University of Alberta

MINIMAL HELLINGER DEFLATORS AND HARA FORWARD UTILITIES WITH APPLICATIONS: HEDGING WITH VARIABLE HORIZON

by

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Abstract

This thesis develops three major essays on the topic of horizon-dependence for optimal portfolio. The first essay contributes extensively to the newest concept of forward utilities. In this essay, we describe explicitly three classes of forward utilities–that we call HARA forward utilities–as well as their corresponding optimal portfolios. The stochastic tool behind our analysis lies in the concept of Minimal Hellinger Martingale densities (called MHM densities hereafter), introduced and developed recently by Choulli and his collaborators. The obtained results for HARA forward utilities by using MHM densities are derived under assumptions on the market model. The relaxation of some of these assumptions leads to introduce the new concept of Minimal Hellinger Deflator in order to characterize HARA forward utilities. The second essay addresses the problem of finding horizon-unbiased optimal portfolio from the perspective of contract theory. In fact, we consider an agent with classical exponential utility and describe–as explicit as possible–the payoff process for which there exists a horizon-unbiased optimal hedging portfolio. The last essay focuses on the financial problem that we call optimal sale problem. This problem consists of an agent who is investing in stocks and possesses a non-tradable asset that she aims to sell. The goal of this investor is to find the optimal portfolio–from her investment in stock market–and optimal time to liquidate all her assets (tradable or not).

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Chapter 1

Introduction

This chapter constitutes a general introduction of the thesis. Herein, I will introduce the reader to the birth of the area of mathematical finance in Section 1.A. Afterwards, I will address the topics of optimal portfolio and random horizon in Section 1.B. These two topics are the core financial topics that the thesis deals with. For the convenience of the reader, I will conclude this chapter by a summary of the thesis in Section 1.C.

1.A Historical Facts for Mathematical Finance and Modern Finance

Mathematical Finance and Modern Finance were born in 1900, when Louis Bachelier –a French mathematician– defended his PhD thesis at Sorbonne University (Paris). In his thesis, Bachelier simultaneously elaborated the first building blocks for Mathematical Finance and Modern Finance and for Continuous-time stochastic processes by discovering the Brownian motion five years before Albert Einstein. Unfortunately, this foundational and revolutionary work was forgotten for more than fifty years. It is in mid-fifties that Paul Samuleson who translated and highlighted the tremendous importance of Bachelier's work. Then, many foundational and important works were derived afterwards, such as the historical works of the Nobel Prize winners Black-

Scholes (see [7]), Robert Merton (see [58]), Harry Markowitz (see [55]),..., etcetera. For more details about the evolution and the birth of Mathematical Finance and Modern Finance, we refer the reader to [3] and [20].

1.B Optimal Portfolio and Random Horizon

The optimal portfolio problem is an old and important problem in Finance and Economics. The seminal and foundational works on optimal portfolios are the works of the two Nobel Prize winners Markowitz and Robert Merton. In Markowitz analysis (see [55]), the author addressed the optimal portfolio by using the variance as a measure for the risk, while in Merton's works, the author used the utility of the agent to deal with the risk and address the issue of optimal portfolio. In Mathematical Finance, both Markowitz' and Merton's works have been extended to the most general market models and in many directions due to the rich theories of martingale and convex analysis. For the topic on mean-variance portfolio optimization, we refer the readers to [10], [32], [71] (and the references therein), while for the case of utility maximization, see [46], [48], [45], [70] (and the references therein).

All these highly interesting works consider either infinite horizon or a finite horizon that is a constant real time fixed at the beginning. This assumption excludes the situation where the agent may suddenly liquidate all her assets due to the occurrence of a random event for instance. Another case of random horizon is the one where the agent looks for the optimal portfolio and optimal time to liquidate all her assets (tradable or not).

Besides the optimal portfolio problem *à la* Merton (or utility maximization), this thesis deals with the issue of *random exit time*, or *random time-horizon*. As a basic terminology in finance, a *time-horizon* is a time interval during which an investment lasts. When an investment is created or selected, the time-horizon could be fixed constant, such as the fixed income financial products of bonds, the insurance contracts for retirement plan and the financial derivatives of European options,..., etcetera. However, there are also plenty of economic and financial problems where the time-horizon needs (or even has) to be variable and/or random. When incorporated into the market model, a random time-horizon would have a pronounced effect on investment/portfolio selection, hedging and/or pricing problems. This fact was observed and conjectured since early twentieth century by Irving Fisher. In [30], Fisher discussed "General Income Risks" and wrote

> *" Even when there is no risk (humanly speaking) in the loan itself, the rate realized on it is affected by risk in other connections. The uncertainty of life itself casts a shadow on every business transaction into which time enters. Uncertainty of human life increases the rate of preference for present over future income for many people, although for those with loved dependents it may decrease impatience. Consequently, the rate of interest, even on the safest loans, will, in general, be raised by the existence of such life risks. The sailor or soldier who looks forward to a short or precarious existence will be less likely to make permanent investments, or, if he should make them, is less likely to pay a high price for them. Only a low price, that is, a high rate of interest, will induce him to invest for long ahead".*

This Fisher's conjecture was established by Yaari in [73] for the discrete-time market models. Around that time, there were an increase interest in investigating the effect of the Fisherian random time-horizon (a time-horizon that is related to the death of a life) by many economists. Among these, we cite Champernowne, Hakansson, Levhari, Mirman and Yaari (see [9], [33], [51], [73], and the references therein). For this type of random horizon, researchers appeal to life insurance and actuarial sciences to deal with the risk intrinsic to this horizon. While in economics and empirical studies researchers have been actively discussing this issue of variable horizon, the mathematical structure/foundation that drives this impact of the horizon on market models was left open—up to our knowledge—and only recently the literature starts growing with the works of Choulli–Schweizer and Larsen–Hang, see [14] and [50] for details. Furthermore, during the recent decade, this horizon-dependence problem has been addressed in a different perspective. One of these problems constitutes one of the main Leitmotif-goal of this thesis, to which we refer as "optimal sale problem", and can be described as follows. Consider an agent with utility function U, who possesses an asset (tradeable or not) that we model by a stochastic process, $(P_t)_{t\geq 0}$. The aim of this agent is to find the optimal portfolio and the optimal time to liquid all her assets. This can be translated, mathematically, to

$$\max_{\tau\in\mathcal{T},\ \theta\in\Theta} E\left[U(W^{\theta}_{\tau} - P_{\tau})\right].$$

Here $\mathcal{T}$ is the set of stopping times, Θ is the set of all admissible portfolios, and W^{θ} is the wealth process associated to the portfolio $\theta \in \Theta$. This problem was considered also by Henderson and Hobson in the real option context, see [36] for details. By considering this problem, the authors contributed to the birth of a new concept called *Forward Utilities*.

These forward utilities/performances were fathered and baptized (with their current name) by Musiela and Zariphopoulou in a series of papers starting with the multiperiod incomplete binomial model in [62]. Then, the concept was extended to diffusion models in [61]. For motivations behind this concept, those authors wrote in [60]:

> *" Firstly, fixing the trading horizon makes the valuation of claims of arbitrary maturities impossible...."*

> *" Secondly, the fact that, from one hand, the utility is exogenously chosen far ahead in the future, and on the other, it is used to make investment decisions for today, does not appear very natural. Besides, the optimal expected utility is generated backwards in time while the market moves in the opposite direction (forward), an apparently not very intuitive situation."*

For more economic motivations on forward utilities, we refer the reader to the numerous papers of Musiela and Zariphopoulou on this topic (see [59], [61], [62] and [63] for details). Therein, the authors also introduce applications of forward utilities on indifference pricing and optimal asset allocation. Around the birth time of these forward utilities, Choulli and Stricker introduced and constructed in [16] and [17] a class of optimal martingale measures that possesses the feature of being robust with respect to the variation of the horizon. The authors also linked these optimal martingale measures to optimal portfolios that are horizon-independent for a simple example of utilities. At the same time, and independently, Henderson and Hobson proposed the concept of horizon-unbiased as the solution to the optimal sale problem. These three independent groups of researcher contributed then to the birth of the forward utilities in a way or another. For more details about this fact, we refer the reader to [75].

1.C Summary of the Thesis

The thesis contains seven other chapters besides the current one, and is organized as follows. The next chapter (Chapter 2) will recall some stochastic tools (martingale theory and stochastic calculus) that will be very useful through out the thesis. I give a short review on some important results on Minimal Entropy Hellinger martingale densities in Section 2.D.1 and on Minimal Hellinger Martingale densities of order q in Section 2.D.2. My original contribution in Chapter 2 lies in extending this theory to include the case where one faces a change of measure. The main and original contributions of the thesis are detailed in chapters 3–8.

Chapters 3, 4 and 5 describe as explicit as possible the HARA (log-type, power-type and exponential-type) forward utilities under mild assumptions on the market model. Furthermore, their optimal portfolio is described via pointwise equations in $\mathbb{R}^d$. The analysis uses the powerful tool of semimartingale characteristics and the interesting concept of minimal Hellinger densities. For the needs in following chapters, we list some important definitions and theorems on stochastic processes and utility functions. In these chapters, the explicit parametrization or characterization is achieved for the most general semimartingale model that satisfies some mild assumptions. Illustrations of the main results on different practical market models (such as discrete-time markets, discrete market models, volatility models, and market driven by Lévy processes) are also detailed.

In Chapter 6, one of the assumptions imposed on the market model in previous chapters (mainly Chapters 3 and 4) is relaxed. This leads naturally to the birth of the concept of Minimal Hellinger Deflator. This concept extends in more general context the previous concept defined by Dr. Choulli and his collaborators in [16], [17] and [18]. We prove the existence of this minimal deflator and establish the duality between the obtained deflator and HARA forward utilities. This gives a new characterization for HARA forward utilities in more complex market models with much less technical assumptions (without any technical assumptions for many practical market models). The analysis used for this study involves stochastic optimization, convex analysis and martingale theory.

Chapter 7 addresses the horizon-unbiased hedging problem for exponential utilities. We find out the necessary and sufficient conditions on the payoff process such that the optimal portfolio–that hedges this payoff dynamically in time–exists and does not depend on the horizon. Meanwhile, this optimal portfolio is again described explicitly. Herein, we consider the usual exponential utility.

Chapter 8 focuses on the optimal sale problem where the agent with exponential utility is looking for the optimal portfolio and the optimal time to liquidate her assets (tradable or not). The optimal pair constituted by the optimal investment timing and the optimal portfolio is described as explicit as possible using the previous results on forward utilities for the case of general semimartingales. When the market model is Markovian, this optimal sale problem is investigated using the variational inequalities and Hamilton-Jacob-Bellman (HJB hereafter) equations. We proved that the value function is the unique viscosity of the HJB equation written in the form of variational inequalities.

Chapter 2

Elements from Stochastics and Martingale Theory

In this chapter, we will review some fundamental concepts and properties on stochastic processes and utility functions. We will start with the stochastic basis and some useful σ-fields. Then, we will put emphasis individually on four topics in the following sections: semimartingale and its characteristics in Section 1.A, local martingale and its Jacod decomposition in Section 1.B, utility functions in Section 1.C and Hellinger process of local martingale densities in 1.D. For more details on these topics, we refer the reader to [39], [66], [26] and [34].

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, called stochastic basis. Here, Ω is the sample space and $\mathbb{F}$ is the filtration, which is right continuous and complete (i.e. satisfies the usual conditions). For each $t \in [0, T]$, the σ-field (or σ-algebra), $\mathcal{F}_t$, represents the aggregate public information up to time t. P is the real-world probability measure and we further denote by $\mathbb{P}_a$ (respectively $\mathbb{P}_e$) the set of all probability measures that are absolutely continuous with respect to (respectively equivalent to) P. T represents a fixed horizon for investments.

Definition: A process X is called càdlàg, or RCLL, if all its paths are right-continuous and admit left-hand limits.

On the product space $\Omega \times [0,\ T]$, we define two σ-fields: The optional σ-field denoted by $\mathcal{O}$ and the predictable σ-field denoted by $\mathcal{P}$. Moreover, on the set $\Omega \times [0,\ T] \times \mathbb{R}^d$, we consider the extended σ-field $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ (resp. $\widetilde{\mathcal{O}} = \mathcal{O} \otimes \mathcal{B}(\mathbb{R}^d)$), where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ-field for $\mathbb{R}^d$.

Definition: The optional (respectively predictable) σ-field is the σ-field $\mathcal{O}$ (respectively $\mathcal{P}$) on $\Omega \times \mathbb{R}_+$ that is generated by all adapted and RCLL processes (respectively continuous processes).

In the sequel, a process X that is $\mathcal{O}$ (respectively $\mathcal{P}$)-measurable is called optional (respectively predictable) and frequently it will be denoted by $X \in \mathcal{O}$ (respectively $X \in \mathcal{P}$).

Definition: A random variable $\tau : \Omega \to [0, +\infty]$ is a stopping time if the event $\{\tau \leq t\} \in \mathcal{F}_t$, for every $t \in [0, +\infty[$.

The set of all stopping time, τ, such that $t \leq \tau \leq T$, P-a.e., will be denoted by $\mathcal{T}_{t,T}$. For simplicity, when $t = 0$, we write this set as $\mathcal{T}_T$.
For any process X and stopping time τ, we denote by X^τ the stopped process of X at τ, given by

$$X_t^\tau = \begin{cases} X_t, & \text{when } t \leq \tau; \\ X_\tau, & \text{otherwise.} \end{cases}$$

Definition: Let τ be a stopping time. Then, the σ-field $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \text{ for all } \ t \geq 0.\}$$

One way to interpret σ-field $\mathcal{F}_\tau$ is to compare it with the σ-field $\mathcal{F}_t$ at fixed time t: $\mathcal{F}_\tau$ represents the aggregate information up to τ.

Definition: We denote by $\mathcal{V}^+$ (respectively $\mathcal{V}$) the set of all real-valued processes X that are RCLL, adapted, with $X_0 = 0$, and whose each path $t \to X_t(\omega)$ is non-decreasing (respectively has a finite variation over the interval $[0, T]$).

For any $X \in \mathcal{V}$, we denote by $Var(X)$ the variation process of X, which is clearly non-decreasing. Therefore, its terminal variable, $Var(X)_T$, exists. We denote by $\mathcal{A}$ the set of all $A \in \mathcal{V}$ that have integrable variation, that is,

$$\mathcal{A} := \{A \in \mathcal{V} : \quad E(Var(A)_T) < +\infty.\}$$

Also, we put $\mathcal{A}^+ := \mathcal{A} \cap \mathcal{V}^+$, which represents the set of all $A \in \mathcal{V}^+$ that are integrable, that is,

$$E(A_T) < +\infty.$$

Moreover, we denote by $L(A)$ the set of all predictable processes, H, satisfying

$$\int_0^T |H_s| dVar(A)_s < +\infty, \quad P - a.s. \tag{2.1}$$

For any $H \in L(A)$, we denote the resulting integral of H with respect to A by $H \cdot A$, which is an element of $\mathcal{V}$. More details on this integration and the set $L(A)$ (especially for the case of multi-dimension) can be found in [39] (see page 206).

Throughout this thesis, if $\mathcal{C}$ is a class of processes, we denote by $\mathcal{C}_0$ the set of processes X with $X_0 = 0$ and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence of stopping times, $(T_n)_{n \geq 1}$, increasing stationarily to T (i.e., $P(T_n = T) \to 1$ as $n \to \infty$) and the stopped process X^{T_n} belongs to $\mathcal{C}$. This sequence of stopping times are called a localizing sequence for X. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.

Remark that by following this notation, we can write the set of locally integrable increasing processes by $\mathcal{A}^+_{loc}$ and the set of processes with locally integrable variation by $\mathcal{A}_{loc}$.

The following lemma is borrowed from [23] (see Lemma A1.1). It provides a technique to construct an almost surely convergent sequence via a convex combination.

Lemma 2.1: *Let (f_n) be a sequence of $[0, +\infty[$ valued measurable functions. There is a sequence $g_n \in conv(f_n, f_{n+1}, ...)$ such that (g_n) converges almost* *surely to a $[0, +\infty]$ valued function g, and the following properties hold:*

(1) If $conv(f_n, n \geq 1)$ is bounded in L^0, then g is finite almost surely,

(2) If there are $c > 0$ and $\delta > 0$ such that for all n

$$P(f_n > c) > \delta,$$

then $P(g > 0) > 0$.

Remark that the notation L^0 used in this lemma is the abbreviation of the space $L^0(\Omega, \mathcal{F}, P)$, which represents the set of all $\mathcal{F}$-measurable and real-valued random variables.

2.A Semimartingales and Its Characteristics

The use of semimartingale characteristics in mathematical finance can be traced back to Yuri Kabanov in [31] and [40]. The latest work on its applications can be found in [15], [11], [16], [18]...,etcetera.

We start this section with the most fundamental concepts in mathematical finance: martingale, submartingale and supermartingale.

Definition: A martingale (respectively, submartingale, supermartingale) is an adapted and RCLL process X, such that X_t is integrable for any $t \in [0, T]$, and that for $s \leq t$:

$$X_s = E(X_t|\mathcal{F}_s), \quad \text{(respectively, } X_s \leq E(X_t|\mathcal{F}_s), \quad X_s \geq E(X_t|\mathcal{F}_s)).$$

The set of martingales under the probability Q is denoted by $\mathcal{M}(Q)$ whereas the set of all local martingales is denoted by $\mathcal{M}_{loc}(Q)$. Especially when $Q = P$, we write them as $\mathcal{M}$ and $\mathcal{M}_{loc}$ for short. Furthermore, we denote by $\mathcal{M}^2$ the set of all square-integrable martingales, which is given by

$$\mathcal{M}^2 := \{X \in \mathcal{M} : \sup_{t \in [0,T]} E(X_t^2) < +\infty.\}$$

And, the set of locally square-integrable martingales is denoted by $\mathcal{M}^2_{loc}$. Furthermore, we denote by $L^2_{loc}(X)$ the set of all predictable processes H satisfying $H^2 \cdot [X, X] \in \mathcal{A}^+_{loc}$ such that the integration $H \cdot X$ is well defined and $H \cdot X \in \mathcal{M}^2_{loc}$. More details on this integration can be found in [39] (see page 204).

The following theorem is well-known and useful when manipulating martingales (respectively submartingales and supermartingales) on stopping times.

Theorem 2.1: **(Optional Sampling Theorem)** *Let $X = (X_t, \mathcal{F}_t)_{t\in[0,T]}$ be a martingale (respectively submartingale). Let τ_1 and τ_2 be two stopping times such that $\tau_1 \leq \tau_2$, P-a.s. Then, the following hold:*

(i) If X is a martingale, then

$$X_{\tau_1} = E(X_{\tau_2}|\mathcal{F}_{\tau_1}).$$

(ii) If X is a submartingale, then

$$X_{\tau_1} \leq E(X_{\tau_2}|\mathcal{F}_{\tau_1}).$$

The compensators of the increasing processes are frequently used in this thesis and defined in the following.

Definition: Let $X \in \mathcal{A}^+_{loc}$ (resp. $\mathcal{A}_{loc}$). Then the compensator, or dual predictable projection of X, denoted by X^p, is a predictable process that belongs to $\mathcal{A}^+_{loc}$ (resp. $\mathcal{A}_{loc}$) such that $X - X^p$ is a local martingale.

Remark that if the process X is predictable with finite variation, then $X^p = X$.

Semimartingale is the central concept in this section and even in the whole thesis. Its definition will be given in the following.

Definition: A semimartingale is a process X of the form $X = X_0 + M + A$ where X_0 is finite-valued and $\mathcal{F}_0$-measurable, M is a local martingale and A has finite variation (i.e. $A \in \mathcal{V}$).

Remark that the decomposition of semimartingale in above definition is not unique. However, a subclass of semimartingales, called special semimartingale, admits a unique decomposition. Its definition is given in the following.

Definition: A special semimartingale is a semimartingale X which admits a decomposition $X = X_0 + M + A$, where $A \in \mathcal{V} \cap \mathcal{P}$ and M is a local martingale.

For any d-dimensional semimartingale S, the random measure μ associated to its jumps is defined by

$$\mu(dt,\ dx) = \sum I_{\{\Delta S_s \neq 0\}} \delta_{(s,\ \Delta S_s)}(dt,\ dx),$$

where δ_a is the Dirac measure at point a. For any $\mathcal{B}[0,T] \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}$-measurable and non-negative functional W, $W = (W(t,x,\omega),\ x \in \mathbb{R}^d,\ t \in [0,T],\ \omega \in \Omega)$, we denote by $W \star \mu$ the following non-decreasing process given by

$$(W \star \mu)_t := \sum_{0<s\le t} I_{\{\Delta S_s \neq 0\}} W(s, \Delta S_s), \quad 0 \le t \le T. \tag{2.2}$$

Furthermore, for every $\widetilde{\mathcal{P}}$-measurable functional K satisfying $|K| \star \mu \in \mathcal{A}^+_{loc}$, there exists a random measure, called the *compensator of* μ and denoted by ν such that $|K| \star \nu \in \mathcal{A}^+_{loc}$ and $K \star \nu$ is the compensator of $K \star \mu$ (or equivalently, $K \star \mu - K \star \nu$ is a local martingale).

Also, for any measurable functional W on $\Omega \times [0,\ T] \times \mathbb{R}^d$, we associate the process

$$\widehat{W}_t(\omega) := \int_{\mathbb{R}^d} W(\omega, t, x)\nu(\omega, \{t\} \times dx),$$

if $\int_{\mathbb{R}^d} |W(\omega, t, x)|\nu(\omega, \{t\} \times dx) < +\infty$ and $\widehat{W}_t(\omega) := +\infty$, otherwise.

Consider the set, $\mathcal{G}^1_{loc}(\mu)$, given by

$$\mathcal{G}^1_{loc}(\mu) := \left\{ W \in \widetilde{\mathcal{P}} : \left[\sum_{0<s\le \cdot} (W_s(\Delta S_s) I_{\{\Delta S_s \neq 0\}} - \widehat{W}_s)^2 \right]^{1/2} \in \mathcal{A}^+_{loc} \right\}. \tag{2.3}$$

Then, for any functional $W \in \mathcal{G}^1_{loc}(\mu)$, the integral of W with respect to $(\mu - \nu)$, denoted $W\star(\mu-\nu)$, is well-defined and is purely discontinuous local martingale. These formulations lead to the following decomposition of semimartingale S, called "*the canonical representation*" (see Theorem 2.34, Section II.2 of [39]), namely,

$$S = S_0 + S^c + h(x) \star (\mu - \nu) + (x - h(x)) \star \mu + B, \tag{2.4}$$

where S^c is the continuous local martingale part of S and $h(x)$ is the truncation function which is usually given by $h(x) = xI_{\{|x|\leq 1\}} \in \mathcal{G}^1_{loc}(\mu)$. For the matrix C with entries $C^{ij} := \langle S^{c,i}, S^{c,j}\rangle$, the triple $(B,\ C,\ \nu)$ is called *predictable characteristics* of S. Furthermore, we can find a version of the characteristics triplet satisfying

$$B = b\cdot A,\quad C = c\cdot A \quad\text{and}\quad \nu(\omega,\ dt,\ dx) = dA_t(\omega)F_t(\omega,\ dx). \tag{2.5}$$

Here A is an increasing and predictable process , b and c are predictable processes (A is continuous if and only if S is quasi-left continuous), $F_t(\omega,\ dx)$ is a predictable kernel, $b_t(\omega)$ is a vector in $\mathbb{R}^d$, and $c_t(\omega)$ is a symmetric $d\times d$-matrix, for all $(\omega,\ t)\in\Omega\times[0,\ T]$. In the sequel we will often drop ω and t and write, for instance, $F(dx)$ as a shorthand for $F_t(\omega, dx)$.
The characteristics $B,\ C$, and ν, satisfy

- $F_t(\omega,\ \{0\}) = 0$;
- $\displaystyle\int (|x|^2\wedge 1)F_t(\omega,\ dx) \leq 1$;
- $\displaystyle\Delta B_t = \int h(x)\nu(\{t\}, dx)$;
- $c = 0 \quad \text{on} \quad \{\Delta A \neq 0\}$.

The measure $\nu(\{t\}, dx)$ will appear from time to time in this thesis, we write it in a compact way as $\nu_t(dx)$. As well, we denote the quantity

$$a_t := \nu_t(\mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1$$

which will be commonly used within semimartingale framework.

For any $\widetilde{\mathcal{O}}$-measurable functional, g, we define

$$M_\mu^P(g) := E(g \star \mu_T) = E\left(\int_0^T \int_{\mathbb{R}^d} g(s,x)\mu(ds,dx)\right).$$

Furthermore, we define $M_\mu^P(g \mid \widetilde{\mathcal{P}})$ to be the unique $\widetilde{\mathcal{P}}$-measurable functional, when it exists, such that for any bounded $W \in \widetilde{\mathcal{P}}$,

$$M_\mu^P(Wg) := E\Big(\int_0^T \int_{\mathbb{R}^d} W(s,x)g(s,x)\mu(ds,dx)\Big) = M_\mu^P\Big(W M_\mu^P(g \mid \widetilde{\mathcal{P}})\Big).$$

The integration with respect to semimartingales is another main topic in this section. We formulate it in two cases. First of all, for any locally bounded and predictable process H and semimartingale X, the following integration is well defined:

$$Y_t := \int_0^t H_u dX_u = H \cdot X_t,$$

which is also a semimartingale. More properties on this integration can be found in related reference books, particularly in [39] (see page 46 for more details). As an extension of above integration, the class of integrands can be enlarged to some non-locally bounded processes. Precisely, we say that a predictable process H is integrable with respect to a semimartingale X if there exists a decomposition of X, given by

$$X = X_0 + M + A, \quad M \in \mathcal{M}^2_{loc},\ A \in \mathcal{V}.$$

such that $H \in L^2_{loc}(M) \cap L(A)$. In this case we define the integral of H with respect to X as

$$H \cdot X := H \cdot M + H \cdot A,$$

and denote by $L(X)$ the set of all predictable processes that are X-integrable.

Finally in this section, we focus on an important class of semimartingales that are locally bounded. They have the following properties.

Proposition 2.1: *Suppose that S is locally bounded with the following decom-*

position

$$S = S_0 + S^c + z \star (\mu - \nu) + b \cdot A.$$

Let θ be an S-integrable process, and $\alpha \in (0, +\infty)$. Then the following assertions hold.

(i) The process

$$X^\theta := \theta \cdot S - \sum \theta^T \Delta S I_{\{|\theta^T \Delta S| > \alpha\}}, \tag{2.6}$$

is a locally bounded semi-martingale.

(ii) If we denote

$$\xi^\theta := \theta^T b - \int (\theta^T z) I_{\{|\theta^T z| > \alpha\}} F(dz),$$

then $|\xi^\theta| \cdot A \in \mathcal{A}^+_{loc}$.

(iii) The process

$$X^\theta - \xi^\theta \cdot A, \tag{2.7}$$

is a local martingale.

Proof. The proof of assertion (i) is classic, and can be found in [26] or [39]. Now, we will focus on proving simultaneously the remaining assertions.

Since S is locally bounded, then it is clear that $\theta I_{\{|\theta| \le n\}} \cdot S$ and $I_{\{|\theta| \le n\}} \cdot X^\theta$ are locally bounded semimartingales. Therefore, $\sum \theta^T \Delta S I_{\{|\theta^T \Delta S| > \alpha,\ |\theta| \le n\}}$ is a locally bounded process with finite variation, and its compensator is given by

$$V^{\theta,n} := (\theta^T z) I_{\{|\theta^T z| > \alpha,\ |\theta| \le n\}} \star \nu.$$

It is obvious that the two processes

$$\theta^T I_{\{|\theta| \le n\}} \cdot S - \theta^T b I_{\{|\theta| \le n\}} \cdot A, \quad \text{and} \quad \sum \theta^T \Delta S I_{\{|\theta^T \Delta S| > \alpha,\ |\theta| \le n\}} - V^{\theta,n}$$

are local martingales. Since $\widetilde{X}^\theta$ —the compensator of X^θ— exists and is a locally integrable process, then–due to $Var\left(I_{\{|\theta| \le n\}} \cdot \widetilde{X}^\theta\right) = |\xi^\theta| I_{\{|\theta| \le n\}} \cdot A$–we derive

$$Var\left(\widetilde{X}^{\theta}\right) = \lim_{n\to+\infty} I_{\{|\theta|\leq n\}}\cdot Var\left(\widetilde{X}^{\theta}\right) = \lim_{n\to+\infty} Var\left(I_{\{|\theta|\leq n\}}\cdot \widetilde{X}^{\theta}\right) = |\xi^{\theta}|\cdot A.$$

This proves both assertions (ii) and (iii). □

2.B Local Martingales and Jacod Decomposition

In this section, we will focus on local martingales and their Jacod decomposition. Additionally, the concept of σ-martingale is also introduced.

Definition: Let X be a RCLL semimartingale, and Q be a probability measure.

(i) X is called a σ-martingale under Q if there exists a bounded and positive predictable process ϕ such that $\phi\cdot X$ is a Q-local martingale. The set of all σ-martingales under Q will be denoted by $\mathcal{M}_{\sigma}(Q)$.

(ii) X is said to be locally integrable if there exists a sequence of stopping times $(T_n)_{n\geq 1}$, that increases stationarily to T, such that

$$E\left[\sup_{0\leq t\leq T_n}|X_t|\right] < +\infty, \quad \forall\, n\geq 1.$$

For the following representation theorem, we refer to [37] (Theorem 3.75, page 103) and to [39] (Lemma 4.24, page 185) and recent result in [15]. Consider a set $\mathcal{H}^1_{loc}(\mu)$, given by

$$\mathcal{H}^1_{loc}(\mu) := \Big\{ g:\Omega\times[0,+\infty)\times\mathbb{R}^d\to\mathbb{R},\, g\in\widetilde{\mathcal{O}},\;\; M^P_{\mu}(g\mid\widetilde{\mathcal{P}})=0, \\ \text{and } \sqrt{g^2\star\mu}\in\mathcal{A}^+_{loc}.\Big\}$$

Then, the Jacod decomposition of a local martingale is given by:

Theorem 2.2: (**Jacod Decomposition**) *Let $N\in\mathcal{M}_{0,loc}$. Then, there exist* *a predictable and S^c-integrable process β, $N' \in \mathcal{M}_{0,loc}$ with $[N', S] = 0$, functionals $f \in \widetilde{\mathcal{P}}$ and $g \in \mathcal{H}^1_{loc}(\mu)$ such that $\left(\sum_{s=0}^t f(s, \Delta S_s)^2 I_{\{\Delta S_s \neq 0\}}\right)^{1/2} \in \mathcal{A}^+_{loc}$ and*

$$N = \beta \cdot S^c + W \star (\mu - \nu) + g \star \mu + N', \quad W = f + \frac{\widehat{f}}{1-a} I_{\{a<1\}} \in \mathcal{G}^1_{loc}(\mu). \quad (2.8)$$

Here $\widehat{f}_t = \int f_t(x)\nu_t(dx)$ and f has a version such that $\{a = 1\} \subset \{\widehat{f} = 0\}$. Moreover

$$\Delta N_t = \Big(f_t(\Delta S_t) + g_t(\Delta S_t)\Big) I_{\{\Delta S_t \neq 0\}} - \frac{\widehat{f}_t}{1-a_t} I_{\{\Delta S_t = 0\}} + \Delta N'_t. \qquad (2.9)$$

And if $\Delta N > -1$, f can be selected to satisfy $f + 1 > 0$.

In the sequel, we shall call (β, f, g, N'), the Jacod components/parameters of N (under P, with respect to S).

2.C Utility Functions

The development of utility functions in economics and finance has its intrinsic reasons. Undoubtedly, all agents in market aim to maximize their wealth. However, their expectation over wealth can not be unlimited because of the tradeoff between return and risk. A rule of thumb can explain it: “ *Higher return is typically accompanied by higher risk”*. Therefore, the value of a potential investment is affected by investor’ preference or tolerance towards risks. For different investors, such tolerance is typically different. In mathematical finance, such preference is measured by a utility and is described by a mathematical function equipped with some basic properties: strictly increasing, strictly concave and twice differentiable.

Definition: A (deterministic) utility is a function $U(x)$, $x \in dom(U)$, that is strictly increasing, strictly concave and twice differentiable.

In particular, a family of utilities called HARA (hyperbolic absolute risk aversion) utilities is extensively developed and widely used in finance and economics (see [1], [57], [65] and the reference therein). Its definition is given as follows.

Definition: A utility $U(x)$ is a member of HARA utilities if its absolute risk aversion $A(x) := \frac{-U''(x)}{U'(x)}$ is a hyperbolic function.

It is easy to check that the power utility, x^p/p, exponential utility $-e^{-x}$ and logarithm utility $\log(x)$ are all HARA utilities.

Forward utility is one of the main topics in current thesis. These utilities have the basic properties of conventional deterministic utility functions. Furthermore, it is endowed with more features to deal with more complicated problems. In this section, these features will be exhibited one by one as we review its definition and some related concepts. First of all, we recall an extension of the notion of deterministic utility functions. This extension is called random field utility and appears in [75].

Definition: We call a random field utility, any $\mathcal{B}([0,T]) \otimes \mathcal{B}(dom(U)) \otimes \mathcal{F}$-measurable functional, $U(t,x,\omega)$, such that, for any fixed x, the process $U(t,x,\omega)$ is a RCLL adapted process, and for any fixed (t,ω) the function $x \mapsto U(t,x,\omega)$ is a utility.

Clearly from above definition, random field utilities are functionals of t and ω. This is actually a feature of random field utilities, which reflects the variation of investors' preference over time and scenarios. Another important concept associated with forward utilities is the set of admissible portfolios or admissible portfolios. The following definition is written in general form but will appear in different specific forms in following chapters.

Definition: For a random field utility, $U(t,x,\omega)$, any probability measure Q, any semimartingale X, and $x \in \mathbb{R}$ such that $U(t,x,\omega) < +\infty$ we denote by

$$\mathcal{A}_{adm}(x,X,Q) := \left\{\pi \in L(X) \quad \sup_{\tau \in \mathcal{T}_T} E^Q \left[U\Big(\tau, x + (\pi \cdot X)_\tau\Big)^-\right] < +\infty\right\}, \tag{2.10}$$

the set of admissible portfolios for the model (x, X, Q, U). Here $\mathcal{T}_T$ is the set of stopping time, τ, such that $\tau \leq T$. When $X = S$ and $Q = P$, we simply write $\mathcal{A}_{adm}(x)$ and, furthermore, when $x = 1$, we write it as $\mathcal{A}_{adm}$.

The forward utilities are built up on the basis of random field utilities. But it has nicer properties, which make them powerful tools for dealing with financial models under random horizon.

Definition: Consider a RCLL semimartingale, X, and a probability measure, Q. Then, we call a forward (dynamic) utility for (X, Q), any random field utility, $U(t, \omega, x)$, fulfilling the following self-generating property:
a) The function $U(0, x)$ is a deterministic utility function.
b) There exists an admissible portfolio π^* (i.e. $\pi^* \in \mathcal{A}_{adm}(x, X, Q)$) such that

$$U\Big(s, x + (\pi^* \cdot X)_s\Big) = E^Q\left[U\Big(t, x + (\pi^* \cdot X)_t\Big)|\mathcal{F}_s\right], \quad \forall\, T \geq t \geq s \geq 0.$$

c) For any admissible portfolio π, for any $T \geq t \geq s$, we have

$$U\Big(s, x + (\pi \cdot X)_s\Big) \geq E^Q\left[U\Big(t, x + (\pi \cdot X)_t\Big)|\mathcal{F}_s\right].$$

When $X = S$ and $Q = P$, we simply call U a forward dynamic utility.

Definition: For any $\pi \in L(S)$ and any $x > 0$ satisfying

$$x + \pi \cdot S > 0, \quad \text{and} \quad x + (\pi \cdot S)_- > 0, \tag{2.11}$$

we associate it with the *portfolio rate* that we denote by $\widetilde{\pi}_x$, and is given by

$$\widetilde{\pi}_x := \Big(x + (\pi \cdot S)_-\Big)^{-1} \pi. \tag{2.12}$$

Lemma 2.2: *For any $\pi \in L(S)$ and any initial capital $x > 0$ satisfying (2.11), its portfolio rate $\widetilde{\pi}_x$ satisfies:*
(i) $\widetilde{\pi}_x$ is S-integrable and $\mathcal{E}(\widetilde{\pi}_x \cdot S) > 0$.

(ii) There is one-to-one correspondence between the portfolio π and its portfolio rate $\widetilde{\pi}_x$ via (2.12) and

$$\pi = x\mathcal{E}_-(\widetilde{\pi}_x \cdot S)\widetilde{\pi}_x. \tag{2.13}$$

Proof. The proof of this lemma is obvious and will be skipped. □

For any portfolio $\pi \in \mathcal{A}_{adm}(x)$, the associated wealth process is $X^\pi = x + \pi \cdot S$ and the portfolio rate is $\theta_t = \pi_t / X^\pi_{t-}$, $0 \le t \le T$ such that the wealth process can also be rewritten as $X^\theta := x\mathcal{E}(\theta \cdot S)$. In the same spirit as $\mathcal{A}_{adm}(x, X, Q)$, $\mathcal{A}_{adm}(x, S, P)$ and $\mathcal{A}_{adm}$, we denote the set of portfolio rates by $\Theta(x, X, Q)$, $\Theta(x, S, P)$ and Θ, respectively. Remark that in many places of this thesis, we shall use portfolio rate θ instead of π for convenience.

In the forthcoming analysis, both the stopping rule and the change of probability measures play crucial roles. Thus, the following lemma states how robust of the forward property under these two operations.

Lemma 2.3: *Let $U := U(t, \omega, x)$ be a random field utility and S be a semimartingale. Then the following hold.*
(i) If U is a forward utility for (S, P), then for any stopping time $\tau \in \mathcal{T}_T$, the functional

$$\overline{U}(t, \omega, x) := U(t \wedge \tau(\omega), \omega, x), \tag{2.14}$$

is a forward dynamic utility for (S^τ, P).
(ii) Consider a probability measure Q that is absolutely continuous with respect to P with the density process denoted by Z. Then, the random field utility

$$U^Q(t, \omega, x) := U(t, \omega, x) Z_t(\omega), \tag{2.15}$$

is a forward dynamic utility for (S, P) if and only if U is forward dynamic utility for (S, Q).

Proof. The proof of this lemma is straightforward and will be omitted.

□

2.D Hellinger Process of Local Martingale Density

The concept of Hellinger process was introduced for the first time by Kabanov, Y. M., Liptser, R. S. and Shiryaev, A. N. in [41] (See also in the following papers, [42], [43]). Afterwards, it is further developed by Jacod, J., Choulli, T. and Stricker, C. in [39], [16], [17] and [18].

In this section, the classical concept and properties on Minimal entropy-Hellinger Martingale density (called MEH hereafter) and Minimal Hellinger Martingale density of order q (called MHM of order q hereafter) will be reviewed quickly in the beginning. Afterwards, we will introduce the generalized version of these concepts and properties under change of measure, which is the main purpose of this section. This section prepares the ground for Chapters 3, 4 and 5 when we will focus on characterizing forward utilities.

2.D.1 Minimal Entropy-Hellinger Martingale Densities

When focusing on the exponential utility, we consider the σ-martingale measures with finite entropy. The set of these measures is given by

$$\mathcal{M}_f^e(S) = \left\{ Q \in \mathbb{P}_e \mid \ S \in \mathcal{M}_\sigma(Q), \text{ and } \ E\left[\frac{dQ}{dP}\log\Big(\frac{dQ}{dP}\Big)\right] < +\infty \right\}. \quad (2.16)$$

Very frequently, throughout this section and Chapter 5, we will work with densities instead of probabilities. For this, we will use the following set

$$\mathcal{Z}_{loc}^e(S) := \{Z \in \mathcal{M}_{loc}(P)\,|\, Z > 0, Z\log(Z) \text{ is locally integrable}, ZS \in \mathcal{M}_\sigma(P)\}. \quad (2.17)$$

The following function will be used from time to time,

$$f_1(x) := \begin{cases} (x+1)\log(x+1) - x, & \text{if } x > -1; \\ +\infty, & \text{otherwise.} \end{cases} \quad (2.18)$$

Throughout this section and Chapter 5, the main assumption on S is

$$\int_{\{|x|>1\}} |x| e^{\lambda^T x} F(dx) < +\infty, \quad P \otimes A - a.s., \qquad \text{for all} \quad \lambda \in \mathbb{R}^d. \quad (2.19)$$

The next proposition will provide a necessary and sufficient characterization on σ-martingale density, which is expressed by Jacod parameters.

Proposition 2.2: *Let $Z = \mathcal{E}(N)$ be a positive local martingale and (β, f, g, N') be the Jacod components of N. Then Z is a σ-martingale density for (S, P) if and only if the following hold:*
(i) We have

$$\int |x(1 + f(x)) - h(x)| F(dx) < +\infty, \quad P \otimes A - a.e.$$

(ii) and

$$b \cdot A + c\beta \cdot A + (x - h(x) + xf(x)) \star \nu = 0. \quad (2.20)$$

Furthermore, if Z is a σ-martingale density for (S, P), then the following holds:

$$\int x(1 + f(x)) F(dx) \Delta A = 0, \quad P - a.s. \quad (2.21)$$

Proof. Thanks to Ito's formula, we deduce that ZS is a σ-martingale if and only if $S + [S, N]$ is a σ-martingale. The last statement is equivalent to say that there exists a bounded and predictable positive process ϕ such that

$$\phi \cdot (S + [S, N]) \text{ is a local martingale.} \quad (2.22)$$

Due to Theorem 2.2 (precisely the representation of ΔN given by (2.9)) and the representation of S given by (2.4), we derive that

$$S + [S, N] = S_0 + S^c + h(x) \star (\mu - \nu) + c\beta \cdot A + b \cdot A + [x - h(x) + x(f(x) + g(x))] \star \mu$$

Therefore, (2.22) holds if and only if the following two conditions hold:

$$\phi[x - h(x) + x(f(x) + g(x))] \star \mu \text{ is locally integrable} \tag{2.23}$$

$$\phi b \cdot A + \phi c\beta \cdot A + \phi(x - h(x) + xf(x)) \star \nu = 0. \tag{2.24}$$

Since ϕ is predictable, positive and bounded, it is easy to deduce that (2.24) is equivalent to (ii). While, (2.23) is equivalent to

$$\int_0^T \phi_t \int [x - h(x) + xf(x)] F_t(dx) dA_t < +\infty, \quad P - a.s.$$

which holds if and only if (i) is true.
Furthermore, by taking jumps on both sides of (2.24) and using $\Delta A b = \int x F(dx) \Delta A$, $\Delta A c = 0$ (see the properties of predictable characteristics of S in Section 2.A for details), we have (2.21) immediately. This ends the proof of this proposition. □

The following definitions on entropy-Hellinger process can be found in [16] and [17], to which we refer the readers for more details about the entropy-Hellinger process of a probability measure (which is also called Leibler-Kullback process).

Definition: (i) Let $N \in \mathcal{M}_{0,\ loc}(P)$ such that $1 + \Delta N \geq 0$. Then, if the non-decreasing adapted process

$$V_t^{(E)}(N) := \frac{1}{2}\langle N^c \rangle_t + \sum_{0<s\leq t} \Big[(1 + \Delta N_s)\log(1 + \Delta N_s) - \Delta N_s\Big] \tag{2.25}$$

is locally integrable (i.e. $V^E(N) \in \mathcal{A}^+_{loc}(P)$), then its compensator (with respect to the probability P) is called the entropy-Hellinger process of N, and is denoted by $h^E(N, P)$.
(ii) Let $Q \in \mathbb{P}_a$ with density $Z = \mathcal{E}(N)$. Then, we define the entropy-Hellinger process of Q with respect to P by

$$h_t^E(Q, P) := h_t^E(Z, P) := h_t^E(N, P), \quad 0 \leq t \leq T.$$

The expression of entropy-Hellinger process for a positive σ-martingale and its jump will be shown explicitly in the next lemma.

Lemma 2.4: *Consider a positive σ-martingale density $Z = \mathcal{E}(N)$ satisfying*

$$\begin{aligned} N &= \lambda \cdot S^c + W \star (\mu - \nu), \\ W_t(x) &= (e^{\lambda_t^T x} - 1)\Big(1 - a_t + \int e^{\lambda_t^T x} \nu(\{t\}, dx)\Big)^{-1}. \end{aligned} \tag{2.26}$$

Then, we have

$$h^E(Z,P) = \frac{\lambda^T c \lambda}{2} \cdot A + \frac{1}{\gamma} f_1\left(e^{\lambda^T x} - 1\right) \star \nu - \sum \left(\log(\gamma) - 1 + \frac{1}{\gamma}\right) \tag{2.27}$$

$$= \frac{\lambda^T c \lambda}{2} \cdot A + f_1(\frac{1}{\gamma} e^{\lambda^T x} - 1) \star \nu + \sum (1-a) f_1(\gamma^{-1} - 1) \tag{2.28}$$

and

$$\Delta h^E(Z,P) = -\log(\gamma), \tag{2.29}$$

where $\gamma_t = 1 - a_t + \int e^{\lambda_t^T x} \nu(\{t\}, dx)$..

Proof. Notice that $h^E(Z,P)$ is the compensator of $V^E(N)$, where

$$V^E(N) = \frac{1}{2}\langle N^c \rangle + \sum \left[(1+\Delta N)\log(1+\Delta N) - \Delta N\right]. \tag{2.30}$$

Then, from (2.26), we derive

$$1 + \Delta N_t = \frac{e^{\lambda_t^T \Delta S_t}}{\gamma_t} I_{\{\Delta S_t \neq 0\}} + \frac{1}{\gamma_t} I_{\{\Delta S_t = 0\}}.$$

After simplification, this leads to

$$\sum \left[(1+\Delta N)\log(1+\Delta N) - \Delta N\right] = \left[\tfrac{e^{\lambda^T x}}{\gamma} \log\left(\tfrac{e^{\lambda^T x}}{\gamma}\right) - \tfrac{e^{\lambda^T x}}{\gamma} + 1\right] \star \mu$$

$$+ \sum \left[\tfrac{1}{\gamma} \log\left(\tfrac{1}{\gamma}\right) - \tfrac{1}{\gamma} + 1\right] I_{\{\Delta S = 0\}}.$$

Then, by plugging this equation in (2.30) and compensating, we obtain

$$h^E(Z,P) = \frac{1}{2}\lambda^T c\lambda \cdot A + \sum \left(\frac{1}{\gamma}\log\left(\frac{1}{\gamma}\right) - \frac{1}{\gamma} + 1\right)(1-a) + \Sigma_1, \qquad (2.31)$$

where

$$\Sigma_1 = \left[\tfrac{e^{\lambda^T x}}{\gamma}\log\left(\tfrac{e^{\lambda^T x}}{\gamma}\right) - \tfrac{e^{\lambda^T x}}{\gamma} + 1\right] \star \nu = \left(\tfrac{e^{\lambda^T x}}{\gamma}\log\left(\tfrac{1}{\gamma}\right) + \tfrac{\lambda^T x}{\gamma}e^{\lambda^T x} - \tfrac{e^{\lambda^T x}}{\gamma} + 1\right) \star \nu$$

$$= \sum \left[\tfrac{1}{\gamma}\log\left(\tfrac{1}{\gamma}\right)\int e^{\lambda^T x}F(dx)\Delta A + \left(1 - \tfrac{1}{\gamma}\right)a\right] + \tfrac{1}{\gamma}\left(\lambda^T x e^{\lambda^T x} - e^{\lambda^T x} + 1\right) \star \nu.$$

Hence, after simplification, (2.27) follows.
By taking the jumps in both sides of (2.27), we get

$$\begin{aligned}\Delta h^E(Z,P) &= \frac{1}{\gamma}\left(a - \int e^{\lambda^T x}F(dx)\Delta A\right) - \frac{\gamma\log(\gamma) - \gamma + 1}{\gamma} \\ &= \frac{-\gamma+1}{\gamma} - \frac{\gamma\log(\gamma) - \gamma + 1}{\gamma} = -\log(\gamma).\end{aligned}$$

Note that in this equality, we used that $\int x e^{\lambda^T x}\nu(\{t\}, dx) = 0$. This follows from the fact that Z is a σ-martingale density (see (2.21)). This completes the proof. $\square$

Theorem 2.3: *Suppose that $\mathcal{Z}^e_{loc}(S) \neq \emptyset$ and that (2.19) holds. If $\widetilde{Z} \in \mathcal{Z}^e_{loc}(S)$ is the MEH σ-martingale density, then, there exists $\widetilde{H} \in L(S)$ such that*

$$\log(\widetilde{Z}) = \widetilde{H} \cdot S + h^E(\widetilde{Z}, P). \qquad (2.32)$$

Proof. Notice that the assumptions of Theorem 3.3 in [17] are fulfilled. Hence, a direct application of this theorem implies

$$\widetilde{Z} = \mathcal{E}(\widetilde{N}), \qquad \widetilde{N} := \widetilde{\beta} \cdot S^c + \widetilde{W} \star (\mu - \nu),$$

$$\widetilde{W}_t(x) := (\widetilde{\gamma}_t)^{-1}\left(e^{\widetilde{\beta}_t^T x} - 1\right), \qquad \widetilde{\gamma}_t := 1 - a_t + \int e^{\widetilde{\beta}_t^T x}\nu(\{t\}, dx).$$

Thus,

$$\log(\widetilde{Z}) = \widetilde{N} - \tfrac{1}{2}\langle\widetilde{N}\rangle + \sum[\log(1+\Delta\widetilde{N}) - \Delta\widetilde{N}]$$

$$= \widetilde{\beta}\cdot S^c + \widetilde{W}\star(\mu-\nu) - \tfrac{1}{2}\widetilde{\beta}^T c\widetilde{\beta}\cdot A + \sum[\log(\tfrac{e^{\widetilde{\beta}^T x}}{\widetilde{\gamma}}) - \tfrac{e^{\widetilde{\beta}^T x}}{\widetilde{\gamma}} + 1]I_{\{\Delta S\neq 0\}}$$

$$+ \sum[\log(\tfrac{1}{\widetilde{\gamma}}) - \tfrac{1}{\widetilde{\gamma}} + 1]I_{\{\Delta S=0\}}$$

$$= \widetilde{\beta}\cdot S^c + \widetilde{W}\star(\mu-\nu) - \tfrac{1}{2}\widetilde{\beta}^T c\widetilde{\beta}\cdot A + \sum[\tfrac{-\widetilde{\gamma}\log(\widetilde{\gamma})+\widetilde{\gamma}-1}{\widetilde{\gamma}}] + \tfrac{\widetilde{\gamma}\widetilde{\beta}^T x - e^{\widetilde{\beta}^T x}+1}{\widetilde{\gamma}}\star\mu.$$

Remark that

$$\frac{1}{\widetilde{\gamma}}(\widetilde{\gamma}\widetilde{\beta}^T x - e^{\widetilde{\beta}^T x}+1)\star\mu = \widetilde{\beta}^T(x-h(x))\star\mu + \frac{1}{\widetilde{\gamma}}(\widetilde{\gamma}\widetilde{\beta}^T h(x) - e^{\widetilde{\beta}^T x}+1)\star(\mu-\nu)+$$

$$+\widetilde{\gamma}^{-1}(\widetilde{\gamma}\widetilde{\beta}^T h(x) - e^{\widetilde{\beta}^T x}+1)\star\nu,$$

since the functional $\widetilde{\gamma}^{-1}(\widetilde{\gamma}\widetilde{\beta}^T h(x) - e^{\widetilde{\beta}^T x}+1)$ is $(\mu-\nu)$-integrable which is due to the $(\mu-\nu)-$integrability of $\widetilde{\gamma}^{-1}(e^{\widetilde{\beta}^T x}-1) = W(x)$ and the boundedness of $h(x)$. Therefore, we get

$$\log(\widetilde{Z}) = \widetilde{\beta}\cdot S^c + \widetilde{\beta}^T h(x)\star(\mu-\nu) + \widetilde{\beta}^T(x-h(x))\star\mu +$$

$$+\widetilde{\gamma}^{-1}(\widetilde{\gamma}\widetilde{\beta}^T h(x) - e^{\widetilde{\beta}^T x}+1)\star\nu - \tfrac{1}{2}\widetilde{\beta}^T c\widetilde{\beta}\cdot A + \sum\widetilde{\gamma}^{-1}(-\widetilde{\gamma}\log(\widetilde{\gamma})+\widetilde{\gamma}-1).$$

Equivalently, we deduce that

$$\log(\widetilde{Z}) = \widetilde{\beta}\cdot S + \frac{1}{2}\widetilde{\beta}^T c\widetilde{\beta}\cdot A + \frac{\widetilde{\beta}^T x e^{\widetilde{\beta}^T x} - e^{\widetilde{\beta}^T x}+1}{\widetilde{\gamma}}\star\nu + \sum\widetilde{\gamma}^{-1}(-\widetilde{\gamma}\log(\widetilde{\gamma})+\widetilde{\gamma}-1), \tag{2.33}$$

due to

$$\widetilde{\beta}\cdot S = \widetilde{\beta}\cdot S^c + \widetilde{\beta}^T b\cdot A + \widetilde{\beta}^T h(x)\star(\mu-\nu) + \widetilde{\beta}^T(x-h(x))\star\mu.$$

Therefore, a direct application of Lemma 2.4 for $\lambda = \widetilde{\beta}$, (2.32) follows immediately. This ends the proof of the theorem. □

Next, we will give the variation of this entropy-Hellinger concept towards the change of probability measures.

Definition: (i) Let Q be a probability measure and Y be a Q-local martingale such that $1+\Delta Y \geq 0$. Then, if the RCLL nondecreasing process

$$V^E(Y) = \frac{1}{2}\langle Y^c\rangle + \sum \left[(1+\Delta Y)\log(1+\Delta Y) - \Delta Y\right], \tag{2.34}$$

is Q-locally integrable (i.e. $V^E(Y) \in \mathcal{A}^+_{loc}(Q)$), then its Q-compensator is called the entropy-Hellinger process of Y (or equivalently of $\mathcal{E}(Y)$) with respect to Q, and is denoted by $h^E(Y,Q)$ (respectively $h^E(\mathcal{E}(Y),Q)$).

(ii) Let $N \in \mathcal{M}_{0,\ loc}(P)$ such that $1+\Delta N > 0$ and Y is a semimartingale such that $Y\mathcal{E}(N)$ is a P-local martingale and $1+\Delta Y \geq 0$. Then, if the process

$$\frac{1}{2}\langle Y^c\rangle + \sum (1+\Delta N)\Big[(1+\Delta Y)\log(1+\Delta Y) - \Delta Y + 1\Big], \tag{2.35}$$

is P-locally integrable, then its P-compensator is called the entropy-Hellinger process of $\mathcal{E}(Y)$ with respect to $\mathcal{E}(N)$, and is denoted by $h^E\left(\mathcal{E}(Y),\mathcal{E}(N)\right)$.

Remark that the first definition above in (i) is a natural extension in probability as well as in mathematical finance areas, due to the popular and useful technique of change of probability measures. The second definition in (ii), which we will use throughout the thesis, extends (i) to the case when the uniform integrability of the nonnegative local martingale $\mathcal{E}(N)$ may not hold. The relationship between the two definitions is obvious. Indeed, let $(T_n)_{n\geq 1}$ be a sequence of of stopping times that increases stationarily to T such that $\mathcal{E}(N)^{T_n}$ is a true martingale. Then, by putting $Q_n := \mathcal{E}_{T_n}(N)\cdot P$, we obtain

$$h^E_{t\wedge T_n}(\mathcal{E}(Y),\mathcal{E}(N)) = h^E_t(\mathcal{E}(Y^{T_n}), Q_n), \qquad 0 \leq t \leq T.$$

What we actually need in current thesis is the MEH local martingale density under change of probability measure. In the remaining part of this section, we focus on describing the MEH σ-martingale density when we change probability. This case can be derived easily from the more general case where one works with respect to a positive local martingale density, Z, that may not be uniformly integrable. First, we generalize the characterization of the MEH σ-martingale density for the case when S may not be bounded nor quasi-left continuous. For the case of bounded and quasi-left continuous S, a more elaborate result is given in [16].
In what follows, we denote by Z a positive local martingale given by

$$Z := \mathcal{E}(N),\ N := \beta\cdot S^c + W\star(\mu-\nu) + g\star\mu + \overline{N},\ W_t(x) := f_t(x) + \frac{\widehat{f}_t}{1-a_t} I_{\{a_t<1\}}, \tag{2.36}$$

where $\left(\beta, f, g, \overline{N}\right)$ are the Jacod components of N. Here, we define:

$$\mathcal{Z}^e_{loc}(S,Z) := \{\overline{Z} \mid \overline{Z} > 0,\ \ \overline{Z}Z \in \mathcal{Z}^e_{loc}(S)\}, \tag{2.37}$$

where $\mathcal{Z}^e_{loc}(S)$ is given by (2.17).

Theorem 2.4: *Consider Z defined in (2.36) and suppose that*

$$\mathcal{Z}^e_{loc}(S,Z) \neq \emptyset, \quad \textit{and} \quad \int_{\{|x|>1\}} e^{\lambda^T x}(1+f(x))F(dx) < +\infty, \quad \forall\ \lambda \in I\!R^d.$$

Then, the minimization problem

$$\min_{\overline{Z}\in\mathcal{Z}^e_{loc}(S,Z)} h^E(\overline{Z}, Z), \tag{2.38}$$

admits a solution $\widetilde{Z} = \mathcal{E}(\widetilde{N})$ given by

$$\widetilde{N} = \widetilde{\beta}\cdot S^{c,Z} + \widetilde{W}\star(\mu-\nu^Z),\ \widetilde{W}_t(x) = \frac{e^{\widetilde{\beta}_t^T x} - 1}{1 - a_t^Z + \int e^{\widetilde{\beta}_t^T y}\nu^Z(\{t\}, dy)},$$

where $\widetilde{\beta}$ *is the root of*

$$0 = b^Z + c\lambda + \int (e^{\lambda^T x} x - h(x)) F^Z(dx). \tag{2.39}$$

Here $S^{c,Z}$, b^Z, a^Z, ν^Z *and* F^Z *are given by*

$$S^{c,Z} := S^c - c\beta \cdot A, \quad b^Z := b + c\beta - \int f(x)h(x)F(dx), \quad a^Z_t := \nu^Z(\{t\}, I\!R^d \setminus \{0\})$$

$$\text{and} \quad \nu^Z(dt, dx) := F^Z_t(dx)dA_t, \qquad F^Z_t(dx) := (1 + f_t(x))F_t(dx).$$

Proof. Consider a sequence of stopping times, $(T_n)_{n\geq 1}$, stationarily increasing to T (i.e., $P(T_n = T) \to 1$ as $n \to \infty$) such that Z^{T_n} is a true martingale, for a fixed but arbitrary n we denote $Q := Z_{T_n} \cdot P$. Remark that all the equations in the theorem are robust with stopping. Then due to Lemma 2.5, it is enough to prove that the theorem is valid on $[\![0, T_n]\!]$. Then, we obtain that $\nu^Q(dt, dx) = (1 + f_t(x))I_{\{t \leq T_n\}}\nu(dt, dx)$, $\mathcal{Z}^e_{loc}(S, Q) \neq \emptyset$ and that $\int_{\{|x|>1\}} e^{\lambda^T x} F^Q(dx) < +\infty$ for $\lambda \in \mathbb{R}^d$.

Therefore, the assumptions of Theorem 3.3 in [17] are fulfilled. Hence a direct application of this theorem for S^{T_n} and under the measure $Q = Z_{T_n} \cdot P$, we deduce that the problem defined in (2.38) admits a solution $\widetilde{Z}^Q = \mathcal{E}\left(\widetilde{N}^Q\right)$, where $\widetilde{N}^Q$ is given, on $[\![0, T_n]\!]$, by

$$\widetilde{N}^Q = \widetilde{\beta} \cdot S^{c,Q} + \widetilde{W} \star (\mu - \nu^Q), \; \widetilde{W}_t(x) = \frac{e^{\widetilde{\beta}_t^T x} - 1}{1 - a^Q_t + \int e^{\widetilde{\beta}_t^T y} \nu^Q(\{t\}, dy)}.$$

Herein $S^{c,Q}$ is the continuous local martingale part of S under Q and ν^Q is the Q-compensator measure of μ, and $a^Q_t = \nu^Q\left(\{t\}, \mathbb{R}^d \setminus \{0\}\right)$. Moreover, $\widetilde{\beta}$ is given by the equation

$$\begin{aligned} 0 &= b^Q + c\lambda + \int \left(e^{\lambda^T x} x - h(x)\right) F^Q(dx) \\ &= \left[b^Z + c\lambda + \textstyle\int (e^{\lambda^T x} x - h(x)) F^Z(dx)\right] I_{[\![0,T_n]\!]}. \end{aligned} \tag{2.40}$$

It is then clear that $\widetilde{N}^Q$ coincides with $\widetilde{N}$ of the theorem and that the equation (2.40) is exactly the equation (2.39) of the theorem. This ends the proof of theorem. □

Theorem 2.5: *Let Z be a positive local martingale and $\widetilde{Z} \in \mathcal{Z}^e_{loc}(S, Z)$. If the assumptions of Theorem 2.4 are fulfilled, and $\widetilde{Z}$ is the MEH local martingale density with respect to Z, then*

$$\log(\widetilde{Z}) = \widetilde{\beta} \cdot S + h^E(\widetilde{Z}, Z). \tag{2.41}$$

Proof. The proof of this theorem follows the same arguments as in the proofs of Theorems 2.3 and 2.4. □

2.D.2 Minimal Hellinger Martingale Density of Order q

Consider the following function $f_q(x)$ and the set $\mathcal{D}$, which will be used frequently in this section and Chapter 3 and 4:

$$f_q(x) := \begin{cases} \frac{(1+x)^q - 1 - qx}{q(q-1)}, & \text{if } x > -1 \text{ and } q \notin \{0, 1\}; \\ x - \log(1+x), & \text{if } x > -1 \text{ and } q = 0; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.42}$$

$$\mathcal{D} := \left\{ \theta \in \mathbb{R}^d \ : \ 1 + \theta^T x > 0, \qquad F - a.e. \right\}. \tag{2.43}$$

Also, considering the following set of local martingale densities

$$\mathcal{Z}^e_{q,loc}(S) := \{ Z = \mathcal{E}(N) > 0 \, | N \in \mathcal{M}_{loc}(P), \ ZS \in \mathcal{M}_{loc}(P), \ f_q(\Delta N) \in L^1_{loc} \}. \tag{2.44}$$

Now, we introduce the central concept in this chapter on Hellinger processes of order q.

Definition: (i) Let $N \in \mathcal{M}_{0,\ loc}(P)$ such that $1 + \Delta N \geq 0$ and $q \neq 1$. Then, if the non-decreasing adapted process

$$V_t^{(q)}(N) := \frac{1}{2}\langle N^c\rangle_t + \sum_{0<s\le t} f_q(\Delta N_s), \quad 0 \le t \le T, \tag{2.45}$$

is locally integrable (i.e. $V^{(q)}(N) \in \mathcal{A}^+_{loc}(P)$), then its compensator (with respect to the probability P) is called the Hellinger process of order q ($q \neq 0$) of the local martingale N, and is denoted by $h^{(q)}(N, P)$.

(ii) Let $Q \in \mathbb{P}_a$ with density $Z = \mathcal{E}(N)$. Then, we define the entropy-Hellinger process of Q with respect to P by

$$h_t^{(q)}(Q, P) := h_t^{(q)}(Z, P) := h_t^{(q)}(N, P), \quad 0 \le t \le T.$$

From time to time, we need to stop the local martingale densities. The following lemma will help us a lot when we go back from "local" to "global".

Lemma 2.5: *Let $(T_n)_{n\ge0}$ ($T_0 = 0$) be a sequence of stopping times that increases stationarily to T. Suppose that for each n, S^{T_n} admits the MHM density of order q, denoted by $\widetilde{Z}^{(n)}$. Then, S admits the MHM density of order q, $\widetilde{Z}$, given by*

$$\widetilde{Z} := \mathcal{E}(\widetilde{N}) \quad \text{and} \quad \widetilde{N} := \sum_{n\ge1} I_{]\!]T_{n-1},T_n]\!]} \frac{1}{\widetilde{Z}^{(n)}_-} \cdot \widetilde{Z}^{(n)}.$$

The next proposition provides an important characterization of the local martingale density, given by (2.47). Furthermore, the representation of Hellinger processes of local martingale densities and the associated jumps are given in assertions (2), (3) and (3′). Remark that we consider the process S in the general framework as a semimartingale. However, when we apply them in our main results in Chapter 3 and 4, we are only interested with the case that S is locally bounded.

Proposition 2.3: *Let $p \in (-\infty, 1)$, $q = \frac{p}{p-1}$, and $Z = \mathcal{E}(N)$ be a positive* *martingale density given by*

$$N = \beta \cdot S^c + W \star (\mu - \nu), \tag{2.46}$$

$$\text{where} \qquad W_t(x) = k_t(x) + \frac{\widehat{k}_t}{1-a_t} I_{\{a_t<1\}} \quad \text{and} \quad \widehat{k}_t := \int k_t(x)\nu(\{t\}, dx).$$

Here $\beta \in L(S)$ *and* $\left(\sum_{0<t\leq \cdot} k_t(\Delta S_t)^2 I_{\{\Delta S_t \neq 0\}} \right)^{1/2} \in \mathcal{A}^+_{loc}$. *Then, the following hold:*

(1) *The process* $|x(k(x)+1) - h(x)| \star \nu$ *has a finite variation (*h *is the truncation function such as* $h(x) = xI_{\{|x|\leq 1\}}$*) and* Z *(or equivalently* (β, k)*) satisfies*

$$b \cdot A + c\beta \cdot A + [x(k(x)+1) - h(x)] \star \nu = 0 \tag{2.47}$$

(2) *The Hellinger process of order* q *for* Z *is given by*

$$h^{(q)}(Z, P) = \frac{1}{2}\beta^T c\beta \cdot A + f_q(k) \star \nu + \sum (1-a) f_q\left(-\frac{\widehat{k}}{1-a} \right), \tag{2.48}$$

where f_q *is a function defined in (2.42).*

(3) *Let* $q \neq 0$ *and suppose that there exists* $\widetilde{\lambda} \in L(S)$ *such that* $1 + \widetilde{\lambda}^T x > 0$, $F(dx) \otimes dA \otimes P(d\omega)$*-a.e., and*

$$k_t(z) = \widetilde{\gamma}_t^{-1}\left(1 + \widetilde{\lambda}_t^T z\right)^{1/(q-1)} - 1, \quad \widetilde{\gamma}_t := 1 - a_t + \int (1 + \widetilde{\lambda}_t^T y)^{1/(q-1)} \nu(\{t\}, dy). \tag{2.49}$$

Then, the Hellinger process of order q *and its jumps are given by*

$$h^{(q)}(\widetilde{Z}, P) = \frac{\widetilde{\lambda}^T c \widetilde{\lambda}}{2(q-1)^2} \cdot A + f_q(k) \star \nu + \sum (1-a) f_q\left(\frac{1}{\widetilde{\gamma}} - 1 \right) \tag{2.50}$$

$$\text{and} \qquad \Delta h^{(q)}(\widetilde{Z}, P) = \frac{\widetilde{\gamma}^{1-q} - 1}{q(q-1)}. \tag{2.51}$$

(3′) *Suppose that there exists* $\widetilde{\lambda} \in L(S)$ *such that* $1 + \widetilde{\lambda}^T x > 0$, $F(dx) \otimes dA \otimes$

$P(d\omega)$-a.e., and

$$k_t(x) = (1 + \widetilde{\lambda}_t^T x)^{-1} - 1. \tag{2.52}$$

Then, the Hellinger process of order zero, $h^{(0)}(Z, P)$, and its jump process, $\Delta h^{(0)}(Z, P)$ are given by

$$h^{(0)}(Z, P) = \frac{1}{2}\widetilde{\lambda}^T c \widetilde{\lambda} \cdot A + f_0 \left((1 + \widetilde{\lambda}^T y)^{-1} - 1 \right) \star \nu \tag{2.53}$$

and $$\Delta h^{(0)}(Z, P) = \int \log(1 + \widetilde{\lambda}^T x) \nu_t(dx). \tag{2.54}$$

Proof. The proof of the assertions (1) and (2) follow from Lemma 2.4 and Proposition 3.5 respectively in [18].
It is obvious that (2.50) follows from (2.47) using (2.49).
Since $\widetilde{\gamma} \neq 0$, then, by taking the jumps ion both sides of (2.48), and using (2.49), we obtain

$$\int y \left(1 + \widetilde{\lambda}_t^T y\right)^{p-1} \nu(\{t\}, dy) = 0, \quad p - 1 = \frac{1}{q-1}.$$

Hence, (2.51) is derived from taking jumps in both sides of (2.50), inserting the above equation in the resulting equality, and using the expression of $\widetilde{\gamma}$ of (2.49) afterwards.
Assertion $(3')$ corresponds to the case of $p = 0$. Remark that in this case, the quantity $\widetilde{\gamma}$ given in (2.49) is 1. Then, a similar calculation as (2.50) and (2.51) leads to (2.53) and (2.54). This ends the proof of the proposition. □

Consider the following assumption

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P \otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $P \otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P \otimes A - a.e.$,

$$\lim_{n\to+\infty} \int K_p(\lambda_n^T x)F(dx) = \begin{cases} +\infty, & \text{on } \Gamma; \\ \int K_p(\lambda^T x)F(dx), & \text{on } \Gamma^c. \end{cases} \tag{2.55}$$

where $K_p(y) := |y||(1+y)^{p-1} - 1|$ and $\Gamma := \{F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

What follows below is very useful proposition which is slightly different formulation of Corollary 4.7 in [18]. Indeed, in the following proposition we explicitly provide the integrand in the stochastic integral with respect to S.

Proposition 2.4: *Suppose that (2.55) holds. Let $\widetilde{Z} \in \mathcal{Z}^e_{q,loc}(S)$ be the minimal Hellinger martingale density of order q. Then, the following hold.*
(1) There exists $\widetilde{\lambda} \in L(S)$ such that $1 + \widetilde{\lambda}^T z > 0$ $F(dz) \otimes dA \otimes P(d\omega)$-a.e. and

$$\widetilde{Z} := \mathcal{E}(\widetilde{N}), \quad \widetilde{N} := \frac{\widetilde{\lambda}}{q-1} \cdot S^c + \widetilde{W} \star (\mu - \nu), \quad \widetilde{W}_t(x) = \frac{1}{\widetilde{\gamma}_t}\left(\left(1 + \widetilde{\lambda}_t^T x\right)^{\frac{1}{q-1}} - 1\right),$$

where

$$\widetilde{\gamma}_t := 1 - a_t + \int (1 + \widetilde{\lambda}_t^T y)^{1/(q-1)} \nu(\{t\}, dy).$$

(2) Furthermore, $\widetilde{Z}$ satisfies

$$\begin{aligned} \widetilde{Z}^{q-1} &= \mathcal{E}\Big((\widetilde{\gamma}^{1-q}\widetilde{\lambda}) \cdot S + q(q-1)h^{(q)}(\widetilde{Z}, P)\Big), \\ &= \mathcal{E}\Big(\widetilde{\lambda} \cdot S\Big)\mathcal{E}\Big(q(q-1)h^{(q)}(\widetilde{Z}, P)\Big) \end{aligned} \tag{2.56}$$

Corollary 2.5.1: Suppose (2.55) holds for $p = 0$. Let $\widetilde{Z} \in \mathcal{Z}^e_{0,loc}(S)$ be the minimal Hellinger martingale density of order 0. Then, Then, the following hold.

(1) There exists $\widetilde{\lambda} \in L(S)$ such that $1 + \widetilde{\lambda}^T z > 0$ $F(dz) \otimes dA \otimes P(d\omega)$-a.e. and $\widetilde{Z}$ satisfies

$$\widetilde{Z} := \mathcal{E}(\widetilde{N}), \quad \widetilde{N} := -\widetilde{\lambda} \cdot S^c + \widetilde{W} \star (\mu - \nu), \quad \widetilde{W}_t(x) = \left(1 + \widetilde{\lambda}_t^T x\right)^{-1} - 1.$$

(2) Furthermore, $\widetilde{Z}$ satisfies

$$\widetilde{Z}^{-1} = \mathcal{E}\left(\widetilde{\lambda} \cdot S\right). \tag{2.57}$$

This section extents the MHM density concept to the case where one is facing a local change of probability. Through out this section, consider a positive local martingale, Z, given by

$$Z := \mathcal{E}(N), \; N := \beta \cdot S^c + W \star (\mu - \nu) + g \star \mu + \overline{N}, \; W_t(x) := f_t(x) - 1 + \frac{\widehat{f}_t - a_t}{1 - a_t} I_{\{a_t < 1\}}. \tag{2.58}$$

Here $\left(\beta, f, g, \overline{N}\right)$ are the Jacod components of N. Through out this section, we will frequently use the set of martingale density with respect to the density Z defined by

$$\mathcal{Z}^e_{q,loc}(S, Z) := \left\{ \overline{Z} \quad | \quad \overline{Z} > 0, \;\; \overline{Z} Z \in \mathcal{Z}^e_{q,loc}(S) \right\}, \tag{2.59}$$

where $\mathcal{Z}^e_{q,loc}(S)$ is given by (2.44). Then, the minimal Hellinger martingale density of order q with respect to Z is given by the following.
Next, we will give the variation of this Hellinger concept towards the change of probability measures.

Definition: (i) Let Q be a probability measure and Y be a Q-local martingale such that $1 + \Delta Y \geq 0$. Then, if the RCLL nondecreasing process

$$V^{(q)}(Y) = \frac{1}{2} \langle Y^c \rangle + \sum f_q(\Delta Y), \tag{2.60}$$

is Q-locally integrable (i.e. $V^{(q)}(Y) \in \mathcal{A}^+_{loc}(Q)$), then its Q-compensator is called the Hellinger process of order q of the local martingale Y (or equivalently of $\mathcal{E}(Y)$) with respect to Q, and is denoted by $h^{(q)}(Y,Q)$ (respectively $h^{(q)}(\mathcal{E}(Y),Q)$).

(ii) Let $N \in \mathcal{M}_{0,\ loc}(P)$ such that $1+\Delta N > 0$ and Y is a semimartingale such that $Y\mathcal{E}(N)$ is a P-local martingale and $1+\Delta Y \geq 0$. Then, if the process

$$\frac{1}{2}\langle Y^c\rangle + \sum(1+\Delta N) f_q(\Delta Y), \tag{2.61}$$

is P-locally integrable, then its P-compensator is called the Hellinger process of order q of the local martingale $\mathcal{E}(Y)$ with respect to $\mathcal{E}(N)$, and is denoted by $h^{(q)}\left(\mathcal{E}(Y),\mathcal{E}(N)\right)$.

Consider the following assumption.

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P\otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $P\otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P\otimes A - a.e.$,

$$\lim_{n\to+\infty}\int K_p(\lambda_n^T x) f(x) F(dx) = \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \int K_p(\lambda^T x) f(x) F(dx), & \text{on} \quad \Gamma^c. \end{cases} \tag{2.62}$$

where $K_p(y) := |y||(1+y)^{p-1} - 1|$ and $\Gamma := \{F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

Theorem 2.6: *Let $\mathcal{D}$ be the set defined in (2.43), $1 \neq p \in \mathbb{R}$, and $q = \frac{p}{p-1}$. Suppose that (2.62) holds, $\mathcal{Z}^e_{q,loc}(S) \neq \emptyset$, $int(\mathcal{D}) \neq \emptyset$, and for any $\lambda \in int(\mathcal{D})$ $P\otimes A - a.e.$,*

$$\int_{\{|x|>1\}} |x|(1+\lambda^T x)^{p-1} F^Z(dx) := \int_{\{|x|>1\}} |x| f(x)(1+\lambda^T x)^{p-1} F(dx) < +\infty. \tag{2.63}$$

Then, the minimization problem

$$\min_{\overline{Z}\in\mathcal{Z}^e_{q,loc}(S,Z)} h^{(q)}(\overline{Z}, Z), \tag{2.64}$$

admits a solution $\widetilde{Z} = \mathcal{E}(\widetilde{N})$ *given by*

$$\widetilde{N} := \frac{1}{q-1}\widetilde{\beta}\cdot S^{c,Z} + \widetilde{W}\star(\mu-\nu^Z),\ \widetilde{W}_t(x) := (\widetilde{\gamma}^Z)^{-1}\Big((1+\widetilde{\beta}_t^T x)^{p-1} - 1\Big), \tag{2.65}$$

where $\widetilde{\gamma}^Z := 1 - a_t^Z + \int (1+\widetilde{\beta}_t^T y)^{p-1} y \nu^Z(\{t\}, dy)$*, and* $\widetilde{\beta}$ *is the root of*

$$0 = b^Z + (p-1)c\lambda + \int \left[(1+\lambda^T x)^{p-1}x - h(x)\right] F^Z(dx). \tag{2.66}$$

Here $S^{c,Z}$*,* b^Z*,* a^Z*,* ν^Z *and* F^Z *are given by*

$$\begin{aligned} &S^{c,Z} := S^c - c\beta\cdot A, \quad a_t^Z := \nu^Z(\{t\}, I\!R^d),\\ &b^Z := b + c\beta + \int (f(x)-1)h(x)F(dx),\\ &\nu^Z(dt,dx) := F_t^Z(dx)dA_t, \quad F_t^Z(dx) := f_t(x)F_t(dx). \end{aligned} \tag{2.67}$$

Proof. Consider a sequence of stopping times, $(T_n)_{n\geq 1}$, that increases stationarily to T (i.e., $P(T_n = T) \to 1$ as $n \to \infty$) such that Z^{T_n} is a true martingale, and denote $Q := Z_{T_n}\cdot P$ (for n fixed but arbitrary). Remark that all the equations in the theorem are robust with respect to (stable under) stopping, and due to Lemma 2.5, it is enough to prove the theorem on $[\![0, T_n]\!]$. If, we put

$$\nu^Q(dt,dx) = f_t(x)I_{\{t\leq T_n\}}\nu(dt,dx) =: F_t^Q(dx)dA_t,$$

then it is clear from the assumption of the theorem that $\mathcal{Z}^e_{q,loc}(S^{T_n}, Q) \neq \emptyset$ and that $\int_{\{|x|>1\}}(1+\lambda^T x)^{p-1}|x|F^Q(dx) < +\infty$ for $\lambda \in \mathrm{I\!R}^d$. Hence, the assumption of Theorem 4.3 in [18] are fulfilled for the model $(S^{T_n}, Q = Z_{T_n}\cdot P)$. Thus, a direct application of this theorem leads to the existence of the minimal Hellinger martingale density of order q for (S^{T_n}, Q), that we will denote by $\widetilde{Z}^Q = \mathcal{E}\left(\widetilde{N}^Q\right)$, and is given, on $[\![0, T_n]\!]$, by

$$\widetilde{N}^Q := \frac{1}{q-1}\widetilde{\beta}\cdot S^{c,Q} + \widetilde{W}\star(\mu-\nu^Q),\ \widetilde{W}_t(x) = \frac{(1+\widetilde{\beta}_t^T x)^{p-1} - 1}{1 - a_t^Q + \int(1+\widetilde{\beta}_t^T y)^{p-1}\nu^Q(\{t\}, dy)}.$$

Herein $(S^{c,Q}, b^Q, a^Q, \nu^Q, F^Q)$ coincides with $(S^{c,Z}, b^Z, a^Z, \nu^Z, F^Z)$ (defined in (2.67)) on $[\![0, T_n]\!]$. Moreover, $\widetilde{\beta}$ is given by the equation

$$
\begin{aligned}
0 &= b^Q + (p-1)c\lambda + \int \Big((1+\lambda^T x)^{p-1} x - h(x) \Big) F^Q(dx) \\
&= \Big[b^Z + (p-1)c\lambda + \int \Big((1+\lambda^T x)^{p-1} x - h(x) \Big) F^Z(dx) \Big] I_{[\![0,T_n]\!]}.
\end{aligned}
$$

Therefore, it is obvious that the above equation coincides with (2.66). This ends the proof. □

Proposition 2.5: *Let $p \in (-\infty, 1)$ and q be the conjugate number. Consider a positive local martingale, Z, and $\widetilde{Z} \in \mathcal{Z}^e_{q,loc}(S, Z)$. If the assumptions of Theorem 2.6 are fulfilled, and $\widetilde{Z}$ denotes the minimal Hellinger martingale density of order q with respect to Z, given by (2.65), then*

$$
\begin{aligned}
\widetilde{Z}^{q-1} &= \mathcal{E}\left(\widetilde{H}^Z \cdot S + q(q-1) h^{(q)}(\widetilde{Z}, Z) \right) \\
&= \mathcal{E}\left(\widetilde{\beta} \cdot S \right) \mathcal{E}\left(q(q-1) h^{(q)}(\widetilde{Z}, Z). \right)
\end{aligned}
\tag{2.68}
$$

Here $\widetilde{H}^Z := (\gamma^Z)^{1-q} \widetilde{\beta}$, $\widetilde{\beta}$ is root of (2.66), and γ^Z satisfies

$$
\gamma^Z_t := 1 - a^Z_t + \int (1 + \widetilde{\beta}^T y)^{p-1} \nu^Z(\{t\}, dy) = \Big(1 + q(q-1) \Delta h^{(q)}(\widetilde{Z}, Z) \Big)^{p-1}, \tag{2.69}
$$

while a^Z and ν^Z are given by (2.67).

Proof. The proof of this proposition follows the same arguments as in the proof of Proposition 2.4 after stopping and under a suitable change of probability. □

Chapter 3

Log-Type Forward Utilities

This chapter focuses on our first class of forward utilities that we will parameterize/characterize fully. This class of forward utilities will be called log-type forward utilities and is defined in the following.

Definition: Let X be a RCLL semimartingale and Q be a probability measure. Then, we call log-type forward utility for (X, Q), any forward dynamic utility for (X, Q), $U_0(t, x, \omega)$, given by

$$U_0(t, \omega, x) = D_1(t, \omega) + D_0(t, \omega)\log(x), \quad x \in (0, +\infty). \tag{3.1}$$

Here $D_0(t)$ and $D_1(t)$ are two stochastic processes.

The first contribution of this chapter lies in characterizing the two processes D_0 and D_1 of (3.1) such that U_0 is a forward utility. While, the second contribution deals with the explicit description of the optimal portfolio for U_0. These two important contributions will be first elaborated for the general semimartingale framework in Section 3.A and, afterwards, illustrated on many particular examples for different market models in Section 3.B, 3.C, 3.D and 3.E.

3.A The Semimartingale Framework

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration is complete and right continuous. Here, T represents a fixed horizon for investments. In this setup, we consider a d-dimensional **locally bounded** semimartingale $S = (S_t)_{0\leq t\leq T}$ which represents the discounted price processes of d risky assets. Our goal in this section is to describe the processes D_0 and D_1 in (3.1) such that $U_0(t, x, \omega)$ is a forward utility.

First, we recall the predictable characteristics of S, ($B := b\cdot A$, $C := c\cdot A$, $\nu(dt, dx) := F_t(dx)dA_t$), that are defined in (2.5) such that S can be represented as

$$S = S_0 + S^c + x \star (\mu - \nu) + b \cdot A. \tag{3.2}$$

Throughout the analysis, the set $\mathcal{D}$ and the function Φ_0 (that are dependent on (ω, t)) defined below will play important roles. The set $\mathcal{D}$ is given by

$$\mathcal{D} := \left\{\theta \in \mathbb{R}^d \;\;:\;\; 1 + \theta^T x > 0, \qquad F - a.e.\right\}. \tag{3.3}$$

The function Φ_0 takes values in $(-\infty, +\infty]$ and is given by

$$\Phi_0(\lambda) := -b^T\lambda + \frac{1}{2}\lambda^T c\lambda + \int f_0(\lambda^T x)F(dx), \quad \forall\, \lambda \in \mathbb{R}^d, \tag{3.4}$$

where, the function f_0 is defined by

$$f_0(x) := \begin{cases} x - \log(1 + x), & \text{if } x > -1; \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

The set of admissible portfolios for U_0 (see (3.1)) is denoted by $\mathcal{A}_{adm}(x, U_0, P)$ (or simply $\mathcal{A}_{adm}(x)$), which is already defined in (2.10) that I recall below

$$\mathcal{A}_{adm}(x) := \Big\{\pi \in L(S) \,|\; 1+\pi\cdot S \geq 0 \;\text{ and }\; \sup_{\tau\in\mathcal{T}_T} EU_0\Big(\tau, x+(\pi\cdot S)_\tau\Big)^- < +\infty\Big\}. \tag{3.6}$$

Remark: Since $U_0(t,\omega,x)$ is a random field utility, for any fixed $x > 0$, the functional $U_0(t,\omega,x)$ is adapted and RCLL. As a result, by taking $x = 1$ and $x = e$, we deduce that the processes D_0 and D_1 are adapted and RCLL processes. Furthermore, $D_0(\omega,t) > 0$, for all $(\omega,t) \in \Omega \times [0,T]$, due to the strict increasingness of $x \mapsto U_0(t,\omega,x)$.

The characterization of U_0 in the full generality will be achieved after two steps. In the first step, we assume that the process D_0 is constant and equal to 1, that is,

$$U_0(t,\omega,x) := \log(x) + D_1(t,\omega), \quad x > 0. \tag{3.7}$$

Here, we suppose that D_1 satisfies the condition

$$\sup_{\tau \in \mathcal{T}_T} E\left(|D_1(\tau)|\right) < +\infty. \tag{3.8}$$

The second step will relax this assumption and work towards the general result. For the first step, our main assumption on the model is intimately related to the random measure F. Below, we state this assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P \otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $P \otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P \otimes A - a.e.$,

$$\lim_{n\to+\infty} \int K_0(\lambda_n^T x)F(dx) = \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \int K_0(\lambda^T x)F(dx), & \text{on} \quad \Gamma^c. \end{cases} \tag{3.9}$$

where $K_0(y) := \dfrac{|y|^2}{1+y}$ and

$$\Gamma := \{(\omega,t) \in \Omega \times [0,T] : F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}.$$

Theorem 3.1: *Consider the functional U_0 given in (3.7). Suppose that S is locally bounded and the assumption (3.8) and (3.9) hold. Then, the following two assertions are equivalent.*

(i) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(ii) The following properties hold:

(ii.1) The optimal portfolio rate $\widehat{\theta}$ belongs to $int(\mathcal{D})$ and is a root of

$$b - c\lambda - \int \frac{\lambda^T x}{1 + \lambda^T x} x F(dx) = 0, \tag{3.10}$$

(ii.2) The MHM density of order zero exists, denoted by $\widetilde{Z}$, and there exists a local martingale M such that

$$D_1(t) = D_1(0) + M_t - h_t^{(0)}(\widetilde{Z}, P), \quad 0 \leq t \leq T. \tag{3.11}$$

(ii.3) The process $\widehat{N} := D_1 - \log(\widetilde{Z})$ is a martingale.

The proof of this theorem is long and requires some technical lemmas and propositions. In the following, we will start by stating and proving these intermediate results that are also interesting in themselves while the proof of Theorem 3.1 will be provided after them.

Lemma 3.1: *Suppose S is locally bounded. Then, we have*

$$0 \in int(\mathcal{D}) \subseteq \mathcal{D}_1,$$

$$\text{where} \qquad \mathcal{D}_1 := \{\lambda \in \mathcal{D} : \ \exists \ \ \delta > 0, \ \ 1 + \lambda^T x \geq \delta, \ \ F - a.e.\}. \tag{3.12}$$

Proof. For any $\lambda_0 \in int(\mathcal{D})$, there exists $\varepsilon > 0$ such that for any λ satisfying $|\lambda_0 - \lambda| \leq \varepsilon$, we have $1 + \lambda^T x > 0$, $F - a.e.$

Now, we put $\lambda := \lambda_0 / a$, where

$$a = \frac{1}{1 + \varepsilon / |\lambda_0|} \in (0, 1).$$

It is easy to check that $|\lambda - \lambda_0| \leq \varepsilon$ and

$$1 + \lambda_0^T x = 1 + a\lambda^T x = a(1 + \lambda^T x) + (1 - a) \geq 1 - a > 0, \quad F - a.e..$$

Hence, $\lambda_0 \in \mathcal{D}_1$. To prove $0 \in int(\mathcal{D})$, we follow the localizing procedure and–

without loss of generality–assume that S is bounded, i.e. $|S| \leq K$. Then, there exists $\varepsilon := \frac{1}{2K}$ such that

$$\forall \quad \lambda \in B(0,\varepsilon), \quad 1 + \lambda^T x \geq 1 - K\frac{1}{2K} = 1/2 > 0.$$

Therefore, the neighborhood $B(0,\varepsilon) \subseteq \mathcal{D}$ and thus $0 \in int(\mathcal{D})$. This completes the proof of this lemma. □

Lemma 3.2: *Suppose S is locally bounded, then for any $\lambda \in I\!R^d$ and $\delta > 0$, we have*

$$\int_{\{\lambda^T x \geq \delta - 1\}} f_0(\lambda^T x) F(dx) < +\infty \quad P \otimes A - a.e. \tag{3.13}$$

Proof. Thanks to Taylor's expansion of f_0, we have

$$f_0(\lambda^T x) = \frac{(\lambda^T x)^2}{2}(1 + r\lambda^T x)^{-2}, \quad \text{for} \quad 0 < r < 1.$$

For $\delta > 0$ such that $\lambda^T x \geq \delta - 1$, we put $\overline{\delta} := \delta \wedge 1$ and obtain

$$1 + r\lambda^T x \geq 1 + r(\delta - 1) \geq \delta \wedge 1 = \overline{\delta}. \tag{3.14}$$

Therefore, we obtain that

$$\int_{\{\lambda^T x \geq \delta - 1\}} f_0(\lambda^T x) F(dx) \leq \frac{1}{2}\overline{\delta}^{-2}|\lambda|^2 \int |x|^2 F(dx). \tag{3.15}$$

Since S is locally bounded, it is easy to see that $[S,S] \in \mathcal{A}^+_{loc}$. As a result, we have $x^2 \star \nu_T < +\infty$, P-a.s., (terminal value of the compensator of $\sum |\Delta S|^2$) and hence

$$\int |x|^2 F(dx) < +\infty, \quad P \otimes A - a.e.$$

By combining this with (3.15), (3.13) follows immediately. This completes the proof of this lemma.

□

Lemma 3.3: *Suppose S is locally bounded. Then, the following two assertions hold, $P \otimes A$-a.e.*

(i) For any $\lambda \in int(\mathcal{D})$,

$$\int |x| \frac{|\lambda^T x|}{1+\lambda^T x} F(dx) < +\infty. \tag{3.16}$$

(ii) $\Phi_0(\lambda)$ *is differentiable on* $int(\mathcal{D})$ *and for any* $\lambda_0 \in int(\mathcal{D})$,

$$\Phi_0'(\lambda_0) = b - c\lambda_0 - \int x \frac{\lambda_0^T x}{1+\lambda_0^T x} F(dx).$$

Proof. (i) For any $\lambda \in int(\mathcal{D})$, due to Lemma 3.1, there exists $\delta \in (0,1)$ such that $1 + \lambda^T x \geq \delta > 0$ $F - a.e.$ Thus,

$$\int_{\{\lambda^T x \geq \delta - 1\}} |x| \frac{|\lambda^T x|}{1+\lambda^T x} F(dx) \leq \frac{|\lambda|}{\delta} \int |x|^2 F(dx)$$

which is finite since $[S,S] \in \mathcal{A}_{loc}^+$.
(ii) Let $\lambda_0 \in int(\mathcal{D})$. Then, for any $y \in \mathbb{R}^d$, thanks to Lemmas 3.1 and 3.2, there exists $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, $\lambda_0 + \varepsilon y \in dom(\Phi_0)$.
Due to Taylor's expansion of the function $g_0(\lambda^T x) := \lambda^T x - \log(1+\lambda^T x)$, we deduce the existence of $r \in (0,1)$ such that

$$k_\varepsilon(x) := \frac{g_0(\lambda_0^T x + \varepsilon y^T x) - g_0(\lambda_0^T x)}{\varepsilon} = y^T x \frac{\lambda_0^T x + r\varepsilon y^T x}{1 + \lambda_0^T x + r\varepsilon y^T x}.$$

Meanwhile, notice that $(|k_\varepsilon(x)|)_\varepsilon$ is bounded from above by

$$k(x) := |y||x| \max\left(\frac{|\lambda_0^T x|}{1+\lambda_0^T x}, \frac{|\lambda_0^T x + \varepsilon_0 y^T x|}{1 + \lambda_0^T x + \varepsilon_0 y^T x} \right).$$

Thanks to Lemma 3.3–(i), $k(x)$ is integrable since $\lambda_0, \lambda_0 + \varepsilon_0 y \in int(\mathcal{D}) \subseteq dom(\Phi_0)$. It allows us to apply Dominated Convergence Theorem to $(|k_\varepsilon(x)|)_\varepsilon$, which leads to

$$\lim_{\varepsilon \to 0} \frac{\Phi_0(\lambda_0 + \varepsilon y) - \Phi_0(\lambda_0)}{\varepsilon} = y^T \Phi^*, \tag{3.17}$$

where Φ^* is given by

$$\Phi^* := b - c\lambda_0 - \int x \frac{\lambda_0^T x}{1+\lambda_0^T x} F(dx).$$

It is clear from (3.17) that $y^T\Phi^*$ is the directional derivative of $-\Phi_0$ at λ_0, which is linear in y. Thus, due to Theorem 25.2 in [68], Φ_0 is differentiable on $int(\mathcal{D})$. This completes the proof. □

Lemma 3.4: *Suppose S is locally bounded. Then, the interior of the effective domain of Φ_0 coincides with $int(\mathcal{D})$, that is,*

$$int(dom(\Phi_0)) = int(\mathcal{D}), \quad P \otimes A - a.e.$$

Proof. For any $\lambda_0 \in int(dom(\Phi_0))$, there exists a neighborhood $B(\lambda_0, \varepsilon)$, such that $B(\lambda_0, \varepsilon) \subseteq dom(\Phi_0)$. Let $\delta = \varepsilon/2$ and $\lambda \in B(\lambda_0, \varepsilon/2)$ such that

$$\lambda \pm \delta e_i \in B(\lambda_0, \varepsilon).$$

Due to the convexity of the function $f_0(x)$, we have

$$\delta |x| \frac{|\lambda^T x|}{1+\lambda^T x} \leq \sum_{i=1}^{d} \left[f_0(\lambda^T x + \delta e_i^T x) + f_0(\lambda^T x - \delta e_i^T x) \right],$$

where e_i is the vector of $\mathbb{R}^d$ whose i^{th} component equals one, and the others are null. Therefore, we have

$$\delta \int |x| \frac{|\lambda^T x|}{1+\lambda^T x} F(dx) < +\infty, \quad P \otimes A - a.e.$$

It implies that $1+\lambda^T x > 0$, $F-a.e.$, from which we clearly deduce that $\lambda \in \mathcal{D}$. Therefore, $\lambda_0 \in int(\mathcal{D})$.

On the other hand, for any $\lambda_0 \in int(\mathcal{D})$, there exists a neighborhood $B(\lambda_0, \varepsilon) \subseteq int(\mathcal{D})$. Then, due to Lemmas 3.1 and 3.2, we have $B(\lambda_0, \varepsilon) \subseteq dom(\Phi_0)$. Hence, $\lambda_0 \in int(dom(\Phi_0))$. This completes the proof of this lemma. □

Proposition 3.1: *Suppose S is locally bounded and assumption (3.9) holds. If $\Phi_0(\lambda)$ attains its minimum at $\widetilde{\lambda}$, then $\widetilde{\lambda} \in int(\mathcal{D})$. Furthermore,*

$$\Phi_0'(\widetilde{\lambda}) = b - c\widetilde{\lambda} - \int \frac{\widetilde{\lambda}^T x}{1+\widetilde{\lambda}^T x} x F(dx) = 0. \tag{3.18}$$

Proof. Suppose that the minimum of $\Phi_0(\lambda)$ is attained at $\widetilde{\lambda}$. Then, it is easy to deduce that $\widetilde{\lambda} \in \mathcal{D}$. For any $r \in (0,1)$, we put $\overline{\lambda} := (1-r)\widetilde{\lambda}$, which is convex combination of 0 and $\widetilde{\lambda}$, that belongs to $int(\mathcal{D})$. Since $\widetilde{\lambda}$ is the minimum of $\Phi_0(\lambda)$ over $\mathcal{D}$, we have

$$\Phi_0(\widetilde{\lambda}) \le \Phi_0(\overline{\lambda}), \quad P \otimes A - a.e. \tag{3.19}$$

On the other hand, due to the convexity of f_0, we have

$$\frac{f_0(\widetilde{\lambda}^T x) - f_0(\overline{\lambda}^T x)}{r} \ge f_0(\widetilde{\lambda}^T x).$$

Since $f_0(\widetilde{\lambda}^T x)$ is integrable, the above inequality allows us to apply Fatou's Lemma ,and get

$$-\widetilde{\lambda}^T G(\widetilde{\lambda}) \le \lim_{r\to 0} \frac{\Phi_0(\widetilde{\lambda}) - \Phi_0(\overline{\lambda})}{r} \le 0, \quad P \otimes A - a.e.,$$

Here

$$G(\lambda) := b - c\lambda - \int \frac{\lambda^T x}{1+\lambda^T x} x F(dx).$$

After rearranging the terms, it is clear to see that

$$0 \le \int \frac{|\widetilde{\lambda}^T x|^2}{1+\widetilde{\lambda}^T x} F(dx) \le \widetilde{\lambda}^T b - \widetilde{\lambda}^T c \widetilde{\lambda} < +\infty. \tag{3.20}$$

For some $\lambda_0 \in int(\mathcal{D})$, it is easy to check that the convex combination

$$\widetilde{\lambda}_n := (1 - \frac{1}{n})\widetilde{\lambda} + \frac{1}{n}\lambda_0 \in int(\mathcal{D}) \quad \text{and} \quad \widetilde{\lambda}_n \to \widetilde{\lambda}.$$

Suppose that $\widetilde{\lambda} \notin int(\mathcal{D})$ and put

$$l_n(x) := \frac{|\widetilde{\lambda}_n^T x|^2}{1+\widetilde{\lambda}_n^T x} \quad l(x) := \max\left\{\frac{|\widetilde{\lambda}^T x|^2}{1+\widetilde{\lambda}^T x}, \frac{|\lambda_0^T x|^2}{1+\lambda_0^T x}\right\}.$$

It is easy to see that the function $k(y) := \frac{y^2}{1+y}$, $y > -1$, is positive and convex. Thus, we have $l_n(x) \leq l(x)$. Meanwhile, $l(x)$ is integrable due to (3.20) and Lemma 3.3–(i). This allows us to apply Dominated Convergence Theorem to $l_n(x)$, i.e.

$$\int \frac{|\widetilde{\lambda}^T x|^2}{1+\widetilde{\lambda}^T x} F(dx) = \lim_{n\to+\infty} \int \frac{|\widetilde{\lambda}_n^T x|^2}{1+\widetilde{\lambda}_n^T x} F(dx),$$

which is $+\infty$ due to assumption (3.9). This is a contradiction with (3.20). Thus, we can conclude $\widetilde{\lambda} \in int(\mathcal{D})$. Meanwhile, due to Lemma 3.3–(ii), Φ_0 is differentiable at $\widetilde{\lambda}$. Therefore, we have (3.18) by recalling Lemma 3.3–(ii). This ends the proof. □

Lemma 3.5: *Let $\widehat{\theta} \in L(S)$ and suppose that*

$$\Big((1+\widehat{\theta}^T z)^{-1} - 1 + \log(1+\widehat{\theta}^T z)\Big)\star\mu \in \mathcal{A}_{loc}^+, \ \textit{and} \ \int \frac{\widehat{\theta}^T x}{1+\widehat{\theta}^T x}\nu(\{t\}, dx) = 0. \tag{3.21}$$

Then, the functional W, given by

$$W_t(z) := (1+\widehat{\theta}^T z)^{-1} - 1$$

belongs to $\mathcal{G}_{loc}^1(\mu)$ and the process $\widetilde{Z}$, given by

$$\widetilde{Z} = \mathcal{E}\left(\widetilde{N}\right), \quad \widetilde{N} = -\widehat{\theta}\cdot S^c + W \star (\mu - \nu) \tag{3.22}$$

is a well-defined local martingale.

Proof. We need to show that every component in this construction is well-defined. For $\widehat{\theta} \in L(S)$ and S being locally bounded, thanks to Proposition 2.1, we have $\widehat{\theta} \in L(S^c)$. Thus, the integral $-\widehat{\theta}\cdot S^c$ is a well-defined continuous local martingale. For the purely discontinuous part, we need to prove W is

$(\mu - \nu)$-integrable, i.e.

$$\left(\sum (W_t(\Delta S) I_{\{\Delta S_t \neq 0\}} - \widehat{W}_t)^2\right)^{1/2} \in \mathcal{A}^+_{loc}.$$

Due to $\displaystyle\int \frac{\widehat{\theta}^T x}{1 + \widehat{\theta}^T x} \nu(\{t\}, dx) = 0$ assumed in (3.21), we deduce that

$$\widehat{W}_t = \int W_t(x) \nu(\{t\}, dx) = - \int \frac{\widehat{\theta}^T x}{1 + \widehat{\theta}^T x} \nu(\{t\}, dx) = 0.$$

Hence, $W \in \mathcal{G}^1_{loc}(\mu)$ if and only if

$$\left(\sum (W_t(\Delta S) I_{\{\Delta S_t \neq 0\}})^2\right)^{1/2} \in \mathcal{A}^+_{loc}.$$

Since $\widehat{\theta} \cdot S$ is a RCLL semimartingale, then, for any $\alpha \in (0,1)$, the non-decreasing process $I_{\{|\widehat{\theta}^T \Delta S| \leq \alpha\}} \cdot [\widehat{\theta} \cdot S, \widehat{\theta} \cdot S]$ is locally bounded[1] and, hence, it is locally integrable. Then, due to the inequalities

$$\begin{aligned} \sum \left(\left(1 + \widehat{\theta}^T \Delta S\right)^{-1} - 1 \right)^2 I_{\{|\widehat{\theta}^T \Delta S| \leq \alpha\}} &\preceq \frac{1}{(1-\alpha)^2} \sum (\widehat{\theta}^T \Delta S)^2 I_{\{|\widehat{\theta}^T \Delta S| \leq \alpha\}} \\ &\preceq \frac{1}{(1-\alpha)^2} I_{\{|\widehat{\theta}^T \Delta S| \leq \alpha\}} \cdot [\widehat{\theta} \cdot S, \widehat{\theta} \cdot S], \end{aligned}$$

we deduce that $\sum ((1 + \widehat{\theta}^T \Delta S)^{-1} - 1)^2 I_{\{|\widehat{\theta}^T \Delta S| \leq \alpha\}}$ is locally integrable.
Put $A = (1-\alpha)\log(1-\alpha) + \alpha \in (0,1)$ and remark that

$$|(1 + \widehat{\theta}^T \Delta S)^{-1} - 1| I_{\{|\widehat{\theta}^T \Delta S| > \alpha\}} \leq \begin{cases} I_{\{|\widehat{\theta}^T \Delta S| > \alpha\}}, & \widehat{\theta} \Delta S \geq 0; \\ \left(A K(\widehat{\theta}^T \Delta S) - 1 \right) I_{\{|\widehat{\theta}^T \Delta S| > \alpha\}}, & \widehat{\theta} \Delta S < 0. \end{cases}$$

where $K(y) := (1+y)^{-1} - 1 + \log(1+y)$. Then, we derive that

$$\sum |(1 + \widehat{\theta}^T \Delta S)^{-1} - 1| I_{\{|\widehat{\theta}^T \Delta S| > \alpha\}} \preceq 2 I_{\{|\widehat{\theta}^T x| > \alpha\}} \star \mu + A |K(\widehat{\theta}^T x)| I_{\{|\widehat{\theta}^T x| > \alpha\}} \star \mu.$$

[1]Fact: any semimartingale with bounded jumps is locally bounded.
The process $[\widehat{\theta} \cdot S, \widehat{\theta} \cdot S]$ is non-decreasing and has finite variation with bounded jumps, hence, it is locally bounded.

Hence, due to (3.21), $\sum |(1+\widehat{\theta}^T \Delta S)^{-1} - 1| I_{\{|\widehat{\theta}^T \Delta S|>\alpha\}}$ is locally integrable. [2]
Finally, using the inequality

$$\Big(\sum ((1+\widehat{\theta}^T \Delta S)^{-1} - 1)^2 I_{\{|\widehat{\theta}^T \Delta S|>\alpha\}}\Big)^{1/2} \preceq \sum |(1+\widehat{\theta}^T \Delta S)^{-1} - 1| I_{|\{\widehat{\theta}^T \Delta S|>\alpha\}},$$

we obtain the local integrability of $\Big(\sum ((1+\widehat{\theta}^T \Delta S)^{-1} - 1)^2\Big)^{1/2}$.
This ends the proof of the $(\mu-\nu)$-integrability of W, and hence $W \star (\mu - \nu)$ is a local martingale and the process $\widetilde{Z}$ constructed in (3.22) is well defined. □

Proof of Theorem 3.1:

The proof of this theorem will be carried out in four steps, where the implication of $(i) \Rightarrow (ii)$ will be given in Step 1–3, while Step 4 will focus on the proof of $(ii) \Rightarrow (i)$.

Step 1: Suppose (i) holds. Due to assumption (3.12), $\theta = 0$ is admissible. We deduce that $D(t)$ is a supermartingale, which can be written in the form of $D(t) = D(0) + M_t + a^D_t$, where M is a local martingale and a^D is predictable with finite variation. By Ito's formula, we get for any admissible portfolio rate θ,

$$X^\theta := \log(\mathcal{E}(\theta \cdot S)) = \theta \cdot S - \frac{1}{2}\theta^T c \theta \cdot A + \big(\log(1+\theta^T z) - \theta^T z\big) \star \mu.$$

Hence, for any admissible portfolio rate θ (respectively, the optimal portfolio rate $\widehat{\theta}$),

$$U(t, x\mathcal{E}_t(\theta \cdot S)) = U(t, x\mathcal{E}(\theta \cdot S)) = D(0) + \log(x) + M_t + X^\theta_t + a^D_t \qquad (3.23)$$

[2] For the process $\sum I_{\{|\widehat{\theta}^T \Delta S|>\alpha\}} = I_{\{|\widehat{\theta}^T x|>\alpha\}} \star \mu$, it is non-decreasing, and note a fact that, it is right continuous and , hence, the number of jump is finite (RCLL process $(\widetilde{\theta} \cdot S)$ has finite number of lower bounded jump and countable total jumps), which implies it is real valued and has finite variation. Moreover, its jump is bounded and, hence, it is locally bounded. When $-1 < x < -\alpha$, we choose A such that $f(x) < 0$, where

$$f(x) := A(\log(1+x) + \frac{1}{1+x} - 1) - 1 - (\frac{1}{1+x} - 1).$$

Note that the function $g(x) := (1+x)\log(1+x) - x$ is decreasing on $(-1, -\alpha)$ and $g(0) = 0$.

is a supermartingale (respectively, a local martingale) if and only if the process $X^\theta + a^D$ is a local supermartingale (respectively, is a local martingale). It implies that

$$|\log(1+\theta^T z) - \theta^T z| \star \mu \in \mathcal{A}^+_{loc}, \tag{3.24}$$

and the process

$$a^D - \Phi_0(\theta) \cdot A \quad \text{is non-increasing (respectively, is null),} \tag{3.25}$$

where $\Phi_0(\theta)$ is given in (3.4). From (3.25), it is clear that when $\theta = \widehat{\theta}$,

$$a^D = \Phi_0\left(\widehat{\theta}\right) \cdot A, \tag{3.26}$$

that is, the optimal portfolio rate $\widehat{\theta}$ will minimize the functional $\Phi_0(\theta)$. Recalling the assumption (3.9) and Proposition 3.1, we can conclude that $\widehat{\theta} \in int(\mathcal{D})$ and it is a root of equation (3.10). This completes this proof of (ii.1).

Step 2: By inserting the equation (3.10) to the right hand side of (3.26)(recall the integrability (3.24)), we have,

$$-a^D = \frac{1}{2}\widehat{\theta}^T c \widehat{\theta} \cdot A + \int \left(\frac{1}{1+\widehat{\theta}^T z} - 1 + \log(1+\widehat{\theta}^T z) \right) F(dz) \cdot A \tag{3.27}$$

From (3.27), we deduce that the process $\left((1+\widehat{\theta}^T z)^{-1} - 1 + \log(1+\widehat{\theta}^T z)\right) \star \mu$ is locally integrable. Meanwhile, by taking jumps in (3.10), we deduce that

$$\int \frac{\widehat{\theta}^T x}{1+\widehat{\theta}^T x} \nu(\{t\}, dx) = 0.$$

This fulfills the condition of Lemma 3.5, where we have proved that $\widetilde{Z}$, given in (3.22), is well defined. Moreover, it is a local martingale density due to (3.10) and Proposition 2.2.

It remains to prove the optimality of $\widetilde{Z}$. Thanks to Proposition 3.2 in [17] (see also Proposition 4.2 in [16] for the case of quasi-left continuity), it is enough to consider a positive martingale density $Z = \mathcal{E}(N)$ of the form

$$N = \beta{\cdot}S^c + Y\star(\mu-\nu), \quad Y_t(x) = k_t(x) + \frac{\widehat{k}_t}{1-a_t}I_{\{a_t<1\}}, \quad \widehat{k}_t := \int k_t(x)\nu(\{t\},dx),$$

where $\beta \in L(S)$ and $\left(\sum k_t(\Delta S_t)^2 I_{\{\Delta S_t\neq 0\}}\right)^{1/2} \in \mathcal{A}^+_{loc}$. Then, due to the convexity of $z^T cz$ and $f_0(z) := z - \log(1+z)$, we obtain on $\{\Delta A = 0\}$,

$$\begin{aligned}
&\frac{dh^{(0)}(Z,P)}{dA} - \frac{dh^{(0)}(\widetilde{Z},P)}{dA} && (3.28)\\
&= \int \Big[f_0(k(x)) - f_0\Big((1+\widehat{\theta}^T x)^{-1} - 1\Big)\Big]F(dx) + \frac{1}{2}(\beta^T c\beta - \widehat{\theta}^T c\widehat{\theta}) \\
&\geq \widehat{\theta}^T c(\beta - \widehat{\theta}) + \int \widehat{\theta}^T x\Big(k(x) + 1 - (1+\widehat{\theta}^T x)^{-1}\Big)F(dx) = 0. && (3.29)
\end{aligned}$$

The equality (3.29) is derived from a combination of (3.10) for $\widetilde{Z}$ and a similar equation for Z based on (2.47) that

$$b\cdot A + c\beta\cdot A + x(k(x)+1)\star\nu = 0.$$

On the other hand, due to (2.53) and (2.54) in Proposition 2.3 and the convexity of $f_0(z)$, we get

$$\begin{aligned}
&\Delta h^{(0)}_t(Z,P) - \Delta h^{(0)}_t(\widetilde{Z},P) \\
&= \int \Big[f_0(k_t(x)) - f_0\Big((1+\widehat{\theta}^T_t x)^{-1} - 1\Big)\Big]\nu_t(dx) + (1-a_t)f_0\Big(\frac{-\widehat{k}_t}{1-a_t}\Big) && (3.30)\\
&\geq \int \Big(k_t(x) + 1 - (1+\widehat{\theta}^T_t x)^{-1}\Big)\widehat{\theta}^T_t x\nu_t(dx) = 0. && (3.31)
\end{aligned}$$

Indeed, the equation (3.30) comes from Proposition 2.3, while the equation (3.31) follows from (3.10) for $\widetilde{Z}$ on the set $\{\Delta A \neq 0\}$ and a similar equation from (2.47) for Z that

$$\int x(1+k_t(x))\nu_t(dx) = 0.$$

Thus, by combining (3.29) and (3.31), we deduce that $\widetilde{Z}$ is the MHM density of order 0.

Step 3: In the step, we will characterize a^D and then conclude (3.11). Furthermore, the proof of (ii.3) is given in the end.

By considering (3.27), on the set $\{\Delta A = 0\}$, we have

$$\begin{aligned} -I_{\{\Delta A=0\}} \cdot a^D &= I_{\{\Delta A=0\}} \left[\frac{1}{2}\widetilde{\theta}^T c \widetilde{\theta} + \int f_0 \left((1+\widetilde{\theta}^T y)^{-1} - 1\right)\right] \cdot A \\ &= I_{\{\Delta A=0\}} \cdot h^{(0)}(\widetilde{Z}, P) \end{aligned} \tag{3.32}$$

where the Hellinger process $h^{(0)}(\widetilde{Z}, P)$ are given in Proposition 2.3. On the set $\{\Delta A \neq 0\}$, (3.10) implies

$$\int \frac{x}{1+\widehat{\theta}^T x} \nu_t(dx) = 0. \tag{3.33}$$

With the help of (3.33), we take jump on both sides of (3.27) and use Proposition 2.3, we derive

$$-\Delta a^D = \int \log(1+\widehat{\theta}^T z)\nu_t(dz) = \Delta h^{(0)}(\widetilde{Z}, P). \tag{3.34}$$

Therefore, following (3.32) and (3.34), we can conclude immediately (3.11). Finally, due to the forward property of U_0, $U_0(t, \mathcal{E}(\widehat{\theta} \cdot S))$ is a martingale. We combine this fact with the result in Corollary 2.5.1 and derive

$$\widehat{N} := D_1 - \log(\widetilde{Z}) = U_0(t, \mathcal{E}(\widehat{\theta} \cdot S)).$$

Therefore, (ii.3) holds immediately.

Step 4: Now, we suppose (ii) holds. For the particular portfolio rate $\widehat{\theta}$, we apply the result in Proposition 2.4,

$$U_0\left(\cdot, x\mathcal{E}\left(\widehat{\theta} \cdot S\right)\right) = \log(x) - \log\left(\widetilde{Z}\right) + D_1 \tag{3.35}$$

which is a martingale from assertion (ii.3). Meanwhile, for any admissible portfolio rate θ, we have

$$U_0\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right) = U_0\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right) + \log\left(\frac{\mathcal{E}(\theta\cdot S)}{\mathcal{E}(\widehat{\theta}\cdot S)}\right)$$

By applying Jensen's inequality, we have

$$E\left[\log\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}\right)|\mathcal{F}_s\right] \leq \log\left[E\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}|\mathcal{F}_s\right)\right]. \tag{3.36}$$

Thanks to Corollary 2.5.1, we have $\widetilde{Z} = \mathcal{E}^{-1}(\widehat{\theta}\cdot S)$ and hence

$$\log\left[E\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}|\mathcal{F}_s\right)\right] = \log\left[E(\widetilde{Z}_t\mathcal{E}_t(\theta\cdot S)|\mathcal{F}_s)\right]. \tag{3.37}$$

It is easy to see that $\widetilde{Z}\mathcal{E}(\theta\cdot S)$ is a positive local martingale hence a supermartingale. As a result,

$$\log\left[E(\widetilde{Z}_t\mathcal{E}_t(\theta\cdot S)|\mathcal{F}_s)\right] \leq \log\left(\widetilde{Z}_s\mathcal{E}_s(\theta\cdot S)\right) = \log\left(\frac{\mathcal{E}_s(\theta\cdot S)}{\mathcal{E}_s(\widehat{\theta}\cdot S)}\right)$$

Therefore, $\log\left(\mathcal{E}(\theta\cdot S)/\mathcal{E}(\widehat{\theta}\cdot S)\right)$ is a supermartingale, and $U_0\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right)$ is a supermartingale too. This completes the proof. $\square$

Now, we are ready to characterize the log-type forward utility, U_0, given by (3.1), for the most general case without any assumption on D_0 and D_1. This will be achieved in two steps. First, an equivalent statement on the forward property of U_0 is established in Theorem 3.2. Then, in Theorem 3.3, we focus on this equivalent statement and give a full characterization of D_0 and D_1.

Theorem 3.2: *Suppose that the processes D_0 and D_1 of (3.1) satisfy*

$$\sup_{\tau\in\mathcal{T}_T} E\left(|D_0(\tau)|+|D_1(\tau)|\right) < +\infty. \tag{3.38}$$

Then the following two assertions are equivalent:

(i) $U_0(t,x)$ is a forward utility.

(ii) $(D_0(t))_{t\geq 0}$ is a positive martingale, $(D_1(t))_{t\geq 0}$ is a supermartingale and the functional U_0^Q, given by

$$U_0^Q(t,x) := \log(x) + \frac{D_1(t)}{D_0(t)} \tag{3.39}$$

is a forward utility for the model (S,Q), where $Q := \frac{D_0(T)}{D_0(0)} \cdot P$.

Proof. Suppose that (i) holds. Then, under the assumption (3.38), the null portfolio rate is admissible, and hence $U_0(t,x) = D_1(t) + D_0(t)\log(x)$ is a supermartingale for any $x > 0$. By considering the cases of $x = 1$ and $x \neq 1$, we conclude that the process D_1 is a supermartingale and the process D_0 is a special semimartingale.

The Doob-Meyer decomposition of D_0 and D_1 are

$$D_1 = D_1(0) + M^{D_1} + A^{D_1}, \quad D_0 = D_0(0) + M^{D_0} + A^{D_0},$$

where M^{D_1} and M^{D_0} are local martingales, and, A^{D_1} and A^{D_0} are predictable processes with finite variation such that

$$M_0^{D_1} = A_0^{D_1} = M_0^{D_0} = A_0^{D_0} = 0.$$

By inserting the above decompositions into the expression of U_0, we get for any $x > 0$

$$U_0(t,x) = D_1(0) + D_0(0)\log(x) + M_t^{D_1} + M_t^{D_0}\log(x) + A_t^{D_0}\log(x) + A_t^{D_1}. \tag{3.40}$$

Since $U_0(t,x)$ is a supermartingale, the predictable part of $U_0(t,x)$ is non-increasing and hence

$$A^{D_0}\log(x) + A^{D_1} \leq 0, \quad \forall\ x > 0. \tag{3.41}$$

The inequality in (3.41) holds only if $A^{D_0} = 0$ and $A^{D_1} \leq 0$ P-a.s. Therefore, the process D_0 is a local martingale. To prove that D_0 is a true martingale, we note that for any $x > 0$, there exists a portfolio rate $\widehat{\pi}_x$ (we explicitly denote its dependence on x) such that $(U_0(t, x + \widehat{\theta}_x \cdot S_t))_{t \in [0,T]}$ is a martingale. Thus, $\widehat{\pi}_x$ is the solution of the following maximization problem for any $\tau \in \mathcal{T}_T$

$$\max_{\pi \in \mathcal{A}_{adm}(x)} E(U_0(\tau, x + \pi \cdot S_\tau)) = E\left(U_0(\tau, x + \widehat{\pi}_x \cdot S_\tau)\right) = U_0(0, x). \tag{3.42}$$

Meanwhile, we notice the equality

$$\max_{\pi \in \mathcal{A}_{adm}(x)} E\left(D_0(\tau) \log(1 + \frac{\pi}{x} \cdot S_\tau)\right) = \max_{\pi \in \mathcal{A}_{adm}(1)} E\left(D_0(\tau) \log(1 + \pi \cdot S_\tau)\right). \tag{3.43}$$

A combination of (3.42) and (3.43) leads to

$$\begin{aligned} & E(D_0(\tau) \log(x)) - D_0(0) \log(x) \\ = & -E\left(D_0(\tau) \log(1 + \frac{\widehat{\pi}_x}{x} \cdot S_\tau)\right) + E\left(D_0(\tau) \log(1 + \widehat{\pi}_1 \cdot S_\tau)\right) \\ = & \ 0. \end{aligned}$$

Since x can be taken arbitrarily, it is easy to derive that

$$E(D_0(\tau)) = D_0(0), \quad \forall\, \tau \in \mathcal{T}_T.$$

Hence, we conclude that the process D_0 is a true martingale.
Now, an application of Lemma 2.3-(ii) can complete the proof of (ii). Furthermore, (ii)⇒(i) follows again from this lemma. This completes the proof of this theorem. □

In Theorem 3.2, the process D_0 is characterized as a positive martingale. Thus, we can write it in the form of stochastic exponential, given by

$$D_0 = D_0(0)\mathcal{E}(N),$$

where N is a local martingale and we let $(\beta, f, g, \overline{N}')$ be the Jacod components of N. We consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P \otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $P \otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P \otimes A - a.e.$,

$$\lim_{n\to+\infty} \int K_0(\lambda_n^T x)(1+f(x))F(dx) = \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \int K_0(\lambda^T x)(1+f(x))F(dx), & \text{on} \quad \Gamma^c. \end{cases} \tag{3.44}$$

where $K_0(y) := \dfrac{|y|^2}{1+y}$ and $\Gamma := \{F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

Now, thanks to Theorem 3.1, the full description of D_0 and D_1 can be derived via working on the equivalent statement on the forward property of U_0^Q, given by Theorem 3.2–(ii).

Theorem 3.3: *Let D_0 be a positive martingale (put $Q = \frac{D_0(T)}{D_0(0)} \cdot P$) and D_1 be a supermartingale. Suppose that S is locally bounded and the assumption (3.44) holds. Then the following two assertions are equivalent.*

(1) The functional U_0^Q given by (3.39) is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The following properties hold:

(2.a) The MHM density of order zero with respect to Q exists, denoted by $\widetilde{Z}^Q$, and there exists a Q-local martingale M^Q such that

$$D_1(t) = D_0(t)\left(\frac{D_1(0)}{D_0(0)} + M_t^Q - h_t^{(0)}(\widetilde{Z}^Q, Q)\right), \quad 0 \leq t \leq T. \tag{3.45}$$

(2.b) The optimal portfolio rate $\widehat{\theta}$ is a root for

$$b^Q - c\lambda + \int \left(\frac{1}{1+\lambda^T x} - 1\right) x F^Q(dx) = 0, \tag{3.46}$$

where b^Q, $F^Q(dx)$ are the predictable characteristics of S under Q.

(2.c) The process $\widehat{N} := D_1 - D_0 \log(\widetilde{Z}^Q)$ is a martingale.

Proof. (1) $\Rightarrow$ (2). Suppose that (1) holds, that is, the functional $U^Q(t,\omega,x) := \log(x) + \frac{D_1(t)}{D_0(t)}$ is a forward utility for (S,Q) with the optimal portfolio rate $\widehat{\theta}$ and the process D_1/D_0 is a Q-supermartingale. Hence, we can apply Theorem 3.1 directly on the model (S,Q,U_0^Q) under the assumption (3.44). It states that there exists the MHM density of order 0, denoted by $\widetilde{Z}^Q$, such that D_1/D_0 can be characterized as (3.45). Furthermore, the optimal portfolio rate $\widehat{\theta}$ is a root of the equation (3.46).

It remains to prove (2.c). Due to Proposition 2.5, we have

$$\widehat{N} := D_1 + D_0 \log(\mathcal{E}(\widehat{\theta}\cdot S)) = -D_0\log(x) + U(t, x\mathcal{E}_t(\widehat{\theta}\cdot S)).$$

Hence, $\widehat{N}$ is a true martingale, and we complete the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). Since $\widetilde{Z}^Q$ is the MHM density of order 0, we apply the result in Lemma 2.5 and derive

$$U_0^Q\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right) = \log(x) - \log\left(\widetilde{Z}^Q\right) + \frac{D_1}{D_0} = \log(x) + \frac{\widehat{N}}{D_0}. \tag{3.47}$$

Then, due to assertion (2.c), we conclude that $U_0^Q\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right)$ is a Q-martingale. Now, for any admissible portfolio rate θ, we have

$$U_0^Q\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right) = U_0^Q\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right) + \log\left(\frac{\mathcal{E}(\theta\cdot S)}{\mathcal{E}(\widehat{\theta}\cdot S)}\right)$$

By applying Jensen's inequality, we have

$$E^Q\left[\log\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}\right)|\mathcal{F}_s\right] \leq \log\left[E^Q\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}|\mathcal{F}_s\right)\right]. \tag{3.48}$$

Thanks to Proposition 2.5, $\widetilde{Z}^Q = 1/\mathcal{E}(\widehat{\theta}\cdot S)$ and thus $\mathcal{E}(\theta\cdot S)/\mathcal{E}(\widehat{\theta}\cdot S)$ is a positive Q-local martingale. Hence, it is a Q-supermartingale which leads to

$$\log\left[E^Q\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}|\mathcal{F}_s\right)\right]\leq\log\left(\frac{\mathcal{E}_s(\theta\cdot S)}{\mathcal{E}_s(\widehat{\theta}\cdot S)}\right). \tag{3.49}$$

By (3.48) and (3.49), it is clear that $\log\left(\mathcal{E}(\theta\cdot S)/\mathcal{E}(\widehat{\theta}\cdot S)\right)$ is a Q-supermartingale and hence, $U_0^Q\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right)$ is a Q-supermartingale as well. This completes the proof.

□

Remark: The proof of (2) $\Rightarrow$ (1) does not require the assumption (3.44). Thus, the assertion (2) is the full description of the processes D_0 and D_1 such that $U_0(t,x)$, given by (3.1), is a forward utility.

3.B Discrete-Time Market Models

This section is devoted to illustrate the general results of Section 3.A on Discrete-time market models. Precisely, the trading times are countably many, say, $j = 0, 1, ..., N$, and the information flow of the market model is given by $\mathbb{F} = (\mathcal{F}_j)_{j=0,1,...,N}$. The d-dimensional stock price process is denoted by $S = (S_j)_{j=0,1,...,N}$, where S_i is a d-dimensional, **bounded** and $\mathcal{F}_j$ random variable.

In this setup, the random utility U_0 in the log-type given by (3.1) becomes

$$U_0(j,x) = D_0(j)\log(x) + D_1(j), \quad j = 0, 1, ..., N, \quad \forall\, x > 0, \tag{3.50}$$

Here, $D_0 = (D_0(j))_{j=0,1,...,N}$ and $D_1 = (D_1(j))_{j=0,1,...,N}$ are stochastic processes satisfying

$$\sup_{0\leq j\leq N} E\left(|D_0(j)|+|D_1(j)|\right) < +\infty. \tag{3.51}$$

Similarly as the continuous-time case, we denote the jumps of the price process by $\Delta S_j := S_j - S_{j-1}$, $j = 1, 2, ...N$, and their corresponding conditional distribution given $\mathcal{F}_{j-1}$ is denoted by $G_j(dx)$, i.e.

$$G_j(dx) := P(\Delta S_j \in dx | \mathcal{F}_{j-1}).$$

Then the sets $\mathcal{D}_j$, $j = 1, 2, ..., N$, take the following form

$$\mathcal{D}_j := \left\{ \theta \in \mathbb{R}^d \;\; | \;\; 1 + \theta^T x > 0, \;\; G_j(dx) - a.e. \right\}. \tag{3.52}$$

By following a similar argument as Lemma 3.1, the interior of $\mathcal{D}$, $int(\mathcal{D})$, can be expressed as

$$int(\mathcal{D}) = \left\{ \theta \in \mathbb{R}^d \;\; | \; \exists \;\; \delta > 0 \;\; s.t. \;\; 1 + \theta^T x \geq \delta, \;\; G_j(dx) - a.e. \right\}.$$

Associated with the log-type forward utility (3.50), we denote the set of admissible portfolio rates for the j^{th} period of time, $j = 1, 2, ..., N$, by $\Theta_j^{(0)}$, which is given by

$$\Theta_j^{(0)} := \left\{ \theta \in L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j | E\left(|D_0(j) \log(1 + \theta^T \Delta S_j)| | \mathcal{F}_{j-1} \right) < +\infty \right\}. \tag{3.53}$$

The main assumption imposed on this model is given in the following:

Assumption: For any $j = 1, ..., N$, $\theta \in \mathcal{D}_j$, P-a.e., and every sequence $(\theta_n)_{n \geq 1}$, $\theta_n \in int(\mathcal{D}_j)$, P-a.e., and converges to θ, we have, $P - a.e.$

$$\lim_{n \to +\infty} E\Big(\frac{|D_0(j) \theta_n^T \Delta S_j|}{1 + \theta_n^T \Delta S_j} \Big| \mathcal{F}_{j-1} \Big) = \begin{cases} +\infty, & \text{on } \Gamma_j; \\ E\Big(\frac{|D_0(j) \theta^T \Delta S_j|}{1 + \theta^T \Delta S_j} \Big| \mathcal{F}_{j-1} \Big), & \text{on } \Gamma_j^c. \end{cases} \tag{3.54}$$

where $\Gamma_j := \{ G_j(\mathbb{R}^d) > 0 \;\; \text{and} \;\; \theta \notin int(\mathcal{D}_j) \}$.

The next theorem states our parametrization algorithm for log-type forward utilities having the form of (3.50).

Theorem 3.4: *Suppose that S is bounded and the assumptions (3.51) and*

(3.54) hold. Then, the followings are equivalent.

(i) The functional U_0, given by (3.50), is a forward utility with the optimal portfolio rate $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The following two properties hold:

(ii.1) D_0 is a positive martingale and the process $\widehat{\theta}$ for $j = 1, 2, ..., N$ satisfies

$$\widehat{\theta}_j \in \Theta_j^{(0)} \quad \textit{and is a root for} \quad E\left(\frac{D_0(j)}{1+\theta^T \Delta S_j} \Delta S_j | \mathcal{F}_{j-1}\right) = 0. \tag{3.55}$$

(ii.2) The process D_1 is a supermartingale with the predictable part A^{D_1}, given by

$$A_j^{D_1} = -\sum_{k=1}^{j} E\left(D_0(k)\log(1+\widehat{\theta}_k^T \Delta S_k)|\mathcal{F}_{k-1}\right), \quad j = 0, 1, ..., N.^3 \tag{3.56}$$

Proof. Suppose that assertion (i) holds. Then, for any $x \in (0, +\infty)$, the functional $U_0(j, x)$ is strictly concave with respect to x. Hence, $D_0(j)$ is positive almost surely for any $j = 0, 1, ..., N$. Meanwhile, the processes [4]

$$U_0\left(j, x\prod_{k=1}^{j}(1+\widehat{\theta}_k^T \Delta S_k)\right), \quad j = 0, 1, ..., N, \text{ is a martingale} \tag{3.57}$$

and for any admissible $\theta = (\theta_j)_{j=1,...,N}$ (i.e. $\theta_j \in \Theta_j^{(0)}$),

$$U_0\left(j, x\prod_{k=1}^{j}(1+\theta_k^T \Delta S_k)\right), \quad j = 0, 1, ..., N, \quad \text{is a supermartingale.} \tag{3.58}$$

It is easy to see that the null portfolio rate $\theta = 0$ is admissible due to (3.50) (i.e. $0 \in \Theta_j^{(0)}$). Then, a direct application of (3.58) implies that for any $x > 0$,

$$U_0(j, x) = D_0(j)\log(x) + D_1(j), \quad j = 0, 1, ..., N, \quad \text{is a supermartingale.} \tag{3.59}$$

Due to the arbitrariness of x in (3.59), we put $x = 1$ and $x = e$ respectively

[3]By convention, the sum of empty, when $j = 0$, is zero.

[4]By convention, the product of empty, when $j = 0$, is one.

and deduce that D_1 is a supermartingale and D_0 is a special semimartingale with the following Doob-Meyer decompositions

$$D_1 = D_1(0) + M^{D_1} + A^{D_1}, \quad D_0 = D_0(0) + M^{D_0} + A^{D_0},$$

$$\text{with} \;\; M_0^{D_1} = A_0^{D_1} = M_0^{D_0} = A_0^{D_0} = 0,$$

where M^{D_1} and M^{D_0} are local martingales, and, A^{D_1} and A^{D_0} are predictable with finite variation. Accordingly, the supermartingale U_0 in (3.59) can be rewritten as

$$U_0 = D_1(0) + D_0(0)\log(x) + M^{D_1} + M^{D_0}\log(x) + A^{D_0}\log(x) + A^{D_1}. \quad (3.60)$$

where its predictable part is $A^{D_0}\log(x) + A^{D_1}$. For any $x > 0$, the supermartingale property of U_0 implies that

$$A_j^{D_0}\log(x) + A_j^{D_1} \leq 0, \quad \forall \; x > 0, \quad j = 1, 2, ..., N. \quad (3.61)$$

It is a linear function in $\log(x)$ and holds true only if $A_j^{D_0} = 0$ and $A_j^{D_1} \leq 0$, P-a.s. We therefore proved that the process D_0 a local martingale. Then, an application of Theorem 2 in [38] leads to that D_0 is a martingale.
To prove (ii.2), we consider a process $M = (M_j)_{j=0,1,...,N}$, given by

$$M_j := D_1(j) + \sum_{k=1}^{j} E\left(D_0(k)\log(1 + \widehat{\theta}_k^T \Delta S_k)|\mathcal{F}_{k-1}\right).$$

After simplification and application of the martingale property of D_0, the conditional expectation, $E(M_j|\mathcal{F}_{j-1})$, can be calculated as follows

$$E(M_j \mid \mathcal{F}_{j-1}) = E\left(D_1(j) + D_0(j)\sum_{k=1}^{j}\log\left((1 + \widehat{\theta}_k^T \Delta S_k)\right)|\mathcal{F}_{j-1}\right) -$$
$$D_0(j-1)\sum_{k=1}^{j-1}\log(1 + \widehat{\theta}_k^T \Delta S_k) + \sum_{k=1}^{j-1} E\left(D_0(k)\log(1 + \widehat{\theta}_k^T \Delta S_k)|\mathcal{F}_{k-1}\right).$$

Remark that the process

$$D_1(j) + D_0(j) \sum_{k=1}^{j} \log\left(1 + \widehat{\theta}_k^T \Delta S_k\right),\ j = 0, ..., N$$

is a martingale due to (3.57), thus,

$$E(M_j|\mathcal{F}_{j-1}) = D_1(j-1) + \sum_{k=1}^{j-1} E\left(D_0(k)\log(1 + \widehat{\theta}_k^T \Delta S_k)|\mathcal{F}_{k-1}\right) = M_{j-1}.$$

This proved that $(M_j)_{j=0,1,\dots,N}$ is a martingale and hence A^{D_1} is the predictable part of D_1. This completes the statement (ii.2).

Now, it remains to prove (3.55) in order to complete the proof of (ii). Again, due to the martingale property of D_0 together with (3.57) and (3.58), we derive

$$E\left(D_0(j)\log(1 + \widehat{\theta}_j^T \Delta S_j)|\mathcal{F}_{j-1}\right) = D_1(j-1) - E(D_1(j)|\mathcal{F}_{j-1}), \quad \text{and}$$
$$E\left(D_0(j)\log(1 + \theta_j^T \Delta S_j)|\mathcal{F}_{j-1}\right) \leq D_1(j-1) - E(D_1(j)|\mathcal{F}_{j-1}), \quad \forall\ \ \theta_j \in \Theta_j^{(0)}.$$

Then, by putting $Q := \frac{D_0(N)}{D_0(0)} \cdot P$ and

$$\Phi_j(y) := \int \log(1 + y^T x)\widetilde{G}_j(dx), \quad \widetilde{G}_j(dx) := Q(\Delta S_j \in dx|\mathcal{F}_{j-1}), \quad y \in \mathcal{D}_j, \tag{3.62}$$

we obtain

$$\Phi_j(\theta_j) \leq \Phi_j(\widehat{\theta}_j), \quad \forall\ \ \theta_j \in \Theta_j^{(0)}, \quad \forall\ \ j = 1, 2, ..., N. \tag{3.63}$$

Thanks to the assumption (3.54) and Proposition 3.1, we can conclude that for any $j = 1, 2, ..., N$, $\widehat{\theta}_j$ is a root for

$$\int \frac{x}{1 + \theta^T x}\widetilde{G}_j(dx) = E\left(\frac{D_0(j)}{1 + \theta^T \Delta S_j}\Delta S_j|\mathcal{F}_{j-1}\right) = 0.$$

This proves (3.55) and completes the proof of (i)⇒(ii).

To prove the reverse, we suppose that assertion (ii) holds. Since the processes $M := D_1 - A^{D_1}$ and D_0 are martingales, we obtain

$$
\begin{aligned}
& E\left(D_1(j) + D_0(j)\sum_{k=1}^{j}\log\left((1+\widehat{\theta}_k^T\Delta S_k)\right)|\mathcal{F}_{j-1}\right) \\
= & E\left(D_1(j) + \sum_{k=0}^{j} E\left(D_0(k)\log(1+\widehat{\theta}_k^T\Delta S_k)|\mathcal{F}_{k-1}\right)|\mathcal{F}_{j-1}\right) \\
& + D_0(j-1)\sum_{k=0}^{j-1}\log(1+\widehat{\theta}_k^T\Delta S_k) - \sum_{k=0}^{j-1} E\left(D_0(k)\log(1+\widehat{\theta}_k^T\Delta S_k)|\mathcal{F}_{k-1}\right) \\
= & D_1(j-1) + D_0(j-1)\sum_{k=0}^{j-1}\log(1+\widehat{\theta}_k^T\Delta S_k).
\end{aligned}
$$

Hence, in one hand, the process

$$
U_0\left(j, x\prod_{k=1}^{j}(1+\widehat{\theta}_k^T\Delta S_k)\right),\ \ j = 0, 1, ..., N
$$

is a martingale and we have

$$
E\left(D_0(j)\log(1+\widehat{\theta}_j^T\Delta S_j)|\mathcal{F}_{j-1}\right) = D_1(j-1) - E(D_1(j)|\mathcal{F}_{j-1}) \tag{3.64}
$$

On the other hand, for any admissible portfolio rate $\theta = (\theta_j)_{j=1,2,...,N}$, due to the concavity of the function $\phi(y) := D_0\log(1+y)$, we derive

$$
D_0(j)\log(1+\theta_j^T\Delta S_j) - D_0(j)\log(1+\widehat{\theta}_j^T\Delta S_j) \leq \frac{D_0(j)}{1+\widehat{\theta}_j^T\Delta S_j}(\theta_j - \widehat{\theta}_j)^T\Delta S_j.
$$

By taking conditional expectation in both sides above and using (3.55), we obtain

$$
E\left(D_0(j)\log(1+\theta_j^T\Delta S_j)|\mathcal{F}_{j-1}\right) \leq E\left(D_0(j)\log(1+\widehat{\theta}_j^T\Delta S_j)|\mathcal{F}_{j-1}\right). \tag{3.65}
$$

Then, adding $D_0(j-1)\sum_{k=1}^{j-1}\log(1+\theta_k^T\Delta S_k) + E(D_1(j)|\mathcal{F}_{j-1})$ on both sides of

(3.65) and using the martingale property of D_0 together with (3.64), we have

$$
\begin{aligned}
& E\left(U_0\left(j, x\sum_{k=0}^{j}\log(1+\theta_k^T\Delta S_k)\right)|\mathcal{F}_{j-1}\right) \\
& \leq E\left(D_0(j)\log(1+\widehat{\theta}_j^T\Delta S_j)+D_1(j)|\mathcal{F}_{j-1}\right)+D_0(j-1)\sum_{k=1}^{j-1}\log(1+\theta_k^T\Delta S_k) \\
& = U_0\left(j-1, x\sum_{k=0}^{j-1}\log(1+\theta_k^T\Delta S_k)\right).
\end{aligned}
$$

Clearly, $U(j, x\sum_{k=0}^{j}\log(1+\theta_k^T\Delta S_k))$, $j = 0, 1, ..., N$, is a supermartingale for any admissible $\theta = (\theta_j)_{j=0,...,N}$ and this completes the whole proof. □

3.C Discrete Markets Models

This section will investigate two particular examples of the general discrete-time market model. More precisely, it will contain two subsections by considering respectively two cases:

- The stock price process is binomial (i.e. branches into two values at any time)
- The stock price process branches into n ($n > 2$) possible values at any time.

We pay attention to these two examples for their common feature that they are not relying on the assumption (3.54). The technical reason behind this will be explained in their proofs.

3.C.1 One-Dimensional Binomial Model

In this subsection, we will consider the binomial model. Let ξ_j be a $\mathcal{F}_j$-measurable random variable, which has two values, ξ_j^u and ξ_j^d satisfying $0 < \xi_j^d < 1 < \xi_j^u$ for any $j = 1, ..., N$. Then, given the price of the stock at time

$j-1$ (i.e. S_{j-1}), the price at time j will either go up to $S_{j-1}\xi_j^u$ or go down to $S_{j-1}\xi_j^d$. Therefore, we get

$$S_j = S_{j-1}\xi_j = S_0 \prod_{k=1}^{j} \xi_k.$$

Assume that $S > 0$, P-a.s., then $\xi_{j+1} = \frac{S_{j+1}}{S_j}$. We denote a sequence of events, $(A_j)_{j=1,\ldots,N}$

$$A_j := \{\xi_j = \xi_j^u\} \in \mathcal{F}_j. \tag{3.66}$$

For this current binomial model, the size of the sample space, $\#(\Omega) = 2^N$, is finite.

Similarly as in the discrete-time model in the general framework discussed in last section, we calculate the jump of S as follows

$$\Delta S_j := S_j - S_{j-1} = (\xi_j - 1)S_{j-1}, \quad j = 1, 2, \ldots N.$$

Therefore, the set $\mathcal{D}_j$, $j = 1, 2, \ldots, N$, becomes

$$\mathcal{D}_j := \left\{\theta \in \mathbb{R} \middle| \ 1 + (\xi_j^u - 1)\theta S_{j-1} > 0 \text{ and } 1 + (\xi_j^d - 1)\theta S_{j-1} > 0.\right\}.$$

Or, equivalently,

$$\mathcal{D}_j = \Big] 1/(1-\xi_j^u)S_{j-1}, \quad 1/(1-\xi_j^d)S_{j-1} \Big[. \tag{3.67}$$

Therefore, one of the features of this model lies in the fact that given $\mathcal{F}_{j-1}$, $\mathcal{D}_j$ is an open set in $\mathbb{R}$ and hence

$$int(\mathcal{D}_j) = \mathcal{D}_j, \quad P-a.e., \quad \forall \ \ j = 1, \ldots, N. \tag{3.68}$$

Furthermore, due to $\#(\Omega) < +\infty$, it is obvious that any real-valued random variable is integrable and its conditional expectation is finite as well. Thus, we conclude that the admissible sets, $\Theta_j^{(0)}$, defined in (3.53) take the following forms

$$\Theta_j^{(0)} = L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j, \quad j = 1, ..., N. \tag{3.69}$$

The characterization of the logarithm forward utilities in binomial model is stated in the following theorem.

Theorem 3.5: *The following two assertions are equivalent.*

(i) The functional $U_0(t,x)$, defined in (3.50), is a forward utility with the optimal portfolio rate denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The following properties hold:

(ii.1) D_0 is a positive martingale and $\widehat{\theta}_j$ is given by

$$\widehat{\theta}_j = \frac{(\xi_j^u - 1)Q_j - (1 - \xi_j^d)(1 - Q_j)}{(\xi_j^u - 1)(1 - \xi_j^d)S_{j-1}} \in \mathcal{D}_j, \tag{3.70}$$

where $Q_j := Q(A_j|\mathcal{F}_{j-1})$, $Q := \frac{D_0(N)}{D_0(0)} \cdot P$ and A_j is given by (3.66).

(ii.2) D_1 is a supermartingale with predictable part given by

$$-\sum_{k=1}^{j} \Big[\log\Big(\frac{\xi_k^u - \xi_j^d}{1 - \xi_k^d} Q_j\Big) Q_j + \log\Big(\frac{\xi_k^u - \xi_j^d}{\xi_k^u - 1}(1 - Q_j)\Big)(1 - Q_j)\Big]. \tag{3.71}$$

Proof. Remark that the one-dimensional binomial model is only a particular example of the discrete-time markets models. As a result, the proof of the current theorem would be very similar with–but simpler than–the proof of Theorem 3.4 due to the following three remarks.

a) Assumptions (3.54) and (3.51) are automatically satisfied for the current case due to (3.68) and $\#(\Omega) < +\infty$, respectively.

b) The function Φ_j given by (3.62) becomes

$$\Phi_j(\theta) = \log\big((1 + (\xi_j^u - 1)\theta S_{j-1})\big) Q_j + \log\big((1 + (\xi_j^d - 1)\theta S_{j-1})\big)(1 - Q_j), \tag{3.72}$$

which is differentiable for any $\theta \in \mathcal{D}_j$ and its derivative is given by

$$\Phi_j'(\theta) = \frac{(\xi_j^u - 1)S_{j-1}}{1 + (\xi_j^u - 1)\theta S_{j-1}} Q_j + \frac{(\xi_j^d - 1)S_{j-1}}{1 + (\xi_j^d - 1)\theta S_{j-1}} (1 - Q_j).$$

Clearly the solution of $\Phi_j'(\theta) = 0$ is given by (3.70).

c) It is clear that the predictable part of D_1 takes the exact form of (see (3.56))

$$-\sum_{k=1}^{j} \Big[\log \Big(1 + (\xi_j^u - 1)\widehat{\theta}_k S_{k-1}\Big) Q_j + \log \Big(1 + (\xi_j^u - 1)\widehat{\theta}_k S_{k-1}\Big)(1 - Q_j)\Big].$$

Afterwards, by plugging (3.70) into the above expression, (3.71) follows immediately. This ends the proof of the theorem.

□

3.C.2 Multi-Dimensional Discrete Model

This subsection will extend the one-dimensional binomial model given in Subsection 3.C.1 to multi-dimensional discrete model. Precisely, we will consider a market with d stocks, $(S_j^i)_{j=0,1,...,N}$, $i = 1, ..., d$. Same as before, we assume $S^i > 0$, $P-a.s.$. Moreover, at any time j, the stock price takes up to n, $n \geq 2$, possible values. In other words, for $i = 1, ..., d, \quad j = 0, 1, ..., N$,

$$S_{j+1}^i = \xi_{j+1}^i S_j^i, \quad \xi_{j+1}^i \in \{\xi_{j+1}^i(1),\ \xi_{j+1}^i(2),\ \dots\ , \xi_{j+1}^i(n)\},$$

where $\xi_{j+1}^i > 0$ is $\mathcal{F}_{j+1}$-measurable. In vector form, we put $\xi_j = (\xi_j^1, \xi_j^2, ..., \xi_j^d)^T$, $j = 1, ..., N$. Remark that for current multi-dimensional discrete model, the size of the sample space, Ω, is finite, i.e., $\#(\Omega) = (dn)^N$.

We denote the set of the index of different scenarios by $\mathcal{N} := \{1, 2, ..., n\}$. Then, for d stocks, when the time goes from j to $j + 1$, there are totally n^d combinations of different scenarios. If we represent one of such scenario by $\{n_1, n_2, ..., n_d\}$ with $n_1, n_2, ..., n_d \in \mathcal{N}$, then it is an element of the d-dimensional product space $\widetilde{\mathcal{N}} := \mathcal{N} \otimes \cdots \otimes \mathcal{N}$. Meanwhile, the price for d stocks, $S_{j+1} = (S_{j+1}^1, ..., S_{j+1}^d)^T$ at time $j + 1$ for a particular scenario $\{n_1, n_2, ..., n_d\}$, can be represented in a matrix form, i.e.

$$S_{j+1} = \Xi_{j+1}(n_1, n_2, ..., n_d) S_j,$$

$$\text{where} \qquad \Xi_{j+1}(n_1, n_2, ..., n_d) := \text{diag}(\xi^1_{j+1}(n_1), \xi^2_{j+1}(n_2), ..., \xi^d_{j+1}(n_d)).$$

We denote the event that the scenario $(n_1, ..., n_d) \in \widetilde{\mathcal{N}}$ occurs by $A_j(n_1, ..., n_d)$, $j = 1, ..., N$, given by

$$A_j(n_1, ..., n_d) := \{\xi^1_j = \xi^1_j(n_1),\ \xi^2_j = \xi^2_j(n_2),\ ...,\ \xi^d_j = \xi^d_j(n_d)\} \in \mathcal{F}_j, \quad (3.73)$$

which, naturally, should satisfy

$$\sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} P(A_j(n_1, ..., n_d)) = 1, \quad j = 1, ..., N.$$

As well, we can write the jump process of the price process in the matrix form as follows

$$\Delta S_j := S_j - S_{j-1} = (\Xi_j - I_{d\times d}) S_{j-1}, \quad j = 1, 2, ...N,$$

where $I_{d\times d}$ is the $d \times d$ identity matrix. Furthermore, we denote the d-dimensional vector with all entries equal to 1 by I_d.
In this setup, the set $\mathcal{D}_j$, $j = 1, 2, ..., N$, becomes

$$\mathcal{D}_j := \left\{\theta \in \mathbb{R}^d |\ 1+\theta^T(\Xi_j(n_1, ..., n_d)-I_{d\times d})S_{j-1} > 0, \forall\, (n_1, ..., n_d) \in \widetilde{\mathcal{N}}\right\}. \quad (3.74)$$

Remark that the set $\mathcal{D}_j$ is open. Therefore, we have

$$int(\mathcal{D}_j) = \mathcal{D}_j, \quad P-a.e., \quad \forall\ \ j = 1, ..., N. \quad (3.75)$$

Again, due to $\#(\Omega) < +\infty$, the admissible sets, $\Theta^{(0)}_j$, defined in (3.53), becomes

$$\Theta^{(0)}_j = L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j, \quad j = 1, ..., N. \quad (3.76)$$

The characterization of the logarithm forward utilities given by (3.50) in multi-dimensional discrete model is stated in the following theorem.

Theorem 3.6: *The following two assertions are equivalent.*

(i) The functional $U_0(t,x)$, defined by (3.50), is a forward utility with the optimal portfolio rate denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The following properties hold:

(ii.1) D_0 is a positive martingale and $\widehat{\theta}_j \in \mathcal{D}_j$ is a root of the equation

$$\sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \frac{(\Xi_j(n_1,...,n_d) - I_{d\times d})I_d}{1+\theta^T(\Xi_j(n_1,...,n_d) - I_{d\times d})S_{j-1}} Q(A_j(n_1,...,n_d)|\mathcal{F}_{j-1}) = 0, \tag{3.77}$$

where $Q := \frac{D_0(N)}{D_0(0)} \cdot P$ and A_j is given by (3.73).

(ii.2)D_1 is a supermartingale with predictable part given by

$$-\sum_{k=1}^{j} \sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \log\left(1+\theta^T(\Xi_k(n_1,...,n_d) - I_{d\times d})S_{k-1}\right) Q(A_k(n_1,...,n_d)|\mathcal{F}_{k-1}). \tag{3.78}$$

Proof. The proof of this theorem would be very similar with the proof of one-dimensional binomial model and a special case of the proof of Theorem 3.4. We omit the details of this proof, but only comment on the following.

Assumptions (3.54) and (3.51) are automatically satisfied for the current case due to (3.75) and $\#(\Omega) < +\infty$, respectively. Furthermore, in current model, the functional Φ_j given by (3.62) becomes

$$\Phi_j(\theta) := \sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \log(1+\theta^T(\Xi_j(n_1,...,n_d) - I_{d\times d})S_{j-1})Q(A_j(n_1,...,n_d)|\mathcal{F}_{j-1}),$$

whose derivative, $\Phi_j'(\theta)$, is exact same with the left-hand-side of (3.77). Finally, it has been proved in Theorem 3.4 that the process D_1 is a supermartingale and its predictable part is given by (3.78) by referring to (3.56). □

3.D Lévy Market Models

This section illustrates the results of Section 3.A on Lévy market models. I will discuss the general Lévy market model and proceed with characterizing the log-type forward utilities. This section contains two subsections, namely, Subsections 3.D.1 and 3.D.2 respectively, that address two popular and practical models.

Consider a financial market where the stock price process, S_t, $t \in [0,T]$, is given by

$$S_t = S_0 \exp(X_t), \tag{3.79}$$

where X is a **locally bounded** Lévy process.
For any Lévy process X, the Lévy-Khintchine formula allows us to decompose it as

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\leq 1} x\widetilde{N}(dt,dx) + \int_0^t \int_{|x|\geq 1} xN(dt,dx), \tag{3.80}$$

$$\text{where} \qquad \widetilde{N}(dt,dx) = N(dt,dx) - F^X(dx)dt,$$

Here, γ and σ are positive constant; $W = (W_t)_{t\in[0,T]}$, represents a Brownian motion; $N(dt,dx)$ is a random measure on $[0,T] \times \mathbb{R} \setminus \{0\}$, called Poisson random measure; $\widetilde{N}(dt,dx)$ is the compensated Poisson measure with the intensity measure $F^X(dx)dt$, where $F^X(dx)$ is called the Lévy measure defined on $\mathbb{R} \setminus \{0\}$, satisfying

$$\int_{\mathbb{R}\setminus\{0\}} (|x|^2 \wedge 1) F^X(dx) < +\infty. \tag{3.81}$$

For more details about Lévy processes, we refer the reader to [69]. Particularly, for its applications in mathematical finance, we refer the reader to [4], [19], [27], [28] and the references therein.

The Lévy processes treated here are semimartingales. Compared with the decomposition of general semimartingale given in (2.4), Lévy process inherits the same structure but has more concrete components. One of main features for Lévy model is that it is quasi-left continuous. For locally bounded Lévy process, we don't need to introduce the truncation function $h(x)$ since there is no "large jump". Also, the condition (3.81) is satisfied since the process $[X, X]$ is locally integrable and hence $W_t(x) := x$ is $\widetilde{N}(dt, dx)$-integrable on $[0, T] \times \mathbb{R} \setminus \{0\}$ due to

$$\left(\sum_s (\Delta X_s)^2\right)^{1/2} \leq ([X, X])^{1/2} \in \mathcal{A}^+_{loc}.$$

Therefore, the process X can be represented as

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} x\widetilde{N}(dt, dx). \tag{3.82}$$

In the same spirit as Theorem 2.2, any local martingale N in this model can be decomposed as follows

$$N = \beta \cdot S^c + (Y - 1) \star (\mu - \nu) + V \star \mu + N', \quad \nu(dt, dx) = F^S(dx)dt. \tag{3.83}$$

Here, the vector of processes, (β, Y, V, N'), is the Jacod components of N.
For this Lévy market model, our goal is to characterize the logarithm-type forward utility, $U_0(t, x)$, given by

$$U_0(t, x) = D_0(t)\log(x) + D_1(t), \tag{3.84}$$

Here we suppose that the processes D_0 and D_1 of (3.84) satisfy

$$\sup_{\tau\in\mathcal{T}_T} E\left(|D_0(\tau)|+|D_1(\tau)|\right) < +\infty. \tag{3.85}$$

Precisely, we need to describe as explicitly as possible the processes D_0 and D_1 such that $U_0(t, x)$ is a forward utility. The set $\mathcal{D}$ given by (3.3) becomes

$$\mathcal{D} := \left\{\theta \in \mathbb{R} \ : \ \ 1 + e^X\theta(e^x - 1) > 0, \qquad F^X - a.e.\right\}. \tag{3.86}$$

For any probability measure P, stock price process S, and $x > 0$ such that $U_0(t, x, \omega) < +\infty$ we denote by

$$\mathcal{A}_{adm}(x) := \Big\{ \pi \in L(S) \mid \sup_{\tau \in \mathcal{T}_T} E\left(|D_0(\tau) \log(x + \pi \cdot S_\tau) + D_1(\tau)| \right) < +\infty \Big\}, \tag{3.87}$$

the set of admissible portfolios and we denoted by $\Theta(x)$ the corresponding set of portfolio rates. Here $\mathcal{T}_T$ is the set of stopping time, τ, such that $\tau \leq T$. Remark that under the assumption (3.85), Proposition 3.2 implies the description on D_0 and D_1 that

$$D_0 \text{ is a positive martingale and } D_1 \text{ is a supermartingale.} \tag{3.88}$$

are necessary conditions for U_0 being a forward utility. Thus, without loss of any generality, we suppose that (3.88) holds. It allows us to write D_0 in the form of stochastic exponential, given by $D_0 = D_0(0)\mathcal{E}(N)$, where N is a local martingale and we let $(\beta, Y, V, \overline{N}')$ be the Jacod components of N. Also, we consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $dP \otimes dt$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n \geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $dP \otimes dt$-a.e., and $\lambda_n \to \lambda$, we have, $dP \otimes dt - a.e.$,

$$\lim_{n \to +\infty} \int K_0(e^{X_-}(e^x - 1)\lambda_n) Y(e^{X_-}(e^x - 1)) F^X(dx) \tag{3.89}$$
$$= \begin{cases} +\infty, & \text{on } \Gamma; \\ \int K_0(e^{X_-}(e^x - 1)\lambda) Y(e^{X_-}(e^x - 1)) F^X(dx), & \text{on } \Gamma^c. \end{cases}$$

where $K_0(y) := \dfrac{|y|^2}{1+y}$ and $\Gamma := \{F^X(\mathbb{R}) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

The main result in this section is given in the following theorem.

Theorem 3.7: *Consider the functional $U_0(t, \omega, x)$ defined in (3.84). Suppose that (3.85), (3.88) and (3.89) hold. Then the following two assertions are* *equivalent.*

(1) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) Let $Q := \frac{D_0(T)}{D_0(0)} \cdot P$. Then, the following properties hold:

(2.a) D_1/D_0 is a Q-supermartingale and its predictable and finite variation part is given by

$$-\int_0^{\cdot} [\frac{\sigma^2 e^{2X_u}}{2} \widehat{\theta}_u^2 + \widetilde{\xi}_u] du. \tag{3.90}$$

where

$$\widetilde{\xi}_u := \int_{\mathbb{R}\setminus\{0\}} f_0((1 + e^{X_u}(e^x - 1)\widehat{\theta}_u)^{-1} - 1) Y(e^{X_u}(e^x - 1)) F_u^X(dx).$$

(2.b) The optimal portfolio rate $\widehat{\theta} \in int(\mathcal{D})$ is a root for

$$\gamma + \frac{1}{2}\sigma^2 + e^X \sigma^2(\beta - \lambda) + \int_{\mathbb{R}\setminus\{0\}} \frac{e^x - 1}{1 + e^X(e^x - 1)\lambda} Y(e^X(e^x - 1)) F^X(dx) = 0. \tag{3.91}$$

(2.c) The local martingale $\widehat{N} := D_1 - D_0 \log(\mathcal{E}(\widehat{\theta} \cdot S))$ is a true martingale.

Proof. Remark that for locally bounded X, $S = e^X$ is also locally bounded and is a Lévy process. Thus, by Ito's formula, the dynamics of S can be presented as

$$\frac{dS_t}{S_{t-}} = (\gamma + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1) F_t^X(dx)) dt + \sigma dW_t + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1) \widetilde{N}(dt, dx).$$

We can find out the predictable characteristics of S from this representation, given by

- $S_t^c = \int_0^t e^{X_u} \sigma dW_u, \quad c_t = e^{2X_{t-}} \sigma^2,$
- $b_t = e^{X_{t-}} (\gamma + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^x - 1) F_t^X(dx))$
- $\nu(dt, dx) = F_t^S(dx) dt$

Furthermore, for any measurable and non-negative/integrable function $k(x)$, the two measures $F^X(dy)$ and $F^S(dx)$ are related in the following manner

$$\int_{\mathbb{R}\setminus\{0\}} k(x)F^S(dx) = \int_{\mathbb{R}\setminus\{0\}} k(e^{X_-}(e^y - 1))F^X(dy).$$

Then the characteristics of S under Q are given by

- $S^{c,Q} = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_u}\sigma^2\beta_u du,$
- $b^Q = e^{X_-}(\gamma + e^{X_-}\sigma^2\beta + \frac{\sigma^2}{2}) + e^{X_-}\int_{\mathbb{R}\setminus\{0\}} Y(e^{X_-}(e^x - 1))(e^x - 1)F^X(dx)$
- $F^Q(dx) = Y(x)F^S(dx).$

The assumption (3.89) allows us to apply Proposition 3.1 for the model (S, Q), which implies that $\widehat{\theta}$ is the root of the equation (3.91).
The MHM of order 0, $\widetilde{Z}^Q$, exists and the Minimal Hellinger process of order 0, $h^{(0)}(\widetilde{Z}^Q, Q)$ is given by

$$h^{(0)}(\widetilde{Z}^Q, Q) = \int_0^{\cdot} [\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2 + \widetilde{\xi}_u]du$$

We thus have (3.90).
For anything else on this proof that is not provided here can be derived in the same way as the proof of Theorem 3.3 and will be omitted here.

□

3.D.1 Jump-Diffusion Model

In this subsection, we will simplify more the Lévy market model in the following way. Let the stock price process S be given by

$$S = S_0 e^X, \quad X_t = \gamma t + \sigma W_t + \widetilde{N}_t, \quad \widetilde{N}_t = N_t - \lambda t. \tag{3.92}$$

Here, N is a simple Poisson process with rate λ ($\lambda > 0$) and $\widetilde{N}$ is the compensated Poisson process. In this model, if X has a jump, its size is 1 and the Poisson process N counts the number of jumps of X. Compared with the

Lévy measure $F^X(dx)$, the parameter λ can also be explained as the expected number of jumps per unit time. Here it is treated as a constant.

Consider the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$ and let the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by the Brownian Motion $W = (W_t)_{t\in[0,T]}$ and the Poisson process $N = (N_t)_{t\in[0,T]}$. Thus, any local martingale Y in this model can be represented as the sum of stochastic integrals with respect to W and $\widetilde{N}$. Indeed, we have

$$Y_t = Y_0 + \int_0^t \alpha_u dW_u + \int_0^t \eta_u d\widetilde{N}_u, \quad t \in [0, T], \tag{3.93}$$

where α and η are predictable processes such that

$$\int_0^T (\alpha_u^2 + \eta_u^2) du < +\infty, \quad P - a.s.$$

In the current context, the set $\mathcal{D}$ becomes

$$\mathcal{D} := \{\theta \in \mathbb{R} : \quad 1 + \theta e^{X_-}(e-1) > 0\} =]-\frac{e^{X_-}}{e-1}, \ +\infty[= int(\mathcal{D}).$$

Our main result in this subsection is put in the following theorem.

Theorem 3.8: *Consider the functional $U_0(t, \omega, x)$ defined in (3.84) satisfying (3.85). Then the following two assertions are equivalent.*

(1) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The processes D_0 and D_1 satisfy the following conditions :

(2.a) D_0 is a positive martingale and we put $Q := \frac{D_0(T)}{D_0(0)} \cdot P$.

(2.b) D_1/D_0 is a Q-supermartingale and its predictable and finite variation part is given by

$$-\int_0^{\cdot} [\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2 + \lambda(1+\eta_u) f_0((1 + e^{X_u}(e-1)\widehat{\theta}_u)^{-1} - 1)] du. \tag{3.94}$$

(2.c) The optimal portfolio rate $\widehat{\theta} \in \mathcal{D}$ is a root for

$$\gamma + \frac{1}{2}\sigma^2 + e^{X_-}\sigma^2(\alpha - \theta) + \lambda(e-1)\frac{1+\eta}{1 + e^{X_-}(e-1)\theta} = 0. \tag{3.95}$$

(2.d) The local martingale $\widehat{N} := D_1 - D_0 \log(\mathcal{E}(\widehat{\theta} \cdot S))$ *is a true martingale.*

Proof. The dynamics of S can be written as

$$\frac{dS_t}{S_{t-}} = (\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda)dt + \sigma dW_t + (e-1)d\widetilde{N}_t.$$

Its predictable characteristics are

$$S_t^c = \int_0^t e^{X_u}\sigma du, \quad c = e^{2X}\sigma^2, \quad A_t = t,$$

$$b_t = (\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda)\int_0^t e^{X_u} du.$$

The jumps of S can be calculated by

$$\Delta S_t = \begin{cases} e^{X_{t-}}(e-1), & \text{on } \{\Delta X_t = 1\}; \\ 0, & \text{on } \{\Delta X_t = 0\}. \end{cases}$$

Suppose (i) holds and we apply Theorem 3.2, the process D_0 is a positive martingale. We write it as $D_0 = \mathcal{E}(N)$, where the local martingale N can be represented as

$$N(t) = N(0) + \int_0^t \alpha_u dW_u + \int_0^t \eta_u d\widetilde{N}_u, \quad t \in [0, T]$$

Then, the characteristics of S with respect to Q are

$$S^{c,Q} = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_u}\sigma^2\alpha_u du,$$

$$b^Q = e^X(\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda) + e^{2X}\sigma^2\alpha + e^X\eta\lambda(e-1).$$

And, the dynamics of S under Q can be written as

$$dS_t = dS_t^{c,Q} + b_t^Q dt + e^X(e-1)d(N_t - (1+\eta_t)\lambda dt).$$

Therefore, $\widehat{\theta}$ will minimize the function

$$\Phi_0(\theta) := -\theta b^Q + \frac{1}{2}c\theta^2 + \lambda(1+\eta)\left(\theta e^{X_-}(e-1) - \log(1+\theta e^{X_-}(e-1))\right),$$

which is differentiable on $\mathcal{D}$, thus, $\widehat{\theta}$ is a root of the equation (3.95). The Minimal Hellinger process of order 0 can be calculated as

$$\int_0^{\cdot} [\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2 + \lambda(1+\eta_u)f_0((1+e^{X_u}(e-1)\widehat{\theta}_u)^{-1} - 1)]du$$

such that (3.94) would be the predictable part of D_1/D_0.
The remaining parts of the proof of this theorem follows from the proof of Theorem 3.3 and will be omitted. □

3.D.2 Black-Scholes Model

Here, we consider the Black-Scholes model where there are no jumps and the only source of uncertainty is from the Brownian Motion. Same as before, the price process $S = S_0 e^X$, where X is an Ito process, given by

$$X_t = \gamma t + \sigma W_t, \quad t \in [0,T].$$

The filtration is generated by $W = (W_t)_{t\in[0,T]}$ such that any local martingale, Y, can be represented as

$$Y_t = Y_0 + \int_0^t \alpha_u dW_u, \quad t \in [0,T],$$

where α is progressively measurable process satisfying

$$\int_0^T \alpha_u^2 du < +\infty, \quad P - a.s.$$

Consider the characterization problem of the log-type forward utility, given by (3.84), we derive the following result.

Theorem 3.9: *Consider the functional $U_0(t,\omega,x)$ defined by (3.84) satisfying (3.85). Then the following two assertions are equivalent.*

(1) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The processes D_0 and D_1 satisfy the following conditions :

(2.a) D_0 is a positive martingale and we put $Q := \frac{D_0(T)}{D_0(0)} \cdot P$.

(2.b) D_1/D_0 is a Q-supermartingale and its predictable and finite variation part is given by

$$-\frac{1}{2}\sigma^2 \int_0^t e^{2X_u}\widehat{\theta}_u^2 du, \quad t \in [0,T]. \tag{3.96}$$

(2.c) The optimal portfolio rate $\widehat{\theta}$ is given by

$$\widehat{\theta} = \alpha + e^{-X}(\gamma\sigma^{-2} + \frac{1}{2}). \tag{3.97}$$

(2.d) The local martingale $\widehat{N} := D_1 - D_0 \log(\mathcal{E}(\widehat{\theta} \cdot S))$ is a true martingale.

Proof. The proof of this theorem follows from Theorem 3.8 by putting $\lambda = 0$ and $\eta = 0$. Only note that the assumption (3.89) is automatically satisfied since this model is continuous and hence $F = 0$. □

3.E Volatility Models

The volatility models are of great interest in financial industry. The volatility is a stochastic process and follows its own dynamic. Here we consider two classes of volatility models, Corrected Stein and Stein Model and Barndorff-Nielsen Shephard Model. The first one has no jumps, while the later model permits jumps of Lévy type.

3.E.1 Corrected Stein and Stein Model

In the corrected Stein and Stein model, the price process S follows the dynamic

$$dS_t = \mu V_t^2 S_t dt + \sigma V_t S_t dB_t, \quad t \in [0,T], \tag{3.98}$$

and the volatility process V follows the dynamic

$$dV_t = (m - \alpha V_t)dt + \beta dW_t, \quad t \in [0, T]. \tag{3.99}$$

Here, all the parameters μ, σ, m, α and β are positive constants. The processes B and W are two Brownian Motions with the correlation coefficient $\rho \in (-1, +1)$. The filtration is generated by the Brownian Motions $B = (B_t)_{t\in[0,T]}$ and $W = (W_t)_{t\in[0,T]}$, i.e., $\mathbb{F} := (\mathcal{F}_t^{B,W})_{t\in[0,T]}$.

Theorem 3.10: *Consider the functional $U_0(t, \omega, x)$ defined in (3.84) satisfying (3.85). Then the following two assertions are equivalent.*

(1) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The processes D_0 and D_1 satisfy the following conditions :

(2.a) D_0 is a positive martingale and we put $Q := \frac{D_0(T)}{D_0(0)} \cdot P$.

(2.b) D_1/D_0 is a Q-supermartingale with the predictable and finite variation part given by

$$-\frac{1}{2}\sigma^2 \int_0^t (\mu\sigma^{-2} + \alpha S_u)^2 V_u^2 du. \tag{3.100}$$

(2.c) The optimal portfolio rate $\widehat{\theta}$ is given by

$$\widehat{\theta}_t = \mu\sigma^{-2} S_t^{-1} + \alpha_t. \tag{3.101}$$

(2.d) The local martingale $\widehat{N} := D_1 - D_0 \log(\mathcal{E}(\widehat{\theta} \cdot S))$ is a true martingale.

Proof. From the dynamics of stock price process S in (3.98), the predictable characteristics of S are

$$b_t = \mu V_t^2 S_t, \quad c_t = \sigma^2 V_t^2 S_t^2, \quad F = 0, \quad A_t = t.$$

Thanks to Theorem 3.2, the process D_0 is a positive martingale and hence it has the form of $D_0 = \mathcal{E}(N)$. Since the filtration is generated by the Brownian motions W and B, the local martingale N can be represented as

$$N(t) = N(0) + \int_0^t \alpha_u dB_u + \int_0^t \beta_u dW_u, \quad t \in [0,T],$$

where α and β are progressively measurable processes such that

$$\int_0^T (\alpha_u^2 + \beta_u^2) du < +\infty, \quad P - a.s.$$

The characteristics of S with respect to Q are

$$S_t^{c,Q} = \int_0^t \sigma V_u S_u dB_u - \int_0^t \sigma^2 V_u^2 S_u^2 \alpha_u du, \;\; b_t^Q = \mu V_t^2 S_t + \sigma^2 V_t^2 S_t^2 \alpha_t.$$

The optimal portfolio rate $\widehat{\theta}$ is the root of the following equation (note that the assumption (3.9) is satisfied since it is continuous (thus $F = 0$))

$$b^Q - c\theta = 0,$$

that obviously leads to (3.101). The Minimal Hellinger process of order 0 can be written as

$$\frac{1}{2}\sigma^2 \int_0^t S_u^2 V_u^2 \widehat{\theta}_u^2 du = \frac{1}{2}\sigma^2 \int_0^t (\mu\sigma^{-2} + \alpha S_u)^2 V_u^2 du.$$

Thus we have (3.100). We omit the proof of the remaining parts as they are straightforward from the proof of Theorem 3.3. □

3.E.2 Barndorff-Nielsen Shephard Model

The model considered in this subsection is known in the literature as the Barndorff-Nielsen-Shephard model (see [5] and [67]). Precisely, the flow of public information is represented by the filtration generated by a one dimensional Lévy process Y, $Y = Y^c + \widetilde{Y}^d$, where Y^c denotes the continuous part and its pure discontinuous part is driven by the random measure $\widetilde{\mu}(dt \times dz)$ which measures the jump of Y and has a compensator measure $\widetilde{\nu}(dt \times dz) = \widetilde{F}(dz) dA_t$.

Remark that this model is a quasi-left-continuous and for simplicity, we put $A_t = t$ since it is continuous and $\langle Y^c \rangle_t = t$.
The price process S is **locally bounded** and defined by $S_t = \exp(X_t)$, with X following the dynamics

$$dX_t = (\mu + \beta\sigma_t^2)dt + \sigma_t dY_t^c + d(\rho z \star \widetilde{\mu}_Y)_t, \tag{3.102}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + d(z \star \widetilde{\mu}_Y)_t, \tag{3.103}$$

where the parameters $\mu, \beta, \rho, \lambda$ are real constants with $\lambda > 0$ and $\rho < 0$.
The set $\mathcal{D}$ is given by

$$\mathcal{D} := \left\{ \theta \in \mathbb{R} \ : \ 1 + S_{-}\theta(e^{\rho x} - 1) > 0, \qquad \widetilde{F} - a.e. \right\}. \tag{3.104}$$

Thanks to the assumption (3.85) and Proposition 3.2, the following conditions

$$D_0 \text{ is a positive martingale and } D_1 \text{ is a supermartingale.} \tag{3.105}$$

are necessary conditions for U_0 being a forward utility. Thus, without loss of any generality, we suppose that (3.105) holds. It allows us to write D_0 as $D_0 = D_0(0)\mathcal{E}(N)$, where N is a local martingale and we let $(\beta, Y, V, \overline{N}')$ be the Jacod components of N. We consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $dP \otimes dt$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $dP \otimes dt$-a.e., and $\lambda_n \to \lambda$, we have, $dP \otimes dt - a.e.$,

$$\lim_{n\to+\infty} \int K_0(e^{X_-}(e^x - 1)\lambda_n) Y(e^{X_-}(e^x - 1))\widetilde{F}(dx) \tag{3.106}$$
$$= \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \displaystyle\int K_0(e^{X_-}(e^x - 1)\lambda) Y(e^{X_-}(e^x - 1))\widetilde{F}(dx), & \text{on} \quad \Gamma^c. \end{cases}$$

where $K_0(y) := \dfrac{|y|^2}{1+y}$ and $\Gamma := \{\widetilde{F}(\mathbb{R}) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

For this model, we have the following characterization of the log-type forward utility.

Theorem 3.11: *Consider the functional $U_0(t,\omega,x)$ defined in (3.84) satisfying (3.85). Suppose that (3.105) and (3.106) hold. Then the following two assertions are equivalent.*

(1) The functional U_0 is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) Let $Q := \frac{D_0(T)}{D_0(0)} \cdot P$. Then, the followings hold.

(2.a) D_1/D_0 is a Q-supermartingale and its predictable and finite variation part is given by

$$-\int_0^{\cdot} \left[\frac{1}{2}S_u^2\sigma_u^2\widehat{\theta}_u^2 + \widetilde{\xi}_u\right] du, \tag{3.107}$$

where

$$\widetilde{\xi}_u := \int f_0((1+S_u(e^{\rho x}-1)\widehat{\theta}_u)^{-1}-1)Y(S_u(e^{\rho x}-1))\widetilde{F}_u(dx).$$

(2.b) The optimal portfolio rate $\widehat{\theta}$ is a root for

$$\mu+(\beta+\frac{1}{2})\sigma^2 - S_-\sigma^2\theta + \int \frac{e^{\rho x}-1}{1+S_-(e^{\rho x}-1)\theta} Y(S_-(e^{\rho x}-1))\widetilde{F}(dx) = 0. \tag{3.108}$$

(2.c) The local martingale $\widehat{N} := D_1 - D_0 \log(\mathcal{E}(\widehat{\theta}\cdot S))$ is a true martingale.

Proof. By Ito's formula, the dynamics of S can be represented as

$$\frac{dS_t}{S_{t-}} = \left(\mu+\sigma_t^2(\beta+\frac{1}{2}) + \int(e^{\rho z}-1)\widetilde{F}_t(dz)\right)dt + \sigma_t dY_t^c + d(e^{\rho z}-1)\star(\widetilde{\mu}_Y - \widetilde{\nu}_Y)_t. \tag{3.109}$$

We have $S_t^c = S_-\sigma\cdot Y_t^c$, $\Delta S_t = S_{t-}(e^{\rho\Delta\sigma_t^2}-1)$. We start by calculating the predictable characteristics, (b,c,ν^S), of S from the dynamics (3.109) as follows,

$$\begin{aligned} b_t &= S_{t-}\left(\mu+\sigma_t^2(\beta+\frac{1}{2}) + \int(e^{\rho x}-1)\widetilde{F}_t(dx)\right), \\ c_t &= S_{t-}^2\sigma_t^2, \quad \nu^S(dt\times dz) = F_t^S(dz)dt. \end{aligned} \tag{3.110}$$

Furthermore, for any measurable and non-negative/integrable function $k(x)$, the two measures F^S and $\widetilde{F}$ are related in the following manner

$$\int_{\mathbb{R}\setminus\{0\}} k(x)F^S(dx) = \int_{\mathbb{R}\setminus\{0\}} k(e^{X_-}(e^{\rho y}-1))\widetilde{F}(dy).$$

Then the characteristics of S with respect to Q are

$$S^{c,Q} = \int_0^t S_{u-}\sigma_u dY_u^c - \int_0^t S_{u-}^2\sigma_u^2\alpha_u du,$$

$$b^Q = S_-\left(\mu + \sigma^2(\beta + \frac{1}{2}) + \int Y((e^{\rho x}-1)S_-)(e^{\rho x}-1)\widetilde{F}(dx)\right) + S_-^2\sigma^2\alpha,$$

$$F^Q(dx) = Y(x)F^S(dx) = Y((e^{\rho x}-1)S_-)\widetilde{F}(dx).$$

Then $\widehat{\theta}$ is the root of the equation (3.108). The Minimal Hellinger process of order 0, $h^{(0)}(\widetilde{Z}^Q, Q)$ can be derived as

$$\frac{1}{2}\int_0^t \left[S_{u-}^2\sigma_u^2\widehat{\theta}_u^2 + \widetilde{\xi}_u\right] du$$

Thus, we have (3.107). We omit the remaining proof of this theorem. They are same with the proof of Theorem 3.3. □

Chapter 4

Power-Type Forward Utilities

This chapter focuses on the description of forward utilities that have the form of power function. Their features lie in the randomness of their risk-aversion parameter p and a factor D.

Definition: Let X be a RCLL semimartingale and Q be a probability measure. Then, we call power/power-type forward utility for (X, Q), any forward dynamic utility for (X, Q) that takes the following form

$$U_p(t, \omega, x) := D(t, \omega)x^{p(t,\omega)}, \qquad \text{with} \quad \inf_{0\leq t\leq T}|p(t, \omega)| > 0, \quad P - a.s. \tag{4.1}$$

Here $D = (D(t))_{0\leq t\leq T}$ and $p = (p(t))_{0\leq t\leq T}$ are stochastic processes.

There are several sections in this chapter where we consider different models to exhibit our results. The forthcoming section addresses the most general case, where the stochastic processes p and D are treated separately.

4.A The Semimartingale Framework

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration is complete and right continuous. Here, T represents a fixed horizon for investments. In this setup, we consider a d-dimensional **locally bounded** semimartingale $S = (S_t)_{0\leq t\leq T}$ which represents the discounted price processes of d risky assets.

The set of equivalent local martingale measures is given by

$$\mathcal{M}^e_{loc}(S) := \Big\{Q \in \mathbb{P}_e : \quad S \text{ is a local martingale under } Q\Big\}. \tag{4.2}$$

For any probability measure Q, any stock price process X, and $x \in \mathbb{R}$ such that $U_p(t,x,\omega) < +\infty$, P-a.s., $\forall\, t \in [0,T]$, we denote by

$$\mathcal{A}_{adm}(x,X,Q) := \Big\{\pi \in L(X) \mid \sup_{\tau \in \mathcal{T}_T} E^Q\left[D(\tau)^-(x+\pi\cdot X_\tau)^{p(\tau)}\right] < +\infty\Big\}, \tag{4.3}$$

the set of admissible portfolios for the model (x,X,Q,U). Here $\mathcal{T}_T$ is the set of stopping time, τ, such that $\tau \leq T$. When $X = S$ and $Q = P$, we simply write $\mathcal{A}_{adm}(x)$.

For any random field utility, $\overline{U}(t,x)$, $x \in \mathbb{R}^+$ we define its Frenchel-Legendre conjugate (called hereafter its random field conjugate), $\overline{V}(\omega,t,y)$, by

$$\overline{V}(t,\omega,y) := \sup_{x>0}\left(\overline{U}(t,\omega,x) - xy\right), \quad for \quad t \geq 0,\ y > 0. \tag{4.4}$$

In particular, for a random field utility U_p, of the form (4.1), we calculate its conjugate, V_p as follows

$$V_p(t,y) = -(D(t)p(t))^{1-q(t)}\frac{y^{q(t)}}{q(t)},$$

where q is the conjugate process of p and is given by $q(t) := \frac{p(t)}{p(t)-1}$, $t \geq 0$.

Our first result in this section is simple but very useful, and it deals with deriving the semimartingale structures for both processes p and D as a result of the forward property of the utility.

Proposition 4.1: *Consider a random utility, U_p, having the form (4.1). Then,* *if $U_p(t,x)$ is a forward utility and satisfies*

$$\sup_{\tau \in \mathcal{T}_T} E\left[D(\tau)^-\right] < +\infty. \tag{4.5}$$

Then $(D(t))_{t\geq 0}$ is a RCLL supermartingale and $(p(t))_{t\geq 0}$ is a RCLL semimartingale satisfying $p(t) < 1,\ P-a.s.$

Proof. From the definition of U_p given by (4.1), we deduce that for any $x > 0$, $U_p(t,x)$ is an adapted and RCLL process . Thus, by taking $x = 1$, we deduce that $D(t) = U(t,1)$ is an adapted and RCLL process. Also, the function $x \mapsto U_p(t,x)$ is strictly increasing and strictly concave, hence by taking $x = 1$, $U_p'(t,1) = D(t)p(t)$ is positive and $U_p''(t,1) = D(t)p(t)(p(t)-1)$ is negative. This implies $D(t) \neq 0$ and $p(t) < 1$, P-a.s. Thus, by taking $x = e$, we get

$$p(t) = \log\left(\frac{U_p(t,e)}{D(t)}\right). \tag{4.6}$$

Hence, p is an adapted and RCLL process.
On the other hand, under the assumption (4.5), the null portfolio is admissible and $U_p(t,x) = D(t)x^{p(t)}$ is a supermartingale for any $x > 0$. Again, we set $x = 1$ and conclude that $D(t)$ is a supermartingale. In (4.6), a direct application of Ito's formula implies $p(t)$ is a semimartingale. This completes the proof of this proposition. □

Proposition 4.2: *Suppose that S is locally bounded and $\mathcal{M}^e_{loc}(S) \neq \emptyset$. Let p be a real number such that $p \in (-\infty, 0) \cup (0,1)$, and consider*

$$U_p(t,x) = D(t)x^p. \tag{4.7}$$

For any $x \in (0,+\infty)$, consider the following maximization problem

$$\max_{\theta \in \mathcal{A}_{adm}(x)} E\Big[U_p\left(T, x + (\theta \cdot S)_T\right)\Big], \tag{4.8}$$

where the set $\mathcal{A}_{adm}(x)$ is defined in (4.3). Then following assertions hold.

(1) For any $x \in (0, +\infty)$, if the solution to (4.8) —that we denote by $\widetilde{\theta}_x$— exists, then

$$x + \widetilde{\theta}_x \cdot S > 0, \qquad \textit{and} \quad x + (\widetilde{\theta}_x \cdot S)_- > 0. \tag{4.9}$$

(2) Furthermore, the optimal portfolio rate for U_p with initial capital x, that we denote by $\widehat{\theta}_x := \Big(x + (\widetilde{\theta}_x \cdot S)_- \Big)^{-1} \widetilde{\theta}_x$, is independent of $x \in (0, +\infty)$ (or equivalently $\widetilde{\theta}_x / x$ is independent of x).

Proof. It is clear from [48], that the random variable $x + (\widetilde{\theta}_x \cdot S)_T$ is positive, and the process $(x + \widetilde{\theta}_x \cdot S)Z$ is a supermartingale, for any density process Z of $Q \in \mathcal{M}^e_{loc}(S) \neq \emptyset$. These imply that both processes $x + \widetilde{\theta}_x \cdot S$ and $x + (\widetilde{\theta}_x \cdot S)_-$ are positive and assertion (1) follows. To prove assertion (2), it is enough to remark that for any $x \in (0, +\infty)$, $x\widetilde{\theta}_1 \in \mathcal{A}_{adm}(x)$, and for any $\theta \in \mathcal{A}_{adm}(x)$, we have $x^{-1}\theta \in \mathcal{A}_{adm}(1)$. This ends the proof of the proposition. □

This subsection formulates our main results that explicitly parameterize forward utilities having the form of (4.1). In fact, through out the rest of the chapter, we denote by $U_p(t, x)$ the functional defined in (4.1). This functional depends, also, on the uncertainty $\omega \in \Omega$, while through out the chapter we use the shorthand $U_p(t, x)$ for the sake of simplifying notations.

Our first step —within the explicit parametrization of forward utilities of the form of (4.1)— consists of describing the dynamic of the process $(p(t))_{0 \leq t \leq T}$. This is the aim of the following subsection.

4.A.1 The Dynamic of the Risk-Aversion Process p

The following theorem is the main result in this subsection. It describes the dynamic of the risk-aversion process p under the forward property.

Theorem 4.1: *Suppose that S is locally bounded. Let $U_p(t, x)$ be defined in (4.1), such that $p = (p(t))_{0 \leq t \leq T}$ is locally bounded, satisfies (4.5) and*

$$p(t) < 1, \quad \textit{and} \quad \inf_{0 \leq t \leq T} |p(t)| > 0 \quad P - a.s. \tag{4.10}$$

If $U_p(t,x)$ is a forward utility, then the process p is constant in (ω,t), i.e.

$$p(\omega,t) = p(0), \quad 0 \leq t \leq T, \qquad P - .a.s. \tag{4.11}$$

The proof of Theorem 4.1 requires a number of intermediary steps. We will start by recalling a useful result of [6], where the author tried to measure the effect of forward property on the random field conjugate, $V(t,y)$.

Proposition 4.3: *If $U(t,x)$ is a forward utility, then for any T', $0 \leq t \leq T' \leq T$ and any $\eta \in L^0_+(\mathcal{F}_t)$, we have*

$$V(t,\eta) \leq \operatorname*{ess\,inf}_{Z \in \mathcal{Z}^e_{loc}(S)} E\left(V(T', \eta \frac{Z_{T'}}{Z_t}) \Big| \mathcal{F}_t \right), \quad P - a.s. \tag{4.12}$$

Proof. The proof of this proposition can be found in [6]. □

Lemma 4.1: *Suppose that the assumptions of Theorem 4.1 are fulfilled and the process $p = (p(t))_{t \geq 0}$ is positive. Then, the process p is constant in (ω,t) (i.e. Theorem 4.1 holds true in this case).*

Proof. Recall that any random field utility $U_p(t,x)$ is strictly increasing as a function of x for any $(t,\omega) \in [0,T] \times \Omega$. It implies the product $pD > 0$, $P - a.s.$ and, hence $D > 0$, $P - a.s.$, since $p > 0$, $P - a.s.$ Furthermore, it is easy to see that the null portfolio rate $\theta = 0$ is admissible (that is, $0 \in \Theta(x)$) for any $x > 0$. Therefore, an application of the optional sampling theorem to the supermartingale $U_p(t,x)$ leads to

$$E\left(D(\sigma)x^{p(\sigma)} | \mathcal{F}_\tau\right) \leq D(\tau)x^{p(\tau)}, \quad \forall\ x > 0, \ \ \forall\ \ \sigma,\ \tau \in \mathcal{T}_T,\ \sigma > \tau. \tag{4.13}$$

By putting

$$Q := \frac{D(\sigma)/D(\tau)}{E(D(\sigma)/D(\tau))} \cdot P \ \text{ and } \ \Delta := p(\sigma) - p(\tau),$$

the inequality (4.13) becomes

$$E^Q\left(e^{\log(x)\Delta} - 1|\mathcal{F}_\tau\right) \le C_Q, \quad C_Q := \frac{D(\tau)}{E(D(\sigma)|\mathcal{F}_\tau)} - 1, \quad \forall\, x > 0. \tag{4.14}$$

The random variable Δ can be written in the form of $\Delta = \Delta^+ - \Delta^-$ and thus

$$e^{\log(x)\Delta} - 1 = e^{\log(x)\Delta^+} + e^{-\log(x)\Delta^-} - 2.$$

By inserting this into (4.14), we get

$$E^Q\left(e^{\log(x)\Delta^+} + e^{-\log(x)\Delta^-}|\mathcal{F}_\tau\right) \le C_Q + 2, \quad \forall\, x > 0. \tag{4.15}$$

A direct application of Jesen's inequality in (4.15) yields

$$\max\left\{\exp\left(\log(x)E^Q(\Delta^+|\mathcal{F}_\tau)\right), \exp\left(-\log(x)E^Q(\Delta^-|\mathcal{F}_\tau)\right)\right\} \le C_Q + 2,\ \forall\, x > 0. \tag{4.16}$$

Therefore, (4.16) is possible only if

$$E^Q(\Delta^+|\mathcal{F}_\tau) = E^Q(\Delta^-|\mathcal{F}_\tau) = 0 \quad \text{or equivalently} \quad \Delta^+ = \Delta^- = 0, \quad P - a.s. \tag{4.17}$$

Otherwise, there would be a contradiction by sending $x \to +\infty$ and $x \to 0$, respectively. This proves that the process p is a constant and the proof of this lemma is completed. □

The following lemma deals with the case when the process p is negative.

Lemma 4.2: *Suppose that the process $p = (p(t))_{0\le t\le T}$ is negative. Then $U_p(t,x) := D(t)x^{p(t)}$ is a forward utility satisfying (4.5), only if p is constant in (t,ω), i.e., $p(t) = p(0)$, $P - a.s.$*

Proof. Consider $t \ge 0$, arbitrary but fixed. For any $T' \in [t, +\infty)$, $Z \in \mathcal{Z}^e_{loc}(S)$, and $\eta \in L^0_+(\mathcal{F}_t)$, a direct application of Proposition 4.3 to $U_p(t,x) = D(t)x^{p(t)}$

leads to

$$E\left(\frac{(D(T')p(T'))^{1-q(T')}}{q(T')}(\eta\frac{Z_{T'}}{Z_t})^{q(T')}|\mathcal{F}_t\right)\leq\frac{(D(t)p(t))^{1-q(t)}}{q(t)}\eta^{q(t)}. \tag{4.18}$$

By choosing $\eta = Z_t e^{\alpha}$, $\alpha \in \mathbb{R}$, and putting $X_s := \frac{(D(s)p(s))^{1-q(s)}}{q(s)} Z_s^{q(s)}$, the equation (4.18) becomes

$$E\left(X_{T'}e^{\alpha(q(T')-q(t))}|\mathcal{F}_t\right)\leq X_t. \tag{4.19}$$

Since X is positive (which is due to $q(t) = \frac{p(t)}{p(t)-1} > 0$), we derive

$$\max\left\{e^{\alpha^+\varepsilon}E\left(X_{T'}I_{\{q(T')-q(t)\geq\varepsilon\}}|\mathcal{F}_t\right), e^{\alpha^-\varepsilon}E\left(X_{T'}I_{\{q(T')-q(t)\leq-\varepsilon\}}|\mathcal{F}_t\right)\right\}\leq X_t, \tag{4.20}$$

for any $\alpha \in \mathbb{R}$, $\alpha = \alpha^+ - \alpha^-$, and any $\varepsilon > 0$. Therefore, again, we deduce that (4.20) holds only if

$$q(T') = q(t), \quad P-a.s. \quad \forall\ T' \geq t \geq 0.$$

This proves that $q(t) = q(0)$, $P-a.s.$ $\forall\, t \geq 0$ and hence $p(t)$ has a constant version, i.e.

$$p(t) = p(0), \quad P-a.s. \quad \forall\, t \geq 0.$$

This ends the proof of the lemma. □

Proof. of Theorem 4.1: If the process p is either positive or negative, then the proof of this theorem follows from Lemma 4.1 and Lemma 4.2 respectively. Thus, the proof of this theorem will immediately follow once we prove that the process p has a constant sign (i.e. $p(t)p(0) > 0$, $P-a.s.$ for all $t \in [0, T]$). To this end, we consider the following stopping time

$$\tau := \inf\left\{t \geq 0 \;\middle|\; p(t)p(0) < 0\right\} \wedge T.$$

Remark that, due to the right continuity of p and $\inf_{0\leq t\leq T}|p(t)| > 0$, the process

p has a constant sign if and only if

$$\tau = T, \quad p(0)p(T) > 0, \quad P - .a.s. \tag{4.21}$$

Since p is locally bounded, there is no loss of generality in assuming p bounded. Let $\widehat{\theta}$ be the optimal portfolio rate for the initial capital $x = 1$. Then, $D_t\mathcal{E}_t(\widehat{\theta} \cdot S)^{p(t)}$ is a true martingale and for any $x > 0$, the process $D_t\mathcal{E}_t(\widehat{\theta} \cdot S)^{p(t)}x^{p(t)}$ is a supermartingale (since $\widehat{\theta}$ is also admissible to $x > 0$ due to the boundedness of p). Below, we consider two cases of $p(0) < 0$ and $p(0) > 0$ in parts a) and b), respectively.

a) Suppose that $p(0) < 0$, and hence $D(0) < 0$. Then, due to the assumptions on the processes p and D, we deduce that the null portfolio is admissible for any $x > 0$. Hence, the process $D(t)x^{p(t)}$ is a supermartingale, for any $x > 0$, and we have

$$E\left(D(\tau)x^{p(\tau)}I_{\{p(\tau)>0\}} \;\middle|\; \mathcal{F}_0\right) \leq -E\left(D(\tau)x^{p(\tau)}I_{\{p(\tau)<0\}} \;\middle|\; \mathcal{F}_0\right) + D(0)x^{p(0)}.$$

By letting x goes to infinity and using Fatou's lemma we conclude that we should have $P\left(p(\tau) > 0\right) = 0$ (otherwise we will have a contradiction from the above inequality). This proves that $p(\tau) < 0$, $P - a.s.$ or equivalently (4.21) holds.

b) Suppose that $p(0) > 0$, or equivalently $D(0) > 0$. Then for any $n \geq 1$, there exists $\theta_n \in \mathcal{A}_{adm}(n^{-1})$ such that

$$E\left\{D(\tau)n^{-p(\tau)}(1 + (\theta_n \cdot S)_\tau)^{p(\tau)}\right\} = D(0)n^{-p(0)}. \tag{4.22}$$

Thanks to Lemma 2.1, there exists a sequence of non-negative real numbers,

$(\alpha_k)_{k=n,\ldots,N_n}$, such that

$$\sum_{k=n}^{N_n} \alpha_k = 1 \ \text{ and } \ Y_n := 1 + \sum_{k=n}^{N_n} \alpha_k (\theta_k \cdot S)_\tau \quad \text{converges almost surely to } \ Y \geq 0.$$

Furthermore, we can deduce that $Y < +\infty$, P-a.s., after an application of Fatou's Lemma as follows

$$E\left(ZY\right) \leq \lim_{n\to+\infty} E\left(ZY_n\right) \leq 1, \quad \forall \quad Z \in \mathcal{Z}^e_{loc}(S).$$

Now, we consider a sequence of random variables, $(X_n)_{n\geq 1}$, given by

$$X_n := D(\tau) n^{-p(\tau)} Y_n^{p(\tau)} - D(\tau) Y_n^{p(\tau)}.$$

It is easy to check that

$$X_n \leq 0, \quad P - a.s. \tag{4.23}$$

By considering the cases of $\{p(\tau) > 0\}$ and $\{p(\tau) < 0\}$ separately, we obtain

$$\lim_{n\to+\infty} X_n = \begin{cases} -D(\tau) Y^{p(\tau)}, & \text{if } \ p(\tau) > 0; \\ -\infty, & \text{if } \ p(\tau) < 0. \end{cases} \tag{4.24}$$

Moreover, we have

$$\begin{aligned} X_n &\geq \sum_{k=n}^{N_n} \alpha_k n^{-p(\tau)} D(\tau) (1 + \theta_k \cdot S_\tau)^{p(\tau)} - D(\tau) Y_n^{p(\tau)} \\ &\geq \sum_{k=n}^{N_n} \alpha_k k^{-p(\tau)} D(\tau) (1 + \theta_k \cdot S_\tau)^{p(\tau)} - D(\tau) Y_n^{p(\tau)}. \end{aligned} \tag{4.25}$$

Then, by taking expectation on both sides of (4.25), and recalling (4.22) and the supermartingale property of $U_p(\tau, 1 + \sum_{k=n}^{N_n} \alpha_k \theta_k \cdot S_\tau)$, we derive

$$E(X_n) \geq D(0) \left[\sum_{k=n}^{N_n} \alpha_k k^{-p(0)} - 1 \right]. \tag{4.26}$$

Since X_n is negative (see (4.23), we apply Fatou's Lemma to the left-hand-side term in (4.26) and obtain

$$E(\lim_{n\to+\infty} X_n) \geq -D(0) > -\infty. \tag{4.27}$$

On the other hand, by considering (4.24), we deduce that $P(p(\tau) < 0) > 0$ implies

$$-\infty = E(I_{\{p(\tau)<0\}} \lim_{n\to+\infty} X_n) \geq E(\lim_{n\to+\infty} X_n).$$

This is a contradiction with (4.27). Hence, we conclude that

$$p(\tau) > 0, \quad P - a.s.,$$

which is equivalent to (4.21). This ends the proof of the theorem. □

4.A.2 The Dynamic of the Process D

Our next step focuses on describing the process, $(D(t))_{0\leq t\leq T}$, and the optimal portfolio in the utility maximization problem associated to $U_p(t,x)$. This step contains two theorems that are stated in the increasing order of generality. First, we describe the process D that is predictable with finite variation. Afterwards, we drop the predictability and the finite variation assumptions, and determine the general form of D.

Through out the analysis, the following set, $\mathcal{D}$, will play important roles in our analysis

$$\mathcal{D} := \left\{\theta \in \mathbb{R}^d \ : \ 1 + \theta^T x > 0, \qquad F - a.e.\right\}. \tag{4.28}$$

Also, the function Φ_p will be used from time to time, which takes values in $(-\infty, +\infty]$ and is given by

$$\Phi_p(\lambda) := \frac{b^T\lambda}{p-1} + \frac{1}{2}\lambda^T c\lambda + \int f_p(\lambda^T x)F(dx), \ \ \forall\ \lambda \in \mathbb{R}^d,\ p \in (-\infty,0)\cup(0,1), \tag{4.29}$$

where the non-negative function f_p is defined by (2.42).
The next assumption is crucial in our proof of Theorem 4.2. It excludes the situation where the optimal portfolio rate may belong to the boundary of $\mathcal{D}$.

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P \otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $P \otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P \otimes A - a.e.$,

$$\lim_{n\to+\infty} \int K_p(\lambda_n^T x)F(dx) = \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \int K_p(\lambda^T x)F(dx), & \text{on} \quad \Gamma^c. \end{cases} \tag{4.30}$$

where $K_p(y) := |y||(1+y)^{p-1} - 1|$ and $\Gamma := \{F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

Theorem 4.2: *Let p be a real number such that $p \in (-\infty, 0) \cup (0, 1)$, q is its conjugate ($q := \frac{p}{p-1}$), and the set $\mathcal{D}$ is given by (4.28). Suppose that $D(t)$ is a RCLL predictable process with finite variation, S is locally bounded and assumption (4.30) holds. Then the following assertions are equivalent.*
(1) The random field utility, $U(t,x) = D(t)x^p$, is a forward utility with the optimal portfolio rate $\widehat{\theta}$.
(2) The minimal Hellinger martingale density of order q, $\widetilde{Z}$, exists and satisfies:
(2.a) The process $\widehat{Z} := \widetilde{Z}\mathcal{E}\left(\widehat{\theta} \cdot S\right)$ is a true martingale.
(2.b) The process D is given by

$$D = D_0 \mathcal{E}\left(q(q-1)h^{(q)}(\widetilde{Z}, P)\right)^{p-1}. \tag{4.31}$$

(2.c) The optimal portfolio rate, $\widehat{\theta}$, belongs to $int(\mathcal{D})$, and is a root for

$$b + (p-1)c\theta + \int \left[(1+\theta^T x)^{p-1} - 1\right] xF(dx) = 0, \qquad P \otimes A - a.e. \tag{4.32}$$

The proof of this theorem is long and requires a number of intermediary results that are interesting in themselves. Technically, Theorem 4.2 is the back-bone of this subsection. Thus, for the sake of clear exposition of our ideas and results, we will postpone the proof of this theorem. In the following, we will highlight the importance of Theorem 4.2 on particular case of continuous process S, and afterwards we will deal with describing D in the general case.

Corollary 4.2.1: Suppose that D is predictable with finite variation, and S is continuous. Then the following are equivalent.

(1) The random field utility, $U(t,x) = D(t)x^p$, is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The optimal portfolio rate, $\widehat{\theta}$, is a root for

$$b + (p-1)c\theta = 0, \qquad P \otimes A - a.e, \tag{4.33}$$

and the following properties hold:

(2.a) The process D is given by

$$D_t = D_0 \exp\left(\frac{q}{2}\int_0^t \widehat{\theta}_u^T c_u \widehat{\theta}_u dA_u\right). \tag{4.34}$$

(2.b) The process $\widehat{Z} := \mathcal{E}\left((p-1)\widehat{\theta}\cdot M\right)\mathcal{E}\left(\widehat{\theta}\cdot S\right)$ is a true martingale, where M is the local martingale part of S.

Proof. The proof of this corollary is straightforward from Theorem 4.2, and from the fact that when S is continuous, assumption (4.30) is fulfilled due to $F = 0$, and all the minimal Hellinger densities of any order q coincide with the minimal martingale density. For this last fact we refer the reader to [18]. This ends the proof of this corollary. □

Proposition 4.4: *Let $p \in (-\infty, 0) \cup (0, 1)$, $\widetilde{Z}$ be a martingale density, and $\widehat{\theta} \in \mathcal{A}_{adm}(1)$ such that $\widehat{Z} := \widetilde{Z}\mathcal{E}(\widehat{\theta}\cdot S)$ is a true martingale. If we denote $\widehat{Q} := \widehat{Z}_T \cdot P$ and consider $\theta \in \mathcal{A}_{adm}(1)$ satisfying*

$$\sup_{\tau \in \mathcal{T}_T} E^{\widehat{Q}}\left(\frac{\mathcal{E}_\tau(\theta\cdot S)^p}{\mathcal{E}_\tau(\widehat{\theta}\cdot S)^p}\right) < +\infty, \tag{4.35}$$

then $\operatorname{sign}(p)\left(\frac{\mathcal{E}(\theta\cdot S)}{\mathcal{E}(\widehat{\theta}\cdot S)}\right)^p$ *is a $\widehat{Q}$-supermartingale.*

Proof. Notice that the case when $p \in (0,1)$, the proposition is trivial and (4.35) is always true.

In the remaining part of the proof we assume that $p < 0$, and we consider $(T_n)_{n\geq 1}$ a sequence of stopping times that increases stationarily to T such that $\widetilde{Z}^{T_n}$ is a true martingale. Therefore, since $\widetilde{Z}\mathcal{E}(\theta\cdot S)$ is a supermartingale, by putting $\widetilde{Q}_n := \widetilde{Z}_{T_n}\cdot P$ and using Jensen's inequality we derive

$$E^{\widehat{Q}}\left[\left(\frac{\mathcal{E}_{t\wedge T_n}(\theta\cdot S)}{\mathcal{E}_{t\wedge T_n}(\widehat{\theta}\cdot S)}\right)^{p/2}\Big|\mathcal{F}_s\right] \geq \left(E^{\widehat{Q}}\left[\frac{\mathcal{E}_{t\wedge T_n}(\theta\cdot S)}{\mathcal{E}_{t\wedge T_n}(\widehat{\theta}\cdot S)}\Big|\mathcal{F}_s\right]\right)^{p/2}$$

$$\text{and}\quad E^{\widehat{Q}}\left[\frac{\mathcal{E}_{t\wedge T_n}(\theta\cdot S)}{\mathcal{E}_{t\wedge T_n}(\widehat{\theta}\cdot S)}\Big|\mathcal{F}_s\right] = \frac{E^{\widetilde{Q}_n}\left(\mathcal{E}_{t\wedge T_n}(\theta\cdot S)|\mathcal{F}_s\right)}{\mathcal{E}_{s\wedge T_n}(\widehat{\theta}\cdot S)} \leq \frac{\mathcal{E}_{s\wedge T_n}(\theta\cdot S)}{\mathcal{E}_{s\wedge T_n}(\widehat{\theta}\cdot S)},$$

for $0 \leq s < t \leq T$. This proves that $\left(\mathcal{E}(\theta\cdot S)/\mathcal{E}(\widehat{\theta}\cdot S)\right)^{p/2}$ is a nonnegative $\widehat{Q}$-local submartingale. Then, due to (4.35) and de la Vallée Poussin's argument, we deduce that this process is a true $\widehat{Q}$-submartingale. Again, an application of Jensen's inequality leads to,

$$E^{\widehat{Q}}\left[\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}\right)^p\Big|\mathcal{F}_s\right] \geq \left(E^{\widehat{Q}}\left[\left(\frac{\mathcal{E}_t(\theta\cdot S)}{\mathcal{E}_t(\widehat{\theta}\cdot S)}\right)^{p/2}\Big|\mathcal{F}_s\right]\right)^2 \geq \left(\frac{\mathcal{E}_s(\theta\cdot S)}{\mathcal{E}_s(\widehat{\theta}\cdot S)}\right)^p,$$

which is finite due to (4.35). Hence, $-\left(\mathcal{E}(\theta\cdot S)/\mathcal{E}(\widehat{\theta}\cdot S)\right)^p$ is a $\widehat{Q}$-supermartingale and this ends the proof. □

Now, we are ready to state our parametrization result in its full generality.

Theorem 4.3: *Suppose that S is locally bounded. Let $U_p(t,x)$ be the random field utility defined in (4.1) such that p is locally bounded, and (4.5)–(4.10) hold. Let q be the conjugate process of p (i.e. $q := \frac{p}{p-1}$). Then the following assertions are equivalent.*

(1) U_p is a forward utility with optimal portfolio rate $\widehat{\theta}_x$ for any $x \in (0,+\infty)$.

(2) The process p satisfies (4.11) (i.e. $p(t) = p(0)$, $P-a.s.$ for any $t \in [0,T]$), and hence $\widehat{\theta}_x = \widehat{\theta}$ does not depend on initial capital $x \in (0,+\infty)$. The process D is a RCLL supermartingale and there exists a positive local martingale $Z^D = \mathcal{E}(N^D)$ and a predictable process a^D with finite variation such that

$$D = D_0 Z^D \exp(a^D), \tag{4.36}$$

and, the functional $\overline{U}_p(t,x) = D_0 Z^D \exp(a^D) x^p$ is a forward utility.

Proof. Suppose that assertion (1) holds. Then Theorem 4.1 implies that the process p is constant in (ω, t). This together with the fact that U_p is a random field utility, implies that $D/D(0)$ is a positive RCLL supermartingale (take $x = 1$) (see Lemma 4.1 and its proof). A combination of this with (4.5), leads to the multiplicative Doob-Meyer decomposition of D. Hence, there exists a positive local martingale $Z^D = \mathcal{E}(N^D)$ and a predictable process with finite variation, a^D, such that

$$D_t = D_0 Z_t^D \exp(a_t^D), \quad 0 \le t \le T. \tag{4.37}$$

Consequently, the functional $\overline{U}_p(\cdot, x) := D_0 Z^D \exp(a^D) x^p = U_p(\cdot, x)$ is a forward utility. This ends the proof of (1) $\Rightarrow$ (2).
The proof of (2) $\Rightarrow$ (1) is obvious and the theorem is proved completely. $\square$

Let $\left(\beta, f, g, \overline{N}^D\right)$ denote the Jacod components for N^D and consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $P \otimes A$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\ge1}$, such that $\lambda_n \in int(\mathcal{D})$, $P \otimes A$-a.e., and $\lambda_n \to \lambda$, we have, $P \otimes A - a.e.$, (4.38)

$$\lim_{n\to+\infty} \int K_p(\lambda_n^T x)(1 + f(x))F(dx) = \begin{cases} +\infty, & \text{on} \quad \Gamma; \\ \int K_p(\lambda^T x)(1+f(x))F(dx), & \text{on} \quad \Gamma^c. \end{cases}$$

where $K_p(y) := |y||(1+y)^{p-1} - 1|$ and $\Gamma := \{F(\mathbb{R}^d) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

Theorem 4.4: *Consider the functional* $\overline{U}_p$ *given by Theorem 4.3–(2). Suppose that* S *is locally bounded and (4.38) hold. Then, the following two assertions are equivalent:*

(1) $\overline{U}_p$ *is a forward utility with the optimal portfolio rate* $\widehat{\theta}$*.*

(2) The following hold:

(2.a) The minimal Hellinger martingale density of order q *with respect to* Z^D*, denoted by* $\widetilde{Z}^D$*, exists and*

$$D = D_0 Z^D \mathcal{E}\left(q(q-1)h^{(q)}(\widetilde{Z}^D, Z^D)\right)^{p-1}. \tag{4.39}$$

(2.b) The optimal portfolio rate, $\widehat{\theta}$*, belongs to* $int(\mathcal{D})$*, and is a root for*

$$b^D + (p-1)c\theta + \int \left[(1+\theta^T x)^{p-1} - 1\right] x F^D(dx) = 0, \quad P\otimes A\text{−}a.e. \tag{4.40}$$

$$\text{Here,}\quad b^D := b + c\beta + \int f(x)xF(dx), \quad F^D(dx) := (1+f(x))F(dx). \tag{4.41}$$

(2.c) The process $\widehat{Z} := Z^D\widetilde{Z}^D\mathcal{E}(\widehat{\theta}\cdot S)$ *is a true martingale.*

Proof. We start by proving (1) ⇒ (2). Let $(T_n)_{n\geq 1}$ be an increasing sequence of stopping times that increases stationarily to T and $(Z^D)^{T_n}$ is a true martingale. Put $Q_n := Z^D_{T_n}\cdot P$. Then, due to Lemma 2.3, we conclude that $U_n(t,\omega,x) := D_0\exp(a^D_{t\wedge T_n})x^p$ is a forward dynamic utility for (S^{T_n}, Q_n) with the optimal portfolio rate $\widehat{\theta}_n := \widehat{\theta} I_{[0,T_n]}$. Hence, a direct application of Theorem 4.2 to $(S^{T_n}, Q_n, U_n, \widehat{\theta}_n)$ implies the existence of the minimal Hellinger martingale density for this model, denoted by $\widetilde{Z}^{D,n}$, that satisfies

$$\exp(a^D_{t\wedge T_n}) = \mathcal{E}_{t\wedge T_n}\left(q(q-1)h^{(q)}(\widetilde{Z}^{D,n}, Q_n)\right)^{1/(q-1)}, \qquad 0\leq t\leq T, \tag{4.42}$$

and $\widehat{\theta}$ is a root of (4.40) on $[0,T_n]$. Thus, it is clear that, this last statement implies assertion (2.b). By virtue of Lemma 2.5, we deduce that the minimal Hellinger martingale density of order q with respect to Z^D (denoted by $\widetilde{Z}^D$)

exists and

$$h_t^{(q)}(\widetilde{Z}^{D,n}, Q_n) = h_{t\wedge T_n}^{(q)}(\widetilde{Z}^D, Z^D).$$

Therefore, a combination of this equality with (4.42) leads to the assertion (2.a).

Due to Proposition 2.5 (see formula (2.68) and notice that our $\widehat{\theta}$ here is a version of $\widetilde{\beta}$ of that proposition) and (4.39), we derive

$$\begin{aligned}\widehat{Z} = Z^D\widetilde{Z}^D\mathcal{E}(\widehat{\theta}\cdot S) &= Z^D\mathcal{E}\left(\widetilde{H}^D\cdot S + q(q-1)h^{(q)}(\widetilde{Z}^D, Z^D)\right)^{p-1}\mathcal{E}(\widehat{\theta}\cdot S)\\ &= Z^D\mathcal{E}(\widehat{\theta}\cdot S)^{p-1}\mathcal{E}\left(q(q-1)h^{(q)}(\widetilde{Z}^D, Z^D)\right)^{p-1}\mathcal{E}(\widehat{\theta}\cdot S)\\ &= Z^D\mathcal{E}\left(q(q-1)h^{(q)}(\widetilde{Z}^D, Z^D)\right)^{p-1}\mathcal{E}(\widehat{\theta}\cdot S)^p\\ &= \tfrac{1}{D_0x^p}\overline{U}_p(t, x\mathcal{E}_t(\widehat{\theta}\cdot S)).\end{aligned} \tag{4.43}$$

This proves that $\widehat{Z}$ is a true martingale, since $\overline{U}_p(t,x)$ is a forward utility with optimal portfolio rate $\widehat{\theta}$. This ends the proof of (1) $\Rightarrow$ (2).

In the remaining part of this proof, we will address (2) $\Rightarrow$ (1). Suppose that assertion (2) is fulfilled. Remark that (4.43) remains valid as long as assertion (2–a) holds. Thus, we obtain

$$\overline{U}_p\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right) = D_0x^p\widehat{Z},$$

and due to assertion (2-c), we conclude that $\overline{U}_p\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right)$ is a martingale for any $x > 0$. Furthermore, for any admissible portfolio rate θ, we have

$$\overline{U}_p\Big(t, x\mathcal{E}_t\left(\theta\cdot S\right)\Big) = D_0x^p\widehat{Z}_t\left(\frac{\mathcal{E}_t\left(\theta\cdot S\right)}{\mathcal{E}_t\left(\widehat{\theta}\cdot S\right)}\right)^p. \tag{4.44}$$

Thanks to $pD_0 > 0$ since $\overline{U}_p(t,x)$ is a random field utility, Proposition 4.4 (take $\widetilde{Z} := Z^D\widetilde{Z}^D$ which is a martingale density for S by definition of $\widetilde{Z}^D$), and

$$\sup_{\tau\in\mathcal{T}_T} E^{\widehat{Q}}\left\{\left(\frac{\mathcal{E}_\tau(\theta\cdot S)}{\mathcal{E}_\tau\left(\widehat{\theta}\cdot S\right)}\right)^p\right\} = -\frac{1}{D_0 x^p}\sup_{\tau\in\mathcal{T}_T} E\Big[U\Big(\tau, x\mathcal{E}_\tau(\theta\cdot S)\Big)^-\Big] < +\infty,$$

we deduce that $\overline{U}_p(t, x\mathcal{E}_t(\theta\cdot S))$ is a supermartingale for any admissible strategy θ and any $x>0$. This ends the proof of the theorem. □

The rest of this section is to prove Theorem 4.2. To this end, some useful technical lemmas are required and will be detailed first.

Lemma 4.3: *Suppose S is locally bounded. Then, the interior of $\mathcal{D}$ satisfies*

$$0\in int(\mathcal{D}) = \mathcal{D}_1$$

$$\textit{where}\qquad \mathcal{D}_1 := \{\lambda\in\mathcal{D}:\ \exists\ \ \delta>0, 1+\lambda^T x\geq\delta,\quad F-a.e.\}. \tag{4.45}$$

Proof. The fact that $0\in int(\mathcal{D})\subseteq\mathcal{D}_1$ has already been proved in Lemma 3.1. Here, we focus on the remaining part that $int(\mathcal{D})\supseteq\mathcal{D}_1$. By virtue of the argument of localizing procedure, without loss of generality, we suppose that S is bounded, for any $\lambda_0\in\mathcal{D}_1$, there exists $\delta>0$ such that $1+\lambda_0^T x\geq\delta$ $F-a.e.$ and let K be the bound of S (i.e. $|S|\leq K$). Consider a neighborhood at λ_0, $B(\lambda_0,\varepsilon)$ with radius $\varepsilon := \frac{\delta}{2K+1}$, then for any $\lambda\in B(\lambda_0,\varepsilon)$, we have

$$1+\lambda^T x\geq\frac{\delta}{2K+1}>0.$$

Hence, $\lambda_0\in int(\mathcal{D})$. This ends the proof of this lemma. □

The following two lemmas (Lemmas 4.4 and 4.5) are generalizations of Lemmas 3.2 and 3.3 to the case of f_p and Φ_p ($p\neq 0$).

Lemma 4.4: *Suppose S is locally bounded, then for any $\lambda\in I\!R^d$ and $\delta>0$, we have*

$$\int_{\{\lambda^T x\geq\delta-1\}} f_p(\lambda^T x)F(dx) < +\infty\quad P\otimes A-a.e. \tag{4.46}$$

Proof. Thanks to Taylor's expansion of f_q, we have

$$f_p(\lambda^T x) = \frac{(\lambda^T x)^2}{2}(1 + r\lambda^T x)^{p-2}, \quad \text{for} \quad 0 < r < 1.$$

For $\delta > 0$ such that $\lambda^T x \geq \delta - 1$, we put $\overline{\delta} := \delta \wedge 1$ and have

$$1 + r\lambda^T x \geq 1 + r(\delta - 1) \geq \delta \wedge 1 = \overline{\delta}. \tag{4.47}$$

Therefore, we obtain that

$$\int_{\{\lambda^T x \geq \delta - 1\}} f_p(\lambda^T x) F(dx) \leq \frac{1}{2}\overline{\delta}^{p-2}|\lambda|^2 \int |x|^2 F(dx). \tag{4.48}$$

Since S is locally bounded, it is easy to see that $[S, S] \in \mathcal{A}^+_{loc}$. As a result, we have $x^2 \star \nu_T < +\infty$, P-a.s., and hence

$$\int |x|^2 F(dx) < +\infty, \quad P \otimes A - a.s.$$

By combining this with (4.48), we can conclude (4.46) immediately. This completes the proof of this lemma.

□

Lemma 4.5: *Suppose S is locally bounded. Then, the following two assertions hold, $P \otimes A$-a.e.*

(i) For any $\lambda \in int(\mathcal{D})$,

$$\int |x||(1 + \lambda^T x)^{1/(q-1)} - 1|F(dx) < +\infty. \tag{4.49}$$

(ii) $\Phi_p(\lambda)$ is differentiable on $int(\mathcal{D})$ and for any $\lambda_0 \in int(\mathcal{D})$,

$$\Phi'_p(\lambda_0) = b + \frac{c\lambda_0}{q-1} + \int \left[x(1 + \lambda_0^T x)^{1/(q-1)} - x\right] F(dx).$$

Proof. (i) For any $\lambda \in int(\mathcal{D})$, due to Lemma 4.3, there exists $\delta \in (0, 1)$ such that $1 + \lambda^T x \geq \delta > 0$, $F - a.e.$

An application of Taylor's expansion to $(1+\lambda^T x)^{1/(q-1)}$-1 leads to the existence of $r \in (0,1)$ such that

$$(1+\lambda^T x)^{1/(q-1)} - 1 = \frac{\lambda^T x}{q-1}(1+r\lambda^T x)^{\frac{2-q}{q-1}}. \qquad (4.50)$$

It is easy to see that

$$1 + r\lambda^T x \geq 1 + r(\delta - 1) \geq \delta. \qquad (4.51)$$

By combining (4.51) and (4.50), we have

$$\int |x||(1+\lambda^T x)^{1/(q-1)} - 1|F(dx) \leq \frac{\delta^{\frac{2-q}{q-1}}|\lambda|}{1-q}\int |x|^2 F(dx). \qquad (4.52)$$

Since S is locally bounded, it is easy to deduce that (see the proof of Lemma 4.4 for details)

$$\int |x|^2 F(dx) < +\infty, \quad P \otimes A - a.s.$$

By combining this with (4.52), we can conclude (4.49) immediately. This completes the proof of this lemma.

(ii) Let $\lambda_0 \in int(\mathcal{D})$. Then, for any $y \in \mathbb{R}^d$, thanks to Lemmas 4.3 and 4.4, there exists $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, $\lambda_0 + \varepsilon y \in dom(\Phi_p)$.
An application of Taylor's expansion of the function $g_p(\lambda^T x) := \frac{(1+\lambda^T x)^p - 1 - p\lambda^T x}{q(q-1)}$ implies the existence of $r \in (0,1)$ such that

$$k_\varepsilon(x) := \frac{g_p(\lambda_0^T x + \varepsilon y^T x) - g_p(\lambda_0^T x)}{\varepsilon} = y^T x\left((1+\lambda_0^T x + r\varepsilon y^T x)^{1/(q-1)} - 1\right).$$

Meanwhile, notice that $(|k_\varepsilon(x)|)_\varepsilon$ is bounded from above by

$$k(x) := |y||x|\max\Big(|(1+\lambda_0^T x)^{1/(q-1)} - 1|, |(1+\lambda_0^T x + \varepsilon_0 y^T x)^{1/(q-1)} - 1|\Big).$$

Thanks to Lemma 4.5–(i), $k(x)$ is integrable since $\lambda_0, \lambda_0 + \varepsilon_0 y \in int(\mathcal{D}) \in dom(\Phi_p)$. It allows us to apply Dominated Convergence Theorem to $(|k_\varepsilon(x)|)_\varepsilon$, which leads to

$$\lim_{\varepsilon \to 0} \frac{\Phi_p(\lambda_0 + \varepsilon y) - \Phi_p(\lambda_0)}{\varepsilon} = y^T \Phi^*, \tag{4.53}$$

where Φ^* is given by

$$\Phi^* := b + \frac{c\lambda_0}{q-1} + \int \left[x(1 + \lambda_0^T x)^{1/(q-1)} - x \right] F(dx).$$

It is clear from (4.53) that $y^T \Phi^*$ is the directional derivative of Φ_p at λ_0, which is linear in y. Thus, due to Theorem 25.2 in [68], Φ_p is differentiable at $\lambda_0 \in int(\mathcal{D})$. This completes the proof. □

Lemma 4.6: *Suppose S is locally bounded. Then, the interior of the effective domain of Φ_p coincides with $int(\mathcal{D})$, that is,*

$$int(dom(\Phi_p)) = int(\mathcal{D}).$$

Proof. $\Rightarrow$: For any $\lambda_0 \in int(dom(\Phi_p))$, there exists a neighborhood $B(\lambda_0, \varepsilon)$, such that for any λ satisfying $|\lambda - \lambda_0| \leq \varepsilon$, $\lambda \in int(dom(\Phi_p))$. Let $\delta \in (0, \varepsilon)$ and apply the convexity of the function $f_p(x)$, we calculate

$$\delta |x| \left| \frac{(1 + \lambda^T x)^{p-1} - 1}{p-1} \right| \leq \sum_{i=1}^{d} \left[f_p(\lambda^T x + \delta e_i^T x) + f_p(\lambda^T x - \delta e_i^T x) \right],$$

where e_i is the vector of $\mathbb{R}^d$ whose i^{th} component equals one, and the others are null. Then from the above inequality, we clearly deduce that $\lambda \in \mathcal{D}$. Therefore, $\lambda_0 \in int(\mathcal{D})$.

$\Leftarrow$: For any $\lambda_0 \in int(\mathcal{D})$, there exists a neighborhood $B(\lambda_0, \varepsilon) \subseteq int(\mathcal{D})$. Then, due to Lemmas 4.3 and 4.4, we have $B(\lambda_0, \varepsilon) \subseteq dom(\Phi_p)$. Hence, $\lambda_0 \in int(dom(\Phi_p))$. This completes the proof of this lemma. □

Proposition 4.5: *Suppose S is locally bounded and assumption (4.30) holds. If $\Phi_p(\lambda)$ attains its minimum at $\widetilde{\lambda}$, then $\widetilde{\lambda} \in int(\mathcal{D})$. Furthermore,*

$$\Phi_p'(\widetilde{\lambda}) = b + \frac{c\widetilde{\lambda}}{q-1} + \int \left[x(1 + \widetilde{\lambda}^T x)^{1/(q-1)} - x \right] F(dx) = 0. \tag{4.54}$$

Proof. Consider the case of S being bounded and adopt the argument of contradiction. In other words, we suppose that $\Phi_p(\lambda)$ attains its minimum at $\widetilde{\lambda}$ but $\widetilde{\lambda} \notin int(\mathcal{D})$. For any $\lambda \in \mathcal{D}$, put $\lambda_n := (1-1/n)\lambda$, which converges to λ and, due to Lemma 4.4, satisfies

$$\int f_p(\lambda_n^T x)F(dx) < +\infty. \tag{4.55}$$

For any $r \in (0,1)$, the convex combination

$$\overline{\lambda} := r\lambda_n + (1-r)\widetilde{\lambda} = \widetilde{\lambda} + r(\lambda_n - \widetilde{\lambda}) \in \mathcal{D},$$

thus we have

$$\Phi_p(\widetilde{\lambda}) \le \Phi_p(\overline{\lambda}), \quad P \otimes A - a.e. \tag{4.56}$$

On the other hand, the integrability of $\frac{f_p(\widetilde{\lambda}^T x) - f_p(\overline{\lambda}^T x)}{r}$ follows from

$$f_p(\widetilde{\lambda}^T x) - f_p(\overline{\lambda}^T x) \ge r(f_p(\widetilde{\lambda}^T x) - f_p(\lambda_n^T x))$$

which is due to the convexity of f_p and (4.55). This allows us to apply Fatou's Lemma, which leads to

$$(\lambda_n - \widetilde{\lambda})^T G(\widetilde{\lambda}) \le \lim_{r\to 0} \frac{\Phi_p(\widetilde{\lambda}) - \Phi_p(\overline{\lambda})}{r} \le 0, \quad P \otimes A - a.e.,$$

where

$$G(\lambda) := b + \frac{c\lambda}{q-1} + \int \left[x(1+\lambda^T x)^{1/(q-1)} - x\right] F(dx).$$

As a result, we obtain

$$(\lambda_n - \widetilde{\lambda})^T G(\widetilde{\lambda}) \le 0, \quad \forall\, n \ge 1, \quad P \otimes A - a.e.$$

By sending $n \to +\infty$ and taking sup for λ over $\mathcal{D}$, we get

$$\widetilde{\lambda}^T G(\widetilde{\lambda}) \ge \sup_{\lambda \in \mathcal{D}} \lambda^T G(\widetilde{\lambda}), \quad P \otimes A - a.e.$$

As an direct application of above inequality by taking $\lambda = 0 \in \mathcal{D}$, it yields

$$\widetilde{\lambda}^T G(\widetilde{\lambda}) \geq 0, \quad P \otimes A - a.e. \tag{4.57}$$

We rearrange the terms in (4.57) and get

$$0 \leq \widetilde{\lambda}^T \int (x - x(1 + \widetilde{\lambda}^T x)^{p-1}) F(dx) \leq \widetilde{\lambda}^T b + (p-1)c\widetilde{\lambda} < +\infty, \tag{4.58}$$

which is due to the function $g(y) := y - y(1+y)^{p-1} > 0$, $1 + y \geq 0$ and $p < 1$.

Let $(\widetilde{\lambda}_n)_n$ be given by $\widetilde{\lambda}_n := (1 - 1/n)\widetilde{\lambda}$. Then, due to Lemma 4.3, $\widetilde{\lambda}_n \in int(\mathcal{D})$ and converges to $\widetilde{\lambda}$. Put

$$l_n(x) := |(1 + \widetilde{\lambda}_n^T x)^{1/(q-1)} - 1|, \quad l(x) := |(1 + \widetilde{\lambda}^T x)^{1/(q-1)} - 1|$$

and consider two sets

$$\Gamma^+(\lambda) := \{x : \quad \lambda^T x \geq 0\}, \quad \Gamma^-(\lambda) := \{x : \quad \lambda^T x < 0\}.$$

By studying $l_n(x)$ on $\Gamma^+(\widetilde{\lambda}_n)$ and $\Gamma^-(\widetilde{\lambda}_n)$ respectively, we obtain

$$\begin{aligned} l_n(x) I_{\Gamma^+(\widetilde{\lambda})} &= I_{\Gamma^+(\widetilde{\lambda}_n)}(1 - (1 + \widetilde{\lambda}_n^T x)^{1/(q-1)}) \\ &\leq I_{\Gamma^+(\widetilde{\lambda}_n)}(1 - (1 + \widetilde{\lambda}^T x)^{1/(q-1)}) \\ &= I_{\Gamma^+(\widetilde{\lambda}_n)}|1 - (1 + \widetilde{\lambda}^T x)^{1/(q-1)}|, \end{aligned}$$

and

$$\begin{aligned} l_n(x) I_{\Gamma^-(\widetilde{\lambda})} &= I_{\Gamma^-(\widetilde{\lambda}_n)}((1 + \widetilde{\lambda}_n^T x)^{1/(q-1)} - 1) \\ &\leq I_{\Gamma^-(\widetilde{\lambda}_n)}((1 + \widetilde{\lambda}^T x)^{1/(q-1)} - 1) \\ &= I_{\Gamma^-(\widetilde{\lambda}_n)}|1 - (1 + \widetilde{\lambda}^T x)^{1/(q-1)}|. \end{aligned}$$

These imply that

$$0 \leq l_n(x) \leq l(x), \quad P \otimes A \otimes F - a.e.$$

Thus, the sequence $(|\widetilde{\lambda}_n^T x|l_n)_n$ is bounded by $|\widetilde{\lambda}^T x|l$, which is integrable due to (4.58). Therefore, an application of the Dominated Convergence Theorem and assumption (4.30) lead to

$$\int |\widetilde{\lambda}^T x||(1+\widetilde{\lambda}^T x)^{\frac{1}{q-1}}-1|F(dx) = \lim_{n\to+\infty}\int |\lambda_n^T x||(1+\widetilde{\lambda}_n^T x)^{\frac{1}{q-1}}-1|F(dx) = +\infty,$$

which contradicts with (4.58). Hence, $\widetilde{\lambda} \in int(\mathcal{D})$. Recalling Lemma 4.5, we deduce that Φ_p is differentiable at $\widetilde{\lambda}$ and (4.54) follows immediately. □

Lemma 4.7: *Suppose that the assumptions and assertion (1) of Theorem 4.2 are fulfilled. Then, the process D satisfies*

$$D = D_0 \exp(a^D) = D_0 \mathcal{E}(X^D), \qquad X^D := a^D + \sum (e^{\Delta a^D} - 1 - \Delta a^D), \tag{4.59}$$

and the following assertions hold.

(i) For any $\alpha \in (0,1)$, the processes

$$\frac{(1+\widehat{\theta}^T z)^p - 1 - p\widehat{\theta}^T z}{p(p-1)} I_{\{|\widehat{\theta}^T z|\leq\alpha\}} \star \mu, \quad and \quad (1+\widehat{\theta}^T z)^p I_{\{|\widehat{\theta}^T z|>\alpha\}} \star \mu, \tag{4.60}$$

are non decreasing and locally integrable.

(ii) $P\otimes A$-almost all $(\omega,t)\in\Omega\times[0,+\infty[$, $\widehat{\theta}\in int\,(dom(\Phi_p))$.

(iii) The optimal portfolio rate, $\widehat{\theta}$, is a root of (4.32). That is $P\otimes A$-a.e.

$$0 = \frac{1}{p-1} b + c\widehat{\theta} + \int \frac{(1+\widehat{\theta}^T z)^{p-1} - 1}{p-1} z F(dz). \tag{4.61}$$

(iv) The optimal portfolio rate, $\widehat{\theta}$, satisfies

$$e^{-\Delta a^D}\cdot X^D = \frac{q}{2}\widehat{\theta}^T c\widehat{\theta}\cdot A + \left[\int \frac{(1+\widehat{\theta}^T z)^p - 1 - q(1+\widehat{\theta}^T z)^{p-1} + q}{q-1} F(dz)\right]\cdot A. \tag{4.62}$$

Here X^D is given by (4.59).

(v) If we denote $u(t,\omega,x) := \left(1 + x^T\widehat{\theta}_t(\omega)\right)^{p-1} - 1$, *then*

$$1 + \widehat{u}_t := 1 + \int u(t,x)\nu(\{t\},dx) = 1 - \exp(-\Delta a^D)\Delta X^D = \exp(-\Delta a^D), \tag{4.63}$$

and, as a consequence, the nonnegative predictable process, $(1+\widehat{u})^{-1}$*, is locally bounded.*

Proof. Since $pD(t) > 0$ for all $(t,\omega) \in [0,T] \times \Omega$ —$U(t,x)$ is a random field utility—, it is obvious to see that $D(t)/D_0$ is a positive and predictable process with finite variation. Therefore, the decomposition in (4.59) follows from Ito's formula.

Then, for any admissible portfolio rate θ,

$$\begin{aligned} U(t, x\mathcal{E}_t(\theta \cdot S)) &= D_0 x^p \exp\left(a_t^D\right) \mathcal{E}_t\left(\theta \cdot S\right)^p = D_0 x^p \mathcal{E}_t\left(X^D\right) \mathcal{E}_t\left(X^\theta\right) \\ &= D_0 x^p \mathcal{E}_t\left((1+\Delta X^D) \cdot X^\theta + X^D\right). \end{aligned} \tag{4.64}$$

where X^D is defined in (4.59) and X^θ is given by

$$X^\theta = p\theta \cdot S + \frac{p(p-1)}{2}\theta^T c\theta \cdot A + \left((1+\theta^T z)^p - 1 - p\theta^T z\right) \star \mu.$$

Therefore, for any admissible portfolio rate θ, $U(t, x\mathcal{E}_t(\theta \cdot S))$ is a local supermartingale (respectively is a local martingale) if and only if the process

$$\frac{1}{p(1-p)}\left(e^{\Delta a^D} \cdot X^\theta + X^D\right),$$

is a local supermartingale (respectively is a local martingale). Then, due to Ito's formula, we easily deduce that this fact is equivalent to

$$|(1+\theta^T z)^p - 1 - p\theta^T z I_{\{|\theta^T z| \le \alpha\}}| \star \mu \in \mathcal{A}^+_{loc}, \quad \alpha \in (0,1), \tag{4.65}$$

and $\frac{\exp(-\Delta a^D)}{p(1-p)} \cdot X^D - \Phi_p(\theta) \cdot A$ is non-increasing (respectively is null for $\theta = \widehat{\theta}$), or equivalently, (4.65) and the following equalities hold,

$$\frac{\exp(-\Delta a^D)}{p(1-p)} \cdot X^D = \Phi_p\left(\widehat{\theta}\right) \cdot A, \tag{4.66}$$

$$\text{and} \qquad \min_{\theta \in \mathbb{R}^d} [\Phi_p(\theta)] = \Phi_p(\widehat{\theta}), \tag{4.67}$$

where Φ_p is given by (4.29).

Due to $I_{\{|\theta^T z| > \alpha\}} \star \mu = \sum I_{\{|\theta^T \Delta S| > \alpha\}} \in \mathcal{A}^+_{loc}$, and

$$|(1+\theta^T z)^p - 1 - p\theta^T z I_{\{|\theta^T z| \leq \alpha\}}| \star \mu = |(1+\theta^T z)^p - 1 - p\theta^T z| I_{\{|\theta^T z| \leq \alpha\}} \star \mu +$$

$$+ |(1+\theta^T z)^p - 1| I_{\{|\theta^T z| > \alpha\}} \star \mu,$$

we deduce that (4.65) is equivalent to the assertion (i) of the lemma.

Then, by combining (4.67) with Proposition 4.5 and recalling Lemma 4.6, we deduce that $\widehat{\theta} \in \text{int}(\mathcal{D}) = \text{int}\,(dom(\Phi_p))$, and the proof of assertion (ii) and (iii) are completed.

A direct implication of (4.61) yields

$$\frac{1}{p-1}\widehat{\theta}^T b = -\widehat{\theta}^T c \widehat{\theta} - \int \frac{(1+\widehat{\theta}^T z)^{p-1} - 1}{p-1} \widehat{\theta}^T z F(dz) \tag{4.68}$$

As a consequence, by inserting (4.68) into the equation (4.66), assertion (iv) follows immediately.

Assertion (v) of the lemma is a direct consequence of (4.61) and (4.62). Indeed, by multiplying (4.61) with ΔA, using $b\Delta A = \int x F(dx) \Delta A$, $c\Delta A = 0$ (see the properties of predictable characteristics of S in Section 2.A for details), we obtain

$$\int (1+\widehat{\theta}^T z)^{p-1} z \nu(\{t\}, dz) = 0, \tag{4.69}$$

By taking jumps in (4.62), we get

$$\exp(-\Delta a^D)\Delta X^D = \int \frac{(1+\widehat{\theta}^T z)^p - q(1+\widehat{\theta}^T z)^{p-1}}{q-1}\nu(\{t\},dz) + a. \qquad (4.70)$$

Then, by combining (4.69), (4.70) and

$$(1+\widehat{\theta}^T z)^p = (1+\widehat{\theta}^T z)^{p-1} + \widehat{\theta}^T z(1+\widehat{\theta}^T z)^{p-1}, \quad \text{and} \quad \Delta X^D = \exp(\Delta a^D) - 1,$$

assertion (v) follows immediately. This ends the proof of the lemma. □

The following lemma will show how the minimal Hellinger martingale density of order q is built-up and is related to the optimal portfolio rate, $\widehat{\theta}$, when U is a forward utility.

Lemma 4.8: *Suppose that the assumptions and assertion (1) of Theorem 4.2 are fulfilled. Then, the following properties hold:*
(i) The following $\widetilde{\mathcal{P}}$-measurable functional

$$W_t(z) := \frac{\left(1+\widehat{\theta}_t^T z\right)^{1/(q-1)} - 1}{1 - a_t + \int \left(1+\widehat{\theta}_t^T y\right)^{1/(q-1)} \nu(\{t\},dy)} = \frac{u(t,z)}{1+\widehat{u}}, \qquad (4.71)$$

is $(\mu-\nu)$-integrable, i.e. $W \in \mathcal{G}^1_{loc}(\mu)$.
(ii) The process, $\widetilde{Z}$, defined by

$$\widetilde{Z} := \mathcal{E}\left(\widetilde{N}\right), \quad \widetilde{N} := \frac{1}{q-1}\widehat{\theta}\cdot S^c + W \star (\mu-\nu), \qquad (4.72)$$

is a martingale density for S.
(iii) The following

$$(q-1)X^D + \sum \left[\left(1+\Delta X^D\right)^{q-1} - 1 - (q-1)\Delta X^D\right] = q(q-1)h^{(q)}(\widetilde{Z},P) \qquad (4.73)$$

holds, where X^D is defined in (4.59).

Proof. Thanks to Lemma 4.7–(v), we deduce that $(\widetilde{\gamma}_t)^{-1} = \frac{1}{1+\widehat{u}} = \exp(\Delta a^D)$

is locally bounded, and

$$\sum(\widehat{W}_t)^2 = \sum\left(\frac{\widehat{u}}{1+\widehat{u}}\right)^2 \preceq e^{3|\Delta a^D|}\cdot|a^D|_{var}.$$

Hence $\sum(\widehat{W}_t)^2$ is locally bounded process. Therefore, it is easy to see that $W\in\mathcal{G}^1_{loc}(\mu)$ if and only if the process

$$\left[\sum\left(W_t(\Delta S_t)\right)^2 I_{\{\Delta S_t\neq 0\}}\right]^{1/2} = \left[(\widetilde{\gamma})^{-2}\Big((1+\widehat{\theta}^T x)^{p-1}-1\Big)^2\star\mu\right]^{1/2},$$

is locally integrable. Since $(\widetilde{\gamma})^{-2}=(1+\widehat{u})^{-2}=e^{2\Delta a^D}$ is locally bounded, then this is equivalent to

$$\left[\Big((1+\widehat{\theta}^T x)^{p-1}-1\Big)^2\star\mu\right]^{1/2}\in\mathcal{A}^+_{loc}. \tag{4.74}$$

If we put $\Gamma:=\{z\in\mathbb{R}^d|\ |\widehat{\theta}^T z|\leq\alpha\}$, then it is easy to check that (4.74) is equivalent to the local integrability of

$$V_1 := \Big((1+\widehat{\theta}^T z)^{p-1}-1\Big)^2 I_\Gamma\star\mu,\ \text{ and }\ V_2 := |(1+\widehat{\theta}^T z)^{p-1}-1|I_{\Gamma^c}\star\mu. \tag{4.75}$$

The local integrability of V_1 follows directly from $I_{\{|\widehat{\theta}^T\Delta S|\leq\alpha\}}\cdot[\widehat{\theta}\cdot S,\widehat{\theta}\cdot S]\in\mathcal{A}^+_{loc}$ (since $\widehat{\theta}$ is S-integrable and hence $\widehat{\theta}\cdot S$ is a RCLL semimartingale), and

$$\frac{(q-1)^2}{(1-\alpha)^{2(p-2)}}V_1 \preceq \sum(\widehat{\theta}^T\Delta S)^2 I_{\{|\widehat{\theta}^T\Delta S|\leq\alpha\}} \preceq I_{\{|\widehat{\theta}^T\Delta S|\leq\alpha\}}\cdot[\widehat{\theta}\cdot S,\widehat{\theta}\cdot S].$$

To prove the local integrability of V_2, it is enough to prove

$$|\int(1+\widehat{\theta}^T z)^{p-1}\widehat{\theta}^T z I_{\{|\widehat{\theta}^T z|>\alpha\}}F(dz)|\cdot A\in\mathcal{A}^+_{loc}. \tag{4.76}$$

Indeed, by combining (4.76) with $(1+\widehat{\theta}^T z)^p I_{\{|\widehat{\theta}^T z|>\alpha\}}\star\mu\in\mathcal{A}^+_{loc}$ (see Lemma

4.7–(i)), and

$$(1+\widehat{\theta}^T z)^{p-1} I_\Gamma \star \nu = -\int_\Gamma (1+\widehat{\theta}^T z)^{p-1}\widehat{\theta}^T z F(dz)\cdot A + (1+\widehat{\theta}^T z)^p I_\Gamma \star \nu,$$

we deduce that $(1+\widehat{\theta}^T z)^{p-1} I_{\{|\widehat{\theta}^T z|>\alpha\}} \star \mu$ is locally integrable (since it is non-decreasing and its compensator is locally integrable). Finally, due to

$$I_{\{|\widehat{\theta}^T z|>\alpha\}} \star \mu = \sum I_{\{|\widehat{\theta}^T \Delta S|>\alpha\}} \in \mathcal{A}^+_{loc},$$

which follows from the fact that $\widehat{\theta}\cdot S$ is a RCLL semimartingale, we conclude that V_2 is locally integrable. In the remaining part of this proof, we will prove (4.76).

Thanks to Proposition 2.1, we have

$$\widehat{\theta}^T c\widehat{\theta}\cdot A \in \mathcal{A}^+_{loc}, \quad \text{and} \quad |\widehat{\xi}|\cdot A := |\widehat{\theta}^T b - \int \widehat{\theta}^T z I_{\{|\widehat{\theta}^T z|>\alpha\}} F(dz)|\cdot A \in \mathcal{A}^+_{loc}. \tag{4.77}$$

Since $\widehat{\theta}$ satisfies (4.61), then we get

$$-\widehat{\xi} - \widehat{\theta}^T c\widehat{\theta} = \frac{1}{p-1}\int_{\Gamma^c} (1+\widehat{\theta}^T z)^{p-1}\widehat{\theta}^T z F(dz) + \int_\Gamma \frac{(1+\widehat{\theta}^T z)^{p-1}-1}{p-1}\widehat{\theta}^T z F(dz). \tag{4.78}$$

Then, by combining

$$\begin{aligned} 0 &\preceq \tfrac{(1+\widehat{\theta}^T z)^{p-1}-1}{p-1}(\widehat{\theta}^T z) I_{\{|\widehat{\theta}^T z|\leq\alpha\}} \star \mu \preceq (1-\alpha)^{p-2}(\widehat{\theta}^T z)^2 I_{\{|\widehat{\theta}^T z|\leq\alpha\}} \star \mu \\ &\preceq (1-\alpha)^{p-2} I_{\{|\widehat{\theta}^T \Delta S|\leq\alpha\}}\cdot[\widehat{\theta}\cdot S, \widehat{\theta}\cdot S] \in \mathcal{A}^+_{loc}, \end{aligned}$$

(4.77) and (4.78), we conclude that (4.76) holds. This ends the proof of assertion (i) of the lemma.

The second assertion (i.e. assertion (ii)) of the lemma, follows directly from (4.61) by recalling Proposition 2.2.

The equality (4.73) is derived from (4.62) by using (2.50)—(2.51) depending whether $\Delta A = 0$ or $\Delta A \neq 0$ respectively. Precisely, on $\{\Delta A = 0\}$, a combination of (4.62) and (2.50) leads to

$$
\begin{aligned}
& I_{\{\Delta A=0\}} \cdot \Big((q-1)X^D + \sum \Big[\big(1+\Delta X^D\big)^{q-1} - 1 - (q-1)\Delta X^D \Big] \Big) \\
&= I_{\{\Delta A=0\}} e^{-\Delta a^D} \cdot X^D \qquad (4.78) \\
&= q(q-1) I_{\{\Delta A=0\}} \cdot h^{(q)}(\widetilde{Z}, P).
\end{aligned}
$$

While, on $\{\Delta A \neq 0\}$, by combining (4.62) and (2.51), we have

$$
\begin{aligned}
& \Delta \Big((q-1)X^D + \sum \Big[\big(1+\Delta X^D\big)^{q-1} - 1 - (q-1)\Delta X^D \Big] \Big) \\
&= \widetilde{\gamma}^{1-q} - 1 = q(q-1)\Delta h^{(q)}(\widetilde{Z}, P).
\end{aligned}
$$

This ends the proof of the lemma. $\square$

Lemma 4.9: *The process $\widetilde{Z}$ defined in Lemma 4.8 is the minimal Hellinger martingale density of order q. That is, $\widetilde{Z}$ is a martingale density (belongs to $\mathcal{Z}^e_{q,loc}(S,P)$) satisfying*

$$
h^{(q)}(\widetilde{Z}, P) \preceq h^{(q)}(Z, P), \quad \textit{for any } Z \in \mathcal{Z}^e_{q,loc}(S,P). \qquad (4.76)
$$

Proof. Thanks to Lemma 4.8–(ii), the proof of the lemma will follow from proving the optimality of $\widetilde{Z}$. In virtue of Proposition 3.2 in [17], it is enough to prove (4.76) for any positive martingale density $Z = \mathcal{E}(N)$ of the form

$$
N = \beta \cdot S^c + Y \star (\mu - \nu), \;\; Y_t(x) = k_t(x) + \frac{\widehat{k}_t}{1-a_t} I_{\{a_t<1\}}, \;\; \widehat{k}_t := \int k_t(x)\nu(\{t\}, dx),
$$

where $\beta \in L(S)$ and $\big(\sum k_t(\Delta S_t)^2 I_{\{\Delta S_t \neq 0\}}\big)^{1/2} \in \mathcal{A}^+_{loc}$. Due to the convexity of $z^T c z$ and $\phi(z) := \frac{(1+z)^q - qz - 1}{q(q-1)}$, on the set $\{\Delta A = 0\}$ we derive

$$
\begin{aligned}
\frac{dh^{(q)}(Z,P)}{dA} - \frac{dh^{(q)}(\widetilde{Z},P)}{dA} &= \frac{1}{2}(\beta^T c \beta - \widetilde{\theta}^T c \widetilde{\theta}) + \int \Big[\phi(k(x)) - \phi(\widetilde{k}(x)) \Big] F(dx) \\
&\geq \widetilde{\theta}^T c(\beta - \widetilde{\theta}) + \int \widetilde{\theta}^T x \Big(k(x) - \widetilde{k}(x) \Big) F(dx) = 0.
\end{aligned}
\qquad (4.77)
$$

Here $\widetilde{\theta} = (p-1)\widehat{\theta}$, $\widetilde{k}(x) := (1+\widehat{\theta}^T x)^{p-1} - 1$ and $\phi'(k((x)) = (p-1)\widehat{\theta}^T x = \widetilde{\theta}^T x$. The last equality in (4.77) is obtained from the fact that both $\widetilde{Z}$ and Z belong to $\mathcal{Z}^e_{q,loc}(S)$, which, due to Proposition 2.2, is equivalent to

$$b + c\beta + \int xk(x)F(dx) = 0, \quad \text{and} \quad b + (p-1)c\widehat{\theta} + \int x\widetilde{k}(x)F(dx) = 0. \tag{4.78}$$

On the other hand, due to (2.48), (2.49), and (2.51) in Proposition 2.3 and the convexity of $\phi(z)$, we get

$$\begin{aligned}
&\Delta h^E_t(Z,P) - \Delta h^E_t(\widetilde{Z},P) = (1-a_t)\left[\phi\Big(-\frac{\widehat{k}_t}{1-a_t}\Big) - \phi\Big(\widetilde{\gamma}_t^{-1} - 1\Big)\right] \\
&+ \int \Big[\phi(k_t(x)) - \phi\Big((1+\widehat{\theta}_t^T x)^{p-1}\widetilde{\gamma}_t^{-1} - 1\Big)\Big]\nu_t(dx) \\
&\geq (1-a_t)(1 - \frac{\widehat{k}_t}{1-a_t} - \frac{1}{\widetilde{\gamma}_t})\frac{\widetilde{\gamma}_t^{1-q} - 1}{q-1} \\
&+ \int \Big[k_t(x) + 1 - \widetilde{\gamma}_t^{-1}(1+\widehat{\theta}_t^T x)^{p-1}\Big]\frac{(\widehat{\theta}_t^T x + 1)\widetilde{\gamma}_t^{1-q} - 1}{q-1}\nu_t(dx) \\
&= \frac{\widetilde{\gamma}_t^{1-q}}{q-1}\int \Big[(k_t(x)+1) - (\widetilde{\gamma}_t)^{-1}(1+\widehat{\theta}^T x)^{p-1}\Big]\widehat{\theta}_t^T x \nu_t(dx) = 0.
\end{aligned} \tag{4.79}$$

The equation (4.79) follows from Proposition 2.2, which lead to two equations

$$\int x(k(x)+1)\nu(\{t\},dx) = 0, \quad \text{and} \quad 0 = \int \widetilde{\gamma}^{-1} x(1+\widehat{\theta}^T x)^{p-1}\nu(\{t\},dx).$$

Thus, by combining (4.77) and (4.79), we deduce that $\widetilde{Z}$ is the minimal Hellinger martingale density of order q for S. This achieves the proof of the lemma. □

Proof of Theorem 4.2:

We start proving (1) $\Longrightarrow$ (2). Thus, suppose that assertion (1) holds. Therefore, Proposition 4.5 and Lemmas 4.7, 4.8, 4.9 are valid, and the minimal Hellinger martingale density, $\widetilde{Z}$ exists (it is given by Lemma 4.8). Furthermore, an application of Ito's formula to $\mathcal{E}(X^D)^{q-1}$ combined with (4.37) and (4.73), will easily lead to (4.31). This proves assertions (2.b) and (2.c) of the theorem. To conclude that assertion (2) is satisfied, we need to prove the assertion (2.a). This follows from the forward property of U and

$$\begin{aligned} & U\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right) \\ &= D_0 x^p \mathcal{E}\left(q(q-1)h^{(q)}\left(\widetilde{Z}, P\right)\right)^{p-1} \mathcal{E}\left(\widehat{\theta}\cdot S\right)^p && (4.80)\\ &= D_0 x^p \mathcal{E}\left(\widehat{\theta}\cdot S\right) \mathcal{E}\left(\widetilde{\gamma}^{1-q}\widehat{\theta}\cdot S + q(q-1)h^{(q)}\left(\widetilde{Z}, P\right)\right)^{p-1} && (4.81)\\ &= D_0 x^p \mathcal{E}\left(\widehat{\theta}\cdot S\right)\widetilde{Z} = D_0 x^p \widehat{Z}. && (4.82)\end{aligned}$$

It is clear that (4.80) follows from (4.37) and $p-1 = \frac{1}{q-1}$, while (4.81) and (4.82) follows from (2.56) whenever the MHM density of order q exists and assertion (2.b) holds. This proves assertion (2).

In the remaining part of this proof, we focus on proving (2) $\Longrightarrow$ (1). Thus, we suppose that assertion (2) is fulfilled. Then, it is obvious that (4.80), (4.81), and (4.82) always hold as long as the MHM density of order q exists and assertion (2.b) is valid. As a consequence, a combination of these equalities with assertion (2.a) imply that $U\left(\cdot, x\mathcal{E}\left(\widehat{\theta}\cdot S\right)\right)$ is a martingale. Furthermore, for any admissible θ, we have

$$\begin{aligned}\frac{U\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right)}{D_0 x^p} &= \frac{\mathcal{E}\left(\theta\cdot S\right)^p}{\mathcal{E}\left(\widehat{\theta}\cdot S\right)^p}\mathcal{E}\left(q(q-1)h^{(q)}\left(\widetilde{Z}, P\right)\right)^{p-1}\mathcal{E}\left(\widehat{\theta}\cdot S\right)^{p-1}\mathcal{E}\left(\widehat{\theta}\cdot S\right)\\ &= \mathcal{E}\left(\widehat{\theta}\cdot S\right)\widetilde{Z}\left(\frac{\mathcal{E}\left(\theta\cdot S\right)}{\mathcal{E}\left(\widehat{\theta}\cdot S\right)}\right)^p = \widehat{Z}\left(\frac{\mathcal{E}\left(\theta\cdot S\right)}{\mathcal{E}\left(\widehat{\theta}\cdot S\right)}\right)^p.\end{aligned}$$

Then, due to this equality, the equivalence between the admissibility of θ and (4.35), and Proposition 4.4, we conclude that $U\left(\cdot, x\mathcal{E}\left(\theta\cdot S\right)\right)$ is a supermartingale for any admissible θ. Hence, U is a forward utility and assertion (1) holds true. This ends the proof of the theorem. $\square$

4.B Discrete-Time Market Models

Consider the discrete-time market model introduced in Section 3.B. Let p be a real number such that $p \in (-\infty, 0) \cup (0, 1)$. Here, consider the following utilities:

$$U_p(j, x) := D(j)x^p, \quad \text{for any } x \in (0, +\infty) \text{ and } j = 0, ..., N. \tag{4.83}$$

Here $D = (D(j))_{j=0,...,N}$ is a process satisfying

$$\sup_{0 \le j \le N} E\Big[|D(j)|\Big] < +\infty. \tag{4.84}$$

The set $\mathcal{D}_j$ and measure $G_j(dx)$ are same as given in (3.52) that I recall below

$$\mathcal{D}_j := \Big\{\theta \in \mathbb{R}^d \Big| \ 1 + \theta^T x > 0,\ G_j(dx) - a.e\Big\},\ \ G_j(dx) := P(\Delta S_j \in dx \mid \mathcal{F}_{j-1}). \tag{4.85}$$

For any process $X = (X_j)_{j=0,...,N}$, we associate to it the set of admissible portfolio rates for the j^{th} period of time, denoted by $\Theta_j^{(p)}(X)$, given by

$$\Theta_j^{(p)}(X) := \Big\{\theta \in L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j \Big| E\left(|X_j|(1 + \theta^T \Delta S_j)^p \ \Big| \mathcal{F}_{j-1}\right) < +\infty\Big\}. \tag{4.86}$$

Consider the following assumption:

Assumption: For any $j = 1, ..., N$, $\theta \in \mathcal{D}_j$, P-a.e., and every sequence $(\theta_n)_{n \ge 1}$, $\theta_n \in int(\mathcal{D}_j)$, P-a.e., and converges to θ, we have, $P - a.e.$ (4.87)

$$\lim_{n \to +\infty} E\Big(|D(j)K_p(\theta_n^T \Delta S_j)||\mathcal{F}_{j-1}\Big) = \begin{cases} +\infty, & \text{on } \Gamma_j; \\ E\Big(|D(j)K_p(\theta^T \Delta S_j)||\mathcal{F}_{j-1}\Big), & \text{on } \Gamma_j^c. \end{cases}$$

where $K_p(y) := y(1 + y)^{1/(q-1)}$ and $\Gamma_j := \{G_j(\mathbb{R}^d) > 0 \ \text{ and } \ \theta \notin int(\mathcal{D}_j)\}$.

Below, we state our parametrization algorithm for forward utilities having the form of (4.83).

Theorem 4.5: *Suppose that S is bounded, D satisfies (4.84) and assumption (4.87) holds. Let $p \in (-\infty, 0) \cup (0, 1)$. Then, the following are equivalent.*
(i) The functional $U_p(t, x)$, defined in (4.83), is a forward utility with the optimal portfolio rate denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,...,N}$.
(ii) The two processes D and $\widehat{\theta}$ are given by

$$\widehat{\theta}_j \in \Theta_j^{(p)}(D) \quad \textit{is a root of} \quad E\left(D(j)\Delta S_j(1+\theta^T \Delta S_j)^{p-1} \;\Big|\; \mathcal{F}_{j-1}\right) = 0, \tag{4.88}$$

$$\textit{and} \qquad D(j-1) = E\left(D(j)(1+\widehat{\theta}_j^T \Delta S_j)^{p-1} \Big| \mathcal{F}_{j-1}\right), \tag{4.89}$$

for all $j = 1, ..., N$.

Remark: Theorem 4.5 completely parameterizes the forward utilities of (4.83) in the discrete time setting. In fact, the unique parameter for these forward utilities is the terminal value of the process D, which is $D(N)$. Given this random variable, we calculate the optimal portfolio rate for the N^{th}-period of time, $\widehat{\theta}_N$ as a root of equation (4.88). Afterwards, we calculate D_{N-1} from (4.89). Then, we repeat this procedure over and over again until we completely determine the two processes D and $\widehat{\theta}$.

Proof of Theorem 4.5:
Remark that, due to (4.84), the process D can be represented by

$$D(j) = D(0) Z_j^D \exp(a_j^D), \qquad Z_j^D := \prod_{i=1}^{j} \frac{D(i)}{E(D(i)|\mathcal{F}_{i-1})},$$

$$a_j^D := \sum_{i=1}^{j} \log\left[E\left(\frac{D(i)}{D(i-1)}\Big|\mathcal{F}_{i-1}\right)\right], \qquad j = 1, ..., N; \quad Z_0^D = 1, \quad a_0^D = 0.$$

Here it is easy to check that Z^D is a true positive (since $pD(j) > 0$) martingale and a^D is predictable. Thus, through out the proof, we consider the probability measure $Q := Z_n^D \cdot P$. We will start by proving $(i) \Longrightarrow (ii)$. Thus, suppose that (i) holds. Then there exists an admissible portfolio rate $\widehat{\theta}$ such that for any other admissible portfolio rate θ, the processes $U_p\left(j, \prod_{k=1}^{j}(1+\widehat{\theta}_k^T \Delta S_k)\right)$ and $U_p\left(j, \prod_{k=1}^{j}(1+\theta_k^T \Delta S_k)\right)$ are martingale and supermartingale respectively. This implies that for any $j = 1, ..., N$,

$$D_0 E^Q\left((1+\theta_j^T \Delta S_j)^p \Big| \mathcal{F}_{j-1}\right) \leq e^{-a_j^D + a_{j-1}^D} = D_0 E^Q\left((1+\widehat{\theta}_j^T \Delta S_j)^p \Big| \mathcal{F}_{j-1}\right). \tag{4.90}$$

Then the equality in the right hand side of (4.90) together with Bayes' rule and

$$\frac{D(j)}{D(j-1)} = \frac{Z_j^D}{Z_{j-1}^D} e^{a_j^D - a_{j-1}^D}$$

implies (4.89). While the whole inequality (4.90) can be transformed into

$$D_0 \int (1+\theta_j^T x)^p G_j^Q(dx) \leq D_0 \int (1+\widehat{\theta}_j^T x)^p G_j^Q(dx),$$

where $G_j^Q(dx)$ is the random measure give by $G_j^Q(dx) := Q\left(\Delta S_j \in dx \Big| \mathcal{F}_{j-1}\right)$. Considering the assumption (4.87), by virtue of Proposition 4.5, it is clear that the function

$$\Psi_j(\lambda) := D_0 \int (1+\lambda^T x)^p G_j^Q(dx), \quad \lambda \in \mathcal{D}_j, \tag{4.91}$$

is differentiable on $int(\mathcal{D}_j)$, and attains its maximum at $\widehat{\theta}_j$. This implies that $\widehat{\theta}_j$ is a root for

$$0 = \nabla \Psi_j(\lambda) = p D_0 \int (1+\lambda^T x)^{p-1} x G_j^Q(dx).$$

This is equivalent to (4.88), and assertion (ii) follows.

To prove the reverse (i.e. $(ii) \Longrightarrow (i)$), we suppose that assertion (ii) holds. Then by multiplying both sides of (4.89) by $x^p \prod_{k=1}^{j-1}(1+\widehat{\theta}_j^T \Delta S_k)^p$, we obtain

$$D(j-1) x^p \prod_{k=1}^{j-1}(1+\widehat{\theta}_k^T \Delta S_k)^p = E\left(D(j) x^p \prod_{k=1}^{j}(1+\widehat{\theta}_k^T \Delta S_k)^p \,\Big| \mathcal{F}_{j-1}\right).$$

This proves that for any $x \in (0, +\infty)$ the process $U_p\left(j, x\prod_{k=1}^{j}(1+\widehat{\theta}_k^T\Delta S_k)\right)$, $j = 0, 1, ..., N$, is a martingale. Since $pD_j > 0$ and $p < 1$ for any $j = 0, ..., , N$, then for any admissible portfolio rate θ, we derive

$$D(j)(1+\theta_j^T\Delta S_j)^p - D(j)(1+\widehat{\theta}_j^T\Delta S_j)^p \leq D(j)\left(\theta_j - \widehat{\theta}_j\right)^T \Delta S_j(1+\widehat{\theta}_j^T\Delta S_j)^{p-1}.$$

Then, by taking conditional expectation in both sides above and using (4.88) and afterwards (4.89), we obtain

$$E\left(x^p D(j)(1+\theta_j^T\Delta S_j)^p \middle| \mathcal{F}_{j-1}\right) \leq E\left(x^p D(j)(1+\widehat{\theta}_j^T\Delta S_j)^p \middle| \mathcal{F}_{j-1}\right) = D(j-1)x^p.$$

Then by multiplying both sides of this inequality with $\prod_{k=1}^{j-1}(1+\theta_k^T\Delta S_k)^p$, we conclude that the process

$$U_p\left(j, x\prod_{k=1}^{j}(1+\theta_k^T\Delta S_k)\right) = x^p D(j)\prod_{k=1}^{j}(1+\theta_k^T\Delta S_k)^p, \quad j = 0, ..., N$$

is a supermartingale. This ends the proof of the theorem. □

4.C Discrete Market Models

Recall the discrete market models introduced in Section 3.C, including the one-dimensional binomial model in Subsection 3.C.1 and the multi-dimensional model in Subsection 3.C.2. In this section, I will consider these models, for which the power-type forward utilities having the form of (4.84) will be characterized.

4.C.1 One-Dimensional Binomial Model

First of all, for the one-dimensional binomial model, we recall the set $\mathcal{D}_j$, $j = 1, 2, ..., N$, (see (3.67))

$$\mathcal{D}_j = \Big]1/(1-\xi_j^u)S_{j-1}, \quad 1/(1-\xi_j^d)S_{j-1}\Big[. \tag{4.92}$$

It is clear that $\mathcal{D}_j$ is an open set in $\mathbb{R}$ and hence

$$int(\mathcal{D}_j) = \mathcal{D}_j, \quad P-a.e., \quad \forall \;\; j = 1, ..., N. \tag{4.93}$$

Furthermore, recall that $\#(\Omega) < +\infty$, then the admissible sets, $\Theta_j^{(p)}$, for the j^{th} period of time, $j = 1, 2, ..., N$, defined in (4.86) take the following forms

$$\Theta_j^{(p)} = L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j, \quad j = 1, ..., N. \tag{4.94}$$

Furthermore, remark that in current setting, the assumption (4.84) and (4.87) are satisfied.

The characterization of the power-type forward utilities in binomial model is stated in the following theorem.

Theorem 4.6: *Then, the following two assertions are equivalent.*

(i) The functional $U_p(t,x)$, defined in (4.83), is a forward utility with the optimal portfolio rate denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The process D is a supermartingale with the multiplication Doob-Meyer decomposition, $D = D_0 M \exp(a^D)$ (M is a positive martingale and a^D is predictable with finite variation) such that the following properties hold:

(ii.1) By putting $Q := \frac{M(N)}{M(0)} \cdot P$, then $\widehat{\theta}_j$ is given by for $j = 1, ..., N$

$$\widehat{\theta}_j = \frac{\gamma_j - 1}{(\xi_j^u - 1 - \gamma_j \xi_j^d + \gamma_j)S_{j-1}} \in \mathcal{D}_j, \;\; \gamma_j := \left(\frac{(\xi_j^u - 1)Q(A_j|\mathcal{F}_{j-1})}{(1-\xi_j^d)Q(A_j^c|\mathcal{F}_{j-1})} \right)^{1-q} \tag{4.95}$$

(ii.2) The predictable process a^D is given by

$$a_j^D = -\sum_{k=1}^{j} \log\Big(\frac{(\gamma_k^{p-1}Q(A_k|\mathcal{F}_{k-1}) + Q(A_k^c|\mathcal{F}_{k-1}))(\xi_k^u - \xi_k^d)^{p-1}}{(\xi_k^u - 1 - \gamma_k\xi_k^d + \gamma_k)^{p-1}}\Big).$$

Proof. This theorem can be treated as an application of Theorem 4.5. Thus, we will avoid to repeat the same proof again, but only give some remarks emphasizing its nice features that simplify tremendously the proof.

Since $\mathcal{D}_j$ is an open and $\#(\Omega) < +\infty$, the assumptions (4.84) and (4.87) are automatically fulfilled.

The function Ψ_j given by (4.91) becomes

$$\Psi_j(\lambda) = Q(A_j|\mathcal{F}_{j-1})(1 + (\xi_j^u - 1)\lambda S_{j-1})^p + Q(A_j^c|\mathcal{F}_{j-1})(1 + (\xi_j^d - 1)\lambda S_{j-1})^p$$

which is differentiable on $\mathcal{D}_j$. Thus, $\widehat{\theta}_j$ is the solution of the equation, $\Psi'(\lambda) = 0$, which leads to (4.95).

Finally, a_j^D is derived by plugging (4.95) into (4.89) and apply the decomposition of D. This ends the proof of the theorem. □

4.C.2 Multi-Dimensional Discrete Model

Now, we turn to the multi-dimensional discrete market models described in Subsection 3.C.2. Let Ω be the sample space, which is finite in current model and the number of its elements is $(dn)^N$. We recall the set $\mathcal{D}_j$, $j = 1, 2, ..., N$, defined in (3.74) and the event A_j, given in (3.73). Remark that the set $\mathcal{D}_j$ is open. Therefore, we have

$$int(\mathcal{D}_j) = \mathcal{D}_j, \quad P - a.e., \quad \forall \;\; j = 1, ..., N. \tag{4.96}$$

Again, due to $\#(\Omega) < +\infty$, the admissible sets, $\Theta_j^{(p)}$, defined in (4.86) take the following forms

$$\Theta_j^{(p)} = L^0(\mathcal{F}_{j-1}) \cap \mathcal{D}_j, \quad j = 1, ..., N. \tag{4.97}$$

The characterization of the logarithm forward utilities in multi-dimensional discrete model is stated in the following theorem.

Theorem 4.7: *Then, the following two assertions are equivalent.*

(i) The functional $U_p(t,x)$, defined in (4.83), is a forward utility with the optimal portfolio rate denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The process D is a supermartingale with the multiplication Doob-Meyer decomposition, $D = D_0 M \exp(a^D)$, (M is a positive martingale and a^D is predictable with finite variation) such that the following properties hold:

(ii.1) By putting $Q := \frac{M(N)}{M(0)} \cdot P$, for $j = 1, 2, ..., N$, $\widehat{\theta}_j \in \mathcal{D}_j$ and is a root of

$$\sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \frac{(\Xi_j(n_1,...,n_d) - I_{d\times d})I_d Q(A_j(n_1,...,n_d)|\mathcal{F}_{j-1})}{(1+\theta^T(\Xi_j(n_1,...,n_d) - I_{d\times d})S_{j-1})^{1/(1-q)}} = 0. \qquad (4.98)$$

(ii.2) The predictable part $a^D = (a_j^D)_{j=1,...,N}$ is given by

$$a_j^D = -\sum_{k=1}^{j} \log\left[K_k\right], \qquad (4.99)$$

where

$$K_k := \sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \left(1+\theta^T(\Xi_k(n_1,...,n_d) - I_{d\times d})S_{k-1}\right)^{p-1} Q(A_k(n_1,...,n_d)|\mathcal{F}_{k-1})$$

Proof. The proof would be a generalization of the proof of Theorem 4.6 to vectors and matrices. For the same reason as indicated in the proof of Theorem 4.6, the assumptions (4.84) and (4.87) are automatical satisfied.

Herein, the function $\Psi(\lambda)$ given by (4.91) becomes

$$\Psi(\lambda) = \sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \left(1+\theta^T(\Xi_k(n_1,...,n_d) - I_{d\times d})S_{k-1}\right)^{p} Q(A_k(n_1,...,n_d)|\mathcal{F}_{k-1}).$$

Therefore, we can derive (4.98) and (4.99) immediately. □

4.D Lévy Market Models

Consider the Lévy market models given in Section 3.D. The process of the stock price is presented by $S = S_0 \exp(X)$, which is semimartingale. X is modeled by a locally bounded Lévy process, given by (3.82). Some usual notation will be put in the same way as Section 3.D. In particular, W_t, $t \in [0,T]$, represents a Brownian motion; $N(dt, dx)$ is Poisson random measure on $[0,T] \times \mathbb{R} \setminus \{0\}$, used to measure the jumps of X; $\widetilde{N}(dt, dx)$ is the compensated Poisson measure with the intensity measure $F^X(dx)dt$, where $F^X(dx)$ is called the Lévy measure defined on $\mathbb{R} \setminus \{0\}$.
Since X is locally bounded, S is **locally bounded** as well. Thus, $\int \int_{\mathbb{R}\setminus\{0\}}(e^x - x - 1)F_t^X(dx)dt$ is locally integrable and $W(t,x) := e^x - x - 1$ is $\widetilde{N}$-integrable. By virtue of Ito's formula, S is also a Lévy process. Let F^S is the intensity of S, then, Furthermore, for any measurable and non-negative/integrable function $k(x)$, the two measures $F^X(dy)$ and $F^S(dx)$ are related in the following manner

$$\int_{\mathbb{R}\setminus\{0\}} k(x)F^S(dx) = \int_{\mathbb{R}\setminus\{0\}} k(e^{X_-}(e^y - 1))F^X(dy).$$

Remark that this model is quasi-left continuous such that any local martingale N follows a decomposition given by (3.83) and we let (β, Y, V, N') be the Jacod components of N. For more details on the properties of each component, the reader can find them in Theorem 2.2, where they are given in the most general semimartingale framework.

In this section, we will investigate the characterization of the power-type forward utilities, $U_p(t,x)$, given by

$$U_p(t,x) := D(t)x^p, \qquad p \in (-\infty, 0) \cup (0,1). \tag{4.100}$$

As usual, an integrability condition imposed on the process D is

$$\sup_{\tau \in \mathcal{T}_T} E\Big[|D(\tau)|\Big] < +\infty. \tag{4.101}$$

Also, we recall the set $\mathcal{D}$ given by (4.28) and the functional $\Phi_p(\lambda)$ given by (4.29). For any probability measure Q, any stock price process X, and $x \in \mathbb{R}$ such that $U_0(t, x, \omega) < +\infty$ we denote by

$$\mathcal{A}_{adm}(x, X, Q) := \Big\{ \pi \in L(X) \mid \sup_{\tau \in \mathcal{T}_T} E^Q \left[|D(\tau)(x + \pi \cdot X_\tau)^p| \right] < +\infty \Big\}, \tag{4.102}$$

the set of admissible portfolios for the model (x, X, Q, U). Here $\mathcal{T}_T$ is the set of stopping time, τ, such that $\tau \leq T$. When $X = S$ and $Q = P$, we simply write $\mathcal{A}_{adm}(x)$.

Recall Theorem 4.3, the following condition is necessary for U_p being forward utility:

$$D = D_0 \mathcal{E}(N)\mathcal{E}(V), \text{ where } \mathcal{E}(N) > 0,\ N \in \mathcal{M}_{loc}(P) \text{ and } V \in \mathcal{P} \cap \mathcal{V}. \tag{4.103}$$

Let $N = (\beta, Y, V, N')$ be the Jacod components and consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $dP \otimes dt$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $dP \otimes dt$-a.e., and $\lambda_n \to \lambda$, we have, $dP \otimes dt - a.e.$,

$$\lim_{n\to+\infty} \int K_p(e^{X_-}(e^x - 1)\lambda_n) Y(e^{X_-}(e^x - 1)) F^X(dx) \tag{4.104}$$
$$= \begin{cases} +\infty, & \text{on } \ \Gamma; \\ \int K_p(e^{X_-}(e^x - 1)\lambda) Y(e^{X_-}(e^x - 1)) F^X(dx), & \text{on } \ \Gamma^c. \end{cases}$$

where $K_p(y) := y((1 + y)^{p-1} - 1)$ and $\Gamma := \{F^X(\mathbb{R}) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

Our main result in this section is presented in the following theorem.

Theorem 4.8: *Consider the functional $U_p(t, \omega, x)$ defined in (4.100) satisfying (4.101). Suppose that the assumption (4.103) and (4.104) hold. Then the* *following two assertions are equivalent.*

(1) The functional U_p is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) The following properties hold.

(2.a) The predictable process V is

$$q\int_0^{\cdot}[\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2+\widetilde{\xi}_u]du, \tag{4.105}$$

where

$$\widetilde{\xi}_u := \int_{I\!R\setminus\{0\}} f_p((1+e^{X_u}(e^x-1)\widehat{\theta}_u)^{p-1}-1)Y(e^{X_u}(e^x-1))F_u^X(dx).$$

(2.b) The optimal portfolio rate $\widehat{\theta}$ is a root for

$$\begin{aligned}\int_{I\!R\setminus\{0\}}(e^x-1)(1+e^X(e^x-1)\lambda)^{p-1}Y(e^X(e^x-1))F^X(dx)\\ +\gamma+\frac{1}{2}\sigma^2+e^X\sigma^2(\beta+(p-1)\lambda)=0\end{aligned} \tag{4.106}$$

(2.c) The local martingale $\widehat{Z}$, given by

$$\widehat{Z} := \mathcal{E}(N)\mathcal{E}\left(\frac{\widehat{\theta}}{q-1}\cdot\overline{S}^c+\left((1+e^X(e^x-1)\widehat{\theta})^{p-1}-1\right)\star\overline{N}\right)\mathcal{E}(\widehat{\theta}\cdot S),$$

is a true martingale. Here,

$$\overline{S}^c=\int_0^t e^{X_u}\sigma dW_u-\int_0^t e^{2X_u}\sigma^2\beta_u du,$$

$$\overline{N}(dt,dx)=N(dt,dx)-Y(e^{X_t}(e^x-1))F_t^Y(dx)dt.$$

Proof. This theorem can be viewed as an application of Theorems 4.3 and 4.4. We can apply their results directly. Furthermore, most of the calculations for Lévy market model has been given in the proof of Theorem 3.7. They include the dynamics of S, the predictable characteristics of S, (b,c,F^S) under P and, (b^Q,c,F^Q), under Q. We only need to mention that the function Φ_p given by

(4.29) becomes

$$\Phi_p(\lambda) := \frac{(b^Q)^T\lambda}{p-1} + \frac{1}{2}\lambda^T c\lambda + \int f_p(\lambda^T x)F^Q(dx), \quad \forall\, \lambda \in \mathbb{R}^d,\ p \in (-\infty,0)\cup(0,1), \tag{4.107}$$

Assumption (4.104) will guarantee its differentiability on $int(\mathcal{D})$, which leads to (4.106). Furthermore, (4.105) would be a direct application of (4.39). Finally, the assertion (2.c) can be derived after some simple calculation by using the predictable characteristics of S. □

4.D.1 Jump-Diffusion Model

Here in this subsection, we consider the model where the stock price process S is given by $S = e^X$, and X is a jump-diffusion process following dynamics

$$X_t = \gamma t + \sigma W_t + \widetilde{N}_t, \quad \widetilde{N}_t = N_t - \lambda t. \tag{4.108}$$

Here, W is a standard Brownian Motion, N is a simple Poisson process with rate $\lambda > 0$, and $\widetilde{N}$ is the compensated Poisson process ($\widetilde{N}$ is a martingale). $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by the Brownian Motion W and the Poisson process N. In this model, for any local martingale Y, there exists two predictable processes, α and η, such that $\int_0^T (\alpha_u^2 + \eta_u^2)du < +\infty$, P-a.s., and

$$Y_t = Y_0 + \int_0^t \alpha_u dW_u + \int_0^t \eta_u d\widetilde{N}_u, \quad t \in [0,T] \tag{4.109}$$

Then, the characterization/paramaterization of the power-type forward utilities, U_p, defined in (4.100), becomes as follows.

Theorem 4.9: *Consider the functional $U_p(t,\omega,x)$ defined in (4.100) satisfying (4.101), and the stock price process $S = e^X$ with X given by (4.108). Then the following two assertions are equivalent.*

(1) The functional U_p is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2) $D = D_0\mathcal{E}(M)\mathcal{E}(V)$ is a supermartingale (M is a local martingale fol-

lowing the decomposition (4.109) and V is non-decreasing and continuous), satisfying the following properties:
(2.a) The process V coincides with

$$q\int_0^t [\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2 + \lambda(1+\eta_u)f_q((1+e^{X_u}(e-1)\widehat{\theta}_u)^{\frac{1}{q-1}} - 1)]du, \quad 0 \le t \le T. \tag{4.110}$$

(2.b) The optimal portfolio rate $\widehat{\theta}$ is a root for

$$\gamma + \frac{1}{2}\sigma^2 + e^{X_-}\sigma^2(\alpha + \frac{\theta}{q-1}) + \lambda(e-1)\frac{1+\eta}{(1+e^{X_-}(e-1)\theta)^{1/(1-q)}} = 0. \tag{4.111}$$

(2.c) The local martingale

$$\widehat{Z} := D_0\mathcal{E}(M)\mathcal{E}\left(\frac{1}{q-1}\widehat{\theta}\cdot\overline{S}^c + \left[(1+e^{X_-}(e-1)\widehat{\theta})^{1/(q-1)} - 1\right]\cdot\overline{N}\right)\mathcal{E}(\widehat{\theta}\cdot S)$$

is a true martingale. Here

$$\overline{S}_t^c = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_u}\sigma^2\alpha_u du, \quad d\overline{N}_t = dN_t - \lambda(1+\eta)dt,$$

Proof. This theorem is a direct application of Theorem 4.8 and the following remarks.
a) All quantities required for this proof are already given in the proof of Theorem 3.8, especially the characteristics of S under P and Q.
b) The assumption (4.104) is satisfied here since the set $\mathcal{D}$ is open and is given by

$$\mathcal{D} := \{\theta \in \mathbb{R} : \quad 1 + \theta e^{X_-}(e-1) > 0\} =]-\frac{e^{X_-}}{e-1}, \ +\infty[= int(\mathcal{D}).$$

c) The function Φ_p becomes

$$\Phi_p(\theta) := \frac{\theta b^Q}{p-1} + \frac{1}{2}c\theta^2 + \lambda(1+\eta)f_p(\theta e^{X_-}(e-1)),$$

which is differentiable on $\mathcal{D}$. Therefore, (2.a), (2.b) and (2.c) will follow after some simple calculations. This ends the proof of the theorem. □

4.D.2 Black-Scholes Model

Finally, we consider the Black-Scholes model where there are no jumps and the only source of uncertainty is from the Brownian Motion. Same as before, the price process $S = e^X$, where X is an Ito process. It can be written as

$$X_t = \gamma t + \sigma W_t, \quad t \in [0, T]. \tag{4.112}$$

The filtration is generated by W such that any local martingale, Y, can be represented as

$$Y_t = \int_0^t \alpha_u dW_u, \quad Y_0 = 0, \quad t \in [0, T], \tag{4.113}$$

where α is a progressively measurable process such that $\int_0^T \alpha_u^2 du < +\infty$, P-a.s. Hence, the characterization of the power-type forward utilities under this setup becomes as follows.

Theorem 4.10: *Consider the functional $U_p(t, \omega, x)$ defined in (4.100) satisfying (4.101) and the stock prices process is given as $S = e^X$, where X follows the dynamics (4.112). Then the following two assertions are equivalent.*
(1) The functional U_p is a forward utility with the optimal portfolio rate $\widehat{\theta}$.
(2) The processes $D = D_0 \mathcal{E}(M) \exp(a^D)$ is a supermartingale (M is a local martingale and a^D is continuous with finite variation), satisfying the following:
(2.a) The process a^D is given by

$$\frac{q}{2}\sigma^2 \int_0^{\cdot} e^{2X_u} \widehat{\theta}_u^2 du. \tag{4.114}$$

(2.b) Put $M = \alpha \cdot W$, then the optimal portfolio rate $\widehat{\theta}$ is given by

$$\widehat{\theta} = (1 - q) \left[e^{-X}(\gamma \sigma^{-2} + \frac{1}{2}) + \alpha \right]. \tag{4.115}$$

(2.c) The local martingale $\widehat{Z}$, given by

$$\widehat{Z} := \mathcal{E}(M)\mathcal{E}\left(\frac{\widehat{\theta}}{q-1}e^{X}\alpha \cdot W - \frac{\sigma^2}{q-1}\int_0^{\cdot} \widehat{\theta}_u e^{2X_u}\alpha_u du\right)\mathcal{E}(\widehat{\theta}\cdot S)$$

is a true martingale.

Proof. Note that the assumption (4.104) is automatically satisfied when S is continuous due to $F = 0$. Then, this theorem follows immediately from Theorem 4.9 by putting $\lambda = \eta = 0$. □

4.E Volatility Market Models

In this section, we will describe the power-type forward utilities in two volatility models: The corrected Stein and Stein model and the Barndorff-Nielsen-Shephard model. Detailed formulations on these models have been provided in Section 3.E.

4.E.1 Corrected Stein and Stein Model

I will start this subsection by recalling the corrected Stein and Stein model as follows. The stock price process, S, follows the dynamics as

$$dS_t = \mu V_t^2 S_t dt + \sigma V_t S_t dB_t, \quad t \in [0,T], \tag{4.116}$$

where V is the volatility process described by

$$dV_t = (m - \alpha V_t)dt + \beta dW_t. \tag{4.117}$$

Here, all the parameters μ, σ, m, α and β are positive constants.

Theorem 4.11: *Consider two $(\mathcal{F}_t)_{0\le t\le T}$ progressively measurable processes,* *D and p, such that*

$$\sup_{\tau\in\mathcal{T}_T} E\left[|D(\tau)|+|p(\tau)|\right] < +\infty, \text{ and } \quad \inf_{0\leq t\leq T}|p(t)|> 0 \quad P-a.s. \tag{4.118}$$

Then, the following are equivalent.

(i) The functional, $U(t,x) := D(t)x^{p(t)}$, is a forward utility with optimal portfolio rate, $\widehat{\theta}_x$, for any initial capital $x \in (0,+\infty)$.

(ii) The process $p(t)$ is constant in (ω,t) (i.e. $p(t,\omega) = p(0) = p < 1$), and there exist two progressively measurable processes, ϕ and ψ, such that

$$\int_0^T \left[(\phi_u)^2 + (\psi_u)^2\right] du < +\infty, \ P-a.s.,$$

and the following properties hold.

(ii.a) The process D is described by

$$D(t) = D(0)\mathcal{E}_t\left(\phi\cdot B + \psi\cdot B^{\perp}\right)\exp\left[\frac{p}{2(p-1)}\int_0^t\left(\frac{\mu V_u}{\sigma}+\phi_u\right)^2 du\right]. \tag{4.119}$$

(ii.b) The optimal portfolio rate, $\widehat{\theta}$, is given by

$$\widehat{\theta}_t = -\frac{\mu V_t}{\sigma} - \phi_t. \tag{4.120}$$

(ii.c) If we put $\Psi(\theta) := (2+\sigma SV)\theta + \frac{\mu}{\sigma}V(1+\sigma SV)$, then

$$\mathcal{E}\left(\Psi(\phi)\cdot B + \psi\cdot B^{\perp}\right) \quad \textit{is a martingale.} \tag{4.121}$$

Proof. Remark that, in the current framework, any positive local martingale Z is given by two progressively measurable processes ϕ and ψ as follows

$$Z^D := \mathcal{E}\left(\phi\cdot B + \psi\cdot B^{\perp}\right), \quad \int_0^T \left[(\phi_u)^2 + (\psi_u)^2\right] du < +\infty, \quad P-.a.s.,$$

and any local martingale is continuous. Thus, any special semimartingale is locally bounded (since its local martingale part is continuous), and hence the process p is locally bounded. Furthermore, the assumption (4.30) is automatically satisfied in current continuous model since the measure $F = 0$. Given these details, it is obvious that the proof of this theorem follows immediately from Theorem 4.3. $\square$

Remark: Since the model in (4.116)—(4.117) is a Markovian model, then one can probably characterize the forward utilities when p is a real constant by putting $D(t) = g(S_t, V_t)$ and g will be then solution to a Hamilton-Jacobi-Bellman equation (HJB equation hereafter). The main difficulty in this method (as well as other methods proposed in the literature) lies in solving the obtained HJB which there is no reason to have an explicit solution. Hence, this will directly impact negatively our hope to get examples of forward utilities or explicitly describe this class of forward utilities.

4.E.2 Barndorff-Nielsen Shephard Model

Now, we turn to the Barndorff-Nielsen-Shephard model. The stock price process is assumed to be the exponential of a Lévy process and is defined by $S_t = \exp(X_t)$, where X satisfies

$$dX_t = (\mu + \beta\sigma_t^2)dt + \sigma_t dY_t^c + d(\rho z \star \tilde{\mu}_Y)_t, \tag{4.122}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + d(z \star \tilde{\mu}_Y)_t, \tag{4.123}$$

Here, the parameters $\mu, \beta, \rho, \lambda$ are real constants with $\lambda > 0$ and $\rho < 0$. Then, a simple application of Ito's formula gives us the following dynamic of S

$$\frac{dS_t}{S_{t-}} = \left(\mu + \sigma_t^2(\beta + \frac{1}{2}) + \int (e^{\rho z} - 1)\widetilde{F}_t(dz)\right) dt + \sigma_t dY_t^c + d(e^{\rho z} - 1) \star (\mu_Y - \nu_Y)_t. \tag{4.124}$$

The set $\mathcal{D}$ is given by

$$\mathcal{D} := \left\{\theta \in \mathbb{R} \ : \quad 1 + S\theta(e^{\rho x} - 1) > 0, \qquad \widetilde{F} - a.e.\right\}. \tag{4.125}$$

Recall Theorem 4.3, consider a process, D, such that

$$\sup_{\tau \in \mathcal{T}_T} E\left[|D(\tau)|\right] < +\infty. \tag{4.126}$$

If U_p is a forward utility, there exist a predictable process, $\phi \in L(Y^c)$, and a positive and $\widetilde{\mathcal{P}}$-measurable functional, $f \in \mathcal{G}^1_{loc}(\mu)$, $f > 0$, such that the process D is given by

$$D(t) = D(0)\mathcal{E}_t(N) \exp\Big(V_t(\phi, f)\Big), \tag{4.127}$$

where N is given by

$$N := \phi \cdot Y^c + (f-1) \star (\mu - \nu), \tag{4.128}$$

Due to Theorem 4.3, (4.127–4.128) are necessary conditions for U_p to be a forward utility.
We consider the following assumption:

Assumption: For any predictable process λ such that $\lambda \in \mathcal{D}$, $dP \otimes dt$-a.e., and every sequence of predictable processes, $(\lambda_n)_{n\geq 1}$, such that $\lambda_n \in int(\mathcal{D})$, $dP \otimes dt$-a.e., and $\lambda_n \to \lambda$, we have, $dP \otimes dt - a.e.$,

$$\begin{aligned}&\lim_{n\to+\infty} \int K_p(e^{X_-}(e^{\rho x}-1)\lambda_n) f(e^{X_-}(e^{\rho x}-1))\widetilde{F}(dx) \qquad (4.129)\\ &= \begin{cases} +\infty, & \text{on } \quad \Gamma; \\ \int K_p(e^{X_-}(e^{\rho x}-1)\lambda) f(e^{X_-}(e^{\rho x}-1))\widetilde{F}(dx), & \text{on } \quad \Gamma^c. \end{cases}\end{aligned}$$

where $K_p(y) := y((1+y)^{p-1}-1)$ and $\Gamma := \{\widetilde{F}(\mathbb{R}) > 0 \text{ and } \lambda \notin int(\mathcal{D})\}$.

The main result in this subsection is given in the following theorem.

Theorem 4.12: *Let $p \in (-\infty, 0) \cup (0,1)$, q is its conjugate number and (4.129) holds. D is given by (4.127–4.128) satisfying (4.126). Then, the following* *are equivalent.*

(i) $U(t,x) := D(t)x^p$ *is a forward utility with optimal portfolio rate,* $\widehat{\theta}$.

(ii) The following holds:

(ii.a) $V(\phi, f)$ *is coincides with*

$$\frac{q}{2(q-1)^2}\int_0^t \widehat{\theta}_u^2\sigma_u^2 e^{2X_{u-}}du + qf_q\left((1+\widehat{\theta}(e^{\rho z}-1)e^{X_-})^{\frac{1}{q-1}}-1\right)f(z)\star\nu_t. \tag{4.130}$$

(ii.b) The optimal portfolio rate, $\widehat{\theta}$ *is a root for*

$$0 = \mu+\sigma^2(\beta+\frac{1}{2})+\sigma\phi+\frac{\sigma^2 e^{X_-}}{q-1}\theta+\int\left(1+\theta(e^{\rho z}-1)e^{X_-}\right)^{\frac{1}{q-1}}(e^{\rho z}-1)f(z)\widetilde{F}(dz). \tag{4.131}$$

(ii.c) The local martingale

$$\widehat{Z} := \mathcal{E}(N)\mathcal{E}\left(\frac{\widehat{\theta}}{q-1}\cdot\overline{S}^c + \left((1+e^{X_-}(e^{\rho x}-1)\widehat{\theta})^{\frac{1}{q-1}}-1\right)\star\overline{N}(dt,dx)\right)\mathcal{E}(\widehat{\theta}\cdot S),$$

is a true martingale. Here,

$$\overline{S}^c = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_{u-}}\sigma^2\alpha_u du,$$

$$\overline{N}(dt,dx) = \mu(dt,dx) - f(e^{X_{t-}}(e^{\rho x}-1))F_t(dx)dt,$$

Proof. The proof of this theorem is immediate by virtue of Theorem 4.2. We will only give short remarks to clarify more the connection between the current theorem and Theorem 4.2.

From the dynamic of S given by (4.124), we can find the predictable characteristics of S, (b, c, ν^S) under P and, (b^Q, c, F^Q), under Q (see the proof of Theorem 3.11 for details). Furthermore, the function Φ_p given by (4.29) becomes

$$\Phi_p(\lambda) := \frac{(b^Q)^T\lambda}{p-1}+\frac{1}{2}\lambda^T c\lambda+\int f_p(\lambda^T x)F^Q(dx), \quad p\in(-\infty,0)\cup(0,1), \tag{4.132}$$

Assumption (4.129) will guarantee its differentiability on $int(\mathcal{D})$, which leads to (4.131). The assertion (ii.c) is a direct application of Theorem 4.2–(2.a) together with some calculations based on the predictable characteristics of S. $\square$

Chapter 5

Exponential-Type Forward Utilities

This chapter will address the third and last class of forward utilities. We start by providing its definition as follows.

Definition: Let X be a RCLL semimartingale and Q be a probability measure. Then, we call exponential-type forward utility (called exp-type forward utility hereafter) for (X, Q), any forward dynamic utility for (X, Q), $U(t, x, \omega)$, given by

$$U_1(t, x, \omega) = -\exp\left(-\frac{x - B_t(\omega)}{N_t(\omega)}\right), \quad x \in \mathbb{R}, \tag{5.1}$$

where N is a positive process and B is a process.

Herein, the process N can be seen as the risk aversion coefficient of the random field utility. The interplay between the stochastic risk aversion and the forward property will be investigated in the following. Indeed, this is an extension of the case when $U_1(\omega, t, x)$ is independent of (ω, t) and takes the form of $-e^{-rx}$.

Remark that one of the main difference between the exp-type forward utilities and the log-type or power-type forward utilities lies in the effective domain and its impact on the analysis. Moreover, when S is locally bounded, we actually don't require any technical assumption to characterize the exp-type forward utilities.

5.A The Semimartingale Framework

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration is complete and right continuous (i.e. satisfies the usual conditions). In this setup, we consider a d-dimensional semimartingale $S = (S_t)_{0\leq t\leq T}$ which represents the discounted price processes of d risky assets. The Canonical representation of S is given by

$$S = S_0 + S^c + h(x) \star (\mu - \nu) + (x - h(x)) \star \mu + B. \tag{5.2}$$

More details on this representation can be found in Chapter 1. Throughout this chapter, the main assumption imposed on S is

$$\int_{\{|x|>1\}} |x| e^{\lambda^T x} F(dx) < +\infty, \qquad P \otimes A - a.e., \tag{5.3}$$

for all $\lambda \in \mathbb{R}^d$. Remark that this assumption is satisfied automatically when S is locally bounded.

The following intermediary lemmas play important roles in simplifying our forthcoming analysis. Furthermore, these lemmas are also interesting on their own right.

Lemma 5.1: *Let Q be a σ-martingale measure for S, and $\theta \in L(S)$ be such that*

$$\sup_{\tau\in\mathcal{T}_T} E^Q \exp\Big[(\theta \cdot S)_\tau\Big] < +\infty. \tag{5.4}$$

Then, the process $\theta \cdot S$ is a Q-local martingale and the process $\exp[\theta \cdot S]$ *is a positive Q-submartingale.*

Proof. Since Q is a σ-martingale measure for S, then, there exists a positive, bounded and predictable process ϕ such that $\phi \cdot S$ is a Q-local martingale. As a result, $\theta \cdot S$ is σ-martingale under Q on one hand. On the other hand, it is clear that

$$X_t := \exp\left(\frac{1}{2}(\theta \cdot S)_t\right)$$

is a positive and special semimartingale with the Doob-Meyer decomposition given by

$$X = X_0 + \overline{N} + \overline{B}. \tag{5.5}$$

Here, $\overline{N}$ is a local martingale and $\overline{B}$ is predictable with finite variation such that $\overline{N}_0 = \overline{B}_0 = 0$. Let $(T_n)_{n\geq 1}$ be a sequence of stopping times that increases stationarily to T and

$$E\left([\overline{N}, \overline{N}]_{T_n}^{1/2} + Var_{T_n}(\overline{B})\right) < +\infty. \tag{5.6}$$

Then, for any predictable process φ such that $|\varphi|\leq 1$, we have

$$E|\varphi \cdot X_{T_n}|\leq cE\left([\overline{N}, \overline{N}]_{T_n}^{1/2} + Var_{T_n}(\overline{B})\right), \tag{5.7}$$

where c is a constant that does not depend on φ.

Thanks to Ito's formula, we get

$$X = 1 + \frac{1}{2}X_- \cdot (\theta \cdot S) + X_- \cdot V(\theta),$$

where $V(\theta)$ is a non-decreasing process given by

$$V(\theta) := \frac{1}{8}\theta^T c\theta \cdot A + \left(\exp\left(\frac{1}{2}\theta^T x\right) - 1 - \frac{1}{2}\theta^T x\right) \star \mu. \tag{5.8}$$

Since $\theta \cdot S$ is a σ-martingale under Q, then there exists $0 < \phi \leq 1$ such that $\phi\theta \cdot S$ is a Q-local martingale. Consider a sequence of stopping times, $(\sigma_n)_{n\geq 1}$, that increases stationarily to T such that $(\phi\theta \cdot S)^{\sigma_n}$ is a true Q-martingale. Then, for any $\varepsilon > 0$, $\left(\frac{\phi}{\phi+\varepsilon X_-}X_-\theta \cdot S\right)^{\sigma_n}$ is also a true Q-martingale. As a result, we derive

$$\begin{aligned} E\int_0^{\sigma_n\wedge T_n} X_{s-}dV_s(\theta) &= \lim_{\varepsilon\downarrow 0} E\int_0^{\sigma_n\wedge T_n} \frac{\phi_s}{\phi_s + \varepsilon X_{s-}} X_{s-}dV_s(\theta) && (5.9)\\ &= \lim_{\varepsilon\downarrow 0} E\left(\frac{\phi}{\phi + \varepsilon X_-} \cdot X_{\sigma_n\wedge T_n}\right) < +\infty. && (5.10)\end{aligned}$$

The first equality follows from the monotone convergence theorem, while the finiteness of the last quantity is due to (5.7).
Hence, $V(\theta)$ is locally integrable and, thus, $(\theta \cdot S)$ is a Q-locally integrable. This proves that $(\theta \cdot S)$ is really a Q-local martingale. Furthermore, $\exp\left(\frac{1}{2}\theta \cdot S\right)$ is a positive Q-local submartingale. Then, the condition (5.4) and de la Vallée Poussin's argument imply that $\exp\left(\frac{1}{2}\theta \cdot S\right)$ is a positive Q-submartingale which is square integrable. Hence the lemma follows from Jensen's inequality. $\square$

Lemma 5.2: *Suppose that (5.3) holds. Then the function,* $\overline{K} : I\!R^d \to I\!R \cup \{-\infty, +\infty\}$, *given by*

$$\overline{K}(\lambda) := b^T\lambda + \frac{1}{2}\lambda^T c\lambda + \int \left(e^{\lambda^T x} - 1 - \lambda^T h(x)\right) F(dx), \quad \lambda \in I\!R^d,$$

is convex, proper, closed, and continuously differentiable with

$$\nabla\overline{K}(\lambda) = b + c\lambda + \int \left(xe^{\lambda^T x} - h(x)\right) F(dx), \quad \lambda \in I\!R^d. \tag{5.11}$$

Proof. Due to assumption (5.3), we deduce that

$$\int_{\{|x|>1\}} \left(e^{\lambda^T x} + 1\right) F(dx) < +\infty, \quad \text{for any } \lambda \in \mathbb{R}^d, \quad P \otimes A - a.e. \tag{5.12}$$

On the other hand, it is easy to check that

$$0 \le e^\alpha - 1 - \alpha \le \frac{\alpha^2}{2} e^{|\alpha|}, \quad \forall \ \alpha \in \mathbb{R},$$

from which we get

$$\int_{\{|x|\le 1\}} \left(e^{\lambda^T x} - 1 - \lambda^T x\right) F(dx) \le \frac{1}{2} e^{|\lambda|} |\lambda|^2 \int_{\{|x|\le 1\}} |x|^2 F(dx). \tag{5.13}$$

Meanwhile, since S is a semimartingale, we deduce that $I_{\{|\Delta S|\le 1\}} \cdot [S,S] \in \mathcal{A}^+_{loc}$, which implies

$$\int_{\{|x|\le 1\}} |x|^2 F(dx) < +\infty, \quad P \otimes A - a.e. \tag{5.14}$$

A combination of (5.12), (5.13) and (5.14), we obtain

$$\int \left(e^{\lambda^T x} - 1 - \lambda^T h(x)\right) F(dx) < +\infty, \quad P \otimes A - a.e. \tag{5.15}$$

Therefore, $\overline{K}(\lambda)$ is a well defined real-valued function. Hence, $\overline{K}$ is proper and closed with effective domain

$$dom(\overline{K}) = \mathbb{R}^d.$$

It is obvious that $\overline{K}$ is convex due to the convexity of $\lambda^T x \lambda$ and $\int e^{\lambda^T x} F(dx)$. To prove the differentiability of $\overline{K}$, we notice that the function

$$\lambda \to b^T \lambda + \frac{1}{2}\lambda^T c \lambda + \int_{\{|x|>1\}} e^{\lambda^T x} F(dx)$$

is continuous differentiable. Hence, it is enough to prove that the function

$$K_0(\lambda) := \int_{\{|x|\leq 1\}} \left(e^{\lambda^T x} - 1 - \lambda^T x\right) F(dx)$$

is continuously differentiable on $\mathbb{R}^d$. To this end, let $\lambda, \gamma \in \mathbb{R}^d$ and notice that

$$\begin{aligned} &|\gamma|^{-1} \left| K_0(\lambda+\gamma) - K_0(\lambda) - \int_{\{|x|\leq 1\}} \gamma^T x \left(e^{\lambda^T x} - 1\right) F(dx) \right| \\ &= |\gamma|^{-1} \left| \int_{\{|x|\leq 1\}} e^{\lambda^T x} \left(e^{\gamma^T x} - 1 - \gamma^T x\right) F(dx) \right| \\ &\leq e^{|\lambda|+|\gamma|} \frac{|\gamma|}{2} \int_{\{|x|\leq 1\}} |x|^2 F(dx) \to 0, \quad \text{as } |\gamma| \downarrow 0. \end{aligned} \tag{5.15}$$

Therefore, the function K_0 is continuous differentiable at λ with derivative, $\triangledown K_0(\lambda)$, given by

$$\nabla K_0(\lambda) = \int_{\{|x|\leq 1\}} x \left(e^{\lambda^T x} - 1\right) F(dx).$$

And hence $\overline{K}$ is also continuous differentiable at $\lambda \in \mathbb{R}^d$ with derivative, $\nabla\overline{K}(\lambda)$, given by (5.11). □

Lemma 5.3: *The following assertions (i) and (ii) are equivalent:*

(i) For any $\lambda \in \mathbb{R}^d$,

$$\int_{\{|x|>1\}} e^{\lambda^T x} F(dx) < +\infty. \tag{5.16}$$

(ii) For any $\lambda \in \mathbb{R}^d$,

$$\int_{\{|x|>1\}} |x| e^{\lambda^T x} F(dx) < +\infty. \tag{5.17}$$

As a result, if assertion (i) holds, then for any $\lambda \in \mathbb{R}^d$, any $q \in (0, +\infty)$,

$$\int_{\{|x|>1\}} |x|^q e^{\lambda^T x} F(dx) < +\infty. \tag{5.18}$$

Proof. The implication $(ii) \Longrightarrow (i)$ is obvious. We focus on proving the reverse. Let e_i be the element of $\mathbb{R}^d$ that has the i^{th} component equal to one and the other components null. Then, due to the equivalence between norms in $\mathbb{R}^d$, it is enough to consider the norm $|x|= \sum_{i=1}^{d} |x_i|$. Then, we get that

$$\begin{aligned}\int_{\{|x|>1\}} |x| e^{\lambda^T x} F(dx) &= \sum_{i=1}^{d} \int_{\{|x|>1\}} \left((e_i^T x)^+ + (-e_i^T x)^+\right) e^{\lambda^T x} F(dx) \\ &\leq \sum_{i=1}^{d} \int_{\{|x|>1\}} e^{(e_i+\lambda)^T x} F(dx) + \sum_{i=1}^{d} \int_{\{|x|>1\}} e^{(-e_i+\lambda)^T x} F(dx).\end{aligned} \tag{5.19}$$

Thus, the last term in the right hand side of the above string is finite for any $\lambda \in \mathbb{R}^d$, due to assertion (i). The proof of the remaining part of the lemma follows from the same arguments. □

Next, we will state our main results of this section. To this end, we first assume that the process $N = 1$ and B is predictable with finite variation.

Remark that this assumption may sound restrictive, but it leads to some kind of "uniqueness" of the forward utility.

Theorem 5.1: *Suppose that (5.3) holds and* $B = (B_t)_{0\leq t\leq T}$ *is a RCLL predictable process with finite variation. Then, the following assertions are equivalent:*

(i) The random field utility , $U(t,\omega,x) := -\exp(-x + B_t(\omega))$*, is a forward utility.*

(ii) The minimal entropy-Hellinger σ*-martingale measure,* $\widetilde{Q}$*, exists and*

$$B = B_0 + h^E(\widetilde{Q}, P). \tag{5.20}$$

Proof. The proof of this theorem will be achieved in two steps. The first step (part **1)**) will prove $(ii) \Longrightarrow (i)$, while the second step (part **2)**) will prove the reverse implication.

1) In this part, we will prove $(ii) \Longrightarrow (i)$. Suppose that assertion (ii) holds. Then, thanks to Theorem 2.3 (or see Theorem 4.6 in [16]), the MEH σ-martingale measure $\widetilde{Q}$ with the density process, $\widetilde{Z}$, satisfies

$$\log(\widetilde{Z}) = \widetilde{\theta}\cdot S + h^E(\widetilde{Z}, P).$$

Thus, it is easy to check that $-\widetilde{\theta}$ is admissible and $U(\cdot, -\widetilde{\theta}\cdot S) = -e^{B_0}\widetilde{Z}$ is a true martingale. Thanks to Lemma 5.1, it is also clear that for any admissible portfolio $\theta \in \mathcal{A}_{adm}(x)$, the process

$$U(\cdot, \theta\cdot S) = -e^{B_0}\widetilde{Z}\exp\left[-(\theta + \widetilde{\theta})\cdot S\right]$$

is a supermartingale. Hence assertion (i) follows immediately.

2) In this part, we prove $(i) \Longrightarrow (ii)$ in four steps (**(a)**–**(d)**). Precisely, the first step (part **(a)**) will show that the optimal portfolio $\widehat{\theta}$ satisfies the pointwise equation that characterizes the MEH σ-martingale density when it exists. The second step (part **(b)**) is devoted to the construction of a positive local martingale, $\widetilde{Z}$, candidate to the MEH σ-martingale density. The third

step (part **(c)**) will prove (5.20). Finally, the last step (part **(d)**) will prove the optimality of $\widetilde{Z}$, and, in turn, conclude the whole part **2)**.
To this end, we suppose that assertion (i) holds. Then, there exists $\widehat{\theta} \in \mathcal{A}_{adm}(x)$ such that $-\exp\Big(-(\widehat{\theta}\cdot S)_t + B_t\Big)$ is a true martingale and for any $\theta \in \mathcal{A}_{adm}(x)$, the process $-\exp\Big(-(\theta\cdot S)_t + B_t\Big)$ is a supermartingale.
a) Thanks to Ito's formula, for any $\theta \in L(S)$,

$$-\exp\Big(-(\theta\cdot S)_t + B_t\Big) = -e^{-(\theta\cdot S)_t}e^{B_t} = -e^{B_0}\mathcal{E}_t(X^\theta)\mathcal{E}_t(X^B),$$

$$X^\theta := -\theta\cdot S + \tfrac{1}{2}\theta^T c\theta\cdot A + (e^{-\theta^T x} - 1 + \theta^T x)\star\mu, \tag{5.21}$$

$$X^B := B - B_0 + \sum(e^{\Delta B} - 1 - \Delta B).$$

Therefore, for any admissible portfolio θ, the process $-\exp\Big(-(\theta\cdot S)_t + B_t\Big)$ is a local supermartingale (respectively, a local martingale) if and only if the process $e^{\Delta B}\cdot X^\theta + X^B$ is a local submartingale (respectively, is a local martingale). This fact is equivalent to the statements (a.1) and (a.2) given by:
(a.1) the process $|e^{-\theta^T x} - 1 + \theta^T h(x)|\star\mu$ is locally integrable,
(a.2) and the process $e^{-\Delta B}\cdot X^B - K(\theta)\cdot A$ is nondecreasing (respectively is null), where

$$K(\theta) := \theta^T b - \frac{1}{2}\theta^T c\theta + \int\Big(-e^{-\theta^T x} + 1 - \theta^T h(x)\Big)F(dx), \quad \theta\in\mathbb{R}^d. \tag{5.22}$$

As a result, the optimal admissible portfolio for the forward utility, $\widehat{\theta}$, maximizes the functional K over the set of admissible portfolios, and

$$K(\widehat{\theta})\cdot A = e^{-\Delta B}\cdot X^B = e^{-\Delta B}\cdot B + \sum(1 - e^{-\Delta B} - \Delta B e^{-\Delta B}). \tag{5.23}$$

Then, using the optimality of $\widehat{\theta}$ together with Lemma 5.2 (note that $K(\theta) =$

$-\overline{K}(-\theta))$ we conclude that $-\widehat{\theta}$ is a root of the equation

$$b + c\theta + \int [e^{\theta^T x} x - h(x)] F(dx) = 0. \tag{5.24}$$

Furthermore, by combining (5.23) and (5.24), and putting $\widetilde{\theta} := -\widehat{\theta}$, we get

$$\left[\frac{1}{2}\widetilde{\theta}^T c\widetilde{\theta} + \int (\widetilde{\theta}^T x e^{\widetilde{\theta}^T x} - e^{\widetilde{\theta}^T x} + 1) F(dx)\right] \cdot A = e^{-\Delta B} \cdot B + \sum (1 - e^{-\Delta B} - \Delta B e^{-\Delta B}). \tag{5.25}$$

b) Since $\widetilde{\theta} := -\widehat{\theta} \in L(S^c)$, the process $\widetilde{\theta} \cdot S^c$ is a well-defined continuous local martingale that will constitute the continuous part of $\widetilde{N} := \frac{1}{\widetilde{Z}_-} \cdot \widetilde{Z}$. Hence, to define the pure discontinuous ingredient of $\widetilde{N}$, we consider the $\widetilde{\mathcal{P}}$-measurable functional

$$W_t(x) := (\widetilde{\gamma}_t)^{-1} \left(e^{\widetilde{\theta}_t^T x} - 1\right), \quad \widetilde{\gamma}_t := 1 - a_t + \int e^{\widetilde{\theta}_t^T x} \nu(\{t\}, dx) \tag{5.26}$$

and we will prove that W is $(\mu - \nu)$-integrable. This will be carried out in several steps, see (b.1)–(b.5).

(b.1) Since $\widehat{\theta} \cdot S$ is a RCLL semimartingale, then the process $I_{\{|\widetilde{\theta}^T \Delta S| \leq \alpha\}} \cdot [\widetilde{\theta} \cdot S, \widetilde{\theta} \cdot S]$ is locally bounded and, hence, locally integrable. Then, due to

$$\begin{aligned} \sum (e^{\widetilde{\theta}^T \Delta S} - 1)^2 I_{\{|\widetilde{\theta}^T \Delta S| \leq \alpha\}} &\preceq e^{2\alpha} \sum (\widetilde{\theta}^T \Delta S)^2 I_{\{|\widetilde{\theta}^T \Delta S| \leq \alpha\}} \\ &\preceq e^{2\alpha} I_{\{|\widetilde{\theta}^T \Delta S| \leq \alpha\}} \cdot [\widetilde{\theta} \cdot S, \widetilde{\theta} \cdot S], \end{aligned} \tag{5.27}$$

we deduce that $\sum (e^{\widetilde{\theta}^T \Delta S} - 1)^2 I_{\{|\widetilde{\theta}^T \Delta S| \leq \alpha\}}$ is locally integrable.

(b.2) Thanks to the equation (5.25) and the local integrability of both processes $e^{-\Delta B} \cdot B + \sum (1 - e^{-\Delta B} - \Delta B e^{-\Delta B})$ and $\widetilde{\theta}^T c\widetilde{\theta} \cdot A$, we deduce that the non-decreasing process

$$\left(\widetilde{\theta}^T x e^{\widetilde{\theta}^T x} - e^{\widetilde{\theta}^T x} + 1\right) \star \mu$$

is locally integrable. A combination of this and

$$\sum|e^{\widetilde{\theta}^T\Delta S}-1|I_{\{|\widetilde{\theta}^T\Delta S|>\alpha\}} \preceq \sum\left[\frac{e^\alpha-1}{\alpha}+\frac{\widetilde{\theta}^T\Delta S e^{\widetilde{\theta}^T\Delta S}-e^{\widetilde{\theta}^T\Delta S}+1}{\alpha}\right]I_{\{|\widetilde{\theta}^T\Delta S|>\alpha\}}, \tag{5.28}$$

implies the local integrability of $\sum|e^{\widetilde{\theta}^T\Delta S}-1|I_{\{|\widetilde{\theta}^T\Delta S|>\alpha\}}$.

(b.3) Using the inequality

$$\left(\sum(e^{\widetilde{\theta}^T\Delta S}-1)^2 I_{\{|\widetilde{\theta}^T\Delta S|>\alpha\}}\right)^{1/2} \preceq \sum|e^{\widetilde{\theta}^T\Delta S}-1|I_{|\{\widetilde{\theta}^T\Delta S|>\alpha\}}$$

and parts (b.1)–(b.2), we obtain the local integrability of $\left(\sum(e^{\widetilde{\theta}^T\Delta S}-1)^2\right)^{1/2}$.

(b.4) Due to (5.24) and the properties of the predictable characteristics of S given in Section 2.A (precisely, $c=0$ on $\{\Delta A\neq 0\}$ and $\Delta B_t=\int h(x)\nu(\{t\},dx)$), we obtain

$$\int x e^{\widetilde{\theta}_t^T x}\nu_t(dx)=0. \tag{5.29}$$

By combining (5.29) and (5.25), we derive

$$\widetilde{\gamma}=1+\int(e^{\widetilde{\theta}_t^T x}-1)\nu_t(dx)=e^{-\Delta B}. \tag{5.30}$$

Hence, $\widetilde{\gamma}^{-1}$ is locally bounded.

(b.5) Defining $\Gamma:=\{x\in\mathbb{R}^d \mid |\widetilde{\theta}^T x|\leq\alpha\}$ and using the notations of (5.26), we derive

$$\begin{aligned}
\frac{1}{2}\sum(\widehat{W}_t)^2 &\preceq \sum\left(\frac{1}{\widetilde{\gamma}_t}\int_\Gamma(e^{\widetilde{\theta}_t^T x}-1)\nu_t(dx)\right)^2+\sum\left((\widetilde{\gamma}_t)^{-1}\int_{\mathbb{R}^d\setminus\Gamma}(e^{\widetilde{\theta}_t^T x}-1)\nu_t(dx)\right)^2\\
&\preceq \sum(\widetilde{\gamma}_t)^{-2}\int_\Gamma(e^{\widetilde{\theta}_t^T x}-1)^2\nu_t(dx)+\left(\sum(\widetilde{\gamma}_t)^{-1}\int_{\mathbb{R}^d\setminus\Gamma}|e^{\widetilde{\theta}_t^T x}-1|\nu_t(dx)\right)^2\\
&\preceq (\widetilde{\gamma})^{-2}\left(e^{\widetilde{\theta}^T x}-1\right)^2 I_{\{|\widetilde{\theta}^T x|\leq\alpha\}}\star\nu+\left((\widetilde{\gamma})^{-1}|e^{\widetilde{\theta}^T x}-1|I_{\{|\widetilde{\theta}^T x|>\alpha\}}\star\nu\right)^2.
\end{aligned}$$

Due to (b.1)–(b.2), the predictable nondecreasing processes $(e^{\widetilde{\theta}^T x}-1)^2 I_{\{|\widetilde{\theta}^T x|\leq\alpha\}}\star\nu$ and $|e^{\widetilde{\theta}^T x}-1|I_{\{|\widetilde{\theta}^T x|>\alpha\}}\star\nu$ have finite variation and, thus, are locally bounded.

This follows from the fact that these processes are the compensators of the two processes discussed in (b.1) and (b.2) respectively. Due to the locally boundedness of $\widetilde{\gamma}^{-1}$ proved in part (b.4), the process $\sum(\widehat{W}_t)^2$ is locally bounded.

(b.6) Using once more the local boundedness of $\widetilde{\gamma}^{-1}$, parts (b.1)–(b.4), and

$$\begin{aligned}
&\Big(\sum(W_t(\Delta S)I_{\{\Delta S_t\neq 0\}} - \widehat{W}_t)^2\Big)^{1/2}\\
\preceq\ &\Big(2\sum(W_t(\Delta S))^2 I_{\{\Delta S_t\neq 0\}}\Big)^{1/2} + \Big(2\sum(\widehat{W}_t)^2\Big)^{1/2}\\
=\ &\Big[2(\widetilde{\gamma})^{-2}\Big(e^{\widetilde{\theta}^T x} - 1\Big)^2 \star \mu\Big]^{1/2} + \Big(2\sum(\widehat{W}_t)^2\Big)^{1/2}.
\end{aligned}$$

we deduce the locally integrability of $\Big(\sum(W_t(\Delta S)I_{\{\Delta S_t\neq 0\}} - \widehat{W}_t)^2\Big)^{1/2}$. This ends the proof of the $(\mu-\nu)$-integrability of W (i.e. $W\in\mathcal{G}^1_{loc}(\mu)$, see (2.3) for details), and hence $W\star(\mu-\nu)$ is a local martingale and the process $\widetilde{Z} := \mathcal{E}(\widetilde{N})$ such that

$$\widetilde{N} := \widetilde{\theta}\cdot S^c + W\star(\mu-\nu), \qquad \widetilde{\theta} = -\widehat{\theta}, \qquad W_t(x) := \frac{e^{\widetilde{\theta}_t^T x} - 1}{1 - a_t + \int e^{\widetilde{\theta}_t^T y}\nu(\{t\},dy)}, \tag{5.31}$$

is well defined and is a σ-martingale density for S due to the equation (5.24) satisfied by $\widetilde{\theta}$ and Proposition 2.2.

c) Considering (5.25) and (2.27), on $\{\Delta A = 0\}$, we derive

$$\begin{aligned}
I_{\{\Delta A=0\}}\cdot B &= \big(e^{-\Delta B} I_{\{\Delta A=0\}}\big)\cdot X^B\\
&= I_{\{\Delta A=0\}}\left[\frac{\widetilde{\theta}^T c\widetilde{\theta}}{2} + \int\Big(\widetilde{\theta}^T x e^{\widetilde{\theta}^T x} - e^{\widetilde{\theta}^T x} + 1\Big)F(dx)\right]\cdot A \qquad (5.32)\\
&= I_{\{\Delta A=0\}}\cdot h^E(\widetilde{Z},P).
\end{aligned}$$

Due to (5.30), we have $\Delta B = -\log(\widetilde{\gamma})$. Hence, by combining this with (2.29) (here $\lambda = \widetilde{\theta}$ and, hence, $\gamma = \widetilde{\gamma}$), we obtain

$$\Delta B = \Delta h^E(\widetilde{Z},P). \tag{5.33}$$

Therefore, (5.20) follows immediately from (5.32) and (5.33).

d) Thanks to Proposition 3.2 in [17] (see also Proposition 4.2 in [16] for the case of quasi-left continuity), it is enough to consider a positive σ-martingale density $Z = \mathcal{E}(N)$ of the form

$$N = \beta\cdot S^c + Y\star(\mu-\nu), \quad Y_t(x) = k_t(x) + \frac{\widehat{k}_t}{1-a_t} I_{\{a_t<1\}}, \quad \widehat{k}_t := \int k_t(x)\nu(\{t\}, dx),$$

where $\beta \in L(S)$ and $\left(\sum k_t(\Delta S_t)^2 I_{\{\Delta S_t \neq 0\}}\right)^{1/2} \in \mathcal{A}^+_{loc}$. Then, due to the convexity of $z^T cz$ and $\phi(z) := (1+z)\log(1+z) - z$, we obtain on $\{\Delta A = 0\}$ on one hand

$$\begin{aligned} & \frac{dh^E(Z,P)}{dA} - \frac{dh^E(\widetilde{Z},P)}{dA} \\ = & \int \left[\phi(k(x)) - \phi\Big(e^{\widetilde{\theta}^T x} - 1\Big)\right] F(dx) + \frac{1}{2}(\beta^T c\beta - \widetilde{\theta}^T c\widetilde{\theta}) \qquad (5.34) \\ \geq & \ \widetilde{\theta}^T c(\beta - \widetilde{\theta}) + \int \widetilde{\theta}^T x\Big(k(x) + 1 - e^{\widetilde{\theta}^T x}\Big) F(dx) = 0. \qquad (5.35) \end{aligned}$$

Indeed, the equation (5.34) comes from Lemma 2.4 and Proposition 3.5 in [18], while the equality (5.35) is derived from a combination of (5.24) for $\widetilde{Z}$ and a similar equation for Z, i.e.

$$0 = b + c\beta + \int [x(k(x)+1) - h(x)]F(dx)$$

since Z is a σ-martingale density for S (see Proposition 2.2). On the other hand, due to the convexity of ϕ, we get

$$\begin{aligned} & \Delta h^E(Z,P) - \Delta h^E(\widetilde{Z},P) \\ = & \int \left[\phi(k(x)) - \phi\Big(\frac{e^{\widetilde{\theta}^T x}}{\widetilde{\gamma}} - 1\Big)\right]\widetilde{\nu}(dx) + (1-a)\left[\phi\Big(\frac{-\widehat{k}}{1-a}\Big) - \phi\Big(\frac{1}{\widetilde{\gamma}} - 1\Big)\right] \qquad (5.36) \\ \geq & \int [k(x) + 1 - \frac{e^{\widetilde{\theta}^T x}}{\widetilde{\gamma}}](\widetilde{\theta}^T x + \log(\frac{1}{\widetilde{\gamma}}))\widetilde{\nu}(dx) + (1-a)(1 - \frac{\widehat{k}}{1-a} - \frac{1}{\widetilde{\gamma}})\log(\frac{1}{\widetilde{\gamma}}). \\ = & \int \left[(k(x)+1) - (\widetilde{\gamma})^{-1}e^{\widetilde{\theta}^T x}\right]\widetilde{\theta}^T x\widetilde{\nu}(dx) = 0, \qquad (5.37) \end{aligned}$$

where $\widetilde{\nu}_t(dx) := \nu(\{t\}, dx)$. Indeed, the equation (5.36) is derived from Lemma 2.4 and Proposition 3.5 in [18]. The equation (5.37) follows from the fact that Z and $\widetilde{Z}$ are σ-martingale densities for S and an application of Proposition 2.2. Thus, by combining (5.35) and (5.37), we deduce that $\widetilde{Z}$ is the MEH σ-martingale density for S. Furthermore, due to Theorem 2.3, (5.20) and $\widetilde{\theta} = -\widehat{\theta}$ (see part a), (b.1)–(b.6) and c)), we get

$$\widetilde{Z} = e^{-B_0} \exp\left[B - (\widehat{\theta} \cdot S)\right] = -e^{-B_0} U_1(\cdot, \widehat{\theta} \cdot S).$$

Hence, it is a true martingale and this implies the existence of the MEH σ-martingale measure, $\widetilde{Q}$. This proves assertion (ii) and the proof of the theorem is complete. □

Remark: 1. It is clear that the proof of the part $(ii) \Longrightarrow (i)$ of Theorem 5.1 follows easily from [16] and [17]. In fact, it was clearly stated in those papers that this kind of robustness with respect to the horizon is one of the important features of the minimal entropy-Hellinger σ-martingale measure that other σ-martingale measures lack to possess; see, also, [18] for a more explicit relationship between this horizon-robustness for σ-martingale measures and utility maximization for all HARA utilities.

2. The highly original part of Theorem 5.1 lies in proving that the only forward utility of this kind (i.e. when B is predictable with finite variation) is the one given through the MEH σ-martingale measure and this σ-martingale measure exists in fact. Furthermore, this original part of the theorem also gives necessary and sufficient conditions for the existence of MEH σ-martingale measure via the utility maximization problem with weaker conditions on S.

Theorem 5.1 sounds restrictive due to the assumption on B, while—as we will illustrate in the proof of the next theorem—it is crucial and constitutes an important step for proving our general result. This result requires some preparations.

Definition: A RCLL semimartingale B is called exponentially special, if $\exp(B)$ is a special semimartingale, i.e.

$$\exp(B) = \exp(B_0) + M^{(B)} + A^{(B)}, \tag{5.38}$$

where $M^{(B)}$ is a local martingale, $A^{(B)}$ is predictable with finite variation such that $M_0^{(B)} = A_0^{(B)} = 0$.

Lemma 5.4: *Let B be a RCLL semimartingale. Then, the following hold.*

(i) If B is exponentially special, then there exists a unique positive local martingale, $Z^{(B)}$, and predictable process, B', with finite variation such that

$$e^B = e^{B_0 + B'} Z^{(B)}, \qquad B'_0 = 0, \quad Z_0^{(B)} = 1. \tag{5.39}$$

(ii) Suppose that pB is exponentially special, for some $p \in (1, +\infty)$, $Z^{(B)}$ is a true martingale, and that (5.3) holds. Then,

$$\int_{\{|x|>1\}} |x| e^{\lambda^T x} F^Q(dx) < +\infty, \quad \text{for all } \lambda \in I\!R^d, \tag{5.40}$$

where F^Q is the kernel measure for the jumps sizes of S under $Q := Z_T^{(B)} \cdot P$.

Proof. Since e^B is a special semimartingale, then $e^{-B_-} \cdot e^B$ is also a special semimartingale. Then, there exist unique local martingale, $N^{(B)}$, and a predictable process, $C^{(B)}$, with finite variation such that

$$e^{-B_-} \cdot e^B = N^{(B)} + C^{(B)} \qquad \text{and} \qquad C_0^{(B)} = N_0^{(B)} = 0.$$

Then, the above equation implies

$$e^B = e^{B_0} \mathcal{E}\left(N^{(B)} + C^{(B)}\right), \quad 1 + \Delta C^{(B)} > 0 \;\text{ and }\; 1 + \frac{\Delta N^{(B)}}{1 + \Delta C^{(B)}} > 0.$$

As a result, the process $\frac{1}{1+\Delta C^{(B)}} \cdot N^{(B)}$ is a local martingale, $\mathcal{E}\left(\frac{1}{1+\Delta C^{(B)}} \cdot N^{(B)}\right) > 0$, and $\mathcal{E}\left(C^{(B)}\right)$ is a positive predictable process with finite variation. Then, due to Yor's formula (i.e. $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y+[X,Y])$ for any semimartingales X, Y), we write

$$e^B = e^{B_0}\mathcal{E}\left(\frac{1}{1+\Delta C^{(B)}}\cdot N^{(B)}\right)\mathcal{E}\left(C^{(B)}\right).$$

Thus, assertion (i) follows directly from putting $Z^{(B)} := \mathcal{E}\left(\frac{1}{1+\Delta C^{(B)}}\cdot N^{(B)}\right)$ and $B' := \log\left[\mathcal{E}(C^{(B)})\right]$.

Next, we will prove the assertion (ii). To this end, we suppose that pB is exponentially special. Thus, B is exponentially special and, hence, assertion (i) holds. On the other hand, it is clear that $(Z^{(B)})^p$ is locally integrable (i.e. a special semimartingale), and $F^Q(dx) = (1+f(x))F(dx)$, if $(\beta, f, g, \overline{M})$ is Jacod components for $M^{(B)} := \frac{1}{Z^{(B)}_-}\cdot Z^{(B)}$. Then, due to Lemma 5.3, we deduce

$$\begin{aligned}\int_{\{|x|>1\}}|x|e^{\lambda^T x}F^Q(dx) &= \int_{\{|x|>1\}}|x|e^{\lambda^T x}(1+f(x))F(dx)\\ &\leq \left(\int I_{\{|x|>1\}}|x|^q e^{q\lambda^T x}F(dx)\right)^{\frac{1}{q}}\left(\int I_{\{|x|>1\}}(1+f(x))^p F(dx)\right)^{\frac{1}{p}} < +\infty.\end{aligned}$$

This proves assertion (ii), and the proof of the lemma is complete. □

Now, we will state our main and general result of this section.

Theorem 5.2: *Suppose that S satisfies (5.3) and consider a RCLL semimartingale, B, such that pB is exponentially special for some $p \in (1,+\infty)$. Then, the following results hold.*

1) The following assertions, (i) and (ii), are equivalent:

(i) The random field utility, $U_1(t,\omega,x) = -\exp\left(-x + B_t(\omega)\right)$, is a forward utility with optimal portfolio $\widehat{\theta}$.

(ii) There exists a unique positive local martingale $Z^{(B)}$ satisfying:

(ii.a) The MEH σ-martingale density with respect to $Z^{(B)}$ exists. It is denoted by $\widetilde{Z}^{(B)}$ and satisfies

$$B - B_0 = \log\left[Z^{(B)}\right] + h^E\left(\widetilde{Z}^{(B)}, Z^{(B)}\right). \tag{5.41}$$

(ii.b) The process $\widehat{Z}^{(B)} := \widetilde{Z}^{(B)} Z^{(B)}$ *is a true martingale,* $\widehat{Q}^{(B)} := \widehat{Z}_T^{(B)} \cdot P$ *is a* σ*-martingale measure, and* $\widehat{Z}^{(B)} \log\left[\widehat{Z}^{(B)}\right]$ *is locally integrable (i.e a special semimartingale).*

(ii.c) We have

$$\log(\widetilde{Z}^{(B)}) = \widetilde{\theta}^{(B)} \cdot S + h^E\left(\widetilde{Z}^{(B)}, Z^{(B)}\right) \qquad \textit{and} \qquad \widetilde{\theta}^{(B)} = -\widehat{\theta}. \tag{5.42}$$

2) If assertion (i) holds and furthermore, B satisfies

$$\sup_{\tau \in \mathcal{T}_T} E\left[e^{pB_\tau}\right] < +\infty \quad \textit{for some} \;\; p \in (1, +\infty), \tag{5.43}$$

then $Z^{(B)}$ *is a true martingale. Moreover, the probability* $\widehat{Q}^{(B)}$ *has finite* P*-entropy, i.e.* $\widehat{Q}^{(B)} \in \mathcal{M}_f^e(S)$*.*

Proof. The proof of this theorem will be given in three parts. The first part (part I)) will prove $(i) \Longrightarrow (ii)$, the second part (part II)) will prove the reverse, while the last part (part III)) will prove assertion 2). First, notice that under the assumptions of this theorem, the assertions of Lemma 5.4 hold.

I) Suppose that assertion (i) holds and consider a sequence of stopping times, $(T_n)_{n\geq 1}$, that increases stationarily to T such that $(Z^{(B)})^{T_n}$ is a true martingale and $B'_{t\wedge T_n}$ is bounded. Then, by putting $Q_n := Z_{T_n}^{(B)} \cdot P$, and using Lemma 2.3, we deduce that the process $U_n(t, \omega, x) := -\exp(-x + B'_{t\wedge T_n})$ is a forward dynamic utility for (S^{T_n}, Q_n). Therefore, assertion (ii) of Lemma 5.4 (precisely condition (5.40)) guarantee a direct application of Theorem 5.1 on the model (S^{T_n}, Q_n, U_n). This implies the existence of the MEH σ-martingale measure with respect to Q_n, denoted by $\widetilde{Q}^n$, whose density $\widetilde{Z}^{(B,n)}$ satisfies

$$B'_{t\wedge T_n} = h_t^E\left(\widetilde{Z}^{(B,n)}, Q_n\right).$$

Using Lemma 2.5, we conclude that the MEH σ-martingale density with respect to $Z^{(B)}$, denoted by $\widetilde{Z}^{(B)}$, exists and satisfies

$$B'_t = h^E_t\left(\widetilde{Z}^{(B)}, Z^{(B)}\right).$$

Then, plugging this equation into (5.39), the assertion (ii)-(a) follows immediately.
From Theorem 2.5, we have

$$\log(\widetilde{Z}^{(B)}) = \widetilde{\theta}^{(B)} \cdot S + h^E\left(\widetilde{Z}^{(B)}, Z^{(B)}\right),$$

where the process $\widetilde{\theta}^{(B)}$ is explicitly described and coincides with $-\widehat{\theta}$; this follows as a consequence of Theorem 5.1 on the model (S^{T_n}, Q_n, U_n). This proves (ii)-(c).
To prove assertion (ii)-(b), it is easy to note that—due to the definition of $\widetilde{Z}^{(B)}$—$\widehat{Z}^{(B)}$ is a σ-martingale density for S, and $Z^{(B)}\widetilde{Z}^{(B)}\log(\widetilde{Z}^{(B)})$ is locally integrable.
Now, we will prove that $\widehat{Z}^{(B)}\log(\widehat{Z}^{(B)})$ is locally integrable. Consider a sequence of stopping times, $(T_n)_{n\geq 1}$, that increases stationarily to T such that $E\left[(Z^{(B)}_{T_n})^p\right] < +\infty$ (this is possible since pB is exponentially special) and $h^E_{t\wedge T_n}\left(\widetilde{Z}^{(B)}, Z^{(B)}\right)$ is bounded. Then, by putting $\varepsilon = p-1$ and $Q_n := Z^{(B)}_{T_n}\cdot P$, and using Young's inequality (i.e. $xy \leq y\log(y) - y + e^x$), we derive

$$\begin{aligned}
E\left[Z^{(B)}_{T_n}\widetilde{Z}^{(B)}_{T_n}\log(Z^{(B)}_{T_n})\right] &= E^{Q_n}\left(\frac{1}{\varepsilon}\widetilde{Z}^{(B)}_{T_n}\log[(Z^{(B)}_{T_n})^{\varepsilon}]\right)\\
&\leq E^{Q_n}\left[\frac{\widetilde{Z}^{(B)}_{T_n}}{\varepsilon}\log(\frac{\widetilde{Z}^{(B)}_{T_n}}{\varepsilon}) - \frac{\widetilde{Z}^{B}_{T_n}}{\varepsilon}\right] + E^{Q_n}\left[(Z^{(B)}_{T_n})^{\varepsilon}\right]\\
&\leq E^{Q_n}\left[\frac{\widetilde{Z}^{(B)}_{T_n}}{\varepsilon}\log(\frac{\widetilde{Z}^{B}_{T_n}}{\varepsilon})\right] - \frac{1}{\varepsilon} + E\left[(Z^{(B)}_{T_n})^p\right].
\end{aligned}$$

This proves that $Z^{(B)}\widetilde{Z}^{(B)}\log(Z^{(B)})$ is locally integrable. Thus, by putting

$$\widehat{Z}^{(B)}\log(\widehat{Z}^{(B)}) = Z^{(B)}\widetilde{Z}^{(B)}\log(\widetilde{Z}^{(B)}) + Z^{(B)}\widetilde{Z}^{(B)}\log(Z^{(B)}), \tag{5.44}$$

we deduce that $\widehat{Z}^{(B)}\log(\widehat{Z}^{(B)})$ is locally integrable.

II) Suppose that assertion (ii) holds. Then, assertions (ii)-(b) and (ii)-(c) imply that $-\widetilde{\theta}^{(B)}$ is an admissible portfolio, the process

$$U\left(t, -(\widetilde{\theta}^{(B)}\cdot S)_t\right) = -\exp\left((\widetilde{\theta}^{(B)}\cdot S)_t + B_t\right) = -e^{B_0}\widehat{Z}^{(B)}_t,$$

is a true martingale, and $\widehat{Q} := \widehat{Z}^{(B)}_T \cdot P$ is a σ-martingale measure for S. Then, for any admissible portfolio θ, we have

$$\sup_{\tau\in\mathcal{T}_T} E^{\widehat{Q}}\exp\Big[-(\theta+\widetilde{\theta}^{(B)})\cdot S_\tau\Big] = e^{-B_0}\sup_{\tau\in\mathcal{T}_T} E\exp\Big[B_\tau - (\theta\cdot S)_\tau\Big] < +\infty.$$

Thus, thanks to Lemma 5.1, we deduce that the process

$$-\exp\Big(-\theta\cdot S + B\Big) = -e^{B_0}\widehat{Z}^{(B)}\exp\left(-(\widetilde{\theta}+\theta)\cdot S\right),$$

is a supermartingale. This proves assertion (i).

III) Thanks to (5.41), (5.44), and assertion (ii)-c), we obtain

$$\begin{aligned}
&E\left(\widehat{Z}^{(B)}_{\tau\wedge T_n}\log\left[\widehat{Z}^{(B)}_{\tau\wedge T_n}\right]\right)\\
&= E^{Q_n}\left(\widetilde{Z}^{(B)}_{\tau\wedge T_n}\log\left[\widetilde{Z}^{(B)}_{\tau\wedge T_n}\right]\right) + E^{Q_n}\left[\widetilde{Z}^{(B)}_{\tau\wedge T_n}\left(B_{T_n\wedge\tau} - B_0 - \log\widetilde{Z}^{(B)}_{\tau\wedge T_n}\right)\right]\\
&= E\left[\widehat{Z}^{(B)}_{\tau\wedge T_n}B_{\tau\wedge T_n}\right] - B_0.
\end{aligned}$$

Hence, again Young's inequality yields

$$E\left(\widehat{Z}^{(B)}_{\tau\wedge T_n}\log\Big[\widehat{Z}^{(B)}_{\tau\wedge T_n}\Big]\right) \le \frac{p}{p-1}\sup_{\sigma\in\mathcal{T}_T} E\left(e^{pB_\sigma}\right) - \frac{p}{p-1}B_0.$$

Then, using Fatou's lemma, the above inequality leads to $\widehat{Q}^{(B)} \in \mathcal{M}^e_f(S)$ and, hence, assertion 2) of the theorem follows. This ends the proof of the theorem. □

Theorem 5.3: *Let B be a RCLL semimartingale and $N := \mathcal{E}(\pi\cdot S)$ a numéraire.*

Then, there is equivalence between:

(i) The random field utility $U(t,\omega,x) = -\exp\left(-\frac{x+B_t(\omega)}{N_t(\omega)}\right)$ *is a forward utility for the assets* S*, and*

(ii) The random field utility $\overline{U}(t,\omega,x) := -\exp\left(-x + \frac{B_t(\omega)}{N_t(\omega)}\right)$ *is a forward utility for the assets*

$$\overline{S} := S - \frac{1}{1+\pi^T \Delta S} \cdot [S, \pi \cdot S]. \tag{5.45}$$

Proof. Due to Yor's formula, we deduce that

$$\frac{1}{N} = \mathcal{E}\left(-\pi \cdot S + \frac{1}{1+\pi^T \Delta S} \cdot [\pi \cdot S, \pi \cdot S]\right).$$

On the other hand, Ito's formula yields

$$d\left(\frac{\theta \cdot S}{N}\right) = \phi(\theta) d\overline{S}, \tag{5.46}$$

where $\phi(\theta)$ is given by

$$\phi(\theta) := \frac{\theta - (\theta \cdot S)_{-}\pi}{N_{-}}, \quad \text{for any} \quad \theta \in L(S). \tag{5.47}$$

As a result, we get

$$U\left(t, x + (\theta \cdot S)_t\right) = \overline{U}\left(t, x + (\phi(\theta) \cdot \overline{S})_t\right), \quad \text{for any} \quad \theta \in L(S).$$

Therefore, for any process θ, $\theta \in \mathcal{A}_{adm}(x, S, U)$ if and only if $\phi(\theta) \in \mathcal{A}_{adm}(x, \overline{S}, \overline{U})$. The proof of the theorem follows easily. □

Remark: 1. Theorem 5.3 yields our complete and explicit parametrization for the exponential forward utilities. In fact, using a nice result of [75] that states that if $-\exp(-\frac{x+B}{N})$ is a forward utility, then N is a numéraire and B is a semimartingale. This gives us the first parametrization through the description of N. Then, by using Theorem 5.3, we transfer the self-generating property to the model $\overline{S}$ and the payoff $\overline{B} = \frac{B}{N}$ instead, and Theorem 5.2 completes the explicit parametrization of the utility by describing the structure of $\overline{B}$. Thus, the parameters of a forward utility are $(\pi, N^{(B)}) \in L(S) \times \mathcal{M}_{loc}(P)$ or equivalently $(\pi, \beta, f, g, \overline{N}^{(B)})$.

2. The semimartingale property for B becomes obvious from the definition of forward dynamic utility if the set of admissible portfolios $\mathcal{A}_{adm}(x)$ contains the null portfolio for some $x \in \mathbb{R}$. This situation is realizable when more integrability conditions are imposed on the payoff B such as boundedness for instance.

The following remark discusses the originality of this section, and compare its results (mainly Theorems 5.1–5.2) with the most recent literature on the exponential forward dynamic utilities.

Remark: This remark, as suggested by an anonymous referee, discusses the originality of the results of this section and compares them with those of Zitkovic obtained in [75] (especially Theorem 4.4 of that paper). To this end, we focus on the case of $N = 1$ for simplicity. The result of Zitkovic in [75] characterizes the exponential forward utility relying on the relative conditional entropy concept. Precisely, for any $Q \in \mathbb{P}_a$ with density process Z^Q, and any $0 \leq t \leq T < +\infty$,

$$H(Q, t, T) := E\left(\frac{Z^Q_T}{Z^Q_t} \log\Big(\frac{Z^Q_T}{Z^Q_t}\Big) \Big| \mathcal{F}_t\right),$$

denotes the relative conditional entropy of Q with respect to P. Using this concept, Zitkovic derived the following characterization

$$B_t = -\operatorname*{ess\,inf}_{Q \in \mathcal{M}^a_f(S)} H(Q, t, T), \tag{5.48}$$

for the case of $N = 1$, for any $0 \leq t \leq T < +\infty$. Here, $\mathcal{M}^a_f(S)$ denotes the set of $Q \in \mathbb{P}_a$ with finite entropy (i.e. $H(Q, 0, T) < +\infty$ for any T) such that S is a Q-local martingale.

It is very clear—up to our knowledge—that for any T, the essential inf in the RHS term of (5.48) is attainable under Zitkovic's assumptions (i.e. S locally bounded and $\mathcal{M}_f^a(S) \neq \emptyset$) by the minimal entropy martingale measure for the model S^T. It is, also, very clear that there is no single result in the literature that describes **explicitly** this optimal martingale measure for the general semimartingale S. Thus, in our view, (5.48) is a characterization that is not applicable (at least we do not see how to apply it) and it is not explcit for general case of locally bounded semimartingale S. Thus, This result does not parameterize the exponential forward utility, while our results (of this section) give a clear and explicit parametrization.

Furthermore,—as mentioned by an anonymous referee—our assumptions on the model S are much more general than those of Zitkovic. Indeed, in [75], the author assumed that S is locally bounded and $\mathcal{M}_f^a(S) \neq \emptyset$, while we designed our parametrization under the assumption (5.3) which is much more weaker.

In our view, the most practical result of [75]—beside the section that deals with the easiest case of Ito processes—is Proposition 4.7, where the author proved that the process N (denoted by γ in his chapter) should be a numéraire. In other words, there exists $\pi \in L(S)$ such that $N = N_0\mathcal{E}(\pi \cdot S)$. Herein, we use this nice result to complete our full parametrization.

5.B Discrete-Time Market Models

Recall the discrete-time market models described in Section 3.B. Now, we consider the family of exponential utilities, given by

$$U_1(j, x) = -\exp(-x + B_j), \quad j = 0, 1, ..., N, \quad , x \in \mathbb{R}. \tag{5.49}$$

Suppose that the process B satisfies

$$\sup_{0 \le j \le N} E(e^{B_j}) < +\infty. \tag{5.50}$$

Define the set of admissible portfolios for the j^{th} period of time ($j = 1, 2, ..., N$), denoted by $\Theta_j^{(1)}$, and is given by

$$\Theta_j^{(1)} := \Big\{\theta \in L^0(\mathcal{F}_{j-1}) \mid E\Big(\exp(-\theta^T \Delta S_j + B_j) \mid \mathcal{F}_{j-1}\Big) < +\infty\Big\}. \tag{5.51}$$

The next theorem will state our parametrization algorithm for the subclass of exp-type forward utilities having the form of (5.49).

Theorem 5.4: *Suppose that (5.50) holds. Then, the following are equivalent.*

(i) The functional $U_1(j, x)$, defined in (5.49), is a forward utility with the optimal portfolio denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The following two properties hold:

(ii.1)The process $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,...,N}$ satisfies $\widehat{\theta}_j \in \Theta_j^{(1)}$ and $\widehat{\theta}_j$ is a root for the equation

$$E\left(\exp(-\lambda^T \Delta S_j + B_j)\Delta S_j|\mathcal{F}_{j-1}\right) = 0, \quad j = 1, 2, ..., N. \tag{5.52}$$

(ii.2) There exists a positive martingale $M = (M_j)_{j=0,...,N}$ such that $M_0 = 1$ and

$$B_j = B_0 + \log(M_j) - \sum_{k=1}^{j} \log\left[E^Q\left(\exp(-\widehat{\theta}_k^T \Delta S_k)|\mathcal{F}_{k-1}\right)\right], \quad j = 0, 1, ..., N,^{1} \tag{5.53}$$

where $Q := M_N \cdot P$.

Proof. Suppose that assertion (i) holds, then [2]

$$U_1\left(j, \sum_{k=1}^{j} \widehat{\theta}_k^T \Delta S_k\right) = -\exp(-\sum_{k=1}^{j} \widehat{\theta}_k^T \Delta S_k + B_j), j = 0, 1, ..., N, \tag{5.54}$$

[1] By convention, the sum over empty set is zero.
[2] By convention, the product over empty set is one.

is a martingale and

$$U_1\left(j,\sum_{k=1}^{j}\theta_k^T\Delta S_k\right)-\exp(-\sum_{k=1}^{j}\theta_k^T\Delta S_k+B_j),\quad j=0,1,...,N, \tag{5.55}$$

is a supermartingale for any admissible portfolio θ (i.e. $\theta_j\in\Theta_j^{(1)}$, $j=1,...,N$). By combining (5.55) and the assumption (5.50), we deduce that the process $-\exp(B)$ is a supermartingale. Remark that in discrete time models, any positive special semimartingale can be decomposed explicitly as follows:

$$\exp(B_j)=\exp(B_0)M_jA_j, \tag{5.56}$$

where

$$M_j=\prod_{k=1}^{j}\frac{\exp(B_k)}{E\left(\exp(B_k)|\mathcal{F}_{k-1}\right)},\quad A_j=\prod_{k=1}^{j}E\left(\frac{\exp(B_k)}{\exp(B_{k-1})}\Big|\mathcal{F}_{k-1}\right),$$

It is clear that M is a positive martingale and A is a predictable process. Put $Q:=M_N\cdot P$. Then, by combining Bayes' rule with (5.54) and (5.55) respectively, we obtain

$$E^Q\left(\exp(-\widehat{\theta}_j^T\Delta S_j)|\mathcal{F}_{j-1}\right)=\frac{A_{j-1}}{A_j},\quad j=1,...,N \tag{5.57}$$

$$\text{and}\quad E^Q\left(\exp(-\theta_j^T\Delta S_j)|\mathcal{F}_{j-1}\right)\geq\frac{A_{j-1}}{A_j},\ \ \forall\ \theta_j\in\Theta_j^{(1)},\quad j=1,...,N. \tag{5.58}$$

From (5.57) and (5.56), we derive (ii.2) directly. Now, we combine (5.57) and (5.58) together and obtain for every $j=1,...,N$,

$$E^Q\left(\exp(-\widehat{\theta}_j^T\Delta S_j)|\mathcal{F}_{j-1}\right)\leq E^Q\left(\exp(-\theta_j^T\Delta S_j)|\mathcal{F}_{j-1}\right),\ \ \forall\ \theta_j\in\Theta_j^{(1)}. \tag{5.59}$$

By considering the functional

$$\Phi_j(y):=\int\exp(-y^Tx)\widetilde{G}_j(dx),\quad \widetilde{G}_j(dx):=Q(\Delta S_j\in dx|\mathcal{F}_{j-1}), \tag{5.60}$$

the inequality (5.59) is equivalent to

$$\Phi_j(\theta_j) \geq \Phi_j(\widehat{\theta}_j), \quad \forall \ \ \theta_j \in \Theta_j^{(1)}, \quad j = 1, 2, ..., N. \tag{5.61}$$

Based on the definition of admissible portfolio in (5.51), the functional Φ_j is proper, closed, convex and differentiable on $\mathbb{R}^d$. Therefore, $\widehat{\theta}_j$ is the root of

$$\int \exp(-\lambda^T x) x \widetilde{G}_j(dx) = E^Q \left(\exp(-\lambda^T \Delta S_j) \Delta S_j | \mathcal{F}_{j-1} \right) = 0, \quad j = 1, 2, ..., N..$$

This proved (5.52) and completes the proof of (i)⇒(ii).
To prove the reverse, we suppose that assertion (ii) holds. Put $Q := M_N \cdot P$. Then, for $j = 0, 1, ..., N$,

$$\begin{aligned} U_1 \left(j, \sum_{k=1}^{j} \widehat{\theta}_k^T \Delta S_k \right) &= -\exp(-\sum_{k=1}^{j} \widehat{\theta}_k^T \Delta S_k + B_j) \\ &= -M_j \exp(-\sum_{k=1}^{j} \widehat{\theta}_k^T \Delta S_k) \left[\prod_{k=1}^{j} E^Q(\exp(-\widehat{\theta}_k^T \Delta S_k) | \mathcal{F}_{k-1}) \right]^{-1}, \end{aligned}$$

which is a martingale by a simple calculation. For any admissible portfolio $(\theta_j)_{j=1,2,...,N}$, the convexity of the function $\exp(-y)$ implies

$$\exp(-\theta_j^T \Delta S_j + B_j) - \exp(-\widehat{\theta}_j^T \Delta S_j + B_j) \geq \exp(-\widehat{\theta}_j^T \Delta S_j + B_j)(\widehat{\theta}_j - \theta_j)^T \Delta S_j.$$

By taking conditional expectation in both sides above and using (5.52), we obtain

$$E \left(\exp(-\widehat{\theta}_j^T \Delta S_j + B_j) | \mathcal{F}_{j-1} \right) \leq E \left(\exp(-\theta_j^T \Delta S_j + B_j) | \mathcal{F}_{j-1} \right). \tag{5.62}$$

By multiplying both sides of (5.62) with $\prod_{k=1}^{j-1} \exp(-\theta_k^T \Delta S_k)$ and using

$$E \left(\exp(-\widehat{\theta}_j^T \Delta S_j + B_j) | \mathcal{F}_{j-1} \right) = \exp(B_{j-1})$$

which follows from the martingale property of $U_1(j, \sum_{k=1}^{j} \widehat{\theta}_j^T \Delta S_j)$, we derive

$$E\left(U_1\Big(j, x\sum_{k=0}^{j} \theta_k^T \Delta S_k\Big)|\mathcal{F}_{j-1}\right) \leq U_1\left(j-1, x\sum_{k=0}^{j-1} \theta_k^T \Delta S_k\right).$$

This proves that $U\left(j, x\sum_{k=0}^{j} \log(1+\theta_k^T \Delta S_k)\right)$, $j = 0, 1, ..., N$, is a supermartingale for any admissible θ and this completes the proof of this theorem. □

5.C Discrete Market Models

To further exhibit the characterization of exponential forward utilities in discrete-time markets, we will consider two particular cases. Namely, the binomial model and the multi-dimensional discrete model. For details on the setup of these two models, I refer the readers to Subsections 3.C.1 and 3.C.2.

5.C.1 One-Dimensional Binomial Model

In binomial model, the stock price process, S, satisfies for $j = 1, ..., N$,

$$S_{j+1} = S_0 \prod_{k=1}^{j+1} \xi_k, .$$

where ξ_{k+1} is a random variable that takes either ξ_{k+1}^d or ξ_{k+1}^u. Here, ξ_{k+1}^d and ξ_{k+1}^u are real numbers such that $0 < \xi_{k+1}^d < 1 < \xi_{k+1}^u$. We denote a sequence of events for $j = 1, ..., N$,

$$A_j := \{\xi_j = \xi_j^u\} \in \mathcal{F}_j.$$

Furthermore, due to $\#(\Omega) < +\infty$, it is obvious that any real-valued random variable is integrable and its conditional expectation is finite as well. Thus, we conclude that the admissible sets, $\Theta_j^{(1)}$, defined in (5.51) take the following forms

$$\Theta_j^{(1)} = L^0(\mathcal{F}_{j-1}), \quad j = 1, ..., N. \tag{5.63}$$

Theorem 5.5: *The following are equivalent.*

(i) The functional $U_1(j,x)$, defined in (5.49), is a forward utility with the optimal portfolio denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) There exists a martingale $M = (M_j)_{j=0,...,N}$ such that $M_0 = 1$ and for $j = 0, ..., N$, the following two properties hold:

(ii.1) $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,...,N}$ is given by

$$\widehat{\theta}_j = \frac{\log(Q_j(\xi_j^u - 1)) - \log((1-Q_j)(1-\xi_j^d))}{(\xi_j^u - \xi_j^d)S_{j-1}}, \quad j = 1, ..., N, \tag{5.64}$$

where $Q_j := Q(A_j|\mathcal{F}_{j-1})$ and $Q := M_N \cdot P$.

(ii.2) B can be described as follows for $j = 0, 1, ..., N$

$$B_j = B_0 + \log(M_j) - \sum_{k=1}^{j} \log\Big(\exp((1-\xi_k^u)\widehat{\theta}S_{k-1})Q_k + \exp((1-\xi_k^d)\widehat{\theta}S_{k-1})(1-Q_k)\Big). \tag{5.65}$$

Proof. The proof is straightforward from the proof of Theorem 5.4 and the following remarks.

a) The assumption (5.50) is automatically satisfied since the sample space is finite.

b) The function given by (5.60) becomes

$$\Phi_j(\theta) = \exp[(1-\xi_j^u)\lambda S_{j-1}]Q(A_j|\mathcal{F}_{j-1}) + \exp[(1-\xi_j^d)\lambda S_{j-1}]Q(A_j^c|\mathcal{F}_{j-1}),$$

and its derivative is given by

$$\Phi_j'(\lambda) = S_{j-1}(\xi_j^u - 1)\exp\Big[(1-\xi_j^u)\ \lambda S_{j-1}\Big]Q_j + \\ S_{j-1}(\xi_j^d - 1)\exp\Big[(1-\xi_j^d)\lambda S_{j-1}\Big](1-Q_j).$$

The root of $\Phi_j'(\lambda) = 0$ is obviously given by (5.64).

c) Assertion (ii.2) is a direct application of (5.53).

This ends the proof of this theorem. □

5.C.2 Multi-Dimensional Discrete Model

The setting of the model is same with the model introduced in Section 3.C.2. To this end, we use the same notations as there and skip their definitions. We only remark that due to $\#(\Omega) < +\infty$, the admissible sets, $\Theta_j^{(1)}$, defined in (5.51) take the following forms

$$\Theta_j^{(1)} = L^0(\mathcal{F}_{j-1}), \quad j = 1, ..., N. \tag{5.66}$$

The main result in this subsection is given the following theorem.

Theorem 5.6: *The following are equivalent.*

(i) The functional $U_1(j,x)$, defined in (5.49), is a forward utility with the optimal portfolio denoted by $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,2,...,N}$.

(ii) The following two properties hold:

(ii.1)There exists a martingale $M = (M_j)_{j=0,...,N}$, $M_0 = 1$, such that, for $j = 0, ..., N$,

$$B_j = B_0 + \log(M_j) - \sum_{k=1}^{j} \log\left[Y_k\right]. \tag{5.67}$$

where we put $Q := M_N \cdot P$ and

$$Y_k := \sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \frac{Q(A_k(n_1, ..., n_d)|\mathcal{F}_{k-1})}{\exp\left(\widehat{\theta}^T(\Xi_k(n_1, ..., n_d) - I_{d\times d})S_{k-1}\right)}.$$

(ii.2) $\widehat{\theta} = (\widehat{\theta}_j)_{j=1,...,N}$ is such that $\widehat{\theta}_j \in L^0(\mathcal{F}_{j-1})$ and $\widehat{\theta}_j$ is a root of the equation for $j = 1, ..., N$

$$\sum_{(n_1,...,n_d)\in\widetilde{\mathcal{N}}} \frac{(\Xi_j(n_1, ..., n_d) - I_{d\times d})I_d}{\exp(\theta^T(\Xi_j(n_1, ..., n_d) - I_{d\times d})S_{j-1})} Q(A_j(n_1, ..., n_d)|\mathcal{F}_{j-1}) = 0. \tag{5.68}$$

Proof. This theorem is a particular case of Theorem 5.4 by considering finite sample space. Note that the assumption (5.50) is automatically satisfied since the sample space is finite. Therefore, (5.67) and (5.68) can be derived by writing the expectation in (5.53) and (5.52) for discrete sample space. □

5.D Lévy Market Models

Recall the model described in Section 3.D, the stock price process S is given by $S = S_0 \exp(X)$, where X is a Lévy process. We consider a class of exponential forward utilities given in the form of

$$U_1(t, \omega, x) := -\exp(-x + B_t(\omega)). \tag{5.69}$$

For any probability measure Q, any stock price process X, and $x \in \mathbb{R}$ such that $U_0(t, x, \omega) < +\infty$ we denote by

$$\mathcal{A}_{adm}(X, Q) := \left\{ \pi \in L(X) \mid \sup_{\tau \in \mathcal{T}_T} E^Q \left[\exp(-\pi \cdot X_\tau + B_\tau) \right] < +\infty \right\}, \tag{5.70}$$

the set of admissible portfolios for the model (X, Q, U). Here $\mathcal{T}_T$ is the set of stopping time, τ, such that $\tau \le T$. When $X = S$ and $Q = P$, we simply write $\mathcal{A}_{adm}$.

Theorem 5.7: *Suppose that S is locally bounded and consider a RCLL semimartingale, B, which is exponentially special. Then the following two assertions are equivalent.*

(1) The functional U_1 given by (5.69) is a forward utility with the optimal portfolio $\widehat{\theta}$.

(2) There exists a local martingale $M := \mathcal{E}(N)$ (let (β, Y, V, N') denote the Jacod components of N) such that

(2.a) B can be written as

$$\begin{aligned} B = B_0 + \log(M) - \int_0^{\cdot} [\frac{1}{2}\sigma^2 e^{2X_u} \widehat{\theta}_u^2 + \\ + \int_{\mathbb{R}\setminus\{0\}} f_1(\exp(-e^{X_u}(e^x - 1)\widehat{\theta}_u) - 1) Y(e^{X_u}(e^x - 1)) F_u^X(dx)] du. \end{aligned} \tag{5.71}$$

(2.c) The optimal portfolio $\widehat{\theta}$ *is a root for*

$$\gamma+\frac{1}{2}\sigma^2+e^{X_-}\sigma^2(\beta-\lambda)+\int_{\mathbb{R}\setminus\{0\}}\frac{(e^x-1)Y(e^{X_-}(e^x-1))}{\exp(e^{X_-}(e^x-1)\lambda)}F^X(dx)=0. \quad (5.72)$$

(2.d) The local martingale $\exp(-\widehat{\theta}\cdot S+B)$ *is a true martingale.*

Proof. By Ito's formula, the dynamics of S can be presented as

$$\frac{dS_t}{S_{t-}}=(\gamma+\frac{1}{2}\sigma^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)F_t^X(dx))dt+\sigma dW_t+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)\widetilde{N}(dt,dx).$$

The triplet of predictable characteristics of S can be written as follows,

$$S_t^c=\int_0^t e^{X_u}\sigma dW_u, \quad c_t=e^{2X_{t-}}\sigma^2,$$

$$b_t=e^{X_{t-}}(\gamma+\frac{1}{2}\sigma^2+\int_{\mathbb{R}\setminus\{0\}}(e^x-1)F_t^X(dx))$$

For any measurable and non-negative/integrable function $k(x)$, F^X is related to F^S as follows

$$\int_{\mathbb{R}\setminus\{0\}}k(x)F^S(dx)=\int_{\mathbb{R}\setminus\{0\}}k(e^{X_-}(e^y-1))F^X(dy).$$

Since B is assumed to be exponentially special, due to Lemma 5.4, it can be decomposed as

$$\exp(B)=\exp(B_0)M\exp(B'),$$

where M is a positive local martingale with $M_0=1$. Hence, $M=\mathcal{E}(N)$, where the local martingale N can be represented as

$$N=\beta\cdot S^c+(Y-1)\star(\mu-\nu)+V\star\mu+N', \quad \nu(dt,dx)=F^S(dx)dt.$$

By the procedure of localization, it is enough to consider M to be a true martingale and define $Q:=M_T\cdot P$, then the characteristics of S with respect to Q are

$$S^{c,Q} = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_u}\sigma^2\beta_u du,$$

$$b^Q = e^{X_-}\gamma + e^{2X_-}\sigma^2\beta + \frac{1}{2}e^{X_-}\sigma^2 + e^{X_-}\int_{\mathbb{R}\setminus\{0\}} Y(e^{X_-}(e^x-1))(e^x-1)F^X(dx)$$

$$F^Q(dx) = Y(x)F^S(dx).$$

For locally bounded S, the assumption (5.3) is satisfied under measure Q. Hence, based on Theorem 5.2, $\widehat{\theta}$ is the root of the equation (5.72). The Minimal Entropy Hellinger process, $h^E(\widetilde{Z}^Q, Q)$ helps us to find the representation of B in (5.71). Anything else that is not proved here is straightforward and the reader can find hints in the proof of Theorems 5.1 and 5.2.

□

5.D.1 Cramér-Lundberg Risk Model

Consider the classical Cramér-Lundberg risk process $(R_t)_{t\geq 0}$, given by

$$R_t = x + \gamma t - \sum_{i=1}^{N_t} Y_i. \tag{5.73}$$

Here $(Y_i)_{i\geq 1}$ are i.i.d positive random variables representing the claims, $N = (N_t)_{t\geq 0}$ is an independent Poisson process with intensity $\lambda > 0$ modeling the times at which the claims occur, $x > 0$ denotes the initial surplus, and γ is a premium intensity. The stock price process is given by $S = \exp(R)$. In this model, there is no Brownian Motion and the source of risk coming from the compound Poisson process.

We assume that each claim follows normal distribution, $N(\mu, \sigma)$ with density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Theorem 5.8: *Consider the functional $U_1(t,\omega,x)$ defined in (5.69). Suppose that pB is exponential special for some $p > 1$. Then the following two assertions are equivalent.*

(1) The functional U_1 is a forward utility with the optimal portfolio $\widehat{\theta}$.

(2) There exists a local martingale $M := \mathcal{E}(N)$ (let (β, Y, V, N') denote the Jacod components of N) such that

(2.a) B can be written as

$$B = B_0 + \log(M) - \lambda \int_0^{\cdot} \left[\int f_1(\exp(-S_{u-}(e^x - 1)\widehat{\theta}_u) - 1) Y(S_{u-}(e^x - 1)) f(x) dx \right] du. \tag{5.74}$$

(2.b) The optimal portfolio $\widehat{\theta}$ is a root for

$$\gamma + \lambda \int \frac{e^x - 1}{\exp(S_-(e^x - 1)\theta)} Y(S_-(e^x - 1)) f(x) dx = 0. \tag{5.75}$$

(2.c) The local martingale $\exp(-\widehat{\theta} \cdot S + B)$ is a true martingale.

Proof. Notice that for normal distribution, the intensity $\nu(dt \times dx) = F_t(dx)dt$ becomes

$$F(dx) = \lambda f(x)dx.$$

Furthermore, the assumption (5.3) is satisfied. Thus, by associating the assumption that pB is exponential special for some $p > 1$, the functional

$$\Phi(\theta) := \theta b - \frac{1}{2} c\theta^2 + \lambda \int f(x)(-e^{-\theta x} + 1 - \theta h(x)) dx$$

is proper, closed, convex and differentiable. If (i) holds, $\widehat{\theta}$ is the minimum of $\Phi(x)$. That is, the root of (5.75).

Anything else on the proof–that is not provided here–is straightforward and the reader can find them in the proof of Theorems 5.1 and 5.2. □

5.D.2 Jump-Diffusion Model

Consider the jump-diffusion model of Subsection 3.D.1. In current subsection, we will characterize the exponential forward utilities for this model.

Theorem 5.9: *Consider the functional $U_1(t,\omega,x)$ defined in (5.69). Suppose that B satisfies*

$$\sup_{\tau\in\mathcal{T}_T} E(e^{B_\tau}) < +\infty. \tag{5.76}$$

Then the following two assertions are equivalent.

(1) The functional U_1 is a forward utility with the optimal portfolio $\widehat{\theta}$.

(2) U_1 has the following properties:

(2.a) There exists a positive local martingale M, and predictable processes α and η satisfying $\int_0^T(\alpha_u^2+\eta_u^2)du < +\infty$, $P-a.s.$, such that

$$M_t = \int_0^t \alpha_u dW_u + \int_0^t \eta_u d\widetilde{N}_u, \quad M_0 = 0, \quad t\in[0,T]$$

and

$$B = B_0 + \log(M) - \int_0^{\cdot}\Big[\frac{\sigma^2 e^{2X_u}\widehat{\theta}_u^2}{2} + \lambda(1+\eta_u)f_1(\exp(-e^{X_u}(e-1)\widehat{\theta}_u)-1)\Big]du. \tag{5.77}$$

(2.b) The optimal portfolio $\widehat{\theta}$ is a root for

$$\gamma + \frac{1}{2}\sigma^2 + e^{X_-}\sigma^2(\alpha-\theta) + \lambda(e-1)\frac{1+\eta}{\exp(e^{X_-}(e-1)\theta)} = 0. \tag{5.78}$$

(2.d) The local martingale $\exp(-\widehat{\theta}\cdot S + B)$ is a true martingale.

Proof. The dynamics of S can be written as

$$\frac{dS_t}{S_{t-}} = (\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda)dt + \sigma dW_t + (e-1)d\widetilde{N}_t.$$

The predictable characteristics of S are

$$S_t^c = \int_0^t e^{X_u}\sigma du, \quad c = e^{2X_-}\sigma^2, \quad A_t = t,$$

$$b_t = (\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda)\int_0^t e^{X_u}du.$$

$$F_t(dx) = \delta_1(dx).$$

$$\mu^S(dt, dx) = (e-1)S_{t-}\sum I_{\{\Delta X_s \neq 0\}}\delta_{(s,\Delta X_s)}(ds, dx)$$

The jumps of S can be calculated by

$$\Delta S_t = \begin{cases} e^{X_{t-}}(e-1), & \text{on } \{\Delta X_t = 1\}; \\ 0, & \text{on } \{\Delta X_t = 0\}. \end{cases}.$$

Under assumption (5.76), the null portfolio is admissible and the process e^B is a positive submartingale. It can be decomposed as

$$e^B = e^{B_0}Me^{B'}, \quad M = \mathcal{E}(N), \quad M_0 = 1.$$

Due to procedure of localization, it is enough to consider M to be a true martingale. Put $Q := M_T \cdot P$, then the characteristics of S with respect to Q are

$$S^{c,Q} = \int_0^t e^{X_u}\sigma dW_u - \int_0^t e^{2X_u}\sigma^2\alpha_u du,$$

$$b^Q = e^{X_-}(\gamma + \frac{1}{2}\sigma^2 + (e-1)\lambda) + e^{2X_-}\sigma^2\alpha + e^{X_-}\eta\lambda(e-1).$$

$$F_t^Q(dx) = (1+\eta_t)F_t(dx).$$

And, the dynamics of S under Q can be written as

$$dS_t = dS_t^{c,Q} + b_t^Q dt + e^{X_{t-}}(e-1)d(N_t - (1+\eta_t)\lambda dt).$$

If (1) holds, then $\widehat{\theta}$ will minimize the following equation

$$\Phi(\theta) := -\theta b^Q + \frac{1}{2}c\theta^2 + \lambda(1+\eta)\left(\theta e^{X_-}(e-1) - 1 + \exp(-\theta e^{X_-}(e-1))\right).$$

It is clear that ϕ is differentiable, and hence $\widehat{\theta}$ is a root of the equation (5.78). The Minimal Hellinger process of order 0 can be calculated as

$$\int_0^{\cdot} [\frac{1}{2}\sigma^2 e^{2X_u}\widehat{\theta}_u^2 + \lambda(1+\eta_u)f_1(\exp(-e^{X_u}(e-1)\widehat{\theta}_u) - 1)]du$$

such that (5.77) will be the predictable and finite variation part of D_1 with respect to Q. $\square$

5.D.3 Black-Scholes Model

In Black-Scholes model, the only source of risk come from the Brownian Motion W. The optimal portfolio $\widehat{\theta}$ can be obtained explicitly.

Theorem 5.10: *Consider the functional $U_1(t, \omega, x)$ defined in (5.69). Suppose that B satisfies (5.76). Then the following two assertions are equivalent.*

(1) The functional U_1 is a forward utility with the optimal portfolio $\widehat{\theta}$.

(2) There exists a local martingale $M_t := \int_0^t \alpha_u dW_u$ such that

(2.a) B can be described as

$$B = B_0 + \log(M) - \frac{1}{2}\sigma^2 \int_0^{\cdot} e^{2X_u}\widehat{\theta}_u^2 du. \tag{5.79}$$

(2.b) The optimal portfolio $\widehat{\theta}$ is given by

$$\widehat{\theta} = e^{-X}(\gamma\sigma^{-2} + \frac{1}{2}) + \alpha. \tag{5.80}$$

(2.c) The local martingale

$$\widehat{Z} := M\mathcal{E}\left(-\sigma\widehat{\theta}e^X \cdot W + \sigma^2 \int_0^{\cdot} \widehat{\theta}_u e^{2X_u}\alpha_u du\right)$$

is a true martingale.

Proof. The proof is a direct application of Theorem 5.9 by putting $\lambda = \eta = 0$. $\square$

Chapter 6

Minimal Hellinger Deflator

In this chapter, we combine the concept of deflators (see its definition in Section 6.A) with the concept of Hellinger processes and look for the minimal Hellinger deflator (called MHD hereafter). This idea is a natural extension of the main idea developed in [16], [17] and [18]. In these papers, the authors created the concept of entropy-Hellinger process and Hellinger process of order q for local martingale densities and obtained the minimal Hellinger martingale densities. The definitions and properties on minimal entropy-Hellinger martingale density and minimal Hellinger martingale density of order q are briefly reviewed in Section 2.D.

Here, we extend these concepts and build up the new concept of MHD. Precisely, we define the Hellinger distance for deflators that are supermartingales. Then, we focus on minimizing this distance to obtain the MHD. Finally, we conclude our study by deriving properties for this MHD and applying these results to HARA forward utilities.

6.A Mathematical Model and Preliminaries

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration satisfies the usual conditions and is quasi-left continuous. The quasi-left-continuity of the filtration is equivalent to the continuity of any

RCLL predictable process. We consider a d-dimensional stock price process, S, which is a semimartingale with the Canonical representation

$$S = S_0 + S^c + h(x) \star (\mu - \nu) + (x - h(x)) \star \mu + B. \tag{6.1}$$

More details on this representation can be found in Chapter 2.
Throughout this chapter, the set of integrands, $\mathcal{L}$, given by

$$\mathcal{L} := \{\pi \in L(S) : 1 + \pi^T x > 0, \quad P \otimes F \otimes A - a.e.\} \tag{6.2}$$

will play a crucial role in our analysis. Moreover, the set of bounded elements in $\mathcal{L}$ will be denoted by $\mathcal{L}_b$, i.e.

$$\mathcal{L}_b := \{\lambda \in \mathcal{L} \ : \text{ there exists } C > 0, \, C \in \mathbb{R}, \text{ such that } |\lambda| \leq C.\} \tag{6.3}$$

For $p \in (-\infty, 0) \cup (0, 1)$, our main assumption in this chapter is given by

$$(A1) : \int \left| x(1+\lambda^T x)^{p-1} - h(x) \right| F(dx) < +\infty, \quad P \otimes A - a.e., \; \forall \; \lambda \in \mathcal{L}. \tag{6.4}$$

Remark: It is easy to check that the null element $0 \in \mathcal{L}$. Therefore, the assumption (6.4) implies

$$\int_{\{|x|>1\}} |x| F(dx) < +\infty, \quad P \otimes A - a.e. \tag{6.5}$$

The condition (6.5) will be used from time to time in this chapter as an assumption.
Furthermore, the set $\mathcal{L}$ (defined in (6.2)) is intimately related to the set of wealth processes, $\mathscr{X}_+(x)$, defined below

$$\mathscr{X}_+(x) = \{X : \text{ there exists } \theta \in L(S), \; X = x + \theta \cdot S, \; X > 0 \text{ and } X_- > 0.\} \tag{6.6}$$

The relationship between $\mathcal{L}$ and $\mathscr{X}_+(x)$ is described in the following.

Lemma 6.1: *For $x > 0$ and $\theta \in L(S)$, we consider $X = x + \theta \cdot S$. Then, $X \in \mathscr{X}_+(x)$ if and only if there exists $\pi \in \mathcal{L}$ such that $X = x\mathcal{E}(\pi \cdot S)$.*

Proof. For $X = x + \theta \cdot S \in \mathscr{X}_+(x)$, consider the sequence of stopping times, $(R_n)_{n\geq 1}$, given by

$$R_n := \inf \ \{t : \quad X_t \leq 1/n\} \wedge T.$$

Since $X > 0$ and $X_- > 0$, it is clear that R_n increases to T stationarily and on $[\![0, R_n]\!]$, for $n \geq 1$,

$$X_{t-} \geq 1/n, \quad P - a.s.$$

This implies that the process $1/X_-$ is locally bounded. Hence, the integral $Y := \frac{1}{X_-} \cdot X$ is well-defined, which allows us to write X in the stochastic exponential form as follows

$$X = x\mathcal{E}(\pi \cdot S).$$

Then, it is easy to check that $\pi := \frac{\theta}{x+\theta \cdot S_-} \in L(S)$ and $1 + \pi^T \Delta S = \frac{1+\theta \cdot S}{1+\theta \cdot S_-} > 0$ since $\theta \in L(S)$ and $\frac{1}{x+\theta \cdot S_-}$ is locally bounded.
The remaining part of this proof follows from the following equivalences.

$$1 + \pi^T \Delta S > 0, \quad P - a.s. \Longleftrightarrow P - a.s. \quad 1 + \pi^T x > 0, \quad \mu - a.e. \quad (6.7)$$
$$\Longleftrightarrow 1 + \pi^T x > 0, \quad P \otimes A \otimes F - a.e. \quad (6.8)$$

The equivalence (6.7) is easy to see, while the equivalence (6.8) comes from the following fact:

$$\begin{aligned} P \otimes A \otimes F\Big(\{(\omega, t, x) : 1 + \pi^T x \leq 0\}\Big) &= E\left(\int_0^T F_s(\{x : 1 + \pi_s^T x \leq 0\}) dA_s\right) \\ &= E\left(I_{\{(t,x):1+\pi_t^T x \leq 0\}} \star \nu_T\right) \\ &= E\left(I_{\{(t,x):1+\pi_t^T x \leq 0\}} \star \mu_T\right). \end{aligned}$$

From the equivalences in (6.7) and (6.8), we have immediately that $\pi \in \mathcal{L}$.

To prove the reverse sense, it is enough to prove that $X = x\mathcal{E}(\pi \cdot S) = x + \theta \cdot S$, $\theta := \pi X_-$, satisfies $X > 0$ and $X_- > 0$. This follows from $1 + \pi^T \Delta S > 0$, which is induced by $\pi \in \mathcal{L}$ and (6.8). Then, we get

$$X > 0 \quad \text{and} \quad X_- = \frac{X}{1 + \pi^T \Delta S} > 0, \quad P - a.s.$$

This completes the proof of this lemma. □

Remark: This lemma allows us to write the wealth process in $\mathscr{X}_+(x)$ in two forms:

$$X = x + \theta \cdot S, \quad \text{or equivalently} \quad X = x\mathcal{E}(\pi \cdot S), \quad \pi := \frac{\theta}{x + \theta \cdot S_-} \in \mathcal{L}.$$

Most of the time throughout our analysis in this chapter, the second one is more convenient and hence frequently adopted.

Definition: A stochastic process Y is called a deflator if YX is a supermartingale for any $X \in \mathscr{X}_+(x)$.

Lemma 6.2: *Let V be a RCLL, non-decreasing and predictable process with $V_0 = 0$. Then, there exist two predictable and non-negative processes, α and $V^\perp$, such that $V^\perp$ is non-decreasing,*

$$\alpha \cdot A_T < +\infty, \quad P - a.s, \tag{6.9}$$

$$\textit{and} \qquad V = \alpha \cdot A + V^\perp, \qquad supp(V^\perp) \cap supp(A) = \emptyset. \tag{6.10}$$

Proof. Thanks to the Lebesgue Decomposition Theorem (see [2], page 115) which is applied path-by-path, we deduce the existence of processes, V_a and $V^\perp$,[1] such that $dV_a \ll dA$, $V^\perp \perp A$ and

$$V = V_a + V^\perp.$$

This proves (6.9) and it is obvious that V_a and $V^\perp$ are RCLL, predictable and

[1]For instance, we can put $\Gamma := \text{supp}(A)$, $V_a := 1_\Gamma \cdot V$ and $V^\perp := 1_{\Gamma^c} \cdot V$.

non-decreasing with $V_0^{\perp} = 0$. Then, the Radon-Nikodym Theorem implies the existence of a non-negative and predictable process α such that

$$V_a = \alpha \cdot A.$$

This ends the proof of the lemma. $\square$

Throughout this thesis, the pair $(\alpha, V^{\perp})$ will be called the Lebesgue-Radon-Nikodym components of V.

Definition: Let X and Y be two processes such that $X_0 = Y_0$. Then, we write

$$X \preceq Y$$

if $Y - X$ is a nondecreasing process.

Proposition 6.1: *Let $N \in \mathcal{M}_{0,\ loc}(P)$ with Jacod components (β, f, g, N') and V be a RCLL, predictable and nondecreasing process with Lebesgue-Radon-Nikodym components $(\alpha, V^{\perp})$ such that*

$$1 + \Delta N - \Delta V > 0. \tag{6.11}$$

Then, the process $Y := \mathcal{E}(N)\mathcal{E}(-V)$ is a deflator if and only if the following conditions hold:
(i) For any $\theta \in \mathcal{L}$, $\left|\theta^T \left[(1 - \Delta V)x(1 + f(x) + g(x)) - h(x)\right]\right| \star \mu$ is locally integrable, or equivalently, we have

$$\left|\theta^T x(1 - \Delta V)(1 + f(x)) - \theta^T h(x)\right| \star \nu_T < +\infty, \quad P - a.s., \tag{6.12}$$

(ii) For any $\theta \in \mathcal{L}$, we have

$$\theta^T G(\beta, f, \alpha) \leq \alpha, \quad P \otimes A - a.e. \tag{6.13}$$

where

$$G(\beta, f, \alpha) := b + c\beta + \int \Big[(1-\alpha\Delta A)x(1+f(x)) - h(x)\Big] F(dx). \quad (6.14)$$

Proof. Consider the wealth process X of the form $X = \mathcal{E}(\theta \cdot S)$, where $\theta \in \mathcal{L}$. Then, Y is a deflator if and only if

$$Y\mathcal{E}(\theta \cdot S) \text{ is a supermartingale for any } \theta \in \mathcal{L}. \quad (6.15)$$

Put $\theta_k := \theta I_{\{|\theta| \leq k\}}$, $k \geq 1$. Then, $\theta_k \in \mathcal{L}$, bounded and converges to θ. In virtue of Fatou's lemma, (6.15) is equivalent to the fact that

$$Y\mathcal{E}(\theta \cdot S) \text{ is a supermartingale for any } \theta \in \mathcal{L}_b. \quad (6.16)$$

Moreover, due to Ito's formula and the Canonical decomposition of S given in (2.4), (6.16) holds if and only if the process

$$\begin{aligned} X &:= \theta \cdot (S + [N,S]) - V - [N + \theta \cdot (S + [N,S]), V] \\ &= \theta^T x \star (\mu - \nu) + \theta \cdot S^c - \Delta V \cdot N + \theta^T b \cdot A \\ &\qquad + \theta^T c\beta \cdot A - V + \theta^T[(1-\Delta V)x(1+f(x)+g(x)) - h(x)] \star \mu \end{aligned}$$

is a local supermartingale for any $\theta \in \mathcal{L}_b$. This is equivalent to the following two conditions:

$$|\theta^T[(1-\Delta V)x(1+f(x)+g(x)) - h(x)]| \star \mu \in \mathcal{A}^+_{loc}, \quad (6.17)$$

$$\text{and} \qquad \theta^T G(\beta, f, \alpha) \leq \alpha, \quad P \otimes A - a.e., \quad (6.18)$$

for any $\theta \in \mathcal{L}_b$. This equivalence is derived via compensating X using $M^P_\mu(g|\widetilde{P}) = 0$ and applying Lemma 6.2. Finally, note that for any $\theta \in \mathcal{L}$, there exists a sequence $\theta_k := \theta I_{\{|\theta| \leq k\}}$, $k \geq 1$ such that $\theta_k \in \mathcal{L}_b$ and converges to θ. Therefore, (6.17)–(6.18) are equivalent to (6.12)–(6.13) immediately. This completes the proof of this proposition.

□

Remark: In Lemma 6.1, the characterization of deflators is given in the general semimartingale framework. When the filtration is quasi-left continuous, the functional $G(\beta, f, \alpha)$ in (6.14) becomes

$$G(\beta, f) = b + c\beta + \int (x - h(x) + xf(x))F(dx), \tag{6.19}$$

and the condition (6.12) becomes

$$|\theta^T x(1 + f(x)) - \theta^T h(x)| \star \nu_T < +\infty, \quad P - a.s.. \tag{6.20}$$

Furthermore, the conditions (6.17) and (6.18) in the proof of Proposition 6.1 also play important role and can be used as alternative results for (6.12) and (6.13).

Lemma 6.3: *For any positive supermartingale Y, there exists a predictable and non-decreasing processes V^Y and a local martingale M^Y such that*

$$Y = Y_0 \mathcal{E}(M^Y)\mathcal{E}(-V^Y) = Y_0\mathcal{E}(M' - V^Y), \quad M' := (1 - \Delta V^Y) \cdot M^Y. \tag{6.21}$$

Proof. The proof of this lemma can be found in [11] and [12]. □

The following function will be used frequently in the rest of this chapter:

$$f_r(x) := \frac{(1+x)^r - 1 - rx}{r(r-1)}, \quad r \in (-\infty, 0) \cup (0, 1) \text{ and } x \geq -1; \tag{6.22}$$

Let p and q are conjugate numbers ($p = \frac{q}{q-1}$, $q \in (-\infty, 0) \cup (0, 1)$), which will be used from time to time.

In the current section, we are interested in the set of positive deflators satisfying an integrability condition,

$$\mathcal{Y}_q(S) := \Big\{ Y > 0 \mid \ Y \text{ is a deflator } \text{ and } \sum f_q(\Delta Y / Y_-) \in \mathcal{A}^+_{loc} \Big\}. \tag{6.23}$$

Definition: Let $q \in (-\infty, 0) \cup (0, 1)$ and $Y = \mathcal{E}(N - V)$ where $N \in \mathcal{M}_{0,\ loc}(P)$ and $V \in \mathcal{P} \cap \mathcal{V}^+$ such that $1 + \Delta N - \Delta V > 0$. If the non-decreasing process $V^{(q)}(Y)$, given by

$$V_t^{(q)}(Y) := \frac{1}{2}\langle N^c \rangle_t + \sum_{0<s\leq t} f_q(\Delta N_s) + \frac{V_t}{1-q}, \quad 0 \leq t \leq T, \tag{6.24}$$

is locally integrable (i.e. $V^{(q)}(Y) \in \mathcal{A}^+_{loc}(P)$), then its compensator (dual predictable projection) will be called the Hellinger process of order q for Y and will be denoted by $h^{(s,q)}(Y, P)$.

Lemma 6.4: *Let $q \in (-\infty, 0) \cup (0, 1)$ and $Y = \mathcal{E}(N - V)$ where $N \in \mathcal{M}_{0,\ loc}(P)$ and $V \in \mathcal{P} \cap \mathcal{V}^+$ such that $1 + \Delta N - \Delta V > 0$.*
Then, $h^{(s,q)}(Y, P)$ exists if and only if $h^{(q)}(N, P)$ exists.
Furthermore, when $h^{(s,q)}(Y, P)$ exists, it is given by

$$h^{(s,q)}(Y, P) = h^{(q)}(N, P) + \frac{V}{1-q}. \tag{6.25}$$

Proof. Recall the definition of Hellinger processes of order q for local martingales given in [18]: $h^{(q)}(N, P)$ exists if $N \in \mathcal{M}_{0,\ loc}(P)$ satisfying $1 + \Delta N > 0$ and

$$\frac{1}{2}\langle N^c \rangle + \sum_{0<s\leq \cdot} f_q(\Delta N_s) \in \mathcal{A}^+_{loc}. \tag{6.26}$$

Under the conditions of this lemma, we have $V \in \mathcal{P} \cap \mathcal{V}^+$ and

$$1 + \Delta N > \Delta V \geq 0.$$

Thus, (6.26) is fulfilled if and only if

$$V^{(q)}(Y) = \frac{1}{2}\langle N^c \rangle + \sum_{0<s\leq \cdot} f_q(\Delta N_s) + \frac{V}{1-q} \in \mathcal{A}^+_{loc}.$$

This gives us the equivalence between the existence of $h^{(s,q)}(Y, P)$ and the existence of $h^{(q)}(N, P)$. Therefore, by compensating $V^{(q)}(Y)$, (6.25) follows immediately. This completes the proof of the lemma. □

6.B Existence of Minimal Hellinger Deflator

This section will discuss the existence of the minimal Hellinger deflator as well as its description in an explicit way. To this end, we present the following definition on the "smallest" Hellinger process among all Hellinger processes for deflators.

Definition: A deflator $\widetilde{Y} \in \mathcal{Y}_q(S)$ is called the minimal Hellinger deflator of order q (called MHD of order q hereafter) if

$$h^{(s,q)}(\widetilde{Y}, P) \preceq h^{(s,q)}(Y, P), \quad \text{for any } Y \in \mathcal{Y}_q(S).$$

In virtue of this definition, the MHD of order q is the solution to the following minimization problem

$$\min_{Y \in \mathcal{Y}_q(S)} \; h^{(s,q)}(Y, P). \tag{6.27}$$

Therefore, the remaining part of this section will focus on investigating the existence of solution to (6.27). Thanks to Theorem 2.2, for any local martingale $N \in \mathcal{M}_{0,loc}(P)$ with Jacod components (β, f, g, N') can be written as

$$N = N^1 + g \star \mu + N',$$

where

$$N^1 = \beta \cdot S^c + f \star (\mu - \nu). \tag{6.28}$$

The following lemma characterizes a class of deflators whose local martingale part has the form of N^1.

Lemma 6.5: *Let $q \in (-\infty, 0) \cup (0, 1)$ and $Y = \mathcal{E}(N - V)$ where $N \in \mathcal{M}_{0,\ loc}(P)$ with Jacod components (β, f, g, N') and $V \in \mathcal{P} \cap \mathcal{V}^+$ with Lebesgue-Radon-*

Nikodym components $(\alpha, V^{\perp})$. *If* $Y \in \mathcal{Y}_q(S)$, *then* $Y^1 := \mathcal{E}(N^1 - \alpha \cdot A) \in \mathcal{Y}_q(S)$ *where* N^1 *has the form of (6.28).*

Proof. For $Y = \mathcal{E}(N - V) \in \mathcal{Y}_q(S)$, we have $1 + \Delta N - \Delta V > 0$, which implies

$$1 + \Delta N > \Delta V \geq 0.$$

Then, an application of Theorem 2.2 (precisely $\Delta N > -1$ implies that f can be selected to satisfy $f + 1 > 0$), we get

$$1 + \Delta N^1 - \Delta(\alpha \cdot A) = 1 + f(\Delta S) I_{\{\Delta S \neq 0\}} = I_{\{\Delta S = 0\}} + (1 + f(\Delta S)) I_{\{\Delta S \neq 0\}} > 0,$$

which implies $Y^1 > 0$, P-a.s. On the other hand, thanks to Proposition 6.1 and $\Delta V = 0$ (due to quasi-left continuity), we have

$$|\theta^T x(1 + f(x)) - \theta^T h(x)| \star \nu_T < +\infty, \quad P - a.s. \quad and$$

$$\theta^T G(\beta, f) \leq \alpha, \quad A \otimes P - a.s.$$

for any $\theta \in \mathcal{L}$ (see (6.19) and (6.20)). Notice that these two conditions– that completely characterize deflators in the quasi-left continuous context– are independent with g, $V^{\perp}$ and N'. Hence, these two conditions, in turn, guarantee that Y^1 is also a deflator. Moreover, $Y \in \mathcal{Y}_q(S)$ satisfies $X := \sum f_q(\Delta N) \in \mathcal{A}^+_{loc}$, whose compensator, X^p, is (see (3.10) in [18] for details)

$$X^p := f_q(f) \star \nu + (1 + f)^q M^P_{\mu}(f_q(\frac{g}{1 + f}) | \widetilde{\mathcal{P}}) \star \nu \in \mathcal{A}^+_{loc}.$$

Observe that

$$f_q(f(x)) \star \nu_T \leq X^p_T < +\infty, \quad P - a.s.,$$

and $f_q(f(x)) \star \nu_T$ is the terminal value of the compensator of $\sum f_q(\Delta N^1)$. Hence

$$\sum f_q(\Delta N^1) \in \mathcal{A}^+_{loc}. \tag{6.29}$$

Therefore, Y^1 is a positive deflator and satisfies (6.29), from which we can deduce that $Y^1 \in \mathcal{Y}_q(S)$. This ends the proof of the lemma. □

The subset of $\mathcal{Y}_q(S)$ whose elements have the form of Y^1 will be denoted by

$$\mathcal{Y}_q^1(S) = \{Y = \mathcal{E}(N - V) \in \mathcal{Y}_q(S) | N \text{ has the form of (6.28) and } dV << dA.\} \quad (6.30)$$

Proposition 6.2: *Let $q \in (-\infty, 0) \cup (0, 1)$ and $Y^1 := \mathcal{E}(N^1 - \alpha \cdot A) \in \mathcal{Y}_q^1(S)$, where N^1 has the form of (6.28). Then, the Hellinger process of order q, $h^{s,q}(Y^1, P)$ for Y^1, is given by*

$$h^{(s,q)}(Y^1, P) = \frac{1}{2}\beta^T c\beta \cdot A + \int f_q(f) F(dx) \cdot A + \frac{\alpha}{1-q} \cdot A. \quad (6.31)$$

Proof. This proof comes from a combination of Lemma 6.4 and Proposition 3.5 in [18]. First of all, based on Lemma 6.4, we have

$$h^{(s,q)}(Y^1, P) = h^q(N^1, P) + \frac{\alpha}{1-q} \cdot A.$$

Furthermore, $h^q(N^1, P)$ has already been derived from Proposition 3.5 in [18] as (for quasi-left continuous model)

$$h^q(N^1, P) = \frac{1}{2}\beta^T c\beta \cdot A + \int f_q(f) F(dx) \cdot A.$$

Therefore, (6.31) follows immediately. □

In the following, we will prove that the solution to (6.27) – when it exists – will belong to $\mathcal{Y}_q^1(S)$.

Proposition 6.3: *The following equivalence holds:*

$$\min_{Y \in \mathcal{Y}_q(S)} h^{(s,q)}(Y, P) = \min_{Y \in \mathcal{Y}_q^1(S)} h^{(s,q)}(Y, P). \quad (6.32)$$

Proof. Thanks to Lemma 6.4, for any $Y = \mathcal{E}(N - V) \in \mathcal{Y}_q(S)$, where

$$N = N^1 + g \star \mu + N' \quad \text{and} \quad V = \alpha \cdot A + V^{\perp},$$

its Hellinger process $h^{(s,q)}(Y, P)$ can be represented as

$$h^{(s,q)}(Y, P) = h^{(q)}(N, P) + \frac{V}{1-q}.$$

Following a similar argument as Proposition 4.2 in [18], we can easily prove that (recall that $X \preceq Y$ means that $Y - X$ is non-decreasing)

$$h^{(q)}(N^1, P) \preceq h^{(q)}(N, P).$$

Meanwhile, due to Lemma 6.2, we have

$$V \succeq \alpha \cdot A.$$

Thus, for any $Y \in \mathcal{Y}_q(S)$, there always exists a process $Y^1 := \mathcal{E}(N^1 - V^1)$, which belongs to $\mathcal{Y}_q^1(S)$ deduced from Lemma 6.5 and whose Hellinger process, $h^{(s,q)}(Y^1, P)$, satisfies

$$h^{(s,q)}(Y^1, P) \preceq h^{(s,q)}(Y, P).$$

This leads to

$$\min_{Y \in \mathcal{Y}_q(S)} h^{(s,q)}(Y, P) \succcurlyeq \min_{Y \in \mathcal{Y}_q^1(S)} h^{(s,q)}(Y, P).$$

Therefore, (6.32) follows immediately due to $\mathcal{Y}_q^1(S) \subseteq \mathcal{Y}_q(S)$. This ends the proof of this proposition. □

By observing closely the elements in $\mathcal{Y}_q^1(S)$, one may find that any deflator $Y \in \mathcal{Y}_q^1(S)$ can be uniquely determined by a triplet (β, f, α). Particularly, we call β the *principal component* for reasons that will be explained later on when we are looking for the MHD of order q ($q \in (-\infty, 0) \cup (0, 1)$). Moreover, due to positivity of any deflator in $\mathcal{Y}_q(S)$, we have $f > -1$. Here, we denote the set of all such triplets by $J^1(S)$, which can be defined precisely as

$$J^1(S) := \{(\beta, f, \alpha) \ : \ \beta \in L(S^c), \ 0 \leq \alpha \in \mathcal{P} \cap L(A) \text{ and } -1 < f(x) \in \mathcal{G}^1_{loc}(\mu) \\ \text{satisfying (6.20) and } f_q(f) \star \nu_T < +\infty\}.$$

Thanks to Proposition 6.3, the minimization (6.27) becomes

$$\min_{(\beta,f,\alpha)\in J^1(S)} K(\beta, f, \alpha), \quad \text{subject to} \quad \sup_{\theta\in\mathcal{L}} \theta^T G(\beta, f) \leq \alpha, \ \ P \otimes A - a.e. \tag{6.33}$$

$$\text{where} \quad K(\beta, f, \alpha) := \frac{1}{2}\beta^T c\beta + \int f_q(f)F(dx) + \frac{\alpha}{1-q}.$$

The next theorem states our main result of this section. We provide – under some no-arbitrage assumptions on the model – sufficient and necessary conditions for the existence of the MHD of order q, as well as its explicit description.

Theorem 6.1: *Let $q \in (-\infty, 0) \cup (0, 1)$ and suppose that (6.5) holds. Then, the MHD of order q exists if and only if $\mathcal{Y}_q(S) \neq \emptyset$.*
Furthermore, if the MHD (denoted by $\widetilde{Y}$) exists, then there exists $\widetilde{\lambda} \in \mathcal{L}$ such that

$$\widetilde{Y} = \mathcal{E}(\widetilde{N} - \widetilde{\alpha} \cdot A), \quad \widetilde{N} := \frac{\widetilde{\lambda}}{q-1} \cdot S^c + \widetilde{f} \star (\mu - \nu), \tag{6.34}$$

where

$$\widetilde{f}(x) := \left(1 + \widetilde{\lambda}^T x\right)^{1/(q-1)} - 1, \quad \textit{and} \quad \widetilde{\alpha} := \widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) = \sup_{\lambda\in\mathcal{L}} \lambda^T G(\widetilde{\beta}, \widetilde{f}).$$

The proof of this theorem is long and requires numerous lemmas. The first one introduces a new minimization problem by considering a set $J^2(S)$, defined by

$$J^2(S) := \{(\beta, f) \ : \ \beta \in L(S^c) \ \text{ and } \ -1 < f(x) \in \mathcal{G}^1_{loc}(\mu) \ \text{ satisfying } \ (6.20), \\ \sup_{\lambda \in \mathcal{L}} \theta^T G(\beta, f) \in L(A) \text{ and } f_q(f) \star \nu_T < +\infty\}. \tag{6.35}$$

The new minimization problem has close relationship with the minimization of (6.33), which is described in the following.

Lemma 6.6: *The optimization problem (6.33) admits a solution $(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha})$ if and only if the following optimization problem:*

$$\min_{(\beta,f)\in J^2(S)} \ H(\beta, f), \tag{6.36}$$

admits a solution $(\widehat{\beta}, \widehat{f})$. Here

$$H(\beta, f) := \frac{1}{2}\beta^T c\beta + \int f_q(f)F(dx) + \frac{1}{1-q} \sup_{\theta\in\mathcal{L}} \ \theta^T G(\beta, f), \tag{6.37}$$

Furthermore, the two solutions–when they exist–are connected as follows

$$\widetilde{\beta} = \widehat{\beta}, \quad \widetilde{f} = \widehat{f}, \quad \widetilde{\alpha} = \sup_{\theta\in\mathcal{L}} \ \theta^T G(\widetilde{\beta}, \widetilde{f}). \tag{6.38}$$

Proof. We start by supposing that the minimization (6.36) admits a solution, denoted by $(\widehat{\beta}, \widehat{f})$. Thus, for any $(\beta, f) \in J^2(S)$, we obtain

$$K(\widehat{\beta}, \widehat{f}, \widehat{\alpha}) = H(\widehat{\beta}, \widehat{f}) \leq H(\beta, f), \quad \text{where} \quad \widehat{\alpha} := \sup_{\theta\in\mathcal{L}} \ \theta^T G(\widehat{\beta}, \widehat{f}). \tag{6.39}$$

On the other hand, for any $(\beta, f, \alpha) \in J^1(S)$ satisfying the constraint

$$\sup_{\theta\in\mathcal{L}} \theta^T G(\beta, f) \leq \alpha, \quad P \otimes A - a.e.,$$

we have

$$H(\beta, f) \leq K(\beta, f, \alpha). \tag{6.40}$$

By combining (6.39) and (6.40), we deduce that $(\widehat{\beta}, \widehat{f}, \widehat{\alpha})$ is solution of (6.33). To prove the reverse, suppose that (6.33) has a solution, denoted by $(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha})$. Then, the following holds

$$\widetilde{\alpha} \geq \overline{\alpha} := \sup_{\theta \in \mathcal{L}} \theta^T G(\widetilde{\beta}, \widetilde{f}), \quad P \otimes A - a.e.$$

We start by proving that $\widetilde{\alpha} = \overline{\alpha}$. Indeed, if $\widetilde{\alpha} > \overline{\alpha}$, then $K(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha}) > K(\widetilde{\beta}, \widetilde{f}, \overline{\alpha})$, which is a contradiction with the fact that $(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha})$ is the solution of (6.33).

Now, for any $(\beta, f) \in J^2(S)$, we put $\alpha = \sup_{\theta \in \mathcal{L}} \theta^T G(\beta, f)$ such that

$$H(\beta, f) = K(\beta, f, \alpha). \tag{6.41}$$

It is easy to see that $(\beta, f, \alpha) \in J^1(S)$ and by taking $\widehat{\beta} := \widetilde{\beta}$, $\widehat{f} := \widetilde{f}$, we have

$$H(\widehat{\beta}, \widehat{f}) = K(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha}) \leq K(\beta, f, \alpha) = H(\beta, f). \tag{6.42}$$

This proves that $(\widehat{\beta}, \widehat{f}) \in J^2(S)$ is the solution of (6.36). This completes the proof of the lemma. □

The next lemma is a technical result that is needed in the forthcoming analysis.

Lemma 6.7: *For any $q \in (-\infty, 0) \cup (0, 1)$, the conjugate function of the convex function, $f_q(x)$, defined in (6.22), is given by*

$$f_q^*(y) = \frac{1 - (1 + (1-q)y)^{\frac{q}{q-1}}}{q} - y, \quad \textit{for} \quad y > \tfrac{1}{q-1} = p - 1. \tag{6.43}$$

Proof. In general, for a convex function $k(y)$, its conjugate function, $k^*(y)$, is defined by

$$k^*(y) = \inf_{x > -1} (k(x) + xy).$$

Thus, a simple calculation leads to (6.43). □

Associated with the set $\mathcal{L}$, we introduce another set of $\overline{\mathcal{L}}$, defined by

$$\overline{\mathcal{L}} := \{\lambda \in L(S) : 1 + \lambda^T x \geq 0, \quad P \otimes F \otimes A - a.e.\}$$

The following lemma defines a functional on $\overline{\mathcal{L}}$ and provides its upper bound over $\mathcal{L}$.

Lemma 6.8: *Let $q \in (-\infty, 0) \cup (0, 1)$ and suppose that (6.5) holds. Then, the functional*

$$L(\lambda) = \frac{\lambda^T b}{1-q} - \frac{1}{2}\frac{\lambda^T c\lambda}{(1-q)^2} + \int \left(\frac{1-(1+\lambda^T x)^p}{q} + \frac{\lambda^T h(x)}{q-1}\right) F(dx), \ \ \lambda \in \overline{\mathcal{L}}, \tag{6.44}$$

is well defined [2] *($p = \frac{q}{q-1}$).*
Furthermore, for any $(\beta, f) \in J^2(S)$ and $\forall\ \lambda \in \mathcal{L}$, we have

$$L(\lambda) \leq H(\beta, f), \qquad P \otimes A - a.e.. \tag{6.45}$$

Proof. We first check that the function L is well defined on $\overline{\mathcal{L}}$. Indeed, under the assumption (6.5), we can rewrite the integral part of L as

$$\int \left(\frac{1-(1+\lambda^T x)^p}{q} + \frac{\lambda^T x}{q-1}\right) F(dx) + \int_{\{|x|>1\}} \frac{\lambda^T x}{q-1} F(dx).$$

Meanwhile, notice that the function $g(y) := \frac{1-(1+y)^p}{q} + \frac{y}{q-1}$ satisfies that $g(y) \leq 0$ for any $y \geq -1$. Hence, the integral $\displaystyle\int g(\lambda^T x)F(dx)$ is well defined for any $\lambda \in \overline{\mathcal{L}}$ and thus the function L is also well defined on $\overline{\mathcal{L}}$.
On one hand, for any $(\beta, f) \in J^2(S)$ and $\lambda \in \mathcal{L}$, thanks to Lemma 6.7, we get

$$\frac{1}{q}(1-(1+\lambda^T x)^p) - \frac{1}{1-q}\lambda^T x \leq \frac{(1+f(x))^q - 1 - qf(x)}{q(q-1)} + \frac{1}{1-q}\lambda^T x f(x). \tag{6.46}$$

[2] by convention $1/0 = +\infty$

On the other hand, since the following quadratic formula always holds,

$$\left(\frac{\lambda}{1-q}+\beta\right)^T c\left(\frac{\lambda}{1-q}+\beta\right)\geq 0,$$

we have

$$-\frac{1}{2}\frac{1}{(1-q)^2}\lambda^T c\lambda \leq \frac{1}{2}\beta^T c\beta + \frac{1}{1-q}\lambda^T c\beta. \tag{6.47}$$

Then, by combining (6.46) and (6.47), we get

$$L(\lambda)\leq \frac{1}{2}\beta^T c\beta + \frac{\lambda^T b}{1-q} + \frac{\lambda^T c\beta}{1-q} + \int\left(f_q(f(x)) + \frac{\lambda^T x(f(x)+1)-\lambda^T h(x)}{1-q}\right)F(dx). \tag{6.48}$$

Due to the definition of $J^2(S)$ in (6.35), for any $\lambda\in\mathcal{L}$

$$\int|\lambda^T(1+f(x))x-\lambda^T h(x)|F_T(dx)<+\infty,\quad P\otimes A-a.e. \tag{6.49}$$

This allows us to rewrite the integral in the right-hand-side term of (6.48) as follows

$$\frac{1}{2}\beta^T c\beta + \int f_q(f)F(dx) + \frac{\lambda^T G(\beta,f)}{1-q},$$

where G is defined in (6.19). Therefore, it is clear that

$$L(\lambda)\leq \frac{1}{2}\beta^T c\beta + \int f_q(f)F(dx) + \frac{1}{1-q}\lambda^T G(\beta,f)\leq H(\beta,f).$$

This completes the proof of this lemma. □

Lemma 6.9: *For any $r\in(-\infty,0)\cup(0,1)$, the function, $\widetilde{f}_r(y)$, given by*

$$\widetilde{f}_r(y):=\begin{cases}\frac{(1+y)^r-1-ry}{r(r-1)}, & y\geq -1;\\ +\infty, & \textit{otherwise}.\end{cases} \tag{6.50}$$

is lower semi-continuous on $\mathbb{R}^d$.

Proof. For any $y\in\mathbb{R}^d$ and an arbitrary sequence in $\mathbb{R}^d$, $(y_i)\to y$, such that $\widetilde{f}_r(y_i)$ converges to l, we consider three cases: $y>-1$, $y<-1$ and $y=-1$.

For $y > -1$, there exists an integer N such that for any $i \geq N$, $y_i > -1$. Hence, due to continuity of $\widetilde{f}_r(y)$ for $y > -1$, we get

$$\widetilde{f}_r(y) = \lim_{i \to +\infty} \widetilde{f}_r(y_i) = l.$$

For $y < -1$, $\widetilde{f}_r(y) = +\infty$ and there exists an integer N such that for any $i \geq N$, $y_i < -1$ in which case

$$\widetilde{f}_r(y) = \lim_{i \to +\infty} \widetilde{f}_r(y_{m_i}) = +\infty = l.$$

For $y = -1$, we can either find an integer N such that for any $i \geq N$, $y_i + 1 \geq 0$ in which case

$$\widetilde{f}_r(y) = \lim_{i \to +\infty} \widetilde{f}_r(y_{m_i}) = l,$$

or, otherwise, there exists a subsequence of y_{m_i} satisfying $y_{m_i} < -1$ such that $y_{m_i} \to y$, in which case,

$$\widetilde{f}_r(y) \leq \lim_{i \to +\infty} \widetilde{f}_r(y_{m_i}) = +\infty = l.$$

Thus, we always have $\widetilde{f}_r(y) \leq l$ for any $y \in \mathbb{R}^d$. Hence, $\widetilde{f}_r$ is lower semi-continuous. □

Remark: As stated in [68], page 51, an alternative definition of lower semi-continuity for $\widetilde{f}_r$ is:

$$\widetilde{f}_r(y) = \liminf_{z \to y} \widetilde{f}_r(z) = \lim_{\epsilon \to 0} (\inf\{\widetilde{f}_r(z) | \, |z - y| \leq \epsilon\}), \quad \forall \quad y \in \mathbb{R}^d.$$

Thus, for any sequence of $(y_i) \to y$, we have

$$\widetilde{f}_r(y) = \liminf_{z \to y} \widetilde{f}_r(z) \leq \liminf_{y_i \to y} \widetilde{f}_r(y_i).$$

We also introduce the following technical result. This result provides a sufficient condition for $\mathcal{L}$ to be included in the effect domain of the functional L

(defined in (6.44)), denoted by $dom(L)$. It also gives us a way to approximate the element in $\mathcal{L}$ by a sequence of elements in $dom(L)$.

Lemma 6.10: *Let $q \in (-\infty, 0) \cup (0, 1)$ and suppose that (6.5) holds. Then, for any $\lambda \in I\!\!R^d$ and $\delta \in (0, 1)$, we have*

$$\int_{\{\lambda^T x \geq \delta - 1\}} f_p(\lambda^T x) F(dx) < +\infty \quad P \otimes A - a.e. \tag{6.51}$$

Proof. Consider the following two sets:

$$\Gamma_\lambda^+ := \{x : \lambda^T x \geq 0\}, \quad \Gamma_\lambda^- := \{x : \lambda^T x < 0\}, \quad \forall\, \lambda \in \mathbb{R}^d.$$

For any $\lambda \in \mathbb{R}^d$, the integral $\lambda \cdot S = \lambda^T S$ is a well-defined semimartingale. Thus, we have

$$\sum I_{\{|\lambda^T \Delta S| \leq \alpha\}} (\lambda^T \Delta S)^2 \leq I_{\{|\lambda^T \Delta S| \leq \alpha\}} \cdot [\lambda \cdot S, \lambda \cdot S] \in \mathcal{A}_{loc}^+, \quad \forall\, \alpha > 0. \tag{6.52}$$

Hence its compensator $I_{\{|\lambda^T x| \leq \alpha\}} (\lambda^T x)^2 \star \nu$ exists and belongs to $\mathcal{A}_{loc}^+$. Then, in one hand, by putting $\alpha = 1 - \delta$, we derive

$$\int_{\{\lambda^T x \geq \delta - 1\} \cap \Gamma_\lambda^-} f_p(\lambda^T x) F(dx) \leq (1 + \delta^{p-2}) \int_{\{|\lambda^T x| \leq \alpha\}} (\lambda^T x)^2 F(dx) < +\infty. \tag{6.53}$$

On the other hand, due to assumption (6.5) and $\{\lambda^T x \geq \delta - 1\} \subseteq \Gamma_\lambda^+$, we have

$$\int_{\Gamma_\lambda^+ \cap \{|x| > 1\}} f_p(\lambda^T x) F(dx) \leq \frac{|\lambda|}{1 - p} \int_{\{|x| > 1\}} |x| F(dx) < +\infty, \tag{6.54}$$

and due to (6.52), we get

$$\int_{\Gamma_\lambda^+ \cap \{|x| \leq 1\}} f_p(\lambda^T x) F(dx) \leq |\lambda|^2 \int_{\{|x| \leq 1\}} |x|^2 F(dx) < +\infty. \tag{6.55}$$

Therefore, by combining (6.53), (6.54) and (6.55), we deduce that (6.51) holds. □

The following lemma will play an important role when proving our main results in this chapter.

Proposition 6.4: *Let $q \in (-\infty, 0) \cup (0, 1)$. Suppose that $\mathcal{Y}_q(S) \neq \emptyset$ and (6.4) holds. Then, the following arise.*
(i) The functional $L(\lambda)$ given in (6.44) attains its maximum at some $\widetilde{\lambda} \in \mathcal{L}$.
(ii) We have

$$\widetilde{\beta} := \frac{1}{q-1}\widetilde{\lambda} \in L(S^c), \quad \widetilde{f} := \left(1 + \widetilde{\lambda}^T x\right)^{1/(q-1)} - 1 \in \mathcal{G}^1_{loc}(\mu), \tag{6.56}$$

$$f_q(\widetilde{f}) \star \nu \in \mathcal{A}^+_{loc}, \tag{6.57}$$

$$\widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) = \sup_{\lambda \in \mathcal{L}} \lambda^T G(\widetilde{\beta}, \widetilde{f}), \quad \textit{and} \quad \widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) \cdot A \in \mathcal{A}^+_{loc}. \tag{6.58}$$

Proof. (i) Consider the function $\widetilde{f}_p$ defined in (6.50) by putting $r = p$ and the functional $\widetilde{L}$, given by

$$\widetilde{L}(\lambda) := \Gamma(\lambda) + (p-1)^2 \int \widetilde{f}_p(\lambda^T x) F(dx), \quad \lambda \in \mathbb{R}^d. \tag{6.59}$$

Here $\Gamma(\lambda)$ is a quadratic function of λ, given by

$$\Gamma(\lambda) := (p-1)\lambda^T b + \frac{1}{p-1} \int_{\{|x|>1\}} \lambda^T x F(dx) + \frac{1}{2}(p-1)^2 \lambda^T c \lambda.$$

Due to the assumption (6.5), it is clear that

$$\widetilde{L}(\lambda) = \begin{cases} -L(\lambda), \text{ on } \overline{\mathcal{L}}; \\ +\infty, \quad \text{otherwise.} \end{cases}$$

Next, we are going to prove that the functional $\widetilde{L}$ attains its minimum at $\widetilde{\lambda} \in \overline{\mathcal{L}}$. This proof relies on Theorem 27.1–(b) in [68], which states that a convex, proper and closed function would attain its minimum if the set of recession is contained in the set of directions in which $\widetilde{L}$ is constant.
First of all, it is easy to see that $\widetilde{L}$ is convex and proper. Meanwhile, note a fact that for convex function, the closeness is equivalent to its lower semi-

continuity. Thus, in order to prove $\widetilde{L}$ being closed, it is enough to focus on its lower semi-continuity in the following.

For any sequence of $(\lambda_i) \to \lambda$ such that $\widetilde{L}(\lambda_i)$ converges. In virtue of Lemma 6.9 and the remark after it, the sequence $y_i := \lambda_i^T x$ converges to $y := \lambda^T x$ and we have

$$\widetilde{f}_p(y) \leq \liminf_{y_i \to y} \widetilde{f}_p(y_i).$$

Then, an application of Fatou's lemma leads to

$$\widetilde{L}(\lambda) \leq \liminf_{\lambda_i \to \lambda} \widetilde{L}(\lambda_i) = \lim_{\lambda_i \to \lambda} \widetilde{L}(\lambda_i).$$

This proved the closeness of $\widetilde{L}$.

In the remaining part of this proof, we will calculate the set of recession for $\widetilde{L}$. First of all, we calculate its recession function $\widetilde{L}0^+(\delta)$, which is defined by

$$\widetilde{L}0^+(\delta) := \lim_{\alpha \to +\infty} \frac{\widetilde{L}(\lambda + \alpha\delta) - \widetilde{L}(\lambda)}{\alpha}, \quad \delta \in \mathbb{R}^d.$$

By considering different cases, we obtain

$$\widetilde{L}0^+(\delta) = \begin{cases} +\infty, & F(\Gamma_\delta^-) > 0; \\ +\infty, & F(\Gamma_\delta^-) = 0 \text{ and } c\delta \neq 0; \\ \frac{b^T\delta - \int \delta^T h(x) F(dx)}{q-1}, & F(\Gamma_\delta^-) = c\delta = 0 \text{ and } F(\Gamma_\delta^+) \geq 0; \end{cases} \tag{6.60}$$

where

$$\Gamma_\delta^+ := \{z \in \mathbb{R}^d : z^T\delta > 0\}, \quad \Gamma_\delta^- := \{z \in \mathbb{R}^d : z^T\delta < 0\}.$$

We need to work more on the third case in (6.60). Remark that the assumption $\mathcal{Y}_q(S) \neq \emptyset$ implies that there exists a local martingale density $Z_0 = (\beta_0, Y_0 > 0)$ (see [45] and [18]) such that

$$b + c\beta_0 + \int (xY_0(x) - h(x))F(dx) = 0. \tag{6.61}$$

By virtue of (6.61), the third case in (6.60) can be rewritten as

$$\frac{b^T\delta - \int \delta^T h(x)F(dx)}{q-1} =$$
$$\begin{cases} +\infty, & F(\Gamma_\delta^-) = c\delta = 0,\ F(\Gamma_\delta^+) > 0 \text{ and } \int \delta^T h(x)F(dx) = +\infty; \\ \frac{-\int \delta^T x Y_0(x)F(dx)}{q-1} > 0, & F(\Gamma_\delta^-) = c\delta = 0,\ F(\Gamma_\delta^+) > 0 \text{ and } \int \delta^T h(x)F(dx) < +\infty. \\ \frac{b^T\delta}{q-1}, & F(\Gamma_\delta^-) = c\delta = F(\Gamma_\delta^+) = 0; \end{cases}$$

Furthermore, using (6.61) again, we deduce that

$$b^T\delta = 0, \quad P \otimes A - a.e., \quad \text{on} \quad \{F(\Gamma_\delta^-) = c\delta = F(\Gamma_\delta^+) = 0\}.$$

Therefore, the set of recession for $\widetilde{L}$ (i.e. $\{\delta : \widetilde{L}0^+(\delta) \leq 0\}$), is

$$\mathcal{R} := \{\delta \in \mathbb{R}^d \ : \ c\delta = 0, \ \ b^T\delta = F(\Gamma_\delta^+) = F(\Gamma_\delta^-) = 0\}.$$

It is easy to check that $\delta \in \mathcal{R}$ if and only if $-\delta \in \mathcal{R}$. As a result, the set of directions in which $\widetilde{L}$ is a constant (i.e. $\{\delta : \widetilde{L}0^+(\delta) \leq 0$ and $\widetilde{L}0^+(-\delta) \leq 0\}$), coincides with $\mathcal{R}$. Hence, due to Theorem 27.1–(b) in [68], we deduce that the functional $\widetilde{L}(\lambda)$ attains its minimum in $\overline{\mathcal{L}}$, we denote it as $\widetilde{\lambda}$.

(ii) Thanks to measurable selection theorem, $\widetilde{\lambda}$ can be selected to be predictable. For any $\lambda \in \overline{\mathcal{L}}$, put $\lambda_n := (1 - 1/n)\lambda$, that converges to λ. Due to Lemma 6.10 (consider $\delta = 1/n$), we have

$$\int f_p(\lambda_n^T x)F(dx) < +\infty, \quad n \geq 1. \tag{6.62}$$

For any $r \in (0, 1)$, the convex combination

$$\overline{\lambda} := r\lambda_n + (1-r)\widetilde{\lambda} = \widetilde{\lambda} + r(\lambda_n - \widetilde{\lambda}) \in \mathcal{L},$$

thus we have

$$\widetilde{L}(\widetilde{\lambda}) \leq \widetilde{L}(\overline{\lambda}), \quad P \otimes A - a.e. \tag{6.63}$$

The convexity of f_p and (6.62) implies

$$\frac{f_p(\widetilde{\lambda}^T x) - f_p(\overline{\lambda}^T x)}{r} \geq f_p(\widetilde{\lambda}^T x) - f_p(\lambda_n^T x).$$

This allows us to apply Fatou's Lemma and get

$$(1-p)(\lambda_n - \widetilde{\lambda})^T G(\widetilde{\beta}, \widetilde{f}) \leq \lim_{r\to 0} \frac{\widetilde{L}(\widetilde{\lambda}) - \widetilde{L}(\overline{\lambda})}{r} \leq 0, \quad P \otimes A - a.e.$$

Thus, we have

$$(\lambda_n - \widetilde{\lambda})^T G(\widetilde{\beta}, \widetilde{f}) \leq 0, \quad \forall\ n \geq 1, \quad P \otimes A - a.e.$$

By sending $n \to +\infty$ and taking sup for λ over $\overline{\mathcal{L}}$, we get

$$\widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) \geq \sup_{\lambda \in \overline{\mathcal{L}}} \lambda^T G(\widetilde{\beta}, \widehat{f}), \quad P \otimes A - a.e.$$

As an direct application of above inequality by taking $\lambda = 0 \in \overline{\mathcal{L}}$, it yields

$$\widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) \geq 0, \quad P \otimes A - a.e. \tag{6.64}$$

Then, by rearranging the terms in (6.64), we get

$$0 \leq \widetilde{\lambda}^T \int (x - x(1+\widetilde{\lambda}^T x)^{p-1}) F(dx) \leq \widetilde{\lambda}^T \int_{|x|\geq 1} x F(dx) + \widetilde{\lambda}^T b + (p-1)\widetilde{\lambda}^T c \widetilde{\lambda} < +\infty, \tag{6.65}$$

due to our assumption (6.5) and the positivity of the function $g(y) := y - y(1+y)^{p-1}$, $1 + y \geq 0$ and $p < 1$. Thus, $F(\{x : 1 + \widetilde{\lambda}^T x = 0\}) = 0$ which implies $\widetilde{\lambda} \in \mathcal{L}$. This proves the first part of (6.58).

Consider $\widetilde{\beta}$ and $\widetilde{f}$ defined in (6.56). Then, a simple calculation leads to

$$\frac{1 - (1 + \widetilde{\lambda}^T x)^p}{q} + \frac{\widetilde{\lambda}^T h(x)}{q-1} = f_q(\widetilde{f}(x)) + \frac{1}{1-q} \widetilde{\lambda}^T (x(1 + \widetilde{f}(x)) - h(x)) \tag{6.66}$$

Then, (6.65) allows us to rewrite $L(\widetilde{\lambda})$ as

$$L(\widetilde{\lambda}) = \frac{1}{2}\widetilde{\beta}^T c\widetilde{\beta} + \int f_q(\widetilde{f})F(dx) + \frac{1}{1-q}\widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}). \tag{6.67}$$

Observe that the first two terms of $L(\widetilde{\lambda})$ expressed in (6.67) are non-negative and the last one, due to (6.64), is non-negative as well. Meanwhile, due to $\mathcal{Y}_q(S) \neq \emptyset$, there exists $(\beta_0, f_0) \in J^2(S)$. Then, thanks to Lemma 6.8, we have

$$L(\widetilde{\lambda}) \cdot A \leq H(\beta_0, f_0) \cdot A \in \mathcal{A}^+_{loc},$$

which implies

$$\widetilde{\beta}^T c\widetilde{\beta} \cdot A \in \mathcal{A}^+_{loc}, \quad f_q(\widetilde{f}) \star \nu \in \mathcal{A}^+_{loc} \quad \text{and} \quad \widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) \cdot A \in \mathcal{A}^+_{loc}.$$

This completes the proof of the first part of (6.56) and (6.58). It remains to prove (6.57). To prove it, we apply Theorem 1.33–d) in [39] which states $\widetilde{f}(x) \in \mathcal{G}^1_{loc}(\mu)$ if and only if

$$C(\widetilde{f}) := \left(1 - \sqrt{1 + \widetilde{f}}\right)^2 \star \nu \in \mathcal{A}^+_{loc}.$$

For $x > -1$ arbitrary and fixed, a study on the functions $f_1(q) := \frac{(1+x)^q - 1 - qx}{q}$ and $f_2(q) := \frac{(1+x)^q - 1 - qx}{q-1}$ reveals that $f_1(q)$ is increasing for $0 \leq q \leq 1/2$, thus

$$-f_1(q) \geq -2(\sqrt{1+x} - 1 - \frac{1}{2}x) = (1 - \sqrt{1+x})^2.$$

and $f_2(q)$ is increasing for $1/2 \leq q \leq 1$, thus

$$f_2(q) \geq -2(\sqrt{1+x} - 1 - \frac{1}{2}x) = (1 - \sqrt{1+x})^2.$$

By putting $K := \max(q, 1-q)$, we have

$$\left(1 - \sqrt{1 + \widetilde{f}}\right)^2 \star \nu \leq K f_q(\widetilde{f}) \star \nu \in \mathcal{A}^+_{loc}.$$

This completes the proof. $\square$

Proof of Theorem 6.1:

Suppose that $\mathcal{Y}_q(S) \neq \emptyset$. Thanks to Proposition 6.4, the functional $L(\lambda)$ attains its maximum at $\widetilde{\lambda} \in \mathcal{L}$. And, for any $(\beta, f) \in J^2(S)$, we deduce from Lemma 6.8 that

$$L(\widetilde{\lambda}) \leq H(\beta, f), \quad P \otimes A - a.e.$$

By taking minimum over (β, f), we derive

$$L(\widetilde{\lambda}) \leq \min_{(\beta,f)\in J^2(S)} H(\beta, f). \tag{6.68}$$

By putting

$$\widetilde{\beta} := \frac{1}{q-1}\widetilde{\lambda}, \quad \widetilde{f}(x) := (1 + \widetilde{\lambda}^T x)^{1/(q-1)} - 1 \quad \text{and} \quad \widetilde{\alpha} := \widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f})$$

and a direct application of Proposition 6.4 lead to

$$\widetilde{\beta} \in L(S^c), \quad f_q(\widetilde{f}) \star \nu \in \mathcal{A}^+_{loc}, \quad \widetilde{f} \in \mathcal{G}^1_{loc}(\mu) \tag{6.69}$$

$$\widetilde{\alpha} = \sup_{\lambda\in\mathcal{L}} \lambda^T G(\widetilde{\beta}, \widetilde{f}) \in L(A). \tag{6.70}$$

We deduce from (6.70) that $\forall \;\; \lambda \in \mathcal{L}$

$$\lambda^T G(\widetilde{\beta}, \widetilde{f}) \leq \widetilde{\alpha}, \quad P \otimes A - a.e. \tag{6.71}$$

Meanwhile, due to assumption (6.4), we have $\forall \;\; \lambda \in \mathcal{L}_b$

$$|\lambda^T x(1 + \widetilde{f}(x)) - \lambda^T h(x)| \star \nu_T < +\infty, \quad P - a.s. \tag{6.72}$$

Then, thanks to Proposition 6.1 (and the remark after it), (6.71) and (6.72) allow us to construct a deflator

$$\widetilde{Y} := \mathcal{E}(\widetilde{N}), \quad \widetilde{N} = \widetilde{\beta} \cdot S^c + \widetilde{f} \star (\mu - \nu) - \widetilde{\alpha} \cdot A.$$

Moreover, by combining (6.69) and (6.70), we deduce that $(\widetilde{\beta}, \widetilde{f}) \in J^2(S)$ and $L(\widetilde{\lambda})$ can be rewritten as

$$L(\widetilde{\lambda}) = \frac{1}{2}\widetilde{\beta}^T c\widetilde{\beta} + \int f_q(\widetilde{f})F(dx) + \frac{1}{1-q}\widetilde{\lambda}^T G(\widetilde{\beta}, \widetilde{f}) = H(\widetilde{\beta}, \widetilde{f}). \tag{6.73}$$

Thus, by recalling (6.68), it is clear that $(\widetilde{\beta}, \widetilde{f})$ is the solution of (6.36). Furthermore, from Lemma 6.6, the triplet $(\widetilde{\beta}, \widetilde{f}, \widetilde{\alpha})$ will be the solution of (6.33).

It remains to verify that $\widetilde{Y}$ is the MHD. Note that due to Proposition 6.3, it is enough to consider this problem over $\mathcal{Y}_q^1(S)$. We let Y^1 be an arbitrary element in $\mathcal{Y}_q^1(S)$, given by

$$Y^1 := \mathcal{E}(N^1 - \alpha \cdot A), \quad N^1 := \beta \cdot A + f \star (\mu - \nu).$$

Recall Proposition 6.31, the Hellinger process of order q, $h^{(s,q)}(Y^1, P)$, is

$$h^{(s,q)}(Y^1, P) = \frac{1}{2}\beta^T c\beta \cdot A + \int f_q(f)F(dx) \cdot A + \frac{\alpha}{1-q} \cdot A. \tag{6.74}$$

Recalling (6.73), since $(\widetilde{\beta}, \widetilde{f}) \in J^2(S)$ is the solution of (6.36), we get

$$h^{(s,q)}(\widetilde{Y}, P) = L(\widetilde{\lambda}) \cdot A = H(\widetilde{\beta}, \widetilde{f}) \cdot A \preceq h^{(s,q)}(Y^1, P) = K(\beta, f, \alpha) \cdot A.$$

This proves the minimality of $\widetilde{Y}$ and completes the proof of this theorem. $\square$

6.C A Useful Characterization of MHD

In this section, we will elaborate a very important property on MHD.

Theorem 6.2: *Let $\widetilde{Y} \in \mathcal{Y}_q(S)$ and $\widetilde{\lambda} \in \mathcal{P}$ be such that $1+\widetilde{\lambda}^T z > 0$, $P \otimes F \otimes A$-a.e. and*

$$\widetilde{Y} = \mathcal{E}(\widetilde{N} - \widetilde{\alpha} \cdot A), \quad \widetilde{N} = \frac{\widetilde{\lambda}}{q-1} \cdot S^c + \widetilde{f} \star (\mu - \nu). \tag{6.75}$$

Here,

$$\widetilde{f}_t(x) = \left(1 + \widetilde{\lambda}_t^T x\right)^{1/(q-1)} - 1, \quad \widetilde{\alpha} = \widetilde{\lambda}^T G(\frac{\widetilde{\lambda}}{q-1}, \widetilde{f}). \tag{6.76}$$

Then, $\widetilde{\lambda} \in L(S)$ *and*

$$\widetilde{Y}^{q-1} = \mathcal{E}\left(\widetilde{\lambda} \cdot S + q(q-1)h^{(s,q)}(\widetilde{Y}, P)\right). \tag{6.77}$$

Proof. In virtue of Ito's formula, we have

$$\widetilde{Y}^{q-1} = \mathcal{E}(\widetilde{X}),$$

where $\widetilde{X}$ is given by

$$\begin{aligned} \widetilde{X} = (q-1)\widetilde{N} + \frac{(q-1)(q-2)}{2}\langle \widetilde{N}^c \rangle + \sum \Big(&(1+\Delta\widetilde{N})^{q-1} \\ &-1 - (q-1)\Delta\widetilde{N}\Big) - (q-1)\widetilde{\alpha} \cdot A. \end{aligned} \tag{6.78}$$

Therefore, to prove (6.77), it is enough to show

$$\widetilde{X} = \widetilde{\lambda} \cdot S + q(q-1)h^{(s,q)}(\widetilde{Y}, P). \tag{6.79}$$

First of all, notice that

$$\Delta\widetilde{N} = f(\Delta S) I_{\{\Delta S \neq 0\}} = (1 + \widetilde{\lambda}^T \Delta S)^{1/(q-1)} - 1.$$

By inserting $\Delta\widetilde{N}$ into (6.78) and integrating $I_{\{|\widetilde{\lambda}| \leq n\}}$ on both sides of the resulting equality, we obtain

$$\begin{aligned} &I_{\{|\widetilde{\lambda}| \leq n\}} \cdot \widetilde{X} \qquad (6.80) \\ = \; &\widetilde{\lambda}^{(n)} \cdot S^c + \frac{q-2}{2(q-1)}(\widetilde{\lambda}^{(n)})^T c \widetilde{\lambda}^{(n)} \cdot A + (q-1)((1 + (\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1) \star (\mu - \nu) \\ &+ \left((\widetilde{\lambda}^{(n)})^T x - (q-1)((1 + (\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1)\right) \star \mu - (q-1) I_{\{|\widetilde{\lambda}| \leq n\}} \widetilde{\alpha} \cdot A, \end{aligned}$$

where $\widetilde{\lambda}^{(n)} := \widetilde{\lambda} I_{\{|\widetilde{\lambda}| \leq n\}}$. Recall the $(\mu - \nu)$-integrability of $\widetilde{f}$ and the boundedness of $h(x)$, we have the $(\mu-\nu)$-integrability of the process

$$(\widetilde{\lambda}^{(n)})^T h(x) - (q-1)\left((1+(\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1\right).$$

Thus, the fourth term in (6.80) can be rewritten as

$$\begin{aligned}
&\left((\widetilde{\lambda}^{(n)})^T x - (q-1)((1+(\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1)\right) \star \mu \\
=\ & (\widetilde{\lambda}^{(n)})^T (x - h(x)) \star \mu + \left((\widetilde{\lambda}^{(n)})^T h(x) - (q-1)\left((1+(\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1\right)\right) \star \nu \\
&+ \left((\widetilde{\lambda}^{(n)})^T h(x) - (q-1)\left((1+(\widetilde{\lambda}^{(n)})^T x)^{\frac{1}{q-1}} - 1\right)\right) \star (\mu-\nu), \qquad (6.81)
\end{aligned}$$

On the other hand, note that $\widetilde{\alpha}$ satisfies the equation

$$\widetilde{\lambda}^T b \cdot A + \frac{1}{q-1}\widetilde{\lambda}^T c \widetilde{\lambda} \cdot A + \left(\widetilde{\lambda}^T x (1+\widetilde{\lambda}^T x)^{\frac{1}{q-1}} - \widetilde{\lambda}^T h(x)\right) \star \nu = \widetilde{\alpha} \cdot A. \quad (6.82)$$

Hence, due to (6.81), (6.82) and recalling the decomposition of S in (2.4), (6.80) can be rewritten as

$$\begin{aligned}
I_{\{|\widetilde{\lambda}| \le n\}} \cdot \widetilde{X} = I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\lambda} \cdot S + \frac{q}{2(q-1)} I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\lambda}^T c \widetilde{\lambda} \cdot A - q I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\alpha} \cdot A \\
+ q(q-1) I_{\{|\widetilde{\lambda}| \le n\}} f_q(\widetilde{f}(x)) \star \nu,
\end{aligned} \qquad (6.83)$$

Since $\widetilde{X}$ is a semimartingale and the processes $\widetilde{\alpha} \cdot A$, $\widetilde{\lambda}^T c \widetilde{\lambda} \cdot A$ and $f_q(\widetilde{f}(x)) \star \nu$ are non-decreasing and locally bounded, we deduce that $I_{\{|\widetilde{\lambda}| \le n\}} \cdot \widetilde{X}$ converges in the semimartingale topology to $\widetilde{X}$, and the random variables $I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\lambda}^T c \widetilde{\lambda} \cdot A_T$, $I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\alpha} \cdot A_T$ and $I_{\{|\widetilde{\lambda}| \le n\}} f_q(\widetilde{f}(x)) \star \nu_T$ all converges in probability. This implies that $I_{\{|\widetilde{\lambda}| \le n\}} \widetilde{\lambda} \cdot S$ converges in semimartingale topology. Hence $\widetilde{\lambda} \in L(S)$. Thus, by sending $n \to +\infty$ in (6.83), we derive

$$\begin{aligned}
\widetilde{X} &= \widetilde{\lambda} \cdot S + \frac{q}{2(q-1)} \widetilde{\lambda}^T c \widetilde{\lambda} \cdot A + q(q-1) f_q\left((1+\widetilde{\lambda}^T x)^{\frac{1}{q-1}} - 1\right) \star \nu - q\widetilde{\alpha} \cdot A \\
&= \widetilde{\lambda} \cdot S + q(q-1) h^{(s,q)}(\widetilde{Y}, P).
\end{aligned}$$

It completes the proof of this proposition. □

6.D Duality with HARA Forward Utilities

In this section, we will investigate an important application of the MHD. Precisely, we will establish a duality between the MHD of order q and the HARA forward utilities.

Considering the HARA forward utilities, $U(t,x)$, having the power form as follows

$$U(t,x) = D(t)x^p, \quad x > 0. \tag{6.84}$$

where p is a constant, $p \in (-\infty, 0) \cup (0,1)$, and D is a process. The following theorem is our main result of this section.

Theorem 6.3: *Consider the quasi-left continuous model defined in Section 6.A and $q \in (-\infty, 0) \cup (0,1)$. Suppose $\mathcal{Y}_q(S) \neq \emptyset$ and (6.4) holds. Let $D(t)$ is a RCLL predictable process with finite variation. Then, the following assertions are equivalent.*

(1) The random field utility, $U(t,x)$, given by (6.84), is a forward utility with the optimal portfolio rate $\widehat{\theta}$.

(2)The minimal Hellinger deflator of order q, $\widetilde{Y}$, exists with the principal component $\widehat{\theta}/(q-1)$ such that

(2.a) $\widehat{Y} := \widetilde{Y}\mathcal{E}\left(\widehat{\theta} \cdot S\right)$ is a true martingale.

(2.b) The process D is given by

$$D(t) = D_0 \mathcal{E}\left(q(q-1)h^{(s,q)}(\widetilde{Y}, P)\right)^{1/(q-1)}. \tag{6.85}$$

Proof. We will start proving (2) $\Rightarrow$ (1). Suppose assertion (2) holds. By combining (6.85) and Lemma 6.2 , for the particular portfolio rate $\widehat{\theta}$, we have

$$\begin{aligned} U\left(\cdot, x\mathcal{E}\left(\widehat{\theta} \cdot S\right)\right) &= D_0 x^p \mathcal{E}\left(q(q-1)h^{(s,q)}\left(\widetilde{Y}, P\right)\right)^{1/(q-1)} \mathcal{E}\left(\widehat{\theta} \cdot S\right)^p \\ &= D_0 x^p \mathcal{E}\left(\widehat{\theta} \cdot S\right) \mathcal{E}\left(\widehat{\theta} \cdot S + q(q-1)h^{(s,q)}\left(\widetilde{Y}, P\right)\right)^{1/(q-1)} \\ &= D_0 x^p \mathcal{E}\left(\widehat{\theta} \cdot S\right) \widetilde{Y} \\ &= D_0 x^p \widehat{Y}, \end{aligned}$$

which is a true martingale due to assertion (2.a). This also allows us to define a new martingale measure $\widehat{Q} := \widehat{Y}_T \cdot P$. Then, for any admissible portfolio rate θ, we have

$$\begin{aligned}
& U\left(\cdot, x\mathcal{E}\left(\theta \cdot S\right)\right) \\
&= D_0 x^p \frac{\mathcal{E}\left(\theta \cdot S\right)^p}{\mathcal{E}\left(\widehat{\theta} \cdot S\right)^p} \mathcal{E}\left(q(q-1)h^{(s,q)}\left(\widetilde{Y}, P\right)\right)^{1/(q-1)} \mathcal{E}\left(\widehat{\theta} \cdot S\right)^{\frac{q}{q-1}} \qquad (6.86) \\
&= D_0 x^p \mathcal{E}\left(\widehat{\theta} \cdot S\right) \widetilde{Y} \left(\frac{\mathcal{E}\left(\theta \cdot S\right)}{\mathcal{E}\left(\widehat{\theta} \cdot S\right)}\right)^p = D_0 x^p \widehat{Y} \left(\frac{\mathcal{E}\left(\theta \cdot S\right)}{\mathcal{E}\left(\widehat{\theta} \cdot S\right)}\right)^p .
\end{aligned}$$

Since θ is admissible (see (2.10)) and $D_0 p > 0$, (6.86) implies

$$\sup_{\tau \in \mathcal{T}_T} E^{\widehat{Q}} \left(\frac{\mathcal{E}_\tau\left(\theta \cdot S\right)}{\mathcal{E}_\tau\left(\widehat{\theta} \cdot S\right)}\right)^p = -\frac{1}{D_0 x^p} \sup_{\tau \in \mathcal{T}_T} E\left[U\left(\tau, x\mathcal{E}_\tau\left(\theta \cdot S\right)\right)^-\right] < +\infty. \qquad (6.87)$$

Since $\widehat{Y}$ is a martingale density, we combine (6.87) with Proposition 4.4 and deduce that $U(\cdot, x\mathcal{E}(\theta \cdot S))$ is a supermartingale. This ends the proof of (2) $\Rightarrow$ (1).

In the rest of the proof, we focus on (1)$\Rightarrow$ (2). Suppose (1) holds. Since $D(t)$ is predictable with finite variation and never vanish, we can write it in the form of $D(t) = D_0 \exp(a_t^D)$. In virtue of Ito's formula, the forward property of $U(t,x) = D(t)x^p$ leads to

$$\frac{1}{q} a^D \geq L(\theta) \cdot A, \quad \forall \ \theta \in \mathcal{A}_{adm}, \qquad (6.88)$$

$$\text{and} \qquad \frac{1}{q} a^D = \max_{\theta \in \mathcal{L}} L(\theta) \cdot A = L(\widehat{\theta}) \cdot A. \qquad (6.89)$$

Here, for any $\lambda \in \mathcal{L}$, $L(\lambda)$ is given by

$$L(\lambda) = \frac{\lambda^T b}{1-q} - \frac{1}{2} \frac{\lambda^T c \lambda}{(1-q)^2} + \int \left(\frac{1 - (1 + \lambda^T x)^p}{q} + \frac{\lambda^T h(x)}{q-1}\right) F(dx).$$

Remark that $\widehat{\theta} \in \mathcal{L}$ since $\widehat{\theta}$ is admissible and $\widehat{\theta}$ is the maximizer of L over $\mathcal{L}$.

Thus, by putting

$$\widetilde{\beta} := \frac{1}{q-1}\widehat{\theta}, \quad \widetilde{f} := \left(1 + \widehat{\theta}^T x\right)^{1/(q-1)} - 1 \quad \text{and} \quad \widetilde{\alpha} := \widehat{\theta}^T G(\widetilde{\beta}, \widetilde{f}),$$

we follow the same argument carried out in (6.70)–(6.72) and deduce that the process

$$\widetilde{Y} := \mathcal{E}(\widetilde{\beta} \cdot S^c + \widetilde{f} \star (\mu - \nu) - \widetilde{\alpha} \cdot A) \tag{6.90}$$

is a deflator. Furthermore, thanks to (6.57), we have $f_q(\widetilde{f}) \star \nu \in \mathcal{A}^+_{loc}$. Hence, $\widetilde{Y} \in \mathcal{Y}^1_q(S)$.

In the following, we will prove that $\widetilde{Y}$ is the minimal Hellinger deflator of order q. To this end, we consider any deflator $Y \in \mathcal{Y}^1_q(S)$ (it is enough to consider $Y \in \mathcal{Y}^1_q(S)$ instead of $\mathcal{Y}_q(S)$ due to Proposition 6.3), given by

$$Y = \mathcal{E}(N - V), \quad N = \beta \cdot S^c + f \star (\mu - \nu), \quad V = \alpha \cdot A.$$

Thanks to (6.31), the Hellinger processes of order q, $h^{(s,q)}(Y, P)$ and $h^{(s,q)}(\widetilde{Y}, P)$, are given by

$$h^{(s,q)}(Y, P) = \frac{1}{2}\beta^T c \beta \cdot A + \int f_q(f) F(dx) \cdot A + \frac{\alpha}{1-q} \cdot A. \tag{6.91}$$

$$h^{(s,q)}(\widetilde{Y}, P) = \frac{1}{2}\widetilde{\beta}^T c \widetilde{\beta} \cdot A + \int f_q(\widetilde{f}(x)) F(dx) \cdot A + \frac{\widetilde{\alpha}}{1-q} \cdot A. \tag{6.92}$$

By plugging $\widetilde{\beta}$, $\widetilde{f}$ and $\widetilde{\alpha}$ into $h^{(s,q)}(\widetilde{Y}, P)$, we obtain

$$h^{(s,q)}(\widetilde{Y}, P) = L(\widehat{\theta}) \cdot A.$$

Moreover, an application of Lemma 6.8, we deduce that

$$L(\widehat{\theta}) \cdot A \preceq H(\beta, f) \cdot A = h^{(s,q)}(Y, P), \quad P - a.s., \quad \forall \ Y \in \mathcal{Y}^1_q(S).$$

This proves that $\widetilde{Y}$ is the MHD of order q. Furthermore, recalling (6.89), we have

$$D = D_0 \exp(a^D) = D_0 \mathcal{E}\left(q(q-1)h^{(s,q)}(\widetilde{Y}, P)\right)^{1/(q-1)}.$$

Finally, due to Proposition 6.2, we have

$$\begin{aligned}
\widetilde{Y}\mathcal{E}(\widehat{\theta}\cdot S) &= \mathcal{E}\left(\widehat{\theta}\cdot S + q(q-1)h^{(s,q)}(\widetilde{Y}, P)\right)^{1/(q-1)} \mathcal{E}(\widehat{\theta}\cdot S) \\
&= \mathcal{E}(\widehat{\theta}\cdot S)^{1/(q-1)} \mathcal{E}\left(q(q-1)h^{(s,q)}(\widetilde{Y}, P)\right)^{1/(q-1)} \mathcal{E}(\widehat{\theta}\cdot S) \\
&= \mathcal{E}(\widehat{\theta}\cdot S)^{p} \mathcal{E}(qh^{(s,q)}(\widetilde{Y}, P)) \\
&= \frac{1}{D_0 x^p} U(t, x\mathcal{E}_t(\widehat{\theta}\cdot S)).
\end{aligned}$$

Hence, $\widehat{Y} := \widetilde{Y}\mathcal{E}(\widehat{\theta}\cdot S)$ is a true martingale. This completes the proof of this theorem. □

Chapter 7

Horizon-Unbiased Hedging

In general, the optimal portfolio depends on the variation of horizon. The difficulty lies in how we can materialize this dependence and there is no single result in the literature addressing this issue. As explained before, the notion of forward utilities cancels out the effect of horizon on optimal portfolio. Here, in this chapter, we explore the case where the optimal portfolio is horizon-unbiased (independent of horizon) from a different perspective. In fact, we consider an agent with classical exponential utility. We want to characterize all payoff processes that this agent can hedge with an optimal portfolio which is horizon-unbiased. Basically, we are approaching this horizon-unbiased optimal portfolio from the view of contract theory.

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration is complete and right continuous. We consider a d-dimensional semimartingale stock price process, S. The set of σ-martingale measures with finite entropy is denoted by

$$\mathcal{M}_f^e(S) = \left\{ Q \in \mathbb{P}_e \mid \ S \in \mathcal{M}_\sigma(Q), \text{ and } \ E\left[\frac{dQ}{dP}\log\Big(\frac{dQ}{dP}\Big)\right] < +\infty \right\}. \quad (7.1)$$

Throughout this chapter, we assume that

$$S \text{ locally bounded and } \ \mathcal{M}_f^e(S) \neq \emptyset. \quad (7.2)$$

Remark that under assumption (7.2), $\mathcal{M}_\sigma(Q) = \mathcal{M}_{loc}(Q)$.
In this chapter, we are interested with the following exponential random field utility,

$$U(t,x) = -\exp(-x + B_t), \quad x \in \mathbb{R}, \quad t \in [0,T]. \tag{7.3}$$

For the process B and any stopping time τ, we denote by $Q^{(\tau,B)}$ the minimal entropy martingale measure for S^τ with respect to $P^{(\tau,B)}$, where $P^{(\tau,B)} := \frac{e^{B_\tau}}{E(e^{B_\tau})} \cdot P$. Then, the set of admissible portfolios that we will consider in this chapter is given by

$$\Theta(S,B) := \left\{ \theta \in L(S) |\ (\theta \cdot S)^\tau \in \mathcal{M}(Q^{(\tau,B)}), \quad \text{for all} \quad \tau \in \mathcal{T}_T \right\}, \tag{7.4}$$

where $\mathcal{T}_T$ denotes the set of all stopping times that are bounded from above by T. This definition of portfolio extends slightly the definition given by [24] to the case of a dynamic payoff B. For other sets of portfolios, we refer the reader to this seminal paper.
For the random field utility $U(t,x)$ defined in (7.3), we will focus on describing the process B such that the optimal hedging portfolio, $\widehat{\theta}$, is horizon-unbiased (independent of horizon). In other words, we will find B such that for any $\tau \in \mathcal{T}_T$, the following holds

$$\max_{\theta \in \Theta(S,B)} E\left[-\exp\Big(B_\tau - (\theta \cdot S)_\tau \Big) \right] = E\left[-\exp\Big(B_\tau - (\widehat{\theta} \cdot S)_\tau \Big) \right]. \tag{7.5}$$

As a consequence of our method, the optimal portfolio $\widehat{\theta}$ will be described as explicit as possible.

In what follows, we will solve this problem in two steps. First of all, a particular case when B is predictable with finite variation is investigated and the necessary and sufficient conditions for horizon-unbiased hedging are derived. Afterwards, the most general case when B is a semimartingale is considered and is fully solved. In both cases, the optimal horizon-unbiased hedging portfolio is presented explicitly.

For a random variable H, we denote by $\widetilde{Q}^{(H)}$ the minimal entropy martingale measure for S with respect to $P^{(H)} := e^H \left(E(e^H)\right)^{-1} \cdot P$. Moreover, Θ_1 denotes the set of portfolios considered in [24], and is given by

$$\Theta_1 := \left\{ \theta \in L(S) \quad \middle| \quad (\theta \cdot S) \in \mathcal{M}(\widetilde{Q}^{(H)}) \right\}. \tag{7.6}$$

Lemma 7.1: *Suppose that (7.2) holds. Let H be a random variable that is bounded from below and*

$$E\left[e^{pH}\right] < +\infty, \tag{7.7}$$

for some $p \in (1, +\infty)$, and $\widetilde{\theta} \in \Theta_1$.

Then, the following are equivalent:

(i) We have

$$1 - u_0 := \inf_{\theta \in \Theta_1} E\left(\exp\Big[H - (\theta \cdot S)_T\Big]\right) = E\left(\exp\left[H - (\widetilde{\theta} \cdot S)_T\right]\right).$$

(ii) For any stopping time $\sigma \le T$, we have

$$\begin{aligned} 1 - u_\sigma &:= \underset{\theta \in \Theta_1}{ess\,inf}\; E\left(\exp\Big[H - \int_\sigma^T \theta_u dS_u\Big]\Big|\mathcal{F}_\sigma\right) \\ &= E\left(\exp\Big[H - \int_\sigma^T \widetilde{\theta}_u dS_u\Big]\Big|\mathcal{F}_\sigma\right). \end{aligned}$$

Proof. Using the results in [24], we change the probability and work under Q instead of P, where

$$Q := \frac{\exp(H)}{E[\exp(H)]} \cdot P.$$

Suppose that assertion (i) holds, and put

$$J_t := \underset{Z \in Z^e_f(S,Q)}{ess\,inf}\; E^Q\left(\frac{Z_T}{Z_t} \log \frac{Z_T}{Z_t}\Big|\mathcal{F}_t\right), \tag{7.8}$$

where $\mathcal{Z}^e_f(S,Q)$ is given by

$$\mathcal{Z}^e_f(S,Q) := \Big\{ Z > 0 | Z \in \mathcal{M}_{loc}(Q), ZS \in \mathcal{M}_{loc}(Q) \text{ and } E^Q[Z_T \log(Z_T)] < +\infty \Big\}. \tag{7.9}$$

Due to the assumption (7.2), Proposition 3.1 in [44] implies the existence of $\underline{\xi}$ that belongs to the set

$$\Xi := \Big\{\xi > 0 \mid E(\xi) = 1, \quad E(\xi\eta) = 0, \quad \text{for any} \quad \eta := (\theta \cdot S)_T, \ \theta \in \Theta_1\Big\}$$

and $\underline{\xi}$ satisfies

$$J_0 = \min_{\xi \in \Xi} E^Q(\xi \log \xi) = E^Q(\underline{\xi} \log \underline{\xi}).$$

Furthermore, an application of Theorem 3.5 in [44] leads to

$$\underline{\xi} = \exp\Big(-\log\Big(E^Q e^{-(\widetilde{\theta}\cdot S)_T}\Big) - \widetilde{\theta} \cdot S_T\Big) \quad \text{and} \quad u_0 = 1 - e^{-J_0}. \tag{7.10}$$

It is clear that the set $\mathcal{Z}_f^e(S, Q)$ is stable under concatenation [1], and due to Proposition 4.1 in [44], we conclude that the optimizer of J_t is given by $Z_t^* := E^Q(\underline{\xi}|\mathcal{F}_t) \in \mathcal{Z}_f^e(S, Q)$. Then, by considering $P^* := Z_T^* \cdot Q$ and applying Bayes' rule, it is easy to derive the following two equalities

$$E^Q\left(\frac{Z_T^*}{Z_\sigma^*} e^{-J_\sigma} \log\left(\frac{Z_T^*}{Z_\sigma^*} e^{-J_\sigma}\right) \Big|\mathcal{F}_\sigma\right) = 0, \tag{7.11}$$

$$E^Q\left(\Big(\int_\sigma^T \theta_u dS_u\Big) \frac{Z_T^*}{Z_\sigma^*} e^{-J_\sigma} \Big|\mathcal{F}_\sigma\right) = 0. \tag{7.12}$$

Moreover, the first equation in (7.10) implies

$$J_t = E^Q\left(\frac{Z_T^*}{Z_t^*} \log \frac{Z_T^*}{Z_t^*} \Big|\mathcal{F}_t\right) = J_0 - \widetilde{\theta} \cdot S_t - \log Z_t^*.$$

Equivalently, for any stopping time $\sigma \le T$, we have

$$\frac{Z_T^*}{Z_\sigma^*} = \exp\left[-\int_\sigma^T \widetilde{\theta}_u dS_u + J_\sigma\right]. \tag{7.13}$$

And by taking conditional expectation, it is easy to see that

$$E^Q\left(1 - e^{-\int_\sigma^T \widetilde{\theta}_u dS_u} \Big|\mathcal{F}_\sigma\right) = 1 - e^{-J_\sigma} \tag{7.14}$$

[1] We refer for more details on this concept to [44]

Thanks to Young's inequality (i.e. $xy \leq e^x + y\log(y) - y$), for any $\theta \in \Theta_1$, we obtain that

$$\left(-\int_\sigma^T \theta_u dS_u\right)\left(\frac{Z_T^*}{Z_\sigma^*}e^{-J_\sigma}\right) \leq e^{-\int_\sigma^T \theta_u dS_u} + \frac{Z_T^*}{Z_\sigma^*}e^{-J_\sigma}\log\left(\frac{Z_T^*}{Z_\sigma^*}e^{-J_\sigma}\right) - \frac{Z_T^*}{Z_\sigma^*}e^{-J_\sigma}.$$

Therefore, by taking conditional expectation above on both sides, and using (7.11), (7.12) and (7.13), we derive

$$E^Q\left(1 - e^{-\int_\sigma^T \theta_u dS_u}\middle|\mathcal{F}_\sigma\right) \leq 1 - e^{-J_\sigma}. \tag{7.15}$$

By combining (7.14) and (7.15), assertion (ii) follows immediately. The converse is immediate by putting $\sigma = 0$. This ends the proof of the lemma. □

In the following, we start addressing the horizon-unbiased hedging problem for the case when the payoff process B is predictable with finite variation. The next proposition characterizes the optimal portfolio $\widehat{\theta}$ being the solution of an equation.

Proposition 7.1: *Suppose that (7.2) is satisfied and let B be a bounded predictable process with finite variation. Then, if there exists $\widehat{\theta} \in \Theta(S, B)$ such that for any stopping time τ,*

$$\min_{\theta\in\Theta(S,B)} E\left[\exp\Big(B_\tau - (\theta\cdot S)_\tau\Big)\right] = E\left[\exp\Big(B_\tau - (\widehat{\theta}\cdot S)_\tau\Big)\right], \tag{7.16}$$

then, $-\widehat{\theta}$ is a pointwise root of

$$0 = \begin{cases} b + c\theta + \displaystyle\int x(e^{\theta^T x} - 1)F(dx), & \text{on } \{\Delta A = 0\}; \\ \displaystyle\int x e^{\theta^T x} F(dx), & \text{on } \{\Delta A \neq 0\}. \end{cases} \tag{7.17}$$

Proof. Notice–see Lemma 7.1 for details–that (7.16) is equivalent to the fact that for any stopping times $T \geq \tau \geq \sigma$ and any $\theta \in \Theta(S,B)$, we have

$$E\left(\exp\Big[B_\tau - \int_\sigma^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_\sigma\right) \geq E\left(\exp\Big[B_\tau - \int_\sigma^\tau \widehat{\theta}_u dS_u\Big]\Big|\mathcal{F}_\sigma\right), \quad P-a.s. \tag{7.18}$$

Put $X_t^\theta := \exp\Big[B_t - (\theta\cdot S)_t\Big]$, and for any $\theta \in \Theta(S,B)$ consider a stationarily increasing sequence of stopping times $(T_n)_{n\geq 1}$ such that $X^{\widehat{\theta}}_{t\wedge T_n}$ and $X^{\theta}_{t\wedge T_n}$ are both special semimartingales with integrable martingale and predictable parts, and their left limit processes are bounded from below by $1/n$. Then for each T_n, (7.18) implies that for any nonnegative left continuous and bounded process H, and any subdivision $\rho := (\tau_i)_{0\leq i\leq m+1}$ with $\tau_0 = 0$ and $\tau_{m+1} = T_n$, which is composed by a finite and increasing sequence of stopping times, we have

$$\begin{aligned}&\sum_{i=0}^m H_{\tau_i} E(\exp[B_{\tau_{i+1}} - \int_{\tau_i}^{\tau_{i+1}} \theta_u dS_u]|\ \mathcal{F}_{\tau_i})\\ \geq &\sum_{i=0}^m H_{\tau_i} E(\exp[B_{\tau_{i+1}} - \int_{\tau_i}^{\tau_{i+1}} \widehat{\theta}_u dS_u]\mid \mathcal{F}_{\tau_i}).\end{aligned} \tag{7.19}$$

Due to (7.19), it is easy to deduce that:

$$E\left(\sum_{i=0}^m \frac{H_{\tau_i}}{X^\theta_{\tau_i}}(X^\theta_{\tau_{i+1}} - X^\theta_{\tau_i})\right) \geq E\left(\sum_{i=0}^m \frac{H_{\tau_i}}{X^{\widehat{\theta}}_{\tau_i}}(X^{\widehat{\theta}}_{\tau_{i+1}} - X^{\widehat{\theta}}_{\tau_i})\right) \tag{7.20}$$

Let $K^m(\theta) := \sum_{i=0}^m \frac{H_{\tau_i}}{X^\theta_{\tau_i}} I_{]\!]\tau_i,\tau_{i+1}]\!]}$. Then, the sums in (7.20) can be written in the form of integrals and consequently (7.20) can be written as follows

$$E\left(\int_0^{T_n} K^m_u(\theta) dX^\theta_u\right) \geq E\left(\int_0^{T_n} K^m_u(\widehat{\theta}) dX^{\widehat{\theta}}_u\right) \tag{7.21}$$

Due to the arbitrariness of the subdivision ρ, we let $|\rho|\to 0$.[2] Then, $K^m(\theta) \to \frac{H}{X^\theta_-}$ and (7.21) becomes

$$E\Big(\int_0^{T_n} \frac{H_u}{X^\theta_{u-}} dX^\theta_u\Big) \geq E\Big(\int_0^{T_n} \frac{H_u}{X^{\widehat{\theta}}_{u-}} dX^{\widehat{\theta}}_u\Big). \tag{7.22}$$

[2]Here, we put $|\rho| := \sup_{0\leq i\leq m}(\tau_{i+1} - \tau_i)$

An application of Ito's formula to $(X^{\theta})^{T_n}$ and $(X^{\widehat{\theta}})^{T_n}$, and recalling the boundedness of H, $(X^{\theta}_{-})^{T_n}$ and $(X^{\widehat{\theta}}_{-})^{T_n}$, we have

$$E\Big[\int_0^{T_n} H_u dA^{\theta}_u\Big] = E\Big[\int_0^{T_n} \frac{H_u}{X^{\theta}_{u-}} dX^{\theta}_u\Big], \text{ and } E\Big[\int_0^{T_n} \frac{H_u}{X^{\widehat{\theta}}_{u-}} dX^{\widehat{\theta}}_u\Big] = E\Big[\int_0^{T_n} H_u dA^{\widehat{\theta}}_u\Big], \tag{7.23}$$

where A^{θ} is a predictable process with finite variation given by

$$\begin{aligned} A^{\theta} := B - \theta^T b \cdot A + \frac{1}{2}\theta^T c\theta \cdot A + \sum (e^{\Delta B} - 1 - \Delta B)(1-a) + \\ + \left(e^{\Delta B - \theta^T x} - 1 - \Delta B + \theta^T x\right) \star \nu. \end{aligned} \tag{7.24}$$

Since the process H is arbitrary, we let $n \to +\infty$ in (7.23) and deduce that

$$A^{\widehat{\theta}} \preceq A^{\theta}, \qquad \text{for any } \theta \in \Theta(S, B).$$

Or, equivalently, for any $\theta \in \Theta(S, B)$,

$$\overline{f}(\theta) \geq \overline{f}(\widehat{\theta}), \quad \text{where } \overline{f}(\lambda) := -\lambda^T b + \frac{1}{2}\lambda^T c\lambda + \int (e^{\Delta B - \lambda^T x} - 1 - \Delta B + \lambda^T x) F(dx).$$

We easily deduce that, on the set $\{\Delta A = 0\}$, the function $\overline{f}(\theta)$ coincides with $\overline{K}(-\theta)$ of Lemma 5.2 (note that, in our current situation, the truncation function can be taken $h(x) = x$ due to the local boundedness of S), where

$$\overline{K}(\lambda) := b^T\lambda + \frac{1}{2}\lambda^T c\lambda + \int \left(e^{\lambda^T x} - 1 - \lambda^T x\right) F(dx), \quad \lambda \in \mathbb{R}^d.$$

Hence, we deduce that $-\widehat{\theta}$ is a root of the first equation in (7.17). On the set $\{\Delta A \neq 0\}$, we obtain

$$\begin{aligned} \overline{f}_t(\theta_t)\Delta A_t &= e^{\Delta B_t} \int e^{-\theta_t^T x} \nu(\{t\}, dx) - (1 + \Delta B_t) a_t \\ &= e^{\Delta B_t}(\overline{K}(-\theta_t)\Delta A_t + a_t) - (1 + \Delta B_t) a_t. \end{aligned}$$

Hence $-\widehat{\theta}$ is a root of the second equation in (7.17). This completes the proof of the proposition. □

Our main result is given in the following theorem.

Theorem 7.1: *Suppose that (7.2) is satisfied and let B be a bounded predictable process with finite variation. Then, the following are equivalent:*
(i) There exists $\widehat{\theta} \in \Theta(S,B)$ such that for any stopping time τ,

$$\min_{\theta\in\Theta(S,B)} E\left[\exp\Big(B_\tau - (\theta\cdot S)_\tau\Big)\right] = E\left[\exp\Big(B_\tau - (\widehat{\theta}\cdot S)_\tau\Big)\right]. \tag{7.25}$$

(ii) For any $\theta \in \Theta(S,B)$

$$I_{\{(\theta\cdot S)_- \neq 0\}} \cdot B = I_{\{(\theta\cdot S)_- \neq 0\}} \cdot h^E(\widetilde{Z}, P), \tag{7.26}$$

where $\widetilde{Z} = \exp\left(\widetilde{\theta}\cdot S + h^E(\widetilde{Z},P)\right)$ is minimal entropy-Hellinger local martingale density.
Furthermore, the optimal portfolio $\widehat{\theta}$ coincides with $-\widetilde{\theta}$ obtained explicitly from $\widetilde{Z}$, i.e. $-\widehat{\theta}$ is a pointwise root of

$$0 = \begin{cases} b + c\theta + \displaystyle\int x(e^{\theta^T x} - 1)F(dx), & \text{on } \{\Delta A = 0\}; \\ \displaystyle\int x e^{\theta^T x} F(dx), & \text{on } \{\Delta A \neq 0\}. \end{cases} \tag{7.27}$$

Proof. In Proposition 7.1, we have proved that the optimal portfolio $\widehat{\theta}$ in (7.25)–when it exists–can be derived from (7.27). Therefore, in the remaining part, we focus on the equivalence between assertions (i) and (ii). First, we assume that assertion (i) holds, and put

$$\Theta_b := \{\theta \in L(S) \mid (\theta\cdot S)_t(\omega) \quad \text{is uniformly bounded in t and } \omega\}. \tag{7.28}$$

Then, for any $\theta \in \Theta_b$, both $(\theta\cdot S)^\tau$ and $(\widehat{\theta}\cdot S)^\tau$ are true $Q^{(\tau,B)}$-martingales, where $Q^{(\tau,B)}$ is given by

$$Q^{(\tau,B)} := \frac{\exp[B_\tau - (\widehat{\theta}\cdot S)_\tau]}{E\exp[B_\tau - (\widehat{\theta}\cdot S)_\tau]}\cdot P.$$

Therefore, for any $\theta \in \Theta_b$ and any stopping time τ, we obtain

$$E\Big[(\theta \cdot S)_\tau \exp\Big(B_\tau - (\widehat{\theta} \cdot S)_\tau\Big)\Big] = E\Big[(\widehat{\theta} \cdot S)_\tau \exp\Big(B_\tau - (\widehat{\theta} \cdot S)_\tau\Big)\Big] = 0.$$

Hence, the process $(\theta \cdot S)\exp(B - (\widehat{\theta} \cdot S)) = e^{B - h^E(\widetilde{Z},P)}\widetilde{Z}(\theta \cdot S)$ is a local martingale. Then, a direct application of Ito's formula leads to (7.26), and assertion (ii) follows.

Now, suppose that assertion (ii) holds. Thanks to a direct application of Ito's formula, assertion (ii) is equivalent to the statement that, for any $\theta \in \Theta(S, B)$, the process $Y^\theta := \exp\left[B + \widetilde{\theta} \cdot S\right](\theta \cdot S)$ is a local martingale. Indeed, this equivalence follows immediately from the fact that

$$\exp\Big[B + \widetilde{\theta} \cdot S\Big](\theta \cdot S) = \widetilde{Z}\exp\Big[B - h^E(\widetilde{Z}, P)\Big](\theta \cdot S).$$

Let $\theta \in \Theta_b$, and $(T_n)_{n\geq 1}$ be a sequence of stopping times that increases stationarily to T and $Y^{\widetilde{\theta}}_{t\wedge T_n}$ and $Y^{\theta}_{t\wedge T_n}$ are true martingales. Then, for any stopping time τ, we put $\tau_n := \tau \wedge T_n$ and obtain

$$E[e^{B_{\tau_n} - (\theta \cdot S)_{\tau_n}}] - E[e^{B_{\tau_n} + (\widetilde{\theta} \cdot S)_{\tau_n}}] \geq E\left\{((-\theta - \widetilde{\theta}) \cdot S)_{\tau_n} e^{B_{\tau_n} + (\widetilde{\theta} \cdot S)_{\tau_n}}\right\} = 0$$

Thanks to Fatou's lemma and the boundedness of $\exp[B + (\theta \cdot S)]$ for any $\theta \in \Theta_b$, we get

$$E\left(\exp\Big[B_\tau - (\theta \cdot S)_\tau\Big]\right) \geq E\left(\exp\Big[B_\tau + (\widetilde{\theta} \cdot S)_\tau\Big]\right). \tag{7.29}$$

Due to Theorem 2.1–(c) in [44], there exists a sequence of $(\theta^n)_{n\geq 1}$, $\theta^n \in \Theta_b$, such that

$$\lim_{n\to+\infty} E\left(\exp\Big[B_\tau - (\theta^n \cdot S)_\tau\Big]\right) = \inf_{\theta \in \Theta(S,B)} E\left(\exp\Big[B_\tau - (\theta \cdot S)_\tau\Big]\right) \tag{7.30}$$

By combining (7.29) and (7.30), and putting $\widehat{\theta} := -\widetilde{\theta}$, the proof of the assertion

(i) is complete. □

Remark: Theorem 7.1 determines explicitly the optimal portfolio in the horizon-unbiased exponential hedging when it exists. Furthermore, the theorem clearly illustrates the relationship between horizon-unbiased hedging and a forward utility. In fact, we can easily conclude that, in general, the horizon-unbiased problem in (7.25) admits a solution while the corresponding random field utility , $U(t, x) = -\exp(-x + B_t)$, may not be a forward utility. A simple example is when S is constant in a neighborhood of zero (i.e $S_t = S_0$ for t close to zero), and B is neither increasing nor constant process. Furthermore, the equivalence between the existence of solution to the horizon-unbiased hedging problem and the property that $-\exp(-x + B)$ is a forward utility only holds only if there exists a portfolio θ such that

$$\{(\omega, t) | (\theta \cdot S)_{t-}(\omega) \neq 0\} = \Omega \times]0, T].$$

In general, this equality does not hold. In fact if S is constant in a neighborhood of zero (i.e $S_t = S_0$ for t close to zero), then this equality is violated. Hence, in this case, the two concepts of forward utility and horizon-unbiased utility differ.

Also, it is easy to see that the horizon-unbiased hedging problem admits a solution and its value function $v(\tau) := \min_{\theta \in \Theta(S,B)} E\left[\exp\Big(B_\tau - (\theta \cdot S)_\tau\Big)\right]$ is constant, i.e. $v(\tau) = v(T)$, if and only if $-\exp(B_t - x)$ is a forward dynamic utility.

Now, we turn to the general case, where B is semimartingale satisfying the following condition:

$$\sup_{\tau \in \mathcal{T}_T} E\Big[e^{pB_\tau}\Big] < +\infty \quad \text{for some} \quad p \in (1, +\infty). \tag{7.31}$$

Our main result is formulated in the following theorem.

Theorem 7.2: *Suppose that (7.2) holds and consider a semimartingale, B,* *satisfying (7.31) with the Doob-Meyer multiplicative decomposition given by*

$$e^B = e^{B_0} Z^{(B)} e^{B'}. \tag{7.32}$$

Here $Z^{(B)}$ *is a positive local martingale* ($Z_0^{(B)} = 1$) *and* B' *is a predictable process with finite variation* ($B'_0 = 0$).

Then, there is equivalence between assertions (i) and (ii):

(i) There exists $\widehat{\theta} \in \Theta(S, B)$ *such that for any stopping time* τ,

$$\min_{\theta \in \Theta(S,B)} E\Big[\exp(B_\tau - (\theta \cdot S)_\tau)\Big] = E\Big[\exp(B_\tau - (\widehat{\theta} \cdot S)_\tau)\Big]. \tag{7.33}$$

(ii) The MEH local martingale density with respect to $Z^{(B)}$, *denoted by* $\widetilde{Z}^{(B)}$, *exists and satisfies*

$$I_{\{(\theta \cdot S)_- \neq 0\}} \cdot B' = I_{\{(\theta \cdot S)_- \neq 0\}} \cdot h^E\left(\widetilde{Z}^{(B)}, Z^{(B)}\right), \tag{7.34}$$

for any $\theta \in \Theta(S, B)$.

Furthermore, the optimal portfolio in (7.33) is given by

$$\log(\widetilde{Z}^{(B)}) = -\widehat{\theta} \cdot S + h^E\left(\widetilde{Z}^{(B)}, Z^{(B)}\right). \tag{7.35}$$

Proof. Consider a sequence of stopping times, $(T_n)_{n\geq 1}$, that increases stationarily to T and such that $(B')^{T_n}$ is bounded and $(Z^{(B)})^{T_n}$ is a true martingale. Then, by putting $Q_n := Z_{T_n}^{(B)} \cdot P$, assertion (i) implies that the horizon-unbiased hedging problem for $(S^{T_n}, (B')^{T_n}, Q_n)$ has a solution. Thus, a direct application of Theorem 7.1—to the model $(S^{T_n}, (B')^{T_n}, Q_n)$—implies that (7.34) holds for any $\theta \in \Theta(S, B)$, and the optimal portfolio $\widehat{\theta}$ in (7.33) coincides with $-\widetilde{\theta}$, where $\widetilde{\theta}$ is the integrand that appears in the expression of $\widetilde{Z}^{(B)}$. This proves assertion (ii).

Next, assume that assertion (ii) holds, and notice that this assertion is equivalent to the statement that $\exp(B - \widehat{\theta} \cdot S)(\theta \cdot S)$ is a local martingale for any $\theta \in \Theta(S, B)$. Let $\theta \in \Theta_b$ and $(T_n)_{n\geq 1}$ be a stationarily increasing sequence of stopping times such that $((\widehat{\theta}-\theta)\cdot S)^{T_n}\exp\left(B^{T_n}-\left(\widehat{\theta}\cdot S\right)^{T_n}\right)$ is a true martingale.

Then, for any stopping time τ, we derive

$$\begin{aligned} 0 &= E\left[\exp\Big(B_{\tau\wedge T_n}-(\widehat{\theta}\cdot S)_{\tau\wedge T_n}\Big)\Big((\widehat{\theta}-\theta)\cdot S\Big)_{\tau\wedge T_n}\right] \\ &\leq E\left[\exp\Big(B_{\tau\wedge T_n}-(\theta\cdot S)_{\tau\wedge T_n}\Big)\right]-E\left[\exp\left(B_{\tau\wedge T_n}-(\widehat{\theta}\cdot S)_{\tau\wedge T_n}\right)\right]. \end{aligned} \tag{7.36}$$

Thus, due to Fatou's lemma we get

$$\begin{aligned} E\left[\exp\Big(B_\tau-(\widehat{\theta}\cdot S)_\tau\Big)\right] &\leq \liminf_{n\to+\infty} E\left[\exp\Big(B_{\tau\wedge T_n}-(\theta\cdot S)_{\tau\wedge T_n}\Big)\right] \\ &= E\left[\exp\Big(B_\tau-(\theta\cdot S)_\tau\Big)\right]. \end{aligned}$$

The last equality above follows from the fact that $\{\exp(B_\tau-(\theta\cdot S)_\tau),\ \ \tau\in\mathcal{T}_T\}$ is uniformly integrable. Indeed, this fact follows from

$$\int_{\{B_\tau>c\}} e^{B_\tau}dP \leq e^{-(p-1)c}E\left[e^{pB_\tau}\right] \leq e^{-(p-1)c}\sup_{\tau\in\mathcal{T}_T} E\left[e^{pB_\tau}\right],$$

and for any $\theta\in\Theta_b$,

$$e^{B_\tau-(\theta\cdot S)_\tau} \leq e^{B_\tau}\exp\left[\|\sup_{t\in[0,T]}|(\theta\cdot S)_t|\|_\infty\right].$$

Hence, again due to Theorem 2.1–(c) in [44], we obtain that

$$\begin{aligned} E\left[\exp\left(B_\tau-(\widehat{\theta}\cdot S)_\tau\right)\right] &= \min_{\theta\in\Theta(S,B)} E\left[\exp\left(B_\tau-(\theta\cdot S)_\tau\right)\right] \\ &= \inf_{\theta\in\Theta_b} E\left[\exp\left(B_\tau-(\theta\cdot S)_\tau\right)\right]. \end{aligned}$$

This proves assertion (i), and the proof of the theorem is complete. □

Chapter 8

Optimal Investment Timing and Optimal Portfolio

In this chapter, we focus on the problem where the investor is seeking the optimal portfolio from her investment in stocks and the optimal time to liquidate her assets (tradable or not). In previous chapters, we have introduced and developed two powerful approaches when dealing with this problem (i.e. forward utilities and horizon-unbiased hedging). In fact, these two methods solve this problem partially. This chapter will exhibit direct approaches to solve this problem in two contexts. Precisely, we will use the martingale theory for general market model with exponential utility, and control theory when the market model is Markovian.

The martingale approach is detailed in Section 8.A. Therein, we focus on the exponential random utility and look for the explicit description of the optimal portfolio and the optimal stopping time via studying the value process. The second approach is investigated in Section 8.B and deals with the Markovian framework. Therein, the value function is proved to be the unique viscosity solution of the variational inequalities.

8.A Exponential Hedging with Variable Horizon

Consider a filtered probability space denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ where the filtration is complete and right continuous. In this setup, we consider a d-dimensional semimartingale $S = (S_t)_{0\leq t\leq T}$ which represents the discounted price processes of d risky assets. The set of σ-martingale measures with finite entropy is given by

$$\mathcal{M}^e_f(S) = \left\{ Q \in \mathbb{P}_e \mid \ S \in \mathcal{M}_\sigma(Q), \text{ and } \ E\left[\frac{dQ}{dP}\log\Bigl(\frac{dQ}{dP}\Bigr)\right] < +\infty \right\}. \quad (8.1)$$

Throughout this section, we assume the following:

$$\mathcal{M}^e_f(S) \neq \emptyset, \quad \text{and} \ \forall\, \lambda \in \mathbb{R}^d, \int_{\{|x|>1\}} |x| e^{\lambda^T x} F(dx) < +\infty, \quad P\otimes A{-}a.e. \quad (8.2)$$

Consider the following exponential random field utility,

$$U(t,x) = -\exp(-x + B_t), \quad x \in \mathbb{R}, \quad t \in [0,T], \quad (8.3)$$

where the process B is a RCLL semimartingale and the set of admissible portfolios is given by

$$\Theta := \left\{ \theta \in L(S) \ \middle| \ \sup_{\tau\in\mathcal{T}_T} E\left(\exp[B_\tau - (\theta\cdot S)_\tau]\right) < +\infty \right\}. \quad (8.4)$$

Occasionally in our analysis, there are integrability conditions imposed on B, such as

$$\sup_{\tau\in\mathcal{T}_T} E\Big[\exp(B_\tau)\Big] < +\infty. \quad (8.5)$$

$$\text{and} \qquad \sup_{\tau\in\mathcal{T}_T} E\Big(e^{pB_\tau}\Big) < +\infty \quad \text{for some } \ p \in (1,+\infty). \quad (8.6)$$

This section is devoted to the following optimization problem. Throughout this thesis, we often call it the optimal sale problem.

Problem 8.1: Find the pair $(\theta^*, \tau^*) \in \Theta \times \mathcal{T}_T$ such that

$$\min_{\theta\in\Theta,\tau\in\mathcal{T}_T} E\left(\exp\Big[B_\tau - (\theta\cdot S)_\tau\Big]\right) = E\left(\exp\Big[B_{\tau^*} - (\theta^*\cdot S)_{\tau^*}\Big]\right). \qquad (8.7)$$

Remark that the solution of this problem has two components. Namely, the optimal portfolio θ^* and optimal investment timing τ^*. Our contribution lies in describing—as explicitly as possible—the optimal solution (θ^*, τ^*) to Problem 8.1. The description is essentially based on the characterization of the optimal value process via a dynamic programming equation. This will constitute our first result in this section and is given by Theorem 8.1. The latter is based on the following lemma.

Lemma 8.1: *Suppose that the payoff process, B, satisfies (8.5). Then, for any $\theta \in \Theta$, the process*

$$L_t(\theta) := V(t)\exp\left(-\int_0^t \theta_u dS_u\right), \quad 0 \le t \le T, \qquad (8.8)$$

is a supermartingale. Here, V is the value process, given by

$$V(t) := \operatorname*{ess\,sup}_{\theta\in\Theta,\ \tau\ge t} E\left(-\exp\Big[B_\tau - \int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right), \qquad 0 \le t \le T. \quad (8.9)$$

Proof. For any $\theta \in \Theta$ and any stopping time $\tau \ge t$, we put

$$j_t(\theta,\tau) := E\left(-e^{-\int_t^\tau \theta_u dS_u + B_\tau}\Big|\mathcal{F}_t\right)$$

and $$J_t(\theta) := \operatorname*{ess\,sup}_{\tau\ge t}\ E\left(-e^{-\int_t^\tau \theta_u dS_u + B_\tau}\Big|\mathcal{F}_t\right).$$

Notice that the process $J(\theta)\exp(-\theta\cdot S)$ is the Snell envelope of $-\exp(-\theta\cdot S + B)$ and, hence, it is a RCLL supermartingale (see [56], and [26]). Furthermore, we have for any $t \in [0,T]$, $\tau \ge t$, and any $\theta \in \Theta$,

$$V(t) \ge J_t(\theta) \ge j_t(\theta,\tau) \quad \text{and} \quad j_t(\theta,\tau) = j_t(\theta I_{]\!]t,T]\!]}, \tau).$$

Consider $t \geq s \geq 0$, $\tau \geq t$, and $\theta,\ \overline{\theta} \in \Theta$ such that $\theta I_{[\![s,t]\!]} = \overline{\theta} I_{[\![s,t]\!]}$. Then, due to the above facts, we derive

$$\begin{aligned} V(s) \geq J_s(\overline{\theta}) \geq E\left(J_t(\overline{\theta}) e^{-\int_s^t \overline{\theta}_u dS_u} \Big| \mathcal{F}_s \right) &= E\left(J_t(\overline{\theta}) e^{-\int_s^t \theta_u dS_u} \Big| \mathcal{F}_s \right) \\ &\geq E\left(j_t(\overline{\theta}, \tau) e^{-\int_s^t \theta_u dS_u} \Big| \mathcal{F}_s \right). \end{aligned} \tag{8.10}$$

Remark that for two pairs (θ_1, τ_1) and (θ_2, τ_2), there exists (θ_3, τ_3) such that

$$\max(j_t(\theta_1, \tau_1); j_t(\theta_2, \tau_2)) = j_t(\theta_3, \tau_3).$$

In fact, it is enough to consider

$$\theta_3 := \theta_1 I_{\{j_t(\theta_1,\tau_1) \geq j_t(\theta_2,\tau_2)\} \otimes]\!]t,T]\!]} + \theta_2 I_{\{j_t(\theta_1,\tau_1) < j_t(\theta_2,\tau_2)\} \otimes]\!]t,T]\!]},$$

which is predictable and belongs to Θ, and

$$\tau_3 = \begin{cases} \tau_1, & \text{on } \{j_t(\theta_1, \tau_1) \geq j_t(\theta_2, \tau_2)\}; \\ \tau_2, & \text{otherwise,} \end{cases}$$

which is a stopping time satisfying $\tau_3 \geq t$. Then—an application of Zorn's lemma leads to—for any t there exists a sequence of pairs (θ_n, τ_n) such that $j_t(\theta_n, \tau_n)$ increases to $V(t)$. By combining this fact with (8.10), we obtain

$$V(s) \geq E\left(j_t(\theta_n, \tau_n) e^{-\int_s^t \theta_u dS_u} \Big| \mathcal{F}_s \right).$$

Thus, due to the monotone convergence theorem, we deduce that

$$V(s) \geq E\left(V(t) e^{-\int_s^t \theta_u dS_u} \Big| \mathcal{F}_s \right).$$

This ends the proof of the lemma. □

Theorem 8.1: *Suppose that (8.5) holds. Then, the following assertions hold:*

(i) V admits a RCLL modification and satisfies the following dynamic pro-

gramming equation

$$V(t) = \max\left[-e^{B_t};\ \operatorname*{ess\,sup}_{\theta\in\Theta,\tau>t} E\left(V(\tau)\exp\Big[-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right)\right]. \tag{8.11}$$

(ii) If B is bounded from below and assumption (8.2) holds, then V is a RCLL negative semimartingale that has the following decomposition

$$V(t) = V(0)\mathcal{E}_t(M^V)e^{A_t^V} \tag{8.12}$$

$$\begin{aligned} \textit{where} \qquad M^V &= \beta\cdot S^c + W\star(\mu-\nu) + g\star\mu + \overline{M^V} \\ \textit{and} \qquad W_t(x) &:= f_t(x) + \frac{\widehat{f}_t}{1-a_t} I_{\{a_t<1\}}. \end{aligned} \tag{8.13}$$

Here, A^V is a predictable process with finite variation, M^V is a local martingale, and $(\beta, f, g, \overline{M^V})$ are its Jacod components.

Proof. (i) It is clear from Lemma 8.1, that the process $L(\theta)$ is a supermartingale for any $\theta\in\Theta$. Then, thanks to Theorem 2 in [26] (page 73), we deduce that the process $L(\theta)$ admits right and left limits along the rationals and, the process $L_{t+}(\theta)$ is a RCLL supermartingale with respect to the filtration $\mathcal{F}_{t+} = \mathcal{F}_t$. These imply that both processes $V(t+)$ and $V(t-)$ exist, and moreover that

$$V(t) \geq V(t+), \quad P-a.s. \tag{8.14}$$

On the other hand, since $L_{t+}(\theta) = V(t+)\exp\left[-(\theta\cdot S)_t\right]$ is a RCLL supermartingale and $V(t+)\geq -e^{B_t}$, then an application of the optional sampling theorem for supermartingales leads to the inequalities

$$V(t+) \geq E\left(V(\tau+)\exp(-\int_t^\tau \theta_u dS_u)\Big|\mathcal{F}_t\right) \geq E\left(-\exp\Big[-\int_t^\tau \theta_u dS_u + B_\tau\Big]\Big|\mathcal{F}_t\right).$$

Then, by taking the essential sup, we obtain that

$$V(t+) \geq V(t).$$

Thus, a combination of this with (8.14), the right continuity of V follows immediately. This proves that the process V admits a modification that is RCLL. We will consider this modification throughout the rest of this chapter. As a result, the two processes $L(\theta)$ and $(\theta\cdot S)$ are RCLL semimartingales for any $\theta\in\Theta$. Therefore, an application of the optional sampling theorem implies

$$V(t)\geq \operatorname*{ess\,sup}_{\theta\in\Theta,\tau>t}\ E\left(V(\tau)\exp\Big[-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right).$$

Combining the above with $V(t)\geq -e^{B_t}$, we conclude that

$$V(t)\geq \max\left[-e^{B_t};\ \operatorname*{ess\,sup}_{\theta\in\Theta,\tau>t}\ E\left(V(\tau)\exp\Big[-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right)\right].$$

To prove the reverse inequality, we write

$$\begin{aligned}
V(t) &= \operatorname*{ess\,sup}_{\theta\in\Theta,\tau\geq t} E\left(-\exp\Big[B_\tau-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right)\\
&= \max\left[-e^{B_t};\operatorname*{ess\,sup}_{\theta\in\Theta,\tau>t} E\left(-\exp\Big[B_\tau-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right)\right] \qquad (8.15)\\
&\leq \max\left[-e^{B_t};\operatorname*{ess\,sup}_{\theta\in\Theta,\tau>t} E\left(V(\tau)\exp\Big[-\int_t^\tau \theta_u dS_u\Big]\Big|\mathcal{F}_t\right)\right].
\end{aligned}$$

This ends the proof of assertion (i).

(ii) Suppose that B is bounded from below by a constant $-C$. Then, we derive

$$\begin{aligned}
-V(t) &= \operatorname*{ess\,inf}_{\theta\in\Theta,\tau\in\mathcal{T}_T}\ E\left(\exp\left(B_\tau-\int_t^\tau \theta_u dS_u\right)\Big|\mathcal{F}_\tau\right)\\
&\geq e^{-C}\operatorname*{ess\,inf}_{\theta\in\Theta,\tau\in\mathcal{T}_T}\ E\left(\exp(-\int_t^\tau \theta_u dS_u)\Big|\mathcal{F}_\tau\right)\\
&= e^{-C}\operatorname*{ess\,inf}_{\theta\in\Theta}\ E\left(\exp(-\int_t^T \theta_u dS_u)\Big|\mathcal{F}_t\right)\\
&= e^{-C}\exp\left(-\operatorname*{ess\,inf}_{Z\in\mathcal{Z}^e_f(S)}\ E\Big(\frac{Z_T}{Z_t}\log\Big(\frac{Z_T}{Z_t}\Big)\Big|\mathcal{F}_t\Big)\right).
\end{aligned}$$

Here $\mathcal{Z}^e_f(S)$ denotes the set of martingale densities, Z, such that $Z\log(Z)$ is an integrable submartingale. Due to the assumption that $\mathcal{M}_f^e(S)$ is not empty, or equivalently $\mathcal{Z}_f^e(S) \neq \emptyset$, we have

$$\operatorname*{ess\,inf}_{Z\in\mathcal{Z}_f^e(S)} E\left(\frac{Z_T}{Z_t}\log\Big(\frac{Z_T}{Z_t}\Big)\Big|\mathcal{F}_t\right) < +\infty, \quad P-a.s.$$

Hence, this together with the right continuity of V prove that the process V is a negative supermartingale (take $\theta = 0$ in Lemma 8.1) or, equivalently, $\frac{V}{V(0)}$ is a positive exponential local submartingale. This leads to the existence of a local martingale M^V and a predictable process, A^V, with finite variation such that $V = V(0)\mathcal{E}(M^V)e^{A^V}$. These facts follow from the Doob-Meyer decomposition and the fact that $\frac{1}{V_-}\cdot V$ is a local submartingale. The decomposition for the local martingale M^V follows from Jacod Theorem; see Theorem 2.2. This completes the proof of the theorem. □

Remark: The equation (8.11) describes the optimal cost process/optimal value process. This description resembles the dynamic maximum principle, which will lead, in the Markovian case, to a HJB equation. In a model driven by Brownian motions, this HJB equation can be solved explicitly such as the case in [35]. The derivation of these HJB in a more general case than the Brownian as well as their investigations, and their relationship to backward stochastic differential equations (BSDEs) are not the scope of this thesis and are left to future research.

Once the process V is determined, then the optimal investment timing and the optimal portfolio can be derived in the general semimartingale framework, as it will be illustrated in the following.

Theorem 8.2: *Consider the process V defined in (8.9) and its Jacod components $(\beta, f, g, \overline{M^V}, A^V)$ given by (8.12)–(8.13). Suppose that Problem 8.1 admits a solution (θ^*, τ^*), and that the assumptions (8.2) and (8.6) are fulfilled. Then, the following assertions hold.*

(i) There exists a probability measure $Q_V \sim P$ such that the MEH martin-

gale measure with respect to Q_V*—that we denote by* $\widetilde{Q}_V$ *and its density by* $\widetilde{Z}^V_t := E(\frac{d\widetilde{Q}_V}{dQ_V}|\ \mathcal{F}_t)$*—exists and satisfies*

$$A^V_{t\wedge\tau^*} = h^E_{t\wedge\tau^*}(\widetilde{Q}_V, Q_V), \quad \textit{and} \quad \log(\widetilde{Z}^V) = \widetilde{\theta}^V\cdot S + h^E(\widetilde{Q}_V, Q_V). \quad (8.16)$$

(ii) The optimal controls, (θ^*, τ^*)*, solution to (8.7) can be described as follows:*

a) The optimal investment θ^* *coincides with* $-\widetilde{\theta}^V$ *on* $[\![0, \tau^*]\!]$*, i.e.* $-\theta^*$ *is a pointwise root to*

$$-b+c(\theta-\beta)+\int\left[h(x)-(f(x)+1)e^{-\theta^T x}x\right]F(dx) = 0, \ \textit{on}\ \{\Delta A = 0\}\cap[\![0,\tau^*]\!] \quad (8.17)$$

$$\textit{and} \qquad \int(f(x)+1)e^{-\theta^T x}xF(dx) = 0, \qquad \textit{on}\ \ \{\Delta A \neq 0\}\cap[\![0,\tau^*]\!]. \quad (8.18)$$

b) The stopping time τ^* *satisfies* $\tau^* \geq \widetilde{\tau}\ P-a.s.$*, where* $\widetilde{\tau}$ *is the smallest stopping times such that* $(\theta^* I_{[\![0,\widetilde{\tau}]\!]}, \widetilde{\tau})$ *is a solution to (8.7), and is given by*

$$\widetilde{\tau} = \inf\{0\leq t < T \mid V(t) = -e^{B_t}, \quad \textit{or} \quad V(t-) = -e^{B_{t-}}\ \}\wedge T, \quad (8.19)$$

i.e. $V(0) = \sup_{\theta\in\Theta} E\left[-e^{-(\theta\cdot S)_{\widetilde{\tau}}+B_{\widetilde{\tau}}}\right]$*. More generally, we have*

$$V(t) = \operatorname*{ess\,sup}_{\theta\in\Theta}\ E\left(-\exp\Big[-\int_t^{\tau_t}\theta_u dS_u + B_{\tau_t}\Big]\Big|\mathcal{F}_t\right),$$

$$\textit{where} \quad \tau_t := \inf\{u\in[t,T[\ \mid V(u) = -e^{B_u}, \quad \textit{or} \quad V(u-) = -e^{B_{u-}}\ \}\wedge T. \quad (8.20)$$

Proof. First, recall that due to the main result of [15], we deduce that the set $\mathcal{Z}^e_{f,loc}(S,Q) \neq \emptyset$, for any $Q \sim P$ if and only if $\mathcal{Z}^e_{loc}(S) \neq \emptyset$. Here the set $\mathcal{Z}^e_{f,loc}(S,Q)$ denotes the set of positive Q-local martingale, Z^Q, (i.e. $Z^Q \in \mathcal{M}_{loc}(Q)$, $Z^Q > 0$) such that $Z^Q S$ is a σ-martingale under Q, (i.e. $Z^Q S \in \mathcal{M}_{loc}(Q)$) and $Z^Q \log\left(Z^Q\right)$ is Q-locally integrable.
It is obvious that $(-V(0))^{-1}V$ is a positive local submartingale and the inequality, $\frac{V}{V(0)} = \mathcal{E}\left(M^V\right)e^{A^V} \leq \frac{e^B}{-V(0)}$, holds. Thus, under assumption (8.6),

we derive

$$\sup_{\tau\in\mathcal{T}_T} E\left[\mathcal{E}_\tau\left(M^V\right)^p\right] \leq (-V(0))^{-p} \sup_{\tau\in\mathcal{T}_T} E\left(e^{pB_\tau}\right) < +\infty,$$

and the uniform integrability of $\mathcal{E}\left(M^V\right)$ follows. Hence, $Q_V := \mathcal{E}_T\left(M^V\right)\cdot P \sim P$ is a probability measure. Furthermore, due to Lemma 5.3 and assumption (8.2), we get

$$\int_{\{|x|>1\}} e^{\lambda^T x} F^{Q_V}(dx) < +\infty, \tag{8.21}$$

where $F^{Q_V}(dx)$ is the kernel corresponding to the jumps of S under the measure Q_V. Thus, under assumption (8.2) and (8.6), we deduce that

$$\mathcal{Z}^e_{f,loc}(S, Q_V) \neq \emptyset$$

and, thus, we can apply Theorem 3.3 of [17] for the model (S, Q_V). This proves the existence of the MEH σ-martingale density $\widetilde{Z}^V := \widetilde{Z}^{Q_V}$ with respect to Q_V, and moreover, that, $\log(\widetilde{Z}^V) = \widetilde{\theta}^V \cdot S + h^E\left(\widetilde{Z}^V, Q_V\right)$.
Since $L(\theta) = Ve^{-\theta\cdot S}$ is a supermartingale for any $\theta \in \Theta$, we deduce that the process $\exp\left(A^V - h^E(\widetilde{Q}_V, Q_V) - (\theta + \widetilde{\theta}^V)\cdot S\right)$ is a $\widetilde{Q}_V$-submartingale. As a result, the process

$$L\left(-\widetilde{\theta}^V\right) = Ve^{\widetilde{\theta}^V\cdot S} = V(0)\mathcal{E}\left(M^V\right)e^{A^V+\widetilde{\theta}^V\cdot S} = V(0)\mathcal{E}\left(M^V\right)\widetilde{Z}^V e^{A^V - h^E\left(\widetilde{Q}_V, Q_V\right)}$$

is a local supermartingale or, equivalently, the process $A^V - h^E\left(\widetilde{Z}^V, Q_V\right)$ is nondecreasing. Furthermore, a combination of the inequalities

$$E^{Q_V}\left(e^{h^E_T\left(\widetilde{Z}^V, Q_V\right)}\right) \leq E^{Q_V}\left(e^{A^V_T}\right) = E\left(\frac{V(T)}{V(0)}\right) = E\left(\frac{-e^{B_T}}{V(0)}\right) < +\infty,$$

and Theorem III.1 of [53], we deduce that $\widetilde{Z}^V$ is a true Q_V-martingale. This proves the existence of the MEH σ-martingale measure for (S, Q_V) denoted by $\widetilde{Q}_V$. This proves the assertion (i) without the first equality of (8.16).

By combining the equality

$$V\exp(-(\theta^*\cdot S)) = V_0\mathcal{E}(M^V)\widetilde{Z}^V\exp\left(A^V - h^E(\widetilde{Q}_V, Q_V) - (\widetilde{\theta}^V + \theta^*)\cdot S\right),$$

the $\widetilde{Q}_V$-submartingale property of

$$A^V_{t\wedge\tau^*} - h^E_{t\wedge\tau^*}\left(\widetilde{Q}_V, Q_V\right) - (\widetilde{\theta} + \theta^*)\cdot S_{t\wedge\tau^*}, \tag{8.22}$$

and the strict convexity of e^z, we deduce that $V(t\wedge\tau^*)\exp(-(\theta^*\cdot S)_{t\wedge\tau^*})$ is a true martingale if and only if the process (8.22) is null, or, equivalently, that

$$A^V_{t\wedge\tau^*} = h^E_{t\wedge\tau^*}\left(\widetilde{Q}_V, Q_V\right), \qquad \widetilde{\theta} I_{[\![0,\tau^*]\!]} = -\theta^* I_{[\![0,\tau^*]\!]}.$$

This ends, simultaneously, the proof of assertion (i) and assertion (ii)-a).

Next, we will prove assertion (ii)-b). To this end, we consider the process $\widetilde{Y}$ and the stopping time $\widetilde{\tau}^*$ given, respectively, by

$$\widetilde{Y}_t := \operatorname*{ess\,sup}_{\tau\geq t}\ E\left(-\exp\Big[B_\tau + \int_t^\tau \widetilde{\theta}_u dS_u\Big] \ \Big|\ \mathcal{F}_t\right) \quad \text{and}$$

$$\widetilde{\tau}^* := \inf\left\{t\in[0,T[\ :\ \widetilde{Y}_t = -e^{B_t}, \quad \text{or} \quad \widetilde{Y}_{t-} = -e^{B_{t-}}\ \right\}\wedge T. \tag{8.23}$$

Then, it is obvious to note that for any $t\in[0,T]$,

$$V(t) \geq \widetilde{Y}(t) \geq -\exp(B_t), \quad P-a.s. \tag{8.24}$$

Furthermore, since

$$\begin{aligned} V(0) = E\left[-V(\tau^*)\exp\Big(-(\theta^*\cdot S)_{\tau^*}\Big)\right] &= E\left[-\exp\Big(B_{\tau^*} - (\theta^*\cdot S)_{\tau^*}\Big)\right] \\ &\leq \sup_{\tau\in\mathcal{T}_T} E\left[-\exp\Big(B_\tau - (\theta^*\cdot S)_\tau\Big)\right] \\ &=: \widetilde{Y}(0), \end{aligned}$$

we derive

$$V(0) = \widetilde{Y}(0) \quad \text{and} \qquad \tau^* \geq \widetilde{\tau} \geq \widetilde{\tau}^* \quad P - a.s.$$

Combining these inequalities with the fact that $V(t \wedge \widetilde{\tau}^*) \exp(-(\theta^* \cdot S)_{t\wedge\widetilde{\tau}^*})$ and $\widetilde{Y}_{t\wedge\widetilde{\tau}^*} \exp(-(\theta^* \cdot S)_{t\wedge\widetilde{\tau}^*})$ are martingales, we deduce that

$$EV(t \wedge \widetilde{\tau}^*) \exp(-(\theta^* \cdot S)_{t\wedge\widetilde{\tau}^*}) = EY(t \wedge \widetilde{\tau}^*) \exp(-(\theta^* \cdot S)_{t\wedge\widetilde{\tau}^*}).$$

This equality together with (8.24) prove that the two processes $V(t \wedge \widetilde{\tau}^*)$ and $\widetilde{Y}(t \wedge \widetilde{\tau}^*)$ coincide. Thus, the two stopping times $\widetilde{\tau}$ and $\widetilde{\tau}^*$ coincide also. Thanks, to the result of [54] (see Théorème 4 therein), we deduce that the stopping time $\widetilde{\tau}$ is the smallest optimal stopping time, and the assertion (ii)-b) follows. This ends the proof of the theorem. □

Remark: 1. Our main results of this section (Theorems 8.1 and 8.2) contribute by giving the structure of the optimal value process V, and the explicit description of τ^* and θ^* when they exist.

2. The financial problem that we consider in this section is the same as the one of [35]. Therefore, our two theorems generalize the results of that paper to the semimartingale framework. See also [36], [29], and the reference therein for the same financial problem with other utilities.

3. Concerning the mathematical formulation and/or technical aspects, Problem 8.1 is very close to the one considered in [47]. However, there are fundamental differences:

   (a) Our running reward function (e^x) is multiplied to the terminal reward function ($g(S_t) = -e^{B_t}$), while in [47] they add-up. Furthermore, the control θ appears in the expectation operation which is not the case in our situation.

   (b) The terminal reward function, $g(x)$, is assumed to be bounded from below (positive), which does not correspond to our case ($g(S_t) = -e^{B_t} < 0$ might be unbounded from below). It is important to mention that this positivity assumption is crucial in the analysis of [47].

(c) Our framework is very general by dealing with semimartingales in which the predictable representation property may never hold. Furthermore, the additional feature of jumps in the model may add tremendous technical difficulties to the method used in [47].

Comment that the optimal sale problem with investments (i.e. Problem 8.1) was the main motivation for the horizon-unbiased utility concept of Henderson-Hobson. Herein, Theorem 8.2—and mainly its proof—establishes the connection between the existence of solution to Problem 8.1 and the forward utility concept of Musiela-Zariphopoulou. This can be stated as follows.

Corollary 8.2.1: Suppose that assumptions of Theorem 8.2 hold, and consider the following random field utility

$$U(t, \omega, x) = V_t(\omega) \exp(-x). \tag{8.25}$$

Then, there exists a stopping time τ such that the random field utility $U(t \wedge \tau(\omega), x)$ is an exponential forward dynamic utility for the model S^τ, and $\widetilde{\tau}$—defined in (8.19)—is the smallest stopping time satisfying this property.

Proof. The proof of this Corollary follows directly from the proof of Theorem 8.2. □

8.B Markovian Case

In Markovian case, the value process V (see (8.9)) can be written as

$$V(t, x, p) := \operatorname*{ess\,sup}_{\theta \in \Theta(x,p),\ \tau \geq t} E\left(e^{-\int_t^\tau \zeta_u(X_u^\theta) du} U(\tau, X_\tau^\theta - \xi(P_\tau)) \middle| X_t^\theta = x, P_t = p \right). \tag{8.26}$$

Here, $U(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, is deterministic utility function. We assume that $U(t,x)$ is global Lipschitz continuous, i.e. $\exists\ K$ such that

$$|U(t,x) - U(s,y)| \leq K(|t-s| + |x-y|). \tag{8.27}$$

We also assume the function $x \mapsto \xi(x)$ is global Lipschitz continuous, i.e.

$$|\xi(x) - \xi(y)| \leq K|x-y|, \quad \forall\ x, y \in \mathbb{R}_+, \tag{8.28}$$

satisfying $\xi(0) = 0$. [1]

ζ is the discount rate process. $\Theta(x,p)$ is the set of admissible portfolios (see (2.10)), where, in particular, we require that the portfolios are bounded, i.e. there exists $K > 0$ such that $|\theta_t| \leq K$, P-a.s.

X^θ is the wealth process under the trading strategy θ which represents the proportion of wealth invested in the stock, $\frac{dX_t^\theta}{X_{t-}^\theta} = \theta_t \frac{dS_t}{S_{t-}}$. Consider the dynamics of the stock price process S as follows

$$\frac{dS_t}{S_{t-}} = \mu dt + \sigma dW_t + \int_{\mathbb{R}\setminus\{0\}} \psi(z) \widetilde{N}(dz, dt). \tag{8.29}$$

Here, $N(dz,dt)$ is a Poisson random measure on the Borel sets of $\mathbb{R}_+ \times \mathbb{R}\setminus\{0\}$ with intensity $dt \times n(dz)$. $n(dz)$ is the Lévy measure which is positive and σ-finite on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\{|z| \geq 1\}} n(dz) < +\infty$$

$\widetilde{N}$ is the compensated Poisson measure given by $\widetilde{N}(dz,dt) = N(dz,dt) - n(dz) \times dt$. $\psi(z)$ is assumed to be Borel measurable on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} \psi^2(z) dz < +\infty. \tag{8.30}$$

[1] For example, the call option $\xi(x) = (x-K)^+$.

Moreover, $\psi(z)$ is assumed to be bounded in a neighborhood of $z = 0$.
Therefore, the dynamics of the wealth process X^θ can be expressed by

$$dX_t^\theta = \mu\theta_t X_t^\theta dt + \sigma\theta_t X_t^\theta dW_t + \int_{\mathbb{R}\setminus\{0\}} \theta_t \psi(z) X_{t-}^\theta \widetilde{N}(dz, dt). \tag{8.31}$$

Let the payoff process P follow the dynamics

$$dP_t = P_t\kappa dt + P_t\eta dB_t + \int_{\mathbb{R}\setminus\{0\}} P_{t-}\ell(z)\widetilde{N}(dz, dt), \quad P_0 > 0, \tag{8.32}$$

which satisfies $P_t > 0$, $P - a.s.$. Here, $\ell(z)$ is assumed to be Borel measurable on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}\setminus\{0\}} \ell^2(z)dz < +\infty. \tag{8.33}$$

Also, it is assumed to be bounded in a neighborhood of $z = 0$.
The Brownian motions W and B are correlated with $\rho \in [-1, 1]$ and we can write

$$dW_t = \rho dB_t + \sqrt{1-\rho^2} dZ_t$$

for another Brownian motion Z which is independent of B.

Note that in above setup, the coefficients of X^θ satisfies the global Lipschitz conditions, i.e. there exists $K > 0$ such that:

$$|\mu\theta x - \mu\theta y| \leq K|x - y|, \tag{8.34}$$

$$|\sigma\theta x - \sigma\theta y| \leq K|x - y|, \tag{8.35}$$

$$|\psi(z)\theta x - \psi(z)\theta y| \leq K|\psi(z)||x - y|, \tag{8.36}$$

And, the global linear growth conditions hold:

$$|\mu\theta x| \leq K|x|, \tag{8.37}$$

$$|\sigma\theta x| \leq K|x|, \tag{8.38}$$

In particular, by following the above conditions, the wealth dynamics (8.31) has a unique strong solution.

8.B.1 Hamilton-Jacobi-Bellman equation

For any $\phi \in C^2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, we define two differential operators

$$D_{xp}\phi(t,x,p) := \begin{pmatrix} \frac{\partial \phi}{\partial x}(t,x,p) \\ \frac{\partial \phi}{\partial p}(t,x,p) \end{pmatrix}, \quad D^2_{xp}\phi(t,x,p) := \begin{pmatrix} \frac{\partial^2 \phi}{\partial x^2}(t,x,p) & \frac{\partial^2 \phi}{\partial x \partial p}(t,x,p) \\ \frac{\partial^2 \phi}{\partial x \partial p}(t,x,p) & \frac{\partial^2 \phi}{\partial p^2}(t,x,p) \end{pmatrix}$$

And, for any $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$ and any admissible strategy $\theta \in \Theta(x,p)$, we define the operator

$$A^\theta(t,x,p,m,M) := \begin{pmatrix} \mu\theta x & \kappa p \end{pmatrix} .m + tr\Big\{ \begin{pmatrix} \frac{\sigma^2}{2}\theta^2 x^2 & \frac{\sigma\eta\rho}{4}\theta xp \\ \frac{\sigma\eta\rho}{4}\theta xp & \frac{\eta^2}{2}p^2 \end{pmatrix} M \Big\}$$

Furthermore, for $\delta \in (0,1)$ and $\phi \in C^2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, we define

$$B^\theta_{\delta-}(t,x,\phi) := \int_{\{|z| \le \delta\}} \Big[\phi(t, x + \theta\psi(z)x, p + \ell(z)p) - \phi(t,x,p) \\ - \theta\psi(z)x\frac{\partial \phi}{\partial x}(t,x,p) - \ell(z)p\frac{\partial \phi}{\partial p}(t,x,p)\Big] n(dz)$$

Remark that under our assumptions on $n(z)$, $\psi(z)$ and $\ell(z)$, $B^\theta_{\delta-}(t,x,\phi)$ is well defined, bounded uniformly in θ and

$$\lim_{\delta \to 0} \sup_\theta B^\theta_{\delta-}(t,x,\phi) = 0. \tag{8.39}$$

Define a set of continuous functions, C_2, that grow in quadratic rate:

$$C_2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+) := \\ \Big\{\phi \in C^0([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+) : \sup_{[0,T] \times \mathbb{R}_+ \times \mathbb{R}_+} \frac{\phi(t,x,p)}{1 + x^2 + p^2} < +\infty. \Big\}$$

For $\phi \in C_2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, we define an operator

$$B^\theta_{\delta+}(t,x,m,\phi) := \int_{\{|z|\geq\delta\}} \Big[\phi(t, x+\theta\psi(z)x, p+\ell(z)p) \\ - \phi(t,x,p) - \Big(\theta\psi(z)x \quad \ell(z)p\Big).m\Big] n(dz)$$

Also, under our assumptions on $n(z)$, $\psi(z)$ and $\ell(z)$, $B^\theta_{\delta+}(t,x,m,\phi)$ is well defined and bounded uniformly in θ.

By putting $B^\theta(t,x,m,\phi) = B^\theta_{\delta-}(t,x,\phi)+B^\theta_{\delta+}(t,x,m,\phi)$, we define an operator

$$\mathcal{L}^\theta v(t,x,p) := -\zeta v + v_t + A^\theta(t,x,Dv,D^2v) + B^\theta(t,x,Dv,v)$$

If we denote $\widetilde{U}(t,x,p) := U(t, x-\xi(p))$, the HJB equation can be formulated in the form of variational inequality

$$0 = \min\left\{-\sup_\theta \mathcal{L}^\theta v(t,x,p),\ v(t,x,p) - \widetilde{U}(t,x,p)\right\},\ (t,x,p) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}_+, \tag{8.40}$$

with the terminal condition

$$v(T,x,p) = \widetilde{U}(T,x,p), \quad \forall\ (x,p) \in \mathbb{R}_+ \times \mathbb{R}_+. \tag{8.41}$$

8.B.2 Viscosity solution

We start by putting the definition of a viscosity solution.

Definition: (i) Any $v \in C^0([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ is a viscosity supersolution (subsolution) of (8.40) if

$$\min\left\{v - \widetilde{U}, \zeta v - \varphi_t - \sup_\theta \left(A^\theta(t,x,D\varphi,D^2\varphi) + B^\theta(t,x,D\varphi,\varphi)\right)\right\} \geq 0$$

(≤ 0)whenever $\varphi \in C^2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap C_2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ and $v - \varphi$ has global minimum (maximum) at $(t,x,p) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}_+$.

(ii) v is a viscosity solution of (8.40) if it is both supersolution and subsolution.

We recall some usual notions that appears in different books and literatures. Given $v \in C^0([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ and $(t,x,p) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}_+$, we define the parabolic superjet:

$$\mathcal{P}^{2,+}v(t,x,p) := \{(m_0,m,M) \in \mathbb{R} \times \mathbb{R}^2 \times S^2 : v(s,y,q) \leq v(t,x,p)+ \\ m_0(s-t) + m.(y-x\ q-p) + \frac{1}{2}(y-x\ q-p)'M(y-x\ q-p) \\ +o(|s-t|^2+|y-x|^2+|q-p|^2),\ as\ (s,y,q) \to (t,x,p)\}.$$

and its closure

$$\bar{\mathcal{P}}^{2,+}\ v\ (t,x,p) := \\ \{(m_0,m,M) \in \mathbb{R} \times \mathbb{R}^2 \times S^2 : (m_0,m,M) = \lim_{n\to+\infty}(m_0^n,m^n,M^n) \\ with\,(m_0^n,m^n,M^n) \in \mathcal{P}^{2,+}v(t^n,x^n,p^n) \\ and\ \lim_{n\to+\infty}(t^n,x^n,p^n,v(t^n,x^n,p^n)) = (t,x,p,v(t,x,p))\}.$$

Parabolic subjet can be defined as $\mathcal{P}^{2,-}v(t,x,p) = -\mathcal{P}^{2,+}(-v(t,x,p))$ and its closure $\bar{\mathcal{P}}^{2,-}v(t,x,p) = -\bar{\mathcal{P}}^{2,+}(-v(t,x,p))$. One fact which is proved by P.L.Lions is that

$$\mathcal{P}^{2,+(-)}v(t,x,p) = \{(\frac{\partial\phi}{\partial t}(t,x), D_{xp}\phi(t,x,p), D^2_{xp}\phi(t,x,p)) : \phi \in C^2\,and \\ v-\phi\ \text{ has a global maximum (minimum) at}(t,x)\}$$

In the set $C_2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$, the viscosity solution can be characterized in another way as follows. It is convenient to use this formulation when we study its uniqueness.

Lemma 8.2: *Let $v \in C_2([0,T] \times I\!R_+ \times I\!R_+)$ be a viscosity supersolution (resp. subsolution). Then, for all $(t,x,p) \in [0,T) \times I\!R_+ \times I\!R_+$ and for any $(m_0,m,M) \in \overline{\mathcal{P}}^{2,-}v(t,x,p)$ (resp. $\overline{\mathcal{P}}^{2,+}v(t,x,p)$), there exists $\varphi \in C^2$*

such that

$$\min\left\{v-\widetilde{U},\zeta\geq v-m_0-\sup_\theta(A^\theta(t,x,p,m,M)+B^\theta_{\delta-}(t,x,p,\varphi)\right.$$
$$\left.+B^\theta_{\delta+}(t,x,p,m,v))\right\}\geq 0\ (\textit{resp.}\ \ \leq 0).$$

Proof. The proof of this lemma is referred to Lemma 2.1 in [72]. □

Proposition 8.1: *Under our assumptions on U, ξ, ψ and ℓ, i.e. (8.27),(8.28), (8.30) and (8.33), the value process satisfies the growth property,*

$$|V(t,x,p)|\leq K(1+|x|+|p|),\quad K>0,$$

and it is Lipschitz continuous in x and p uniformly in t, i.e.

$$|V(t,x,p)-V(t,y,q)|\leq K(|x-y|+|p-q|),\quad K>0.$$

Proof. Due to the Lipschitz continuity of $U(t,x)$ with x uniformly in t we have its linear growth rate

$$|U(t,x)|\leq K(1+|x|).$$

The lipschitz continuity of $\xi(x)$ also implies its linear growth rate,

$$|\xi(x)|\leq K(1+|x|).$$

Applying these results, we have

$$\begin{aligned}|V(t,x,p)|\leq&\operatorname*{ess\,sup}_{\theta\in\Theta(x,p),\ \tau\geq t}E\left(e^{-\int_t^\tau\zeta_u(X_u^\theta)du}|U(\tau,X_\tau^\theta-\xi(P_\tau)|\middle|X_t^\theta=x,P_t=p\right)\\ \leq&\operatorname*{ess\,sup}_{\theta\in\Theta(x,p),\ \tau\geq t}E\left(2+|X_\tau^\theta|+|P_\tau|\middle|X_t^\theta=x,P_t=p\right).\end{aligned}$$

Due to Gronwall's lemma, we can prove that

$$E(|X_\tau^\theta|^2)\leq C|x|^2,\quad E(|P_\tau|^2)\leq C|p|^2. \tag{8.42}$$

Thus, by applying Holder's inequality, we have

$$|V(t,x,p)| \leq K(1+|x|+|p|).$$

Similarly, we can show the Lipschitz continuity,

$$|V(t,x,p)-V(t,y,q)| \leq K(|x-y|+|p-q|).$$

□

The following Dynamic Programming equation will play an important role in our analysis:

Proposition 8.2: *Suppose the optimization for V admits a solution (θ^*, τ^*). Let $\epsilon > 0$. For all $(t,x,p) \in [0,T] \times I\!R_+ \times I\!R_+$ and for each admissible strategy $\theta \in \Theta(x,p)$, define the stopping time*

$$\tau^{\epsilon}_{t,x,p,\theta} = \inf\left\{ t \leq s \leq T : \quad V(s, X^{\theta}_s, P_s) \leq U(s, X^{\theta}_s + \xi(P_s))) \right\} \tag{8.43}$$

Then, if $t \leq \tau_\theta \leq \tau^{\epsilon}_{t,x,p,\theta}$ for all $\theta \in \Theta(x,p)$, we have

$$V(t,x,p) = \operatorname*{ess\,sup}_{\theta\in\Theta(x,p)} \; E\left(e^{-\int_t^\tau \zeta_u(X^\theta_u)du} V(\tau, X^\theta_\tau, P_\tau) \middle| X^\theta_t = x, P_t = p \right)$$

Proof. Let (θ^*, τ^*) be the optimizer of the value function $V(t,x)$, then we have $V(\tau^*, X^{\theta^*}_{\tau^*}) = U(\tau^*, X^{\theta^*}_{\tau^*} + \xi(P_{\tau^*}))$ and the process $V(t \wedge \tau^*, X^{\theta^*}_t)$ is a true martingale. Then, the claim follows immediately by optional sampling theorem. □

The above dynamic programming equation implies another version which we need in the following.

Proposition 8.3: *For all* $(t, x, p) \in [0, T] \times I\!R_+ \times I\!R_+$, $h \in T_{t,T}$, *we have*

$$V(t, x, p) = \operatorname*{ess\,sup}_{\theta \in \Theta(x,p), \tau \geq t} E\left(I_{\{\tau < h\}} e^{-\int_t^\tau \zeta_u(X_u^\theta)du} U(\tau, X_\tau^\theta + \xi(P_\tau)) \right.$$
$$\left. + I_{\{\tau \geq h\}} e^{-\int_t^h \zeta_u(X_u^\theta)du} V(h, X_h^\theta, P_h) \middle| X_t^\theta = x, P_t = p\right)$$

Proof. The implication of the proof from Proposition 8.2 refers to [49].

□

The continuity of the value function is stated in the following.

Proposition 8.4: *Under assumption that every* $\theta \in \Theta(x, p)$ *is bounded and* $U(t, x)$ *is Lipschitz continuous, the value function* $V \in C^0([0, T] \times I\!R_+ \times I\!R_+)$.

Proof. We first focus on the continuity of $V(t, x, p)$ in t. From dynamic programming equation in Proposition 8.3, for $0 \leq t \leq s \leq T$, let $h = s - t$, then

$$\begin{aligned}
0 &\leq V(t, x, p) - V(s, x, p) \\
&\leq \operatorname*{ess\,sup}_{\theta \in \Theta(x,p), \tau \geq t} E\Big(I_{\{\tau < h\}} e^{-\int_t^\tau \zeta_u(X_u^\theta)du} (U(\tau, X + \xi(P)) - U(t, x + \xi(p))) \\
&\quad + I_{\{\tau < h\}} e^{-\int_t^\tau \zeta_u(X_u^\theta)du} (U(t, x + \xi(p)) - V(s, x, p)) \\
&\quad + I_{\{\tau < h\}} (e^{-\int_t^\tau \zeta_u(X_u^\theta)du} - 1) V(s, x, p) + I_{\{\tau \geq h\}} (e^{-\int_t^h \zeta_u(X_u^\theta)du} - 1) V(s, x, p) \\
&\quad + I_{\{\tau \geq h\}} e^{-\int_t^h \zeta_u(X_u^\theta)du} (V(h, X_h, P_h) - V(s, x, p))\Big)
\end{aligned}$$

which implies

$$\begin{aligned}
|V(t, x, p) - V(s, x, p)| &\leq K\Big(\operatorname*{ess\,sup}_{\theta \in \Theta(x,p), \tau \geq t} E(|X_\tau - x| + |P_\tau - p|) \\
&\quad + \operatorname*{ess\,sup}_{\theta \in \Theta(x,p)} E(|X_h - x| + |P_h - p|) + (1 + |x| + |p|)h\Big) \\
&\leq K(1 + |x| + |p|)(|s - t|^{1/2} + |s - t|)
\end{aligned}$$

Then, for any $(t, x, p), (s, y, q) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$, we obtain

$$|V(t, x, p) - V(s, y, q)| \leq K(|x - y| + |p - q| + (1 + |x| + |p|)(|s - t|^{1/2} + |s - t|))$$

This completes the proof of this proposition. $\square$

Here we provide a technical result which is needed when we characterize V as the viscosity solution in Theorem 8.3.

Lemma 8.3: *For any $(t,x,p) \in [0,T] \times I\!R_+ \times I\!R_+$, $\theta \in \Theta(x,p)$ and $h \in [t,T]$, there exists some $\varepsilon > 0$ and constant $C > 0$ such that*

$$P(h > \tau^{\varepsilon}_{t,x,p,\theta}) \leq Ch. \tag{8.44}$$

where $\tau^{\varepsilon}_{t,x,p,\theta}$ (we will write it as τ^{ε} for short) is a stopping time defined as (8.43).

Proof. We define ε in an explicit form as

$$\varepsilon := \begin{cases} \frac{1}{2}(V(t,x,p) - U(t,x+\xi(p))), & \text{if } V(t,x,p) > U(t,x+\xi(p)); \\ \delta, & \text{if } V(t,x,p) = U(t,x+\xi(p)). \end{cases}$$

for some $\delta > 0$. Then, by the definition of τ^{ε} and Markov's inequality, we have

$$\begin{aligned}
&P(h > \tau^{\varepsilon}) \\
&\leq P\left(\sup_{t\leq s\leq h}|V(s,X^{\theta}_s,P_s) - U(s,X^{\theta}_s+\xi(P_s)) - V(t,x,p) + U(t,x+\xi(p))|\geq \varepsilon\right) \\
&\leq \frac{1}{\varepsilon^2}E\left(\sup_{t\leq s\leq h}|V(s,X^{\theta}_s,P_s) - U(s,X^{\theta}_s+\xi(P_s)) - V(t,x,p) + U(t,x+\xi(p))\right)^2 \\
&\leq \frac{1}{\varepsilon^2}E\Big(\sup_{t\leq s\leq h}|V(s,X^{\theta}_s,P_s) - V(t,x,p)| + \\
&\qquad\qquad \sup_{t\leq s\leq h}|U(s,X^{\theta}_s+\xi(P_s)) - U(t,x+\xi(p))|\Big)^2.
\end{aligned}$$

Recall the Lipschitz continuity of $V(t,x,p)$ proved in Proposition 8.1 and assumed for functions $U(t,x)$ and $\xi(x)$, we have

$$P(h > \tau^{\varepsilon}) \leq \frac{K}{\varepsilon^2}E\left(\sup_{t\leq s\leq h}|X^{\theta}_s - x| + \sup_{t\leq s\leq h}|P_s - p|\right)^2.$$

From the dynamics of X^{θ} given in (8.31), we can solve X^{θ} explicitly and then combine it with the inequality derived in (8.42), we have

$$E\left(\sup_{t\le s\le h}|X^\theta_s - x|\right)^2 \le C_1 E\left(\int_t^h X^\theta_s ds\right)^2 \le C_1\int_t^h E(X^\theta_s)^2 ds \le C_1'|x|^2 h$$

By performing a similar argument for P, we can conclude (8.44) immediately.

□

Now, we are ready to state our first main result for this section.

Theorem 8.3: *Under assumptions (8.27),(8.28), (8.30) and (8.33), the value function V is a viscosity solution of (8.40).*

Proof. Proposition 8.4 has proved the continuity of V, hence, it remains to prove that V is both a supersolution and subsolution. We start from the supersolution.

For any $(t,x,p)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}_+$, we let φ be a function in $C^2([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)\cap C_2([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ such that $V-\varphi$ has global minimum at (t,x,p). Without loss of generality, we assume

$$\min_{(s,y,q)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}_+}(V(s,y,q)-\varphi(s,y,q)) = V(t,x,p)-\varphi(t,x,p)=0.$$

Thus, a direct application of the dynamic programming principle derived in Proposition 8.3 by putting $\tau = h$ (we choose stopping time h as a constant in $(t,T]$), we have

$$V(t,x,p)=\varphi(t,x,p)\ge \operatorname*{ess\,sup}_{\theta\in\Theta(x,p)} E\left(e^{-\int_t^h\zeta_u(X^\theta_u)du}\varphi(h,X^\theta_h,P_h)\Big|X^\theta_t = x, P_t = p\right).$$

Applying Ito's formula to the process $e^{-\int_t^s\zeta_u(X^\theta_u)du}\varphi(s,X^\theta_s,P_s)$, $s\in[t,h]$ and then rearranging the terms and dividing it by $(h-t)$ in above inequality, we have

$$0\ge\frac{1}{h-t}E\left[\int_t^h e^{-\int_t^s\zeta_u(X^\theta_u)du}G^\theta(s,X^\theta_s,P_s)ds|X^\theta_t = x, P_t = p\right],\ \forall\,\theta\in\Theta(t,p) \tag{8.45}$$

where

$$G^{\theta}(\varphi;t,x,p) = -\zeta\varphi + \varphi_t + \kappa p\varphi_p + \frac{\eta^2}{2}\varphi_{pp} + \mu\theta x\varphi_x + \frac{\sigma^2}{2}\theta^2x^2\varphi_{xx} + \frac{\sigma\eta\rho}{2}\theta xp\varphi_{xp}$$
$$- \int \left[-\varphi(x) + \varphi(x-\theta\psi(z)x) + \varphi_x(x-\theta\psi(z)x)\theta\psi(z)x\right] n(dz)$$

In (8.45), we let $h \to t$ and notice the differentiability of the integral

$$\int_t^h G^{\theta}(\varphi; s, X_s^{\theta}, P_s)ds$$

with respect to h, $h \in [t,T]$, and the right continuity of $G^{\theta}(\varphi; s, X_s^{\theta}, P_s)$ with respect to s, $s \in [t,h]$, we have

$$0 \geq G^{\theta}(\varphi; t, X_t^{\theta}, P_t), \quad \forall\, \theta \in \Theta(x,p).$$

Due to the arbitrariness of θ, and note the fact $V(t,x,p) \geq U(t, x+\xi(p))$ for any $(t,x) \in [0,T] \times \mathbb{R}_+$, we have

$$\min\Big\{V(t,x,p) - U(t,x+\xi(p)), \zeta V - \varphi_t - \kappa p\varphi_p - \frac{\eta^2}{2}\varphi_{pp} -$$
$$\sup_{\theta}\Big[\mu\theta x\varphi_x + \frac{\sigma^2}{2}\theta^2x^2\varphi_{xx} + \frac{\sigma\eta\rho}{2}\theta xp\varphi_{xp} - \int [-\varphi(x) + \varphi(x-\theta\psi(z)) +$$
$$\varphi_x(x-\theta\psi(z))\theta\psi(z)\Big]n(dz)]\Big\} \geq 0$$

which proves that the value process V is a supersolution.

It remains to prove the value process V is also a subsolution. For any $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$, we let φ be a function in $C^2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap C_2([0,T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ such that $V - \varphi$ has global maximum at (t,x,p). Without loss of generality, we assume

$$\max_{(s,y,q)\in[0,T]\times\mathbb{R}_+\times\mathbb{R}_+} (V(s,y,q) - \varphi(s,y,q)) = V(t,x,p) - \varphi(t,x,p) = 0.$$

For any $\varepsilon > 0$, $\theta \in \Theta(x,p)$, define the stopping time

$$\tau^{\epsilon} = \inf\left\{t \le s \le T, V(s, X_s^{\theta}, P_s) \le U(s, X_s^{\theta} + \xi(P_s)))\right\}.$$

Thus, for any $h \in [t,T]$, since $h \wedge \tau^{\epsilon} \le \tau^{\epsilon}$, the dynamic programming principle stated in Proposition 8.2 implies,

$$V(t,x,p) = \operatorname*{ess\,sup}_{\theta\in\Theta(x,p)} E\left(e^{-\int_t^{h\wedge\tau^{\varepsilon}} \zeta_u(X_u^{\theta})du} V(h\wedge\tau^{\varepsilon}, X_{h\wedge\tau^{\varepsilon}}^{\theta}, P_{h\wedge\tau^{\varepsilon}})\Big| X_t^{\theta} = x, P_t = p\right). \tag{8.46}$$

Then, for any $\delta > 0$, there exists some strategy $\theta \in \Theta(x,p)$ which depends on δ and h such that

$$\begin{aligned} &-\delta(h-t) \\ &\le E\left(e^{-\int_t^{h\wedge\tau^{\varepsilon}} \zeta_u(X_u^{\theta})du} V(h\wedge\tau^{\varepsilon}, X_{h\wedge\tau^{\varepsilon}}^{\theta}, P_{h\wedge\tau^{\varepsilon}})\Big| X_t^{\theta} = x, P_t = p\right) - V(t,x,p) \\ &\le E\left(e^{-\int_t^{h\wedge\tau^{\varepsilon}} \zeta_u(X_u^{\theta})du} \varphi(\tau, X_{\tau}^{\theta}, P_{\tau}) - \varphi(t,x,p)\Big| X_t^{\theta} = x, P_t = p\right). \end{aligned} \tag{8.47}$$

Applying Ito's formula to the process $e^{-\int_t^s \zeta_u(X_u^{\theta})du}\varphi(s, X_s^{\theta}, P_s)$, $s \in [t,T]$, and noticing that $I_{\{s\le t^{\varepsilon}\}} = 1 - I_{\{s>t^{\varepsilon}\}}$, the inequality (8.47) can be improved as

$$\begin{aligned} -\delta \le &\frac{1}{h-t} E\left(\int_t^h e^{-\int_t^s \zeta_u(X_u^{\theta})du} G^{\theta}(s, X_s^{\theta}, P_s)ds\Big| X_t^{\theta} = x, P_t = p\right) \\ &+\frac{1}{h-t} E\left(\int_t^h -I_{\{s>t^{\varepsilon}\}} G^{\theta}(s, X_s^{\theta}, P_s)ds\Big| X_t^{\theta} = x, P_t = p\right). \end{aligned} \tag{8.48}$$

Now, let's focus on the second term in (8.48). Firstly, by using Hölder's inequality, we derive

$$\begin{aligned} &\frac{1}{h-t} E\left(\int_t^h -I_{\{s>t^{\varepsilon}\}} G^{\theta}(s, X_s^{\theta}, P_s)ds\Big| X_t^{\theta} = x, P_t = p\right) \\ &\le \sqrt{E\Big|\frac{1}{h-t}\int_t^h G^{\theta}(s, X_s^{\theta}, P_s)ds\Big|^2}\sqrt{P(h > t^{\varepsilon})}. \end{aligned}$$

Meanwhile, we recall the result proved in Lemma 8.3,

$$P(h > t^\varepsilon) \leq Ch.$$

Therefore, by letting $h \to t$ and $\delta \to 0$ in (8.48), we have immediately that

$$-\sup_\theta (G^\theta(t,x,p)) \leq -G^\theta(t,x,p) \leq 0.$$

Combining this with the fact that $V(t,x,p) \geq U(t, x+\xi(p))$, we can conclude that the value process V is a subsolution. □

We have verified that the value process $(V_t)_{0\leq t\leq T}$ is a viscosity solution of the HJB equation (8.40). It remains to show it is the unique one. We will prove the uniqueness by the comparison principle of the second-order integrodifferential PDE. This result is presented in next theorem. Since the value process $(V_t)_{0\leq t\leq T}$ has been characterized to be uniformly continuous in x and p, uniformly in t, it is now enough to show the uniqueness in the set, $UC(x,p)$, which is a collection of all uniformly continuous functions in x and p.

Theorem 8.4: *Under our assumptions (8.27) and (8.28) on the Lipschitz continuity of $\xi(x)$ and $U(t,x)$, let u (resp. v) be a subsolution (resp. supersolution) of* (8.40) *in $UC(x,p)$. If $u(T,x,p) \leq v(T,x,p)$ for all $(x,p) \in \mathbb{R}_+ \times \mathbb{R}_+$, then*

$$u(t,x,p) \leq v(t,x,p), \quad \forall\ (t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{8.49}$$

Proof. Remark that it is enough to prove (8.49) for $t \in (0,T]$ since the trivial case of $t = 0$ will follow immediately by the right continuity.
For any $t \in (0,T]$, $x,y,p,q \in \mathbb{R}$ and $\beta, \varepsilon, \alpha, \lambda > 0$, we define a function

$$\phi(t,x,y,p,q) := \frac{\beta}{t} + \frac{(x-y)^2 + (p-q)^2}{2\varepsilon} + \alpha e^{\lambda(T-t)}(x^2+y^2+p^2+q^2),$$

which is second order differentiable and has second order growth rate in t, x, y, p, q. For u and v in $UC(x,p)$ defined on $[0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$, their modulus are sublinear and, therefore, have linear growth rate. Thus, the function $\Phi(t, x, y, p, q)$ defined as

$$\Phi(t, x, y, p, q) := u(t, x, p) - v(t, y, q) - \phi(t, x, y, p, q)$$

admits an maximum on $(0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. Let the maximum be reached at $(\bar{t}, \bar{x}, \bar{y}, \bar{p}, \bar{q})$, which is dependent on $\beta, \varepsilon, \alpha, \lambda$. To see such dependence closely, we notice that

$$\Phi(\bar{t}, \bar{x}, \bar{x}, \bar{p}, \bar{p}) + \Phi(\bar{t}, \bar{y}, \bar{y}, \bar{q}, \bar{q}) \leq 2\Phi(\bar{t}, \bar{x}, \bar{y}, \bar{p}, \bar{q})$$

which, after simplification, is

$$\frac{(\bar{x}-\bar{y})^2 + (\bar{p}-\bar{q})^2}{\varepsilon} \leq u(\bar{t}, \bar{x}, \bar{p}) - u(\bar{t}, \bar{y}, \bar{q}) + v(\bar{t}, \bar{x}, \bar{p}) - v(\bar{t}, \bar{y}, \bar{q}).$$

Thus, due to the uniformly continuity of $u(t, x, p)$ and $v(t, x, p)$ in (x, p), uniformly in t, there exists $\widetilde{\delta} > 0$ such that

$$(\bar{x}-\bar{y})^2 + (\bar{p}-\bar{q})^2 \leq \min(\varepsilon\widetilde{\delta}, \widetilde{\delta}). \tag{8.50}$$

Therefore, when $\varepsilon \to 0$, we have $(\bar{x}, \bar{p}) \to (\bar{y}, \bar{q})$. Furthermore, due to

$$\Phi(T, 0, 0, 0, 0) \leq \Phi(\bar{t}, \bar{x}, \bar{y}, \bar{p}, \bar{q})$$

we have

$$\delta(\bar{x}^2 + \bar{y}^2 + \bar{p}^2 + \bar{q}^2) \leq C(1 + \bar{x} + \bar{y} + \bar{p} + \bar{q}).$$

This fact conflicts with their growth rates, which implies that $\bar{x}, \bar{y}, \bar{q}, \bar{q}$ are bounded by some constant depending on α. Thus, for the sequence (t, x, y, p, q) parameter ε, by sending $\varepsilon \to 0$, there exists a subsequence converging to $(t_0, x_0, y_0, p_0, q_0)$.

For the extreme case that $\bar{t} = T$, the fact $\phi(t,x,x,p,p) \le \Phi(T,\bar{x},\bar{y},\bar{p},\bar{q})$ yields

$$\begin{aligned} u(t,x,p) - v(t,x,p) - \tfrac{\beta}{t} - 2\alpha e^{\lambda(T-t)}(x^2+p^2) &\le u(T,\bar{x},\bar{p}) - v(T,\bar{y},\bar{q}) \\ &\le v(T,\bar{x},\bar{p}) - v(T,\bar{y},\bar{q}), \end{aligned} \tag{8.51}$$

where in the last inequality, we have used the condition that $u(T,\bar{x},\bar{p}) \le v(T,\bar{x},\bar{p})$. Recall the dependence of $(T,\bar{x},\bar{y},\bar{p},\bar{q})$ on ε in (8.50), we let $\beta,\varepsilon,\alpha \to 0$ in (8.51). Thus, the uniformly continuity of v implies

$$u(t,x,p) \le v(t,x,p).$$

We now focus on the case $\bar{t} \in (0,T)$, which allows us to apply directly Theorem 9 in [21]. It states that there exist $m_1, m_2 \in \mathbb{R}$ and 2×2 matrix M, N such that

$$(m_1, D_{xp}\phi(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q}), M) \in \bar{\mathcal{P}}^{2,+}u(\bar{t},\bar{x},\bar{p}), \tag{8.52}$$

$$(m_2, D_{yq}\phi(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q}), N) \in \bar{\mathcal{P}}^{2,+}(-v(\bar{t},\bar{y},\bar{q})), \tag{8.53}$$

and satisfies

$$m_1 + m_2 = \frac{\partial \phi}{\partial t}(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) = -\frac{\beta}{\bar{t}^2} - \lambda\alpha e^{\lambda(T-\bar{t})}(\bar{x}^2+\bar{y}^2+\bar{p}^2+\bar{q}^2) \tag{8.54}$$

$$\begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \le \begin{pmatrix} D^2_{xp}\phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) & 0 \\ 0 & D^2_{yq}\phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) \end{pmatrix}. \tag{8.55}$$

Recall the formulation of viscosity solution in Lemma 8.2. For the subsolution $u \in C_2$, there exists $\phi_1 \in C^2$ such that (I omit the dependence of m_1, ϕ and M on $(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q})$)

$$\min\Big\{u(\bar{t},\bar{x},\bar{p}) - \widetilde{U}(\bar{t},\bar{x},\bar{p}), G(\bar{t},\bar{x},\bar{p},u,\varphi_1,m_1,D_{xp}\phi,M)\Big\} \le 0. \tag{8.56}$$

where

$$G(\bar{t},\bar{x},\bar{p},u,\varphi_1,\ m_1,D_{xp}\phi,M) = \zeta u(\bar{t},\bar{x},\bar{p}) - m_1 - \sup_\theta(A^\theta(\bar{t},\bar{x},\bar{p},D_{xp}\phi,M) + B^\theta_{\delta-}(\bar{t},\bar{x},\bar{p},\varphi_1(\bar{t},\bar{x},\bar{p})) + B^\theta_{\delta+}(\bar{t},\bar{x},\bar{p},D_{xp}\phi,u(\bar{t},\bar{x},\bar{p}))).$$

For the supersolution $v \in C_2$, we have

$$(-m_2, -D_{yq}\phi(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q}), -N) \in \bar{\mathcal{P}}^{2,-}v(\bar{t},\bar{y},\bar{q})$$

such that there exists $\phi_2 \in C^2$ [2]

$$\min\left\{v(\bar{t},\bar{y},\bar{q}) - \widetilde{U}(\bar{t},\bar{y},\bar{q}), G(\bar{t},\bar{y},\bar{q},v,\varphi_2,-m_2,-D_{yq}\phi,-N)\right\} \geq 0. \quad (8.57)$$

Combining (8.57) and (8.56) and noting one simple fact that $\min\{a,b\} \leq \min\{c,d\} \to a \leq c \ or \ b \leq d$, we have two cases: either

$$u(\bar{t},\bar{x},\bar{p}) - \widetilde{U}(\bar{t},\bar{x},\bar{p}) \leq v(\bar{t},\bar{y},\bar{q}) - \widetilde{U}(\bar{t},\bar{y},\bar{q}), \quad (8.58)$$

or,

$$G(\bar{t},\bar{x},\bar{p},u,\varphi_1,m_1,D_{xp}\phi,M) \leq G(\bar{t},\bar{y},\bar{q},v,\varphi_2,-m_2,-D_{yq}\phi,-N). \quad (8.59)$$

For the first case in (8.58), recall the Lipschitz continuity of U and ξ and the result in (8.50), we have

$$u(\bar{t},\bar{x},\bar{p}) - v(\bar{t},\bar{y},\bar{q}) \leq C\min(\varepsilon\widetilde{\delta},\widetilde{\delta}).$$

Also, recall that the function $\phi(t,x,y,p,q)$ reaches its maximum at $(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q})$, thus, for any $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$,

$$u(t,x,p) - v(t,x,p) \leq -\phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) + C\min(\varepsilon\widetilde{\delta},\widetilde{\delta}).$$

[2] we omit the dependence of m_2, ϕ and N on $(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q})$

By sending $\varepsilon, \beta, \alpha \to 0$, we have

$$u(t,x,p) \le v(t,x,p).$$

For the case in (8.59), we rearrange the terms and recall the result in (8.54), meanwhile, note one simple fact that $\sup(a+b) \le \sup(a) + \sup(b)$, we have

$$\zeta(u(\bar{t},\bar{x},\bar{p}) - v(\bar{t},\bar{y},\bar{q})) + \frac{\beta}{\bar{t}^2} + \lambda\alpha e^{\lambda(T-\bar{t})}(\bar{x}^2+\bar{y}^2+\bar{p}^2+\bar{q}^2) \le I_1 + I_2 + I_3 \quad (8.60)$$

where

$$I_1 := \sup_\theta (A^\theta(\bar{t},\bar{x},\bar{p},D_{xp}\phi,M) - A^\theta(\bar{t},\bar{y},\bar{q},-D_{yq}\phi,-N))$$

$$I_2 := \sup_\theta (B^\theta_{\delta-}(\bar{t},\bar{x},\bar{p},\varphi_1(\bar{t},\bar{x},\bar{p})) - B^\theta_{\delta-}(\bar{t},\bar{y},\bar{q},\varphi_2(\bar{t},\bar{y},\bar{q})))$$

$$I_3 := \sup_\theta (B^\theta_{\delta+}(\bar{t},\bar{x},\bar{p},D_{xp}\phi,u(\bar{t},\bar{x},\bar{p})) - B^\theta_{\delta+}(\bar{t},\bar{y},\bar{q},-D_{yq}\phi,v(\bar{t},\bar{y},\bar{q})))$$

$$D_{xp}\phi(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q}) = \begin{pmatrix} \frac{x-y}{\varepsilon} + 2\alpha e^{\lambda(T-\bar{t})}\bar{x} & \frac{p-q}{\varepsilon} + 2\alpha e^{\lambda(T-\bar{t})}\bar{p} \end{pmatrix}'$$

$$D_{yq}\phi(\bar{t},\bar{x},\bar{p},\bar{y},\bar{q}) = \begin{pmatrix} -\frac{x-y}{\varepsilon} + 2\alpha e^{\lambda(T-\bar{t})}\bar{y} & -\frac{p-q}{\varepsilon} + 2\alpha e^{\lambda(T-\bar{t})}\bar{q} \end{pmatrix}'.$$

$$D^2_{xp}\phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) = D^2_{yq}\phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q}) = \begin{pmatrix} 1/\varepsilon + 2\alpha e^{\lambda(T-\bar{t})} & -1/\varepsilon \\ -1/\varepsilon & 1/\varepsilon + 2\alpha e^{\lambda(T-\bar{t})} \end{pmatrix}$$

For I_1, recalling the boundedness of θ, we have

$$I_1 \le C_1 \frac{(\bar{x}-\bar{y})^2 + (\bar{p}-\bar{q})^2}{\varepsilon} + C_2\alpha e^{\lambda(T-\bar{t})}(\bar{x}^2+\bar{y}^2+\bar{p}^2+\bar{q}^2).$$

For I_2, recalling what we have discussed before in (8.39), both of $B^\theta_{\delta-}$ and $B^\theta_{\delta+}$ are bounded uniformly in θ, thus

$$\lim_{\delta\to 0} I_2 \le 0.$$

For I_3, we combine the two integrals and calculate its integrand,

$$\begin{aligned}&B^{\theta}_{\delta+}(\bar{t},\bar{x},\bar{p},D_{xp}\phi,u(\bar{t},\bar{x},\bar{p}))-B^{\theta}_{\delta+}(\bar{t},\bar{y},\bar{q},-D_{yq}\phi,v(\bar{t},\bar{y},\bar{q}))\\=&\int_{\{|z|\geq\delta\}}H(z,\theta,u,v)n(dz)\end{aligned}$$

where

$$\begin{aligned}&H(z,\theta,u,v):=\\&\Phi(\bar{t},\bar{x}+\theta\psi(z)\bar{x},\bar{p}+\ell(z)\bar{p},\bar{y}+\theta\psi(z)\bar{y},\bar{q}+\ell(z)\bar{q})-\Phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q})+\\&\frac{(\bar{x}-\bar{y})^2\theta^2\psi^2(z)+(\bar{p}-\bar{q})^2\ell^2(z)}{2\varepsilon}+\alpha e^{\lambda(T-\bar{t})}[(\bar{x}^2+\bar{y}^2)\theta^2\psi^2(z)+(\bar{p}^2+\bar{q}^2)\ell^2(z)]\end{aligned}$$

Due to the boundedness of θ and the assumption (8.30) on ψ and (8.33) on ℓ, we have

$$\int_{\{|z|\geq\delta\}}H(z,\theta,u,v)n(dz)\leq C_1\frac{(\bar{x}-\bar{y})^2+(\bar{p}-\bar{q})^2}{\varepsilon}+C_2\alpha e^{\lambda(T-\bar{t})}(\bar{x}^2+\bar{y}^2+\bar{p}^2+\bar{q}^2)$$

Since the function $\Phi(t,x,y,p,q)$ has its maximum at $(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q})$, we have

$$\Phi(t,x,x,p,p)\leq\Phi(\bar{t},\bar{x},\bar{y},\bar{p},\bar{q})\leq u(\bar{t},\bar{x},\bar{p})-v(\bar{t},\bar{y},\bar{q}),\quad\forall\ (t,x,p)$$

By applying the inequality in (8.60) and after some simplifications, this implies

$$u(t,x,p)-v(t,x,p)\leq\frac{\beta}{t}+2\alpha e^{\lambda(T-t)}x^2+\frac{1}{\zeta}(I_1+I_2+I_3)-\frac{\lambda\alpha}{\zeta}e^{\lambda(T-\bar{t})}(\bar{x}^2+\bar{y}^2+\bar{p}^2+\bar{q}^2)$$

Let $\varepsilon\to0$ and recall that $(\bar{t},\bar{x},\bar{x},\bar{p},\bar{p})\to(t_0,x_0,y_0,p_0,q_0)$ which may depend on α, thus,

$$u(t,x,p)-v(t,x,p)\leq\frac{\beta}{t}+2\alpha e^{\lambda(T-t)}x^2+\frac{2\alpha}{\zeta}e^{\lambda(T-\bar{t})}(C-\lambda)(x_0^2+p_0^2)$$

We choose $\lambda>C$ and let $\beta,\alpha\to0$, it yields

$$u(t,x,p)\leq v(t,x,p).$$

The uniqueness of the viscosity of the HJB equation (8.40) associated with the terminal condition (8.41) is straightforward by using above theorem. Precisely, assume that u_1 and u_2 are two viscosity solutions of (8.40) satisfying

$$u_1(T, x, p) = u_2(T, x, p) = \widetilde{U}(T, x, p), \quad \forall\ (x, p).$$

They are both subsolutions and supersolutions. Then, by applying Theorem 8.4, we have both $u_1 \leq u_2$ and $u_2 \leq u_1$, hence, $u_1 = u_2$.

$\square$

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