A Theory of Net Convergence with Applications to Vector Lattices

by

Michael John O'Brien

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Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

The theory of convergence structures in [BB02] delivers a promising foundation on which to study general notions of convergence. However, that theory has one striking feature that stands out against all others: it is described using the language of filters. This is contrary to how convergence is used in functional analysis, where one often prefers to work with nets, and this thesis reconciles the issue by introducing an equivalent theory of net convergence structures.

Our approach has several advantages over other efforts to develop a convergence theory using nets. Most notably, we are able to translate between the languages of filter and net convergence structures. These results make the theory in [BB02] more accessible for the working mathematician and provide a new angle for studying aspects of vector lattice theory. We demonstrate the value of this approach using order convergence in vector lattices. This leads to the novel concept of order compactness, and it is shown that order compact sets satisfy an analogue of the Heine-Borel theorem in atomic order complete vector lattices.

Preface

Some of the results in Chapters 2, 3 and 5 have been submitted for peer review; the work in which they are involved is available on arXiv: *Net convergence structures with applications to vector lattices* by M. O'Brien, V.G. Troitsky and J.H. Van der Walt, arXiv:2103.01339v1 [math.FA]. The third author independently discovered alternate axioms for net convergence structures that are not discussed in this thesis; the paper [OTW] represents our collective efforts to unify these theories.

While the work in Chapter 4 has not yet been submitted for publication, there are plans to do so.

Acknowledgments

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I would also like to thank my wife, Faye, for the pure joy she brings to everything she does. Although we both agree that pursuing what you love is not always easy, loving you has always been easy and I am forever grateful to share every day with you.

And to every one of my friends and family: thank you for being you.

Contents

1. Introduction

Convergence of sequences and nets are fundamental tools in many areas of mathematics. While topologies are often given the task to model these concepts, a simple example due to [Ord66] shows that no topology can witness almost everywhere convergence of measurable functions. Similarly, there are generally no topologies that can model order convergence, *uo*-convergence, and relative uniform convergence in vector lattices. This growing list of important "non-topological" convergences necessitates the study of general convergence theory, and the goal of this thesis is to develop such a theory *using nets*.

The theory of net convergence structures is developed in Chapter 2 and they are shown to be equivalent to filter convergence structures. This observation has several advantages over other attempts to develop a convergence theory for nets. In particular, it allows us to translate between filters and nets when applying abstract convergence theory. These results are applied in subsequent chapters to study convergences in vector lattices that are defined in terms of nets. Chapter 3 illustrates that our theory is a suitable model for order convergence in vector lattices. The value of this approach is demonstrated in Chapter 4 when the concept of order compactness is introduced. Chapter 5 focuses on relative uniform convergence and its relationship with order convergence from the viewpoint of convergence theory.

1.1. **Preliminaries.** The theory of convergence structures was introduced in [Fis59] and provides an axiomatic framework for studying convergence. Because this material is not standard, I will provide a brief overview of important background and results. The convergence theory portion of this thesis is self-contained, but the interested reader can see [BB02] for further details and applications.

Throughout this section X will denote a non-empty set. A ***filter*** $\mathcal{F}$ on X is a subset of the power set of X, $\mathcal{P}(X)$, that satisfies the following properties:

(F1) $\varnothing \notin \mathcal{F}$ and $X \in \mathcal{F}$;

(F2) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;

(F3) $A \in \mathcal{F}$ and $A \subseteq B \Rightarrow B \in \mathcal{F}$.

The *trivial filter* is defined to be $\mathcal{F} = \{X\}$

At first glance, the definition of a filter may seem unmotivated. However, it generalizes the concept of neighborhood from topology.

Example 1.1. If (X, τ) is a topological space then $\mathcal{N}_x$, the set of all neighborhoods of x, is a filter on X for each $x \in X$.

It is often easier to build a filter from a simpler collection of sets. A non-empty collection of subsets $\mathcal{B}$ of X satisfying

(FB1) $B \neq \varnothing$ for every $B \in \mathcal{B}$;

(FB2) for every $B_1, B_2 \in \mathcal{B}$ there is a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$

is called a ***filter base***. If $\mathcal{B}$ is a filter base, there is a least filter that contains $\mathcal{B}$, denoted by

$$[\mathcal{B}] := \{A \subseteq X : B \subseteq A \text{ for some } B \in \mathcal{B}\},$$

and is called the ***filter generated by*** $\mathcal{B}$. If $\mathcal{F}$ is another filter on X with $\mathcal{B} \subset \mathcal{F}$, note that $[\mathcal{B}] \subseteq \mathcal{F}$.

Example 1.2. Given $x \in X$ the collection $\mathcal{B} = \{x\}$ is a filter base. In this case we denote $[\{x\}]$ by $[x]$ and call it the ***point-filter*** generated by x. Note that $[x] = \{A \subseteq X : x \in A\}$.

Example 1.3. In topological spaces, a neighborhood base at a point x is a filter base for $\mathcal{N}_x$.

Example 1.4. Let X and Y be two non-empty sets, $f : X \to Y$ a set mapping and $\mathcal{F}$ a filter on X. The collection $\mathcal{B} := \{f(A) : A \in \mathcal{F}\}$ is a filter base on Y. The ***image filter*** of $\mathcal{F}$ induced by f is given by $f(\mathcal{F}) := [\mathcal{B}]$.

A relation $\leq$ on a set A is called a ***pre-order*** if $\leq$ is reflexive and transitive. In this case, we call the pair $(A, \leq)$ a ***directed set*** if each pair of elements has a common descendant; that is,

$$\forall a, b \in A \quad \exists c \in A \text{ such that } a \leq c \text{ and } b \leq c.$$

A ***net*** in X is a function $x : A \to X$ whose domain is a directed set; A is called the ***index set*** of the net. It is more common to denote a net by the notation $(x_\alpha)_{\alpha \in A}$, and we will often write (x_α) when there is no reason to emphasize the index set.

There is a close connection between nets and filters.

Example 1.5. Each net $(x_\alpha)_{\alpha \in A}$ can be used to define a filter base in the following way. For each $\alpha_0 \in A$ the set $\{x_\alpha : \alpha \geq \alpha_0\}$ is called a ***tail set*** of the net (x_α). The set of all tails sets

$$\mathcal{T} = \Big\{\{x_\alpha : \alpha \geq \alpha_0\} : \alpha_0 \in A\Big\}$$

of (x_α) is a filter base and $[x_\alpha] := [\mathcal{T}]$ and call it the ***tail filter*** of (x_α).

$\mathcal{U}$ is an ***ultrafilter*** on X if it is not properly contained in any other filter on X. Clearly every point-filter is an ultrafilter; these are called ***fixed*** ultrafilters. Ultrafilters that are not fixed are called ***free***. It is a consequence of Zorn's Lemma that every infinite set admits a free ultrafilter.

1.2. **Filter Convergence Structures.** Let X be a set and let $\mathfrak{F}(X)$ denote the set of all non-trivial filters on X. If a function $\lambda : X \to \mathcal{P}(\mathfrak{F}(X))$ satisfies

(C1) $[x] \in \lambda(x)$;

(C2) $\mathcal{F} \in \lambda(x)$ and $\mathcal{F} \subset \mathcal{G} \Rightarrow \mathcal{G} \in \lambda(x)$; and

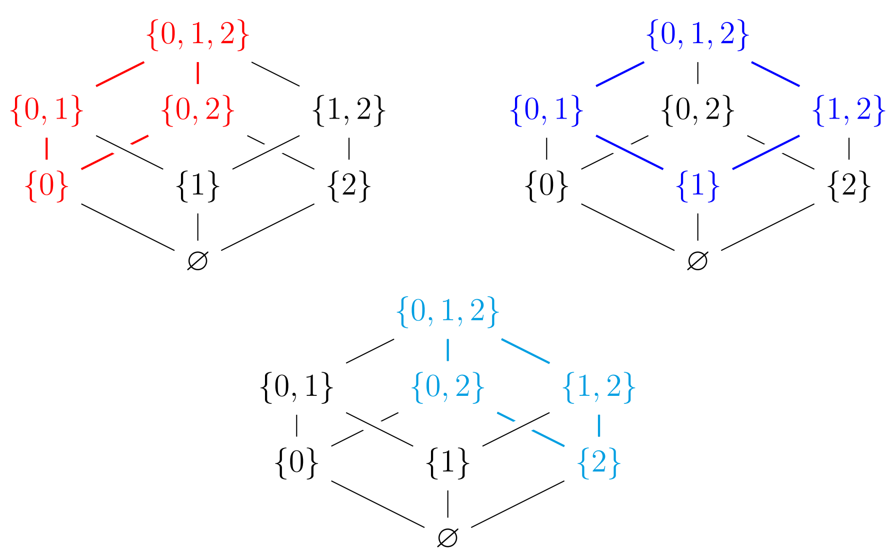
(C3) $\mathcal{F}, \mathcal{G} \in \lambda(x) \Rightarrow \mathcal{F} \cap \mathcal{G} \in \lambda(x)$

for each $x \in X$ then we call λ a ***convergence structure***, and the pair (X, λ) is called a ***convergence space***. We will write $\mathcal{F} \xrightarrow{\lambda} x$ instead of $\mathcal{F} \in \lambda(x)$ and say "$\mathcal{F}$ converges to x with respect to λ" where x is called a ***limit*** of $\mathcal{F}$. We may drop the explicit reference to the convergence structure and refer to X as a convergence space when there is no chance of confusion.

Example 1.6. Let $X = \{0, 1, 2\}$ and write

$$\mathcal{F} \to 0 \quad \text{if } \{0,1\} \in \mathcal{F},$$
$$\mathcal{F} \to 1 \quad \text{if } \{1,2\} \in \mathcal{F}, \text{ and}$$
$$\mathcal{F} \to 2 \quad \text{if } \{0,2\} \in \mathcal{F}.$$

This convergence is easier to visualize with Hasse diagrams:



The filter generated by 0 is highlight in red, the filter generated by 1 in blue, and the filter generated by 2 in cyan. Clearly $[0] \to 0$, $[1] \to 1$ and $[2] \to 2$ in X. Since every filter on X contains at most one singleton, it is easy to verify that the only remaining non-trivial filters on X are the filters $\{\{0,1\},X\}, \{\{1,2\},X\}$ and $\{\{0,2\},X\}$. The remaining convergence structure axioms are now easily verified.

Convergences may have undesirable properties. The convergence from the previous example satisfies $[0] \to 0$ and $[0] \to 2$, so the limit of a convergent filter is generally not unique. A convergence space in which every convergent filter has a unique limit is called a ***Hausdorff*** space.

Example 1.7. If X is a topological space, define $\mathcal{F} \to x$ if $\mathcal{N}_x \subset \mathcal{F}$. It is easily checked that this defines a convergence structure on X; we call it the ***natural convergence structure*** on a topological space. It is easy to see this convergence space is Hausdorff precisely when the topology on X is Hausdorff.

If λ and μ are two convergence structures on a set X such that $\mathcal{F} \xrightarrow{\lambda} x$ implies $\mathcal{F} \xrightarrow{\mu} x$ for each $x \in X$, we say λ is ***stronger*** than μ; dually, we say μ is ***weaker*** than λ. A convergence structure λ is called ***topological*** if λ agrees with the natural convergence of a topology; if no such topology exists, it is said to be ***non-topological***. It was mentioned at the beginning that convergence almost everywhere (a.e.) is a non-topological convergence. This statement can now be made precise.

Example 1.8. Let (Ω, Σ, μ) be a measure space and let $X = L_0(\Omega, \Sigma, \mu)$ be the space of all Σ-measurable functions on X identified up to equivalence almost everywhere. Almost everywhere convergence arises as the convergent sequences of the following convergence structure: We say that a filter $\mathcal{F} \xrightarrow{\text{a.e.}} f$ if there exists a sequence (f_n) in X such that $f_n \xrightarrow{\text{a.e.}} f$ and $[f_n] \subset \mathcal{F}$. This defines a non-topological convergence structure on $L_0(\Omega, \Sigma, \mu)$. Moreover, a sequence $f_n \to f$ almost everywhere if and only if $[f_n] \xrightarrow{\text{a.e.}} f$.

In light of the previous two examples, convergence structures appear to be generalizations of topologies. We begin to deepen this analogy.

1.3. **Basic Properties of Convergence Spaces.** Topology is often described as a general framework for modeling continuous functions, but the definition of continuity in terms of open sets is a bit removed from our intuitive understanding. In contrast, convergence spaces have an obvious framework for discussing continuous functions. Fix two convergence spaces (X, λ) and (Y, μ). Using the notation from Example 1.4, a function $f : X \to Y$ is said to be ***continuous at x*** if $f(\mathcal{F}) \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X. If f is continuous at all points $x \in X$ then f is called ***continuous***. It is called a ***homeomorphism*** if it is bijective, continuous, and f^{-1} is also continuous. Each of these terms is an extension of its topological counterpart.

For each $x \in X$ the ***neighborhood filter at x*** is the filter

$$\mathcal{U}_x = \bigcap \Big\{ \mathcal{F} : \mathcal{F} \to x \Big\},$$

and the sets $U \in \mathcal{U}_x$ are called ***neighborhoods*** of x. U is said to be ***open*** if it is a neighborhood of each of its points.

For $A \subseteq X$ the ***adherence*** of A taken in X is the set

$$a_\lambda(A) := \Big\{ x \in X : \exists \mathcal{F} \xrightarrow{\lambda} x \text{ with } A \in \mathcal{F} \Big\}.$$

Again, if there is no chance for ambiguity, we will drop the reference to the convergence structure and only write $a(A)$. We think of adherence as the convergence space analogue of closure from topology. A subset F is called ***closed*** if $a(F) = F$ and ***dense*** if $a(F) = X$. A convergence space is said to be ***regular*** if $a(\mathcal{F}) \to x$ whenever $\mathcal{F} \to x$ where $a(\mathcal{F})$ is the filter generated by the sets $\{a(F) : F \in \mathcal{F}\}$.

Lemma 1.9. [BB02, Lemma 1.3.4] *Let X be a convergence space. A subset U is open if and only if $X \setminus U$ is closed.*

Given a convergence space (X, λ), the collection of all open subsets of X forms a topology that we denote by $\tau(\lambda)$, and we call the pair $(X, \tau(\lambda))$ the ***topological modification of*** λ. Clearly (X, λ) and $(X, \tau(\lambda))$ have the same open and closed sets. The most important result we need about $\tau(\lambda)$ is the following; see [BB02] Proposition 1.3.9.

Proposition 1.10. *Let (X, λ) be a convergence space. Then the identity map $id : (X, \lambda) \to (X, \tau(\lambda))$ is continuous. Moreover, $\tau(\lambda)$ is the finest topology on X with this property.*

The identity map $id : (X, \lambda) \to (X, \tau(\lambda))$ is not generally a homeomorphism. In fact, it is a homeomorphism precisely when λ is topological; that is, when $\mathcal{F} \xrightarrow{\lambda} x$ agrees with the natural convergence of a topology; see Example 1.7.

Given a family of convergence spaces and a family $\mathcal{A}$ of functions from a set X to those spaces, $\mathcal{A}$ induces a natural convergence structure on X. Put $\mathcal{F} \to x$ in X whenever $f(\mathcal{F}) \to f(x)$ for every $x \in X$ and every $f \in \mathcal{A}$; this is called the ***convergence induced by*** $\mathcal{A}$ or the ***initial convergence structure*** on X with respect to $\mathcal{A}$; it is the weakest convergence on X that makes all $f \in \mathcal{A}$ continuous. We will look at two important applications of this construction.

If X is any convergence space and $S \subseteq X$, one may use the construction in the previous paragraph to define a convergence structure on S induced by the inclusion map $\iota_S : S \to X$; this is called the ***subspace structure*** on S induced by X. We will use s_λ to denote convergence in the subspace structure. By construction, s_λ is the weakest convergence on S making ι_S continuous. For a filter $\mathcal{F}$ on S we have $\mathcal{F} \xrightarrow{s_\lambda} x$ if and only if $[\mathcal{F}]_X \xrightarrow{\lambda} x$ where $[\mathcal{F}]_X$ denotes the filter generated by $\mathcal{F}$ in X.

If $\{(X_i, \lambda_i) : i \in I\}$ is an indexed family of convergence spaces, the Cartesian product $X = \prod_{i \in I} X_i$ with the convergence induced by the family $\mathcal{A}$ of coordinate projections

$\pi_i : \prod_{i\in I} X_i \to X_i$ is called the ***product convergence structure*** on X. This convergence is denoted using $\prod_{i\in I} \lambda_i$; it is the weakest convergence on X such that each coordinate projection is continuous. For convenience, in cases where I is sufficiently small, say when $I = \{1, 2\}$, this convergence will be denoted by $\lambda_1 \times \lambda_2$.

Let $(V, +, \cdot)$ be a $\mathbb{F}$-vector space where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. A ***convergence vector space*** is a pair (V, λ) where λ is a convergence structure on V and the vector space operations are continuous. To be clear, we are dealing with *joint* continuity of the vector space operations; that is, $V \times V$ carries $\lambda \times \lambda$ convergence and $\mathbb{F} \times V$ carries $\tau \times \lambda$ convergence where τ denotes the standard topological convergence of filters on $\mathbb{F}$.

We are now in a position to define compact and bounded sets in convergence spaces. The remaining details in this section may be skipped on first reading; we will return to these concepts for some results in later chapters.

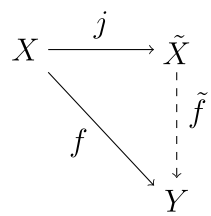
A convergence space X is called ***compact*** if every ultrafilter on X converges. A *subset* A is called compact if it is compact in the subspace structure induced by X. If every convergent filter on X contains a compact set then X is said to be ***locally compact***. It should be noted that even though these definitions extend the corresponding definitions from topology, many powerful theorems from topology may fail to hold in convergence spaces.

For the remainder of this section we let X denote a convergence vector space. A subset $B \subset X$ is said to be ***bounded*** if $\mathcal{N}_0 B \to 0$ where $\mathcal{N}_0$ denotes the neighborhood filter of 0 in the underlying scalar field, $\mathbb{R}$ or $\mathbb{C}$, and where $\mathcal{N}_0 B = [\mathcal{B}]$ for the filter base $\mathcal{B} = \{UB : U \in \mathcal{N}_0\}$. X is said to be ***locally bounded*** if every convergent filter contains a bounded set. Locally bounded spaces are useful for linking compactness and boundedness.

Proposition 1.11 ([BB02] Proposition 3.7.10)**.** *In a locally bounded convergence vector space every compact subset is bounded.*

In the theory of convergence vector spaces, a filter $\mathcal{F}$ is called ***Cauchy*** if $\mathcal{F} - \mathcal{F} \to 0$ where $\mathcal{F} - \mathcal{F}$ is the filter generated by the filter base $\mathcal{B} = \{A - B : A, B \in \mathcal{F}\}$. X is called ***complete*** if every Cauchy filter on X converges. The following result on completions of convergence vector spaces is due to [GGK76].

Theorem 1.12. *For any Hausdorff convergence vector space X there are a complete Hausdorff convergence vector space $\tilde{X}$ and a continuous linear function $j : X \to \tilde{X}$ such that $j(X)$ is dense in $\tilde{X}$ with the following universal property: For every continuous linear function f of X into a complete Hausdorff convergence vector space Y there is a unique continuous linear function $\tilde{f} : \tilde{X} \to Y$ such that the following diagram commutes*



Furthermore, $\tilde{X}$ is uniquely determined up to linear homeomorphism. If X is locally bounded then j is a linear homeomorphism onto $j(X)$.

In Chapter 2 we will introduce another definition of convergence defined in terms of nets. In order to distinguish between these two theories of convergence, we will always modify the term ‘convergence structure’ appropriately. For example, the notion of convergence structures given above, which is expressed using the language of filters, will be referred to as ***filter convergence structures***. In Chapter 2 we introduce a theory of *net* convergence structures.

2. A Theory of Net Convergence Structures

In analysis, one often deals with convergences that are defined in terms of nets or sequences and it is natural to ask if filter convergence structures can be applied to study their properties. While filter convergence is equivalent to net convergence in topological spaces, it is not known whether this equivalence extends to the non-topological setting. This issue is further complicated by the fact that there is no consensus on how to define abstract convergence for nets; see, for example, [Kel55, p. 73], [Kat67, AA72, Ars77, Pea88], [Sch97, pp. 168-170], [HZW10] and most recently [AEG21]. In this chapter we introduce yet another definition of convergence for nets that we hope will resolve these longstanding issues. First and foremost, our theory is equivalent to the theory of filter convergence structures presented in [BB02]. As a consequence, all results from the theory of filter convergence structures remain applicable to nets. Our definition also handles several set-theoretic subtleties.

2.1. **Definition of a Net.**

There is some inconsistency in the literature about the exact definition of a net. While some authors require their index sets to be partially ordered, others allow for more general pre-ordered index sets. We will only require our nets to be indexed by a pre-ordered directed set. For those accustomed to partially ordered index sets, note that every pre-ordered directed set $(A, \preceq)$ can be made into a partially ordered directed set by defining $a \sim b \iff a \preceq b$ and $b \preceq a$ and considering the quotient $A/\sim$ with the partial order $[a] \leq [b] \iff a \preceq b$.

Throughout this chapter X will denote an arbitrary set. Recall from the previous chapter that a function from a directed set A to X is referred to as a ***net*** in X indexed by A. In this case, instead of writing $x\colon A \to X$ we write $(x_\alpha)_{\alpha\in A}$ or just (x_α) if there is no reason to highlight the index set. A ***sequence*** is simply a net indexed by $\mathbb{N}$ with the standard order.

Let $(x_\alpha)_{\alpha\in A}$ be a net in X. We write $\{x_\alpha\}_{\alpha\in A}$ for the set of all terms of the net, $\{x_\alpha : \alpha \in A\}$, which is just the range of x when viewed as a function. If $\alpha_0 \in A$ is fixed and we put $A_0 = \{\alpha \in A : \alpha \geq \alpha_0\}$ then A_0 is again a directed set under the pre-order induced from A. The restriction of the function x to A_0 is called a ***tail*** of (x_α), and it is denoted by $(x_\alpha)_{\alpha\geq\alpha_0}$. The range of this new net will be denoted by $\{x_\alpha\}_{\alpha\geq\alpha_0}$; this is the ***tail set*** of $(x_\alpha)_{\alpha\in A}$ introduced in Example 1.5.

2.2. **Abstract Convergence for Nets.** One barrier to a well-defined notion of convergence for nets is the absence of a set of all nets in X. By the Well-Ordering Principle, every set can be totally ordered and, therefore, can be viewed as a directed set. It follows that any set Y can be viewed as the index set for a net $x : Y \to X$. Thus, there are at least as many nets in X as there are sets and, since there is no set of all sets, there cannot be a set of all nets in X. This illustrates the main issue with nets: the choice of index sets for an arbitrary net in X is too large. We will limit our choice of index sets to a collection that is still a set.

The aim of the following discussion is to motivate our restrictions on index sets. Recall that a net $(x_\alpha)_{\alpha\in A}$ in a topological space (X, τ) is said to converge to x if for every $U \in \tau$ with $x \in U$ we have some $\alpha_0 \in A$ such that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$. This definition of convergence is intuitive and very reminiscent of sequential convergence in metric spaces; however, unlike sequences in general topological spaces, nets are able to detect important topological properties because the more general index set allows them to "see" the whole topology — $\mathbb{N}$ is generally too poor a guide to describe all the neighborhoods.

Example 1.7 shows us how convergent filters strike a middle ground between nets and sequences in topological spaces: their guide sets, which are the neighborhood system at a point, are precisely the right tool for describing all the neighborhoods of a topology. The intuition we are after here is that the guide set for convergent filters comes directly from the space itself and does not depend on an auxiliary set. Considering that nets and filters are equivalent for describing convergence in topological spaces, we should be able to describe net convergence by restricting our focus to nets whose index sets can be constructed from the ambient space. The following simple example illustrates this idea.

Example 2.1. Let (X, τ) be a topological space. X is Hausdorff if and only if convergent nets have unique limits.
$(\implies)$: The forward implication does not illustrate our main point and is left as an easy exercise for the reader.

$(\impliedby)$: We prove the contrapositive. Suppose X is not Hausdorff. Then there are distinct points $x, y \in X$ such that for every $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$, $U \cap V \neq \varnothing$. Consider $\mathcal{N}_x$ with the partial order $U \leq V \iff V \subseteq U$. Then $\mathcal{N}_x \times \mathcal{N}_y$ with the induced coordinate-wise order becomes a directed set. For each $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$ we know that $U \cap V \neq \varnothing$, so put $z_{(U,V)} = z$ where $z \in U \cap V$. This defines a net in X and one can show that $z_{(U,V)} \to x$ and $z_{(U,V)} \to y$. So, if X is not Hausdorff, one can find a net that converges to two distinct points. This completes the proof.

The previous example uses an index set that was constructed as product of subsets of the ambient set; specifically, the index set is an element of $\mathcal{P}\big(\mathcal{P}(X) \times \mathcal{P}(X)\big)$. This illustrates a common theme when working with nets: we build our index sets using set theoretic constructions applied to the ambient set. Thus, given a set X, we would like to guarantee that, at the very least, our index sets for nets in X include many of the standard set theoretic constructions arising from X; *e.g.,* subsets of X, products of X, power set of X, products of subsets of X, subsets of products of X, *etc.* In order to make sure we keep a large variety of potential index sets available, we use the following construction from nonstandard analysis.

Definition 2.2. We will only be working with infinite sets so we may assume $\mathbb{N} \hookrightarrow X$. Write $\mathcal{P}(X)$ for the power set of X and define

$$\begin{aligned} V_0(X) &= X; \\ V_n(X) &= V_{n-1}(X) \cup \mathcal{P}\big(V_{n-1}(X)\big), \text{ and} \\ V(X) &= \bigcup_{n=1}^{\infty} V_n(X). \end{aligned}$$

$V(X)$ is called the *superstructure* over X in [LW15, p. 37]. Note that x and $\{x\}$ are considered to be distinct. A net $(x_\alpha)_{\alpha\in A}$ in X is called ***admissible*** if $A \in V(X)$. Unlike all nets in X, the admissible nets in X form a set that we denote by $\mathfrak{N}(X)$.

Another requirement for our convergence theory is to be able to capture the tail behaviour of nets. In practice, convergence is often thought of as a "tail property", which means that altering terms at the "head" of a convergent net does not affect the convergence — so convergence should be determined by tail sets of a net. This idea is embedded into the following definition. Following [Kat67], given two nets $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ we say that $(y_\beta)_{\beta\in B}$ is a ***quasi-subnet*** of $(x_\alpha)_{\alpha\in A}$, and write $(y_\beta)_{\beta\in B} \preceq (x_\alpha)_{\alpha\in A}$, if for every $\alpha_0 \in A$ there exists $\beta_0 \in B$ such that $\{y_\beta\}_{\beta\geq\beta_0} \subseteq \{x_\alpha\}_{\alpha\geq\alpha_0}$; this just means every tail of $(x_\alpha)_{\alpha\in A}$ contains a tail of $(y_\beta)_{\beta\in B}$ as a subset. Recall that $[x_\alpha]$ means the tail filter of the net (x_α); for reference, see Example 1.5. It follows that

$$(y_\beta) \preceq (x_\alpha) \iff [x_\alpha] \subseteq [y_\beta].$$

If both $(y_\beta)_{\beta\in B} \preceq (x_\alpha)_{\alpha\in A}$ and $(x_\alpha)_{\alpha\in A} \preceq (y_\beta)_{\beta\in B}$ hold then we say $(x_\alpha)_{\alpha\in A}$ and $(y_\beta)_{\beta\in B}$ are ***tail equivalent*** and write

$$(x_\alpha)_{\alpha\in A} \sim (y_\beta)_{\beta\in B}.$$

Equivalently,

$$(x_\alpha) \sim (y_\beta) \iff [x_\alpha] = [y_\beta].$$

It is easy to see that $\preceq$ is a pre-order and $\sim$ is an equivalence relation on the set of admissible nets, $\mathfrak{N}(X)$.

Now consider a net $(x_\alpha)_{\alpha \in A}$ in X. Recall from Example 1.5 that the family of tail sets

$$\mathcal{T} = \Big\{ \{x_\alpha : \alpha \geq \alpha_0\} : \alpha_0 \in A \Big\}$$

is a filter base on X. Consequently, each net in X gives rise to a filter, $[x_\alpha]$. The converse is also true: every filter is the tail filter of a net in X; see, for example, [AB06] section 2.6. The proof is presented here for the convenience of the reader and we make an additional observation: the net is also admissible.

Proposition 2.3. *Let $\mathcal{F}$ be a filter on a non-empty set X. There exists an admissible net $(x_\lambda)_{\lambda \in \Lambda}$ such that $[x_\lambda] = \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filter on X. Consider the following subset of $X \times \mathcal{F}$:

$$\Lambda = \Big\{ (y, A) : A \in \mathcal{F} \text{ and } y \in A \Big\}$$

and note that $\Lambda \subset X \times \mathcal{P}(X)$, so $\Lambda \in \mathcal{P}(X \times \mathcal{P}(X))$. Using the properties of filters one can show the relation

$$(y, A) \leq (z, B) \iff B \subseteq A$$

is a pre-order and $(\Lambda, \leq)$ is a directed set. For each $\lambda \in \Lambda$ define $x_\lambda = x_{(y,A)} = y$. Then (x_λ) is an admissible net and the remaining claim is $[x_\lambda] = \mathcal{F}$.

If $U \in [x_\lambda]$ then there is some $\lambda_0 = (x_0, F_0) \in \Lambda$ such that $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq U$; note that $F_0 \in \mathcal{F}$. We will show $U \in \mathcal{F}$ by showing $F_0 = \{x_\lambda : \lambda \geq \lambda_0\}$. To this end, fix any $\lambda = (y, F) \geq \lambda_0$. It follows that $x_\lambda = x_{(y,F)} = y \in F$, and from $F \subseteq F_0$ we obtain

$x_\lambda = y \in F_0$ regardless of the choice of element from F. This implies $x_\lambda \in F_0$ for any $\lambda \geq \lambda_0$; that is, $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq F_0$. Now let $y \in F_0$ be arbitrary and put $\lambda' = (y, F_0)$. Clearly $\lambda' \geq \lambda_0$ and $y = x_{(y,F_0)} = x_{\lambda'}$; hence $y \in \{x_\lambda : \lambda \geq \lambda_0\}$. We have now proved $F_0 = \{x_\lambda : \lambda \geq \lambda_0\}$ and, since the latter is contained in U, $U \in \mathcal{F}$.

Conversely, if $U \in \mathcal{F}$ then $U \neq \varnothing$ and we can find some $x_0 \in U$. Letting $\lambda_0 = (x_0, U)$ gives $\{x_\lambda : \lambda \geq \lambda_0\} \subseteq U$. □

The following result is now immediate.

Theorem 2.4. *Every net in X is tail equivalent to an admissible net.*

Proof. Let $(x_\alpha)_{\alpha \in A}$ be any net in X. Apply the previous result to the filter $[x_\alpha]$ to find an admissible net $(x_\lambda)_{\lambda \in \Lambda}$ such that $[x_\lambda] = [x_\alpha]$. This yields $(x_\alpha) \sim (y_\lambda)$. □

While there is no set of all nets, each net in X is tail equivalent to an admissible net. We will start by defining convergence on the *set* of all admissible nets in X. This definition can then be extended to all nets provided the convergence respects tail equivalence. This is the final ingredient needed to motivate the following definition.

Fix a set X. A ***net convergence structure*** on X is a function $\eta \colon X \to \mathcal{P}(\mathfrak{N}(X))$ satisfying certain axioms that will be given below. Instead of writing $(x_\alpha)_{\alpha \in A} \in \eta(x)$, we write $(x_\alpha)_{\alpha \in A} \xrightarrow{\eta} x$ and say that $(x_\alpha)_{\alpha \in A}$ η-converges to x. For convenience, and when there is no risk of confusion, we may de-emphasize η by writing $x_\alpha \to x$, and say (x_α) converges to x. In all these cases, x is called a ***limit*** of (x_α). Here are the axioms:

- (N1) Constant nets converge: if $x_\alpha = x$ for every α then $x_\alpha \to x$;
- (N2) If $(y_\beta) \preceq (x_\alpha)$ and $x_\alpha \to x$ then $y_\beta \to x$;
- (N3) Suppose that $(x_\alpha)_{\alpha \in A} \to x$ and $(y_\alpha)_{\alpha \in A} \to x$. Let $(z_\alpha)_{\alpha \in A}$ be a net in X such that $z_\alpha \in \{x_\alpha, y_\alpha\}$ for every α. Then $z_\alpha \to x$.

The pair (X, η) is a called a ***net convergence space***.

We call the net (z_α) in axiom (N3) a ***braiding*** of (x_α) with (y_α). It can be interpreted in the following way: Given nets $(x_\alpha)_{\alpha\in A}$ and $(y_\alpha)_{\alpha\in A}$, for any $B\in\mathcal{P}(A)$

$$z_\alpha = \begin{cases} x_\alpha & \alpha\in B \\ y_\alpha & \alpha\in A\setminus B. \end{cases}$$

If we want to highlight the rule for assigning the terms of (z_α) then the braiding will be denoted by $z_\alpha = x_\alpha \otimes_B y_\alpha$, which is called a B***-braiding of*** (x_α) ***with*** (y_α).

While axioms (N1) and (N2) are the same axioms introduced in [HZW10], note the inclusion of the third axiom (N3). We will see later that (N3) allows us to make a close connection between net and filter convergence structures.

Example 2.5. Consider the set $X=\{0,1,2\}$. For a net (x_α) in X write

$$\begin{aligned} x_\alpha \to 0 &\quad \text{if a tail of } (x_\alpha) \text{ is contained in } \{0,1\} \\ x_\alpha \to 1 &\quad \text{if a tail of } (x_\alpha) \text{ is contained in } \{1,2\} \\ x_\alpha \to 2 &\quad \text{if a tail of } (x_\alpha) \text{ is contained in } \{0,2\} \end{aligned}$$

This defines a net convergence structure on X with some counter-intuitive properties. For example $(0,0,0,0,...)$ converges to both 0 and 2, but not to 1, while $(1,1,1,1,...)$ converges to 0 and 1, but not to 2. Also $(0,1,0,1,0,...)$ converges to 0 but not to 1 or 2.

Example 2.6. For $X=\mathbb{R}$ define convergence of nets by

$$x_\alpha \xrightarrow{\eta} x \iff x\neq 0 \text{ and } x_\alpha \to x \text{ in the standard topology on } \mathbb{R}$$

$$x_\alpha \xrightarrow{\eta} 0 \iff \begin{cases} \bullet \quad x_\alpha \to 0 \text{ in the standard topology on } \mathbb{R} \text{ and} \\ \bullet \quad \text{there is a tail of } (x_\alpha) \text{ consists entirely of rational numbers.} \end{cases}$$

This defines a net convergence structure on $\mathbb{R}$.

The next example shows that axiom (N3) is independent of axioms (N1) and (N2) — so our definition of net convergence is more restrictive than the one given by [HZW10].

Example 2.7. For $X = \mathbb{R}$ define

$$x_\alpha \xrightarrow{\eta} x \iff \begin{cases} \bullet \quad x_\alpha \to x \text{ in the standard topology on } \mathbb{R} \\ \bullet \quad \text{a tail of } (x_\alpha) \text{ consists entirely of rational} \\ \quad \text{numbers or entirely of irrational numbers} \end{cases}$$

It is straightforward to verify this definition satisfies (N1) and (N2). However, taking $x_n = 1$ and $y_n = 1 - \frac{\sqrt{2}}{n}$ for all n, and braiding by taking B to be the set of all positive even integers gives

$$x_n \otimes_B y_n = (1 - \sqrt{2}, 1, 1 - \frac{\sqrt{2}}{3}, 1, 1 - \frac{\sqrt{2}}{5}, \dots).$$

Notice that every tail of $(x_n \otimes_B y_n)$ contains both rational and irrational numbers, so it cannot converge to 1 in this convergence.

There are several natural approaches to define net convergence structures in the literature, and many of them include an axiom related to convergence of *subnets*. However, there are several non-equivalent definitions of the term subnet so we take some time to distinguish them from quasi-subnets.

Let $(x_\alpha)_{\alpha \in A}$ be a net, B be a directed set, and $\varphi\colon B \to A$ such that $\operatorname{Range}\varphi$ is *co-final* in A, meaning for every $\alpha_0 \in A$ there is $\beta_0 \in B$ with $\varphi(\beta) \geq \alpha_0$ whenever $\beta \geq \beta_0$. The composition $x \circ \varphi\colon B \to X$ is a net in X indexed by B and is called a ***subnet*** of $(x_\alpha)_{\alpha \in A}$ according to Kelley in [Kel55]. Willard in [Wil70] requires the additional assumption that φ be monotone. Clearly every Willard-subnet is a Kelley-subnet.

Proposition 2.8. *Every Kelley-subnet is a quasi-subnet.*

Proof. Let $(x_\alpha)_{\alpha \in A}$ be a net in a set X and suppose that $(y_\beta)_{\beta \in B}$ is a Kelley-subnet of (x_α). Then there is a map $\phi : B \to A$ with $\phi(B)$ cofinal in A and $y_\beta = x_{\phi(\beta)}$ for each $\beta \in B$. We will show $(y_\beta) \preceq (x_\alpha)$ by showing $[x_\alpha] \subseteq [y_\beta]$. Let $U \in [x_\alpha]$. Then there is some α_0 such that $x_\alpha \in U$ for every $\alpha \geq \alpha_0$. Use the fact that $\phi(B)$ is cofinal in A to find $\beta_0 \in B$ such that $\phi(\beta) \geq \alpha_0$ whenever $\beta \geq \beta_0$. This implies

$$y_\beta = x_{\phi(\beta)} \in U \quad \text{for all } \beta \geq \beta_0$$

and shows that the tail $(y_\beta)_{\beta \geq \beta_0}$ is in U; hence $U \in [y_\beta]$. □

Since every Willard-subnet is automatically a Kelley-subnet, Proposition 2.8 has an analogue for Willard-subnets. The next result shows that our definition of net convergence encompasses the natural idea that subnets of convergent nets must converge to the same points.

Corollary 2.9. *Suppose X is a net convergence space. Then every Kelley-subnet of a convergent net is also convergent to the same limits.*

Proof. Let (y_β) be a Kelley-subnet of (x_α). By Proposition 2.8 $(y_\beta) \preceq (x_\alpha)$. Now $x_\alpha \to x$ implies $y_\beta \to x$ by axiom (N2). □

A similar result holds for Willard-subnets. The next example demonstrates that the converse to Proposition 2.8 is generally false.

Example 2.10. The following example can be found in [Kat67]. Let Ω denote an uncountable set and let Λ be the set of all finite subsets of Ω ordered by $A \leq B \iff A \subseteq B$. Then $(\Lambda, \subseteq)$ is a directed set. For $\lambda \in \Lambda$ put $x_\lambda = |\lambda|$, the cardinality of λ, and put $y_n = n$. Then (y_n) is a quasi-subnet of (x_λ) th at is not a Kelley-subnet.

Lemma 2.11. *Let X be a net convergence space. If (x_α) and (y_β) are tail equivalent in X then $x_\alpha \to x$ if and only if $y_\beta \to x$.*

Proof. Apply axiom (N2) and the fact that $(x_\alpha) \sim (y_\beta)$ means $(x_\alpha) \preceq (y_\beta)$ and $(y_\beta) \preceq (x_\alpha)$. □

Remark 2.12. In particular, any net in a net convergence space is tail equivalent to all of its tails. Applying this observation with Lemma 2.11 means passing to the tail of a net does not affect convergence — this is precisely the intuition we are after: forgetting the terms at the head of a convergent net does not spoil the convergence.

Note that quasi-subnets are used with axiom (N2) to ensure that tail equivalence respects convergence. The following result will be used to show how one can adapt the definition of net convergence structure to only deal with subnets, if desired.

Proposition 2.13. *Every quasi-subnet of a net is tail equivalent to a Willard-subnet.*

Proof. Let $(y_\beta)_{\beta\in B}$ be a quasi-subnet of $(x_\alpha)_{\alpha\in A}$. Put $C = \big\{(\alpha,\beta) \in A \times B \,:\, x_\alpha = y_\beta\big\}$. For $(\alpha,\beta) \in C$, put $\varphi(\alpha,\beta) = \alpha$. It is straightforward that C is directed under the product order induced from $A \times B$, and $x \circ \varphi$ is a subnet of $(x_\alpha)_{\alpha\in A}$ that is tail equivalent to $(y_\beta)_{\beta\in B}$. □

A similar result holds for Kelley-subnets.

It follows that axiom (N2) in the definition of net convergence structure can be replaced with both of the following axioms.

(N2a) If a net converges to x then so does each of its Kelley-subnets;

(N2b) If (x_α) converges to x and (y_β) is tail equivalent to (x_α) then (y_β) converges to x.

Again, one can recover an analogous result for Willard-subnets. The rest of this thesis will only be concerned with quasi-subnets since they are more general and much easier to work with.

To this point we have only considered admissible nets. Lemma 2.11 allows us to extend convergence to non-admissible nets. For an arbitrary net (x_α) in X we say $x_\alpha \to x$ if (x_α) is tail equivalent to an admissible net that converges to x. Combining this with Theorem 2.4 means that without loss of generality we may restrict our attention to admissible nets. In view of this, we will often identify an arbitrary net $(x_\alpha)_{\alpha \in A}$ in X with its equivalence class in $\mathfrak{N}(X)/\!\sim$ consisting of all the admissible nets that are tail equivalent to $(x_\alpha)_{\alpha \in A}$. Thus, we may quantify over "all nets in X" but what we really mean is "all nets in $\mathfrak{N}(X)$ up to tail equivalence". Going forward, the term net will implicitly mean *admissible* net unless stated otherwise.

The next result ensures any given pair of nets can be re-indexed over a common index set.

Lemma 2.14. *Let $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$ be two nets. There are quasi-subnets $(\tilde{x}_\gamma)_{\gamma \in \Gamma}$ and $(\tilde{y}_\gamma)_{\gamma \in \Gamma}$ of (x_α) and (y_β), respectively, that are defined over a common index set Γ.*

Proof. Given $(x_\alpha)_{\alpha \in A}$ and $(y_\beta)_{\beta \in B}$, take $\Gamma = A \times B$. The partial order $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$ if $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ turns $(\Gamma, \leq)$ into a directed set. For $\gamma = (\alpha, \beta) \in \Gamma$ define nets

$$\tilde{x}_\gamma = x_\alpha \text{ and } \tilde{y}_\gamma = y_\beta.$$

We claim $(\tilde{x}_\gamma) \preceq (x_\alpha)$; that is, $[x_\alpha] \subseteq [\tilde{x}_\gamma]$. If $U \in [x_\alpha]$ then $(x_\alpha)_{\alpha \geq \alpha_0} \in U$ for some α_0. Choose any $\beta_0 \in B$ and put $\gamma_0 = (\alpha_0, \beta_0)$. Then for every $\gamma \geq \gamma_0$ we have $\tilde{x}_\gamma = x_\alpha \in U$; hence $(\tilde{x}_\gamma)_{\gamma \geq \gamma_0} \in U$. This shows $U \in [\tilde{x}_\gamma]$. A similar argument can be used to confirm $(\tilde{y}_\gamma) \preceq (y_\beta)$. ◻

Corollary 2.15. *If $x_\alpha \to x$ and $y_\beta \to x$ then there are quasi-subnets $(\tilde{x}_\gamma)_{\gamma \in \Gamma}$ and $(\tilde{y}_\gamma)_{\gamma \in \Gamma}$ of (x_α) and (y_β), respectively, such that $\tilde{x}_\gamma \to x$ and $\tilde{y}_\gamma \to x$.*

Proof. Apply Lemma 2.14 and axiom (N2). ◻

We have not yet made use of the third axiom for net convergence structures. It becomes a necessary ingredient for building a close connection between net and filter convergence structures, as we now demonstrate.

Theorem 2.16. *Let* (X, η) *be a net convergence space. For a filter* $\mathcal{F}$ *on* X*, define* $\mathcal{F} \xrightarrow{\lambda_\eta} x$ *if there is a net* $x_\alpha \xrightarrow{\eta} x$ *with* $[x_\alpha] \subseteq \mathcal{F}$*. This defines a filter convergence structure on* X*, called the* ***associate filter convergence structure****, denoted by* λ_η*.*

Proof. (F1) Consider a constant net $x_\alpha = x$ that satisfies $x_\alpha \xrightarrow{\eta} x$. Then $[x_\alpha] = [x]$, so $[x] \xrightarrow{\lambda_\eta} x$.

(F2) If $\mathcal{F} \xrightarrow{\lambda_\eta} x$ and $\mathcal{F} \subseteq \mathcal{G}$ then there is a net $x_\alpha \xrightarrow{\eta} x$ such that $[x_\alpha] \subseteq \mathcal{F} \subseteq \mathcal{G}$. So $\mathcal{G} \xrightarrow{\lambda_\eta} x$.

(F3) Suppose $\mathcal{F} \xrightarrow{\lambda_\eta} x$ and $\mathcal{G} \xrightarrow{\lambda_\eta} x$. Then there are nets $x_\alpha \xrightarrow{\eta} x$ and $y_\beta \xrightarrow{\eta} x$ with $[x_\alpha] \subseteq \mathcal{F}$ and $[y_\beta] \subseteq \mathcal{G}$. By Corollary 2.15 we may assume the nets (x_α) and (y_β) are indexed over the same set, say A. Put $\Lambda = A \times \{1, 2\}$ and define $(\alpha_1, i) \leq (\alpha_2, j)$ if $\alpha_1 \leq \alpha_2$. This pre-order makes Λ into a directed set and we will use this to define the net $(z_\lambda)_{\lambda \in \Lambda}$ *via*

$$z_{(\alpha,1)} = x_\alpha \text{ and } z_{(\alpha,2)} = y_\alpha.$$

We begin by showing $(x_\alpha) \preceq (z_\lambda)$ and $(y_\alpha) \preceq (z_\lambda)$, which can be achieved by showing both $[z_\lambda] \subseteq [x_\alpha]$ and $[z_\lambda] \subseteq [y_\alpha]$. To this end, let $U \in [z_\lambda]$. Then $(z_\lambda)_{\lambda \geq \lambda_0} \in U$ for some $\lambda_0 \in \Lambda$. Write $\lambda_0 = (\alpha_0, i_0)$ for $\alpha_0 \in A$ and $i_0 \in \{1, 2\}$. Now $\lambda = (\alpha, i) \geq (\alpha_0, i_0) = \lambda_0$ implies $z_{(\alpha,i)} \in U$. Since the pre-order is independent of the term from $\{1, 2\}$, we have shown there is $\alpha_0 \in A$ such that $(z_{(\alpha,i)})_{\alpha \geq \alpha_0} \in U$ for each $i \in \{1, 2\}$. Taking $i = 1$ gives $(x_\alpha)_{\alpha \geq \alpha_0} \in U$ and $U \in [x_\alpha]$, and setting $i = 2$ gives $(y_\alpha)_{\alpha \geq \alpha_0} \in U$ and $U \in [y_\alpha]$.

It follows from the previous paragraph, and $[x_\alpha] \subseteq \mathcal{F}$ and $[y_\beta] \subseteq \mathcal{G}$, that $[z_\lambda] \subseteq \mathcal{F} \cap \mathcal{G}$. Moreover, the net $z_\lambda \xrightarrow{\eta} x$. To see this, for $\lambda \in \Lambda$ define

$$\tilde{x}_\lambda = x_\alpha \text{ and } \tilde{y}_\lambda = y_\alpha.$$

Then $\tilde{x}_\lambda \xrightarrow{\eta} x$ and $\tilde{y}_\lambda \xrightarrow{\eta} x$ by the proof of Lemma 2.14 and its corollary. Notice that $z_\lambda \in \{\tilde{x}_\lambda, \tilde{y}_\lambda\}$ for each $\lambda \in \Lambda$ — so it is a braiding of $(\tilde{x}_\lambda)$ and $(\tilde{y}_\lambda)$. It follows from the third axiom of net convergence structures that $z_\lambda \xrightarrow{\eta} x$.

All in all, we have shown that there is a net $z_\lambda \xrightarrow{\eta} x$ with $[z_\lambda] \subseteq \mathcal{F} \cap \mathcal{G}$; hence $\mathcal{F} \cap \mathcal{G} \xrightarrow{\lambda_\eta} x$. □

There is a natural analogue for Theorem 2.16 in the setting of filter convergence spaces. That is, to each filter convergence structure there is an associated convergence for nets.

Theorem 2.17. *Let (X, λ) be a filter convergence space. For a net (x_α) in X, define $x_\alpha \xrightarrow{\eta_\lambda} x$ if $[x_\alpha] \xrightarrow{\lambda} x$. This defines a net convergence structure on X, called the* ***associate net convergence structure****, denoted by η_λ.*

Proof. (N1) Consider a constant net $x_\alpha = x$ for every α. Then $[x_\alpha] = [x]$ and $[x] \xrightarrow{\lambda} x$ implies $x_\alpha \xrightarrow{\eta_\lambda} x$.

(N2) Assume $y_\beta \xrightarrow{\eta_\lambda} x$ and $(x_\alpha) \preceq (y_\beta)$. Then $[y_\beta] \xrightarrow{\lambda} x$. Now $(x_\alpha) \preceq (y_\beta)$ means $[y_\beta] \subseteq [x_\alpha]$, so it follows from the second axiom of filter convergence structures that $[x_\alpha] \xrightarrow{\lambda} x$. Therefore $x_\alpha \xrightarrow{\eta_\lambda} x$.

(N3) Suppose $(x_\alpha)_{\alpha \in A} \xrightarrow{\eta_\lambda} x$ and $(y_\alpha)_{\alpha \in A} \xrightarrow{\eta_\lambda} x$. For any $B \in \mathcal{P}(A)$ consider the B-braiding of x_α with y_α defined by

$$z_\alpha = x_\alpha \otimes_B y_\alpha = \begin{cases} x_\alpha & \alpha \in B \\ y_\alpha & \alpha \in A \setminus B. \end{cases}$$

As $[x_\alpha] \xrightarrow{\lambda} x$ and $[y_\alpha] \xrightarrow{\lambda} x$, we must have $[x_\alpha] \cap [y_\alpha] \xrightarrow{\lambda} x$. By the second axiom of filter convergence structures, we will be finished if we can show $[x_\alpha] \cap [y_\alpha] \subseteq [z_\alpha]$. To this end, let $U \in [x_\alpha] \cap [y_\alpha]$. Then there are α_1, α_2 such that

$$\{x_\alpha : \alpha \geq \alpha_1\} \subseteq U \text{ and } \{y_\alpha : \alpha \geq \alpha_2\} \subseteq U.$$

The index set A is directed, so we can find $\alpha_0 \geq \alpha_1, \alpha_2$. Then

$$\{x_\alpha : \alpha \geq \alpha_0\} \subseteq \{x_\alpha : \alpha \geq \alpha_1\} \subseteq U$$

and

$$\{y_\alpha : \alpha \geq \alpha_0\} \subseteq \{y_\alpha : \alpha \geq \alpha_1\} \subseteq U.$$

$z_\alpha \in \{x_\alpha, y_\alpha\}$ for each α gives $\{z_\alpha : \alpha \geq \alpha_0\} \subseteq U$, confirming that $U \in [z_\alpha]$. □

The theories of filter and net convergences are known to be equivalent in the context of topology. We can now extend this result beyond topology.

Theorem 2.18. *Let (X, η) be a net convergence space and (x_α) a net in X. Then $\eta = \eta_{\lambda_\eta}$.*

Proof. Suppose $x_\alpha \xrightarrow{\eta} x$. Taking $\mathcal{F} = [x_\alpha]$ in the definition of the associate filter convergence structure yields $[x_\alpha] \xrightarrow{\lambda_\eta} x$. This means $x_\alpha \xrightarrow{\eta_{\lambda_\eta}} x$.

Conversely, $x_\alpha \xrightarrow{\eta_{\lambda_\eta}} x$ means $[x_\alpha] \xrightarrow{\lambda_\eta} x$. Then there is a net $y_\beta \xrightarrow{\eta} x$ with $[y_\beta] \subseteq [x_\alpha]$; that is, $(x_\alpha) \preceq (y_\beta)$. Since $y_\beta \xrightarrow{\eta} x$, the second axiom of net convergence structures implies $x_\alpha \xrightarrow{\eta} x$. □

Theorem 2.19. *Let (X, λ) be a filter convergence space and $\mathcal{F}$ a filter on X. Then $\lambda = \lambda_{\eta_\lambda}$.*

Proof. First assume $\mathcal{F} \xrightarrow{\lambda_{\eta_\lambda}} x$. Then there is a net $x_\alpha \xrightarrow{\eta_\lambda} x$ with $[x_\alpha] \subseteq \mathcal{F}$. But $x_\alpha \xrightarrow{\eta_\lambda} x$ if and only if $[x_\alpha] \xrightarrow{\lambda} x$. It now follows from the second axiom of filter convergence structures that $\mathcal{F} \xrightarrow{\lambda} x$.

For the converse, suppose $\mathcal{F} \xrightarrow{\lambda} x$. By Proposition 2.3 there is a net (x_α) in X with $[x_\alpha] = \mathcal{F}$. It follows that $[x_\alpha] \xrightarrow{\lambda} x$; hence $x_\alpha \xrightarrow{\eta_\lambda} x$. As $[x_\alpha] = \mathcal{F}$, this shows $\mathcal{F} \xrightarrow{\lambda_{\eta_\lambda}} x$. $\square$

Corollary 2.20. *There is a one-to-one correspondence between net and filter convergence structures.*

Proof. Given a net convergence structure η, consider the associate filter convergence structure λ_η. By Theorem 2.18 η convergence agrees with the associate net convergence of λ_η.

Similarly, for a filter convergence structure λ, consider the associate net convergence structure η_λ. By Theorem 2.19 λ convergence agrees with the associate filter convergence of η_λ. $\square$

Remark 2.21. Corollary 2.20 can be strengthened. In the next section we will introduce the concept of continuous functions between net convergence spaces. These functions satisfy several familiar properties; for example, the composition of continuous functions is continuous. This allows one to define the category of net convergence spaces. The remainder of this remark is devoted to showing that this category is equivalent to, in a very strong sense, the category of filter convergence spaces. Here are the details.

Let $\mathscr{N}$ denote the category whose objects are net convergence spaces with morhphisms given by (net) continuous functions, and let $\mathscr{C}$ denote the category whose objects are filter convergence spaces with morphisms given by (filter) continuous functions. Define the maps

$$F_\lambda : \mathscr{N} \to \mathscr{C} \text{ by } F_\lambda\Big((X, \eta)\Big) := (X, \lambda_\eta) \text{ and}$$

$$F_\eta : \mathscr{C} \to \mathscr{N} \text{ by } F_\eta\Big((X, \lambda)\Big) := (X, \eta_\lambda).$$

It will be shown in Theorem 2.30 that associate convergence spaces share the exact same continuous functions; so $\mathrm{Hom}_{\mathscr{N}} = \mathrm{Hom}_{\mathscr{C}}$. Therefore, if we put $F_\lambda(f) = f$ for all $f \in \mathrm{Hom}_{\mathscr{N}}$

and $F_\eta(f) = f$ for all $f \in \mathrm{Hom}_{\mathscr{C}}$, we immediately see that F_λ and F_η are functors. By Theorem 2.18 we have

$$F_\eta\big(F_\lambda\big((X,\eta)\big)\big) = F_\eta\big((X,\lambda_\eta)\big) = (X,\eta_{\lambda_\eta}) = (X,\eta)$$

for each $(X,\eta) \in \mathrm{ob}(\mathscr{N})$ — so $F_\eta \circ F_\lambda = \mathrm{id}_{\mathscr{N}}$. Similarly, $F_\lambda \circ F_\eta = \mathrm{id}_{\mathscr{C}}$ by Theorem 2.19. This implies F_η and F_λ provide an isomorphism of categories and, in particular, $\mathscr{N}$ and $\mathscr{C}$ are equivalent categories.

In light of the observations in this section, the theory of filter convergence structures can be described using the language of nets. This means that results about filter convergence can be translated into ones about net convergence, and vice versa. The upshot to this approach is that nets are more intuitive for certain concepts in functional analysis. Moreover, many interesting convergences are defined using nets and the results in this chapter allow us to apply convergence theory to study them. In the next section we will develop several basic properties of net convergence structures. Also, in light of the equivalence discussed in Remark 2.21, if we write *convergence structure* or *convergence space* then the reader may assume the result applies to both filter and net convergence structures.

2.3. **Basic Properties of Net Convergence Spaces.** The concepts introduced in this section have analogues for filter convergence structures and are, therefore, generalizations of topology. We will focus on translating topics from filter convergence theory that will be useful in our applications with nets.

It is natural to begin by discussing *the* limit of a convergent net. A net convergence structure will be called ***Hausdorff*** if limits of convergent nets are unique. Hausdorff net convergence structures are easy to come by; take, for example, convergence of nets in any

Hausdorff topological space. Note that Example 2.5 is a non-Hausdorff net convergence structure.

Proposition 2.22. *For a pair of associate convergence spaces, one is Hausdorff if and only if the other is.*

Proof. First we claim that if (X,η) is a Hausdorff net convergence space then (X,λ_η) is a Hausdorff filter convergence space. To see this, suppose we have $\mathcal{F}\xrightarrow{\lambda_\eta}x$ and $\mathcal{F}\xrightarrow{\lambda_\eta}y$. By definition there is a net (x_α) in X with $x_\alpha\xrightarrow{\eta}x$ and $x_\alpha\xrightarrow{\eta}y$; hence $x=y$.

Secondly, if (X,λ) is a Hausdorff filter convergence space then (X,λ_η) is a Hausdorff net convergence space. Indeed, if $x_\alpha\xrightarrow{\eta_\lambda}x$ and $x_\alpha\xrightarrow{\eta_\lambda}y$ then $[x_\alpha]\xrightarrow{\lambda}x$ and $[x_\alpha]\xrightarrow{\lambda}y$; hence $x=y$.

If (X,η) is Hausdorff then the first claim yields (X,λ_η) is Hausdorff. For the reverse implication, suppose (X,λ_η) is Hausdorff. By the second claim and Theorem 2.18 we must have $(X,\eta_{\lambda_\eta})=(X,\eta)$. □

For a net convergence space (X,η) and $U\subseteq X$ the ***adherence*** or ***closure*** of U in X with respect to η is defined by

$$\overline{U}^{\eta}:=\{x\in X:\ \text{there is a net } (x_\alpha) \text{ in } U \text{ with } x_\alpha\xrightarrow{\eta}x \text{ in } X\}.$$

We say that U is ***closed*** in X if $U=\overline{U}^{\eta}$ and U is ***dense*** in X if $\overline{U}^{\eta}=X$. There are several results about this "closure" operation that resemble its topological counterpart. For examples, the empty set is closed, $U\subseteq\overline{U}^{\eta}$, and $A\subseteq B$ implies $\overline{A}^{\eta}\subseteq\overline{B}^{\eta}$. However, the closure of a set need not be closed.

Example 2.23. Consider the net convergence space $(\mathbb{R},\eta)$ defined in Example 2.6. Set $U=\mathbb{R}\setminus\mathbb{Q}$ and note that $\overline{U}^{\eta}=\mathbb{R}\setminus\{0\}$ but $\overline{\overline{U}^{\eta}}^{\eta}=\mathbb{R}$.

Given this example, the use of the term closure may be misleading in the general context of net convergence structures and it is best practice to use a different term, like adherence, to avoid potential confusion. Still, this notion of closure agrees with the one that is commonly used in vector lattice theory (c.f. [GL18]), and it agrees with the topological closure when the convergence comes from a topology. The next result gives more motivation for using the term adherence instead of closure in general net convergence spaces.

Lemma 2.24. *Given a pair of associate convergence spaces, the adherence of a subset with respect to one structure agrees with the adherence of that subset with respect to the other structure.*

Proof. Let $x \in \overline{U}^{\eta}$. Then there is a net $(x_\alpha) \in U$ such that $x_\alpha \xrightarrow{\eta} x$. This is equivalent, by Theorem 2.18, to $x_\alpha \xrightarrow{\eta_{\lambda_\eta}} x$ hence $[x_\alpha] \xrightarrow{\lambda_\eta} x$. Now $U \in [x_\alpha]$ implies $x \in a_{\lambda_\eta}(U)$. Conversely, $x \in a_{\lambda_\eta}(U)$ yields a filter $\mathcal{F} \xrightarrow{\lambda_\eta} x$ with $U \in \mathcal{F}$. Then there is a net $(x_\alpha)_{\alpha \in A}$ in X such that $x_\alpha \xrightarrow{\eta} x$ and $[x_\alpha] \subseteq \mathcal{F}$. It follows that for each α_0

$$\{x_\alpha : \alpha \geq \alpha_0\} \cap U \neq \varnothing.$$

That is, for each α_0 there is some $\beta \in A$ with $\beta \geq \alpha_0$ and $x_\beta \in U$.

Set $B = \{\alpha \in A : x_\alpha \in U\}$. We claim that B is directed by the order induced by A. Indeed, given $\alpha_1, \alpha_2 \in B$ there is some $\alpha_0 \in A$ with $\alpha_0 \geq \alpha_1, \alpha_2$. For this choice of α_0, $\{x_\alpha : \alpha \geq \alpha_0\} \cap U \neq \varnothing$, so there is some $\beta_0 \geq \alpha_0 \geq \alpha_1, \alpha_2$ such that $x_{\beta_0} \in U$. This shows there is $\beta_0 \in B$ with $\beta_0 \geq \alpha_1, \alpha_2$, so B is directed. Now consider $(x_\beta)_{\beta \in B}$. Clearly $(x_\beta) \preceq (x_\alpha)$, so $x_\alpha \xrightarrow{\eta} x$ implies $x_\beta \xrightarrow{\eta} x$. Since $(x_\beta) \in U$, we have $x \in \overline{U}^{\eta}$.

Moreover, if (X, λ) is a filter convergence space then $\overline{U}^{\eta_\lambda} = a_{\lambda_{\eta_\lambda}}(U) = a_\lambda(U)$, where the last equality follows from Theorem 2.19. ∎

Corollary 2.25. *Associate convergence spaces have the same closed sets.*

Proof. Closed sets in a convergence space are defined in terms of the adherence operation. The result is now an immediate application of the previous lemma. □

A ***neighborhood of*** x is a subset U such that for each net $x_\alpha \xrightarrow{\eta} x$ there exists α_0 such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$; that is, each net that converges to x has a tail in U. A set U is called ***open*** if it is a neighborhood of each of its points. The following result reveals a close connection between open and closed sets of net convergence spaces.

Proposition 2.26. *Let (X, η) be a net convergence space and $U \subseteq X$. Then U is open if and only if $X \setminus U$ is closed.*

Proof. Suppose $U \subsetneq X$ is open. Let (x_α) be a net in $X \setminus U$ such that $x_\alpha \xrightarrow{\eta} x$. We will show that $x \in X \setminus U$. Indeed, if $x \notin X \setminus U$ then $x \in U$. As U is open, it is a neighborhood of each of its points. In particular, U is a neighborhood of x. Then there exists α_0 such that $x_\alpha \in U$ whenever $\alpha \geq \alpha_0$, a contradiction.

Conversely, suppose that $X \setminus U$ is closed and assume for the sake of contradiction that U is not open. Then there is some $x_0 \in U$ such that U is not a neighborhood of x_0. Thus, there is some net $(x_\alpha)_{\alpha \in A}$ in X with $x_\alpha \xrightarrow{\eta} x_0$, yet no tail of x_α lies entirely inside of U. That is, for each $\alpha_0 \in A$ one can find some $\beta \in A$ with $\beta \geq \alpha_0$ and $x_\beta \in X \setminus U$. Set $B = \{\alpha \in A : x_\alpha \in X \setminus U\}$. By a similar argument used in the proof of Lemma 2.24, B is directed. Now $(x_\beta)_{\beta \in B}$ is a net in $X \setminus U$ and it is easy to verify $(x_\beta) \preceq (x_\alpha)$. It follows from $x_\alpha \xrightarrow{\eta} x_0$ that $x_\beta \xrightarrow{\eta} x_0$. As $X \setminus U$ was assumed to be closed, we must have $x_0 \in X \setminus U$, a contradiction. □

Corollary 2.27. *Associate convergence spaces have the same open sets.*

Proof. This is immediate from Lemma 1.9, Corollary 2.25 and Proposition 2.26. □

Corollary 2.28. *The collection of all open subsets of a net convergence space forms a topology. Moreover, it is the finest topology whose convergence is weaker than the net convergence.*

Proof. While this result can be proved directly, we can also use the established theory to avoid the extra computations. Apply the previous result to the associate filter convergence space and recall from the paragraph before Proposition 1.10 that the collection of all open subsets of a filter convergence space is a topology on X; it is called the topological modification of the structure. The moreover part follows from Proposition 1.10. $\square$

Since we have shown that the open subsets of a net convergence space (X, η) form a topology (call it $\tau(\eta)$) that agrees with the topological modification of λ_η, we call $\tau(\eta)$ the ***topological modification of*** η. We can now use topology to study certain concepts in net convergence spaces. The next result demonstrates the ease of working with net convergences.

Corollary 2.29. *Every finite subset in a Hausdorff net convergence space is closed.*

Proof. Clearly $\{x\} \subseteq \overline{\{x\}}^\eta$. If $y \in \overline{\{x\}}^\eta$ then there is a net (x_α) in $\{x\}$ that converges to y. There is no choice but to have $x_\alpha = x$ for every α. Now $x_\alpha \to x$ and $x_\alpha \to y$ gives $x = y$, so $y \in \{x\}$. This shows that any singleton in a Hausdorff space is closed. In particular, any finite union of singletons is closed by Corollary 2.28. $\square$

A function $f\colon X \to Y$ between two net convergence spaces is said to be ***continuous at*** x if $x_\alpha \to x$ in X implies $f(x_\alpha) \to f(x)$ in Y for every admissible net (x_α) in X. Note that the net $\big(f(x_\alpha)\big)$ need not be admissible in Y; this is not really an issue since it suffices that is it tail equivalent to an admissible net in Y that converges to $f(x)$. We say that f is ***continuous*** if it is continuous at every $x \in X$. It is straightforward to verify that the composition of two continuous functions is again a continuous function. We use the term ***homeomorphism*** to mean an invertible continuous mapping between two net convergence spaces whose inverse is also continuous.

Theorem 2.30. *Associate convergence spaces have the same continuous functions.*

Proof. We first claim that if (X, η_1), (Y, η_2) are two net convergence spaces and $f : (X, \eta_1) \to (Y, \eta_2)$ is continuous then $f : (X, \lambda_{\eta_1}) \to (Y, \lambda_{\eta_2})$ is continuous. To see this, we begin with a filter $\mathcal{F} \xrightarrow{\lambda_{\eta_1}} x$ and find a net (x_α) in X with $x_\alpha \xrightarrow{\eta_1} x$ and $[x_\alpha] \subseteq \mathcal{F}$. Our goal is to show $f(\mathcal{F}) \xrightarrow{\lambda_{\eta_2}} f(x)$. Since the continuity of f implies $f(x_\alpha) \xrightarrow{\eta_2} f(x)$, we will be finished if we can show $[f(x_\alpha)] \subseteq f(\mathcal{F})$. To this end, let α_0 be arbitrary and consider the tail sets

$$F_{\alpha_0} = \{f(x_\alpha) : \alpha \geq \alpha_0\} \text{ and } T_{\alpha_0} = \{x_\alpha : \alpha \geq \alpha_0\}.$$

Clearly $F_{\alpha_0} = f(T_{\alpha_0}) \in f(\mathcal{F})$ because $T_{\alpha_0} \in \mathcal{F}$. As α_0 was arbitrary, this shows $f(\mathcal{F})$ contains each base set of $[f(x_\alpha)]$ and, therefore, $[f(x_\alpha)] \subseteq f(\mathcal{F})$.

Our next claim is that if (X, λ_1) and (Y, λ_2) are two filter convergence spaces and $f : (X, \lambda_1) \to (Y, \lambda_2)$ is continuous then $f : (X, \eta_{\lambda_1}) \to (Y, \eta_{\lambda_2})$ is also continuous. To see this, consider a net $x_\alpha \xrightarrow{\eta_{\lambda_1}} x$. Then $[x_\alpha] \xrightarrow{\lambda_1} x$ and $f([x_\alpha]) \xrightarrow{\lambda_2} f(x)$. A simple computation shows $f([x_\alpha]) = [f(x_\alpha)]$ and we conclude $f(x_\alpha) \xrightarrow{\eta_{\lambda_2}} f(x)$.

Finally, given net convergence spaces (X, η_1) and (Y, η_2) and a continuous function $f : (X, \eta_1) \to (Y, \eta_2)$ the first claim yields $f : (X, \lambda_{\eta_1}) \to (Y, \lambda_{\eta_2})$ is continuous. Conversely, if η_1 and η_2 are two net convergence structures on X and Y, respectively, such that $f : (X, \lambda_{\eta_1}) \to (Y, \lambda_{\eta_2})$ is continuous then $f : (X, \eta_{\lambda_{\eta_1}}) \to (Y, \eta_{\lambda_{\eta_2}})$ is continuous by the claim in the second paragraph. Continuity of $f : (X, \eta_1) \to (Y, \eta_2)$ follows from Theorem 2.18. The remainder of the proof can be deduced using the above claims and Theorem 2.19. □

If η_1 and η_2 are two net convergence structures on a set X then we say that η_1 is ***stronger*** than η_2 (or η_2 is ***weaker*** than η_1) if the identity map $id : (X, \eta_1) \to (X, \eta_2)$ is continuous; that is, if $x_\alpha \xrightarrow{\eta_1} x$ implies $x_\alpha \xrightarrow{\eta_2} x$ for each net $(x_\alpha) \in X$. Two net convergence structures are said to be ***comparable*** if either convergence is stronger than the other. A net convergence structure is said to be ***topological*** if it is equal to the net convergence of some topology. Our next result shows that any net convergence structure is comparable with the net convergence of a topology.

Corollary 2.31. *If* (X, η) *is a net convergence space then the identity map*

$$id_X : (X, \eta) \to (X, \tau(\eta))$$

is continuous.

Proof. Note that $id_X : (X, \lambda_\eta) \to (X, \tau(\lambda_\eta))$ is continuous by Proposition 1.10, where $\tau(\lambda_\eta)$ denotes the topological modification of λ_η. Corollary 2.25 and Proposition 2.26 show $\tau(\lambda_\eta) = \tau(\eta)$, so Theorem 2.30 gives $id_X : (X, \eta) \to (X, \tau(\eta))$ is continuous. □

Corollary 2.32. *A net convergence structure* η *is topological if and only if*

$$id_X : (X, \eta) \to (X, \tau(\eta))$$

is a homeomorphism.

Proof. If id_X is a homeomorphism then $\tau(\eta)$-convergence is stronger than η-convergence; hence they agree by Corollary 2.31. This shows η is a topological net convergence.

For the converse, it suffices to show id_X^{-1} is continuous when η is a topological net convergence. Let σ be a topology whose net convergence agrees with η convergence. Then all the σ-open subsets are η-open and hence $\tau(\eta)$-open by the definition of $\tau(\eta)$. So any net satisfying $x_\alpha \xrightarrow{\tau(\eta)} x$ must satisfy $x_\alpha \xrightarrow{\sigma} x$, and the latter is equivalent to $x_\alpha \xrightarrow{\eta} x$. Thus, id_X^{-1} is continuous and id_X is a homeomorphism by Corollary 2.31. □

The last result shows that if η is a topological net convergence then the topology that gives rise to η must be $\tau(\eta)$.

We now examine a few constructions to create new net convergence spaces from old ones.

Proposition 2.33. *Let X be a set and $(X_i, \eta_i)_{i\in I}$ an indexed family of net convergence spaces with mappings $f_i : X \to X_i$ for each i. The **initial net convergence structure** on X with respect to $(X_i, \eta_i, f_i)_{i\in I}$ is defined by*

$$x_\alpha \to x \text{ if } f_i(x_\alpha) \xrightarrow{\eta_i} f_i(x) \text{ for each } i \in I.$$

This defines a net convergence structure on X. Moreover, it is the weakest net convergence structure on X making all the f_i continuous.

Proof. If $x_\alpha = x$ for all α then $f_i(x_\alpha) = f_i(x)$ is a constant net for each i. It follows that $f_i(x_\alpha) \xrightarrow{\eta_i} f_i(x)$ for each i, so $x_\alpha \to x$.

Suppose $(y_\beta) \preceq (x_\alpha)$ and $x_\alpha \to x$. It is easy to verify that $(f_i(y_\beta)) \preceq (f_i(x_\alpha))$ for each i. Now $f_i(x_\alpha) \xrightarrow{\eta_i} f_i(x)$ for each i implies $f_i(y_\beta) \xrightarrow{\eta_i} f_i(x)$ for each i, so $y_\beta \to x$.

For an index set A with $x_\alpha \to x$, $y_\alpha \to x$ and $B \in \mathcal{P}(A)$, consider a B-braiding of x_α with y_α given by $(z_\alpha) = (x_\alpha \otimes_B y_\alpha)$. It follows that

$$f_i(z_\alpha) = f_i(x_\alpha) \otimes_B f_i(y_\alpha)$$

for each i. Now $f_i(x_\alpha) \xrightarrow{\eta_i} f_i(x)$ and $f_i(y_\alpha) \xrightarrow{\eta_i} f_i(x)$ for each i gives $f_i(z_\alpha) \to f_i(x)$ for each i; hence, $z_\alpha \to x$.

Moreover, if (X, η) is a net convergence space where $f_i : (X, \eta) \to (X_i, \eta_i)$ is continuous for each i then $x_\alpha \xrightarrow{\eta} x$ implies $f_i(x_\alpha) \xrightarrow{\eta_i} f_i(x)$ for each i and, therefore, $x_\alpha \to x$ with respect to the initial net convergence structure. $\square$

Proposition 2.33 gives allows one to define products and subspaces of net convergence spaces.

Corollary 2.34. *Let $(X_i, \eta_i)_{i\in I}$ be an indexed family of net convergence spaces. The **product net convergence structure** on $\prod_{i\in I} X_i$ is the initial net convergence structure with respect to the family $(X_i, \eta_i, \pi_i)_{i\in I}$ where $\pi_i : \prod_{i\in I} X_i \to X_i$ denotes the standard projection. This is a net convergence structure by Proposition 2.33.*

Remark 2.35. (i)If necessary, we use $\prod \eta_i$ to denote the product net convergence structure of the η_i. We use $\eta_1 \times \eta_2$ when dealing with two net convergence structures η_1 and η_2.

(ii) It also follows from Proposition 2.33 that $\prod \eta_i$ is the weakest net convergence structure on $\prod_{i\in I} X_i$ that makes all the projections $\pi_i : \prod_{i\in I} X_i \to (X_i, \eta_i)$ continuous.

Corollary 2.36. *Let (X, η) be a net convergence space and $A \subseteq X$. The **net convergence subspace structure** on A is the initial net convergence structure on A with respect to the inclusion mapping $\iota_A : A \to (X, \eta)$. This is a net convergence structure on A by Proposition 2.33.*

Remark 2.37. (i) Given a net convergence space (X, η) and $(x_\alpha), x \in A \subseteq X$, we will denote convergence with respect to the net convergence subspace structure on A by $x_\alpha \xrightarrow{s_\eta} x$; hence $x_\alpha \xrightarrow{s_\eta} x$ in A means $(x_\alpha), x \in A$ and $x_\alpha \xrightarrow{\eta} x$ in X.

(ii) It follows from Proposition 2.33 that s_η is the weakest net convergence structure on A that makes the inclusion map continuous.

The next result shows that passing between associate convergence structures preserves products.

Lemma 2.38. *The product net convergence structure commutes with the operation of passing to the associate filter convergence structure. That is, $\eta_{\prod \lambda_i} = \prod \eta_{\lambda_i}$ and $\lambda_{\prod \eta_i} = \prod \lambda_{\eta_i}$. A similar result holds if we swap the instances of nets and filters in this statement.*

Proof. Let $(X_i, \lambda_i)_{i\in I}$ be an indexed family of filter convergence spaces. The first claim is that $\eta_{\prod \lambda_i}$ is weaker than $\prod \eta_{\lambda_i}$. Let $(x_\alpha) \in \prod_{i\in I} X_i$ such that $x_\alpha \xrightarrow{\prod \eta_{\lambda_i}} x$. This means

$\pi_i(x_\alpha) \xrightarrow{\eta_{\lambda_i}} \pi_i(x)$ for each i. Then for each i we have $\pi_i([x_\alpha]) = [\pi_i(x_\alpha)] \xrightarrow{\lambda_i} \pi_i(x)$, hence $[x_\alpha] \xrightarrow{\prod \lambda_i} x$. So $x_\alpha \xrightarrow{\eta_{\prod \lambda_i}} x$.

Since $\prod \eta_{\lambda_i}$ is uniquely defined as the weakest net convergence structure that makes all the projections $\pi_i : \prod_{i \in I} X_i \to (X_i, \eta_{\lambda_i})$ continuous, in order to prove $\eta_{\prod \lambda_i} = \prod \eta_{\lambda_i}$ it suffices to show $\pi_i : (\prod_{i \in I} X_i, \eta_{\prod \lambda_i}) \to (X_i, \eta_{\lambda_i})$ is continuous for each i. To this end, recall that the product filter convergence structure $\prod \lambda_i$ on $\prod_{i \in I} X_i$ makes all the projections $\pi_i : (\prod_{i \in I} X_i, \prod \lambda_i) \to (X_i, \lambda_i)$ continuous. Now Theorem 2.30 yields $\pi_i : (\prod_{i \in I} X_i, \eta_{\prod \lambda_i}) \to (X_i, \eta_{\lambda_i})$ is continuous for each i.

We apply a similar strategy to prove the remaining result. Let $(X_i, \eta_i)_{i \in I}$ be an indexed family of net convergence spaces. Firstly, $\lambda_{\prod \eta_i}$ is weaker than $\prod \lambda_{\eta_i}$. Suppose $\mathcal{F} \xrightarrow{\prod \lambda_{\eta_i}} x$ in $\prod_{i \in I} X_i$. Use Proposition 2.3 to find a net $(y_\beta) \in \prod_{i \in I} X_i$ such that $[y_\beta] = \mathcal{F}$. Then $\pi_i([y_\beta]) \xrightarrow{\lambda_{\eta_i}} \pi_i(x)$ for each i. That is, for each i there is a net $(x_\alpha^{(i)})$ in X_i (where α depends on i) such that $x_\alpha^{(i)} \xrightarrow{\eta_i} \pi_i(x)$ and $[x_\alpha^{(i)}] \subseteq \pi_i([y_\beta]) = [\pi_i(y_\beta)]$. Then for each i we have $(\pi_i(y_\beta))$ is a quasi-subnet of $(x_\alpha^{(i)})$; hence $\pi_i(y_\beta) \xrightarrow{\eta_i} \pi_i(x)$ for each i. This shows $y_\beta \xrightarrow{\prod \eta_i} x$ so that $[y_\beta] = \mathcal{F} \xrightarrow{\lambda_{\prod \eta_i}} x$.

Now the fact that $\pi_i : (\prod_{i \in I} X_i, \lambda_{\prod \eta_i}) \to (X_i, \lambda_{\eta_i})$ is continuous for each i follows easily from the continuity of $\pi_i : (\prod_{i \in I} X_i, \prod \eta_i) \to (X_i, \eta_i)$ for each i and Theorem 2.30. Since $\prod \lambda_{\eta_i}$ is the weakest convergence with this property, we have shown $\lambda_{\prod \eta_i} = \prod \lambda_{\eta_i}$. □

The next lemma is often useful for working with convergence subspaces.

Lemma 2.39. *Let A be a filter convergence subspace and $x \in A$. For a net (x_α) in A we have $x_\alpha \xrightarrow{\eta_{s_\lambda}} x$ implies $x_\alpha \xrightarrow{s_{\eta_\lambda}} x$. Similarly, for a net convergence subspace A and a filter $\mathcal{F}$ on A we have $\mathcal{F} \xrightarrow{\lambda_{s_\eta}} x$ implies $\mathcal{F} \xrightarrow{s_{\lambda_\eta}} x$.*

Proof. By definition s_{η_λ} and s_{λ_η} are the weakest net (respectively filter) convergence structures on A that make the maps $\iota_A : A \to (X, \eta_\lambda)$ and $\iota_A : A \to (X, \lambda_\eta)$ continuous. Thus it suffices to show that both $\iota_A : (A, \eta_{s_\lambda}) \to (X, \eta_\lambda)$ and $\iota_A : (A, \lambda_{s_\eta}) \to (X, \lambda_\eta)$ are continuous. This is now a triviality since $\iota_A : (A, s_\lambda) \to (X, \lambda)$ and $\iota_A : (A, s_\eta) \to (X, \eta)$ are continuous by the definition of convergence subspaces; now apply Theorem 2.30. □

A net convergence space (X, η) is called ***compact*** if each net (x_α) in X has a η-convergent quasi-subnet. A subset A of X is called compact if it is compact in the net convergence subspace structure.

Proposition 2.40. *The following are true in any net convergence space.*

(a) *A closed subspace of a compact space is compact.*

(b) *A compact subspace of a Hausdorff space is closed.*

(c) *The continuous image of a compact set is compact.*

Proof. (a) Let A be closed subset of the compact net convergence space X, and let (x_α) be a net in A. Then, as a net in X, we must have a convergent quasi-subnet (y_β); say $y_\beta \to x$ in X. Since $x_\alpha \in A$ for every α, it follows that $(y_\beta) \in A$ and, as A is closed, $y_\beta \to x$ in A.

(b) Suppose A is a compact subspace of a Hausdorff space X and assume that $(x_\alpha) \in A$ satisfies $x_\alpha \to x$ in A for some $x \in X$. Since A is compact, there is an element $y \in A$ and a quasi-subnet (y_β) such that $y_\beta \to y$ in A. Axiom (N2) of net convergence structures also implies $y_\beta \to x$ in A. The inclusion map into X is continuous; hence $y_\beta \to y$ in X and $y_\beta \to x$ in X. Then we must have $x = y \in A$.

(c) Let X and Y be net convergence spaces, and let $f : X \to Y$ be a continuous function. Furthermore, assume X is compact, and let (z_α) be a net in $f(X)$. Then there is a net $(x_\alpha) \in X$ such that $z_\alpha = f(x_\alpha)$ for each α. Since X is compact, there is some $x \in X$ and a quasi-subnet $(y_\beta) \preceq (x_\alpha)$ such that $y_\beta \to x$ in X. It follows that $f(y_\beta) \to f(x)$ in Y. Since $\big(f(y_\beta)\big) \preceq \big(f(x_\alpha)\big) = (z_\alpha)$, we have $f(y_\beta) \to f(x)$ in $f(X)$. □

Corollary 2.41. *If η_1 convergence is stronger than η_2 convergence then every η_1-compact set is η_2-compact.*

Proof. The hypothesis means that the identity map $id : (X, \eta_1) \to (X, \eta_2)$ is continuous. Now apply part (c) of Proposition 2.40. $\square$

Proposition 2.42. *Given a pair of associate convergence spaces, one space is compact if and only if the associate space is compact.*

Proof. If (X, η) is a compact net convergence space then (X, λ_η) is compact. Indeed, let (X, η) be compact and let $\mathcal{U}$ be any ultrafilter on X. Use Proposition 2.3 with $\mathcal{B} = \mathcal{U}$ to find a net (x_α) in X with $[x_\alpha] = \mathcal{U}$. Since X is η-compact, there is a η-convergent net (y_β) in X such that $(y_\beta) \preceq (x_\alpha)$; hence, $\mathcal{U} = [x_\alpha] \subseteq [y_\beta]$. Since $\mathcal{U}$ is maximal, we have $\mathcal{U} = [y_\beta]$. Now $y_\beta \xrightarrow{\eta} x_0$ for some $x_0 \in X$. In particular, $\mathcal{U} = [y_\beta] \xrightarrow{\lambda_\eta} x_0$.

Next we show that if (X, λ) is compact then (X, η_λ) is compact. To see this, let (x_α) be any net in X and put $\mathcal{F} = [x_\alpha]$. Find an ultrafilter $\mathcal{U}$ such that $\mathcal{F} \subset \mathcal{U}$. It follows from compactness of (X, λ) that $\mathcal{U} \xrightarrow{\lambda} x_0$ for some $x_0 \in X$. Now use Proposition 2.3 to find a net (y_β) in X with $[y_\beta] = \mathcal{U}$. As $[y_\beta] \xrightarrow{\lambda} x_0$, we have $y_\beta \xrightarrow{\eta_\lambda} x_0$ and $(y_\beta) \preceq (x_\alpha)$ — so (X, η_λ) is compact.

The above paragraphs together with Theorem 2.18 and Theorem 2.19 yield the desired results. $\square$

Remark 2.43. (i) By a slight variation of the arguments used in Proposition 2.42, one can apply Lemma 2.39 to show that associate convergence spaces share the same compact subsets.

(ii) A net that is the tail filter of an ultrafilter is called an ***ultranet***; they have the property that, for every subset U of X, a tail of the net is either contained entirely in U or $X \setminus U$. The proof of Proposition 2.42 actually shows that compactness for net convergence structures is equivalent to the following criterion: every ultranet in X converges.

2.4. **Net Convergence Vector Spaces.** Let V be a $\mathbb{R}$-vector space with a net convergence structure η. The pair (V, η) is called a ***net convergence vector space*** if the addition and scalar multiplication on V are jointly continuous. That is, the maps

$$f : V \times V \to V \quad f(x, y) = x + y$$

$$g : \mathbb{R} \times V \to V \quad g(\mu, x) = \mu x$$

are $(\eta \times \eta, \eta)$ and $(\tau_{\mathbb{R}} \times \eta, \eta)$-continuous, respectively, where $\tau_{\mathbb{R}}$ denotes the standard topological net convergence on $\mathbb{R}$. The convergence vector space terminology for filters that was introduced in Chapter 2 will be referred to as filter convergence vector spaces in order to distinguish them from the concept of a net convergence vector space.

Proposition 2.44. *Let (V, η) be a net convergence vector space. For each $v \in V$ and $\mu \in \mathbb{R} \setminus \{0\}$ the maps*

$$t_v : V \to V \quad t_v(x) = x + v$$

$$m_\mu : V \to V \quad m_\mu(x) = \mu x$$

are homeomorphisms.

Proof. The joint continuity of the vector space operations gives t_v and m_μ are continuous for each $v \in V$ and $\mu \in \mathbb{R} \setminus \{0\}$. In particular $(t_v)^{-1} = t_{-v}$ and $(m_\mu)^{-1} = m_{\mu^{-1}}$ are continuous. □

Corollary 2.45. *In a net convergence vector space $x_\alpha \to x$ if and only if $x_\alpha - x \to 0$.*

Proof. If $x_\alpha \to x$ then $x_\alpha - x = t_{-x}(x_\alpha) \to x - x = 0$. Conversely, $x_\alpha - x \to 0$ implies $x_\alpha = t_x(x_\alpha - x) \to t_x(0) = x$. □

The next result demonstrates that passing between associate convergence structures preserves the vector space operations.

Theorem 2.46. *Given a pair of associate convergence spaces, one is a convergence vector space if and only if the other space is.*

Proof. Let $f(x,y) = x + y$ and $g(\mu, x) = \mu x$. First, we claim that if (V,η) is a net convergence vector space then (V,λ_η) is a filter convergence vector space. To see this, assume that f and g are $(\eta\times\eta,\eta)$ and $(\tau_{\mathbb{R}}\times\eta,\eta)$-continuous, respectively. Theorem 2.30 implies f and g are $(\lambda_{\eta\times\eta},\lambda_\eta)$ and $(\lambda_{\tau_{\mathbb{R}}\times\eta},\lambda_\eta)$-continuous. Apply Lemma 2.38 to get f is $(\lambda_\eta\times\lambda_\eta,\lambda_\eta)$-continuous and g is $(\tau_{\mathbb{R}}\times\lambda_\eta,\lambda_\eta)$-continuous and, therefore, (V,λ_η) is a filter convergence vector space.

Next we show that if (V,λ) is a filter convergence vector space then (V,η_λ) is a net convergence vector space. Since f and g are $(\lambda\times\lambda,\lambda)$ and $(\tau_{\mathbb{R}}\times\lambda,\lambda)$-continuous then Theorem 2.30 gives f is $(\eta_{\lambda\times\lambda},\eta_\lambda)$-continuous and g is $(\eta_{\tau_{\mathbb{R}}\times\lambda},\eta_\lambda)$-continuous. Applying Lemma 2.38 yields the desired claim.

The final results follow from the above paragraphs and Theorem 2.18 and Theorem 2.19.

□

Since filter and net convergence structures are equivalent, we begin to drop the reference to filters and nets in our terminology. The precise definitions should always be clear from the context. Recall that a subset B of a (filter) convergence vector space is called bounded if $\mathcal{N}_0 B\to 0$ where $\mathcal{N}_0$ denotes the neighborhood filter at 0 in the standard topology on $\mathbb{R}$ and $\mathcal{N}_0 B$ is the filter generated by sets of the form $\{UB : U\in\mathcal{N}_0\}$.

Definition 2.47. A subset B of a convergence vector space is called ***bounded*** if $(\mu_\alpha b)_{(\alpha,b)}\xrightarrow{\eta} 0$ whenever $(\mu_\alpha)_{\alpha\in A}\to 0$ in the usual convergence on $\mathbb{R}$; here we view $(\mu_\alpha b)_{(\alpha,b)}$ as a net indexed by $A\times B$ directed by the pre-order $(\alpha_1,b_1)\preceq(\alpha_2,b_2)\iff\alpha_1\le\alpha_2$.

Proposition 2.48. *A convergence vector space and its associate convergence space share the same bounded subsets.*

Proof. Suppose B is η-bounded. Use Proposition 2.3 to find a net (μ_α) with $[\mu_\alpha] = \mathcal{N}_0$. It follows that $\mu_\alpha \to 0$. Now B is η-bounded implies $(\mu_\alpha b)_{(\alpha,b)} \xrightarrow{\eta} 0$. Since $[\mu_\alpha b] = [\mu_\alpha]B = \mathcal{N}_0 B$, we have $\mathcal{N}_0 B \xrightarrow{\lambda_\eta} 0$ and B is λ_η-bounded.

Assume that B is bounded in (V, λ). Let $\mu_\alpha \to 0$ in $\mathbb{R}$ and consider the net $(\mu_\alpha \cdot b)$ as in Definition 2.47. $[\mu_\alpha] \to 0$ in $\mathbb{R}$ and B is bounded gives $[\mu_\alpha \cdot b] = [\mu_\alpha]B \xrightarrow{\lambda} 0$. So, $\mu_\alpha \cdot b \xrightarrow{\eta_\lambda} 0$ and B is bounded in (V, η_λ).

We have shown that boundedness in convergence vector spaces is preserved by passing to the associate convergence structure. Thus, if B is a subset of (V, η) such that B is bounded in (V, λ_η) then B is η_{λ_η}-bounded. This is equivalent to B being η-bounded by Theorem 2.18. The remaining result follows a similar argument using Theorem 2.19. □

We use Definition 2.47 to define the concept of a bounded net in convergence vector spaces. A net (x_α) is ***bounded*** if $\{x_\alpha : \alpha \in A\}$ is contained in a bounded set. A convergence vector space is called ***locally bounded*** if every convergent net has a bounded tail.

Proposition 2.49. *A convergence vector space is locally bounded if and only if its associate convergence space is locally bounded.*

Proof. Suppose (V, η) is locally bounded. By linearity it suffices to consider convergence at 0. If $\mathcal{F} \xrightarrow{\lambda_\eta} 0$ then there is a net (x_α) in V such that $x_\alpha \xrightarrow{\eta} 0$ and $[x_\alpha] \subseteq \mathcal{F}$. Then a tail of (x_α) is bounded by our assumption. Find α_0 and a η-bounded set B such that $\{x_\alpha : \alpha \geq \alpha_0\} \subseteq B$. Then $B \in [x_\alpha] \subseteq \mathcal{F}$. B is automatically λ_η-bounded by Proposition 2.48. Since $\mathcal{F}$ was arbitrary, we have shown that (V, λ_η) is locally bounded.

Similarly, assume (V, λ) is locally bounded and let $x_\alpha \xrightarrow{\eta_\lambda} 0$. Then $[x_\alpha] \xrightarrow{\lambda} 0$; hence there is a λ-bounded subset B such that $B \in [x_\alpha]$. That is, there is α_0 with $\{x_\alpha : \alpha \geq \alpha_0\} \subseteq B$. B is automatically η_λ-bounded by Proposition 2.48, hence a tail of (x_α) is η_λ-bounded. (x_α) was arbitrary, so we deduce that (V, η_λ) is locally bounded.

The result follows from the above arguments and another routine application of Theorem 2.18 and Theorem 2.19. ☐

A net $(x_\alpha)_{\alpha\in A}$ in a convergence vector space (V,η) is called η-***Cauchy*** if

$$(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A} \xrightarrow{\eta} 0;$$

note that the index set is the coordinate-wise order on $A\times A$. (V,η) is ***complete*** if every η-Cauchy net in V converges.

Proposition 2.50. *A convergence vector space is complete if and only if its associate convergence space is complete.*

Proof. Let $\mathcal{F}$ be a λ_η-Cauchy filter; *i.e.*, $\mathcal{F}-\mathcal{F} \xrightarrow{\lambda_\eta} 0$. Use Proposition 2.3 to find a net $(x_\alpha)_{\alpha\in A}$ with $[x_\alpha]=\mathcal{F}$. It follows from

$$[x_\alpha]-[x_\alpha]=[x_\alpha - x_\beta] \xrightarrow{\lambda_\eta} 0$$

that $(x_\alpha - x_\beta) \xrightarrow{\eta_{\lambda_\eta}} 0$; hence, by Theorem 2.18, $(x_\alpha - x_\beta) \xrightarrow{\eta} 0$. So (x_α) is η-Cauchy and, therefore, η-converges to some x_0. It follows from $[x_\alpha]=\mathcal{F}$ that $\mathcal{F} \xrightarrow{\lambda_\eta} x_0$, so (V,λ_η) is complete.

Now suppose that (V,λ) is a complete convergence vector space. Let (x_α) be η_λ-Cauchy. Since $(x_\alpha - x_\beta) \xrightarrow{\eta_\lambda} 0$, we have

$$[x_\alpha]-[x_\alpha]=[x_\alpha - x_\beta] \xrightarrow{\lambda} 0,$$

i.e., $[x_\alpha]$ is λ-Cauchy. It follows that $[x_\alpha] \xrightarrow{\lambda} x_0$ for some $x_0\in V$. Equivalently $x_\alpha \xrightarrow{\eta_\lambda} x_0$, so (V,η_λ) is complete.

The remaining claim follows from the above arguments and another routine application of Theorem 2.18 and Theorem 2.19. ☐

In this chapter, we introduced a new definition of net convergence and it was shown to be equivalent to filter convergence. This has the benefit that many abstract convergence properties can be phrased in the language of nets, which are more intuitive for certain concepts. Still, the theory of filter convergence structures is well established and concepts from this area can now be translated for nets and used to study non-topological net convergences. The remainder of this thesis investigates such applications.

3. Order Convergence Structure in Vector Lattices

This chapter demonstrates that the theory of net convergence outlined in Chapter 2 is robust enough to model order convergence in vector lattices.

3.1. **Vector Lattices.** This section serves as a quick review of terminology from vector lattices. For more background and details, see [ABP06], [Sch74] or [LT79, Chapter 1].

An ***ordered vector space*** is a $\mathbb{R}$-vector space X with a partial order $\leq$ in which

(i) $x \leq y$ implies $x + z \leq y + z$ and
(ii) $x \leq y$ implies $\lambda x \leq \lambda y$

hold for every $x, y, z \in X$ and $\lambda \in \mathbb{R}_+$. For $a \leq b$, the ***order interval*** $[a, b]$ is the set $\{x \in X : a \leq x \leq b\}$. A set is said to be ***order bounded*** in X if it is a subset of some order interval.

An ordered vector space X that is also a ***lattice***, that is

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y := \inf\{x, y\}$$

exists for every pair of vectors $x, y \in X$, is called a ***vector lattice***. The set $\mathbb{R}$ with the standard ordering is an example of a vector lattice. It is frequently useful to think of vector lattices as spaces of functions. In fact, many function spaces carry a natural vector lattice structure.

Example 3.1. Let Ω be any set. The set of all $\mathbb{R}$-valued functions on Ω, denoted by $\mathbb{R}^\Omega$, is a vector lattice under the natural (pointwise) ordering of functions. That is, for $f, g \in \mathbb{R}^\Omega$ we write $f \leq g$ iff $f(\omega) \leq g(\omega)$ for every $\omega \in \Omega$. $f \vee g$ and $f \wedge g$ are defined pointwise *via*

$$(f \vee g)(\omega) = f(\omega) \vee g(\omega) \qquad\qquad (f \wedge g)(\omega) = f(\omega) \wedge g(\omega)$$

for $\omega \in \Omega$. In particular, setting $\Omega = \mathbb{N}$ yields the set of all real sequences is a vector lattice under pointwise operations.

Example 3.2. This is a special case of the previous example. If Ω is finite then we identify $\mathbb{R}^\Omega$ with $\mathbb{R}^n$ (where $|\Omega| = n$) by viewing $\mathbb{R}$-valued functions on $\{1, \ldots, n\}$ as vectors in $\mathbb{R}^n$; *i.e.*, we identify $x \in \mathbb{R}^\Omega$ with the vector in $\mathbb{R}^n$ whose coordinates are given by $x_i = x(i)$ for $i \in \{1, \ldots, n\}$. It is easy to see that $\mathbb{R}^n$ is a vector lattice under the order from Example 3.1, and, in this case, $x \leq y$ iff $x_i \leq y_i$ for every i. It is also easy to see

$$x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n) \qquad\qquad x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots y_n) \in \mathbb{R}^n$.

Example 3.3. $C(K)$ will always denote the space of all continuous $\mathbb{R}$-valued functions on a compact Hausdorff space, K. $C(K)$ is a vector lattice under the ordering $f \leq g$ iff $f(t) \leq g(t)$ for all $t \in K$. We have

$$(f \vee g)(t) = \max\{f(t), g(t)\} = \frac{f(t) + g(t) + |f(t) - g(t)|}{2}$$

and

$$(f \wedge g)(t) = \min\{f(t), g(t)\} = \frac{f(t) + g(t) - |f(t) - g(t)|}{2}$$

for $t \in K$.

Example 3.4. Let (Ω, Σ, μ) be a measure space and set $L_0(\mu) = \{f : \Omega \to \mathbb{R} : f \text{ is } \Sigma\text{–measurable}\}$. Recall that we identify two functions in $L_0(\mu)$ if they are equal μ-almost everywhere (*a.e.*); hence, elements of $L_0(\mu)$ are actually equivalence classes of functions. For $f, g \in L_0(\mu)$ we set $f \leq g$ iff $f(\omega) \leq g(\omega)$ μ-a.e.. $L_0(\mu)$ is a vector lattice under this order where

$$(f \vee g)(\omega) = f(\omega) \vee g(\omega) \qquad\qquad (f \wedge g)(\omega) = f(\omega) \wedge g(\omega)$$

hold for μ-a.e. $\omega \in \Omega$. Moreover, for $1 \leq p < \infty$ we let $\|f\|_p = (\int_\Omega |f(\omega)|^p d\mu)^{\frac{1}{p}}$ and $L_p(\mu) = \{f \in L_0(\mu) : \|f\|_p < \infty\}$. Then $L_p(\mu)$ is a vector lattice under the order it inherits from $L_0(\mu)$.

For the rest of this section X will denote a vector lattice. A net (x_α) in X is said to be ***increasing*** if $x_\alpha \geq x_\beta$ whenever $\alpha \succeq \beta$. An increasing net will be denoted by $x_\alpha \uparrow$. The notion of ***decreasing*** net is defined in the obvious manner and denoted by $x_\alpha \downarrow$. It is also convenient to write $x_\alpha \uparrow x$ when (x_α) is increasing and $\sup\limits_\alpha x_\alpha = x$. Similarly, we write $x_\alpha \downarrow x$ when (x_α) is decreasing and $\inf\limits_\alpha x_\alpha = x$.

An element $x \in X$ is called ***positive*** if $x \geq 0$. We use X_+ to denote the set of all positive elements in X. X is said to be ***Archimedean*** if for every $x \in X$ and $u \in X_+$ that satisfy $nx \leq u$ for all $n \in \mathbb{N}$ implies $x \leq 0$.

For $x \in X$, the ***positive part***, the ***negative part***, and the ***modulus*** of x are defined by the identities

$$x^+ = x \vee 0 \qquad x^- = (-x) \vee 0 \qquad |x| = x \vee (-x), \text{ respectively.}$$

These operations, along with the operations $x \wedge y$ and $x \vee y$, are collectively referred to as the ***lattice operations***. Note that $x \in X$ implies $x^+, x^-, |x| \in X_+$, and $x = x^+ - x^-$ and $|x| = x^+ + x^-$. This often allows one to reduce an argument down to working with positive vectors.

A linear operator $T : X \to Y$ between two vector lattices is said to be ***positive*** if it maps positive vectors to positive vectors. If T preserves all of the lattice operations then it is called a ***lattice homomorphism***. A ***lattice isomorphism*** is a bijective lattice homomorphism, and two vector lattices that are lattice isomorphic to one another are essentially indistinguishable as vector lattices.

The lattice operations are also used to investigate the structure of vector lattices. Let Y be a linear subspace of X. If, in addition, Y is closed under each of the lattice operations, then we say that Y is a ***sublattice*** of X. Thus a sublattice of a vector lattice is always a vector lattice. The classical sequence spaces c_{00}, c_0, c and ℓ_p for $0 < p \leq \infty$ are all sublattices of $\mathbb{R}^{\mathbb{N}}$.

We say that Y is an ***ideal*** of X if Y is a sublattice of X and $0 \leq x \leq y$ implies $x \in Y$ whenever $y \in Y$. For any subset S of X, the intersection of all ideals containing S is the smallest ideal containing S; it is called the ***ideal generated by S*** and denoted by I_S. This construction can be characterized more explicitly:

$$I_S = \Big\{x : |x| \leq \sum_{i=1}^{k} \lambda_i |x_i|; k \in \mathbb{N}, \lambda_1, \ldots, \lambda_k \in \mathbb{R}_+, x_1, \ldots, x_k \in S\Big\}.$$

Let $e \geq 0$. If $S = \{e\}$ then we write I_e instead of $I_{\{e\}}$; note that

$$I_e = \{x \in X : \exists \lambda \in \mathbb{R}_+ |x| \leq \lambda e\}.$$

3.2. **Order Convergence.** Unless stated otherwise X will denote a vector lattice. A net (x_α) in X ***order converges*** to x if there exists a net $(u_\gamma)_{\gamma \in \Gamma}$ in X such that $u_\gamma \downarrow 0$ and for every γ there exists α_0 such that $|x_\alpha - x| \leq u_\gamma$ whenever $\alpha \geq \alpha_0$; this mode of convergence is denoted by $x_\alpha \xrightarrow{o} x$. Order convergence plays a fundamental role in vector lattice theory.

- A set $A \subset X$ is said to be ***order closed*** if every net (x_α) in A that order converges has its limit in A.
- If Y is another vector lattice then a function $f : X \to Y$ is said to be ***order continuous*** if $f(x_\alpha) \xrightarrow{o} f(x)$ in Y whenever $x_\alpha \xrightarrow{o} x$ in X.
- A vector lattice is said be ***Dedekind complete*** (or ***order complete***) if every bounded above increasing net has a supremum; *i.e.*, $x_\alpha \uparrow \leq u$ implies $\sup_\alpha x_\alpha$ exists.

- A sublattice Y of X is said to be ***order dense*** if for each $x \in X_+$ there is some $y \in Y$ such that $0 < y \leq x$.

Despite the suggestive terminology from vector lattices, order convergence is not a topological convergence and there is generally no topology to model all these concepts. The remainder of this chapter will show how these terms fit naturally into the framework of net convergence structures. Similar ideas are explored in [AVW05], [VdW06] and [VdW11] using σ-order convergence: a sequence (x_n) in X is said to ***σ-order converge*** to x if there is a sequence $u_n \downarrow 0$ in X such that $|x_n - x| \leq u_n$ for each n.

The following convergence structure is given in [Sch74, Defintion 1.7, p. 54] and highlights our motivation for developing a general theory of net convergence structures. In any vector lattice, put $\mathcal{F} \xrightarrow{\lambda_\star} x$ if there is a filter base of order intervals $\mathcal{B}$ such that $[\mathcal{B}] \subseteq \mathcal{F}$ and $\cap\mathcal{B} = \{x\}$. Note that, at this point, it would be a lot of work to investigate properties of this convergence structure directly. For example, try to prove this convergence is translation invariant.

Remark 3.5. One can show that $\mathcal{F} \xrightarrow{\lambda_\star} x$ if there are nets $a_\gamma \uparrow x$ and $b_\gamma \downarrow x$ in X such that $[a_\gamma, b_\gamma] \in \mathcal{F}$ for every γ. This should remind us of the following alternative characterization of order convergence: $x_\alpha \xrightarrow{o} x$ if and only if there are nets $a_\gamma \uparrow x$ and $b_\gamma \downarrow x$ such that for every γ there is α_0 such that $x_\alpha \in [a_\gamma, b_\gamma]$ for all $\alpha \geq \alpha_0$.

It is now easy to see that the associate net convergence of $\lambda_\star$ agrees with order convergence. Indeed, we have $x_\alpha \xrightarrow{o} x$ if and only if there are nets $a_\gamma \uparrow x$, $b_\gamma \downarrow x$ such that for every γ there is α_0 such that $x_\alpha \in [a_\gamma, b_\gamma]$ for all $\alpha \geq \alpha_0$. This is equivalent to saying that for every γ there is α_0 such that $\{x_\alpha : \alpha \geq \alpha_0\} \subseteq [a_\gamma, b_\gamma]$, which means $[a_\gamma, b_\gamma] \in [x_\alpha]$ for every γ. Therefore, $[x_\alpha] \xrightarrow{\lambda_\star} x$. This demonstrates that order convergence is the net convergence of a filter convergence structure and, thanks to the theory from Chapter 2, we can deduce properties for $\lambda_\star$ by analyzing the corresponding properties of order convergence of nets.

Recall that we are identifying nets up to tail equivalence. So concepts that are defined for individual nets need to be adapted for equivalence classes. Since we would like to apply net convergence theory to study order convergence in vector lattices, we need to adapt the notion of monotone net for equivalence classes of nets.

Example 3.6. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{N}$ given by

$$\begin{aligned} (x_n)_{n\in\mathbb{N}} &= (1,2,3,4,5,6,7,8,\dots) \\ \text{and} \quad (y_n)_{n\in\mathbb{N}} &= (2,1,4,3,6,5,8,7,\dots). \end{aligned}$$

It is easy to see that these two sequences are tail equivalent. Note that $(x_n)_{n\in\mathbb{N}}$ is increasing while $(y_n)_{n\in\mathbb{N}}$ is not. So the property of being monotone is not preserved under tail equivalence.

Lemma 3.7. *Let $(x_\alpha)_{\alpha\in A}$ be an increasing net in a partially ordered set X. Then there is an increasing and admissible net $(y_\gamma)_{\gamma\in\Gamma}$ that is tail equivalent to (x_α). An analogous result holds for decreasing nets.*

Proof. For each $\alpha \in A$, let $T_\alpha = \{x_\beta : \beta \geq \alpha\}$ be the α-th tail set of (x_α). Since (x_α) is increasing, x_α is the least element of T_α. Consider the tail filter base

$$\Gamma = \{T_\alpha : \alpha \in A\}$$

ordered by reverse inclusion. This makes Γ into a partially ordered and directed set. Define a net $(y_\gamma)_{\gamma\in\Gamma}$ in X by $y_\gamma :=$ the (unique) least element of γ. It can be easily verified that this net is increasing, and it is admissible because $\Gamma \in \mathcal{P}\big(\mathcal{P}(X)\big)$.

It remains to show $(y_\gamma)_{\gamma\in\Gamma}$ is tail equivalent to (x_α). We will first show $[y_\gamma] \subseteq [x_\alpha]$. Let $U \in [y_\gamma]$. Then there exists some $\gamma_0 \in \Gamma$ such that $y_\gamma \in U$ for all $\gamma \geq \gamma_0$. By the construction of (y_γ), there is some $\alpha_0 \in A$ with $\gamma_0 = T_{\alpha_0}$. For $\alpha \geq \alpha_0$ put $\gamma = T_\alpha$. Then $T_\alpha \subseteq T_{\alpha_0}$. We thus obtain $\gamma \geq \gamma_0$, so $x_\alpha = y_\gamma \in U$ whenever $\alpha \geq \alpha_0$; that is, $U \in [x_\alpha]$.

For the reverse inclusion, suppose $U \in [x_\alpha]$. Then there is $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $x_\alpha \in U$. Put $\gamma_0 = T_{\alpha_0}$, and let $\gamma \geq \gamma_0$. Then we can find some α such that $\gamma = T_\alpha$, so $T_\alpha \geq T_{\alpha_0}$; that is, $T_\alpha \subseteq T_{\alpha_0}$. Now $x_\alpha \in T_\alpha$ implies $x_\alpha \in T_{\alpha_0}$. In particular, $\alpha \geq \alpha_0$. It follows that $y_\gamma = x_\alpha \in U$ whenever $\gamma \geq \gamma_0$. This shows $U \in [y_\gamma]$ and the proof is complete.

The proof for decreasing nets is similar. $\square$

An equivalence class in $\mathfrak{N}(X)/\!\sim$ is said to be ***increasing*** if it contains an increasing net and ***decreasing*** if it contains a decreasing net. So, by passing to a tail equivalent net, we can treat our increasing or decreasing equivalence classes as if they are increasing or decreasing nets.

Proposition 3.8. *Order convergence defines a net convergence structure. We denote it by o.*

Proof. It is immediate that for each $x \in X$ the constant net $x_\alpha = x$ satisfies $x_\alpha \xrightarrow{o} x$.

Suppose $(x_\alpha) \preceq (y_\beta)$ and $y_\beta \xrightarrow{o} x$. The former condition yields $[y_\beta] \subseteq [x_\alpha]$ while the latter condition allows us to find a net $u_\gamma \downarrow 0$ such that for each γ there is a β_0 with $|y_\beta - x| \leq u_\gamma$ whenever $\beta \geq \beta_0$. Let γ be fixed, and find β_0 as in the definition of order convergence. Note that the tail set $\{y_\beta : \beta \geq \beta_0\} \in [y_\beta]$ and hence

$$\{y_\beta : \beta \geq \beta_0\} \in [x_\alpha].$$

Then there is α_0 such that for all $\alpha \geq \alpha_0$ we have $x_\alpha = y_\beta$ for some $\beta \geq \beta_0$. So $|x_\alpha - x| = |y_\beta - x| \leq u_\gamma$ whenever $\alpha \geq \alpha_0$. As γ was arbitrary, $x_\alpha \xrightarrow{o} x$.

Suppose $x_\alpha \xrightarrow{o} x$ and $y_\alpha \xrightarrow{o} x$. For $B \in \mathcal{P}(A)$, put $z_\alpha = x_\alpha \otimes_B y_\alpha$. Find nets $u_\gamma \downarrow 0$ and $s_\omega \downarrow 0$ in X with the following properties:

(i) for each $\gamma \in \Gamma$ there is a $\alpha_1 \in A$ such that $\alpha \geq \alpha_1$ implies $|x_\alpha - x| \leq u_\gamma$ and

(ii) for each $\omega \in \Omega$ there is $\alpha_2 \in A$ such that $\alpha \geq \alpha_2$ implies $|y_\alpha - x| \leq s_\omega$.

Now give $\Lambda = \Gamma \times \Omega$ the coordinate-wise order and, for $\lambda \in \Lambda$, put

$$t_\lambda = t_{(\gamma,\omega)} = u_\gamma \vee s_\omega.$$

It is straightforward to verify $t_\lambda \downarrow$ and we claim that $t_\lambda \downarrow 0$. Indeed, for a fixed γ_0, we have

$$\inf_{\omega\in\Omega}(t_{(\gamma_0,\omega)}) = \inf_{\omega\in\Omega}(u_{\gamma_0} \vee s_\omega) = u_{\gamma_0} \vee \inf_{\omega\in\Omega} s_\omega = u_{\gamma_0} \vee 0 = u_{\gamma_0}.$$

Taking the infimum over all such γ_0 yields $t_\lambda \downarrow 0$.

Finally, using properties (i) and (ii) for (u_γ) and (s_ω), for each $\lambda \in \Lambda$ there exists α_1 and α_2 such that $\alpha \geq \alpha_1$ implies $|x_\alpha - x| \leq u_\gamma$ and $\alpha \geq \alpha_2$ implies $|y_\alpha - x| \leq s_\omega$. Take $\alpha_0 \geq \alpha_1, \alpha_2$. Then $\alpha \geq \alpha_0$ implies either

$$|z_\alpha - x| = |x_\alpha - x| \leq u_\gamma \leq t_\lambda, \quad \text{if } \alpha \in B, \text{ or}$$

$$|z_\alpha - x| = |y_\alpha - x| \leq s_\omega \leq t_\lambda, \quad \text{if } \alpha \in A \setminus B$$

In any case, for each $\lambda \in \Lambda$ there is an α_0 such that $|z_\alpha - x| \leq t_\lambda$ whenever $\alpha \geq \alpha_0$. Thus, $z_\alpha \xrightarrow{o} x$. □

Corollary 3.9. *Order convergence is the net convergence of a filter convergence structure.*

Proof. (X, o) is a net convergence space, so (X, λ_o) is a filter convergence space by Theorem 2.16. Then (X, o_{λ_o}) is a net convergence space by Theorem 2.19. Now $o_{\lambda_o} = o$ by Theorem 2.18. □

We call λ_o the ***order convergence structure***.

Remark 3.10. By the observations following Remark 3.5, $\lambda_\star$ and λ_o share the same associate net convergence. Then $\lambda_\star = \lambda_o$. This direct characterization of order convergence structure is sometimes useful.

The term order convergence structure has been used in the literature to refer to the following notion of convergence: define a filter $\mathcal{F} \to x$ if there are sequences $a_n \uparrow 0$ and $b_n \downarrow 0$ such that $[a_n, b_n] \in \mathcal{F}$ for all n; see for example [AVW05], [VdW06], [VdW11], and [VdW16]. We should mention that our order convergence structure λ_o is not the same as the one appearing in those papers; we will distinguish them by calling their convergence the ***σ-order convergence structure*** and denote it by $\lambda_{\sigma-o}$. This name is fitting since the sequential convergence of $\lambda_{\sigma-o}$ agrees with σ-order convergence whereas the sequential convergence of λ_o is order convergences of sequences. While $x_n \xrightarrow{\sigma-o} x$ implies $x_n \xrightarrow{\text{o}} x$, the converse is generally false.

3.3. **Basic Properties.** In this section we examine some properties of λ_o. Similar work was done for σ-order convergence structure in [VdW06] but the proofs rely heavily on filters. The approach taken here will highlight the utility of net convergence theory for studying filter convergence structures.

Proposition 3.11. *(X, λ_o) is a Hausdorff convergence space.*

Proof. It is well-known that the limit of an order convergent net is unique, so (X, o) is Hausdorff. Then (X, λ_o) is Hausdorff by Proposition 2.22. □

Proposition 3.12. *For vector lattices X and Y a map $f : (X, \lambda_o) \to (Y, \lambda_o)$ is continuous if and only if it is order continuous.*

Proof. The continuous maps $f : (X, o) \to (Y, o)$ are the order continuous maps by definition. Now apply Theorem 2.30. □

Corollary 3.13. *The lattice operations on* (X, λ_o) *are continuous.*

Proof. It is well-known that the lattice operations are order continuous. Proposition 3.12 completes the proof. □

Proposition 3.14. *In* (X, λ_o) *a subset is closed if and only if it is order closed.*

Proof. The order closed subsets of a vector lattice are, by definition, the closed subsets of (X, o). The result is immediate since (X, o) and (X, λ_o) have the same closed sets by Corollary 2.25. □

The collection of order closed sets form the closed sets of a topology; this has been called the ***order topology*** in [LZ73].

Corollary 3.15. *The topological modification of* λ_o *is the order topology. We denote it by* $\tau(o)$.

Proof. The order closed sets are precisely the λ_o-closed sets by Proposition 3.14. It follows from Proposition 2.26 that the open sets of the order topology are the λ_o-open sets. That is, $\tau(o) = \tau(\lambda_o)$. □

Corollary 3.16. *The order topology on a vector lattice* X *is the finest topology on* X *whose convergence is weaker than order convergence.*

Proof. Recall from Corollary 2.31 that $id_X : (X, o) \to (X, \tau(o))$ is always continuous. By Corollary 3.15 we have $\tau(o) = \tau(\lambda_o)$, so $x_\alpha \xrightarrow{\text{o}} x$ implies $x_\alpha \xrightarrow{\tau(o)} x$. That $\tau(o)$ is the finest topology with this property follows from Corollary 2.28. □

We have shown that several terms from vector lattice theory are expressed naturally in the language of convergence structures. However, this is not always the case. For sublattices there are two common notions of density in the literature that, despite their names, are not equivalent. A sublattice Y of X is said to be ***order dense*** if for each $x \in X_+$ with $x \neq 0$ there is some $y \in Y$ such that $0 < y \leq x$. Y is ***dense with respect to order convergence*** in X if for each $x \in X$ there is a net (y_α) in Y with $y_\alpha \xrightarrow{\text{o}} x$ in X. Generally, order dense sublattices are dense with respect to order convergence, but not conversely.

Example 3.17. This example appeared in [GTX17] but we provide the details. In the vector lattice $V = \mathbb{R}^{[0,1]}$ we consider two sublattices given by $Y = C([0,1])$ and $X =$ all those $v \in V$ of the form $v = g + h$ where $g \in Y$ and $h \in V$ is supported at only finitely-many points. Clearly Y is a sublattice of X. It is also dense with respect to order convergence: given $x \in X$ find $g \in Y$ and $h \in V$ with $\operatorname{supp}(h) = \{t_1, ..., t_k\}$ and $x = g + h$. We will construct a sequence (x_n) in Y such that $x_n \xrightarrow{\text{o}} x$ in X.

To simplify the presentation we will assume that $0, 1 \notin \operatorname{supp}(h)$; the argument can easily be adapted to account for these cases. Without loss of generality we may assume $h > 0$ and that $t_1 < t_2 < \cdots < t_k$. Define $t^\star = \min\limits_{1 \leq i \leq k+1} \{\frac{t_i - t_{i-1}}{2}\}$, where we define $t_0 = 0$ and $t_{k+1} = 1$, and construct neighborhoods of t_i of the form $[t_i - t^\star, t_i + t^\star]$ for each $i = 1, ..., k$. Define $g_n(t)$ to be $h(t_i)$ for $t = t_i$, linear for $t \in [t_i - \frac{t^\star}{n}, t_i)$ and $t \in (t_i, t_i + \frac{t^\star}{n}]$; put $g_n(t) = 0$ for all other $t \in [0, 1]$. By construction we have $g_n \in Y$ for all n and $g_n \downarrow h$ in X, hence $g_n \xrightarrow{\text{o}} h$ in X. Now put $x_n = g + g_n$ to get a sequence in Y such that $x_n \xrightarrow{\text{o}} x$ in X. Then Y is dense in Z with respect to order convergence in X. However, note that Y is not order dense in Z since there is no *nonzero* continuous function in $C([0,1])$ that is dominated by $\mathbf{1}_{\{\frac{2}{3}\}} \in Z$.

We have now seen that, unlike order continuous functions and order closed sets, the vector lattice analogue of λ_o-dense subsets remains to be determined. This matter is easily resolved with net convergence theory.

Proposition 3.18. *A subset of a vector lattice is λ_o-dense if and only if it is dense with respect to order convergence.*

Proof. Let $U \subseteq X$. Lemma 2.24 gives $a_{\lambda_o}(U) = \overline{U}^o$. If U is dense then $a_{\lambda_o}(U) = X = \overline{U}^o$. □

There are situations in which these two notions of order density can be reconciled. Recall that a sublattice $Y \subseteq X$ is said to be ***regular*** if the inclusion map $Y \hookrightarrow X$ is order continuous. [GTX17, Lemma 2.5] shows that regular sublattices that are dense with respect to order convergence are order dense. In other words, we have the following basic observation.

Corollary 3.19. *If $Y \subseteq X$ is a regular sublattice then Y is λ_o-dense in X if and only if it is order dense in X.*

Notice that regularity of a sublattice is defined in terms of convergence. Recall that the subspace convergence induced on a subset from a convergence structure is defined to be the weakest convergence on the subset that makes the inclusion map continuous. So, from a convergence theory perspective, it is natural to ask about the relation between regular sublattices and the subspace order convergence structure.

Lemma 3.20. *If $Y \subseteq X$ is a regular sublattice then the subspace order convergence structure on Y induced by order convergence on X is weaker than the order convergence structure on Y.*

Proof. If Y is just a sublattice of X, then it is a vector lattice in its own right and one may consider order convergence on Y. But Y inherits another natural convergence from X, the subspace order convergence structure. Now consider the two convergence spaces (Y, λ_o) and (Y, s_{λ_o}) where s_{λ_o} is the subspace convergence induced by λ_o-convergence on X; it is the weakest convergence on Y that makes the inclusion into X continuous. If

Y is a regular sublattice, then $(Y, o) \hookrightarrow (X, o)$ is continuous; hence $(Y, \lambda_o) \hookrightarrow (X, \lambda_o)$ is continuous by Theorem 2.30. It follows that $id_Y : (Y, \lambda_o) \to (Y, s_{\lambda_o})$ is continuous. □

If Y is an order dense sublattice of X then we think of Y as sitting "nicely" inside of X "from below" in the sense that, for every $x > 0$, there is some $y \in Y$ where $0 < y \leq x$. A sublattice Y is ***majorizing*** in X if for every $x \geq 0$ there is some $y \in Y$ with $x \leq y$. For order dense and majorizing sublattices, the subspace structure induced from order convergence is precisely order convergence on the sublattice.

Corollary 3.21. *If Y is order dense and majorizing in X then $id_Y : (Y, \lambda_o) \to (Y, s_{\lambda_o})$ is a homeomorphism.*

Proof. If Y is order dense in X, then it is regular. So, in light of Lemma 3.20, it suffices to show $id_Y : (Y, s_{\lambda_o}) \to (Y, \lambda_o)$ is continuous. By Theorem 2.8 of [GTX17], if Y is order dense and majorizing in X then $x_\alpha \xrightarrow{o} x$ in Y if and only if $x_\alpha \xrightarrow{o} x$ in X. If $(x_\alpha), x \in Y$ with $x_\alpha \xrightarrow{s_o} x$ then $x_\alpha \xrightarrow{o} x$ in X; hence $x_\alpha \xrightarrow{o} x$ in Y. □

It is an easy fact from vector lattice theory that the Archimedean property on X is equivalent to $\frac{1}{n}u \downarrow 0$ for each $u \in X_+$. This is generally considered to be a very weak property, and many authors restrict themselves to Archimedean spaces. It is interesting that this property is equivalent to the order convergence structure being a convergence vector space.

Proposition 3.22. *(X, λ_o) is a convergence vector space if and only if X is Archimedean.*

Proof. The forward implication is trivial. Indeed, if (X, λ_o) is a convergence vector space then (X, o) has the same property by Theorem 2.46. In particular, the scalar multiplication $g : \mathbb{R} \times X \to X$ is jointly o-continuous. It follows from $\frac{1}{n} \to 0$ in $\mathbb{R}$ that $\frac{1}{n}u \downarrow 0$ for all $u \in X_+$, so X is Archimedean.

The joint o-continuity of addition is well-known and easy to verify. Thus, it suffices to verify that scalar multiplication is jointly o-continuous. To this end, suppose $\mu_k \to \mu$ in $\mathbb{R}$ and $x_\alpha \xrightarrow{o} x$ for a net (x_α) indexed by A. Order convergence of (x_α) yields a net $(u_\gamma)_{\gamma\in\Gamma} \downarrow 0$ in X. Also, every convergent sequence of real numbers is bounded so we can find a $M \geq 0$ such that $\mu_k \leq M$ for all $k \in \mathbb{N}$. Equip the set $\Lambda = \mathbb{N} \times \Gamma$ with the coordinate-wise order and define a net

$$z_\lambda = z_{(n,\gamma)} = \frac{1}{n}|x| + Mu_\gamma \quad \text{for each } \lambda \in \Lambda.$$

It follows from the Archimedean property that $z_\lambda \downarrow 0$ in X.

Now equip the set $B = \mathbb{N} \times A$ with the coordinate-wise order and define

$$\omega_\beta = \omega_{(k,\alpha)} = \mu_k x_\alpha \quad \text{for each } \beta \in B.$$

It follows that for each each $\lambda = (n,\gamma) \in \Lambda$ there exists $\beta_0 = (k_0, \alpha_0)$ such that $\beta = (k,\alpha) \geq \beta_0$ implies

$$\begin{aligned} |\omega_\beta - \mu x| &= |\mu_k x_\alpha - \mu_k x + \mu_k x - \mu x| \\ &\leq |\mu_k - \mu||x| + |\mu_k||x_\alpha - x| \\ &\leq \frac{1}{n}|x| + Mu_\gamma = z_\lambda. \end{aligned}$$

Since $z_\lambda \downarrow 0$ implies $\mu_k x_\alpha \xrightarrow{o} \mu x$, (X,o) is a convergence vector space. Then (X,λ_o) is a convergence vector space by Theorem 2.46. □

The following example shows that, in contrast to the order convergence structure, the order topology on a vector lattice may fail to be linear (and Hausdorff) even when the space is Archimedean; cf. [Vla69, p. 146].

Example 3.23. Let $X = C[0,1]$ and let (q_n) be an enumeration of the rationals in $[0,1]$. $\mathbf{0}$ and $\mathbf{1}$ will be used to denote the functions with constant value 0 and constant value 1 on $[0,1]$, respectively. We define a net in X as follows: For each $k \in \mathbb{N}$ put

$$f_{k,n}(t) = \begin{cases} 0 & t \in [0, q_n - \frac{1}{k}) \cup (q_n + \frac{1}{k}, 1] \\ kt + (1 - kq_n) & t \in [q_n - \frac{1}{k}, q_n) \\ -kt + (1 + kq_n) & t \in (q_n, q_n + \frac{1}{k}] \\ 1 & t = q_n. \end{cases}$$

These are simply the functions that take a maximum value of 1 at q_n, are linear on $[q_n - \frac{1}{k}, q_n)$ and $(q_n, q_n + \frac{1}{k}]$, and vanish everywhere else on $[0,1]$. For fixed n, observe that $f_{k,n} \downarrow \mathbf{0}$. So $f_{k,n} \xrightarrow{o} \mathbf{0}$ for each n. It follows from Corollary 3.16 that $f_{k,n} \xrightarrow{\tau(o)} \mathbf{0}$ for each n.

Let U be any $\tau(o)$-open neighborhood of $\mathbf{0}$. Then there is some k_1 such that $f_{k_1,n} \in U$ for $k \geq k_1$ and all n. Put $g_1 = f_{k_1,1}$. Now $g_1 \vee f_{k,2} \downarrow g_1$ implies $g_1 \vee f_{k,2} \xrightarrow{o} g_1$; hence $g_1 \vee f_{k,2} \xrightarrow{\tau(o)} g_1$. It follows that there is some k_2 such that $g_1 \vee f_{k_2,2} \in U$. Put $g_2 = g_1 \vee f_{k_2,2}$. Continuing inductively yields an increasing sequence $g_n = g_{n-1} \vee f_{k_n,n} \in U$ for all n. Now $g_n(q_n) = 1$ for all n, so g_n attains the value 1 on a dense subset of $[0,1]$. It follows that $g_n \uparrow \mathbf{1}$ and, therefore, $g_n \to \mathbf{1}$ in the order topology. Let V be any $\tau(o)$-open neighborhood of $\mathbf{1}$. Then there is some n_0 such that $n \geq n_0$ implies $g_n \in V$. Since U and V were arbitrary $\tau(o)$-open neighborhoods of $\mathbf{0}$ and $\mathbf{1}$, respectively, we conclude that $\mathbf{0}$ and $\mathbf{1}$ cannot be separated by disjoint $\tau(o)$-open sets. It follows that the order topology on $C[0,1]$ is not Hausdorff.

We can now show the order topology on a vector lattice is generally not a linear topology. Recall that points are closed in $(C[0,1], o)$ by Corollary 2.29 and, therefore, $\tau(o)$-closed. Recall that for linear topological spaces the latter condition is equivalent to the topology being Hausdorff; see for example Theorem 5.1 in [KN76]. The previous example shows the topology $\tau(o)$ on $C[0,1]$ is not Hausdorff and, therefore, cannot be linear.

Thus, there is an advantage to working with convergence structures instead of topologies on vector lattices: convergence vector space theory applies in situations where the theory of topological vector spaces may not. This could explain why many authors work with Archimedean vector lattices. Since we are looking to apply convergence vector space theory to study order convergence, we will assume all our vector lattices are Archimedean.

The ability to use nets to study filter convergence structures highlights the utility of the theory presented in Chapter 2. Even though the net and filter theories of convergence are equivalent, we are free to choose when to use one language over the other. Indeed, there are still situations where it may be easier to work with filters.

A convergence vector space is said to be ***locally convex*** if $co(\mathcal{F}) \to 0$ whenever $\mathcal{F} \to 0$ where $co(\mathcal{F}) = [\mathcal{B}]$ for the filter base

$$\mathcal{B} = \{co(F) : F \in \mathcal{F}\}$$

where $co(F)$ denotes the convex hull of F. For λ_o, we can deduce this property directly instead of translating into the language of nets.

Proposition 3.24. *(X, λ_o) is a locally convex convergence space.*

Proof. Suppose $\mathcal{F} \xrightarrow{\lambda_o} 0$. Then there are nets $a_\gamma \uparrow 0$ and $b_\gamma \downarrow 0$ such that $[a_\gamma, b_\gamma] \in \mathcal{F}$ for all γ. Order intervals are convex, so we get $co([a_\gamma, b_\gamma]) = [a_\gamma, b_\gamma]$ for every γ. It follows that $[a_\gamma, b_\gamma] \in co(\mathcal{F})$ for every γ; hence $co(\mathcal{F}) \xrightarrow{\lambda_o} 0$. □

We can prove that λ_o is a regular convergence space in a similar fashion. Note that there is an unfortunate conflict of terminology with the notion of regular sublattice. A convergence space is said to be ***regular*** if $a(\mathcal{F}) \to x$ whenever $\mathcal{F} \to x$ where $a(\mathcal{F}) = [\{a(F) : F \in \mathcal{F}\}]$.

Proposition 3.25. *(X, λ_o) is a regular convergence space.*

Proof. Suppose $\mathcal{F} \xrightarrow{\lambda_o} x$. Then there are nets $a_\gamma \uparrow x, b_\gamma \downarrow x$ with $[a_\gamma, b_\gamma] \in \mathcal{F}$ for every γ. Order intervals are order closed, so they are closed in (X, λ_o) by Proposition 3.14; hence $a_{\lambda_o}([a_\gamma, b_\gamma]) = [a_\gamma, b_\gamma]$ for every γ. Now $[a_\gamma, b_\gamma] \in \mathcal{F}$ for every γ gives $[a_\gamma, b_\gamma] \in a_{\lambda_o}(\mathcal{F})$ for every γ. So $a_{\lambda_o}(\mathcal{F}) \xrightarrow{\lambda_o} x$. □

Proposition 3.26. *A subset B of X is λ_o-bounded if and only if it is order bounded.*

Proof. Assume that B is order bounded. Then there is some $u \in X_+$ such that $B \subseteq [-u, u]$. Let $(\mu_\alpha)_{\alpha \in A}$ be a net in $\mathbb{R}$ such that $\mu_\alpha \to 0$. We must show the net $\mu_\alpha b \xrightarrow{o} 0$ when viewed as a net indexed over $A \times B$ directed by the first component. Note that $|\mu_\alpha b| \leq |\mu_\alpha| u$ for all α and $b \in B$. Now $|\mu_\alpha| u \xrightarrow{o} 0$ implies $\mu_\alpha b \xrightarrow{o} 0$ and hence B is o-bounded. It is λ_o-bounded by Proposition 2.48.

Now assume that B is λ_o-bounded. Then it must also be o-bounded by Proposition 2.48. In particular, the net $(\frac{1}{n} b)$ must converge in order to zero when viewed as a net indexed by $\mathbb{N} \times B$ directed by the first component. Since every order convergent net has an order bounded tail, it follows that there is some $u \in X_+$ and $n_0 \in \mathbb{N}$ such that $\frac{1}{n} b \in [-u, u]$ for every $b \in B$ whenever $n \geq n_0$. This shows $B \subseteq [-n_0 u, n_0 u]$, so B is order bounded. □

Corollary 3.27. *(X, λ_o) is a locally bounded convergence space.*

Proof. It is well-known that every order convergent net has an order bounded tail. Since order bounded is the same as o-bounded, this means (X, o) is locally bounded. Then (X, λ_o) is locally bounded by Proposition 2.49. □

3.4. **Applications to Completeness and Completions.** We finish this section by highlighting another concept in vector lattices that can be described using convergence theory. Recall that a vector lattice is ***Dedekind complete*** if every positive increasing order bounded net has a supremum.

Every Archimedean vector lattice X is lattice isomorphic to an order dense sublattice of its ***order completion*** X^δ. Since lattice isomorphisms preserve linear, lattice, and order convergence structures, we may identify X with an order dense sublattice of X^δ.

Corollary 3.28. *X is λ_o-dense in X^δ.*

Proof. X is an order dense, and therefore, regular sublattice of X^δ. It follows from Corollary 3.19 that X is λ_o-dense in X^δ. □

The next result says there is no need to distinguish between the order convergence structure on X and X^δ.

Proposition 3.29. *The convergence subspace structure on X induced from order convergence on X^δ is the order convergence from X.*

Proof. Let s_o^δ denote the order convergence subspace structure induced on X from (X^δ, o). It is a standard fact of vector lattices that X is an order dense and majorizing sublattice of X^δ. By Corollary 3.21 $id_X : (X, o) \to (X, s_o^\delta)$ is an linear homeomorphism. □

We will show that Dedekind complete vector lattices are precisely those in which every order Cauchy net converges in order; *i.e.*, they are complete with respect to the order convergence structure.

Lemma 3.30. *Every monotone order bounded net in an Archimedean vector lattice is order Cauchy.*

Proof. Let $(x_\alpha)_{\alpha\in A}$ be a net in X satisfying $x_\alpha \uparrow\leq u$ in X. Then $s = \sup x_\alpha$ exists in X^δ and, therefore, $x_\alpha \xrightarrow{o} s$ in X^δ. Since every order convergent net is also order Cauchy, the double net $(x_\alpha - x_\beta)_{(\alpha,\beta)\in A\times A} \xrightarrow{o} 0$ in X^δ and, therefore, in X. A similar argument works for decreasing nets. □

Proposition 3.31. *An Archimedean vector lattice is Dedekind complete if and only if it is complete with respect to the order convergence structure.*

Proof. Note that by equivalence of net and filter convergence structures it suffices to prove the result using nets. Let (x_α) be an order Cauchy net in an order complete vector lattice. By passing to a tail and using the fact that (X, o) is locally bounded, we may assume that (x_α) is order bounded. Define

$$x = \sup a_\alpha \text{ where } a_\alpha = \inf_{\beta \geq \alpha} x_\beta \text{ and } y = \inf b_\alpha \text{ where } b_\alpha = \sup_{\beta \geq \alpha} x_\beta.$$

Clearly, $a_\alpha \leq x \leq y \leq b_\alpha$ for every α. Since X is order complete, it suffices to show that $x = y$. Let $(v_{\alpha,\beta})$ be a net such that $v_{\alpha,\beta} \downarrow 0$ and $|x_\alpha - x_\beta| \leq v_{\alpha,\beta}$. Fix (α_0, β_0) and take α so that $\alpha \geq \alpha_0$ and $\alpha \geq \beta_0$. For every β with $\beta \geq \alpha$, we have $(\alpha, \beta) \geq (\alpha_0, \beta_0)$; hence $|x_\alpha - x_\beta| \leq v_{\alpha_0,\beta_0}$. It follows that $x_\beta \in [x_\alpha - v_{\alpha_0,\beta_0}, x_\alpha + v_{\alpha_0,\beta_0}]$, which yields $a_\alpha, b_\alpha \in [x_\alpha - v_{\alpha_0,\beta_0}, x_\alpha + v_{\alpha_0,\beta_0}]$ and, therefore, $0 \leq b_\alpha - a_\alpha \leq 2v_{\alpha_0,\beta_0}$, and hence $0 \leq y - x \leq 2v_{\alpha_0,\beta_0}$. If follows that $x - y = 0$.

Conversely, suppose that every order Cauchy net in X is order convergent. Let $0 \leq x_\alpha \uparrow \leq u$. By the lemma, (x_α) is order Cauchy, hence order convergent; it follows that $\sup x_\alpha$ exists. □

The following result is a summary of several basic results that have been covered in this chapter.

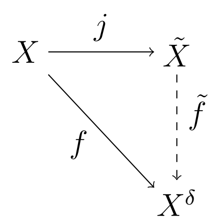
Theorem 3.32. *Let X be an Archimedean vector lattice. Then (X, λ_o) is homeomorphic to a λ_o-dense convergence space of the Hausdorff λ_o-complete convergence vector space (X^δ, λ_o).*

Furthermore, (X^δ, λ_o) satisfies a universal property among such spaces — this is given by Veksler's Theorem; see, for example, Theorem 1.65 in [ABP06, p. 55]. It follows that the order completion is uniquely determined up to lattice isomorphism and, therefore, very deserving of the name "completion". It is interesting to compare these results with the completion of (X, λ_o) as a convergence vector space.

Remark 3.33. If X is Archimedean then (X, λ_o) is a Hausdorff locally bounded convergence vector space. Take $Y = (X^\delta, \lambda_o)$ and $f : X \hookrightarrow X^\delta$ to be the inclusion map in Theorem 1.12. Since X is regular in X^δ (*i.e.* f is order continuous) and X^δ is λ_o-complete, Theorem 1.12 yields a complete Hausdorff convergence vector space $\tilde{X}$, a linear embedding $j : X \to \tilde{X}$ where $j(X)$ is dense in $\tilde{X}$, and a unique continuous linear extension of f

$$\tilde{f} : \tilde{X} \to (X^\delta, \lambda_o)$$

such that the following diagram commutes:



That is, $\tilde{f}$ is the (unique) continuous linear map $\tilde{f} : \tilde{X} \to (X^\delta, \lambda_o)$ with $f = \tilde{f} \circ j$. While there is no reason to expect that $\tilde{X}$ is a vector lattice, the elements of $\tilde{X}$ belong to X^δ. In fact, we will show that $\tilde{f}$ is a bijection.

Proposition 3.34. *Let X be an Archimedean vector lattice and let $\tilde{X}$ denote the completion of X as a convergence vector space. Then the map $\tilde{f} : \tilde{X} \to X^\delta$ highlighted in Remark 3.33 is an algebraic isomorphism.*

Proof. Suppose that X is Archimedean and let $f, j, \tilde{X}$ and $\tilde{f}$ be as in Remark 3.33. Since $\tilde{f}$ is linear, it remains to show that $\tilde{f}$ is a bijection.

For surjectivity, let $z \in X^\delta$. Since X is λ_o-dense in X^δ, there is a net $(x_\alpha) \in X$ such that $f(x_\alpha) \xrightarrow{o} z$ in X^δ. It follows that (x_α) is order Cauchy in X^δ and, therefore, in

X. Then $\big(j(x_\alpha)\big)$ is Cauchy in $\tilde{X}$; hence it converges to some element $y \in \tilde{X}$. Then $f(x_\alpha) = \tilde{f}(j(x_\alpha)) \xrightarrow{o} \tilde{f}(y)$ and $f(x_\alpha) \xrightarrow{o} z$. Thus, $\tilde{f}(y) = z$ for some $y \in \tilde{X}$.

For injectivity, suppose $y \in \ker \tilde{f}$. $j(X)$ is dense in $\tilde{X}$ so we can find a net (x_α) in X such that

$$j(x_\alpha) \to y \text{ in } \tilde{X}.$$

Then $f(x_\alpha) = \tilde{f}(j(x_\alpha)) \xrightarrow{o} \tilde{f}(y) = 0$ in X^δ. But order convergence in X^δ agrees with order convergence in X, so $x_\alpha \xrightarrow{o} 0$ in X. It follows that

$$j(x_\alpha) \to 0 \text{ in } \tilde{X}.$$

Now $\tilde{X}$ is Hausdorff implies $y = 0$ and $\ker \tilde{f} = \{0\}$. □

While the construction of the convergence vector space completion in [GGK76] generalizes the completion of a topological vector space, it is not clear if it generalizes the order completion of an Archimedean vector lattice; it is also unclear from their construction if $\tilde{X}$ even inherits a natural vector lattice structure from X. We hope that it will be possible to describe a general completion procedure using the language of nets. If this can be done, we expect the result will be more intuitive than the construction in [GGK76]. There is promise in this direction.

Recall that the most common construction of the order completion for an Archimedean vector lattice is analogous to the construction of $\mathbb{R}$ using Dedekind cuts; see, for example, [Vul67]. Intuitively, the space is completed by adjoining certain sets that fit together in a nice way. Also recall the alternate construction of $\mathbb{R}$ using equivalence classes of Cauchy sequences of rationals. The latter construction has an analogue for general metric spaces; cf. Exercise 9 in [Mun, p.271]. It also has an analogue for Archimedean vector lattices.

Given an Archimedean vector lattice X, we may view the order convergence structure of its order completion X^δ as a ***Cauchy completion*** of the order convergence structure on

X in the following sense: X is a dense subspace of X^δ and the elements of X^δ are limits of order Cauchy nets in X.

Proposition 3.35. *X^δ is a Cauchy completion of X with respect to the order convergence structure.*

Proof. By Corollary 3.28 and Proposition 3.18 the order closure of X taken in X^δ is equal to all of X^δ. Thus, $x \in X^\delta = \overline{X}^o$ implies there is a net $(x_\alpha) \in X$ such that $x_\alpha \xrightarrow{o} x$ in X^δ. It is easy to see that order convergent nets are o-Cauchy, so every element of X^δ is the order limit of an order Cauchy net in X.

Conversely, every o-Cauchy net in X has an order limit in X^δ. To see this, start with an o-Cauchy net (x_α) in X. X is an order dense and, therefore, regular sublattice of X^δ. It follows that (x_α) is o-Cauchy in X^δ. But X^δ is order complete, so Proposition 3.31 implies that (X^δ, o) is complete; hence $x_\alpha \xrightarrow{o} x$ for some $x \in X^\delta$. □

Inspired by the previous observation, a positive answer to the following question may provide a more intuitive notion of completion in general convergence spaces.

Question 3.36. Consider the collection $\mathscr{C}$ of all Cauchy nets in a Hausdorff locally bounded convergence vector space X. For nets (x_α) and (y_α) in X define $(x_\alpha) \sim (y_\alpha)$ if $(x_\alpha - y_\beta)_{(\alpha,\beta)\in A\times A} \to 0$. $\sim$ is an equivalence relation on $\mathscr{C}$. Put $\tilde{X} = \mathscr{C}/\sim$. Does $\tilde{X}$ correspond to the completion of X?

4. Order Compactness

In Chapter 2 we showed that nets can be used to describe an abstract theory of convergence. We then applied these results to order convergence in vector lattices and demonstrated several known concepts in vector lattice theory are described naturally using convergence structures. In this chapter we introduce something new into the area of vector lattices: *order compactness*. We will show that this notion of compactness satisfies an analogue of the Heine-Borel theorem in atomic order complete vector lattices. It also simplifies certain concepts in Banach lattice theory.

Throughout this chapter X will denote an Archimedean vector lattice. A subset $A \subset X$ is said to be ***order compact*** if it is compact in the order convergence structure. Recall from Remark 2.43 that compact subsets are preserved by passing to the associate convergence. This allows us to use nets or filters to study order compactness whenever it is convenient.

We begin by looking for examples of order compact sets. It is easy to check that order convergence agrees with the standard convergence in $\mathbb{R}^n$; hence order compactness agrees with the notion of norm compactness in this setting. Since every finite dimensional Archimedean vector lattice is lattice isomorphic to some $\mathbb{R}^n$, order and norm compactness agree in this slightly more general setting.

If X has a norm satisfying

(i) $x \leq y$ implies $\|x\| \leq \|y\|$ for every $x, y \in X_+$;

(ii) $\big\||x|\big\| = \|x\|$ for every $x \in X$.

then X is called a ***normed lattice*** and the norm itself is called a ***lattice norm***. If X is complete with respect to a lattice norm then we call it a ***Banach lattice***. Since X is a Banach space, we may discuss convergence in norm on X. In this situation, X is said to be ***order continuous*** (or to have ***order continuous norm***) if $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \to x$ in norm.

Proposition 4.1. *If X is an order continuous Banach lattice then order compact sets are norm compact.*

Proof. Order convergence is stronger than norm convergence in order continuous spaces. Now apply Corollary 2.41. □

In particular, the unit ball B_X of an infinite-dimensional order continuous Banach lattice is never order compact. Later we will see that for an (infinite) compact Hausdorff space K, order compactness of the unit ball in $C(K)$ is more interesting.

Proposition 4.2. *An order compact subset is necessarily order closed and order bounded.*

Proof. (X, λ_o) is Hausdorff by Proposition 3.11, so Proposition 2.40 implies that any order compact set must be λ_o-closed. The latter is equivalent to order closed by Proposition 3.14. To observe the second property we recall from Corollary 3.27 that (X, λ_o) is locally bounded. Now Proposition 1.11 implies that any λ_o-compact set is λ_o-bounded and hence order bounded by Proposition 3.26. □

The following example gives an order closed and order bounded subset of a vector lattice that fails to be order compact, so the converse of Proposition 4.2 is generally false.

Example 4.3. Generally, we cannot expect order intervals to be order compact. We use L_p to denote $L_p[0,1]$ for $1 < p < \infty$. It follows from Lebesgue's Dominated Convergence Theorem that the L_p-norm is order continuous. It follows from Proposition 4.1 that any order compact set in L_p must also be norm compact. As a consequence, if $[-\mathbf{1}, \mathbf{1}]$ were order compact then it would also be norm compact. Consider the Rademacher functions $r_n(t) = \text{sgn} \ \sin(2^n t)$ in $L_\infty[0,1] \subset L_p$. Clearly we have $r_n \in [-\mathbf{1}, \mathbf{1}]$ for all n. Now we use the norm compactness of $[-\mathbf{1}, \mathbf{1}]$ to find positive integers $n_1 < n_2 < \cdots < n_k$ and $r \in [-\mathbf{1}, \mathbf{1}]$ such that $r_{n_k} \to r$ in L_p-norm. Then $r_{n_k} \xrightarrow{w} r$ must also hold. However, note that $r_n \xrightarrow{w} 0$, forcing $r = 0$. This is a contradiction since $\|r_n\|_p = 1$ for all n; so $[-\mathbf{1}, \mathbf{1}]$

cannot be order compact. This also shows that a weakly compact set may fail to be order compact.

Example 4.4. Let $\{e_n : n \in \mathbb{N}\}$ denote the standard unit vector basis of ℓ_1. Consider the sequence $x_n = \frac{1}{n}e_n$. Then $x_n \to 0$ in ℓ_1-norm implies $\{0\} \cup \{\frac{1}{n}e_n : n \in \mathbb{N}\}$ is a norm compact set. However, it is not order compact because it is not even order bounded in ℓ_1.

An ***atom*** in a vector lattice is an element $a \geq 0$ such that $0 \leq x \leq a$ implies $x \in \operatorname{span}\{a\}$; *i.e.*, $I_a = \operatorname{span}\{a\}$.

Proposition 4.5. *If a is an atom in X then $[0, a]$ is order compact.*

Proof. Let a be an atom in X_+. Then $I_a = \operatorname{span}\{a\}$. Define a linear operator $T : \mathbb{R} \to I_a$ *via* $T(1) = a$. Clearly T is a linear isomorphism. It follows that I_a is lattice isomorphic to $\mathbb{R}$. Lattice isomorphisms are automatically order continuous, so we have a convergence space homeomorphism $T : (\mathbb{R}, o) \to (I_a, o)$ given by $T(\mu) = \mu a$. Now I_a is an ideal and, therefore, a regular sublattice of X; this makes the inclusion $\iota : (I_a, o) \hookrightarrow (X, o)$ continuous. We thus obtain a continuous map $\iota \circ T : (\mathbb{R}, o) \to (X, o)$ between Hausdorff convergence spaces. Now apply Proposition 2.40 to get $(\iota \circ T)([0, 1]) = [0, a]$ is order compact in X. □

Since there is a connection between atoms and order compact intervals, it is natural to investigate the converse of Proposition 4.2 in spaces with a large supply of atoms. To do this, we need some more standard terminology. A ***band*** in a vector lattice is an order closed ideal. The smallest band that contains a given subset S is called the ***band generated by*** S; it is the intersection of all the bands containing S and denoted B_S. X is said to be ***atomic*** if $X = B(A)$, the band generated by the set A of all atoms in X. Atomic vector lattices are essentially just order dense sublattices of the function space $\mathbb{R}^A$ for some set A. We will now provide a brief sketch of these ideas.

There is a surprising connection between bands and disjoint vectors. Two vectors x and y are called ***disjoint*** if $|x| \wedge |y| = 0$ and we denote this relationship by $x \perp y$. Given a subset A of a vector lattice the set $A^d = \{x \in X : x \perp a \text{ for all } a \in A\}$ is called the ***disjoint complement of*** A. For any subset S, S^d is always a band. Moreover, we have $B_S = S^{dd}$.

A band B is called a ***projection band*** if $X = B \oplus B^d$; that is, for each $x \in X$ there is a unique $y \in B$ and $z \in B^d$ such that $x = y + z$. This induces a positive linear projection from X onto B. Indeed, for each $x \in X$ there are is a unique $y \in B$ and $z \in B^d$ where $x = y + z$. Then one can define $P : X \to X$ *via* $Px = y$. The resulting operator is the desired positive linear projection and is called the ***band projection*** onto B.

If $a \in X_+$ is an atom then $\text{span}\{a\} = I_a = B_a$ is a projection band. Let P_a denote the corresponding band projection. The projection P_a satisfies

$$P_a(x) \in B_a = \text{ span } \{a\} \quad \forall x \in X.$$

This means $P_a(x) = \lambda a$ for some constant $\lambda = \lambda(a, x) \in \mathbb{R}$ that depends on both a and x. Then the map $\phi_a : X \to \mathbb{R}$ defined by $\phi_a(x) = \lambda$ defines a positive (order continuous) linear functional called the ***coordinate functional of a***.

If X is atomic and A is taken to be a maximal disjoint collection of atoms in X, then we may view X as an order dense sublattice of $\mathbb{R}^A$ *via* the following identification: For each $x \in X$ consider the map

$$\rho_x : A \to \mathbb{R} \qquad \rho_x(a) = \phi_a(x).$$

The map $T : X \to \mathbb{R}^A$ given by $T(x) = \rho_x$ is a lattice isomorphism whose image $T(X)$ is an order dense sublattice of $\mathbb{R}^A$. It is customary to identify X with $T(X)$ and treat X as an order dense sublattice of $\mathbb{R}^A$. If X is also order complete, it is an ideal in $\mathbb{R}^A$.

We are now ready to prove our first major result of this section.

Proposition 4.6. *If X is atomic and order complete then every order interval is order compact.*

Proof. Let $c \in X_+$. It suffices to prove $[0,c]$ is order compact; the general case follows from the fact that, for $a < b$, $[a,b]$ is the image of $[0,b-a]$ under the (order) continuous map $\ell_a(x) = a + x$.

Since X is atomic and order complete, we may identify it with an ideal of $\mathbb{R}^A$ for some maximal disjoint collection of atoms A in X. Now equip $\mathbb{R}^A$ with the product topology and, for each $a \in A$, let π_a denote the standard coordinate projection. Then $[0,\pi_a(c)]$ is a compact subset of $\mathbb{R}$ for each a. It follows that $\prod_{a\in A}[0,\pi_a(c)]$ is compact in $\mathbb{R}^A$.

The first claim is that $[0,c]$ is a compact subset of X when viewed from $\mathbb{R}^A$. Since $[0,c] \subset \prod_{a\in A}[0,\pi_a(c)]$, it remains to show $[0,c]$ is pointwise closed in X. Let (x_α) be a net in $[0,c]$ such that $x_\alpha \to x$ in the topology of pointwise convergence. Then $x_\alpha(a) \to x(a)$ for each $a \in A$. For each a, and for all α, we have $0 \le x_\alpha(a) \le c(a)$. Passing to the limit gives $0 \le x \le c$ and, as X is an ideal, $x \in X$. Thus, as a closed subset of a compact space, $[0,c]$ is a compact subset of X in the topology of pointwise convergence from $\mathbb{R}^A$; see Example 4.7.

Next we show that pointwise convergence in $\mathbb{R}^A$ agrees with order convergence for order bounded nets; in fact, this is true for any set A. One direction is immediate: if $(f_\alpha) \in \mathbb{R}^A$ and $f_\alpha \xrightarrow{o} f$ then $f_\alpha(x) \to f(x)$ for each $x \in A$. For the converse, suppose $f_\alpha \to f$ pointwise and that (f_α) is order bounded. We may assume without loss of generality that all $f_\alpha \ge 0$ and $f = 0$. Since $\mathbb{R}^A$ is order complete and (f_α) is order bounded, it remains to show $\inf_\alpha(\sup_{\beta\ge\alpha} f_\alpha) = 0$; see, for example, Remark 2.2 in [GTX17]. Let $v_\alpha = \sup_{\beta\ge\alpha} f_\beta$. Clearly $0 \le v_\alpha \downarrow$. Suppose $u \le v_\alpha$ for all α, and let $\epsilon > 0$ and $x \in A$ be fixed. Then there is some α_0 such that $\alpha \ge \alpha_0$ implies $f_\alpha(x) < \epsilon$. It follows that

$$v_{\alpha_0}(x) = \sup_{\alpha\ge\alpha_0} f_\alpha(x) \le \epsilon.$$

Since v_α is decreasing, we must have $v_\alpha(x) \le \epsilon$ for all $\alpha \ge \alpha_0$. As ϵ and x were arbitrary, we obtain $v_\alpha \to 0$ pointwise; therefore, $u \le 0$. This shows $0 = \inf_\alpha v_\alpha = \inf_\alpha(\sup_{\beta \ge \alpha} f_\alpha)$ and hence $f_\alpha \xrightarrow{o} 0$.

There is one subtlety remaining: since X is an ideal, and therefore a regular sublattice of $\mathbb{R}^A$, the order convergence on X is equivalent to the order convergence on $\mathbb{R}^A$ for order bounded nets; see, for example, Corollary 2.12 in [GTX17]. Therefore, the order convergence structure on $[0, c]$ inherited from X is equivalent to the order convergence structure on $[0, c]$ inherited from $\mathbb{R}^A$. Since the latter agrees with pointwise convergence and makes $[0, c]$ into a compact space, $[0, c]$ must also be order compact in X. $\square$

This result will be significantly improved in Theorem 4.12.

Example 4.7. The following example shows that if X is a just an order dense sublattice of $\mathbb{R}^A$ (*i.e.* atomic) then, from the point of view of the topology of pointwise convergence on $\mathbb{R}^A$, order intervals in X are "porous". Let e_i denote the i-th standard vector in c and define

$$x_n = \sum_{i=1}^{n} e_{2i}.$$

Then $x_n \in [\mathbf{0}, \mathbf{1}]$ for all n, $x_n \to (0, 1, 0, 1, 0, 1, ...)$ pointwise in $\mathbb{R}^{\mathbb{N}}$, but $(0, 1, 0, 1, 0, 1, ...) \notin c$; here $\mathbf{0}$ and $\mathbf{1}$ denote the constant sequences $(0, 0, 0, ...)$ and $(1, 1, 1, ...)$, respectively. The assumption of order completeness in Proposition 4.6 was used to guarantee that order intervals in X are closed when viewed in $\mathbb{R}^A$.

We have seen how order compactness is (strictly) stronger than norm compactness in the presence of an order continuous norm. In the absence of an order continuous norm, it can be significantly weaker. Indeed, ℓ_∞ is atomic and order complete so the unit ball $B_1^{\ell_\infty} = [-\mathbf{1}, \mathbf{1}]$ is order compact. On the other hand, the unit ball of an infinite-dimensional Banach space is never norm compact.

We present some further applications of Proposition 4.6.

Corollary 4.8. *Every order complete atomic vector lattice is λ_o-locally compact.*

Proof. By Remark 3.10 every λ_o-convergent filter contains an order interval. If X is order complete and atomic then every order interval is order compact; hence every λ_o-convergent filter contains a λ_o-compact subset. □

It follows that $\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}$, c_0 and ℓ_p where $0 < p \leq \infty$ are all λ_o-locally compact.

Remark 4.9. It is a fact of general convergence theory that every locally compact convergence space is necessarily complete; see, for example, [BB02, Proposition 3.1.14, p. 84]. Then the assumption of order completeness in Corollary 4.8 cannot be removed. The same can be said for Proposition 4.6.

It is part of the folklore of Banach lattices that order intervals are norm compact precisely when the space is atomic and has an order continuous norm. The results above make one direction of the proof completely trivial.

Corollary 4.10. *Let X be a Banach lattice. If X is atomic and has an order continuous norm then order intervals are norm compact.*

Proof. If X has order continuous norm then it is order complete. Now X is atomic implies $[a, b]$ is order compact. Since the norm on X is order continuous, it follows from Proposition 4.1 that $[a, b]$ is norm compact. □

Remark 4.11. In the previous proof we used the standard fact that every order continuous Banach lattice is order complete. We give a very natural proof of this fact. Indeed, if $0 \leq x_\alpha \uparrow\leq u$ in X then $0 \leq x_\alpha \uparrow\leq u$ in X^δ. Then, by Lemma 3.30, we know that (x_α) is order Cauchy in X^δ. Since the double net $x_\alpha - x_\beta \xrightarrow{o} 0$ in X^δ, we must have $x_\alpha - x_\beta \xrightarrow{o} 0$ in X. Now order continuity of the norm implies $x_\alpha - x_\beta \to 0$ in norm; hence $x_\alpha \to x$ for some x. It follows from the Monotone Convergence Lemma of vector lattices that $x = \sup x_\alpha$ exists and the proof is complete.

We now come to a remarkable characterization of the order compact subsets in order complete atomic vector lattices; it is an analogue of the Heine-Borel theorem for order compactness.

Theorem 4.12. *If X is atomic and order complete then a subset is order compact if and only if it is order closed and order bounded.*

Proof. The forward implication is just Proposition 4.2 and does not require the space to be atomic or order complete.

For the converse, suppose A is order closed and order bounded. Then there are $a, b \in X$ such that $A \subseteq [a, b]$. Proposition 4.6 now applies to yield $[a, b]$ is order compact. Now we have A is order closed, so applying Proposition 2.40 gives A is order compact. □

It is natural to ask whether the converse is also true.

Question 4.13. Let X be an Archimedean vector lattice. If every interval is order compact, must X be atomic and order complete?

We will show that vector lattices in which order intervals are order compact are necessarily order complete.

Lemma 4.14. *Let (x_α) be a monotone net. If it has an order convergent quasi-subnet then (x_α) converges to the same limit.*

Proof. Let (y_β) be a quasi-subnet of (x_α) with $y_\beta \xrightarrow{o} y$. Without loss of generality $x_\alpha \downarrow$ and $y = 0$. There exists a net (u_γ) such that $u_\gamma \downarrow 0$ and for every γ_0 there exists β_0 such that $y_\beta \in [-u_\gamma, u_\gamma]$ for all $\alpha \geq \alpha_0$.

Fix γ_0 and find β_0 as above. For every α_0, we can find $\alpha_1 \geq \alpha_0$ and $\beta_1 \geq \beta_0$ such that $y_{\beta_1} = x_{\alpha_1}$. It follows that $x_{\alpha_0} \geq x_{\alpha_1} = y_{\beta_1} \geq -u_{\gamma_0}$. Since γ_0 is arbitrary and $-u_\gamma \uparrow 0$, we conclude that $x_{\alpha_0} \geq 0$, hence (x_α) is in X_+. Moreover, for every $\alpha \geq \alpha_1$, we have $0 \leq x_\alpha \leq x_{\alpha_1} = y_{\beta_1} \leq u_{\gamma_0}$. It follows that $x_\alpha \downarrow 0$ and consequently $x_\alpha \xrightarrow{o} 0$. □

Proposition 4.15. *Let X be a vector lattice. If every order interval is order compact then X is order complete.*

Proof. Suppose that $0 \leq x_\alpha \uparrow\leq u$. Since $[0,u]$ is order compact, (x_α) has an order convergent quasi-subnet. By Lemma 4.14, $\sup x_\alpha$ exists. $\square$

Thus, Question 4.16 reduces to the following open question.

Question 4.16. If every order interval in X is order compact, must X be atomic?

The remainder of this chapter is devoted to collecting some simple facts that may be useful for answering Question 4.16.

Even though order compactness is weak enough to allow the unit ball of ℓ_∞ to be order compact, order compactness of the unit ball in a normed lattice is still a rather strong property. A positive vector $e > 0$ is called a ***strong unit*** if $I_e = X$.

Proposition 4.17. *Let X be a normed lattice. If B_X is order compact then X has a strong unit.*

Proof. If B_X is order compact then it must be order bounded by Proposition 4.2. So there is some $e \in X_+$ with $B_X \subseteq [-e,e]$. Then for each nonzero $x \in X$ we have $\frac{1}{\|x\|}x \in [-e,e]$; hence $|x| \leq \|x\|e$. It follows that e is a strong unit. $\square$

Proposition 4.18. *Let $X = C(K)$. If $[\mathbf{0},\mathbf{1}]$ is order compact then every order interval in X is order compact.*

Proof. It suffices to show that every order interval is the image of $[0,\mathbf{1}]$ under an order continuous map. Fix $u \in X_+$. The map $T\colon X \to X$ given via $Tx = u \cdot x$ is positive and order continuous. It now follows from $T[0,\mathbf{1}] = [0,u]$ that $[0,u]$ is compact. It is now easy to see that every order interval is order compact. $\square$

Proposition 4.19. *Let X be a Banach lattice. If B_X is order compact then every order interval in X is order compact.*

Proof. This follows easily from the previous two results. Indeed, the order compactness of the unit ball implies X has a strong unit; hence X is lattice isomorphic to a $C(K)$ space for some compact Hausdorff K. In this setting, B_X corresponds to $[-\mathbf{1}, \mathbf{1}]$ in $C(K)$. It follows that $[-\mathbf{1}, \mathbf{1}]$ is order compact in $C(K)$; hence every order interval in X is order compact. $\square$

In particular, for Banach lattices, B_X is order compact implies X must be an order complete space with a strong unit. Thus X is lattice isomorphic to a $C(K)$ space where K is extremally disconnected. The next example demonstrates that such spaces can have many order compact intervals.

Example 4.20. Let $\beta\mathbb{N}$ denote the Stone-Čech compactification of $\mathbb{N}$. Then $C(\beta\mathbb{N})$ is lattice isometric to ℓ_∞; see for example Example 2 on page 152 in [Car04]. Since ℓ_∞ is atomic and order complete, all order intervals are order compact in $C(\beta\mathbb{N})$.

Remark 4.21. The previous example can be generalized: if $K = \beta\Gamma$ for some discrete space Γ then $C(K)$ is lattice isometric to $\ell_\infty(\Gamma)$. Indeed, suppose $K = \beta\Gamma$ for some discrete topological space Γ. Since Γ is discrete, each $f \in \ell_\infty(\Gamma)$ is bounded and continuous. The universal property of the Stone-Čech compactification gives a unique continuous extension of f to all of $\beta\Gamma$; let $\overline{f} : \beta\Gamma \to \mathbb{R}$ denote this extension. This gives rise to a bijective linear map $\phi : \ell_\infty(\Gamma) \to C(\beta\Gamma)$ given by $\phi(f) = \overline{f}$; the inverse is $\phi^{-1}(f) = f|_\Gamma$. It is straightforward to verify $\phi^{-1} \geq 0$. If we fix some $0 \leq f \in \ell_\infty(\Gamma)$ then, as f is bounded, we may assume without loss of generality that $f(\Gamma) \subseteq [0, 1]$. Then $\phi(f) = \overline{f} : \beta\Gamma \to [0, 1]$ is clearly positive and ϕ is a lattice isomorphism. Moreover, this argument shows these two spaces have the same unit ball, so ϕ is even a lattice isometry.

Our next goal is to show the converse of Remark 4.21 is also true; that is, if $C(K)$ is atomic and order complete then K is the Stone-Čech compactification of some discrete space. We begin with a simple lemma characterizing the atoms in $C(K)$ spaces.

Lemma 4.22. *An atom in $C(K)$ must be a scalar multiple of a characteristic function*

$$\mathbf{1}_{\{t\}}(x) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}$$

for some $t \in K$.

Proof. Let $a > 0$ be an atom in $C(K)$. For the sake of contradiction, assume the support of a contains at least two distinct points, s and t. Since K is a compact Hausdorff space, it is completely regular; that is, points in K can be separated from closed sets by a continuous function. Let U and V be disjoint open neighborhoods of s and t, respectively. Then by the assumption that K is (locally) compact Hausdorff, there exists a closed set F with $s \in F \subset U$; so $t \notin F$. Now K is completely regular allows us to find a continuous function $f : K \to [0, 1]$ such that $f|_F \equiv \mathbf{0}$ and $f(t) = 1$.

Since $a > 0$ and $f(t) = 1$ imply $0 < af$, and $f(K) \subseteq [0, 1]$ implies $af \leq a$, we obtain $0 < af \leq a$; hence $af \in I_a = \text{span}\{a\}$. This yields a scalar $\lambda > 0$ such that $\lambda \cdot af = a$, which is impossible because $f(s) = 0$ implies $0 = \lambda \cdot a(s)f(s) = a(s) \neq 0$. □

The previous result implies that atoms in $C(K)$ correspond to isolated points of K. Thus, in situations where $C(K)$ is atomic one should expect K to have many isolated points.

Proposition 4.23. *If $C(K)$ is atomic then the set of isolated points of K is dense in K.*

Proof. Let Γ denote the set of all isolated points in K, and let $t \in K$ and U denote an open neighborhood of t. Use Urysohn's Lemma to find a continuous function $f : K \to [0, 1]$

that is supported on U. The assumption $C(K)$ is atomic yields an atom $a \in C(K)$ such that $0 \leq a \leq f$. Use Lemma 4.22 to find a point $p \in K$ and $\lambda > 0$ such that $a = \lambda \cdot \mathbf{1}_{\{p\}}$. The continuity of a implies $\{p\}$ is open and hence $p \in \Gamma$. Now observe $f(p) \geq a(p) = \lambda \cdot \mathbf{1}_{\{p\}}(p) = \lambda > 0$. Thus p is in the support of f, which is contained in U. Consequently, $U \cap \Gamma \neq \varnothing$ and Γ is dense in K. □

Given the connection between atoms in $C(K)$ and isolated points in K, it is natural to ask what extent K being a discrete topological space affects the condition that $C(K)$ is an atomic vector lattice (and vice versa). If K is discrete then $C(K) = \mathbb{R}^K$. The assumption K is a compact Hausdorff space now forces K to be finite, hence $C(K)$ would be finite dimensional and, therefore, lattice isomorphic to $\mathbb{R}^n$ for some n; in particular, it is atomic and order complete.

Conversely, if $C(K)$ is atomic then the set Γ of isolated points of K is dense. Since every point of Γ is open in K, the induced topology it inherits as a subset of K is the discrete topology. Thus, we view Γ as a discrete space. If Γ were closed in K, we would have $\Gamma = \overline{\Gamma} = K$, implying that K is discrete, and hence $C(K)$ is lattice isomorphic to $\mathbb{R}^n$ for some n; this is not interesting. What happens if $\Gamma \subsetneq K$? The expectation that K is discrete now seems too strong, but perhaps there are additional conditions on $C(K)$ that force K to be "sufficiently discrete". We thus seek additional conditions on $C(K)$ that will help Γ cover the "gap" between Γ and K. If the gap is "small enough", one would hope that every (bounded) function on Γ extends uniquely to a continuous function on all of K; the Stone-Čech compactification of Γ is the universal space with this property, but such spaces are generally "huge". So we rephrase the question: What are conditions on atomic $C(K)$ spaces that ensure the gap between $\beta\Gamma$ and K is "small"? Since Γ is a discrete space, $\beta\Gamma$ will be ***extremally disconnected***, meaning the closure of every open subset is open; see [Gle58] or [DDLS16, p. 30, Proposition 1.5.9]. It is a well-known fact that $C(K)$ is order complete if and only if K is extremally disconnected, and this additional assumption is all we need to ensure the gap between $\beta\Gamma$ and K is as small as possible.

Proposition 4.24. *$C(K)$ is atomic and order complete if and only if K is homeomorphic to $\beta\Gamma$ for some discrete space Γ. In particular, under these assumptions $C(K)$ is lattice isometric to $\ell_\infty(\Gamma)$.*

Proof. In light of Remark 4.21, it remains to prove the forward implication. Let Γ denote the set of all isolated points in K. If $C(K)$ is atomic then Proposition 4.23 implies Γ is a dense subset of K. Furthermore, the order completeness assumption implies K is extremally disconnected. [1] Now we apply [SW13, Theorem 4.7]: if D is a dense subset of a compact Hausdorff extremally disconnected space E then E is homeomorphic to βD. Thus, K is is homeomorphic to $\beta\Gamma$.

□

Remark 4.25. Order completeness of $C(K)$ implies that K is an extremally disconnected compact Hausdorff space (*i.e.* K is ***Stonean***) and the condition that $C(K)$ is atomic implies the set Γ of isolated points of K is dense in K. Since K is homeomorphic to $\beta\Gamma$ in this setting, K is *hyper-Stonean*; see the paragraph before Proposition 5.1.2 on page 162 in [DDLS16]. Thus, atomic order complete $C(K)$ spaces are, according to [DDLS16, Theorem 6.4.2 on p. 200], von Neumann algebras whose preduals are isometrically isomorphic to the space of normal measures on K, $N(K)$.

[1]We can also continue with a more self-contained proof. It remains to show that every bounded function $f:\Gamma\to\mathbb{R}$ has a continuous extension to a function $\overline{f}:K\to\mathbb{R}$; the uniqueness of the extension follows for free from the density of Γ combined with [Mun, pp. 240, Lemma 38.3]. Let f be such a bounded function. We may assume without loss of generality that $f(\Gamma)\subseteq[0,1]$. Define

$$\Omega=\{f(t)\cdot\mathbf{1}_{\{t\}}:t\in\Gamma\}.$$

By the choice of Γ, this is a subset of $C(K)$; it is also non-empty and bounded above by the constant function $\mathbf{1}$. Now the assumption that $C(K)$ is order complete implies $\overline{f}:=\sup\{\Omega:t\in\Gamma\}$ exists in $C(K)$. It remains to show $\overline{f}$ agrees with f on Γ. For any $s\in\Gamma$ we have

$$\overline{f}\wedge\mathbf{1}_{\{s\}}=\sup\{f(t)\mathbf{1}_{\{t\}}\wedge\mathbf{1}_{\{s\}}:t\in\Gamma\}=f(s)\mathbf{1}_{\{s\}}.$$

It follows that $\overline{f}(s)=(\overline{f}\wedge\mathbf{1}_{\{s\}})(s)=f(s)\mathbf{1}_{\{s\}}(s)=f(s)$.

This chapter introduced the notion of order compactness to vector lattices. We were able to characterize these sets as the closed and bounded sets in the order convergence structure for atomic order complete spaces. This led to an application of order compact intervals to the theory of Banach lattices. We also showed that atomic order complete $C(K)$ spaces are lattice isometric to $\ell_\infty(\Gamma)$ for the set Γ of all isolated points in K. It remains an open question whether or not vector lattices in which every order interval is order compact are necessarily atomic.

5. Relative Uniform Convergence

It is natural to ask how general convergence theory can be applied to other non-topological convergences in vector lattices. In this chapter we will explore some interesting connections between order and relative uniform convergence in vector lattices.

A net x_α in a vector lattice is said to be ***relatively uniformly convergent*** to x, written $x_\alpha \xrightarrow{\text{ru}} x$, if there exists $u \in X_+$ such that for each $\epsilon > 0$ there exists α_0 such that $|x_\alpha - x| \leq \epsilon u$ whenever $\alpha \geq \alpha_0$. This is another example of a non-topological convergence in vector lattices, so we will apply general convergence theory in our approach to studying it.

We begin by reviewing an important construction. Choose any vector $e > 0$ and consider the ideal generated by e

$$I_e = \{x \in X : \exists \lambda \in \mathbb{R}_+ \text{ such that } |x| \leq \lambda e\}.$$

There is a natural choice for a lattice norm on I_e defined by

$$\|x\|_e = \inf\{\lambda \geq 0 : |x| \leq \lambda e\}.$$

Actually, this is only a lattice seminorm on I_e; it will be a norm for every $e > 0$ when X is Archimedean. Then each positive element in an Archimedean vector lattice is contained in a normed vector lattice; so each Archimedean space may be viewed locally as a normed lattice. This perspective is useful for gaining a more intuitive understanding of relative uniform convergence. It is easily shown that $x_\alpha \xrightarrow{\text{ru}} x$ in X if and only if there is some $e > 0$ where $\|x_\alpha - x\|_e \to 0$; that is, $x_\alpha \to x$ in $\|\cdot\|_e$-norm for some $e > 0$. Since $\|x\|_e < \infty$ is equivalent to $x \in I_e$, we see $x_\alpha \xrightarrow{\text{ru}} x$ implies that $\|x_\alpha - x\|_e < 1$ holds eventually, and therefore a tail of $(x_\alpha - x)$ is contained in I_e. So a net that converges relatively uniformly can be "captured" by some normed lattice $(I_e, \|\cdot\|_e)$ in which the norm $\|\cdot\|_e$ witnesses the convergence. The search for this ideal can be greatly simplified when the space has a strong unit.

Lemma 5.1. *If u is a strong unit in X then $x_\alpha \xrightarrow{\text{ru}} x$ if and only if $\|x_\alpha - x\|_u \to 0$.*

Proof. If $\|x_\alpha - x\|_u \to 0$ then clearly $x_\alpha \xrightarrow{\text{ru}} x$. We turn to the converse. Fix $\epsilon > 0$ and suppose $x_\alpha \xrightarrow{\text{ru}} x$. Then there is some $e \in X_+$ such that $\|x_\alpha - x\|_e \to 0$. If u is a strong unit, find $\lambda > 0$ such that $e \leq \lambda u$. It follows that there is α_0 such that $|x_\alpha - x| < \frac{\epsilon}{\lambda} e \leq \epsilon u$ whenever $\alpha \geq \alpha_0$. So $\|x_\alpha - x\|_u < \epsilon$ whenever $\alpha \geq \alpha_0$. □

In particular, when $X = C(K)$ and $u = \mathbf{1}$, the function that is identically 1 on K, we have $\|f\|_{\mathbf{1}} = \sup_{t \in K} |f(t)|$ and hence ru-convergence is just uniform convergence on $C(K)$.

An Archimedean vector lattice X is ***uniformly complete*** if $(I_e, \|\cdot\|_e)$ is complete for every $e > 0$. Uniformly complete spaces play an important role in the theory of vector lattices. At first glance this property seems quite strong, but it is satisfied in a healthy class of spaces: Banach lattices and σ-order complete vector lattices are uniformly complete. A famous result from the theory of vector lattices says that, in a uniformly complete vector lattice, each principal ideal I_e is lattice isomorphic to a $C(K)$ space for some compact Hausdorff space K. This significantly improves previous observations: every uniformly complete Archimedean vector lattice is locally a $C(K)$ space. This idea has many far-reaching consequences and is fundamental to the study of vector lattices.

In this section we will present basic facts about relative uniform convergence of nets; cf. [VdW16]. Note, however, that [VdW16] uses the name order convergence structure to refer to what we have called the σ-order convergence structure. Despite the names, these two convergence structures are not identical since they do not admit the same convergent sequences. Our approach has two advantages over the one taken in [VdW16]: (i) many of our results are generalized to nets instead of sequences and (ii) our arguments use nets instead of filters.

We begin with some general background from convergence theory. Given a convergence vector space (X, λ) consider the collection $\mathfrak{B}$ of all λ-bounded subsets. The ***Mackey modification***[2] of λ is the convergence structure $\mu(\lambda)$ where $\mathcal{F} \xrightarrow{\mu(\lambda)} 0$ if there is some $B \in \mathfrak{B}$ such that $\mathcal{N}_0 B \subseteq \mathcal{F}$; see [BB02] Lemma 3.7.12. We will usually write $\mu(X)$ as shorthand for $(X, \mu(\lambda))$ and make use of the following known properties; see [BB02, pp. 115-116].

Proposition 5.2. *If (X, λ) is a convergence vector space the following are true.*

(i) *$\mu(X)$ is a convergence vector space.*

(ii) *$\mu(\lambda)$ is stronger than λ.*

(iii) *X and $\mu(X)$ share the same bounded sets.*

(iv) *For convergence vector spaces X and Y a linear operator $T : \mu(X) \to \mu(Y)$ is continuous if and only if $T : X \to Y$ maps bounded set to bounded sets.*

(v) *$\mu(X)$ is a first countable convergence space* [3].

For the rest of this section X will denote an Archimedean vector lattice equipped with the order convergence structure. In this context we have the following characterization of the Mackey modification.

Lemma 5.3. *A filter $\mathcal{F} \xrightarrow{\mu(\lambda_o)} 0$ iff there exits $u \in X_+$ such that $(-\frac{1}{n}u, \frac{1}{n}u) \in \mathcal{F}$ for all $n \in \mathbb{N}$.*

Proof. Take $\mathfrak{B}$ to be the collection of all order bounded subsets of X in the definition of Mackey modification and use the fact $\mathcal{N}_0 = [(-\frac{1}{n}, \frac{1}{n})]$. □

A more explicit characterization of Mackey convergence in terms of nets follows.

[2]The definition of the Mackey modification was presented using filters; this modification can be expressed using the language of nets and appears in [OTW].

[3]A convergence space is said to be ***first countable*** if for each filter $\mathcal{F} \to x$ there exists a countable filter base $\mathcal{B}$ such that $[\mathcal{B}] \subseteq \mathcal{F}$ and $[\mathcal{B}] \to x$.

Proposition 5.4. *The associate net convergence of $\mu(\lambda_o)$ agrees with relative uniform convergence.*

Proof. First, it suffices to study convergence at 0 since $\mu(\lambda_o)$ is a convergence vector space by property (i) of Proposition 5.2. Furthermore, passing between associate convergences does not spoil the vector space structure. Start with a net $x_\alpha \xrightarrow{\text{ru}} 0$. Then

$$\exists u \in X_+ \quad \forall \epsilon > 0 \quad \exists \alpha_0 \quad \forall \alpha \geq \alpha_0 \quad |x_\alpha| \leq \epsilon u.$$

In particular, for $n \in \mathbb{N}$ put $\epsilon = \frac{1}{n}$. Then there is some α_0 such that

$$\{x_\alpha : \alpha \geq \alpha_0\} \subseteq [-\frac{1}{n}u, \frac{1}{n}u].$$

Hence, for each n we have $(-\frac{1}{n}u, \frac{1}{n}u) \in [x_\alpha]$. Lemma 5.3 gives $[x_\alpha] \xrightarrow{\mu(\lambda_o)} 0$; so x_α converges to 0 in the associate net convergence of $\mu(\lambda_o)$.

Conversely, consider a net (x_α) such that $[x_\alpha] \xrightarrow{\mu(\lambda_o)} 0$. Then there is $u \in X_+$ such that $(-\frac{1}{n}u, \frac{1}{n}u) \in [x_\alpha]$ for all n. Let $\epsilon > 0$ be arbitrary and choose n_0 large enough so $\frac{1}{n_0} < \epsilon$. Now $(-\frac{1}{n_0}u, \frac{1}{n_0}u) \in [x_\alpha]$ implies there is α_0 such that

$$\{x_\alpha : \alpha \geq \alpha_0\} \subseteq (-\frac{1}{n_0}u, \frac{1}{n_0}u)$$

That is, there is α_0 such that $\alpha \geq \alpha_0$ implies $|x_\alpha| < \frac{1}{n_0}u < \epsilon u$. So $x_\alpha \xrightarrow{\text{ru}} 0$. $\square$

Corollary 5.5. *Relative uniform convergence of nets defines a convergence vector space structure on Archimedean vector lattices.*

Proof. X is Archimedean implies (X, λ_o) is a convergence vector space. Then $\mu(X)$ is a convergence vector space by property (i) of Proposition 5.2. Proposition 5.4 shows that ru-convergence is the associate net convergence of a (filter) convergence vector space, so the result follows from Theorem 2.46. $\square$

Thus, as long as we work with Archimedean vector lattices, ru-convergence is a convergence vector space structure. The next results use net convergence theory to highlight a few basic facts about $\mu(\lambda_o)$.

Corollary 5.6. *Relative uniform convergence is stronger than order convergence.*

Proof. This result is already known and can be proven directly. However, in light of Proposition 5.4 it is a consequence of property (ii) in Proposition 5.2. $\square$

The converse is true if and only if the space has an order continuous norm and was proved in [TT20]. We now deduce several basic properties about $\mu(\lambda_o)$ using facts about ru-convergent nets.

Proposition 5.7. $(X, \mu(\lambda_o))$ *is Hausdorff.*

Proof. It is well-known and easily verified that the limit of a ru-convergent net is unique, so this property passes to the associate space $(X, \mu(\lambda_o))$. $\square$

Proposition 5.8. *A subset is* $\mu(\lambda_o)$*-closed if and only if it is relatively uniformly closed.*

Proof. Relatively uniformly closed sets are the closed sets of (X, ru) by definition. The result follows from Proposition 5.4 and Corollary 2.25. $\square$

As an immediate consequence we obtain that ru-closed sets are characterized sequentially.

Corollary 5.9. *A set is relatively uniformly closed if and only if it is sequentially* ru*-closed.*

Proof. The forward implication is obvious. For the converse suppose $A \subseteq X$ is sequentially ru-closed and let $x \in \overline{A}^{ru}$. Then $\overline{A}^{ru} = a_{\mu(\lambda_o)}(A)$. However, property (v) of Proposition 5.2 gives $\mu(X)$ is first countable, so Proposition 1.6.4 of [BB02] applies. Then there is a sequence $(y_n) \in A$ such that $y_n \xrightarrow{\text{ru}} x$. As A is sequentially ru-closed we have $x \in A$ and hence $A = \overline{A}^{ru}$. $\square$

Remark 5.10. It is interesting that an ideal I is relatively uniformly closed if and only if the quotient vector lattice X/I is Archimedean. Recall that the latter is equivalent to linearity of order convergence. In light of the previous result, this means the convergence vector space structure on X/I is determined by the sequential property that I be ru-closed. This is fitting since the Archimedean property is sequential in nature.

Proposition 5.11. *A subset of X is $\mu(\lambda_o)$-bounded if and only if it is order bounded.*

Proof. Property (iii) of Proposition 5.2 says that $\mu(\lambda_o)$ and λ_o have the same bounded sets. The result now follows from Proposition 3.26. □

Corollary 5.12. *$(X, \mu(\lambda_o))$ is a locally bounded convergence space.*

Proof. It is straightforward to verify every ru-convergent net has ru-bounded (*i.e.* order bounded) tail, so (X, ru) is locally bounded. But ru-convergence is the associate net convergence of $(X, \mu(\lambda_o))$, so the result follows from Proposition 2.49. □

The notion of uniform completeness is also expressed naturally in the language of convergence structures. This result was also obtained in [VdW16], but the following proof avoids using filters.

Proposition 5.13. *X is uniformly complete if and only if it is $\mu(\lambda_o)$-complete. That is, uniformly complete spaces are precisely the spaces that are complete with respect to the ru-convergence structure.*

Proof. Property (v) of Proposition 5.2 says $\mu(\lambda_o)$ is a first countable convergence structure. So [BB02] Proposition 3.6.5 implies $\mu(\lambda_o)$-completeness is characterized by sequences.

For the forward implication assume that X is uniformly complete and let (x_n) be a ru-Cauchy sequence in X. Then there is some $e \in X_+$ such that (x_n) is $\|\cdot\|_e$-Cauchy. The main issue is that (x_n) may not belong to I_e, so we need to find some $u \in X_+$ such that

$(x_n) \in I_u$ and (x_n) is also $\|\cdot\|_u$-Cauchy. As (x_n) is assumed to be $\|\cdot\|_e$-Cauchy, we can find k_0 such that $\|x_n - x_m\|_e < 1$ whenever $m, n \geq k_0$. Now

$$|x_n| \leq |x_n - x_{k_0}| + |x_{k_0}| \leq e + |x_{k_0}|$$

holds for all $n \geq k_0$. In particular, $(x_n)_{n \geq k_0} \in I_{e+|x_{k_0}|}$. Note that (x_n) is also Cauchy with respect to the $\|\cdot\|_{e+|x_{k_0}|}$-norm since $0 \leq e \leq e + |x_{k_0}|$ implies $\|\cdot\|_{e+|x_{k_0}|} \leq \|\cdot\|_e$ and hence

$$\|x_n - x_m\|_{e+|x_{k_0}|} \leq \|x_n - x_m\|_e \to 0 \quad \text{ as } n, m \to \infty.$$

So by taking $u = e + |x_{k_0}|$ we find a tail of (x_n) inside of I_u and (x_n) is $\|\cdot\|_u$-Cauchy. Our initial assumption implies that $(I_u, \|\cdot\|_u)$ is complete, so this tail sequence must converge in $\|\cdot\|_u$-norm. It follows that the original sequence (x_n) converges in $\|\cdot\|_u$-norm; *i.e.*, $x_n \xrightarrow{\text{ru}} x$ for some $x \in X$. This shows X is sequentially ru-complete and, therefore, ru-complete by our opening remarks. That X is $\mu(\lambda_o)$-complete follows from Proposition 2.50.

For the converse assume that X is $\mu(\lambda_o)$-complete and, therefore, ru-complete. Fix $e \in X_+$ and consider a Cauchy sequence in $(I_e, \|\cdot\|_e)$; *i.e.* (x_n) is Cauchy with respect to the $\|\cdot\|_e$-norm. In particular, (x_n) is uniformly Cauchy and hence it ru-converges to some $x \in X$. So there exists $u > 0$ such that $\|x_n - x\|_u \to 0$. It remains to show that $x \in I_e$ and $\|x_n - x\|_e \to 0$. Let $\epsilon > 0$ and find N_0 such that $\|x_n - x_m\|_e < \epsilon$ whenever $n, m > N_0$. Then we have

$$|x_n - x_m| < \epsilon e \quad \text{whenever } n, m > N_0.$$

Let $n > N_0$ be fixed and notice that for every k there exists N_k such that $\|x_m - x\|_u < \frac{1}{k}$ whenever $m > N_k$. Take $N_1 = \max\{n, N_k\}$. This guarantees that for each k there is some $N_1 > N_0$ such that $|x_m - x| < \frac{1}{k}u$ whenever $m > N_1$. It follows that for each k we have

$$|x_n - x| \leq |x_n - x_m| + |x_m - x| \leq \epsilon e + \frac{1}{k}u$$

whenever $m > N_1 \geq n > N_0$. But X is Archimedean gives $\frac{1}{k}u \downarrow 0$ and hence $\|x_n - x\| < \epsilon e$ whenever $n > N_0$. This shows $\|x_n - x\|_e \to 0$ and $x_n - x \in I_e$ for $n > N_0$. Now $x_n \in I_e$ for all n and $x = x_n - (x_n - x)$ implies $x \in I_e$. This shows that $(I_e, \|\cdot\|_e)$ is complete and, since $e \in X_+$ was arbitrary, that X is uniformly complete. □

Next we demonstrate a close connection between the notions of boundedness and continuity for linear operators between Archimedean vector lattices. If X and Y are normed spaces and $T : X \to Y$ is a linear operator, it is well-known that T is (norm) bounded if and only if it is (norm) continuous. So in normed spaces we view boundedness as a form of continuity. It is natural to ask if boundedness and continuity of an operator are equivalent when one deals with order convergence structure.

Let X and Y be Archimedean vector lattices. In light of Proposition 3.26, the most natural definition for a bounded linear operator in the order convergence structure corresponds to the order bounded operators. Recall that a linear operator $T : X \to Y$ is said to be ***order bounded*** if it maps order bounded sets to order bounded sets. T is said to be ***order continuous*** if $x_\alpha \xrightarrow{o} 0$ in X implies $Tx_\alpha \xrightarrow{o} 0$ in Y. While it is clear that order continuous operators are the continuous linear maps with respect to λ_o, it is not clear how order boundedness of an operator is explicitly related to a notion of continuity. It is known that every order continuous operator is order bounded, but the converse is generally false. It would then appear that the connection between bounded and continuous operators fails when dealing with the order convergence structure, so that boundedness of an operator should not be thought of as a form of continuity. However, not all hope is lost! We may view order bounded operators as continuous operators, we just need to use the correct notion of convergence. The following result appeared in [TT20] but here it is deduced from convergence theory. For a more general statement see [OTW].

Theorem 5.14. *The relatively uniformly continuous linear operators between Archimedean vector lattices are precisely the order bounded operators.*

Proof. $T : X \to Y$ is order bounded if and only if it maps order bounded sets to order bounded sets. Since order bounded sets are precisely the λ_o-bounded sets, property (iv) of Proposition 5.2 says this is equivalent to the continuity of $T : \mu(X) \to \mu(Y)$. The continuity of $T : (X, ru) \to (Y, ru)$ follows from Theorem 2.30. □

Even though the analogy between continuous and bounded operators breaks down in the order convergence structure, the bounded operators can still be realized as continuous operators in their own right. If the domain space is a Banach lattice with an order continuous norm then we recover the analogy in full strength.

Corollary 5.15. *If X is an order continuous Banach lattice then $T : X \to Y$ is order bounded if and only if it is order continuous.*

Proof. Every order continuous operator between Archimedean vector lattices is order bounded, so only the converse needs to be shown. Suppose X has an order continuous norm and let $T : X \to Y$ be order bounded. Take $x_\alpha \xrightarrow{\text{o}} 0$ in X. The order continuity of the norm on X implies order convergence agrees with ru-convergence on X; hence $x_\alpha \xrightarrow{\text{ru}} 0$ in X. Now the ru-continuity of $T : (X, ru) \to (Y, ru)$ implies $Tx_\alpha \xrightarrow{\text{ru}} 0$ in Y. But ru-convergence is stronger than order convergence, so $Tx_\alpha \xrightarrow{\text{o}} 0$ in Y. □

Example 5.16. Here is another easy application of this notion of ru-continuity. If X is a normed lattice and Y is an order complete Banach lattice with a strong unit then every norm bounded operator $T : X \to Y$ is regular; that is, it can be expressed as the difference of two positive operators. Let $x_\alpha \xrightarrow{\text{ru}} 0$ in X. Then $x_\alpha \to 0$ in the norm on X and hence $\|Tx_\alpha\| \to 0$ in Y. Let $e \in Y_+$ be a strong unit so that $I_e = Y$. If Y is order complete then it is uniformly complete and hence the lattice norm $\|x\|_e = \inf\{\lambda > 0 : |x| \leq \lambda e\}$ on

$I_e = Y$ is a Banach lattice norm. The Open Mapping Theorem gives the equivalence of any two Banach lattice norms on Y. It follows that $\|Tx_\alpha\|_e \to 0$, thus $Tx_\alpha \xrightarrow{\text{ru}} 0$ in Y. We have now demonstrated that T is (ru, ru)-continuous and, therefore, order bounded. The order completeness of Y and the Riesz-Kantorovich theorem imply the set of regular and order bounded operators from X into Y are equal — so T is regular.

We will now take a brief look at the compact subsets of $(X, \mu(\lambda_o))$. A set A is called ***uniformly compact*** or ru-***compact*** if it is compact with respect to $\mu(\lambda_o)$. An application of Proposition 2.42 gives the equivalent characterization that A is ru-compact if and only if every net in A has a ru-convergent quasi-subnet. Since ru-convergence is stronger than order convergence, every ru-compact set must be order compact.

Proposition 5.17. *A $\mu(\lambda_o)$-compact subset must be relatively uniformly closed and order bounded.*

Proof. The proof is identical to the one given for Proposition 4.2. □

We have already seen that uniform convergence is stronger than order convergence, so every ru-compact set is order compact. In normed lattices uniform convergence is also stronger than norm convergence. It follows that every ru-compact set is norm compact. Since ru-convergence is the same as norm convergence in $C(K)$, ru-compactness must agree with norm compactness in $C(K)$; the latter sets are given by the Arzelà-Ascoli Theorem; see, for example, [Mun].

Theorem 5.18. *(Arzelà-Ascoli) Let $X = C(K)$. Then the ru-compact sets in X are precisely the norm compact sets. That is, $F \subset X$ is ru-compact if and only if it is equicontinuous and $F_t = \{f(t) : f \in F\}$ is closed and relatively compact for each $t \in K$.*

Remark 5.19. The equivalence of ru-convergence with norm convergence in $C(K)$ allows us to deduce that every norm compact set in $C(K)$ is order compact.

In this chapter, we showed that relative uniform convergence is the Mackey modification of order convergence in Archimedean vector lattices. This idea was explored for sequences in [VdW16] but nets have a fundamental advantage here. Notably, we are able to use convergence theory to give a new and short proof of the classical fact that the relatively uniformly continuous operators between vector lattices are precisely the order bounded operators.

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