Observing High-school Students' Mathematical Understanding and Mathematical Proficiency in the Context of Mathematical Modeling

BY

Priscila Marques Dias Corrêa

A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Department of Secondary Education

University of Alberta

ABSTRACT

The use of mathematical modeling in education has been investigated for the last five decades. The benefits of bringing modeling into mathematics classes are well known and well accepted, and modeling is becoming more common and more appealing to mathematics teachers. However, there are still unanswered questions and conjectures to be explored, so as to aid and encourage mathematics teaching through modeling. The present study uses classroom-based research to explore the use of modeling tasks within high-school mathematics classes, in order to provide insight into the teaching of mathematics for understanding. In this study, participants' were engaged in mathematical modeling tasks in which they were required to develop models for mathematical situations, instead of using an already known mathematical model or a given one. This investigation intended to comprehend what forms of mathematical understanding and mathematical proficiency are observed and how they are expressed when high-school students are engaged in this mathematical modeling setting.

The research methodology is founded on design-based research, since it combines theoretical research knowledge with practical experiences, yielding practical knowledge (The Design-Based Research Collective, 2003). The classroom design framework is based on complexity science underpinnings. This is due to the fact that mathematics classes are acknowledged as complex systems, in which students collectively act and interact in order to develop, construct and enhance their mathematical ideas. These actions and interactions are believed to be non-linear, spontaneous and self-organized, characterizing a complex system that allows mathematical understanding to emerge (Davis & Simmt, 2003). In order to investigate students' mathematical understanding and proficiency while engaged in mathematical modeling tasks, four different tasks were proposed to a high-school class taking grade 11 mathematics. The class was composed of 27 students. Although all of them participated in the tasks, data was collected from the 12 students who provided consent. Tasks were applied during a four-month Alberta mathematics course. Audio and video recordings, students' mathematics journals and researcher field notes were collected. Post class sessions, students were invited to participate in recall interviews.

Assuming that students' mathematical understanding is encompassed by students' mathematical proficiency, data analysis was conducted using Kilpatrick, Swafford and Findell's (2001) model of mathematical proficiency, where mathematical proficiency is composed by five strands, namely: *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition*. The basis of the research data analysis framework consists of identifying indicators of each of Kilpatrick et al.'s strands in students' work, and then investigating how students undergo these strands along the modeling tasks.

This research study offers insight into the use of mathematics modeling by: 1) portraying how mathematical modeling tasks foster high-school students' mathematical understanding and proficiency; and 2) showing the feasibility of implementing this kind of task in mathematics classes with time and curriculum constraints. The study revealed that students demonstrate mathematical understanding and proficiency during the course of the modeling tasks, even when they do not come to full resolutions of problems. Research outcomes indicate that mathematical modeling tasks promote students' mathematical understanding and proficiency and can be an important approach in the task of teaching for understanding.

PREFACE

This research is an original study by Priscila Marques Dias Corrêa. However parts of chapters 2, 3 and 4 of this thesis have been published in P. D. Corrêa, 2015, "Teaching mathematics for understanding: Approaching and observing, *Delta-k - Journal of the Mathematics Council of the Alberta Teachers' Association*, vol. 52, number 2, 37-43.

The research project, under the name "High School Students' Mathematical Understanding When Engaged in Modeling Tasks", and its respective data collection, received ethics approval from the University of Alberta Research Ethics Board, on November 12th, 2014, and from the Edmonton Public Schools Ethics Board, on January 8th, 2015.

ACKNOWLEDGMENTS

I would like to thank Dr Elaine Simmt, my supervisor, who welcomed me at the University of Alberta and gave me and my family inestimable assistance. Dr Simmt was always there with thoughtful advice on my work, encouragement in diverse situations, confidence in my capacity, and support on my personal issues. I am very grateful to have her as my supervisor; I am sure I could not have a better choice. I wish to thank Dr Lynn McGarvey and Dr Gerda de Vries, from my supervisory committee. I appreciate the dedication they had to read my work, provide me with attentive comments and suggestions, and spend time in meetings and discussions about my research. I thank Dr Norma Nocente and Dr Rose Mary Zbiek who accepted the invitation to participate on my examining committee, giving me the opportunity to have their relevant feedback regarding my work. I must thank also Dr Florence Glanfield, who was always of great encouragement and help whenever I needed during my doctoral studies. I am glad Dr Glanfield was committed with my examining committee as well, offering me meaningful reflections and suggestions about my work.

I am forever grateful to the teacher, who accepted me and my ideas in his classroom, and gave me full support to develop my research with his students. I definitely could not make it without his belief in my work. I am also thankful to his students (and their parents) who agreed to participate in my research and consented access to their data. Their involvement was fundamental for this study to happen.

I am indebted with the Department of Secondary Education and with the University of Alberta who granted me financial support throughout my doctoral program. I also acknowledge the considerable financial assistance received from CAPES Foundation (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), process number 0318-12-4. By means of this funding I was able to focus on my learning and on my research goals.

I wish to thank my parents and my sister — Ernani, Celeste and Cristina — who supported my plans to move to another hemisphere and came to visit my family and I several times, lessening the distance. My parents help was of huge relevance, in particular, in the last year of my program. They came for a long visit during the summer, and aided me with whatever I needed, while I could focus and work on my final document. I am sure I would not be able to finish within my deadline without their help.

I must acknowledge my family — Leonardo, Natália, Gabriela and Daniel — for their remarkable support. Leo got on board with me in this program since the day he did not hesitate and accepted to abdicate on everything to move to another country with our family. Natália and Gabriela had no idea of what to expect, but they greatly faced a completely new life, learned a new language from scratch, and adapted their lives because of me. They three renounced on family time, and endeavoured to understand my needs, so that I could pursue my doctoral studies. I will never forget what they did; that is of immeasurable value. Last by not least, Daniel, who came in the middle of the program with no understanding at all of what was happening, and even though was extremely supportive. A quiet and calm baby that changed my routine, but brought cheerfulness to our home, and allowed me to keep on doing and finish my work. My deepest gratitude and appreciation for you four.

Finally, I thank God, who is silently guiding my steps and blessing my life.

TABLE OF CONTENTS

List of Tables x

List of Figures xi

1. A possibility of change 1

2. Mathematical modeling: A path to be experienced 8

2.1. Exploring the notion of mathematical modeling 8

2.1.1. Mathematical modeling as “vehicle” or as “content” 12

2.1.2. Developing mathematical models in contrast to using mathematical models 13

2.1.3. Mathematical modeling in contrast to modeling mathematics 14

2.1.4. Modeling cycle, type and competencies 15

2.2. Relevance and advantages of mathematical modeling 16

2.3. Mathematical modeling as an approach for teaching for understanding 18

2.4. Mathematical modeling and its challenges 20

2.5. Mathematical modeling and other learning approaches 25

2.5.1. Problem solving 25

2.5.2. Problem-based learning 27

2.5.3. Case-based, project-based and inquiry-based learning 29

2.5.4. Realistic mathematics 31

2.5.5. Mathematics applications 32

2.6. Mathematical modeling in the school curriculum 35

3. Framing the question: Theoretically and methodologically 43

3.1. Theoretical framework 43

3.1.1. What is mathematical understanding? 43

3.1.2. How to investigate mathematical understanding? 45

3.1.3. Why a mathematical proficiency model? 49

3.1.4. Data analysis framework 50

3.2. Methodological framework 52

3.2.1. Why design-based research? 52

3.2.2. Why complexity science? 55
3.2.3. Classroom design framework 58
4. Approaching the question: The intervention 61
4.1. Participants 61
4.2. Intervention 63
4.3. Data collection methods 67
4.3.1. Audio and video recordings 67
4.3.2. Field notes 67
4.3.3. Mathematics journals 68
4.3.4. Stimulated recall interviews 68
4.4. Modeling tasks 70
4.4.1. Task one: Cost, revenue and profit 75
4.4.2. Task two: Flight simulators 79
4.4.3. Task three: Distance medley relay 83
4.4.4. Task four: Binary search 87
5. Unpacking the question: Data analysis 93
5.1. Data processing 93
5.2. Diagram-based approach and structure 94
5.3. Participants' selection for diagram building 95
5.4. Participants' diagrams and descriptions 96
5.4.1. Amanda's diagram for task one – Cost, revenue and profit 97
5.4.2. Jack's diagram for task two – Flight simulators 99
5.4.3. Leo's diagram for task three – Distance medley relay 102
5.4.4. Clara's diagram for task four – Binary search 106
6. Answering the question: Findings and conjectures 109
6.1. Observed ways of expressing mathematical understanding and proficiency 109
6.2. Observed behaviours of mathematical proficiency strands 120
6.3. Mathematical modeling affordances and constraints on students' mathematical understanding and proficiency 124

6.4. Mathematical modeling as a feasible approach in mathematics classes 128

7. Teaching for understanding and proficiency through modeling: Some considerations 131

Bibliography 137

Appendix A: Nathalie's diagram for task one – Cost, revenue and profit 149

Appendix B: Philip's diagram for task one – Cost, revenue and profit 152

Appendix C: Sophia's diagram for task one – Cost, revenue and profit 155

Appendix D: Clara's diagram for task two – Flight simulators 158

Appendix E: Sophia's diagram for task two – Flight simulators 161

Appendix F: Thomas's diagram for task two – Flight simulators 164

Appendix G: Amanda's diagram for task three – Distance medley relay 167

Appendix H: Philip's diagram for task three – Distance medley relay 170

Appendix I: Rick's diagram for task three – Distance medley relay 173

Appendix J: Leo's diagram for task four – Binary search 176

Appendix K: Rick's diagram for task four – Binary search 179

Appendix L: Thomas's diagram for task four – Binary search 183

LIST OF TABLES

Table 1: Data analysis strands of mathematical proficiency and their respective indicators. 51

Table 2: Formed groups for each task. 62

Table 3: Engagement factors and influence percentages in high-level thinking tasks........................ 73

Table 4: Diagram distribution per participants, tasks and groups. 96

Table 5: Mathematical modeling affordances and respective promoted strands of mathematical proficiency. 125

List of Figures

Figure 1: An overview of modeling features, advantages and challenges. 24

Figure 2: Mathematical modeling and other learning approaches. 35

Figure 3: Framework schema for the classroom design. 60

Figure 4: Intervention cycle. 64

Figure 5: Task one – Cost, revenue and profit. 76

Figure 6: Task two – Flight simulators. 80

Figure 7: Possible triangular representation for task two. 81

Figure 8: Task three – Distance medley relay. 84

Figure 9: Task four – Binary search. 88

Figure 10: Diagram structure. 95

Figure 11: Amanda's diagram for task one. 98

Figure 12: Jack's diagram for task two. 101

Figure 13: Leo's diagram for task three. 105

Figure 14: Clara's diagram for task four. 107

Figure 15: Forms of expressing conceptual understanding. 111

Figure 16: Forms of expressing strategic competence. 113

Figure 17: Forms of expressing procedural fluency. 115

Figure 18: Forms of expressing adaptive reasoning. 117

Figure 19: Forms of expressing productive disposition. 119

Figure 20: Graph of weighted strands over investigation phases. 122

1. A POSSIBILITY OF CHANGE

Students develop their sense of what it means to "do mathematics" from their actual experiences with mathematics, and their primary opportunities to experience mathematics as a discipline are seated in the classroom activities in which they engage. (Henningsen & Stein, 1997, p. 525)

Teaching mathematics can be demanding for different reasons. As a mathematics teacher, I am constantly challenged to make my classes interesting, provoking and inspiring for my students. This is not an easy task, in particular because – as many current mathematics teachers – I come from an education based on rote learning, in which the main goal is to have students reproducing content, no matter if they make sense of it or not. In this context, teaching mathematics for meaning making can be a tricky job for a teacher like me. However, I believe in the potential of a mathematics class to engage students, and to help them develop understanding and knowledge. Hence, my research aims to investigate promising mathematics classroom settings and approaches, which have the potential to support the teaching of mathematics for understanding. This investigation intends to comprehend how mathematical understanding and proficiency are observed and expressed within these settings and approaches.

As a mathematics teacher for more than 10 years, I inevitably had to face and deal with students that failed in mathematics[1]. These students were primarily those who did not develop mathematical skills such as numeracy and reasoning skills, which could benefit them in their daily lives. This is the kind of student that I am most worried about. Their struggle with mathematics resonates with me and encourages me to try harder in my mathematics classes, in order to make their experience with mathematics worthwhile. In this context, I have been wondering about how to help these students to engage in mathematics classes, think mathematically, improve their mathematical skills and finally build a strong and long-lasting mathematical knowledge.

My experience leads me to believe that guiding students through the usefulness of mathematical content can be one of the steps to change the current taken-for-granted scenario of students struggling with mathematics. Today, in general, students are more inquiry-focused than students from my generation. Interacting with today’s students, I have

[1] Although my experience as a high-school teacher is in the Brazilian environment, I believe that my reflections and concerns are applicable in both Brazilian and Canadian high-school settings.

come to believe that many of them do not want to study mathematics just because they have to; they want to know the purpose of studying mathematical content and they want also to understand why studying mathematics is important for them. I agree with them. Hirsch and McDuffie (2016) report teachers' sensitivity to students' perceptions of the lack of usefulness and relevance in mathematics. Indeed, if students do not have answers for their questions about the purpose of studying mathematics, they do not make sense of the process of learning mathematics and, as a consequence, they feel unmotivated and do not commit themselves in mathematics classes. From my teaching practice, I understand that one of the ways of answering students' questions about mathematics relevance and usefulness is by: 1) analyzing pertinent contextualized situations in class; 2) requiring students to develop mathematical models which represent these situations; and 3) prompting students to learn and investigate new mathematical contents that are useful within this process.

Indeed, there is a tendency of bringing mathematical models into mathematics classes, which can be confirmed by the amount of work related to mathematical modeling that has been done up to date (Biembengut, 2009). I believe that if teachers bring these modeling experiences to their classes, they will be able to create a more engaging and motivating environment in which students study mathematics. This may have positive effects in students' long-term mathematical understanding and skills, leading to more engaged participation in society. Therefore, there is relevance in researching students' mathematical understanding when engaged in modeling practices. Consistent with that, Lesh, Young and Fennewald (2013) underline that "[t]he models and ideas of teachers, who are themselves developing models of students modeling, are critically important to understand and investigate if modeling research is to truly impact student learning" (p. 281). Further, Doerr (2016) reinforces that modeling is a relevant aid in observing and understanding students' thinking and reasoning, which are directly related to students' understanding.

From this perspective, the purpose of this research is to investigate students' mathematical understanding and proficiency in a teaching setting that involves modeling experiences. I contend that modeling approaches potentially engage students and trigger their interest in relation to mathematics, given that these approaches are usually related to likely-to-happen contextualized scenarios. Lamon (2003) states that "children are intensely motivated when they are immersed in the problem" (p. 447). As a result, mathematics teaching and learning processes are enhanced, and students better develop the mathematical proficiency they are expected to have. Students also build on their ability to deal with everyday situations in which mathematical thinking and reasoning benefit them. Having said that, I formulate the question that guides my investigation as: ***How are mathematical understanding and mathematical proficiency observed and expressed in the actions and interactions of students when engaged in mathematical modeling approaches in a high-school mathematics class?*** To better explore this question, a secondary question can be asked: ***What are the affordances and constraints on students' mathematical understanding and on students' mathematical proficiency through modeling approaches?***

The concern about the relationship between instructional approaches and students' understanding in mathematics classes is not something new. Silver, Mesa, Morris, Star and Benken (2009) state that "[a]lthough reformers have disagreed on many issues, there is a widely shared concern for enhancing opportunities for students to learn mathematics with understanding and thus a strong interest in promoting teaching mathematics for understanding" (p.503). Nonetheless, while it is clear that current teachers are concerned about teaching mathematics for understanding, they seem not to know exactly where to start and how to do that. This may be one of the reasons why changes in mathematics classes do not appear to be happening. Even though teachers are trying to contextualize mathematics, use different strategies and involve students in their classes, mathematics teaching is still mainly based on old instructional strategies, and founded on procedural knowledge and instrumental understanding (Skemp, 1976/2006). Indeed, Silver et al. (2009) affirm that

> [w]ith amazingly little variation across teachers and over time, research has found that mathematics instruction and instructional tasks tend to emphasize low-level rather than high-level cognitive processes (...), require students to work alone and in silence (...), focus attention on a narrow band of mathematics content (...), and do little to help students develop a deep understanding of mathematical ideas (p.502).

In addition to this difficulty to innovate in mathematics classes, teachers' commitment with a tight curriculum — which is perceived as one they must cover — is also a complicating factor. Dealing with time constraints is another huge challenge. As a consequence, teachers keep teaching in a more procedural and instrumental way that — apparently — is less time consuming and more successful in terms of covering the curriculum. Purpel (2004) highlights that "[s]chools take on the mantle of science and rationality even when they absurdly and crudely distort their essences" (p.69). In other words, because of allegedly time and curriculum issues, the essence of mathematics might not be worked out in class and students might not understand why they are doing mathematics or why they benefit by studying mathematics. Further, procedural and instrumental ways of teaching are unsuccessful in terms of promoting thoughtful inquiry and reasoning abilities, which are fundamental features of mathematics essence. Purpel draws attention to the fact that "[u]rging students to work hard and to do well in areas in which they have little or no interest or ability is a way of encouraging mindless, instrumental behavior" (p.67). That is the reason why I believe mathematics is understood by many learners as a matter of memorizing procedures. Based on my experience as a mathematics teacher, I conjecture that although things appear to be changing in terms of teaching for understanding, there is still a lot to be changed within mathematics classes, especially at the high-school level. This research serves the purpose of offering reasons and strategies for this change.

Change is especially relevant in a moment in which facts are being "normalized" (Jickling, 2013). Based on my experience, I can tell that different normalization processes are happening in many different circumstances. For example, the instrumental way that mathematics education is conducted in high-school classes is already normalized, as if teaching for the sake of covering the curriculum was enough. Many teachers are used to that and keep doing the same. Most parents do not complain as long as their children are able to enter college or the university, as usual. It is all about good marks; the normal expectation is to get good marks, even when good marks do not necessarily represent that students have learned something. As normalized, this way of teaching mathematics does not bother individuals, at least not enough to provoke changes. Society is evolving in such a way that what is important is to succeed in a professional career, regardless of whether you can or cannot use the knowledge you learnt at high-school. It is all about products (such as high grades) and not about processes. I agree with Ben-Hur (2006) in that it should be the opposite. Foerster (2003) complements this discussion introducing the concept of trivialization. He refers to trivialization as the process in which an expected output is returned with a given input; characterizing deterministic events (p. 208), that is, events in which results are known beforehand. The author explains that "trivialization is a dangerous panacea when man applies it to himself" (p. 208). An example of this jeopardy is when the teaching and learning process is trivialized, and students are induced to give the answers the teacher wants to hear (pp. 208-209). In these terms, all the potential that students bring with them goes to waste. These two processes of normalization and trivialization, observed in the educational system, suggest that change is needed and new possibilities are needed too.

Seidel (2006) points out that

> the possibilities for our work with children in schools seem to narrow. (...) Our work will never be enough, our own embodied and experiential wisdom is disregarded, and there will always be another project, another method, another wave of inadequacy and another reform program sold to schools. We feel less freedom to make decisions about children's learning and our teaching (p.1902).

In fact, it is hard to figure out new possibilities that challenge and change old entrenched ways of teaching and learning. Nevertheless, I believe there are possible ways of changing that are worth trying. In this particular research, I argue that one possible way of changing is by modeling contextualized situations that involve mathematics in high-school classes. I believe this approach can be the starting point of an interesting and effective mathematics class. It can be practical, instigating and enlightening. In this research, I investigate students' mathematical understanding and proficiency when submitted to this approach. I do so in order to assess if this form of practice is a possibility for the teaching of mathematics that positively influences students' mathematical learning processes. In accordance with this thought, Lesh and Fennewald (2013) affirm that "[m]odeling is not so much a view of how learning works as it is a methodological approach and framework for investigating learning" (p.8). I wonder if a modeling approach can effectively address learning issues such as those listed below.

- Lack of usefulness of mathematics (Hirsch & McDuffie, 2016).
- Mathematics teaching based on low level cognitive processes that does little to help with students' understanding (Silver, Mesa, Morris, Star, & Benken, 2009).
- Normalization of instrumental teaching and learning processes (Jickling, 2013).
- Focus on products (Ben-Hur, 2006).
- Trivialization of teaching and learning processes, in which students are induced to come up with a determined output given an input (Foerster, 2003).

As already noted, my first effort to make mathematics classes an effective learning environment is motivated by the desire of enhancing my students learning experiences, enabling them to make informed decisions and better face the world outside of school. However, as a mathematics teacher, also committed to the training and professional development of mathematics teachers, my goals go beyond my individual students' improvement. I am also interested in the possibility of developing a different method for mathematics teaching, enabling student-teachers and also skilful practicing teachers to take part in this experience. I understand that this appears to be a way to benefit students from an individual perspective and to benefit society from a whole perspective.

In order to better comprehend this research, Chapter 2 addresses the notion of mathematical modeling, its advantages and challenges. When exploring the notion of mathematical modeling, Chapter 2 highlights the difference between mathematical modeling as "vehicle" or as "content", between *using* and *developing* models, and between mathematical modeling and modeling mathematics. It goes further and analyzes mathematical modeling as a teaching approach to teach for mathematical understanding. This connection with mathematical understanding is pertinent, given that students' mathematical understanding is at stake in this research. A whole section in Chapter 2 refers to other mathematics learning approaches. This section compares these approaches with the mathematical modeling one, emphasizing differences and similarities between them. Finally, Chapter 2 speaks to previous work about modeling in the school mathematics curriculum.

Following, Chapter 3 reviews three different theories that support this research theoretically and methodologically. The chapter details Kilpatrick, Swafford and Findell's (2001) theory of mathematical proficiency, which serves as the foundation of the framework for this research data analysis. As this framework assists the analysis of students' mathematical understanding, this chapter also clarifies what is meant by mathematical understanding and how it can be analyzed in this research context. Moreover, this chapter outlines and explains the used mathematical proficiency framework. Next, Chapter 3 describes the research methodology, namely research-based design, and the theory used to frame the research classroom design, namely, complexity science. The chapter describes what complexity science is and how it relates to educational settings and to this study. Chapter 3 also delineates and explains the implemented classroom design framework.

Chapter 4 is intended to tease out how this study approaches the research question. The goal is to give a clear notion of what has been done prior to and during data collection. The chapter begins by addressing research participants' profile, and then the chapter fully describes the research intervention. All four data collection methods are outlined. The four modeling tasks — created exclusively for this research purpose — are presented and explained as well. A possible solution for each task is detailed, just in the same way that doable prompts for students are suggested. Finally, Chapter 4 briefly speaks to how students and/or groups performed in each task.

Chapter 5 focuses on the research data analysis. As Chapter 3 will depict, this research data analysis is framed by a theory of mathematical proficiency. However, apart from using this theory as the framework underpinnings, a diagram-based approach was used to organize the data, in order to better understand and investigate it. Hence, Chapter 5 starts off by describing the data processing procedures and the diagram-based approach, explaining how it connects to the data analysis framework. Selection of participants for diagram building is also addressed. Finally, four diagrams, concerning four different students and each of the four implemented tasks, are displayed and described, with the aim of elucidating how data analysis was handled.

Chapter 6 speaks to research findings and conjectures. The chapter answers the two research questions and gives extra insight into other issues related to teaching mathematics through modeling. In sum: 1) modeling is claimed to be a comprehensive and worthwhile way of investing in high-school students' mathematical understanding and proficiency; 2) understanding and proficiency behaviours while students are modeling are discussed; 3) modeling is argued to be a viable way of teaching high-school mathematics; and 4) modeling affordances and constraints on students' mathematical understanding and proficiency are discussed.

Ultimately, Chapter 7 presents some considerations regarding the research relevance and suitability, its challenges, outcomes, possible improvements and future work.

2. Mathematical modeling: A path to be experienced

Beyond investigating students' understanding and proficiency while modeling, this research offers a possibility of change. A change in the way mathematics classes are commonly held and a change in the way students' experience mathematics learning. I argue that the richness and the strength of the use of modeling in high-school classes can lead to an enhanced learning process with fruitful outcomes. Mathematics teachers, professors and educational researchers are invited to think of mathematical modeling as a path that needs to be experienced. I invite them to promote encounters with modeling in their classes, in order to sense the benefits that will emerge. I invite them to reflect about the necessity of change in mathematics teaching and learning practices, and about the relevance of going through modeling experiences while mathematical learners.

This chapter is composed of six sections that explore aspects of mathematical modeling. The first section starts by exploring the notion of mathematical modeling; it defines mathematical modeling, explains what mathematical modeling processes involve, gives details about terms or expressions that might be encountered in modeling contexts, and specifies what this research focus is in terms of modeling. The second section speaks to the relevance and to the advantages of modeling to the education realm, both in the teacher's perspective and in the student's perspective. The third section discusses teaching for understanding, and presents mathematical modeling as an option for this way of teaching. The fourth section investigates the challenges mathematical modeling can entail, mainly for teachers, but also for students. Then the fifth section describes and compares different learning approaches with modeling, mentioning in what ways they are similar and in what ways they diverge. Finally, the sixth section speaks to mathematical modeling in the school curriculum context, describing what is been done in different perspectives and contexts.

2.1. Exploring the notion of mathematical modeling

The term mathematical modeling has been used in areas such as engineering and economic sciences since the beginning of the last century (Biembengut, 2009). In mathematics education this term can be found in some American research dating from the late 1950s, and the discussion around this theme increased in the international mathematics education community in the 1960s (Biembengut, 2009). Burkhardt's (2006) work gives an overview of the modeling developments from the 1960s on. Behind the idea of bringing mathematical modeling to educational settings, there was a concern of teaching mathematics in a way that would make its utility become apparent (Biembengut, 2009). This is still one of the goals that guides mathematical modeling into mathematics education nowadays. Although this attempt to incorporate mathematical modeling into educational settings is more than 50 years old, there is a lot to be developed and assimilated in order to have mathematical modeling in secondary school mathematics classes in an effective and worthwhile way.

Lesh (2007) points out mathematical modeling as one of the mathematics areas that ought to have attention. He affirms that "modeling abilities are among the most important proficiencies students need to develop" (p.158). Lesh and Fennewald (2013) also highlight that "research on models and modeling [are] especially important for mathematics education researchers" (p.8). Mathematical modeling is being discussed in different parts of the world and is being recognized as relevant not only for students and education researchers, but also for teachers who can use it as a tool for teaching mathematics. But "*what exactly is mathematical modeling?"* Before researching about mathematical modeling as a possibility of change in the teaching of mathematics, this question needs to be explored.

The *International Community of Teachers of Mathematical Modeling and Applications* (ICTMA) is organized by the *International Commission on Mathematics Instruction* (ICMI) study group, and gathers research on mathematical modeling in different areas of mathematics education. It is interesting to notice that different perspectives and views of modeling can be found within this community. Cirillo, Pelesko, Felton-Koestler and Rubel (2016) explain that an agreed-upon definition for *mathematical modeling* cannot be found, instead there are descriptions, definitions or assumptions made by single authors. Lesh and Fennewald (2013) agree that there is not an established definition of what *modeling* is. However, they assert that there is an agreement that model-development research analyzes relevant contextualized decision-making problems, tending to be more like an engineering approach, in which "realistic solutions" are developed to "realistically complex problems" (p.6). According to Lesh and Doerr (2000), a *model* is a system that encompasses elements, its relationships, the operations that describe elements interactions and the patterns that rule relationships and operations. In addition, they claim that a model is only a model, if its system is somehow related to another system, either by describing it, thinking about it, making sense of it, explaining it, or predicting about it. In a simpler way, Dym (2004) defines a *mathematical model* as "a representation in mathematical terms of the behaviour of real devices and objects" (p. 4). He emphasizes that there are different ways to formulate this representation, which can be in writing, through pictures or prototypes, or using formulas or programs. Viecili (2006) defines *mathematical modeling* as the process that connects a problem situation to a mathematical model. Blum and Ferri (2009) define *mathematical modeling* as a translation of an ordinary situation into mathematical terms or vice-versa. Based on these underpinnings, for this research study, *mathematical modeling* is considered as the ingenious process of developing a *mathematical model*, that is, a mathematical representation that expresses how a situation, object or process runs. It is relevant to note that, although my definition is in tune with other mentioned definitions, I want to emphasize that my understanding of mathematical modeling necessarily includes the *development* of a mathematical model.

Henry Pollak (Teachers College Columbia University, 2012) is very helpful in explaining what mathematical modeling is in practical terms.

> Whether the problem is huge or little, the process of "interaction" between the mathematics and the real world is the same: the real situation usually has so many facets that you can't take everything into account, so you decide which aspects are most important and keep those. At this point, you have an idealized version of the real-world situation, which you can then translate into mathematical terms. What do you have now? A mathematical model of the idealized question. You then apply your mathematical instincts and knowledge to the model, and gain interesting insights, examples, approximations, theorems, and algorithms. You translate all this back into the real-world situation, and you hope to have a theory for the idealized question. But you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is what is called mathematical modeling. (p. viii)

The modeling process itself can vary from a very simple to an extremely complex one, as well as the necessary modeling ability and the involved content knowledge. A mathematical modeling process encompasses mathematical thinking, mathematical reasoning and mathematical knowledge. As so, modeling tasks are considered high-order thinking tasks. Bassanezi (1994) brings up a very important aspect about mathematical modeling. He asserts that working with mathematical models in the teaching and learning processes is not only about expanding students' knowledge, it is overall a matter of structuring the way students think and act. Therefore, mathematical modeling endorses students as agents of change (Viecili, 2006). Cirillo et al. (2016) also draw attention to some relevant features of mathematical modeling, which are helpful in characterizing modeling, they are: 1) the connection with ordinary situations; 2) the ill-defined nature; 3) the necessity of a creative modeler who is able to make assumptions, choices and decisions; 4) the recursive behaviour given that the modeler needs to constantly confront the model and the phenomena to validate the model; and 5) the non-unique or non-strict nature since the modeler can chose from multiple paths and get to different solutions. Lesh and Doerr (2003) add to this description, explaining that modeling usually comprises mathematizing, which requires modelers to quantify, dimension, coordinate, categorize, algebraate, and systematize. All these features together accentuate the richness of mathematical modeling approaches. I argue that mathematical modeling is a potential and valid way of doing and learning mathematics.

Bleiler-Baxter, Barlow and Stephens (2016) use a framework that helps in grasping the idea of mathematical modeling used in this research. This framework is based on the Common Core State Standards for Mathematics — CCSSM (National Governors Association; Council of Chief State School Officers, 2010) description for modeling with mathematics, and assumes that in modeling students are expected to make decisions related to problem simplifications, map variables relationships, and do problem analysis. These features are present when students need to develop models. Dym (2004) adds that a mathematical modeling approach follows some principles. He enumerates some questions that help understanding these principles and, as a consequence, help understanding what students are expected to do during a modeling task. The questions are: 1) "What are we looking for?" 2) "What do we want to know?" 3) "What do we know?" 4) "What can we assume?" 5) "How should we look at this model?" 6) "What will our model predict?" 7) "Are the predictions valid?" 8) "Are the predictions good?" 9) "Can we improve the model?" 10) "How will we exercise the model?" (p. 6 and 7).

This research mathematical modeling approach refers to classes triggered by modeling situations, which require students to work on all six CCSSM (National Governors Association; Council of Chief State School Officers, 2010) modeling actions — *identifying*, *formulating*, *performing*, *interpreting*, *validating* and *reporting* — so that students can work on their mathematical thinking and reasoning skills, and their content knowledge. The main idea is to have students analyzing contextualized situations related to circumstances that they might face within their lives outside school. When dealing with interesting, challenging and motivating situations, students are allegedly more curious about the mathematics that is behind the investigated situation. From there, students collectively explore the given situations, figure out what kind of mathematics they need to model it, look for the necessary mathematical content (what can imply in learning and producing new knowledge), *model* the problem, and finally come up with their conclusions and/or conjectures. Undergoing this approach, students engage and experience rich mathematical thinking and learning processes, building on their mathematical understanding and proficiency.

To be more specific and to clarify what is being named as mathematical modeling, I present an example. In order to have high-school students experience and investigate mathematical modeling in class, I have brought to students' attention the operation of a simulator of movements. This simulator consists of a platform that holds up to 5 tonnes and can make 6 different sorts of movements (Corrêa, Sandres, Nedjah, & Mourelle, 2012). It can be translated up and down, left and right and forward and backward. In addition, it can be rotated along three different axes (x, y and z). This simulator can be used, for example, in flight simulators and in movie recordings. Cars, ships or planes can be put above the platform and the desired movements will be done accordingly. But how can this idea be used in a mathematics class? A film or a simulation with the platform movements can be shown to students; pictures that elucidate its operation can be displayed; examples in which the simulator can be used may be presented; or even a field trip in which students can try a flight simulator can be arranged. It might be also possible to have a mini prototype of the platform. After this initial moment, the idea is to have students figuring out a strategy to actually *model* the platform operation by themselves. The task could be simply that: create a model to make the platform execute the six different desired movements. No extra information is given at first. Students are encouraged to think mathematically, conjecture, try different ideas and make their own conclusions. The platform can be modeled at least by one model that uses vectors, translations, rotations and matrices. This content is part of the high-school curriculum in Brazil and can be generated or taught while the modeling task is being done if that is the case. Other modeling possibilities can emerge during students' investigation as well, and these different options should be acknowledged by the teacher. This is an important step in supporting students throughout modeling processes.

2.1.1. Mathematical modeling as "vehicle" or as "content"

Apart from interpreting the notion of mathematical modeling, it is also relevant to differentiate ways of employing modeling into the classroom; that is, the notion of modeling as "vehicle" or the notion of modeling as "content". Galbraith (2011) explains that when modeling is approached as "vehicle" the focus is on using modeling to learn other curricular contents. In this context, modeling features and processes are means of enhancing the learning of mathematical concepts and contents. On the other hand, Galbraith explains that when modeling is approached as "content" the focus is on learning how to model; in other words, the goal is to develop abilities to investigate and solve life-like problems by means of models. For this research purpose, modeling is used as "vehicle" following Zbiek and Conner (2006).

> The primary goal of including mathematical modeling activities in students' mathematics experiences within our schools typically is to provide an alternative — and supposedly engaging — setting in which students learn mathematics without the primary goal of becoming proficient modelers. We refer to the mathematics to be learned in these classrooms as "curricular mathematics" to emphasize that this mathematics is the mathematics valued in these schools and does not include mathematical modeling as an explicit area of study. (p. 89)

2.1.2. DEVELOPING MATHEMATICAL MODELS IN CONTRAST TO USING MATHEMATICAL MODELS

This research study modeling approach is such that students are expected to learn and/or produce new content by modeling a situation, rather than applying newly learnt content to a modeling situation, as a reinforcement or practice activity. In other words, in the research classroom observed for this study, students were expected to *develop* a mathematical model, engage in the reasoning process and notice the relevance and need of new content. This approach contrasts with the situation in which a pre-given mathematical model is *used*. As a result, in the latter situation students are not taking part in the reasoning process demanded by developing mathematical models, possibly preventing students from the creative work that might lead to positive mathematical experiences and to the expansion of their mathematical knowledge, understanding and proficiency. This difference between *using* mathematical models and *developing* mathematical models can be ratified when looking into Bloom's taxonomy (Bloom, 1956) categories: *remembering*, *understanding*, *applying*, *analyzing*, *evaluating* and *creating*. The *use* of models can be classified under *applying*, while *developing* mathematical models can be classified under *creating*.

Similarly, based on Meyer's (2015) work, Cavey and Champion (2016) assert that "most textbook 'modeling' problems engage students in *using* [emphasis added] a given model, not in *developing* [emphasis added] their own model or thinking about how to make improvements" (p. 132). By challenging students to participate in this whole model development process, they are given the opportunity to think by themselves, make their own conjectures, and elaborate their own strategies, among other relevant abilities. This is different than when students are required just to *use* a model given some input values. Consistent with this thought, Foerster (2003) affirms that:

> We seem to be brought up in a world seen through description by others rather than through our own perceptions. This has the consequence that instead of using language as a tool with which to express thoughts and experience, we accept language as a tool that determines our thought and experience (p. 202).

Any confusion about mathematical modeling (specifically between *using* models and *developing* models) might be due to challenges faced by teachers (more on these challenges in section 2.4). According to Silver et al.'s (2009) research sample, teachers would design tasks that "involved hands-on activities or real-world contexts and technology but rarely required students to provide explanations or demonstrate mathematical reasoning" (p.501). Besides, as mentioned by Stillman (2001), teachers are encouraged to assess students' abilities in *using* models, rather than their abilities in *developing* models. I wonder if mathematical modeling is being mistaken as contextualized problem solving (more on problem solving in sub-section 2.5.1). It could be that teachers are introducing contextualized situations by means of problems that *use* models assuming that they are implementing mathematical modeling. Similarly, Zbiek and Conner (2006) assert that "many modeling tasks in our schooling context are fundamentally applied problems in disguise and are presented to use existing mathematical knowledge rather than to evoke new mathematical knowledge" (p. 100).

2.1.3. MATHEMATICAL MODELING IN CONTRAST TO MODELING MATHEMATICS

It is also necessary, for the conceptualization of this research study, to distinguish the use of the term *mathematical modeling* from the use of the term *modeling mathematics*, because the lack of distinction between these two different terms might lead to misunderstandings. Cirillo et al. (2016) clarify this difference, explaining that *modeling mathematics* refers to the use of manipulatives to model mathematics, that is, the use of concrete materials to represent a mathematical concept, a mathematical relation, or other mathematical situations. For instance, the use of a balance scale to represent an equation, the use of pizza pieces to represent fractions, the use of algebra tiles to represent notable products, these are all examples of *modeling mathematics*. On the other hand, as already described, *mathematics modeling* refers to the use of mathematics to model an ordinary situation, object or process. Cirillo et al. emphasize that the process of *modeling mathematics* derives from the mathematical world, given that the intention is to create a model for a mathematical concept. While the process of *mathematical modeling* derives from outside the mathematical world, given that the intention is to create a mathematical model for an ordinary situation.

I wonder if *modeling mathematics* is being mistaken as *mathematical modeling*. Although the word model is present in both terms, these terms refer to different methods and cannot be used interchangeably. This is clear in Viecili's work. In my perspective, although the author has scathing arguments related to *mathematical modeling* in her dissertation, in her research tasks, she is doing *modeling mathematics* instead. Based on Gould's work, Henry Pollak (2015) discusses some teachers' conceptions about models and modeling that confirm this confusion or misunderstanding between mathematical modeling and other approaches. These teachers believe that: 1) concrete manipulatives or concrete objects can be mathematical models; 2) modeling processes do not always require decision making; 3) modeling does not require validation; and 4) modeling scenarios can be unrealistic ones (Gould, 2013). These conceptions clearly show the difference between what these teachers are doing in class and modeling as defined in my work.

2.1.4. MODELING CYCLE, TYPE AND COMPETENCIES

As a final point, it is worth mentioning some specificities of mathematical modeling, such as *modeling cycle*, *modeling type*, and *modeling competencies*. There are different diagrams that try to translate what a *mathematical modeling cycle* is, that is, the stages a modeler undergoes when modeling. Blum and Ferri (2009), for example, present a diagram which notes seven different stages in a modeling cycle, namely: 1) understanding the task; 2) simplifying/structuring; 3) mathematizing; 4) working mathematically; 5) interpreting; 6) validating; and 7) presenting. In a similar perspective, the Common Core State Standards for Mathematics — CCSSM (National Governors Association; Council of Chief State School Officers, 2010) describes the modeling process as follows: 1) problem and variable analysis; 2) model formulations; 3) computations; 4) results interpretation; 5) model validation; and 6) findings report. Even though modeling cycles have their relevance in modeling research, they are not the focus of this research. Therefore, I am not exploring this topic further. As well, I am not concerned with *mathematical modeling competencies and sub-competencies*, which are usually related to modeling cycle stages. Some researchers, such as Grünewald (2013), focus on investigating the development of modeling competencies while engaged in modeling tasks, but this is not the case for the current research. The same holds true in relation to the different *types of mathematical models*, namely: empirical, mechanistic, deterministic and stochastic (Groshong, 2016). That is, models driven by data, models based on theoretical principles, models that ignore randomness and models that include randomness, respectively. In the current research, the main concern is about the richness of the whole modeling process as a way of enhancing students' mathematical understanding and proficiency, without drawing attention to the modeling cycle, the modeling competencies, or the modeling type.

2.2. RELEVANCE AND ADVANTAGES OF MATHEMATICAL MODELING

Mathematical modeling can be used as a tool for investigating different facts, occurrences, phenomena, projects or experiments. It allows a deeper comprehension of the studied object, leading to predictions, growth or improvements. In accordance with that, Cirillo et al. (2016) assert that "[t]he process of mathematical modeling is intended to help the modeler understand or predict something about the real world and to develop theories and explanations that provide insight and understanding of the original real-world situation" (p. 8). Dym (2004) adds that scientific mathematical modeling is used to model phenomena based on observations, being a helpful tool for future predictions. This is true for forecast prediction, for example. Dym also stresses that models are important to prove if a new project or product is worth implementing or producing. These examples of the use of mathematical modeling portray what mathematics is useful for and, as a result, what is the relevance in doing modeling. Hereupon, mathematical modeling is acknowledged as a relevant tool for the scientific realm. However, mathematical modeling is also relevant for secondary mathematics education, either as a teaching resource, or as a supportive learning approach, as described below.

As a teaching resource, the use of mathematical modeling in class presents advantages to classroom teachers (Biembengut & Hein, 2013). According to Biembengut and Hein (2013), teachers have the opportunity to: 1) research and create their own classroom materials; 2) get a better grasp of the content knowledge involved in the modeling task; 3) implement tasks interacting with other disciplines; 4) be creative and critical in the model development and validation; 5) be more aware of students' work and struggles; and 6) review their assessment resources and criteria. Part of what makes the modeling process implementation a difficult process, also helps teachers evolve as teachers.

When modeling is implemented in high-school settings — apart from drawing attention to mathematics usefulness — mathematical modeling presents other considerable advantages for students. As a high-order thinking activity, modeling prompts students to investigate, analyze, inquiry, reason, model, solve, conjecture, and validate. By working on these abilities, students potentially enhance their mathematical understanding and their mathematical proficiency. Blum and Ferri (2009) point out the advantages of mathematical modeling at the secondary level. They assert that mathematical modeling: 1) helps students in understanding the world; 2) supports students' mathematical learning in that it improves motivation, conceptualization, comprehension, and retention; 3) improves mathematical skills and decision making; and 4) assists in proper framing of mathematics. Biembengut (2009) lists four other different advantages of using mathematical modeling in education; they are: 1) the development of cognitive processes and mental models; 2) the visibility of mathematical applicability and usefulness; 3) the possibility of teaching students how to do research (once the modeling process is a research process per se); and 4) the development of knowledge. Other authors mention benefits of mathematical modeling too. Viecili (2006), for example, stresses that mathematical modeling: 1) motivates students; 2) facilitates students' learning; 3) develops students' skills; 4) gets students ready to use mathematics in diverse areas; and 5) assists students' comprehension of mathematics social-cultural role. Hirsch and McDuffie (2016) indicate that modeling motivates students, given that it shows mathematics applicability, and provides opportunities to integrate other subject-matter and mathematics. Hirsch and McDuffie add that doing modeling at the school level get students ready to modeling in professional work (p. x).

When looking into the relevance and advantages of mathematical modeling from the students' perspective, it is interesting to bring up practical research using mathematical modeling, so as to highlight realistic aspects of this approach. In a three year study, Boaler (2001) compared two groups of students: 1) those who studied from a more conservative teaching approach (over three years); and 2) those who studied under this same conservative approach for two years, and then under a problem solving and mathematical modeling approach during the last year. After the three years, students' knowledge acquisition was compared. Boaler concluded that pedagogical practices impacted in students' knowledge development; in other words, she concluded that knowledge production is related to the use of different practices. She reported that by the end of her research, the students submitted to modeling and problem solving approaches had optimistic beliefs about mathematics usefulness, were more prepared to deal with out-of-school situations, developed more flexible knowledge, were stronger in conceptual knowledge, and outperformed the other group in national examinations. Her findings show functional and concrete benefits of modeling and problem solving approaches in mathematics classrooms.

Due to the valuable advantages modeling can promote in a mathematics classroom, Pollack (Teachers College Columbia University, 2012) underlines that "[t]he particular field of [modeling] application, whether it is everyday life or being a good citizen or understanding some piece of science, is less important than the experience with this thinking process" (p. ix). In sum, modeling is a valuable way of doing mathematics, outlining a potential scenario to investigate mathematical understanding and proficiency.

2.3. MATHEMATICAL MODELING AS AN APPROACH FOR TEACHING FOR UNDERSTANDING

Likewise the acknowledgement of the relevance of modeling in mathematics education, there is also the recognition that mathematics classes should not be solely based on rote learning and drill exercises. Silver et al. (2009) affirm that "there has been increasing emphasis in the mathematics education community on teaching practices that deviate from the canonical version of classroom mathematics instruction (...) and that appear to be more oriented toward the development of students' conceptual understanding" (p.503). Nevertheless, changes towards teaching mathematics for understanding are difficult to implement. When referring to mathematical understanding, different kinds of understanding can be elaborated. Skemp (1976/2006) explains that the word understanding can refer to two different interpretations; accordingly he proposes two different types of understanding: the instrumental and the relational. For Skemp, relational understanding means "knowing both what to do and why" (p. 89), while instrumental understanding is described by "rules without reasons" (p. 89). **In this study, when simply stating the word understanding, the reference is to relational understanding.**

As opposed to teaching for instrumental understanding, teaching for relational understanding has a different focus and presents relevant advantages for the teaching and learning processes. Ben-Hur (2006) discusses concept-rich instruction, which is instruction based on conceptual knowledge. In this sort of instruction, students' obtained knowledge is supposed to be stronger and longer lasting; given that students can always go back to the meanings and understandings they have assimilated. The same is not true for memorized concepts, which students might forget once they are not using them anymore. In addition, when students simply memorize procedures, it is harder to transfer this instrumental knowledge to other situations that require students to make connections and adapt. As Kilpatrick et al. (2001) suggest "[h]ow learners represent and connect pieces of knowledge is a key factor in whether they will understand it deeply and can use it in problem solving" (p.117). That is to say that if students cannot make different associations among the learned concepts, they might not be able to use these concepts in different future situations. In this sense, their mathematical knowledge becomes compromised. Ben-Hur goes further and claims that this kind of short term / range knowledge might lead to an unwise teaching process. He asserts that "[i]n the case of students who have not fully developed concepts and processes, explicitly telling the students the correct answers, concepts, or procedure is a futile exercise of instruction" (p.50).

Based on my experience, teachers may struggle to promote mathematical understanding in class. It is not uncommon to have teachers experimenting with different and new strategies, trying to prompt students' mathematical thinking, and becoming frustrated because their efforts are not successful. This failure might be due to the difficulty in planning this sort of tasks, in implementing them, and in having students engaged and doing the tasks as they are supposed to. Dealing with the unexpected and improvising are almost inevitable in this sort of task. Considering what went wrong, rethinking the tasks and constantly adapting is not simple as well. Indeed, it is a hard job. If teachers fall short in effectively planning their lessons, students might not get involved enough to think hard, reflect, try different strategies, and go over and over the task, until they have final conclusions. Instead, they might just do manipulations based on teacher's guidance, and miss the chance to work on their mathematical thinking skills. It follows that they might miss the chance to enhance their mathematical understanding.

Stein, Grover and Henningsen (1996) assert that:

> Complete understanding [of mathematics] (...) includes the capacity to engage in the processes of mathematical thinking, in essence doing what makers and users of mathematics do: framing and solving problems, looking for patterns, making conjectures, examining constraints, making inferences from data, abstracting, inventing, explaining, justifying, challenging, and so on. (p.456)

In the face of so many challenges, Silver et al. (2009) state that this "complete" understanding of mathematics can be achieved by involving students in mathematics tasks that require high-order thinking; that is, cognitively demanding tasks (see Corrêa, 2015). This seems to be an effective way to teach students for mathematical understanding. That is why modeling tasks were chosen for this research intervention, because they are considered cognitively demanding tasks. Modeling tasks have the potential to involve students in high-level activities and, as so, to promote students' mathematical thinking and understanding. In this sense, mathematical modeling is a good approach for teaching mathematics for relational understanding.

2.4. MATHEMATICAL MODELING AND ITS CHALLENGES

Hernández, Levy, Felton-Koestler and Zbiek (2016) assert that “[a]lthough teachers recognize the value of engaging their students in mathematical modeling, few have had opportunities to experience modeling, and many teachers feel unsure of how to teach it” (p. 337). Indeed, Biembengut (2009) indicates the existence of issues when turning modeling approaches into practice in class. Doerr and Lesh (2011) exemplify: "When students are engaged in modelling activities, teachers are likely to encounter substantial diversity in student thinking. This places new demands on teachers (...)" (p. 267). As a consequence, there is still resistance by teachers to adopt modeling in class (Biembengut, 2009). Biembengut affirms that this is a reality in different countries, such as Germany and Japan. She associates this situation to two different reasons: teachers' training and national standard exams. Doerr (2007) — who writes about the kind of knowledge teachers need to teach mathematics through modeling — confirms that teacher's training ought to have some specificities when preparing teachers to use modeling approaches. She affirms that "pre-service teachers need to be exposed to a range of modelling activities that provide multiple opportunities for explanations and justifications of the modelling decisions that were made" (p. 73). Doerr believes teachers need knowledge about four different teaching features in order to better carry out modeling tasks in class. The features are: 1) paying attention to unexpected ambiguities; 2) offering helpful ways to portray students' ideas; 3) acknowledging unpredicted approaches; and 4) helping students with associations between representations (p. 77). The challenge is how teachers can construct this knowledge. As for national standard exams, these exams impose a demand on teachers' planning that might not favour teaching for understanding and that might require dedicated time for drilling exercises. When the main concern is performance in national standard exams, implementing modeling tasks can represent a drawback from the teachers' perspective.

According to Cirillo et al. (2016), the lack of agreement in what a modeling process consists of and its innate complexity are also reasons that escalate the challenges of teaching and learning through modeling. Almeida, Silva and Virtuan (2013) suggest that the main obstacle to incorporate modeling in the curriculum is the difficulty to introduce and conduct this kind of task (p. 20). Viecili (2006) mentions some extra different challenges, such as: the lack of institutional support to enable modeling practice; the lack of flexibility in learning methodologies; teachers' lack of motivation and time due to their already long work journey (working at two or more schools and in two or three shifts); other teachers' resistance to new and unknown possibilities; the lack of students' interest; and students' disruptive behaviour. In Schmidt's (2011) work about teachers' perspectives on the use of modeling in class, she highlights three main challenges mentioned by teachers: the extended amount of time used to implement tasks in class and also to prepare tasks; the lack of adequate material; and the difficulty in assessing students' performance. In a similar perspective, Silver et al. (2009) state that, in their research outcomes, teachers have listed many and diverse goals to be achieved during their teaching process. According to the authors, sometimes these goals compete, and one goal might have to be put aside in order to have another easier or more manageable goal completed. As high-level mathematics tasks — such as mathematical modeling tasks — might place a big challenge for teachers to deal with, these tasks might be the ones put aside or not implemented as originally planned.

Another challenge to be faced refers to the mismatch between what modeling is and what textbooks offer for teachers. Burak (2004) confirms that modeling problems present different features when compared to textbook problems in general. It is not uncommon to have teachers using textbooks as the basis for lesson plans and teaching strategies without criticizing or modifying textbook content. In this sense, even if a teacher follows the same modeling definition offered by this research, if this teacher decides to stick to a textbook, his/her modeling practice might be impaired. Dan Meyer (2015) analyzes two different textbooks that claim to be aligned with the CCSSM (National Governors Association; Council of Chief State School Officers, 2010), which in turn takes modeling into account. However, when considering the first five CCSSM proposed modeling actions — *identifying*, *formulating*, *performing*, *interpreting* and *validating*, Meyer concluded that the textbooks' focus is on *performing* and *interpreting*, while *identifying* and *formulating* were (mostly) done for students, and *validating* was set aside. Meyer did not consider the sixth of the proposed modelling actions, *reporting* action, because in his opinion this action seemed not to be part of the textbook role. Based on his analysis, the author reinforces that these contemporary textbooks (both published in 2013) continue to draw attention to historically strong features of textbooks: procedures and superficial meaning making. In response to that challenge, Meyer invites teachers to be critical and create opportunities in class to work on the other three modeling actions.

I agree with these arguments and I realize that there are also other reasons that impair the engagement in teaching for understanding through modeling tasks. For instance, even if a teacher has undergone a good teacher training — coming across various mathematical modeling approaches and experiences — she/he can be still challenged when teaching. This might be due to the lack of experience with mathematical modeling as a learner at secondary school. This lack of experience might prompt teachers to teach in the way they have learned. Foerster (2003) calls this phenomenon the "principle of conservation of rule where the future equals the past" (p. 207). Britzman (2003) also addresses this matter in what she analyzes the tension around learning how to teach. She points out that:

> Implicitly, schooling fashions the meanings, realities, and experiences of students; thus those learning to teach draw from their subjective experiences constructed from actually being there. They bring to teacher education their educational biography and some well-worn and commonsensical images of the teacher’s work. In part, this accounts for the persistency of particular world views, orientations, dispositions, and cultural myths that dominate our thinking and, in unintended ways, select the practices that are available in educational life (p.27).

Another aspect that may hinder the process is students' resistance. Certainly, modeling is a challenge for both teachers and students (Vorhölter, Kaiser, & Ferri, 2014). Students are used to teaching and learning processes in which they are expected to use techniques and procedures to solve mathematical questions. They are not used to the mathematical thinking processes that are necessary to go through mathematical modeling tasks. Hence, they might resist engaging in modeling tasks and, as a consequence, they might influence teachers' decisions on further classroom developments. Teachers might also give up on trying if students do not understand the modeling task and/or do not succeed.

Burkhardt's (2006) study addresses some of these challenges involved in the integration between modeling and mathematics curriculum. The author speaks to the needed changes and suggests ways of trying to make these changes a reality. Although Burkhardt's investigation is based on his and Henry Pollak's professional experience — and he does have reasonable arguments — his assertions are predominatly theoretical, in the sense that he does not present empirical evidence for his claims. Therefore, I reinforce the relevance of long term classroom-based research about modeling, in particular, about the forms of understanding and proficiency students undergo while doing mathematical modeling tasks. By proving the use of mathematical modeling worthwhile and reliable, teachers' conflicts and challenges might be diminished and students might benefit from this.

An overview of the mathematical modeling features, advantages and challenges, discussed in Chapter 2 so far, is presented in Figure 1. The literature that underlies the overview is emphasized.

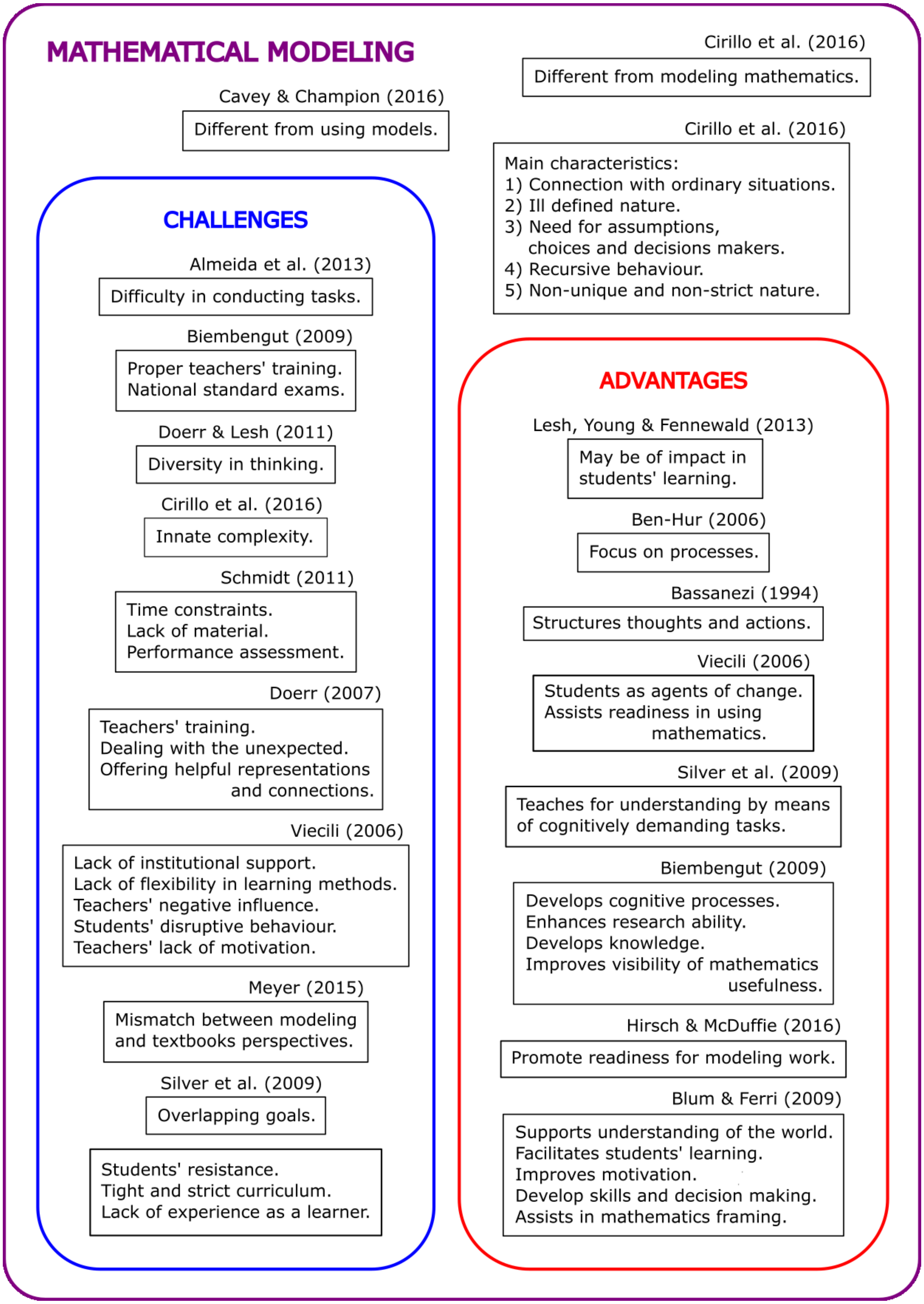


Figure 1: An overview of modeling features, advantages and challenges.

2.5. MATHEMATICAL MODELING AND OTHER LEARNING APPROACHES

This section presents features of other teaching and learning approaches that are commonly compared to mathematical modeling. Describing how these approaches are the same as and how they are different from mathematical modeling can be helpful in better understanding the modeling perspective.

2.5.1. PROBLEM SOLVING

It is not unusual to find people who treat mathematical modeling as problem solving. However, these are different learning approaches. Lesh and Doerr (2003) speak to this matter explaining that in historical perspectives, "[a]pplied problem solving is treated as a special case of traditional problem solving" (p. 4), which means problem solving is seen as a broader category than applied problems. In this case, applied problems can be associated to modeling approaches. On the other hand, in contemporary modeling perspectives, "traditional problem solving is treated as a special case of model-eliciting activities" (p. 4), which means modeling is seen as a broader category than problem solving. In this work, I am aligned with the second perspective, in which modeling encompasses problem-solving — given that in a modeling task students are required to model and solve a problem. That is to say that modeling is a broader activity that includes other thinking processes (Almeida, Silva, & Vertuan, 2013). Almeida, Silva and Vertuan (2013) defend that modeling requires teachers and students to define problems and not only to solve them (p. 9), which reaffirms modeling as a broader activity. In contrast, problem solving does not necessarily involve mathematical modeling. A problem solving task might require students to *use* a model to solve a problem, but not to *develop* a model to solve the problem. In that sense, mathematical modeling and problem solving cannot be treated as the same. Pollak (2012) (quoted in Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016) assertively distinguishes these two approaches:

> a major difference between mathematical modeling and problem solving is that problem solving either does not refer to the real world at all, or, if it does, it usually begins with the idealized real-world situation in mathematical terms and ends with a mathematical result. In contrast, modeling begins in the 'unedited' world, and after engaging in problem formulation and problem solving, the modeler moves back into the real world where the results are considered against the original context (p. 10).

Pollak affirms that in problem solving the problem is usually already given in mathematical terms, which suggests that students may not be required to develop a model of the problem. He also mentions that modeling includes problem formulation and problem solving, emphasizing that modeling encompasses problem solving.

Zawojewski (2013) addresses some differences between problem solving and modeling tasks. She explains that:

> During a problem-solving episode the primary process in which the problem solver engages is a search for a correct procedure(s) that will enable the identification of a solution path that proceeds from the "given information" to the "goals" of problem (p. 239).

In contrast, Zawojewski reports that modeling tasks "require that the modeller interpret the information in the task and interpret the required outcomes (with respect to an articulated function) for the purpose of mathematically modeling the situation" (p. 239). Put in a different way, modeling is a dynamic activity that focuses on processes, while problem solving is a static activity that focuses on procedures (Zawojewski, 2013). In problem-solving activities all the necessary information is usually given in the problem (neither more nor less) and students are essentially supposed to figure out what is the right path to take or the right formula to use. In modeling activities students are supposed to interpret a situation, find out the necessary information, generate a model, find out the solution and validate the model. Doerr (2016) stresses another difference between these two approaches. In problem solving, students usually work on finding a unique solution to a particular question. Whilst, in modeling, students work on finding one or more different solutions that can be generalized and used in other situations.

Hojgaard (2013) presents examples of three different kinds of mathematical tasks that may be implemented in class. The first type involves modeling; the second one involves problem solving; and the third one — that might be the most common in mathematics classes and assessments — involves basically procedural and highly scaffolded mathematics. By classifying modeling and problem solving as different types of mathematical tasks, it is clear that these two learning approaches cannot be considered as the same. The author understands that modeling involves: a) elaborating on the given task in order to understand and identify its individual characteristics; b) selecting the necessary elements and relations to model the situation; c) representing the situation mathematically; d) using mathematical strategies and procedures to solve the problem; e) interpreting results and making conclusions; and f) validating the model. As for problem solving, Hojgaard explains that it might involve only stages c, d and e, given that it is about solving already formulated mathematical problems.

Doerr (2016) explains that in problem solving students are expected to use the mathematical knowledge they already know to solve problems; on the other hand, during modeling, students are expected to build on their mathematical knowledge and on their problem-solving proficiency. In accordance with Doerr, Zawojewski (2013) states that problem solving requires students to use previously learned mathematical knowledge, while modeling prompts students to produce mathematical knowledge. Based on my experience, in problem solving, students are typically presented to mathematical content in advance and by teacher's lecture accompanied by drilling exercises. As so, students are not always invited to think mathematically, only to accept and use the given information. Usually, only after this first teaching moment, students are supposed to apply the learned content in contextualized situations. This structure might raise no doubt about what content is expected to be used by the students in the assigned problems, perhaps requiring no mathematical thinking or reasoning prior to the application of the new learned content. In this research, the methodology intends to have students engaged in modeling tasks at the beginning of the process of teaching new content; that is, when students are first in contact with the new knowledge and not only when they are applying the new knowledge. Students are encouraged to produce the necessary knowledge for the task at hand. If modeling tasks are presented only after the new knowledge is taught, students may be already unmotivated to engage in a mathematical reasoning process, because they probably do not see relevance in the new content. However, if they are invited to investigate a modeling situation at first, they might be challenged enough to work on their mathematical reasoning and thinking, and on the content knowledge needed to explore the situation.

2.5.2. Problem-based learning

Problem-based learning (PBL) is an instructional approach disseminated mainly in the medical field from the 1950's on. This approach gained popularity in elementary and secondary school throughout the 1990's (Hung, Jonassem, & Liu, 2008). Even though, empirical research in the educational level is not extensive (Hung, Jonassem, & Liu, 2008). Hung, Jonassem and Liu (2008) explain that "[t]he primary goal of PBL is to enhance learning by requiring learners to solve problems" (p. 488), but it is not simply about solving problems. It is an instructional approach based on problems, with specific features that are to be employed during the learning process. Problem-based tasks are built around ill-structured problems; are student centered; prompt students to construct knowledge and use it to solve the problem; do not provide students with specific assignments, instead students are supposed to generate and assess their own learning concerns (self-directed); require students to adapt learning strategies according to their needs (self-reflective); and consider tutors as facilitators and not as knowledge providers (Hung, Jonassem, & Liu, 2008).

According to these features, PBL is an instructional approach similar to the mathematical modeling approach done in this research. Modeling tasks are also built around a problem, prompt students to produce content knowledge throughout the task, demand students to adapt strategies, allow students to come up with their own inquiries and conjectures during the process, and have teachers responsible for guiding their learning process instead of leading it. In particular, as mentioned in the previous subsection, not expecting students to learn the required knowledge to do the task in advance is a relevant characteristic of this research modeling approach also present in PBL. This characteristic is not necessarily a modeling requirement, but it was a design decision in the current research. Hung et al. (2008) explain that "[i]n PBL classes, students encounter the problem before learning, which is countered by centuries of formal education practice, where students are expected to master content before they ever encounter a problem and attempt to apply the content" (p. 488). Due to this historical background, teaching by proposing problems before presenting content can be challenging. However, it is a good way to address the importance of mathematics, given that students have the chance to realize the need of mathematics during their investigation.

Presenting students with ill-structured contextualized problems is another similarity when compared to modeling tasks. In mathematical modeling, tasks reflect likely-to-happen situations, and students might have to look for extra information that is not given or discard information that is given. The task needs to be investigated and understood in order to be solved. Playing around with given numbers will probably not lead to an answer. A unique right answer is not always expected at the end of the task, and the task might have different answers or no answers at all. The modeling process itself, and students' reasoning and conjectures along the process are more relevant than a final right answer. Likewise, Woei Hung (2013) explains that:

> All PBL problems are real life complex problems, rather than well-structured, end-of-chapter-type problems. With the complexity and messiness of ill-structured

> problems, the students learn how to deal with the uncertainty, high degree of unknown, and no one-standard-answer nature inherent in real-life problems (p. 31).

PBL presumes that students will build on their problem solving and reasoning skills by doing higher-order thinking tasks. This is also expected from students when doing mathematical modeling tasks. The main difference is that, in the present research, mathematical modeling tasks are higher-order thinking tasks that necessarily involve the development of mathematical models, which is not a condition in PBL tasks. This model development ability is relevant in preparing students to deal with assorted circumstances, in which students are required to analyze ways of describing and representing situations mathematically. Another difference between this research modeling approach and PBL is that the former assigns specific tasks, while the latter does not. It is my opinion that assigning a specific task aids the process of implementing modeling approaches in elementary and secondary education, because when students are not used to this sort of task, they call for some initial direction.

2.5.3. Case-based, project-based and inquiry-based learning

Case-based learning and project-based learning are variations of problem-based learning that focus on case studies and projects respectively. Savery (2015) highlights that both learning approaches engage students in higher-order thinking processes, but they differentiate from each other in what concerns to their distinct focus. By analyzing case studies, students will focus on the context, discipline, terminology, elements and relations related to the specific case under analysis (Savery, 2015). By making a project, students will focus on the strategies and procedures needed to complete the project, usually an end product (Savery, 2015). In both approaches, the teacher's role is more like instructor or coach, who gives skilful assistance and feedback if compared with the role of tutor or facilitator (Savery, 2015). The mathematical modeling approach done in this research is similar to the case-based and to the project-based learning approaches in what refers to being assigned a specific task. On the other hand, modeling is different from these two approaches, in terms of the nature of teacher instructions and interventions. Modeling tasks are more open, less structured, require students to develop a model for the problem, and have teachers act as facilitators (prompt rather than instruct).

Inquiry-based learning (IBL) was also rooted on the problem-based learning approach but intending to be more flexible and holistic (Magnussen, Ishida, & Itano, 2000). Both approaches are based on John Dewey's philosophy, which suggests that a learner’s curiosity triggers the learning process (Savery, 2015). Different from case-based learning and project-based learning, IBL is less directive in what students are expected to do, which gives more room for students to inquire and conjecture, instead of focusing on problem solving. Savery (2015) explains that IBL tasks "begin with a question followed by investigating solutions, creating new knowledge as information is gathered and understood, discussing discoveries and experiences, and reflecting on newfound knowledge" (p. 11). Other than adding the inquiry piece to problem-based learning approaches, according to Savery, the teacher provides students with the necessary information in IBL approaches, while in problem-based learning students are responsible to find out the information they need by themselves. Among the three learning approaches discussed in this subsection, mathematical modeling presents more similarities when compared to IBL. Except for being required to develop a model, the other features of IBL are present in this research modeling approach. Although this difference might be seen as not much variation, as already noted, there is a difference that matters for teaching. Requiring students to develop a model encourages them to create mathematical representations associated to contextualized situations. This difference may be responsible for meaningful changes in mathematics learning processes.

As these three learning approaches (case-based, project-based and inquiry-based) have their foundations in the PBL, except for some of the aforementioned differences, the underpinnings described in the previous subsection are valid for these three instructional methods. Because these four approaches have the same foundations, they undergo the same kind of resistance when employed in elementary and secondary public school (Savery, 2015). This is due to the standardized curricula that should be followed, and to the idea that all students need to present the same achievement goals by the end of their learning processes (Savery, 2015). Besides, teaching approaches at public elementary and secondary schools are supposed to prepare students to succeed in tests, which reinforces drilling practices and memorization as teaching resources (Savery, 2015). Unfortunately, such practices go against the problem-based learning philosophy, in which "high-order thinking skill, self-regulated learning habits, and problem-solving skills" prevails (Savery, 2015, p. 12). Mathematical modeling suffers similar resistance to be accepted as an instructional method in secondary schools, given that modeling is mistakenly taken for granted as prolonged and less able to fulfill curriculum needs.

2.5.4. REALISTIC MATHEMATICS

Realistic mathematics education (RME) dates back to early 1970's and was originated in Netherlands with the intention of bringing realistic situations to mathematics teaching (Van den Heuvel-Panhuizen & Drijvers, 2014). According to Van den Heuvel-Panhuizen and Drijvers (2014), the word realistic remits back to a Dutch expression that means "to imagine", and does not refer only to the use of feasible contextualized situations. It considers also situations which students can imagine, can visualize, even if these situations are not real ones or are even formal mathematics situations. The authors mention that "problems presented to students can come from the real world but also from the fantasy world of fairy tales, or the formal world of mathematics, as long as the problems are experientially real in the student's mind" (p. 521).

RME is supported by three basic principles. The first one is called *reinvention* and presumes that students will be presented to realistic situations (in the sense that they can be imagined), which will allow them to reinvent formal mathematics (Kwon, 2002). In other words, this reinvention process intends to let students "experience a process similar to the process by which the mathematics was invented" (Kwon, 2002). The second principle is called *didactical phenomenology* and refers to the study of the relationship between the mathematical phenomena under analysis and the respective mathematical concept that represents it (Kwon, 2002). The idea is to access the phenomena, interpret and represent it, in order to mathematically analyze and calculate what is needed (Kwon, 2002). The third principle is called *emergent models* and intends to connect informal and formal mathematical knowledge (Kwon, 2002). This modeling stage presumes a dynamic process in which at first there is a model-of the situation under analysis that will become a model-for higher order mathematical reasoning (Kwon, 2002).

Based on these three principles, RME and modeling approaches have similarities, especially because of the use of models in RME. However, RME does not necessarily require students to *develop* a model as the modeling approach in the present research does. Other than that, the modeling approach used in the current research works with feasible contextualized situations, that is, situations that are expected to happen out of the school environment, in which mathematics is genuinely used; whilst RME works with imaginable situations as described above. Although, this might be seem as a minor difference, when students are instigated with likely-to-be experienced situations, I believe that their motivation will increase and, as a consequence, their engagement will become more intense. Catherine Attard (2012) highlights that:

> When viewed through a mathematical lens, engagement occurs when: Mathematics is a subject students enjoy learning; students value their mathematics learning and see its relevance in their own lives now and in the future; students see connections between the mathematics they learn at school and the mathematics they use outside school (p. 10).

Therefore, keeping modeling tasks closely related to doable uses of mathematics is a relevant piece to engage students in the mathematics teaching and learning processes.

2.5.5. MATHEMATICS APPLICATIONS

According to Viecili (2006), an everyday situation that can be managed or treated by mathematics is called a mathematics application. Certainly, the use of mathematics applications is an appealing resource to motivate and engage students in mathematics learning, and it is also a rich way to provoke high-level thinking and reasoning. Mathematics applications can be used within different learning approaches; however, there is a difference when using mathematics applications as a resource and as a learning approach. As a resource, mathematics applications are tools within a learning approach. While, as a learning approach, mathematics applications guide the way the learning is conducted. The use of mathematical modeling and the use of mathematics applications as a learning approach are frequently believed to be the same. However, this comparison is not an accurate one. The difference is usually in the way these two approaches are used. As Stillman, Galbraith, Brown and Edwards (2007) clearly explain:

> Simply put, with *applications* we tend to focus on the direction (mathematics → reality). "Where can I use this particular piece of mathematical knowledge?" On the other hand with *mathematical modelling* we focus on the reverse direction (reality → mathematics). "Where can I find some mathematics to help me with this problem?"(p. 689).

In other words, within mathematics applications, after a specific knowledge is learned, students are invited to apply this knowledge to solve a contextualized problem. On the other hand, within mathematical modeling, before a specific knowledge is learned, students are invited to investigate a contextualized problem that will foster them to learn or produce new knowledge. The latter is the case in the mathematical modeling tasks implemented in this study. Students were not taught the mathematics content required to do the task before being presented to the task. Students were supposed to investigate the task and from there build the necessary knowledge. Therefore this research refers to mathematical modeling and not mathematics applications. It could be the case that any particular student already had the knowledge or concepts needed for the task. For that student, this research modeling tasks could be considered as mathematics application tasks. However, given students' point in the curriculum, knowing the task required knowledge in advance was not a common state for the learners.

A disadvantage usually faced when using mathematics applications is that it is not common to have genuine mathematics applications used in secondary school settings, probably because finding applications that fit this setting might be challenging. Therefore, genuine mathematics applications are usually used at the university level, while pseudo or overly simplified applications are more common at the secondary level. Lesh (2007) emphasizes that life mathematical problems are typically not solved in the simple way that it is shown in mathematics classes. It is typically a much more elaborated process, which can be introduced to students during their schooling process, but that is not. In school, solving mathematics problems — derived from mathematical applications — looks like performing routine procedures within mock situations that will not take place in students' lives. In accordance with this, Cirillo, Bartell and Wager (2016), mention Boaler's (2008) work, saying that "to do well in mathematics, children must suspend reality and accept nonsensical problems where trains travel toward each other on the same tracks, and people paint houses at identical speeds all day long" (Cirillo, Bartell, & Wager, 2016, p. 87). Overall, it is more about problem solving, while what would potentially enrich mathematics classes is more about investigating feasible mathematical applications.

Students might be asked, for example, to use a trigonometric ratio to calculate the ideal distance from the bottom of a ladder to the wall, in order to prevent the ladder from sliding away. This situation can be a genuine one, in which mathematics is contextualized and based on a practical situation, nevertheless — in practice — nobody will calculate this distance before climbing the ladder. Thus, it is not exactly a mathematics application. Moreover, such problem is generally accompanied by a picture illustrating the situation and giving an angle. If that is the case, students are not even trying to visualize what the pseudo application is about. Students are not thinking about the situation, they are just required to apply a trigonometric ratio in a contextualized situation. In this sense, it is not only about using mock applications, it is also about disregarding the opportunity to work on higher order thinking tasks. A more adequate example of the use of mathematics applications could be asking students to design a folding ladder. In this circumstance the ladder manufacturer would have to be worried about the proper angle when the ladder is unfolded, so that the ladder would not represent a risk for the buyer/user. This situation could be used in a modeling approach as well. Whenever a student is learning through mathematics applications and this process involves the development of mathematical models — according to the mathematical modeling definition presented in this research — that could be called as mathematical modeling too.

In this research, modeling tasks are based on genuine contextualized situations that fit the high-school curriculum. It is not about overly simplified situations to the extent that they become artificial ones. In this research intervention, it was necessary to ensure that these situations were experiences one could possibly encounter out of the school settings and that the development of mathematical models was required during the process of investigating and/or solving the task.

The diagram shown in Figure 2 was elaborated in order to illustrate the mentioned differences and similarities between the learning approaches described in this section and the mathematical modeling approach implemented in this research. It is relevant to highlight that this diagram is a particular way in which I see the differences and similarities between these approaches. Other perspectives are possible. In the diagram, characteristics are portrayed inside bubbles, while each learning approach is represented by a rectangle that comprises its respective bubbles (characteristics). The diagram is also useful to underline features of one approach that are not considered in another approach. For example, modeling is supposed to be student-centered and to allow knowledge production within its process. These features are not mentioned in the RME above description, which might indicate that RME can be student-centered or teacher-centered, and can use previous knowledge or produce new knowledge.

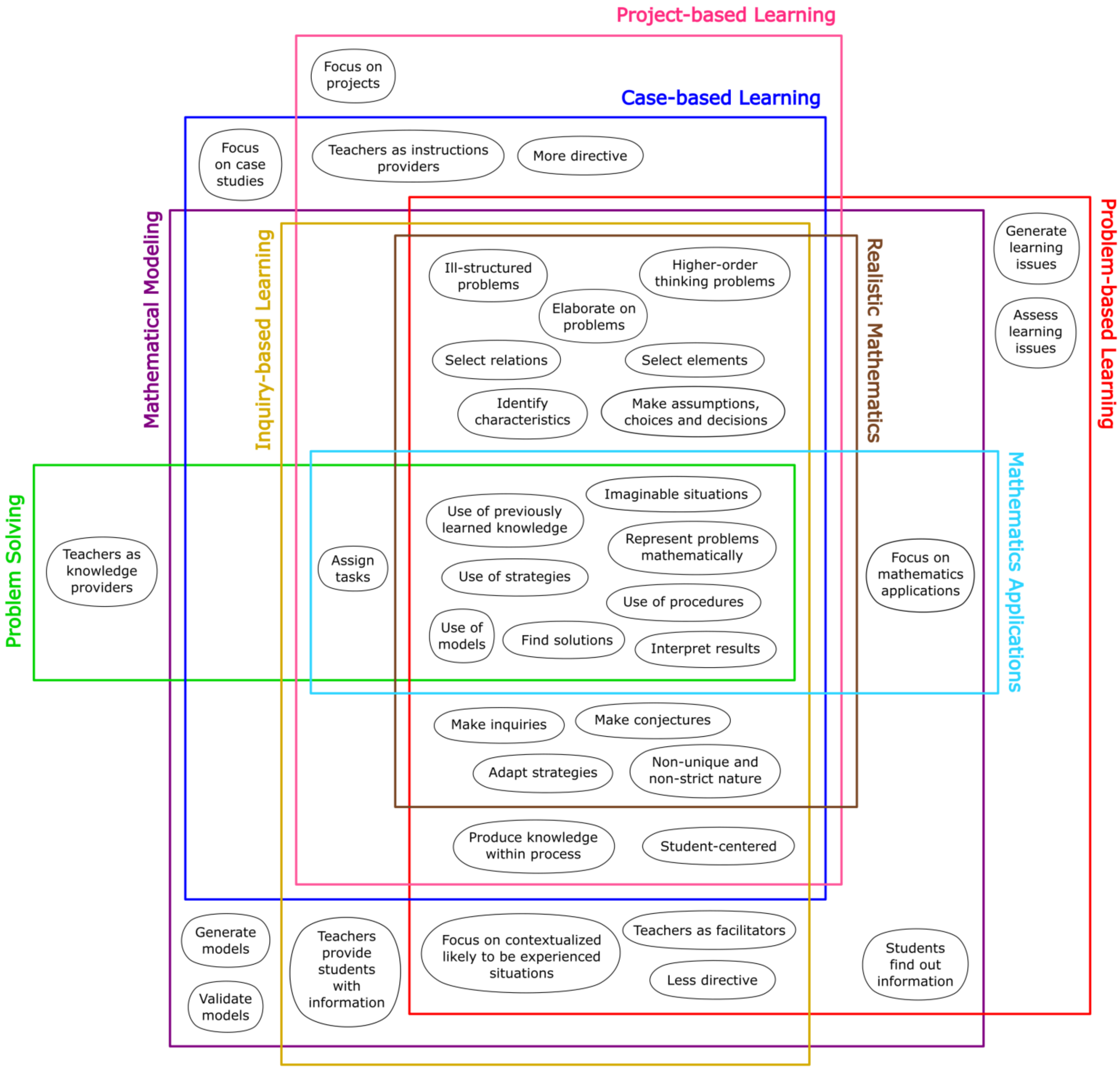


Figure 2: Mathematical modeling and other learning approaches.

2.6. MATHEMATICAL MODELING IN THE SCHOOL CURRICULUM

This section presents an overview on how research is being conducted in the realm of mathematical modeling in relation to the school curriculum. Different perspectives are addressed. Starting with the Brazilian perspective, Biembengut's (2009) work addresses how mathematics modeling evolved in Brazil over the last thirty years prior to 2009. According to her, the mathematics modeling research in Brazil advanced simultaneously with the international research in this field. In the 1980s, the movement in favour of mathematical modeling was all around the country and soon it became a core research area in Brazilian education. Biembengut points out that, although mathematical modeling approaches were being used by important scholars, these approaches were being implemented mainly in undergraduate and graduate courses. In other words, by this time, there was no experience and investment in elementary and secondary education. Even out of the Brazilian perspective, a lot of undergraduate and graduate materials about mathematical modeling can be easily found, for instance in *Mathematical Modeling* by Stefan Heinz (2011) and in *Principles of Mathematical Modeling* by Clive Dym (2004). Nevertheless, the same is not necessarily true in basics education. According to Biembengut, over that past 30 years, a large amount of work has been done in Brazil, including PhD theses, master's dissertations, undergraduate monographs and projects, journal papers and conference proceedings. These works had different objectives, such as the reinforcement of the relevance of mathematical modeling in the teaching process at all learning levels and also in teacher degree courses, and the reporting of teachers' practical experiences (Biembengut, 2009).

The current national curriculum parameters for senior high-school in Brazil (Brasil, 2000) are based on general orientations and competencies that do not specifically guide teachers' practice planning. Although the document states that students need to realize mathematics as a language that allows them to model and interpret reality (p. 40), teachers have no reference or direction to follow. When trying out modeling approaches, challenges such as the ones mentioned in section 2.4 can be easily faced. Also, as the allusion to modeling in the curriculum document is very broad, textbooks do not undergo any major changes in this direction. As a result, mathematical modeling is still not completely integrated in the mathematics curriculum in Brazil.

In their book about modeling in mathematics education, Meyer, Caldeira and Malheiros (2011) explain and discuss many of the facets that are involved in this broad discussion. The authors see modeling as a way of facing mathematics through relevant and applied situations, and not through theoretical problems that are usually found in text books (p.34). They emphasize that, in their point of view, the focus of modeling approaches is not on the content, but rather on using mathematics to discuss and understand daily life problems (p.51). In addition, they understand that mathematical knowledge is not being passed from teacher to students, it is emerging from the interactions among them instead (p.24); and these interactions can be originated through modeling tasks applied in class. Meyer et al. understand that the difference between having mathematical modeling in the midst of pure mathematicians and in the midst of students is that, in the former, the pure mathematicians already know the mathematics content that might be used, whereas in the latter students are supposed to learn mathematics content along with the modeling activity (p.39). This is a considerable difference that may impose some challenges. The authors highlight that this is the scenario in which curriculum issues arise (p.39). Because of these different perspectives, the discussion about incorporating modeling in the curriculum is not an easy one. Apart from that discussion, the authors point out the difficulties that teachers face when implementing modeling activities in their classes, and what kind of change in attitude is necessary to positively influence and aid the use of modeling approaches. Finally, Meyer et al. give some examples on how to do modeling in class as a way of encouraging and supporting teachers in this tough task. Even though this book can serve this purpose, it might be still theoretical for pre-service and in-service teachers.

Almeida et al. (2013) wrote a book about mathematical modeling in elementary and secondary education, which aims to be more focused on practice instead of on theory. A brief section in the book explains what mathematical modeling is, why use it and how to use it. Then 10 different modeling experiences in basics education and six other modeling possibilities that can be worked out in class are presented and discussed. Based on reports about modeling experiences, Almeida et al. acknowledge three different ways of having modeling tasks in class: a) integrated in mathematics classes, b) as extra class activities, or c) as a blend of the two previous situations (p.21). The authors endorse the ideal situation of having modeling tasks precede mathematical content (p.22). In this case, mathematical content would be studied based on students' necessity in order to solve problems (p.22); this scenario might be more easily reproduced when modeling is integrated in mathematics classes. Although this book presents and discusses doable activities to be done in class, it might be the case that teachers still lack some further orientation and motivation to implement modeling activities in class. Instructional materials specifically associated to the mathematics curriculum, and research showing the advantages and the feasibility of this sort of approach might be helpful in this sense.

In the United States, although mathematics modeling is being studied and researched for some decades, it was officially integrated in the USA elementary and secondary curriculum around 6 years ago only. Modeling came into sight in the K–12 curriculum after the publication of the CCSSM in 2010 (National Governors Association; Council of Chief State School Officers, 2010), in which modeling is considered not only one of the eight standards for teaching mathematics, but also a conceptual category at the high-school level (Hirsch & McDuffie, 2016). As the *Mathematical Modeling Handbook* (Teachers College Columbia University, 2012) highlights, "modeling cannot be set aside or employed only when spare time arises" (p. vi); teachers must save class time to work on modeling. When integrating modeling to the curriculum, a transitory period is expected, during which teachers need to accept the challenge and adapt their usual practice. Along this transitory period, proper modeling classroom tasks and materials, as well as teacher accessible research highlighting modeling benefits and classroom possibilities, are definitely desired and required to aid teachers. An example of this kind of research is the one presented by Bleiler-Baxter et al. (2016). The authors implemented a four-day lesson in a high-school setting aiming to have students modeling and also aiming to give teachers insights about mathematical modeling practice. Bleiler-Baxter et al. suggest different ways of scaffolding students when undergoing a modeling task. Another example of research that helps in opening up possibilities for modeling instruction at the elementary and the secondary school level is the one written by Blum and Ferri (2016). The authors present practical examples to be used in primary, lower secondary and upper secondary school. They emphasize advantages of doing modeling, list criteria for designing modeling tasks and give advice for teaching through modeling.

Gann, Avineri, Graves, Hernandez and Teague (2016) share their research experience within a science and mathematics focused high-school setting, addressing students' perspectives and reflections on modeling, as well as their initial struggles with modeling. The authors give structured guidance in what refers to students' scaffolding when doing a modeling task. Another work that focuses on different ways of scaffolding students involved in modeling tasks at the secondary level is Cavey and Champion (2016). All these studies can be helpful in what concerns teachers' adaptation to the use of modeling approaches in class. However, they do not specifically enter the realm of students' understanding and proficiency. Zbiek and Conner’s (2006) work is closer to the current research perspective, given that their work analyzes the mathematical learning that occurs while mathematical modeling tasks are implemented with prospective secondary mathematics teachers. Zbiek and Conner introduce an appealing schema to illustrate mathematical modeling sub-processes and use this schema to investigate learning. They conclude that this sort of “learning potentially involves both deeper understanding of known curricular mathematics and the motivation to learn new curricular mathematics” (p. 110). Zbiek and Conner argue that mathematical modeling provides a potential environment for the development of conceptual and procedural knowledge, which is in tune with these research outcomes. The primary differences between my study and Zbiek and Conner’s study are the grade level and the data analysis, which in their case is based in modeling sub-processes, and not in forms of understanding and proficiency.

In Australia, different studies about modeling have been done. I draw attention to Brown and Edwards' (2011) research, because — similar to the current research — the authors focus on students' mathematical understanding (first with grade 9 students and then with the same group, but at grade 11). They highlight the relevance of the relationship between mathematical modeling and students' mathematical understanding. The authors claim that by communicating non-trivial solutions of complex modeling tasks, students disclose their mathematical understanding. Therefore, Brown and Edwards' work analyzes how mathematical modeling helps in revealing and deepening mathematical understanding. They center their research analysis framework in three aspects present when doing mathematical modeling tasks, which are: 1) students' use of prior knowledge; 2) students' integration between reality and mathematics; and 3) students' high-order thinking. The research analysis looks for evidence that these three aspects were present along students' work. Based on this evidence, Brown and Edwards' conclude that these three aspects are related with the promotion of deep understanding, and with making this deep understanding apparent. The current research acknowledges these outcomes and goes further. It looks into how mathematical understanding and proficiency are expressed through doing mathematical modeling, rather than only exploring if students' understanding is being prompted and clearly communicated while modeling or not.

In other countries around the world, a variety of papers discussing modeling experiences, relevance and implementation issues can be found as well. Nonetheless, some of them seem to be theoretical for practitioners. Bahmaei (2011), a scholar from Iran, discusses advantages and challenges of mathematical modeling at the elementary level. Ikeda and Stephens (2010) discuss a modeling experience with grade 9 students in Japan, in which they conclude that teaching based on three different principles (conflicting situations, repeated connections and spiral reflections) is "effective in fostering students' thinking to promote modelling" (p. 58). Kawasaki, Moriya, Okabe and Maesako (2012) assert that Japanese high-school curriculum underwent modifications in 2012, which involved the inclusion of modeling approaches in the teaching and learning process. Based on teachers' reaction and on the problems that might arise due to this modification, the authors present modeling examples to aid mathematics teachers with the new curriculum. However, emphasis is given to high-school experiences related to students that intend to follow an engineering or scientific career and not to high-school students in general. Bonotto (2010) explains that in Italian schools three underpinnings can be found in teaching practices: 1) problems have only one solution; 2) numbers presented in a problem are sufficient and necessary for solving the problem; and 3) writing down procedures is more important than finding the right answer (p. 19). The author understands that these underpinnings are not appropriate if students are to experiment modeling situations and pose problems. Based on that, she presents some findings from classrooms experiences that intended to modify the teaching and learning processes focusing on "fostering a mindful approach towards realistic mathematical modeling and problem posing" (p. 28). Her results reinforce the idea that modeling should be incorporated in mathematics classes as a teaching approach.

The Alberta curriculum in Canada went through modifications that addressed in part modeling issues at the high-school level. In 2002, a new curriculum (Alberta Curriculum Standards Branch, 2001) was implemented, and students were able to choose a strand called applied mathematics. In this new curriculum, one of the goals was to have students modeling in class. The curriculum said that students would "learn that mathematics is a powerful set of processes, models and skills that can be used to solve non-routine problems, both in and out of the classroom" (Alberta Curriculum Standards Branch, 2001, p. 1). Along with the change in curriculum, textbooks were modified accordingly, and the applied mathematics textbooks were pretty much based on investigations (Alexander, et al., 2000). Nevertheless, in spite of positive learning outcomes that might have been demonstrated during the implementation of this new curriculum, the Alberta mathematics curriculum has changed again in 2007 and further changes are currently being discussed. This is an interesting situation to be analyzed, because Alberta teachers had adequate instructional materials to guide them on implementing the applied mathematics course, which seems not to be true for teachers from other countries. Notwithstanding, in only five years the Alberta curriculum suffered a new change.

In a different perspective, Frejd (2012) investigates about how teachers' conceptions in a Swedish school impact on the use of modeling approaches in high-school classes. The author explains that in the current official Swedish curriculum, modeling is already part of the teaching underpinnings. However, modeling is not defined in the official curriculum and is not mentioned in any of the existent local curriculums (school curriculums). As a result, teachers' interpretation and practice can be fairly different. Frejd affirms that teachers in general do not consider modeling as a priority and they also question some modeling approaches as related to mathematics. To lessen this sort of diverse interpretations and to give proper value to modeling approaches, incorporating the study of modeling in teachers' training might be a good option. In tune with that, based on a research with Austrian pre-service and in-service teachers, Siller and Kuntze (2011) emphasize the need for introducing modeling in teacher professional developments. In recent work, Blum and Ferri (2016) also speak to issues related to the teaching of mathematics. In their work, they underline five different principles that should be taken into account when teaching mathematics through modeling, namely: 1) use of various instructional methods; 2) use of rich content lessons; 3) students cognitive activation; 4) individual solutions encouragement; and 5) students' retrospective reflections encouragement. Blum and Ferri underline that teachers make a meaningful difference in modeling teaching practices. As such, they suggest that modeling should have a main role in pre-service and in-service teacher education.

In a previous work, Blum and Ferri (2009) assert that "the gap between the goals of the educational debate and everyday school practice is that modelling is difficult both for students and for teachers" (p. 45). This is definitely true. Modeling can be challenging. However, whilst mathematical modeling is not brought to teachers and students' daily practice, it will probably remain as a difficult approach that deviate teachers from trying it. If teachers and students do not give modeling a chance, they will miss the opportunity of teaching and learning mathematics in a promising way. Other than that, according to Doerr and English's studies (Doerr H. M., 2006; English & Doerr, 2004), teachers can benefit from teaching by means of modeling approaches. The authors have interesting and useful research studies about how secondary school teachers' interpret and respond to students' thinking while engaged in mathematical modeling tasks. They point out that teachers build on their own knowledge through interacting with students during the modeling process, and also develop a diversity of approaches in response to students' modeling work. Although those studies involve students' way of thinking, their focus is on teachers' knowledge, rather than on students' understanding and proficiency.

A lack of practice-based research, and a lack of instructional materials related to mathematical modeling at the secondary level (to help students and teachers in the challenging task of incorporating modeling approaches into the mathematics curriculum) seemed to be the case in different countries years ago. Fortunately, recently, these resources are more available, and teachers and students are more exposed to the relevance, the feasibility, and the use of mathematical modeling approaches in secondary school settings. Publications such as those by ICTMA (Lesh, Galbraith, Haines, & Hurford, 2013; Kaiser, Blum, Ferri, & Stillman, 2011; Stillman, Kaiser, Blum, & Brown, 2013; Stillman, Blum, & Biembengut, 2015), *Annual Perspectives in Mathematics Education 2016* — APME (Hirsch & McDuffie, 2016), *Guidelines for Assessment & Instruction in Mathematical Modeling Education* - GAIMME (Consortium for Mathematics and its Applications - COMAP & Society for Industrial snf Applied Mathematics - SIAM, 2016), *Mathematics Teacher* (Anhalt & Cortez, 2015), *The Mathematics Teaching in the Middle School* (Baron, 2015; Moore, Doerr, Glancy, & Ntow, 2015; Felton, Anhalt, & Cortez, 2015; Bostic, 2015), and the *Mathematical Modeling Handbook* (Teachers College Columbia University, 2012) present up to date research and instructional materials about the teaching and learning of mathematics at the secondary level through modeling approaches. However, as Lesh, Young and Fennewald (2013) assert in one of the above publications (Lesh, Galbraith, Haines, & Hurford, 2013) — referring to research published about ten years ago only — "most of [the] studies investigate the development of ideas — not the success of treatments and interventions" (p. 280). In this sense, this dissertation research not only draws attention to examples of modeling tasks, it also presents a possible intervention to implement tasks in class, a reflection on possible ways to unpack the tasks, students' actual work on tasks and respective analysis. Then, from students' work analysis, the current research dives into the accomplishment of the intervention in terms of students' mathematical understanding and proficiency. It is fairly complete in portraying the implementation of mathematical modeling tasks in high-school classes, giving teachers a practical sense of what modeling is about, as well as reasons and encouragement to try it.

Finally, a lack of research about the application of modeling interventions in the context of entire regular mathematics high-school courses has also being noticed. Bracke and Geiger's (2011) work contributes to this sort of investigation at the junior high-level. The authors look into the use of modeling tasks during a whole year grade 9 course, and conclude that single modeling events do not offer the same benefits as long-term approaches. In this sense, the present research is useful because it speaks to the viability of the integration of modeling tasks in a regular basis in mathematics entire courses.

3. Framing the question: Theoretically and Methodologically

This chapter explores the theoretical framework used in this research data analysis and also the methodological framework used in this research design. The chapter is divided into two main sections. The first one looks at Kilpatrick et al.'s mathematical proficiency model, which is used to analyze the data collected in terms of students' mathematical understanding and proficiency. The second section addresses design-based research, which is the chosen methodology to design the research intervention; and complexity science, which is used as a methodological framework for the classroom design setting. Complexity science is not used to analyze the data collected[2], but to establish the qualities of a supportive environment for students to engage in mathematical modeling tasks.

3.1. Theoretical framework

The main focus of this research is on students' mathematical understanding, and to better grasp why the theoretical framework was chosen, it is necessary to get a sense of what mathematical understanding is and how it can be investigated. This section is subdivided into four subsections. The first one poses definitions for mathematical understanding, and the second one suggests different ways of investigating understanding. Next, the third subsection presents an explanation for the use of a mathematical proficiency model as the theoretical support for data analysis. Finally, the last subsection discusses and portrays the data analysis framework.

3.1.1. What is mathematical understanding?

Different authors suggest different, yet similar, definitions for understanding. In her book *Understanding in Mathematics*, Sierpinska (1994) explores in depth the meaning of understanding and its multiple facets. She does not consider understanding just as a matter of reasoning. It could be, for example, the understanding that supports an explanation, or the understanding of how to perform a procedure, even if there is no apparently intrinsic reasoning. I agree with her in that understanding is broader than reasoning. I would say that mathematical understanding requires mathematical thinking, and mathematical

---

[2] Zbiek and Conner (2006) observe the complexity nature of mathematical modeling and of mathematical modeling approaches. However, for this research purpose complexity science is not being used as a theory to frame mathematical modeling or to analyze mathematical modeling.

thinking can be expressed in different ways. Reasoning, visualization and pattern noticing are examples of such possible ways. Thus, mathematical understanding, mathematical thinking and mathematical reasoning are not being used in this research interchangeably.

In Sierpinska's (1994) perspective, there are four components in an act of understanding. The first one is the *subject of understanding*, that is, the individual person who actually experiences the act of understanding. The second one is the *object of understanding*, which refers to what the understanding is about. It could be concepts, problems, formalisms and texts. The third component of an act of understanding is the *basis of understanding*, that is, what supports the understanding. It could be representations, mental models, perceptions and thoughts. Finally, the fourth refers to *mental operations* that connect the *object of understanding* with the *basis of understanding*. Sierpinska states that this fourth component encompasses four different *mental operations*, namely: *identification*, *discrimination*, *generalization* and *synthesis*. This sequence of *mental operations* refers respectively to: the recognition of the object of understanding; the differentiation of the object of understanding from other possible objects of understanding; the comprehension that the object of understanding can be a particular case of a different circumstance; and the grasp of the situation as a whole, and not as an isolated situation instead. In sum, an act of understanding is what happens when the *subject of understanding* uses *mental operations* to make sense of the *object of understanding* rooted in its *basis of understanding*.

Based on past descriptions of understanding, Barmby, Harries, Higgins and Suggate (2007) allege that to understand mathematics, one should be able to build connections among different representations of one same mathematical concept (p. 42). The yielded mathematical understanding is what is assimilated by building these connections (Barmby, Harries, Higgins, & Suggate, 2007, p. 42). What is problematic here is that this understanding relies on internal representations, while when accessing and assessing this understanding, external representations are used (Barmby et al., 2007, p. 43). The question is: To what extent do external representations accurately portray internal representations? Barmby et al. also draw attention to the fact that understanding is a broad element that can be in constant development and, when this understanding is accessed, only part of it is actually exposed (p. 44). As such, the difficulty in analyzing or even in detecting students' understanding becomes clear.

3.1.2. HOW TO INVESTIGATE MATHEMATICAL UNDERSTANDING?

After briefly discussing about what understanding means, it is necessary to discuss ways of investigating this understanding. As reviewed in Corrêa (2015), in an analytical perspective, Pirie and Kieren's (1994) model for observing the growth of mathematical understanding suggests that the development of students' mathematical understanding can be observed as having eight different layers, namely: *primitive knowing*, *image making*, *image having*, *property noticing*, *formalising*, *observing*, *structuring* and *inventising*. The first layer, the *primitive knowing*, refers to the "starting place for the growth of any particular mathematical understanding" (p. 170). In other words, it is the background knowledge that students bring with them to start constructing new knowledge. The *image making* layer builds on this previous knowledge, that is, "make distinctions in previous knowing and use it in new way" (p. 170). When students can take actions without directly associating a new situation to the original one, students will have achieved *image having*. That is, they "can use a mental construct about a topic without having to do the particular activities which brought it about" (p. 170). *Property noticing* is observed when students infer properties based on the images they have constructed. Pirie and Kieren say that at this point, students can "manipulate or combine aspects of ones [sic] images to construct context specific, relevant properties" (p. 170). *Formalising* is observed when students abstract "a method or common quality from the previous image dependent knowhow which characterised her noticed properties" (p. 170). When students come up with new understandings by reflecting on and coordinating formal activities and express these understandings as theorems, they are said to be in the *observing* layer (Pirie & Kieren, 1994). When students think in terms of theories, they are in the *structuring* layer, which "occurs when one attempts to think about ones [sic] formal observations as a theory" (p. 171). Finally, when students follow a rationale and pose reasonable new questions, they are *inventising*. That is, they are "able to break away from the preconceptions which brought about this understanding and create new questions which might grow into a totally new concept" (p. 171). Although this model proposes that layers are increasingly more comprehensive than others, the progress of the layers of understanding do not need to follow a linear process. Quite the opposite, it might be a recursive non-linear process that revisits previous layers depending on the situation.

As also previously discussed in Corrêa's (2015) work, Tall (2013) presents an investigative model that can be used for analyzing students' mathematical understanding. Tall's model presents what he calls the three mental worlds of mathematics: the *conceptual* *world* (related to embodiment), the *operational world* (related to symbolism) and the *axiomatic world* (related to formalism). According to the author, these worlds are "based on human recognition, repetition and language to evolve through perception, operation and reason" (p. 153). The *conceptual world* refers to experiences that students have that enable the embodiment of mathematical concepts[3] and, as a result, their better comprehension. These experiences emerge from students' bodily perceptions and actions, and can be associated with concrete materials, schemas, images, gestures, etc. Tall highlights that in this stage the focus is on objects. The second world, that is, the *operational world*, will build on the objects, and the focus will shift to "actions on objects" (p. 141). Thus, students work on procedures related to the concepts acquired in the world of embodiment. These procedures refer to manipulations and calculations, and they might result in new understandings that are not tied to embodiments anymore (p. 145). Finally, when students achieve the third world, the *world of formalism*, they think in terms of mathematical abstraction. At this point, students work on formal definitions and on properties derived from formal proofs (p. 149). These three worlds are likely to blend, yielding combined settings for understanding.

Another way of analyzing students' mathematical understanding is by using Bloom's taxonomy (Bloom, 1956), as discussed in Scott’s work (2007). In her research, in the last lesson of a sequence of four lessons, she proposes three different questions to trigger students' reflections. Scott's goal is to examine if the use of these questions are of any help when looking into students' thinking and understanding. One of the ways she looks into students' levels of understanding is by classifying their written responses according to Bloom's taxonomy, that is, according to the following categories: *remembering*, *understanding*, *applying*, *analyzing*, *evaluating* and *creating*. These categories are named by cognitive activities that basically speak for themselves. *Remembering* refers to retrieving previous knowledge necessary to the task; *understanding* refers to the comprehension of the mathematics involved in or necessary to the task; *applying* refers to the use of mathematics procedures in the task; *analyzing* refers to the mathematical analysis necessary to conjecture, solve or inquire about the task; *evaluating* refers to the assessment of what has been done in the task; and *creating* refers to the production of new content or conjectures as of what has been done in the task. By classifying evidence of

[3] For example, when students gesture with their own arms to retrieve, represent, explain or recognize crescent or decrescent first degree functions.

students' work into these categories, one might be able to better access students' understanding.

Finally, as discussed in Corrêa (2015), Kilpatrick et al.'s (2001) mathematical proficiency model can be used to investigate mathematical understanding too. Based on Kilpatrick et al.'s proficiency model, one can say that mathematical proficiency encompasses mathematical understanding. Kilpatrick et al. posit that successfully learning mathematics is a comprehensive and compound process, which involves "aspects of expertise, competence, knowledge, and facility in mathematics" (p. 5). Kilpatrick et al. use the term mathematical proficiency to refer to all aspects necessary to successfully learn mathematics. Their notion of mathematical proficiency states that students need to accomplish simultaneously five different strands to achieve mathematical proficiency, namely: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*. The authors understand that these five strands of mathematical proficiency are interconnected as a complex whole and all of them influence students' mathematical proficiency.

Kilpatrick et al. (2001) acknowledge that *conceptual understanding* and *procedural fluency* are both essential parts of the learning process. The authors assert that these two strands are interconnected, although these forms of understanding often appear in opposition within the literature (Eisenhart, Borko, Underhill, Brown, & Agard, 1993). In Eisenhart et al.'s practice-based research, procedural knowledge was overvalued when compared to conceptual knowledge in the teaching of mathematics. Nonetheless, Kilpatrick et al. affirm that "it is not always necessary, useful, or even possible to distinguish concepts from procedure because understanding and doing are interconnected in such complex ways" (p. 134). In this sense, Kilpatrick et al. propose a complete and integrated model that values each and all of the skills students need to achieve mathematical proficiency. Still, Kilpatrick et al. recognize the difference between rote learning and learning for understanding, valuing the latter approach over the former one. They assert that "learning with understanding is more powerful than simply memorizing because the organization improves retention, promotes fluency, and facilitates learning related material" (p. 118).

The first strand that Kilpatrick et al. (2001) discuss is *conceptual understanding*, which means a connected and coherent understanding of mathematical ideas. Learning for conceptual understanding enables students to retrieve mathematical content more easily, because they can make sense of mathematics as a whole and not as isolated parts. Kilpatrick et al. emphasize that "knowledge that has been learned with understanding provides the basis for generating new knowledge and for solving new and unfamiliar problems" (p. 119). This is a desired skill for mathematics students. *Procedural fluency* relates to the ability of choosing the right mathematical procedure, and effectively performing it. It is not only about knowing what to do, it is also about knowing when and how to do. Understanding the procedure and why it works is also part of the process of acquiring mathematical proficiency. Next, *strategic competence* refers to the ability of building strategies to understand, represent and solve problems. Kilpatrick et al. point out that this ability is different from trying out some possibilities with the given numbers in a problem, hoping to get the right answer. Instead, *strategic competence* is a complex and also an essential strand for mathematical proficiency. At this point, it is relevant to notice how these first three strands are interconnected. The authors assert that "[t]he development of strategies for solving non-routine problems depends on understanding the quantities involved in the problems and their relationships as well as on fluency in solving routine problems" (p. 127). In other words, *strategic competence* depends on *conceptual understanding* and *procedural fluency*. They are not sufficient on their own; the former ability is supported by the latter ones.

Next, Kilpatrick et al. (2001) suggest that mathematical proficiency requires *adaptive reasoning*, which is "the capacity to think logically about the relationships among concepts and situations" (p. 129). That is to say that *adaptive reasoning* is also the ability of making connections between situations, in order to adapt and transfer concepts from one situation to another. It is the ability of relating concepts as well, in order to adapt and compare situations. This capacity of extending the mathematics that you learn to other contexts is a key step for mastering mathematics, that is, for being proficient in mathematics. Once more, strands are intertwined. *Adaptive reasoning* is likely to occur once the first three skills are surpassed. Finally, *productive disposition* is a personal issue related to students' attitudes towards mathematics. Kilpatrick et al. affirm that if a student perceives mathematics as worthwhile and believes that he/she is capable of doing and learning mathematics, he/she has *productive disposition*. Definitely, *productive disposition* can positively influence mathematical proficiency. Although *productive disposition* is a personal characteristic, I believe it is highly influenced by a teacher’s attitude and teaching style. Unfortunately, as Cirillo et al. (2016) highlight, "[t]he residue left behind from traditional mathematics classes often works against the development of productive dispositions toward mathematics" (p. 87). I trust the change in the way mathematics classes run is necessary to better encourage students' *productive disposition*, and this change is fundamentally based on teacher's attitude. Kilpatrick et al. emphasize that "[h]ow a teacher views mathematics and its learning affects that teacher's teaching practice, which ultimately affects not only what the students learn but how they view themselves as mathematics learners" (p. 132). *Productive disposition* can be prompted and fostered in students during their schooling years.

3.1.3. WHY A MATHEMATICAL PROFICIENCY MODEL?

Analyzing students' mathematical understanding is a challenging task, given that it depends on comprehending in what ways students understand. Due to the difficulties in pointing out students' understanding, Kilpatrick et al.'s (2001) mathematical proficiency model was chosen in order to operationalize this process through some indicators. Pirie and Kieren's (1994), and Tall's (2013) models for mathematical understanding, or Bloom's taxonomy (Bloom, 1956) could have been used in order to describe students' understanding in this research data analysis. Nevertheless, for this research purpose, Pirie and Kieren's model characterizes students' mathematical performance in terms of advanced aspects, which can be more difficult to be observed and identified in ordinary mathematics classes; Tall's three worlds are too general in terms of students' mathematical performance; and Bloom's taxonomy is more general (it can be used within other subject-matter) and refers to students' activities, instead of to the end products of students' activities.

On the other hand, Kilpatrick et al.'s (2001) proficiency model presents some benefits for this research, which are: 1) the inclusion of features of mathematical understanding that are present in the literature; 2) the description of students' mathematical performance in a way that relates to the daily practice of a mathematics teacher (as the current research is grounded in secondary education teaching, the closer the results of this research are to teachers' reality, the better); 3) the possibility of teasing apart students' mathematical performance through a comprehensive model that includes aspects related to concepts, procedures, strategies, and reasoning; and 4) the inclusion of student-related aspects in the model, namely, *productive disposition*. Kilpatrick et al.'s model also presents some limitations, in the sense that it does not specifically address: 1) the nuances of students' mathematical understanding within the mathematical proficiency strands; 2) the diversity of mathematical representations; and 3) the different forms of mathematical communication. As these limitations do not necessarily impair this research study, it is being assumed that Kilpatrick et al. model of proficiency can be used for framing students' mathematical understanding with no harm.

In sum, Kilpatrick et al.'s (2001) proficiency model encompasses strands related to mathematical knowledge (*conceptual understanding* and *procedural fluency*), its use (*strategic competence*) and extensions (*adaptive reasoning*), and also to student-related issues (*productive disposition*). Supporting this choice, Barmby, Harries, Higgins and Suggate (2007) confirm that to evaluate the understanding of a mathematical concept, it is necessary to analyze the concept representation in different aspects, such as the conceptual one, the procedural one and the affective one (p. 46). These different aspects are considered in Kilpatrick et al.'s model. Therefore, this model is chosen as a suitable option to frame this study data analysis.

3.1.4. DATA ANALYSIS FRAMEWORK

The basis of this research data analysis framework consists in identifying indicators of each of Kilpatrick et al.'s (2001) five strands of mathematical proficiency in students' work, and then investigating how students demonstrate these strands during the modeling tasks. Table 1 shows the list of indicators used to detect each proficiency strand. This list was elaborated based on the aspects that Kilpatrick et al. (2001) understand as necessary to characterize each of the five strands of mathematical proficiency. In tune with the employed design-based research methodology (to be discussed in Subsection 3.2.1), this list of indicators was susceptible to changes during the research period. During data analysis, a change under the *adaptive reasoning* strand was effectuated to better express the indicators that could be observed within this strand. Along with the three original indicators (logically relate contents, logically relate situations and transfer content between situations), a fourth indicator was added to the data analysis: logically relate content and situation. This addition means that, instead of only expecting students to connect two or more different contents, or two or more different situations, they are expected to connect content and situation as well. Actually, this was the most common indicator identified for *adaptive reasoning*, and sometimes it preceded a content transfer between two different situations.

Although mathematical proficiency and mathematical understanding refer to different matters, there is a close relationship between them that allows a mathematical proficiency model to be used to explore mathematical understanding with the fewest potential drawbacks. Based on Kilpatrick et al.’s proficiency model, if a student is working on his/her mathematical proficiency, then he/she is working on his/her mathematical understanding as well. Kilpatrick et al.'s (2001) strands of proficiency and respective indicators (Table 1) are helpful when discussing this relationship. It is possible to observe that mathematical proficiency encompasses mathematical understanding, given that the former includes aspects of the latter, such as: understanding mathematical contents, building strategies to understand problems, understanding procedures, and logically relating contents and/or situations. On the other hand, mathematical proficiency presents aspects that are not necessarily directly-related to mathematical understanding, such as: retrieving mathematical content, performing procedures correctly, believing in his/her ability to do and learn mathematics, which in turn can be accomplished without mathematical relational understanding. However, these non-directly-related indicators might still have intrinsic influence on students' mathematical understanding, as described in Section 6.3. Hence, all mathematical proficiency indicators in Table 1 were sought during data analysis. As a result — although this research main motivation has always been about students' mathematical understanding — the focus of this research was not only on mathematical understanding, but on mathematical proficiency as well. The emphasis on students' mathematical understanding remains throughout the thesis, though.

Conceptual Understanding	Connect mathematical content.
	Retrieve mathematical content.
	Understand mathematical content.
Strategic Competence	Build strategy to understand problem.
	Build strategy to represent problem.
	Build strategy to solve problem.
Procedural Fluency	Choose right procedure.
	Choose right moment to apply procedure.
	Perform procedure correctly.
	Understand procedure.
Adaptive Reasoning	Logically relate contents.
	Logically relate situations.
	Logically relate content and situation.
	Transfer content between situations.
Productive Disposition	Perceive mathematics as worthwhile.
	Believe in his/her ability to learn mathematics.
	Believe in his/her ability to do mathematics.

Table 1: Data analysis strands of mathematical proficiency and their respective indicators.

3.2. METHODOLOGICAL FRAMEWORK

This section discusses this study methodological underpinnings. The design-based research methodology was used as a foundation to the research intervention design; and complexity science was used to inform the classroom design framework. This section is subdivided into 3 subsections. The first one presents a literature review about design-based research; it defines this methodology, discusses the appropriateness of using a design-based research approach in a mathematical modeling-related research, and considers some related critiques. The second subsection explains why complexity science is used to support the classroom setting, describes what complexity science is, and explores nuances of complexity science in education. Finally, the last subsection presents and explains the classroom design framework, relating complexity principles and the classroom setting.

3.2.1. WHY DESIGN-BASED RESEARCH?

According to The Design-Based Research Collective (2003) (a group of professors and researchers that investigates, experiences and improves design-based approaches in education), educational design-based research aims to combine theoretical research knowledge with every day practical experiences. This blend is expected to yield practical knowledge, that is, knowledge that can be used in practical situations. According to them, this practical knowledge can be helpful in developing teaching and learning processes in educational settings. Cobb, Confrey, diSessa, Lehrer and Schauble (2003) describe design-based experiments as "extended (iterative), interventionist (innovative and design-based), and theory-oriented enterprises whose 'theories' do real work in practical educational contexts" (p. 13). Indeed, my study brings theoretical knowledge into mathematics classes, by means of an iterative intervention, in order to analyze practical issues related to students' understanding and proficiency.

The Design-Based Research Collective (2003) states that "[d]esign-based research is an emerging paradigm for the study of learning in context through the systematic design and study of instructional strategies and tools" (p.5). In accordance with that, the present research investigates mathematical understanding and proficiency (which are directly related to the learning process), in a classroom context, and using a modeling intervention (which speaks to context and to instructional strategies and tools). In addition, The Design-Based Research Collective (2003) states that design-based research is supposed to present five main characteristics. It should: 1) present dual purpose: design of learning environments and theory development; 2) consider the implementation, analysis and improvement of the planned design; 3) produce supportive theories that help practitioners and educational designers; 4) consider genuine settings and contribute to the understanding of intrinsic learning issues; and 5) serve the purpose of addressing relevant findings. The current research contemplates all these features. The research: 1) designs a complex learning environment, in which students are involved in mathematical modeling tasks; and also speaks to the development of a theory that claims that mathematical modeling is of benefit to students' understanding and proficiency; 2) involves implementation, analysis and improvement of the planned intervention; 3) produces a supportive theory that aids teachers and educational designers in understanding how and why mathematical modeling approaches should be employed in mathematics classes; 4) is implemented in an ordinary mathematical class that presents common learning issues and challenges to be dealt with; and 5) addresses important outcomes to the mathematical teaching community. Hence, design-based research is an appropriate methodology for this research.

The Design-Based Research Collective (2003) highlights that

> design-based research goes beyond merely designing and testing particular interventions. Interventions embody specific theoretical claims about teaching and learning, and reflect a commitment to understanding the relationships among theory, designed artifacts, and practice. At the same time, research on specific interventions can contribute to theories of learning and teaching. (p.6)

In tune with that, the current study design is not limited to a particular mathematical modeling intervention. It is supported by complexity science, and it aims to add to the teaching and learning of mathematics, by theoretically analysing students' mathematical understanding and proficiency in the context of a classroom modeling intervention. Zawojewski (2013) agrees that modeling research can lead to the design of theories related to students' learning, and she also believes that modeling can support the design of instructional tools related to the improvement of students' modeling performances. Also, design-based research focuses on the development of learning models (The Design-Based Research Collective, 2003). Consistent with these thoughts, this research uses modeling to set up a different approach for teaching and learning mathematics, which turns out to be an effective teaching model for mathematics instruction. As The Design-Based Research Collective emphasizes "design-based research goes beyond perfecting a particular product. The intention of design-based research in education is to inquire more broadly into the nature of learning in a complex system and to refine generative or predictive theories of learning" (2003, p. 7). Finally, Zawojewski (2013) points out that the nature of mathematical modeling research is related to design-based research. She asserts that

> one of the most salient distinctions between the research methodology used in problem solving and in modeling concerns the role of changing conditions during a study. In a modeling perspective, the researcher assumes that his or her own way of thinking about students' modeling will evolve as part of the instructional tool design process, and therefore research in modeling necessarily embraces and exploits changing conditions (...) rather than working to insure [sic] pure treatments and controls in static conditions in traditional problem solving studies. (p. 243)

Prediger, Gravemeijer and Confrey (2015) wrote a useful, up-to-date and comprehensive literature review about design-based research. The authors discuss what has been done so far and the challenges faced by design-based researchers. Their paper focuses on learning processes, which is exactly the case in the present research. Prediger et al. highlight the fact that design-based research works on instructional approaches and research simultaneously. This way of researching present benefits, for instance, the fact that the research can be revised and adapted according to the need, and the fact that practical instructional approaches can have support from research theories. The authors explain that design-based research focused on learning processes usually: 1) has its underpinnings in learning theories that treat students as agents of their own; 2) is executed during extended periods of time; and 3) acknowledges the close relationship between action and thoughts. Likewise, this research design: 1) is based on complexity science, which views students as agents; 2) is based on data gathered during a four-month mathematics course; and 3) its data analysis observes students' actions, comments and written records in order to grasp students' thoughts and developments. Another aspect highlighted by Prediger et al. is the necessity to look at as many learning indicators as possible throughout the whole learning process, and not only to the task and its outcomes. As will be described in section 4.3, this research collected data in different ways, so that not only tasks and tasks' outcomes were used to observe students' learning.

In terms of justifying the means in a design-based research, Prediger et al. (2015) mention the work of Cobb, Jackson and Dunlap (2015), in which it is highlighted that the association between data, analysis and final conjectures should be endorsed by: 1) the portrayal of students' specific sort of reasoning; 2) evidence that students' reasoning is due to students' involvement in the designed intervention; 3) the description of the way this sort of reasoning comes forward; and 4) the description of the way the learning ecology supports everything (for more details on learning ecology, refer to Cobb et al. 2015). All these four aspects can be seen in the present research, that is, students' mathematical understanding and proficiency: 1) is portrayed in the diagram-based approach used for data analysis; 2) is due to students' commitment with the intervention, as shown by the collected data; 3) is a result of their work on investigating and solving the modeling tasks, as described by diagram narratives; and 4) is established within a complex learning environment. More on the diagram-based analysis approach and respective narratives to follow in section 5.2.

As a final remark, I mention the critiques that design-based research faces. One of these critiques is related to the difficulty in chasing different goals, that is, a design-related one and a theoretical one (Prediger et al., 2015). Another critique mentioned by Prediger et al. (2015) is connected with the kind of instructional materials teachers use. Design-based research is dynamic and cannot rely on prearranged materials and methods, instead design-based research "requires the teachers to continuously adapt to how the students of their classroom act and reason rather than simply use ready-made instructional materials in prescribed ways" (Prediger et al., 2015, p. 884). Teachers and researchers should not limit the possibilities of their work because of the lack of flexibility in regular instructional materials. Overall, the two mentioned issues deserve attention when implementing a design-based research in order not to compromise it.

3.2.2. WHY COMPLEXITY SCIENCE?

Classrooms have the potential to be complex systems. Nevertheless, the school system imposes a structure and some conditions on classrooms that end up inhibiting the complex nature of classrooms. In the particular case of mathematics classrooms, Ricks (2009) asserts that students' difficulties in mathematics learning are precisely because of this lack of acknowledgment of mathematics classes as complex systems. In view of this, this research recognizes that the complex nature of a mathematics classroom must be respected, so that classroom intrinsic conditions are not suppressed. In this sense, complexity science is used as a theoretical framework for the classroom design; that is, as a framework to set up the classroom environment in which the modeling intervention is held. In the proposed classroom design, students are encouraged to collectively investigate the given tasks and to explore the mathematics behind them. Although students are collectively engaged in modeling tasks, this research analysis is not about collective results. This research analysis looks into the individual outcomes derived from the collective work among students, given that individuals are nested in a collective learning system. The goal is to set up the classroom as a non-linear, spontaneous and self-organizing environment, and ultimately reduce mathematics learning issues commonly witnessed in the more instrumental and didactic classes. Davis and Simmt (2014) assert that complexity science can address some existing issues in the present-day school mathematics curriculum. For example, the contemporary curriculum in Alberta and in Brazil is largely founded on assumed linear relationships between mathematics content and learner development; it has changed little in centuries and it does not acknowledge new mathematics knowledge and new understandings of human development. As such, the use of complexity science as a framework for teaching environments has potential for improvements in the teaching of mathematics. Davis and Simmt note that "complexity has emerged in education (...) as a new sensibility for orienting oneself to the world, and for considering the conditions for emergent possibilities leading to more productive, 'intelligent' classrooms" (p. 90). In agreement with that, this classroom design framework is intended to set the environment as a complex system to allow mathematical understanding to emerge when students are engaged in modeling tasks.

Although there is not a unique definition for complexity science (Davis & Simmt, 2014), complexity can be described as the science responsible for analyzing complex systems, that is, systems in which the simple sum of the system parts does not constitute the whole system. Complex systems have a two-way connection: the whole is dependent on the parts, at the same time as the parts rely on the whole (Davis & Simmt, 2003). In other words, to understand a complex system, it is not only necessary to comprehend its parts and the whole separately, it is also necessary to comprehend the relations between them and how those relations generate new possibilities for the system. This connection between parts and whole creates a profound and significant interdependency that generates a powerful and promising environment for the system. As Davis and Simmt (2003) assert, "under certain circumstances agents can spontaneously cohere into functional collectives — that is, they can come together into unities that have integrities and potentialities that are not represented by the individual agents themselves" (p.141). Complex systems are adaptive, and because of that they cannot be compared to complicated systems, which are mechanical, and non-adaptive (Davis & Simmt, 2014). One could say that complexity is a different way of investigating the relationships that we can find in the natural world and also the relationships that are established among people that inhabit our social world. In these worlds, it is possible to observe spontaneous, self-organizing and adaptive relations, in which non-linearity, openness and interactivity are essential to the formation of its relations. That is essentially what constitutes complex systems, and that is what complexity science investigates.

Chaos theory and complexity science are very much interlaced, as both demonstrate that order can emerge from disorder. The chaos theory has its roots in Poincarè's studies almost one century before the arising of complexity theory (Doll, 2008). Although the instability and/or disorder present in complex systems may be seem as a problem or a challenge, it has been argued that it is in fact a relevant part of the system development (Doll & Trueit, 2012); "this sort of disequilibrium is necessary to keep creativity active" in complex systems (Doll & Trueit, 2012, p. 165). In addition, Davis and Sumara (2000) highlight that "[f]ocused on the ways that order often emerges for free when dynamic forms are allowed to interact with one another, complexivists have demonstrated that life itself seems to be organized fractally" (pp.828). Fractalness and self-organization are two key concepts that are part of the complexity science underpinnings (Davis & Sumara, 2000). Taking these complex features into consideration in a mathematics classroom might be a good way to allow and promote innovative mathematical thinking.

In education, experiencing the curriculum through a complex viewpoint permits teachers to work with the unforeseen, and to value different aspects of students' contributions in the teaching and learning process. Doll (2008) highlights how teaching and learning processes have changed and moved towards a complex approach, in which collaborative investigation is being valued: "Learning now occurs, not through direct transmission from expert to novice, or from teacher to student, but in a non-linear manner through all in a class exploring a situation/problem/issue together (and indeed from multiple perspectives)" (p.202). Following this same idea that learning is not a linear process, Lamon (2003) indicates that "there is a great deal of danger in overworking the assumption that children's learning proceeds through stages that can be observed and reformulated into an instructional sequence" (p. 442). In the late 1980s, Doll (1989) was already looking for the complex process that is established within mathematics classes. In one of his articles he reports an experience in a grade 6 mathematics class. He describes that students were given problems to solve, and they had social and intellectual flexibility to organize strategies and figure out solutions. Although disorder emerged in the mathematics classes described by Doll, coherence was also present in students' thoughts and attitudes. As a result, progressively, order became apparent. Not a linear order, but an order sufficient for problem solving. Doll emphasizes that to assimilate what was happening along the activities, the process should be fully analyzed as a whole. He states that:

> Whether an observer saw randomness or progressive order depended on whether that observer was in the class for a few minutes or for the whole class period. (...) While the process seemed disorderly from a segmented view, it had a unity found only by looking at the *whole* class during the *entire* period (p. 66).

This sort of thinking and experience supports the current research, given that contextualized modeling situations are the inquiry starting point that enables students through a collaborative mathematical investigation, in which students are essentially free to develop their own ideas.

As discussed in Chapter 2, modeling tasks are high-order thinking tasks. As previously discussed in Corrêa (2015), Henningsen and Stein (1997) argue that a different number of factors are necessary to support engagement in cognitively high-level mathematical thinking during mathematical tasks. The authors point out that these factors are not necessarily related to the mathematical task itself, but it can be also related to the environment in which the task is being implemented. In other words, the task itself might not be able to engage students in mathematical thinking if they are not properly provided with a supportive environment, including teacher's specific assistance when needed. Consistent with this finding, Henningsen and Stein assume that students' failures in school are not due to the lack of students' capability, but to the lack of opportunities to engage in adequate learning experiences instead. In this sense, a complexity setting seems to be a good option for generating this supportive atmosphere, as it better respects and preserves the authenticity of a learning system that is desired for a mathematics classroom. This argument reinforces why complex science constitutes a useful foundation for this research classroom design framework.

3.2.3. CLASSROOM DESIGN FRAMEWORK

Davis and Simmt (2003) suggest that there are five necessary conditions to implement, develop and maintain a complex environment within mathematics classes. These conditions are: a) *internal diversity*, b) *redundancy*, c) *decentralized control*, d) *organized randomness*, and e) *neighbour interactions*. *Internal diversity* refers to the necessity of having students from different perspectives and with different backgrounds to generate possibilities for diverse contributions to the class. *Redundancy* is when students have common knowledge, experiences and expectations; in this scenario interactions among students are more likely to happen. Thus, *internal diversity* and *redundancy* should be well balanced in the classroom setting. *Decentralized control* speaks to the necessity of having the teacher stepping aside at various points and for varying length of time to leave students free to lead activities, thinking and mathematics in class. This is a challenging task: since the teacher's role has typically been to control and manage knowledge transfer, student activity, and other issues in class. The same challenge holds true for *organized randomness*, because classes have been strictly planned, not leaving room for disorder or the unforeseen. Allowing students to organize their activity in a task can be pretty challenging for many teachers. However, this condition is essential for the emergence of mathematical understanding, given that it grants students the opportunity to organize their thoughts and reasoning in the way that makes more sense to them, supporting their processes of understanding. Finally *neighbour interactions* is related to the essential notion of having students interacting with each other, the teacher, the mathematics and other useful thoughts. These interactions allow students to share insights, ideas and whatever might help building on their mathematical understanding. I agree with Davis and Simmt in that all these five conditions are important in a mathematics classroom setting if we want to promote a collective learning system among class participants. This research endorses that these conditions can enable the emergence of thoughts that will enable mathematical understanding, mathematical proficiency and mathematical knowledge production.

Ricks (2009) understands mathematics classrooms as complex systems in which "teacher's principle responsibility is to develop and maintain a communal classroom entity that mathematizes" (p. 63). Learning mathematics is a consequence of this process. Ricks recognizes the five aforementioned necessary conditions proposed by Davis and Simmt (2003). He defends that, except for *redundancy*, all the other conditions are not present in conservative mathematics instruction. This would explain why students struggle so much when studying mathematics: because they are immersed in a complex system, in which not all necessary conditions to support it are present. Ricks wisely asserts that "every recommendation to improve the work of teaching in mathematics education is supported by principles of complexity. Similarly, (...) all detrimental effects to student learning are understandable from a complex perspective" (p. 63). In accordance with Ricks (2009), whatever learning approach is being used to teach mathematics — mathematical modeling in this case — the five conditions proposed by Davis and Simmt (2003) should be considered and pursued.

Based on the above aspects and arguments, the proposed framework for the classroom design consists of carrying out mathematical modeling tasks in a classroom environment that sustains Davis and Simmt (2003) five complex conditions. Figure 3 illustrates this framework. The pentagon represents the classroom, in which the five conditions are sought. Within this classroom, a learning context is established. This context refers to the use of mathematical modeling tasks as a resource for teaching and learning mathematics. Within this setting, students' mathematical understanding and proficiency are investigated along the teaching and learning processes. In line with this framework, Davis and Simmt (2014) claim that the use of complexity in classroom

> may have the most potential for affecting school mathematics by offering guidance for structuring learning contexts. In particular, complexity offers direct advice for organizing classrooms to support the individual-and-collective generation of insight — by, for example, nurturing the common experiences and other redundancies of learners while making space for specialist roles, varied interpretations, and other diversities (p. 90).

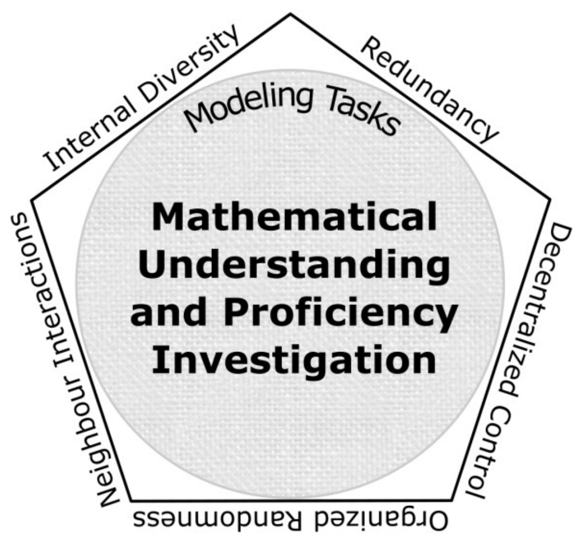


Figure 3: Framework schema for the classroom design.

Section 4.2 describes the research intervention in detail, including how the five complexity conditions were sought and achieved in the classroom setting design.

4. Approaching the question: The intervention

The goal of this chapter is to describe practical details related to the research design methodology. The chapter is divided into four sections. The first one describes the research participants, presents how participants were grouped along research interventions, and discusses ethical issues. Then, the second section is about the research intervention; it describes the intervention and respective stages, relates the intervention classroom design with the five complexity conditions that support the research framework (described in subsection 3.2.3), explains how each condition was achieved, and speaks to the production of knowledge during the intervention. The third section describes and comments on each of the four data collection methods. Finally, the fourth section is about the modeling tasks created for this research purpose; it describes the guiding principles of modeling tasks, discusses this research tasks main aspects, considers engagement factors and hindering aspects involved in high-level thinking tasks (such as the modeling tasks used in this research), ponders about scaffolding practices, and in the end presents each of the four modeling tasks with detailed explanations about possible solutions and prompts.

4.1. Participants

The researched group was composed of a high-school class enrolled in a grade 11 Alberta mathematics course (Mathematics 20-1). The group was an International Baccalaureate (IB) class composed of 27 students. Although all of them participated in the modeling tasks, only 16 brought the consent form back. Among these 16, one parent did not consent complete access to his child's data, one parent did not consent access at all to his child's data, and another parent consented access to his child's data, but the child did not assent to provide complete data. Therefore, these 3 students were not included in data collection. Among the 13 remaining students, one did not show up for the tasks, except for the first day of task one. Hence, he was discarded from data collection, which in turn was based on 12 students. All 27 students were divided into groups to work on the tasks. The 12 participants, who allowed video recordings, were tentatively concentrated into three groups. For task number one, the classroom teacher was more directive when forming the groups. For the other three tasks, students were more or less free to organize themselves into groups, as long as they (preferably) changed previous group configurations. Table 2 presents groups' final configuration for each task. In the case that one student's name is not shown in the table for a specific task, it means this student was either absent, or in a group with research non-participants.

	Group 1	**Group 2**	**Group 3**
Task 1	Leo Philip Rick Maya	Amanda Brenda Nathalie	Diana Jack Sophia Thomas
Task 2	Clara Diana Nathalie Rick	Jack Leo Maya	Brenda Philip Sophia Thomas
Task 3	Brenda Nathalie Rick	Amanda Leo Philip	Clara Jack Sophia Thomas
Task 4	Amanda Diana Rick	Clara Leo Philip	Jack Sophia Thomas

Table 2: Formed groups for each task.

Data collection was done in a high-school setting with youth under the age of legal majority. The core principles of the *Tri-Council Policy Statement: Ethical Conduct for Research Involving Humans*, were respected, namely: respect for persons, concerns for welfare, and justice (Canadian Institutes of Health Research, Natural Sciences and Engineering Research Council of Canada and Social Sciences and Humanities Research Council of Canada, 2010).

Respecting people encompasses respecting their will to participate in the research and their autonomy in relation to research issues. Participation in this research was voluntary and participants were free to withdraw whenever they wanted, including withdrawing data already collected. Students were presented with informed consent documentation explaining the research purpose and clarifying all the details related to their participation. Parental consent was sought for all participants. Student assent was also sought. Further, students were allowed to remove or provide additional information related to their personal data within one month post data collection. Any content on transcripts or in videos that students said was embarrassing was disregarded and/or deleted. Anonymity was a concern as well. No other person, other than the researcher and her supervisor, had access to the identity of the participants. Thus, all participants had a pseudonym, in order to keep them anonymous.

As for concerns for welfare, as this research intends to investigate potential high-school scenarios that might positively influence students' mathematical understanding, proficiency and knowledge production, being part of the research represented having worthwhile learning mathematical experiences. In this sense, participating in the study might have been an advantage. Students' learning processes were at stake and were not neglected because of research goals. However, even in this scenario, there was a minor risk of hindering students' learning processes by research interventions. The classroom teacher was able to call an end to the research in total or for a particular student if needed. According to the teacher's monitoring, that was not necessary.

Finally, regarding justice, it was necessary to have a fair and equitable environment, in which participants felt safe and in equal conditions to engage in the research. To help sustain a just environment, the researcher built teacher-student appropriate relationships with participants, and avoided personal bias and pre-assumptions during classes, on interviews and on data analysis as well.

4.2. INTERVENTION

Based on design-based research underpinnings, in order to investigate students' mathematical understanding and proficiency while engaged in mathematical modeling tasks, an intervention was elaborated to allow students to engage in the modeling process and to work on their mathematical understanding and proficiency while data was being collected. As aforementioned, this research was planned to take place during a four-month grade 11 mathematics course. The course started in February 2015 and ended in early June 2015. The research data collection was based on four tasks done during four different interventions. The first intervention was completed by the end of February, the second by mid March, the third by the end of April, and the fourth at the beginning of June.

According to Biembengut and Hein (2002) (cited in Zorzan, 2007), the implementation of mathematical modeling in class is based on five different stages: 1) diagnosis of students' interest; 2) selection of the mathematical model or theme; 3) development of the content to be studied; 4) students' orientation towards the modeling process; and 5) assessment of the whole process. The intervention cycle of the present research (Figure 4) was based on these implementation stages, except for one extra initial moment, in which students were presented with an illustrative example of modeling in order to understand what was being required for the study. After this first moment, a poll was done with students, so that they could contribute to the choice of themes for the modeling tasks (Stage 1). The four most preferred themes were selected and the researcher formulated four tasks, one for each theme, to be used in four different interventions (Stage 2).

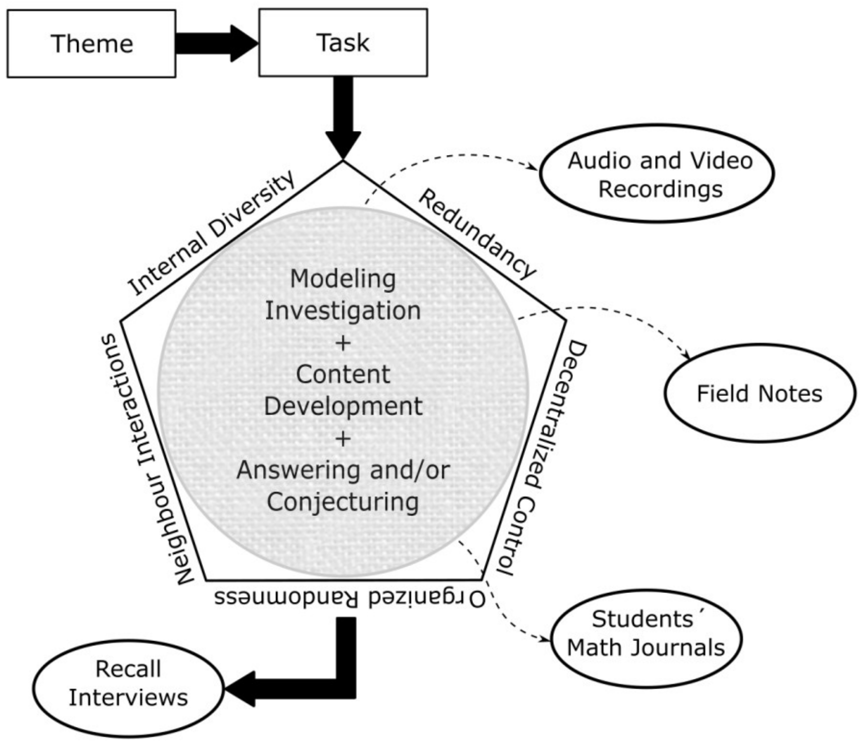


Figure 4: Intervention cycle.

For each task, the class setting was organized according to the classroom design framework (described in subsection 3.2.3); which means the classroom setting was in agreement with the five conditions of Davis and Simmt (2003) to support a complex environment (*internal diversity*, *redundancy*, *decentralized control*, *organized randomness* and *neighbour interactions*). The pentagon in Figure 4 represents this arrangement. Since in this study the mathematics classroom is acknowledged as a complex system, it is assumed that these conditions are intrinsically present in the natural classroom setting — except when the schooling system imposes other circumstances. Considering that these conditions can vary all the time in class, conditions were monitored during the intervention in order to ensure the appropriate setting.

Redundancy was considered to be already established, given that students were at the same grade, experienced common or similar educational settings, and were taught the same contents. As for *internal diversity*, it was not possible to assume that students would always naturally group themselves taking into account their diverse profiles. Hence, the teacher was to interfere as needed in order to group the class favouring diversity in terms of students' profile, life experiences and backgrounds. Nonetheless, the teacher also tried not to make students uncomfortable within their groups, since this could impair students' engagement and task resolution. In this sense, group arrangements were flexible. The possibility for *neighbour interactions* was satisfied as students were allowed and encouraged to share ideas with the teacher, the researcher and their peers, either within their groups or within the whole class. To facilitate group work and interactions, every day, before the beginning of each task, tables were set together according to the number of groups and to the number of students in each group. Finally, the last two conditions — *decentralized control* and *organized randomness* — were more challenging to achieve, but were well managed by the class teacher. Students were encouraged to choose strategies and follow procedures according to their own preferences and thoughts. This openness gave room for random processes to emerge, which does not mean unorganized processes. They were random, but organized according to students' reasoning, which characterizes *organized randomness*. Students also made decisions in relation to the modeling process, even when these decisions were inadequate from the teacher's perspective. They had to look for necessary knowledge (information, processes, concepts, etc.) and to figure out strategies to find answers to the task or to make conjectures. In this way, students were in charge of the modeling process, characterizing *decentralized control*. Naturally, the teacher and the researcher were able to help students, prompting questions and thoughts when requested or needed. In order not to weaken students' thinking and decision-making processes, it was important that teacher and researcher did not make substantial unnecessary contributions. Students were most of the time working by themselves. Teacher and researcher did not interfere a lot, they were mostly observing. It was essential for teacher and researcher to provide a judgment-free environment, in which students felt safe to cooperate and to take risks. It is relevant to note that, although teacher and researcher were scaffolding students and even teaching some content during interventions, centralized control was not in place during interventions. Teacher and researcher were part of the system, and when addressing situations they were agents of the system making contributions. In a complex system, control shifts all the time. That was not different during students' engagement in these modeling tasks. Sometimes the teacher or the researcher had control, sometimes students had; this does not mean that control was centralized in the teacher or in the researcher. Control was distributed within system agents. Definitely, *decentralized control* and *organized randomness* are tricky to maintain, but the teacher and the researcher had them in mind throughout the project. The complex nature of the mathematics learning environment was respected and the investigation process was not impaired.

As long as these conditions were settled and maintained, the modeling task could proceed. According to Biembengut and Hein (2002) (cited in Zorzan, 2007), after the selection of the theme, the content to be studied should be developed (Stage 3) and students should move towards the modeling process (Stage 4). In the current research, students were supposed to work on the content to be studied (Stage 3) and undergo the modeling process (Stage 4) simultaneously and on their own. This means students were not taught the content knowledge necessary to model the task beforehand. Students were invited to investigate the modeling task, work on it, learn or produce new mathematical content if necessary, and answer questions or make conjectures related to the task. The teacher and the researcher were available to scaffold students. It is worth mentioning that having students producing the necessary knowledge during modeling tasks was one of the differences of this research in relation to other studies. Contrast this approach with Bracke and Geiger's (2011) research, in which they inquiry about the viability of the use of modeling tasks in a regular basis during a whole grade 9 mathematics course. The authors report that "students were directed to use methods which have been discussed in the lessons before the start of the respective task" (p. 532). This is not uncommon when there is a concern about fulfilling curriculum demands. In this case, teachers are usually more worried about covering a specific content than about doing modeling activities, given that in this sort of activity students are at risk of not using the desired mathematical content. Zbiek and Conner (2006) affirm that:

> We note that a teacher's or a curriculum's focus on learning curricular mathematics often means the situation is narrowly cast to ensure students use a particular type of entity. In this case, the teacher's purpose for a modeling activity is to use the modeling activity to introduce or use a particular type of mathematical entity, regardless of how this purpose and entity match the students' modeling purposes and known or natural entity choices (p. 92).

The intervention cycle was repeated for each of the four tasks. During tasks and after tasks were completed, the whole intervention cycle was assessed, so that the next intervention could be improved if needed. In fact, according to the design-based foundation of this research, the research intervention cycle was subject to change along the process. Although the core structure of the intervention remained the same, the way the teacher dealt with the implementation of each task was the result of reflective assessments made during and after each task. This assessment refers to Stage 5 of Biembengut and Hein's (2002) (cited in Zorzan, 2007) mathematical modeling stages.

4.3. DATA COLLECTION METHODS

During each intervention, *audio and video recordings* of students' activities, and researcher's *field notes* were gathered, and at the end of each task students' *mathematics journals* were collected. After the modeling task was done and task questions were answered or saved for future investigations, students were invited to participate in *recall interviews* in order to clarify or explain any further questions related to the tasks. These four data collection methods were replicated for every intervention and served the purpose of analyzing students' mathematical understanding and proficiency, as well as of assessing, modifying and improving the intervention process or tasks preparation in case it was necessary. The next subsections detail each of these four methods.

4.3.1. AUDIO AND VIDEO RECORDINGS

Students' classroom activities were video and audio recorded throughout each task, in order to allow recurrent examination. Video records were collected using iPads. The recording setting was prepared in a way that iPads were not directly facing students, so that they did not feel constrained or under pressure during the tasks. Audio records were collected by iPads, and also by audio recorders placed in the middle of table arrangements. As students are used to iPads and cell phones, and are also used to video and audio recordings (their own recordings, for example), in general, students seemed to be comfortable enough to act and interact naturally. These recordings were used to create students' transcripts for tasks. These transcripts were fundamental to the analysis of students' mathematical understanding and proficiency.

4.3.2. FIELD NOTES

While students' activities were being audio and video recorded, the researcher was walking around the classroom, observing students and taking field notes based on students' work, questions and conjectures about tasks. These field notes were to complement perceptions and comments gathered through recordings.

4.3.3. *MATHEMATICS JOURNALS*

During activities, students were encouraged to report their discussion processes and investigation outcomes by writing notes in their journals. They were provided with enough time to do that. Sometimes, it was necessary to prompt students to write or represent their thoughts and procedures in their journals. These written materials helped to portray students' mathematical understanding and proficiency. These documents were compared and contrasted with audio and video recordings, providing adequate warrants for the conclusions made in relation to students' mathematical understanding and proficiency. Mathematics journals were used in stimulated recall interviews as a way of reminding students of what they did during the tasks. Journals served also as underpinnings to the data analysis.

4.3.4. *STIMULATED RECALL INTERVIEWS*

After each intervention was completed, students were invited to participate in stimulated recall (SR) interviews (Anderson, Nashon, & Thomas, 2009) to further discuss what they have experienced in class. Not all invited students were able to participate in the interviews. SR interviews are interviews in which students are asked to recall situations that have happened in class and discuss them. For the SR interviews, it was extremely important to provide students with a comfortable and respectful environment; not to create expectations in relation to what students were to say; not to induce students by researcher thoughts; and not to drive students to say whatever the researcher wanted to hear from them. A relevant aspect of SR interviews is that if students feel comfortable enough to speak freely, they are allowed to drive the interviews, given that this can be helpful in understanding their thoughts. These potential moments let "participants' own discourse to become the subject of self reflection and repeated self analysis" (Anderson et al., 2009, p. 191). This strategy is likely to draw researcher's attention to relevant issues that could not have been noticed before, and can definitely enrich research findings.

SR interview questions were meant to unpack students' work on tasks and also to investigate students' mathematical understanding and proficiency based on Kilpatrick's et al.' proficiency model. Some general questions or prompts were elaborated for discussion in SR interviews. Some other particular questions or prompts were also elaborated, but these were based on student's specific work during the modeling tasks. The combination of asked questions depended on the task, the student and the course of the interview. The general questions or prompts were:

A. Tell me about your experience by doing this mathematics task.
B. Were you confident in doing the task? Did you believe you were able to come up to a conclusion?
C. What kind of connections did you do between the task and your mathematical knowledge?
D. What was the main idea you were working on during the task?
E. What was your main strategy? Why did you decide to do what you did?
F. Did you change your mind/strategy/procedure along the process? Why? What about your peers in the group?
G. Were you able to mathematically formalize your thoughts and come to a conclusion? Were you expecting what you got as a conclusion?
H. Do you have any other comment you would like to add about the task?

Some examples of particular questions or prompts are:

a. Can you explain your thoughts about the pictures you did.
b. Can you clarify the sentence "Make triangles for approximation"?
c. What about the use of the formula $-b/2a$? Where did this formula come from? Do you understand its meaning?
d. Why did you decide to work in three dimensions?
e. Why did you connect the GPS with distances and not with coordinates in your solution, although when asked you said that the GPS gives us coordinates?
f. What did motivate you to find a pattern?
g. I see your strategy was first based on the given numbers. Can you explain your thoughts while investigating the task?
h. Why was it difficult to interact with other group members? Why didn't you share your thoughts?
i. Were you unmotivated? Why were you frustrated and confused?
j. Did the lack of numbers somehow unmotivated you?
k. Can you go over the thoughts on your journal and help me understand how your thinking process changed over time?
l. Did you give up on your attempt with percentages?
m. Why were you rounding the numbers?

n. What and why were you trying to graph?
o. Where did the *x* equation come from?
p. Can you talk a little bit more about the final comment on your journal?

Because SR interviews are based on post-reflections about what students meant or thought, they might not reflect students' activities and opinions with exactness. In this sense, it is essential to consider other methods of data collection, such as the ones described before. Audio and video recordings, field notes and mathematics journals reflect students' activities in the actual intervention setting; therefore students' ideas and thoughts might be more accurate when gathered by these methods. For this research purpose, I decided only to use data specifically connected to the classroom activities. In analyzing and overlapping the data collected from the four different portrayed methods, findings supplement each other, bring up nuances and discrepancies, give insights about students' thinking, and as a result provide a richer picture of the situation.

4.4. MODELING TASKS

Lesh, Yoon and Zawojewski (2007) describe model-eliciting tasks as tasks which illustrate contextualized situations, with the aim of promoting, "powerful, sharable, and re-useable conceptual models" (p. 323). The authors also emphasize that model-eliciting tasks are supposed to be "thought-revealing" (p. 323). Indeed, Doerr (2006) states that "[m]odel eliciting tasks provide students with opportunities to reveal how they are thinking about the situation by representing their ideas" (p. 256). In other words, thought-revealing means that the task incites students to describe, represent, explain and justify the thinking and reasoning processes that emerge while students are working on their modeling tasks and its outcomes (Doerr, 2006). Although throughout this research I was not concerned about naming the kind of task I was using, the created tasks certainly could be categorized as model-eliciting tasks.

Blum and Ferri (2016) present six criteria that should be considered when creating modeling tasks. These criteria are essential for the creation of modeling tasks and should be given the appropriate attention.

1. Focus on the necessity of creating a task that truly addresses *life situation contexts*, instead of unfeasible mock situations.
2. Create an *open task*, in which a single correct answer is not the only possible solution for the task.

3. Make the task *complex enough*, so that the task is not straightforwardly solved without challenging students' thoughts.
4. Create a *tricky task* so that it is problematic enough in order to trigger students' high-level thinking.
5. Create a *cognitively accessible* task that invites students to work within their zone of proximal development (Vygotsky, 1978), so that they do not give up on trying to solve the task.
6. Ensure the task considers *all modeling cycle stages*, in order to have students working on all modeling competencies.

Although these criteria might seem uncomplicated to follow, this is not exactly the reality. Undeniably, a lot of effort and time is necessary when elaborating these modeling tasks. The tasks created for this study followed these six criteria. Nonetheless, there was an extra constraint or criterion to follow; that is, the tasks were to promote the study of a specific content of the mathematics course. This necessity posed an additional challenge for researcher and teacher in the creation of the tasks. The goal was to have tasks created in such way that — although students were allowed to follow their own paths when solving tasks — they would ideally end up working on the desired content.

Based on the above criteria, the research tasks were designed in order to work on students' reasoning skills and mathematical knowledge, by offering students thought-provoking, challenging, and high-order thinking tasks, based on situations likely-to-be experienced. Students were encouraged to collectively analyze tasks, make decisions and assumptions, create models, validate models, and share their ideas, notes, thoughts, and conjectures with the whole group. With this in mind, it follows that four different tasks were specifically elaborated for this research purpose. Each research intervention was based on one task, and tasks aimed to address four different content areas from the teacher's Mathematics 20-1 course planning. Suggestions by the classroom teacher were taken into consideration in the creation of the tasks. The first task was about quadratic functions in a profit context, the second task was about the cosine law in a flight simulator context, the third task was about rational equations in a medley relay context, and the fourth task was about arithmetic and geometric sequences in linear and binary search contexts. The first task was implemented during three consecutive 80-minute classes, while each of the other three tasks was implemented during two consecutive 80-minute classes. In all tasks, students were asked and remembered to record variables, assumptions, strategies, thoughts, changes in reasoning, and doubts in the mathematics journal each of them received. Students were also encouraged to numerate their steps as a way of helping them to organize their thinking process. Although students were encouraged to work collaboratively in teams, at the end of the process they could find different and individual solutions for the same problem.

As mentioned before, for each task, there was at least one not-already-taught mathematical concept or procedure, which was expected to be needed in the solution of the respective modeling task. This mathematical content was expected to be constructed during the modeling task; that is, they were expected to be raised during the time data was actually collected. This expectation is defended by Doerr (2016), who believes that "[l]earning mathematical content occurs through the process of developing an adequate and productive model that can be used and re-used in a range of contexts" (p. 198). This approach was used on purpose, because this sort of encounter with new knowledge during an investigation — in which the new knowledge is useful and necessary — presumably enables mathematics appreciation, and meaningful and longer lasting learning.

Aligned with the design-based underpinnings that support this research methodology — after one or more tasks were applied and based on students' prolonged struggle — whatever was deemed enabling or constraining in promoting students' mathematical activity and modeling was taken into account to modify or adapt the following task. The main challenge in the elaboration and implementation of the tasks was the decision about how much data was just the "right amount" for learners to work through the task relatively autonomously from the teacher and researcher. The first task provided students with a reasonable amount of information. For the second task, the decision was to experiment with almost no data given with the prompt. However, by the end of implementing task number two, the teacher and the researcher both felt the need to give some numerical data to scaffold the student meaning making. Combining the experiences and lessons learned with the first two tasks, the third task was created and implemented. Again, the amount of given information was an issue. For this task, there was not the same amount of data given in task number one, but there was also not the same lack of information as in task number two. Task number three was in the middle, and some extra information was available in case students asked for it. But that was not the case. Finally, all first three tasks experiences were contemplated when finalizing task number four, which was somehow in between tasks number two and three in relation to the amount of given data.

As mentioned in Corrêa (2015), Henningsen and Stein (1997) research (based on 22 high-level tasks) highlights at least seven relevant factors for maintaining engagement in high-level thinking tasks. The authors' concluded that each of these factors influence students' engagement in a different way. The seven factors are shown in the first column of Table 3, while their respective percentages of influence in students' engagement are shown in the second column of Table 3.

1. Building connections with students' background knowledge.	82%
2. Providing students with proper amount of time to do the activity, not too little and not too much.	77%
3. Emphasizing meaning and requiring students to explain their understandings.	77%
4. Having students modeling thinking processes and strategies.	73%
5. Scaffolding when necessary.	73%
6. Enabling students to monitor and question their selves.	36%
7. Drawing conceptual connections.	14%

Table 3: Engagement factors and influence percentages in high-level thinking tasks.

Henningsen and Stein wisely assert that "[n]ot only must the teacher select and appropriately set up worthwhile mathematical tasks, but the teacher must also proactively and consistently support students' cognitive activity without reducing the complexity and cognitive demands of the task" (p. 546). Although this is not a simple job, all these seven factors were sought during the implementation of each of the four tasks, in order to sustain students' high-level mathematical thinking.

Moreover, as discussed in Corrêa (2015), Henningsen and Stein (1997) explain that high-level tasks require students to take risks and this can be a reason to have teachers pressured to reduce the activity to some instructional steps or reduce accountability expectations. The authors emphasize that the demand for right answers in a short period of time might detract the focus on mathematical understanding. As so, Henningsen and Stein highlight issues that might hinder the engagement in the mathematical thinking process and, because of that, deserved attention during tasks implementation. They are: 1) unsuitable tasks; 2) problems with classroom management; 3) inadequate amount of time; 4) accountability is not required; 5) challenges that are no longer challenges; and 6) focus that changes to finding the right answer. Being aware of these factors was a critical aspect during data collection, because if classroom interventions incurred in a few of them, looking for mathematical understanding and proficiency could be an ineffective attempt.

It is relevant to note that when doing all four tasks, the strategy was not to give unnecessary support for students to start with. In other words, students would be given just enough data to start with. During the task, if a group required extra support, they would get an appropriate piece of advice or information, just enough to scaffold their meaning making. But if a group did not require scaffolding, they would not get it and would have the opportunity to face difficulties and challenges on their own. Stillman (2001) asserts that scaffolding is directly related to the number of decisions students are to face and deal with. The lower the scaffolding the higher the number of decisions to be made. The higher the scaffolding, the lower the number of decisions to be made; which can interfere in the modeling process and, as a result, in students' mathematical understanding and proficiency. Vorhölter et al. (2014) highlight that the scaffolding practice cannot be based on the teacher's immediate thoughts about what is being done by students. It needs to be based on the diagnosis of students' understanding instead. Kaiser and Stender (2013) proposes this diagnosis to be done by asking students what their state of work is. This is definitely a good practice to promote students' thinking processes. The intention during the scaffolding in this research task was not to induce students to choose a path according to the available information, and also not to get in the way of their thinking processes by anticipating steps they were supposedly able to achieve. The aim was to prompt "students to move forward without removing [the task] problematic nature" (Cavey & Champion, 2016, p. 132). Allowing students to fully experience this process is a way of promoting high-level mathematics knowledge and reasoning, as well as confidence. Along with that thinking, Gann et al. (2016) claim that

> the productive struggle inherent in modeling is key to students' success in mathematics. It builds their skills in solving problems; gives them experiences in thinking creatively to solve future, more complex problems; and helps them develop an increasingly nimble and flexible mathematical mind. For many students, successfully solving these modeling problems counts among their most memorable experiences in mathematics (p. 105).

Based on the aforementioned reasons, as a rule, students were prompted only after they were given enough time to struggle, investigate and elaborate on tasks. Consistent with that, Kaiser and Stender (2013) point out that a relevant and needed modeling piece is when "students experience long phases of helplessness and insecurity" (p. 280). Stender and Kaiser (2016) draw attention to a relevant feature that should be sought during modeling tasks: students' independence. The authors highlight that out of the school setting, students will not have external advice to do the modeling and will have to search for their answers by themselves. Therefore, Stender and Kaiser suggest that, when modeling, students should get enough help, not too much and not too little. In this way, teachers can still be supportive in terms of student's independence.

The following subsections present the four tasks on their final versions, illustrate possible solutions, suggest possible prompts, and describe what was anticipated by the researcher for each task. Doerr (2016) underlines that teachers need to "anticipate possible solutions and ambiguities that might occur as students express their emerging ideas" (p. 203). Brief explanations about students' work on tasks are also included.

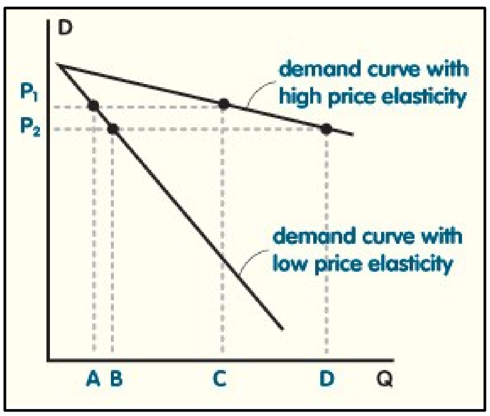
4.4.1. TASK ONE: COST, REVENUE AND PROFIT

Task number one, shown in Figure 5, was intended to have students modeling a quadratic function. At this point, students had not yet been exposed to this content in their mathematics course. The only content they had knowledge about was the parabola. They knew about the main features of the parabola and about the vertex form of its equation. It was anticipated that students would model a quadratic function while handling the problem, and then use their previous knowledge about parabolas to complete the square and find the zeros of the modeled function. The next paragraphs describe what the researcher anticipated students might do before task one was implemented.

As the relationship between profit (P), revenue (R) and cost (C) was mentioned on the text of the task, it was anticipated that students could start off by figuring out an equation to represent that the profit (P) is equal to the revenue (R) minus the cost (C). They could come up with the following equation, for example: $P = R - C$ (1). If not, they could be prompted to do so by being asked: *"Can you justify the relationship between profit, revenue and cost? Do you think that relating these elements through an equation might be helpful?"* Then, students could figure out a relationship between revenue, quantity sold and price per unit, that is, $R = xy$, where y is the price per unit and x is the quantity sold. They could also consider the cost being composed by the cost per unit (c) and the fixed cost (C_0), that is, $C = C_0 + c$. As a result, the profit equation number 1 could turn out to be $P = xy - C_0 - cx$ (2). If students do not figure out this relationship, they could be prompted by being asked: *"Look at the given data in the table and in the problem. Can you relate the given data with the profit equation? How would you do that?"*

Price elasticity tells how much of an impact a change in price will have on the consumers' willingness to buy that item. If the price rises, the law of demand states that the quantity demanded of that item will decrease. Price elasticity of demand tells you how much the quantity demanded decreases. Elastic demand means that the consumers of that good or service are highly sensitive to changes in price. Usually, a good which is not a necessity or has numerous substitutes has elastic demand. Inelastic demand means that the consumers of that good are not highly sensitive to price changes. If the price of an inelastic good, say [bread], rises by 10 percent, maybe sales will only decrease by 1 percent. Consumers will still buy that good, typically because it is essential or has no substitutes. (Tuck, n.d.)

In the graph below, D stands for price and Q stands for quantity demanded.



(Elasticity of demand, n.d.)

Total revenue is calculated as the quantity of a good sold multiplied by its price. It is a measure of how much money a company makes from selling its product, before any costs are considered. Obviously, the goal of a company is to maximize profits, and one way to do this is by increasing total revenue. The company can increase its total revenue by selling more items or by raising the

---

Suppose you are responsible for the accountancy of a book store. You were requested to do a price analysis for two products: a text book and a reading book. In this analysis you should consider cost, revenue and profit. The table below presents some data about variation in quantity sold when prices vary. Other than that, it might be helpful to know that there is a fixed cost of CAD$ 618 to produce either book. Based on your analysis, what is the best option to price each product in order to maximize the store profit?

Text Book			**Reading Book**		
Unit Price	*Quantity Sold*	*Cost/Unit*	*Unit Price*	*Quantity Sold*	*Cost/Unit*
CAD$ 26	102	CAD$ 18	CAD$ 24	112	CAD$ 18
CAD$ 30	100		CAD$ 30	100	
CAD$ 34	98		CAD$ 34	92	
CAD$ 40	95		CAD$ 38	84	

Figure 5: Task one – Cost, revenue and profit.

Once students got to equation 2, they could get stuck. The idea was to find out a quadratic equation by substituting *y* in terms of *x* or vice-versa. However, students could not realize that. Hence, they could be prompted by being asked: *"How many variables are there in equation 2? Would it be helpful to have fewer variables? How many variables would be ideal? Is there any way of reducing the number of variables based on the information you already have?"* If this was not enough, they could need more direct prompts, such as: "*Would it be helpful to find a relationship between x and y?"* This relationship could vary according to the situation they were analyzing: the text book or the reading book. If students were analyzing the text book table, they could get $y = -2x + 230$. If students were analyzing the reading book table, they could get $y = -\frac{x}{2} + 80$. At first, students were expected to find out any of these two relationships without struggling, but that could not be the case for all of them. Hence, students could be given some scaffolding if needed.

With the relationship between *x* and *y* in hand, the idea was to go back to equation 2 and substitute *y*. Students could substitute *x* instead. If they substituted *y*, they could get

for the text book situation

$$P = xy - C_0 - Cx$$
$$P = x(-2x + 230) - 618 - 18x$$
$$P = -2x^2 + 212x - 618$$

or

for the reading book situation

$$P = xy - C_0 - Cx$$
$$P = x\left(-\frac{x}{2} + 80\right) - 618 - 18x \, .$$
$$P = -\frac{x^2}{2} + 62x - 618$$

The next step could be figuring out the maximum profit. Given that students already knew that a parabola has a maximum or a minimum value, it was expected that they could relate the profit quadratic function to a parabola. If they needed help to figure this out, the following prompts could be helpful: *"Can you describe how profit and quantity vary? Would the graph be helpful to visualize this relationship?"* Once students realized the maximum value was related to the vertex of the parabola, they would need to find the vertex coordinates. Students already knew how to find the vertex coordinates when they had the equation in the vertex form. In this case though, they had the equation in the general form, and to get to the vertex form they needed to complete the square. This moment was anticipated and it was left to unfold according to students' thoughts throughout the task. Some teaching moments were expected. At this point, some prompts could be helpful, for instance: *"Are there any similarities between the general form and the vertex form? If the general form was a perfect trinomial would that be helpful? Why? If the general form is not a perfect trinomial, could you make up a perfect trinomial to help you out? How?"* Students could need some other specific prompts as well, depending on group ideas.

If students were working on the text book situation, they could find:

$$P = -2\left(x^2 - 106x\right) - 618$$
$$P = -2\left(x^2 - 2 \cdot x \cdot 53 + 53^2 - 53^2\right) - 618$$
$$P = -2\left(x^2 - 2 \cdot x \cdot 53 + 53^2\right) + 2 \cdot 53^2 - 618$$
$$P = -2\left(x - 53\right)^2 + 5618 - 618$$
$$P = -2\left(x - 53\right)^2 + 5000$$

If students were working on the reading book situation, they could find:

$$P = -\frac{1}{2}\left(x^2 - 124x\right) - 618$$
$$P = -\frac{1}{2}\left(x^2 - 2 \cdot x \cdot 62 + 62^2 - 62^2\right) - 618$$
$$P = -\frac{1}{2}\left(x^2 - 2 \cdot x \cdot 62 + 62^2\right) + \frac{1}{2} \cdot 62^2 - 618$$
$$P = -\frac{1}{2}\left(x - 62\right)^2 + 1922 - 618$$
$$P = -\frac{1}{2}\left(x - 62\right)^2 + 1304$$

Once they had the vertex form, finding the vertex coordinates could be easier if students connected to their previous knowledge. They could get:

for the text book situation $\begin{aligned} x_v &= 53 \\ P_{\text{Max}} &= 5000 \end{aligned}$ or

for the reading book situation $\begin{aligned} x_v &= 62 \\ P_{Max} &= 1304 \end{aligned}$.

That is still not the task final response. The task asks for the ideal price, which is the variable y. Therefore, students could go back to the price function and get the y value corresponding to the x_v. That could be:

for the text book situation $\begin{aligned} y_v &= -2x_v + 230 \\ y_v &= -2 \cdot 53 + 230 \\ y_v &= -106 + 230 = 124 \end{aligned}$ or for the reading book situation

$$y_v = -\frac{x_v}{2} + 80$$
$$y_v = -\frac{62}{2} + 80 \quad .$$
$$y_v = -31 + 80 = 49$$

Some possible extensions were planned for students who finished the task before the end of the given time. Students could be asked: "*Can you justify the difference between the price of the text book and the price of the reading book? Can you justify why the vertex coordinates are* (q, p) *when the parabola equation is* $y = a(x-q)^2 + p$ *? Can you find out the vertex coordinates without using the vertex form?*"

Students' work in general did not correspond to the described expectation. Basically, all three groups started off by analyzing the given tables, which reinforces that students are more used to problems with numerical data. Students' first goal was mainly to find out if there was any pattern ruling the data given in the tables. In general, they did not need prompts to give this first step. After that, their solutions evolved to find equations 1 and 2, and then to find the quadratic function. Most students needed some prompt to figure out that the vertex form of the quadratic equation was the link between the equation they had in hand and their previous knowledge about the maximum point of a parabola. Once the teacher noticed that groups were in need of the vertex form, he prompted students and discussed about the relevance of writing the profit equation into the vertex form. The teacher then taught the whole class how to complete the square. Afterwards, students were able to apply the learned procedure to complete the square, find the vertex form of the quadratic function, and finally finish their solutions. Students from two of the three groups got to a final answer, while other students got close to a final solution, with a decent notion of what their next steps would be.

4.4.2. TASK TWO: FLIGHT SIMULATORS

Task number two, shown in Figure 6, was intended to have students figure out the cosine law. Again, students had not been instructed in the cosine law prior to this lesson. Students were expected to go through all the necessary steps to model the cosine law. The idea was to aid students in realizing that the cosine law is a shortcut to a longer procedure that they were already able to implement. This subsection describes what was anticipated by the researcher before the implementation of task two.

More travellers are flying than ever before, creating a daunting challenge for airlines: keep passengers safe in an ever more crowded airspace. (...)

Sherry Carbary, vice president of Boeing Flight Services, says there is an "urgent demand for competent aviation personnel". (...)

Technological improvements are also helping to lower the accident rate. Cockpits now come with systems that automatically warn if a jet is too low, about to hit a mountain or another plane. Others detect sudden wind gusts that could make a landing unsafe. (...)

"At the end of the day, we're a safety net. We're there to help the flight crew," says Ratan Khatwa, senior chief engineer for human factors at Honeywell. (Press, 2014)

You have been hired to program a flight simulator for the training of beginner pilots. One of the features your team is supposed to develop is the feedback that control towers send to aircrafts, which includes advice about approaching aircrafts and changes in route. Imagine that two aircrafts are approaching the same control tower from different directions and following a straight path.

a) How would you determine the distance between the two aircrafts considering that you have a GPS?
b) How would you determine the distance between the two aircrafts if the GPS was down and you needed a backup safety system?

(Boeing 737-800 and airbus A320 flight simulators, n.d.)

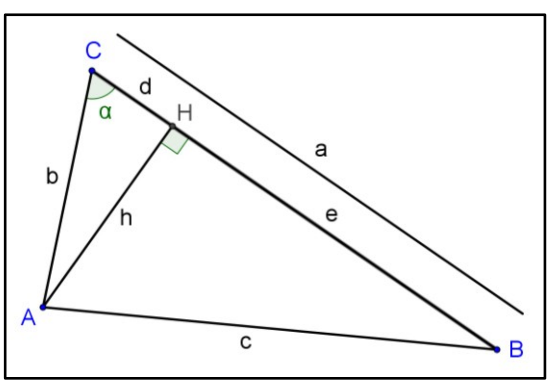
Figure 6: Task two – Flight simulators.

As the first question of the task asks students to consider a GPS as a resource, it was expected that they would think in terms of coordinates. So considering that aircraft A was in position (x_A, y_A) and aircraft B was in position (x_B, y_B), the distance between the two aircrafts would be $d = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$. If students were not thinking like that, they could be prompted by being asked: *"What kind of information does the GPS provide? How can you use that information?"* If they thought about coordinates but did not think in terms of the Cartesian plane or the Pythagoras theorem, they could be prompted as well, by being asked: *"How can you represent this situation using a Cartesian plane? Can you think about a mathematical resource to calculate the distance between these two points?"*

Then, the second question asks students to disregard the GPS. In this case, it was expected that they would think in terms of known distances. Students could think about that in different ways. They could work with average speed, elapsed time since the aircraft left the origin, trip total distance, distance obtained by sonar, etc. In any case, the idea was to have them modeling the scenario by triangulating the situation. If they did not, they could be prompted by being asked: *"Have you tried to illustrate the situation by drawing? Would this be helpful?"* Next, students could need prompts to think in terms of the angles in the triangle, other than only in terms of the distances. So they could be asked: *"Are these known distances enough for you to work with? What else could you work with in a triangle? Would the control tower be able to measure something else in relation to the two aircrafts?"*

Once students had triangulated the situation and had chosen values or variables for known distances and angles, they were expected to work with the triangle they had in hand to figure out the distance between the two aircrafts. At this point, they would probably only know how to work with a right triangle, so they could consider the idea of splitting the triangle they had in hand into two right triangles. If they did not, a prompt could be helpful: *"What kind of triangle are you used to work with? Can you somehow work with this kind of triangle in the situation you have in hand? How?"* It could be the case that they would split the triangle in two by also splitting the supposedly known angle. This was not desired, because they would then have two angles, none of them known. In this case, students could be prompted like that: *"Is it helpful to split the only angle you know into two unknown angles? Could you split the triangle in a different way in order not to do that?"*

As shown in Figure 7, with two right triangles to work on, it would be a matter of working with the Pythagoras theorem and doing some algebraic manipulations as follows. Depending on what students were doing, some specific prompts could be needed.



C – Control Tower

A – Aircraft A

B – Aircraft B

c – Desired distance between aircrafts A and B

Figure 7: Possible triangular representation for task two.

Students could find *c* by crossing over three different results. The first result could emerge from the application of the Pythagoras theorem in triangles ACH and ABH. By applying this theorem, two different equations could be obtained, and these equations could be crossed in order to eliminate variable *h*. This procedure could be done as follows:

$$- \begin{cases} h^2 + d^2 = b^2 \\ h^2 + e^2 = c^2 \end{cases}$$

$$d^2 - e^2 = b^2 - c^2 \quad (1)$$

$$c^2 = b^2 + e^2 - d^2$$

The second result could come out from the cosine relation for the known angle in triangle ACH.

$$\cos\alpha = \frac{d}{b} \quad \Rightarrow \quad d = \mathrm{b}\cos\alpha \quad (2)$$

Finally, the third result could be the relation between *a*, *d* and *e*: $e = a - d$ (3).

By algebraically manipulating results 1, 2 and 3, students could get the cosine law:

$$c^2 = b^2 + e^2 - d^2$$

$$c^2 = b^2 + (a-d)^2 - d^2$$

$$c^2 = b^2 + a^2 - 2ad \cancel{+d^2} \cancel{-d^2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\alpha$$

$$c = \sqrt{a^2 + b^2 - 2ab\cos\alpha}$$

Differently from task one, task two had no numerical values. It was anticipated that this could be an issue for some students. Therefore, in case students did not make up their own numbers to work with, but were in need of numbers, the teacher could provide them with the following values.

- Speed of aircraft A: 828 km/h
- Speed of aircraft B: 792 km/h
- Height of aircraft A: approximately 32 thousand feet
- Height of aircraft B: approximately 36 thousand feet
- Time since aircraft A has left the origin: 10 minutes
- Time since aircraft B has left the origin: 50 minutes
- Total distance to be traveled by aircraft A: 226 km
- Total distance to be traveled by aircraft B: 699 km
- Angle measured between aircraft A, control tower and aircraft B: 65 degrees

Once more, some possible extensions were planned for students who finished the task before the given time was over. Students could be asked: *"Could you find the other two angles in the triangle you have been working with? What if instead of having one angle and two lengths you had two angles and one length?"* This second question would prompt students to investigate the sine law.

Unexpectedly, the first item of task two took students a lot more time than expected. Students struggled to understand what a GPS is and how it works. Some students associated a GPS equipment with a GPS system used in a car. While the former gives coordinates, the latter uses the given coordinates to find out routes and distances. On the other hand, some students brought up the GPS triangulation system, as if they were supposed to model the GPS itself. This discussion was rich, but the teacher had to intervene and make sure everyone understood what a GPS is and what a GPS could provide them with in the task scenario. After that, students figured out the first item of the task more easily and then continued to the second item of the task. For this second item, some students needed prompts to represent the situation through a drawing and to split their triangular representation into two right triangles without splitting the known angle into two unknown angles. Also, the whole class was presented with the above suggested values on the second day of the task. All of the three groups got to a final answer for the task first item, but none of them got to a final solution for the task second item. This was perhaps due to the amount of time spent in the first item, while the main goal of the task was focused on the second item. The researcher had an arranged agreement with the class teacher in relation to the amount of time saved for each task intervention. Unfortunately, it was not possible to extend the set time.

4.4.3. TASK THREE: DISTANCE MEDLEY RELAY

Task number three, shown in Figure 8, was created to introduce students to rational equations. The goal was to set up a situation in which students would be encouraged to use a rational equation to model the problem. Then, they would be prompted to work on their fractions background knowledge to figure out how to manipulate and solve the rational equation. The next paragraphs describe what was anticipated by the researcher before task three was implemented, and how students actually tackled the problem.

The first anticipated idea was that students could start off by trying to relate speed, time and distance, given that this information was given in the problem. In case they did not remember the relationship between these variables, the teacher could suggest that they observe what kind of unit is used to measure speed (in this task, m/s). This would be a hint for them to figure out that speed is given by distance over time. As a consequence, time is given by distance over speed.

The distance medley relay is an athletic event in which four athletes compete as part of a relay. Unlike most track relays, each member of the team runs a different distance. A distance medley relay is made up of a 1200 meter leg, or three laps on a standard 400 meter track; a 400 meter leg, or one lap; an 800 meter leg, or two laps; and a 1600 meter leg, or four laps - in that order. The total distance run is 4000 meters or nearly 2.5 miles. Aside from the 400 meter segment, which is a sprint, all legs are a middle distance run. Prior to going metric, the distance medley relay consisted of a 440 yard leg, an 880 yard leg, a 1320 yard leg and a mile leg. The total distance for the old distance medley relay was 4400 yards and the total distance for the current metric distance medley relay is 4374.45 yards - a little over 25 yards shorter than the old race. (Adapted from Distance medley relay, n.d.)

You are the coach of a women's distance medley relay team. In a few months, your team will be participating in a competition, and your goal is to have them complete the task in 11 minutes and 15 seconds or less. The difference between the speed of your fastest runner and your slowest runner is 1.6 m/s at maximum. Other factors can influence athletes' performance as well. Decide which athlete will run each relay leg. What is the desired minimum speed you should be looking for from each athlete during the training period?

(Derderian, n.d.)

Figure 8: Task three – Distance medley relay.

Then, students would theoretically have to figure out that the sum of all times should be equal to or less than 11 minutes and 15 seconds (675 seconds). Students could need some prompt here, such as: *"How do you get the total relay time? How will your equation reflect 'equal to' or 'less than'?"* Students could also need to be prompted to have all time variables under the same unit. An example of a solution could start with:

$$675 \geq t_1 + t_2 + t_3 + t_4$$

$$675 \geq \frac{1200}{v_1} + \frac{400}{v_2} + \frac{800}{v_3} + \frac{1600}{v_4}$$

As shown above, students could name one different variable for each runner speed. It could be the case that students would need some prompt to write the speed of three of the runners as a function of the fourth one, so that they could use one single variable instead of four different variables. The prompt could be: *"Is there any way to relate runners' speed? Do you have any information that might be helpful to figure out a relation?"* As the problem only states that the maximum difference should not exceed 1.6 m/s, numerous relations could be used. An example of one possibility could be:

$$675 \geq \frac{1200}{v+0.5} + \frac{400}{v+1.5} + \frac{800}{v+1} + \frac{1600}{v} \qquad (1)$$

From there, students could be prompted to work out a way of finding the value of v. As this is a new kind of equation for them, it would be nice to give students the proper amount of time and scaffold as needed, respecting the path they choose to follow. Some prompts could be needed, such as: *"How would you solve this equation in case you had four different numerical fractions? What do you need to do to sum up fractions?"* The idea was to have them looking for one single denominator, in order to get the following equation.

$$675v(v+1)(v+0.5)(v+1.5) \geq 1200v(v+1.5)(v+1) + 400v(v+0.5)(v+1) + 800v(v+0.5)(v+1.5) + 1600(v+0.5)(v+1)(v+1.5)$$

Once students got to this equation, they would have to work algebraically to get a polynomial of degree four. This was a long process.

$$675v(v+1)(v^2+2v+0.75) \geq 1200v(v^2+2.5v+1.5) + 400v(v^2+1.5v+0.5) + 800v(v^2+2v+0.75) + 1600(v+1)(v^2+2v+0.75)$$

$$675v(v^3+3v^2+2.75v+0.75) \geq 1200v^3+3000v^2+1800v+400v^3+600v^2+200v + 800v^3+1600v^2+600v+1600(v^3+3v^2+2.75v+0.75)$$

$$675v^4+2025v^3+1856.25v^2+506.25v \geq 2400v^3+5200v^2+2600v + 1600v^3+4800v^2+4400v+1200$$

$$675v^4+2025v^3+1856.25v^2+506.25v \geq 4000v^3+10000v^2+7000v+1200$$

$$675v^4-1975v^3-8143.75v^2-6493.75v-1200 \geq 0$$

Finally, as students would not have enough knowledge to work out the polynomial of degree four without the help of technology, they would need a scientific calculator to solve it for *v*. There were three different intervals for which the inequality was greater than or equal to zero. However, in only one of them *v* was positive, as it was expected from a speed value. Based on this positive interval for *v*, it would be a matter of finding the other three minimum speeds.

$$v = v_4 \geq 5.5 \text{ m/s}$$
$$v_1 = v_4 + 0.5 \quad \Rightarrow \quad v_1 \geq 6 \text{ m/s}$$
$$v_2 = v_4 + 1.5 \quad \Rightarrow \quad v_2 \geq 7 \text{ m/s}$$
$$v_3 = v_4 + 1 \quad \Rightarrow \quad v_3 \geq 6.5 \text{ m/s}$$

It is worth mentioning that students could simplify equation 1 by having at least two athletes running with the same speed. Thus they would get a third or second degree equation. With this in mind, as an extension or as part of students' investigation, the teacher could prompt students by asking: *"What kind of assumptions can be made to simplify the polynomial of degree four?"*

Task three did not have structured data as task one had, but it was also not like task two, which had no given data. Task three had some given information, but with certain degree of freedom. Students were left to choose the difference between speeds as long as the given maximum difference was respected. This freedom was somehow an issue, given that students are normally not used to that and do not believe they are allowed or even capable of making choices. During the task implementation, the teacher had to make it clear that students were not only allowed to choose their own speeds difference, but that they were expected to do that to solve the task.

In general, students started off task three by relating speed, time and distance. Some groups discussed also the difference between speed and velocity. Another concern that was present among students' thoughts was the idea that they needed to somehow average out runners' speeds. Nonetheless, it was not clear for everybody that they were talking about a weighted average. Some students spent some time working with simple averages before realizing they were not considering the different distances in the medley relay. The idea of summing up times was not easy to occur without prompts. Actually, this idea was not a necessary one, as students could work with average speeds only. After much student investigation, the teacher prompted the whole class to find a time inequality and see if that would be helpful or not. Once students got to an equation or inequality – either involving times or speeds – they proceeded to algebraic manipulations in order to solve it.

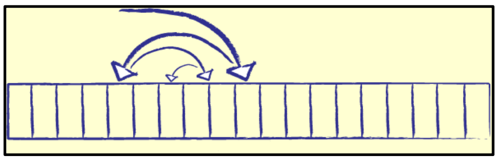
Finding a common denominator and the respective procedure was not hard for students to retrieve and experiment in order to solve the rational equation. Students from two groups got to a final solution, although one of them was acknowledged partially wrong. The other students did not get to a final solution for the task, but they were on the right track.

4.4.4. TASK FOUR: BINARY SEARCH

Task number four is illustrated in Figure 9, and it was designed to introduce arithmetic and geometric sequences. With this in mind, students were introduced to two different sorts of searches: the linear search, which is based on arithmetic sequence principles, and the binary search, which is based on geometric sequence principles. The central purpose was to have students modeling how to count the number of necessary interactions for both kinds of search in order to compare them and justify the claim that the binary search is faster than the linear search. The next paragraphs describe what the researcher anticipated students could do and what students did.

The first concern of the researcher was related to the notion of worst-case scenario. Therefore, the idea was to give some time for the groups to discuss what they thought the worst case scenario was, and then they could share their ideas with the whole class. The worst case scenario would be to find out the desired item in the last interaction only. So it was necessary to figure out the maximum possible number of required interactions. Once students understood what the worst-case scenario was, time could be given for them to think about a strategy to explore the task. Students could get stuck if they did not establish a case example to investigate at first, that is, a determined number of items in a list. Thus they could need some prompt here, such as: *"What is your strategy to explore this task? Would it be helpful to have a concrete example to work with? Consider replacing n with a small value to examine the possible outcomes."* With an example in hand, it was foreseen that students could start counting each interaction. Then it could be good to have them generalizing their thoughts. Students could be prompted by being asked: *"What happens as n becomes larger? Is there a generalization to be made? You might need a generalization in case you have a bigger list of items. You might also need a generalization to program a computer to do one of these searches."*

In computer science, a binary search or half-interval search algorithm finds the position of a specified input value (the search "key") within an array sorted by key value. For binary search, the array should be arranged in ascending or descending order. In each step, the algorithm compares the search key value with the key value of the middle element of the array. If the keys match, then a matching element has been found and its index, or position, is returned. Otherwise, if the search key is less than the middle element's key, then the algorithm repeats its action on the sub-array to the left of the middle element or, if the search key is greater, on the sub-array to the right. If the remaining array to be searched is empty, then the key cannot be found in the array and a special "not found" indication is returned. (Binary search algorithm, n.d.)



(Improving binary search, 2014)

Binary search is a common tool to search data bases such as: dictionaries, library catalogues, phone books, etc. Linear search, which examines a disordered list by looking at each item at a time, is not as common. Binary search seems to be preferred in relation to linear search; based on the claim that binary search is faster than linear search. To better illustrate this idea, consider a system looking for the name *Miller* in a data base with 10 clients' last names. The binary search can be done as follows. The system will retrieve all information in position 6, which corresponds to client Miller.

1	2	3	4	5	6	7	8	9	10
Davis	Harris	Hill	Johnson	**Jones**	**Miller**	**Moore**	**Parker**	**Smith**	**Taylor**
1	2	3	4	5	6	7	8	9	10
Davis	Harris	Hill	Johnson	Jones	**Miller**	**Moore**	**Parker**	Smith	Taylor
1	2	3	4	5	6	7	8	9	10
Davis	Harris	Hill	Johnson	Jones	**Miller**	**Moore**	Parker	Smith	Taylor
1	2	3	4	5	**6**	7	8	9	10
Davis	Harris	Hill	Johnson	Jones	**Miller**	Moore	Parker	Smith	Taylor

Consider you have a phone book list with ***n*** entries and is searching for one specific number. How would you justify the aforementioned claim based on the worst-case scenario?

Figure 9: Task four – Binary search.

Just to demonstrate what was anticipated for students' work, an example of a list with 2000 items to search for was considered. A way to think through the linear search could be the one that follows, but other possibilities could be considered.

1st step: Look at item #1. In the worst case scenario, that wouldn't be the desired item. So look at the next one.

2nd step: Look at item #2. In the worst case scenario, that wouldn't be the desired item. So look at the next one.

3rd step: Look at item #3. In the worst case scenario, that wouldn't be the desired item. So look at the next one.

4th step: Look at item #4. In the worst case scenario, that wouldn't be the desired item. So look at the next one.

...

pth step: Look at item #p. In the worst case scenario, that wouldn't be the desired item. So look at the next one.

...

2000th step: Look at item #2000. In the worst case scenario, that would be the last term, hence the desired item.

Thus, 2000 interactions would be necessary to find the desired item in the worst-case scenario. This is an example of an arithmetic sequence, in which the first term is 1, the last term is 2000 and the common difference is one (given that the total number of interactions increases by one every time the next term is searched). In this case, the sequence is going up. The idea of arithmetic sequence could be explored in class once this stage was accomplished.

A way to think through the binary search could be the one that follows, but other possibilities could be considered as well.

First, sort the list of items in increasing or decreasing order, then:

1st step: Look at the item in the middle of the list. In a list of 2000 items, the middle one could be item #1000 (it could be item #1001 as well). In the worst case scenario, that wouldn't be the desired item. If the desired item is greater than item #1000, disregard all items lesser than item #1000. If the desired item is lesser than item #1000, disregard all items greater than item #1000. Consider the desired item greater than item #1000. Then there would be approximately half of the total items to search for, that is, $\frac{2000}{2} = 1000$ items.

2nd step: Look at the item in the middle of items #1000 and #2000. The middle one could be item #1500 (it could be item #1501 as well). In the worst case scenario, that wouldn't be the desired item. If the desired item is greater than item #1500, disregard all items lesser than item #1500. If the desired item is lesser than item #1500, disregard all items greater than item #1500. Consider the desired item lesser than item #1500. Then there would be approximately half of 1000 items to search for, that is, $\frac{1000}{2} = 500$ items.

3rd step: Look at the item in the middle of items #1000 and #1500. The middle one could be item #1250 (it could be item #1251 as well). In the worst case scenario, that wouldn't be the desired item. If the desired item is greater than item #1250, disregard all items lesser than item #1250. If the desired item is lesser than item #1250, disregard all items greater than item #1250. Consider the desired item lesser than item #1250. Then there would be approximately half of 500 items to search for, that is, $\frac{500}{2} = 250$ items.

4th step: Look at the item in the middle of items #1000 and #1250. The middle one could be item #1125 (it could be item #1126 as well). In the worst case scenario, that wouldn't be the desired item. If the desired item is greater than item #1125, disregard all items lesser than item #1125. If the desired item is lesser than item #1125, disregard all items greater than item #1125. Consider the desired item greater than item #1125. Then there would be approximately half of 250 items to search for, that is, $\frac{250}{2} = 125$ items.

...

pth step: In this interaction, it might be hard to figure out the numbers, as in the first four interactions. However, it is possible to anticipate that there would be approximately $\frac{2000}{2^p}$ items to search for, given that for every interaction the number of items was divided by two.

The binary search could be more challenging, given that it is not as straight forward as the linear search is. In this situation, students could not count the number of interactions as in the linear search; they could count the number of remaining items instead. Some prompts could be necessary then, for instance: *"What is happening with the number of remaining items? What do you want this number to be in the worst case scenario? Can you generalize what is happening with this number?"* Prompts could depend on students' thoughts. Once students figure out a generalization for the number of remaining items, and also that they wanted this number to be one, the idea could be to write this into an equation. So the question to be answered could be: *"When are we going to have only one item left?"* This could be a necessary prompt.

The equation could be:

$$\frac{2000}{2^p} \cong 1 \quad \Rightarrow \quad 2000 \cong 2^p$$
$$2^p \cong 2000 \cong 2048 = 2^{11}$$
$$2^p \cong 2^{11} \quad \Rightarrow \quad p \cong 11$$

Hence, 11 interactions would be approximately enough to find the desired item in the worst case scenario of binary search. This is an example of a geometric sequence, in which the first term is 2000, the last term is 1 and the common ratio is $\frac{1}{2}$ (given that the total number of remaining items is divided by two every time the next term is searched). In this case the sequence is going down. The idea of geometric sequence could be explored once this stage was accomplished.

After this point, students would probably not have issues to justify the claim posed in the task. As 11 is much less than 2000, that would be enough for them to validate the preference for the binary search.

Again, task four was not as structured as task one, but it had enough information for students to explore the task. As in task three, students had also a degree of freedom, given that they were able to make up numbers to better investigate and understand the task. The difference in the degree of freedom was that in task three students were supposed to come up with numbers that would match the given information, while in task four students were supposed to simply come up with numbers to create an example. Based on class field notes, it seems that in task three students were less confident to make their own choices than in task four. It could be the case that after experimenting with task three, students were more comfortable with the idea of making choices in task four.

When first analyzing task four, some students grasped how a binary search runs, while others had some difficulties. The exchange between students to explain a binary search was very rich. For linear search, as a rule, there were no big issues for students to figure it out. Most students' course of action for this task was based on examples. They chose small numbers to perform some examples and to look for patterns and relations. Students' analysis of their own examples was very interesting, diverse and productive. Not surprisingly, students did not follow the anticipated steps. They did not have problems to model the linear search, so they concentrated all efforts to model the binary search. Many students figured out the division-by-two pattern, but putting the pattern into a mathematical equation was challenging. Students from one group got to a final solution, one of them even got the mathematical equation after some prompting. Students from the other two groups did not get to a formal final solution but they were close to it. One of them even developed a computational algorithm to implement the binary search.

5. Unpacking the question: Data analysis

This chapter presents the method used for analyzing data, and some examples of data analysis. It is divided into four sections. The first section speaks to the steps for data processing. The second section explains and describes the method and the structure of the diagram-based approach designed to organize, illustrate and analyze the collected data. The third section justifies the selection of the students that had their diagrams created. Finally, the fourth section presents and describes four students' diagram examples (one for each task), which were created for data analysis.

5.1. Data processing

Five different stages were used to process the collected data. The first stage was reviewing the raw data (both recorded and written data from classroom work). This revision was ideally to be done before SR interviews, so that recorded and written pieces of students' work could be retrieved during interviews. Unfortunately, due to time issues, recorded pieces of students' work were used only in part, while students' written work was used in full during SR interviews. This stage was relevant to realize the amount and the quality of data to be analysed. For instance, by reviewing the raw data a lot of background noise was noticed. Software resources were used to try lessening the noise effect in order to have better data; however, that did not turn out successfully, given that noise was not reduced adequately. SR interviews raw data was not submitted to review.

The second stage of data processing was data transformation, which refers to transcribing the data from classroom activities and from interviews, and also scanning students' mathematical journals (which were given back to students). At first, data transcription was done for one whole group of students working together in one task. However, later the decision to transcribe data from one student at a time was made, given that the analysis was based on students' individual work. Next, the third stage was about data reduction. One full dossier was elaborated for each student doing each task. Dossiers contained relevant fragments from student's transcriptions (both from classroom work and interview) and journal, and intended to aid data organization.

The fourth stage of data processing was related to data organization and data interpretation. Data was organized into diagrams, and interpreted according to Kilpatrick et al. (2001) proficiency model indicators (shown in Table 1). Next section describes and explains how these diagrams were structured. Narratives were written for each diagram explaining the reasoning behind them. Narratives included field notes observations, which were not considered in the diagram building. Finally, the last stage of data processing refers to the organization of the data into a graphic. This graphic intended to portray data from all built diagrams, which contemplate all tasks and all selected participants. This graphic was elaborated with the purpose of observing general patterns or tendencies. Section 5.3 speaks to participants' selection and Section 6.2 describes and explains the obtained graphic.

5.2. DIAGRAM-BASED APPROACH AND STRUCTURE

In order to analyze the data from multiple sources as a whole, an interpretive approach of designing a diagram to illustrate individual student's path and thinking through a task was elaborated and investigated. Figure 10 portrays the structure of diagrams, which present only fragments of student's thoughts and written solutions. Fragments were collected from student's written materials, audio and video recordings transcripts, and stimulated recall interview transcripts. As aforementioned, researcher's field notes are not included in the diagrams, but in the narratives they are included. Fragments are mapped according to student's journey through the task from introduction to completion, and are shown inside rectangles within the diagram. Although rectangles may contain more than one fragment, for the purpose of simplification, each rectangle is being referred to as one fragment. Italics is used to represent student's verbatim quotations. Kilpatrick et al.'s (2001) mathematical proficiency strands are illustrated on the diagram within bubbles. According to the data analysis framework, fragments were categorized according to Kilpatrick et al.'s (2001) strands — based on the list of indicators presented in Table 1 — in order to investigate student's mathematical understanding and proficiency during task. To illustrate the interpretation of a fragment as a particular strand or strands of mathematical proficiency, black arrows connect them. A grey ellipse-shaped arrow shows the chronological order in which task fragments were gathered. The beginning of the process is represented by a dash at one extreme of the arrow, and the end of the process is represented by a dot at the other extreme of the arrow. Fragments are ordered counter-clockwise starting off in the fragment marked by the dashed extreme of the ellipse-shaped arrow. The fragments that are not overlapped by the grey arrow were collected during recall interviews, which were a few weeks after the task was done in class. In order to analyse the data across students, their work needed to be made comparable. Therefore, a decision was made to divide their work into a beginning, middle and end phase. Phases 1, 2 and 3 refer to the beginning, the middle and the end of a student's work respectively. Depending on student's work, these phases slightly varied. Although these phases speak to three different moments of the modeling investigation process, they do not intend to characterize the modeling process as a three-stage process. Fragments pertaining to phases 1 and 3 are numbered in dark gray. Fragments pertaining to phase 2 are numbered in light gray. Fragments pertaining to interviews are numbered in white. Sixteen diagrams (four diagrams for each of the four tasks) were created to analyze the collected data.

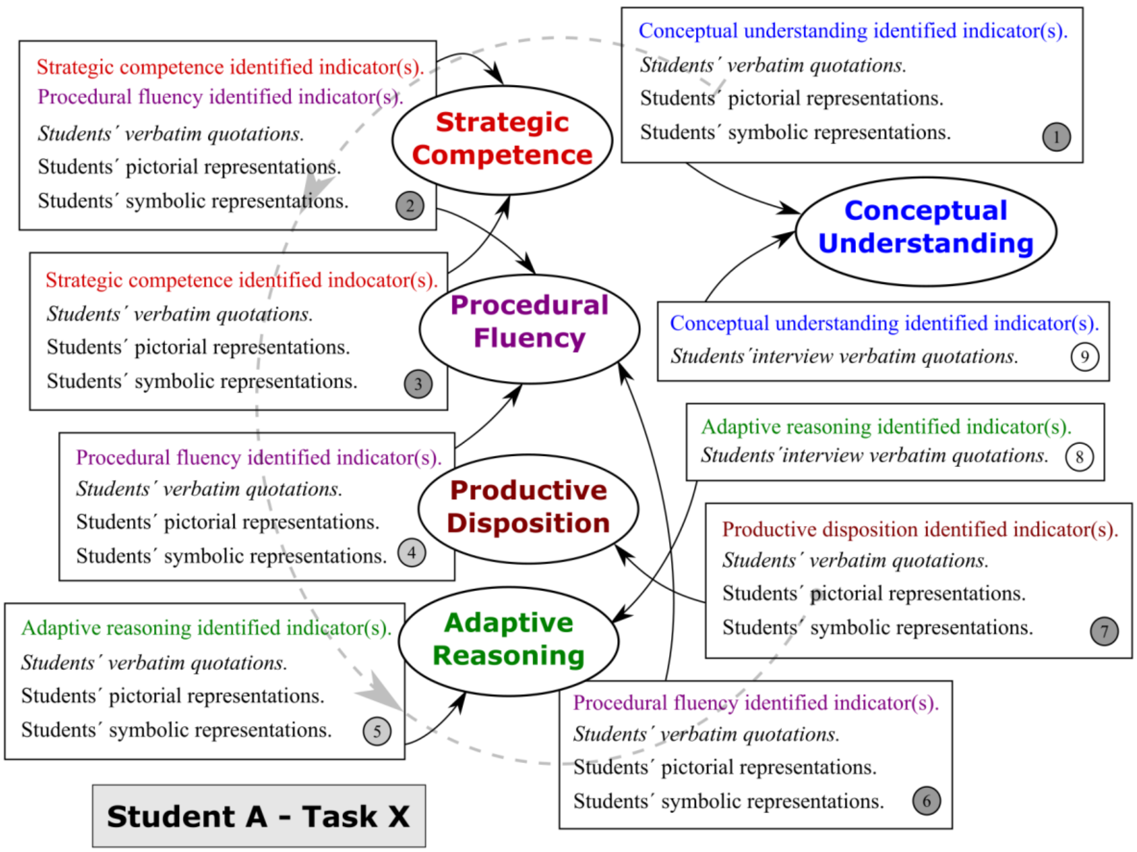


Figure 10: Diagram structure.

5.3. PARTICIPANTS' SELECTION FOR DIAGRAM BUILDING

Among the 12 participant students, there was no selection at first, that is, all of them were considered in the data analysis regardless of their interest, background or readiness in mathematics. However, when examining audio and video recordings, as well as mathematical journals, it was clear that two of the students did not yield enough data for analysis, either because they were basically quiet during the task period, either because they did not have sufficient records in their mathematical journals, or both. Also, these two students did not participate in the recall interviews. Therefore, it was difficult to analyze their understandings and proficiency due to the lack of data. For this reason, these two students were not considered in the data analysis. As for the ten remaining students, diagram distribution was intended to include all of them once or twice, and also all three settled groups for each task. The goal was not to discard any student, but to select data-rich and interesting scenarios for analysis. When examining audio and video recordings, the amount of data and some discourses stood out and interfered in the choice of which task would be analyzed for each student. Based on this, seven of the ten students included in the data analysis had two different tasks assessed, two of them had one task assessed, and one of them was put on hold to be included in case a fifth diagram was done for each task, which was not the case. Table 4 presents the distribution of diagrams according to tasks, participants and groups. Each participant is given a number (first column) to be used in research findings. Numbers in the last four columns refer to the group participants were in.

Participants		**Task 1**	**Task 2**	**Task 3**	**Task 4**
1	*Amanda*	2		2	
2	*Nathalie*	2			
3	*Philip*	1		2	
4	*Sophia*	3	3		
5	*Rick*			1	1
6	*Leo*			2	2
7	*Clara*		1		2
8	*Thomas*		3		3
9	*Jack*		2		

Table 4: Diagram distribution per participants, tasks and groups.

5.4. PARTICIPANTS' DIAGRAMS AND DESCRIPTIONS

Four of the 16 created diagrams — one for each task — are portrayed in this section in order to illustrate how data was analyzed and how the strands of mathematical proficiency are present in students' work. Each of these diagrams is complemented with a descriptive narrative explaining how indicators were identified on students' discourse and journals. The other 12 diagrams and respective narratives can be found in appendixes A to L.

5.4.1. AMANDA'S DIAGRAM FOR TASK ONE – COST, REVENUE AND PROFIT

Amanda's diagram for task one (Figure 11) reveals that she underwent the five strands of mathematical proficiency while modeling task one. As fragments one to four illustrate, she starts off by exploring the task, asking questions and making conjectures, suggesting she is making sense of the task and of the mathematical content related to the task. In fragment two specifically, Amanda brings in ideas of graphing and linear functions, revealing her need to retrieve and make connections with her prior mathematical experiences and knowledge. That is also one first step on her strategy to represent and understand the problem. As fragment four shows, by considering graphing the given data and finding its respective equation, Amanda is delineating her strategy to understand, represent and solve the problem. As so, this first phase of her work can be described as a blend of *conceptual understanding* and *strategic competence*.

After that initial phase, Amanda brings forward mathematical procedures to determine the linear equation $(y = -0.5x + 115)$ that models the textbook data given in the task table, with x being the cost and y being the quantity sold. As shown in fragments six and seven, she chooses an appropriate procedure, a right moment to apply the procedure, and performs it correctly. During this second phase, she also logically relates content and situation. This can be seen in fragments five and nine when she asks "What does the slope mean?" and answers "price elasticity", or when she states "I guess the slope read the demand actually". These examples show her attempt to relate slope (content) with price elasticity and demand (situation). Then, in fragment ten, Amanda tries to figure out if she can graph profit as well, showing her effort to transfer the procedure she has already used when graphing quantity sold in relation to price. Along with Amanda's statement in the recall interview (as shown in fragment eight), she deliberately looks for a quadratic equation to represent the profit, because she is looking for a maximum point and she knows a parabola presents this attribute (retrieving and connecting mathematical content). By doing this, it is possible to infer that she is transferring the content knowledge she had about a parabola's maximum point to this profit situation. Accordingly, this second phase of her work reflects a blend of *conceptual understanding*, *procedural fluency* and *adaptive reasoning*.

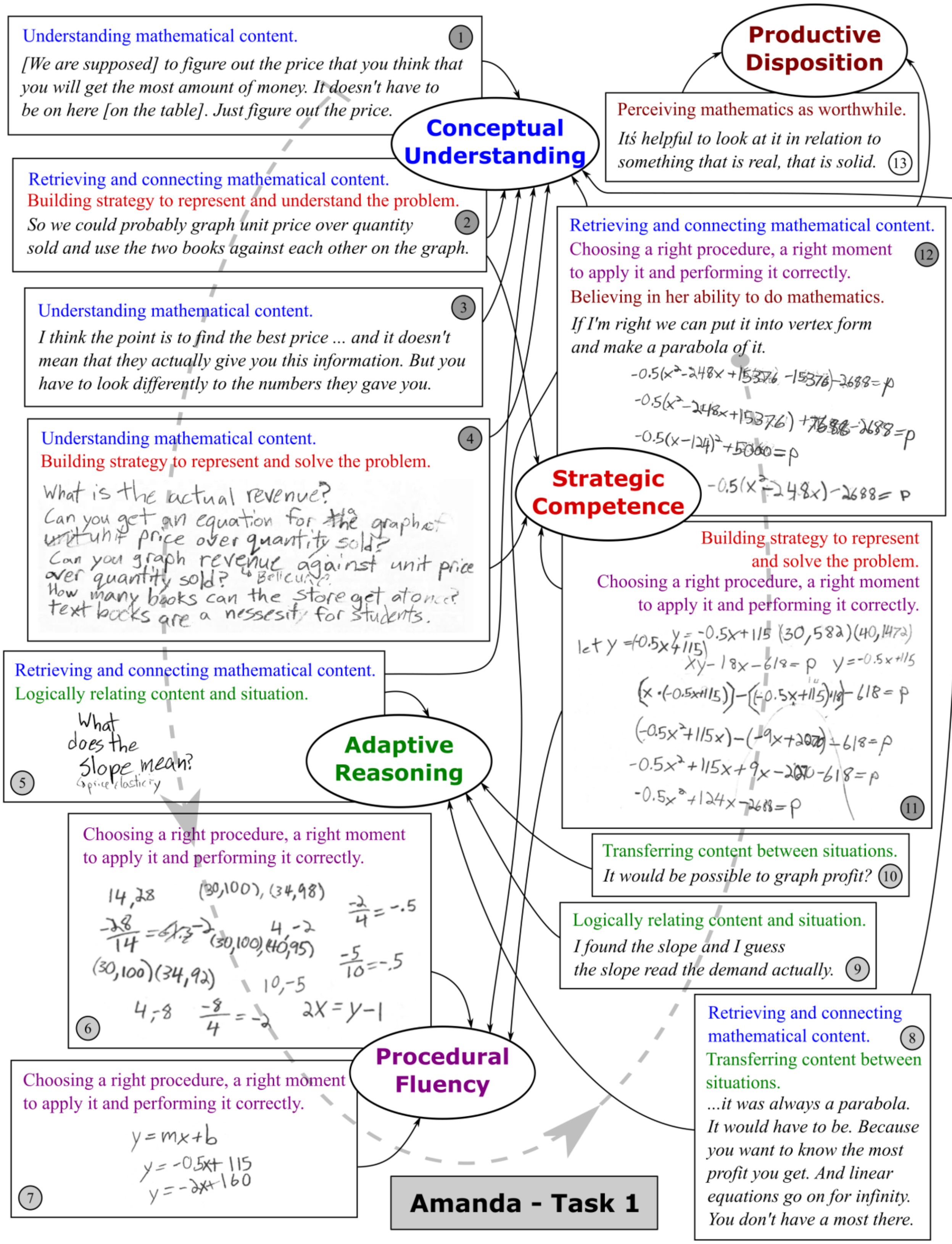


Figure 11: Amanda's diagram for task one.

In the last phase, as shown in fragment eleven, Amanda has an equation for profit $(p = xy - 18x - 618)$ in which one of the variables is quantity sold (y). As she had an equation for textbook quantity sold $(y = -0.5x + 115)$, she plugs this equation into the profit equation to finally get to a model for textbook profit $(p = -0.5x^2 + 124x - 2688)$, which is a quadratic equation that presents profit (p) as a function of price (x). Once more, she correctly performs a right procedure at a right moment. By doing that, she is working on a strategy to represent and solve the problem. Then, in fragment twelve, Amanda works algebraically with the new equation she got to determine its maximum value. As the diagram shows, Amanda was able to correctly apply the taught procedure to complete the square and get the vertex form of the textbook profit equation, making her procedural knowledge explicit $\left(p = -0.5(x - 124)^2 + 5000\right)$. Once Amanda got this final equation, based on her previous knowledge about parabolas and vertex form — that is, by connecting and retrieving mathematical content again — she was able to come up with the textbook price that maximizes the store profit (CAD$ 124). Still in fragment twelve, when Amanda says "If I'm right...", it highlights that she is making choices based on her beliefs. This consistent positive attitude is an evidence of the confidence she has in her ability to learn and to do mathematics. Thus, in this final phase, Amanda was demonstrating a blend of *conceptual understanding*, *strategic competence*, *procedural fluency* and *productive disposition* in her work.

It is worth noticing that *productive disposition* could be seen throughout Amanda's whole work. Researcher's in-class field notes confirm that she was actively engaged and working on the task for basically the entire period. During the interview, as fragment thirteen confirms, she also points out that it is helpful to look at mathematics in relation to concrete situations, emphasizing her perception that mathematics is worthwhile.

5.4.2. JACK'S DIAGRAM FOR TASK TWO – FLIGHT SIMULATORS

Jack starts off his work by building on a strategy to solve the problem based on right triangles, as his diagram portrays (Figure 12). Fragment one confirms he suggests right angles to be forced into the situation, so that right triangles can be created. In this sense, Jack retrieves his previous knowledge about right angles and connects to the task. When he says "That's what I would do", he shows he believes his strategy is worth trying, which speaks to his confidence in his ability to do mathematics. In fragment two, Jack organizes his thoughts in steps, structuring his strategy to solve the problem. He also advances on his strategy to represent the problem, given that he uses a Cartesian plane to locate the planes and the control tower. As such, he retrieves his previous knowledge about Cartesian plane and right triangles, and connects to the task. Fragment three shows Jack retrieving his previous knowledge about Cartesian coordinates and connecting to the task. He also explains how he uses this knowledge to build on his strategy to solve and represent the problem. This explanation also indicates the logical relation between the situation (GPS use) and the content (Cartesian coordinates). The picture in fragment three speaks to the chosen procedure to determine the length of the sides of the right triangle that connects both air planes. Jack chooses a right procedure and a right moment to apply it. In fragment four, Jack explains his strategy and his procedure, showing understanding both of the mathematical content and of the chosen procedure. Therefore, this initial phase of Jack's work merges the five strands of mathematical proficiency, namely: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

In the next phase, Jack works on solving the first question of the task. In fragment five, he retrieves his previous knowledge about angles and connects to the task to conclude he does not need angle measurements to solve the problem at this point. Then, at a proper moment, he correctly performs the chosen procedure to calculate the length of the sides of the triangle. The picture in fragment five illustrates this procedure. Fragment six presents the final step of his strategy, in which Jack retrieves his knowledge about the Pythagoras theorem and connects it to the task. He chooses a right moment to apply the right procedure (the Pythagoras theorem) and performs it correctly. Thus, the middle phase of Jack's work is a blend of *conceptual understanding* and *procedural fluency*.

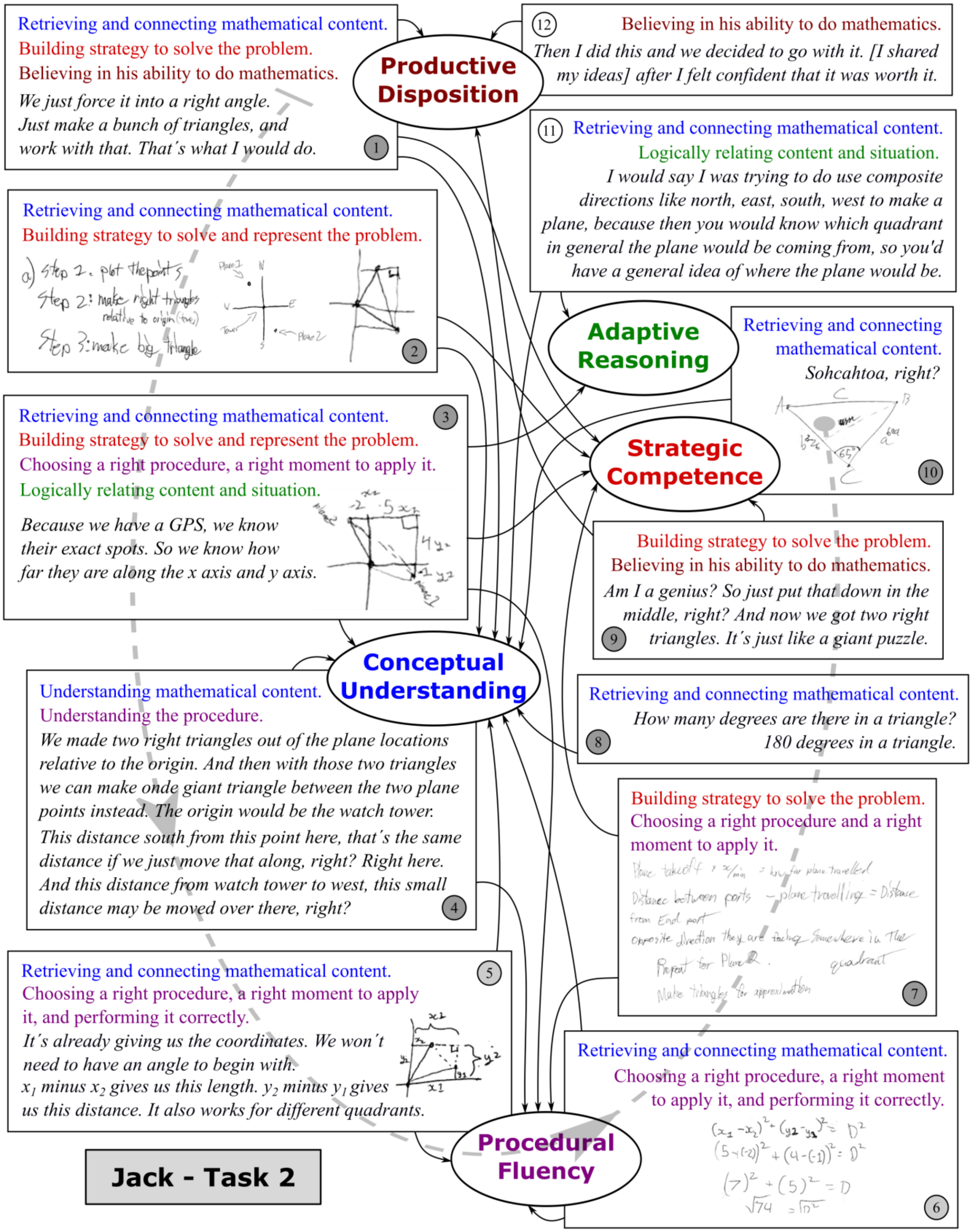


Figure 12: Jack's diagram for task two.

The last phase of Jack's work is based on the second question of the task. As fragment seven shows, he starts off by looking at extra available data, because the GPS is out of order. Jack makes use of the time the plane took off and its speed, to figure out how far the plane has travelled. Then he subtracts this distance from the distance between the two initial airports in order to figure out the remaining distance to the final airport. Jack repeats this procedure for both airplanes. As such, fragment seven speaks to Jack's strategy to solve the problem, as well as to his right choice of procedure and proper moment to apply it. Fragment eight addresses another mathematical content that Jack retrieves and connects to the task. At this point, he is analyzing the triangle he has in hand. For that reason he brings up the sum of the angles in a triangle. Fragment nine points out to Jack's strategy of splitting up the bigger triangle into two right triangles, so that he could use his previous knowledge about right triangles. When Jack asks "Am I a genius?", he is showing his confidence in his capacity of doing mathematics. Anyhow, Jack needs some prompt to visualize he has different ways of splitting up the big triangle. Jack then realizes he does not need to split up a known angle. Finally, in fragment ten, Jack has a big triangle divided into two right triangles. He then retrieves his previous knowledge about trigonometric ratios and connects to the task to finish solving it. Although Jack does not get to a final answer, the final phase of his work is a mix of *conceptual understanding*, *strategic competence*, *procedural fluency* and *productive disposition*.

Afterwards, in the interview, as fragment eleven displays, Jack explains that he has also retrieved his previous knowledge about the compass rose, and connected it to the task. He logically related this content and the Cartesian plane knowledge with the situation he had in hand, in order to locate the air planes and better represent the problem. Still during the interview, as fragment twelve confirms, Jack observes he shared his ideas with the group once he felt confident about them. In other words, at the sharing moment, he believed his thoughts were worth communicating, which supports Jack's belief in his ability to do mathematics.

5.4.3. LEO'S DIAGRAM FOR TASK THREE – DISTANCE MEDLEY RELAY

As can be seen in fragment one of Leo's task three diagram (Figure 13), Leo starts off the task by questioning the usefulness of the data he has in hand (the 4000 meters total distance). He also retrieves his previous mathematical knowledge about speed as distance over time, and connects it to the task. Leo shows understanding of what speed represents when he acknowledges that in the task the speed is the equivalent of running 4000 meters in 675 seconds. He correctly chooses and performs two different procedures at this first phase, one to convert minutes into seconds and the other one to calculate the speed. Both procedures are applied at proper moments. After that initial exploration, Leo continues his inquiring — as shown in fragment two — by questioning if he knows how much each runner runs. In accordance with that, he logically relates the different situations under analysis, that is, runners with different speeds running different distances. As a result of his reasoning, he concludes that the fastest runner is expected to run the shortest distance, while the slowest runner is expected to run the longest distance. His arguments to justify his reasoning do not mention runners' stamina or resistance. The initial phase of Leo's work is a mix of *conceptual understanding*, *procedural fluency* and *adaptive reasoning*.

In the next phase, as fragment three illustrates, Leo attempts to get to an equation which represents the problem and helps in solving it. His strategy is to sum up runners' time and equal it to 675 seconds $\left(675 = \frac{400}{a} + \frac{1200}{b} + \frac{800}{c} + \frac{1600}{p}\right)$. Leo verbalizes that the group is "trying to get *a*, *b* and *c* into one variable". At this point he already has *a* as a function of *p* ($a = p + 1.6$), illustrating his ability to choose a procedure at a right moment and perform it correctly. In fragment four, Leo presents a different way to think about the problem. At this time, he retrieves his knowledge about percentages and connects to the task. Based on ratios he obtained from the task data, Leo calculates the time percentages each runner takes to run each leg, showing his understanding about the content. To calculate these percentages he needs to choose and correctly perform a procedure in an appropriate moment. He also shows content understanding when he acknowledges that these percentages are valid in case all runners are running with the average speed. This acknowledgement reinforces his *adaptive reasoning*, given that he is logically relating content (percentages) and situation (having all runners running with the average speed). This percentage-based way of thinking represents a strategy used to understand the problem. However, Leo does not follow this strategy to solve the problem. Instead, after some teacher's prompting, he goes back to his initial strategy of summing up times, as can be seen in fragment five. Interestingly, Leo changes his initial thoughts and now he considers the fastest runner running the longest distance, while the slowest runner runs the shortest distance. This change might speak to a lack of *conceptual understanding*, as noticed before in his arguments in fragment two. During the recall interview (fragment nine), Leo mentions that the difference in speed between the fastest and the slowest runner would be 0.8, what is not exactly what his equation expresses. He seems to expect the precise difference to be a given. Although he is unsure about the speed of the runners, he tries to relate the given difference (the content) and the situation (runners running at different speeds). Based on these fragments, it is possible to infer that the middle phase of Leo's work is a blend of *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*.

Then in the final phase of Leo's work, his focus is on solving the obtained equation. As such, he chooses a procedure to follow, a right moment to apply it and performs it (fragment six). The way he communicates his solution (by multiplying both equation sides by the different denominators) demonstrates his understanding about the procedure. Nevertheless, he makes a mistake — which he acknowledges during the recall interview (fragment seven) — and restarts the procedure (fragment eight). During the interview, when he states he was trying "to get rid of the denominators", he speaks to his understanding of the procedure as well. Unfortunately, time was not enough for Leo to finish his solution. It seems that he was doing it in a right way and would probably get to an end if he was given more time. This final phase of his work was basically characterized by *procedural fluency*.

As for *productive disposition*, Leo's discourse during the recall interview visibly speaks to it. In fragment ten, he says he likes making assumptions and trying them out, which gives him more options for what to try. This statement supports his belief in his ability to do and learn mathematics, given that he believes he can make different choices to get to an answer. In fragment eleven, he also affirms that this kind of task reflects applied mathematics, which expresses his perception of mathematics as worthwhile. As a final remark based on researcher’s field notes, Leo's interactions with the group as a whole were important for his work, in the sense that the exchange of ideas gave him insights and ideas about how to proceed.

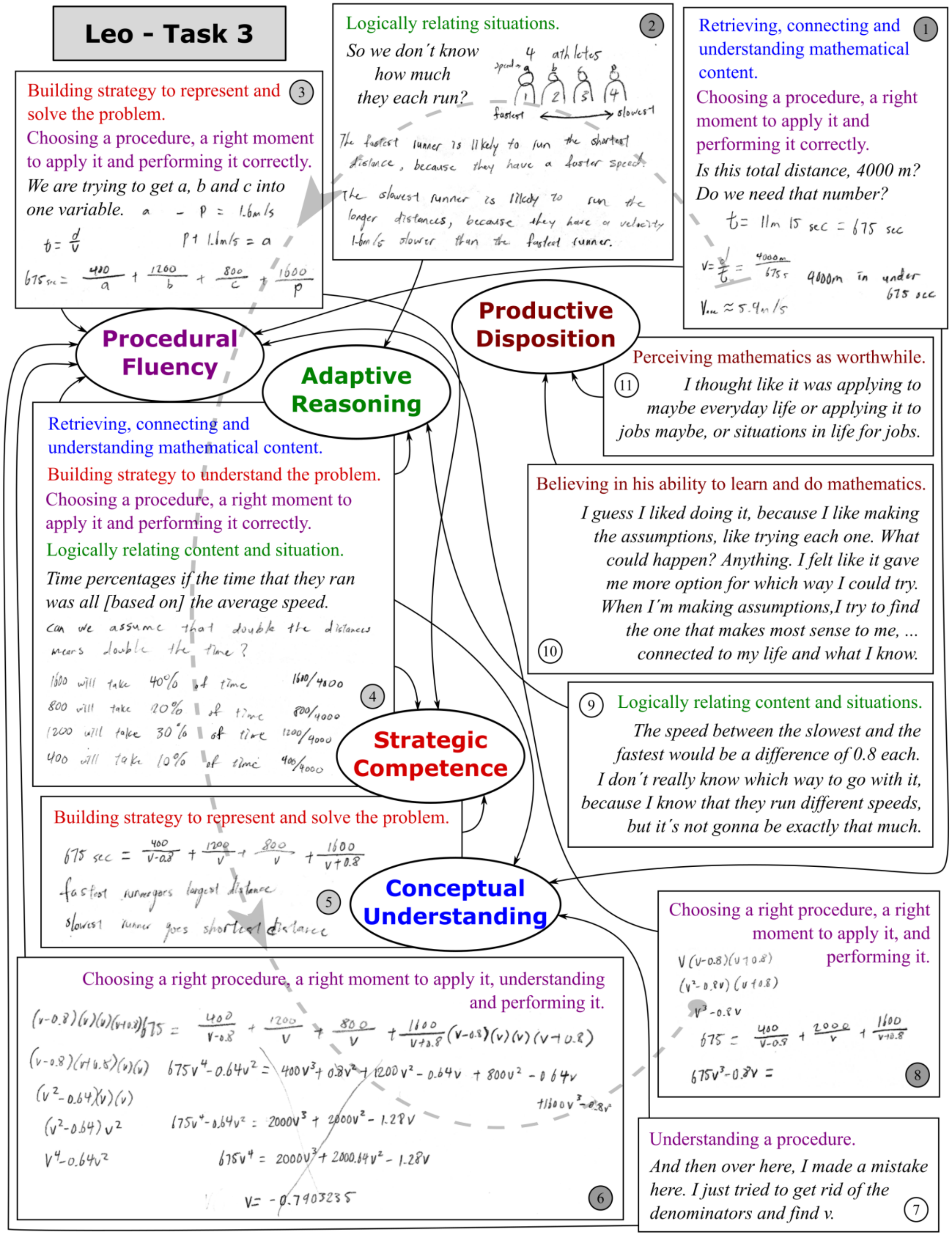


Figure 13: Leo's diagram for task three.

5.4.4. *CLARA'S DIAGRAM FOR TASK FOUR – BINARY SEARCH*

Clara's diagram for task four (Figure 14) is an example of how *procedural fluency* can be present during the initial phase of a task; that is, when understanding a task, and not only during middle and final phases when solving a task. In fragment one, Clara correctly performs a binary search procedure in order to understand the task. She also asks some questions to better understand the mathematical content behind the binary search. When asking if the searched numbers should be in numerical order or not, she logically relates two situations, an ordered binary search and an unordered binary search. In fragment two, Clara again correctly performs a binary and a linear search to better understand the task. She compares the two results as a way of making sense of both; in other words, she logically relates the two situations, the linear and the binary searches. Therefore, this first phase of her work is a combination of *conceptual understanding*, *procedural fluency* and *adaptive reasoning*.

In the next phase, Clara works on building a strategy to understand, represent and solve the task. This attempt is still based on the linear and binary search procedures. In fragment three, she shows understanding about the linear search and she presents her strategy to figure out the number of necessary searches for this case. Then, in fragment four, she shows understanding about the binary search procedure, figuring out the strategy she needs to follow to find out the number of searches in the binary case. In fragment five, Clara chooses do reproduce the binary search procedure again, however, this time, she has a different strategy. She decides to look into the discarded options as well. In this way, she is able to logically compare these situations in order to try to understand the problem and figure out a solution. When she states that it is just about "halving and choosing a side" she demonstrates understanding about the procedure she is implementing. The same is valid for her discourse in fragment six, when she shows understanding both about the procedure and about the content. Hence, this middle phase of her work is described as a blend of *conceptual understanding*, *strategic competence*, *procedural fluency*, and *adaptive reasoning*.

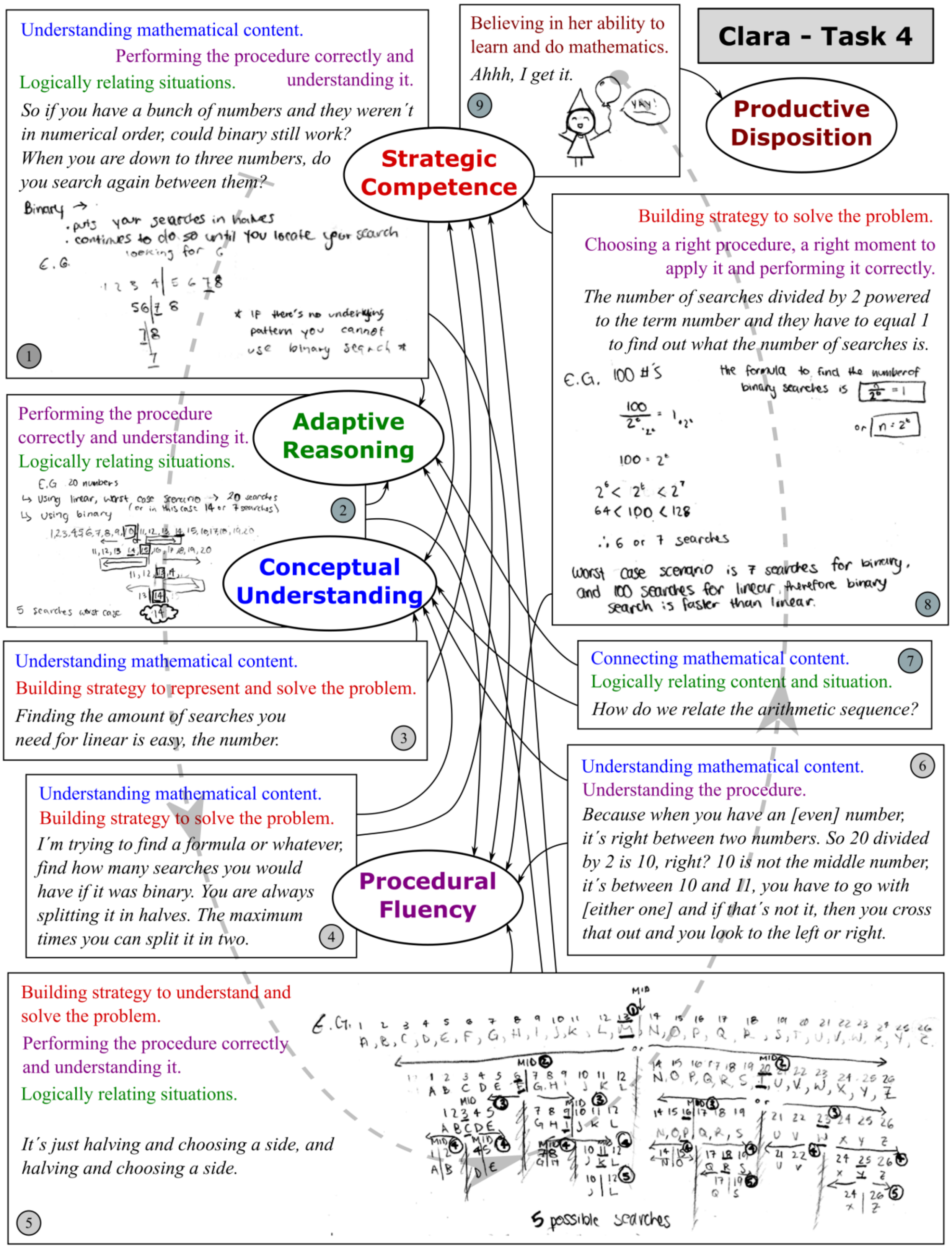


Figure 14: Clara's diagram for task four.

Finally, as fragment seven shows, based on the teacher's input about arithmetic and geometric sequences, Clara wonders how she can connect what she has been doing so far with this new mathematical content (arithmetic sequences). That is, she tries to logically relate this new content to the linear and binary searches situations. Nevertheless, she does not present a conclusion about this possible connection. Next, Clara needs some prompting to formalize her ideas mathematically. After visualizing how she could do that, in fragment eight, she uses an example to validate what she called as "the formula to find the number of binary searches" $\left(n = 2^t\right)$. She chooses a right procedure and a right moment to apply it, and she also describes and performs the procedure correctly. Solving this example is the strategy she uses to answer the task initial question, that is, to justify why the binary search is faster than the linear search in the worst case scenario. As for *productive disposition*, researcher's field notes attest to Clara's interest in the activity. Although she has moments of discouragement, she asks for help and tries to understand the prompts she is given. At the end of the task, she expresses her contentment for being able to learn and do mathematics both by verbalizing and by drawing, as can be seen in the last fragment. Clara did not participate in a recall interview for this task. It is possible to infer that the last phase of Clara's work involved *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning*, and *productive disposition*.

6. ANSWERING THE QUESTION: FINDINGS AND CONJECTURES

This chapter reports on research findings and conjectures raised by crossing and analyzing students' diagrams. The chapter is divided into four sections. The first section presents the outcomes that answer this research main question, which is: ***How are mathematical understanding and mathematical proficiency observed and expressed in the actions and interactions of students when engaged in mathematical modeling approaches in a high-school mathematics class?*** Then, the second section discusses behaviours of mathematical proficiency strands along the modeling tasks. Section three answers this research complementary question, which is: ***What are the affordances and constraints on students' mathematical understanding and on students' mathematical proficiency through modeling approaches?*** Finally, the last section argues about the feasibility of the use of this study modeling approach in high-school mathematics classes.

6.1. OBSERVED WAYS OF EXPRESSING MATHEMATICAL UNDERSTANDING AND PROFICIENCY

One of the main outcomes of this research speaks to the possibility of observing all five Kilpatrick et al.'s (2001) strands of mathematical proficiency when high-school students are engaged in mathematical modeling tasks. Indeed, all 16 diagrams (representing students' work) created for this research data analysis confirm that students worked on *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning*, and *productive disposition* at least once during the mathematical modeling tasks. These diagrams represent only a sample of a specific group of students, working with a specific teacher. In addition, diagrams were based on the relevance and amount of data. That is to say that, other diagrams could be created and maybe not all of them would present all five strands of mathematical proficiency. Nevertheless, I am not claiming that the use of modeling tasks in class will necessarily promote all strands of mathematical proficiency. I am arguing about the richness and the possibilities of the use of mathematical modeling in class. This study confirms that mathematics classes based on modeling can play a relevant role in mathematical understanding and mathematical proficiency. The next paragraphs present a more detailed explanation about how mathematical understanding and proficiency were expressed during students' work.

Students' *conceptual understanding* was identified based on the three indicators shown in Table 1. That is, whenever students' were retrieving mathematical content to investigate the task, connecting mathematical content to the task situation to help solving the task, or showing understanding of the mathematical content used during the task, students were assumed to be working on *conceptual understanding.* But how was this *conceptual understanding* expressed in students' work? In other words, what in students' work was pointing to one of those three mentioned indicators? Figure 15 brings up some examples of fragments that were identified as *conceptual understanding* pieces. As can be seen, most fragments were taken from students' verbal explanations about what they were doing during the task, but there are two written records as well. Students' expressions of *conceptual understanding* were mainly afforded by their discussions and explanations in the group, and on a smaller scale these expressions were afforded by the teacher or by self-talk. Students expressed *conceptual understanding* when: 1) retrieving a concept; 2) explaining a concept they were willing to use; 3) explaining how or why they were willing to use a determined concept; 4) describing a situation that was related to a specific concept; 5) explaining what they were doing; 6) explaining how or why they got to a determined stage; 7) describing a situation under analysis and making a conceptual conclusion; 8) describing a procedural step that was justified based on the concept behind it; 9) representing a situation by using mathematical terms and concepts; 10) representing a situation through a mathematical illustration and mathematical concepts.

Conceptual Understanding

400 out of 4000 can be reduced down to 1/10, it would be 1/10 of the whole race. And I did the same sort of thinking, I'm just creating fractions, fractions of the race [that] would be taken up by each leg.

We found the formula for the number sold. And then using that we put it in the profit revenue cost [equation], and then we find the formula for the profit.

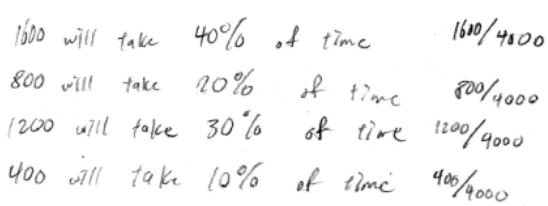
How many degrees are there in a triangle? 180 degrees in a triangle.

I think I know what a binary search is. It splits in the half, so it's either this side or this side, and it says this is the middle, and it splits in half again, and then in half.

For the binary one, I think [the relationship] might be n divided by 2 exponent something. When asked why: *Because each time that you perform the operation you are cutting in half.*

I'm pretty sure the control tower [represents the] origin. So they just need to know how far away from the origin they are and then they know how far they are from each other.

...it was always a parabola. It would have to be. Because you want to know the most profit you get. And linear equations go on for infinity. You don't have a most there.



Did you have three points on the map? Did you figure out where they intersect the straight lines? This is a mini right triangle here.

[W]hatever graph we do make, those price can vary, so the price range (...) of the quantity sold is basically infinite.

Because when you have an [even] number, it's right between two numbers. So 20 divided by 2 is 10, right? 10 is not the middle number, it's between 10 and 11, you have to go with [either one] and if that's not it, then you cross that out and you look to the left or right.

The speed is distance over time. If you divide that over here and then you move over here, time is equal to distance divided by speed.

So if the term you are looking for is the first one, then using the binary is gonna take more moves and if it's linear and the term is the first one, you just need one move.

For every one book that you sell you have to minus 18 dollars, because that's how much it cost to make the actual book.

The 400 meter racer should be running a lot faster than the 1600 meters, so we would have to average out somehow.

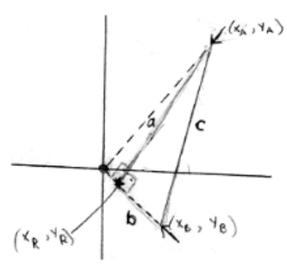


Figure 15: Forms of expressing conceptual understanding.

As for *strategic competence*, as described in Table 1, three different indicators were used to identify this strand of mathematical proficiency. To match one of these three indicators, students should be working on strategies to represent, understand, or solve the task. Figure 16 presents examples of this sort of evidence in students' work, in order to better illustrate how *strategic competence* was expressed. Again, most fragments are transcripts from the students' discourse, but there are also three written records as examples. Some ways of expressing students' *strategic competence* are: 1) describing a strategy; 2) explaining the usefulness of a strategy; 3) describing the approach to solve/model the task; 4) explaining what needs to be done; 5) explaining why following a determined plan; 6) making conjectures about a possible plan; 7) making questions to better organize thoughts and understand a task; 8) representing a strategy by using mathematical terms; 9) representing a strategy through a table; 10) representing a strategy through a mathematical illustration.

Strategic Competence

[We] were working together. He rounded down and I rounded up and this is the worst case. So we tried to get it to one and see how many moves it would take. So we´ve found a pattern, I think.

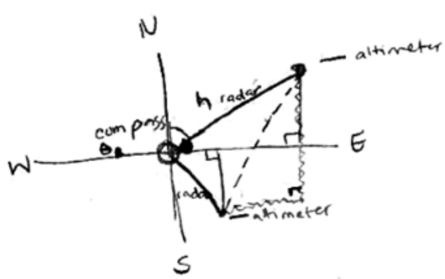
I decided they are evenly spaced apart, so [for] this person x - 0.53 equals their speed and for this person 0.53 + 0.53, 1.06, so x - 1.06 equals their speed.

I´m trying to find the relationship between first odd numbers. Because odd numbers have a definite middle and even numbers don´t have a definite middle. Find the correlation between the odd number and how many tests.

The number of searches divided by 2 powered to the term number and they have to equal 1 to find out what the number of searches is.

You only use the numbers from here or you make up? We can make up numbers, right?

We just force it into a right angle.



But then how do you figure out how far they are from each other? And how do you know the length of this? And how do you know they are over here or over here? What if they are the same distance over here and the same distance over here?

We were expecting that the runner that runs the second leg, so the 400 meter leg, should be running a maximum of 1.6 m/s [faster] than the runner of the fourth leg.

Actually, we should just start with finding the pattern within these numbers.

So we could probably graph unit price over quantity sold and use the two books against each other on the graph.

They are both approaching the control tower and in different directions and straight. So we´ve to determine the distance between the two planes.

We are trying to get a, b and c into one variable.

What is the relation between these numbers? I want a formula.

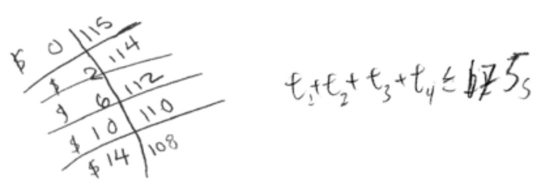


Figure 16: Forms of expressing strategic competence.

Procedural fluency is the easiest strand of mathematical proficiency to be identified in students' work. This is due to the fact that — different from other strands — students usually make note of the procedures they use, given that they typically need to write down their mathematical procedures to work on them. As a consequence, assessments are commonly based on *procedural fluency*. If some students are not prompted to write down their thoughts, conjectures and reasoning, they will probably only write down procedures. Students' also talk about the procedures they are using, commonly to explain them to somebody else. To identify *procedural fluency* on students' work, the four indicators shown in Table 1 were used. Students' work was scrutinized in order to find pieces that indicated they were able to choose an adequate procedure to implement, and a right moment to apply the procedure. Fragments were also expected to show that the procedure was performed correctly and that students understood the procedure. When working on the activities described by these four indicators, students were showing fluency in and comprehension about mathematical procedures. Figure 17 shows examples of how *procedural fluency* was expressed while students' were engaged in the modeling tasks. Some ways of doing that were: 1) writing and working on a mathematical equation (or another procedure) that represented a problem; 2) organizing data in a table or in another format and finding patterns or relations; 3) writing a mathematical algorithm that represented a problem; 4) describing a procedure; 5) explaining a procedure.

Procedural Fluency

We found a linear equation for the relationship between the quantity and the unit price. And then we just [substitute] that to the profit equation. And then we basically find the quadratics equation.

What I did was: I'm in this point, so I'm x_3 and y_3 maybe, and then I subtracted the points from each other here to get what A was and B was, and then square them.

Let's do it again in a different perspective. Does the GPS give height, though? We have to do it three times, right? So we do it from the front, and then we do from the side, and then we do it from above.

This distance south from this point here, that's the same distance if we just move that along, right? Right here. And this distance from watch tower to west, this small distance may be moved over there, right?

I can change minutes into seconds.

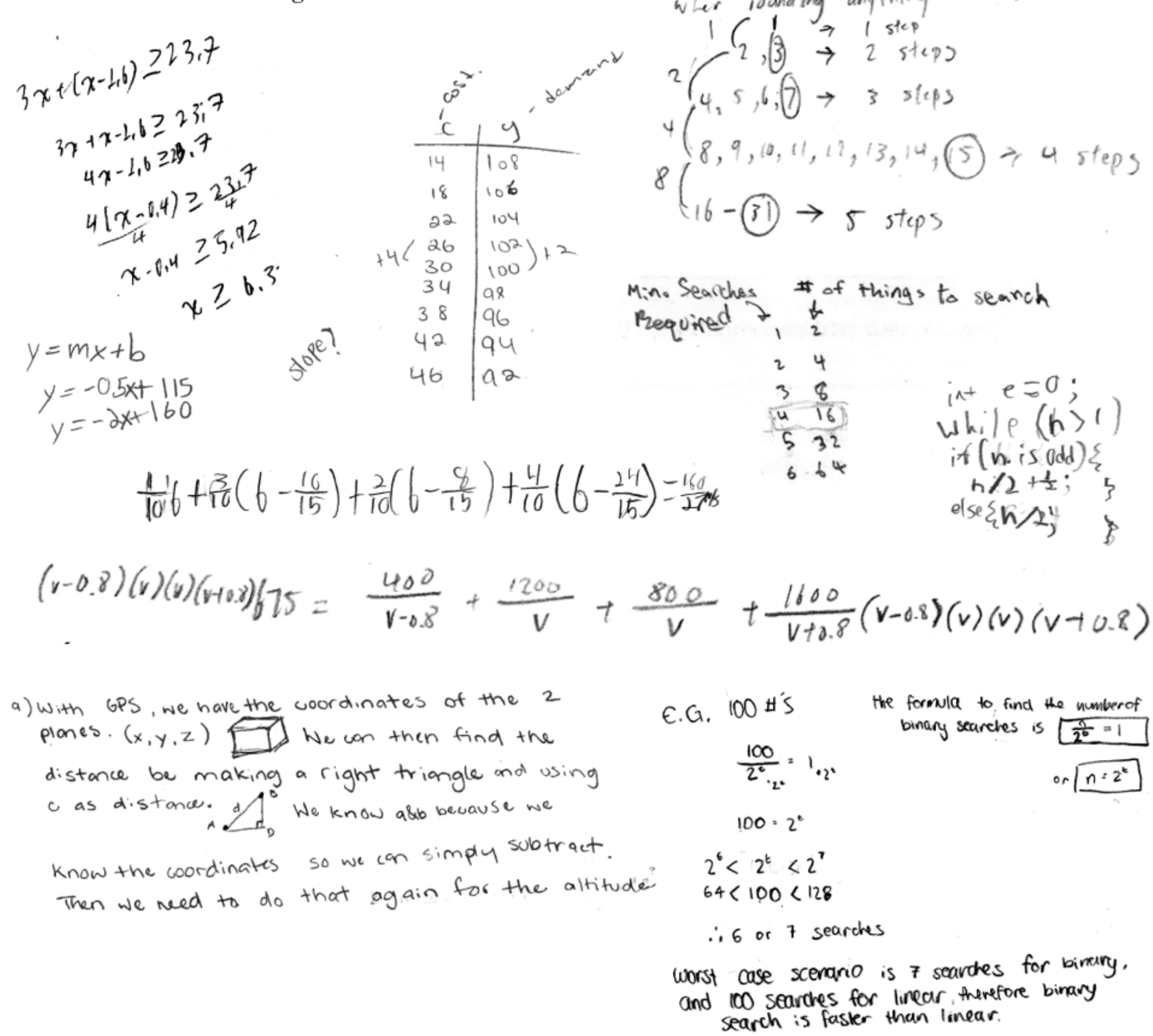


Figure 17: Forms of expressing procedural fluency.

The next strand of mathematical proficiency is *adaptive reasoning*, and the four different indicators shown in Table 1 were used to identify this strand pieces. That is to say that, if students were logically associating two different contents or two different situations, they were allegedly working on *adaptive reasoning*. The same holds true when students were logically relating content and situation or transferring content between two different situations. It is worth noticing that part of students' *adaptive reasoning* fragments was gathered during students' reflections in stimulated recall interviews. This fact confirms that students do not always reveal their thoughts while engaged in classroom mathematical tasks. Sometimes students need to be asked or prompted to reflect about what they are doing, so that their reasoning is disclosed and elucidated. Figure 18 displays several examples of how students expressed their *adaptive reasoning* when doing mathematical modeling tasks. In other words, how students expressed ability and flexibility in reasoning mathematically. Some possibilities were when students: 1) realized a change in a task situation and conjectured about new possibilities; 2) explained why or how a determined content was used to approach the given situation; 3) explained why or how a content interfered in the way a situation unfolded; 3) compared different contents in relation to one same feature; 4) conjectured about how different situations related to each other; 6) explained how or why transferring one content from one situation to another; 7) inquired about the use of a certain content in a certain situation.

Adaptive Reasoning

What would help us to find the coordinates since we didn't have the GPS.

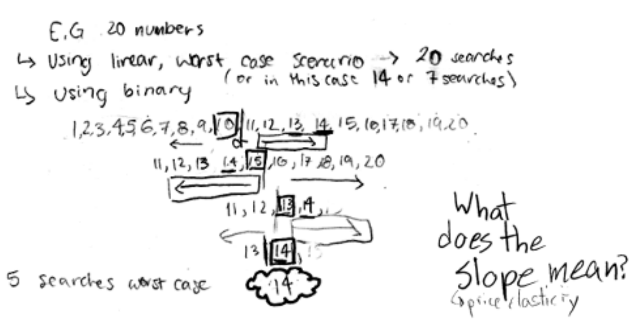
I would say I was trying to use composite directions like north, east, south, west to make a plane, because then you would know which quadrant in general the plane would be coming from, so you'd have a general idea of where the plane would be.

When asked about why using fractions: *That way is exact values instead of rounding with decimals. You get more exact answer.*

I found the equation to map the relationship between [quantity sold and price]. So basically you said that you couldn't find any more data points beyond this, but there is a pattern between them, so you could infer that there are data points beyond them, and look at that relationship.

Do you know like in movies? This is the board and then there is like a dot... Radar? So what's GPS? What is it on a plane? What does it look like? It's just telling you where to turn?

The 400 meter racer should be running a lot faster than the 1600 meters, so we would have to average out somehow.



If the difference in average speed were not equal, then everything kind of breaks down, because that would not work for my equation, because what my equation was doing was assuming that the differences in maximum speed could apply to the differences in average speed as well.

I think that trains have that too, right? Some like a control centre, tell which train to go on the tracks.

When asked about what she was struggling with: *Whether y is the right variable to use here, because the quantity sold isn't the quantity that you make.*

But don't we have to follow the rule? Do you know what I mean? Because when the price goes up by a certain number, a certain number of people drop, stop buying it.

I think for x it should be maybe minus one, because there isn't a zero search.

Binary also need you to arrange it in an order. Linear doesn't need you to arrange it in an order. That's the down side [of binary search].

In programming there are multiple flips you can use so that will repeat the same function.

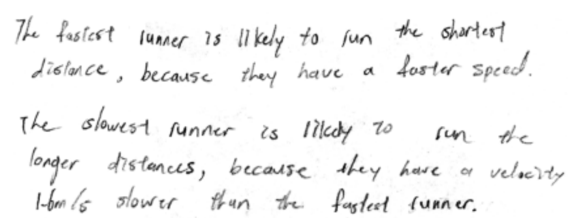


Figure 18: Forms of expressing adaptive reasoning.

Finally, *productive disposition* is the last strand of mathematical proficiency, which was also identified based on Table 1 indicators. The three indicators pointed out in the table assert that when students show belief in their capacity to do or to learn mathematics, they have *productive disposition*. The same is true when students see usefulness and worthiness in mathematics. As illustrated in Figure 19, this propensity towards mathematics was expressed when students: 1) valued figuring out the task on their own, instead of being told what to do; 2) identified what needed to be done and worked towards it; 3) took their own decisions; 4) described what to be done; 5) showed belief in their ability to infer; 6) showed belief in their capacity to understand; 7) approached the hardest first; 8) trusted that at some point they would get to an answer; 9) shared and used their ideas within the group; 10) showed enjoyment when getting to an end; 11) inferred what they would get when following a different path; 12) valued the fact that the task did not limit them; 13) acknowledged they were more confident; 14) pointed up to the relevance of mathematical modeling tasks done in class; 15) underlined the usefulness of associating tasks with contextualized situations.

During the modeling tasks, some forms of apathy and/or boredom with a few specific students were noticed. These circumstances did not seem to be context-related; it seemed to be due to the lack of *productive disposition*. These situations were delicate ones and deserved attention. Students could be motivated by the task context at first, but their lack of confidence in their ability to do and learn mathematics could have contributed to their disengagement in the task. Although this study does not specifically speak to engagement matters, during recall interviews, some students' discourse confirmed this claim. That was the case of Jack, in particular, in tasks three and four. In the recall interview for these two tasks, he was clear about giving up on doing both tasks. Jack said that he was "bad at putting stuff you learn in school to stuff in real life". This idea seemed ingrained enough in his mind, to the point that he did not believe he was able to solve this sort of contextualized problem: "Through me something like this, and tell me to do it; I can't do it". These two fragments of Jack's discourse reinforce his disbelief in his ability to learn mathematics. Jack also had a suggestion to solve task four; he thought that he could use the "percentage of how close they would have to be in this worst case scenario". But then he said "I never followed to that percentage thing anyways. I gave up on that idea." In other words, he did not believe he was able to do mathematics. This disengagement due to the shortage of *productive disposition* led Jack to a poor participation in the modeling tasks and, as a consequence, to deficient mathematical understanding and proficiency.

Productive Disposition

It´s nice because we are kind of essentially finding out how to do it on our own, than being told. We find our own direction and try to see if we could figure it out on our own. I like the challenge of it.

So you will see two dots, but you don´t know one is way above or way under. So we need to figure out, how to figure that?

When asked if he believed he would get an answer: *Yes, eventually. At some point it seemed like it might take a long time to get there.*

I thought like it was applying to maybe everyday life or applying it to jobs maybe, or situations in life for jobs.

Then I did this and we decided to go with it. [I shared my ideas] after I felt confident that it was worth it.

Itś helpful to look at it in relation to something that is real, that is solid.

I decided x would be the fastest.

Ahhh, I get it.

This answer was actually with the assumption that all the weights would be equal, which is wrong. But even if I signed the weights properly, I would still kind of get an answer like this, like the same reasoning.

It helps you think outside the box, instead of setting limits on yourself. After doing the first task, I have more confidence in doing the second task.

Imagine that this was a grid. What I would do is just count the sections down and then count it across. So that would be four and that much would be three, and then just four squared plus three squared.

I think [odd numbers] would be a little bit harder to work with, so I kind of get the harder out of the way.

There is a difference of four for step two and three, a difference of 8 between four and three, after finding that I just predicted.

It works! Look, oh my gosh. That was so much easier.

Oh my God, I got it. I have the solution. Figure out the slope ... and then you can create your own numbers.

I still don´t see the pattern. Explain it to me.

Figure 19: Forms of expressing productive disposition.

The five lists of how strands of mathematical proficiency can be expressed in students' work are not intended to be exhaustive lists. Their main purpose is to show the richness of possibilities that can reveal mathematical understanding and proficiency along students' work while doing modeling tasks. This richness is related to the timing and setting in which the research was implemented and to its participants. Fragments of students' work are snapshots at a particular time in a particular context, with particular content, etc. Therefore, one could argue that these research results cannot be generalized, given that they depend on the situation in which data was collected. I agree that it is a particular case, but I contend that if there were positive and encouraging results for this particular case, mathematical modeling shows potential for the mathematics class.

6.2. OBSERVED BEHAVIOURS OF MATHEMATICAL PROFICIENCY STRANDS

Other than looking at ways of expressing individual strands of mathematical proficiency, it is also relevant to look at proficiency strands in students' work as a whole. By doing this analysis, particular behaviours can be found. All 16 diagrams were analyzed in four different ways: 1) individually; 2) across students; 3) across tasks; and 4) across strands of mathematical proficiency. The outcomes of each of these analyses are described in the following paragraphs.

The first analysis approach, based on an individual student working on an individual task, brought up three specific aspects of students' work. The first one refers to the difficulty in gathering fragments related to *adaptive reasoning* and *productive disposition*. Students did not commonly or openly express these strands, and when they did it was usually by verbal communication. *Adaptive reasoning* was expressed when students explained their thoughts or options to their peers, or when they were thinking out loud as a way of understanding better or clarifying their ideas. It was not usual for students to write down their reasoning. In general, they wrote down the procedures derived from their reasoning processes. Therefore, many times their reasoning was implicit in their procedures. This implicit reasoning might not be hard to assess by analysing students' procedures; however, changes in reasoning might be almost unobservable, in particular when students erase or do not register their attempts to solve the task. As for *productive disposition*, it was mainly observed as student excitement and confidence about what they were doing, or when they verbalized their contentment in learning some useful knowledge or in exploring some likely-to-happen situations. Some students wrote down positive comments about their accomplishments, but this was not common. As a result, opportunities to identify *productive disposition* were scarce. The second aspect observed in students' individual work refers to the use of *procedural fluency* as a way of understanding a task, instead of just a way of solving a task. *Procedural fluency* is typically expected when students are implementing procedures to solve the task. This individual analysis drew attention to the fact that given information in a task was used by the learner to make sense of the task. In other words, *procedural fluency* was not used only to solve the task, but to understand the task too. Lastly, the third noticed aspect in this individual analysis refers to the fact that students did not need to get to a final answer to work on all five strands of mathematical proficiency.

This might be the most important aspect among the three mentioned here. Not needing to get to a final answer reinforces Ben-Hur's (2006) point of view, which supports that processes should be given more value than products. Teachers should ask themselves if a student's correct final answer means that this student understands or is proficient in mathematics. It might only mean that this student knows how to get to a correct answer, regardless if this student knows what he is doing or not. During this research, students were showing understanding and proficiency throughout the modeling processes, even when they did not get to a final answer. It is my opinion that process-based outcomes are of more relevance than a possible right answer.

The second analysis approach was by crossing data over students, that is, by looking at one same task when done by different students. All four diagrams referring to one same task — from four different students — were investigated and mapped in search of common aspects or behaviours. The goal was to figure out if the task was accountable for prompting specific thoughts or raising specific strategies that would result in a mathematical understanding and proficiency pattern. The same procedure was repeated for tasks one, two, three and four. However, no patterns or tendencies triggered by the tasks were found.

The third analysis approach was done by crossing data over tasks, that is, by looking at one same student when doing different tasks. In this case, two different tasks done by the same student were analyzed and mapped looking for similar ways of conducting the tasks. The idea was to identify if the student's mindset could possibly influence the resolution of different tasks in the same way. In other words, if the student could carry out different tasks with similar thoughts, approaches or strategies, due to his or her particular way of doing mathematics. If that was the case, peculiar mathematical understanding and proficiency patterns could be noticed. Nevertheless, that was not the case, which means no particular patterns or tendencies seemed to be generated by students' individualities.

The final analysis approach was by crossing data over the strands of mathematical proficiency, that is, by looking at these strands along time for all students and all tasks. A general tendency was found in this last approach. To better understand this tendency, a distribution for the identified strands of mathematical proficiency, throughout task phases, is portrayed in the graph shown in Figure 20. By task phases, I mean phases 1, 2 and 3 identified in each diagram (refer to Section 5.2). Each dot in the graph is obtained by coordinates, in which x corresponds to [Phase Number . Student Number] and y corresponds to [Strand Number . Task Number]. For this matter, *productive disposition* is considered as strand 0, *adaptive reasoning* as strand 1, *procedural fluency* as strand 2, *strategic competence* as strand 3, and *conceptual understanding* as strand 4. Student numbers are the ones shown in Table 4, and tasks and phases are already numbered. If a dot has coordinates (1.7, 2.4) it means that this dot refers to student number 7 (Clara), doing task number 4, during phase number 1, and working on strand number 2 (*procedural fluency*). In the graph, strands are weighted, which means that if the same strand occurs more than once within the same phase, same task and same student, the respective dot will be increasingly bigger according to the number of occurrences.

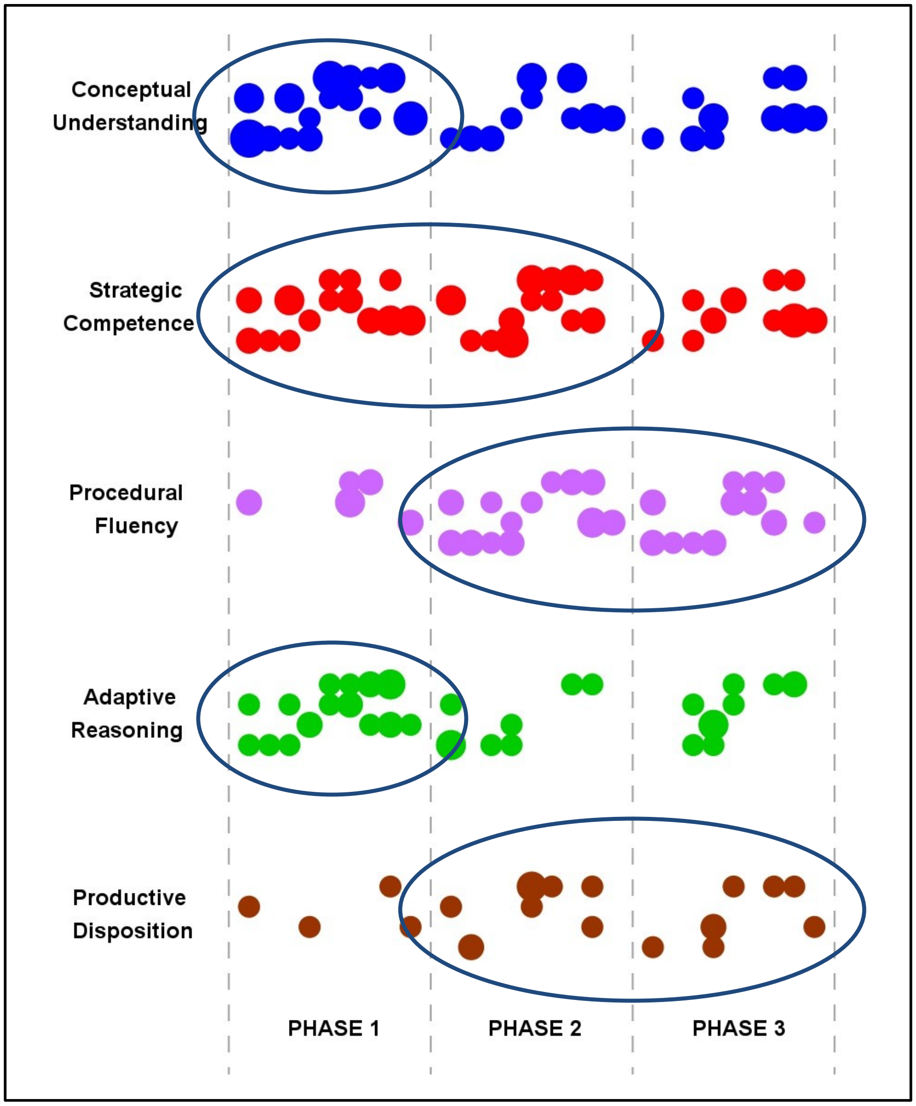


Figure 20: Graph of weighted strands over investigation phases.

Based on this description, Figure 20 shows a proper representation of how mathematical proficiency strands were expressed throughout the modeling tasks. As can be seen in the graph, *conceptual understanding* was usually present during the whole task, which means that students were constantly working on mathematical concepts and their comprehension. Nevertheless, as highlighted in the graph, *conceptual understanding* was more concentrated at phase 1, meaning that at this phase students were more required to retrieve previous concepts and connect to the task, in order to understand the task and establish a working plan. As for *strategic competence*, it was also present all the way through tasks. However, it was more intense in the initial and middle phases. In fact, this is expected, given that in these two first phases, students are presumed to come up with different strategies to represent, understand and solve the problem. These strategies can change during the investigation until a solid and definite strategy is reached. This can take some time, but generally by the middle phase, students already have a strategy to work with. Strategies discussed during the final phase can be more related to further developments of the task. It is also true that some students can be still building strategies by the final phase and do not have enough time to finish the whole task.

As the graph in Figure 20 portrays, *procedural fluency* was mainly concentrated in the middle and final phases. Again, this behaviour is expected, given that in the middle and final phases students are implementing their strategies and, as so, they are required to choose, implement and understand procedures. Usually, students will keep working on the necessary procedures to solve the task until they get to an end. As mentioned before, an interesting point is that some students used their *procedural fluency* skills in the initial phase of their work to better understand the task. Students used procedural examples to illustrate what was being described in the task. By doing that, students were able to embody what they were supposed to visualize and analyze, which made their comprehension process more tangible and easier. As for *adaptive reasoning*, it goes along with a similar pattern as *conceptual understanding*. Indeed, students were involved in a lot of reasoning in the initial phase of the process, due to the need of understanding the task and elaborating possible strategies to be followed. As so, at this phase, students were required to ponder different options, relate contents and situations, transfer content between situations, that is, they needed to reason and adapt their reasoning accordingly. *Adaptive reasoning* was also present in the middle and final phases, due to reflections upon and assessment of what was being done. As a result, strategies and procedures needed to be adapted along the task.

Finally, according to the graph, *productive disposition* was expressed more frequently in the middle and final phases. This makes sense, given that after students have done a bit of work and have experienced some encouraging results, they are more prone to believe in their capacity of doing and learning mathematics, as well as in the usefulness of mathematics. Fragments of students' work showing *productive disposition* were not easy to be gathered. They were limited and some students did not verbalize or write comments that illustrated their *productive disposition*. Therefore, there is less credibility in the pattern of *productive disposition* shown in the graph.

In short, students' understanding and proficiency may unfold along time following the tendency shown in Figure 20. Nevertheless, this is not necessarily a pattern for students' mathematical understanding and proficiency while modeling. This can be confirmed for example in Clara's diagram (Figure 14), which starts off showing *procedural fluency* and not *strategic competence*. In any case, what is being emphasized in this research is not the necessity of identifying a tendency among students' course of action. Instead, this research highlights possible scenarios for the emergence of students' mathematical understanding and proficiency, when mathematical modeling is implemented in class.

6.3. MATHEMATICAL MODELING AFFORDANCES AND CONSTRAINTS ON STUDENTS' MATHEMATICAL UNDERSTANDING AND PROFICIENCY

As section 6.1 describes, the main affordance of the use of mathematical modeling in high-school classes is exactly the investment on students' mathematical understanding and proficiency, by means of working on the five essential strands of mathematical proficiency according to Kilpatrick et al. (2001), namely: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition.* It is important to investigate how these strands function in the context of mathematical modeling. In other words, how mathematical modeling supports these strands. A mathematical modeling task, as implemented in this research, yields affordances by demanding students to engage in activities that are not simply based on the use of procedures. When undergoing these activities, the five strands of mathematical proficiency are promoted. Table 5 presents the modeling affordances, the corresponding activities developed by students, and the respective promoted strands of mathematical proficiency.

Mathematical Modeling Affordances	**Students' Activities**	**Promoted Strands of Mathematical Proficiency**
1) Engagement	Investigating likely-to-happen situations.	*Productive Disposition*
2) Motivation	Realizing mathematics usefulness.	*Productive Disposition*
3) Positive Attitude	Realizing ability to do and learn mathematics, and to solve problems.	*Productive Disposition*
4) Investigation	Exploring task and its goals.	*All five strands.*
5) Conceptual Analysis	Relating task with mathematical concepts.	*Conceptual Understanding*
6) Content Analysis	Relating task with mathematical contents.	*Conceptual Understanding*
7) Strategy Building	Figuring out strategies to solve task.	*Strategic Competence*
8) Decision Making	Choosing appropriate options to solve task.	*Strategic Competence*
9) Choice of Procedures	Determining proper procedures to solve task.	*Procedural Fluency*
10) Use of Procedures	Implementing mathematical procedures.	*Procedural Fluency*
11) Knowledge Production	Producing or researching for necessary knowledge to solve task.	*All five strands.*
12) Mathematical Reasoning	Finding out logical relations.	*Adaptive Reasoning*

Table 5: Mathematical modeling affordances and respective promoted strands of mathematical proficiency.

Each of these promoted proficiency strands presents benefits for students' mathematical understanding. For instance, by working on students' *conceptual understanding*, students are learning to search for their background common or content-related knowledge when presented to new problems or situations. Students are improving their research and investigation abilities, as well as their aptitude to build a knowledge network. These concept-related competencies demonstrate how mathematical modeling affords students' mathematical understanding, given that this understanding relies on conceptual knowledge. When working on and with *strategic competence*, students are primarily learning how to be strategic whenever facing a problem or challenge, which is a relevant skill in students' everyday life. Strategies demand students to analyze a situation and figure out a plan or approach to get to whatever is required. Students enhance their *strategic competency* when required to diversify their strategic abilities, for instance, by creating strategies to understand and to represent problems or situations. All these skills contribute to enhance students' mathematical understanding abilities, once they promote the design of mathematical methods that depend on understanding.

As for *procedural fluency*, although it seems to be a simple or easy skill to be implemented and observed — given that it seems to be just about the application of mathematical procedures — it is not exactly like that. This might be true when students are presented problems right after learning a specific content. In this case, *procedural fluency* is straight forward and does not involve high-order thinking. However, when students are expected to investigate situations that are not directly related to specific learned contents, they will need to consider all possible procedures and identify an appropriate one to be applied at a proper moment. In this case, it is not only about implementing a procedure correctly; it is also about making a right choice. This is a relevant ability, which improves students' understanding as well. Without understanding, students might not be able to identify proper options. Other than that, a usually neglected part of *procedural fluency* is understanding the procedure. In this research modeling approach, students' are encouraged to grasp the mathematics behind each procedure, and not only to apply procedures arbitrarily. This feature is also important to students' mathematical understanding.

Working on *adaptive reasoning* competencies is a significant affordance for students' mathematical understanding. The ability to reason flexibly is not always present in students' work, in particular when they know they are supposed to find one single right answer. Undeniably, it is hard to be flexible when you know that no flexibility is being expected from you. In this kind of situation, students are more worried about finding the one single right answer, instead of allowing them to adapt their reasoning to find their own path to the solution. In this sense, modeling approaches are relevant to set the proper context to encourage and enable students to make their own logical relations and adjust their mathematical reasoning accordingly. This reasoning process relies on mathematical understanding, at the same time that enriches it.

Finally, supporting students' *productive disposition* results in advantages in terms of students' mathematical understanding as well. However, in this case, the advantages result from students' investment on their own understanding. In reality, when students are immersed in a task that is applicable, useful, and motivating, they tend to see worthiness on it and better engage in the task. The same is true when students trust they are capable of learning and doing mathematics. As a consequence, students put more effort in their work, and their mathematical understanding benefit from that. In reality, *productive disposition* can be a result of students' understanding that leads to more investment on students' understanding. It is a feedback loop. Undeniably, the emergence of this sort of belief and investment is a considerable mathematical modeling affordance on students' mathematical understanding.

Another relevant modeling affordance is that students do not need to get to a final answer to work on their mathematical understanding and proficiency. In other words, the modeling process promotes the five strands of mathematical proficiency throughout the modeling task; which means mathematical understanding and proficiency are not dependent on tasks' final results. This is a significant advantage, given that students do not need to be under pressure to reach a final formal solution; as if getting to a final solution was a condition for them to learn. Instead, students are learning and working on mathematical skills throughout the implementation of the task. This affordance can be confirmed by means of students' diagrams. Some of the students did not finish their tasks, mainly because they needed some extra time. But it is clear that all of the students' diagrams portray at least one piece of evidence of all five strands of mathematical proficiency. It follows that students were working on their mathematical understanding and proficiency regardless of completing or not the task. As a final remark, the mathematical modeling affordances surpass the aforementioned mathematics related ones. When modeling, students are encouraged to make their own assumptions, create their own numbers, and/or establish their own conditions. By going through these processes, students increase their overall confidence and become flexible in dealing with different and non-mathematical related circumstances. The capacities that arise from this modeling process are highly demanded in professional contexts (The Conference Board of Canada, n.d.).

In spite of the aforementioned modeling affordances, there is also at least one modeling constraint on students' mathematical understanding and proficiency that must be observed. This constraint is related to students' lack of interest in the modeling task context. This issue may lead to lack of engagement and, as a consequence, it can get in the way of students' mathematical understanding and proficiency. This constraint can be minimized when the teacher has different task options addressing distinct contexts — within the same content-matter — for students to choose from. In this way, students' have the chance to pick the most interesting task in their viewpoint, while still working on the same content.

Apart from that, there are two other constraints that are relevant to be addressed. However, these constraints are not modeling-related; they are respectively group-related and knowledge-related instead. The first constraint is due to the difference between students' reasoning time. Some students reason faster than others and sometimes faster students do not realize that they should give other students the opportunity to think and make conclusions on their own. This might impair slower students' understanding and proficiency. This constraint might be reduced — but not avoided — if students are given some individual time to work on the task before working in groups. The second constraint is unanticipated discrepancy in students' background knowledge. This divergence may lead to a non-participant student, preventing this student to work on his/her mathematical understanding and proficiency. This constraint may be lessened when two complexity conditions are well balanced, namely, *redundancy* and *internal diversity*. Another strategy that may be required from teachers is to have a specific backup plan/task to focus on the necessary missing knowledge before going further with the actual task. This might imply in having students working apart from their original group. However, this might be necessary in order to let them in their groups again, but actively participating this time.

6.4. MATHEMATICAL MODELING AS A FEASIBLE APPROACH IN MATHEMATICS CLASSES

One of the outcomes of this study is revealing the feasibility of modeling approaches in mathematics classrooms. As described in Chapter 4, the modeling tasks used in this research were applied in an Alberta grade 11 mathematics class with 27 students, and faced usual time and curriculum constraints. No special arrangements were made apart from working towards Davis and Simmt's (2003) five complexity conditions (*redundancy*, *internal diversity*, *neighbour interactions*, *decentralized control* and *organized randomness*). In this sense, it can be said that the classroom environment was pretty close to what is commonly expected in a mathematics high-school classroom, and this was not an obstacle to the modeling approach. Although modeling was something new to students in the class, they understood and engaged in the process. As students were from the IB program, some extra commitment with mathematics classes could be expected from them; however no particular behaviour that would characterize a significant difference from a non-IB course was observed. The content that was supposed to be covered according to the teacher's course planning was covered and the teacher did not have to rush because of the modeling approach.

In addition to portraying the feasibility of the use of mathematical modeling as a resource for teaching for understanding, this research reveals a doable and relevant approach to teachers and student teachers, by offering a way of accessing and unpacking students' mathematical understanding and proficiency. This is possible by observing students' modeling processes during tasks, and by means of implementing the diagram-based approach created for this research analysis. Carlson, Larsen and Lesh (2003) underline that the use of modeling practices within school students' is relevant for student teachers, because these practices allow teachers to have regular "access to the developing understandings and reasoning patterns of their students" (p. 476). The use of modeling approaches in class is not only possible, but also supportive of teachers' work.

Certainly, there are challenges inherent to the use of modeling in class, and Chapter 2 presented and discussed the majority of these general challenges. This section looks into some other realistic concerns brought to the researcher's awareness during this study. These concerns are not preventive, but they might be demanding at first, due to the lack of teachers' practice within modeling contexts and tasks. Therefore, they deserve attention. The first concern is that a daily planning is not useful or even necessary during the days in which tasks are carried out. Instead, task planning over a few days is more appropriate. This is due to *decentralized control*, which allows students to follow their own unforeseen paths every day, and not the teacher's path instead. Teacher's planning is supposed to be completed as a whole by the end of the task and not by the end of each day of the task implementation. The second concern refers to whole class instruction, which can be unnecessary or also inappropriate. This is due to the difference in groups' developmental stages during a task. A specific moment of whole class instruction may hinder the work of some groups, given that the teacher can bring up content or an explanation that is still being worked out in a specific group. Therefore, group instruction is more suitable instead. Nonetheless, this does not mean that the teacher cannot bring up discussions within the whole class. The teacher will likely have enough and adequate information from each group to decide if a discussion should be held as a class or as a group.

The third potential concern is the difference between students' class time usage. Some students or some groups may finish the task ahead of time. This may require the teacher to have further developments of the current task or extra tasks for faster groups, as the ones suggested in section 4.4. It is pertinent that these extra tasks are relevant to the main modeling task or to the main content studied by the class. If the extra discussion deviates a lot from the main class content, potential discussions within the whole class might be impaired, given that new and different discussions will be raised within groups. Finally, a last but not least important concern is the need for teachers to deal with the unexpected. Because the unexpected is present in modeling approaches, teachers are required to anticipate diverse scenarios and work with the unanticipated ones directly on site. The main challenge here is that these unpredictable situations might represent the deviation from the main goal of the task, and teachers need to manage these circumstances, without discouraging students. Overall, all these issues can be handled when teachers are predisposed to welcome modeling in their classes and not to face these concerns as obstacles.

7. TEACHING FOR UNDERSTANDING AND PROFICIENCY THROUGH MODELING: SOME CONSIDERATIONS

The use of mathematical modeling in education has been investigated for five decades. Different studies cover the mathematical modeling history within the realm of education and also its advantages. The benefits of bringing modeling into mathematics classes are well known and well accepted, and modeling is becoming more common and more appealing. Research around this topic is increasing, and new understandings and possibilities are emerging. These studies can be of benefit for teachers and students. Nonetheless, as Vorhölter et al. (2014) confirm, teaching through modeling is a debatable topic. Although there is adequate research about this topic — gathering a number of successful examples, there is not enough knowledge to lead to a consensus about how modeling should be taught in order to benefit students (Vorhölter et al., 2014). The authors mention Blum and Niss's work (1991), which sorts the teaching through modeling in four different possible ways: 1) in an applied mathematics strand, separated from the pure mathematics strand; 2) within several different applied mathematics strands, also separated from the pure mathematics strands; 3) mixed in the mathematics program as introductory examples or application examples; 4) integrated in the mathematics program as a means to produce mathematical content. Recent academic work about modeling demonstrates that the last of Blum and Niss's categories is the one that is being predominantly sought by mathematics teachers. However, the practical incorporation of mathematical modeling in classes — in the way suggested by the last category — still poses a big challenge. As Kaiser and Stender (2013) affirm:

> It is especially an open question, how complex authentic modelling problems put forward by the realistic or applied perspective on modelling can be integrated into mathematics education, what kind of learning environment is necessary, whether a change in the role of the teacher to a coach or mentor of the students is needed (p. 279).

The research in this thesis intends to yield some insight into this open question; insight in how to integrate mathematical modeling in a high-school mathematics class in order to promote mathematical understanding, mathematical proficiency, and the production of mathematical content knowledge. The first concern when investigating this scenario was related to the design of the classroom in which students were immersed. As complexity science is gaining space in educational settings, it drew my attention to the fact that a mathematics class is a complex system, and that there is a need to respect and support the complex features inherent to it. Therefore, the idea was to have students modeling relevant contextualized situations by themselves, in a classroom driven by the five conditions proposed by Davis and Simmt (2003) to support a complex system, namely: *internal diversity*, *redundancy*, *decentralized control*, *neighbour interactions* and *organized randomness*. I believe that by acknowledging and accepting these conditions as intrinsic to the learning process, the emergence of mathematical understanding and proficiency is facilitated. Maintaining this supportive environment was essential to obtain the reported outcomes. Furthermore, a complex environment favours and nurtures desired characteristics for students out of the school classroom. Therefore, not only students' mathematical understanding and proficiency benefit from complexity science perspectives, students' attitudes and beliefs benefit as well. As a result, students' change of mindset could positively impacts society. In conformity with that, Davis and Simmt (2014) argue that:

> As complexity becomes more prominent in educational discourses and entrenches in the infrastructure of "classrooms", mathematics education can move from a culture of cooperation to one of collaboration, and that has entailments for the outcomes of schooling — articulated in, e.g., movements from generalist preparation to specialist expertise, from independent workers to team-based workplaces, and from individual knowing to social action." (p. 91)

That said, a complex environment can be taken as one promising possibility or as one relevant requirement for the integration of mathematical modeling with mathematics classes.

The next research concern was observing students' mathematical understanding and proficiency in this complex classroom setting while modeling. This concern is due to the fact that grasping students' mathematical understanding and proficiency helps in guiding teachers' actions. By analyzing these two aspects, teachers can better access and grasp students' thoughts in order to enhance teaching and learning strategies. As a consequence, students' knowledge production improves as well. Palharini and Almeida (2015) state that "it is possible to see that in situations with modelling tasks educators can perceive and study mathematical thinking processes and, by knowing these, it is possible to create learning environments or scenarios to explore the emergence of such processes" (p. 227). In other words, teachers need to know what students are taking out from mathematics classes and tasks in order to manage their own practice. In agreement with that, Doerr and Lesh (2011) raise the question: "How can teachers be expected to teach important concepts or abilities effectively if it is not clear what it means for students to 'understand'?" (p. 248). The authors conclude by indicating the need to clarify: 1) what is the meaning of understanding a specific content teachers want students to learn; and 2) how to measure and assess the progress of this understanding (p. 249). In this study, if a student was said to understand a specific mathematical content, it meant this student was able to handle all five strands of mathematics proficiency in what refers to this specific content, namely: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition* (Kilpatrick et al., 2001). Besides, a diagram-based approach was created to portray how this mathematical proficiency was expressed throughout modeling tasks.

Although mathematical modeling has been the subject of different papers and academic curriculum discussions in the last decades, there are still unanswered questions and conjectures to be explored. As described in Chapter 1, this research seeks to investigate how mathematical understanding and mathematical proficiency are observed and expressed when a mathematical modeling approach is used to teach mathematics. This study differs from previous ones primarily in the two different aspects described above, which are: 1) the classroom design, which is set respecting complex systems features; and 2) the research analysis, which focus on students' mathematical understanding and proficiency.

This research intervention is a challenging one, given that it accounts for teaching mathematical content by means of modeling tasks, with students supposedly producing knowledge, instead of accepting knowledge. Different challenges can be encountered when trying to bring this sort of practice to regular mathematics classes. In the same way, diverse reasons can be mentioned for teachers to avoid it. Three of them might be of more influence when teachers give up, they are: 1) the difficulty in creating mathematical modeling tasks; 2) the difficulty in fitting modeling tasks within curriculum outcomes; and 3) the long time required implementing modeling tasks. As mentioned in the preface of the *Mathematical Modeling Handbook* from Teachers College Columbia University (2012), "[t]he integrated nature of mathematical modeling, and in turn the number of curricular standards covered when working through a modeling activity, make modeling activities a very efficient use of class time" (p. vi). That is to say that the time required for implementation and the connection with the curriculum are not exactly constraints, once modeling successfully integrates content in a timely manner. Actually, these two aspects might be better exploited when employing mathematical modeling in class, so they should not be considered as drawbacks. This same handbook (Teachers College Columbia University, 2012) presents 26 different modeling modules that can be used in mathematics classes, not to mention other publications with the same goal. The difficulty in creating modeling tasks and/or the lack of instructional materials are not as problematic as they were when tasks were not readily available; thus, this difficulty is no longer a sound reason to avoid modeling. In recent academic publications there is a considerable number of studies suggesting modeling tasks for secondary education.

As Chapter 5 describes, this research findings point to modeling as a potential way of working on students' mathematical understanding and proficiency. Some aspects for improvement can be highlighted though. For example, students could have been asked to present formal reports about the work done during the task. Not exactly reports to their teacher, but sort of professional reports to an employer, as if students had been hired to solve each of the tasks. Also, individual microphones would have been helpful to better grasp students' comments during class discussions. The attainment of these two aspects would probably have provided more quality data. Lastly, if the task could have been mentioned and/or explored throughout the whole content lesson — and not only as an introduction for the lesson content, more insights about students' mathematical understanding and proficiency might have been gathered in different moments other than only during the task itself. In addition, more time could have been given for students to solve the tasks, which would definitely be of benefit for students' mathematical understanding and proficiency.

Future developments related to this research can reinforce the benefits of doing modeling in class and open up other relevant perspectives regarding teaching mathematics for understanding through modeling. Future work might include the extension of the research intervention to other settings, such as: a standard mathematics class, instead of an IB class; in rural communities, instead of in an urban community; and in a private school, instead of a public school. Another potential future work can be the implementation of a whole course taught by means of modeling tasks. The idea is to be less concerned about specific curriculum requirements, presuming that, by the conclusion of the course, the entire curriculum would be discussed. Other than working on students' mathematical understanding and proficiency, students would also have more opportunity to build on all the aspects involved in managing likely-to-happen situations. As English (2013) affirms:

> One means of preparing students for existing and future challenges is the inclusion of complex modelling problems within the curriculum. Such problems place students

> in multidimensional situations that require them to make reasoned and sophisticated choices about the knowledge they will apply and how, and ways in which they might communicate and share their products.
>
> The ability to deal with complex, life-based situations requires flexibility, creativity and innovation, critical analysis, empathetical considerations, political awareness, and a commitment to knowledge generation and refinement (p. 502).

An additional future development can be focused on teaching strategies. A promising study could analyze students' mathematical understanding and proficiency as a means of evaluating modeling teaching approaches. If students' mathematical understanding and proficiency is a high-quality one, it can positively speak to the implemented teaching approaches. In a different perspective, the expertise gained in this research tasks creation may be helpful in developing future studies about creating mathematical modeling tasks. Another suggestion for prospective work is based on the teacher's role within this sort of mathematical modeling approach. As Vorhölter et al. (2014) assert

> the role of the teacher within modelling activities has not been researched sufficiently: not enough secure empirical evidence exists on how teachers can support students in independent modelling activities, how they can support them in overcoming cognitive blockages, or how can they foster metacognitive competencies (2014, p. 33).

Repeating the research intervention, but collecting data from the teacher this time, could lead to interesting data regarding teacher's influence and relevance during modeling tasks. It would be interesting and probably necessary to vary teacher's way of intervening in students' work, of suggesting paths to be followed, and of prompting students. Finally, a last idea to be implemented in future research is the use of the diagram-based approach used in this research together with Kilpatrick et al. (2001) model of mathematical proficiency as a way of assessing students' understanding and proficiency in different performance tasks and not only when engaged in modeling tasks. This alternative rubric for assessment seems to have potential when compared with paper and pencil tests that can observe not much more than some *conceptual understanding* and *procedural fluency*.

To conclude, it is worth emphasizing that these research findings are of relevance for the mathematics education research community, as they aim to draw teachers and researchers' attention to helpful aspects related to the challenging task of teaching for understanding through mathematical modeling approaches. Mathematical modeling is confirmed as a promising way of doing mathematics, and of working on students' mathematical understanding and proficiency. The research intervention reinforces the feasibility of conducting modeling approaches along one regular mathematics course, without hindering curriculum goals or wasting classroom time. By accessing students' mathematical understanding and proficiency, this study reveals an important aid for teachers to detect students' thoughts and misconceptions, plan their classes seeking for students' understanding, and enhance teaching strategies for understanding and long lasting knowledge production. Lastly, the extension of Kilpatrick et al.'s (2001) work, by means of a diagram-based approach created to analyze student's work, is also an important contribution to the field, either due to its methodological nature, or due to its assessment value.

BIBLIOGRAPHY

Alberta Curriculum Standards Branch. (2001). *Applied mathematics 10-20-30: Program of studies.* Alberta.

Alexander, R., Bennet, C., Coleborn, R., Davis, G., Glanfield, F., Hope, C. B., et al. (2000). *Applied mathematics.* Toronto: Addison-Wesley.

Almeida, L., Silva, K. d., & Vertuan, R. (2013). *Modelagem matemática na educação básica.* São Paulo, SP: Contexto.

Anderson, D., Nashon, S. M., & Thomas, G. P. (2009). Evolution of research methods for probing and understanding metagognition. *Research in Science Education , 39*, pp. 181-195.

Anhalt, C. O., & Cortez, R. (2015). Mathematical modeling: A structured process. *Mathematics Teacher , 108* (6), 446-452.

Attard, C. (2012). Engagement with mathematics: What does it mean and what does it look like? *Australian Primary Mathematics Classroom , 17* (1), 9-13.

Bahmaei, F. (2011). Mathematical modelling in primary school, advantages and challenges. *Journal of Mathematical Modelling and Application , 1* (9), pp. 3-13.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2007). How can we assess mathematical understanding? In J. H. Woo, H. C. Lew, K. S. Park, & D. Y. Seo (Ed.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education. 2*, pp. 41-48. Seoul: PME.

Baron, L. M. (2015). An authentic task that models quadratics. *The Mathematics Teaching in the Middle School , 20* (6), 334-340.

Bassanezi, R. C. (1994). Modelagem como estratégia metodológica no ensino da matemática. *Boletim de Educação da SBMAC* .

Ben-Hur, M. (2006). *Concept-rich mathematics instruction.* Virgina, USA: Association for Supervision and Curriculum Development.

Biembengut, M. S. (2009). 30 anos de modelagem matemática na educação brasileira: Das propostas primeiras às propostas atuais. *Alexandria Revista de Educação em Ciência e Tecnologia , 2* (2), 7-32.

Biembengut, M. S., & Hein, N. (2013). Mathematical modeling: Implications for teaching. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford, *Modeling students' mathematical modeling competencies* (pp. 481-490). Springer.

Binary search algorithm. (n.d.). Retrieved from Wikipedia: http://en.wikipedia.org/wiki/Binary_search_algorithm

Bleiler-Baxter, S. K., Barlow, A. T., & Stephens, D. C. (2016). Moving beyond context: Challenges in modeling instruction. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 53-64). Reston, VA: The National Council of Teachers of Mathematics.

Bloom, B. S. (1956). *Taxonomy of educational objectives: The classification of educational goals.* New York, NY: David McKay Co.

Blum, W., & Ferri, R. B. (2016). Advancing the teaching of mathematical modeling: Research-based concepts and examples. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 65-76). Reston, VA: The National Council of Teachers of Mathematics.

Blum, W., & Ferri, R. B. (2009). Mathematical modelling: Can it be taught and learnt? *Journal of Mathematical Modelling and Application , 1* (1), pp. 45-58.

Blum, W., & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics , 22*, pp. 37-68.

Boaler, J. (2001). Mathematical modelling and new theories of learning. *Teaching Mathematics and its Applications , 20* (3), pp. 121-127.

Boaler, J. (2008). *What's math got to do with it? Helping children learn to love their least favorite subject - and why it's important for america.* New York: Viking.

Boeing 737-800 and airbus A320 flight simulators. (n.d.). Retrieved from MPL Simulator Solutions: http://www.mplsimulatorsolutions.com/

Bonotto, C. (2010). Engaging students in mathematical modelling and problem posing activities. *Journal of Mathematical Modelling and Application , 1* (3), pp. 18-32.

Bostic, J. D. (2015). A blizzard of a value. *Mathematics Teaching in the Middle School , 20* (6), 350-357.

Bracke, M., & Geiger, A. (2011). Real-world modelling in regular lessons: A long-term experiment. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in* *teaching and learning of mathematical modelling* (pp. 529-549). Dordrecht, SH: Springer.

Brasil. (2000). *Parâmetros curriculares nacionais - Ensino médio.* Brasília, DF.

Britzman, D. (2003). *Practice makes practice: A critical study of learning to teach.* Albany, NY: State University of New York Press.

Brown, J. P., & Edwards, I. (2011). Modelling tasks: Insight into mathematical understanding. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 187-197). Dordrecht, SH: Springer.

Burak, D. (2004). Modelagem matemática e a sala de aula. *Anais do Encontro Paranaense de Modelagem em Educação Matemática*, *1*, pp. 1-10. Joinville.

Burkhardt, H. (2006). Modelling in mathematics classrooms: Reflections on past developments and the future. *ZDM , 38* (2), pp. 178-195.

Canadian Institutes of Health Research, Natural Sciences and Engineering Research Council of Canada and Social Sciences and Humanities Research Council of Canada. (2010). Tri-council policy statement: Ethical conduct for research involving humans. Canada.

Carlson, M., Larsen, S., & Lesh, R. (2003). Integrating a models and modeling perspective with existing research and practice. In R. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism - Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 465-478). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Cavey, L. O., & Champion, J. (2016). Learning secondary school mathematics through authentic mathematical modeling tasks. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 131-141). Reston, VA: The National Council of Teachers of Mathematics.

Cirillo, M., Bartell, T. G., & Wager, A. A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 87-96). Reston, VA: The National Council of Teachers of Mathematics.

Cirillo, M., Pelesko, J. A., Felton-Koestler, M. D., & Rubel, L. (2016). Perspectives on modeling in school mathematics. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 3-16). Reston, VA: The National Council of Teachers of Mathematics.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003, January/February). Design experiments in educational research. *Educational Researcher , 32* (1), pp. 9-13.

Cobb, P., Jackson, K., & Dunlap, C. (2015). Design research: An analysis and critique. In L. English, & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 481-503). New York, NY: Routledge.

Consortium for Mathematics and its Applications - COMAP & Society for Industrial snf Applied Mathematics - SIAM. (2016). *Guidelines for assessment & instruction in mathematical modeling education - GAIMME.* COMAP and SIAM.

Corrêa, P. D. (2015). Teaching mathematics for understanding: Approaching and observing. *Delta-k - Journal of the Mathematics Council of the Alberta Teachers' Association , 52* (2), 37-43.

Corrêa, P. D., Sandres, P. R., Nedjah, N., & Mourelle, L. M. (2012). Modeling, development and implementation of a Stewart platform with six degrees of freedom. *Proceedings of the XIX Brazilian Automation Congress - CBA 2012*, (pp. 1069-1076). Campina Grande.

Davis, B., & Simmt, E. (2014). Complexity in mathematics education. In *Encyclopedia of Mathematics Education* (pp. 87-91). Netherlands: Springer.

Davis, B., & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. *Journal for Research in Mathematics Education , 34* (2), pp. 137-167.

Davis, B., & Sumara, D. (2000). Curriculum forms: On the assumed shapes of knowling and knowledge. *Journal of Curriculum Studies , 32* (6), pp. 821-845.

Derderian, T. (n.d.). Retrieved from Greater Boston Track Club: http://www.gbtc.org/2014/01/9656/dsc_0011/

Distance medley relay. (n.d.). Retrieved from Wikipedia: http://en.wikipedia.org/wiki/Distance_medley_relay

Doerr, H. (2016). Designing sequences of model development tasks. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 197-205). Reston, VA: The National Council of Teachers of Mathematics.

Doerr, H. M. (2006). Teachers' ways of listening and responding to students' emerging mathematical models. *ZDM , 38* (3), pp. 255-268.

Doerr, H. M., & Lesh, R. (2011). Models and modelling perspectives on teaching and learning mathematics in the twenty-first century. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 247-268). Dordrecht, SH: Springer.

Doerr, H. (2007). What knowledge do teachers need for teaching mathematics through applications and modelling? In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (Vol. 10, pp. 69-78). New York, NY: Springer.

Doll, W. (2008). Complexity and the culture of curriculum. *Educational Philosophy and Theory , 40* (1), pp. 190-212.

Doll, W. (1989). Complexity in the classroom. *Educational Leadership , 7* (1), pp. 65-70.

Doll, W., & Trueit, D. (2012). Pragmatism, postmodernism, and complexity theory: The 'fascinating imaginative realm' of William E. Doll, Jr. New York, NY: Routledge.

Dym, C. L. (2004). *Principles of mathematical modeling.* Amsterdam: Academic Press.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., & Agard, D. J. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach. *Journal for Research in Mathematics Education , 24*, pp. 8-40.

Elasticity of demand. (n.d.). Retrieved from Wrights Land of Economics: https://wrightslandofeconomics.wikispaces.com/Elasticity+of+Demand+(Tax+Incidence)

English, L. D. (2013). Complex modelling in the primary and middle school years: An interdisciplinary approach. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 491-505). Dordrecht, SH: Springer.

English, L., & Doerr, H. (2004). Learning through interacting with students' ways of thinking. *Mathematics Education For the Third Millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia. 1*, pp. 215-222. MERGA.

Felton, M. D., Anhalt, C. O., & Cortez, R. (2015). Going with the flow: Challenging students to make assumptions. *Mathematics Teaching in the Middle School , 20* (6), 342-349.

Foerster, H. v. (2003). Perception of the future and the future of perception. In H. v. Foerster, *Understanding understanding: Essays on cybernetics and cognition* (pp. 199-210). New York, NY: Springer-Verlag.

Frejd, P. (2012). Teachers' conceptions of mathematical modelling at Swedish upper secondary school. *Journal of Mathematical Modelling and Application , 1* (5), pp. 17-40.

Galbraith, P. (2011). Models of modelling: Genres, purposes or perspectives. *Journal of Mathematical Modelling and Application , 1*, pp. 3-16.

Gann, C., Avineri, T., Graves, J., Hernandez, M., & Teague, D. (2016). Moving students from remembering to thinking: The power of mathematical modeling. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 97-106). Reston, VA: The National Council of Teachers of Mathematics.

Gould, H. (2013). *Teachers' conceptions of mathematical modeling .* Doctoral Dissertation: Columbia University.

Groshong, K. (2016). Different types of mathematical models. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 17-24). Reston, VA: The National Council of Teachers of Mathematics.

Grünewald, S. (2013). The development of modelling competencies by year 9 students: Effects of a modelling project. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 185-194). Dordrecht, SH: Springer.

Heinz, S. (2011). *Mathematical modeling.* Springer Science & Business Media.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and students cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education , 28* (5), pp. 524-549.

Hernández, M. L., Levy, R., Felton-Koestler, M. D., & Zbiek, R. M. (2016). Mathematical modeling in the high school curriculum. *Mathematics Teacher , 110* (5), 336-342.

Hirsch, C. R., & McDuffie, A. R. (Eds.). (2016). *Annual perspectives in mathematics education: Mathematical modeling and modeling mathematics.* Reston, VA: The National Council of Teachers of Mathematics.

Hojgaard, T. (2013). Communication: The essential difference between mathematical modeling and problem solving. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford, *Modeling students' mathematical modeling competencies* (pp. 255-264). Dordrecht, SH: Springer.

Hung, W. (2013). Problem-based learning: A learning environment for enhancing learning transfer. In *New directions for adult and continuing education* (pp. 27-38).

Hung, W., Jonassem, D. D., & Liu, R. (2008). Problem-based learning. In *Handbook of research on educational communications and technology* (Vol. 3, pp. 485-506).

Ikeda, T., & Stephens, M. (2010). Three teaching principles for fostering students' thinking about modelling: An experimental teaching program for 9th grade students in Japan. *Journal of Mathematics Modelling and Application , 1* (2), pp. 49-59.

Improving binary search. (2014). Retrieved from Code: http://code.scottshipp.com/2014/02/17/the-problem-with-binary-search/

Jickling, B. (2013). Normalizing catastrophe: An educational response. *Environmental Education Research , 19* (2), pp. 161-176.

Kaiser, G., & Stender, P. (2013). Complex modelling problems in co-operative, self-directed learning environments. In G. A. Stillman, G. Kaiser, W. Blum, & J. P. Brown (Eds.), *Teaching mathematical modelling: Connecting to research and practice* (pp. 277-293). Dordrecht, SH: Springer.

Kaiser, G., Blum, W., Ferri, R. B., & Stillman, G. (Eds.). (2011). *Trends in teaching and learning of mathematical modelling.* Dordrecht, SH: Springer.

Kawasaki, T., Moriya, S., Okabe, Y., & Maesako, T. (2012). The problems of mathematical modelling introduction on mathematics education in Japanese school. *Journal of Mathematical Modelling Application , 1* (5), pp. 50-58.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. In J. Kilpatrick, J. Swafford, & B. Findell, *Adding it up [eletronic resource]: Helping children learn mathematics* (pp. 115-155). Washington, DC: National Academy Press.

Kwon, O. N. (2002). Conceptualizing the realistic mathematics education approach in the teaching and learning of ordinary differential equations. *Proceedings of the 2nd International Conference on the Teaching of Mathematics (at the Undergraduate Level).* Greece.

Lamon, S. J. (2003). Beyond constructivism: An improved fitness metaphor for the acquisition of mathematical knowledge. In R. Lesh, & H. M. Doerr (Eds.), *Beyond constructivism - Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 435-447). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Lesh, R. (2007). What changes are occurring in the kind of elementary-but-powerful mathematics concepts that provide new foundations for the future? In R. Lesh, E.

Hamilton, & J. Kaput, *Foundations for the future in mathematics education* (pp. 155-159). Mahwah, NJ: Lawrence Erlbaum S.

Lesh, R., & Doerr, H. M. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh, & H. M. Doerr (Eds.), *Beyond Constructivism - Models and modelling perspectives on mathematics problem solving, learning, and teaching* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Lesh, R., & Doerr, H. (2000). Symbolizing, communicating, and mathematizing: Key components of models and modeling. In P. Cobb, E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating in mathematics classes - Perspectives on discourse, tools, and instructional design* (pp. 361-383). Mahwah, NJ: Lawrence Erlbaum Associates, Publishers.

Lesh, R., & Fennewald, T. (2013). Introduction to part I: What is it? Why do it? In R. Lesh, P. Galbraith, C. Haines, & A. Hurford, *Modeling students' mathematical modeling competencies* (pp. 5-10). Dordrecht, SH: Springer.

Lesh, R., Galbraith, P. L., Haines, C. R., & Hurford, A. (Eds.). (2013). *Modeling students' mathematical modeling competencies.* Dordrecht, SH: Springer.

Lesh, R., Yoon, C., & Zawojewski, J. (2007). John Dewey revisited - Making mathematics practical VERSUS making practice mathematical. In R. Lesh, E. Hamilton, & J. J. Kaput (Eds.), *Foundations for the future in mathematics education* (pp. 315-348). Mahwah, NJ: Lawrence Erlbaum Associates.

Lesh, R., Young, R., & Fennewald, T. (2013). Modeling in K-16 mathematics classrooms – and beyond. In R. Lesh, P. L. Galbraith, C. R. Haines, & A. Hurford (Eds.), *Modeling students' mathematical modeling competencies* (pp. 275-283). Dordrecht, SH: Springer.

Magnussen, L., Ishida, D., & Itano, J. (2000). The impact of the use of inquiry-based learning as a teaching methodology on the development of critical thinking. *Journal of Nursing Education , 39* (8), pp. 360-364.

Meyer, D. (2015). Missing the promisse of mathematical modeling. *Mathematics Teacher , 108* (8), 578-583.

Meyer, J. F., Caldeira, A., & Malheiros, A. P. (2011). *Modelagem em educação matemática.* Belo Horizonte, MG: Autêntica.

Moore, T. J., Doerr, H. M., Glancy, A. W., & Ntow, F. D. (2015). Preserving pelicans with models that make sense. *Mathematics Teaching in the Middle School , 20* (6), 358-364.

National Governors Association; Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Washington, DC: Council of Chief State School Officers and the National Governors Association.

Palharini, B., & Almeida, L. M. (2015). Mathematical modelling tasks and the mathematical thinking of students. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 219-228). Cham, ZG: Springer.

Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics , 26* (2/3), pp. 165-190.

Pollak, H. (2012). Introduction: What is mathematical modeling? In H. Gould, D. R. Murray, & A. Sanfratello (Eds.), *Mathematical modeling handbook* (pp. viii-xi). Bedford, MA: Consortium for Mathematics ans its Applications.

Pollak, H. (2015). The place of mathematical modelling in the system of mathematics education: Perspective and prospect. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 265-276). Cham, ZG: Springer.

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. (Springer, Ed.) *ZDM , 47*, 877-891.

Press, A. (2014). *Avoiding plane crashes as air traffic doubles*. Retrieved from Fox News Travel: http://www.foxnews.com/travel/2014/07/29/avoiding-plane-crashes-as-air-traffic-doubles/

Purpel, D. (2004). The moral and spiritual crisis in American education. In D. Purpel, *Reflections on the moral & spiritual crisis in education* (pp. 42-74). New York, NY: Peter Lang.

Ricks, T. E. (2009). Unifying current trends in mathematics education research using complexity theory. *Southeatern Teacher Education Journal , 2* (2), pp. 63-74.

Savery, J. R. (2015). Overview of problem-based learning: Definitions and distinctions. In A. Walker, H. Leary, C. E. Hmelo-Silver, & P. A. Ertmer, *Essential readings in problem-*

based learning: Exploring and extending the legacy of Howard S. Barrows (pp. 5-15). West Lafayette, Indiana: Purdue University Press.

Schmidt, B. (2011). Modelling in the classroom: Obstacles from the teacher's perspective. In G. Kaiser, W. Blum, R. B. Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modeling* (pp. 641-651). Dordrecht, SH: Springer.

Scott, A. (2007). Seeking evidence of thinking and mathematical understandings in students' writing. In J. Watson, & K. Beswick (Ed.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the mathematics education research group of Australasia. 2*, pp. 641-650. Adelaide: MERGA.

Seidel, J. (2006). Some thoughts on teaching as contemplative practice. *Teachers College Record , 108* (9), pp. 1901-1914.

Sierpinska, A. (1994). *Understanding in mathematics.* London, UK: Press, The Falmer.

Siller, H.-S., & Kuntze, S. (2011). Modelling as a big idea in mathematics - Knowledge and views of pre-service and in-service teachers. *Journal of Mathematical Modelling and Application , 1* (6), pp. 33-39.

Silver, E., Mesa, V. M., Morris, K. A., Star, J. R., & Benken, B. M. (2009). Teaching mathematics for understanding: An analsys of lessons submitted by teachers seeking NBPTS certification. *American Educational Research Journal , 46* (2), pp. 501-531.

Skemp, R. R. (1976/2006). Relational understanding and instrumental understanding. *Mathematics Teaching in the Middle School , 2* (88).

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal , 33* (2), pp. 455-488.

Stender, P., & Kaiser, G. (2016). Fostering modeling competencies for complex situations. In C. R. Hirsch, & A. R. McDuffie (Eds.), *Mathematical modeling and modeling mathematics* (pp. 107-115). Reston, VA: The National Council of Teachers of Mathematics.

Stillman, G. A., Blum, W., & Biembengut, M. S. (Eds.). (2015). *Mathematical modelling in education research and practice: Cultural, social and cognitive influences.* Cham, ZG: Springer.

Stillman, G. A., Kaiser, G., Blum, W., & Brown, J. P. (Eds.). (2013). *Teaching mathematical modelling: Connecting to research and practice.* Dordrecht, SH: Springer.

Stillman, G. (2001). The impact of school-based assessment on the implementation of a modelling/applications-based curriculum: An Australian example. *Teaching Mathematics and its Applications , 20* (3), pp. 101-107.

Stillman, G., Galbraith, P., Brown, J., & Edwards, I. (2007). A framework for success in implementing mathematical modelling in the secondary classroom. In J. Watson, & K. Beswick (Ed.), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the mathematics education research group of Australasia. 2*, pp. 688-697. Adelaide: MERGA.

Tall, D. (2013). How humans learn to think mathematically: Exploring the three worlds of mathematics. Cambridge, UK: Cambridge University Press.

Teachers College Columbia University. (2012). *Mathematical modeling handbook.* (H. Gould, D. R. Murray, & A. Sanfratello, Eds.) Bedford, MA: COMAP - The Consortium for Mathematics and its Applications.

The Conference Board of Canada. (n.d.). *Employability skills 2000+*. Retrieved December 8, 2016, from The Conference Board of Canada: http://www.conferenceboard.ca/topics/education/learning-tools/employability-skills.aspx

The Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher , 32* (1), pp. 5-8.

Tuck, J. (n.d.). *The relationship between price elasticity and total revenue*. Retrieved from Small Business Chron: http://smallbusiness.chron.com/relationship-between-price-elasticity-total-revenue-24544.html

Van den Heuvel-Panhuizen, M., & Drijvers, P. (2014). Realistic mathematics education. In *Encyclopedia of Mathematics Education* (pp. 521-525). Dordrecht, SH: Springer.

Viecili, C. R. (2006). *Modelagem matemática: Uma proposta para o ensino de matemática (Master's Thesis).* Retrieved from: http://hdl.handle.net/10923/3029.

Vorhölter, K., Kaiser, G., & Ferri, R. B. (2014). Modelling in mathematics classroom instruction: An innovative approach for transforming mathematics education. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 21-36). Cham, ZG: Springer.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Zawojewski, J. (2013). Problem solving versus modeling. In R. Lesh, P. Galbraith, C. Haines, & A. Hurford, *Modeling students' mathematical modeling competencies: ICTMA 13* (pp. 237-243). Dordrecht, SH: Springer.

Zbiek, R. M., & Conner, A. (2006). Beyond motivation: Exploring mathematical modeling as a context for deepening students' understandings of curricular mathematics. *Educational Studies in Mathematics , 63*, pp. 89–112.

Zorzan, A. S. (2007). Teaching-learning: Some trends in mathematical education. *Revista de Ciências Humanas , 8* (10), 77-93.

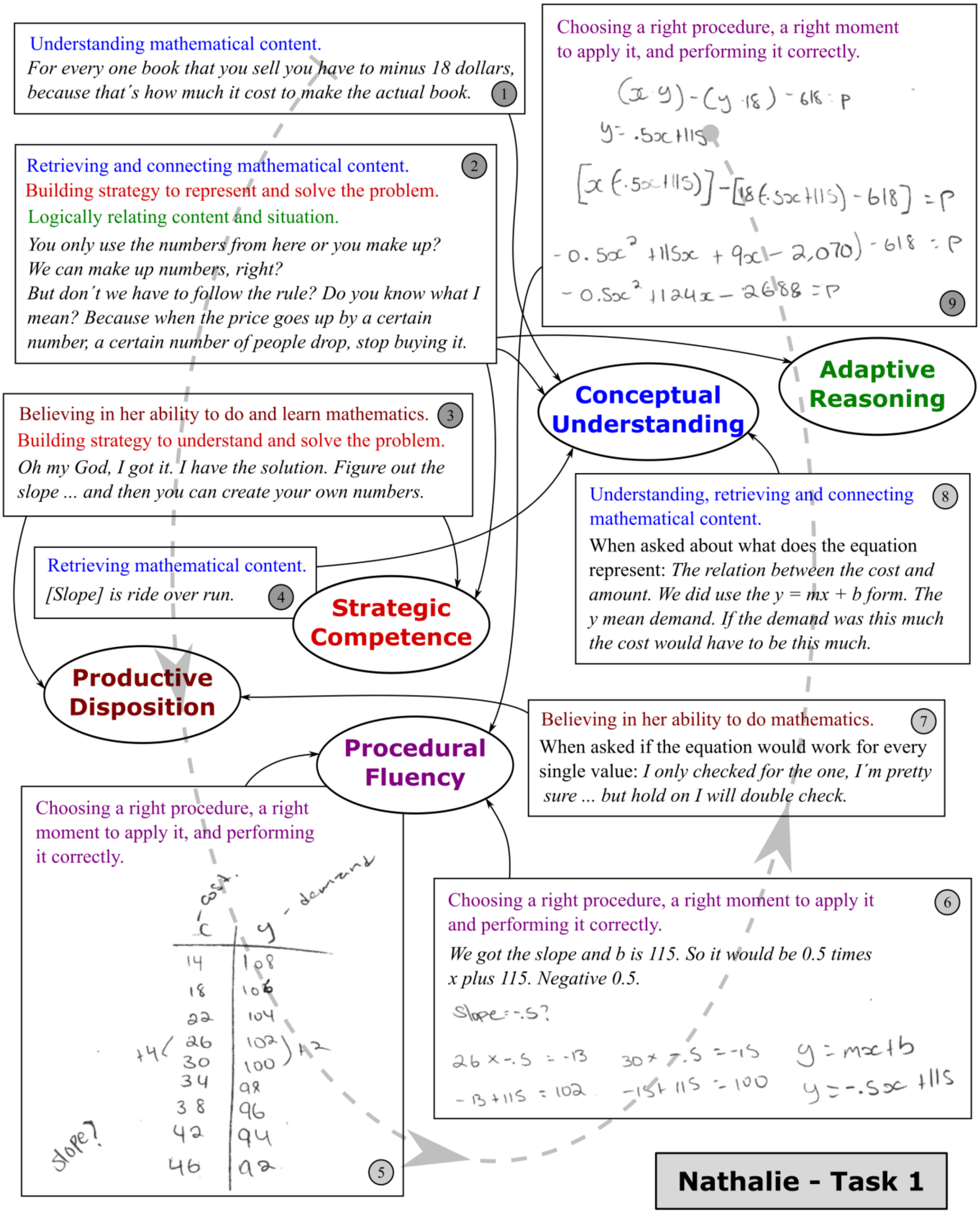
Appendix A: Nathalie's diagram for task one – Cost, revenue and profit

The first phase of Nathalie's work is characterized by her attempt to build a strategy to make up numbers based on the given table in the task. In fragment one, she shows her understanding about the necessity of subtracting the book cost when calculating the profit. Then, in fragment two, when mentioning a rule to be followed, she is retrieving her knowledge about functions and connecting to the task. She conjectures about the rule, logically relating the content (the rule) and the situation (prices going up and buyers dropping down). Other than that, she also starts working on her strategy to understand and solve the problem, given that she wants to make up her own numbers and finds out the rule. In fragment three, she shows her excitement when she realizes the strategy she needs to follow: finding the slope and then creating her own numbers. Apart from highlighting that she figures out the strategy, this fragment also speaks to her belief in her ability to do and learn mathematics. This can be seen when Nathalie says "Oh my God, I got it. I have the solution." Closing this initial phase of her work is fragment four, in which she states that slope is ride over run. This is a sample of her retrieving mathematical content. Hence, this phase of Nathalie's work is portrayed by a blend of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *productive disposition*.

Following this first phase, Nathalie starts to concretely work on her strategy. Therefore, she chooses a right procedure to find out the slope, at a right moment and correctly performs it. This can be seen in fragment five when she observes that for a increment of 4 dollars in the cost, there is a decrement of 2 people in the demand. Then, in fragment six, she chooses another procedure to perform and find out the linear equation that rules the values in the given table ($y = -0.5x + 115$). As this was a collective work, Nathalie might have had access to the thoughts and notes of other group members, and might have had help to figure out this linear equation. Perhaps that is why her notes are not so clear about how she gets the slope ($m = -0.5$) and the equation linear coefficient ($b = 115$). Nathalie reasoning processes are very much based on trying out numbers. In fragment seven, when asked if the equation would work for every single value, Nathalie answered that she only checked for one of them, but that she would double check. Anyway, she says "I'm pretty sure", emphasizing her belief in her ability to do mathematics. Still in this middle phase, she was asked about the meaning of the found equation. As fragment eight shows, she is able to explain that the equation refers to the linear relation between cost and demand in the $y = mx+b$ form, which means she retrieved previous mathematical content and connected to the task. Her explanation speaks to her mathematical understanding about the content as well. Therefore, this phase of Nathalie's work presents a mix of *conceptual understanding*, procedural *fluency* and *productive disposition*.

Finally, by the end of her reasoning processes, as fragment nine illustrates, she plugs the linear equation ($y = -0.5x + 115$) she found before in the profit equation ($P = xy - 18y - 618$). The profit equation may have been found with group's help. Plugging in the linear equation is a right procedure to be applied, and at a right moment to find out the profit equation. She correctly performs it and gets $P = -0.5x^2 + 124x - 2688$. Her notes end up at this point and she does not complete the square to get the task final answer. However, she does get to the desired model. Nathalie's final phase is characterized basically by *procedural fluency*. It is worth highlighting that, even though Nathalie did not finish the task and apparently had help from peers, her work still presents evidences of all five strands of mathematical proficiency.

APPENDIX A: NATHALIE'S DIAGRAM FOR TASK ONE – COST, REVENUE AND PROFIT



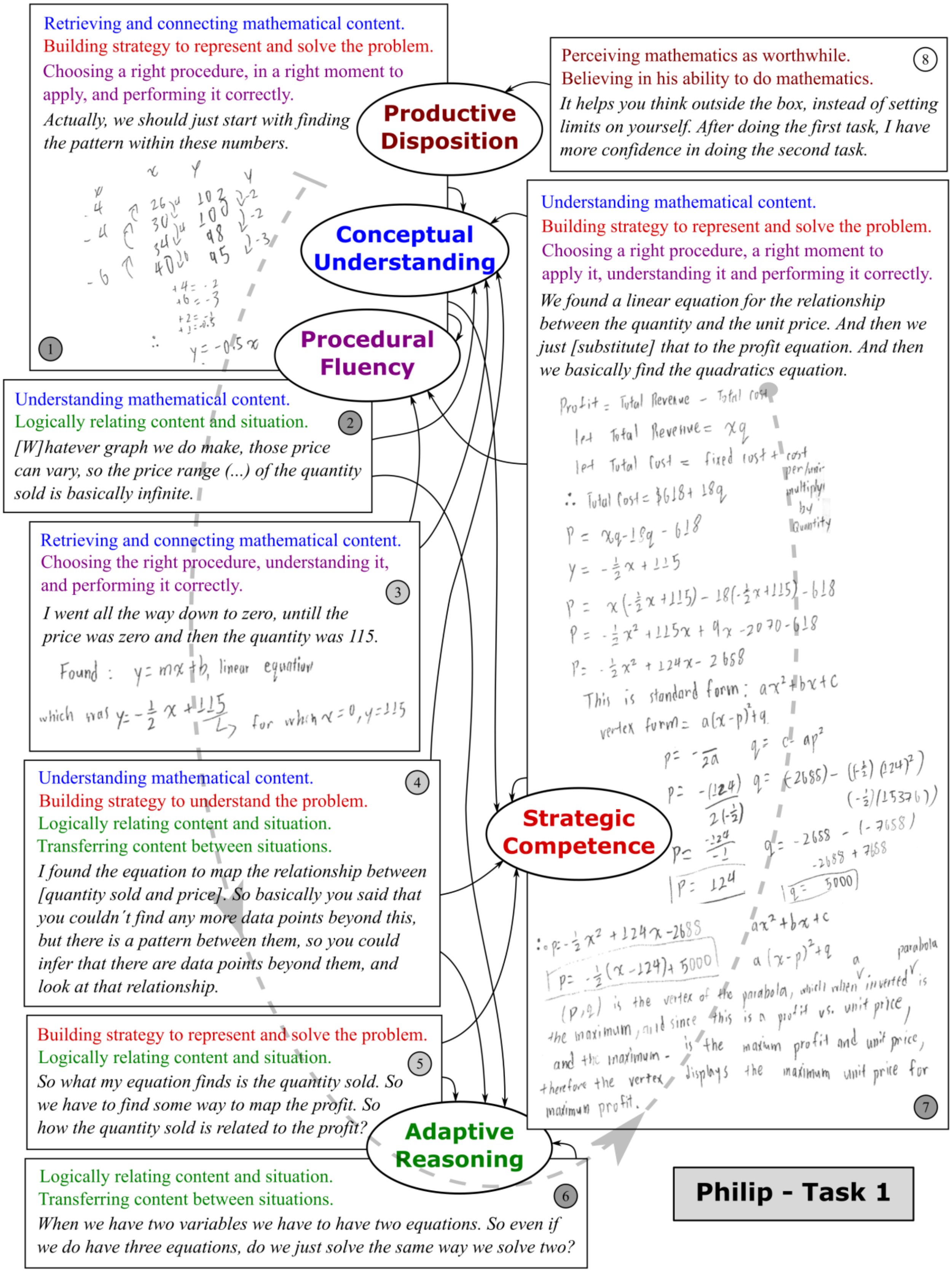
APPENDIX B: PHILIP'S DIAGRAM FOR TASK ONE – COST, REVENUE AND PROFIT

Philip's work has one peculiar characteristic. Almost all of his fragments present conceptual understanding, attesting to his good mathematical underpinnings. In fragment one, Philip starts off by analyzing the given data in the table and verbalizes he intends to find a pattern within those numbers. He retrieves his knowledge about patterns and connects to the problem. Finding a pattern is the strategy he uses to represent and solve the problem. Indeed, he follows this plan and chooses a right procedure, at a right moment, and performs it correctly to find out the slope (–0.5). Then, in fragment two, Philip logically relates the situation under analysis (price range and quantity variation) with linear graphing (content), mentioning that the "price range of the quantity sold is basically infinite". This correlation speaks to his understanding about the range of linear functions and about the notion of infinite. This rich initial phase of Philip's work can be described as a blend of *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*.

In the next phase, Philip retrieves his prior knowledge about linear functions and connects it with his reasoning as can be seen in fragment three. He explains a right chosen procedure to find out the linear coefficient (115), confirming his understanding about the procedure. On his notes, he writes down the correct linear equation found ($y = -\frac{1}{2}x + 115$), in which *y* is quantity sold and *x* is the book price. Still working on this matter, in fragment four, Philip explains the logical correlation between content (the found equation) and situation (relationship between quantity sold and price). He goes further transferring content between situations. This can be seen when he explains that the pattern found can be used to rule the relationship between the given numbers in the table and also to rule the relationship between numbers beyond the ones in the table. This reasoning speaks to his understanding about linear functions and patterns, as well as to a strategy to understand what is going on in the problem. Once Philip has this first equation, in fragment five, he expresses his need to map profit. He explains that as his equation finds quantity sold, he wonders how to connect quantity sold and profit. At this point, he shows his attempt to logically relate content (the quantity sold equation) and situation (the profit). At the same time, this is a strategy to represent and solve the problem, given that he is trying to use what he has in hands to get to the task final answer. As so, this middle phase of Philip's work is characterized by a mix of *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*, just as the initial phase.

In the final phase of Philip's task solution, he starts off by questioning how to solve a system of three equations. Fragment six shows his questionings. Philip logically relates the content he already knows (the solving process of systems of two equations) with the situation he is analyzing, in which he has three equations. He not only relates content and situation, he also wonders if he can transfer this previous content knowledge (the solving process of systems of two equations) between the two situations (having two and three equations). In the end, as fragment seven illustrates, Philip chooses a right procedure to be applied, at a right moment and performs it correctly. He explains the chosen procedure, saying it was just a matter of plugging the linear equation into the profit equation to come up with the quadratic equation, which attests for his understand of the procedure and also for his strategy to solve and represent the problem. As fragment seven shows, the profit equation is $P = xq - 18q - 618$, in which P is profit, x is the book price and q is quantity sold (referred to as y in the linear equation). Philip then substitutes q per y $\left(y = -\frac{1}{2}x + 115 \right)$ and gets to the profit equation in the standard form $P = -\frac{1}{2}x^2 + 124x - 2688$. His next step is to get the profit equation into the vertex form $\left(P = -\frac{1}{2}(x - 124)^2 + 5000 \right)$ and he uses the formula he got in his textbook to find the vertex coordinates. After getting the task solution, Philip gives an explanation relating an inverted parabola and a maximum profit equation, showing his understanding about the involved mathematical content. Once more, four of the five strands of mathematical proficiency can be found in the final phase of Philip's work, namely *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*.

As for *productive disposition*, classroom observations confirm that Philip was engaged in the task for the whole period and was committed to get to an answer. During the recall interview, as fragment eight confirms, he says this kind of task "helps you think outside the box, instead of setting limits on yourself". This statement speaks to Philip's perception that mathematics is worthwhile. He also says that after the first task he was more confident to do the second task, which is an evidence of his belief in his ability to do mathematics.

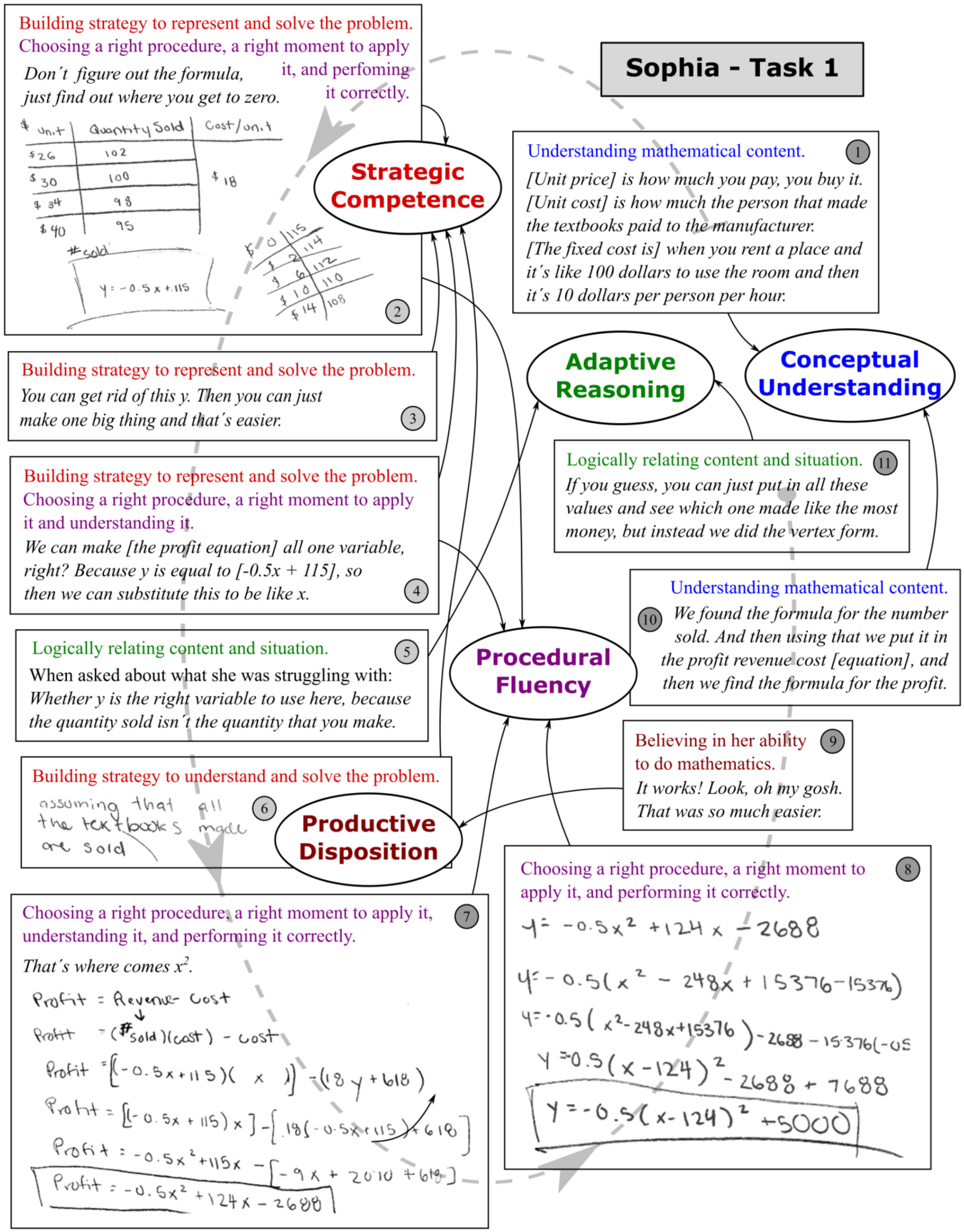

Retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Choosing a right procedure, in a right moment to apply, and performing it correctly.
Actually, we should just start with finding the pattern within these numbers.
1
Productive Disposition
Perceiving mathematics as worthwhile.
Believing in his ability to do mathematics.
It helps you think outside the box, instead of setting limits on yourself. After doing the first task, I have more confidence in doing the second task.
8
Conceptual Understanding
Understanding mathematical content.
Building strategy to represent and solve the problem.
Choosing a right procedure, a right moment to apply it, understanding it and performing it correctly.
We found a linear equation for the relationship between the quantity and the unit price. And then we just [substitute] that to the profit equation. And then we basically find the quadratics equation.
Profit = Total Revenue – Total Cost
let Total Revenue = xq
let Total Cost = fixed cost + cost per unit multiply by Quantity
∴ Total Cost = $618 + 18q
P = xq – 18q – 618
y = –½x + 115
P = x(–½x + 115) – 18(–½x + 115) – 618
P = –½x² + 115x + 9x – 2070 – 618
P = –½x² + 124x – 2688
This is standard form: ax² + bx + c
vertex form = a(x – p)² + q
7
Procedural Fluency
Understanding mathematical content.
Logically relating content and situation.
[W]hatever graph we do make, those price can vary, so the price range (...) of the quantity sold is basically infinite.
2
Retrieving and connecting mathematical content.
Choosing the right procedure, understanding it, and performing it correctly.
I went all the way down to zero, untill the price was zero and then the quantity was 115.
Found : y = mx + b, linear equation
which was y = –½x + 115 for when x = 0, y = 115
3
Understanding mathematical content.
Building strategy to understand the problem.
Logically relating content and situation.
Transferring content between situations.
I found the equation to map the relationship between [quantity sold and price]. So basically you said that you couldn´t find any more data points beyond this, but there is a pattern between them, so you could infer that there are data points beyond them, and look at that relationship.
4
Strategic Competence
Building strategy to represent and solve the problem.
Logically relating content and situation.
So what my equation finds is the quantity sold. So we have to find some way to map the profit. So how the quantity sold is related to the profit?
5
Adaptive Reasoning
Logically relating content and situation.
Transferring content between situations.
When we have two variables we have to have two equations. So even if we do have three equations, do we just solve the same way we solve two?
6
Philip - Task 1

Appendix C: Sophia's diagram for task one – Cost, revenue and profit

Sophia's work starts with some explanations about what unit price, unit cost and fixed cost are, as fragment one shows. These explanations show her understanding about the mathematical content. In the classroom initial discussion, Sophia and other group participants observe that if the book price decreases, more people buy it. On the other hand, if the book price increases, less people buy it. This observation characterizes that at this point group members, including Sophia, retrieved and connected with their previous knowledge about relations and patterns. Therefore, this initial phase of Sophia's work was basically supported by *conceptual understanding*.

Then, in the middle phase of her work, Sophia builds on her strategy to represent and solve the problem. Her first step is finding out a relation between the given table values. As fragment two shows, she has a particular strategy to do that: observing the table pattern and finding out where the table values get to zero. Accordingly, she chooses a right procedure, at a right moment and applies it correctly, finding the equation for quantity sold ($y = -0.5x + 115$). Then in fragment three, still working on her strategy, Sophia wants to eliminate the *y* variable. She further develops this strategy in fragment four, when choosing a right procedure to eliminate *y* and applying it at a right moment. Sophia shows understanding about the procedure by explaining that the profit equation can be written with one single variable (*x*) and that *y* can be substituted by $-0.5x + 115$. Following, in fragment five, Sophia presents an interesting reasoning. She relates the variable *y* in the found equation (content) with two different possible situations: *y* could be quantity sold or *y* could be quantity made. Hence, she struggles to define whether *y* should be quantity sold or quantity made, or if she needs two variables to represent each of these quantities. After being prompted to make her own hypothesis, Sophia assumes that all produced textbooks are sold, as fragment six confirms. This assumption is her strategy to make sense of the task and solve it. Therefore, the middle phase of Sophia's work is characterized by *strategic competence*, *procedural fluency* and *adaptive reasoning*.

Fragments seven and eight are part of the final phase of Sophia's work and they show how Sophia moved from the profit basic equation (*Profit* = *Revenue* – *Cost*) to the profit standard form quadratic equation ($\text{Profit} = -0.5x^2 + 124x - 2688$), and then to the profit vertex form quadratic equation ($y = -0.5(x - 124)^2 + 5000$). She chooses right procedures and correctly performs them at proper moments. In fragment seven, when Sophia indicates how she gets to *x* squared, she is also demonstrating understanding about the procedure she is performing. It is worth noticing that, as the rest of the students, Sophia had some teacher prompts before getting to the vertex form of the profit equation. Fragment nine illustrates her contentment when getting to her final equation, which addresses her belief in her capacity to do mathematics. Then in fragment ten, Sophia expresses her understanding about how she gets to the profit equation, in other words, her understanding about the mathematical content. Finally, the last fragment, illustrates Sophia logically relating the situation under analysis (finding the maximum profit) and different ways of getting to an answer, by guess and check or by finding the vertex form (mathematical content). This final phase of Sophia's work merges *conceptual understanding*, *procedural fluency*, *adaptive reasoning* and *productive disposition*. Sophia did not participate in this task interview.


Sophia - Task 1
Strategic Competence
Adaptive Reasoning
Conceptual Understanding
Procedural Fluency
Productive Disposition
Building strategy to represent and solve the problem.
Choosing a right procedure, a right moment to apply it, and perfoming it correctly.
Don´t figure out the formula, just find out where you get to zero.
$ Unit | Quantity Sold | Cost/unit
$26 | 102
$30 | 100 | $18
$34 | 98
$40 | 95
#sold
y = -0.5x + 115
$ 0 | 115
$ 2 | 114
$ 6 | 112
$ 10 | 110
$ 14 | 108
2
Understanding mathematical content.
1
[Unit price] is how much you pay, you buy it. [Unit cost] is how much the person that made the textbooks paid to the manufacturer. [The fixed cost is] when you rent a place and it´s like 100 dollars to use the room and then it´s 10 dollars per person per hour.
Building strategy to represent and solve the problem.
You can get rid of this y. Then you can just make one big thing and that´s easier.
3
Building strategy to represent and solve the problem.
Choosing a right procedure, a right moment to apply it and understanding it.
We can make [the profit equation] all one variable, right? Because y is equal to [-0.5x + 115], so then we can substitute this to be like x.
4
Logically relating content and situation.
11
If you guess, you can just put in all these values and see which one made like the most money, but instead we did the vertex form.
Logically relating content and situation.
5
When asked about what she was struggling with:
Whether y is the right variable to use here, because the quantity sold isn´t the quantity that you make.
Understanding mathematical content.
10
We found the formula for the number sold. And then using that we put it in the profit revenue cost [equation], and then we find the formula for the profit.
Building strategy to understand and solve the problem.
assuming that all the textbooks made are sold
6
Believing in her ability to do mathematics.
9
It works! Look, oh my gosh. That was so much easier.
Choosing a right procedure, a right moment to apply it, understanding it, and performing it correctly.
7
That´s where comes x².
Profit = Revenue - Cost
Profit = (#sold)(cost) - cost
Profit = [(-0.5x + 115)(x)] - (18y + 618)
Profit = [(-0.5x + 115)x] - [18(-0.5x + 115) + 618]
Profit = -0.5x² + 115x - [-9x + 2070 + 618]
Profit = -0.5x² + 124x - 2688
Choosing a right procedure, a right moment to apply it, and performing it correctly.
8
y = -0.5x² + 124x - 2688
y = -0.5(x² - 248x + 15376 - 15376)
y = -0.5(x² - 248x + 15376) - 2688 - 15376(-0.5)
y = -0.5(x - 124)² - 2688 + 7688
y = -0.5(x - 124)² + 5000

APPENDIX D: CLARA'S DIAGRAM FOR TASK TWO – FLIGHT SIMULATORS

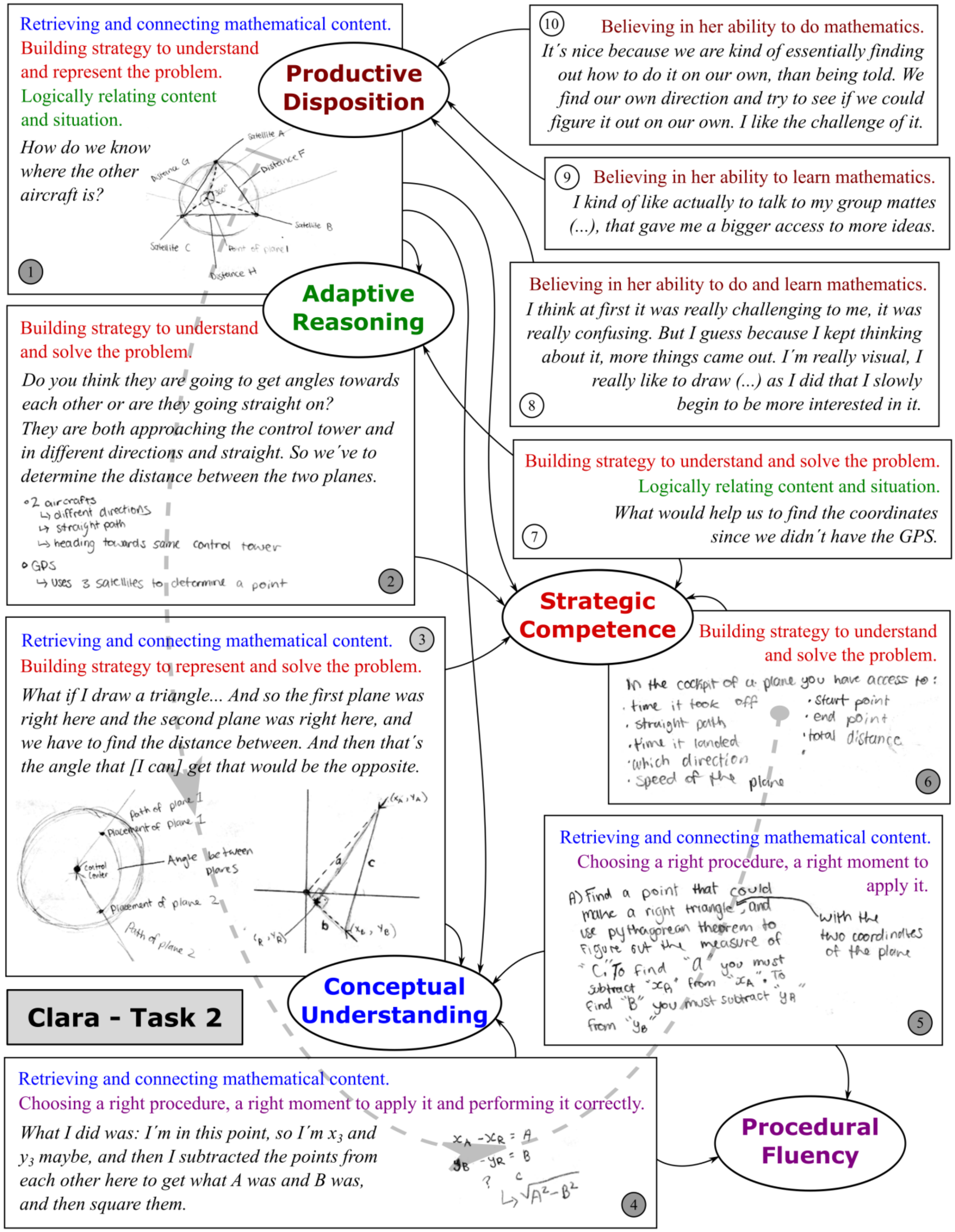
Clara's starts off her investigation by retrieving her knowledge about a GPS operation and connecting it to the task. She logically relates the task situation (two airplanes flying towards a control tower) with the way a GPS runs (content) as a strategy to understand and represent the problem. This first step of her analysis is illustrated in fragment one, in which she presents a visual representation of the three GPS satellite positions when locating air plane 1. She also wonders how they would know where air plane 2 is located. In fragment two, Clara speculates if the path should be straight or not. When she realizes this information is actually given by the task, Clara highlights the conditions to be considered before analyzing the situation, which are: straight paths, different directions and movement towards the same control tower. All these specifications are part of defining a strategy to understand and solve the problem. As so, the first phase of Clara's work is a blend of *conceptual understanding*, *strategic competence* and *adaptive reasoning*.

In the next phase of Clara's work, as fragment three shows, she retrieves her knowledge about triangles and connects it to the task. The triangle serves the purpose of improving her initial representation of the task situation, given that Clara puts each air plane in a triangle vertex. In this sense, she also enhances her strategy to solve the problem. Now she can use her knowledge about triangles to find the distance between the air planes, that is, the distance between two triangle vertices. Therefore, the middle phase of Clara's work is a mix of *conceptual understanding* and *strategic competence*.

The final phase of Clara's work was focused on answering the first item of the task and conjecturing about the second item, once she does not really get into the second item. As fragment four shows, Clara retrieves her knowledge about Cartesian plane and connects with the task. She chooses a right procedure (subtracting coordinates) to be applied at a right moment and correctly performs it. Note that Clara's reasoning in fragment four is based on the same Cartesian plane presented on fragment three. Following, in fragment five, Clara formalizes her thoughts and solution for the first part of the task. She again retrieves her knowledge about the Pythagoras theorem and connects to the task. Then, she chooses a right moment to apply a right procedure (the Pythagoras theorem). Although, Clara does not finish the task, as fragment six illustrates, her second task item investigation starts by analyzing what kind of information she could get from an air plane cockpit. Indeed, this is a good start, given that in this second situation the GPS is supposedly out of order. By doing that, Clara is working on her strategy to understand and solve the problem. This way of thinking is confirmed in the interview fragment number seven, in which Clara states they were looking for some information that would be useful to find out the coordinates they would not get from the GPS. She logically relates the situation she has in hands (GPS down) and the air plane coordinates (content). Hence, the final phase of Clara's work is a combination of *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*.

Clara's *productive disposition* could be seen during classroom observations, given that she was engaged and willing to get to an answer, even when frustrated. Other than that, fragments eight, nine and ten are relevant interview pieces in this matter. In fragment eight, Clara explains that although confused, she kept trying and new thoughts emerged. She also said that visually representing her ideas helped her to get more interested in the task. Her persistence speaks to her belief in her ability to do and learn mathematics, that is, she persevered because she believed she was able to get to an end. In fragment nine, Clara states that sharing thoughts with her peers was helpful because she got access to more ideas. This fact accounts to her belief in her aptitude to learn mathematics from peers' ideas. Finally, in fragment ten, she states she likes being challenged and allowed to think and figure out solutions by herself. Again this is an evidence of her confidence in her capacity to do mathematics on her own.

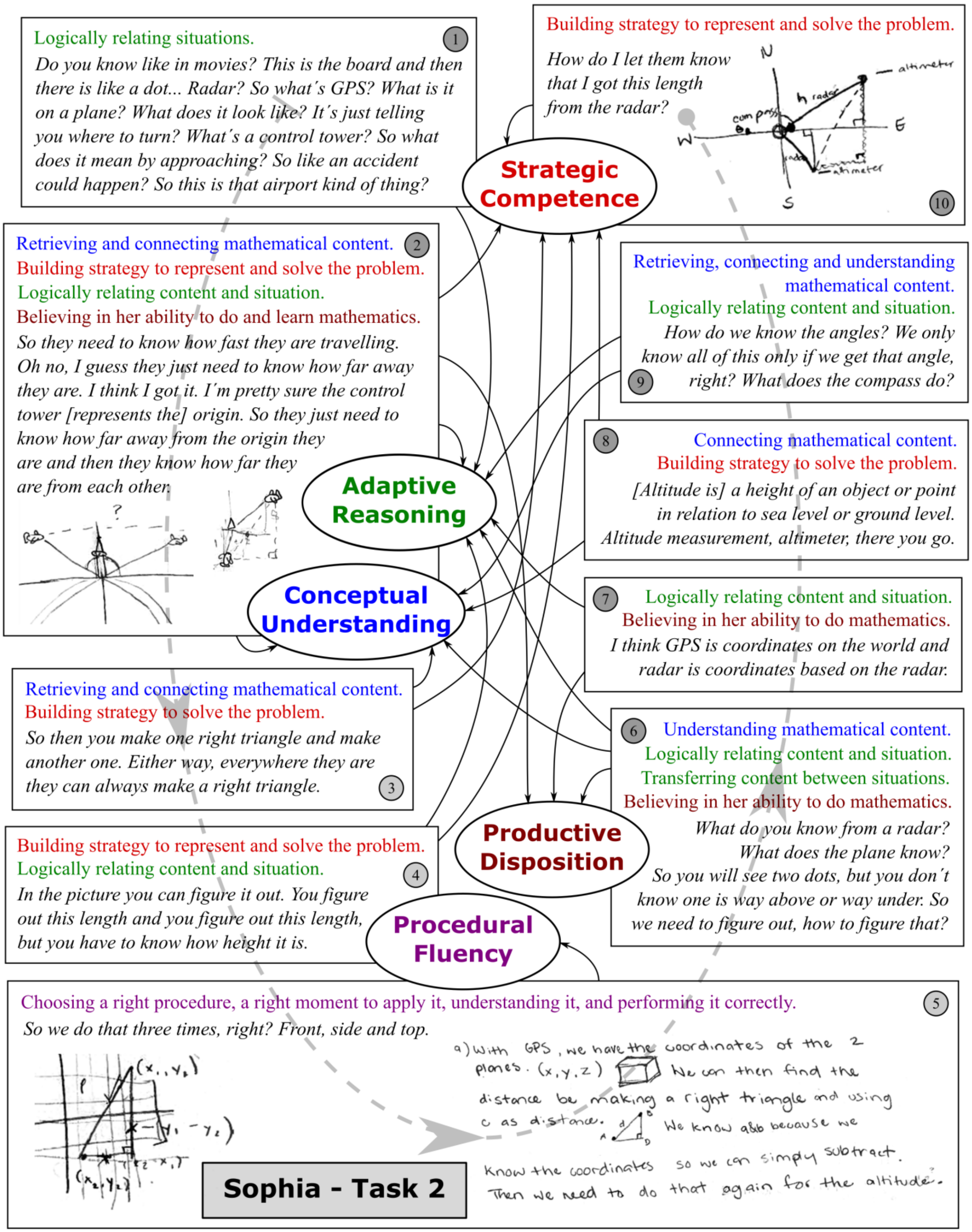
APPENDIX D: CLARA'S DIAGRAM FOR TASK TWO – FLIGHT SIMULATORS



Appendix E: Sophia's diagram for task two – Flight simulators

Sophia's work during task two has an interesting characteristic. It is full of questions that help her understand and conjecture about the situation. As fragment one shows, she starts off by trying to get a sense of what a GPS and a control tower are. To accomplish that, she logically relates the task situation (GPS and control tower) with other familiar situations she has seen in movies and airports. Then, in fragment two, Sophia conjectures that planes' speed is needed, but then she guesses they just need to know how far they are. By saying she has a guess and that she believes she got it, she shows her belief in her ability both to do and learn mathematics. Sophia then retrieves her previous knowledge about Cartesian plane and connects to the situation. She relates the control tower (situation) with the Cartesian plane (content), establishing the control tower as the origin. While doing that, she is working towards a strategy to represent and solve the task. The same is true, when she says she needs to know the planes' distance from the origin to figure out the distance between the planes. Therefore, this first phase of Sophia's mathematical understanding is represented by her conjectures around the problem to understand it and to establish a strategy to solve it. It is a merge of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *productive disposition*.

In the next phase, Sophia goes deeper in her strategy to solve the problem. In fragment three, she retrieves her mathematical knowledge about triangles and connects with the situation. She further develops her strategy to solve the problem by proposing to make triangles in the Cartesian plane. Fragment four still speaks to her strategy, given that Sophia chooses to use a picture (fragment five picture) to represent the problem and figure out the necessary lengths to solve the problem. She mentions that she still needs to figure out the height, what means she is relating the content (2D picture and 2D calculations) with the situation (3D). Then, fragment five presents the aforementioned picture and her explanation about her solution to answer letter *a*. Her records show she chooses a right procedure, at a right moment, performs it correctly and also understands it. She is not explicit about the use of the Pythagoras theorem. Her comment about the necessity of repeating the procedure three times (front, side and top) together with her cube drawing illustrate her 3D reasoning. This was not a common concern in other observed groups. This middle phase of Sophia's work addresses the task first question and is a blend of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *procedural fluency*.


Logically relating situations.
1
Do you know like in movies? This is the board and then there is like a dot... Radar? So what's GPS? What is it on a plane? What does it look like? It's just telling you where to turn? What's a control tower? So what does it mean by approaching? So like an accident could happen? So this is that airport kind of thing?
Building strategy to represent and solve the problem.
How do I let them know that I got this length from the radar?
N
altimeter
radar
compass
W
E
S
altimeter
10
Strategic Competence
Retrieving and connecting mathematical content.
2
Building strategy to represent and solve the problem.
Logically relating content and situation.
Believing in her ability to do and learn mathematics.
So they need to know how fast they are travelling. Oh no, I guess they just need to know how far away they are. I think I got it. I'm pretty sure the control tower [represents the] origin. So they just need to know how far away from the origin they are and then they know how far they are from each other.
?
Retrieving, connecting and understanding mathematical content.
Logically relating content and situation.
How do we know the angles? We only know all of this only if we get that angle, right? What does the compass do?
9
8
Connecting mathematical content.
Building strategy to solve the problem.
[Altitude is] a height of an object or point in relation to sea level or ground level. Altitude measurement, altimeter, there you go.
Adaptive Reasoning
Conceptual Understanding
7
Logically relating content and situation.
Believing in her ability to do mathematics.
I think GPS is coordinates on the world and radar is coordinates based on the radar.
Retrieving and connecting mathematical content.
Building strategy to solve the problem.
So then you make one right triangle and make another one. Either way, everywhere they are they can always make a right triangle.
3
6
Understanding mathematical content.
Logically relating content and situation.
Transferring content between situations.
Believing in her ability to do mathematics.
What do you know from a radar? What does the plane know? So you will see two dots, but you don't know one is way above or way under. So we need to figure out, how to figure that?
Productive Disposition
Building strategy to represent and solve the problem.
Logically relating content and situation.
4
In the picture you can figure it out. You figure out this length and you figure out this length, but you have to know how height it is.
Procedural Fluency
Choosing a right procedure, a right moment to apply it, understanding it, and performing it correctly.
5
So we do that three times, right? Front, side and top.
(x1, y1)
(x2, y2)
a) With GPS, we have the coordinates of the 2 planes. (x, y, z) We can then find the distance by making a right triangle and using c as distance. We know a&b because we know the coordinates so we can simply subtract. Then we need to do that again for the altitude.
Sophia - Task 2

The final phase of Sophia's work focuses on understanding and solving the second question of the task, in which a GPS is not accessible. Her strategy is to use other equipments other than the GPS. So, in fragment six, she inquiries about the available information from a radar and from the plane itself. She logically relates the 3D situation under analysis with the need of some input about the airplanes' height (content). As so she is comparing radar and GPS by transferring content (height input) between the two situations (radar 2D coordinates and GPS 3D coordinates). By doing so, Sophia also demonstrates her understanding about the mathematical knowledge involved in the task. In this same fragment, she states the need to figure out the missing height information from the radar and wonders how to get it. This wonder speaks to her belief in her ability to do mathematics. Still trying to understand the difference between the unavailable GPS and the radar she wants to use instead, in fragment seven, Sophia logically relates a GPS and a radar (situations), with the kind of coordinates (content) she would obtain from each of them. She infers that GPS coordinates are based on an object position in the world, while the radar coordinates are based on an object position in relation to the radar. This inference attests to Sophia's belief in her capacity to do mathematics. Fragment eight illustrates Sophia researching and connecting knowledge about altitude. She finds out which equipment is used for altitude measurement, given that she wants to use data from this equipment to figure out the distance between the two aircrafts. By doing that, Sophia is working on her strategy to solve the problem. The next equipment Sophia's wants to use is a compass to measure angles. Fragment nine points up her inquiry about angles. It is possible to observe that Sophia logically relates content (angles) and the situation under analysis. She retrieves and connects her previous knowledge about angles, showing understanding about it. This understanding is verified once she wants to confirm that all she is doing is only valid in case she knows one of the angles. In other words, she understands that all the data she can figure out depends on this specific angle. Finally, fragment ten presents the picture Sophia does to represent the problem. This same picture shows the lengths and the angle she is expected to know to solve the problem, it also highlights the necessary equipments to figure out each length or angle. Sophia is concerned about how to represent the information provided by the equipments. Therefore, this fragment speaks to her strategy to represent and solve the problem. Nevertheless, due to the lack of time, Sophia does not further advance on how she would get the distance between the aircrafts. This final phase of Sophia's work is characterized by a combination of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *productive disposition*.

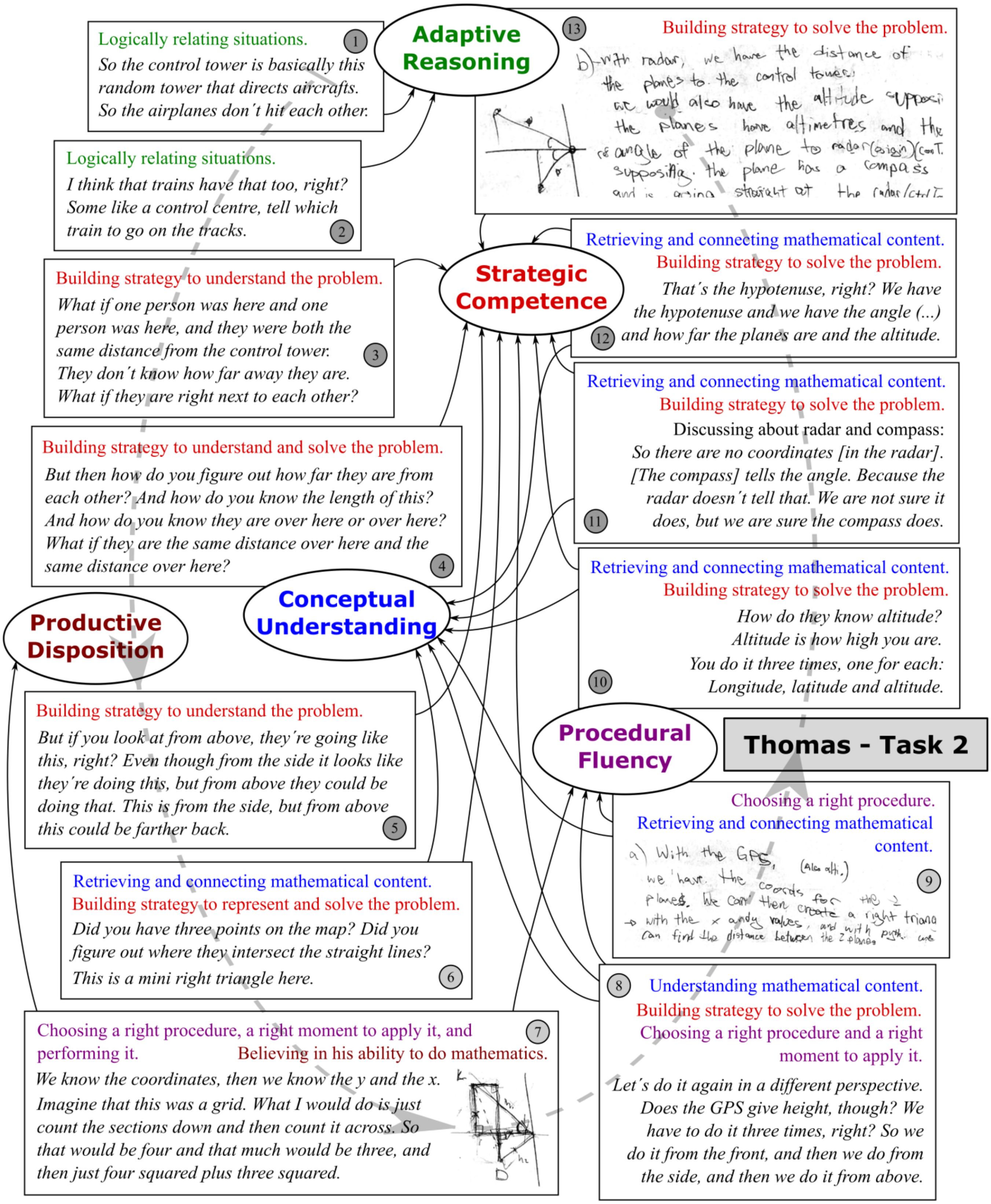
APPENDIX F: THOMAS'S DIAGRAM FOR TASK TWO – FLIGHT SIMULATORS

Thomas's initial phase of his work is based on conjectures, explanations and inquiries about the task. Fragments one and two show Thomas logically relating some situations. He starts off by relating the control tower role of directing aircrafts (one situation) with a possible hit between aircrafts (another situation). He also relates the role of an airplane control tower (one situation) with other possible scenarios, such as a train control centre (another situation). Then, in fragment three, Thomas works on a strategy to understand the problem. He considers different positions for the airplanes, which help him understand the different scenarios that need to be analyzed. Next, in fragment four, Thomas conjectures about how to figure out the distance between the two aircrafts considering the scenarios from fragment three. By doing that he is working on a strategy to help him solve the problem. He also points out some other scenarios to be analyzed to understand the problem. In fragment five, Thomas analyzes the flying airplanes from different perspectives, which might lead to different ways of investigating the task. This approach contributes to his strategy to understand the problem. As so, this first phase of Thomas's work is a mix of *strategic competence* and *adaptive reasoning*.

In the second phase of Thomas's work, he works more specifically in strategies and procedures to solve the problem. Fragment six shows Thomas retrieving and connecting his knowledge about axis intersections and triangles. By having three points in the map and finding out the intersections to build up the triangle, Thomas is working on a strategy to represent and solve the problem. Then, in fragment seven, Thomas explains a right chosen procedure to be performed at a right moment. The procedure starts off from the airplane coordinates and progresses to the Pythagoras theorem in order to find the desired distance. By saying "What I would do is just...", Thomas is also revealing his belief in his ability to do mathematics, given that he presents his idea to the group as a reasonable option. Following, in fragment eight, Thomas reinforces the necessity of doing the procedure in three different perspectives. By envisioning the necessity of three different perspectives, he shows his understanding of the mathematical content, at the same time that he further establishes his strategy to solve the problem. Finally, in fragment nine, Thomas formalizes his solution for the first item of the task. He explains the procedure, retrieving his knowledge about the Pythagoras theorem and connecting to the task. It is worth noticing that, as there is no numeric data in this task, it is expected that students could express their strategies and procedures in words, as Thomas does. This middle phase of Thomas's work is a blend of *conceptual understanding*, *strategic competence, procedural fluency* and *productive disposition*.

Finally, the last phase of Thomas's work focuses on the second item of the task. In this phase, Thomas and the group discuss how to deal with the failure of the GPS. Their main strategy is to use other equipments, such as radar, altimeter and compass. Therefore, in fragments ten, eleven and twelve, Thomas retrieves his previous knowledge about altitude, angles and hypotenuse, respectively, and connects it with the task. At the same time, he builds strategies to solve the problem. For example, in fragment ten, his strategy to solve the problem is based on repeating the chosen procedure three times to consider longitude, latitude and altitude. In fragment eleven, the strategy is based on using an equipment to measure angles. While in fragment twelve, the strategy refers to the available data to solve the problem (hypotenuse, angle, and distances). At last, in fragment thirteen, Thomas partially organizes his strategy based on the available data he has to solve the problem. However, he does not formalize a procedure to be done. The last phase of Thomas's work is a combination of *conceptual understanding* and *strategic competence*.

As for *productive disposition*, throughout the task, Thomas keeps asking questions and conjecturing about the task. This attitude speaks to Thomas belief in his ability to learn mathematics, given that he does not stop trying to understand and solve the task.

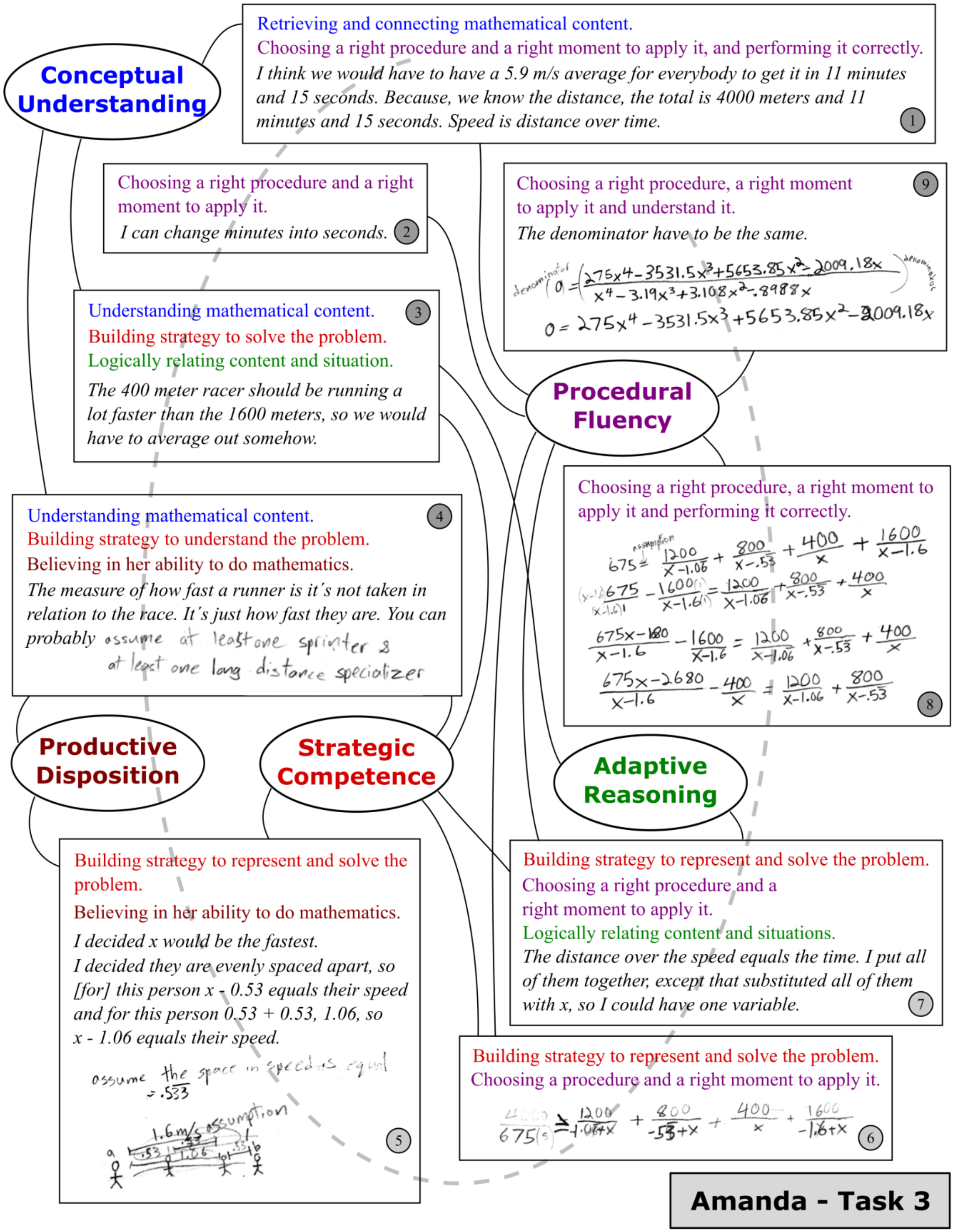

Adaptive Reasoning
Strategic Competence
Conceptual Understanding
Productive Disposition
Procedural Fluency
Thomas - Task 2
1 Logically relating situations.
So the control tower is basically this random tower that directs aircrafts. So the airplanes don't hit each other.
2 Logically relating situations.
I think that trains have that too, right? Some like a control centre, tell which train to go on the tracks.
3 Building strategy to understand the problem.
What if one person was here and one person was here, and they were both the same distance from the control tower. They don't know how far away they are. What if they are right next to each other?
4 Building strategy to understand and solve the problem.
But then how do you figure out how far they are from each other? And how do you know the length of this? And how do you know they are over here or over here? What if they are the same distance over here and the same distance over here?
5 Building strategy to understand the problem.
But if you look at from above, they're going like this, right? Even though from the side it looks like they're doing this, but from above they could be doing that. This is from the side, but from above this could be farther back.
6 Retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Did you have three points on the map? Did you figure out where they intersect the straight lines? This is a mini right triangle here.
7 Choosing a right procedure, a right moment to apply it, and performing it.
Believing in his ability to do mathematics.
We know the coordinates, then we know the y and the x. Imagine that this was a grid. What I would do is just count the sections down and then count it across. So that would be four and that much would be three, and then just four squared plus three squared.
8 Understanding mathematical content.
Building strategy to solve the problem.
Choosing a right procedure and a right moment to apply it.
Let's do it again in a different perspective. Does the GPS give height, though? We have to do it three times, right? So we do it from the front, and then we do from the side, and then we do it from above.
9 Choosing a right procedure.
Retrieving and connecting mathematical content.
10 Retrieving and connecting mathematical content.
Building strategy to solve the problem.
How do they know altitude? Altitude is how high you are. You do it three times, one for each: Longitude, latitude and altitude.
11 Retrieving and connecting mathematical content.
Building strategy to solve the problem.
Discussing about radar and compass:
So there are no coordinates [in the radar]. [The compass] tells the angle. Because the radar doesn't tell that. We are not sure it does, but we are sure the compass does.
12 Retrieving and connecting mathematical content.
Building strategy to solve the problem.
That's the hypotenuse, right? We have the hypotenuse and we have the angle (...) and how far the planes are and the altitude.
13 Building strategy to solve the problem.

APPENDIX G: AMANDA'S DIAGRAM FOR TASK THREE – DISTANCE MEDLEY RELAY

Fragment one of Amanda's diagram shows that she starts off by retrieving her previous knowledge about speed (as distance over time) and about average. She connects this content with the situation and concludes she needs an average of runners' speed of 5.9 m/s. To get the speed of 5.9 m/s, she chooses a right procedure to apply (she verbalizes that speed equals distance over time), at a right moment, performing it correctly (she gets 5.9, which is approximately equal to 4000 meters divided by 675 seconds). To obtain 675 seconds, as can be seen in fragment two, she once more chooses a right procedure (changing minutes into seconds) and a right moment to apply it. Still in the initial phase, as pictured by fragment three, Amanda states that runners should have different speeds, given that they are running different distances and an average is needed. In this sense, she shows understanding of the average content, and she relates the situation under analysis (runners running different distances with different speeds) with the retrieved content (average). Other than that, Amanda is also building a strategy to solve the problem. Continuing with her strategy, in fragment four, she assumes that at least one runner is a sprinter and at least one runner is a long distance specialist. This strategy helps her understand the problem and also shows her understanding about the involved mathematical content, given that she wants to average out the four runners. By making assumptions, Amanda is illustrating her belief in her ability to do mathematics. Therefore, as aforementioned, the first phase of her work presents all five strands of mathematical proficiency, that is, *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

The middle phase of Amanda's work shows the progression of her strategy. At first, as can be seen in fragment five, Amanda decides *x* would represent the speed of the fastest runner and that runners' speed would be evenly spaced apart by 0.53. By making these decisions, Amanda reinforces her belief in her capacity to do mathematics. Indeed, she makes her own choices and do not hesitate to try her own solution out, which means she thought she could be right. Then, as fragment six shows, Amanda sums up runners' race time as if she was summing up speeds. Her intent was to average out speeds, as indicated by her initial strategy in fragment three. However, that is not what she does. Observe that she firstly writes down that the sum of runners' race time should be less than or equal to 4000 divided by 675, which is distance over time, that is, speed. After some prompt, she erased the 4000 (that is why it cannot be seen clearly in the fragment) and realizes she was levelling time with speed. Amanda tries to correct her procedure but she does not consider a weighted average still. In reality, she does not even divide the sum by four, as she should in case it was a simple average. At this point, although she chooses a strategy to represent and solve the problem and chooses a procedure to apply in a convenient moment, she misuses the involved mathematical concepts. After some extra prompt, she figures out what is wrong and adapts her strategy to represent and solve the task. This change can be confirmed in fragment seven, when Amanda states that distance over speed is time, and that she is putting all times together. Amanda realizes what is going on by logically relating the mathematical equation she was working with (the content) and both situations (averaging speeds out and summing up runners' time). Other than that, she advances on her strategy a little more, given that she wants only one variable in the obtained equation. To accomplish that, she chooses a right procedure and a right moment to apply it, once she substitutes x in all speeds. This second phase of her work is also a very rich one, reflecting a blend of *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

Finally, in the last phase, as shown in fragments eight and nine, Amanda presents the second equation that she got, that is, the sum of runners' time (distance over speed) equal to the total desired time $\left(675 = \frac{1200}{x - 1.0\overline{6}} + \frac{800}{x - 0.5\overline{3}} + \frac{400}{x} + \frac{1600}{x - 1.6}\right)$.Then she focus on solving this equation by choosing a right procedure to be applied at a right moment and by performing it correctly. Amanda also shows her understanding of the procedure (in fragment nine) by saying that denominators need to be the same. Hence, the final phase of her work was basically about *procedural fluency*. Unfortunately, Amanda does not get to an end in solving her equation, because the given time for the task was over. Nevertheless, her work was very rich and led her to a comprehensive understanding of the task. As she was engaged and interested in the task, she would probably be able to finish solving the equation in case she was given more time.


Conceptual Understanding
Procedural Fluency
Strategic Competence
Adaptive Reasoning
Productive Disposition
Retrieving and connecting mathematical content.
Choosing a right procedure and a right moment to apply it, and performing it correctly.
I think we would have to have a 5.9 m/s average for everybody to get it in 11 minutes and 15 seconds. Because, we know the distance, the total is 4000 meters and 11 minutes and 15 seconds. Speed is distance over time.
1
Choosing a right procedure and a right moment to apply it.
I can change minutes into seconds.
2
Understanding mathematical content.
Building strategy to solve the problem.
Logically relating content and situation.
The 400 meter racer should be running a lot faster than the 1600 meters, so we would have to average out somehow.
3
Understanding mathematical content.
Building strategy to understand the problem.
Believing in her ability to do mathematics.
The measure of how fast a runner is it's not taken in relation to the race. It's just how fast they are. You can probably assume at least one sprinter & at least one long distance specializer
4
Building strategy to represent and solve the problem.
Believing in her ability to do mathematics.
I decided x would be the fastest.
I decided they are evenly spaced apart, so [for] this person x - 0.53 equals their speed and for this person 0.53 + 0.53, 1.06, so x - 1.06 equals their speed.
assume the space in speed is equal = .533
1.6 m/s assumption
5
Building strategy to represent and solve the problem.
Choosing a procedure and a right moment to apply it.
6
Building strategy to represent and solve the problem.
Choosing a right procedure and a right moment to apply it.
Logically relating content and situations.
The distance over the speed equals the time. I put all of them together, except that substituted all of them with x, so I could have one variable.
7
Choosing a right procedure, a right moment to apply it and performing it correctly.
assumption
8
Choosing a right procedure, a right moment to apply it and understand it.
The denominator have to be the same.
denominator
$0 = 275x^4 - 3531.5x^3 + 5653.85x^2 - 2009.18x$
9
Amanda - Task 3

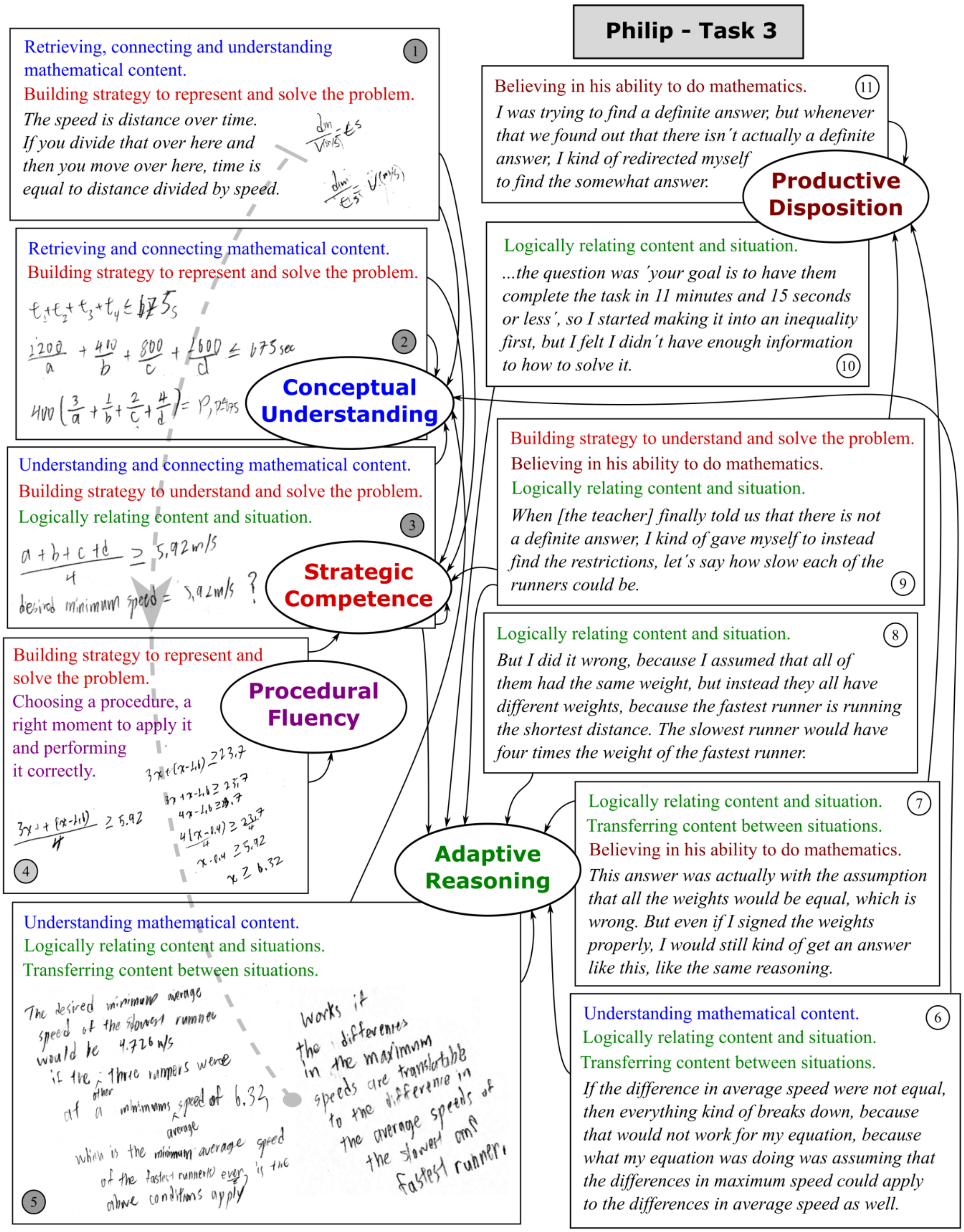
APPENDIX H: PHILIP'S DIAGRAM FOR TASK THREE – DISTANCE MEDLEY RELAY

In the first fragment of Philip's diagram, he retrieves mathematical content from his previous knowledge (relation between speed, distance and time) and connects to the task, trying to build up a strategy to represent and solve the task. When he manipulates the speed equation in fragment one, his discourse shows he understands the mathematical content he is working with. By isolating time instead of speed in the equation, he is trying to find a way to represent the data he has in hands (total time and individual distances) in order to solve the task. Then in fragment two, he retrieves content knowledge about rational inequalities and factoring and connects to the task, still trying to represent and solve the problem through an equation. In fragment three, he changes the strategy to represent the problem, given that he introduces average as a new mathematical content to be worked out in an different equation. At this point, Philip also tries to logically relate content and situation, by conjecturing about the average speed inequality (content) and the desired minimum speed he is looking for (situation). As so, this phase of his work is a mix of *conceptual understanding*, *strategic competence* and *adaptive reasoning*. Indeed, *adaptive reasoning* is strongly present in his solution. This can be confirmed by looking at Philip's interview fragments nine and ten about the first steps of his solution. By recalling he correlated the maximum required time with an inequality (fragment ten) and with minimum speed restrictions (fragment nine), he is actually ratifying he was logically relating situation (maximum time) and content (inequation and restrictions). Working on restrictions is also a way to work on strategies to understand and solve the problem.

After this initial phase, Philip keeps on his second strategy. As illustrated by fragment four, he develops his strategy a little more by establishing that three runners run with speed x and one runner run with speed $x - 1.6$. Then he chooses a procedure to follow and performs it correctly. This phase of his work is characterized basically by *strategic competence* and *procedural fluency*. However, the chosen strategy was not a good one, given that the time average should be a weighted average and not a simple average. As can be seen in the interview fragment eight, Philip realizes he was wrong by logically relating the weighted average (content) with the situation under analysis (different runners running legs different in size).

At the end of the task, as fragment five shows, Philip writes down his conclusion about the problem showing understanding about the used mathematical content. He logically relates content and situation when he compares the difference between maximum speeds and the difference between averages. That is, two different situations (maximum speed and average speed) in which he is admitting he can use the same content (the difference). As so he is also transferring content in between these two situations. Hence, the final phase of his work can be portrayed as a blend of *conceptual understanding* and *adaptive reasoning*. During the interview, as fragment six illustrates, Philip reinforces his reasoning during this final phase, saying that his equation was assuming that the difference could be applied to both situations (maximum speed and average speed). His explanation speaks to his understanding about the involved mathematical content as well. In the next interview fragment (fragment seven), he goes back to the fact that he should have done a weighted average. He logically relates both kinds of average with the situation (different runners running legs different in size) and concludes he would have a similar answer if he had done it right. Therefore, he was transferring content between situations again.

Philip's interview fragments show how much *adaptive reasoning* was involved on his work. This *adaptive reasoning* speaks also to his *conceptual understanding*, given that without *conceptual understanding* it would be more difficult to make the connections he did between content and situation. Finally, *productive disposition* is ingrained during all Philip's work. Researcher observations confirm that he was completely involved in the task seeking for a conclusion about what he was doing. Moreover, fragments seven, nine and eleven (collected during the interview), show some evidence of his belief on his ability to do mathematics. When Philip states "I would still kind of get an answer like this" (fragment seven) he shows confidence about his conjectures. By saying "I kind of gave myself to instead find the restrictions" (fragment nine) he confirms that he gave him a chance to try something else which he believed was worth trying. When he says "I kind of redirected myself to find the somewhat answer" (fragment eleven) he once more shows that he believed he was able to find the "somewhat" answer. All these statements speak to Philip's belief in his ability to do mathematics.



Philip - Task 3
1
Retrieving, connecting and understanding mathematical content.
Building strategy to represent and solve the problem.
The speed is distance over time. If you divide that over here and then you move over here, time is equal to distance divided by speed.
2
Retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Conceptual Understanding
3
Understanding and connecting mathematical content.
Building strategy to understand and solve the problem.
Logically relating content and situation.
desired minimum speed = 5.92 m/s ?
Strategic Competence
4
Building strategy to represent and solve the problem.
Choosing a procedure, a right moment to apply it and performing it correctly.
Procedural Fluency
5
Understanding mathematical content.
Logically relating content and situations.
Transferring content between situations.
The desired minimum average speed of the slowest runner would be 4.726 m/s if the other three runners were at a minimum average speed of 6.32, which is the minimum average speed of the fastest runner(s) ever, if the above conditions apply
works if the differences in the maximum speeds are translatable to the difference in the average speeds of the slowest and fastest runner.
11
Believing in his ability to do mathematics.
I was trying to find a definite answer, but whenever that we found out that there isn't actually a definite answer, I kind of redirected myself to find the somewhat answer.
Productive Disposition
10
Logically relating content and situation.
...the question was 'your goal is to have them complete the task in 11 minutes and 15 seconds or less', so I started making it into an inequality first, but I felt I didn't have enough information to how to solve it.
9
Building strategy to understand and solve the problem.
Believing in his ability to do mathematics.
Logically relating content and situation.
When [the teacher] finally told us that there is not a definite answer, I kind of gave myself to instead find the restrictions, let's say how slow each of the runners could be.
8
Logically relating content and situation.
But I did it wrong, because I assumed that all of them had the same weight, but instead they all have different weights, because the fastest runner is running the shortest distance. The slowest runner would have four times the weight of the fastest runner.
7
Logically relating content and situation.
Transferring content between situations.
Believing in his ability to do mathematics.
This answer was actually with the assumption that all the weights would be equal, which is wrong. But even if I signed the weights properly, I would still kind of get an answer like this, like the same reasoning.
Adaptive Reasoning
6
Understanding mathematical content.
Logically relating content and situation.
Transferring content between situations.
If the difference in average speed were not equal, then everything kind of breaks down, because that would not work for my equation, because what my equation was doing was assuming that the differences in maximum speed could apply to the differences in average speed as well.

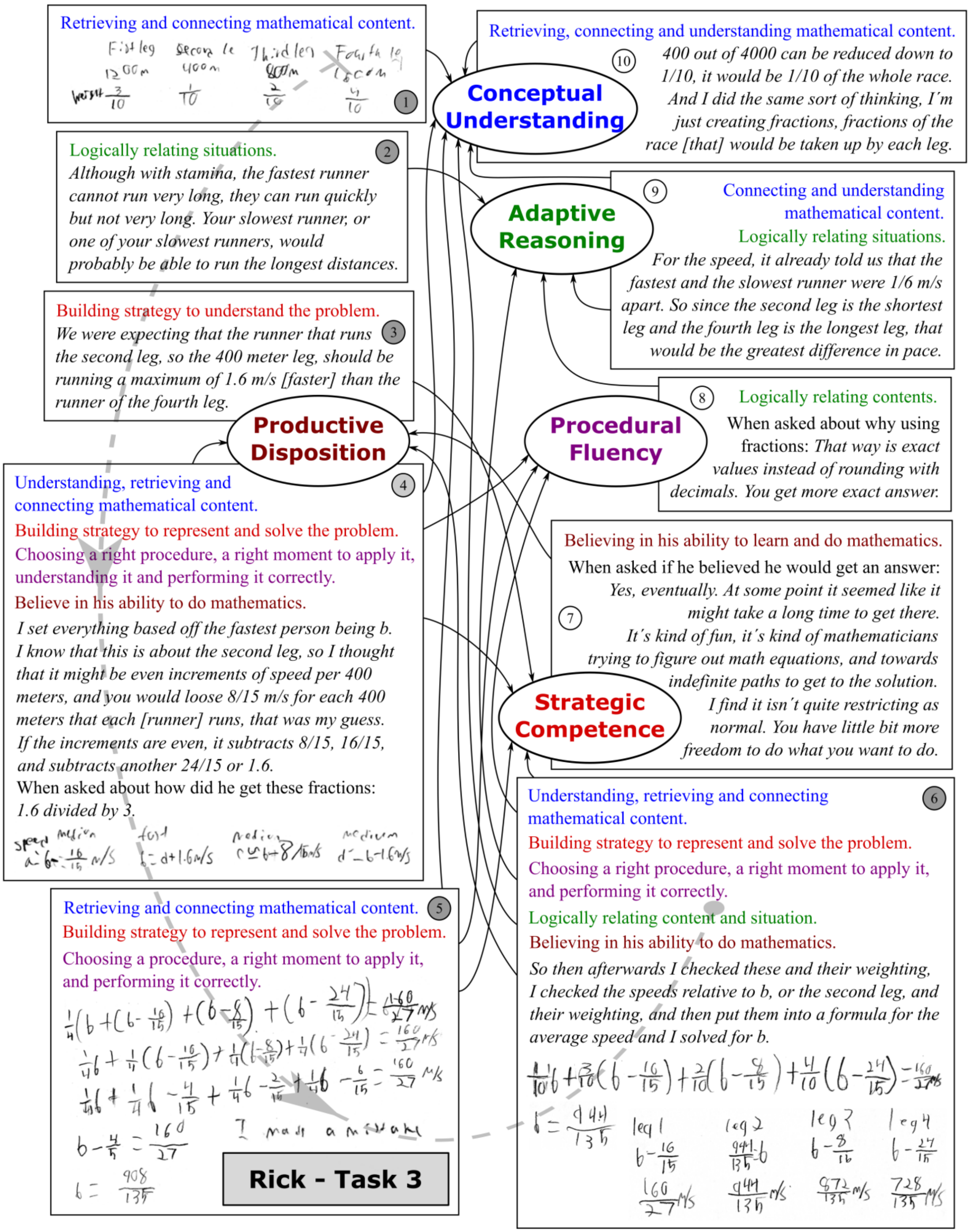
Appendix I: Rick's diagram for task three – Distance medley relay

Rick's work is very interesting and portrays uncommon aspects when compared with other students' work. Rick starts off his investigation by retrieving his mathematical knowledge about ratios and connecting with the given data in the problem (as shown in fragment one). This is a pertinent move, since each runner is supposed to run a different fraction of the total medley distance. During the recall interview, as indicated by fragment ten, Rick verbally explains his notes, showing he not only retrieves his mathematical knowledge and connects to the task, but he also understands the whole process. Also during the interview, when asked about why using fractions (in fragment eight), he clarifies that by using fractions he gets exact values instead of approximate values, which would be the case if he was working with decimals. This explanation shows his ability to logically relate contents, namely, fractions and decimals. Going back to Rick's classroom notes, in fragment two, he has a similar reasoning, but this time logically relating situations (different runners and different distances). He presents an argument to explain why a certain runner should be running a certain distance. Based on this reasoning, Rick builds a strategy to help him make sense of the problem. This strategy can be seen in fragment three. He supposes that the runner who is running the shortest leg will be running at maximum 1.6 m/s faster than the runner who is running the longest leg. In the recall interview, as fragment nine shows, Rick again logically relates the different speeds and the different distances. At this time, he connects these situations with mathematical content, by stating that 1.6 m/s would represent the maximum difference between speeds. His explanation on this matter also speaks to his mathematical understanding of the situation. Based on these observations, it is possible to infer that the initial phase of Rick's work is a mix of *conceptual understanding*, *strategic competence* and *adaptive reasoning*.

The middle phase of Rick's work is characterized basically by the further development of his strategy to represent and solve the problem. This can be seen in fragment four, in which he firstly needs to retrieve his knowledge about fractions and connect to the task. He then chooses a right procedure based on fractions, and correctly applies it at a right moment. Rick explains he has one runner as his main reference and assumes even speed increments for every 400 meters increase in distance. He obtains these speed increments (8/15) by dividing the maximum difference (1.6 m/s) by 3. His explanation confirms his understanding about the required mathematical content and about the procedure as well. By assuming a person to be his reference, affirming he knows what the problem is about, and making informed guesses to solve the task, he is in fact presenting evidences of his belief in his ability to do mathematics. Therefore, this second phase of Rick's work presents a blend of *conceptual understanding*, *strategic competence*, *procedural fluency* and *productive disposition*.

Finally, Rick mathematically formalizes his strategy. As fragment five portrays, he retrieves his previous knowledge about average, connects it to the task and comes up with an equation to balance out runners' speeds. This equation serves the purpose of representing and solving the problem. Fragment five illustrates the equation and the procedure he chooses (at a proper moment) to correctly solve the equation. However, as he acknowledges, he makes a mistake and needs to come up with a new equation. This new equation is presented in fragment six, together with an explanation that makes clear what was wrong about the first equation. The problem is that Rick does not consider a weighted average in his first equation. At this point, he retrieves mathematical content again and connects to the task, but this time mathematical content about weighted average. Balancing runners' speeds with a weighted average is his final strategy to represent and solve the task. To make this move, Rick's notes demonstrate he has to logically relate content (simple average and weighted average) and the situation under analysis (different runners running legs different in size). His explanation speaks to his understanding about weighted average (mathematical content) as well. He chooses a right procedure, at a right moment, and performs it correctly, getting to his final answer this time. By admitting he made a mistake and by restarting his strategy, Rick is showing his belief in his ability to do mathematics. Hence, this final phase of Rick's work is a very rich one, in which all five strands of mathematical proficiency are present, that is, *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

During Rick's recall interview, it was also possible to reinforce his *productive disposition* towards mathematics. As fragment seven shows, he states he believed he would eventually get to an answer, although sometimes it seemed it would take longer. He also says it was fun to engage in a task similar to what mathematicians do. Finally, he affirms that this kind of task is not restricting as usual mathematics tasks, which gives him more autonomy to make his own choices. All this assertions reinforces Rick's belief in his capacity to do and learn mathematics.



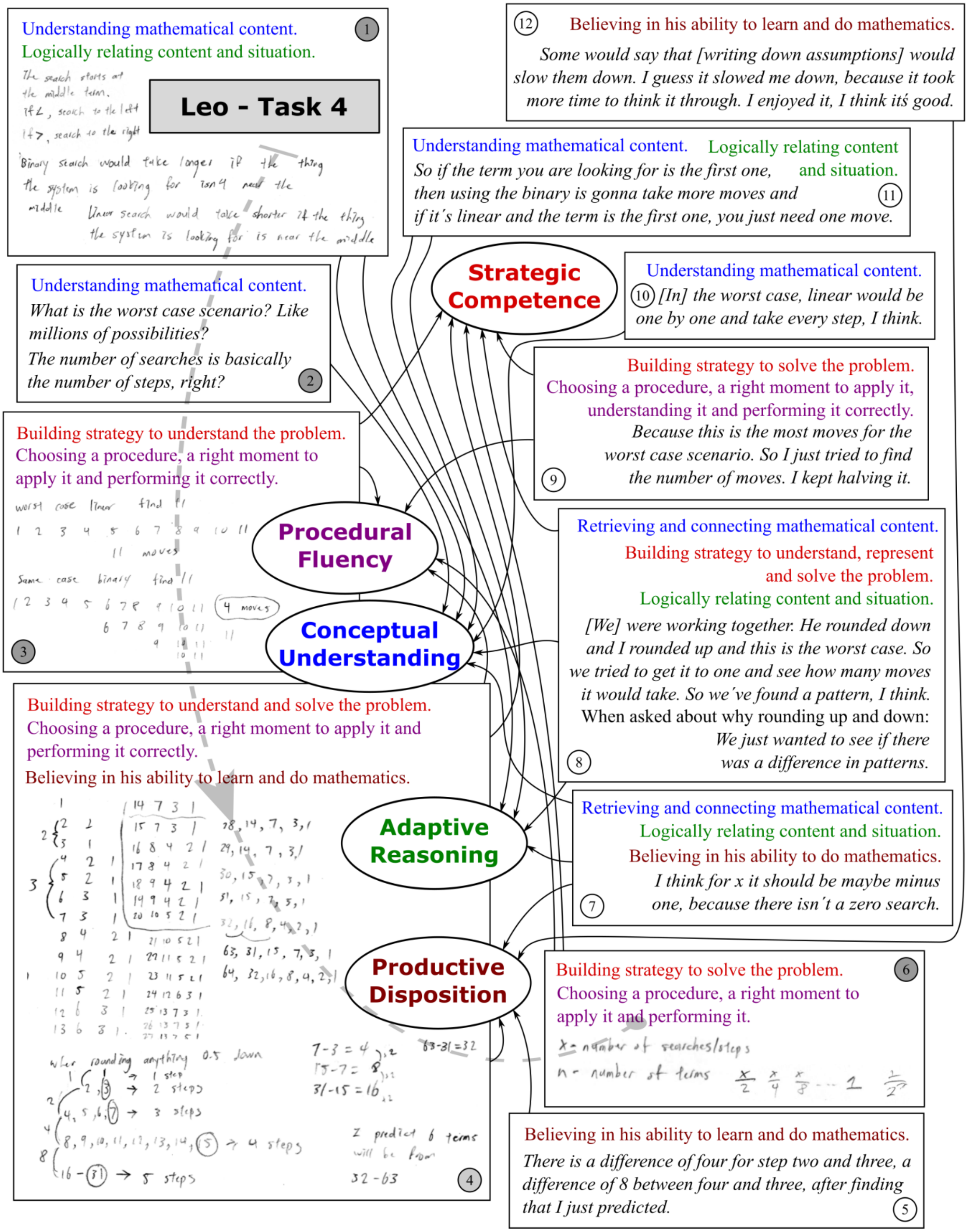
Retrieving and connecting mathematical content.
1
Logically relating situations.
2
Although with stamina, the fastest runner cannot run very long, they can run quickly but not very long. Your slowest runner, or one of your slowest runners, would probably be able to run the longest distances.
Building strategy to understand the problem.
3
We were expecting that the runner that runs the second leg, so the 400 meter leg, should be running a maximum of 1.6 m/s [faster] than the runner of the fourth leg.
Productive Disposition
4
Understanding, retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Choosing a right procedure, a right moment to apply it, understanding it and performing it correctly.
Believe in his ability to do mathematics.
I set everything based off the fastest person being b. I know that this is about the second leg, so I thought that it might be even increments of speed per 400 meters, and you would loose 8/15 m/s for each 400 meters that each [runner] runs, that was my guess. If the increments are even, it subtracts 8/15, 16/15, and subtracts another 24/15 or 1.6.
When asked about how did he get these fractions: 1.6 divided by 3.
5
Retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Choosing a procedure, a right moment to apply it, and performing it correctly.
Rick - Task 3
Conceptual Understanding
10
Retrieving, connecting and understanding mathematical content.
400 out of 4000 can be reduced down to 1/10, it would be 1/10 of the whole race. And I did the same sort of thinking, I´m just creating fractions, fractions of the race [that] would be taken up by each leg.
Adaptive Reasoning
9
Connecting and understanding mathematical content.
Logically relating situations.
For the speed, it already told us that the fastest and the slowest runner were 1/6 m/s apart. So since the second leg is the shortest leg and the fourth leg is the longest leg, that would be the greatest difference in pace.
Procedural Fluency
8
Logically relating contents.
When asked about why using fractions: That way is exact values instead of rounding with decimals. You get more exact answer.
7
Believing in his ability to learn and do mathematics.
When asked if he believed he would get an answer: Yes, eventually. At some point it seemed like it might take a long time to get there. It´s kind of fun, it´s kind of mathematicians trying to figure out math equations, and towards indefinite paths to get to the solution. I find it isn´t quite restricting as normal. You have little bit more freedom to do what you want to do.
Strategic Competence
6
Understanding, retrieving and connecting mathematical content.
Building strategy to represent and solve the problem.
Choosing a right procedure, a right moment to apply it, and performing it correctly.
Logically relating content and situation.
Believing in his ability to do mathematics.
So then afterwards I checked these and their weighting, I checked the speeds relative to b, or the second leg, and their weighting, and then put them into a formula for the average speed and I solved for b.

APPENDIX J: LEO'S DIAGRAM FOR TASK FOUR – BINARY SEARCH

Leo's mathematical understanding initial phase is a rich one, in which he presents four of the five strands of mathematical proficiency. As fragment one illustrates, Leo starts off by writing in his own words what he understands by binary search, and he also conjectures which of the searches will take longer or shorter. By doing that, he shows his understanding about the mathematical content present in the task. He is also logically relating content (linear and binary searches) and the situation under analysis, that is, which one is the faster considering the worst case scenario. Then, fragment two presents some of his questionings when trying to comprehend what the worst case scenario is and what the number of searches represents. At this point, Leo is working on understanding the mathematics content again. During the recall interview, he also presents some ideas related to this initial investigation that are worth looking at. In fragment eleven, he shows *conceptual understanding* when describing the operation of both kinds of search in case the term you are looking for is the first one. By comparing the two different searches (content) when the desired term is the first one (situation), he is also working on *adaptive reasoning*. Then, as fragment ten shows, Leo works towards mathematical understanding by discussing the worst case scenario for the linear search. Going back to the classroom fragments, in the third one, Leo performs a little example as a strategy to better understand the task. He chooses a procedure for each of the searches and performs them for the worst case scenario. Therefore, the first phase of Leo's work is a blend of *conceptual understanding*, *strategic competence*, *procedural fluency* and *adaptive reasoning*.

The middle phase of Leo's solution is illustrated in fragment four, which shows the main part of his reasoning. His strategy is to keep halving the number of items to be searched, until he gets to one. This procedure is further explained during the recall interview, as can be seen in fragments nine and eight. Leo explains he and Philip had a halving strategy and a rounding up or down strategy, which stresses they were retrieving previous mathematical knowledge (rounding approaches) and connecting to the task. This procedural strategy helps him understand and solve the task. The idea was to compare both cases and see if there was a difference in patterns. This comparison highlights they were logically relating content (the found patterns) and situation (rounding up or down). Going back to fragment four, Leo chooses a procedure in an appropriate moment and performs it correctly. He gets the number of searches needed for the binary case in the worst case scenario and predicts a pattern. Finally, by making predictions about the pattern he believes he have found (fragment four bottom right), he shows his belief in his ability to learn and do mathematics. This productive disposition towards mathematics can be confirmed by fragment five — taken from the recall interview, in which Leo explains his arguments to make the prediction he made. This phase of Leo's work is a mix of all five strands, that is, *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

Although Leo's findings are not completely expressed by a formal mathematical equation and he also does not state why or why not binary search is faster than linear search, fragment six illustrates he had an abstract idea of what he was doing. Leo tries to formalize a strategy to solve the problem by creating a general procedure. He first associates the letters x and n to the number of needed searches or steps, and to the number of terms in the search, respectively. Then he writes down what the procedure is supposed to be, that is, keep halving the number of terms (he has the number of searches in his notes though) until it gets to one. He also writes down $\frac{1}{2^x}$, implying he almost gets to a general approach. During the recall interview, as fragment seven illustrates, he further explains his thoughts. He enters an even more precise analysis (by retrieving previous mathematical content and connecting to the task), in which he considers having $x - 1$ instead of x in his general expression, given that there is not a zero search according to him. This fragment makes explicit he was logically relating content (x or $x - 1$) and situation (having or not a zero search). Because Leo allows himself to make this conjecture, it is also possible to infer that he has a positive attitude towards his ability to do mathematics. Fragment twelve speaks to Leo's process of making assumptions and maybe conjectures too. He explains that although this writing down process slows him down, it makes him think through his thoughts and he likes it. Hence, this fragment reinforces Leo's belief in his ability to learn and do mathematics. Once more, all five strands of mathematical proficiency are present in the final phase of Leo's work: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.


Leo - Task 4
1
Understanding mathematical content.
Logically relating content and situation.
The search starts at the middle term.
If <, search to the left
If >, search to the right
Binary search would take longer if the thing the system is looking for isn't near the middle
Linear search would take shorter if the thing the system is looking for is near the middle
2
Understanding mathematical content.
What is the worst case scenario? Like millions of possibilities?
The number of searches is basically the number of steps, right?
3
Building strategy to understand the problem.
Choosing a procedure, a right moment to apply it and performing it correctly.
worst case linear find 11
1 2 3 4 5 6 7 8 9 10 11
11 moves
Same case binary find 11
4 moves
4
Building strategy to understand and solve the problem.
Choosing a procedure, a right moment to apply it and performing it correctly.
Believing in his ability to learn and do mathematics.
when rounding anything 0.5 down
1 step
2 steps
3 steps
4 steps
5 steps
7-3 = 4
15-7 = 8
31-15 = 16
63-31=32
I predict 6 terms will be from
32-63
5
Believing in his ability to learn and do mathematics.
There is a difference of four for step two and three, a difference of 8 between four and three, after finding that I just predicted.
6
Building strategy to solve the problem.
Choosing a procedure, a right moment to apply it and performing it.
x = number of searches/steps
n = number of terms
7
Retrieving and connecting mathematical content.
Logically relating content and situation.
Believing in his ability to do mathematics.
I think for x it should be maybe minus one, because there isn't a zero search.
8
Retrieving and connecting mathematical content.
Building strategy to understand, represent and solve the problem.
Logically relating content and situation.
[We] were working together. He rounded down and I rounded up and this is the worst case. So we tried to get it to one and see how many moves it would take. So we've found a pattern, I think.
When asked about why rounding up and down:
We just wanted to see if there was a difference in patterns.
9
Building strategy to solve the problem.
Choosing a procedure, a right moment to apply it, understanding it and performing it correctly.
Because this is the most moves for the worst case scenario. So I just tried to find the number of moves. I kept halving it.
10
Understanding mathematical content.
[In] the worst case, linear would be one by one and take every step, I think.
11
Understanding mathematical content.
Logically relating content and situation.
So if the term you are looking for is the first one, then using the binary is gonna take more moves and if it's linear and the term is the first one, you just need one move.
12
Believing in his ability to learn and do mathematics.
Some would say that [writing down assumptions] would slow them down. I guess it slowed me down, because it took more time to think it through. I enjoyed it, I think it's good.
Strategic Competence
Procedural Fluency
Conceptual Understanding
Adaptive Reasoning
Productive Disposition

APPENDIX K: RICK'S DIAGRAM FOR TASK FOUR – BINARY SEARCH

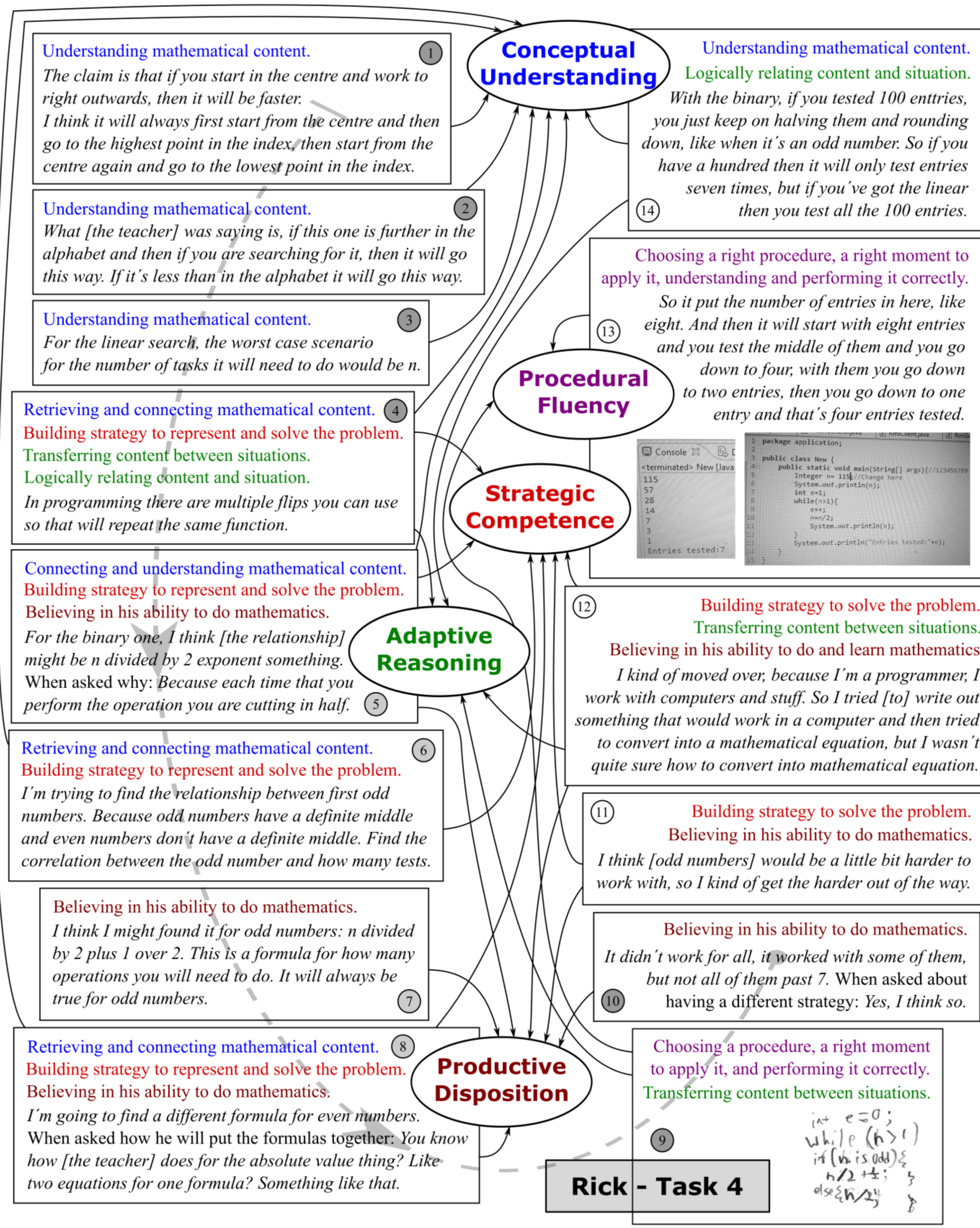
Rick's understanding is pretty much expressed by his comments during the class and during the interview. He does not have a lot of written notes, but he does have a program made after the task was done. Fragment one shows that Rick starts off explaining his understanding about the mathematical content present in the task claim (that is, binary search is faster than linear search), and also explaining his understanding about how a binary search runs. Fragment two also shows his understanding about binary search, and fragment three illustrates his understanding about the linear search worst case scenario. Then, fragment four shows a moment in which Rick retrieves her knowledge about programming and connects to the task. Because he is supposed to repeat the binary search procedure until he gets the desired item, Rick logically relates the programming content (flips that allow the same function to be repeated) to the task situation (data searches). He is not only relating content and situation, he is also transferring the content (flip approach) from previous experienced programming situations to this binary search situation, in order to build a strategy to represent and solve the problem. This first phase of Rick's work can be characterized by a blend of *conceptual understanding*, *strategic competence* and *adaptive reasoning*.

In the next phase, Rick works further on his strategy to represent and solve the problem. He focus on the binary search case and tries to find a relationship between the number of terms being searched (*n*) and the number of necessary searches. In fragment five, he conjectures this relationship can be *n* divided by 2 to the power of something. He connects with powers of 2, because of his understanding that each and every operation cuts the number of terms in half. By investing in an idea he came up with, he attests for his belief in his capacity to do mathematics. Next, in fragment six, Rick deliberately chooses to work with odd numbers first, alleging that odd numbers have a definite middle. This choice represents his work on a strategy again, at the same time that indicates he is retrieving his previous knowledge about even and odd numbers and connecting to the task. During the recall interview, as fragment eleven shows, Rick also comments that he thought odd numbers were harder to work with, and because of that he decided to start from there. This decision is a strategic one and speaks a lot to his *productive disposition* towards mathematics. This is due to the fact that he decided to start from the hardest situation, believing in his capacity to do mathematics. In fragment seven, Rick presents a formula to find the necessary number of operations for an odd number of terms. This fragment also speaks to his belief in his capacity to find a solution, given that he believes he has found the relationship for odd numbers. Finally, as fragment eight shows, Rick decides to find a formula for even numbers. He retrieves his previous knowledge about compound functions and connects to the task to figure out how he is going to put both formulas together. As so, he keeps on making strategic decisions (finding a relationship for even numbers) to represent and solve the task. He also persists being positive about his ability to do mathematics, given that he is confident when saying he is going to find the other formula. This middle phase of Rick's work presents a mix of *conceptual understanding*, *strategic competence* and *productive disposition*.

In the last phase, as shown in fragment nine, Rick procedurally formalizes the formulas he has found to calculate the number of necessary searches in the binary case. He has a procedure, applies it in an appropriate moment and performs it correctly. At this point he transfers content from his programming background knowledge to the task he has in hands (the mathematical problem). This can be seen in the algorithm he writes on his notes. However, as noted on fragment ten, his procedure does not work for numbers past seven. Anyway, Rick reveals he has something else in mind to try out, reinforcing his belief in his capacity to do mathematics. This final phase of Rick's work presents *procedural fluency*, *adaptive reasoning* and *productive disposition*.

Rick's confidence and persistence encouraged him to keep on trying to solve the task at home. In the recall interview, he explains this afterwards process. In fragment twelve, Rick explains he decided to try a strategy based on his experience as a programmer. His strategy was to write a program that would do the binary search and then he would try to convert this program into a mathematical equation. However, his strategy did not turn out as he imagined. Fragment twelve also shows the transference of content (program routine) between situations (algorithm and mathematical equation). His belief in his ability to do mathematics is also noticeable when he says he "moved over", meaning that although his initial mathematical equation was wrong, he would still try something else. Moreover, his belief in his ability to learn mathematics can be observed when he tries to figure out the mathematics equation through his program. In other words, he believed he could learn what the task was expecting from him by converting his program into an equation. Fragment thirteen is specific about the created algorithm. The picture shows Rick chooses a right procedure to be applied at a proper moment, and performs it correctly. His verbal explanation speaks to his understanding of the procedure. In fragment fourteen, the last recall interview fragment, Rick properly describes the binary search process for a given example and compares it with the linear search. He not only logically relates content (both kinds of search) and situation (his given example), but he also communicates his understanding about the task's mathematical content. Therefore, even after the task was done in class, Rick's understanding still unfolded as a mix of the five strands of mathematical proficiency, namely: *conceptual understanding*, *strategic competence*, *procedural fluency*, *adaptive reasoning* and *productive disposition*.

APPENDIX K: RICK'S DIAGRAM FOR TASK FOUR – BINARY SEARCH



Appendix L: Thomas's diagram for task four – Binary search

Thomas does not have lots of notes from task four, however, lots of thinking are going on during his group conversation. As fragment one shows, he starts off by saying he thinks he knows what a binary search is, and explains his reasoning to the rest of the group. His positive discourse speaks to his belief in his capacity to do mathematics, once he believes he knows what a binary search is. When mentioning that binary refers to being one or zero, he is retrieving mathematical content and connecting with the situation under analysis. This sort of information is not given in the task, confirming that he brings up this binary characteristic from his previous knowledge. Fragment two also presents some explanations that address Thomas's understanding about the mathematical content. Still attempting to get a better sense of the task, in fragment three, Thomas tries to relate the content (linear and binary searches) with a known situation (Google searches). He comments that Google is in fact fast, but that he does not know how it works; otherwise he could be able to speculate if binary search is faster than linear search or vice-versa. Fragment four illustrates a comparison between the two sorts of search in what refers to arranging the terms into an order. Thomas is logically comparing both contents (linear and binary searches) in order to facilitate his understanding about how the two different searches run and also to get a sense of which one is faster. Interestingly, classroom observations show that Thomas and his group wonders if arranging terms in order makes binary search slower than linear search, since the latter does not need any arrangement. Then, as shown in fragment five, Thomas makes many questionings still trying to understand the binary search content, at the same time that he tries to build a strategy to understand the problem and maybe solve it. At this point, his strategy is essentially based on the search procedure. Hence he asks procedural questions such as "Why wouldn't go down?" or "Where would we split?", and simultaneously tries out procedures in an example. In this sense, Thomas is logically relating content (binary search) and situation (his example) in order to enquiry about the binary search procedures. As so, this initial phase of Thomas's work is a combination of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *productive disposition*.

In the next phase of Thomas's solution, he focus on finding the minimum and maximum number of binary searches required to search a determined number of terms. In fragment six, his notes show that he chooses a right procedure do find the minimum number of searches required, at a right moment and performs it correctly. Fragment seven shows the same procedure, but to find the maximum number of searches required. In the latter fragment, Thomas makes a comment about his will to find a correlation between the numbers he got (number of searches and number of things to be searched). This observation illustrates he is trying to logically relate content (a mathematical correlation) and situation (the binary search findings). This observation also points up Thomas's attempt to build a strategy to represent and eventually solve the problem. Moreover, the fact that he verbalizes he wants to find this relation speaks to his belief in his ability to find it; that is, to do mathematics. Therefore, this middle phase of Thomas's work is a blend of *strategic competence*, *adaptive reasoning*, *procedural fluency* and *productive disposition*.

Finally, in the last phase of Thomas's solution, his focus is on finding the aforementioned relation. In fragment eight, he retrieves his pattern background knowledge and connects with the numbers he found. He infers there is a pattern between them, but wonders if the pattern is in relation to the number of terms. Again, he is logically relating content (patterns) and situation (the binary search findings). Thomas is still working on a strategy to represent and solve the problem. At some point, as fragment nine shows, a member of the group gives some insight about the pattern that Thomas cannot see. He then asks for help in understanding. This attitude addresses the belief Thomas has in being able to understand the pattern that someone else has found, which means he believes in his capacity to learn mathematics. Although Thomas does not get to a formal final answer, the final fragment shows he was very close to an ending. He retrieves his knowledge about powers and connects with the numbers he is looking at. He attempts to logically relate powers of two (content) and the binary search findings (situation). He understands that he is working with powers of two due to the fact that binary searches imply in halving every time. The final phase of Thomas's work is then a mix of *conceptual understanding*, *strategic competence*, *adaptive reasoning* and *productive disposition*.

APPENDIX L: THOMAS'S DIAGRAM FOR TASK FOUR – BINARY SEARCH