X if and only if the orthogonal complement of $\mathcal{X}$ in X consists of the zero vector alone.*

Let ∂S_* be a simple closed C^2 - curve such that ∂S lies strictly in the domain S_*^+ enclosed by ∂S_*, and let $\{x^{(k)} \in \partial S_*,\ k = 1, 2, ...\}$ be a countable set of points densely distributed on ∂S_*. We set $S_*^- = \mathbb{R}^2 \backslash \overline{S}_*^+$, and denote by $D^{(i)}$ the columns of the fundamental matrix D.

4.3. Theorem. *The set*

$$\{F^{(i)}, \theta^{(jk)}, i, j = 1, 2, 3, k = 1, 2, ...\}, \tag{4.1}$$

where the $F^{(i)}$ are the columns of matrix (3.8)

and

$$\theta^{(jk)}(x) = T(\partial x) D^{(j)}(x, x^{(k)}), \tag{4.2}$$

is linearly independent on ∂S and fundamental in $L^2(\partial S)$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Proof: Suppose that there are a positive integer N and real numbers c_i and c_{jk}, $i, j = 1, 2, 3$, $k = 1, 2, ..., N$, not all zero, such that

$$\sum_{i=1}^{3} c_i F^{(i)}(x) + \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\theta^{(jk)}(x) = 0, \qquad x \in \partial S. \tag{4.3}$$

Then, taking (4.2), (4.3) into consideration and the fact, that the columns of $D(x, y)$ and $P(x, y)$ are solutions of (3.17) at all $x \in \mathbb{R}^2$, $x \neq y$, and for any direction n independent of x, we find that the (3×1) - matrix

$$\varpi(x) = \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk} D^{(j)}(x, x^{(k)}) \tag{4.4}$$

is a regular solution of the interior Neumann problem

$$L(\partial x)\varpi(x) = 0, \quad x \in S^+,$$

$$T(\partial x)\varpi(x) = -\sum_{i=1}^{3} c_i F^{(i)}(x), \quad x \in \partial S.$$

Consequently, by Theorem 3.9(iii)

$$\int_{\partial S} (F^{(l)})^T \left[-\sum_{i=1}^{3} c_i F^{(i)}(x)\right] ds = 0, \quad l = 1, 2, 3,$$

which implies that the coefficients are all equal to zero. This yields

$$T(\partial x)\varpi(x) = 0, \quad x \in \partial S.$$

Hence, by Theorem 3.4 (ii)

$$\varpi(x) = \sum_{i=1}^{3} \beta_i F^{(i)}(x), \quad x \in \overline{S}^+,$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

for some constants β_i, $i = 1, 2, 3$. From this and (4.4) it follows that

$$\varpi(x) = \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} D^{(j)}(x, x^{(k)}) - \sum_{i=1}^{3} \beta_i F^{(i)}(x), \quad x \in \overline{S}^{+}.$$

Then, analyticity requires that $\varpi(x) = 0$ in $\overline{S}_*^+$. Further the linear independence of the set (4.1) on ∂S is established by applying the procedure used in [10] and [46]

Suppose now that for all $i, j = 1, 2, 3$ and $k = 1, 2, \ldots$ the function $\varphi \in L^2(\partial S)$ satisfies

$$\int_{\partial S} (F^{(i)})^T \varphi ds = \int_{\partial S} (\theta^{(jk)})^T \varphi ds = 0.$$

According to (4.2) and (3.12) this means that

$$\int_{\partial S} \varphi_3 ds = 0, \tag{4.5}$$

and

$$\int_{\partial S} P(x^{(k)}, y)\varphi(y) ds = 0, \quad k = 1, 2, \ldots \tag{4.6}$$

By Theorem 3.6 (i), the double layer potential of density φ

$$W(x) = \int_{\partial S} P(x, y)\varphi(y) ds(y)$$

is continuous on ∂S_*. Since the $x^{(k)}$ are densely distributed on ∂S_*, from (4.6) we deduce that $W = 0$ on ∂S_*. Then by Theorem 3.6 (ii), (iii), W is a

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

regular solution of the exterior Dirichlet problem

$$\begin{aligned} L(\partial x)W(x) &= 0, \quad x \in S_*^-, \\ W(x) &= 0, \quad x \in \partial S_*, \\ W &\in \mathcal{A}. \end{aligned}$$

Hence, by the uniqueness result for the exterior Dirichlet problem of antiplane micropolar elasticity, $W = 0$ in $\overline{S_*^-}$. The analyticity of W in $\mathbb{R}^2\backslash\partial S$ now yields $W = 0$ in S^-. Letting $S \ni x' \to x \in \partial S$ along the support line $v(x)$, we find that

$$\frac{1}{2}\varphi(x) + \int_{\partial S} P(x,y)\varphi(y)ds(y) = 0 \tag{4.7}$$

for almost all $x \in \partial S$, where the integral is understood as principal value. If $\varphi(x)$ is the solution of (4.7) then $\varphi \in C^{0,\alpha}(\partial S)$ for any $\alpha \in (0,1)$. [10], which implies that

$$\varphi(x) = \sum_{i=1}^{3} \gamma_i F^{(i)}(x) \quad , \quad x \in \partial S, \quad \gamma_i = \text{const} > 0.$$

Then by (4.5),

$$\int_{\partial S} \left[(F^{(l)})^T \sum_{i=1}^{3} \gamma_i F^{(i)} \right] ds = 0, \qquad l = 1,2,3,$$

and we conclude that all the γ_i are zero, that is, $\varphi = 0$. The desired result now follows from Theorem 4.2.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Let u be a regular solution of the interior Neumann problem (N^+). By Remark 3.2

$$u(x) = -\int_{\partial S} P(x,y)\chi(y)ds(y) + G(x), \quad x \in S^+, \tag{4.8}$$

$$G(x) = \int_{\partial S} P(x,y)\chi(y)ds(y), \quad x \in S^-, \tag{4.9}$$

where

$$G(x) = \int_{\partial S} D(x,y)g(y)ds(y), \quad x \in \mathbb{R}^2 \backslash \partial S, \tag{4.10}$$

$$\chi = u\mid_{\partial S}.$$

and $g(y)-$ boundary conditions prescribed on ∂S.

We rearrange the elements of the subset $\{\theta^{(jk)}, j = 1,2,3,\ k = 1,2,...\}$ of (4.1) in the order

$$\theta^{(11)}, \theta^{(21)}, \theta^{(31)}, ..., \theta^{(1k)}, \theta^{(2k)}, \theta^{(3k)}, ...,$$

denote the new sequence by $\{\theta^{(m)}\}_{m=1}^{\infty}$, and use the Gram-Schmidt procedure to construct the orthonormal sequence $\{\eta^{(n)}\}_{n=1}^{\infty}$ in $L^2(\partial S)$. Thus,

$$\eta^{(n)} = \sum_{m=1}^{n} k_{nm}\theta^{(m)}, \quad n = 1,2,..., \tag{4.11}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where k_{nm} are known coefficients of orthonormalization. Also, let $\{\widetilde{F}^{(i)}\}_{i=1}^{3}$ be the orthonormalized set obtained from $\{F^{(i)}\}_{i=1}^{3}$.

We claim that $\left\{\widetilde{F}^{(i)},\ \eta^{(n)},\ i=1,2,3,\ n=1,2,...\right\}$ is a fundamental orthonormal set in $L^2(\partial S)$. To convince ourselves of this we need only to verify that

$$\int_{\partial S}(\widetilde{F}^{(3)})^T\eta^{(n)}ds=0, \qquad i=1,2,3, \qquad n=1,2,...$$

since

$$F^{(1)}=(0,0,0)^T, \qquad F^{(2)}=(0,0,0)^T.$$

But this is obviously true, since the $\widetilde{F}^{(3)}$ and the $\eta^{(n)}$ are finite linear combinations of the $F^{(3)}$ and $\theta^{(jk)}$, respectively, and,

$$\int_{\partial S}(F^{(3)})^T\theta^{(jk)}ds=\int_{\partial S}T_{3l}D_l^{(j)}(x,x^{(k)})ds=\int_{S^+}L_{3l}D_l^{(j)}(x,x^{(k)})da=0.$$

Let now

$$\chi^{(n)}=\widetilde{q}_3\widetilde{F}^{(3)}+\sum_{r=1}^{n}q_r\eta^{(r)}, \tag{4.12}$$

where

$$\widetilde{q}_3=\int_{\partial S}(\widetilde{F}^{(3)})^T\chi ds,$$

$$q_r=\int_{\partial S}(\eta^{(r)})^T\chi ds=\sum_{m=1}^{r}k_{rm}\int_{\partial S}(\theta^{(m)})^T\chi ds, \quad r=1,2,... \tag{4.13}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Setting

$$u^{(n)}(x) = -\int_{\partial S} P(x,y)\chi^{(n)}(y)ds(y) + G(x), \quad x \in S^+, \tag{4.14}$$

and using (4.8), we find that $u^{(n)} \to u$ as $n \to \infty$, uniformly on any closed subdomain $S' \subset S^+$.

From (4.9) it follows that

$$\int_{\partial S} P(x^{(k)}, y)\chi(y)ds(y) = G_j(x^{(k)}), \quad k = 1, 2..., \quad x \in S^-.$$

By (3.12) and (4.2), this is the same as

$$\int_{\partial S} (\theta^{(jk)})^T \chi ds = G_j(x^{(k)}), \quad j = 1, 2, 3, \quad k = 1, 2, ... \tag{4.15}$$

Applying Remark 3.2 to $\widetilde{F}^{(3)}$ in S^+ from (4.14) and (4.12) we now obtain the approximate solution in the form

$$u^{(n)}(x) = \widetilde{q}_3\widetilde{F}^{(3)} - \sum_{r=1}^{n} q_r \int_{\partial S} P(x,y)\eta^{(r)}(y)ds(y) + G(x), \quad x \in S^+,$$

where the first term on the right-hand side is a rigid displacement independent of n , $G(x)$ is given by (4.10), $\eta^{(r)}$ by (4.11), and the q_r are computed by means of (4.13), (4.15) and (4.10). Since the coefficient $\widetilde{q}_3$ cannot be determined in terms of the boundary data of the problem, we conclude that, in agreement with Theorem 3.4 (ii), the exact solution is determined up to an arbitrary rigid displacement.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

It is obvious that

$$\lim_{n\to\infty} u^{(n)}(x) = u(x), \quad x \in S^+,$$

where $u(x)-$ exact solution given by (4.8).

4.2 The Interior Dirichlet Problem

With the notation introduced in the previous section we can prove the following assertion.

4.4. Theorem. *The set*

$$\{F^{(i)}, \vartheta^{(jk)},\ i, j = 1, 2, 3, k = 1, 2, ...\}, \tag{4.16}$$

where the $F^{(i)}$ are defined by (3.8) and

$$\vartheta^{(jk)} = D^{(j)}(x, x^{(k)}), \tag{4.17}$$

is linearly independent on ∂S and fundamental in $L^2(\partial S)$.

Proof: Suppose that there are a positive integer N and real numbers c_i and c_{jk} $i, j = 1, 2, 3, k = 1, 2, ..., N$, not all zero, such that

$$\sum_{i=1}^{3} c_i F^{(i)}(x) + \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\vartheta^{(jk)}(x) = 0, \quad x \in \partial S. \tag{4.18}$$

Setting

$$\varpi(x) = \sum_{i=1}^{3} c_i F^{(i)}(x) + \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\vartheta^{(jk)}(x), \tag{4.19}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

from (4.17), (4.18) and the fact that the columns of $D(x,y)$ and $P(x,y;n)$ are solutions of (3.17) at all $x \in \mathbb{R}^2$, $x \neq y$, for any direction n independent on x , we see that

$$\begin{aligned} L(\partial x)\varpi(x) &= 0, \quad x \in S^+, \\ \varpi(x) &= 0, \quad x \in \partial S, \end{aligned}$$

that is ϖ is a regular solution of the homogeneous Dirichlet problem. By Theorem 3.4 *(i)* $\varpi = 0$ in S^+. Then using analyticity arguments, we deduce that

$$\varpi(x) = 0, \quad x \in S_*^+. \tag{4.20}$$

Let $x^{(p)}$ be any of the points $x^{(1)}, ..., x^{(N)}$. In view of (4.19) and (4.17) , and taking into account that

$$F^{(1)} = (0,0,0)^T, \quad F^{(2)} = (0,0,0)^T$$

we write

$$\varpi_l(x) = c_3 F_l^{(3)}(x) + \sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} D_{jl}(x, x^{(k)})$$

and remark that, according to (3.14), as $x \to x^{(p)}$ all the terms on the right-hand side remain bounded except $c_{lp} D_{ll}(x, x^{(p)})$, which is of order $O(\ln |x - x^{(p)}|)$. This clearly contradicts the equality (4.20), and we conclude that the c_{jk} in

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

(4.18) must be zero. Since the $F^{(i)}$ are linearly independent, we deduce that the c_3 is also zero. Hence the set (4.16) is linearly independent on ∂S.

Now let $\varphi \in L^2(\partial S)$ be such that for all $i, j = 1, 2, 3$ and $k = 1, 2, ...$

$$\int_{\partial S} (F^{(3)})^T \varphi ds = \int_{\partial S} (\vartheta^{(jk)})^T \varphi ds = 0.$$

By (4.17) and (3.11) this is equivalent to

$$\int_{\partial S} D(x^{(k)}, y)\varphi(y)ds(y) = 0, \quad k = 1, 2, ... \tag{4.21}$$

and

$$\int_{\partial S} \varphi_3 ds(y) = 0. \tag{4.22}$$

Consider the elastic single layer potential of density φ

$$V(x) = \int_{\partial S} D(x, y)\varphi(y)ds(y).$$

Since V is continuous on ∂S_* and the points $x^{(k)}$, $k = 1, 2, ...$, are densely distributed on ∂S_*, from (4.21) it follows that $V = 0$ on ∂S_*. Consequently, we have

$$\begin{aligned} L(\partial x)V(x) &= 0, \quad x \in S_*^-, \\ V(x) &= 0, \quad x \in \partial S_*, \\ V &\in \mathcal{A}. \end{aligned}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

This means that V is a regular solution in $\overline{S}_*^-$ of the homogeneous exterior Dirichlet problem (D^-), consequently, by Theorem 3.4 *(i)*, $V = 0$ in $\overline{S}_*^-$. The analyticity of the elastic single layer potential V in $R^2\backslash\partial S$ now implies that

$$V(x) = 0, \quad x \in S^-. \tag{4.23}$$

In turn, this yields $TV = 0$ in S^-. Letting $S^- \ni x' \to x \in \partial S$ along the support line of $v(x)$, we find that

$$-\frac{1}{2}\varphi(x) + \int_{\partial S} T(\partial x)D(x,y)\varphi(y)ds(y) = 0$$

for almost all $x \in \partial S$, where the integral is understood as principal value. If $\varphi \in C^{0,\alpha}(\partial S)$ with any $\alpha \in (0,1)$, then V is continuous in $\mathbb{R}^2$ and

$$\begin{aligned} L(\partial x)V(x) &= 0, \quad x \in S^+, \\ V(x) &= 0, \quad x \in \partial S, \end{aligned}$$

that is, V is a regular solution in $\overline{S}^+$ of the homogeneous problem (D^+). Consequently, by Theorem 3.4 (i) $V = 0$ in $\overline{S}^+$. From this and (4.23) we deduce that $(TV)^+ = (TV)^- = 0$ on ∂S, and $\varphi = 0$.

Since $L^2(\partial S)$ is a Hilbert space, we now apply Theorem 4.2 to conclude that (4.16) is a fundamental set in $L^2(\partial S)$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Let u be the unique regular solution of (D^+). Using Somigliana representation of the solution (Remark 3.2), we can write that

$$u(x) = \int_{\partial S} D(x,y)\psi(y)ds(y) - F(x), \quad x \in S^+, \tag{4.24}$$

and

$$F(x) = \int_{\partial S} D(x,y)\psi(y)ds(y), \quad x \in S^-, \tag{4.25}$$

where we have used the notation

$$F(x) = \int_{\partial S} P(x,y)f(y)ds(y), \quad x \in \mathbb{R}^2 \backslash \partial S, \tag{4.26}$$

$$\psi(y) = (Tu)(y), \quad y \in \partial S.$$

and f - boundary conditions for (D^+) prescribed on ∂S.

The formula (4.25) yields

$$\int_{\partial S} D(x^{(k)}, y)\psi(y)ds(y) = F(x^{(k)}), \quad k = 1, 2, ...,$$

which by (3.11) and (4.17), is equivalent to

$$\int_{\partial S} (\vartheta^{(jk)})^T \psi ds = F_j(x^{(k)}), \quad j = 1, 2, 3, k = 1, 2...$$

We arrange the elements of (4.16) in the order

$$F^{(1)}, F^{(2)}, F^{(3)}, \vartheta^{(11)}, \vartheta^{(21)}, \vartheta^{(31)}, ..., \vartheta^{(1k)}, \vartheta^{(2k)}, \vartheta^{(3k)}, ...,$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

and denote the new sequence by $\{\vartheta^{(m)}\}_{m=1}^{\infty}$. Let $\{\omega^{(n)}\}_{n=1}^{\infty}$ be the orthonormalized fundamental sequence constructed from the set $\{\vartheta^{(m)}\}_{m=1}^{\infty}$ in $L^2(\partial S)$ by means of the Gram-Schmidt orthonormalization. process. Then

$$\omega^{(n)} = \sum_{m=1}^{n} k_{nm}\vartheta^{(m)}, \quad n = 1, 2, ...,$$

where k_{nm} are well-determined coefficients. Writing

$$\psi^{(n)} = \sum_{r=1}^{n} p_r \omega^{(r)}, \quad n = 1, 2, ..., \tag{4.27}$$

with the coefficients on the right-hand side given by

$$p_r = \int_{\partial S} (\omega^{(r)})^T \psi ds = \sum_{m=1}^{r} k_{rm} \int_{\partial S} (\vartheta^{(m)})^T \psi ds, \quad r = 1, 2, ..., \tag{4.28}$$

and setting

$$u^{(n)}(x) = \int_{\partial S} D(x, y)\psi^{(n)}(y) ds(y) - F(x), \quad x \in S^+, \tag{4.29}$$

from (4.24) we see that for $x \in S^+$

$$\begin{aligned} \left|u(x) - u^{(n)}(x)\right| &\leq \sum_{i=1}^{3} \left|u_i(x) - u_i^{(n)}(x)\right| \\ &\leq \sum_{i=1}^{3} \int_{\partial S} \left|(D^{(i)}(y, x))^T[\psi(y) - \psi^{(n)}(y)]\right| ds(y) \\ &\leq \sum_{i=1}^{3} \left\|D^{(i)}(x, \cdot)\right\|_2 \left\|\psi - \psi^{(n)}\right\|_2. \end{aligned}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Since the $\left\|D^{(i)}(x,\cdot)\right\|_2$ are uniformly bounded on any closed subdomain $S' \subset S^+$ and $\left\|\psi - \psi^{(n)}\right\|_2 \to 0$ as $n \to \infty$, we conclude that $u^{(n)} \to u$ uniformly on S'.

Clearly, each $u^{(n)}$ is a solution of the equation $Lu = 0$ in S^+.

4.3 The Exterior Dirichlet Problem

The construction of a fundamental sequence in the space of the solution for exterior problems meets with the usual difficulties that arise from the behavior of the matrices $D(x,y)$ and $P(x,y)$ for $y \in \partial S$ and $|x|$ large. To overcome these obstacles, we need to establish some auxiliary results.

4.5. Theorem. *If S is a finite domain in $\mathbb{R}^2$, X a space of $(3 \times 1)-$matrix functions defined on ∂S, Φ a linear functional on X, and*

$$F(x) = \Phi_y(D(x,y)\varphi(y)), \quad \varphi \in X,\ x \in S^-, \tag{4.30}$$

where subscript y means that Φ operates with respect to y, then $F \in \mathcal{A}$ if and only if

$$p_3 = \Phi_y(\varphi_3(y)) = 0. \tag{4.31}$$

Proof: Introducing polar coordinates (r,θ) of x, for $|x|$ large we write

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

(4.30) in the form

$$F(x) = \Phi_y(D^\infty(x,y)\varphi(y)) + \Phi_y((D(x,y) - D^\infty(x,y))\varphi(y)) = F^\infty(x) + \widetilde{F}(x), \tag{4.32}$$

where

$$D^\infty(x,y) = \begin{pmatrix} 0 & 0 & r^{-1}\sin\theta \\ 0 & 0 & -r\sin\theta \\ 0 & 0 & 2\ln r \end{pmatrix} \tag{4.33}$$

and $(D(x,y) - D^\infty(x,y)) \in \mathcal{A}$.

Consequently, by direct verification we can conclude that $\widetilde{F}(x) \in \mathcal{A}$, which means $F^\infty(x) = 0$ when (4.31) hold.

4.6. Theorem. *For any fixed* $y \in \partial S$, $L(\partial x) = 0$, $x \in S^-$.

Proof: This assertion can be easily proved by direct verification.

Let now the curve ∂S_* be chosen so that it lies strictly inside the domain S^+.

4.7. Theorem. *The set* (4.16) , *constructed as in Theorem* 4.4 *is linearly independent on* ∂S *and fundamental in* $L^2(\partial S)$.

Proof: Suppose that there are a positive integer N and real numbers c_i and c_{jk}, $i.j = 1,2,3$, $k = 1,2,...,N$, not all zero, such that (4.18) holds, and let ϖ again be defined by (4.19). Since $\varpi \in C^1(S_*^-)$, from the expression

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

for the boundary stress operator we immediately see that

$$T(\partial x)\varpi(x) = 0, \quad x \in \partial S. \tag{4.34}$$

Using the representation

$$\varpi = \varpi^{\infty} + \widetilde{\varpi} + c_3 F^{(3)}, \tag{4.35}$$

where the functions ϖ^{∞} and $\widetilde{\varpi}$ are defined by means of the columns of the matrices D^{∞} and $D - D^{\infty}$, respectively, from Theorem 4.6 and Theorem 4.7 and (4.34) we deduce that $\widetilde{\varpi}$ is a regular solution of the exterior Neumann problem

$$\begin{aligned} L(\partial x)\widetilde{\varpi}(x) &= 0, \quad x \in S^{-}, \\ T(\partial x)\widetilde{\varpi}(x) &= -T(\partial x)\varpi^{\infty}(x), \quad x \in \partial S, \\ \widetilde{\varpi}(x) &\in \mathcal{A}. \end{aligned}$$

According to Theorem 3.9 (iv)

$$\int_{\partial S} (F^{(3)})^T T \varpi^{\infty} ds = 0.$$

Consider a circle Γ_R with the centre in the origin and radius R sufficiently large so that $\overline{S}^{+} \subset \Gamma_R$ strictly. By Remark 3.1 (ii) applied to $F^{(3)}$ and ϖ^{∞} in $\Gamma_R \backslash \overline{S}^{+}$, the above equality yields

$$\int_{\partial \Gamma_R} (F^{(3)})^T T \varpi^{\infty} ds = 0.$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Direct calculation now shows that these relations are equivalent to

$$\sum_{k=1}^{N} c_{3k} = 0. \tag{4.36}$$

Let Φ be the linear functional defined on the space X of bounded (3×1)–matrix functions on ∂S_* by

$$\Phi\varphi = \sum_{k=1}^{N} \varphi(x^{(k)}), \quad \varphi \in X,$$

and φ_c an element of X such that

$$\varphi_c(x^{(k)}) = (c_{1k},\ \ c_{2k},\ \ c_{3k})^T, \quad k = 1, 2, ..., N.$$

Then

$$\begin{aligned}
\sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}\vartheta^{(jk)}(x) &= \sum_{j=1}^{3}\sum_{k=1}^{N} c_{jk}D^{(j)}(x, x^{(k)}) \\
&= \sum_{k=1}^{N} D(x, x^{(k)})\varphi_c(x^{(k)}) \\
&= \Phi_y(D(x, y)\varphi_c(y)).
\end{aligned}$$

In view of the definition of Φ (4.36) is equivalent to (4.31), therefore by Theorem 4.5 and (4.35), $\varpi \in \mathcal{A}^*$. From (4.18) and (4.19) we then see that ϖ is the regular solution in S^- of the homogeneous Dirichlet problem

$$\begin{aligned}
L(\partial x)\varpi(x) &= 0, \quad x \in S^-, \\
\varpi(x) &= 0, \quad x \in \partial S, \\
\varpi &\in \mathcal{A}^*.
\end{aligned}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

By Theorem 3.4 (i), $\varpi = 0$ in $\overline{S}^-$. Due to the analyticity of ϖ, we have $\varpi = 0$ in S_*^-, and the linear independence of the set (4.16) on ∂S is established by the argument used in the proof of Theorem 4.4.

Suppose now that the equalities (4.21), (4.22) hold for some $\varphi \in L^2(\partial S)$. Since the points $x^{(k)}$ are densely distributed on ∂S_*, from the fact if $\varphi \in L^2$ on ∂S then $V(\varphi)$ is analytic in $\mathbb{R}^2 \backslash \partial S$ and $LV(\varphi) = 0$ in $\mathbb{R}^2 \backslash \partial S$ as in [10] , we deduce that the elastic single layer potential V of density φ is a regular solution of the homogeneous interior Dirichlet problem

$$\begin{aligned} L(\partial x)V(x) &= 0, \quad x \in S_*^+, \\ V(x) &= 0, \quad x \in \partial S_*. \end{aligned}$$

Hence, by Theorem 3.4 (i) $V = 0$ in $\overline{S}_*^+$. Due to the analyticity of V in $\mathbb{R}^2 \backslash \partial S$, we conclude that $V = 0$ in S^+, so that $TV = 0$ in S^+. Letting $S^+ \ni x' \to x \in \partial S$ along the support line of $v(x)$ we obtain the equation

$$\frac{1}{2}\varphi(x) + \int_{\partial S} T(\partial x)D(x,y)\varphi(y)ds(y) = 0 \tag{4.37}$$

for almost all $x \in \partial S$, the integral being understood as principal value. If $\varphi(x)$ is the solution of (4.37) then $\varphi \in C^{0,\alpha}(\partial S)$, with any $\alpha \in (0,1)$. Hence, $V \in C(R^2)$, which means that $V = 0$ on ∂S.

Now let Φ be the linear functional defined on the space $X = C(\partial S)$ by

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$\Phi\varphi = \int_{\partial S} \varphi ds$. Then the equality to zero of the integral involving the $F^{(3)}$ in (4.21) is equivalent to (4.31), so, by Theorem 4.5, $V \in \mathcal{A}$. Since V is a regular solution of the homogeneous Dirichlet problem

$$\begin{aligned} L(\partial x)V(x) &= 0, \quad x \in S^-, \\ V(x) &= 0, \quad x \in \partial S, \\ V &\in \mathcal{A}, \end{aligned}$$

by Theorem 3.4 (i), $V = 0$ in S^-. This implies that $(TV)^- = 0$, and $\varphi = 0$. As in the proof of Theorem 4.4 , we finally deduce that (4.16) is a fundamental set in $L^2(\partial S)$.

Let u be the unique regular solution of (D^-). By Theorem 3.9 (iv), we can write

$$u = \widetilde{u} + c_3 F^{(3)}, \tag{4.38}$$

where $\widetilde{u} \in \mathcal{A}$ and

$$c_3 = \int_{\partial S} (F^{(3)})^T q ds.$$

where q - boundary conditions for (D^-) prescribed on ∂S.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

By Remark 3.2 (Somigliana formulae) applied to $\widetilde{u}$,

$$\begin{aligned} \widetilde{u}(x) &= -\int_{\partial S} D(x,y)\psi(y)ds(y) + F(x), \quad x \in S^-, \\ F(x) &= \int_{\partial S} D(x,y)\psi(y)ds(y), \quad x \in S^+, \end{aligned}$$

where

$$\begin{aligned} F(x) &= \int_{\partial S} P(x,y)\left[q(y) - c_3 F^{(3)}(y)\right] ds(y), \quad x \in R^2 \backslash \partial S, \quad (4.39) \\ \psi(y) &= (T\widetilde{u})(y), \quad y \in \partial S. \end{aligned}$$

Since, by (4.38) and (4.39), $\widetilde{u}$ is a regular solution of the exterior Neumann problem

$$\begin{aligned} L(\partial x)\widetilde{u}(x) &= 0, \quad x \in S^-, \\ T(\partial x)\widetilde{u}(x) &= \psi(x), \quad x \in \partial S, \\ \widetilde{u} &\in \mathcal{A}, \end{aligned}$$

it follows from Theorem 3.9 (iv) that

$$\int_{\partial S} (F^{(3)})^T \psi ds = 0,$$

which is equivalent to the fact that

$$\int_{\partial S} \psi_3 ds = 0.$$

. This fact allows us to proceed as in Section 4.2 and construct a similar scheme for the approximation of $\widetilde{u}$.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

4.4 The Exterior Neumann problem

Let the curve ∂S_* and the points $x^{(k)}$ be as described in Section.4.1.

4.8. Theorem. *The set*

$$\{\theta^{(jk)}, \quad j = 1, 2, 3, \quad k = 1, 2, ...\}, \tag{4.40}$$

where the $\theta^{(jk)}$ *are defined by* (4.2), *is linearly independent on* ∂S *and fundamental in* $L^2(\partial S)$.

Proof: Suppose that there are a positive integer N and real numbers c_{jk}, $j = 1, 2, 3, \quad k = 1, 2, .., N$, not all zero, such that

$$\sum_{j=1}^{3} \sum_{k=1}^{N} c_{jk} \theta^{(jk)}(x) = 0, \quad x \in \partial S.$$

Representing ϖ defined by (4.4) in the form $\varpi = \varpi^\infty + \tilde{\varpi}$, where ϖ^∞ and $\tilde{\varpi}$ are constructed in terms of D^∞ and $D - D^\infty$, respectively, just as in the proof of Theorem 4.7 (this time with $\varpi \in \mathcal{A}$ we deduce that the set (4.40) is linearly independent on ∂S.

An argument similar to that used in the proof of Theorem 4.1 now shows that if

$$\int_{\partial S} (\theta^{(jk)})^T \varphi ds = 0, \quad j = 1, 2, 3, k = 1, 2, ...,$$

for some $\varphi \in L^2(\partial S)$, then the elastic double layer potential W of density φ satisfies $W = 0$ in S^+. Hence, as $S^+ \ni x' \to x \in \partial S$ along the support line

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

of $v(x)$, we come to the following integral equation

$$-\frac{1}{2}\varphi(x) + \int_{\partial S} P(x,y)\varphi(y)ds(y0 = 0 \tag{4.41}$$

for almost all $x \in \partial S$, where the integral is understood in the sense of principal value. If $\varphi \in L^2(\partial S)$ is a solution of (4.41), then $\varphi \in C^{0,\alpha}(\partial S)$ and we can deduce that $\varphi(x) = 0, \quad x \in \partial S$.

The fact that (4.40) is a fundamental set in $L^2(\partial S)$ now follows from Theorem.4.2.

The generalized Fourier series approximation $u^{(n)}$ of the unique regular solution u of (N^-) is constructed just as in Section 4.1, the procedure being simplified here by the absence of the rigid displacement $F^{(3)}$ from (4.40).

4.5 Summary

In this chapter we have shown that the method of generalized Fourier series can be employed to approximate numerically the analytical solutions of the boundary value problems of anti-plane micropolar elasticity. Semi-analytic solutions in the form of Fourier series can be used for practical purposes in order to obtain quantitative characteristics of the solutions to the corresponding boundary value problems.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

5 Torsion of Cylindrical Beams with Microstructure

The classical linear elasticity problem of the torsion of a prismatic beam is well-documented. However, only a few papers have been devoted to the investigation of the corresponding problem in linear micropolar elasticity, which seeks to incorporate the effect of material microstructure as a factor in predicting the overall deformation of the beam. The relative lack of progress in this area can be attributed to the fact, as we have already shown in Chapter 3, that the treatment of the problem in micropolar elasticity requires the rigorous analysis of a Neumann-type boundary value problem in which the governing equations are a set of three second order coupled partial differential equations for three unknown anti-plane displacement and microrotation fields. This is in contrast to the relatively simple torsion problem arising in classical linear elasticity in which a single anti-plane displacement is found from the solution of a Neumann problem for Laplace's equation.

Initially, the problem of torsion of micropolar elastic beams had been considered by Smith in [91] and Iesan in [38]. However, Smith's analysis is confined to case of a beam with circular cross-section and Iesan's analy-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

sis overlooks certain differentiability requirements required to establish the rigorous solution of the problem (see, for example, [82]).

In this chapter, we show that the problem of torsion of a micropolar beam of (smooth) *arbitrary* cross-section can be reduced to an interior Neumann boundary value problem in anti-plane micropolar elasticity. Using the real boundary integral equation method, we present a detailed and rigorous solution of this problem including the corresponding uniqueness and existence results in the appropriate function spaces. In addition, we use the method of generalized Fourier series to approximate the unknown density, which is the most important part of the analytical solution, in terms of series. As we have already seen in the previous chapter, the method of generalized Fourier series converges very rapidly and gives excellent results when constructing approximate solutions of boundary value problems of anti-plane micropolar elasticity.

Finally, we consider two numerical examples related to torsion of micropolar beams of circular and elliptic cross-sections. We show that in case of the elliptic cross-section, material microstructure has a significant effect on a warping function of a beam.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

5.1 Formulation of the Problem

Let V be a domain in $\mathbb{R}^3$ occupied by a homogeneous and isotropic linearly elastic micropolar material with elastic constants $\lambda, \mu, \alpha, \beta,\ \gamma$ and κ.whose boundary is denoted by ∂V. The deformation of a micropolar elastic solid can be characterized by a displacement field of the form

$$u(x) = (u_1(x), u_2(x), u_3(x))^T$$

and a microrotation field of the form

$$\varphi(x) = (\varphi_1(x), \varphi_2(x), \varphi_3(x))^T,$$

where $x = (x_1, x_2, x_3)$ is a generic point in $\mathbb{R}^3$.The basic relations describing the deformations of a homogeneous and isotropic, linear elastic Cosserat solid are as follows (see Chapter 2).

The equilibrium equations:

$$\sigma_{ji,j} = 0; \qquad \epsilon_{ijk}\sigma_{jk} + \varrho_{ji,j} = 0, \tag{5.1}$$

where σ_{ji} are the components of the stress tensor, ϱ_{ji} are the components of the couple-stress tensor and ϵ_{ijk} is the alternating tensor.

The constitutive relations:

$$\sigma_{ji} = (\mu + \alpha)\varepsilon_{ji} + (\mu - \alpha)\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}, \tag{5.2}$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

where ε_{ji} are the components of the strain tensor, and δ_{ij} is the Kronecker delta.

The kinematic relations

$$\varepsilon_{ji} = u_{i,j} - \epsilon_{kji}\varphi_k : \quad \varkappa_{ji} = \varphi_{i,j}, \tag{5.3}$$

where $\varkappa_{ji}$ are the components of the torsion tensor.

The surface tractions and surface couples acting at point x on the surface ∂V are given by

$$\sigma_i = \sigma_{ji} n_j, \qquad \varrho_i = \varrho_{ji} n_j \tag{5.4}$$

where $n_j = \cos(n_x, x_j)$ and n_x is the unit vector of the outward normal to ∂V at x.

We consider an isotropic, homogeneous, prismatic micropolar beam bounded by plane ends perpendicular to the generators. A typical cross-section S is assumed to be a simply connected region bounded by a closed C^2- curve ∂S. We suppose that body forces and body couples are absent and the lateral surface is free of applied forces and couples. The axis Ox_3 of the coordinate system is directed along the generators of the beam. The beam is assumed to be of length l , and one of its bases is taken to lie in the Ox_1x_2 plane, while the other is in the plane $x_3 = l$. We suppose that the beam is kept in

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

equilibrium when the end $x_3 = l$ is twisted by the couple of magnitude M. On the plane $x_3 = l$ we have the following conditions:

$$\int_{\partial S} \sigma_{31} d\sigma = 0, \quad \int_{\partial S} \sigma_{32} d\sigma = 0, \quad \int_{\partial S} \sigma_{33} d\sigma = 0, \tag{5.5}$$
$$\int_{\partial S} (x_2 \sigma_{33} + \varrho_{31}) d\sigma = 0, \quad \int_{\partial S} (x_1 \sigma_{33} - \varrho_{32}) d\sigma = 0,$$
$$\int_{\partial S} (x_1 \sigma_{32} - x_2 \sigma_{31} + \varrho_{33}) d\sigma = M.$$

The resultant forces and moments calculated across each cross-section satisfy the equilibrium conditions, so that the conditions (5.5) must be satisfied for $x_3 = h$ $(0 \leq h \leq l)$.

On the lateral surface of the beam we have the following conditions

$$\sigma_{\alpha k} n_\alpha = 0, \qquad \varrho_{\alpha k} n_\alpha = 0. \tag{5.6}$$

The torsion problem consists of equations (5.1) - (5.4) along with boundary conditions (5.5) and (5.6).

In order to solve the torsion problem we assume that the displacement and microrotation fields take the form

$$\begin{aligned} u_1 &= -\tau x_2 x_3, \quad u_1 = \tau x_1 x_3, \quad u_3 = \Phi(x_1, x_2), \\ \varphi_1 &= \varphi_1(x_1, x_2), \quad \varphi_2 = \varphi_2(x_1, x_2), \quad \varphi_3 = \tau x_3, \end{aligned} \tag{5.7}$$

where τ is a constant to be determined below (see, for example, [91]).

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

From (5.2) and (5.7) we obtain that

$$
\begin{aligned}
\sigma_{13} &= (\mu+\alpha)\Phi_{,1} + 2\alpha\varphi_2 - (\mu-\alpha)\tau x_2, \\
\sigma_{31} &= (\mu-\alpha)\Phi_{,1} - 2\alpha\varphi_2 - (\mu+\alpha)\tau x_2, \\
\sigma_{23} &= (\mu+\alpha)\Phi_{,2} - 2\alpha\varphi_1 + (\mu-\alpha)\tau x_1, \\
\sigma_{32} &= (\mu-\alpha)\Phi_{,2} + 2\alpha\varphi_1 + (\mu+\alpha)\tau x_1, \\
\sigma_{33} &= 0, \quad \sigma_{\alpha\beta} = 0, \quad \varrho_{3\alpha} = \varrho_{\alpha 3} = 0, \\
\varrho_{11} &= (2\gamma+\beta)\varphi_{1,1} + \beta(\varphi_{2,2} + \tau), \\
\varrho_{22} &= (2\gamma+\beta)\varphi_{2,2} + \beta(\varphi_{1,1} + \tau), \\
\varrho_{12} &= (\gamma+\kappa)\varphi_{1,2} + (\gamma-\kappa)\varphi_{2,1}, \\
\varrho_{3,3} &= \beta(\varphi_{1,1} + \varphi_{2,2}) + (2\gamma+\beta)\tau.
\end{aligned} \tag{5.8}
$$

The equilibrium equations (5.1) are satisfied if the functions $\Phi(x_1, x_2)$, $\varphi_\alpha(x_1, x_2)$ satisfy the following equations

$$LU = q \text{ in } S \tag{5.9}$$

where $L(\xi) = L(\xi_\alpha) = L(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$

$$
= \begin{pmatrix}
(\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_1^2 & (\beta+\gamma-\kappa)\xi_1\xi_2 & 2\alpha\xi_2 \\
(\beta+\gamma-\kappa)\xi_1\xi_2 & (\gamma+\kappa)\Delta - 4\alpha + (\beta+\gamma-\kappa)\xi_2^2 & -2\alpha\xi_1 \\
-2\alpha\xi_2 & 2\alpha\xi_1 & (\mu+\alpha)\Delta
\end{pmatrix},
$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$U = (\varphi_1, \varphi_2, \Phi)^T$$

and

$$q = (2\alpha\tau x_1, \quad 2\alpha\tau x_1, \quad 0)^T$$

>From (5.4) and (5.8) we obtain the following boundary conditions for

(5.9)

$$Tu = g \text{ on } \partial S, \tag{5.10}$$

where T is the boundary stress operator given by

$$T(\xi) = T(\xi_\alpha)$$

$$= \begin{pmatrix} (2\gamma+\beta)\,\xi_1 n_1 + (\gamma+\kappa)\,\xi_2 n_2 & (\gamma-\kappa)\xi_1 n_2 + \beta\xi_2 n_1 & 0 \\ (\gamma-\kappa)\xi_2 n_1 + \beta\xi_1 n_2 & (\gamma+\kappa)\,\xi_1 n_1 + (2\gamma+\beta)\,\xi_2 n_2 & 0 \\ -2\alpha n_2 & 2\alpha n_1 & (\mu+\alpha)\xi_\alpha n_\alpha \end{pmatrix}$$

$n = (n_1, n_2)$ is the unit outward normal to ∂S and

$$g = (-\beta\tau n_1, \quad -\beta\tau n_1, \quad -(\mu-\alpha)\tau(x_1 n_2 - x_2 n_1))^T.$$

If we now introduce the new torsional functions $\phi(x_1, x_2)$, $\psi_1(x_1, x_2)$, $\psi_2(x_1, x_2)$, such that

$$\Phi(x_1, x_2) = \tau\phi(x_1, x_2), \quad \varphi_\alpha(x_1, x_2) = \tau(\psi_\alpha(x_1, x_2) - \frac{1}{2}x_\alpha)$$

and assume that

$$u = (\psi_1, \psi_2, \phi)^T,$$

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

we can reduce (5.9) to the homogeneous system of the form

$$Lu = 0 \text{ in } S, \tag{5.11}$$

with boundary conditions

$$Tu = f \quad \text{on} \quad \partial S, \tag{5.12}$$

where

$$f = (\gamma n_{1,} \ \ \gamma n_{1,} .. \mu(x_2 n_1 - x_1 n_2))^T.$$

It can now be seen that the problem of torsion of a cylindrical micropolar beam can be reduced to the solution of the system (5.11) in the cross-section S together with boundary conditions (5.12) on the boundary ∂S. This is just the interior Neumann boundary value problem of anti-plane micropolar elasticity (see Chapter 3).

As we have already seen in Chapter 3, the interior Neumann boundary value problem can be successfully solved by means of boundary integral equation method. In addition, the results of Chapter 4 provide us with the effective tool of deriving the semi-analytic solution in terms of series that can be very efficient for establishing results for warping functions of micropolar beams of smooth cross-sections.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

5.2 Numerical Examples

5.2.1 Torsion of Circular Micropolar Beams

As an illustration of the generalized Fourier series method presented in detail in Chapter 4, we, first, consider the torsion of a micropolar beam of circular cross-section. Assume that elastic constants take the following values : $\alpha = 3$, $\beta = 6$, $\gamma = 2$, $\kappa = 1$, $\mu = 1$. Note that in physical problems, elastic constants usually take values of order 10^{-5}. See, for example, [48]. The boundary ∂S of the cross-section is the unit circle, consequently, the boundary conditions are

$$Tu = f \quad \text{on} \quad \partial S, \tag{5.13}$$

where

$$f = (\gamma n_1, \quad \gamma n_2, \quad 0)^T. \tag{5.14}$$

Let ∂S_* be the circle concentric with ∂S and of radius equal to 2. We introduce polar coordinates with the pole at the origin and choose the points $x^{(k)}, k = 1, 2, ...$on ∂S_* to be those corresponding to the polar angles

$$0, \ \pi, \ \frac{3}{2}\pi, \ \frac{1}{2}\pi, \ \frac{1}{4}\pi, \ \frac{3}{4}\pi, \ \frac{5}{4}\pi, \ \frac{7}{4}\pi, \ \frac{1}{8}\pi, \ \frac{3}{8}\pi, \ \frac{5}{8}\pi, \ldots \tag{5.15}$$

Obviously, the set $\{x^{(k)}\}_{k=1}^{\infty}$ is densely distributed on ∂S_*.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Using the Gauss quadrature formula with 16 ordinates to evaluate the integrals over ∂S and following the computational procedure discussed in Chapter 4, we obtain the following approximate values for the solution of the torsion problem at points $(0, 0)$, $(0, 0.5)$, $(0.5, 0)$, $(0.5, 0.5)$ (Table 1):

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Table 1: Approximate Solution of Micropolar Beam with Circular Cross-section

		$(0,0)$	$(0,0.5)$	$(0.5,0)$	$(0.5,0.5)$
	ψ_1	3.39473281	3.04975908	2.44337658	3.32006979
$n=4$	ψ_2	3.99386113	3.84179823	2.94193737	2.76777254
	ϕ	3.53006158	3.79429701	3.30103737	2.82091434
	ψ_1	2.29514136	2.18298043	2.36383833	2.24020310
$n=16$	ψ_2	2.41209050	1.57246625	2.16714609	2.20063132
	ϕ	2.61830667	2.23344182	2.87971480	2.18161064
	ψ_1	1.22947516	1.15576328	1.04023136	1.11921972
$n=32$	ψ_2	1.11855121	1.35325409	1.80063130	1.91018001
	ϕ	2.37680259	2.19944723	2.51001199	2.61311426
	ψ_1	0.00000003	0.00000001	0.24200032	0.23260298
$n=55$	ψ_2	0.00000008	0.24200001	0.00000007	0.23260305
	ϕ	1.94851263	1.94851173	1.94851173	1.94851174
	ψ_1	0.00000000	0.00000000	0.24200001	0.23260300
$n=56$	ψ_2	0.00000000	0.24200000	0.00000000	0.23260301
	ϕ	1.94851178	1.94851178	1.94851178	1.94851178

In Table 1, ϕ represents ant-plane displacement and ψ_1 and ψ_2 represent

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

microrotations about axis x_1 and x_2 correspondingly.

Note, that the solution ϕ, which represents the warping function, converges to the constant 1.94851178 throughout the cross-section. This result is consistent with the exact analytical solution obtained by Smith [91] and the experimental results obtained by Lakes [48] and Gauthier [31]. In other words, the circular cross-section remains flat (no warping) despite the fact that the presence of the microstructure in the model of deformation. This result is not surprising given the symmetry of the cross-section. Consequently, we can conclude that for a circular bar the effect of axial symmetry is stronger than the effect of material microstructure. However, as mentioned above, the numerical method outlined in Chapter 4 is applicable to *any smooth* cross-section ∂S, for example, an elliptic cross-section. To demonstrate this fact we will consider another example.

5.2.2 Torsion of Elliptic Micropolar Beams

Consider the torsion of a micropolar beam of elliptical cross-section in which the elastic constants take the following values : $\alpha = 3$, $\beta = 6$, $\gamma = 2$, $\kappa = 1$, $\mu = 1$. The domain S is bounded by the ellipse

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

$$x_1 = \cos t, \quad x_2 = 1.5 \sin t.$$

As an auxiliary contour ∂S_* we take a confocal ellipse

$$x_1 = 1.1 \cos t, \quad x_2 = 1.6 \sin t.$$

Using the Gauss quadrature formula with 16 ordinates to evaluate the integrals over ∂S and following the computational procedure discussed in Chapter 4, the approximate solution is found to converge to eight decimal place accuracy for $n = 62$ terms of the series. Numerical values are presented below for representative points (0, 0) (0.25, 0.25), (0.25, 0.5), (0.5, 0.75) inside the elliptical cross-section (Table 2):

Table 2: Approximate Solution of Micropolar Beam with Elliptic Cross-section with $n = 62$

		$(0,0)$	$(0.25,0.25)$	$(0.5,0.5)$	$(0.5,0.75)$
	ψ_1	0.74431942	1.17355112	1.24343810	1.82784247
$n = 62$	ψ_2	0.48152259	0.97222035	1.11246544	1.36181203
	ϕ	0.00006160	0.02139392	0.08461420	0.12380739

Note, that if we compare the values of the out-of-plane displacement or torsional function ϕ, with those obtained in the case of a classical elastic

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

elliptic beam (these are 0, 0.02403812, 0.09615251, 0.14422874 at the same points - based on the exact solution for the warping function, see for example [93] or [98]), we conclude that there is up to a 15 percent difference at certain points (In addition to the results presented in the above table we also considered several other points lying within the boundaries of the ellipse).

In contrast to the case of a circular micropolar beam for which the cross-section remains flat (as in the classical case), there is a significant difference in the torsional function for an elliptic beam made of micropolar material when compared to the same beam in which the microstructure is ignored (classical case [98]).

This method is easily extended to the analysis of torsion of micropolar beams of any (smooth) cross-section where one again expect a significant contribution from the material microstructure.

5.3 Summary

In this chapter we have formulated the problem of torsion of a micropolar beam of an arbitrary smooth cross-section. We have shown that the problem can be reduced to the interior Neumann boundary-value problem of anti-plane Cosserat elasticity and the exact analytical solution can be obtained

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

in the form of an integral potential using the boundary integral equation method. The analytical solution has been approximated numerically using the method of generalized Fourier series. The semi-analytic solution in terms of Fourier series has been used to evaluate the amount of warping of micropolar beams of circular and elliptic cross-sections. It has been found that in the case of a circular micropolar beam, an arbitrary cross-section by a plane perpendicular to the generators, as in the classical case, shows no warping. In the case of an elliptic micropolar beam, there can be observed up to 15 percent difference in warping in comparison with the classical case.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

6 Conclusions and Recommendations for Future Work

6.1 Conclusions

The present work has been devoted to investigation of the boundary value problems of antiplane Cosserat elasticity. In spite of the fact that the corresponding three-dimensional and plane theories have been carefully investigated the anti-plane theory remained practically untouched until recently. At the same time, the anti-plane theory plays an important role for applications in mechanics, first of all because the theory can be applied to the solution of the torsion problem and secondly to the solution of the problem of wave propagation in the elastic half space, which, of course, requires the dynamical formulation of the governing equations.

This dissertation is confined to the consideration of only the statical problems with emphasis on the torsion problem. As a result of this work the following results have been obtained:

1. The Dirichlet, Neumann, Robin and mixed interior and exterior boundary value problems have been formulated, shown to be well-posed and rigorously solved by means of the boundary integral equation method. The

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

uniqueness and existence theorems have been proved and the exact analytical (unique) solutions have been obtained in the form of corresponding integral potentials.

2. An efficient numerical scheme, based on the modification of generalized Fourier series method has been devised and successfully applied to construction of the approximate solutions to Dirichlet and Neumann interior and exterior boundary value problems. It has been shown that the method works for domains bounded by any smooth arbitrary twice differentiable curve.

3. As an example intended to demonstrate an important application of the proposed theory, the problem of torsion of micropolar beams has been considered. It has been shown that the assumptions imposed on the torsional functions, in the case of a micropolar beam, introduced as a generalization of the Saint-Venant assumptions for the case of a classical beam [53], allow us to reduce the problem to the interior Neumann boundary value problem of antiplane micropolar elasticity. Using the generalized Fourier method we constructed the approximate solution allowing us to make the following important conclusions valuable for applications:

a. In the case of a micropolar beam of circular cross-section, the solution in the form of generalized Fourier series converges rapidly to a constant. This

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

confirms that in the case of a circular beam, because of the axial symmetry, the cross-section of a beam by a plane perpendicular to generators does not warp (remains flat) which is consistent with the analytical results obtained by Smith [91] and the experimental results obtained by Gauthier [31] and Lakes [48].

b. In the case of an elliptic micropolar beam, it has been found that the solution in the form of generalized Fourier series can differ by up to 15 percent from the solution for an elliptic beam composed of a classical material. This points to strong evidence that the material microstructure does have a significant effect on the torsional function in the case of a beam of non-circular cross-section. The fact that there can be up to 15 percent difference in anti-plane displacement between the micropolar and classical case is consistent with the results obtained by Savin [77] and Mindlin [57] for stress-concentration around holes in micropolar media and by Weitsman [105] and Hartranft [34] for stress concentration around inclusions in micropolar media.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

6.2 Future Work

In this dissertation the major boundary value problems of anti-plane micropolar elasticity has been rigorously solved by means of the boundary integral equation method. In order to demonstrate the importance of the work for applications in mechanics we have also illustrated the effectiveness of the theory on the example of torsion of a cylindrical beam with microstructure. We have shown that the unknown density, the most important part of the analytical solution in the form of an integral potential can be successfully approximated by Fourier series since it cannot be found analytically.

The methodology, analytical and numerical technique introduced in this work can be extended for the solution of a wide class of problems of micropolar elasticity dealing with both structures and three-dimensional and two-dimensional boundary value problems.

First of all, as a direct continuation of the work presented in this dissertation, the method of generalized Fourier series can be applied for the derivation of a semi-analytic solution of mixed boundary value problems of antiplane micropolar elasticity. The treatment of mixed problems is different from the treatment of classical ones. In spite of the fact that we have obtained an analytical solution to both interior and exterior mixed problems in

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

the form of integral potentials in this work we found it necessary to postpone the direct derivation of the unknown density in the form of Fourier series for future work. The reason is, that in this dissertation we wanted to make emphasis more on applications of the technique to the torsion problem and in our opinion if we had introduced the numerical solution to mixed problems it would have obscured the significant results related to interior Neumann (torsion) problem of antiplane micropolar elasticity.

As a second step, it would be possible and at the same time important, to incorporate thermoelastic components into the model. Kupradze [48] has formulated fundamental boundary value problems of three-dimensional thermoelasticity and shown that they can be solved in a rigorous manner using the boundary integral equation method. Same technique, in our opinion, could be applied for the investigation of two-dimensional problems of micropolar thermoelasticity. In addition, the method can be applied for analysis of thermoelastic deformations of plates and shells.

The third direction of the future work could find some applications in the area of composite mechanics. Since we have already considered the classical boundary value problems of anti-plane micropolar elasticity, the next - very challenging step could be an investigation of contact (inclusion, trans-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

mission) boundary value problems. Several authors, for example [8] or [89], have lately considered the problems of stress concentrations around rigid inclusions of different shapes in the case of the three-dimensional theory of micropolar elasticity. Meanwhile, plane transmission problems of Cosserat elasticity remain practically untouched until today. In a very recent paper by Lubarda [54], the problem of stress concentration around rigid inclusion in micropolar media in case of anti-plane was solved under assumptions of the simplified theory of elasticity with couple-stresses. The same methodology and assumptions were used by Weitsmann [105] and Hartranft [34] for solving the problem of stress concentration around rigid inclusions in micropolar media in case of plain strain. Investigation of these problems in a general case, under assumptions when microrotations and displacements are independent is much more complicated and challenging. Unfortunately, consideration of these problems, because of a very complicated nature of the boundary conditions, leads us to completely unknown types of integral equations in terms of complex variables. Hence, the boundary integral equation method, as it was presented in this work, cannot be applied for the treatment of inclusion problems of anti-plane micropolar elasticity, therefore, alternative methods of solution of these problems should be used. The first suggestion is to ex-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

press all the stresses and couple-stresses in terms of two elastic potentials (stress functions). After this, the stress functions can be expressed in terms of complex analytical functions (the treatment of some problems of classical elasticity by means of complex variable method can be found in [63]). The difficulty arising here is that in order to satisfy all the boundary conditions we have to solve a system of algebraic equations of an extremely complicated structure, which requires tremendous efforts to be solved analytically. At the same time the numerical treatment of this system cannot give us a positive answer about the influence of microstructure on stress concentrations around a rigid inclusion.

The alternative approach is to formulate the transmission problem in Sobolev spaces. This method has been recently used for the investigation of stress concentration around rigid inclusions in plates by Chudinovich and Constanda [9]. In all likelihood, this boundary integral equation technique must work also for the problems in micropolar media. The method has wide practical applicability because it also covers domains with reduced boundary smoothness.

There could be other approaches to solve the outlined problems. In any case, this is the objective of the future work, as a result of which, we will defi-

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

nitely obtain a deeper understanding of the effects of material microstructure on deformations, stress concentration factors and elastic behavior of fiber-reinforced composite that will be very helpful for applications in the area of structural mechanics and modern day advanced composite materials.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

References

1. M. Abramowitz, I. Stegun, *Handbook of mathematical functions,* Dover, New York, 1964

2. E. Aero, E. Kuvshinski, Continuum theory of asymmetric elasticity, *Soviet Solid State Phys, 2, (*1964), 1399-1408

3. A. Anthoine, Effect of couple-stresses on the elastic bending of beams, *Int. J. Solids Structures,* **37**, (2000), 1003-1018

4. T. Ariman, Some problems in bending of micropolar plates, *Bull. Pol. Acad. Sci,* ***XVI****,* (1963)*, 295-299*

5. C. Banks, M. Sokolowski, On certain two-dimensional applications of the couple-stress theory, *Int. J. Solids Structures*, **4**, 15-29

6. F. Bouyge, I. Jasiuk, M. Ostoja-Starzewski, A micromechanically based couple-stress model of an elastic two-phase composite, *Int. J. Solids Structures*, **38**, (2001), 1721-1735

7. D. Carlson, Stress functions for plane problems with couple-stresses, *ZAMP*, **17**, (1966), 789-792

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

8. Z. Cheng, L. He, Micropolar elastic fields due to a circular cylindrical inclusion, *Int. J. Engng. Sci.*, **33**, (1995), 389-397

9. I. Chudinovich, C. Constanda, *Variational and potential methods in the theory of bending of plates with transverse shear deformation,* Chapman & Hall/CRC, Boca Raton, London, New York, Washington, 2002

10. C. Constanda, *A mathematical analysis of bending of plates with transverse shear deformation,* Longman Scientific & Technical, Harlow, 1990

11. C. Constanda, Some comments on the integration of certain systems of partial differential equations in continuum mechanics, *J. Appl. Math. Phys.* **29** (1978), 835 - 839

12. E. Cosserat, F. Cosserat, Sur la theorie de l'elasticite, *Ann. de l'Ecole Normale de Toulouse, (*1896), **10, 1, 1**

13. E. Cosserat, F. Cosserat, Sur la mecanique generale, *C. Rend. hebd. des seances del'Acad. des Sci. Paris, (*1907), **145,** 1139

14. E. Cosserat, F. Cosserat, *Theorie des corpes deformables,* A. Herman et Fils, Paris, 1909

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

15. S. Cowin, Stress functions for Cosserat elasticity, *Int. J. Solids Structures*, **6**, (1970), 389-398

16. S. Cowin, An incorrect inequality in micropolar elasticity theory, *ZAMP*, **21**, (1970), 494-497

17. C. Dafermos, On the existence of asymptotic stability of solutions to the equations of thermoelasticity, *Arch. Rat. Mech. Anal.*, **29**, (1968), 34-63

18. R. De Borst, E. van der Giessen, *Material instabilities in solids*, John Wiley, Chichester, 1998

19. S. De Cicco, L. Nappa, On Saint-Venant's principle for micropolar viscoelastic bodies *Int. J. Engng. Sci.,* **37**, (1999), 883-893

20. M. Dokmeci, Theory of micropolar plates and shells, *Recent Advances in Engineering Science* (edited by A.C. Eringen), **Vol.5**, (1970), 189-207

21. J. Dyszlewicz, Stress functions for a second axially-symmetric problem of micropolar elastostatics. *Bull. Acad. Polon. Sci. Ser. Sci. techn.*, **22,** (1974) 847-856

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

22. R. Ellis, C. Smith, A thin plate analysis and experimental evaluation of couple stress effects, *Experimental Mech.*, **7**, (1968), 372-380

23. A.C. Eringen, *Microcontinuum field theories I: Foundations and solids,* Berlin, Springer-Verlag, 1999

24. A.C. Eringen, Linear theory of micropolar elasticity, *J. Math. Mech.*, **15**, (1966) 909 - 923

25. A.C. Eringen, Theory of micropolar plates, *J. Appl. Math. Phys.* **18**, (1967) 12-30

26. A.C. Eringen, *Continuum Physics,* vol. 4. New York: Academic Press, 1976

27. A.C. Eringen, E. Suhubi, Non-linear theory of simple microelastic solids, *Int. J. Engng. Sci.,* **2**, (1964), 389-404

28. G. Fichera, Sul problema della derivata obliqua e sul problema misto per l'equazione di Laplace. *Boll. Un. Mat. Ital. Ser. III*, **7**: 367-377, 1952

29. G. Fichera, Existence theorems in elasticity, *Handbuch der Physik, VIa/2,* Springer-Verlag, 1972

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

30. G. Fichera, Boundary value problems of elasticity with unilateral constraints, *Handbuch der Physik, VIa/2,* Springer-Verlag, 1972

31. R.D. Gauthier, W.E. Jahsman, A quest for micropolar elastic constants, *J.Appl.Mech.(ASME),* **6**, (1975), 369-374

32. G. Grioli, Elasticita asimmetrica, *Ann. Mat. Pura Appl.* (ser 4) **50**, (1960), 389-417

33. A.C. Gunther, Zur Statik und Kinematik des Cosseratschen Kontinuums. *Abh. Braunschweig Wiss. Ges.*, **10**, (1958), 195-213

34. R. Hartranft, G. Sih, The effect of couple -stresses on the stress concentration of a circular inclusion, *J. Appl. Mech.,* **32**, (1965), 429-431

35. C.O. Horgan, Anti-plane shear deformations in linear and nonlinear solid mechanics, *SIAM Review*, **37**, (1995) 53 - 81

36. D. Iesan, On the linear theory of micropolar elasticity, *Int. J. Engng. Sci,* **9**, (1969), 1213-1223

37. D. Iesan, Existence theorems in the theory of micropolar elasticity, *Int. J. Engng. Sci.* **8** (1970) 777-791

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

38. D. Iesan, Torsion of micropolar elastic beams, *Int. J. Engng. Sci.* **9** (1971) 1047-1060

39. D. Iesan, L. Nappa, Saint-Venant's problem for microstretch elastic solids, *Int. J. Engng. Sci.,* **32**, (1994), 229-236

40. S. Itou, The effect of couple-stresses on stress concentration around a rigid circular cylinder in a strip under tension, *Acta Mechanica,* **27**, (1977), 261-268

41. P. Kaloni, T. Ariman, Stress concentration effects in micropolar elasticity, *ZAMP,* **18**, (1968), 136-145

42. S. Kessel, Die Spannungsfunktionen des Cosserat-Kontinuums, *ZAMM,* **47**, (1967), 329-337

43. G. Kluge, Uber den Zusammenhang der allgemeinen Versetzungstheorie mit dem Cosserat-Kontinuum, *ZAMM,* **13**, (1969), 377-380

44. W. Koiter, Couple-stresses in the theory of elasticity, *Koninkl. Nederl. Akad. van Wetenschappen.* Proc. ser. B, I-(1964), **67**, 1-17

45. A. Kolmogorov, S. Fomin, *Elements of the theory of functions and functional analysis*, Rochester, NY, Graylock Press, 1957.

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

46. V.D. Kupradze et al, *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity,* North-Holland, Amsterdam, 1979

47. V.D. Kupradze, *Potential methods in the theory of elasticity,* Israel Program for Scientific Translations, Jerusalem, 1965

48. R. Lakes, Experimental methods for study of Cosserat elastic solids and other generalized elastic continua, in Continuum Models for Materials with Microstructure (edited by H.B. Muhlhaus), John Wiley and Sons, 1995

49. R. Lakes, Experimental micro mechanics methods for conventional and negative Poisson's ratio cellular solids as Cosserat continua, *J. Engng. Materials and Tech.,* **113**, (1991), 148-155

50. R. Lakes, Foam structures with a negative Poisson's ratio, *Science,* **235,** (1987) 1038-1040

51. R. Lakes, Experimental microelasticity of two porous solids, *Int. J. Solids Structures*, **22** (1986), 55-63

52. R. Lakes, Dynamical study of couple stress effects in human compact

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

bone, *J. Biomedical Engng.*, **104**, (1982), 6-11

53. A.E.H. Love, *A treatise on the mathematical theory of elasticity*, Dover Publication, New York, 1944

54. V. Lubarda, Circular inclusions in anti-plane strain couple stress elasticity, *Int. J. Solids Structures*, (2002), (under review)

55. A. Lur'e, *Three-dimensional problems of the theory of elasticity*, New York, Interscience Publishers, 1964

56. S. Mikhlin, *Linear equations of mathematical physics*, New York, Holt, Rinehart and Winston, 1967

57. R. Mindlin, Effect of couple stresses on stress concentrations, *Experimental Mech.*, **3**, (1963), 1-7

58. R. Mindlin, Microstructure in linear elasticity, *Arch. Rat. Mech. Anal.*, **16**, 51-78

59. R. Mindlin, Stress functions for a Cosserat continuum, *Int. J. Solids Structures*, **3**, (1965), 265-285

60. R. Mindlin, H. Tiersten, Effects of couple-stresses in linear elasticity, *Arch. Rational Mech. Anal.*, **11**, (1963), 415-448

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

61. C. Miranda, *Partial differential equations of elliptic type,* Springer-Verlag, Berlin, 1970

62. N. Muskhelishvili, *Singular integral equations*, Noordhoff, Groningen, 1953

63. N. Muskhelishvili, *Some basic problems in the mathematical theory of elasticity,* Noordhoff, Groningen, 1949

64. W. Nowacki, *Theory of asymmetric elasticity,* Polish Scientific Publishers, Warszawa, 1986

65. W. Nowacki, The "second" plane problem of micropolar elasticity, *Bull. Acad. Polon. Sci. Ser. Sci. techn.*, **18,** (1970) 899-906

66. M. Ostoja-Starzewski, I. Jasiuk, Stress invariance in planar Cosserat elasticity, *Proc. R. Soc. London, A,* **451**, (1992) 453-470

67. V. Palmov, Fundamental equations of the theory of asymmetric elasticity (in Russian), *Prikl. Math. Mech,* **28**, (1964), 411-422

68. V. Palmov, Plane problems of the theory of asymmetric elasticity (in Russian), *Prikl. Math. Mech,* **28**, (1964), 1117-1129

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

69. L. Payne, B. Straughan, Order of convergence estimates of the interaction term for a micropolar fluid, *Int. J. Engng. Sci,* **7**, (1989), 837-846

70. L. Payne, B. Straughan, Critical Rayleigh numbers for oscillatory and nonlinear convection in an isotropic thermomicropolar fluid, *Int. J. Engng. Sci,* **7**, (1989), 827-836

71. S. Potapenko, P.Schiavone & A.Mioduchowski, Antiplane shear deformations in a linear theory of elasticity with microstructure, *ZAMP.* (acccepted for publication)

72. S. Potapenko, P.Schiavone & A.Mioduchowski, On the solution of mixed problems in antiplane micropolar elasticity, *Math. Mech. Solids (*accepted for publication*)*

73. S. Potapenko, P. Schiavone, A. Mioduchowski, On the solution of torsion of cylindrical beams with microstructure, (under review)

74. S. Potapenko, P. Schiavone, A. Mioduchowski, Generalized Fourier series solution of torsion of an elliptic beam with microstructure, *Appl. Math. Letters* (accepted for publication)

75. Y. Povstenko, Stress functions for continua with couple-stresses, *J.*

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

Elasticity, **36** : (1994), 99-116

76. N. Sandru, On some problems of the linear theory of the asymmetric elasticity, *Int. J. Engng. Sci.*, **4**, 1966, 81-100

77. G. Savin, *Foundations of plane problems of the theory of elasticity with couple-stress (*in Russian), Kiev, 1965

78. H. Schaeffer, Das Cosserat-Kontinuum, *ZAMM*, **47**, 1967, 487-499

79. H. Schaefer, Analysis der Motorfelder im Cosserat-Kontinuum, *ZAMM*, **47**, (1967), 319-328

80. P. Schiavone, On existence theorems in the theory of extensional motions of thin micropolar plates, *Int. J. Engng. Sci.* **27** (1989), 1129 - 1133

81. P. Schiavone, A generalized Fourier approximation in micropolar elasticity, *Z. angew. Math. Phys.* **40** (1989), 839 - 845

82. P. Schiavone, Fundamental sequences of functions in the approximation of solutions to exterior problems in the bending of thin micropolar plates, *Appl. Anal.* **35** (1990), 263 - 274

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

83. P. Schiavone, Generalized Fourier series for exterior problems in extensional motions of thin micropolar plates, *Int. J. Engng. Sci.* **28** (1990), 1067 - 1072

84. P. Schiavone, Uniqueness in dynamic problems of thin micropolar plates, *Appl. Math. Lett.* 4 (1991), 81 - 83

85. P. Schiavone, Integral equation methods in plane asymmetric elasticity, *Journal of Elasticity*, **43** (1996) 31 - 43

86. P. Schiavone and C. Constanda, Existence theorems in the theory of thin micropolar plates, *Int. J. Engng. Sci.* **27** (1989), 463 - 468

87. P. Schiavone and C. Constanda, Uniqueness in the elastostatic problem of bending of thin micropolar plates, *Arch. Mech.* **41**, 5 (1989), 785 - 791

88. P. Schiavone and C.Q. Ru, On the exterior mixed problem in plane elasticity, *Math. Mech. Solids* **1** (1996) 335 - 341

89. P. Sharma, A. Dasgupta, Eshelby's problem for a cuboidal inclusion in a micropolar medium, *Int. J. Solids Structures*, (to appear)

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

90. V. Smirnov, *A course of higher mathematics*, Pergamon Press, Oxford, 1964

91. A.C. Smith, Torsion and vibrations of cylinders of a micro-polar elastic solid, *Recent Advances in Engineering Science* (edited by A.C. Eringen), Vol.5, (1970), 129-137

92. A. C. Smith, Deformations of micropolar elastic solids, *Int. J. Engng. Sci,* ***16****, 1967, 637-649*

93. I. Sokolnikoff, *Mathematical theory of elasticity,* McGraw Hill, New York-Toronto-London, 1956

94. M. Sokolowski, Couple-stresses in Problems of torsion of prismatic bars, *Bull. Acad. Polon. Sci., Ser. Sci. Tech.,* **16**, (1968), 547-554

95. E. Sternberg, R. Muki, The influence of couple-stresses on singular stress concentrations in elastic bodies, *ZAMP*, **16**, (1965), 611-648

96. E. Sternberg, R. Muki, The effect of couple-stresses on the stress concentration around a crack, *Int. J. Solids Structures*, **3**, (1967), 69-95

97. A. Tikhonov, A. Samarskii, *Equations of mathematical physics,* New York, Macmillann, 1963

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

98. S. Timoshenko, J. Goodier, *Theory of elasticity*, New York, McGraw-Hill, 1970

99. R. Toupin, Elastic materials with couple-stresses, *Arch. Rat. Mech. Anal.,* **17**, 1964, 101-134

100. R. Toupin, Perfectly elastic materials with couple-stresses, *Arch. Rat. Mech. Anal.*, **11**, (1962), 385-414

101. C. Truesdell, R. Toupin, The classical field theories, *Encyclopedia of Physics III/1*, secs. 200, 203, 205, Berlin- Gottingen-Heidelberg, Springer Verlag, 1960

102. N.P. Vekua, *Systems of singular integral equations*, Noordhoff, Groningen 1967.

103. W. Voigt, Theoretische Studien uber die Elastizitatsverhaltnisse der Krystalle, I, II, *Abh. Der Konigl. Ges. Der Wiss.,* Gottingen, 1887

104. F. Wang, On the solutions of Eringen's micropolar plate equations and of other approximate equations, *Int. J. Engng. Sci.*, **28**, (1992), 919-925

105. Y. Weitsmann, Couple-stress effects on stress concentration around a

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.

cylindrical inclusion in a field of uniaxial tension, *J. Appl. Mech* , **6**, (1965), 424-427

106. W.L. Wendland, E. Stephan and G.C. Hsiao, On the integral equation method for the plane mixed boundary value problem of the laplacian, *Math. Mech. Appl. Sci.* **1,**

(1979), 265 - 321.

107. A. Yavari, S. Sarkani, E. Moyer, On fractal cracks in micropolar elastic solids, *J. Appl. Mech.*, **69**, (2002), 45-54

Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.