UNIVERSITY OF ALBERTA

EQUILIBRIUM ANALYSIS OF ELASTIC AND ELASTO-PLASTIC CABLE NETWORKS

BY

ALI ASGHAR ATAI

A thesis submitted to the faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of **Master of Science.**

DEPARTMENT OF MECHANICAL ENGINEERING

Edmonton, Alberta
Fall 1994

ISBN 0-315-94999-6

UNIVERSITY OF ALBERTA

RELEASE FORM

NAME OF AUTHOR: **Ali Asghar Atai**

TITLE OF THESIS: **Equilibrium Analysis of Elastic and Elasto-Plastic Cable Networks**

DEGREE: **Master of Science**

YEAR THIS DEGREE GRANTED: **1994**

A. A. Atai

#2107 Galbraith House

Michener Park

Edmonton, Alberta T6H 5B5

Date May 11, '94

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Equilibrium Analysis of Elastic and Elasto-Plastic Cable Networks** submitted by **Ali Asghar Atai** in partial fulfilment of the requirements for the degree of **Master of Science**.

Dr. A. Mioduchowski (Supervisor)

Dr. D.J. Steigmann (Co-Sup.)

Dr. G. Faulkner

Dr. A.E. Elwi

Date : May 6th, 1994

This work is dedicated to my parents

Abstract

Non-linear equations of equilibrium are derived using the theory of elastic cables. Weierstrass necessary conditions for stability of the equilibrium configuration are discussed and it is proved that they are also sufficient for global stability of cables. The theory is then applied to a general cable network. A method called Dynamic Relaxation (DR) is used to solve the nonlinear equations of equilibrium and some examples are presented which are analyzed by this method. The theory of elasto-plastic cables is then discussed and it is proved that the standard DR method is not applicable to elasto-plastic cables without further modification. An incremental technique is then discussed which is applicable to this class of cables and some examples are presented based on this method.

Acknowledgement

I would like to express my heartfelt thanks to my supervisors Dr. A. Mioduchowski and Dr. D.J. Steigmann for their guidance and support. I wish them a life full of joy and happiness.

Table of Contents

List of Tables

List of Figures

INTRODUCTION

The study of cable networks has long interested the structural engineering community because of their high strength, low weight and low cost. In recent years cable nets have been widely used in suspension structures and tension roofs. Examples of these include the German pavilion at EXPO 67, the Diplomatic Quarters Club in Riyadh, the Hong Kong aviary, and the CFAN (Christ For All Nations mission) transportable tent. Reference [1] provides some examples of cable structures. The above mentioned features have created a great deal of interest in the static analysis of cable nets. The problem that arises in this analysis is the non-linear behaviour of the structure which consists of two parts : geometrical non-linearity due to large deformations and material non-linearity.

Steigmann [2] has shown that a configuration of a dead-loaded elastic cable network minimizes the potential energy absolutely if and only if the network is in equilibrium, the cable forces are non-negative and the cable stretches belong to domains of convexity of the cable strain energy functions. In these conditions, the equilibrated network is globally stable and that's the configuration we are looking for in the analysis.

Several techniques have been developed and used to solve the non-linear equations of equilibrium or to find the absolute minimum of the potential energy and the corresponding configuration. These methods are classified into three principal groups [5]:

(1) Iterative methods

(2) Incremental techniques

(3) Minimization and relaxation methods

The first two groups make use of the overall or tangent stiffness matrix. In the third group, however, a vector formulation of the problem is employed which reduces the data storage requirements for calculations.

The most widely used iterative method for the study of geometrically non-linear problems is the Newton-Raphson method. In this method, however, problems arise when the stiffness matrix becomes positive semi-definite or indefinite [8].

Incremental techniques may be necessary for structures subject to both material and geometric non-linearity with path dependent behaviour. In this technique, loading and prescribed displacements are considered functions of a time-like variable. The time domain is then divided into small increments and at the beginning and end of each increment, the structure is equilibrated by solving a set of linear equations.

Minimization techniques formulate the problem as an optimization problem and find the absolute minimum of the potential energy. Several minimization methods applied to cable networks are reviewed in [8].

One of the methods widely applied to cable networks, and to non-linear problems in general, is Dynamic Relaxation which uses a vector formulation of the problem. In this method, the solution is found as the steady state part of the response of the dynamic problem formed by adding virtual mass and damping to the structure. A review of the method and its application to cable networks can be found in [3] to [7].

In the current work we report on the static analysis of cable networks. We consider cables as discrete elements in the structure. Furthermore, we make the following assumptions :

(1) There are no distributed loads (including gravity) along the cable.

(2) Loads and prescribed displacements are applied at the cable joints (nodes).

(3) Joints are considered frictionless and pinned.

(4) Bending stiffness is negligible.

Later in this work, we will show that based on the above assumptions, cables will remain straight between joints throughout the analysis.

This report is organized in three chapters. Chapter 1 deals with the formulation of the problem based on the above assumptions. In chapter 2, the Dynamic Relaxation method is formulated to solve the non-linear equilibrium equations for both linear and non-linear elastic materials.In chapter 3, first the theory of elasto-plastic cables is developed and then it is shown by a counter example that the conventional DR method without further modification, does not work for elasto-plastic cables because of path dependency in the solution. An incremental technique is developed and applied to elasto-plastic cables. Reference [12] shows that DR can be used for elasto-plastic cables if it's applied in each increment of the incremental technique. Chapters 2 and 3 contain graphical results for some interesting cable networks.

CHAPTER 1 : FORMULATION OF THE PROBLEM

In this chapter we discuss the theory of elastic cables. The equilibrium configuration is then found based on the minimization of the potential energy of the cable and we derive the necessary and sufficient conditions for stability of the equilibrium configuration. The theory is then expanded for a general cable net.

1.1 Elastic cables theory

We consider a cable with initial unstretched length L that undergoes a deformation in three dimensional space (Figure 1.1)

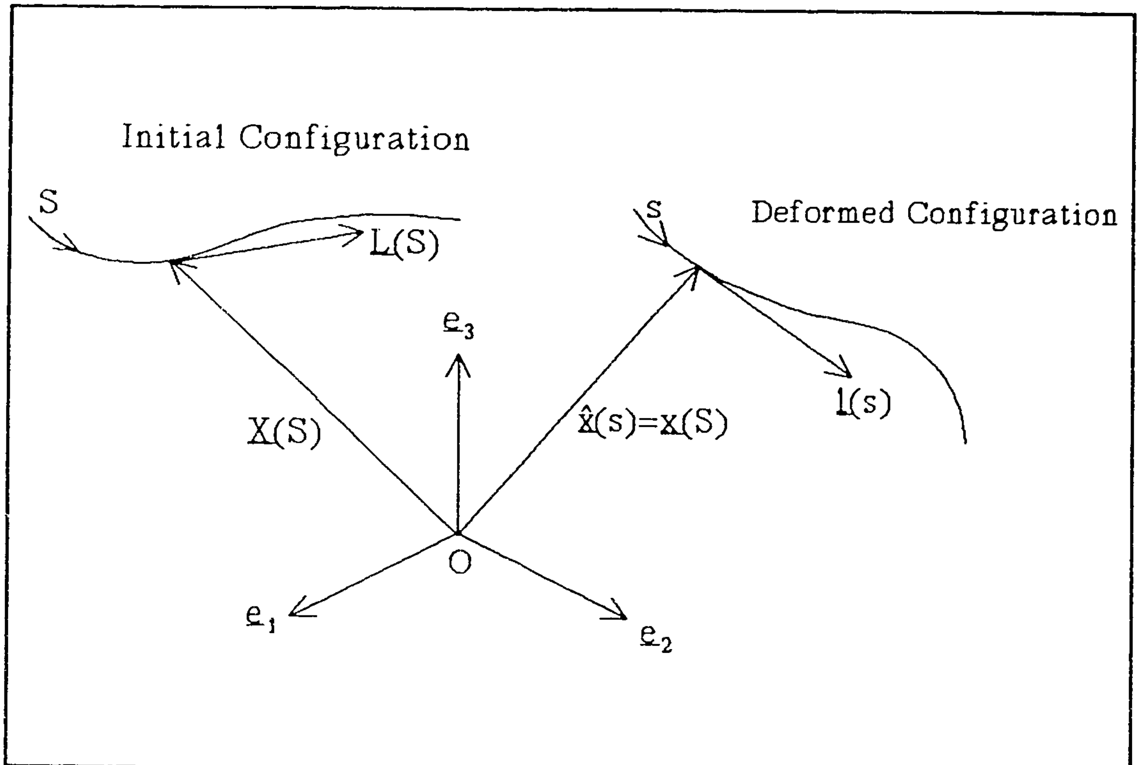

Figure 1.1 : A single cable undergoing a general deformation

In this diagram :

$\underline{e}_i$ (i=1,2,3) are the orthonormal vectors of the basis of the fixed frame of reference

S is the initial arc length that varies in the domain of [0,L]

s(S) is the deformed arc length

$\underline{X}(S)$ is the position vector of a point on the initial configuration of the cable with respect to the fixed frame

$\hat{\underline{x}}(s)=\hat{\underline{x}}(s(S))=\underline{x}(S)$ is the position vector of a point on the deformed configuration with respect to the fixed frame

The unit tangent to the cable in the initial configuration is defined by

$$\underline{L}(S)=\frac{d\underline{X}(S)}{dS} \tag{1.1}$$

Similarly in the deformed configuration, the unit tangent to the cable is

$$\underline{l}(s)=\frac{d\hat{\underline{x}}(s)}{ds} \tag{1.2}$$

The deformation gradient is defined by

$$\frac{d\underline{x}(S)}{dS}=\frac{d\hat{\underline{x}}(s)}{dS}=\frac{d\hat{\underline{x}}(s)}{ds}\frac{ds}{dS} \tag{1.3}$$

or

$$\frac{d\underline{x}(S)}{dS}=\lambda\underline{l} \tag{1.4}$$

where

$$\lambda(S) = \frac{ds}{dS} = \left| \frac{d\underline{x}(S)}{dS} \right| \tag{1.5}$$

is the stretch of the cable. Note that in the unstretched configuration $\lambda(S)=1$.
From now on, we use the (′) symbol to denote differentiation with respect to initial arc length S.

Now let us assume the existence of a strain energy per unit initial cable length of the form $W(\underline{x}'(S))$. It is reasonable to further assume that W is insensitive to rigid motions of the cable defined by

$$\underline{x} \rightarrow \underline{Q}\underline{x} + \underline{C} \tag{1.6}$$

where $\underline{Q}$ is a rotation tensor and $\underline{C}$ is a translation vector and both are constant. So

$$W((\underline{Q}\underline{x} + \underline{C})') = W(\underline{Q}\underline{x}') = W(\underline{x}') \tag{1.7}$$

for all rotations. This is satisfied if and only if W is a function of $\underline{x}'(S)$ through its magnitude λ, i.e.

$$W(\underline{x}'(S)) = w(|\underline{x}'(S)|) = w(\lambda) \tag{1.8}$$

Now we define

$$\underline{t} = \frac{\partial W}{\partial x_i'} \underline{e}_i \tag{1.9}$$

where x'_i $(i=1,2,3)$ are components of $\underline{x}'(S)$ on the fixed rectangular basis $\{\underline{e}_i\}$ and repeated index (i) implies summation and from now on, the summation rule is applied on any repeated index. Later this quantity will be interpreted as force in the cable. From (1.8) and (1.9) we can write

$$t_i = \frac{\partial W}{\partial x_i'} = \frac{dw(\lambda)}{d\lambda} \frac{\partial \lambda}{\partial x_i'} \tag{1.10}$$

But we have

$$\lambda^2 = \underline{x}' \cdot \underline{x}' = x_j' x_j' \tag{1.11}$$

Taking the derivative of both sides with respect to x'_i we get

$$\frac{\partial \lambda}{\partial x_i'} = \lambda^{-1} x_i' \qquad i = 1,2,3 \tag{1.12}$$

Hence

$$\begin{aligned} t_i &= \lambda^{-1} \frac{dw(\lambda)}{d\lambda} x_i' \\ &= \lambda^{-1} f(\lambda) x_i' \qquad i=1,2,3 \end{aligned} \tag{1.13}$$

where

$$f(\lambda) = \frac{dw(\lambda)}{d\lambda} \tag{1.14}$$

is the force in the cable. From (1.4) and (1.13) we can write

$$\underline{t}(S) = \lambda^{-1} f(\lambda) \underline{x}'(S) = f(\lambda) \underline{l} \tag{1.15}$$

The above expression states that $\underline{t}(S)$ is the force exerted by part (S,L] on the [0,S] part of the cable.

Now consider a cable under distributed dead load with density $\underline{a}(S)$ per unit initial cable length and point loads $\underline{F}_0$ and $\underline{F}_L$ corresponding to $S=0$ and $S=L$ respectively (Figure 1.2)

The total potential energy of the cable $E[\underline{x}]$ for the configuration defined by $\underline{x}(S)$ is the sum of strain energy and the potential energy of external loading. That is

$$E[\underline{x}] = \int_0^L (W - \underline{a} \cdot \underline{x}) dS - [\underline{F}_L \cdot \underline{x}(L) + \underline{F}_0 \cdot \underline{x}(0)] \tag{1.16}$$

We would like to minimize $E[\underline{x}]$ to get the equilibrium configuration. To do this, we give the current configuration of the cable defined by $\underline{x}(S)$ a small perturbation

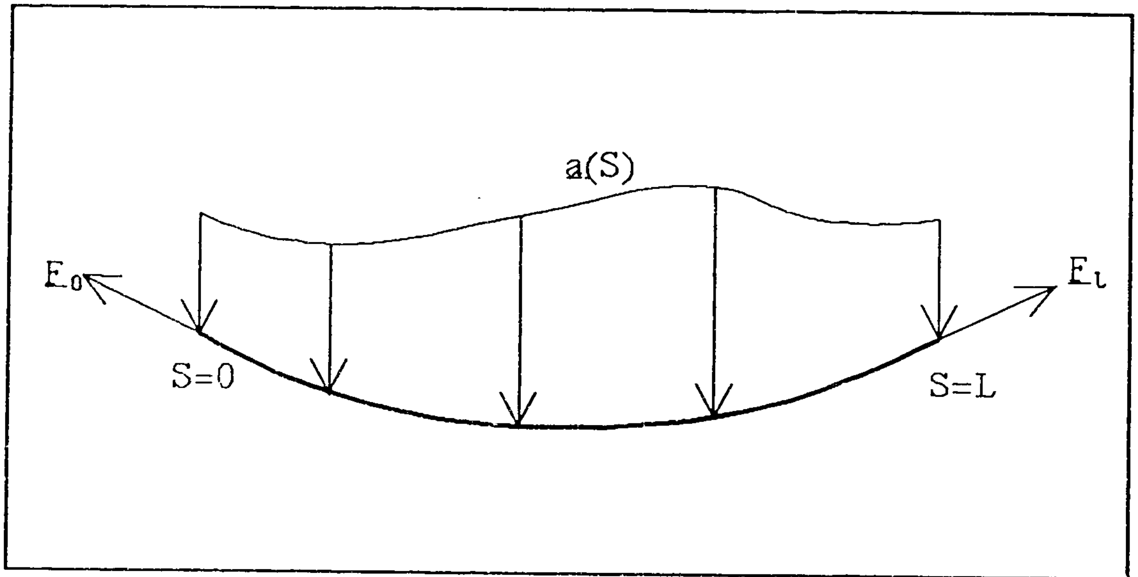

Figure 1.2 A single cable under distributed dead load

$$\underline{x}(S) \rightarrow \underline{x}(S) + \varepsilon \underline{u}(S) \tag{1.17}$$

where $|\varepsilon| << 1$ and $\underline{u}(S)$ satisfies the kinematic boundary conditions but is arbitrary elsewhere. Defining $F(\varepsilon)=E[\underline{x}+\varepsilon\underline{u}]$ and writing the Taylor expansion of $F(\varepsilon)$ about $\varepsilon=0$ we get

$$F(\varepsilon) = F(0) + \varepsilon \frac{dF}{d\varepsilon}\bigg|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2 \frac{d^2F}{d\varepsilon^2}\bigg|_{\varepsilon=0} + o(\varepsilon^2) \tag{1.18}$$

Moreover, we expand the strain energy W for the perturbed configuration in terms of ε

$$\begin{aligned} W((\underline{x}+\varepsilon\underline{u})') &= W(\underline{x}') + \varepsilon u_i' \frac{\partial W}{\partial x_i'}\bigg|_{\varepsilon=0} + \frac{1}{2}\varepsilon^2 u_i' u_j' \frac{\partial^2 W}{\partial x_i' \partial x_j'}\bigg|_{\varepsilon=0} + o(\varepsilon^2) \\ &= W(\underline{x}') + \varepsilon \underline{u}' . \underline{t} + \frac{1}{2}\varepsilon^2 \underline{u}' . \underline{C}(\underline{x}')\underline{u}' + o(\varepsilon^2) \end{aligned} \tag{1.19}$$

where the tensor $\underline{C}$ is defined as

$$\underline{C}(\underline{x}') = \left(\frac{\partial^2 W}{\partial x_i' \partial x_j'}\Big|_{\varepsilon=0}\right)\underline{e}_i \otimes \underline{e}_j = \underline{C}^T(\underline{x}') \tag{1.20}$$

From (1.10), (1.12) and (1.14) this can be written as

$$\underline{C} = \frac{d f(\lambda)}{d\lambda}\underline{1}\otimes\underline{1} + \lambda^{-1} f(\lambda)(\underline{I} - \underline{1}\otimes\underline{1}) \tag{1.21}$$

where $\underline{I}$ is the unit tensor. Now, the potential energy for the perturbed configuration can be written as

$$\begin{aligned} F(\varepsilon) &= \int_0^L [W(\underline{x}') - \underline{a}.\underline{x}]dS - [\underline{F}_L.\underline{x}(L) + \underline{F}_0.\underline{x}(0)] \\ &+ \varepsilon\{\int_0^L (\underline{t}.\underline{u}' - \underline{a}.\underline{u})dS - [\underline{F}_L.\underline{u}(L) + \underline{F}_0.\underline{u}(0)]\} \\ &+ \frac{1}{2}\varepsilon^2 \int_0^L \underline{u}'.\underline{C}(\underline{x}')\underline{u}' dS + o(\varepsilon^2) \end{aligned} \tag{1.22}$$

If configuration $\underline{x}$ is a minimizer of the potential energy we must have

$$\frac{dF}{d\varepsilon}\Big|_{\varepsilon=0} = 0 \tag{1.23}$$

or

$$\int_0^L (\underline{t}.\underline{u}' - \underline{a}.\underline{u})dS - [\underline{F}_L.\underline{u}(L) + \underline{F}_0.\underline{u}(0)] = 0 \tag{1.24}$$

Integrating by parts, we can rewrite this as

$$[\underline{t}(L) - \underline{F}_L].\underline{u}(L) - [\underline{t}(0) + \underline{F}_0].\underline{u}(0) - \int_0^L (\underline{t}' + \underline{a}).\underline{u}\, dS = 0 \tag{1.25}$$

This should be true for all $\underline{u}(S)$ that satisfy the kinematic boundary conditions. From the fundamental lemma of variational calculus, this can be true only if

$$\begin{cases} \underline{t}' + \underline{a} = \underline{0} \\ \underline{t}(0) = -\underline{F}_0 \\ \underline{t}(L) = \underline{F}_L \end{cases} \tag{1.26}$$

Now, if there is no distributed load ($\underline{a} = \underline{0}$) we conclude that $\underline{t}' = \underline{0}$. This means that $\underline{t}$ is constant along the cable and from (1.15) we can say that the cable is a straight line.

Another necessary condition for $\underline{x}$ to be a minimizer of E is that

$$\left.\frac{d^2F}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_0^L \underline{u}' \cdot \underline{C}(\underline{x}')\underline{u}' \, dS \geq 0 \tag{1.27}$$

Again this should be true for any $\underline{u}$ that satisfies the kinematic boundary conditions.This leads us to the well known Weierstrass Necessary Conditions [2,9] which are discussed next.

1.2 Weierstrass necessary conditions

We consider a kinematically admissible configuration of the cable $\underline{x}^*(S)$ that is equal to $\underline{x}(S)$ everywhere in the domain of $S \in [0,L]$ except in an interval $S_1 \leq S \leq S_3$. In one dimension, it looks like a bump (Figure 1.3).

We define $\underline{x}^*(S)$ as

$$\underline{x}^*(S) = \begin{cases} \underline{\psi}(S) = \underline{x}(S) + (S - S_1)\underline{b} & S_1 \leq S \leq S_2 \\ \underline{\phi}(S) = \underline{x}(S) + (S_3 - S)\left(\frac{S_2 - S_1}{S_3 - S_2}\right)\underline{b} & S_2 \leq S \leq S_3 \\ \underline{x}(S) & \text{elsewhere} \end{cases} \tag{1.28}$$

where $\underline{b}$ is an arbitrary constant vector. Note that $\underline{x}^*(S)$ is continuous over $S \in [0,L]$. Let us define

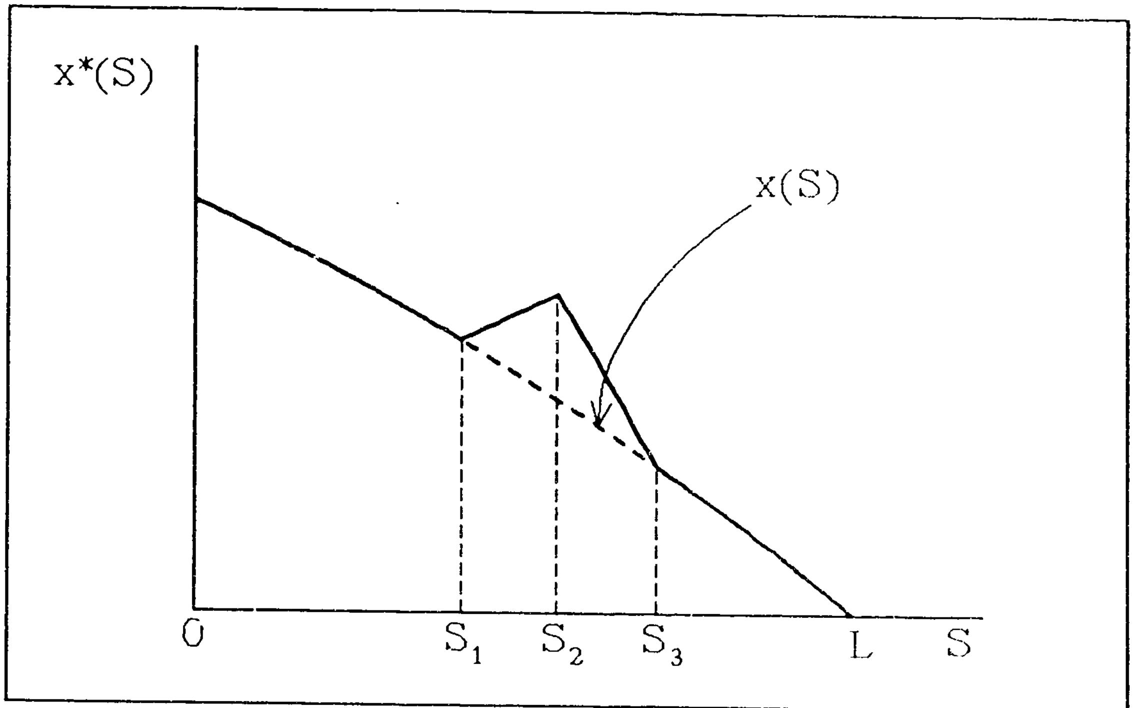

Figure 1.3 Comparison of $x^*(S)$ and x(S) over $S \in [0,L]$

$$\begin{cases} l = S_3 - S_1 \\ l_1 = S_2 - S_1 = \theta l \\ l_2 = S_3 - S_2 = (1-\theta) l \end{cases} \tag{1.29}$$

where

$$0 < \theta = \frac{S_2 - S_1}{S_3 - S_1} < 1 \tag{1.30}$$

Thus from (1.28) we can write

$$\underline{\phi}(S) = \underline{x}(S) - \frac{\theta}{1-\theta}(S - S_3)\underline{b} \tag{1.31}$$

In order for x(S) to be a minimizer of E we require

$$E[\underline{x}^*] \geq E[\underline{x}] \tag{1.32}$$

or from (1.16)

$$l^{-1}\{\int_{S_1}^{S_2}[W(\underline{x}^{*\prime})-W(\underline{x}')]dS-\int_{S_1}^{S_2}\underline{a}.(\underline{x}^{*}-\underline{x})dS\}\geq 0 \tag{1.33}$$

Note that boundary terms cancel here. In the limit $l\to 0$, $\underline{x}^{*}\to\underline{x}$ everywhere and we only need to consider

$$l^{-1}\int_{S_1}^{S_2}[W(\underline{x}^{*\prime})-W(\underline{x}')]dS\geq 0 \tag{1.34}$$

which can be written as

$$\frac{\theta}{l_1}\int_{S_1}^{S_2}W(\underline{x}'+\underline{b})dS+\frac{1-\theta}{l_2}\int_{S_1}^{S_2}W(\underline{x}'-\frac{\theta}{1-\theta}\underline{b})dS-\frac{1}{l}\int_{S_1}^{S_2}W(\underline{x}')dS\geq 0 \tag{1.35}$$

Now we let $l\to 0$ which implies that $l_1,l_2\to 0$ and use the mean value theorem to get

$$\theta W(\underline{x}'+\underline{b})+(1-\theta)W(\underline{x}'-\frac{\theta}{1-\theta}\underline{b})-W(\underline{x}')\geq 0 \qquad \forall\ S\in[0,L] \tag{1.36}$$

Dividing by θ, using Taylor expansion of the second term on the left hand side and letting $\theta\to 0$ we get the Weierstrass condition

$$W(\underline{x}'+\underline{b})-W(\underline{x}')-\underline{b}.\underline{t}(\underline{x}')\geq 0 \qquad \forall\ \underline{b}\ ,\ \forall\ S\in[0,L] \tag{1.37}$$

To interpret the Weierstrass condition, we write (1.37) as

$$W(\underline{v})-W(\underline{x}')\geq\underline{t}(\underline{x}').(\underline{v}-\underline{x}') \qquad \forall\ \underline{v} \tag{1.38}$$

Now we define $\mu=|\underline{v}|$ and from (1.8) and (1.15) we can write

$$w(\mu)-w(\lambda)\geq f(\lambda)(\underline{l}.\underline{v}-\lambda) \tag{1.39}$$

To get a necessary condition for this we write

$$\underline{v}=(\underline{v}.\underline{l})\underline{l}+(\underline{I}-\underline{l}\otimes\underline{l})\underline{v} \tag{1.40}$$

which is basically expressing $\underline{v}$ as the sum of its component along $\underline{l}$ and its projection on the plane perpendicular to $\underline{l}$. Since (1.39) must hold for all $\underline{v}$, it must hold in particular for the following choices

$$\begin{cases}(\underline{I}-\underline{l}\otimes\underline{l})\underline{v}\neq\underline{0}\\ \underline{v}.\underline{l}=\lambda\end{cases} \tag{1.41}$$

Thus $\mu=|\underline{v}|>\underline{v}.\underline{l}=\lambda$ and (1.39) can be written as

$$w(\mu)-w(\lambda)\geq 0 \qquad \forall\ \mu>\lambda \tag{1.42}$$

Writing Taylor expansion of $w(\mu)$-$w(\lambda)$ and letting $\mu\rightarrow\lambda$ we get

$$f(\lambda)\geq 0 \qquad \forall\ S\in[0,L] \tag{1.43}$$

So, to be stable, the cable must be under tension.

To get another necessary condition, take $\underline{v}=\mu\underline{l}$. So (1.39) can be written as

$$w(\mu)-w(\lambda)\geq f(\lambda)(\mu-\lambda) \qquad \forall\ \mu \tag{1.44}$$

which means that λ should be a point of convexity of the strain energy function $w(\lambda)$ of the cable.

We note that if (1.43) and (1.44) are satisfied, the eigenvalues of the tensor $\underline{C}$ defined in (1.21) become non-negative and $\underline{C}$ is a positive semi-definite tensor and (1.27) is satisfied. We show that conditions (1.43) and (1.44) are also sufficient for (1.39). We suppose that these two conditions are satisfied. For any $\underline{v}$ we have

$$\mu=|\underline{v}|\geq\underline{v}.\underline{l} \tag{1.45}$$

Thus

$$\mu-\lambda\geq\underline{v}.\underline{l}-\lambda \tag{1.46}$$

Then from (1.43) we can write

$$f(\lambda)(\mu-\lambda) \geq f(\lambda)(\underline{v}.\underline{1}-\lambda) \tag{1.47}$$

and therefore from (1.44)

$$w(\mu)-w(\lambda) \geq f(\lambda)(\underline{v}.\underline{1}-\lambda) \tag{1.48}$$

which is identical to (1.39). So, conditions (1.44) and (1.45) are also sufficient for (1.39). Therefore, if (1.26), (1.43) and (1.44) are satisfied, the cable is equilibrated and globally stable. To satisfy (1.43) automatically during the equilibrium analysis of the elastic cable we define the force in the cable as

$$f=\begin{cases} f(\lambda) & \lambda>1 \\ 0 & \lambda\leq 1 \end{cases} \tag{1.49}$$

This guarantees that cable would never be under compression.

1.3 Expansion of theory to a general cable net

We consider a network consisting of n cables, each of unstretched length L_j ; $j=1,\ldots,n$. These cables are connected at l unconstrained nodes located at the unknown positions $\underline{y}_k$; $k=1,\ldots,l$ in the deformed configuration of the network. The set of all unconstrained nodes is denoted by K. We let m nodes, belonging to the set H, be fixed at prescribed positions $\underline{z}_h$; $h=1,\ldots,m$. At each of the unconstrained nodes, a dead load $\underline{q}_k$, $k\in K$ is applied. We introduce the following sets [2,11]

$$\begin{cases} I^k=\{j:S_j=0 \text{ at node } k\in K\} \\ E^k=\{j:S_j=L_j \text{ at node } k\in K\} \\ I^h=\{j:S_j=0 \text{ at node } h\in H\} \\ E^h=\{j:S_j=L_j \text{ at node } h\in H\} \end{cases} \tag{1.50}$$

The deformed configuration of the j^{th} cable is described by the vector-valued position function $\underline{x}_j(S_j)$ where S_j is the arc length along the j^{th} cable in its unstretched state varying in the domain of $[0,L_j]$. We define $\underline{x}^0_j=\underline{x}_j(0)$ and $\underline{x}^L_j=\underline{x}_j(L_j)$. These are subjected to the following compatibility constraints

$$\begin{cases} \underline{x}_j^0 = \underline{y}_k \ , \ j \in I^k \\ \underline{x}_j^L = \underline{y}_k \ , \ j \in E^k \end{cases} \tag{1.51}$$

and

$$\begin{cases} \underline{x}_j^0 = \underline{z}_h \ , \ j \in I^h \\ \underline{x}_j^L = \underline{z}_h \ , \ j \in E^h \end{cases} \tag{1.52}$$

We suppose the j^{th} cable is subjected to a distributed dead load with density $\underline{a}_j$. The total potential energy of the net is then expressed by

$$E = \sum_{j=1}^{n} \int_0^{L_j} (W - \underline{a}_j \cdot \underline{x}_j) dS - \sum_{k=1}^{l} \underline{q}_k \cdot \underline{y}_k \tag{1.53}$$

We would like to minimize E to get the equilibrium configuration. We follow the same procedure we did for a single cable in section 1.1 and apply some perturbations that satisfy (1.52) but we apply (1.51) by minimizing the unconstrained functional

$$E^* = E - \sum_k \left\{ \sum_{j \in I^k} \underline{F}_j^0 \cdot (\underline{x}_j^0 - \underline{y}_k) + \sum_{j \in E^k} \underline{F}_j^L \cdot (\underline{x}_j^L - \underline{y}_k) \right\} \tag{1.54}$$

where $\underline{F}_j^0$ and $\underline{F}_j^L$ are Lagrange multipliers.

Following the procedure we used for a single cable, the conditions of equilibrium are found to be (see [2])

$$\begin{cases} \underline{t}'_j + \underline{a}_j = \underline{0} & j = 1, \ldots, n \\ \underline{F}_j^0 = -\underline{t}_j^0 & j \in I^k \\ \underline{F}_j^L = \underline{t}_j^L & j \in E^k \end{cases} \tag{1.55}$$

and the equilibrium equations at the nodes are

$$\sum_{j\in I^*} \underline{F}_j^0 + \sum_{j\in E^*} \underline{F}_j^L = \underline{q}_k \qquad k\in K \tag{1.56}$$

Again if $\underline{a}_j=\underline{0}$, from (1.55) we see that $\underline{t}'_j=\underline{0}$; $j=1,\ldots,n$ and the cables remain straight throughout the analysis. Using (1.55), the nodal equilibrium equations (1.56) can be written as

$$-\sum_{j\in I^*} \underline{t}_j^0 + \sum_{j\in E^*} \underline{t}_j^L = \underline{q}_k \qquad k\in K \tag{1.57}$$

This expression gives us 3×1 nonlinear equations that must be solved for unknown position vectors $\underline{y}_k$; $k=1,\ldots,n$.

It can be concluded that if the Weierstrass necessary (and sufficient) conditions are satisfied for each cable in the net, the potential energy for each cable and therefore for the whole network is minimized and the equilibrium configuration is stable (see [2]).

Summary

In this chapter, we developed the theory of elastic cables and derived the necessary and sufficient conditions for equilibrium of a single cable by minimizing the potential energy. We expanded the theory for a general cable network and derived the nonlinear equations of equilibrium.

CHAPTER 2 : DYNAMIC RELAXATION

In dealing with iterative solvers that use the tangent stiffness matrix directly, e.g. Newton-Raphson, problems arise when some cables become slack and the tangent stiffness matrix becomes singular or ill-conditioned and we get wrong results. So we need a method that is stable and works well even under the situation that some cables are slack. In this chapter we introduce the DR method which is a stable method and doesn't make use of tangent stiffness matrix directly in calculations. We then apply it to the problem at hand. Some examples of cable nets are analyzed and their results are presented.

2.1 Theory of DR

We consider a linear system of equations $\underline{K}\ \underline{x} = \underline{F}$ for which a solution $\underline{x}^* = \underline{K}^{-1}\underline{F}$ is sought. Now we transform the original equations into equations of motion by introducing virtual mass and damping at the unknowns

$$\underline{M}\,\underline{\ddot{x}} + \underline{C}\,\underline{\dot{x}} + \underline{K}\,\underline{x} = \underline{F} \tag{2.1}$$

where $\underline{M}$ and $\underline{C}$ are virtual mass and damping diagonal matrices and as is shown later, are found so that the number of iterations required for convergence is minimized. Here dots indicate differentiation with respect to time. The response for this motion is the sum of homogeneous part (transient response) and particular part (steady state response). If the transient part dies out, we are left with the particular solution $\underline{x}^* = \underline{K}^{-1}\underline{F}$ which is what we want.

We intend to solve (2.1), for fixed $\underline{F}$, in increments of time. For the n^{th} increment this equation can be written as

$$\underline{M}\,\underline{\ddot{x}}^n + \underline{C}\,\underline{\dot{x}}^n + \underline{K}\,\underline{x}^n = \underline{F} \tag{2.2}$$

Using the finite difference technique with the central difference scheme we can write

$$\begin{cases} \dot{\underline{x}}^{n-\frac{1}{2}} = \dfrac{(\underline{x}^{n} - \underline{x}^{n-1})}{h} \\ \ddot{\underline{x}}^{n} = \dfrac{(\dot{\underline{x}}^{n+\frac{1}{2}} - \dot{\underline{x}}^{n-\frac{1}{2}})}{h} \end{cases} \tag{2.3}$$

where h is the fixed time increment. The expression for $\dot{\underline{x}}^n$ is obtained by the average

$$\dot{\underline{x}}^{n} = \frac{(\dot{\underline{x}}^{n-\frac{1}{2}} + \dot{\underline{x}}^{n+\frac{1}{2}})}{2} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) we get the following recurrence formula

$$\begin{cases} \dot{\underline{x}}^{n+\frac{1}{2}} = (\frac{1}{h}\underline{M} + \frac{1}{2}\underline{C})^{-1}[(\frac{1}{h}\underline{M} - \frac{1}{2}\underline{C})\dot{\underline{x}}^{n-\frac{1}{2}} + (\underline{F} - \underline{K}\underline{x}^{n})] \\ \underline{x}^{n+1} = \underline{x}^{n} + h\dot{\underline{x}}^{n+\frac{1}{2}} \end{cases} \tag{2.5}$$

For the DR method, we assume that damping matrix $\underline{C}$ is proportional to mass matrix $\underline{M}$. That is

$$\underline{C} = c\underline{M} \tag{2.6}$$

where c is a constant for each increment to be found later. Substituting (2.6) into (2.5) we get

$$\begin{cases} \dot{\underline{x}}^{n+\frac{1}{2}} = \left(\dfrac{2-ch}{2+ch}\right)\dot{\underline{x}}^{n-\frac{1}{2}} + \dfrac{2h}{2+ch}\underline{M}^{-1}(\underline{F} - \underline{K}\underline{x}^{n}) \\ \underline{x}^{n+1} = \underline{x}^{n} + h\dot{\underline{x}}^{n+\frac{1}{2}} \end{cases} \tag{2.7}$$

The initial conditions for DR are of the form [3]

$$\begin{cases} \underline{x}^{0} \neq \underline{0} \\ \dot{\underline{x}}^{0} = \underline{0} \end{cases} \tag{2.8}$$

Using (2.4) and the second of (2.8) gives

$$\underline{\dot{x}}^{-\frac{1}{2}}=-\underline{\dot{x}}^{\frac{1}{2}} \tag{2.9}$$

then from the first of (2.7) we get

$$\underline{\dot{x}}^{\frac{1}{2}}=\frac{h}{2}\underline{M}^{-1}(\underline{F}-\underline{K}\underline{x}^{0}) \tag{2.10}$$

Now we can summarize the recurrence formula as

$$\begin{cases} \underline{\dot{x}}^{\frac{1}{2}}=\frac{h}{2}\underline{M}^{-1}(\underline{F}-\underline{K}\underline{x}^{0}) & \text{for } n=0 \\ \underline{\dot{x}}^{n+\frac{1}{2}}=\frac{2-ch}{2+ch}\underline{\dot{x}}^{n-\frac{1}{2}}+\frac{2h}{2+ch}\underline{M}^{-1}(\underline{F}-\underline{K}\underline{x}^{n}) & \text{for } n\neq 0 \\ \underline{x}^{n+1}=\underline{x}^{n}+h\underline{\dot{x}}^{n+\frac{1}{2}} & \text{for all } n \end{cases} \tag{2.11}$$

The DR algorithm then can be written as the following steps [3]

(a) choose ν ($\nu=ch$) and $\underline{M}$; $n=0$; $\underline{x}^0$ given ; $\underline{\dot{x}}^0=\underline{0}$

(b) $\underline{r}^n=\underline{F}-\underline{K}\underline{x}^n$

(c) if $\underline{r}^n\approx\underline{0}$ stop, otherwise continue

(d) if $n=0$ then

$$\underline{\dot{x}}^{\frac{1}{2}}=\frac{h}{2}\underline{M}^{-1}\underline{r}^{0}$$

otherwise

$$\underline{\dot{x}}^{n+\frac{1}{2}}=\frac{2-\nu}{2+\nu}\underline{\dot{x}}^{n-\frac{1}{2}}+\frac{2h}{2+\nu}\underline{M}^{-1}\underline{r}^{n} \tag{2.12}$$

(e) $\underline{x}^{n+1}=\underline{x}^{n}+h\underline{\dot{x}}^{n+\frac{1}{2}}$

(f) $n=n+1$; go to (b)

We are seeking the values of c, h and $\underline{M}$ that make the iterations stable and minimize the number of iterations. To do this, we define

$$\begin{cases} \alpha=\dfrac{2h^2}{2+\nu} \\ \beta=\dfrac{2-\nu}{2+\nu} \\ \underline{A}=\underline{M}^{-1}\underline{K} \\ \underline{b}=\underline{M}^{-1}\underline{F} \end{cases} \tag{2.13}$$

Then from (2.7) we get

$$\underline{x}^{n+1}=\underline{x}^{n}+\beta(\underline{x}^{n}-\underline{x}^{n-1})-\alpha\underline{A}\,\underline{x}^{n}+\alpha\underline{b} \tag{2.14}$$

To study the convergence of iterations, we define the error in the n^{th} iteration as

$$\underline{\epsilon}^{n}=\underline{x}^{n}-\underline{x}^{*} \tag{2.15}$$

Substituting (2.15) in (2.14) gives

$$\underline{\epsilon}^{n+1}=\underline{\epsilon}^{n}+\beta(\underline{\epsilon}^{n}-\underline{\epsilon}^{n-1})-\alpha\underline{A}\,\underline{\epsilon}^{n} \tag{2.16}$$

We define λ as the rate of decay of error vector (note that this is different from stretch defined in chapter 1), i.e.

$$\underline{\epsilon}^{n+1}=\lambda\underline{\epsilon}^{n} \tag{2.17}$$

Substituting this into (2.16) we get

$$\lambda^2-(1+\beta-\alpha\lambda_A)\lambda+\beta=0 \tag{2.18}$$

where λ_A is any eigenvalue of $\underline{A}$. This is a quadratic equation in λ and its roots give the values of the error decay rate λ. Let us consider different possible cases :

- For $(1+\beta-\alpha\lambda_A)^2<4\beta$, the roots are a complex conjugate pair with a norm $|\lambda|$ equal to

$$|\lambda| = \beta^{\frac{1}{2}} \tag{2.19}$$

- For $(1+\beta-\alpha\lambda_A)^2 = 4\beta$, the roots are equal and real and we have the same expression for $|\lambda|$ as in (2.19).

- For $(1+\beta-\alpha\lambda_A)^2 > 4\beta$ the roots are real and distinct and the norm of the larger root is given by

$$|\lambda| = \left[\frac{1}{2}(1+\beta-\alpha\lambda_A)^2 - \beta + \frac{1}{2}(1+\beta-\alpha\lambda_A)\sqrt{(1+\beta-\alpha\lambda_A)^2 - 4\beta}\ \right]^{\frac{1}{2}} \tag{2.20}$$

Figure 2.1 shows that for a given value of $\lambda_A h^2$, the minimum $|\lambda|$ is obtained when the roots of (2.18) are equal.

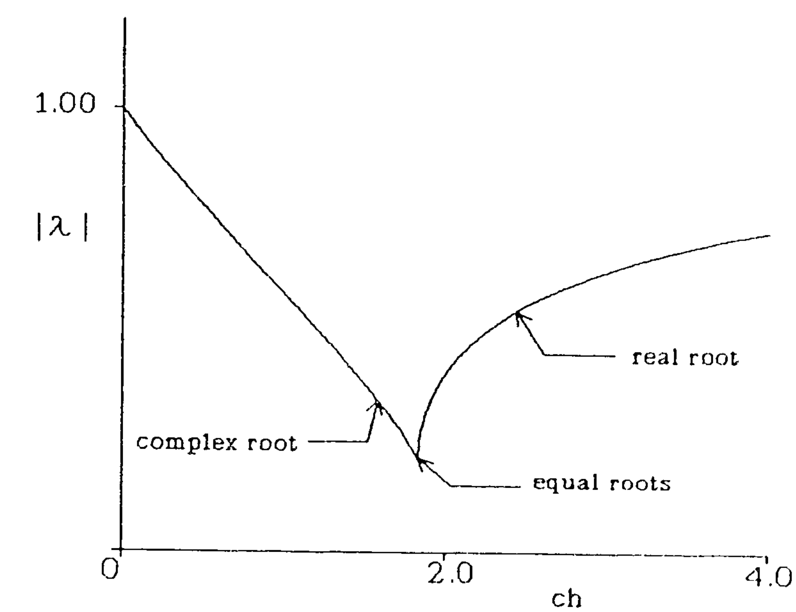

Figure 2.1: Relation between $|\lambda|$ and ch

Now let us examine the relation between $|\lambda|$ and $\lambda_A h^2$ when the roots of (2.18) are equal. This relation is expressed as

$$|\lambda| = \sqrt{\frac{2-\sqrt{\lambda_A h^2(4-\lambda_A h^2)}}{2+\sqrt{\lambda_A h^2(4-\lambda_A h^2)}}} \tag{2.21}$$

which is symmetric about $\lambda_A h^2=2$. If we consider the case when $\lambda_A h^2>4$, it can be seen from (2.21) that $|\lambda| > 1$ which causes instability problems. Since we know that eigenvalues of $\underline{A}$ are positive, we choose the parameter h such that $\lambda_{Amin}h^2$ and $\lambda_{Amax}h^2$ be symmetric about $\lambda_A h^2=2$ where λ_{Amin} and λ_{Amax} are the minimum and maximum eigenvalues of $\underline{A}$ respectively. This assures us that $\lambda_{Amax}h^2<4$ and the iterations are stable. So, for optimum value of h we have

$$(h^2)_{opt}=\frac{4}{\lambda_{Amax}+\lambda_{Amin}} \tag{2.22}$$

Using this with (2.19) and (2.13) we get

$$(\nu)_{opt}=(ch)_{opt}=\frac{4\sqrt{\lambda_{Amin}\cdot\lambda_{Amax}}}{\lambda_{Amin}+\lambda_{Amax}} \tag{2.23}$$

For convenience, we assume that $\lambda_{Amin} << \lambda_{Amax}$. If that's not the case, we just need more iterations for convergence. By this assumption, (2.22) and (2.23) become

$$(h^2)_{opt}\approx\frac{4}{\lambda_{Amax}} \tag{2.24}$$

and

$$(c)_{opt}\approx 2\sqrt{\lambda_{Amin}} \tag{2.25}$$

An upper bound for the eigenvalues of $\underline{A}$ may be determined from Gershgorin bound theorem [3,4], which states that

$$|\lambda_{Amax}| < \max_i \sum_j |A_{ij}| \tag{2.26}$$

From (2.13) and knowing that $\underline{M}$ is diagonal we can write

$$\sum_j |A_{ij}| = \frac{1}{m_{ii}} \sum_j |K_{ij}| \tag{2.27}$$

where m_{ii} is the i^{th} diagonal term of $\underline{M}$. So from (2.27), (2.26) and (2.24) we can write

$$m_{ii} > \frac{1}{4} h^2 \sum_j |K_{ij}| \tag{2.28}$$

Using Rayleigh's Quotient to find an estimate of λ_{Amin} along with (2.25) we get

$$c_n \approx 2 \sqrt{\frac{(\underline{x}^n)^T \underline{K} \underline{x}^n}{(\underline{x}^n)^T \underline{M} \underline{x}^n}} \tag{2.29}$$

where c_n gives the optimum value for c in (2.6) for n^{th} iteration.

The discussion we made sofar about DR is also applicable to nonlinear sets of equations. Consider the following set of nonlinear equations

$$\underline{P}(\underline{x}) = \underline{F} \tag{2.30}$$

where $\underline{P}(\underline{x})$ is the left hand side of equations vector nonlinear in $\underline{x}$ and $\underline{F}$ is the known right hand side vector. We define the incremental tangent stiffness matrix $\underline{K}^n$ as

$$K_{ij}^n = \frac{\partial P_i}{\partial x_j^n} \qquad i, j = 1, \ldots, \text{number of unknowns} \tag{2.31}$$

and use it to calculate elements of matrix $\underline{M}$ from (2.28). We also define the diagonal matrix ${}^{I}\underline{K}^n$ [3] with elements

$$
{}^{1}K_{ii}^{n} = \frac{-P_i(\underline{x}^{n-1}) + P_i(\underline{x}^{n})}{h\dot{x}_i^{n-\frac{1}{2}}} \tag{2.32}
$$

and use this matrix instead of $\underline{K}$ in (2.29).

Usually we pick a value of $h=1.0$ for calculations but to make sure that the value given by (2.26) is an upper bound for eigenvalues of $\underline{A}$, we use a value of $h^{*} > h$ to determine $\underline{M}$ from (2.28). Finally, the adaptive DR algorithm for nonlinear set of equations is expressed in the following steps

(a) $\underline{x}^{0}$ given, $\dot{\underline{x}}^{0}=\underline{0}, n=0$, choose h (usually 1)

(b) determine $\underline{M}$ using (2.28) and (2.31) with $h^{*}>h$ instead of h

(c) $\underline{r}^{n}=\underline{F}-\underline{P}(\underline{x}^{n})$

(d) if $\underline{r}^{n}\approx\underline{0}$ stop, otherwise continue

$$
\text{(e)} \begin{cases} \dot{\underline{x}}^{\frac{1}{2}} = \frac{1}{2}h\underline{M}^{-1}\underline{r}^{0} & n=0 \\ \dot{\underline{x}}^{n+\frac{1}{2}} = \frac{2-\nu_n}{2+\nu_n}\dot{\underline{x}}^{n-\frac{1}{2}} + \frac{2h}{2+\nu_n}\underline{M}^{-1}\underline{r}^{n} & n\neq 0 \end{cases} \tag{2.33}
$$

(f) $\underline{x}^{n+1}=\underline{x}^{n}+h\dot{\underline{x}}^{n+\frac{1}{2}}$

(g) $n=n+1$

(h) $\nu_n=c_n h$ using (2.29) and (2.32)

(i) go to (b)

2.2 Application to cable nets problem

We recall from chapter 1 that equations of equilibrium (1.57) are nonlinear in terms of the components of the unknown position vector $\underline{y}_k, k\in K$. We can treat the left hand side of (1.57) as a vector of nonlinear expressions similar to $\underline{P}(\underline{x})$ and its right hand side as a known vector $\underline{F}$ in our discussion in section 2.1. We use the DR adaptive algorithm (2.33) to solve these nonlinear equations. In the next section, We will apply DR to a number of cable nets and discuss the results.

2.3 Examples

In this section, some examples of cable nets under dead loading or prescribed displacements, which have been analyzed by DR, are discussed. The material used for cable nets is rubber. The reason is that rubber cables can have very large deformation which makes it easy to have a clear picture of deformation of the net and to distinguish between initial and deformed configurations. A useful constitutive relation for rubber is the constitutive relation for the neo-Hookian material which is

$$f(\lambda)=GA(\lambda-\lambda^{-2}) \tag{2.34}$$

where G is the shear modulus and A is the cross section of the cable. The values used in the following examples are G=0.35 MPa and A=4 mm^2. As it is seen, here the material non-linearity is added to the geometrical non-linearity. In the tables that are presented in this chapter and next one, stress is defined as the force in the cable divided by the cross sectional area and strain is defined by λ-1.

Example 2.1

Figure 2.2 shows a 13×13 square mesh. Cables are of equal stretched length of 1.05 m and initial tension of 5 N. A dead load F=100 N is applied normal to the plane of mesh at the central load. The edges of the mesh are fixed. Figure 2.3 shows the deformed configuration. As it is expected, the node under the loading has relatively large deformation.

Example 2.2

This example and next one deal with prescribed deformations of some nodes and show the path independency of the solution.

Figure 2.4 shows a spider net which consists of circumferential and radial set of cables. The outer boundary is fixed and the inner boundary is a rigid frame. We give the

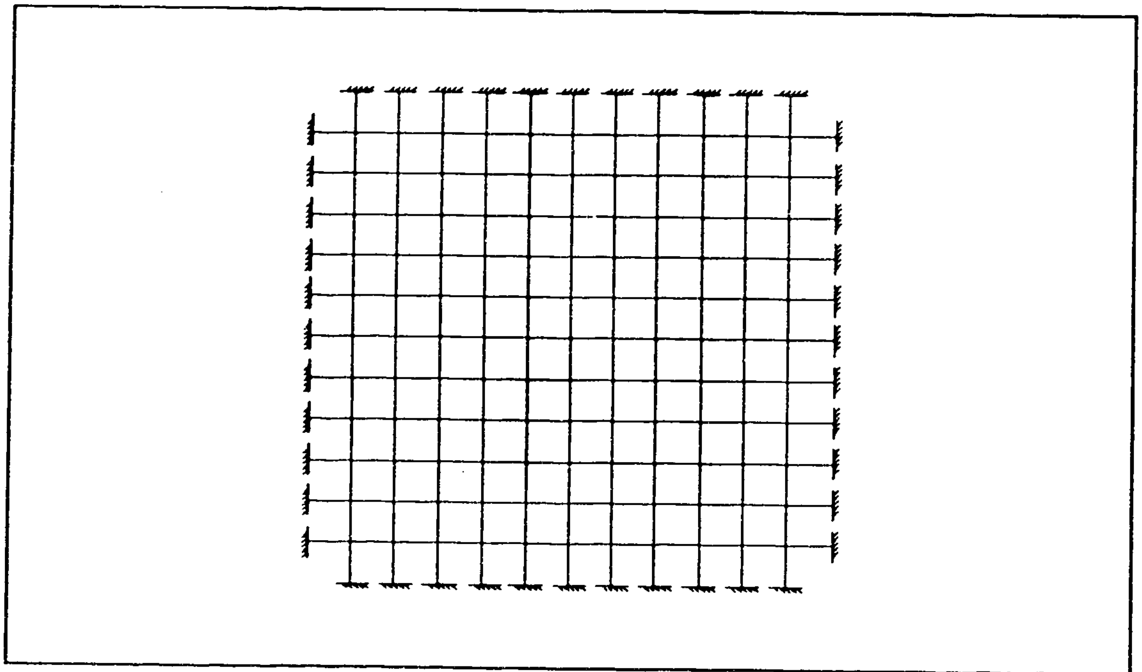

Figure 2.2 : Initial configuration for example 2.1

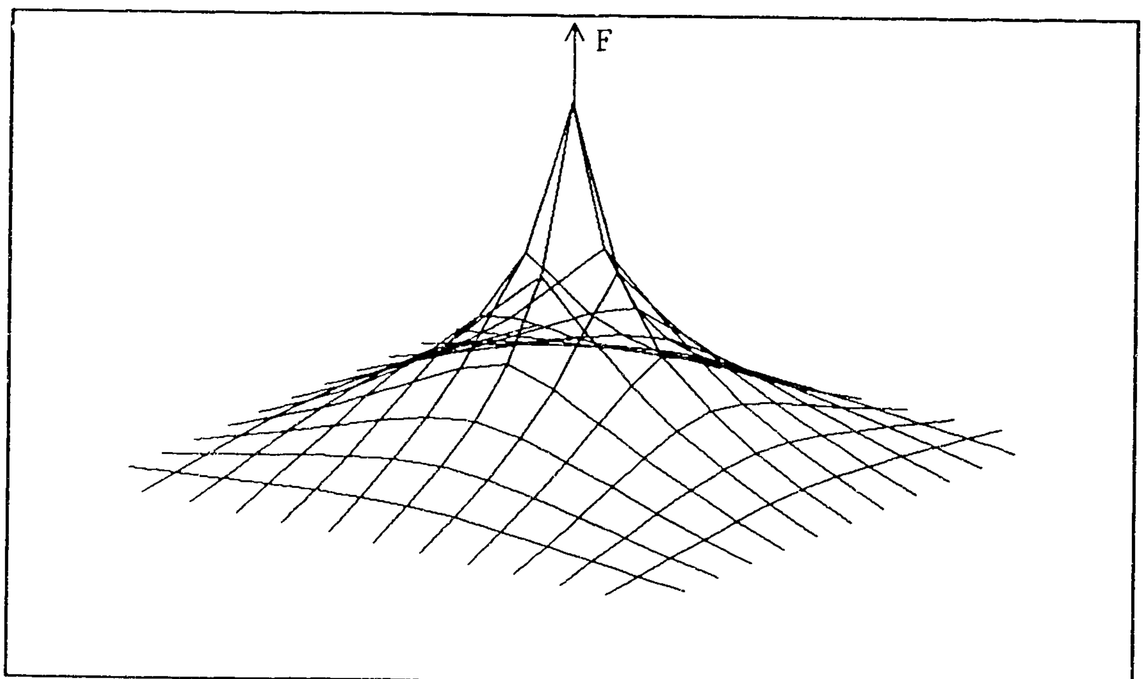

Figure 2.3 : Deformed configuration for example 2.1

rigid frame the following rigid motions separately

(a) Pull the rigid frame out of the plane of the net by some amount.

(b) Twist the rigid frame in the plane of the net by 45° counter clockwise

(c) deformations (a) + (b)

(d) deformations (b) + (a)

Figures 2.5 and 2.6 show the outcome of motions (a) and (b) respectively. Figure 2.7 shows the result of motion (c) which comes out to be identical to the result of motion (d) as is expected.

Table 2.1 compares the stress and strain in some cables for cases (c) and (d) which shows the path independency of the solution.

Cable No. (fig.2.4)	deformation (c)		deformation (d)	
	Stress(MPa)	Strain	Stress(MPa)	Strain
1	3.3100143	8.46834	3.3100145	8.46834
2	0.72085085	1.25605	0.72085083	1.25605
3	1.83139713	4.26859	1.83139743	4.26859
4	0.74748005	1.32125	0.74748008	1.32125

Table 2.1 : Comparison of stress and strain in some cables for deformations (c) and (d) of example 2.2

The values in the table correspond to an error tolerance of 10^{-6} for $\underline{r}^n$ in algorithm (2.33).

Example 2.3

Figure 2.8 shows a 21×21 square mesh. Cables are of equal stretched length of

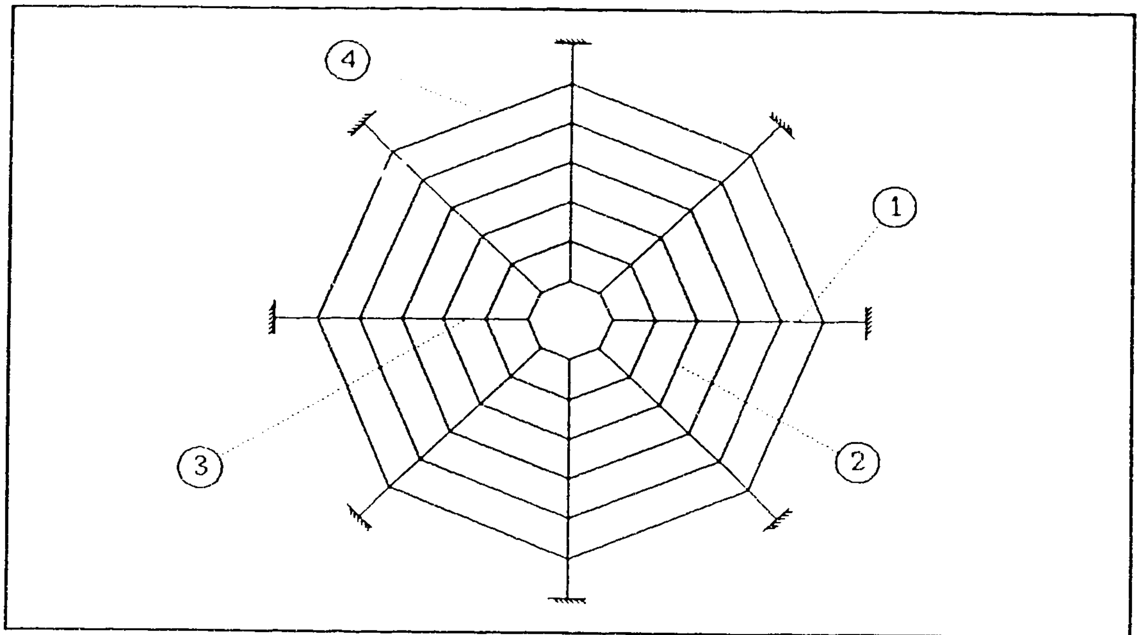

Figure 2.4 : Initial configuration of the spider net of example 2.2

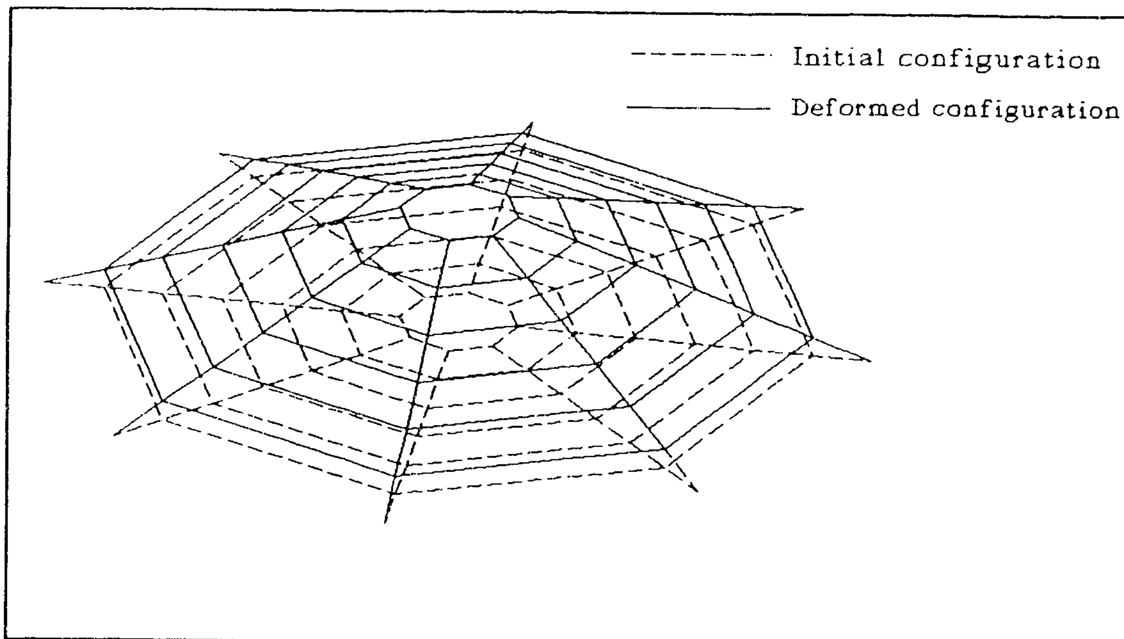

Figure 2.5 : Deformed and undeformed configurations under deformation (a) of example 2.2

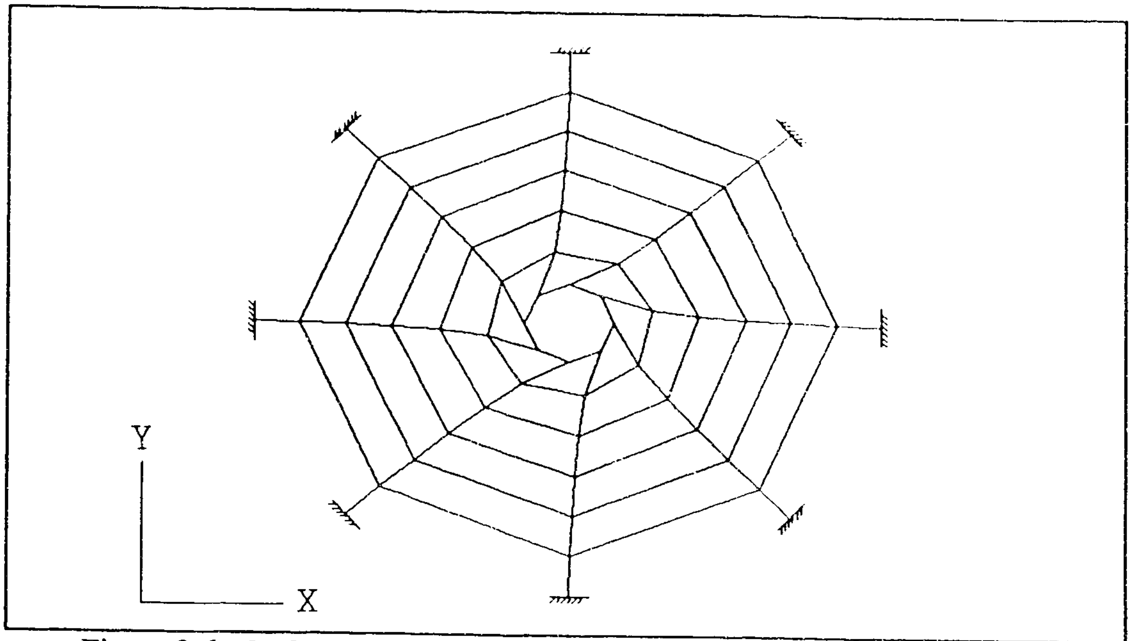

Figure 2.6 : Deformed configuration under deformation (b) of example 2.2

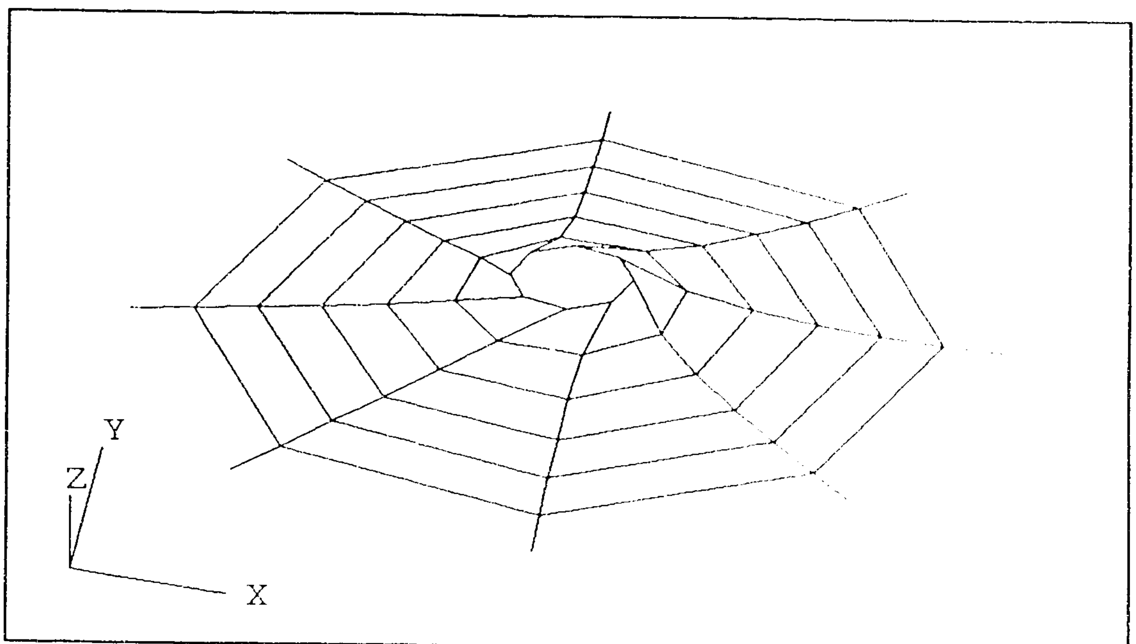

Figure 2.7 : Deformed configuration under deformation (c) or (d) of example 2.2

2 m and initial tension of 5 N. We consider the following displacements of the edges

(a) Map all edges to semicircles out of the plane of the net and perpendicular to it.

(b) Map edges AB and CD to semicircles out of the plane of net and perpendicular to it.

(c) Deformation (b) + map edges BC and DA to semicircles out of the plane of the net and perpendicular to it.

Figures 2.9 and 2.10 show the result of motions (b) and (a) respectively. If we apply motion (c) we end up with the same configuration as in motion (a). Table 2.2 shows stress and strain in some cables for motions (a) and (c). Again in this case we see the path independency of the solution.

Cable No.	deformation (a)		deformation (c)	
(Fig. 2.8)	Stress (MPa)	Strain	Stress (MPa)	Strain
1	1.21039	2.53814	1.21038	2.53812
2	1.2701936	2.70209	1.2701924	2.70209
3	1.326458	2.85710	1.326456	2.85709
4	1.501511	3.34305	1.501514	3.34306

Table 2.2 : Comparison of stress and strain in some cables for deformations (a) and (c) of example 2.3

Table 2.3 shows the number of iterations required for convergence for examples 2.1 to 2.3.

Summary

In this chapter we developed the DR theory for nonlinear set of equations and

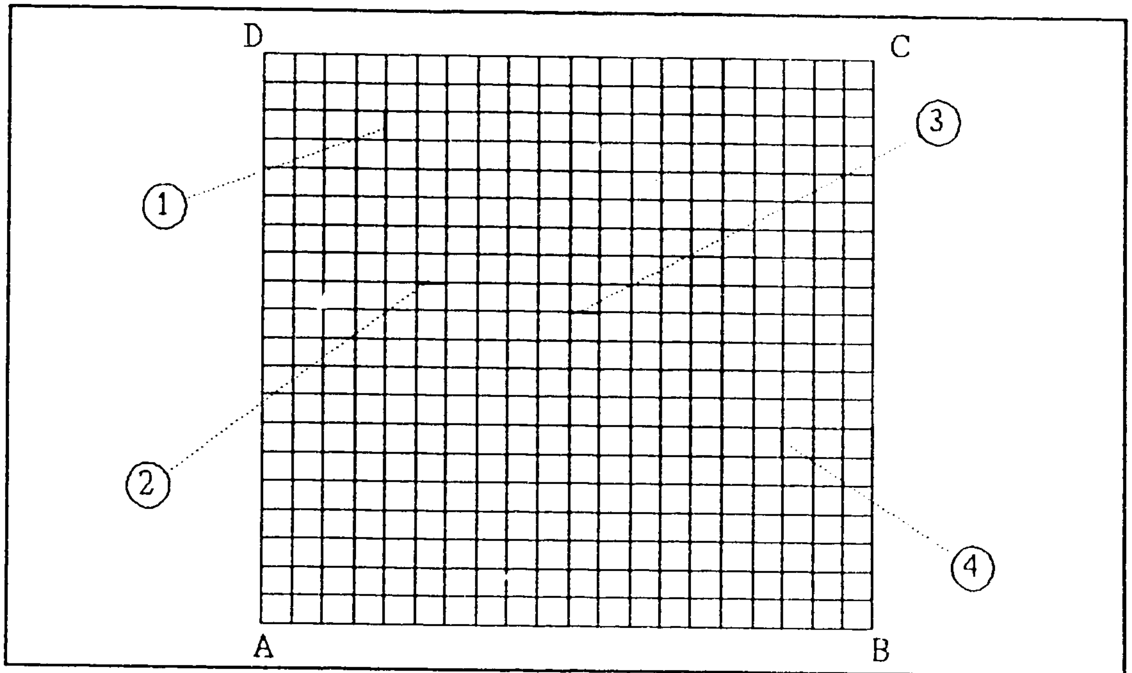

Figure 2.8 : Initial configuration of the square mesh of example 2.3

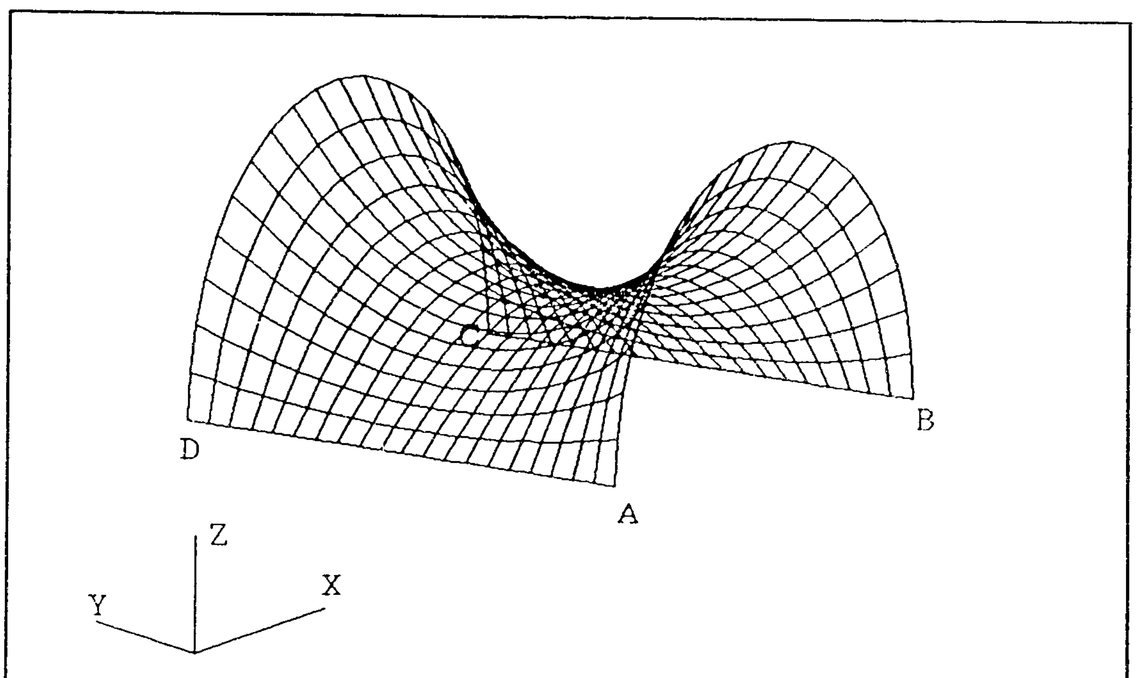

Figure 2.9 : Deformed configuration under deformation (b) of example 2.3

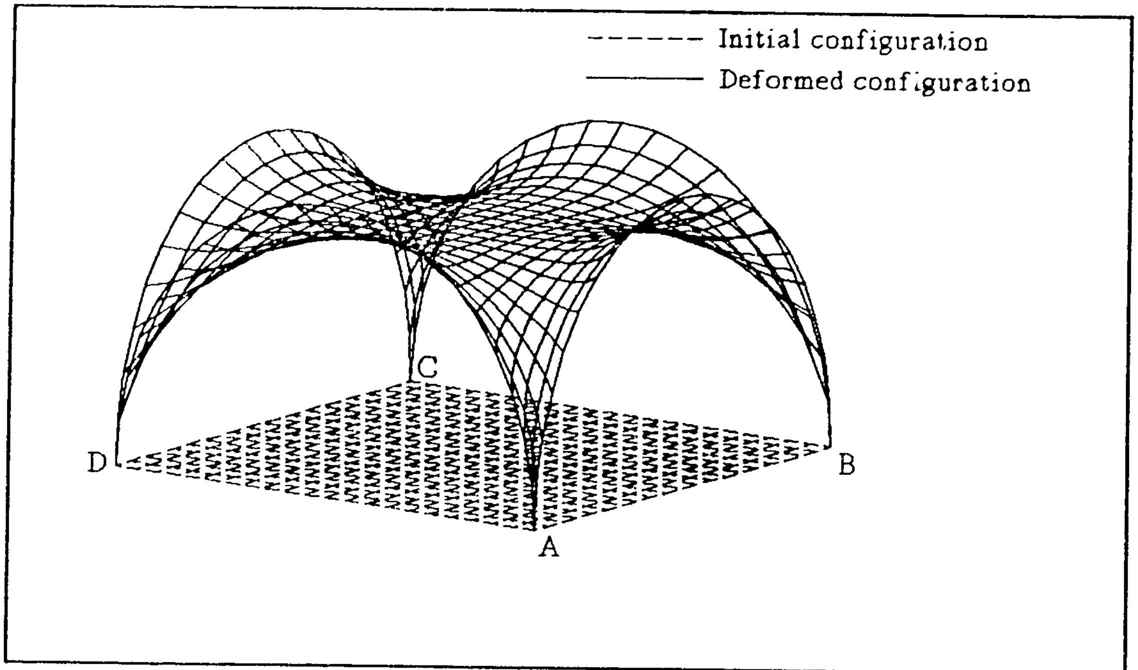

Figure 2.10 : Initial configuration of the square mesh and the deformed configuration under deformation (a) or (c) of example 2.3

Example	# of unknowns	# of iterations
2.1	363	862
2.2	120	300
2.3	1083	2699

Table 2.3 : Required number of iterations for convergence for examples 2.1 to 2.3

applied this technique to the problem of cable nets. Some networks were analyzed using this method and they showed the path independency of the solution for elastic cable nets as expected.

CHAPTER 3 : ELASTO-PLASTIC CABLES

In this chapter we develop the theory for elasto-plastic cables. We then show that DR fails to handle the equilibrium problem of elasto-plastic cables by itself. Finally we introduce the incremental technique and apply it to this class of cables. A number of examples analyzed by this method are also presented.

3.1 Development of elasto-plastic cables theory

The basic concept in dealing with elasto-plastic cables is that the tangent modulus E_t may change during the analysis. For example, the tangent modulus in plastic range is different from that in elastic range or in unloading. This can be defined as the following

$$E_t = \begin{cases} E_1 & \text{for elastic range} \\ \text{Tangent modulus,} & \text{for plastic range} \\ E_1 & \text{for unloading and/or reloading} \end{cases} \qquad (3.1)$$

where E_1 is the elastic Young modulus. Hence we consider the equilibrium problem as "time" dependent and this allows us to consider a "time" dependent tangent modulus. We recall the definition of cable force $\underline{t}$ from chapter 1 (eq. (1.15)) as the following

$$\underline{t} = f\underline{l} \qquad (3.2)$$

Taking the time derivative of both sides we get

$$\underline{\dot{t}} = \dot{f}\underline{l} + f\underline{\dot{l}} \qquad (3.3)$$

The deformation gradient $\underline{x}'(S)$ is (eq.(1.4))

$$\underline{x}'(S) = \lambda \underline{l} \qquad (3.4)$$

Taking the time derivative of both sides gives

$$\underline{\hat{u}}'(S)=\dot{\lambda}\underline{l}+\lambda\dot{\underline{l}} \tag{3.5}$$

where $\underline{\hat{u}}'(S)=\dot{\underline{x}}'(S)$ is the velocity gradient. From (3.5) we can write

$$\dot{\underline{l}}=\lambda^{-1}(\underline{I}-\underline{l}\otimes\underline{l})\underline{\hat{u}}' \tag{3.6}$$

Substituting (3.6) into (3.3) and noting that

$$\dot{f}=E_t A\dot{\lambda}=E_t A\underline{l}.\underline{\hat{u}}' \tag{3.7}$$

where A is the cross section of the cable, we get

$$\begin{aligned}\dot{\underline{t}}&=[E_t A(\underline{l}\otimes\underline{l})+f\lambda^{-1}(\underline{I}-\underline{l}\otimes\underline{l})]\underline{\hat{u}}'\\&=\underline{\underline{C}}\,\underline{\hat{u}}'\end{aligned} \tag{3.8}$$

where

$$\underline{\underline{C}}=E_t A(\underline{l}\otimes\underline{l})+f\lambda^{-1}(\underline{I}-\underline{l}\otimes\underline{l}) \tag{3.9}$$

Now let us consider the equations of equilibrium (1.26) in the following incremental form

$$\begin{cases}\dot{\underline{t}}'+\dot{\underline{a}}=\underline{0}\\ \dot{\underline{F}}_0=-\dot{\underline{t}}(0)\\ \dot{\underline{F}}_L=\dot{\underline{t}}(L)\end{cases} \tag{3.10}$$

We take these to be the incremental equations of equilibrium and we show that they can also be derived by minimizing the following functional associated with the potential energy of the cable [10]

$$P[\underline{\hat{u}}]=\int_0^L[(\frac{1}{2}\underline{\hat{u}}'.\underline{\underline{C}}\,\underline{\hat{u}}')-\dot{\underline{a}}.\underline{\hat{u}}]dS-[\dot{\underline{F}}_L.\underline{\hat{u}}(L)+\dot{\underline{F}}_0.\underline{\hat{u}}(0)] \tag{3.11}$$

We follow the same minimization procedure we used in Chapter 1 by applying the perturbation

$$\underline{\hat{u}} \rightarrow \underline{\hat{u}} + \varepsilon\underline{\hat{v}} \tag{3.12}$$

where epsilon is a small parameter and $\underline{\hat{v}}$ satisfies the kinematic boundary conditions but otherwise is arbitrary.Substituting (3.12) into (3.11) and arranging we get

$$\begin{aligned} G(\varepsilon) = P[\underline{\hat{u}} + \varepsilon\underline{\hat{v}}] = & \int_0^L \left(\frac{1}{2}\underline{\hat{u}}'.\underline{C}\underline{\hat{u}}' - \underline{\dot{a}}.\underline{\hat{u}}\right)dS - [\underline{\dot{F}}_L.\underline{\hat{u}}(L) + \underline{\dot{F}}_0.\underline{\hat{u}}(0)] \\ & + \varepsilon\left\{ \int_0^L (\underline{\hat{v}}'.\underline{C}\underline{\hat{u}}' - \underline{\dot{a}}.\underline{\hat{v}})dS - [\underline{\dot{F}}_L.\underline{\hat{v}}(L) + \underline{\dot{F}}_0.\underline{\hat{v}}(0)] \right\} \\ & + \frac{1}{2}\varepsilon^2 \int_0^L \underline{\hat{v}}'.\underline{C}\underline{\hat{v}}'dS \end{aligned} \tag{3.13}$$

where the symmetrical property of $\underline{C}$ has been used. If $\underline{\hat{u}}$ is a minimizer of P, we must have

$$\left.\frac{dG(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^L (\underline{\hat{v}}'.\underline{C}\underline{\hat{u}}' - \underline{\dot{a}}.\underline{\hat{v}})dS - [\underline{\dot{F}}_L.\underline{\hat{v}}(L) + \underline{\dot{F}}_0.\underline{\hat{v}}(0)] = 0 \tag{3.14}$$

which can be written as

$$-\int_0^L (\underline{\dot{t}}' + \underline{\dot{a}}).\underline{\hat{v}}dS - \left\{[\underline{\dot{F}}_L - \underline{\dot{t}}(L)].\underline{\hat{v}}(L) + [\underline{\dot{F}}_0 + \underline{\dot{t}}(0)].\underline{\hat{v}}(0)\right\} = 0 \tag{3.15}$$

Applying the fundamental lemma, we get exactly the incremental equations of equilibrium (3.10). We now get the necessary conditions for the equilibrium of the cable by applying the same procedure we used to get the Weierstrass conditions in Chapter 1. Let us consider a kinematically admissible $\underline{\hat{v}}(S)$ defined as the following over the region $S \in [0,L]$

$$\underline{\hat{v}}(S) = \begin{cases} \underline{\psi}(S) = (S - S_1)\underline{b} & S_1 \leq S \leq S_2 \\ \underline{\phi}(S) = (S_3 - S)\left(\dfrac{S_2 - S_1}{S_3 - S_2}\right)\underline{b} & S_2 \leq S \leq S_3 \\ 0 & \text{elsewhere} \end{cases} \tag{3.16}$$

where $0 < S_1 < S_2 < S_3 < L$ and $\underline{b}$ is an arbitrary vector. We define the parameters

$$\begin{cases} l = S_3 - S_1 \\ l_1 = S_2 - S_1 = \theta l \\ l_2 = S_3 - S_2 = (1-\theta) l \end{cases} \tag{3.17}$$

where

$$0 < \theta = \left(\frac{S_2 - S_1}{S_3 - S_1}\right) < 1 \tag{3.18}$$

Hence from (3.16) we can write

$$\underline{\phi}(S) = -\frac{\theta}{1-\theta}(S - S_3)\underline{b} \tag{3.19}$$

If $\hat{\underline{u}}$ is a minimizer of $P[\hat{\underline{u}}]$, we must have

$$\left.\frac{d^2G(\varepsilon)}{d\varepsilon^2}\right|_{\varepsilon=0} = \int_0^L \hat{\underline{v}}' . \underline{C}\hat{\underline{v}}' dS \geq 0 \tag{3.20}$$

which can be written as

$$l^{-1}\left[\int_{S_1}^{S_2} \underline{\psi}' . \underline{C}\underline{\psi}' dS + \int_{S_2}^{S_3} \underline{\phi}' . \underline{C}\underline{\phi}' dS\right] \geq 0 \tag{3.21}$$

Substituting from (3.16) and (3.19) we can write

$$\frac{\theta}{l_1}\int_{S_1}^{S_2} \underline{b} . \underline{C}\underline{b} dS + \frac{\theta^2}{l_2(1-\theta)}\int_{S_2}^{S_3} \underline{b} . \underline{C}\underline{b} dS \geq 0 \tag{3.22}$$

Letting $l \rightarrow 0$ ($l_1, l_2 \rightarrow 0$) and using the mean value theorem we get

$$\frac{\theta}{1-\theta}\underline{b}.\underline{C}\underline{b} \geq 0 \tag{3.23}$$

Dividing by $\theta/(1-\theta)$ we get

$$\underline{b}.\underline{C}\underline{b} \geq 0 \tag{3.24}$$

for any arbitrary vector $\underline{b}$. This is the definition of positive semi-definiteness of tensor $\underline{C}$ which means that the eigenvalues of $\underline{C}$ should be non-negative. Hence from (3.9) we can write

$$\begin{cases} E_t \geq 0 \\ f \geq 0 \end{cases} \tag{3.25}$$

These are equivalent to the Weierstrass conditions in chapter 1. These necessary conditions are also sufficient for stability of an equilibrium configuration because they guarantee the positive semi-definiteness of tensor $\underline{C}$. To satisfy the second of (3.25), we set the cable force equal to zero whenever we get a negative stress in a cable during the analysis. So one of the necessary conditions for stability of equilibrium configuration will be automatically satisfied.

Now we consider the case of a cable network. The incremental form of the equations of equilibrium (1.55) and (1.56) for a cable net can be written as

$$\begin{cases} \dot{\underline{t}}'_j + \dot{\underline{a}}_j = \underline{0} & j=1,2,\ldots,n \\ \dot{\underline{F}}^0_j = -\dot{\underline{t}}_j(0) & j \in I^k \\ \dot{\underline{F}}^L_j = \dot{\underline{t}}_j(L) & j \in E^k \end{cases} \tag{3.26}$$

and

$$\sum_{j \in I^k} \dot{\underline{F}}^0_j + \sum_{j \in E^k} \dot{\underline{F}}^L_j = \dot{\underline{q}}_k \qquad k \in K \tag{3.27}$$

respectively. The compatibility constraints (1.51) and (1.52) can also be written in the following incremental forms

$$\begin{cases} \underline{\hat{u}}_j^0 = \dot{\underline{y}}_k & , j \in I^k \\ \underline{\hat{u}}_j^L = \dot{\underline{y}}_k & , j \in E^k \end{cases} \tag{3.28}$$

and

$$\begin{cases} \underline{\hat{u}}_j^0 = \dot{\underline{z}}_h & , j \in I^h \\ \underline{\hat{u}}_j^L = \dot{\underline{z}}_h & , j \in E^h \end{cases} \tag{3.29}$$

respectively. We note that in the incremental procedure, the external loading vector $\underline{q}_k$, $k \in K$, the unknown position vectors $\underline{y}_k$, $k \in K$, and the prescribed position vectors $\underline{z}_h$, $h \in H$ are considered as a function of time. The necessary and sufficient conditions (3.25) still hold for each cable to make the whole network globally stable. In section 3.3 we make use of equilibrium and compatibility equations to develop the incremental technique.

3.2 Failure of conventional DR method for elasto-plastic cables

In this section, we present a counter example that shows the standard DR doesn't work for elasto-plastic materials.

Let us consider a single cable under uniaxial tensile load F as shown in figure 3.1

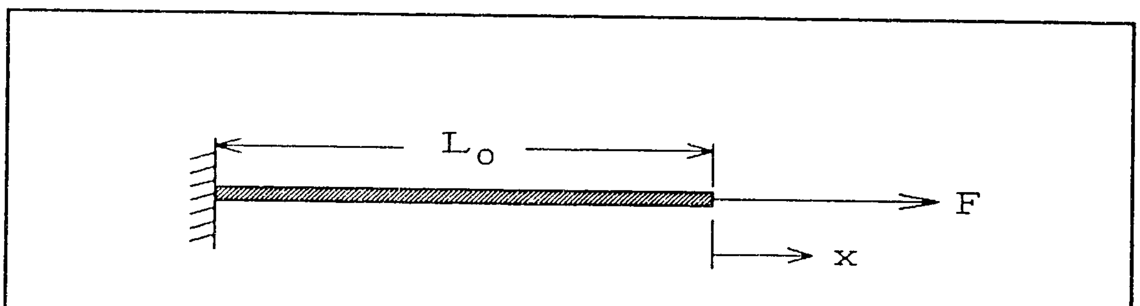

Figure 3.1 : A single cable under uniaxial tension

The cable has an unstretched length $L_0 = 1$ m and a cross section $A = 4$ mm^2. The material is elastic hardening with the constitutive force-stretch diagram shown in figure 3.2

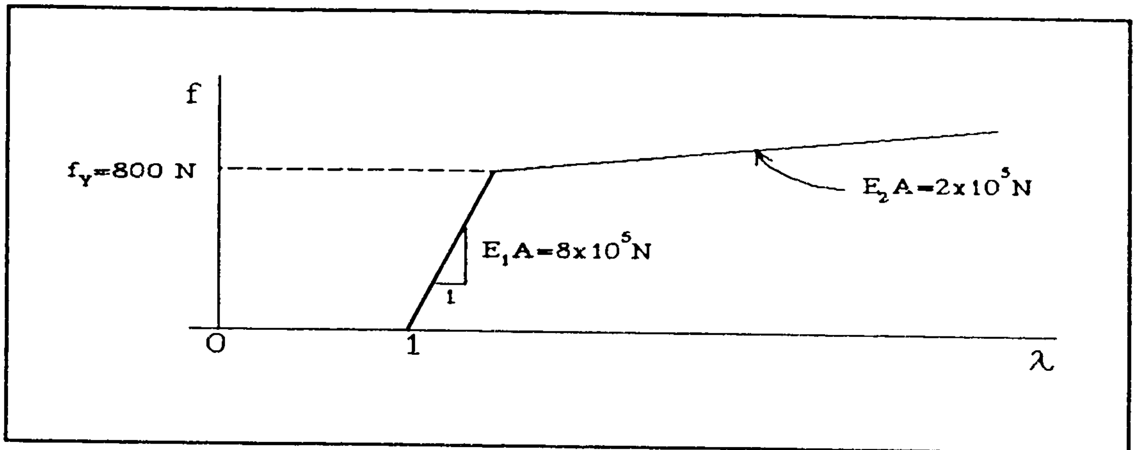

Figure 3.2 : Force-stretch diagram for the cable of fig. 3.1

The applied force F is 400 N. Since $F<F_Y$, the cable is in elastic range and the displacement of the cable a the loaded end is

$$X_{eq}=\frac{FL_0}{E_1A}=5\times10^{-4}m \qquad (3.30)$$

where X_{eq} is the displacement at the loaded end to maintain equilibrium.

Now let us use DR to solve this problem. We use the adaptive DR algorithm (2.33) along with different initial values for $X(0)= X^0$. Figures 3.3 to 3.5 show the force-stretch curve and the path of marching DR towards the solution for these different initial guesses X^0.

As it is seen in these figures, the internal force is equal to the applied external force for the converged value X_{eq} in all three cases. But we see that only for the first case the converged value is correct. We know that the external load F is applied statically and we never go into the plastic range and the values of X_{eq} shown in figures 3.4 and 3.5 are wrong. This example shows that when there is path dependency in the solution, DR is sensitive to initial guesses and hence it can not be used for such a case without further modification. Reference [12] shows that DR can be used along with the incremental

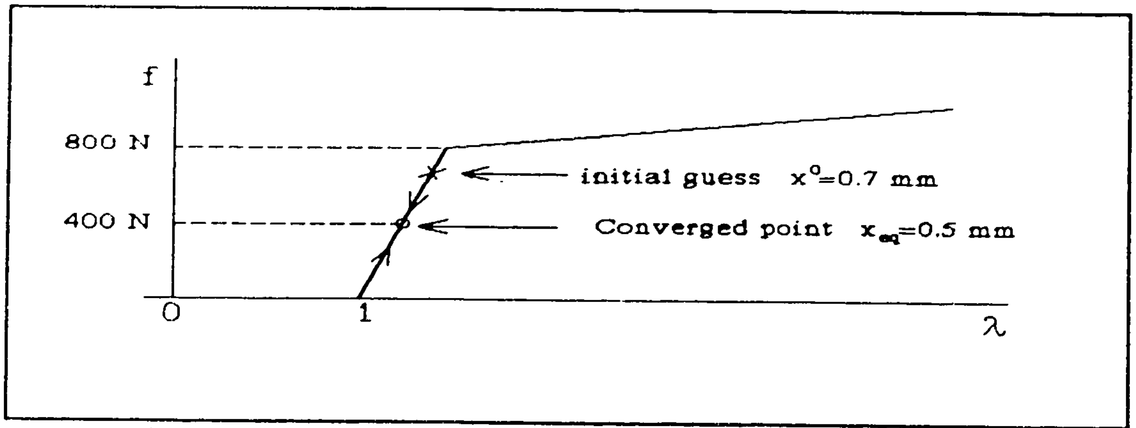

Figure 3.3 : Convergence value of X_{eq} for initial guess $X^0=0.7$ mm

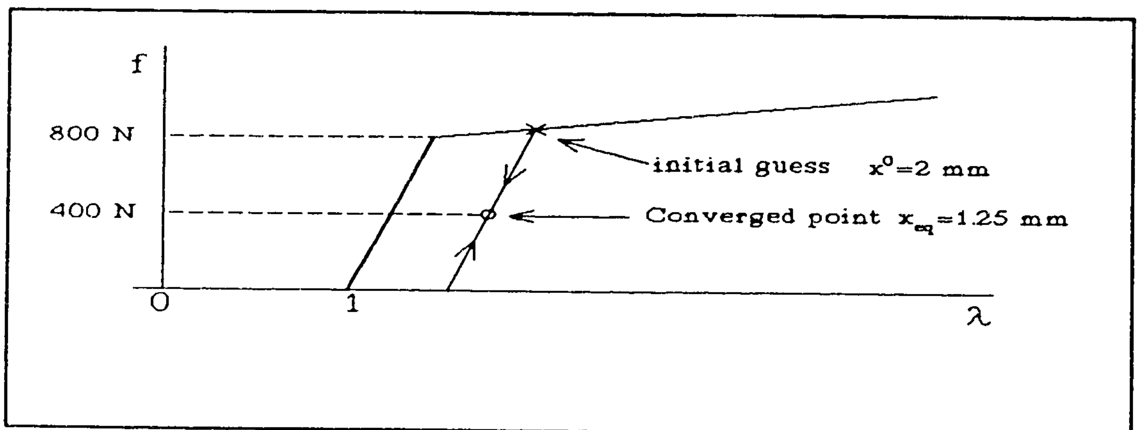

Figure 3.4 : Convergence value of X_{eq} for initial guess $X^0=2$ mm

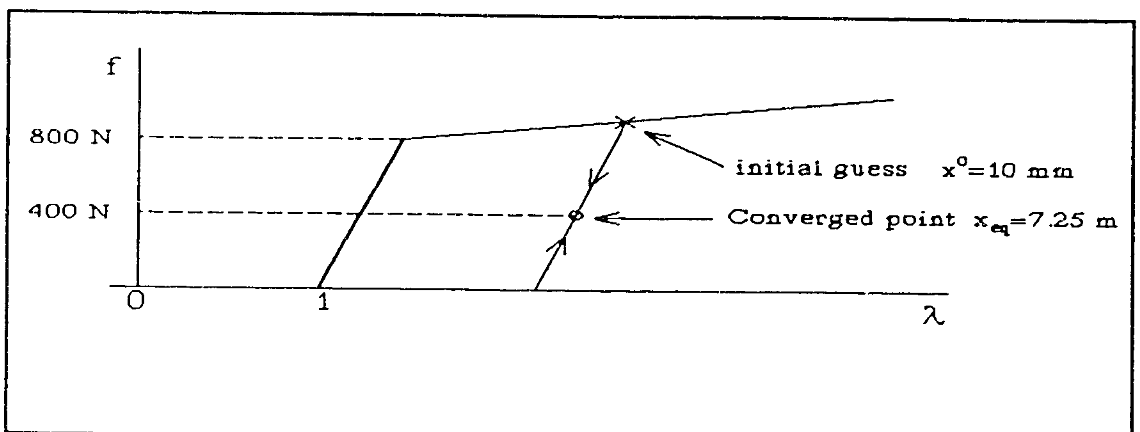

Figure 3.5 : Convergence value of X_{eq} for initial guess $X^0=10$ mm

technique, which is discussed next, to solve problems involving elasto-plastic cables.

3.3 Incremental Technique

In this section we introduce the incremental technique based on the equilibrium equations (3.26) and (3.27) and the compatibility constraints (3.28) and (3.29). Again for the case of no distributed loading for the jth cable, i.e. $\underline{a}_j=\underline{0}$, from the first of (3.26) we get

$$\dot{\underline{t}}_j'=\underline{0} \tag{3.31}$$

which means that

$$\dot{\underline{t}}_j=\hat{\underline{t}}_j \tag{3.32}$$

where $\hat{\underline{t}}_j$ is a constant vector. Considering (3.9) we see that at each increment, the tensor $\underline{C}_j$ is constant for each cable. So from (3.8) we can write

$$\hat{\underline{u}}'_j=\underline{r}_j \tag{3.33}$$

where $\underline{r}_j$ is a constant vector and since the cable is straight at each increment, this vector can be expressed as

$$\hat{\underline{u}}'_j=\underline{r}_j=\frac{\hat{\underline{u}}_j^L-\hat{\underline{u}}_j^0}{L_j} \tag{3.34}$$

Substituting from (3.26) into (3.27) and considering (3.8) we can write

$$-\sum_{j\in I^k}\underline{C}_j\hat{\underline{u}}_j'+\sum_{j\in E^k}\underline{C}_j\hat{\underline{u}}_j'=\dot{\underline{q}}_k \qquad k\in K \tag{3.35}$$

and from (3.34) we get

$$-\sum_{j\in I^k}\underline{C}_j\frac{\hat{\underline{u}}_j(L)-\hat{\underline{u}}_j(0)}{L_j}+\sum_{j\in E^k}\underline{C}_j\frac{\hat{\underline{u}}_j(L)-\hat{\underline{u}}_j(0)}{L_j}=\dot{\underline{q}}_k \qquad k\in K \tag{3.36}$$

This is a set of equations linear in $\underline{u}_j$. Once we've solved for $\underline{u}_j$'s, we can get the new unknown position vectors $\underline{y}_{k(new)}$ using (3.28) as the following

$$\underline{y}_{k(new)} = \begin{cases} \underline{y}_{k(old)} + \Delta t \underline{\hat{u}}_j(0) & k \in K, j \in I^k \\ \underline{y}_{k(old)} + \Delta t \underline{\hat{u}}_j(L) & k \in K, j \in E^k \end{cases} \tag{3.37}$$

and the prescribed nodal displacements are found using (3.29) as the following

$$\underline{z}_{h(new)} = \begin{cases} \underline{z}_{h(old)} + \Delta t \underline{\dot{z}}_h(0) & h \in H, j \in I^h \\ \underline{z}_{h(old)} + \Delta t \underline{\dot{z}}_h(L) & h \in H, j \in E^h \end{cases} \tag{3.38}$$

where Δt is the small time step. It should be noted that at the beginning and end of each time step, the cable net is in equilibrium. This procedure is continued over a time interval T and at the end of this time interval, the structure is in its final loaded configuration or has the prescribed displacement and is in equilibrium. The time step Δt should be small enough to give the required accuracy. It should also be noted that when the eigenvalues of $\underline{C}_j$ are zero, i.e. $f_j=0$ and/or $(E_t)_j=0$ (perfectly plastic cable), from the linear equations (3.36) we may end up with the coefficient matrix that is singular and therefore we don't get the correct solution for $\underline{\hat{u}}_j$'s. Even for the cases with $(E_t)_j$ close to zero, we get a coefficient matrix that is ill conditioned and the results from that are not correct. So we don't consider such cases for the application of the incremental technique. The method described in reference [12] however, can be used for such cases and this could be one of the future works.

We can summarize the incremental technique in the following steps

(a) Start with the cable network initially in equilibrium

(b) Pick the time interval T (usually T=1) and the time step Δt. Set t=0.

(c) Express external loading and prescribed displacement as a function of time (usually linear) over the time interval

(d) Form the linear equations (3.36) and solve for $\underline{\hat{u}}_j$'s.

(e) Update the unknown position vectors $\underline{y}_k$ using (3.37)

(f) Update the prescribed position vectors $\underline{z}_h$ using (3.38)

(g) update time : $t=t+\Delta t$

(h) if $t=T$ then stop otherwise go to (d)

We can do this procedure for one time step and then repeat the procedure with a decreased time step successively until there is not much difference in the results.

3.4 Examples

In this section, some examples are analyzed by incremental technique and the results are discussed. Although this technique can handle elastic cables too, we just present some examples with elastic hardening materials which have a constitutive force-stretch diagram shown in figure 3.2. The first example deals with external loading case while others involve prescribed displacements.

Example 3.1:

Figure 3.6 shows a 13x13 square mesh with fixed boundaries. All cables have equal stretched length of 1.05 m and they are initially under the same tension of 400 N in the elastic range. Therefore the cable net is initially in equilibrium. An external force Q_0 is applied at the central node perpendicular to the plane of net. We take the following loading functions to use with the incremental technique (Figures 3.7 and 3.8)

$$Q_{1,2}=Q_0 \pm Q_a \mathrm{Sin}(\pi t) \quad , \quad Q_0>Q_a \quad , \quad t \in [0,1] \qquad (3.39)$$

The loading is large enough to bring some cables to the plastic range (For this example $Q_0=1500$ N and $Q_a=400$ N). It can be seen that although at the end of time interval, the final loading is $Q_1=Q_2=Q_0$, the results for the two loading cases are not the same because of the path dependency of the solution. Figure 3.9 shows the deformed configuration for both loading cases with a deformation scale of 10. As it is seen, the results are totally different as expected. Table 3.1 provides the stress and strain in some cables for both cases from which we see that the results are not the same. This example shows that it is important how to express loading as a function of time. We should note that during the analysis, some cables may become plastic and if the value of the external loading function decreases, those cable may be unloaded which might not be the case if the external loading was always increasing to reach its final value. The external loading should be applied quasi-statically to avoid this problem.

Cable No. (Fig. 3.6)	Loading Q_1		Loading Q_2	
	Stress (MPa)	Strain	Stress (Mpa)	Strain
1	454.105	0.007664	484.367	0.006696
2	144.533	0.000723	139.087	0.000695
3	253.688	0.002859	275.471	0.002523
4	207.666	0.001616	223.184	0.001467

Table 3.1 : Comparison of stress and strain in some cables for loadings Q_1 and Q_2 of example 3.1

Example 3.2:

Figure 3.10 shows a 12x12 square mesh fixed at the boundaries. Cables have the same stretched length of 1.05 m and initial tension of 400 N in the elastic range. We apply the following rigid motions to the central square separately

(a) Pull the central square up and out of the net plane by one unit
(b) Twist the central square in the plane of net 45 deg. counter clockwise
(c) motions (a) + (b)
(d) motions (b) + (a)

Figures 3.11 and 3.12 show the plan view of the deformed configuration for

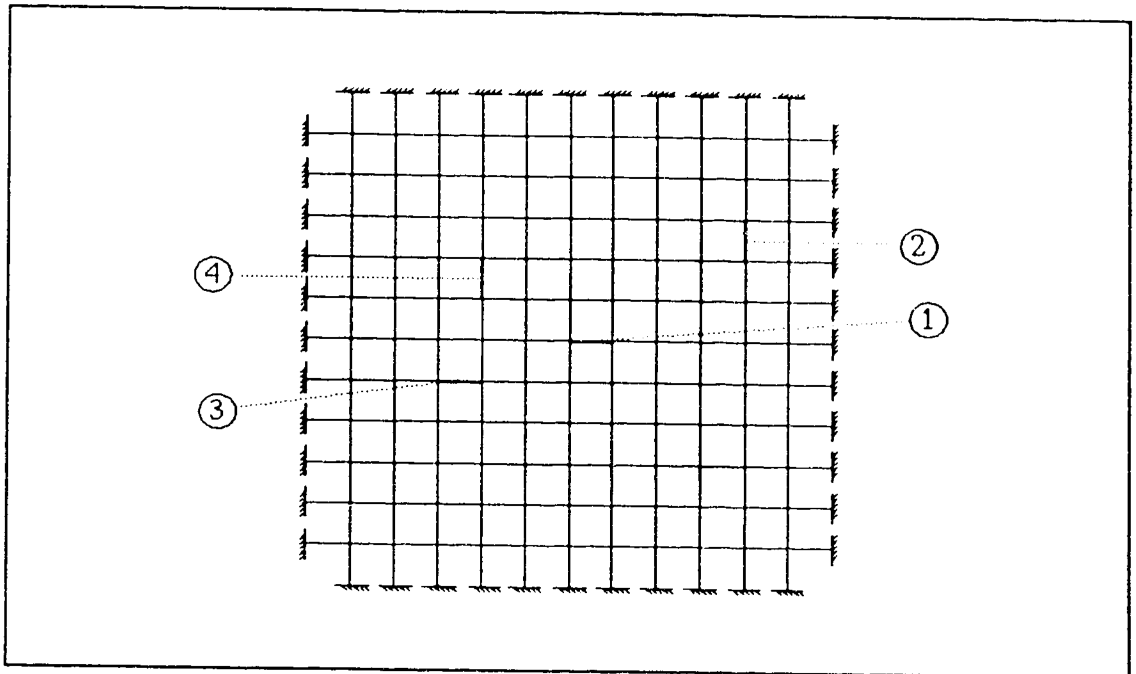

Figure 3.6 : 13x13 square mesh under dead load of example 3.1

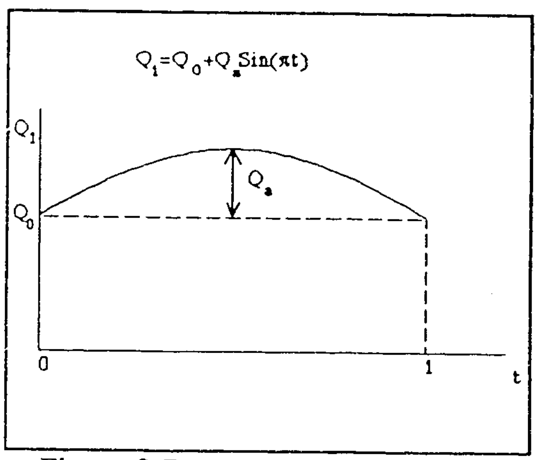

Figure 3.7 : External loading $Q_1(t)$

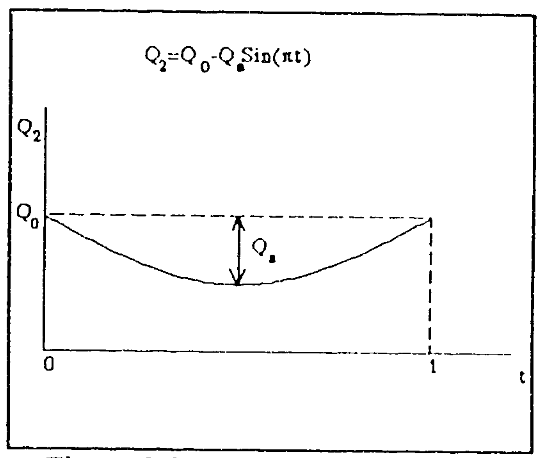

Figure 3.8 : External loading $Q_2(t)$

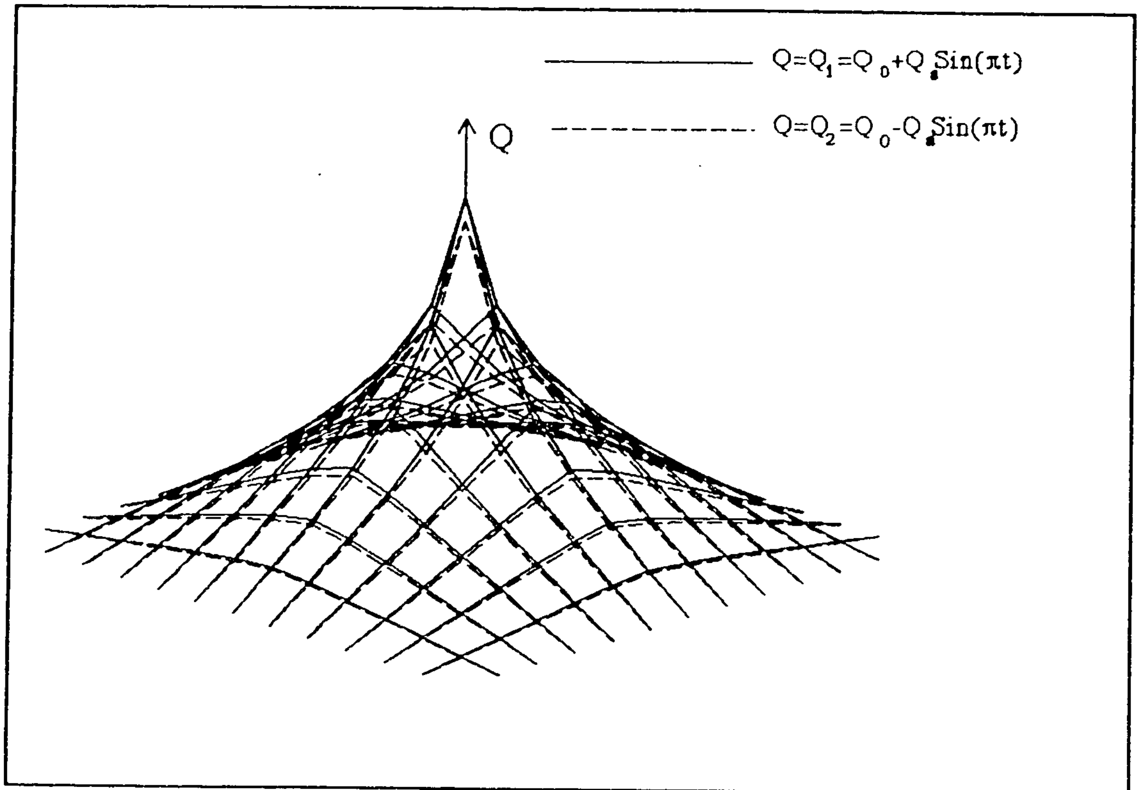

Figure 3.9 : Deformed configurations under external loadings $Q_1(t)$ and $Q_2(t)$ (Scale 1:10)

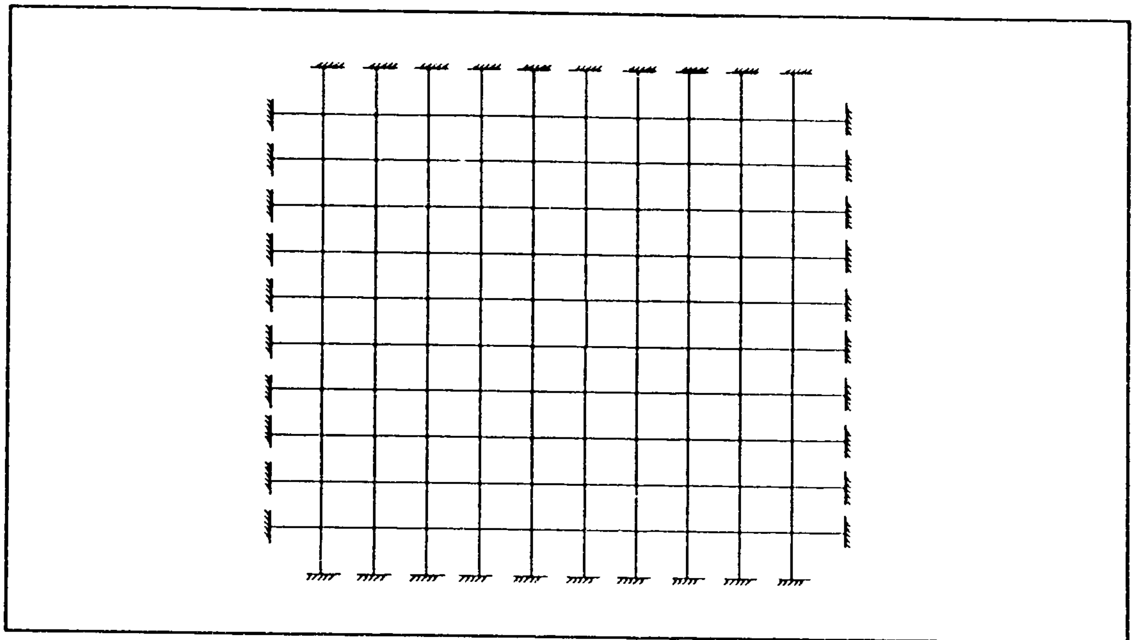

Figure 3.10 : 12x12 square mesh with rigid central square frame

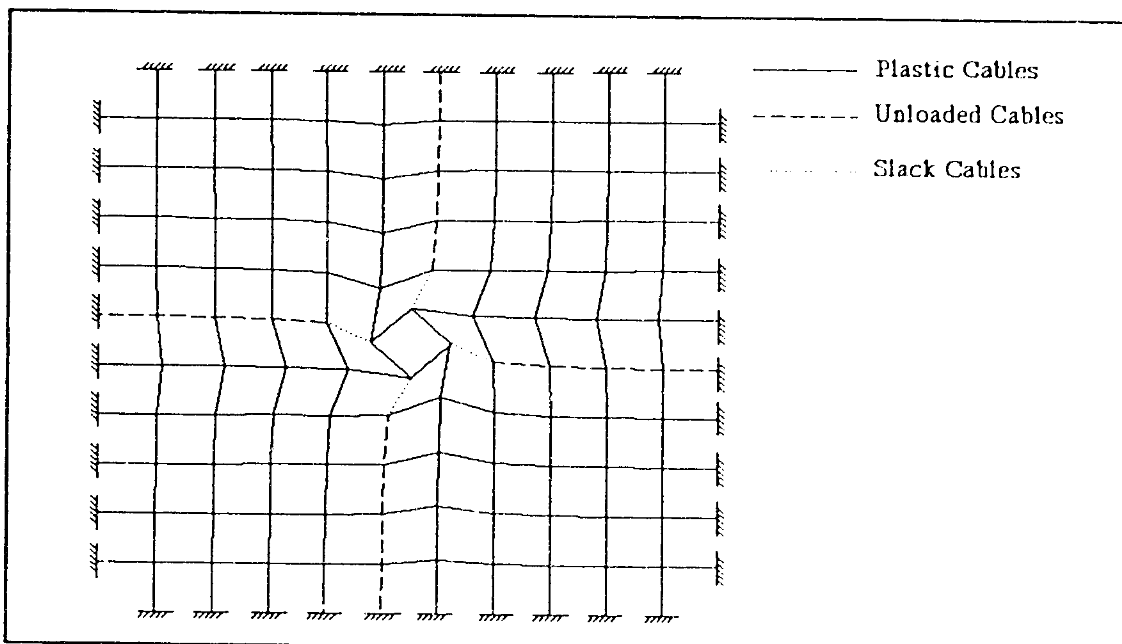

Figure 3.11 : 12x12 square mesh under deformation (c) of example 3.2 (Pullup + twist of central frame)

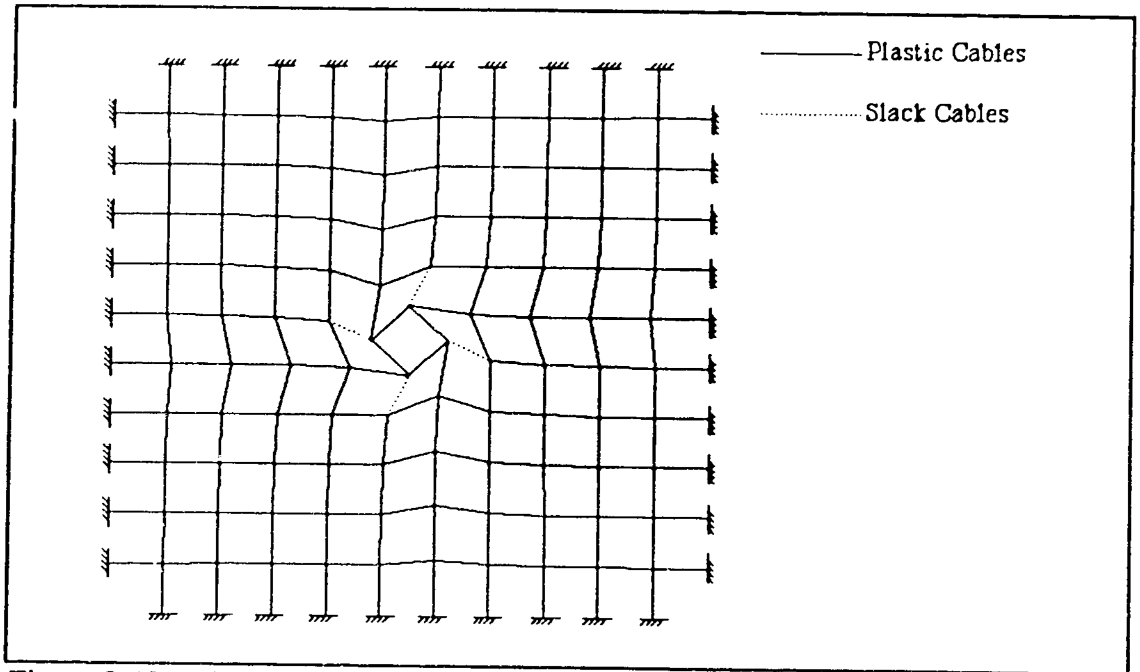

Figure 3.12 : 12x12 square mesh under deformation (d) of example 3.2 (twist+pullup of central frame)

deformations (c) and (d) respectively. As we see again, the results are totally different because of the path dependency in the solution.

Example 3.3:

Figure 3.13 shows a 7x7 square mesh. Cables are of the same stretched length of 2 m and initial tension of 2600 N in the plastic range. We map the boundaries to four straight lines out of the plane of net so that the final configuration is a "Hyperbolic Paraboloid" net which is shown in figure 3.14. Because of the symmetry in mapping the boundaries in this example, cables along AA and BB (figure 3.13) do not deform and remain neutral. Other cables are getting more strained in the plastic range but the strains in the cables are not the same. Now if we remap the boundaries to those of the initial configuration, cables are unloaded except the ones along AA and BB (figure 3.15) and we end up with a configuration identical to the initial one but less stressed. Table 3.2 shows the stress and strain in some cables in the initial and final (boundaries are remapped to the original ones) configurations.

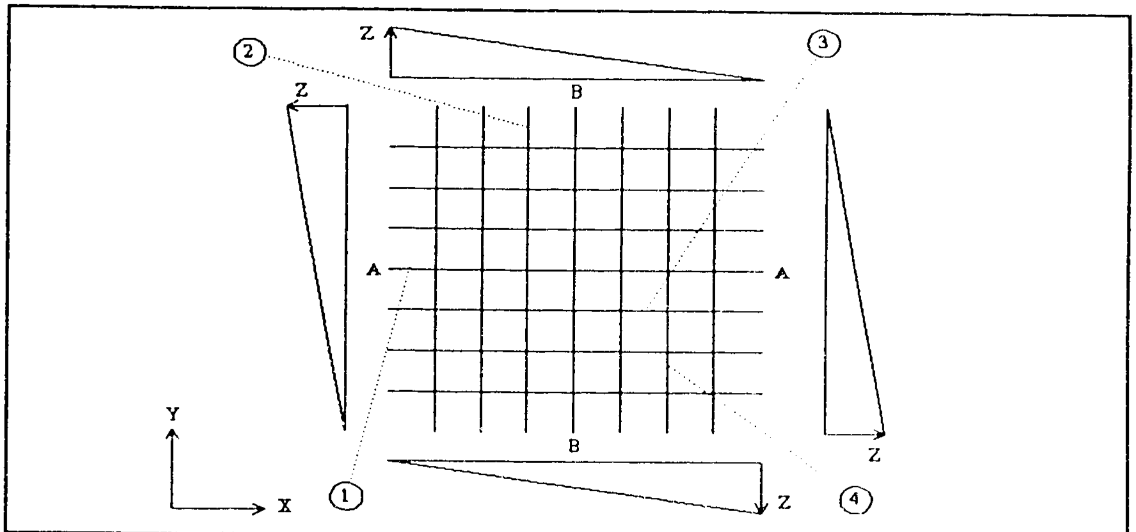

Figure 3.13 : 9x9 square mesh with edges undergoing linear deformation perpendicular to the plane on net

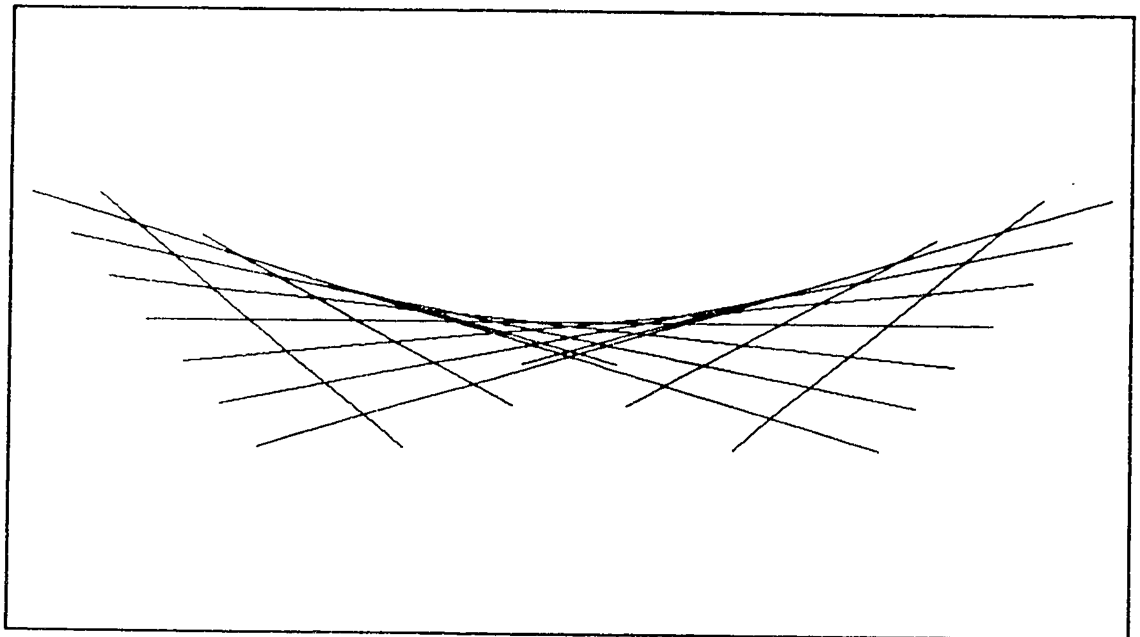

Figure 3.14 : Hyperbolic-Paraboloid net

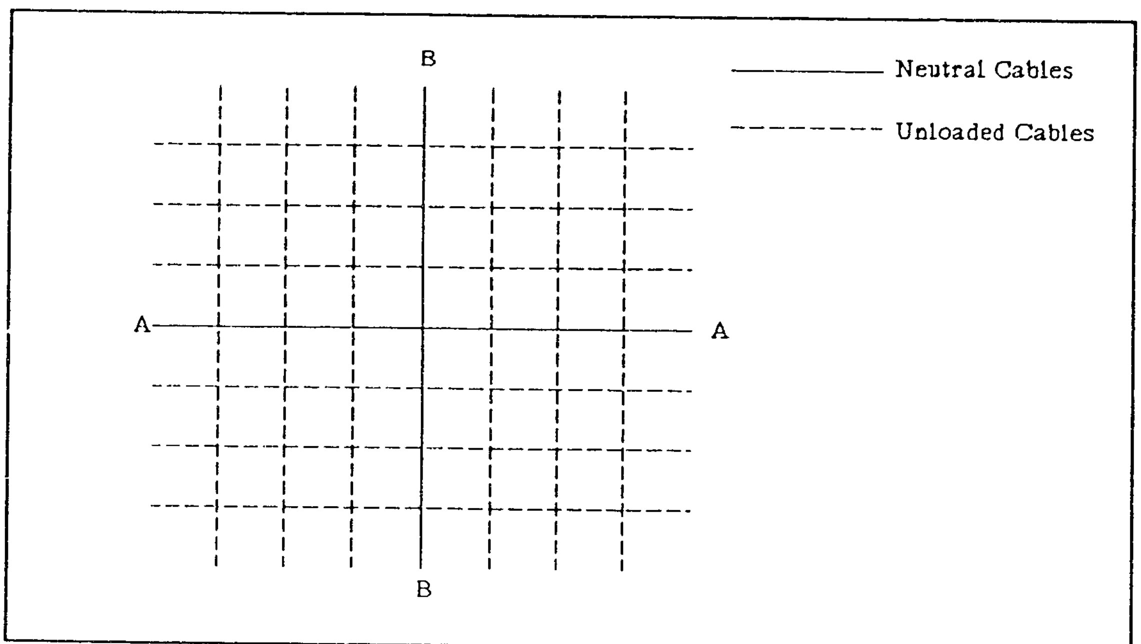

Figure 3.15 :Remap of the edges to the initial configuration of the example 3.3

Cable No. (Fig. 3.13)	Initial configuration		Final configuration	
	Stress (Mpa)	Strain	Stress(MPa)	Strain
1	650.00	0.01	543.51	0.01
2	650.00	0.01	650.00	0.01
3	650.00	0.01	638.16	0.01
4	650.00	0.01	602.66	0.01

Table 3.2 : Comparison of stress and strain in some cables for initial and final configuration of example 3.3

Example 3.4:

In this example, we consider the spider net we had in example 2 of chapter 2 with the same deformations considered there except the material that is elastic hardening here. Figure 3.16 shows the initial configuration. Figures 3.17 and 3.18 show the deformed configurations for deformations (a) and (b) respectively. As it is seen in figure 3.18, the circumferential cables are slack. Figure 3.19 shows the deformed configurations for deformations (c) and (d). Again the circumferential cables are slack and it seems that the problem has been converted to a one dimensional case for the radial cables. Although the deformed configurations for deformations (c) and (d) are identical, but the stresses and strains are different in circumferential cables for the two deformations (Table 3.3). Again in this example, the solution is path dependent.

Example 3.5:

We repeat here example 2.3 with the elastic hardening material. Cables are of equal tension of 100600 N and have a stretched length of 2m and a strain of 0.5 in the

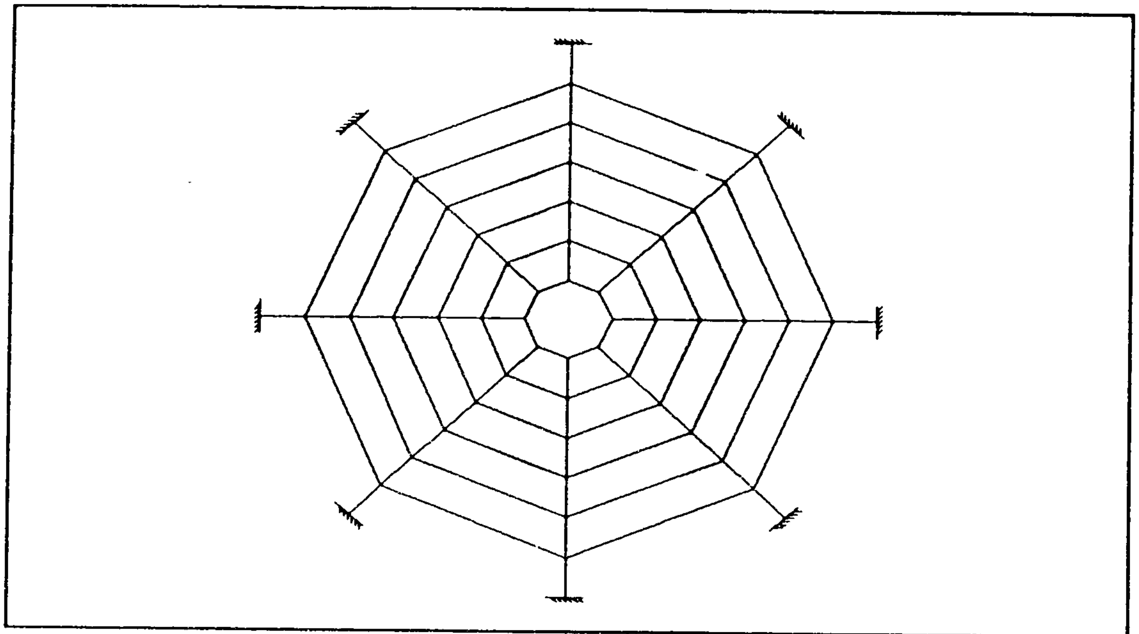

Figure 3.16 : Spider net of example 3.4

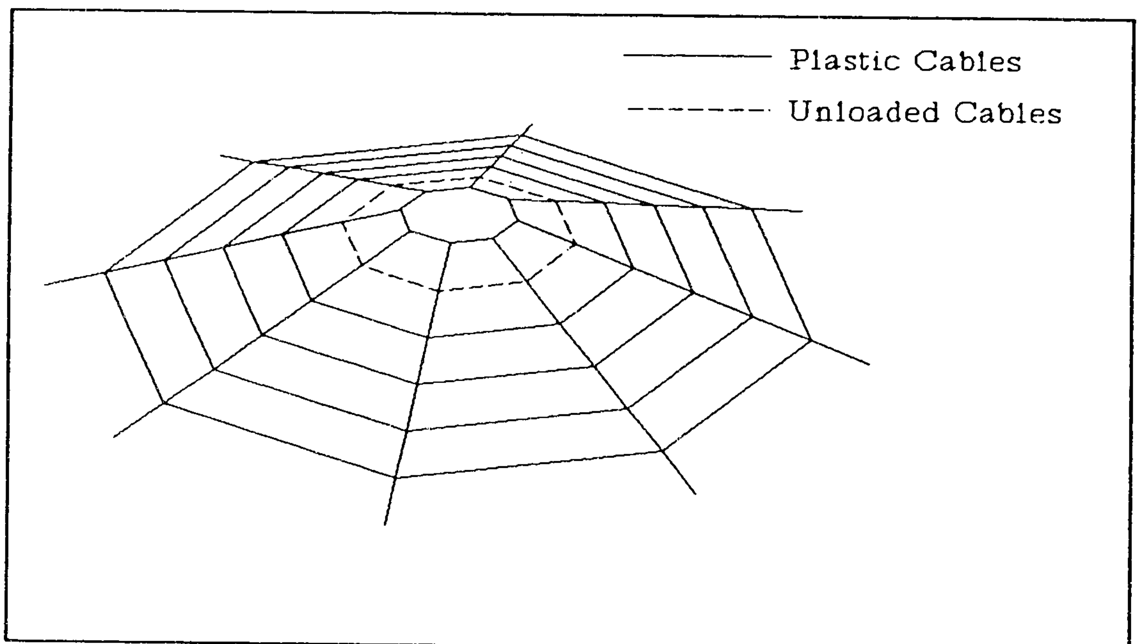

Figure 3.17 : Spider net under deformation (a) of example 3.4 (Pullup of central frame)

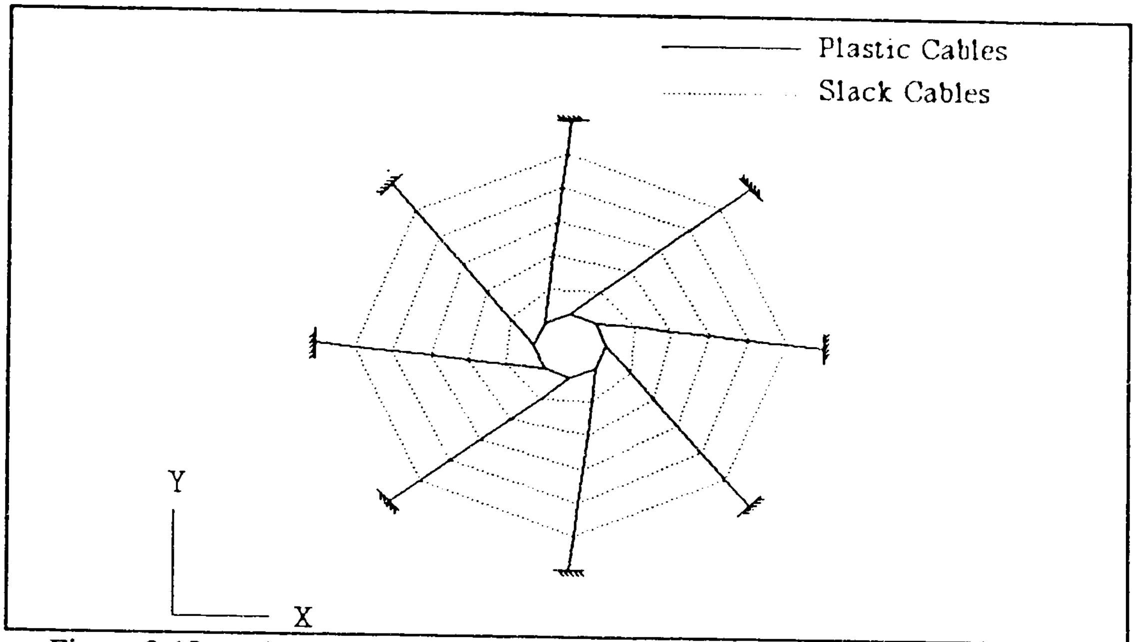

Figure 3.18 : Spider net under deformation (b) of example 3.4 (Twist of central frame)

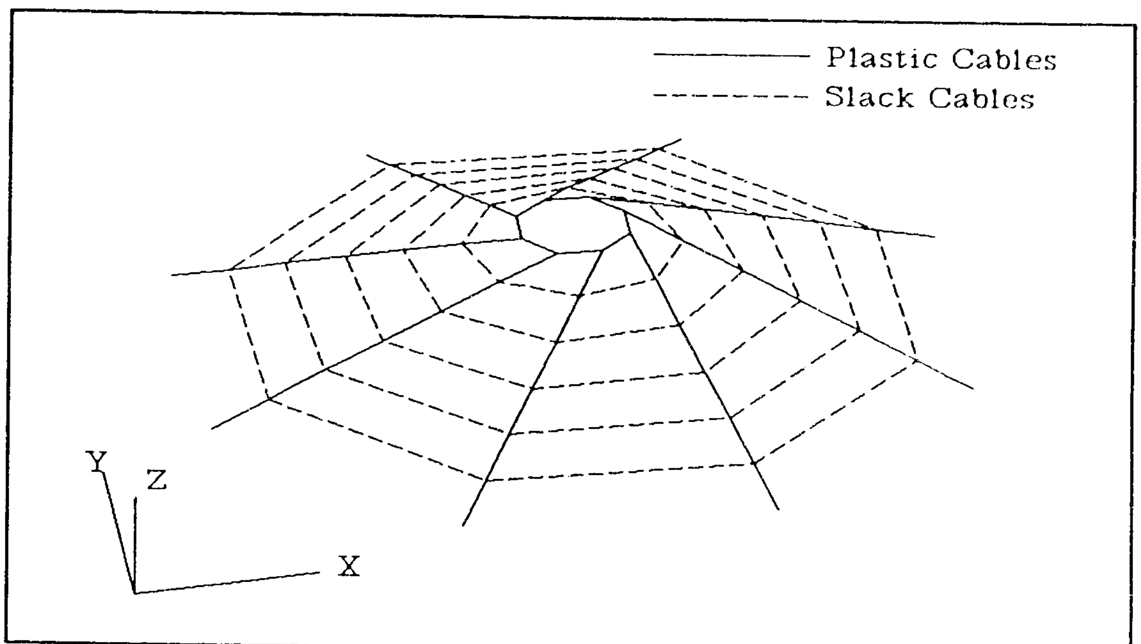

Figure 3.19 : Spider net under deformation (c) or (d) of example 3.4

Cable No. (Fig. 2.4)	Deformation (c)		Deformation (d)	
	Stress (MPa)	Strain	Stress (Mpa)	Strain
1	13142.445	0.259849	13142.869	0.259857
2	0	0.000372	0	0
3	13144.77025	0.259895	13144.7125	0.259894
4	0		0	0

Table 3.3 : Comparison of stress and strain in some cables for deformations (c) and (d) of example 3.4

plastic range. Figure 3.20 shows the deformed model for deformation (b). If the cable net undergoes deformation (a) or (c) the result is almost what we see in figure 3.21. But if we take a look at the plan view of the deformed configurations for these two deformations (Figures 3.22 and 3.23), we see that cables are in different situations for the two deformations and once again we see the path dependency of the solution.

Summary

In this chapter, we developed the theory of plastic cables and noted that conditions of equilibrium and global stability are the same as what we had for elastic cables in chapter 1. The only difference was that we considered the problem as time dependent used a different constitutive law that allowed us to use a tangent modulus E_t which varied based on the deformation that the cable was undergoing at each increment of time. We then presented a counter example and proved that conventional DR method cannot be used for elasto-plastic cables because of the path dependency of the solution. However reference [12] makes use of this method along with the incremental technique for application to elasto plastic including perfect plasticity. We then introduced the

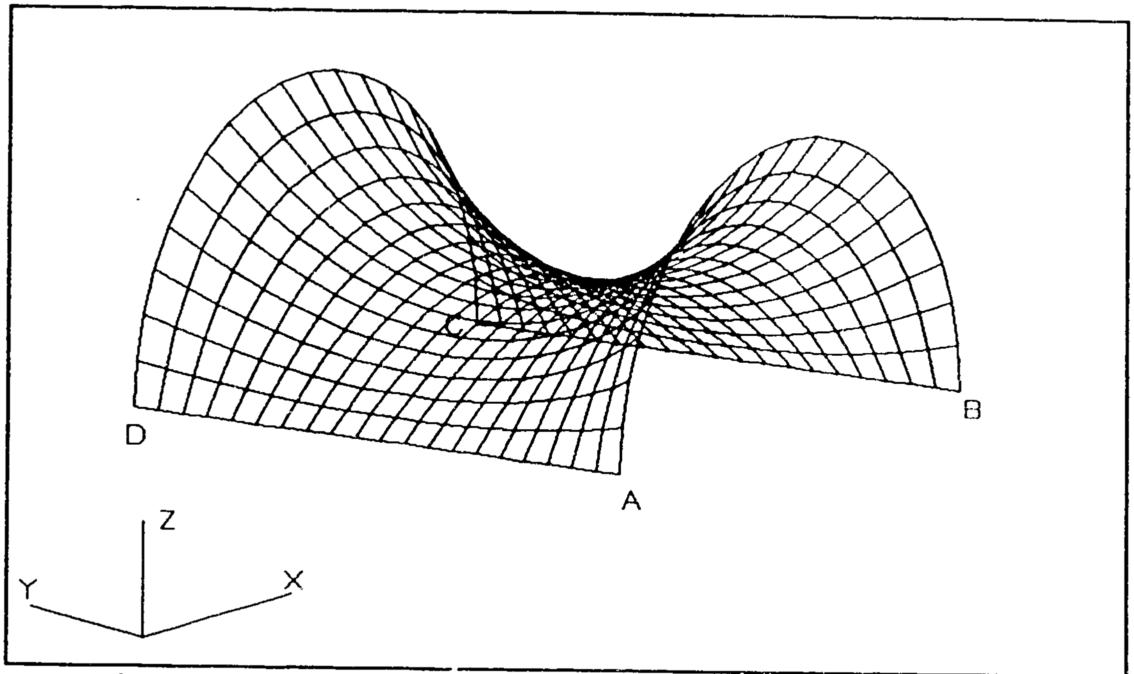

Figure 3.20 : 21x21 square mesh under deformation (b) of example 3.5

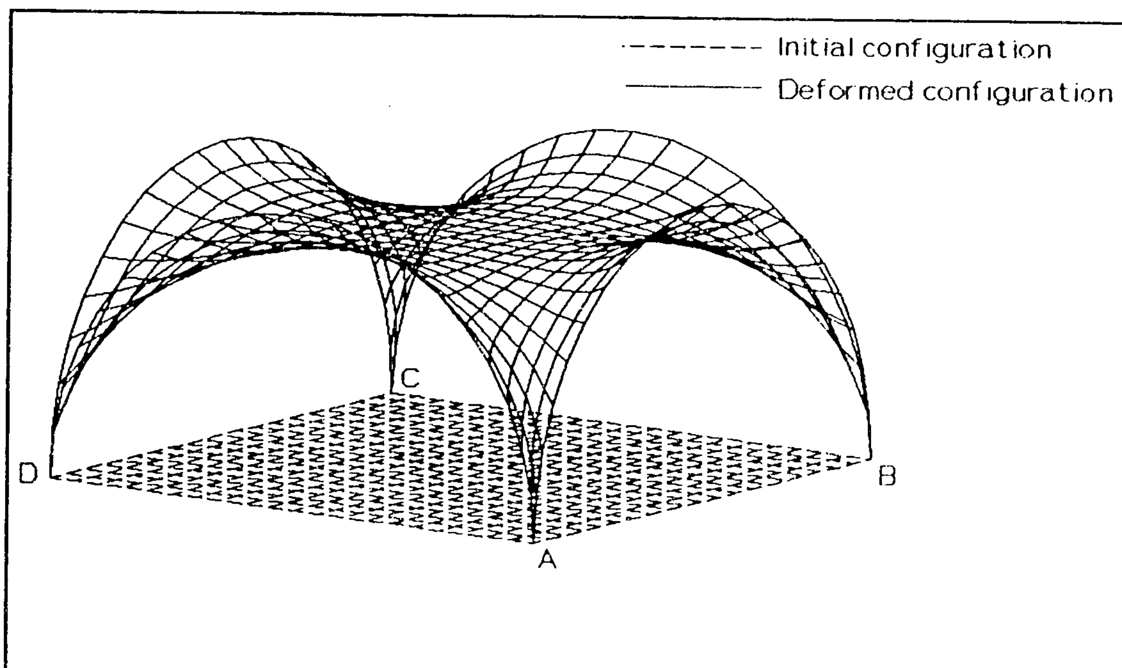

Figure 3.21 : 21x21 square mesh under deformation (a) or (c) of example 3.5

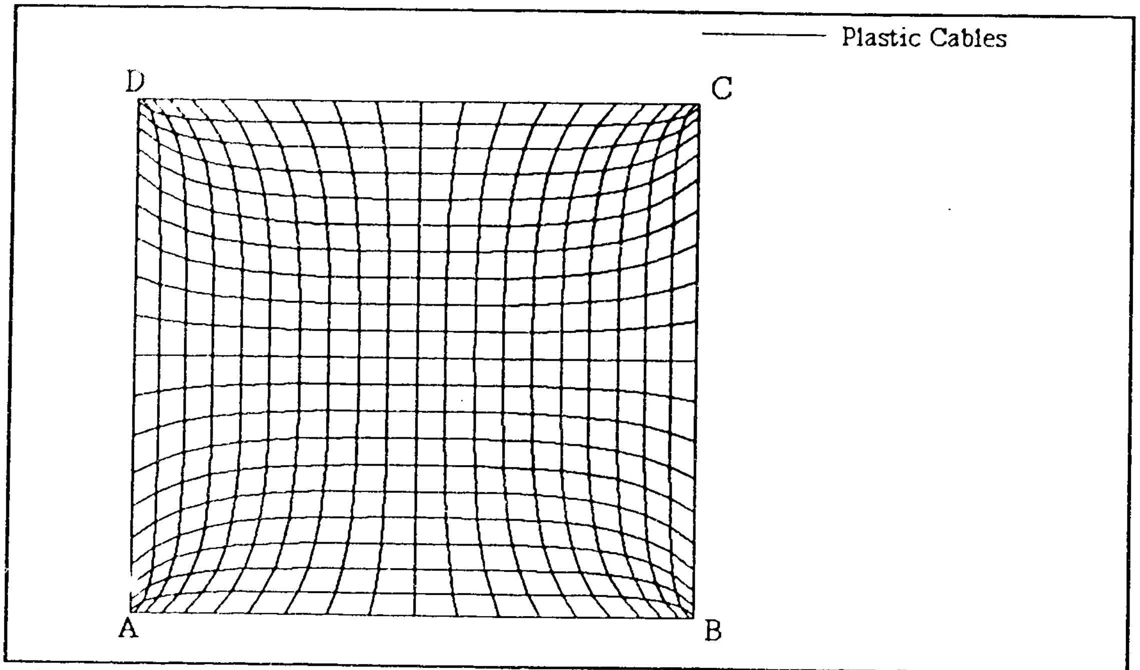

Figure 3.22 : Plan view of 21x21 square mesh under deformation (a) of example 3.5

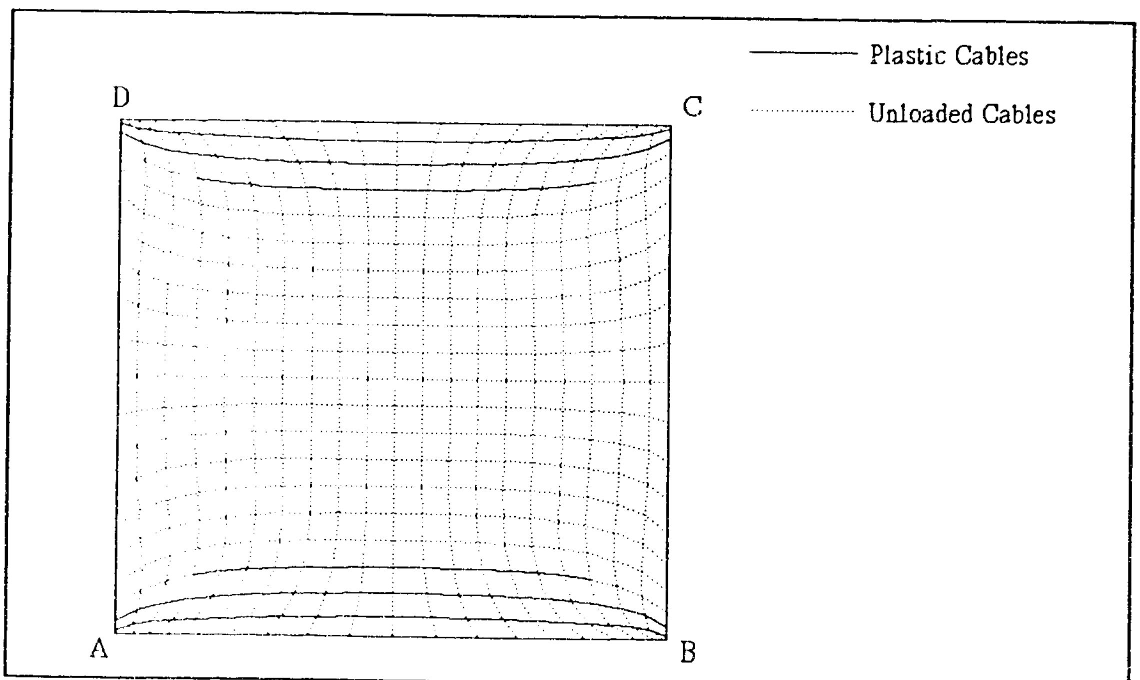

Figure 3.23 : Plan view of 21x21 square mesh under deformation (c) of example 3.5

incremental technique which is capable of handling problems involving elastic and elasto-plastic cables with tangent modulus not close to zero. We also provided some examples showing the path dependency of the solution and this fact that it is important how to express the external loading and prescribed displacements as a function of time.

In the end, we summarize some characteristics of the DR method and incremental technique

DR method:

- Nonlinear equations of equilibrium
- Iterative
- Elastic (linear and/or non-linear) cables only
- Initial configuration doesn't need to be in equilibrium
- Convergence problems may arise
- Convergence rate can be improved by changing the diagonal mass and damping matrices

Incremental technique:

- Linear equations of equilibrium
- Incremental
- Elastic and elasto-plastic materials with non-zero tangent modulus
- Initial configuration must be in equilibrium
- Must adjust time step to get the required accuracy

References

1. F. Otto, "Tensile Structures", 1982, MIT press

2. D.J. Steigmann, "Minimum-Energy Configurations of Elastic Cable Networks", Transactions of CSME, Vol. 14, No. 3, 1990, pp 47-50

3. P. Underwood, "Dynamic Relaxation", Computational methods for transient analysis, Edited by T. Bleytschko and T.J.R. Hughes, 1983, Elsevier Science Publishers B.V., pp 245-265

4. M. Papadrakakis, "A method for the automatic evaluation of the Dynamic Relaxation parameters", Computer methods in applied mechanics and engineering Vol. 25, 1981, pp 35-48

5. M.R. Barnes, "Non-linear numerical solution methods for static and dynamic analysis of tension structures", Appl. Mech. Rev. Vol. 44, No. 7, July 1991, pp 38-56

6. M.R. Barnes, "Application of Dynamic Relaxation to the topology design and analysis of cable, membrane and pneumatic structures", Proceedings of 2nd international , conference on the structures,Guildford Eng., Sept. 1975, pp 211-219

7. M.R. Barnes, "Dynamic Relaxation analysis of tension networks", Proceedings of international conference on tension roof structures, London, April 1974

8. H.A. Buchholdt and B.R. McMilan, "Iterative methods for the solution of pretensioned cable structures and pin-joined assemblies having significant geometrical displacements", Proc. 1971 IASS Pacific Symposium Part II on Tension structures and space frames, Tokyo and Kyoto pp 305-316, paper No. 3-7, 1972, Architectural

institute of Japan

9. D.J. Steigmann, "Elastic Stability", Class notes, Spring 1993, University of Alberta, Edmonton, Alberta, Canada

10. W.T. Koiter, "Minimum principles", Chapter IV, Progress in solid mechanics, Edited by I.N. Sneddon and R. Hill, North-Holland publishing company, Amsterdam 1960

11. M. Cannarozzi, "Stationary and extremum variational formulations for the elastostatics of cable networks", Meccanica, Vol. 20, 1985, pp 136-143

12. S.W. Key, C.M. Stone, R.D. Krieg, "Dynamic Relaxation applied to the quasi-static, large deformation, inelastic response of axisymmetric solids", Non-linear Finite Element Analysis in Structural mechanics, Edited by Wunderlich, Stein, Bathe, Springer-Verlag, Proceedings of the Europe U.S. Workshop, Ruhr-Universität Bochum, Germany, July 28,31 1980, pp 585-620