ON SECTIONS OF CONVEX BODIES IN HYPERBOLIC SPACE

by

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Abstract

The Busemann-Petty problem asks the following: if $K, L \subset \mathbb{R}^n$ are origin-symmetric convex bodies such that

$$\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \leq \mathrm{vol}_{n-1}(L \cap \xi^{\perp}) \qquad \forall\, \xi \in S^{n-1},$$

is it necessary that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$? This problem received a lot of attention, and many analogues have been considered. For origin-symmetric convex bodies K and L in hyperbolic space $\mathbb{H}^n$, we find a suitable condition which guarantees $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$.

Origin-symmetry is important in many problems in convex geometry. By Brunn's Theorem, each central hyperplane section of an origin-symmetric convex body $K \subset \mathbb{R}^n$ has maximal volume amongst all parallel sections of K. Makai, Martini and Ódor proved the converse of this statement for star bodies. Again working in $\mathbb{H}^n$, we prove an analogue of this result.

Acknowledgements

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Chapter 1

Introduction

1.A Background

The Busemann-Petty problem (BP), first posed in [4] in 1956, asks the following: for origin-symmetric convex bodies K and L in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \leq \mathrm{vol}_{n-1}(L \cap \xi^{\perp}) \qquad \forall\, \xi \in S^{n-1}, \tag{1.1}$$

where $\xi^{\perp}$ is the central hyperplane in $\mathbb{R}^n$ orthogonal to ξ, is it necessary that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$? This problem is trivially true for $n = 2$, but otherwise it remained largely open for many years. Larman and Rogers proved in [15] that the answer is negative for $n \geq 12$. Ball [2] verified that the volume of every central hyperplane section of the unit cube is bounded above by $\sqrt{2}$. Using this bound, he then showed that appropriate dilations of the n-dimensional cube and ball provide a counterexample to BP for $n \geq 10$. Giannopoulos [10] and Bourgain [3] constructed counterexamples for $n \geq 7$. Subsequently, Gardner [7] and Zhang [22] showed BP is negative for $n \geq 5$. Finally, Gardner [5] and

Zhang [22] proved that BP is affirmative for $n = 3$ and $n = 4$, respectively. Independently, Gardner, Koldobsky, and Schlumprecht gave a unified solution to BP in [8]: it is true for $n \leq 4$, and false for $n \geq 5$.
The section function of a convex body $K \subset \mathbb{R}^n$ is defined by

$$S_K(\xi) = \mathrm{vol}_{n-1}(K \cap \xi^{\perp}) \quad \xi \in S^{n-1}.$$

It is then extended to $\mathbb{R}^n \setminus \{0\}$ as a homogeneous function of degree $n-1$. Let Δ denote the Laplacian operator in $\mathbb{R}^n$. For $\alpha \in \mathbb{R}$ and a function f on $\mathbb{R}^n$, define the fractional power $(-\Delta)^{\alpha/2}$ by

$$(-\Delta)^{\alpha/2} f = \frac{1}{(2\pi)^n} \left(|x|_2^{\alpha} \widehat{f}(x) \right)^{\wedge},$$

with the Fourier transform taken in the sense of distributions. It is proven in [14] that if $K, L \subset \mathbb{R}^n$ are origin-symmetric convex bodies such that

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi) \quad \forall \xi \in S^{n-1} \tag{1.2}$$

for some $\alpha \in \mathbb{R}$ with $n - 4 \leq \alpha \leq n - 1$, then $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$. For $0 \leq \alpha < n - 4$, there are origin-symmetric convex bodies $K, L \subset \mathbb{R}^n$ such that equation (1.2) holds, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$. Observe that $\alpha = 0$ corresponds to the classic BP.

Variations of BP have also been studied in non-Euclidean spaces. In [21], Yaskin completely solved BP in the spherical (S^n) and hyperbolic ($\mathbb{H}^n$) spaces. The solution to the BP problem in S^n is exactly the same as in Euclidean space, but it is different in hyperbolic space. For $n \geq 3$, Yaskin constructed origin-symmetric convex bodies $K, L \subset \mathbb{H}^n$ so that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H) \tag{1.3}$$

for every central totally-geodesic hyperplane H in $\mathbb{H}^n$, but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$. BP is trivially true in $\mathbb{H}^2$.

Using the idea from [14], we prove a modified version of BP in hyperbolic space. Let $K, L \subset \mathbb{H}^n$ be convex bodies. The section function $S_K(\xi)$ for K is defined as before, with $\xi^\perp$ denoting the totally-geodesic hyperplane in $\mathbb{H}^n$ passing through O (a fixed origin) and perpendicular to $\xi \in S^{n-1}$ in the tangent space to $\mathbb{H}^n$ at O. We show that equation (1.2), interpreted in the setting of hyperbolic space, ensures $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$ when $n-2 \leq \alpha < n-1$. For $0 \leq \alpha < n-2$, we find counterexamples. Our proof is based on the study of the Fourier transform of the distribution

$$\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2}.$$

For $\xi \in S^{n-1}$, let

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \quad t \in \mathbb{R},$$

be the parallel section function of a convex body $K \subset \mathbb{R}^n$. Brunn's theorem (See Theorem 2.A.1) implies that if K is origin-symmetric and convex then $\max\limits_{t\in\mathbb{R}} A_{K,\xi}(t) = A_{K,\xi}(0)$. A natural question is whether the converse is true and it was affirmatively answered by Makai, Martini and Ódor [16]: If K is a convex body in $\mathbb{R}^n$ such that $A_{K,\xi}(0) = \max\limits_{t\in\mathbb{R}} A_{K,\xi}(t)$ for all $\xi \in S^{n-1}$, then K is origin-symmetric. Ryabogin and Yaskin gave an alternative proof of this result using Fourier transform techniques. In fact they prove the result not for convex bodies, but for star bodies. Of course, for star bodies there is no analogue of Brunn's theorem, so they use the assumption that $A_{K,\xi}(t)$ has a critical point at $t = 0$ for every $\xi \in S^{n-1}$. Using similar techniques, we prove the corresponding result in $\mathbb{H}^n$.

Chapter 2

Preliminaries

2.A Some Definitions from Geometry

We give some definitions and theorems as defined in [13].

Definition 2.A.1 (Minkowski Functional):

A set K in $\mathbb{R}^n$ is called a body if it is compact and equal to the closure of its interior. The Minkowski functional of the body K is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}.$$

Definition 2.A.2 (Star body):

A body K in $\mathbb{R}^n$ is called a star body if for every $x \in K$, the interval $[0, x)$ is in the interior of K, and the Minkowski functional of K is continuous on $\mathbb{R}^n$.

Definition 2.A.3 (Radial Function):

The radial function of a star body K is defined by

$$\rho_K(x) = \max\{a \geq 0 : xa \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

Remark: We observe that

$$\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If $\xi \in S^{n-1}$, then $\rho_K(\xi)$ gives the distance from the origin to the boundary of K in the direction of ξ.

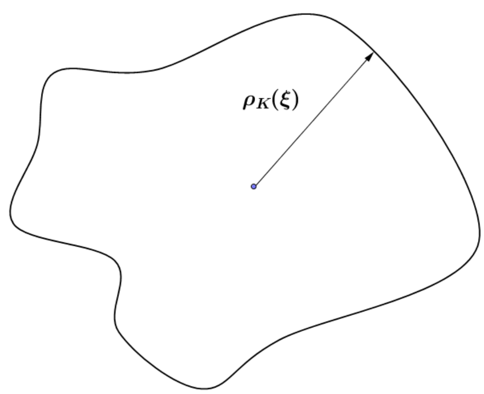

Figure 2.1: Radius of K in the direction of $\xi \in S^{n-1}$.

We say that the body K is C^k-smooth if $\rho_K(x) \in C^k(\mathbb{R}^n \setminus \{0\})$.

If we have a star body K in $\mathbb{R}^2$, we can think of ρ_K as a function of the polar angle, *i.e* a function on $[0, 2\pi]$. The curvature of the boundary curve is given by [see [6]]

$$\frac{2(\rho'_K)^2 - \rho_K \rho''_K + \rho_K^2}{\Big((\rho'_K)^2 + \rho_K^2\Big)^{3/2}} \tag{2.1}$$

Definition 2.A.4 (Convex bodies):

A set $K \subset \mathbb{R}^n$ is called convex if $(1-\lambda)x + \lambda y \in K$ whenever $x, y \in K$ and $0 \le \lambda \le 1$. A convex body is a body which is also convex.

Definition 2.A.5 (Origin-symmetric):

A body K is origin-symmetric if $K = -K$.

Remark: The Minkowski functional becomes a norm on $\mathbb{R}^n$ if the body K is convex and origin-symmetric.

Definition 2.A.6 (Parallel section function):

For $\xi \in S^{n-1}$, the parallel section function $A_{K,\xi}(t)$ is defined by

$$A_{K,\xi}(t) = \mathrm{vol}_{n-1}(K \cap (\xi^\perp + t\xi)), \quad t \in \mathbb{R},$$

where $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$.

We will often use the following formula in polar coordinates:

$$\int_{\mathbb{R}^n} f(x)dx = \int_{S^{n-1}} \int_0^\infty r^{n-1} f(r\theta) dr d\theta.$$

Then we have

$$A_{K,\xi}(t) = \int_{\langle x, \xi \rangle = t} \chi(\|x\|_K) dx.$$

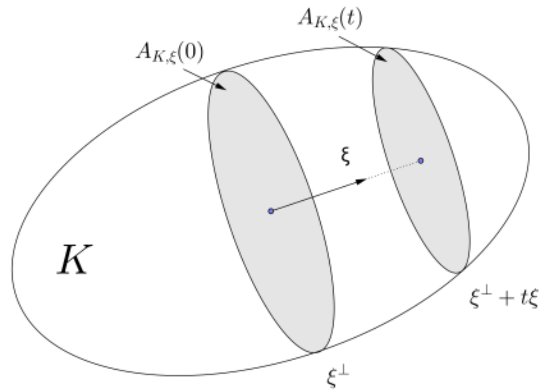

Figure 2.2: Parallel section function $A_{K,\xi}(t)$

Here χ is the characteristic function of $[0,1]$. Thus,

$$\begin{aligned} A_{K,\xi}(0) = \mathrm{vol}_{n-1}(K \cap \xi^\perp) &= \int_{\langle x,\xi\rangle=0} \chi(\|x\|_K)dx \\ &= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^\infty r^{n-2}\chi(r\|\theta\|_K)dr \right) d\theta \\ &= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-2}dr \right) d\theta \\ &= \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)d\theta. \end{aligned}$$

Theorem 2.A.1 (Brunn's Theorem):

Let K be a convex body in $\mathbb{R}^n$. Then for a fixed direction $\xi \in S^{n-1}$:

1. *The function $A_{K,\xi}^{\frac{1}{n-1}}$ is concave on its support.*

2. *If K is origin-symmetric, then*

$$\max_{t\in\mathbb{R}} A_{K,\xi}(t) = A_{K,\xi}(0).$$

3. *If K is origin-symmetric and 2-smooth, then $A''_{K,\xi}(0) \le 0$.*

Proof. See [13, Theorem 2.3]. □

2.B Gamma function, Fourier transform of distributions and Fractional derivatives

Definition 2.B.1 (Gamma function):

For $z \in \mathbb{C}$ with positive real part, the gamma function Γ is defined by :

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.2}$$

Integrating equation (2.2) by parts, we get

$$\Gamma(z+1) = z\Gamma(z). \tag{2.3}$$

This shows that Γ generalizes the factorial function.

Gamma function is analytic in the domain $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Using the formula (2.3) we can extend the gamma function in to an analytic function in the domain $\mathbb{C} \setminus (-\mathbb{N} \cup \{0\})$.

Definition 2.B.2 (Space of test functions):

We consider test functions from the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing, infinitely differentiable functions.

Definition 2.B.3 (Fourier transform):

The Fourier transform of a function $\phi \in \mathcal{S}$ is defined by

$$\mathcal{F}\phi(y) = \hat{\phi}(y) = \int_{\mathbb{R}^n} \phi(x) e^{-i\langle x,y\rangle} dx, \quad y \in \mathbb{R}^n.$$

Remark: The map $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a bijection.

A distribution is an element of the continuous dual $\mathcal{S}'$ of $\mathcal{S}$. The action of a distribution f on a test function ϕ is denoted by $\langle f, \phi \rangle$. The Fourier transform of a distribution f is the distribution $\hat{f}$ defined by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle, \quad \forall\ \phi \in \mathcal{S}.$$

For $\phi \in \mathcal{S}$, we have the identity

$$(\hat{\phi})^\wedge(\xi) = (2\pi)^n \phi(-\xi).$$

If ϕ is even, then

$$(\hat{\phi})^\wedge = (2\pi)^n \phi \quad \text{and} \quad \langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle.$$

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if for every test function ϕ

$$\langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle, \quad \alpha \in \mathbb{R} \setminus \{0\}.$$

We say that a distribution f is **positive definite** if its Fourier transform is a positive distribution; that is $\langle \hat{f}, \phi \rangle \geq 0$ whenever $\phi \in \mathcal{S}$ is such that $\phi \geq 0$.

Lemma 2.B.1: *Let f be an even homogeneous functions of degree $-n+1$ on $\mathbb{R}^n$, continuous on the sphere S^{n-1}. Then the Fourier transform of f is an even homogenous of degree -1, continuous on $\mathbb{R}^n \setminus \{0\}$ function such that, for every $\xi \in S^{n-1}$,*

$$\int_{S^{n-1}\cap\xi^{\perp}} f(\theta)d\theta = \frac{1}{\pi}\hat{f}(\xi).$$

As an application we get Theorem 2.B.2.

Theorem 2.B.2: *Let K be origin-symmetric star body in $\mathbb{R}^n$. The Fourier transform of the function $\|x\|_K^{-n+1}$ is a homogeneous of degree -1 on $\mathbb{R}^n$, continuous on $\mathbb{R}^n \setminus \{0\}$ and such that,*

$$A_{K,\xi}(0) = \mathrm{vol}_{n-1}(K \cap \xi^{\perp}) = \frac{1}{\pi(n-1)}\left(\|\cdot\|_K^{-n+1}\right)^{\wedge}(\xi).$$

Definition 2.B.4 (Fractional derivatives):

Fractional derivatives generalize derivatives to non-integer orders. Let ϕ be a continuous, integrable function on $\mathbb{R}$ which is m-smooth in a neighbourhood of 0, and let $q \in \mathbb{C}$, $-1 < \mathrm{Re}(q) < m, q \neq 0, 1, ..., m-1$. The fractional derivative of ϕ of order q at 0 is defined by

$$\phi^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^1 t^{-1-q}\left(\phi(t) - \phi(0) - ... - \phi^{m-1}(0)\frac{t^{m-1}}{(m-1)!}\right)dt$$
$$+ \frac{1}{\Gamma(-q)}\int_1^{\infty} t^{-1-q}\phi(t)dt + \frac{1}{\Gamma(-q)}\sum_{k=0}^{m-1}\frac{\phi^k(0)}{k!\,(k-q)}.$$

Moreover, if q is a non negative integer, one can observe that the fractional derivative of integer orders coincide with usual derivatives up to a sign (as a limit of the latter expression as $q \to k$).

$$\phi^{(k)}(0) = (-1)^k \frac{d^k}{dt^k}\phi(t)|_{t=0}, \quad k \in \mathbb{N} \cup \{0\}.$$

Theorem 2.B.3: *Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$. Then for every $q \in (-1, \infty)$, $q \neq n-1$, the fractional derivative of the order q of the parallel section function at zero can be expressed in the form*

$$A^{(q)}_{D,\xi}(0) = \frac{cos(\pi q/2)}{\pi(n-q-1)}(\|\cdot\|_D^{-n+q+1})^\wedge(\xi).$$

Moreover, if $k \geq 0$, $k \neq n-1$, is an even integer, then

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A^{(k)}_{D,\xi}(0)$$

and if $k \geq 1$, $k \neq n-1$, is an odd integer, then

$$\begin{aligned}
&(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) \\
&= (-1)^{(k+1)/2}2(n-1-k)k! \\
&\quad \times \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - A''_{D,\xi}(0)\frac{z^2}{2} - ... - A^{k-1}_{D,\xi}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}dz.
\end{aligned}$$

Proof. [13, Theorem 3.18] □

We will often use the following version of Parseval's formula :

Let $f, g \in C^\infty(S^{n-1})$ and $0 < p < n$. Then

$$\int_{S^{n-1}} \left(f\Big(\frac{x}{|x|}\Big)|x|^{-p}\right)^\wedge(\theta) \cdot \left(g\Big(\frac{x}{|x|}\Big)|x|^{-n+p}\right)^\wedge(\theta)d\theta = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta)d\theta.$$

See [13, Lemma 3.22] or [17]

2.C Hyperbolic and Spherical Geometry

We recall some facts on spherical and hyperbolic geometry, as given in [1] and [21].

Let S^n be the unit sphere in $\mathbb{R}^{n+1}$. Using the stereographic projection (from the north pole onto the hyperplane $P = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} = 0\}$) we can think of S^n as $\mathbb{R}^n$ with the metric:

$$ds^2 = 4\frac{dx_1^2 + ... + dx_n^2}{1 + (x_1^2 + ... + x_n^2)^2}. \tag{2.4}$$

To define convexity in spherical space, we need the the geodesic joining any two points to be unique. However, this is not true on the full sphere. Thus, we will work within the open hemisphere, where geodesics are unique. Note that under the stereographic projection, the south hemisphere gets mapped onto the open unit ball B^n in $\mathbb{R}^n$.

We will identify the hemisphere with B^n equipped with the metric (2.4). Hyperbolic space can also be identified with the open ball B^n equipped with the metric :

$$ds^2 = 4\frac{dx_1^2 + ... + dx_n^2}{1 - (x_1^2 + ... + x_n^2)^2}. \tag{2.5}$$

In this metric, geodesic segments are in fact arcs of the circles orthogonal to the boundary of the ball B^n. If a segment passes through the origin, then the segment becomes a straight line.

We will treat spherical, hyperbolic and Euclidean cases simultaneously by considering B^n with the metric :

$$ds^2 = 4\frac{dx_1^2 + ... + dx_n^2}{1 + \delta(x_1^2 + ... + x_n^2)^2}, \tag{2.6}$$

where $\delta = 1, -1, 0$ correspond to spherical, hyperbolic and Euclidean cases respectively.

To distinguish between different types of convexity, we will adopt the following notation. Let K be a body in the open unit ball B^n. The body K is s - convex (+1 - convex) if the body K is convex under the spherical metric defined in the ball B^n. Similarly, h - convexity (-1 - convexity) and e - convexity (0 - convexity) are defined with respect to the hyperbolic and Euclidean metrics, respectively .

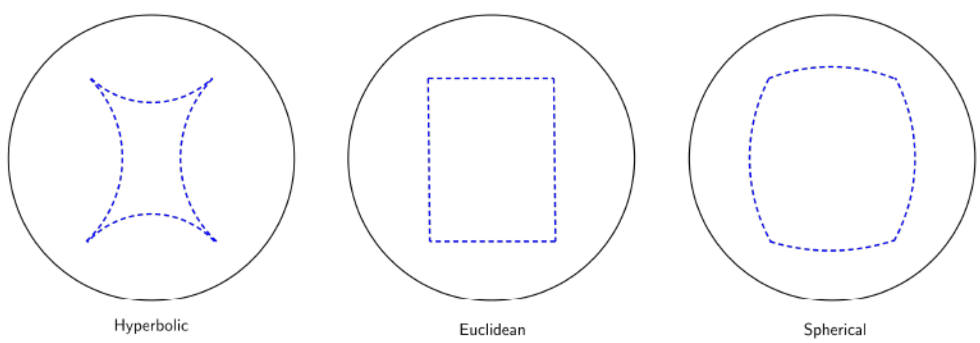

Figure 2.3: Convex hulls in different metrics

Some examples for convex hulls of four points are shown in the figure (2.3) for different metrics.

Clearly any e-convex body containing the origin is a h-convex body. Also any s-convex body containing the origin is e-convex.

A submanifold $\mathcal{F}$ in a Riemannian space $\mathcal{R}$ is called totally geodesic, if every geodesic in $\mathcal{F}$ is also a geodesic in $\mathcal{R}$. In the Euclidean space and spherical space these totally geodesic submanifolds are represented by Euclidean planes and great subspheres respectively. In the Poincaré model of the hyperbolic space these submanifolds are spheres orthogonal to the boundary of the unit ball B^n and Euclidean planes through the origin.

For the purposes of our calculations in the following chapters, we find the volume element $d\mu_n$ corresponding to the metric

$$ds^2 = 4\frac{dx_1^2 + ... + dx_n^2}{1-(x_1^2 + ... + x_n^2)^2}.$$

Since the metric is diagonal,

$$d\mu_n = 2^n \frac{dx_1...dx_n}{\Big(1-(x_1^2 + ... + x_n^2)\Big)^n} = 2^n \frac{dx}{(1-|x|_2^2)^n}.$$

The volume of a body K is then given by

$$vol_n(K) = \int_K d\mu_n = 2^n \int_K \frac{dx}{(1-|x|_2^2)^n}.$$

In polar coordinates we have,

$$vol_n(K) = 2^n \int_{S^{n-1}} \int_0^{\|\theta\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^n} dr d\theta.$$

Similarly, the volume element of the hypersurface $\xi^{\perp}$ is,

$$d\mu_{n-1} = 2^{n-1} \frac{dx}{(1 - |x|_2^2)^{n-1}}$$

and the $(n-1)$-volume of the section of K will be

$$S_K(\xi) = \int_{K \cap \langle x, \xi \rangle = 0} d\mu_{n-1} = 2^{n-1} \int_{K \cap \langle x, \xi \rangle = 0} \frac{dx}{(1 - |x|_2^2)^{n-1}}.$$

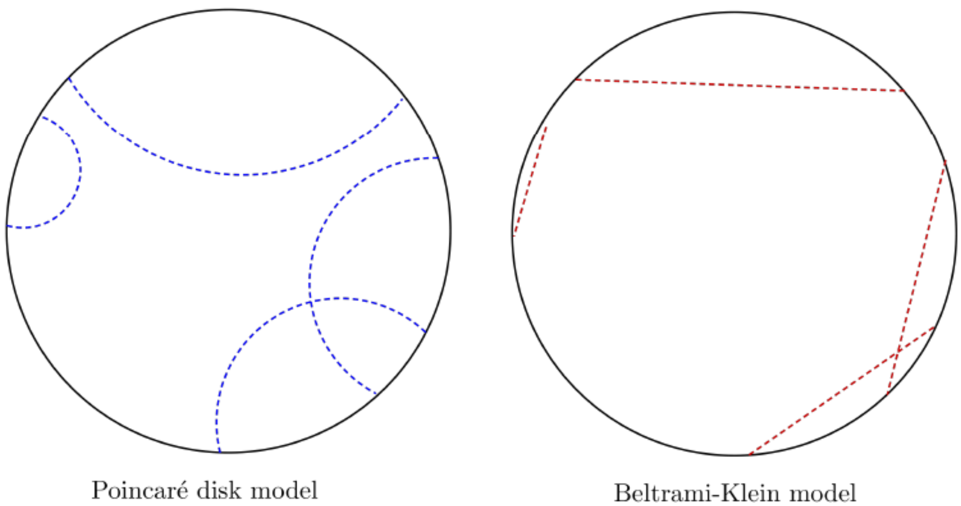

Figure 2.4: Disk models

In Chapter 3 we will be working with a different model called the Beltrami - Klein model. Euclidean geodesics and hyperbolic geodesics are the same in this hyperbolic model. In a certain way, this is an advantage for calculations. There is an isomorphism between the Poincaré and Beltrami - Klein models. The mapping between these models can be described as two projections, illustrated in Figure 2.5.

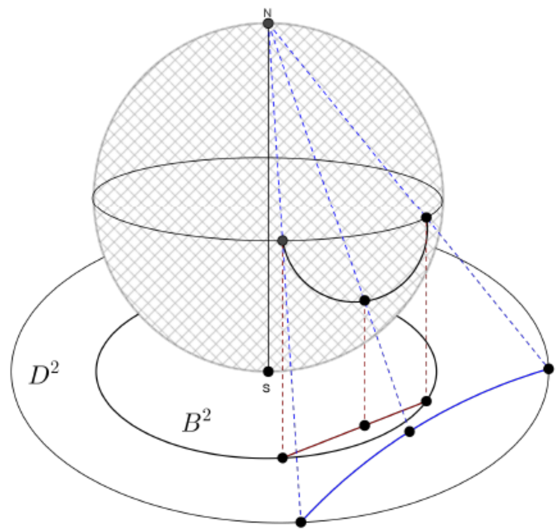

Figure 2.5: An isomorphism between Klein and Poincaré model

Chapter 3

Modified BP problem in Hyperbolic Space

In this chapter we will show that, for an origin-symmetric convex bodies K, L in hyperbolic space $\mathbb{H}^n$, if the distribution

$$\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2} \tag{3.1}$$

is positive definite, then it follows that $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$ whenever,

$$(-\Delta)^{\alpha/2}S_K(\xi) \leq (-\Delta)^{\alpha/2}S_L(\xi) \qquad \forall\ \xi \in S^{n-1}$$

Therefore, the problem is reduced to finding $\alpha \in \mathbb{R}$ such that the distribution in (3.1) is positive definite.

The following lemma is an analogue of Theorem 2.B.2 in non-Euclidean settings.

Lemma 3.1: *Let K be an origin symmetric h-convex body in B^n. Let $\xi^\perp$ be the totally geodesic hyperplane through the origin perpendicular to $\xi \in S^{n-1}$. Then the volume of the section of the body K by the hyperplane $\xi^\perp$ will be :*

$$S_K(\xi) = \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\xi), \tag{3.2}$$

and

$$(-\Delta)^{\alpha/2}S_K(\xi) = \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\xi), \quad \xi \in S^{n-1}. \tag{3.3}$$

K is assumed to be smooth enough. So that the latter is a continuous function on S^{n-1}.

Proof. We give the proof as in [21, Lemma 2.2].

Using polar coordinates we obtain:

$$\begin{aligned} S_K(\xi) &= 2^{n-1}\int_{\xi^\perp} \chi(\|x\|_K)\frac{dx}{(1-|x|_2^2)^{n-1}} \\ &= 2^{n-1}\int_{S^{n-1}\cap\xi^\perp}\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}dr}{(1-r^2)^{n-1}}d\theta. \end{aligned}$$

Since $|\theta|_2 = 1$ for $\theta \in S^{n-1}$, then :

$$S_K(\xi) = 2^{n-1}\int_{S^{n-1}\cap\xi^\perp}|\theta|_2^{-n+1}\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}dr}{(1-r^2)^{n-1}}d\theta.$$

Then using Lemma 2.B.1 :

$$S_K(\xi) = \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\xi).$$

Now we prove (3.3). This result follows immediately from (3.2).

By definition :

$$(-\Delta)^{\alpha/2} S_K(\xi) = \frac{1}{(2\pi)^n}(|x|_2^\alpha \widehat{S_K}(x))^\wedge$$
$$= \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^\wedge(\xi), \quad \xi \in S^{n-1}.$$

Note that $(-\Delta)^{\alpha/2} S_K(\xi)$ is a homogeneous function of degree $-1-\alpha$ on $\mathbb{R}^n \setminus \{0\}$. □

Theorem 3.1: *Let K and L be two h-convex origin symmetric bodies in B^n and suppose $\frac{|x|^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2}$ is a positive definite distribution. If*

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi),$$

for every $\xi \in S^{n-1}$, then

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

Proof. First we prove the following inequality [21].

For $a, b \in (0,1)$

$$\frac{a}{1-a^2}\int_a^b \frac{r^{n-2}}{(1-r^2)^{n-1}}dr \leq \int_a^b \frac{r^{n-1}}{(1-r^2)^n}dr.$$

Observe that $\frac{r}{1-r^2}$ is increasing in the interval $(0,1)$. Therefore,

$$\frac{a}{1-a^2}\int_a^b \frac{r^{n-2}}{(1-r^2)^{n-1}}dr = \int_a^b \frac{r^{n-1}}{(1-r^2)^n}\frac{a}{1-a^2}\left(\frac{r}{1-r^2}\right)^{-1}dr$$
$$\leq \int_a^b \frac{r^{n-1}}{(1-r^2)^n}dr.$$

(Note that $a \leq b$ is not necessary)

We let $a = \|x\|_K^{-1}$ and $b = \|x\|_L^{-1}$. Then

$$\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1-\|x\|_K^{-2}} \int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr dx \leq \int_{S^{n-1}} \int_{\|x\|_K^{-1}}^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1-r^2)^{n}} dr dx.$$

If the left hand side of the inequality is non negative it follows that,

$$\int_{S^{n-1}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-1}}{(1-r^2)^{n}} dr dx \leq \int_{S^{n-1}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-1}}{(1-r^2)^{n}} dr dx$$

That is,

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L).$$

Therefore to complete the proof we only need to show the following inequality

$$\int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1-\|x\|_K^{-2}} \int_0^{\|x\|_K^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr dx$$
$$\leq \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1-\|x\|_K^{-2}} \int_0^{\|x\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr dx.$$

To prove the inequality we use Parseval's formula, the equation (3.3) and our assumption $(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi)$. Note that if K is sufficiently smooth, then

$$\left(\frac{|x|_2^{-\alpha} \|x\|_K^{-1}}{1-(\frac{|x|}{\|x\|_K})^2} \right)^\wedge,$$

restricted to the sphere is a continuous function. Moreover if K is not smooth, then the latter Fourier transform may not be a function, but still there is a positive measure γ on S^{n-1}, that corresponds to the restriction of this Fourier transform to the sphere. See [13, Corollary 2.26]. Thus :

$$
\begin{aligned}
&(2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr dx \\
&= (2\pi)^n \int_{S^{n-1}} \left(\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2} \right) \cdot \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right) dx \\
&= \int_{S^{n-1}} \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right)^{\wedge} (\theta) d\gamma(\theta) \\
&= \int_{S^{n-1}} \frac{\pi}{2^{n-1}} (-\Delta)^{\alpha/2} S_K(\xi) d\gamma(\theta) \\
&\leq \int_{S^{n-1}} \frac{\pi}{2^{n-1}} (-\Delta)^{\alpha/2} S_L(\xi) d\gamma(\theta) \\
&= \int_{S^{n-1}} \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right)^{\wedge} (\theta) d\gamma(\theta) \\
&= (2\pi)^n \int_{S^{n-1}} \left(\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2} \right) \cdot \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right) dx \\
&= (2\pi)^n \int_{S^{n-1}} \frac{\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr dx.
\end{aligned}
$$

□

Theorem 3.2: *Let L be a h-convex infinitely smooth origin-symmetric body in B^n and suppose $\frac{|x|^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}$ is not a positive definite distribution. Then there exists an h-convex body K in B^n such that*

$$(-\Delta)^{\alpha/2} S_K(\xi) \leq (-\Delta)^{\alpha/2} S_L(\xi),$$

for every $\xi \in S^{n-1}$, but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

Proof. By continuity of $\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2} \right)^{\wedge}$ there is a neighbourhood of ξ where this function is negative.

$$\Omega = \left\{ \xi \in S^{n-1} : \left(\frac{|x|_2^{-\alpha} \|x\|_L^{-1}}{1 - (\frac{|x|_2}{\|x\|_L})^2} \right)^{\wedge} (\xi) < 0 \right\}$$

Choose a non positive infinitely smooth even function υ supported on $\Omega \cup \{-\Omega\}$. Now we extend υ to a homogeneous function $r^{-1-\alpha}\upsilon(\theta)$ of degree$-1-\alpha$ on $\mathbb{R}^n$. By [13, Lemma 3.16], the Fourier transform of $r^{-1-\alpha}\upsilon(\theta)$ is equal to $r^{-n+1+\alpha}g(\theta)$ for some infinitely differentiable function g on S^{n-1}.

Then we construct a body K such that,

$$\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr = \int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr + \epsilon g(\theta)$$

For some small $\epsilon > 0$ (to make sure the body K is still convex). Now define $\alpha_\epsilon(\theta)$ such that

$$\int_0^{\|\theta\|_L^{-1}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr + \epsilon g(\theta) = \int_0^{\|\theta\|_L^{-1}+\alpha_\epsilon(\theta)} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr$$

It follows that :

$$\|\theta\|_K^{-1} = \|\theta\|_L^{-1} + \alpha_\epsilon(\theta).$$

Therefore :

$$\begin{aligned}
(-\Delta)^{\alpha/2} S_K(\xi) &= \frac{2^{n-1}}{\pi} \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right)^{\wedge} (\xi) \\
&= \frac{2^{n-1}}{\pi} \left(|x|_2^{-n+1+\alpha} \int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}} dr \right)^{\wedge} (\xi) \\
&\quad + \frac{2^{n-1}}{\pi} \left(|x|_2^{-n+1+\alpha} \epsilon g(\theta) \right)^{\wedge} (\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2^{n-1}}{\pi}\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\xi) \\
&\quad + \frac{2^{n-1}(2\pi)^n}{\pi}\epsilon|x|_2^{-1-\alpha}\upsilon(\xi) \\
&\leq (-\Delta)^{\alpha/2}S_L(\xi)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&(2\pi)^n\int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}drdx \\
&= (2\pi)^n\int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)\cdot\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)dx \\
&= \int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)^{\wedge}(\theta)\cdot\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\theta)d\theta \\
&= \int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)^{\wedge}(\theta)\cdot\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\theta)d\theta \\
&\quad + \int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)^{\wedge}(\theta)\left(\epsilon|x|_2^{-n+1+\alpha}g(x/|x|_2)\right)^{\wedge}(\theta) \\
&= \int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)^{\wedge}(\theta)\cdot\left(|x|_2^{-n+1+\alpha}\int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}}dr\right)^{\wedge}(\theta)d\theta \\
&\quad + (2\pi)^n\int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)^{\wedge}(\theta)\left(\epsilon|x|_2^{-1-\alpha}\upsilon(\theta)\right)d\theta \\
&> (2\pi)^n\int_{S^{n-1}}\left(\frac{|x|_2^{-\alpha}\|x\|_L^{-1}}{1-(\frac{|x|_2}{\|x\|_L})^2}\right)\int_0^{\frac{|x|_2}{\|x\|_L}} \frac{r^{n-2}}{(1-r^2)^{n-1}}drdx.
\end{aligned}
$$

As in the proof of Theorem 3.1, this means $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.

□

In order to find α such that the distribution

$$\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2}$$

is positive definite we use ellipsoids. With the help of ellipsoids proof follows from defining a body which is obtained by perturbing the Euclidean ball.

For small $\epsilon > 0$ and $k > 2$ define a body K such that

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^k \|x\|_E^{-1}, \quad \forall x \in \mathbb{R}^n \backslash \{0\},$$

with the ellipsoid E and the norm

$$\|x\|_E = \left(x_1^2 + x_2^2 + ... + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2}\right)^{1/2}. \tag{3.4}$$

Theorem 3.3: *The body K is well defined and convex for small ϵ.*

Proof. See [20]. This is a standard perturbation argument. We do a similar proof in Theorem 3.8.

□

Lemma 3.2: *Let e_n be the standard n^{th} coordinate vector. If $0 \leq \alpha < n - 4$, then*

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge (e_n) \sim C\epsilon^{-n+\alpha+2},$$

where C is a positive constant and the notation $a(\epsilon) \sim b(\epsilon)$, means $\lim\limits_{\epsilon \to 0} a(\epsilon)/b(\epsilon) = 1$

Proof. When α is an integer, it follows from part (iii) of [20, Lemma 3.3].

We will give the proof when α is not an integer.

If $k > \frac{n-\alpha-2}{2}$ by [20, Lemma 3.1], we get

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n) = C(k,\alpha,n)\int_{S^{n-1}}|(x,e_n)|^{2k-n+\alpha+1}\Delta^k\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)dx,$$

where Δ is the Laplace operator on $\mathbb{R}^n$, and

$$C(k,\alpha,n) = \frac{(-1)^{k+1}\pi}{2\Gamma(2k-n+\alpha+2)\sin(\pi(2k-n+\alpha+1)/2)}.$$

Now use the following formula (see [12, p.9])

$$\int_{S^{n-1}} f((x,\theta))d\theta = |S^{n-2}|\int_{-1}^{1}(1-t^2)^{(n-3)/2}f(t)dt, \quad x \in S^{n-1}. \tag{3.5}$$

Then it follows

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n) \sim$$
$$C\int_0^1 x_n^{2k-n+\alpha+1}\cdot(1-x_n^2)^{\frac{n-3}{2}}\Delta^k(\|x\|_E^{-1}|x|_2^{-\alpha})\Big|_{x_1^2+\cdots+x_{n-1}^2=1-x_n^2}dx_n.$$

Now we introduce a variable z where $x_n = \epsilon\cdot z$. Then

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n)$$
$$\sim C\epsilon^{2k-n+\alpha+2}\int_0^{1/\epsilon} z^{2k-n+\alpha+1}\cdot(1-\epsilon^2z^2)^{\frac{n-3}{2}}\Delta_\epsilon^k(|x|_2^{-1}\|x\|_{E^*}^{-\alpha})\Big|_{x_1^2+\cdots+x_{n-1}^2=1-\epsilon^2z^2,x_n=z}dz,$$

where E^* is the ellipsoid given by

$$\|x\|_{E^*}= \left(x_1^2+\cdots+x_{n-1}^2+\epsilon^2x_n^2\right)^{1/2}, \tag{3.6}$$

and

$$\Delta_\epsilon = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{1}{\epsilon^2}\frac{\partial^2}{\partial x_n^2}.$$

The largest term is obtained when we apply $\frac{1}{\epsilon^2}\frac{\partial^2}{\partial x_n^2}$ to $|x|_2^{-1}$ successively k times.

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n)$$
$$\sim C\epsilon^{-n+\alpha+2}\int_0^{1/\epsilon} z^{2k-n+\alpha+1}\cdot(1-\epsilon^2 z^2)^{\frac{n-3}{2}}\frac{\partial^{2k}}{\partial x_n^{2k}}(|x|_2^{-1})\Big|_{x_1^2+\cdots+x_{n-1}^2=1-\epsilon^2z^2, x_n=z} dz.$$

It is enough to show that

$$\int_0^{1/\epsilon} z^{2k-n+\alpha+1}\cdot(1-\epsilon^2 z^2)^{\frac{n-3}{2}}\frac{\partial^{2k}}{\partial x_n^{2k}}(|x|_2^{-1})\Big|_{x_1^2+\cdots+x_{n-1}^2=1-\epsilon^2z^2, x_n=z} dz$$

has a finite nonzero limit as $\epsilon \to 0^+$.

Observe

$$\int_0^{1/\epsilon} z^{2k-n+\alpha+1}\cdot(1-\epsilon^2 z^2)^{\frac{n-3}{2}}\frac{\partial^{2k}}{\partial x_n^{2k}}(|x|_2^{-1})\Big|_{x_1^2+\cdots+x_{n-1}^2=1-\epsilon^2z^2, x_n=z} dz$$
$$= \int_0^{1/\epsilon} z^{2k-n+\alpha+1}\cdot(1-\epsilon^2 z^2)^{\frac{n-3}{2}}\sum_{m=0}^{2k} b_m z^{2m}(1-\epsilon^2 z^2+z^2)^{\frac{-1-2k-2m}{2}},$$

where b_m is a constant.

We will show that

$$f_\epsilon(z) = z^{2k-n+\alpha+1}\cdot(1-\epsilon^2 z^2)^{\frac{n-3}{2}} z^{2m}(1-\epsilon^2 z^2+z^2)^{\frac{-1-2k-2m}{2}}\chi[0,1/\epsilon](z)$$

is bounded by an integrable function.

We split the integral in to $[0,1]$ and $[1,1/\epsilon]$. In $[0,1]$:

$$|f_\epsilon(z)| \le z^{2k-n+\alpha+1} z^{2m} (0.5 + z^2)^{\frac{-1-2k-2m}{2}} = g(z)$$

and $g(z)$ is integrable.

In $[1, 1/\epsilon]$:

$$f_\epsilon(z)\chi[1,1/\epsilon](z) = z^{2k-n+\alpha+1} z^{2m} z^{-1-2k-2m} \chi[1,\infty)(z) = z^{-n+\alpha}\chi[1,\infty)(z)$$

and $-n+\alpha < -4$.

Therefore from Dominated Convergence Theorem one can see that

$$\lim_{\epsilon\to 0^+} \int_0^{1/\epsilon} z^{2k-n+\alpha+1} \cdot (1-\epsilon^2 z^2)^{\frac{n-3}{2}} \frac{\partial^{2k}}{\partial x_n^{2k}} (|x|_2^{-1}) \Big|_{x_1^2+\cdots+x_{n-1}^2 = 1-\epsilon^2 z^2, x_n = z} dz$$

$$= \int_0^\infty z^{2k-n+\alpha+1} \cdot \frac{\partial^{2k}}{\partial z^{2k}} (1+z^2)^{-1/2} dz.$$

To finish the proof, we need to show that the latter integral is not equal to zero. Let $P(z^2) = c_0 + c_1 z^2 + \cdots + c_{k-1} z^{2k-2}$ be the Taylor polynomial of $(1+z^2)^{-1/2}$ at zero of order $2k-2$. Then clearly,

$$\int_0^\infty z^{2k-n+\alpha+1} \cdot \frac{\partial^{2k}}{\partial z^{2k}} (1+z^2)^{-1/2} dz = \int_0^\infty z^{2k-n+\alpha+1} \cdot \frac{\partial^{2k}}{\partial z^{2k}} \left((1+z^2)^{-1/2} - P(z^2) \right) dz.$$

After integration by parts $2k$ times and the change of the variable $t = z^2$ the integral becomes

$$= (2k-n+\alpha+1)...(-n+\alpha+2) \int_0^\infty z^{-n+\alpha+1} \left((1+z^2)^{-1/2} - P(z^2) \right) dz$$

$$= \frac{(2k-n+\alpha+1)...(-n+\alpha+2)}{2} \int_0^\infty t^{\frac{-n+\alpha}{2}} \left((1+t)^{-1/2} - P(t) \right) dt.$$

Using integration by parts in the opposite order and observing that $P(t)$ is the Taylor polynomial of $(1+t)^{-1/2}$, we get

$$= \frac{(2k-n+\alpha+1)...(-n+\alpha+2)}{2(\frac{-n+\alpha}{2}+k)...(\frac{-n+\alpha}{2}+2)(\frac{-n+\alpha}{2}+1)} \times \int_0^\infty t^{\frac{-n+\alpha}{2}+k} \frac{\partial^k}{\partial t^k} \left((1+t)^{-1/2} - P(t)\right) dt.$$

$$= \frac{(2k-n+\alpha+1)...(-n+\alpha+2)}{2(\frac{-n+\alpha}{2}+k)...(\frac{-n+\alpha}{2}+2)(\frac{-n+\alpha}{2}+1)} \times \int_0^\infty t^{\frac{-n+\alpha}{2}+k} \frac{\partial^k}{\partial t^k} (1+t)^{-1/2} dt.$$

The latter is clearly a nonzero constant.

□

Rest of the chapter consists with few theorems which will be proved for Euclidean space. Those results will play an important role in determining the sign of the distribution $\left(\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-(\frac{|x|_2}{\|x\|_K})^2}\right)^\wedge$ for an h-convex body K; see Theorem 3.6.

Theorem 3.4: *Let $n-4<\alpha<n-2$. Then there exists an origin-symmetric body L in $\mathbb{R}^n$, $n \geq 4$, such that $|x|_2^{-\alpha-2}\|x\|_L$ is not a positive definite distribution.*

Proof. For small $\epsilon > 0$ let L be an ellipsoid with the norm

$$\|x\|_L = \left(x_1^2 + \cdots + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2}\right)^{1/2}.$$

Now, we define a star body $K \subset \mathbb{R}^n$:

$$\rho_K(\theta) = \rho_L^{\frac{1}{-1-\alpha}}(\theta), \quad \theta \in S^{n-1}.$$

Then, we observe

$$|x|_2^{-\alpha-2}\|x\|_L = \left(|x|_2^{\frac{-\alpha-2}{-1-\alpha}}\|x\|_L^{\frac{1}{-1-\alpha}}\right)^{-\alpha-1} = \|x\|_K^{-1-\alpha}, \quad x \in \mathbb{R}^n \setminus \{0\}$$

Using [19, Theorem 2.2] with $q = n - \alpha - 2 \in (0, 2)$ we have

$$(\|x\|_K^{-1-\alpha})^\wedge(\xi) = \frac{\pi(\alpha+1)}{\Gamma(-n+\alpha+2)\cos\frac{\pi(n-\alpha-2)}{2}} \int_0^\infty t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt,$$

case $n - \alpha - 2$ can be obtained from the part (c) of [19, Theorem 2.2].

Since, for $\alpha \in (n-4, n-2)$, $\Gamma(-n+\alpha+2)\cos\frac{\pi(n-\alpha-2)}{2} \leq 0$, we only need to prove that for some ξ

$$\int_0^\infty t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt > 0. \tag{3.7}$$

Note that case $\alpha = n - 3$ can be obtained by the limits.

Let ξ be the direction of the x_n-axis. Let $[-t_0, t_0]$ be the support of $A_{K,\xi}(t)$, then

$$\int_0^\infty t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt$$

$$\begin{aligned} &= \int_0^{t_0} t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt - \int_{t_0}^\infty t^{-n+\alpha+1}A_{K,\xi}(t)dt \\ &= \int_0^{t_0} t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt - t_0^{-n+\alpha+2}\frac{A_{K,\xi}(0)}{n-\alpha-2} \end{aligned} \tag{3.7}$$

Introducing new coordinates on the sphere S^{n-1} we get

$$\theta = \cos\phi \cdot \eta + \sin\phi \cdot \xi, \quad \theta \in S^{n-1}$$

$$(-\pi/2 \le \phi \le \pi/2 \quad \text{and} \quad \eta \in S^{n-1} \cap \xi^{\perp})$$

Since we are interested in sections of K perpendicular to ξ, its axis of revolution, we denote $\rho_K(\phi)$ as the radial function of those $\theta \in S^{n-1}$ that makes an angle ϕ with the plane $\xi^{\perp}$ and therefore

$$\rho_K(\phi) = \left(\cos^2\phi + \frac{\sin^2\phi}{\epsilon^2} \right)^{\frac{1}{2+2\alpha}}$$

Observe that $t = \sin\phi \cdot \rho_K(\phi)$ is an increasing function of the angle $\phi \in (0, \pi/2)$, therefore all the sections of K by hyperplane orthogonal to ξ are $(n-1)$-dimensional disks. Also, $t_0 = \epsilon^{-\frac{1}{4(1+\alpha)}}$ goes to infinity as ϵ tends to zero. Hence, last term in (3.7) approaches to zero as ϵ goes to zero.

Now, we have

$$\begin{aligned} A_{K,\xi}(t(\phi)) &= \omega_{n-1}(\cos\phi \cdot \rho_K(\phi))^{n-1} \\ &= \omega_{n-1} \cos^{n-1}\phi \left(\cos^2\phi + \frac{\sin^2\phi}{\epsilon^2} \right)^{\frac{n-1}{2+2\alpha}} \end{aligned}$$

where, ω_{n-1} is the volume of the unit $(n-1)$-dimensional Euclidean ball. Introduce a new variable $t = \sin\phi \left(\cos^2\phi + \frac{\sin^2\phi}{\epsilon^2} \right)^{\frac{1}{2+2\alpha}}$.

Thus,

$$\int_0^{t_0} t^{-n+\alpha+1}(A_{K,\xi}(t) - A_{K,\xi}(0))dt$$

$$= \omega_{n-1} \int_0^{\pi/2} (\sin\phi)^{-n+\alpha+1} \cdot (\cos^2\phi + \epsilon^{-2}\sin^2\phi)^{\frac{-n+\alpha+1}{2+2\alpha}}$$
$$\times \left(\cos^{n-1}\phi(\cos^2\phi + \epsilon^{-2}\sin^2\phi)^{\frac{n-1}{2+2\alpha}} - 1 \right)$$
$$\times (\cos^2\phi + \epsilon^{-2}\sin^2\phi)^{\frac{1}{2+2\alpha}-1} \left(\cos^3\phi + \epsilon^{-2} + \left(\frac{\epsilon^{-2}-1}{1+\alpha} \right) \cos\phi \sin^2\phi \right) d\phi. \tag{3.7}$$

Now, we will find the intervals where the integrand is positive and negative. This depends on the sign of the equation
$\left(\cos^{n-1}\phi(\cos^2\phi + \epsilon^{-2}\sin^2\phi)^{\frac{n-1}{2+2\alpha}} - 1 \right) = 0.$
In fact we need to solve

$$f(\phi) = (\cos\phi)^{4+2\alpha} + \epsilon^{-2}(\cos\phi)^{2+2\alpha}\sin^2\phi = 1 \tag{3.8}$$

Left hand side of this equation first increases and then decreases to zero. Therefore the equation has two roots, namely ϕ_1 and ϕ_2. Observe that $\phi_1 = 0$. One can show that if ϵ is small, the maximum of the function $f(\phi)$ roughly attained at $\phi = \arccos\sqrt{1 - \frac{1}{2+\alpha}}$. From (3.8) and $\phi_2 \gtrsim \arccos\sqrt{1-\frac{1}{2+\alpha}}$ we get

$$\epsilon^{-2}(\cos\phi)^{2+2\alpha} \leq C(\alpha) \quad \text{and} \quad \phi_2 = \pi/2 - o(\epsilon^{\frac{2}{2+\alpha}})$$

Now, we see that integrand is positive in the interval $(0, \pi/2 - o(\epsilon^{\frac{2}{2+\alpha}}))$ and negative in the interval $(\pi/2 - o(\epsilon^{\frac{2}{2+\alpha}}), \pi/2)$. Splitting the integral into two according to these intervals we see that, in the negative interval absolute value of the integral is bounded above by

$$C\epsilon^{\frac{n-\alpha-2}{1+\alpha}},$$

which approaches to zero as ϵ approaches to zero.

To estimate the positive part of the integral it is sufficient to consider the interval $[\pi/4, \pi/3]$. For small ϵ, the integral has order

$$C\epsilon^{-1},$$

which approaches infinity as ϵ gets smaller. Thus we have (3.7).

□

Corollary 3.4.1: Suppose $q \leq 2k, k \in \mathbb{N} \cup \{0\}$.

1. If $f \in C^{2k}(S^{n-1})$, then the Fourier transform $(f(\theta)r^{-n+q+1})^\wedge$ is a homogeneous of degree $-1-q$ continuous function on $\mathbb{R}^n \setminus \{0\}$.
2. If $f_m, f \in C^{2k}(S^{n-1})$ and f_m converges to f in $C^{2k}(S^{n-1})$, then the Fourier transform $(f_m(\theta)r^{-n+q+1})^\wedge$ converges to $(f(\theta)r^{-n+q+1})^\wedge$ in the space $C(S^{n-1})$.

Proof. See [13, Corollary 3.17] □

In [14] following theorem has been proved.

Theorem 3.5: *Let K be an origin-symmetric convex body in $\mathbb{R}^n$. If $\alpha \in [n-4, n-1)$, then $\|x\|_K^{-1} \cdot |x|_2^{-\alpha}$ is a positive definite distribution.*

However in the sense of hyperbolic space we have a different answer as below.

Theorem 3.6:

1. *Let K be an origin-symmetric h-convex body in B^n. If $\alpha \in [n-2, n-1)$, then*

$$\frac{\|x\|_K^{-1} \cdot |x|_2^{-\alpha}}{1 - \frac{|x|_2^2}{\|x\|_K^2}}$$

is a positive definite distribution.

2. *For $\alpha \in [n-4, n-2)$, there exists an origin-symmetric h-convex body K in B^n such that*

$$\frac{\|x\|_K^{-1} \cdot |x|_2^{-\alpha}}{1 - \frac{|x|_2^2}{\|x\|_K^2}}$$

is not a positive definite distribution.

Proof.

1. For any h - convex body K in B^n define a star body L as follows :

$$\|x\|_L^{-1-\alpha} = \frac{\|x\|_K^{-1} \cdot |x|_2^{-\alpha}}{1 - \frac{|x|_2^2}{\|x\|_K^2}}. \tag{3.9}$$

We use Theorem 2.B.3. Then,

for $q \in (-1, 0)$, $\cos \frac{q\pi}{2}$ and $\Gamma(-q)$ are positive, which gives,

$$A_{L,\xi}^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_{\infty}^{\infty} |t|^{-q-1} A(t) dt \geq 0,$$

for $q = 0$,$\cos \frac{q\pi}{2} = 1$.

$$A_{L,\xi}^{(0)}(0) = (-1)^0 A_{L,\xi}(0) \geq 0.$$

Therefore, $\left(\frac{\|x\|_K^{-1}\cdot|x|_2^{-\alpha}}{1-\frac{|x|_2^2}{\|x\|_K^2}}\right)^\wedge = (\|x\|_L^{-1-\alpha})^\wedge$ is positive for all $\xi \in S^{n-1}$.

2. Let K be defined as follows:

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^3 \|x\|_E^{-1},$$

where E is the ellipsoid

$$\|x\|_E = \left(x_1^2 + \cdots + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2}\right)^{1/2}.$$

The radius of K is less than 1, and so K lies in B^n. Also one can check that K is e-convex, and therefore it is h-convex.

We need to show that the distribution

$$\frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-|x|_2^2\|x\|_K^{-2}}$$

is not positive definite.

We have

$$\begin{aligned} \frac{|x|_2^{-\alpha}\|x\|_K^{-1}}{1-|x|_2^2\|x\|_K^{-2}} &= \epsilon^{-3}|x|_2^{-\alpha-2}\|x\|_E \cdot \frac{1-\epsilon^3|x|_2\|x\|_E^{-1}}{2-\epsilon^3|x|_2\|x\|_E^{-1}} \\ &= \frac{\epsilon^{-3}}{2}|x|_2^{-\alpha-2}\|x\|_E - \frac{|x|_2^{-\alpha-1}}{2(2-\epsilon^3|x|_2\|x\|_E^{-1})} \end{aligned} \tag{3.10}$$

It is not hard to see that the last fraction of (3.10) is close to $1/4|x|_2^{-\alpha-1}$ in $C^2(S^{n-1})$ and its Fourier transform is a constant on the sphere. Therefore, for sufficiently small $\epsilon > 0$ the term $\frac{\epsilon^{-3}}{2}|x|_2^{-\alpha-2}\|x\|_E$ is dominating. So from part (ii) of Corollary 3.4.1 we can just look at the distribution $|x|_2^{-\alpha-2}\|x\|_E$, but it is not positive definite from Theorem 3.4 for $\alpha \in (n-4, n-2)$.

For $\alpha = n-4$ we define a body M such that $|x\|_M^{-\alpha-1} = |x|_2^{-\alpha-2}\|x\|_E$. Then using a similar proof as in [19, Lemma 3.2], we get

$$(|x\|_M^{-\alpha-1})^\wedge(\xi) = -\pi(n-3)A_{M,\xi}^{(2)}(0) < 0.$$

$\square$

In the previous Theorem we covered the case where $\alpha \in [n-4, n-2)$. To cover the remaining case where $\alpha \in [0, n-4)$ we need some results in Euclidean space.

Theorem 3.7: *For $0 \le \alpha < n-4$ there exists an infinitely smooth origin-symmetric convex body K with positive curvature, so that*

$$\|x\|_K^{-1} \cdot |x|_2^{-\alpha}$$

is not a positive definite distribution.

Proof. If $\alpha < n-4$, then for some small $\delta > 0$ we also have $\alpha < n-4-\delta$. Now define K

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^{n-\alpha-2-\delta}\|x\|_E^{-1}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where E is the ellipsoid with the norm

$$\|x\|_E = \left(x_1^2 + \cdots + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2}\right)^{1/2}.$$

Writing the equality using Fourier transforms :

$$(\|x\|_K^{-1}\cdot|x|_2^{-\alpha})^\wedge(\xi) = (|x|_2^{-\alpha-1})^\wedge(\xi) - \epsilon^{n-\alpha-2-\delta}(\|x\|_E^{-1}\cdot|x|_2^{-\alpha})^\wedge(\xi),$$

$$(\|x\|_K^{-1}\cdot|x|_2^{-\alpha})^\wedge(\xi) = C(\alpha+1,n) - \epsilon^{n-\alpha-2-\delta}(\|x\|_E^{-1}\cdot|x|_2^{-\alpha})^\wedge(\xi), \quad \xi \in S^{n-1}. \tag{3.11}$$

The latter comes from the Fourier transform of $|x|_2^{-\alpha-1}$, [9, p. 363].

For $0 < k < n$

$$(|x|_2^{-k})^\wedge(\xi) = C_{k,n}|\xi|_2^{-n+k}$$

Where,

$$C_{k,n} = \frac{2^{n-k}\pi^{n/2}\Gamma((n-k)/2)}{\Gamma(k/2)}$$

Using Lemma 3.2, for direction e_n we get,

$$\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n) \sim C\epsilon^{-n+\alpha+2},$$

where C is a positive constant.

Therefore,

$$\begin{aligned}\left(\|x\|_K^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n) &= \left(|x|_2^{-1-\alpha}\right)^\wedge(e_n) - \epsilon^{n-\alpha-2-\delta}\left(\|x\|_E^{-1}|x|_2^{-\alpha}\right)^\wedge(e_n)\\ &\sim C(\alpha,n) - C\epsilon^{n-\alpha-2-\delta}\epsilon^{-n+\alpha+2}\\ &= C(\alpha,n) - C\epsilon^{-\delta} < 0,\end{aligned}$$

for small enough ϵ. □

To construct a counter example for those values of α in h-space we use the previous result.

Theorem 3.8: *For $0 \leq \alpha < n-4$ there exists an infinitely smooth origin-symmetric, h-convex body L in B^n, so that*

$$\frac{\|x\|_L^{-1} \cdot |x|_2^{-\alpha}}{1 - \frac{|x|_2^2}{\|x\|_L^2}}$$

is not a positive definite distribution.

Proof. Let us define a body L implicitly by the formula :

$$\|x\|_K^{-1} = \frac{\|x\|_L^{-1}}{1 - \frac{|x|_2^2}{\|x\|_L^2}}$$

and let K be the body defined by,

$$\|x\|_K^{-1} = |x|_2^{-1} - \epsilon^{n-\alpha-2-\delta} \|x\|_E^{-1} \quad , \forall x \in \mathbb{R}^n \backslash \{0\},$$

With the ellipsoid E and the norm,

$$\|x\|_E = \left(x_1^2 + x_2^2 + ... + x_{n-1}^2 + \frac{x_n^2}{\epsilon^2} \right)^{1/2}.$$

By Theorem 3.3 K is e-convex. From the previous Theorem 3.7 we have $\frac{\|x\|_L^{-1} \cdot |x|_2^{-\alpha}}{1 - \frac{|x|_2^2}{\|x\|_L^2}}$ is not a positive definite distribution.

So it remains to prove that the body L is e-convex, and therefore h-convex. Since K and L are bodies of revolution, we will only look at their generating curves.

Passing it on to polar coordinates we get:

$$\rho_K = 1 - \epsilon^{n-\alpha-2-\delta}\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-1/2},$$

which implies,

$$\rho_K' = \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2} - 1\right)\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-3/2} \sin 2t$$

and

$$\begin{aligned}\rho_K'' &= \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2} - 1\right) \\ &\quad \times \left[2\cos 2t\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-3/2} - \frac{3}{2}\left(\frac{1}{\epsilon^2} - 1\right)\sin^2 2t\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-5/2}\right] \\ &= \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2} - 1\right)\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-3/2} \\ &\quad \times \left[2\cos 2t - \frac{3}{2}\left(\frac{1}{\epsilon^2} - 1\right)\sin^2 2t\left(\cos^2 t + \frac{\sin^2 t}{\epsilon^2}\right)^{-1}\right]\end{aligned}$$

Using the formula

$$\rho_K = \frac{\rho_L}{1 - \rho_L^2}$$

we get

$$\rho_L = \frac{-1 + \sqrt{1 + 4\rho_K^2}}{2\rho_K}.$$

Therefore:

$$\rho_L' = \frac{1}{2}\frac{\left(-1 + \sqrt{1 + 4\rho_K^2}\right)}{\rho_K^2\sqrt{1 + 4\rho_K^2}}\rho_K'$$

and,

$$\rho_L'' = \frac{1}{2\delta}\left[\left(-\frac{1}{\sqrt{1+4\rho_K^2}}+1\right)'\frac{\rho_K'}{\rho_K^2} + \left(-\frac{1}{\sqrt{1+4\rho_K^2}}+1\right)\frac{\rho_K\rho_K''-2(\rho_K')^2}{\rho_K^3}\right]$$

Now, for small $\epsilon > 0$,

$$\rho_K = 1-\epsilon^{n-\alpha-2-\delta}\left(\cos^2 t+\frac{\sin^2 t}{\epsilon^2}\right)^{-1/2}$$

gives

$$|\rho_K - 1| \leq \epsilon^{n-\alpha-2-\delta}.$$

Also we have,

$$\begin{aligned}
\rho_K' &= \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2}-1\right)\left(\cos^2 t+\frac{\sin^2 t}{\epsilon^2}\right)^{-3/2}\sin 2t \\
|\rho_K'| &\leq \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2}\right) \\
&= \frac{\epsilon^{n-\alpha-4-\delta}}{2},
\end{aligned}$$

and

$$\begin{aligned}
\rho_K'' &= \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2}-1\right)\left(\cos^2 t+\frac{\sin^2 t}{\epsilon^2}\right)^{-3/2} \\
&\quad\times\left[2\cos 2t-\frac{3}{2}\left(\frac{1}{\epsilon^2}-1\right)\sin^2 2t\left(\cos^2 t+\frac{\sin^2 t}{\epsilon^2}\right)^{-1}\right] \\
|\rho_K''| &\leq \frac{\epsilon^{n-\alpha-2-\delta}}{2}\left(\frac{1}{\epsilon^2}\right)\left[2+\frac{3}{2}\frac{1}{\epsilon^2}\left(\frac{1}{\epsilon^2}\right)^{-1}\right] = \frac{7}{4}\epsilon^{n-\alpha-4-\delta}.
\end{aligned}$$

Since, ρ_K is close to 1, we have

$$|\rho_L - \frac{(-1+\sqrt{5})}{2}| < \epsilon^{n-\alpha-2-\delta}.$$

Using those bounds we can again bound ρ'_L and ρ''_L as below.

$$\rho'_L = \frac{1}{2}\frac{\left(-1+\sqrt{1+4\rho_K^2}\right)}{\rho_K^2\sqrt{1+4\rho_K^2}}\rho'_K$$

$$|\rho'_L| \leq D_1\epsilon^{n-\alpha-4-\delta}, \quad \text{where } D_1 \text{ is a constant.}$$

$$\rho''_L = \frac{1}{2}\left[\left(-\frac{1}{\sqrt{1+4\rho_K^2}}+1\right)'\frac{\rho'_K}{\rho_K^2} + \left(-\frac{1}{\sqrt{1+4\rho_K^2}}+1\right)\frac{\rho_K\rho''_K - 2(\rho'_K)^2}{\rho_K^3}\right]$$

$$|\rho''_L| \leq \frac{1}{2}\left[D_2\epsilon^{n-\alpha-4-\delta} + D_3\frac{\rho_K C_2\epsilon^{n-\alpha-4-\delta} + 2\left(C_1\epsilon^{n-\alpha-4-\delta}\right)^2}{\rho_K^3}\right]$$

$$\leq \frac{1}{2}\epsilon^{n-\alpha-4-\delta}\left[D_2 + D_4\right] = D_5\epsilon^{n-\alpha-4-\delta},$$

Where C_1, C_2, D_2, D_3, D_4 and D_5 are constants.

From the curvature formula we observe the body L is e-convex for small $\epsilon > 0$. Hence the result follows from Theorem 3.7.

□

Chapter 4

Converse of Brunn's Theorem

Theorem 4.1: *Let K be a C^1 star body in $\mathbb{H}^n$. If for every $\xi \in S^{n-1}$ the function $A_{K,\xi}(t)$ has a critical point at $t = 0$, then K is origin symmetric.*

This theorem has been proved in [16] for Euclidean space; see also [18]. To prove the theorem in hyperbolic space $\mathbb{H}^n$ we will use the Beltrami- Klein model as it looks more natural. This model can be identified with the interior of the unit ball in $\mathbb{R}^n$ with the metric

$$ds^2 = 4\frac{|dx|_2^2}{(1 - |x|_2^2)} + 4\frac{(x \cdot dx)^2}{(1 - |x|_2^2)^2}. \tag{4.1}$$

In order to compute the volume element of this metric, we write

$$
\begin{aligned}
&ds^2\\
&= 4\frac{(x_1dx_1 + x_2dx_2 + \cdots + x_ndx_n)^2}{\Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)^2} + 4\frac{dx_1^2+dx_2^2+\cdots+dx_n^2}{\Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)}\\
&= 4\frac{(x_1dx_1 + x_2dx_2 + \cdots + x_ndx_n)^2 + \Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)(dx_1^2+dx_2^2+\cdots+dx_n^2)}{\Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)^2}\\
&= 4\frac{\sum\limits_i x_i^2dx_i^2 + 2\sum\limits_{\substack{i,j\\ i\neq j}} x_ix_jdx_idx_j + \sum\limits_i dx_i^2 - \sum\limits_{\substack{i,j\\ i\neq j}} x_i^2dx_j^2 - \sum\limits_i x_i^2dx_i^2}{\Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)^2}\\
&= 4\frac{(1-\sum\limits_{i\neq 1}x_i^2)dx_1^2 + (1-\sum\limits_{i\neq 2}x_i^2)dx_2^2 + \cdots + (1-\sum\limits_{i\neq n}x_i^2)dx_n^2 + 2\sum\limits_{\substack{i,j\\ i\neq j}} x_ix_jdx_idx_j}{\Big(1-(x_1^2+x_2^2+\cdots+x_n^2)\Big)^2}\\
&= \sum_{i,j} g_{ij}dx_idx_j
\end{aligned}
$$

We let G be the determinant of the matrix $[g_{ij}]_{i,j=1}^n$. Then we compute G as below.

$$
G = \frac{4^n}{(1-|x|_2^2)^{2n}}
\begin{vmatrix}
1-\sum\limits_{\substack{i\\ i\neq 1}} x_i^2 & x_1x_2 & x_1x_3 \dots & x_1x_n\\
x_1x_2 & 1-\sum\limits_{\substack{i\\ i\neq 2}} x_i^2 & x_2x_3 \dots & x_2x_n\\
\vdots & \vdots & \vdots \ \ddots & \vdots\\
x_nx_1 & x_nx_2 & x_nx_3 \dots & 1-\sum\limits_{\substack{i\\ i\neq n}} x_i^2
\end{vmatrix}
$$

$$= \frac{4^n}{(1-|x|_2^2)^{2n}} x_1^2 x_2^2 \ldots x_n^2 \begin{vmatrix} \frac{1-\sum\limits_{\substack{i \\ i\neq 1}} x_i^2}{x_1^2} & 1 & 1 \ldots & 1 \\ 1 & \frac{1-\sum\limits_{\substack{i \\ i\neq 2}} x_i^2}{x_2^2} & 1 \ldots & 1 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 1 & 1 & 1 \ldots & \frac{1-\sum\limits_{\substack{i \\ i\neq n}} x_i^2}{x_n^2} \end{vmatrix}$$

$$\xrightarrow[]{\substack{R_1 \Longleftrightarrow R_1 - R_n \\ R_2 \Longleftrightarrow R_2 - R_n \\ \cdots \\ R_{n-1} \Longleftrightarrow R_{n-1} - R_n}}$$

$$= \frac{4^n}{(1-|x|_2^2)^{2n}} x_1^2 x_2^2 \ldots x_n^2 \begin{vmatrix} \frac{1-|x|^2}{x_1^2} & 0 & 0 \ldots & -\frac{1-|x|^2}{x_1^2} \\ 0 & \frac{1-|x|^2}{x_2^2} & 0 \ldots & -\frac{1-|x|^2}{x_1^2} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ 1 & 1 & 1 \ldots & \frac{1-\sum\limits_{\substack{i \\ i\neq n}} x_i^2}{x_n^2} \end{vmatrix}$$

$$= \frac{4^n}{(1-|x|_2^2)^{2n}} \begin{vmatrix} 1-|x|^2 & 0 & 0 & \ldots & -(1-|x|^2) \\ 0 & 1-|x|^2 & 0 & \ldots & -(1-|x|^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^2 & x_2^2 & x_3^2 & \ldots & 1-\sum\limits_{\substack{i \\ i\neq n}} x_i^2 \end{vmatrix}$$

$$= \frac{4^n}{(1-|x|_2^2)^{2n}} (1-|x|^2)^{n-1} \begin{vmatrix} 1 & 0 & 0 & \ldots & -1 \\ 0 & 1 & 0 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^2 & x_2^2 & x_3^2 & \ldots & 1-\sum\limits_{\substack{i \\ i\neq n}} x_i^2 \end{vmatrix}$$

$$\xrightarrow{C_n \Longleftrightarrow C_n + \sum_{i=1}^{n-1} C_i}$$

$$= \frac{4^n}{(1-|x|_2^2)^{n+1}} \underbrace{\begin{vmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^2 & x_2^2 & x_3^2 & \ldots & 1 \end{vmatrix}}_{\text{Lower triangular matrix}}$$

Which implies

$$G = \frac{4^n}{(1-|x|_2^2)^{n+1}}$$

Now, we consider the volume element with respect to the metric (4.1).

$$\text{Volume element} = d(vol) = \sqrt{G} dx_1 dx_1 \ldots dx_n = \frac{2^n}{(1-|x|_2^2)^{\frac{n+1}{2}}} dx_1 dx_1 \ldots dx_n.$$

Proof of Theorem 4.1. For $-1 < \mathrm{Re}(q) < 0$,

$$
\begin{aligned}
A_{K,\xi}^{(q)}(0) &= \frac{1}{\Gamma(-q)} \int_0^\infty z^{-1-q} A_{K,\xi}(z) dz \\
&= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}} (z^{-1-q} + z^{-1-q} sgn\, z) A_{K,\xi}(z) dz \\
&= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}} (z^{-1-q} + z^{-1-q} sgn\, z) \\
&\quad \times \int_{\langle x,\xi\rangle = z} 2^{n-1} \frac{\chi(\|x\|_K)}{(1-|x|_2^2)^{\frac{n}{2}}}\, dxdz
\end{aligned}
$$

$$
= \frac{2^{n-2}}{\Gamma(-q)} \int_K \frac{(\langle x,\xi\rangle^{-1-q} + \langle x,\xi\rangle^{-1-q} sgn\, \langle x,\xi\rangle)}{(1-|x|_2^2)^{\frac{n}{2}}}\, dx.
$$

Passing to polar coordinates we obtain :

$$
\begin{aligned}
A_{K,\xi}^{(q)}(0) &= \frac{2^{n-2}}{\Gamma(-q)} \int_{S^{n-1}} (\langle \theta,\xi\rangle^{-1-q} + \langle \theta,\xi\rangle^{-1-q} sgn\, \langle x,\xi\rangle) \\
&\quad \times \int_0^{\|\theta_K\|^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, drd\theta.
\end{aligned}
$$

writing

$$
\int_0^{\|\theta_K\|^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr
$$

as a sum of its odd and even parts we get :

$$
\begin{aligned}
&A_{K,\xi}^{(q)}(0)\\
&= \frac{2^{n-3}}{\Gamma(-q)} \int_{S^{n-1}} \langle \theta, \xi \rangle^{-1-q} \bigg(\int_0^{\|\theta\|_K^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr + \int_0^{\|-\theta_K\|^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr \bigg) d\theta\\
&\quad + \frac{2^{n-3}}{\Gamma(-q)} \int_{S^{n-1}} \langle \theta, \xi \rangle^{-1-q} sgn\, \langle x, \xi \rangle\\
&\quad \times \bigg(\int_0^{\|\theta_K\|^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr - \int_0^{\|-\theta_K\|^{-1}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr \bigg) d\theta.
\end{aligned}
$$

Next we will use the following formula from [11, Theorem 3.1].

If $0 < \mathrm{Re}(p) < 1$, then the Fourier transform of f_p is a homogeneous function of degree $-p$ on $\mathbb{R}^n \setminus \{0\}$ given by

$$
\begin{aligned}
\hat{f}_p &= \Gamma(p) \cos(p\pi/2) \int_{S^{n-1}} \langle x, \theta \rangle^{-p} f(\theta) d\theta\\
&\quad - i\Gamma(p) \sin(p\pi/2) \int_{S^{n-1}} \langle x, \theta \rangle sgn(x \cdot \theta) f(\theta) d\theta.
\end{aligned}
$$

Where, f_p is a homogeneous degree $-n+p$ extension of $f \in C(S^{n-1})$ to $\mathbb{R}^n \setminus \{0\}$ with,

$$
f_p = |x|_2^{-n+p} f(x/|x|).
$$

Hence,

$$
\begin{aligned}
A_{K,\xi}^{(q)}(0) = \frac{2^{n-2} \cos(q\pi/2)}{\pi} &\bigg(|x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr\\
&+ |x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|-x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr \bigg)^{\wedge} (\xi)\\
&- i \frac{2^{n-2} \sin(q\pi/2)}{\pi} \bigg(|x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr\\
&- |x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|-x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr \bigg)^{\wedge} (\xi).
\end{aligned}
$$

Using the fact $\forall \xi \in S^{n-1}$, $A'_{K,\xi}(0) = 0$ we get,

$$\left(|x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr - |x|_2^{-n+1+q} \int_0^{\frac{|x|_2}{\|-x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr \right)^{\wedge} (\xi) = 0.$$

Thus,

$$\int_0^{\frac{|x|_2}{\|x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr = \int_0^{\frac{|x|_2}{\|-x\|_K}} \frac{r^{n-q-2}}{(1-r^2)^{\frac{n}{2}}}\, dr,$$

which means $\quad \|x\|_K = \|-x\|_K \quad \forall x \in \mathbb{R}^n \setminus \{0\}$.

Therefore, K is origin symmetric.

□

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