University of Alberta

Optimal Portfolio Management When There Are Taxes and Transaction Costs

by

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in

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Abstract

We consider a financial market with a riskless asset (bank account or bond) and a risky asset (stock). The bond has a deterministic exponential evolution, and the stock is modelled as a geometric Brownian motion. For each transaction, brokerage fees are paid as a fixed proportion of the portfolio value. If the transaction incurs a profit, then this profit is taxed at a fixed rate, while for losses the investor obtains a tax credit. The objective is to maximize the long-run growth rate of the wealth.

We apply the theory of optimal stopping to determine the optimal investment strategy. For the case when the bond has zero interest rate we obtain analytical solutions, and for the case with a positive interest rate we apply the Markov chain approximation technique to obtain numerical solutions.

There are two surprising results: first, it is optimal to make a transaction not only when there is a loss but also when there is a gain, and secondly, the investor sometimes prefers a positive tax rate. We provide economic interpretation of the results.

"In mathematics you don't understand things, you just get used to them." G. Zukav

To my mother

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Chapter 1

Introduction

The modern development of mathematical sciences allows mathematicians to suggest improved models of the real world phenomena. In finance, the problem of optimally managing an investment portfolio receives increasing attention, as mathematical results are used to validate, and sometimes challenge, the intuition and experience of investors in the market.

Merton (1969, 1971) was the first to study a consumption-investment problem using continuous time stochastic processes. In his model the prices of the risky assets (stocks) follow geometric Brownian motions. The optimal strategy is that of continuous trading: at each moment of time a transaction is made on the market such that the proportions of wealth invested in the stocks are kept constant. While this model of a perfect market entails an infinite number of transactions, in a more realistic market model where brokerage fees (transaction costs) have to be paid for each transaction, this strategy would lead to ruin.

To account for this, Magill and Constantinides (1976) considered a market model with friction by incorporating transaction costs proportional to the amount of stock traded. Taksar, Klass and Assaf (1988) were the first to apply the theory of stochastic singular control to a model with proportional transaction costs for a market with one stock and one bond. Their optimal policy is to make no transaction when the proportion of wealth invested in the stock lies within a "no-trade" interval, and to make infinitesimal trades when this proportion attempts to escape the interval. These results were generalized by Akian, Sulem and Taksar (2001) for the case of multiple stocks and one bond. Their objective is to maximize the asymptotic rate of growth of portfolio value, while in Magill and Constantinides (1976), Constantinides (1983, 1984, 1986) and Davis and Norman (1990) (see also Shreve and Soner (1994)), the objective is to maximize the total discounted utility of consumption.

A more realistic structure of the transaction costs contains fixed transaction costs in addition to the proportional transaction costs. Eastham and Hastings (1988) were the first to apply the theory of stochastic impulse control to a consumption-investment problem with finite time horizon. Korn (1998) considered an infinite time horizon version of this problem, using quasi-variational inequalities. Variations of these models have been studied by Hastings (1992), Korn (1997), Øksendal and Sulem (2002), Chancelier, Øksendal and Sulem (2002) and others. For a survey of applications of stochastic impulse control to consumption-investment problems with transaction costs see Korn (1999).

Finally, transaction costs can be paid as a fixed proportion of wealth (portfolio management fee). Duffie and Sun (1990) consider such a model in which the wealth can be observed at transaction times. Morton and Pliska (1995) (see also Pliska and Selby (1994)) allow the investor to continuously observe the evolution of the stock prices, and the objective is to maximize the asymptotic rate of growth of portfolio value. Using the theory of semi-Markov decision processes and optimal stopping, they obtain explicit solutions for the case of one stock and one bond, and numerical solutions for the case of two stocks and one bond. An asymptotic analysis of this model is performed by Atkinson and Wilmott (1995), and a generalization to multiple stocks is done by Atkinson, Pliska and Wilmott (1997). A recent development is the introduction of risk-sensitive control techniques by Bielecki and Pliska (2000), such that the growth rate maximization criterion becomes just a special case.

The models above are presented from the point of view of the structure of the transaction costs. Other perspectives include: finite-time horizon versus infinite-time horizon, consumption versus no consumption, maximizing the utility of the consumption or final wealth versus maximizing the long-run average growth rate of the portfolio value. For a survey of papers on transaction costs see Cadenillas (2000). A presentation of the methods used in mathematical finance is given in Karatzas and Shreve (1998).

In the real world, capital gains and losses resulting from the portfolio management are taxed, so that the actual amount available to the investor is the after-tax money. It is, therefore, of great relevance to study the role that taxes play in consumption-investment problems. The case when the short-term and long-term tax rates are equal (symmetric taxation) is studied by Constantinides (1983), and the case when they are different (asymmetric taxation) by Constantinides (1984) and Dammon and Spatt (1996). The optimal strategy in these cases is to sell the stock when there is a loss and to defer the gains as long as possible (cut-losses-short-and-let-profits-run). Based on Constantinides (1984) and Dammon and Spatt (1996), Hur (2002) considers a three-period model with the stock having binomial evolution, and allows short sales in the *absence* of transaction costs. His optimal strategy under symmetric taxation is "cut-loss-keep-gain policy", and he mentions that perhaps the absence of transaction costs plays a determining role.

Hur (2002) observes that there are two approaches to finding the optimal investment strategy: the no-arbitrage approach, based on pricing (valuation of the tax-timing option resulting from the difference between long-term and short-term tax rates), which cannot fully describe the optimal trading strategy, and the utility approach based on investor's preference. Jensen (2002) investigates whether the no-arbitrage condition before the taxes also holds on an after-tax basis, in particular for a linear and symmetric taxation model.

Tompaidis, Gallmeyer and Kaniel (2002) extend Dammon, Spatt and Zhang (2001) to the case of two stocks and one bond. The optimal strategy for the case when short selling is prohibited is similar to that obtained by

Dammon, Spatt and Zhang (2001) in the case of one risky asset. Leland (2000) considers the objective of minimizing the deviation from exogenous constant portfolio weights, subject to proportional transaction costs and capital gains taxes. The form of the optimal strategy is similar to that of Taksar, Klass and Assaf (1988), and is consistent qualitatively with the strategy obtained by Dammon and Spatt (1996).

Considering the case of symmetric taxation for a portfolio of one security with the objective of maximizing the asymptotic growth rate of the portfolio value, Cadenillas and Pliska (1999) obtain two surprising results. First, it is optimal to sell not only when there is a loss, but also when there is a gain! Even more surprising is that in some cases the investor might be better off with a positive tax rate than in the absence of taxes! These are the results, in conjunction with the paper by Morton and Pliska (1995), that motivated the current research.

In this thesis, we investigate whether these surprising results also hold in the more realistic case in which the financial market consists of a bond in addition to the stock. We find that they do hold when the bond pays no interest, and that they hold with restrictions when the bond has a positive interest rate.

In Chapter 2, we present the stochastic model of the financial market, with the stock price following a geometric Brownian motion. Transaction costs are paid as a fraction of the entire wealth (portfolio management fee), while gains are taxed at a fixed rate with symmetric taxation. In the case of losses, we assume that the tax credits are realized as actual cash payments. The objective of the investor is to maximize the long-run growth rate of the portfolio wealth (Kelly criterion), which is consistent with the HARA logarithmic utility. We assume that there is no consumption, and that the stock pays no dividend.

We apply renewal theory in Chapter 3 to reduce the problem to that of optimally managing the portfolio in the first transaction cycle (semi-Markov decision processes).

Special cases of this problem have been studied by Morton and Pliska (1995) in the absence of taxes, and by Cadenillas and Pliska (1999) in the absence of a bond. We present the results of these papers in Chapter 4, where we also recover the problem studied by Merton (1971) in the absence of both taxes and transaction costs.

The case when the portfolio has one stock and one bond with zero interest rate is studied in Chapter 5. Using the dynamic programming approach as in Cadenillas and Pliska (1999), the optimal stopping problem is reduced to a moving boundary problem involving an ordinary differential equation (Hamilton-Jacobi-Bellman) that has an explicit analytical solution. The resulting optimal strategy realizes both losses and gains, and we also give a numerical example where the investor prefers a positive tax rate. An alternative approach to solving the optimal stopping problem via dynamic programming is to maximize a function of several variables; it was first suggested by Cadenillas and Pliska (1999), and we generalize it to this case.

If the strategy of continuous trading is used in the absence of transaction costs, Cadenillas and Pliska (1999) show that the taxes reduce the volatility of the investment. We generalize this result when there is also a zero-interest bond, and also show that when the volatility of the stock increases, the investor prefers a larger tax rate. This enables us to provide intuitive explanation of the results.

In the beginning of Chapter 6 we derive theoretical bounds for the long-run growth rate when the bond in the portfolio has a positive interest rate. This time the resulting moving boundary problem involves a partial differential equation with no explicit analytical solution, so numerical methods are required to approximate the solution. The optimal long-run growth rate also needs to be determined numerically, and to that end, an iterative algorithm is developed in Section 6.3. Finally, the Markov chain approximation technique of Kushner and Dupuis (1992) is used to obtain numerical solutions, which show that the optimal strategy realizes both losses and gains. Furthermore, for a relatively small interest rate we provide numerical examples where the investor is better off with a positive tax rate, while for a significant interest rate the investor prefers a world without taxes. Again, we analyze the results from an economic perspective, trying to understand the intuition behind them.

A generalization of our model to a portfolio holding multiple stocks and a bond is given in Chapter 7. For the general case when the interest rate of the bond is positive, the problem is reduced to a moving boundary problem with a partial differential equation in $n+1$ variables with constant coefficients, where n is the number of stocks. To apply the Markov chain approximation technique, a certain condition needs to be verified which depends on the particular values chosen for the parameters. When the bond has a zero interest rate, the equivalent moving boundary problem is different, as illustrated by the case with two stocks. The numerical methods are similar to those used in the previous cases, so numerical examples will be derived in future research.

Why does the optimal strategy that we obtain in the presence of taxes and transaction costs realize both losses and gains, when other papers show that it is optimal to realize only losses? To answer this question we concentrate on the model of Dammon and Spatt (1996), which considers transaction costs under asymmetric taxation with the objective of maximizing the total discounted utility of wealth. Their portfolio has only one stock (modelled, just like the dividends, by a binomial process), and no bond. We find that their model is valid only when the mean growth rate of the stock is smaller than the interest rate used for discounting. In this situation the optimal strategy should be to invest all the money in the bond, but their portfolio has no bond! Also, they use a risk-neutral utility in their objective, which is less realistic than the HARA logarithmic utility with which our model is consistent. This explains why we obtain different results, as our models are not compatible; it is clear, for the above reasons, that the model of Dammon and Spatt (1996) is less realistic than our model.

A review of the main results along with directions for future research conclude the thesis. Various codes used for the numerical examples of this thesis are provided in the appendices.

The contributions of this thesis are two-fold: mathematical and financial. The main mathematical contributions include an explicit solution to an optimal stopping problem and a numerical solution to another optimal stopping problem. From a financial point of view, we develop a more realistic model under taxes and transaction costs, and show that the optimal strategy is to realize both losses and gains. Furthermore, we show that sometimes the investor is better off with a positive tax rate. In short, we show how an astute investor can take advantage of taxes to reduce the risk of the investment.

Chapter 2

The Financial Market Model

Let $\{\Omega, \mathcal{F}, P\}$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that is the P-augmentation of the filtration generated by a one-dimensional standard Brownian motion $\{W_t\}_{t\geq 0}$ with $W_0 = 0$. In this financial market we consider one stock whose price is modelled by a geometric Brownian motion with dynamics

$$dS_t^1/S_t^1 = \mu dt + \sigma dW_t, \;\; t \geq 0, \tag{2.1}$$

and a bond with exponential form

$$dS_t^0 = rS_t^0 dt, \;\; t \geq 0. \tag{2.2}$$

In the above dynamics the drift coefficient μ and the volatility parameter σ are positive constants, and the interest rate r is a non-negative constant.

The price of the stock follows a geometric Brownian motion

$$S_t^1 = S_0^1 \, \exp\left\{\mu t + \sigma W_t - \frac{1}{2}\sigma^2 t\right\}, \;\; \forall t \geq 0, \tag{2.3}$$

while the price of the bond is a deterministic exponential given by

$$S_t^0 = S_0^0 e^{rt}, \;\; \forall t \geq 0. \tag{2.4}$$

In this financial market the investor chooses the times $0 = \tau_0 \leq \tau_1 \leq \tau_2 \ldots$ to trade shares of the two assets. Suppose that the initial investment at time $\tau_0 = 0$ is $\$V_0$. The investor designates a proportion $\pi_{\tau_0} \in [0,1]$ of the initial wealth to be invested in the stock. That is, an amount $\$\pi_{\tau_0} V_0$ is used to buy shares of the stock and the remaining $\$(1 - \pi_{\tau_0})V_0$ is placed in the bond. This portfolio is held (although the fraction of wealth invested in each asset will fluctuate as prices change) until some stopping time $\tau_1 \geq \tau_0$, when the investor decides to make a transaction.

At time τ_1-, just before making the transaction, the price of the stock is $S^1_{\tau_1-}$ and that of the bond is $S^0_{\tau_1-}$. By continuity we have $S^1_{\tau_1-} = S^1_{\tau_1}$ and $S^0_{\tau_1-} = S^0_{\tau_1}$. If V_t denotes the value of the portfolio at time $t \geq 0$, the value of the investment just before the transaction can be written as

$$V_{\tau_1-} = \frac{\pi_{\tau_0} V_0}{S^1_0} S^1_{\tau_1} + \frac{(1 - \pi_{\tau_0})V_0}{S^0_0} S^0_{\tau_1}. \tag{2.5}$$

Suppose that at time τ_1 the investor makes a transaction on the market, which incurs a cost equal to a fixed fraction $\alpha \in [0, 1)$ of the portfolio value. The remaining proceedings $\$(1 - \alpha)V_{\tau_1-}$ generate a profit (or a loss) $\$(1 - \alpha)V_{\tau_1-} - \$V_0$ which is taxed at a constant rate $\beta \in [0, 1)$.

Therefore, with each transaction the investor pays $\$\alpha V_{\tau_1-}$ in brokerage fees and $\$\beta[(1 - \alpha)V_{\tau_1-} - V_0]$ in taxes. The after-tax value of the investment is then

$$\begin{aligned}V_{\tau_1} &= (1-\alpha)V_{\tau_1-} - \beta[(1-\alpha)V_{\tau_1-} - V_0] = V_0\left[\beta + (1-\alpha)(1-\beta)\frac{V_{\tau_1-}}{V_{\tau_0}}\right] \\ &= V_0\left\{\beta + (1-\alpha)(1-\beta)\left[\pi_{\tau_0}\frac{S^1_{\tau_1}}{S^1_{\tau_0}} + (1-\pi_{\tau_0})\frac{S^0_{\tau_1}}{S^0_{\tau_0}}\right]\right\},\end{aligned}$$

so the overall factor by which wealth is increased over the first cycle is

$$M_1 := \frac{V_{\tau_1}}{V_{\tau_0}} = \beta + (1-\alpha)(1-\beta)\left[\pi_{\tau_0}\frac{S^1_{\tau_1}}{S^1_{\tau_0}} + (1-\pi_{\tau_0})\frac{S^0_{\tau_1}}{S^0_{\tau_0}}\right]. \qquad (2.6)$$

In (2.6) we refer to π_{τ_0} as the proportion that has been chosen at time $\tau_0 = 0$.

Note that transaction costs are paid only when the shares of the assets are redeemed, and not on their purchase. If, after paying transaction costs, the profit $\$(1-\alpha)V_{\tau_1-} - V_0$ is positive, then the investor *pays* a tax equal to β times that profit, whereas if the sale incurs a loss, then the investor *receives* a tax credit equal to β times the absolute value of the loss. (We assume there is other income, perhaps from the sale of other stock or from regular income, so the tax credit from a loss can be realized as an actual cash payment.)

We ignore complications of the tax laws such as distinctions between regular and investment income, limitations on tax credits, *wash* rules concerning the length of time between the time when a stock is sold and then re-purchased, and so forth. We also assume, for simplicity, that tax payments and receipts are made at the time when the stock transaction occurs, not, for example, at the end of some tax year.

At time τ_1 a proportion $\pi_{\tau_1} \in [0, 1]$ of the current wealth is invested in the stock, and the remaining proportion $1-\pi_{\tau_1}$ in the bond. This transaction cycle is repeated at times $\tau_2, \tau_3, \cdots$, with corresponding proportions $\pi_{\tau_2}, \pi_{\tau_3}, \cdots$.

Definition 2.1. *An* admissible strategy *is a sequence of pairs :*

$$\{(\tau_n, \pi_{\tau_n})\}_{n\geq 0},$$

where each τ_n is a stopping time such that $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \ldots$, and the $\mathcal{F}_{\tau_n}$-measurable random variable π_{τ_n} is the proportion chosen at time τ_n such that $P\{\pi_{\tau_n} \in [0,1]\} = 1$. Let us denote by $\mathcal{A}$ the class of admissible strategies.

The cash value of the investment at the end of any transaction cycle $[\tau_{n-1}, \tau_n)$ is given by

$$V_{\tau_n} = V_{\tau_{n-1}} M_n = \cdots = V_{\tau_0} M_1 \ldots M_n, \tag{2.7}$$

where M_n is the factor by which wealth was increased over the n-th cycle

$$M_n := \beta + (1-\alpha)(1-\beta)\left[\pi_{\tau_{n-1}} \frac{S^1_{\tau_n}}{S^1_{\tau_{n-1}}} + (1-\pi_{\tau_{n-1}})e^{r(\tau_n - \tau_{n-1})}\right]. \tag{2.8}$$

Generally, for $t \in [\tau_n, \tau_{n+1})$ we have

$$V_t = V_{\tau_n}\left[\pi_{\tau_n} \frac{S^1_t}{S^1_{\tau_n}} + (1-\pi_{\tau_n})e^{r(t-\tau_n)}\right].$$

Definition 2.2. *A measure of the growth of the investor's wealth is the* long-run growth rate *of the investment portfolio, defined as*

$$\liminf_{t\to\infty} \frac{1}{t} E[\log V_t]. \tag{2.9}$$

Since the long-run growth rate depends on the particular investment strategy chosen, different strategies will result in different growth rates. The focus of this thesis is finding the strategy that maximizes the long-run growth rate of the portfolio.

The long-run growth rate of the stock is denoted by

$$\lambda := \mu - \frac{1}{2}\sigma^2. \tag{2.10}$$

In the case of a portfolio with no bond this quantity needs to be positive (see Cadenillas and Pliska (1999)), while in the case when there is also a bond there is no such restriction (see Morton and Pliska (1995)).

The long-run growth rate is consistent with the logarithmic utility. To see this we review some utility functions according to Section 2.3 of Föllmer and Schied (2002).

Definition 2.3. *A function* $u : (0, \infty) \longrightarrow \mathbb{R}$ *is called a* utility function *if it is strictly concave, strictly increasing and continuous.*

Definition 2.4. *Assume the utility function* u *is twice differentiable. Then*

$$\alpha(x) := -\frac{u''(x)}{u'(x)}$$

is called the Arrow-Pratt coefficient of absolute risk aversion of u at level x.

Based on this coefficient we can give some examples of utility functions.

Example 2.1. *a)* **CARA** (constant absolute risk aversion) utility functions

For a constant coefficient $\alpha(x) = \alpha > 0$ *we obtain*

$$u(x) = 1 - e^{-\alpha x}, \; x > 0.$$

b) **HARA** (hyperbolic absolute risk aversion) utility functions

For a coefficient $\alpha(x) = (1-\gamma)/x$ *for* $x > 0$ *and* $\gamma \in [0, 1)$ *we obtain*

$$u(x) = \log x \;\; \textit{for} \;\; \gamma = 0,$$

$$u(x) = \frac{1}{\gamma} x^{\gamma} \;\; \textit{for} \;\; 0 < \gamma < 1.$$

For $\gamma = 1$ *we have the* risk-neutral case, *with linear utility function (for example* $u(x) = x$*).*

In Chapter 8 we will refer to a less realistic model from literature which is based on risk-neutral utility, but throughout this thesis our model will be concerned only with the long-run growth rate of the investment portfolio.

Chapter 3

The Problem

Problem 3.1. *The investor wants to determine the admissible strategy that maximizes the long-run growth rate of his or her investment. The strategy that maximizes this criterion (known as* the Kelly criterion*) will be called the* optimal strategy*, and the corresponding long-run growth rate will be denoted by R.*

The investor plans to be in the market for a long time, so we assume an infinite time horizon. When $\beta = 0$ we recover the model with no taxes of Morton and Pliska (1995) when there is only one stock and one bond. If we take $\pi_{\tau_n} \equiv 1$, for all $n \in \mathbb{N}$, we get the model of a single stock of Cadenillas and Pliska (1999).

Recall that the random variable M_n given in (2.8) represents the multiplicative factor by which the value of the investment increases over the n-th transaction cycle, so that $V_{\tau_0} M_1 M_2 \ldots M_n$ is the value of the investment just after the n-th transaction. Note that M_n has the same probability distribution as the random variable

$$\beta + (1-\alpha)(1-\beta)\left[(1-\pi_{\tau_{n-1}})\, e^{r(\tau_n - \tau_{n-1})} + \pi_{\tau_{n-1}}\, S^1_{(\tau_n - \tau_{n-1})}\right], \tag{3.1}$$

since $W_{t+s} - W_s$ and W_t are identically distributed.

We want to choose the optimal strategy $(\tau_0, \pi_{\tau_0}), (\tau_1, \pi_{\tau_1}), \cdots$. We first choose the optimal proportion $\pi_{\tau_0} = \pi$ at time $\tau_0 = 0$ in some optimal fashion. Then optimally stop at time τ_1, pay transaction costs and taxes and collect the after-tax wealth.

At the beginning of the second cycle we have to choose again an optimal value for the proportion π_{τ_1} of current wealth to be invested in the same stock

$$S^1_t = S^1_{\tau_1} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t - \tau_1) + \sigma(W_t - W_{\tau_1})\right\}, \quad t \geq \tau_1,$$

which has identical distribution to

$$S^1_{\tau_1} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t - \tau_1) + \sigma W_{t-\tau_1}\right\},$$

and the remaining wealth will be invested in the same bond

$$S^0_t = S^0_{\tau_1} e^{r(t-\tau_1)}, \quad t \geq \tau_1.$$

We are thus in the same context as before (see (2.3), (2.4)), so we choose the optimal proportion $\pi_{\tau_2} \equiv \pi$, and the stopping time τ_2 so that $\tau_2 - \tau_1, \tau_1 - \tau_0$ are independent and identically distributed (i.i.d.) random variables.

Generally, at the beginning of the transaction cycle $[\tau_{n-1}, \tau_n)$ we choose the *proportion* $\pi_{\tau_{n-1}} \equiv \pi$, and we stop the process at time τ_n following the same policy employed in the previous transaction cycles, thus having $\tau_n - \tau_{n-1}, \tau_{n-1} - \tau_{n-2}, \ldots, \tau_1 - \tau_0$ i.i.d.

Remark 3.1. *From the above discussion, it follows that it is optimal to select an admissible strategy such that* $\{(\tau_n - \tau_{n-1}, M_n), n \in \mathbb{N}\}$ *is a sequence of i.i.d. random vectors.*

Suppose at initial time $\tau_0 = 0$ the investor allocates a proportion $\pi_{\tau_0} = \pi$ of the wealth in the stock, and the remaining proportion $1 - \pi_{\tau_0} = 1 - \pi$ of the wealth in the bond. For this fixed constant $\pi \in [0,1]$ we define, for $t \geq 0$

$$\begin{aligned} I_t(\pi) &:= (1-\pi)\frac{S_t^0}{S_0^0} + \pi\frac{S_t^1}{S_0^1}, \\ &= (1-\pi)\, e^{rt} + \pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \qquad (3.2) \\ I_0(\pi) &:= 1. \end{aligned}$$

According to (2.5), this models the factor by which wealth is increased before paying taxes and transaction costs.

Let us denote by $\mathcal{S}$ the class of positive stopping times, and by

$$\tilde{\mathcal{S}} := \{\tau \in \mathcal{S} : E[\tau] \in (0,\infty)\} \qquad (3.3)$$

the subclass of stopping times with positive and finite expectation.

Define the function $g : [0,\infty) \to \mathbb{R}$ by

$$g(x) := \log\{\beta + (1-\beta)(1-\alpha)x\}. \qquad (3.4)$$

Lemma 3.1. *For every $\eta \in (0,\infty)$, $\tau \in \tilde{\mathcal{S}}$, and $\pi \in [0,1]$*

$$E[g(I_\tau(\pi)) - \eta\tau] = g(1) + E\left[\int_0^\tau Y_t dt\right]. \tag{3.5}$$

Here $Y = Y(\eta)$ is the stochastic process defined by

$$\begin{aligned} Y_t : &= \left[\frac{(1-\alpha)(1-\beta)}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right][\mu I_t(\pi) - (\mu - r)(1-\pi)e^{rt}] \\ &- \frac{\sigma^2}{2}\left[\frac{(1-\alpha)(1-\beta)}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right]^2 [I_t(\pi) - (1-\pi)e^{rt}]^2 - \eta, \end{aligned} \tag{3.6}$$

where π is the proportion corresponding to the transaction cycle $[0,\tau)$.

Proof. According to (3.2) we can write $I_t(\pi) = f(t, W_t)$, where

$$f(t,x) := (1-\pi)e^{rt} + \pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma x\right\}.$$

Applying Itô's formula we get for every $\tau \in \tilde{\mathcal{S}}$ and $\pi \in [0,1]$

$$I_\tau(\pi) = 1 + \int_0^\tau \frac{\partial f}{\partial t}(t, W_t)dt + \int_0^\tau \frac{\partial f}{\partial x}(t, W_t)dW_t + \frac{1}{2}\int_0^\tau \frac{\partial^2 f}{\partial x^2}(t, W_t)dt,$$

with

$$\begin{aligned} \frac{\partial f}{\partial t}(t, W_t) &= r(1-\pi)e^{rt} + \left(\mu - \frac{1}{2}\sigma^2\right)\pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \\ \frac{\partial f}{\partial x}(t, W_t) &= \sigma\pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \\ \frac{\partial^2 f}{\partial x^2}(t, W_t) &= \sigma^2\pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}. \end{aligned}$$

Rewriting the above derivatives in terms of $I_t(\pi)$ we get

$$I_\tau(\pi) = 1 + \int_0^\tau \{\mu I_t(\pi) - (\mu - r)(1-\pi)e^{rt}\}dt + \int_0^\tau \sigma[I_t(\pi) - (1-\pi)e^{rt}]dW_t.$$

The above semimartingale form of $I(\pi)$ allows us to apply Itô's formula to the function $f_1(t, I_t(\pi)) = g(I_t(\pi)) - \eta t$ and get

$$\begin{aligned} g(I_\tau(\pi)) - \eta\tau &= g(1) + \int_0^\tau \frac{\partial f_1}{\partial t}(t, I_t(\pi))dt + \int_0^\tau \frac{\partial f_1}{\partial x}(t, I_t(\pi))dI_t \\ &+ \frac{1}{2}\int_0^\tau \frac{\partial^2 f_1}{\partial x^2}(t, I_t(\pi))d\langle I(\pi)\rangle_t, \end{aligned}$$

where $d\langle I(\pi)\rangle_t = \sigma^2[I_t(\pi) - (1-\pi)e^{rt}]^2 dt$.

We have

$$\begin{aligned} g(I_\tau(\pi)) - \eta\tau &= g(1) + \int_0^\tau (-\eta)dt + \frac{1}{2}\int_0^\tau g''(I_t(\pi))\sigma^2[I_t(\pi) - (1-\pi)e^{rt}]^2 dt \\ &+ \int_0^\tau g'(I_t(\pi))\{\mu I_t(\pi) - (\mu - r)(1-\pi)e^{rt}\}dt \\ &+ \int_0^\tau g'(I_t(\pi))\sigma[I_t(\pi) - (1-\pi)e^{rt}]dW_t, \end{aligned}$$

or

$$\begin{aligned} g(I_\tau(\pi)) - \eta\tau = g(1) + \int_0^\tau Y_t dt \\ + \int_0^\tau \frac{(1-\alpha)(1-\beta)\sigma}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}[I_t(\pi) - (1-\pi)e^{rt}]dW_t. \end{aligned} \tag{3.7}$$

We observe now that the expected value of the stochastic integral in (3.7) is zero when $E[\tau] < \infty$. In fact, if $\beta = 0$ and $\pi = 1$, then the stochastic integral is just σW_τ, and we may apply Wald's identity for Brownian motion (see Section 4.1.6 of Shiryayev (1978)). Otherwise the integrand of the stochastic integral is bounded, so the claim is still valid. Hence, taking the expected value in (3.7), we obtain (3.5).

□

Lemma 3.2. *If the pairs* $(\tau_i - \tau_{i-1}, M_i)$, $i \in \mathbf{N}$, *are independent and identically distributed with* $E[\tau_1] < \infty$, *then*

$$\liminf_{t\to\infty} \frac{1}{t} E[\log V_t] = \frac{E[\log M_1]}{E[\tau_1]} = \frac{E[\log\{V_{\tau_1}/V_0\}]}{E[\tau_1]} = \frac{E[g(I_{\tau_1}(\pi))]}{E[\tau_1]}.$$

Proof. We check that the stochastic process Y of Lemma 3.1 is bounded. Note that the right coefficient of the first term of (3.6) can be bounded

$$\begin{aligned} 0 &< \mu I_t(\pi) - (\mu - r)(1-\pi)e^{rt} = \mu\pi \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\} + r(1-\pi)e^{rt} \\ &\le \max\{\mu, r\}\left\{(1-\pi)e^{rt} + \pi\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\right\} = \max\{\mu, r\} I_t(\pi). \end{aligned}$$

This guarantees that the first term of (3.6) is bounded. For the second term we have

$$0 < [I_t(\pi) - (1-\pi)e^{rt}]^2 \le I_t(\pi)^2,$$

thus the second term is also bounded. Hence the process Y is bounded.

Using Lemma 3.1 with a bounded process Y and a stopping time τ with $E[\tau] < \infty$, we obtain $E[g(I_\tau(\pi))] < \infty$. The definition of the function g given in (3.4) and that of $I_t(\pi)$ given in (3.2) on one hand, and the overall factor M_1 given in (2.6) by which wealth is increased after paying taxes and transaction costs - on the other hand, show that

$$g(I_{\tau_1}(\pi)) = \log M_1 = \log \frac{V_{\tau_1}}{V_0}.$$

This implies $E[\log M_1] < \infty$, and we can apply renewal theory (see Theorem 3.6.1 of Ross (1996)) to get the conclusion of this Lemma.

□

In view of Remark 3.1 and Lemma 3.2, we can characterize the admissible strategies and the corresponding long-run growth rates.

Corollary 3.1. *In terms of Definition* 2.1, *the pairs that define the admissible strategies have the form:*

$$(\tau_0 = 0, \pi), (\tau_1, \pi), (\tau_2, \pi), (\tau_3, \pi), \ldots.$$

Here $\{\tau_n - \tau_{n-1};\ n \in \mathbb{N}\}$ *is the set of optimal times between transactions, and* $\pi \in [0, 1]$ *is the proportion of wealth to be invested in the stock.*

Corollary 3.2. *The* long-run growth rate (2.9) *corresponding to an admissible strategy* $\{(\tau_0 = 0, \pi), (\tau_1, \pi), (\tau_2, \pi), (\tau_3, \pi), \ldots\}$ *as in Corollary* 3.1 *is*

$$J(\tau, \pi) := \frac{E[\log\{V_\tau / V_0\}]}{E[\tau]} = \frac{E[g(I_\tau(\pi))]}{E[\tau]} = \frac{E[\log\{\beta + (1-\beta)(1-\alpha)I_\tau(\pi)\}]}{E[\tau]}, \tag{3.8}$$

with $I_\tau(\pi)$ *given by (3.2).*

The investor's portfolio management problem can be restated as follows.

Problem 3.2. *Select a stopping time* $\hat{\tau} \in \tilde{\mathcal{S}}$ *and a constant* $\hat{\pi} \in [0, 1]$ *that maximize* $J(\tau, \pi)$*:*

$$\sup_{\pi \in [0,1], \tau \in \tilde{\mathcal{S}}} J(\tau, \pi) = J(\hat{\tau}, \hat{\pi}) = R. \tag{3.9}$$

Remark 3.2. *The optimal strategy in terms of Corollary 3.1 will be to invest a proportion* $\hat{\pi}$ *of the wealth in the stock at the beginning of each transaction cycle, and hold that portfolio for an amount of time* $\tau_n - \tau_{n-1}$ *corresponding to the n-th transaction cycle, such that* $\tau_n - \tau_{n-1}$ *and* $\hat{\tau}$ *are identically distributed.*

We show next that when the stock has a large drift coefficient the maximum long-rung growth rate is at most λ.

Theorem 3.1. *If $\mu \geq r + \sigma^2$, then $J(\tau, \pi) \leq \lambda$ for every $\tau \in \tilde{S}$, $\pi \in [0,1]$, and $\beta \in [0,1]$.*

Proof. Lemma 3.1 gives, with $\eta = \lambda = \mu - \sigma^2/2 \geq r + \sigma^2/2 > 0$ and $\beta > 0$

$$
\begin{aligned}
Y_t &= \frac{\mu\beta + \mu(1-\alpha)(1-\beta)I_t(\pi) - \mu\beta}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} - \frac{(\mu - r)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} \\
&\quad - \frac{\sigma^2}{2}\left[\frac{(1-\alpha)(1-\beta)[I_t(\pi) - (1-\pi)e^{rt}]}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right]^2 - \mu + \sigma^2/2 \\
&= \mu - \frac{\mu\beta}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} - \frac{(\mu - r)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} \\
&\quad - \frac{\sigma^2}{2}\left[\frac{\beta + (1-\alpha)(1-\beta)I_t(\pi) - \beta - (1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right]^2 \\
&\quad - \mu + \sigma^2/2 \\
&= -\frac{\mu\beta + (\mu - r)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} \\
&\quad - \frac{\sigma^2}{2}\left[1 - \frac{\beta + (1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right]^2 + \sigma^2/2 \\
&= -\frac{\mu\beta + (\mu - r)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} + \sigma^2/2 \\
&\quad - \frac{\sigma^2}{2}\left[1 - 2\frac{\beta + (1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right. \\
&\qquad \left. + \left(\frac{\beta + (1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}\right)^2\right] \\
&= -\frac{\mu\beta + (\mu - r)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)} + \sigma^2/2 - \sigma^2/2 \\
&\quad + \sigma^2\frac{\beta + (1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta + (1-\alpha)(1-\beta)I_t(\pi)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\sigma^2}{2}\left(\frac{\beta+(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta+(1-\alpha)(1-\beta)I_t(\pi)}\right)^2 \\
= \; &-\frac{(\mu-\sigma^2)\beta+(\mu-r-\sigma^2)(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta+(1-\alpha)(1-\beta)I_t(\pi)} \\
&-\frac{\sigma^2}{2}\left(\frac{\beta+(1-\alpha)(1-\beta)(1-\pi)e^{rt}}{\beta+(1-\alpha)(1-\beta)I_t(\pi)}\right)^2.
\end{aligned}
$$

For every $t \geq 0$ we have

$$
(1-\alpha)(1-\beta)(1-\pi)e^{rt} \geq 0 \text{ and } \beta+(1-\alpha)(1-\beta)I_t(\pi) \geq 0.
$$

Using $\mu \geq r+\sigma^2 \geq \sigma^2$ we get, for every $t \geq 0$, $Y_t \leq 0$.

When $\beta = 0$ the process Y becomes

$$
\begin{aligned}
Y_t &= -\frac{(\mu-r-\sigma^2)(1-\alpha)(1-\pi)e^{rt}}{(1-\alpha)I_t(\pi)} - \frac{\sigma^2}{2}\left(\frac{(1-\alpha)(1-\pi)e^{rt}}{(1-\alpha)I_t(\pi)}\right)^2 \\
&= -\frac{(\mu-r-\sigma^2)(1-\pi)e^{rt}}{I_t(\pi)} - \frac{\sigma^2}{2}\left(\frac{(1-\pi)e^{rt}}{I_t(\pi)}\right)^2.
\end{aligned}
$$

Again, $\mu \geq r+\sigma^2 \geq \sigma^2$ gives, for every $t \geq 0$, $Y_t \leq 0$.

Applying Lemma 3.1 yields

$$
E[g(I_\tau(\pi)) - \lambda\tau] \leq g(1) = \log\{\beta+(1-\alpha)(1-\beta)\} \leq 0.
$$

□

In particular, Theorem 3.1 is valid for any $r \geq 0$ and $\beta \geq 0$, thus it motivates the following assumption.

Assumption 3.1. *We shall assume that*

$$\mu < r + \sigma^2. \tag{3.10}$$

This means that the stock has a moderate drift coefficient.

Problem 3.1 has been transformed by Markov renewal theory (see Ross (1996)) into a problem of selecting a single stopping time $\hat{\tau}$ and a constant $\hat{\pi} \in [0, 1]$. Based on the previous remark, the pair $(\hat{\tau}, \hat{\pi})$ completely describes the investment strategy of Corollary 3.1.

For fixed $\pi \in [0, 1]$ we denote the maximum long-run growth rate of the portfolio by

$$R_\pi := \sup_{\tau \in \tilde{S}} J(\tau, \pi) = \sup_{\tau \in \tilde{S}} \frac{E[\log\{V_\tau / V_0\}]}{E[\tau]} = \sup_{\tau \in \tilde{S}} \frac{E[g(I_\tau(\pi))]}{E[\tau]}. \tag{3.11}$$

Thus

$$R = \sup_{\pi \in [0,1]} R_\pi, \tag{3.12}$$

and for any $\pi \in [0, 1] : R_\pi \leq R$, with equality for $\pi = \hat{\pi}$.

From (3.11) it follows that for fixed $\pi \in [0, 1]$ and for every $\tau \in \tilde{S}$ we have

$$\frac{E[g(I_\tau(\pi))]}{E[\tau]} \leq R_\pi, \text{ with equality for some } \tau_\pi \in \tilde{S},$$

or, equivalently, for fixed $\pi \in [0, 1]$ and for every $\tau \in \tilde{S}$

$$E[g(I_\tau(\pi)) - R_\pi \tau] \leq 0, \text{ with equality for the same } \tau_\pi \in \tilde{S}.$$

We write these comments formally in the following theorem.

Theorem 3.2. *For each fixed* $\pi \in [0,1]$, $R_\pi \in [0,\infty)$ *is characterized by*

$$\sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - R_\pi \tau] = 0. \tag{3.13}$$

Furthermore, the maximum long-run growth rate R is the supremum given in (3.12), *and the optimal proportion* $\hat{\pi}$ *is the one that achieves that supremum.*

Remark 3.3. *For* $\beta = 0$ *(no taxes), this is Proposition 3.1 of Morton and Pliska (1995) in the case of one bond and only one stock. Furthermore, if* $\pi \equiv 1$, *then* $J(\tau, 1)$ *is* $J(\tau)$ *of Problem* 3.2 *of Cadenillas and Pliska (1999), and problem* (3.13) *reduces to Problem 5.1 in their paper. Thus, Problem* 3.1 *is more general than the problems studied by Cadenillas and Pliska (1999), and by Morton and Pliska (1995) when there is only one stock.*

Remark 3.4. *One admissible strategy as described in Remark 3.2 is to select* $\bar{\pi} = 1$ *and the optimal stopping times between transactions independent random variables, having identical distribution to the optimal stopping time of Theorem 5.2 of Cadenillas and Pliska (1999)*

$$\bar{\tau} = \inf\{t \in [0,\infty) : S_t^1/S_0^1 \notin (a,b)\}, \text{ with } a, b \in \mathbb{R}.$$

Another policy considered by Cadenillas and Pliska (1999) is to never make a transaction (buy-and-hold), *characterized by*

$$\bar{\tau}_1 = \inf\{t \in [0,\infty) : S_t^1/S_0^1 \notin (0,\infty)\} = \infty,$$

which results in a long-run growth rate equal to λ *(see (2.10)).*

They show that $J(\bar{\tau}, \bar{\pi}) \geq \lambda$. Since the optimal strategy is chosen among all the admissible strategies in $\mathcal{A}$, the long-run growth rate $J(\hat{\tau}, \hat{\pi})$ satisfies

$$R = \sup_{\mathcal{A}} J(\tau, \pi) \geq J(\bar{\tau}, \bar{\pi}) \geq \lambda. \tag{3.14}$$

This leads us to expect a value of R such that $R \geq \lambda$. When the inequality $R > \lambda$ holds we expect that $R_\pi \geq \lambda$ for a whole range of values of π in a neighbourhood of $\hat{\pi}$.

The next remark will be instrumental in deriving an iterative algorithm in Section 6.3.

Remark 3.5. *Let $\pi \in [0, 1]$ be a fixed constant. Then*

$$\theta \longmapsto \sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - \theta\tau]$$

is a decreasing function of θ, which equals zero when $\theta = R_\pi$. Therefore we have

$$\begin{aligned} &\sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - \theta\tau] > 0, \quad \text{if } \theta < R_\pi, \\ &\sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - \theta\tau] = 0, \quad \text{if } \theta = R_\pi, \\ &\sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - \theta\tau] < 0, \quad \text{if } \theta > R_\pi. \end{aligned} \tag{3.15}$$

According to Theorem 3.2, to solve problem (3.13) we first fix $\pi \in [0, 1]$ and determine the corresponding R_π, then determine R using (3.12).

We consider now determining R_π for given $\pi \in [0, 1]$. By Theorem 3.2 and definitions (3.2) and (3.4) this is equivalent to the following problem.

Problem 3.3. *For each fixed* $\pi \in [0,1]$, *determine the value* R_π *of* θ *for which the following optimal stopping problem has value zero*

$$\sup_{\tau \in \tilde{S}} E[g(I_\tau(\pi)) - R_\pi \tau] = 0. \tag{3.16}$$

That is, for each fixed $\pi \in [0,1]$ *and each fixed* θ, *solve the optimal stopping problem with value*

$$H(\theta) := \sup_{\tau \in \tilde{S}} E\Big[\int_0^\tau (-\theta)\, du + g(I_\tau(\pi))\Big]. \tag{3.17}$$

Then, for that fixed $\pi \in [0,1]$, *determine the value* R_π *such that*

$$H(R_\pi) = 0. \tag{3.18}$$

This is a problem of maximizing the average return per unit of time, when we have to pay a continuation fee at a rate θ per unit of time, and the reward collected for stopping is $g(I_\tau(\pi))$.

We can also write our problem in terms of the fraction process $\{B_t\}_{t\geq 0}$ used by Morton and Pliska (1995). They define B_t as the ratio of the monetary value of the stock position at time t to the total wealth at time t. Then the fraction $B_0 = \pi$ of the initial wealth is used to buy shares in the stock, and the remaining $1-\pi$ fraction of the initial wealth is invested in the bond. At time τ the investor makes a transaction on the market to rebalance the portfolio. The number of shares of bond in the portfolio is the same at initial time 0 and at any moment of time t just before the transaction time τ

$$\frac{(1-B_0)V_0}{S_0^0} = \frac{(1-B_t)V_t}{S_t^0},$$

so the increase (3.2) in the wealth at any time t before the transaction is

$$\frac{V_t}{V_0} = \frac{(1 - B_0)e^{rt}}{1 - B_t}. \tag{3.19}$$

The above formula will help us in Section 4.2.2 to relate our results to those of Morton and Pliska (1995) (see (4.9)).

We conclude this chapter with a general notation that shows the dependence of the best long-run growth rate on the parameters of the model.

Notation 3.1. *Each combination of the parameters* (μ, σ, r) *of the two assets, transaction cost rate* $\alpha \geq 0$, *tax rate* $\beta \geq 0$ *and proportion of money invested in the stock* $\pi \in [0, 1]$, *will result in a best long run growth rate denoted by*

$$R^{(r,\mu,\sigma)}_{\pi,\alpha,\beta} := \sup_{\tau \in \tilde{S}} J(\tau, \pi).$$

Chapter 4

Particular Cases

4.1 No Bond

When $\pi_{\tau_n} \equiv 1$ for all $n \in \mathbb{N}$, the portfolio consists only of shares of the stock. This case is solved by Cadenillas and Pliska (1999) for a positive long-run growth rate of the stock: $\lambda > 0$.

For a stock with a medium appreciation rate

$$\frac{\sigma^2}{2} < \mu < \sigma^2$$

the real constants a, b, c, d, θ with $0 < a < 1 < b < \infty$, $\theta \geq \lambda$ are obtained from the system of equations

$$\begin{aligned} s(1) &= 0, \\ s(a) &= g(a), \\ s(b) &= g(b), \\ s'(a) &= g'(a), \\ s'(b) &= g'(b), \end{aligned}$$

where the functions $s, g : (0, \infty) \longrightarrow \mathbb{R}$ are given by

$$s(x) = c + dx^{1-\frac{2\mu}{\sigma^2}} + \frac{\theta}{\lambda}\log x, \tag{4.1}$$

$$g(x) = \log\{\beta + (1-\alpha)(1-\beta)x\}. \tag{4.2}$$

Theorem 4.1. *When $\sigma^2/2 < \mu < \sigma^2$, the optimal strategy is given by*

$$\hat{\tau} = \inf\left\{t \geq 0 : \frac{S_t^1}{S_0^1} \notin (a, b)\right\},$$

with resulting long-run growth rate of θ. Furthermore, for a stock with a large appreciation rate $\mu \geq \sigma^2$, the optimal strategy is buy-and-hold

$$\hat{\tau} = \inf\left\{t \geq 0 : \frac{S_t^1}{S_0^1} \notin (0, \infty)\right\} = \infty.$$

This strategy is equivalent to the strategy cut-losses-short-and-let-profits-run

$$\hat{\tau} = \inf\left\{t \geq 0 : \frac{S_t^1}{S_0^1} \notin (a, \infty)\right\}, \ \forall a \in (0, 1),$$

in the sense that both of them yield a long-run growth rate equal to λ.

Proof. The first part of this theorem is proved in Theorem 5.2 of Cadenillas and Pliska (1999), and the second part in Theorem 4.1 of Cadenillas and Pliska (1999).

□

4.2 No Taxes

4.2.1 Neither Taxes Nor Transaction Costs

In the case with no transaction costs and no taxes ($\alpha = \beta = 0$), Problem 3.2 becomes:

Problem 4.1. *Select a stopping time $\hat{\tau} \in \tilde{S}$ and a constant $\hat{\pi} \in [0,1]$ that maximize*

$$J(\tau, \pi) := \frac{E[\log\{I_\tau(\pi)\}]}{E[\tau]}, \tag{4.3}$$

with $I_\tau(\pi)$ given by (3.2).

The solution to this problem is due to Merton (1971), and is presented in Section 2 of Morton and Pliska (1995).

Theorem 4.2. *The optimal policy is to continually rebalance the portfolio* (continuous trading), *with optimal proportion given by*

$$\tilde{\pi} = \frac{\mu - r}{\sigma^2}, \tag{4.4}$$

and corresponding maximum growth rate given by

$$\tilde{R} = (1 - \tilde{\pi})r + \tilde{\pi}\mu - \frac{\sigma^2}{2}\tilde{\pi}^2 = r + \frac{(\mu - r)^2}{2\sigma^2}. \tag{4.5}$$

Example 4.1. *Let us consider the case when*

$$\mu = 0.065, \quad \sigma = 0.3, \quad \alpha = 0.0 \quad \textit{and} \quad \beta = 0.0. \tag{4.6}$$

For $r = 0$ the optimal policy is continuous trading with the investor rebalancing the portfolio to maintain a proportion

$$\tilde{\pi} = \frac{\mu - r}{\sigma^2} = \frac{\mu}{\sigma^2} = 0.065/0.09 = 0.72222$$

of the wealth in the stock, resulting in a long-run growth rate of

$$\tilde{R} = r + \frac{(\mu - r)^2}{2\sigma^2} = \frac{\mu^2}{2\sigma^2} = 0.023472222.$$

For $r = 0.015$ the optimal policy is continuous trading with optimal proportion

$$\tilde{\pi} = \frac{\mu - r}{\sigma^2} = 0.05/0.09 = 0.55556$$

and resulting long-run growth rate

$$\tilde{R} = r + \frac{(\mu - r)^2}{2\sigma^2} = 0.0288889.$$

4.2.2 The Case of No Taxes, but Positive Transaction Costs

In this section we assume that $\beta = 0$ and $\alpha \in (0, 1)$. This case was studied by Morton and Pliska (1995) for a portfolio with one bond and multiple stocks. We show next that when the portfolio has one stock and one bond the problem is equivalent to that of a portfolio with a stock with a different drift coefficient and a *zero-interest* bond.

Proposition 4.1. *In the absence of taxes ($\beta = 0$), the optimal strategy for a portfolio with a stock with drift coefficient μ and a bond with interest rate $r > 0$ is identical to the one for a portfolio with a stock with drift coefficient $\mu - r$ and a zero-interest bond, and*

$$R^{(r,\mu,\sigma)}_{\hat{\pi},\alpha,0} = r + R^{(0,\mu-r,\sigma)}_{\hat{\pi},\alpha,0}.$$

Proof. Taking $\beta = 0$ in (3.8) we get

$$R^{(r,\mu,\sigma)}_{\hat{\pi},\alpha,0} = \sup_{\pi\in[0,1],\tau\in\tilde{S}} \frac{E\Big[\log\{(1-\alpha)\big[(1-\pi)e^{r\tau} + \pi e^{(\mu-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)}$$

$$
\begin{aligned}
&= \sup_{\pi\in[0,1],\tau\in\tilde{S}} \frac{E\Big[\log\{(1-\alpha)e^{r\tau}\big[(1-\pi)+\pi e^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)} \\
&= \sup_{\pi\in[0,1],\tau\in\tilde{S}} \frac{E\Big[r\tau+\log\{(1-\alpha)\big[(1-\pi)+\pi e^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)} \\
&= r + \sup_{\pi\in[0,1],\tau\in\tilde{S}} \frac{E\Big[\log\{(1-\alpha)\big[(1-\pi)+\pi e^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)},
\end{aligned}
$$

or

$$
R^{(r,\mu,\sigma)}_{\hat{\pi},\alpha,0} = r + R^{(0,\mu-r,\sigma)}_{\hat{\pi},\alpha,0}.
$$

□

In Chapter 5 we solve the problem for the case of one stock and one bond with zero interest rate under positive taxes: $\beta \in [0,1)$. Here we present only the particular case $\beta = 0$. This time the stock has drift $\mu - r$ and volatility σ, so we use $\{e^{-rt}S^1_t\}_{t\geq 0}$ to model it.

Fix $\pi \in (0,1)$. Define the function $\tilde{g}_0 : (0,\infty) \longrightarrow \mathbb{R}$ by

$$
\tilde{g}_0(x) := \log(1-\pi+\pi x) + \log(1-\alpha),
$$

and recall that the function s was defined in (4.1) as

$$
s(x) = c + dx^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{\lambda}\log x.
$$

Let $a \in (0,1)$, $b \in (1,\infty)$, $\theta \in (0,\infty)$, $c \in \mathbb{R}$ and $d \in \mathbb{R}$ be the solution of the following system of five equations

$$\begin{aligned} s(1) &= 0, \\ s(a) &= \tilde{g}_0(a), \\ s(b) &= \tilde{g}_0(b), \\ s'(a) &= \tilde{g}_0'(a), \\ s'(b) &= \tilde{g}_0'(b). \end{aligned}$$

Define the function $\tilde{v}_0 : (0,\infty) \longrightarrow \mathbb{R}$ by

$$\tilde{v}_0(x) := \begin{cases} s(x) & \text{if } x \in (a,b) \\ \tilde{g}_0(x) & \text{if } x \notin (a,b) \end{cases}.$$

Theorem 4.3. *If, for every $x \in (a,b)$ we have $s(x) \geq \tilde{g}_0(x)$, and*

$$a \leq \left(\frac{1-\pi}{\pi}\right) \frac{(\mu - r) - 2\theta - \sqrt{(\mu-r)^2 - 2\theta\sigma^2}}{2[\theta - (\mu - r) + \frac{1}{2}\sigma^2]}, \tag{4.7}$$

and

$$b \geq \left(\frac{1-\pi}{\pi}\right) \frac{(\mu - r) - 2\theta + \sqrt{(\mu-r)^2 - 2\theta\sigma^2}}{2[\theta - (\mu - r) + \frac{1}{2}\sigma^2]}, \tag{4.8}$$

then

$$\tilde{v}_0(x) = \sup_{\tau \in \tilde{S}} E^x[\tilde{g}_0(e^{-r\tau} S_\tau^1 / S_0^1) - \theta\tau],$$

and

$$\hat{\tau} = \inf\{t \geq 0 : e^{-rt} S_t^1 / S_0^1 \notin (a,b)\}.$$

Proof. Take $\beta = 0$ in Theorem 5.1.

□

For each $\pi \in (0,1)$ the above procedure yields a value θ which, according to (3.11), is R_π. Maximizing R_π over $\pi \in (0,1)$ gives the optimal $\hat{\pi}$, while the stopping time τ corresponding to $\hat{\pi}$ is $\hat{\tau}$.

Morton and Pliska (1995) show that the optimal strategy is to make a transaction whenever the fraction B_t of the wealth currently invested in the stock leaves a certain interval (b_l, b_u), where $0 < b_l \le b_u < 1$:

$$\hat{\tau} = \inf\{t \ge 0 : B_t \notin (b_l, b_u)\}.$$

From (3.2) and (3.19) we have, for every $t < \tau_1$,

$$\frac{V_t}{V_0} = (1-\hat{\pi})e^{rt} + \hat{\pi}\frac{S_t^1}{S_0^1} = \frac{(1-\hat{\pi})e^{rt}}{1-B_t},$$

so the fraction process B of Morton and Pliska (1995) can be written in terms of our model as

$$B_t = \frac{\hat{\pi}S_t^1/S_0^1}{\hat{\pi}S_t^1/S_0^1 + (1-\hat{\pi})e^{rt}}.$$

We can write the optimal stopping time of Morton and Pliska (1995) in an equivalent form

$$\begin{aligned}\hat{\tau} &= \inf\left\{t \ge 0 : S_t^1/S_0^1 \notin \left(\frac{(1-\hat{\pi})b_l}{\hat{\pi}(1-b_l)}e^{rt}, \frac{(1-\hat{\pi})b_u}{\hat{\pi}(1-b_u)}e^{rt}\right)\right\} \\ &= \inf\left\{t \ge 0 : e^{-rt}S_t^1/S_0^1 \notin \left(\frac{(1-\hat{\pi})b_l}{\hat{\pi}(1-b_l)}, \frac{(1-\hat{\pi})b_u}{\hat{\pi}(1-b_u)}\right)\right\}.\end{aligned}$$

Comparing this equivalent form of $\hat{\tau}$ with the one in Theorem 4.3, we deduce the following equivalence equations between the model of Morton and Pliska (1995) and our model:

$$a = \frac{(1-\hat{\pi})b_l}{\hat{\pi}(1-b_l)}, \quad b = \frac{(1-\hat{\pi})b_u}{\hat{\pi}(1-b_u)},$$

or

$$b_l = \frac{\hat{\pi} a}{1 - \hat{\pi} + \hat{\pi} a}, \quad b_u = \frac{\hat{\pi} b}{1 - \hat{\pi} + \hat{\pi} b}. \tag{4.9}$$

Remark 4.1. *It is interesting to note that the continuation region has an upper bound and a lower bound, just like in Cadenillas and Pliska (1999); the only difference is in the shape of these bounds: now they are curves that are exponential in time, as opposed to horizontal lines that were constant in time.*

Example 4.2. *Consider the following case:* $\mu = 0.065$, $\sigma = 0.3$, $\alpha = 0.02$, $\beta = 0$, $r = 0$. *Since the bond has* zero interest rate *and there are no taxes, the solution obtained using the method of Morton and Pliska (1995) should agree with our solution.*

On one hand, the method of Morton and Pliska (1995) gives

$$\begin{aligned} \hat{\pi} &= 0.7307228085, \quad R = 0.02211996491, \\ \hat{\tau} &= \inf\{t \geq 0 : B_t \notin (0.3827298508,\ 0.9539750604)\}. \end{aligned}$$

According to (4.9), $\hat{\tau}$ *can be also written as*

$$\hat{\tau} = \inf\{t \geq 0 : S_t^1 / S_0^1 \notin (0.2284883,\ 7.638194261)\}.$$

On the other hand, Theorem 4.3 gives

$$\begin{aligned} \hat{\pi} &= 0.73, \quad R = 0.0221199588, \\ \hat{\tau} &= \tau(0.229, 7.666) \\ &= \inf\{t \in [0, \infty) : e^{-rt} S_t^1 / S_0^1 \notin (0.229, 7.666)\} \\ &= \inf\{t \in [0, \infty) : S_t^1 / S_0^1 \notin (0.229, 7.666)\}. \end{aligned}$$

These values satisfy conditions (4.7)-(4.8)

$$a = 0.229 \leq 0.45, \quad b = 7.666 \geq 3.172.$$

We see that, indeed, the two approaches lead to the same results.

Example 4.3. *Let us consider the numerical example studied by Morton and Pliska (1995) for a portfolio of one stock and one bond with* positive interest rate*:* $\mu = 0.182$*,* $\sigma = 0.4$*,* $r = 0.07$*,* $\alpha = 0.001$ *and* $\beta = 0$*.*

They obtain

$$\begin{aligned} R^{(0.7,0.182,0.4)}_{\hat{\pi},0.001,0} &= 0.1085975824, \\ \hat{\pi} &= 0.701, \\ \hat{\tau} &= \inf\{t \geq 0 : B_t \notin (0.54, 0.837)\}. \end{aligned}$$

For the equivalent problem when the portfolio consists of a zero-interest bond and a stock with drift $\mu - r = 0.112$ *and volatility* $\sigma = 0.4$*, we use Theorem 4.3 to find (see Appendix A)*

$$\begin{aligned} R^{(0,0.112,0.4)}_{\hat{\pi},0.001,0} &= 0.038597524604994, \\ \hat{\pi} &= 0.7, \\ \hat{\tau} &= \inf\{t \geq 0 : e^{-rt} S^1_t / S^1_0 \notin (0.65222, 1.84028)\}. \end{aligned}$$

Conditions (4.7)-(4.8) are satisfied

$$a = 0.65222 \leq 0.6794753, \quad b = 1.84028 \geq 1.581434.$$

We observe that the optimal proportion $\hat{\pi}$ is almost the same, and that

$$R^{(0,0.112,0.4)}_{0.7,0.001,0} + 0.07 = 0.1085975246 \approx 0.1085975824 = R^{(0.7,0.182,0.4)}_{0.701,0.001,0},$$

just as expected from Proposition 4.1.

Also, taking into account the computational error of $\hat{\pi}$, equations (4.9) are satisfied

$$\begin{aligned} \frac{\hat{\pi}a}{1-\hat{\pi}+\hat{\pi}a} &= 0.6 \approx 0.54 = b_l, \\ \frac{\hat{\pi}b}{1-\hat{\pi}+\hat{\pi}b} &= 0.811 \approx 0.837 = b_u. \end{aligned}$$

Remark 4.2. *The results obtained using the method of Morton and Pliska (1995) are strikingly similar to those obtained by our method, which is a generalization of Cadenillas and Pliska (1999) (see Theorem 5.2). Therefore, our hopes for unifying the two approaches are high. Recall that we have shown in Proposition 4.1 that they are equivalent in the absence of taxes, and have found the link between them in* (4.9).

Chapter 5

The Case of One Stock and A Zero Interest Rate Bond

5.1 Deriving the Optimal Strategy

We consider the case when the interest rate of the bond is zero, and there are taxes and transaction costs ($\beta \in (0,1), \alpha \in (0,1)$). We denote

$$\forall t \geq 0, \;\; \tilde{S}_t^1 = S_t^1 / S_0^1.$$

For fixed $\pi \in [0,1]$ we define the function $\tilde{g} : (0, \infty) \longrightarrow \mathbb{R}$ by

$$x \longmapsto \tilde{g}(x) = \log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi x)\}. \tag{5.1}$$

For $r = 0$, Problem 3.3 can be written in the following way:

Problem 5.1. *For each fixed* $\pi \in (0,1)$ *and each* $\theta > 0$ *solve the optimal stopping problem with value*

$$H(\theta) := \sup_{\tau \in \tilde{\mathcal{S}}} E[\tilde{g}(\tilde{S}_\tau^1) - \theta\tau]. \tag{5.2}$$

Then determine the value $R_\pi > 0$ such that

$$H(R_\pi) = 0. \tag{5.3}$$

The growth rate corresponding to the optimal strategy is obtained using (3.12)

$$R = \sup_{\pi \in [0,1]} R_\pi = R_{\hat{\pi}}.$$

The optimal strategy is given, in terms of Remark 3.2, by the above $\hat{\pi}$ and by the stopping time that solves the optimal stopping problem with value $H(R_{\hat{\pi}})$.

First we fix $\pi \in [0,1]$ and determine the corresponding R_π. Applying the principle of dynamic programming (see Shiryayev (1978)), we get that, with initial condition $\tilde{S}_0^1 = x$, the value function $v : (0, \infty) \longrightarrow \mathbb{R}$ given by

$$v(x) = \sup_{\tau \in \tilde{S}} E^x[\tilde{g}(\tilde{S}_\tau^1) - \theta\tau], \tag{5.4}$$

satisfies the following Stefan problem

$$\mu x \frac{\partial v}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} = \theta, \quad \text{if} \quad x \in \mathcal{C}, \tag{5.5}$$

$$v(x) = \tilde{g}(x), \quad \text{if} \quad x \in \Sigma, \tag{5.6}$$

where

$$\mathcal{C} = \{x \in (0, \infty) : v(x) > \tilde{g}(x)\},$$

$$\Sigma = \{x \in (0, \infty) : v(x) = \tilde{g}(x)\}.$$

In order to identify the unique solution that coincides with the payoff v, we conjecture that the usual *smooth-fit condition* holds, i.e., that the derivatives of v and $\tilde{g}$ are equal on the boundary of Σ. Following Cadenillas and

Pliska (1999), we conjecture that the continuation region is an open interval $\mathcal{C} = (a, b)$, with $0 < a < 1 < b < \infty$. (We prove both conjectures in Theorem 5.1.)

The general solution of the ordinary differential equation in x of (5.5) is the function s previously defined in (4.1) (see Section 3.6 of Boyce and DiPrima (1992))

$$s(x) = c + dx^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} \log x, \tag{5.7}$$

where c and d are real numbers.

To find the values of the parameters a, b, c, d and θ we need five equations. The first equation is given by condition (5.3), which says that the value of θ that we are computing is R_π

$$s(1) = 0. \tag{5.8}$$

Next we have the boundary conditions of the Stefan problem

$$s(a) = \tilde{g}(a), \tag{5.9}$$

$$s(b) = \tilde{g}(b), \tag{5.10}$$

and the *smooth-fit conditions*

$$s'(a) = \tilde{g}'(a), \tag{5.11}$$

$$s'(b) = \tilde{g}'(b). \tag{5.12}$$

Equations (5.8)-(5.12) are equivalent to

$$c + d = 0, \tag{5.13}$$

$$c + da^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} \log a = \log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)\}, \quad (5.14)$$

$$c + db^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} \log b = \log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi b)\}, \quad (5.15)$$

$$d\Big(1 - \frac{2\mu}{\sigma^2}\Big)a^{(-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\frac{1}{a} = \frac{(1-\alpha)(1-\beta)\pi}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)}, \quad (5.16)$$

$$d\Big(1 - \frac{2\mu}{\sigma^2}\Big)b^{(-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\frac{1}{b} = \frac{(1-\alpha)(1-\beta)\pi}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi b)}, \quad (5.17)$$

respectively. For computational purposes, we rewrite equations (5.13)-(5.17) as

$$c = -d, \quad (5.18)$$

$$d = \left\{\frac{(1-\alpha)(1-\beta)\pi}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)} - \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\frac{1}{a}\right\} \times \Big(1 - \frac{2\mu}{\sigma^2}\Big)^{-1} a^{(\frac{2\mu}{\sigma^2})}, \quad (5.19)$$

$$\frac{(1-\alpha)(1-\beta)\pi a^{(\frac{2\mu}{\sigma^2})}}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)} - \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} a^{(\frac{2\mu}{\sigma^2}-1)} = \frac{(1-\alpha)(1-\beta)\pi b^{(\frac{2\mu}{\sigma^2})}}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi b)} - \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} b^{(\frac{2\mu}{\sigma^2}-1)}, \quad (5.20)$$

$$\left\{\frac{(1-\alpha)(1-\beta)\pi}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)} - \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\frac{1}{a}\right\} \times \Big(1 - \frac{2\mu}{\sigma^2}\Big)^{-1}\Big(a - a^{(\frac{2\mu}{\sigma^2})}\Big) + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} \log a - \log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi a)\} = 0, \quad (5.21)$$

$$\left\{\frac{(1-\alpha)(1-\beta)\pi}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi b)} - \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\frac{1}{b}\right\} \times \Big(1 - \frac{2\mu}{\sigma^2}\Big)^{-1}\Big(b - b^{(\frac{2\mu}{\sigma^2})}\Big) + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)} \log b - \log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi b)\} = 0. \quad (5.22)$$

To solve this system of equations, we first find a, b, and θ from equations (5.20)-(5.22), then c and d from equations (5.18)-(5.19). The following verification theorem shows under which conditions the solution of this system of equations is also a solution of Problem 5.1.

Theorem 5.1. *Let a, b, c, d, and θ, where $0 < a < 1 < b < \infty$, $\theta > 0$, and $c, d \in \mathbb{R}$, be a solution of equations (5.18)-(5.22). We define the function $\tilde{v}$ by*

$$\tilde{v}(x) := \begin{cases} s(x) & \text{if } x \in (a,b) \\ \tilde{g}(x) & \text{if } x \notin (a,b) \end{cases}. \tag{5.23}$$

If

$$\forall x \in (a,b), \quad s(x) \geq \tilde{g}(x), \tag{5.24}$$

$$a \leq \frac{\{\beta + (1-\alpha)(1-\beta)(1-\pi)\}\{\mu - 2\theta - \sqrt{\mu^2 - 2\theta\sigma^2}\}}{2\pi(1-\alpha)(1-\beta)(\theta - \mu + \frac{1}{2}\sigma^2)}, \tag{5.25}$$

and

$$b \geq \frac{\{\beta + (1-\alpha)(1-\beta)(1-\pi)\}\{\mu - 2\theta + \sqrt{\mu^2 - 2\theta\sigma^2}\}}{2\pi(1-\alpha)(1-\beta)(\theta - \mu + \frac{1}{2}\sigma^2)}, \tag{5.26}$$

then

$$\tilde{v}(x) = \sup_{\tau \in \tilde{S}} E^x[\tilde{g}(\tilde{S}^1_\tau) - \theta\tau]. \tag{5.27}$$

Furthermore, the solution of Problem 5.1 for fixed $\pi \in (0,1]$ is

$$\hat{\tau} = \tau(a,b) := \inf\{t \in [0,\infty) : \tilde{S}^1_t = S^1_t / S^1_0 \notin (a,b)\}. \tag{5.28}$$

Proof. Here we generalize the proof of Theorem 5.2 of Cadenillas and Pliska

(1999). Consider the function $q : [0, \infty) \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} q(x) &:= \left\{ \frac{(1-\beta)(1-\alpha)\pi x}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi x)} \right\} \mu - \theta \\ &\quad -\frac{1}{2}\sigma^2 \left\{ \frac{(1-\beta)(1-\alpha)\pi x}{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi x)} \right\}^2 \\ &= \frac{mx}{mx+n}\mu - \theta - \frac{1}{2}\sigma^2 \left(\frac{mx}{mx+n} \right)^2 \\ &= \frac{\mu mx(mx+n) - \theta(mx+n)^2 - (\sigma mx)^2/2}{(mx+n)^2} \\ &= \frac{(\mu - \frac{1}{2}\sigma^2 - \theta)m^2x^2 + (\mu - 2\theta)mnx - \theta n^2}{(mx+n)^2}, \end{aligned} \tag{5.29}$$

where we denote

$$m := \pi(1-\alpha)(1-\beta),$$

and

$$n := \beta + (1-\alpha)(1-\beta)(1-\pi).$$

We observe that $s \in \mathbb{R}$ satisfies

$$q(x) \leq 0,$$

if and only if

$$\left(\mu - \frac{1}{2}\sigma^2 - \theta\right) m^2x^2 + (\mu - 2\theta)mnx - \theta n^2 \leq 0,$$

which, since $\mu - \frac{1}{2}\sigma^2 - \theta = \lambda - \theta < 0$ (see Remark 3.4), is true if and only if

$$\begin{aligned} x &\leq \frac{-(\mu - 2\theta)mn + \sqrt{(\mu-2\theta)^2m^2n^2 + 4\theta m^2n^2(\mu - \sigma^2/2 - \theta)}}{2(\mu - \sigma^2/2 - \theta)m^2} \\ &= \frac{-(\mu - 2\theta) + \sqrt{(\mu - 2\theta)^2 + 4\theta(\mu - \sigma^2/2 - \theta)}}{2(\mu - \sigma^2/2 - \theta)} \frac{n}{m} \end{aligned}$$

$$= \frac{(\mu-2\theta)-\sqrt{\mu^2-4\mu\theta+4\theta^2+4\mu\theta-2\theta\sigma^2-4\theta^2}}{2(\theta-\mu+\sigma^2/2)}\frac{n}{m}$$
$$= \frac{(\mu-2\theta)-\sqrt{\mu^2-2\theta\sigma^2}}{2(\theta-\mu+\sigma^2/2)}\frac{n}{m}$$

or

$$x \geq \frac{-(\mu-2\theta)mn-\sqrt{(\mu-2\theta)^2m^2n^2+4\theta m^2n^2(\mu-\sigma^2/2-\theta)}}{2(\mu-\sigma^2/2-\theta)m^2}$$
$$= \frac{(\mu-2\theta)+\sqrt{\mu^2-2\theta\sigma^2}}{2(\theta-\mu+\sigma^2/2)}\frac{n}{m},$$

if and only if

$$x \leq \frac{\{\beta+(1-\alpha)(1-\beta)(1-\pi)\}\{\mu-2\theta-\sqrt{\mu^2-2\theta\sigma^2}\}}{2\pi(1-\alpha)(1-\beta)(\theta-\mu+\frac{1}{2}\sigma^2)},$$

or

$$x \geq \frac{\{\beta+(1-\alpha)(1-\beta)(1-\pi)\}\{\mu-2\theta+\sqrt{\mu^2-2\theta\sigma^2}\}}{2\pi(1-\alpha)(1-\beta)(\theta-\mu+\frac{1}{2}\sigma^2)}.$$

In particular, from conditions (5.25) – (5.26), we see that

$$q(x) \leq 0, \quad \forall x \notin (a,b). \tag{5.30}$$

Let us define the adapted process K by

$$K_t = \tilde{v}(\tilde{S}_t^1) - \theta t. \tag{5.31}$$

From equation (2.1) we have the dynamics

$$d\tilde{S}_t^1 = \mu\tilde{S}_t^1 dt + \sigma\tilde{S}_t^1 dW_t,$$

with initial condition resulting from the application of the dynamic programming principle

$$\tilde{S}_0^1 = x.$$

In the region $\mathcal{D} := \{(t,w) \in [0,\infty) \times \Omega : \tilde{S}^1_t(\omega) \notin (a,b)\}$ we have

$$K_t = \log\{\beta + (1-\beta)(1-\alpha)(1-\pi+\pi\tilde{S}^1_t)\} - \theta t =: i(t, \tilde{S}^1_t).$$

Using the derivatives of the function $i(t,x) = \log\{mx+n\} - \theta t$

$$\begin{aligned} \frac{\partial i}{\partial t}(t,x) &= -\theta, \\ \frac{\partial i}{\partial x}(t,x) &= \frac{\partial}{\partial x}\log\{mx+n\} = \frac{m}{mx+n}, \\ \frac{\partial^2 i}{\partial x^2}(t,x) &= -\Big(\frac{m}{mx+n}\Big)^2, \end{aligned}$$

with Itô's formula we obtain

$$\begin{aligned} dK_t &= \frac{\partial i}{\partial t}(t,\tilde{S}^1_t)dt + \frac{\partial i}{\partial x}(t,\tilde{S}^1_t)d\tilde{S}^1_t + \frac{1}{2}\frac{\partial^2 i}{\partial t^2}(t,\tilde{S}^1_t)d\langle\tilde{S}\rangle_t \\ &= -\theta dt + \frac{m}{m\tilde{S}^1_t+n}(\mu\tilde{S}^1_t dt + \sigma\tilde{S}^1_t dW_t) - \frac{1}{2}\Big(\frac{m}{m\tilde{S}^1_t+n}\Big)^2\sigma^2(\tilde{S}^1_t)^2 dt \\ &= q(\tilde{S}^1_t)dt + \left\{\frac{(1-\beta)(1-\alpha)\pi\tilde{S}^1_t}{\beta+(1-\alpha)(1-\beta)(1-\pi+\pi\tilde{S}^1_t)}\right\}\sigma dW_t. \end{aligned}$$

According to equation (5.30), the term multiplying dt is non-positive. In addition, the term multiplying dW_t is bounded. Thus, K is a supermartingale in the region $\mathcal{D}$.

In the region $\mathcal{E} := \{(t,w) \in [0,\infty) \times \Omega : \tilde{S}^1_t(\omega) \in (a,b)\}$,

$$K_t = c + d\big(\tilde{S}^1_t\big)^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu - \frac{1}{2}\sigma^2)}\log\tilde{S}^1_t - \theta t =: j(t, \tilde{S}^1_t),$$

and

$$\frac{\partial j}{\partial t}(t,x) = -\theta, \quad \frac{\partial j}{\partial x}(t,x) = w'(x), \quad \frac{\partial^2 j}{\partial x^2} = w''(x).$$

Applying Itô's formula we obtain

$$
\begin{aligned}
dK_t &= \Big(\mu\tilde{S}_t^1 w'(\tilde{S}_t^1) + \frac{1}{2}\sigma^2(\tilde{S}_t^1)^2 w''(\tilde{S}_t^1) - \theta\Big)dt \\
&\quad + \Big\{d\Big(1-\frac{2\mu}{\sigma^2}\Big)\big(\tilde{S}_t^1\big)^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu-\frac{1}{2}\sigma^2)}\Big\}\sigma dW_t.
\end{aligned}
$$

We observe that the term multiplying dW_t is bounded in $\mathcal{E}$, while the term multiplying dt is zero by (5.5) and (5.7). Thus, K is a martingale in $\mathcal{E}$.

Hence, the process K is a supermartingale in $[0,\infty)\times\Omega$. Furthermore,

$$
\begin{aligned}
K_t &= K_0 + \int_0^t q(\tilde{S}_u^1)1_{\{\tilde{S}_u^1 \notin (a,b)\}}du \\
&\quad + \int_0^t \Big\{\Big[\frac{(1-\beta)(1-\alpha)\pi\tilde{S}_u^1}{\beta+(1-\alpha)(1-\beta)(1-\pi+\pi\tilde{S}_u^1)}\Big]1_{\{\tilde{S}_u^1\notin(a,b)\}} \\
&\quad + \Big[d(1-\frac{2\mu}{\sigma^2})\big(\tilde{S}_u^1\big)^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu-\frac{1}{2}\sigma^2)}\Big]1_{\{\tilde{S}_u^1\in(a,b)\}}\Big\}\sigma dW_u.
\end{aligned}
\tag{5.32}
$$

Since the first integrand is non-positive,

$$
\forall\tau\in\tilde{\mathcal{S}},\ \ E^x\Big[\int_0^\tau q(\tilde{S}_u^1)1_{\{\tilde{S}_u^1\notin(a,b)\}}du\Big] \le 0,
$$

with equality for $\tau=\tau(a,b)$. In addition, for every $\tau\in\tilde{\mathcal{S}}$,

$$
\begin{aligned}
E^x\Big[\int_0^\tau &\Big\{\Big[\frac{(1-\beta)(1-\alpha)\pi\tilde{S}_u^1}{\beta+(1-\alpha)(1-\beta)(1-\pi+\pi\tilde{S}_u^1)}\Big]1_{\{\tilde{S}_u^1\notin(a,b)\}} \\
&+ \Big[d(1-\frac{2\mu}{\sigma^2})\big(\tilde{S}_u^1\big)^{(1-\frac{2\mu}{\sigma^2})} + \frac{\theta}{(\mu-\frac{1}{2}\sigma^2)}\Big]1_{\{\tilde{S}_u^1\in(a,b)\}}\Big\}\sigma dW_u\Big] = 0,
\end{aligned}
$$

because the integrand of this stochastic integral is bounded. Thus, an application of the optional sampling theorem gives (see Theorem 1.3.22 of Karatzas and Shreve (1991))

$$
\forall\tau\in\tilde{\mathcal{S}},\ \ E^x[K_\tau] \le E^x[K_0],
$$

with equality for $\tau = \tau(a,b)$. From condition (5.24), we then obtain

$$\begin{aligned} E^x[\tilde{g}(\tilde{S}^1_\tau) - \theta\tau] &\leq E^x[\tilde{v}(\tilde{S}^1_\tau) - \theta\tau] = E^x[K_\tau] \\ &\leq E^x[K_0] = \tilde{v}(x), \end{aligned}$$

with equality for $\tau = \tau(a,b)$. Therefore,

$$\sup_{\tau \in \tilde{\mathcal{S}}} E^x[\tilde{g}(\tilde{S}^1_\tau) - \theta\tau] = \tilde{v}(x) = E^x[\tilde{g}(\tilde{S}^1_{\tau(a,b)}) - \theta\tau(a,b)].$$

□

Example 5.1. *Let us consider the numerical Example 5.1 of Cadenillas and Pliska (1999) with a zero-interest bond*

$$\mu = 0.065, \quad \sigma = 0.3, \quad \alpha = 0.02 \quad \textit{and} \quad \beta = 0.3. \tag{5.33}$$

We solve the system of equations (5.18) – (5.22) *using* C++ *and determine the long-run growth rate* R_π *corresponding to each proportion* π. *We consider as initial values the solution for the case* $\pi = 1$, *and then iterate the procedure for* $\pi \in [0,1]$ *with a step of 0.01. The solutions are found for* $\pi \in [0.46;1]$ *(this interval contains the solution) and plotted using* S-plus *in Figure 5.1.*

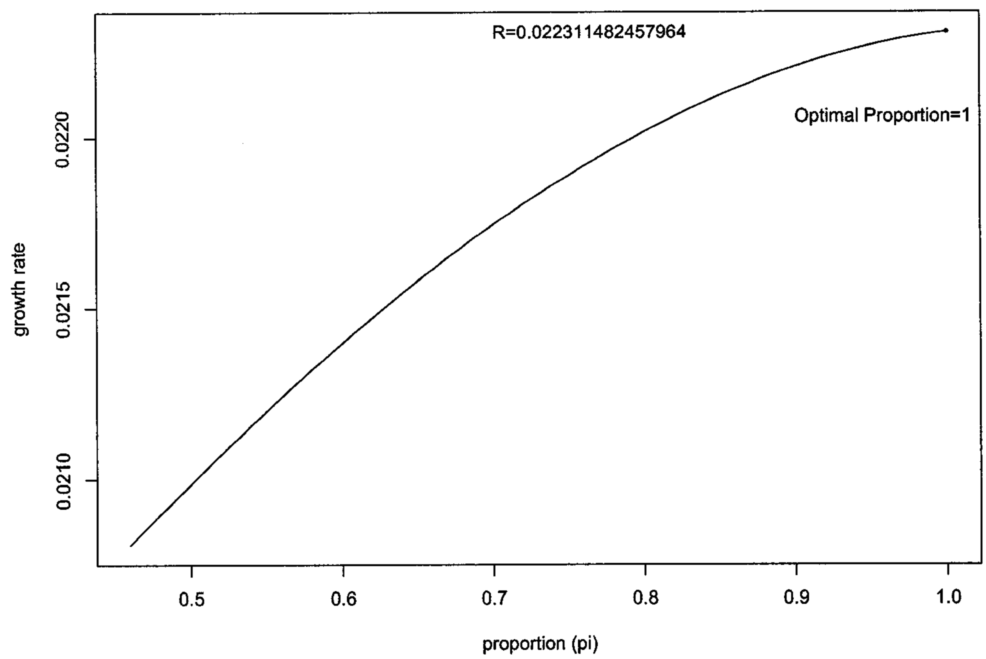

Figure 5.1: Determining the Optimal Proportion and R when $r = 0$

First we check that conditions (5.25) *and* (5.26) *are both satisfied*

$$a = 0.303293472736538 \leq 0.5602, \quad b = 7.186002173447605 \geq 3.2949.$$

The solution to our problem, satisfying the conditions of Theorem 5.1, *is then*

$$\hat{\pi} = 1, \quad R = 0.022311482457964 < \frac{\mu^2}{2\sigma^2} = 0.02347222222,$$
$$\hat{\tau} = \tau(0.3, 7.186) = \inf\{t \in [0, \infty) : \tilde{S}_t^1 = S_t^1 / S_0^1 \notin (0.3, 7.186)\}.$$

This means that the solution of Cadenillas and Pliska (1999) is also the optimal solution for the case when the investor holds in the portfolio a riskless *asset in addition to the stock shares.*

Furthermore, we consider the parameter β variable, and try to find the optimal value of β that maximizes our R. The C++ *code is run with β taking values in* $[0.291, 0.309]$. *The solution is (see Figure 5.2)*

$$\hat{\beta} = 0.299, \quad \hat{\pi} = 1.0, \quad R = 0.022311487480764.$$

$$a = 0.3018522020, \quad b = 7.1517891752.$$

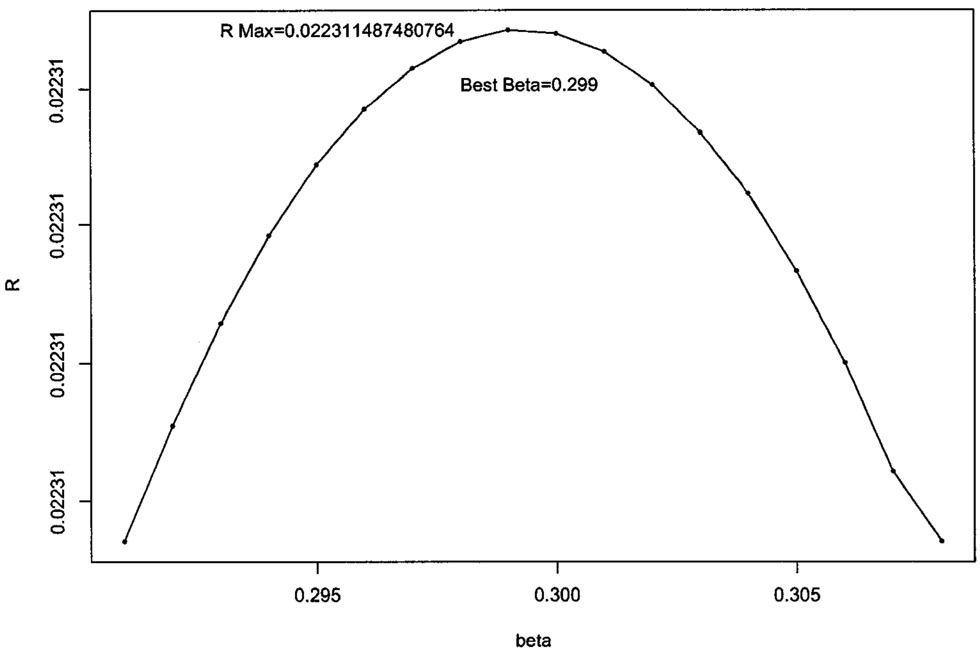

Figure 5.2: Determining the Best Tax Rate when r=0

We observe that the two surprising results obtained by Cadenillas and Pliska (1999) extend to this more general case. First, it is optimal to make a transaction not only when the investor has a loss, but also a gain. Secondly, the best tax rate for the investor can be positive!

We have considered many numerical examples and have observed that whenever the best tax rate is positive, the optimal investment strategy has $\pi = 1$, that is, all the money is invested in the stock.

This means that once the drift and the volatility of the stock are known and the best tax rate has been identified, the investor is going to assume the entire risk of the stock ($\pi = 1$) since it is backed by the cushion of the tax credits (positive tax rate) he or she receives in the trading cycles where a loss is incurred on the investment.

The safety of the tax credits is more important in maximizing the growth rate than the safety provided by cash (the zero-interest bond). We will analyze later if this is still the case when the bond has a positive interest rate, i.e. it provides a significant safety to the investor and might prove to be more appealing than the cushion of tax credits.

In the case of no transaction costs ($\alpha = 0$) and a portfolio having only the stock ($\pi \equiv 1$), Cadenillas and Pliska (1999) show for the strategy of continuous trading that a larger tax rate reduces the volatility of the after-tax portfolio, hence maximizing the growth rate.

With this in mind, we allow the volatility parameter σ to vary and investigate how the preferred tax rates modify. Note that for $\sigma < \sqrt{\mu} = 0.25495$ we shall see in the next section that the optimal strategies have a long-run growth rate independent of the tax rate.

For $\sigma > \sqrt{\mu}$ the results are presented in Figure 5.3.

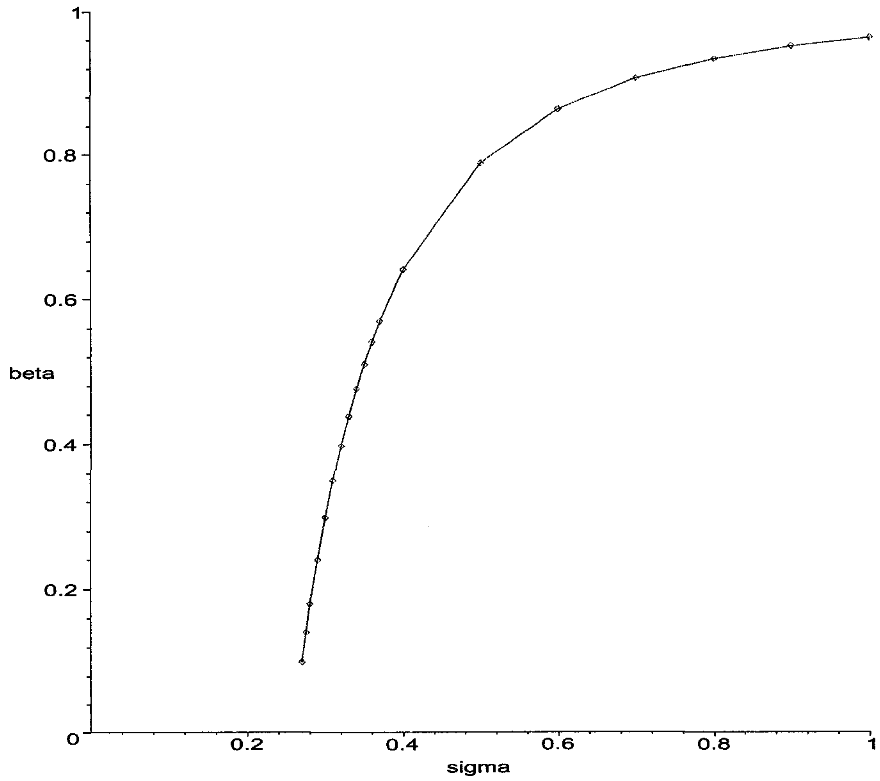

Figure 5.3: Preferred Tax Rate versus Stock Volatility

As the volatility of the stock increases the tax rate that maximizes the rate of return on the investment also increases.

The two ideas are in agreement: when the volatility increases, its erratic effect on the return also increases, and the investor needs a larger tax rate to counteract it. This provides the intuition for the case when the investor can take advantage of a positive tax rate.

5.2 An Alternative Approach

We now consider an alternative approach for the case when the bond has a zero interest rate ($r = 0$). We will assume that the long-run growth rate of the stock is positive: $\lambda > 0$.

Cadenillas and Pliska (1999) consider stopping times of the form

$$\tau(a,b) \quad = \quad \inf\{t \in [0,\infty) : \tilde{S}_t \notin (a,b)\},$$

which form a class

$$\tilde{\mathcal{T}} = \{\tau(a,b) : 0 < a < 1 < b < \infty\} \subset \tilde{\mathcal{S}}.$$

Using this class they give an alternative approach to Problem 3.2 when $\pi = 1$. We generalize their solution of Problem 3.2 to $\pi \in (0,1]$ and $r = 0$.

Theorem 5.2. *The solution to Problem* 3.2 *for* $r = 0$ *in the subclass* $\tilde{\mathcal{T}} \subset \tilde{\mathcal{S}}$ *is given by*

$$R = \sup_{\pi\in(0,1]} \sup_{\tau\in\tilde{\mathcal{T}}} J(\tau,\pi) = \sup_{\pi\in(0,1]} \sup_{0<a<1<b<\infty} h(a,b,\pi), \tag{5.34}$$

where

$$h(a,b,\pi) = \lambda \frac{g(1-\pi+\pi a)(b^{-\frac{2\lambda}{\sigma^2}} - 1) + g(1-\pi+\pi b)(1-a^{-\frac{2\lambda}{\sigma^2}})}{(\log a)\ b^{-\frac{2\lambda}{\sigma^2}} - (\log b)\ a^{-\frac{2\lambda}{\sigma^2}} + \log b - \log a}, \tag{5.35}$$

and g *is defined in (3.4).*

Proof. Note that

$$\begin{aligned} J(\tau(a,b),\pi) &= \frac{E[\log\{\beta + (1-\alpha)(1-\beta)I_{\tau(a,b)}\}]}{E[\tau(a,b)]} \\ &= \frac{E[\log\{\beta + (1-\alpha)(1-\beta)(1-\pi+\pi\tilde{S}^1_{\tau(a,b)})\}]}{E[\tau(a,b)]} \end{aligned}$$

$$= \frac{g(1-\pi+\pi a)P\{\tilde{S}^1_{\tau(a,b)} = a\} + g(1-\pi+\pi b)P\{\tilde{S}^1_{\tau(a,b)} = b\}}{E[\tau(a,b)]}. \quad (5.36)$$

As in Theorem 5.1 of Cadenillas and Pliska (1999), we use Theorem 7.5.2 of Karlin and Taylor (1975) for a Brownian motion with drift to get

$$\begin{aligned} P\{\tilde{S}^1_{\tau(a,b)} = b\} &= P\left\{\lambda\tau(a,b) + \sigma W_{\tau(a,b)} = \log b\right\} \\ &= \frac{1 - a^{-\frac{2\lambda}{\sigma^2}}}{b^{-\frac{2\lambda}{\sigma^2}} - a^{-\frac{2\lambda}{\sigma^2}}}, \end{aligned} \quad (5.37)$$

and similarly

$$P\{\tilde{S}^1_{\tau(a,b)} = a\} = \frac{b^{-\frac{2\lambda}{\sigma^2}} - 1}{b^{-\frac{2\lambda}{\sigma^2}} - a^{-\frac{2\lambda}{\sigma^2}}}. \quad (5.38)$$

Also, Ross (1996) gives

$$E[\tau(a,b)] = \frac{(\log a)(b^{-\frac{2\lambda}{\sigma^2}} - 1) + (\log b)(1 - a^{-\frac{2\lambda}{\sigma^2}})}{\lambda\left(b^{-\frac{2\lambda}{\sigma^2}} - a^{-\frac{2\lambda}{\sigma^2}}\right)}. \quad (5.39)$$

Substituting equations (5.37)-(5.39) in (5.36) gives the result.

□

Example 5.2. *We use this alternative approach to recover the results of Example 5.1. We have*

$$\begin{aligned} \pi = 1.00: \quad & R_1 = \sup_{0<a<1<b<\infty} h(a,b,1) = 0.0223114825, \\ & \hat{\tau}_1 = \inf\{t \geq 0 : \tilde{S}^1_t = S^1_t/S^1_0 \notin (0.30329, 7.1860019)\}, \\ \pi = 0.99: \quad & R_{0.99} = \sup_{0<a<1<b<\infty} h(a,b,0.99) = 0.02230558494, \\ & \hat{\tau}_{0.99} = \inf\{t \geq 0 : \tilde{S}^1_t = S^1_t/S^1_0 \notin (0.319, 7.412236)\}, \end{aligned}$$

so

$$R = \sup_{\pi\in(0,1]} R_\pi, \quad \textit{with } \hat{\pi} = 1 \textit{ and } \hat{\tau} = \hat{\tau}_1.$$

Proposition 5.1. *When* $\lambda > 0$, *the strategy of* cut-losses-short-and-let-profits-run *is equivalent to* buy-and-hold *in terms of the long-run growth rate, that is,*

$$\forall a \in (0,1), \forall \pi \in (0,1]: \quad \lim_{b \to \infty} h(a,b,\pi) = \lambda. \tag{5.40}$$

Proof. When $\lambda > 0$ the exponent $-2\lambda/\sigma^2$ in (5.35) is negative, so

$$\lim_{b \to \infty} b^{-\frac{2\lambda}{\sigma^2}} = 0.$$

For fixed $\pi \in (0,1)$ and $a \in (0,1)$ we have

$$\begin{aligned}
\lim_{b\to\infty} h(a,b,\pi) &= \lim_{b\to\infty} \lambda \frac{g(1-\pi+\pi a)(b^{-\frac{2\lambda}{\sigma^2}}-1) + g(1-\pi+\pi b)(1-a^{-\frac{2\lambda}{\sigma^2}})}{(\log a)\ (b^{-\frac{2\lambda}{\sigma^2}}-1) + (\log b)\ (1-a^{-\frac{2\lambda}{\sigma^2}})} \\
&= \lambda \lim_{b\to\infty} \frac{g(1-\pi+\pi b)(b^{-\frac{2\lambda}{\sigma^2}}-1)}{(\log b)\ (b^{-\frac{2\lambda}{\sigma^2}}-1)}
\end{aligned}$$

$$\times \lim_{b\to\infty} \frac{g(1-\pi+\pi a)/g(1-\pi+\pi b) + (1-a^{-\frac{2\lambda}{\sigma^2}})/(b^{-\frac{2\lambda}{\sigma^2}}-1)}{(\log a)/(\log b) + (1-a^{-\frac{2\lambda}{\sigma^2}})/(b^{-\frac{2\lambda}{\sigma^2}}-1)}$$

$$\begin{aligned}
&= \lambda \lim_{b\to\infty} \frac{g(1-\pi+\pi b)(b^{-\frac{2\lambda}{\sigma^2}}-1)}{(\log b)\ (b^{-\frac{2\lambda}{\sigma^2}}-1)} \times \frac{0+(1-a^{-\frac{2\lambda}{\sigma^2}})/(0-1)}{0+(1-a^{-\frac{2\lambda}{\sigma^2}})/(0-1)} \\
&= \lambda \lim_{b\to\infty} \frac{\log\{\beta+(1-\alpha)(1-\beta)(1-\pi+\pi b)\}}{(\log b)}.
\end{aligned}$$

Applying the l'Hospital's rule gives

$$\lim_{b\to\infty} h(a,b,\pi) = \lambda \lim_{b\to\infty} \frac{(1-\alpha)(1-\beta)\pi}{\beta+(1-\alpha)(1-\beta)(1-\pi+\pi b)} \times \frac{b}{1} = \lambda.$$

□

Remark 5.1. *Here the strategy of* buy-and-hold *means that a positive fraction of the initial investment is used to buy shares in the stock and the rest is invested in the bond, without making further transactions in the market. This corresponds to* $(a, b) = (0, \infty)$ *(we know that for every* $t \geq 0, S_t^1 \in (0, \infty)$*).*

For a stock with high appreciation rate ($\mu \geq r + \sigma^2$), the strategies of *buy-and-hold* and *cut-losses-short-and-let-profits-run* are both optimal. To see this recall that by Proposition 5.1 both strategies yield a growth rate equal to λ (for $\mu \geq r + \sigma^2$ we have $\lambda = \mu - \sigma^2/2 \geq r + \sigma^2/2 > 0$). Also, by Theorem 3.1 the maximum value of the best long-run growth rate is λ, thus giving the result.

Example 5.3. *Consider the parameter values* $\mu = 0.065$, $\sigma = 0.3$, $\alpha = 0$. *For* $r = 0$ *we have seen in Example* 4.1 *that when* $\beta = 0$ *the optimal strategy is continuous trading and the corresponding long-run growth rate is*

$$\frac{\mu^2}{2\sigma^2} = 0.023472222.$$

For a positive tax rate ($\beta = 0.3$) *Cadenillas and Pliska (1999) find a strategy that is better than continuous trading. In the notation of equation* (5.35) *this strategy is given by*

$$h(0.9999999, 1.3, 1) = 0.027311 > \frac{\mu^2}{2\sigma^2} = 0.023472222.$$

Taking advantage of the new parameter representing the proportion π *of wealth to be invested in the stock and keeping* $\beta = 0.3$, *we can find an even better strategy than the one of Cadenillas and Pliska (1999) presented above.*

This strategy is given by

$$h(0.9999999, 1.3, 0.91) = 0.0273394938 > h(0.9999999, 1.3, 1) = 0.027311$$
$$> \frac{\mu^2}{2\sigma^2} = 0.023472222.$$

Example 5.4. *Consider the case when*

$$\mu = 0.05, \quad \sigma = 0.3, \quad \beta = 0.3, \quad \text{and} \quad \alpha = 0.000000001. \tag{5.41}$$

We have seen in Example 4.1 that the optimal policy when $\alpha = 0$ and $\beta = 0$ is continuous trading. We can find a strategy that gives a larger long-run growth rate when $\alpha > 0$, namely:

$$h(.999999, 1.3, 0.75) = 0.01414206250 > \frac{\mu^2}{2\sigma^2} = 0.01388888889.$$

Remark 5.2. *Example 5.4 shows that sometimes an investor who pays taxes and transaction costs can do better than an investor who does not pay any of them! This observation was first made by Cadenillas and Pliska (1999).*

5.3 An Asymptotic Result in the Absence of Transaction Costs

For $\alpha = 0$ and fixed $\pi \in (0, 1)$ we approximate the strategy of continuous trading by allowing the investor to trade at times

$$\frac{1}{n} < \frac{2}{n} < \cdots < \frac{k}{n} < \cdots < \infty.$$

We consider the process describing the portfolio's value at some fixed time t as $n \uparrow \infty$.

Fix $n \in \mathbb{N}$ and denote, for each $k \in \{1, 2, \ldots, n\}$, $s \geq \frac{(k-1)}{n}$, and $\pi \in (0, 1)$

$$\begin{aligned} Y_k(s) &:= \frac{\tilde{S}^1_s}{\tilde{S}^1_{\frac{(k-1)}{n}}} = \frac{S^1_s}{S^1_{\frac{(k-1)}{n}}} \\ &= \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\left(s - \frac{(k-1)}{n}\right) + \sigma\left(W_s - W_{\frac{(k-1)}{n}}\right)\right\}, \\ \tilde{Y}_k(s) &:= 1 - \pi + \pi Y_k(s), \end{aligned}$$

and, for each $a \in \mathbb{R}$, its integer part

$$\lfloor a \rfloor := \text{the greatest integer not exceeding } a.$$

Then the value of the investment at time t is (see (2.7)-(2.8))

$$\begin{aligned} V_n(t) &= \left\{\beta + (1-\beta)\left(1 - \pi + \pi \tilde{S}^1_{1/n}\right)\right\}\left\{\beta + (1-\beta)\left(1 - \pi + \pi\frac{\tilde{S}^1_{2/n}}{\tilde{S}^1_{1/n}}\right)\right\} \\ &\quad \cdots\left\{\beta + (1-\beta)\left(1 - \pi + \pi\frac{\tilde{S}^1_t}{\tilde{S}^1_{\frac{\lfloor tn \rfloor}{n}}}\right)\right\} V_n(0) \\ &= V_n(0)\prod_{k=1}^{\lfloor tn \rfloor}\left\{\beta + (1-\beta)\tilde{Y}_k\left(\frac{k}{n}\right)\right\}\left\{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor + 1}(t)\right\}, \end{aligned}$$

since $\frac{\lfloor tn \rfloor}{n} \leq t < \frac{\lfloor tn \rfloor + 1}{n}$.

We show that the sequence of adapted stochastic processes $\{V_n\}$ defined by $V_n := \{V_n(t) : t \in [0, \infty)\}$ converges in distribution (see, for instance, Ethier and Kurtz (1986) for the definition and some results on convergence in distribution of stochastic processes) as $n \uparrow \infty$ to a geometric Brownian motion with parameters $\hat{\mu} := \pi(1-\beta)\mu$ and $\hat{\sigma} := \pi(1-\beta)\sigma$. In particular, for arbitrary $t \in [0, \infty)$, the sequence of random variables $\{V_n(t)\}$ converges in distribution to a log-normal random variable.

Proposition 5.2. *When $\alpha = 0$ and $\pi \in (0,1)$ is fixed,*

$$V_n \xrightarrow{d} GBM(\hat{\mu}, \hat{\sigma}). \tag{5.42}$$

Here $\xrightarrow{d}$ denotes convergence in distribution, and $GBM(\hat{\mu}, \hat{\sigma})$ denotes a geometric Brownian motion with parameters $\hat{\mu} := \pi(1-\beta)\mu$ and $\hat{\sigma} := \pi(1-\beta)\sigma$.

Proof. This is a generalization of the proof of Cadenillas and Pliska (1999). We observe that

$$\log \frac{V_n(t)}{V_n(0)} = \sum_{k=1}^{\lfloor tn \rfloor} \log \left\{ \beta + (1-\beta) Y_k \left(\frac{k}{n} \right) \right\} + \log \left\{ \beta + (1-\beta) Y_{\lfloor tn \rfloor + 1}(t) \right\}.$$

It suffices to prove that the sequence of adapted stochastic processes $\{\log V_n\}$ converges in distribution to a Brownian motion with constant drift $\hat{\mu} - \frac{1}{2}\hat{\sigma}^2$ and constant volatility $\hat{\sigma}^2$.

Let us consider the sequence of stochastic processes $\{\hat{M}_n\}$ defined by

$$\begin{aligned}
&\hat{M}_n(t) \\
&:= \sum_{k=1}^{\lfloor tn \rfloor} \log \left\{ \beta + (1-\beta) \tilde{Y}_k \left(\frac{k}{n} \right) \right\} + \log \left\{ \beta + (1-\beta) \tilde{Y}_{\lfloor tn \rfloor + 1}(t) \right\} \\
&- \sum_{k=1}^{\lfloor tn \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \left[\frac{\pi(1-\beta) Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta) Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right]^2 \right\} ds. \\
&- \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \left[\frac{\pi(1-\beta) Y_{\lfloor tn \rfloor + 1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor + 1}(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta) Y_{\lfloor tn \rfloor + 1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor + 1}(s)} \right]^2 \right\} ds.
\end{aligned} \tag{5.43}$$

Our first objective is to apply the Martingale Central Limit Theorem to prove that $\{M_n\}$ converges in distribution to the stochastic process $\pi(1-\beta)\sigma W$.

According to Itô's formula applied to the function $\log\{\beta + (1-\beta)(1 - \pi + \pi x\}$ at $Y_k(r)$, for every $n \in \mathbb{N}$, $k \in \{1, 2, \ldots, n\}$, $r \geq \frac{(k-1)}{n}$, and $\pi \in (0,1)$,

$$\begin{aligned}
&\log\left\{\beta + (1-\beta)\tilde{Y}_k(r)\right\} = \log\{\beta + (1-\beta)(1-\pi+\pi Y_k(r))\} \\
&= \int_{\frac{(k-1)}{n}}^{r} \left\{ \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)}\right]\mu - \frac{\sigma^2}{2}\left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)}\right]^2 \right\} ds \\
&+ \int_{\frac{(k-1)}{n}}^{r} \left\{ \frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right\} \sigma dW_s.
\end{aligned}$$

Subtracting the first integral on the right hand side from the expression on the left hand side, and then adding the resulting expressions, we obtain that for each $n \in \mathbb{N}$ and $t \in [0, \infty)$,

$$\begin{aligned}
\tilde{M}_n(t) &= \sum_{k=1}^{\lfloor tn \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right\} \sigma dW_s \\
&\quad + \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \frac{\pi(1-\beta)Y_{\lfloor tn \rfloor + 1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor + 1}(s)} \right\} \sigma dW_s \\
&= \int_0^t \tilde{U}_n(s) \sigma dW_s,
\end{aligned}$$

where

$$\tilde{U}_n(s) = \frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} I_{\{\frac{(k-1)}{n} \leq s \leq \frac{k}{n}\}}.$$

Thus, for arbitrary $n \in \mathbb{N}$, $\tilde{M}_n = \{\tilde{M}_n(t) : t \in [0, \infty)\}$ is a martingale with continuous sample paths and $\tilde{M}_n(0) = 0$. Let us consider now the sequence of stochastic processes $\{\tilde{A}_n\}$ defined by

$$\tilde{A}_n(t) := \langle \tilde{M}_n \rangle_t, \quad t \in [0, \infty).$$

That is, $\tilde{A}_n$ is the quadratic variation of the continuous martingale $\tilde{M}_n$. To apply the Martingale Central Limit Theorem to $\{\tilde{M}_n\}$, we need to consider an arbitrary $t \in [0, \infty)$, and study the convergence in probability of the sequence of random variables $\{\tilde{A}_n(t)\}$.

We observe that, for every $t \in [0, \infty)$,

$$\begin{aligned}
\tilde{A}_n(t) &= \int_0^t \tilde{U}_n^2(s)\sigma^2 ds \\
&= \sum_{k=1}^{\lfloor tn \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right\}^2 \sigma^2 ds \\
&\quad + \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \frac{\pi(1-\beta)Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right\}^2 \sigma^2 ds \\
&= \pi^2(1-\beta)^2\sigma^2 \sum_{k=1}^{\lfloor tn \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right\}^2 ds \\
&\quad + \pi^2(1-\beta)^2\sigma^2 \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \frac{Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right\}^2 ds \\
&= \pi^2(1-\beta)^2\sigma^2 \left[\sum_{k=1}^{\lfloor tn \rfloor} \tilde{Z}_{nk} + \zeta_n \right], \qquad (5.44)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{Z}_{nk} &:= \frac{1}{n}\tilde{R}_{nk}, \\
\tilde{R}_{nk} &:= \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right\}^2 ds, \\
\zeta_n &:= \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \frac{Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right\}^2 ds.
\end{aligned}$$

Thus, for each $n \in \mathbb{N}$, the stochastic process $\tilde{A}_n = \{\tilde{A}_n(t) : t \in [0, \infty)\}$ has continuous sample paths and satisfies, for each $t \geq s \geq 0$, $\tilde{A}_n(t) - \tilde{A}_n(s) \geq 0$. We want to prove that, for arbitrary $t \in [0, \infty)$, the sequence of random variables $\{\tilde{A}_n(t)\}$ converges in probability to $\pi^2(1-\beta)^2\sigma^2 t$. Since the random variable ζ_n in (5.44) converges in probability to 0, this is equivalent to proving that $\sum_{k=1}^{\lfloor tn \rfloor} \tilde{Z}_{nk}$ converges in probability to t.

We cannot apply the standard version of the Weak Law of Large Numbers to prove that $\sum_{k=1}^{\lfloor tn \rfloor} \tilde{Z}_{nk} = \frac{1}{n}\sum_{k=1}^{\lfloor tn \rfloor} \tilde{R}_{nk}$ converges in probability to t, because $\tilde{R}_{nk}$ depends not only on k but also on n. Nevertheless, we observe that for each $n \in \mathbb{N}$ and $t \in [0, \infty)$, making the change of variable $h = s - (k-1)/n$,

$$\begin{aligned}\tilde{R}_{nk} &= \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{Y_k(s)}{\beta + (1-\beta)(1 - \pi + \pi Y_k(s))} \right\}^2 ds \\ &= \frac{1}{\frac{1}{n}} \int_0^{\frac{1}{n}} \left\{ \frac{\exp\{(\mu - \sigma^2/2)h + \sigma(W_{\frac{k-1}{n}+h} - W_{\frac{k-1}{n}})\}}{\beta + (1-\beta)\left[1 - \pi + \pi Y_k(\frac{k-1}{n} + h)\right]} \right\}^2 dh,\end{aligned}$$

and since $W_{\frac{k-1}{n}+h} - W_{\frac{k-1}{n}}$ and W_h are identically distributed, we get that $\{\tilde{Z}_{nk} : k \in \{1, 2, \ldots, \lfloor tn \rfloor\}\}$ are independent and identically distributed positive random variables. Furthermore, for every $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}|\tilde{Z}_{nk}| &= \frac{1}{n}\left| \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{Y_k(s)}{\beta + (1-\beta)\left[1 - \pi + \pi Y_k(s)\right]} \right\}^2 ds \right| \\ &= \frac{1}{n}\left| \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \frac{Y_k(s)}{\beta + (1-\beta)(1-\pi) + \pi(1-\beta)Y_k(s)} \right\}^2 ds \right| \\ &\leq \frac{1}{n}\frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left(\frac{1}{\pi(1-\beta)} \right)^2 ds = \frac{1}{n}\left(\frac{1}{\pi(1-\beta)} \right)^2. \end{aligned} \tag{5.45}$$

In addition, as $n \uparrow \infty$,

$$\tilde{R}_{n1} = n\tilde{Z}_{n1} \xrightarrow{a.s.} \left\{ \frac{Y_1(0)}{\beta + (1-\beta)\tilde{Y}_1(0)} \right\}^2 = 1. \tag{5.46}$$

We also observe that, for every $t \in [0, \infty)$ and $\epsilon > 0$, the Markov inequality gives

$$\sum_{k=1}^{\lfloor tn \rfloor} P\{\tilde{Z}_{nk} \geq \epsilon\} \leq \frac{1}{\epsilon}\sum_{k=1}^{\lfloor tn \rfloor} E[\tilde{Z}_{nk} I_{\{\tilde{Z}_{nk} \geq \epsilon\}}] = \frac{1}{\epsilon}\sum_{k=1}^{\lfloor tn \rfloor} E[\tilde{Z}_{n1} I_{\{\tilde{Z}_{n1} \geq \epsilon\}}]$$
$$= \frac{1}{\epsilon}\frac{\lfloor tn \rfloor}{n} E[\tilde{R}_{n1} I_{\{\tilde{Z}_{n1} \geq \epsilon\}}]. \qquad (5.47)$$

Inequality (5.45) implies that $\tilde{R}_{n1}$ is bounded by $[\pi(1-\beta)]^{-2}$. The limit in (5.46) implies not only $\tilde{R}_{n1} \xrightarrow{a.s.} 1$, but also $I_{\{\tilde{Z}_{n1} \geq \epsilon\}} = I_{\{\tilde{R}_{n1} \geq n\epsilon\}} \xrightarrow{a.s.} 0$. Thus, we may apply the Lebesgue Dominated Convergence Theorem (see Corollary 4.2.3 of Chow and Teicher (1988)) to the right hand side of inequality (5.47). Hence, for every $t > 0$ and $\epsilon > 0$,

$$\lim_{n \to \infty} \sum_{k=1}^{\lfloor tn \rfloor} P\{\tilde{Z}_{nk} \geq \epsilon\} = 0.$$

Another application of the Lebesgue Dominated Convergence Theorem gives

$$\begin{aligned} \sum_{k=1}^{\lfloor tn \rfloor} E[\tilde{Z}_{nk} I_{\{\tilde{Z}_{nk} < 1\}}] &= \lfloor tn \rfloor E[\tilde{Z}_{n1} I_{\{\tilde{Z}_{n1} < 1\}}] \\ &= \frac{\lfloor tn \rfloor}{n} E[\tilde{R}_{n1} I_{\{\tilde{Z}_{n1} < 1\}}] \\ &\longrightarrow t, \end{aligned}$$

because $\tilde{R}_{n1} \xrightarrow{a.s.} 1$, and $I_{\{\tilde{Z}_{n1} < 1\}} \xrightarrow{a.s.} 1$. Thus, the Weak Law of Large Numbers for independent random variables (see Corollary 10.1.2 of Chow and Teicher (1988)), implies that, for each $t \in (0, \infty)$,

$$\sum_{k=1}^{\lfloor tn \rfloor} \tilde{Z}_{nk} \xrightarrow{P} 1, \quad \text{as } n \uparrow \infty.$$

According to equation (5.44), this means that, for each $t \in (0, \infty)$,

$$\tilde{A}_n(t) \xrightarrow{P} \pi^2(1-\beta)^2\sigma^2 t, \quad \text{as } n \uparrow \infty. \qquad (5.48)$$

Here, $\xrightarrow{P}$ denotes convergence in probability.

In summary, for each $n \in \mathbb{N}$, $\tilde{M}_n$ is a martingale with continuous sample paths and $\tilde{M}_n(0) = 0$. Furthermore, the sequence of stochastic processes $\{\tilde{A}_n\}$ defined by $\tilde{A}_n(t) = \langle \tilde{M}_n \rangle_t$ also has continuous sample paths and satisfies, for each $t \geq s \geq 0$, $\tilde{A}_n(t) - \tilde{A}_n(s) \geq 0$. Thus, we may apply the Martingale Central Limit Theorem (see Theorem 7.1.4 of Ethier and Kurtz (1986)). From equation (5.48) we then conclude that

$$\tilde{M}_n \xrightarrow{d} \pi(1-\beta)\sigma W. \tag{5.49}$$

In a similar fashion to the derivation of (5.48), we can also prove that for each $t \in [0, \infty)$,

$$\tilde{C}_n(t) \xrightarrow{P} \pi(1-\beta)\mu t - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2 t, \quad \text{as} \quad n \uparrow \infty, \tag{5.50}$$

where

$$\begin{aligned}
&\tilde{C}_n(t) \\
&:= \sum_{k=1}^{\lfloor tn \rfloor} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right]^2 \right\} ds \\
&+ \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \left[\frac{\pi(1-\beta)Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right]^2 \right\} ds. \\
&= \sum_{k=1}^{\lfloor tn \rfloor} Z_{nk}^c + \zeta_n^c,
\end{aligned} \tag{5.51}$$

with

$$
\begin{aligned}
Z_{nk}^c &:= \frac{1}{n} R_{nk}^c, \\
R_{nk}^c &:= \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right]^2 \right\} ds,
\end{aligned}
$$

$$
\zeta_n^c := \int_{\frac{\lfloor tn \rfloor}{n}}^{t} \left\{ \left[\frac{\pi(1-\beta)Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_{\lfloor tn \rfloor+1}(s)}{\beta + (1-\beta)\tilde{Y}_{\lfloor tn \rfloor+1}(s)} \right]^2 \right\} ds.
$$

Thus, for each $n \in \mathbb{N}$, the stochastic process $\tilde{C}_n = \{\tilde{C}_n(t) : t \in [0,\infty)\}$ has continuous sample paths and satisfies, for each $t \geq s \geq 0$, $\tilde{C}_n(t) - \tilde{C}_n(s) \geq 0$. We want to prove that, for arbitrary $t \in [0,\infty)$, the sequence of random variables $\{\tilde{C}_n(t)\}$ converges in probability to $\pi(1-\beta)\mu t - \pi^2(1-\beta)^2\sigma^2 t/2$. Since the random variable ζ_n^c in (5.51) converges in probability to 0, this is equivalent to proving that $\sum_{k=1}^{\lfloor tn \rfloor} Z_{nk}^c$ converges in probability to $\pi(1-\beta)\mu t - \pi^2(1-\beta)^2\sigma^2 t/2$.

For each $n \in \mathbb{N}$ and $t \in [0,\infty)$, $\{Z_{nk}^c : k \in \{1,2,\ldots,\lfloor tn \rfloor\}\}$ are independent and identically distributed positive random variables. Furthermore, for every $n \in \mathbb{N}$ and $k \in \{1,2,\ldots,n\}$,

$$
\begin{aligned}
|Z_{nk}^c| &= \frac{1}{n} \left| \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left\{ \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right]^2 \right\} ds \right| \\
&\leq \frac{1}{n} \left| \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right] \mu \, ds \right| \\
&\quad + \frac{1}{n} \left| \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_k(s)}{\beta + (1-\beta)\tilde{Y}_k(s)} \right]^2 \right| ds \\
&\leq \frac{1}{n} \frac{1}{\frac{1}{n}} \int_{\frac{(k-1)}{n}}^{\frac{k}{n}} \left(\mu + \frac{\sigma^2}{2} \right) ds = \frac{1}{n} \left(\mu + \frac{\sigma^2}{2} \right).
\end{aligned} \tag{5.52}
$$

In addition, as $n \uparrow \infty$,

$$\begin{aligned} R_{n1}^c &= nZ_{n1}^c = n\int_0^{\frac{1}{n}} \left\{ \left[\frac{\pi(1-\beta)Y_1(s)}{\beta+(1-\beta)\tilde{Y}_1(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_1(s)}{\beta+(1-\beta)\tilde{Y}_1(s)} \right]^2 \right\} ds \\ &\xrightarrow{a.s.} \left[\frac{\pi(1-\beta)Y_1(s)}{\beta+(1-\beta)\tilde{Y}_1(s)} \right] \mu - \frac{1}{2}\sigma^2 \left[\frac{\pi(1-\beta)Y_1(s)}{\beta+(1-\beta)\tilde{Y}_1(s)} \right]^2 \\ &= \pi(1-\beta)\mu - \pi^2(1-\beta)^2\frac{\sigma^2}{2}. \end{aligned} \tag{5.53}$$

For every $t \in [0, \infty)$ and $\epsilon > 0$, the Markov inequality gives

$$\begin{aligned} \sum_{k=1}^{\lfloor tn \rfloor} P\{Z_{nk}^c \geq \epsilon\} &\leq \frac{1}{\epsilon}\sum_{k=1}^{\lfloor tn \rfloor} E[Z_{nk}^c I_{\{Z_{nk}^c \geq \epsilon\}}] = \frac{1}{\epsilon}\sum_{k=1}^{\lfloor tn \rfloor} E[Z_{n1}^c I_{\{Z_{n1}^c \geq \epsilon\}}] \\ &= \frac{1}{\epsilon}\frac{\lfloor tn \rfloor}{n} E[R_{n1}^c I_{\{Z_{n1}^c \geq \epsilon\}}]. \end{aligned} \tag{5.54}$$

Inequality (5.52) implies that R_{n1}^c is bounded by $\mu + \sigma^2/2$, while the limit in (5.53) guarantees that $R_{n1}^c \xrightarrow{a.s.} 1$, and $I_{\{Z_{n1}^c \geq \epsilon\}} = I_{\{R_{n1}^c \geq n\epsilon\}} \xrightarrow{a.s.} 0$. Thus, we may apply the Lebesgue Dominated Convergence Theorem (see Corollary 4.2.3 of Chow and Teicher (1988)) to the right hand side of inequality (5.54). Hence, for every $t > 0$ and $\epsilon > 0$,

$$\lim_{n \to \infty} \sum_{k=1}^{\lfloor tn \rfloor} P\{Z_{nk}^c \geq \epsilon\} = 0.$$

Another application of the Lebesgue Dominated Convergence Theorem gives

$$\begin{aligned} \sum_{k=1}^{\lfloor tn \rfloor} E[Z_{nk}^c I_{\{Z_{nk}^c < 1\}}] &= \lfloor tn \rfloor E[Z_{n1}^c I_{\{Z_{n1}^c < 1\}}] \\ &= \frac{\lfloor tn \rfloor}{n} E[R_{n1}^c I_{\{Z_{n1}^c < 1\}}] \\ &\longrightarrow t\left\{ \pi(1-\beta)\mu - \pi^2(1-\beta)^2\frac{\sigma^2}{2} \right\}, \end{aligned}$$

because $R^c_{n1} \xrightarrow{a.s.} \pi(1-\beta)\mu - \pi^2(1-\beta)^2\frac{\sigma^2}{2}$, and $I_{\{Z^c_{n1}<1\}} \xrightarrow{a.s.} 1$. Thus, the Weak Law of Large Numbers for independent random variables (see Corollary 10.1.2 of Chow and Teicher (1988)), implies that, for each $t \in (0, \infty)$,

$$\sum_{k=1}^{\lfloor tn \rfloor} Z^c_{nk} \xrightarrow{P} t\left\{\pi(1-\beta)\mu - \pi^2(1-\beta)^2\frac{\sigma^2}{2}\right\}, \quad \text{as } n \uparrow \infty.$$

According to equation (5.51), this means that, for each $t \in (0, \infty)$,

$$\tilde{C}_n(t) \xrightarrow{P} \pi(1-\beta)\mu t - \pi^2(1-\beta)^2\frac{\sigma^2}{2}t, \quad \text{as } n \uparrow \infty. \tag{5.55}$$

From the Lebesgue Dominated Convergence Theorem, we observe that for each $t \in [0, \infty)$

$$E[\tilde{C}_n(t)] \longrightarrow \pi(1-\beta)\mu t - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2 t, \quad \text{as} \quad n \uparrow \infty.$$

We want to prove that the sequence of stochastic processes $\{\tilde{C}_n\}$ defined by (5.51) converges in distribution to the deterministic process $\tilde{C}$ defined by

$$\tilde{C}(t) := \pi(1-\beta)\mu t - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2 t. \tag{5.56}$$

Let us define the sequence of stochastic processes $\{\tilde{D}_n\}$ by

$$\tilde{D}_n(t) := \tilde{C}_n(t) - E[\tilde{C}_n(t)], \quad n \in \mathbb{N}, t \in [0, \infty),$$

and the sequence of deterministic processes $\{\tilde{G}_n\}$ by

$$\tilde{G}_n(t) = E[\tilde{D}_n(t)^2] = \mathrm{VAR}[\tilde{C}_n(t)].$$

We observe that, for each $n \in \mathbb{N}$, $\tilde{D}_n$ is a martingale with continuous sample paths and $\tilde{D}_n(0) = 0$. Furthermore, the sequence of deterministic processes

$\{\tilde{G}_n\}$ also has continuous sample paths and satisfies, for each $t \geq s \geq 0$, $\tilde{G}_n(t) - \tilde{G}_n(s) \geq 0$. In addition, for each $t \in [0, \infty)$, $\tilde{D}_n(t) \xrightarrow{P} 0$ and $\{\tilde{D}_n^2(t)\}$ is uniformly integrable.

An application of the Mean Convergence Criterion (see Theorem 4.2.3 of Chow and Teicher (1988)) then gives, for each $t \in [0, \infty)$, $\tilde{G}_n(t) \longrightarrow 0$ as $n \uparrow \infty$. Then, the Martingale Central Limit Theorem (see Theorem 7.1.4 of Ethier and Kurtz (1986)) implies that $\tilde{D}_n \xrightarrow{d} 0$. Hence,

$$\tilde{C}_n \xrightarrow{d} \tilde{C}. \tag{5.57}$$

Finally, equations (5.43), (5.49), and (5.57) imply that the sequence of stochastic processes $\{\log V_n\}$ given by

$$\log V_n(t) \quad = \quad \log V_n(0) + \tilde{M}_n(t) + \tilde{C}_n(t)$$

converges in distribution to a Brownian motion with positive drift $\pi(1-\beta)\mu - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2$ and volatility $\pi(1-\beta)\sigma$. Therefore,

$$V_n \xrightarrow{d} \text{GBM}(\pi(1-\beta)\mu, \pi(1-\beta)\sigma).$$

□

It follows from Proposition 5.2 that with this continuous trading strategy the value of the investment has a long-run growth rate equal to

$$\hat{\mu} - \frac{1}{2}\hat{\sigma}^2 = \pi(1-\beta)\mu - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2.$$

Notice that $\hat{\mu} - \frac{1}{2}\hat{\sigma}^2$ is consistent with the following limiting value of $h(a, b, \pi)$ as $a \uparrow 1$ and $b \downarrow 1$.

Proposition 5.3. *When $\alpha = 0$, for fixed $\pi \in (0, 1)$,*

$$\begin{aligned}\rho(\pi) &:= \lim_{a\uparrow 1, b\downarrow 1} h(a, b, \pi) = \frac{\lambda\pi(1-\beta)}{\gamma}[\gamma - 1 + \pi(1-\beta)] \\ &= \pi(1-\beta)\mu - \frac{1}{2}\pi^2(1-\beta)^2\sigma^2, \end{aligned} \tag{5.58}$$

where $\gamma := -2\lambda/\sigma^2$.

Proof. For a small number $\epsilon > 0$, we have

$$\begin{aligned} &h(1-\epsilon, 1+\epsilon, \pi) \\ &= \lambda\left\{\frac{\log\{1-\pi(1-\beta)\epsilon\}[(1+\epsilon)^\gamma - 1]}{\log(1-\epsilon)[(1+\epsilon)^\gamma - 1)] + \log(1+\epsilon)[1-(1-\epsilon)^\gamma]}\right. \\ &\quad \left. + \frac{\log\{1+\pi(1-\beta)\epsilon\}[1-(1-\epsilon)^\gamma]}{\log(1-\epsilon)[(1+\epsilon)^\gamma - 1)] + \log(1+\epsilon)[1-(1-\epsilon)^\gamma]}\right\}. \end{aligned}$$

By doing a Taylor series expansion around $\epsilon = 0$, we have

$$\begin{aligned} (1+\epsilon)^\gamma - 1 &= \gamma\epsilon + \frac{\gamma(\gamma-1)}{2}\epsilon^2 + O(\epsilon^3), \\ 1-(1-\epsilon)^\gamma &= \gamma\epsilon - \frac{\gamma(\gamma-1)}{2}\epsilon^2 + O(\epsilon^3), \\ \log\{1+\epsilon\} &= \epsilon - \frac{1}{2}\epsilon^2 + O(\epsilon^3), \\ \log\{1-\epsilon\} &= -\epsilon - \frac{1}{2}\epsilon^2 + O(\epsilon^3), \\ \log\{1+\pi(1-\beta)\epsilon\} &= \pi(1-\beta)\epsilon - \frac{1}{2}\pi^2(1-\beta)^2\epsilon^2 + O(\epsilon^3), \\ \log\{1-\pi(1-\beta)\epsilon\} &= -\pi(1-\beta)\epsilon - \frac{1}{2}\pi^2(1-\beta)^2\epsilon^2 + O(\epsilon^3). \end{aligned}$$

The common denominator in the above two terms is equal to

$$\begin{aligned}
&\left[-\epsilon-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]\left[\gamma\epsilon+\frac{\gamma(\gamma-1)}{2}\epsilon^2+O(\epsilon^3)\right]\\
&+\left[\epsilon-\frac{1}{2}\epsilon^2+O(\epsilon^3)\right]\left[\gamma\epsilon-\frac{\gamma(\gamma-1)}{2}\epsilon^2+O(\epsilon^3)\right]\\
=\;&[-\gamma-\gamma(\gamma-1)]\epsilon^3+O(\epsilon^5)=-\gamma^2\epsilon^3+O(\epsilon^5).
\end{aligned}$$

Similarly, the sum of the numerators can be expressed as

$$\begin{aligned}
&\lambda\left[-\pi(1-\beta)\epsilon-\frac{1}{2}\pi^2(1-\beta)^2\epsilon^2+O(\epsilon^3)\right]\left[\gamma\epsilon+\frac{\gamma(\gamma-1)}{2}\epsilon^2+O(\epsilon^3)\right]\\
&+\left[\pi(1-\beta)\epsilon-\frac{1}{2}\pi^2(1-\beta)^2\epsilon^2+O(\epsilon^3)\right]\left[\gamma\epsilon-\frac{\gamma(\gamma-1)}{2}\epsilon^2+O(\epsilon^3)\right]\\
=\;&\lambda[-\pi^2(1-\beta)^2\gamma\epsilon^3-\pi(1-\beta)\gamma(\gamma-1)\epsilon^3+O(\epsilon^4)]\\
=\;&-\lambda\pi(1-\beta)\gamma\epsilon^3[\gamma-1+\pi(1-\beta)]+O(\epsilon^4).
\end{aligned}$$

Applying l'Hospital's rule, we obtain the desired limit as $\epsilon \downarrow 0$. The last equality in the Proposition is obtained immediately by substitution.

□

The investor wants to maximize the growth rate (Kelly criterion)

$$\liminf_{t\to\infty}\frac{E[\log V_t]}{t}=\hat{\mu}-\frac{1}{2}\hat{\sigma}=\pi(1-\beta)\mu-\frac{1}{2}\pi^2(1-\beta)^2\sigma^2.$$

This can be viewed as a second degree polynomial function in π, whose maximum $\mu^2/(2\sigma^2)$ is achieved at $\pi=\mu/[\sigma^2(1-\beta)]$. We assumed $\lambda=\mu-\sigma^2/2>0$, so the value of π that achieves this maximum can be greater than 1.

Proposition 5.4. *When $\alpha = 0$ and the tax rate $\beta \in (0,1)$ is fixed, the optimal proportion of wealth to be invested in the stock is*

$$\hat{\pi}_\beta := \min\left\{\frac{\mu}{\sigma^2(1-\beta)}, 1\right\}. \tag{5.59}$$

When $\hat{\pi}_\beta < 1$, the resulting growth rate is

$$R_\beta = \frac{\mu^2}{2\sigma^2}, \tag{5.60}$$

and when $\hat{\pi}_\beta = 1$,

$$R_\beta = (1-\beta)\mu - \frac{1}{2}\sigma^2(1-\beta)^2. \tag{5.61}$$

It is interesting to note that when $\hat{\pi}_\beta < 1$ the value of the best growth rate is independent of the particular tax rate β in effect. As β increases, the proportion of money to be invested in the stock also increases, suggesting that an increased tax rate makes the stock more attractive. Its appreciation rate decreases from μ to $\pi(1-\beta)\mu$, but its volatility parameter also decreases from σ to $\pi(1-\beta)\sigma$. This is consistent with the result presented in Figure 5.3 with positive transaction costs.

Alternatively, for $\mu \geq \sigma^2(1-\beta)$, that is $\beta \geq 1-\mu/\sigma^2$, we can determine the tax rate that maximizes R_β:

$$\hat{\beta} = 1 - \frac{\mu}{\sigma^2} \in (0,1), \quad R_{\hat{\beta}} = \frac{\mu^2}{2\sigma^2}.$$

Thus, for a tax rate $\beta \in [0, \hat{\beta})$, the optimal proportion is the first term of the minimum in (5.59), and for $\beta \in [\hat{\beta}, 1)$, the optimal proportion is 1.

Remark 5.3. *When the strategy of continuous trading is employed in the absence of transaction costs, the investor is indifferent to the tax rate up to a level $\hat{\beta}$, beyond which the smallest one would be preferable.*

The above discussion applies to the strategy of continuous trading, as $a \uparrow 1$ and $b \downarrow 1$. However, we know from Cadenillas and Pliska (1999) that when $\pi \equiv 1$ the strategy of continuous trading is not always optimal.

Chapter 6

The General Case of One Stock and A Positive Interest Rate Bond

6.1 Bounds for the Long-Run Growth Rate

Let us consider the case when there is only one bond and no stock (i.e. $\pi \equiv 0$). Common sense tells us that the optimal strategy for a portfolio with a bond but no stock is *buy-and-hold*, which corresponds to an infinite optimal stopping time. Therefore we consider for this case the set of stopping times $\mathcal{S}$ instead of $\tilde{\mathcal{S}}$ (see (3.3)).

In terms of Notation 3.1, we have, for fixed $\alpha \in (0,1)$,

$$R_{0,\alpha,\beta}^{(r,\mu,\sigma)} = \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E\Big[\log\Big\{\beta + (1-\alpha)(1-\beta)e^{r\tau}\Big\}\Big]}{E[\tau]}.$$

Using the inequality $\beta \geq 0$ we obtain a lower bound for $R_{0,\beta}^{(r,\mu,\sigma)}$:

$$R_{0,\alpha,\beta}^{(r,\mu,\sigma)} = \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E\Big[\log\Big\{\beta + (1-\alpha)(1-\beta)e^{r\tau}\Big\}\Big]}{E[\tau]}$$

$$\geq \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{(1-\alpha)(1-\beta)e^{r\tau}\}\Big]}{E[\tau]}$$

$$= r + \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{(1-\alpha)(1-\beta)\}\Big]}{E[\tau]}.$$

For every $\tau \in \tilde{\mathcal{S}}$, we have $\beta \leq \beta\, e^{r\tau}$. This gives an upper bound

$$R_{0,\alpha,\beta}^{(r,\mu,\sigma)} = \sup_{\tau\in\tilde{\mathcal{S}}} \frac{E\Big[\log\Big\{\beta + (1-\alpha)(1-\beta)e^{r\tau}\Big\}\Big]}{E[\tau]}$$

$$\leq \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{\beta e^{r\tau} + (1-\alpha)(1-\beta)e^{r\tau}\}\Big]}{E[\tau]}$$

$$= \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\Big\{e^{r\tau}[\beta + (1-\alpha)(1-\beta)]\Big\}\Big]}{E[\tau]}$$

$$= r + \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{\beta + (1-\alpha)(1-\beta)\}\Big]}{E[\tau]}.$$

Clearly $\log\{(1-\alpha)(1-\beta\} \leq 0$, and $\log\{\beta+(1-\alpha)(1-\beta)\} \leq 0$ since $\alpha \in [0,1]$ and $\beta \in [0,1]$ give $\alpha(1-\beta) \geq 0$. Thus the following suprema are zero

$$\sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{(1-\alpha)(1-\beta)\}\Big]}{E[\tau]} = \sup_{\tau\in\mathcal{S}} \frac{E\Big[\log\{\beta + (1-\alpha)(1-\beta)\}\Big]}{E[\tau]} = 0.$$

Since both the lower and upper bounds of $R_{0,\alpha,\beta}^{(r,\mu,\sigma)}$ are r, we obtain the following natural result:

Remark 6.1. *For fixed $\alpha \in (0,1)$ and fixed $\sigma > 0$, the long-run growth rate of a portfolio with one bond but no stock ($\pi \equiv 0$) is*

$$R_{0,\alpha,\beta}^{(r,\mu,\sigma)} = r.$$

Proposition 6.1. *For fixed $\pi \in (0,1]$, we have the following bounds*

$$R_{\pi,\alpha,\beta}^{(0,\mu,\sigma)} \leq R_{\pi,\alpha,\beta}^{(r,\mu,\sigma)} \leq r + R_{\pi,\alpha,\beta}^{(0,\mu-r,\sigma)}. \tag{6.1}$$

Proof. Using (3.2), (3.8) and (3.9), the growth rate $R^{(r,\mu,\sigma)}_{\pi,\alpha,\beta}$ can be written as

$$R^{(r,\mu,\sigma)}_{\pi,\alpha,\beta} = \sup_{\tau\in\tilde{S}} \frac{E\Big[\log\{\beta + (1-\alpha)(1-\beta)\big[(1-\pi)e^{r\tau} + \pi e^{(\mu-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)}.$$

Since $e^{0\tau} = 1 \leq e^{r\tau}$, the inequality involving the lower bound follows. For the inequality involving the upper bound, use $\beta \leq \beta e^{r\tau}$:

$$\begin{aligned}
&\frac{E\Big[\log\{\beta + (1-\alpha)(1-\beta)\big[(1-\pi)e^{r\tau} + \pi e^{(\mu-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)} \\
&\quad\leq \frac{E\Big[\log\{\beta e^{r\tau} + (1-\alpha)(1-\beta)\big[(1-\pi)e^{r\tau} + \pi e^{(\mu-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)} \\
&\quad= \frac{E\Big[\log\Big\{e^{r\tau}\{\beta + (1-\alpha)(1-\beta)\big[(1-\pi) + \pi e^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big\}\Big]}{E(\tau)} \\
&\quad= r + \frac{E\Big[\log\{\beta + (1-\alpha)(1-\beta)\big[(1-\pi) + \pi e^{(\mu-r-\sigma^2/2)\tau+\sigma W_\tau}\big]\}\Big]}{E(\tau)}
\end{aligned}$$

which gives, by taking $\sup_{\tau\in\tilde{S}}$,

$$R^{(r,\mu,\sigma)}_{\pi,\alpha,\beta} \leq r + R^{(0,\mu-r,\sigma)}_{\pi,\alpha,\beta}.$$

□

For fixed tax rate $\beta > 0$ and transaction cost $\alpha > 0$, the above bounds involve three values for the triplet (interest rate, drift, volatility): $(0, \mu, \sigma)$, (r, μ, σ) and $(0, \mu - r, \sigma)$. It is unlikely that the corresponding optimal proportions are all equal, so let us assume them to be π_1, π_2 and π_3. Then, for every $\pi \in (0, 1]$ we have

$$R^{(0,\mu,\sigma)}_{\pi,\alpha,\beta} \leq R^{(0,\mu,\sigma)}_{\pi_1,\alpha,\beta}, \quad R^{(r,\mu,\sigma)}_{\pi,\alpha,\beta} \leq R^{(r,\mu,\sigma)}_{\pi_2,\alpha,\beta}, \quad R^{(0,\mu-r,\sigma)}_{\pi,\alpha,\beta} \leq R^{(0,\mu-r,\sigma)}_{\pi_3,\alpha,\beta}.$$

In particular we have

$$R^{(r,\mu,\sigma)}_{\pi_1,\alpha,\beta} \le R^{(r,\mu,\sigma)}_{\pi_2,\alpha,\beta}, \quad R^{(0,\mu-r,\sigma)}_{\pi_2,\alpha,\beta} \le R^{(0,\mu-r,\sigma)}_{\pi_3,\alpha,\beta}.$$

By Proposition 6.1 we also have

$$R^{(0,\mu,\sigma)}_{\pi_1,\alpha,\beta} \le R^{(r,\mu,\sigma)}_{\pi_1,\alpha,\beta}, \quad R^{(r,\mu,\sigma)}_{\pi_2,\alpha,\beta} \le r + R^{(0,\mu-r,\sigma)}_{\pi_2,\alpha,\beta}.$$

Combining these inequalities gives the range of the optimal growth rate for the triplet (r, μ, σ), without restricting it to a particular proportion.

Corollary 6.1. *The range of the long-run growth rate when $r > 0$ is*

$$R^{(0,\mu,\sigma)}_{\pi_1,\alpha,\beta} \le R^{(r,\mu,\sigma)}_{\pi_2,\alpha,\beta} \le r + R^{(0,\mu-r,\sigma)}_{\pi_3,\alpha,\beta}.$$

Since both $R^{(0,\mu,\sigma)}_{\pi_1,\alpha,\beta}$ and $R^{(0,\mu-r,\sigma)}_{\pi_3,\alpha,\beta}$ correspond to cases with zero interest rates, they can be computed by solving the system of equations (5.18)-(5.22). This yields the expected range for any $R^{(r,\mu,\sigma)}_{\pi_2,\alpha,\beta}$ with $r > 0$.

Example 6.1. *If we take $r = 0.015$ in Example 5.1 we get the following range:*

$$R^{(0,0.065,0.3)}_{1,0.02,0.3} = 0.02231148 \le R^{(0.015,0.065,0.3)}_{\hat{\pi},0.02,0.3} \le 0.015 + R^{(0,0.05,0.3)}_{0.8,0.02,0.3} = 0.027449,$$

Here $\hat{\pi}$ is the unknown optimal proportion to be invested in the stock.

We now summarize the case $\mu = 0.065$, $\sigma = 0.3$, $\alpha = 0.02$. For $\pi \in (0, 1)$, we solved the case of zero interest rate (Example 4.2 for $\beta = 0$ and Example 5.1 for $\beta = 0.3$) and the case of a positive interest rate (Theorem 4.3 for $\beta = 0$ and Example 6.1 for $\beta = 0.3$).

When $\pi = 0$, all the money is invested in the bond, hence the optimal long-run growth rate is given by the interest rate of the bond, say $r = \tilde{r}$ (Remark 6.1). The case $\pi = 1$ (all the money goes into the stock, so the interest rate of the bond is irrelevant), was solved by Cadenillas and Pliska (1999) for $\beta = 0$ ($R = \lambda = 0.02$) and $\beta = 0.3$ (their Example 5.1). These results are presented in Table 6.1.

R	$\pi = 0$	$\pi \in (0,1)$		$\pi = 1$
		$r = 0$	$r = 0.015$	
$\beta = 0.0$	$\tilde{r}$	0.022119	0.0271848	0.02
$\beta = 0.3$	$\tilde{r}$	0.022311	$[0.022311, 0.027449]$	0.022311

Table 6.1: Optimal Long-Run Growth Rate R

Remark 6.2. *We notice that when $\beta = 0$ the optimal proportions are numerically the same for $\mu = 0.065$, $r = 0.015$ and $\mu = 0.05$, $r = 0$ (all other parameters being the same).*

In fact, the continuation regions are identical when written in terms of the fraction process $\{B_t\}_{t\geq 0}$. This verifies numerically Proposition 4.1.

Example 6.2. *Consider again Example* 5.1, *but this time with $r = 0.00075$. Using only the lower bound of Corollary* 6.1 *(see Table 6.2), we conclude that for $r > 0$ an investor is sometimes better off with taxes than without taxes!*

The same qualitative result was obtained for a portfolio with only one stock by Cadenillas and Pliska (1999). According to Example 5.1 this is also true when the portfolio consists of one stock and a zero interest rate bond.

Example 6.2	$r=0$	$r=0.00075$
$\beta=0$	R=0.02211996	R=0.0223037
$\beta=0.3$	R=0.02231148	$R \geq 0.02231148$

Table 6.2: In this example the investor is better off with a positive tax rate

6.2 Solving the Optimal Stopping Problem

We want to solve the general case in the form of Problem 3.3, so we want to find (see (3.17)), for each $\pi \in [0,1]$ and $\theta \in (0,\infty)$:

$$H(\theta) = \sup_{\tau \in \tilde{S}} E\left[\int_0^\tau (-\theta)du + g(I_\tau(\pi)) \right].$$

For fixed value of $\pi \in [0,1]$ we will later choose $\hat{\theta} = R_\pi$ such that

$$H(\hat{\theta}) = 0.$$

This is a problem of maximizing the average reward when costs are incurred at a rate θ per unit of time and a reward $g(I_\tau(\pi))$ is collected at time τ.

Recall that (3.2) gives

$$\begin{aligned} I_\tau(\pi) &= (1-\pi)\frac{S_\tau^0}{S_0^0} + \pi \frac{S_\tau^1}{S_0^1} \\ &= (1-\pi)\, e^{r\tau} + \pi \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W_\tau \right\}. \end{aligned}$$

The explicit dependence on time via the exponential bond price shows that this is an inhomogeneous case. Even written in terms of Morton and Pliska (1995), it is still non-homogeneous (see 3.19)

$$I_\tau(\pi) = \frac{(1-B_0)e^{r\tau}}{1-B_\tau} = \frac{(1-\pi)e^{r\tau}}{1-B_\tau}.$$

To transform this into a homogeneous case we follow the method established in Section 4.6 of Dynkin (1963), and presented with adjustments in Krylov (1980, p. 14), Shiryayev (1978, p. 23), and Section 10.2 of Øksendal (1998).

That is, using (3.2) to express the increase V_t/V_0, we consider the two dimensional process $\{X_t\}_{t\geq 0}$ that has as components time and the homogeneous process $\{S_t^1/S_0^1\}_{t\geq 0}$.

Define the function $G : [0,\infty) \times (0,\infty) \longrightarrow \mathbb{R}$ by

$$G(x_1, x_2) := \log\Big\{\beta + (1-\beta)(1-\alpha)\Big[(1-\pi)\exp(rx_1) + \pi x_2\Big]\Big\}. \tag{6.2}$$

The principle of dynamic programming gives that, with initial condition $X_0 = x := (s,p) \in [0,\infty) \times [0,\infty)$, the value function $v : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ given by

$$v(s,p) := \sup_{\tau \in \tilde{S}} E^{(s,p)}\left[\int_0^\tau (-\theta)du + G(X_\tau^{(s,p)})\right] \tag{6.3}$$

satisfies the following moving boundary problem (see also Crank (1984)),

$$\frac{\partial v}{\partial s}(s,p) + \mu p \frac{\partial v}{\partial p}(s,p) + \frac{\sigma^2 p^2}{2}\frac{\partial^2 v}{\partial p^2}(s,p) = \theta, \ \text{ if } (s,p) \in \tilde{C}, \tag{6.4}$$

$$v(s,p) = G(s,p), \ \text{ if } (s,p) \in \tilde{\Sigma}, \tag{6.5}$$

where

$$\begin{aligned}\tilde{C} &= \{(s,p) \in [0,T] \times [0,\infty) : v(s,p) > G(s,p)\},\\ \tilde{\Sigma} &= \{(s,p) \in [0,T] \times [0,\infty) : v(s,p) = G(s,p)\}.\end{aligned}$$

Since the reward for stopping $G(X^{(s,p)})$ depends only on the increments of the two-dimensional process $\{X_t^{(s,p)}\}_{t\geq 0}$, which depend only on the time increment, and not the absolute position of time itself, we find that this is a homogeneous case.

The stochastic process $\{X_t^{(s,p)}\}_{t\geq 0}$ is given, for every $t \geq 0$, by

$$\begin{aligned} X_t^{(s,p)} : \; \Omega & \longrightarrow [0,\infty) \times \mathbb{R}_+ \\ \omega & \longmapsto X_t^{(s,p)}(\omega) = (t+s, S_{t+s}^1(\omega)/S_0^1(\omega)). \end{aligned} \tag{6.6}$$

Note that taking $p := S_s^1/S_0^1$ gives $X_0 = (s,p) = x$.

The dynamics of $\{X_t = (X_t^{(1)}, X_t^{(2)})\}_{t\geq 0}$ are given, for every $t \geq 0$, by

$$X_t^{(1)} = s + \int_0^t 1\, du + \int_0^t 0\, dW_u, \tag{6.7}$$

$$X_t^{(2)} = p + \int_0^t \mu X_u^{(2)} du + \int_0^t \sigma X_u^{(2)} dW_u. \tag{6.8}$$

To recover the solution of our original stopping problem we use

$$H(\theta) = v(0,1),$$

where v is defined in (6.3).

If $\tilde{\tau}$ is the optimal stopping time for the particular values of $\pi \in [0,1]$ and $\theta \geq 0$, then

$$\begin{aligned} \tilde{\tau} &= \inf\{t_1 \geq 0 : (X_{t_1}^{(1)}, X_{t_1}^{(2)}) \notin \tilde{C}\} = \inf\{t_1 \geq 0 : (t_1 + s, S_{t_1+s}^1/S_0^1) \notin \tilde{C}\} \\ &= \inf\{t = t_1 + s \geq s : (t_1 + s, S_{t_1+s}^1/S_0^1) \notin \tilde{C}\} = \inf\{t \geq s : (t, S_t^1/S_0^1) \notin \tilde{C}\}. \end{aligned}$$

The coefficients of the Hamilton-Jacobi-Bellman partial differential equation in (6.4) depend on the state p, but we can obtain an equivalent problem with constant coefficients by defining

$$Y_t := (Y_t^{(1)}, Y_t^{(2)}) = (X_t^{(1)}, \log X_t^{(2)}), \quad t \geq 0,$$
$$Y_0 = (s, \log p). \tag{6.9}$$

Note that Itô's formula gives, for every $t \geq 0$,

$$S_{t+s}^1 = S_s^1 \exp\{\lambda t + \sigma(W_{t+s} - W_s)\}.$$

Dividing by S_0^1 and using the fact that $W_{t+s} - W_s$ and W_t are identically distributed random variables, we get

$$X_t^{(2)} = X_0^{(2)} \exp\{\lambda t + \sigma W_t\},$$

hence

$$Y_t^{(2)} = Y_0^{(2)} + \int_0^t \lambda du + \int_0^t \sigma dW_u.$$

Define the function $\tilde{G} : [0, \infty) \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\tilde{G}(y_1, y_2) := \log \left\{ \beta + (1-\beta)(1-\alpha)\Big[(1-\pi)\exp(ry_1) + \pi \exp(y_2)\Big] \right\}.$$

We observe that $\tilde{G}(Y_t^y) = G(X_t^x)$.

The value function becomes $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$, defined as

$$u(y) := \sup_{\tau \in \tilde{S}} E^y \left[\int_0^\tau (-\theta) du + \tilde{G}(Y_\tau^y) \right]. \tag{6.10}$$

In view of Remark 4.1, we expect the continuation region to have both lower and upper boundaries.

Then the free boundary problem can be written as

$$\frac{\partial u}{\partial y_1}(y_1, y_2) + \lambda \frac{\partial u}{\partial y_2}(y_1, y_2) + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial y_2^2}(y_1, y_2) = \theta, \text{ if } u(y_1, y_2) > \tilde{G}(y_1, y_2), \tag{6.11}$$

$$u(y_1, y_2) = \tilde{G}(y_1, y_2), \text{ otherwise.}$$

We need to use condition (3.18), that is

$$u(0,0) = 0,$$

in order to uniquely identify our solution.

The partial differential equation (6.11) has constant coefficients and no cross terms. This means that condition (5.3.12) of Kushner and Dupuis (1992)

$$\forall 1 \le i \le n, \forall x: \ a_{ii}(x) - \sum_{j:j\neq i} |a_{ij}(x)| \ge 0, \tag{6.12}$$

where a_{ij} are the coefficients of the partial differential operator

$$\mathcal{L} = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$

is satisfied, so that we can apply their Markov chain approximation method.

We consider the domain of the process Y as $[0, T] \times [-M/2, M/2]$, with $T \in \mathbb{R}$ large enough, and $M \in \mathbb{R}_+$. We use a grid of length h_1 and height h_2

$$\{(h_1 i, h_2 j), i = 0, 1, \ldots, T/h_1, j = 0, \pm 1, \ldots, \pm M/(2h_2)\},$$

chosen such that T/h_1 and $M/(2h_2)$ are integers.

The transition probabilities for the two-dimensional approximating Markov chain are obtained via an implicit finite difference scheme

$$\begin{aligned}
\frac{\partial u}{\partial y_1}(y_1, y_2) &\approx \frac{u(y_1 + h_1, y_2) - u(y_1, y_2)}{h_1}, \\
\frac{\partial u}{\partial y_2}(y_1, y_2) &\approx \frac{u(y_1, y_2 + h_2) - u(y_1, y_2)}{h_2}, \\
\frac{\partial^2 u}{\partial y_2^2}(y_1, y_2) &\approx \frac{u(y_1, y_2 + h_2) + u(y_1, y_2 - h_2) - 2u(y_1, y_2)}{h_2^2}.
\end{aligned}$$

The partial differential equation in (6.11) can then be discretized as

$$\begin{aligned}
&\frac{u(y_1 + h_1, y_2) - u(y_1, y_2)}{h_1} + \lambda \frac{u(y_1, y_2 + h_2) - u(y_1, y_2)}{h_2} \\
&+ \frac{\sigma^2}{2} \frac{u(y_1, y_2 + h_2) + u(y_1, y_2 - h_2) - 2u(y_1, y_2)}{h_2^2} = \theta,
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
u(y_1 + h_1, y_2) \; &+ \; \Big(\lambda \frac{h_1}{h_2} + \frac{\sigma^2}{2} \frac{h_1}{h_2^2}\Big)(y_1, y_2 + h_2) + \frac{\sigma^2}{2} \frac{h_1}{h_2^2} u(y_1, y_2 - h_2) \\
&= \Big(1 + \lambda \frac{h_1}{h_2} + \sigma^2 \frac{h_1}{h_2^2}\Big) u(y_1, y_2) + h_1 \theta,
\end{aligned}$$

which gives

$$u(y_1, y_2) = p_r u(y_1 + h_1, y_2) + p_u u(y_1, y_2 + h_2) + p_d u(y_1, y_2 - h_2) - \Delta t \, \theta,$$

where

$$\begin{aligned}
p_r &= p^h((y_1, y_2) \mapsto (y_1 + h_1, y_2)) = \frac{1}{q}, \\
p_u &= p^h((y_1, y_2) \mapsto (y_1, y_2 + h_2)) = \Big(\lambda \frac{h_1}{h_2} + \frac{\sigma^2}{2} \frac{h_1}{h_2^2}\Big)/q, \\
p_d &= p^h((y_1, y_2) \mapsto (y_1, y_2 - h_2)) = \Big(\frac{\sigma^2}{2} \frac{h_1}{h_2^2}\Big)/q, \\
q &= 1 + \lambda \frac{h_1}{h_2} + \sigma^2 \frac{h_1}{h_2^2}, \\
\Delta t &= h_1 / q.
\end{aligned}$$

Here the notation $(y_1, y_2) \mapsto (y_1, y_2 + h_2)$ means that we have a transition from the two-dimensional point (y_1, y_2) on the grid to the point $(y_1, y_2 + h_2)$.

This means that from any point on the grid we can move to the right in direction y_1 with probability p_r, up in direction y_2 with probability p_u and down in direction y_2 with probability p_d.

Clearly these probabilities add up to 1, so these are the only possibilities. For each transition we have to pay a continuation fee of θ per unit Δt of time elapsed.

These probabilities can be viewed as transition probabilities of a Markov chain $\{\xi_m^h\}_{m\geq 0}$ with $\xi_0^h = y$. The continuous time parameter approximating process is a piecewise constant interpolation of this chain defined by

$$\xi^h(t) = \xi_m^h, \; t \in [t_m^h, t_{m+1}^h), \;\; t_m^h = \sum_{i=0}^{m-1} \Delta t = m\Delta t.$$

Define the stopping times $\tau^h = t_{N_h}^h$, $N_h = \inf\{m \geq 0 : u(\xi_m^h) \leq \tilde{G}(\xi_m^h)\}$. Then we have the weak convergence (see Theorem 9.4.2 and Theorem 10.4.2 of Kushner and Dupuis (1992)): $\tau^h \Longrightarrow \hat{\tau}$ and $\xi^h \Longrightarrow Y$.

Then the $h = (h_1, h_2)$-approximation u^h of the value function u satisfies the equation

$$u^h(y) := \begin{cases} \max\{\sum_{\tilde{y}} p^h(y, \tilde{y}) u^h(\tilde{y}) - \theta\Delta t, \; \tilde{G}(y)\} & \text{if } y \in G^h, \\ \tilde{G}(y) & \text{if } y \notin G^h, \end{cases}$$

where $G^h = \{y \in [0, T] \times [-M/2, M/2] : u(y) > \tilde{G}(y)\}$.

This means that we consider all possible transitions from the point y to points $\tilde{y}$ on the grid, and pay cost at a rate θ proportional to the amount of time Δt elapsed.

The continuation region G^h is unbounded on the time axis, so we use an artificial cut-off point T where we impose boundary conditions. By Theorem IX.5.3 and the discussion of Fleming and Soner (1993, p. 370), the convergence is achieved independent of the particular boundary conditions chosen, as long as $T \to \infty$.

By Theorem 10.6.2 of Kushner and Dupuis (1992) we have the convergence $u^h(y) \longrightarrow u(y)$ as the norm of h goes to zero.

The classical method to compute u^h is the Jacobi iterative method:

$$u^h_{k+1}(y) = \max\left\{\sum_{\tilde{y}} p^h(y,\tilde{y})u^h_k(\tilde{y}) - \theta\Delta t,\ \tilde{G}(y)\right\},\ k \geq 0,$$

with very small initial value: $u^h_0(y) = -9999999$ (see Kushner and Dupuis (1992, p. 391)).

Therefore, for fixed $\pi \in [0,1]$ and fixed θ we obtain a numerical approximation of the value function, as well as the continuation region G^h. In terms of Problem 3.3, we have

$$H(R_\pi) = u(0,0) \approx u^h_k(0,0).$$

We need to determine the optimal growth rate R_π that satisfies (3.18).

6.3 Iterative Algorithm Identifying the Growth Rate

From (3.15) we know that $H(\theta)$ is decreasing in θ, and $H(\theta) = 0$ at $\theta = R_\pi$.

Consider a starting value for θ, say $\theta_0 \in (0, \lambda)$ (or $\theta_0 = 0$ if $\lambda \leq 0$). We derive a formula for updating θ_0. From (3.11), (3.2) and (6.10) we have

$$R_\pi = \sup_{\tau \in \tilde{\mathcal{S}}} J(\tau, \pi) = \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E[g(I_\tau(\pi))]}{E[\tau]} = \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E[\log\{V_\tau/V_0\}]}{E[\tau]} = \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E[\tilde{G}(Y_\tau)]}{E[\tau]}.$$

This implies

$$\begin{aligned} R_\pi &= R_\pi + \sup_{\tau \in \tilde{\mathcal{S}}} \left(\frac{E[\tilde{G}(Y_\tau)]}{E[\tau]} - R_\pi \right) \\ &= R_\pi + \sup_{\tau \in \tilde{\mathcal{S}}} \frac{E[\tilde{G}(Y_\tau) - R_\pi \tau]}{E[\tau]}, \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} &\frac{E[\tilde{G}(Y_\tau) - R_\pi \tau]}{E[\tau]} \leq 0, \quad \forall \tau \in \tilde{\mathcal{S}}, \\ &\frac{E[\tilde{G}(Y_{\tau^*}) - R_\pi \tau^*]}{E[\tau^*]} = 0, \quad \text{for some } \tau^* \in \tilde{\mathcal{S}}. \end{aligned}$$

But $E(\tau) > 0, \forall \tau \in \tilde{\mathcal{S}}$, so we get

$$\begin{aligned} &E[\tilde{G}(Y_\tau) - R_\pi \tau] \leq 0, \quad \forall \tau \in \tilde{\mathcal{S}}, \\ &E[\tilde{G}(Y_{\tau^*}) - R_\pi \tau^*] = 0, \quad \text{for some } \tau^* \in \tilde{\mathcal{S}}, \end{aligned}$$

which means

$$\sup_{\tau \in \tilde{\mathcal{S}}} E[\tilde{G}(Y_\tau) - R_\pi \tau] = 0.$$

Then (6.13) becomes

$$R_\pi = R_\pi + \frac{\sup_{\tau \in \tilde{\mathcal{S}}} E[\tilde{G}(Y_\tau) - R_\pi \tau]}{E[\tau^*]} = R_\pi + \frac{H(R_\pi)}{E[\tau^*]}.$$

This suggests the following formula for updating the value of θ

$$\theta_{n+1} = \theta_n + \frac{H(\theta_n)}{E[\tau_n^*]} \approx \theta_n + \frac{u_{k,n}^h(0,0)}{E_k[\tau_n^*]}, \tag{6.14}$$

with τ_n^* corresponding to θ_n. Here $u_{k,n}^h(0,0)$ is the value obtained after the k-th Jacobi iteration using the h-approximating Markov chain for fixed θ_n.

Denote by $e(y_1, y_2)$ the expected amount of time it takes the process Y, defined in (6.9), to reach the boundary of the continuation region starting from initial value (y_1, y_2) . Then $e_{k,n}(y)$ is computed in a similar fashion to $u_{k,n}^h(y)$, and the value at $(0,0)$ will be $E_k[\tau_n^*]$:

$$
\begin{aligned}
e_{0,n}^h(y_1, y_2) &= 0, \qquad \forall (y_1, y_2), \\
e_{k+1,n}^h(y_1, y_2) &= p_r\, e_{k,n}^h(y_1 + h_1, y_2) + p_u\, e_{k,n}^h(y_1, y_2 + h_2) \\
&\quad + p_d\, e_{k,n}^h(y_1, y_2 - h_2) + \Delta t, \text{ if } u_{k+1,n}^h(y_1, y_2) > \tilde{G}(y_1, y_2), \\
e_{k+1,n}^h(y_1, y_2) &= 0, \text{ if } u_{k+1,n}^h(y_1, y_2) = \tilde{G}(y_1, y_2), \\
E_k[\tau_n^*] &= e_{k,n}^h(0,0).
\end{aligned}
$$

First we update the initial value θ_0:

$$\theta_1 = \theta_0 + u_{k,0}^h(0,0)/E_k[\tau_0^*]. \tag{6.15}$$

Since $\theta_0 < \lambda < R_\pi$ the monotonicity property (3.15) gives $H(\theta_0) > 0$, hence $\theta_1 > \theta_0$. We repeat this until the sequence $\{\theta_n\}_{n \geq 1}$ converges to the real R_π (we can devise a procedure such that the sequence is monotonically increasing and bounded from above, hence convergent).

Using again (3.15) we get, as $\theta_n \to R_\pi$

$$H(\theta_n) \to 0,$$

hence satisfying condition (3.18).

For the limiting value R_π we determine the continuation region; the optimal stopping time will be the time of the first exit from the continuation region.

6.4 Numerical Examples

The overall step by step algorithm is presented schematically in Figure 6.1. Its implementation in *C++* can be found in Appendix B.

MAX=-10000.0 *(initialize the variable* $MAX = \sup_{\pi \in (0,1]} R_\pi$*)*
$\hat{\pi} = -1.0$ *(initialize the optimal proportion* $\hat{\pi}$*)*
FOR $\pi \in (0,1]$
{
  $\theta = \theta_0$ *(initialize the parameter* θ*)*
  REPEAT
  {
    approximate $H(\theta) \approx u^h_{k,n}(0,0)$ and get $E_k(\tau^*_n)$
    $\theta + = u^h_{k,n}(0,0)/E_k(\tau^*_n)$ *(update* θ*)*
  }
  UNTIL $(|u^h_{k,n}(0,0)| < \epsilon)$
  OUTPUT $\theta = R_\pi$
  IF $(R_\pi > MAX)$ THEN
  {
    $MAX = R_\pi$
    $\hat{\pi} = \pi$
  }
}
OUTPUT $(R = MAX, \hat{\pi})$ and the corresponding continuation region

Figure 6.1: The overall Markov chain approximation algorithm

In the following examples we implement this algorithm numerically with

$$h = (h_1, h_2) = (0.247, 0.05), \; k = 2003, \; \epsilon = 10^{-8},$$

and a grid of 160 by 160 points. The typical value for the number of updates of θ is $n = 5$.

Example 6.3. *First we consider the case of no taxes and a zero-interest bond* ($\beta = 0, r = 0$), *solved in Example 4.2. The approach of Morton and Pliska (1995) gave*

$$R = 0.0221199641, \; \hat{\pi} = 0.7307228, \; a = 0.22848, \; b = 7.63819.$$

In the context of a zero-interest bond ($r = 0$) *we obtained*

$$R = 0.0221199588, \; \hat{\pi} = 0.73, \; a = 0.22932, \; b = 7.66633.$$

The Markov chain approximation method gives

$$R = 0.0220532, \; \hat{\pi} = 0.73, \; a = 0.229, \; b = 7.576.$$

The continuation region is presented in Figure 6.2.

Example 6.4. *The previous example can be generalized by allowing a positive interest rate* $r = 0.015$ *and different tax rates.*

For $\beta = 0$ *we get, using the same method as in Example 4.3,*

$$R = 0.0271848, \; \hat{\pi} = 0.557616,$$

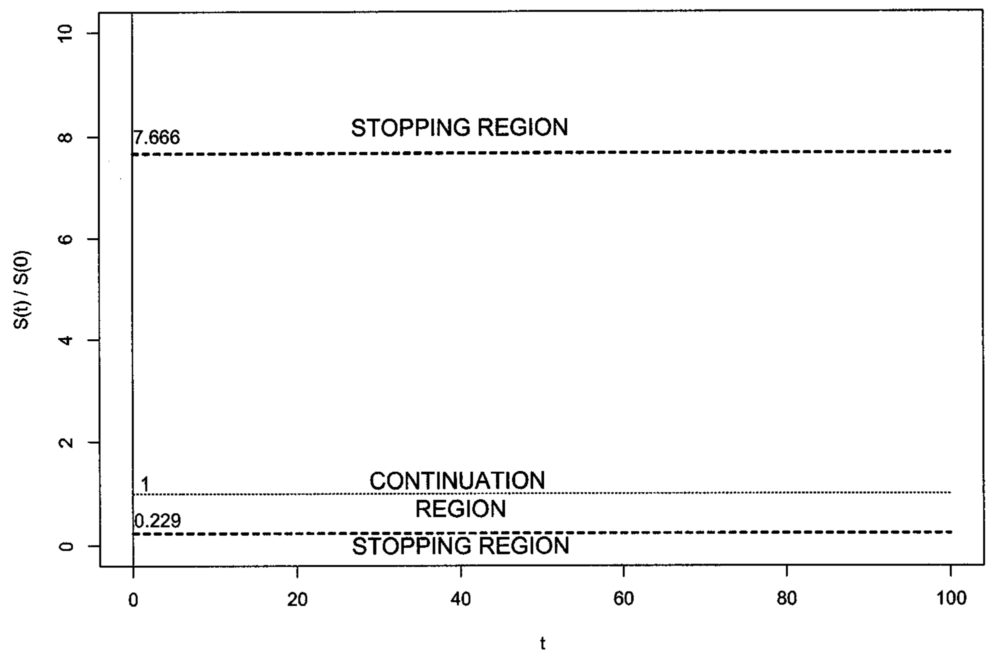

Figure 6.2: Continuation Region for $\beta = 0$, $r = 0$

while the Markov chain approximation method gives

$$R = 0.02718278, \; \hat{\pi} = 0.557.$$

For a tax rate $\beta = 0.3$ *we get* $R = 0.02399063$, $\hat{\pi} = 0.64$ *(see Appendix B); note that the growth rate is within the range obtained in Example 6.1.*

The continuation region is presented in Figure 6.3 for the case of no taxes, and in Figure 6.4 for the case of a positive tax rate (we consider the smaller domain $t \in [0, 20]$*, since beyond that the approximation errors due to* $T = 40$ *start playing a role).*

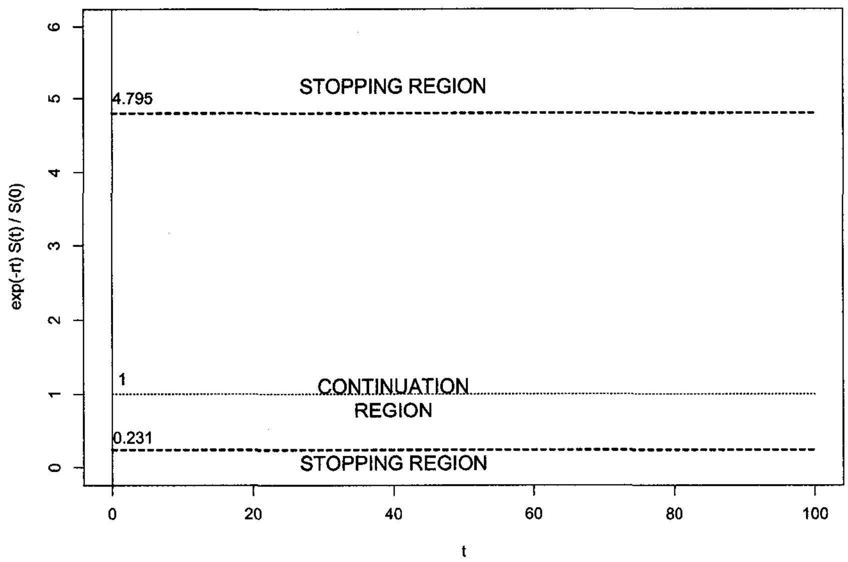

Figure 6.3: Continuation Region for $\beta = 0$, $r = 0.015$

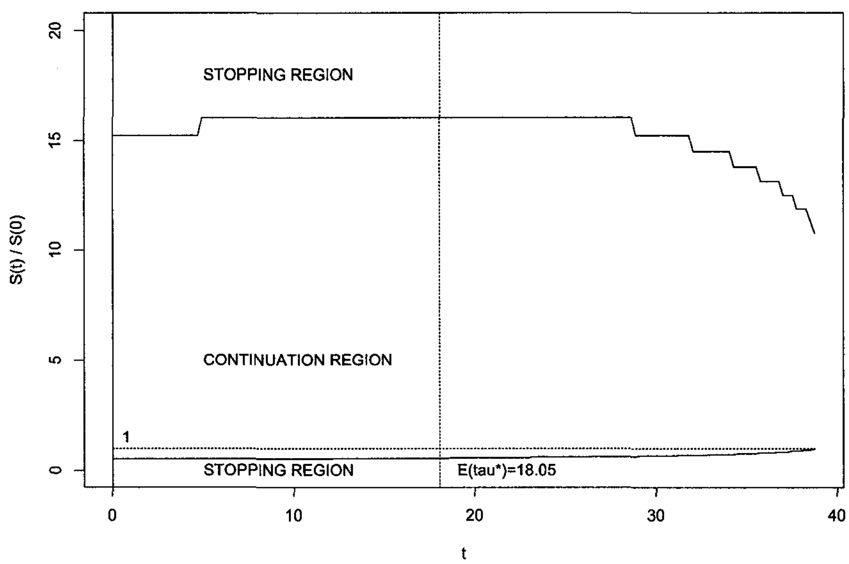

Figure 6.4: Continuation Region for $\beta = 0.3$, $r = 0.015$

In the above example we observe that as the tax rate increases so does the optimal proportion of money to be invested in the stock. This can be better seen by completing Example 6.2 as follows.

Example 6.5. *Let us determine the tax rate preferable to the investor when*

$$\mu = 0.065, \ \sigma = 0.3, \ r = 0.00075, \alpha = 0.02.$$

It turns out that the optimal tax rate is $\hat{\beta} = 0.3$ *(see Figure 6.5).*

Furthermore, the larger the tax rate, the better the cushion against the volatility of the stock, so the investor is willing to allocate an increasing proportion of the initial investment in the stock (see Figure 6.6).

For the best tax rate the optimal strategy is to invest all the money in the stock, so the interest rate of the bond makes no difference (provided r *is small enough) and we recover the result of Example 5.1 (see Figure 5.2).*

We know from Example 6.2 that the investor can prefer a positive tax rate. We consider the effect that the interest rate has on the best tax rate.

Example 6.6. *Consider Example 4.2 with two different tax rates:* $\beta = 0$ *and* $\beta = 0.3$. *We plot in Figure 6.7 the growth rate versus the interest rate in the two cases.*

Obviously, the optimal growth rate is an increasing function of the interest rate. Figure 6.7 is especially interesting for the following reason: if r *is small with respect to* μ *(or* λ*), then the investor will put more money in the*

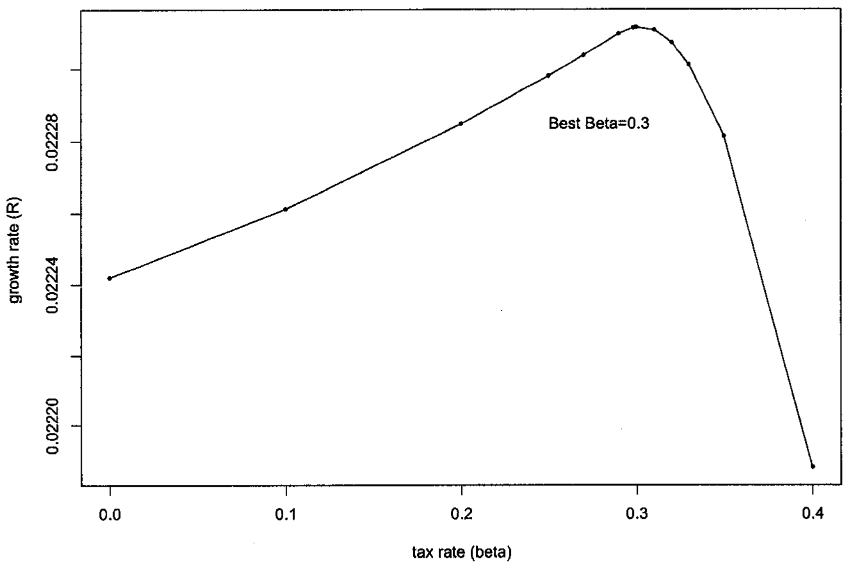

Figure 6.5: Determining the Best Tax Rate when $r = 0.00075$

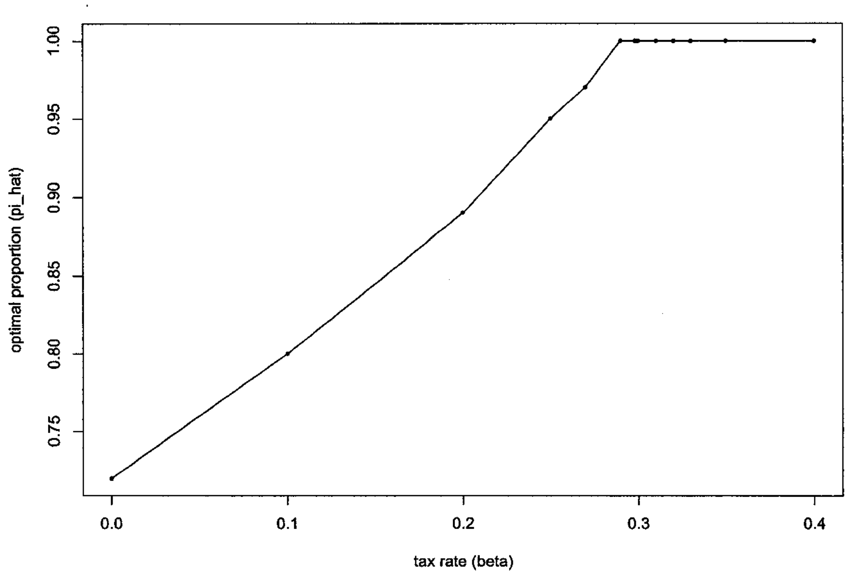

Figure 6.6: Optimal Proportion versus Tax Rate when $r = 0.00075$

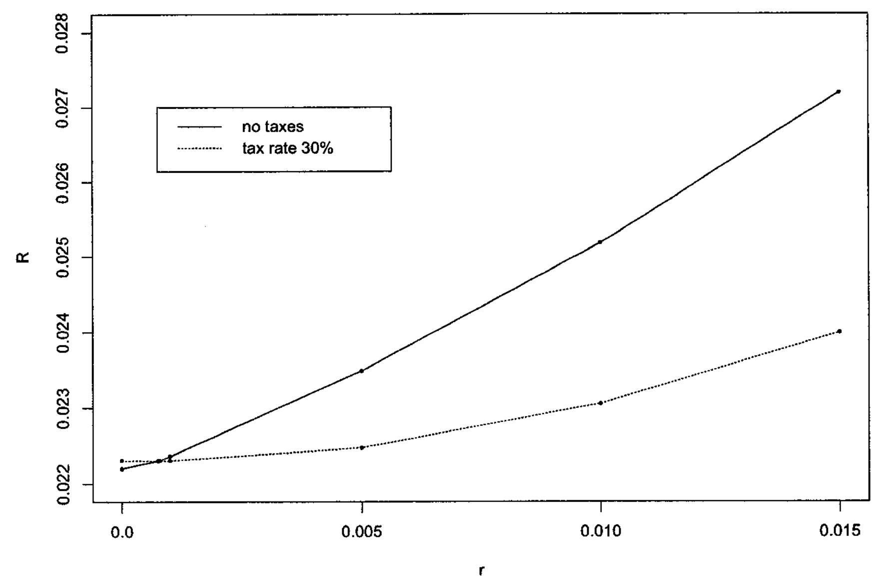

Figure 6.7: Growth Rate versus Interest Rate

stock. However, this is a risky investment, so the investor can do better in the sense of increasing the growth rate in the presence of a positive tax rate.

Furthermore, the optimal proportion of initial wealth invested in the stock is a decreasing function of the interest rate r *(see Figure 6.8).*

In conclusion, Figure 6.6 confirms that the tax rate reduces the risk. That is, the higher the tax rate β*, the lower the risk, so the investor can select a higher proportion* π*.*

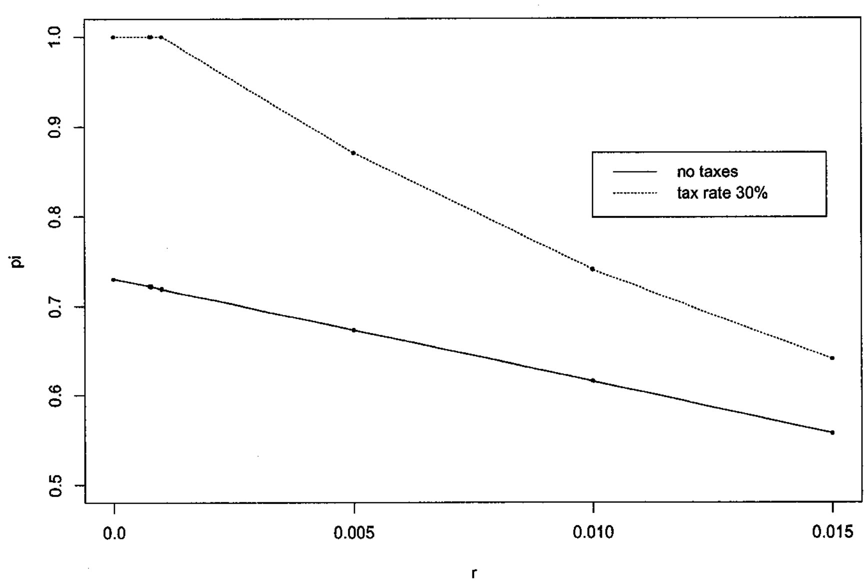

Figure 6.8: Optimal Proportion versus Interest Rate

Chapter 7

A Generalization to Multiple Stocks and One Bond

7.1 Theoretical Approach

In the financial market described in Chapter 2, consider the same bond with dynamics

$$dS_t^0 = rS_t^0 dt, \;\; t \geq 0, \tag{7.1}$$

and n stocks with prices evolving as

$$dS_t^i/S_t^i = \mu_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j, \;\; t \geq 0, \;\; i = 1, 2, \ldots, n, \tag{7.2}$$

where $\{W^i\}_{i=1,2,\ldots,n}$ are independent standard Brownian motions with $W_0^i = 0$ for all $i = 1, 2, \ldots, n$, and μ_i and σ_{ij} are the drift coefficient and the volatilities of the i-th stock. For each $i \in \{1, 2, \ldots, n\}$ denote $\lambda_i := \mu_i - \sum_{j=1}^{n} \sigma_{ij}^2/2$.

At initial time $\tau_0 = 0$, a proportion $\pi_i \in [0, 1]$ of the wealth is invested in the i-th stock, and the remaining proportion $\pi_0 := 1 - \sum_{i=1}^{n} \pi_i$ of wealth is invested in the bond.

Definition 7.1. *The vector of proportions* $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_n)$ *satisfying*

$$\forall i \in \{0, 1, 2, \ldots, n\}: \ \ 0 \leq \pi_i \leq 1,$$

with

$$\sum_{i=0}^{n} \pi_i = 1,$$

defines an admissible portfolio *of the investor. Denote the class of all admissible portfolios by* $\mathcal{P}$.

At time τ_1 the investor makes a transaction, pays taxes and transaction costs, and obtains an amount

$$V_{\tau_1} = V_0 \left\{ \beta + (1-\alpha)(1-\beta) \left[\sum_{i=0}^{n} \pi_i \frac{S^i_{\tau_1}}{S^i_0} \right] \right\}. \tag{7.3}$$

Repeating the procedure of this transaction cycle and using the same reasoning as in Chapters 2 and 3, we obtain the following generalization of Problem 3.3. (According to Corollary 3.2, instead of $g(I_\tau(\pi))$ we now have $\log\{V_\tau/V_0\}$, with V_τ/V_0 given in (7.3)).

Problem 7.1. *For each fixed portfolio* $\underline{\pi} = (\pi_0, \pi_1, \pi_2, \ldots, \pi_n) \in \mathcal{P}$, *determine the value* $R_{\underline{\pi}}$ *of* θ *for which the following optimal stopping problem has value zero*

$$\sup_{\tau \in \tilde{S}} E\Big[\log\left\{\frac{V_\tau}{V_0}\right\} - R_{\underline{\pi}}\tau \Big] = 0, \tag{7.4}$$

where V_τ/V_0 *is given in (7.3).*

That is, for each fixed $\underline{\pi} \in \mathcal{P}$ and each fixed θ solve the optimal stopping problem with value

$$H(\theta) = \sup_{\tau \in \tilde{S}} E\left[\int_0^\tau (-\theta) du + \log\left\{ \frac{V_\tau}{V_0} \right\} \right], \tag{7.5}$$

with V_τ / V_0 given in (7.3).

Then, for that fixed $\underline{\pi} \in \mathcal{P}$, determine the value $R_{\underline{\pi}}$ such that

$$H(R_{\underline{\pi}}) = 0. \tag{7.6}$$

Finally, select the value of $\underline{\pi}$ that maximize $\{R_{\underline{\pi}};\ \underline{\pi} \in \mathcal{P}\}$. That is, find $\hat{\underline{\pi}} \in \mathcal{P}$ such that

$$R_{\hat{\underline{\pi}}} = \sup_{\underline{\pi} \in \mathcal{P}} R_{\underline{\pi}}.$$

To solve (7.5) we proceed as in Section 6.2. We consider the homogenized $(n+1)$-dimensional process $\{\underline{X}_{\,t}\}_{t \geq 0}$ having time as the first component and the natural logarithm of the normalized prices of the stocks as the other n components.

An application of the Itô formula gives, for $i = 1, \ldots, n$ and $t \geq 0$,

$$\begin{aligned} d \log S_t^i / S_0^i &= \frac{1}{S_t^i}\, dS_t^i - \frac{1}{2(S_t^i)^2}\, d\langle S^i \rangle_t \\ &= \frac{1}{S_t^i}\, S_t^i (\mu_i dt + \sum_{j=1}^n \sigma_{ij} dW_t^j) - \frac{1}{2(S_t^i)^2} (S_t^i)^2 \langle \sum_{j=1}^n \sigma_{ij} W^j \rangle_t \\ &= (\mu_i - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2) dt + \sum_{j=1}^n \sigma_{ij} dW_t^j \\ &= \lambda_i\, dt + \sum_{j=1}^n \sigma_{ij}\, dW_t^j. \end{aligned}$$

The dynamics of the process $\underline{X}$ are given, for each $t \geq 0$, by

$$\begin{aligned} d\underline{X}_t^0 &= dt, \\ d\underline{X}_t^i &= \lambda_i\, dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j, \quad \forall i \in \{1, 2, \ldots, n\}. \end{aligned}$$

Generalize $\tilde{G}$ of (6.11) by defining $\underline{G} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ as

$$\begin{aligned} \underline{G}\ & (y_0, y_1, \ldots, y_n) \\ &= \log \left\{ \beta + (1-\alpha)(1-\beta) \left[\pi_0 \exp(r y_0) + \sum_{i=1}^{n} \pi_i \exp(y_i) \right] \right\}. \end{aligned} \tag{7.7}$$

Applying the principle of dynamic programming, we have that, with initial condition $\underline{X}_0 = (y_0, y_1, y_2, \ldots, y_n) =: \mathbf{y} \in \mathbb{R}^{n+1}$, the value function $u : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ defined by

$$u(\mathbf{y}) := \sup_{\tau \in \tilde{S}} E^{\mathbf{y}} \left[\int_0^{\tau} (-\theta) dw + \underline{G}(\underline{X}_\tau) \right] \tag{7.8}$$

satisfies the following moving boundary problem (see (6.4) and (6.11))

$$\frac{\partial u(\mathbf{y})}{\partial y_0} + \sum_{i=1}^{n} \lambda_i \frac{\partial u(\mathbf{y})}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \sigma_{ik} \sigma_{jk} \right) \frac{\partial^2 u(\mathbf{y})}{\partial y_i \partial y_j} = \theta, \quad \text{if } \mathbf{y} \in \mathcal{C} \tag{7.9}$$

$$u(\mathbf{y}) = \underline{G}(\mathbf{y}), \text{ if } \mathbf{y} \in \Sigma, \tag{7.10}$$

where

$$\begin{aligned} \mathcal{C} &:= \{\mathbf{y} \in \mathbb{R}^{n+1} \ : \ u(\mathbf{y}) > \underline{G}(\mathbf{y})\}, & (7.11) \\ \Sigma &:= \{\mathbf{y} \in \mathbb{R}^{n+1} \ : \ u(\mathbf{y}) = \underline{G}(\mathbf{y})\}. & (7.12) \end{aligned}$$

The partial differential equation (7.9) is obtained by applying the multidimensional Itô formula (see Theorem 3.6 of Karatzas and Shreve (1991)) to the function u evaluated at $\underline{X}$

$$\begin{aligned}
du(\underline{X}_t) &= -\theta dt + \sum_{i=0}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} d\underline{X}_t^i + \frac{1}{2}\sum_{i,j=0}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} d\langle \underline{X}^i, \underline{X}^j \rangle_t \\
&= -\theta dt + \frac{\partial u(\underline{X}_t)}{\partial y_0} d\underline{X}_t^0 + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} d\underline{X}_t^i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} d\langle \underline{X}^i, \underline{X}^j \rangle_t \\
&= -\theta dt + \frac{\partial u(\underline{X}_t)}{\partial y_0} dt + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} (\lambda_i \, dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} d\langle \sum_{k=1}^{n} \sigma_{ik} W^k, \sum_{l=1}^{n} \sigma_{jl} W^l \rangle_t \\
&= -\theta dt + \frac{\partial u(\underline{X}_t)}{\partial y_0} dt + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} (\lambda_i \, dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} \sum_{k=1}^{n} \sigma_{ik} \sum_{l=1}^{n} \sigma_{jl} d\langle W^k, W^l \rangle_t .
\end{aligned}$$

Using

$$d\langle W^k, W^l \rangle_t = \delta_{kl} dt$$

we get

$$\begin{aligned}
du(\underline{X}_t) &= -\theta dt + \frac{\partial u(\underline{X}_t)}{\partial y_0} dt + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} (\lambda_i \, dt + \sum_{j=1}^{n} \sigma_{ij} dW_t^j) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} dt \\
&= \Big(-\theta + \frac{\partial u(\underline{X}_t)}{\partial y_0} + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} \lambda_i + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 u(\underline{X}_t)}{\partial y_i \partial y_j} \sum_{k=1}^{n} \sigma_{ik}\sigma_{jk} \Big) dt \\
&\quad + \sum_{i=1}^{n} \frac{\partial u(\underline{X}_t)}{\partial y_i} \sum_{j=1}^{n} \sigma_{ij} dW_t^j .
\end{aligned}$$

The process with the above dynamics is a martingale in the continuation region if and only if the coefficient of dt is zero, resulting in equation (7.9).

The solution to (7.5) is then

$$H(\theta) = u(\underbrace{0, 0, \ldots, 0}_{n+1 \text{ times}}).$$

To obtain numerical approximations of the value function u, the Markov chain approximation method of Kushner and Dupuis (1992) can be used if their condition (5.3.12) (see (6.12)) is satisfied, which involves the coefficients of the terms

$$\frac{\partial^2 u(\mathbf{y})}{\partial y_i \partial y_j}, \quad i \neq j,$$

in equation (7.9). This generalizes the procedure of Chapter 6 to the case of multiple stocks, but the complexity of the computations grows exponentially in comparison to the case of only one stock and one bond.

Proposition 7.1. *Assume that the volatility parameters of (7.2) satisfy the relationship*

$$\forall\, i, j \in \{1, 2 \ldots, n\}: \quad \sigma_{ij} = \sigma_j, \tag{7.13}$$

with $\sigma_j \in \mathbb{R}_+, \forall j \in \{1, 2, \ldots, n\}$*. Then this condition is sufficient for the investor to prefer a positive tax rate.*

Proof. Condition (7.13) ensures that the n stocks have identical randomness, and that the only thing that differentiates them is the drift coefficient of the deterministic part of their stochastic evolution given in (7.2).

Select the largest drift coefficient. Then there are two cases: either there is only one stock with this maximum drift coefficient and all the other $n - 1$

stocks have lower drift coefficients, or there are two or more stocks that share this maximum drift coefficient and the remaining stocks have lower drifts.

First, if one stock, the k-th, has the largest drift coefficient, then all the other $n-1$ stocks will have lower returns on a path-by-path basis (see (7.2))

$$\forall i \in \{1, \ldots, n\} - \{k\} : \ \frac{S_t^k}{S_0^k} > \frac{S_t^i}{S_0^i}.$$

Recall that Problem 7.1 is the multi-dimensional equivalent of Problem 3.3, which originates in Problem 3.2. Therefore we need to select a stopping time $\hat{\tau} \in \tilde{\mathcal{S}}$ and a vector of constants $\hat{\underline{\pi}} \in \mathcal{P}$ that maximize the multi-dimensional equivalent of $J(\tau, \pi)$, namely $E[\log\{V_\tau/V_0\}]/E[\tau]$, with V_τ/V_0 given by (7.3):

$$\sup_{\tau \in \tilde{\mathcal{S}}, \underline{\pi} \in \mathcal{P}} \frac{E\Big[\log\Big\{\beta + (1-\alpha)(1-\beta)\Big[\sum_{i=0}^{n} \pi_i S_{\tau_1}^i / S_0^i\Big]\Big\}\Big]}{E[\tau]}.$$

For every $\tau \in \tilde{\mathcal{S}}$ we notice that

$$\sum_{i=1}^{n} \pi_i \frac{S_\tau^i}{S_0^i} < \Big(\sum_{i=1}^{n} \pi_i\Big) \frac{S_\tau^k}{S_0^k}.$$

Thus the optimal strategy in this case is to invest no money in the $n-1$ stocks with lower returns, while dividing the money between the bond and the k-th stock with largest drift coefficient.

Secondly, if two or more stocks share the largest drift coefficient, their contribution to $\log\{V_\tau/V_0\}$ will be identical, so the money can be invested in these stocks in any proportions and they will have the same return. For simplicity we will assume that the money allocated to these "good" stocks will be invested in only one of them. Then, as before, the remaining stocks with smaller drift coefficients will have lower returns on a path-by-path basis, so the optimal strategy in this case is to divide the money between the bond and one of the stocks sharing the largest drift.

But we have seen in this case (see Example 6.5) that the investor might prefer a positive tax rate, and thus the conclusion of the proposition follows.

□

Remark 7.1. *Proposition 7.1 shows that the second surprising result seen in Chapter 6, that an investor might be better off with a positive tax rate, is also true for the multi-dimensional case when the market consists of multiple stocks and one bond.*

7.2 The Case of Two Stocks and A Zero-Interest Bond

Let us consider the case of two stocks and a zero-interest bond. We rewrite Problem 7.1 when $n = 2$ and $r = 0$. Note that for $t \geq 0$ we have

$$\frac{V_t}{V_0} = \beta + (1-\alpha)(1-\beta)\left(1 - \pi_1 - \pi_2 + \pi_1 \frac{S_t^1}{S_0^1} + \pi_2 \frac{S_t^2}{S_0^2}\right). \tag{7.14}$$

Problem 7.2. *For each fixed portfolio $\underline{\pi} = (\pi_0 = 1 - \pi_1 - \pi_2, \pi_1, \pi_2) \in [0,1]^3$, determine the value $R_{\underline{\pi}}$ of θ for which the following optimal stopping problem has value zero*

$$\sup_{\tau \in \tilde{S}} E\left[\log \frac{V_\tau}{V_0} - R_{\underline{\pi}} \tau\right] = 0. \tag{7.15}$$

That is, for each fixed $\underline{\pi} = (\pi_0 = 1 - \pi_1 - \pi_2, \pi_1, \pi_2) \in [0,1]^3$ and each fixed θ solve the optimal stopping problem with value

$$H(\theta) = \sup_{\tau \in \tilde{S}} E\left[\int_0^\tau (-\theta) du + \log\left\{ \frac{V_\tau}{V_0} \right\} \right]. \tag{7.16}$$

Then, for that fixed $\underline{\pi} \in \mathbb{R}^n$, determine the value $R_{\underline{\pi}}$ such that

$$H(R_{\underline{\pi}}) = 0. \tag{7.17}$$

The function $\underline{G}$ of (7.7) becomes

$$\underline{G}(y_1, y_2) := \log\left[\beta + (1-\alpha)(1-\beta)(1 - \pi_1 - \pi_2 + \pi_1 \exp y_1 + \pi_2 \exp y_2)\right]. \tag{7.18}$$

Consider the two dimensional homogenous process $\underline{X}$ having dynamics like $\{\log S_t^1/S_0^1, \log S_t^2/S_0^2\}_t$:

$$\begin{aligned} d\underline{X}_t^1 &= \lambda_1 dt + \sigma_{11} dW_t^1 + \sigma_{12} dW_t^2, \\ d\underline{X}_t^2 &= \lambda_2 dt + \sigma_{21} dW_t^1 + \sigma_{22} dW_t^2. \end{aligned}$$

Define also

$$m_1 := \sigma_{11}^2 + \sigma_{12}^2, \quad m_2 := \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}, \quad m_3 := \sigma_{21}^2 + \sigma_{22}^2. \tag{7.19}$$

Applying the dynamic programming principle we obtain that the value function $u : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$u(y_1, y_2) := \sup_{\tau \in S} E^{(y_1, y_2)}\left[\int_0^\tau (-\theta) du + \underline{G}(\underline{X}_\tau) \right], \tag{7.20}$$

satisfies the moving boundary problem

$$\lambda_1 \frac{\partial u}{\partial y_1}(y_1, y_2) + \lambda_2 \frac{\partial u}{\partial y_2}(y_1, y_2) + \frac{1}{2} m_1 \frac{\partial^2 u}{\partial y_1^2}(y_1, y_2)$$
$$+ m_2 \frac{\partial^2 u}{\partial y_1 \partial y_2}(y_1, y_2) + \frac{1}{2} m_3 \frac{\partial^2 u}{\partial y_2^2}(y_1, y_2) = \theta, \quad \text{if } (y_1, y_2) \in \mathcal{C}, \quad (7.21)$$

$$u(y_1, y_2) = \underline{G}(y_1, y_2), \text{ if } (y_1, y_2) \in \Sigma, \quad (7.22)$$

where

$$\mathcal{C} := \{(y_1, y_2) \in \mathbb{R}^2 \ : \ u(y_1, y_2) > \underline{G}(y_1, y_2)\}, \quad (7.23)$$

$$\Sigma := \{(y_1, y_2) \in \mathbb{R}^2 \ : \ u(y_1, y_2) = \underline{G}(y_1, y_2)\}. \quad (7.24)$$

The partial differential equation (7.21) has constant coefficients, but the presence of cross terms does not guarantee that condition (5.3.12) of Kushner and Dupuis (1992) (see (6.12)) is always satisfied. When it is satisfied, the Markov chain approximation technique of Kushner and Dupuis (1992) can be used to obtain numerical solutions; when it is not satisfied, the technique of Pliska and Selby (1994) can be used to obtain an equivalent problem that satisfies this condition.

Chapter 8

A Model With Different Results

We investigate the model of Dammon and Spatt (1996), whose optimal trading strategy of a security in discrete time under taxes and transaction costs differs qualitatively from our solution by realizing only losses, but no gains. We show that their model is less realistic than our model.

The techniques of their model are similar to those used by Williams (1985) to obtain the optimal trading strategy of a depreciable asset in continuous time. We first present this auxiliary continuous time model, followed by the analysis of the discrete time model of Dammon and Spatt (1996).

To avoid any confusion between the notation of these two papers and the notation of this thesis, we changed the original notation (for example, the process X of Williams (1985) becomes $\tilde{X}$, while the process X of Dammon and Spatt (1996) becomes $\check{X}$).

8.1 An Auxiliary Model: Trading A Depreciable Asset

We consider the model of Williams (1985). An investor buys a depreciable asset at a price $\tilde{P}$ and receives an immediate proportional tax credit $\tilde{\alpha}\tilde{P}$, $0 \leq \tilde{\alpha} < 1$ (in their numerical example $\tilde{\alpha} = 0$). This purchase price becomes the buyer's initial basis $\tilde{B}$; subsequently the basis $\tilde{B}$ decreases in time at a constant depreciation rate δ:

$$d\tilde{B}/\tilde{B} = -\delta\tilde{B}, \quad \delta \geq 0.$$

(In the model we study in this thesis the basis is not depreciable, so $\delta = 0$.) This results in a depreciation tax shelter of $\delta\tilde{\tau}\tilde{B}$, where $\tilde{\tau}$ is the marginal tax rate on ordinary income.

The depreciable asset generates cash inflows from operations $\tilde{X}$ (for example, the rent from a real estate), evolving as a geometric Brownian motion

$$d\tilde{X}/\tilde{X} = \tilde{\mu}dt + \tilde{\sigma}dZ,$$

where Z is a standard Brownian motion, and $\tilde{\mu}, \tilde{\sigma}$ are constants.

Assume there is a portfolio of risky securities with no dividends, traded continuously without transaction costs, with market price P_s following a geometric Brownian motion

$$dP_s/P_s = \mu_s dt + \sigma_s dZ.$$

In addition, there is a riskless government bond with interest rate r.

Assume that the capital market is sufficiently complete so that the instantaneous return on the investment in the depreciable asset can be replicated by the return on a portfolio of riskless and risky securities. Construct such a portfolio consisting of q units of the bond, with interest taxed at rate $\tilde{\tau}$, and q_s units of the portfolio of risky securities, with gains taxable at rate $0 \leq \tau_s < 1$.

The current value of the investment is denoted by $F(\tilde{B}, \tilde{P}, \tilde{X})$. With $\tilde{P} = \Pi\tilde{X}$, the value $F(\tilde{B}, \tilde{P}, \tilde{X})$ can be rewritten as $V(\tilde{X}, \tilde{B})$. The objective of the investor is to maximize the market value of the investment.

The instantaneous return on the investment in the depreciable asset with current value $V(\tilde{X}, \tilde{B})$ can be computed using Itô's formula (see Black and Scholes (1973) and Scholes (1976) for details)

$$\left[\frac{1}{2}\tilde{\sigma}^2\tilde{X}^2V_{11} + \tilde{\mu}\tilde{X}V_1 - \delta\tilde{B}V_2 + (1-\tilde{\tau})\tilde{X} + \delta\tilde{\tau}\tilde{B}\right]dt + \tilde{\sigma}\tilde{X}V_1 dZ.$$

This reflects the stochastic evolution of the asset price (which is identical to the stochastic evolution of the net operating cash inflow)

$$\left(\frac{1}{2}\tilde{\sigma}^2\tilde{X}^2V_{11} + \tilde{\mu}\tilde{X}V_1\right)dt + \tilde{\sigma}\tilde{X}V_1 dZ,$$

the decrease in the investor's basis

$$-\delta\tilde{B}V_2 dt,$$

the after-tax cash inflows

$$(1-\tilde{\tau})\tilde{X}dt,$$

and the depreciation tax shelter

$$\delta\tilde{\tau}\tilde{B}dt.$$

Here the indices of the function V denote partial derivatives.

The instantaneous after-tax return of the replicating portfolio is

$$q(1-\tilde{\tau})rdt + q_s(1-\tau_s)dP_s,$$

(q units of bond, after-tax factor $(1-\tilde{\tau})$, increase in bond value rdt, and q_s units of stock, after-tax factor $(1-\tau_s)$, increase in stock value dP_s).

Choosing the weights q and q_s appropriately

$$\begin{aligned} q &:= \frac{1}{(1-\tau)r}\Big[\frac{1}{2}\tilde{\sigma}^2\tilde{X}^2V_{11} + \Big(\tilde{\mu} - \frac{\tilde{\sigma}}{\sigma_s}\mu_s\Big)\tilde{X}V_1 - \delta\tilde{B}V_2 + (1-\tilde{\tau})X + \delta\tilde{\tau}\tilde{B}\Big], \\ q_s &:= \frac{\tilde{\sigma}\tilde{X}V_1}{(1-\tau_s)\sigma_s P_s}, \end{aligned}$$

the instantaneous after-tax return of the replicating portfolio is the same as that of the investment in the depreciable asset.

Now

$$V = q + q_s P_s$$

(the depreciable asset is substituted by a portfolio with q units of bond of unit value, and q_s units of stock with price P_s).

Replacing q and q_s in the above equation leads to a PDE. Substituting $x = \tilde{X}/\tilde{B}$ and $v(x) = V(\tilde{X}, \tilde{B})/\tilde{B}$ gives an equivalent ODE

$$\frac{1}{2}\tilde{\sigma}^2x^2v''(x) + (\delta + \tilde{\mu} - \kappa\tilde{\sigma})xv' - [\delta + (1-\tilde{\tau})r]v + (1-\tilde{\tau})x + \delta\tilde{\tau} = 0, \quad (8.1)$$

where κ includes information about μ_s, σ_s, τ_s:

$$\kappa := \frac{(1-\tau_s)\mu_s - (1-\tilde{\tau})r}{(1-\tau_s)\sigma_s}.$$

Here $\tilde{\mu}$ is the mean rate of growth, κ is the standardized excess cost of capital after tax, $\tilde{\sigma}^2$ is the variance of the growth rate, $\tilde{\tau}$ is the tax rate and r, the interest rate. The above ordinary differential equation holds in a non-trading interval.

The general solution of the ordinary differential equation has the form

$$v(x) = \chi + \psi x + Ax^{\zeta} + Bx^{\eta}, \quad A, B \in \mathbb{R}. \tag{8.2}$$

The constants χ and ψ are determined such that $\chi + \psi x$ is a particular solution of the above differential equation:

$$\begin{aligned} \chi &= \frac{\delta\tilde{\tau}}{\delta + (1-\tilde{\tau})r}, \\ \psi &= \frac{1-\tilde{\tau}}{(1-\tilde{\tau})r + \kappa\tilde{\sigma} - \tilde{\mu}} = -\frac{1-\tilde{\tau}}{(\tilde{\tau}+\lambda)r}, \end{aligned} \tag{8.3}$$

where $\lambda := (\tilde{\mu} - \kappa\tilde{\sigma} - r)/r$ is the excess mean rate of growth relative to the interest rate .

The exponents ζ and η are obtained from Ax^{ζ} and Bx^{η} being solutions of the homogeneous differential equation

$$\frac{1}{2}\tilde{\sigma}^2 x^2 v''(x) + [\delta + (\lambda+1)r]xv'(x) - [\delta + (1-\tilde{\tau})r]v(x) = 0$$

as

$$\zeta = \frac{1}{2} - \frac{\delta + (\lambda+1)r}{\tilde{\sigma}^2} - \sqrt{\left\{\frac{1}{2} - \frac{\delta + (1+\lambda)r}{\tilde{\sigma}^2}\right\}^2 + 2\frac{\delta + (1-\tilde{\tau})r}{\tilde{\sigma}^2}} < 0,$$

$$\eta = \frac{1}{2} - \frac{\delta + (\lambda+1)r}{\tilde{\sigma}^2} + \sqrt{\left\{\frac{1}{2} - \frac{\delta + (1+\lambda)r}{\tilde{\sigma}^2}\right\}^2 + 2\frac{\delta + (1-\tilde{\tau})r}{\tilde{\sigma}^2}}.$$

We notice that $0 < \eta$ since $\delta + (1-\tilde{\tau})r \geq 0$, and $\eta < 2$ when $\lambda > -(1+\tilde{\tau})/2 - (\tilde{\sigma}^2 + \delta)/2r$, which holds in their numerical examples.

When selling the depreciable asset brokerage fees are paid at a rate $\tilde{\beta}$, and the capital gains or losses are taxed or credited at rate $\gamma\tilde{\tau}$, where γ is the rate at which capital gains are included in the adjusted gross income. This results in the boundary condition

$$v(x) = (1-\tilde{\beta})(1-\gamma\tilde{\tau})\Pi x + \gamma\tilde{\tau}, \tag{8.4}$$

where Π is chosen such that the following market clearing condition holds

$$v(1/\Pi) = 1 - \tilde{\alpha}. \tag{8.5}$$

In particular, the cash inflows are proportional to the value of the depreciable asset: $\tilde{X} = \tilde{P}/\Pi$.

In addition, the "tight-fit" condition is assumed on the boundary

$$v'(x) = (1-\tilde{\beta})(1-\gamma\tilde{\tau})\Pi. \tag{8.6}$$

The following remark presents the main assumption of the model of Williams (1985).

Remark 8.1. *Williams (1985) obtains a solution to this problem only when the condition*

$$\tilde{\mu} - \kappa\tilde{\sigma} < (1 - \tilde{\tau})r$$

is satisfied, meaning that "the cash inflow $\tilde{X}$ does not grow too rapidly" with respect to the bond. In other words, the cash inflow $\tilde{X}$ has a growth rate that is smaller than the after-tax growth rate of the bond.

This condition ensures that $\eta > 1$.

Let us consider now the solution to this problem when the above condition holds. The condition can be written equivalently as

$$\lambda < -\tilde{\tau} \leq 0.$$

Remark 8.2. *The numerical examples presented in Tables 2 and 3 of Williams (1985) have $\tilde{\tau} > 0$ and $\lambda > 0$, violating the above assumption. This means that $0 < \eta < 1$, which leads to a negative pricing coefficient Π (contrary to the results of their Table 2) and to negative cut-off points (contrary to the values of their Table 3). It is puzzling how the results in Tables 2 and 3 were obtained!*

Assuming that the condition $\tilde{\mu} - \kappa\tilde{\sigma} < (1 - \tilde{\tau})r$ holds, Williams (1985) proves in the Lemma of the Appendix that there can be at most one optimal cut-off point.

Lemma 8.1. *Given $\tilde{\mu} - \kappa\tilde{\sigma} < (1 - \tilde{\tau})r$, any solution to (8.1), (8.4)-(8.6) has at most one optimal trading point: either $x_1 = 0$ or $x_2 = \infty$.*

The proof given to this lemma in Williams (1985) is wrong, but we were able to find a different proof.

They start with the general solution (8.2) to (8.1) on $x_1 < x < x_2$. If $0 < x_1 < x_2 < \infty$, then (8.2) satisfies the border conditions (8.4), (8.6). Since $v(x) > \gamma\tilde{\tau}+\phi\Pi x$ for $x_1 < x < x_2$, we have $v''(x_i) > 0$ at $i = 1, 2$. This results in the existence of two inflection points x_3, x_4 for v such that $x_1 < x_3 < x_4 < x_2$. Differentiating (8.2) twice, multiplying the derivative by x^2 and evaluating the result at $x_i, i = 3, 4$, gives

$$0 = \xi(\xi - 1)Ax_i^{\xi} + \eta(\eta - 1)Bx_i^{\eta}, \quad i = 3, 4, \tag{8.7}$$

and thus

$$0 = \xi(\xi - 1)A(x_3^{\xi} - x_4^{\xi}) + \eta(\eta - 1)B(x_3^{\eta} - x_4^{\eta}). \tag{8.8}$$

Given $v''(x_i) > 0$ at $i = 1, 2$, the constants A and B must have opposite signs in (8.7) and the same sign in (8.8). This produces a contradiction that proves the lemma.

The mistake happens when multiplying the derivative by x^2. In fact, (8.7) should be

$$0 = \xi(\xi - 1)Ax_i^{\xi-2} + \eta(\eta - 1)Bx_i^{\eta-2}, \; i = 3, 4,$$

and thus

$$0 = \xi(\xi - 1)A(x_3^{\xi-2} - x_4^{\xi-2}) + \eta(\eta - 1)B(x_3^{\eta-2} - x_4^{\eta-2}).$$

Since $x_3 < x_4$ and $\eta < 2$, we now have $x_3^{\eta-2} - x_4^{\eta-2} > 0$ instead of $x_3^{\eta} - x_4^{\eta} < 0$, which leads to no contradiction.

Proof. As before it can be shown that there exist two inflection points $x_3 < x_4$ which are solutions of the equation

$$0 = \xi(\xi-1)Ax^{\xi-2} + \eta(\eta-1)Bx^{\eta-2}.$$

This is equivalent to

$$B\eta(\eta-1)x^{\eta-\xi} = -A\xi(\xi-1) \implies x^{\eta-\xi} = -\frac{A\xi(\xi-1)}{B\eta(\eta-1)}.$$

Define the constant

$$c := -\frac{A\xi(\xi-1)}{B\eta(\eta-1)},$$

and the function $m : (0,\infty) \to \mathbb{R}$ given by

$$m(y) := y^{\eta-\xi} - c.$$

We see that $m(0) = -c$ and $m'(y) = (\eta-\xi)y^{\eta-\xi-1} > 0$. Since m is a strictly increasing function, the equation $m(y) = 0$ has a unique solution, contradicting $0 < x_3 < x_4$.

□

Remark 8.3. *Proposition 2 of Williams (1985) holds only for $\delta = \chi = 0$. We could not justify the condition $\chi > \gamma\tilde{\tau} + \tilde{\beta}(1-\gamma\tilde{\tau}) - \tilde{\alpha}$ of Proposition 2.*

To see that $\delta = \chi = 0$, start from the condition $v(x)/x$ bounded as $x \to 0$, with $v(x) = \chi + \psi x + Ax^{\xi} + Bx^{\eta}$. This gives $\chi = 0$, which implies $\delta = 0$ (see the definition of χ in (8.3)). (The restriction $v(x)/x$ bounded by a constant comes from $V(\tilde{X}, \tilde{B}) < \tilde{P}$; the value of the investment in the depreciable asset, after taxes and transaction costs, is less than its market price.)

Remark 8.4. *In Proposition 1 of Williams (1985) the negative sign inside the square root of (21) should be a "+", and the coefficient $\zeta/(1-\zeta)$ of (22) should be $(\gamma\tilde{\tau}-\chi)/(1-\zeta)$ of (23).*

In view of the previous remarks, we summarize next the solution of this problem.

The optimal trading strategy for the depreciable asset falls in one of three cases. If $\chi < \gamma\tilde{\tau}$, it is optimal to realize all losses below an optimal cut-off point (Proposition 1 of Williams (1985)). Otherwise, if $\chi = \delta = \gamma\tilde{\tau} = 0$ and the market clearing condition

$$\chi + \frac{\psi}{\Pi} + \frac{\gamma\tilde{\tau}-\chi}{1-\eta}\left(\frac{\eta-1}{\eta}\frac{\phi-\psi/\Pi}{\chi-\gamma\tilde{\tau}}\right)^{\eta} = 1-\tilde{\alpha}$$

has a unique solution, it is optimal to realize all gains above a certain optimal cut-off point (Proposition 2 of Williams (1985)). In all other cases the strategy of buy-and-hold (make no transaction) is the optimal strategy (Proposition 3 of Williams (1985)).

8.2 Analysis of The Discrete Time Equivalent Model

The model of Dammon and Spatt (1996) is intended to be the discrete time version of that of Williams (1985), but instead of a depreciable asset the portfolio consists of one share of stock. This stock pays dividends $\breve{X}_t$ according to a binomial process with "up factor" $u > 1$, "down factor" u^{-1}, probability to go up q, and probability to go down $1 - q$, with $0 < q < 1$.

Remark 8.5. *1) In this market there is a riskless bond with interest rate* $r = \breve{R} - 1$*, but it* does not *enter the investment portfolio. Its interest rate is used as the rate of discount for intertemporal utility of the investment.*

2) In addition, there is a tax-exempt security generating dividends according to the same binomial process. This security is used only to give an economical interpretation to a constant $\hat{\Pi}$ *that arises in the calculations (this constant will be the tax-exempt pricing operator).*

The share of the stock in the portfolio has current price $\breve{P}_t$, but for an investor with tax basis $\breve{B}$ and holding period h, it is worth $W(\breve{P}_t, \breve{B}, h)$ after-tax dollars. This personal valuation of the share of the stock will generally differ from the market price:

$$W(\breve{P}_t, \breve{B}, h) \leq \breve{P}_t.$$

Initially the investor purchases the share of the stock at market price $\breve{P}_0$, which becomes the tax basis $\breve{B}$. After holding the portfolio for h periods and cashing in dividends in each of these periods according to the formula

$$\breve{X}_t = \frac{1}{\Pi} \breve{P}_t,$$

taxed at rate τ_D, the investor sells the security, pays brokerage fees $c\breve{P}_t$ proportional to the stock price, and the profit or the loss is taxed or credited at a rate $\breve{\tau}$. Here $\breve{\tau} = \tau_s$ in the short-term region, and $\breve{\tau} = \tau_L$ in the long-term region.

The objective of the investor is to maximize the personal valuation $W(\breve{P}_t, \breve{B}, h)$.

Using $\breve{P}_t = \Pi \breve{X}_t$ and defining $x = \breve{X}_t/\breve{B}$ and $v(x,h) = W(\breve{P}_t, \breve{B}, h)/\breve{B}$, in the non-trading periods the personal valuation is the discounted value of the one-period-ahead cumulative dividend payoffs:

$$v(x,h) = \frac{q}{\breve{R}}[ux(1-\tau_D) + v(ux, h+1)] + \frac{1-q}{\breve{R}}[u^{-1}x(1-\tau_D) + v(u^{-1}x, h+1)].$$

In the trading periods the value of the investor's position after paying taxes and transaction costs is

$$v(x,h) = \breve{\tau} + (1-c)(1-\breve{\tau})\Pi x.$$

The general solution of the second order difference equation does not depend on the time parameter h

$$v(x,h) = v(x) = A_1 x^m + A_2 x^n + A_3 x, \;\; A_1, A_2, A_3 \in \mathbb{R},$$

where $A_1 x^m$ and $A_2 x^n$ satisfy the homogenous equation

$$v(x) - \frac{q}{\breve{R}}\, v(ux) - \frac{1-q}{\breve{R}}\, v(u^{-1}x) = 0,$$

and $A_3 x$ is a particular solution of the full difference equation.

This leads to

$$\begin{aligned} m &= \{\log[1 - \sqrt{1 - 4\pi_u \pi_d}] - \log \pi_u - \log 2\}/\log u < 0, \\ n &= \{\log[1 + \sqrt{1 - 4\pi_u \pi_d}] - \log \pi_u - \log 2\}/\log u, \end{aligned}$$

with

$$\pi_u = \frac{q}{1+r}, \quad \pi_d = \frac{1-q}{1+r}.$$

Also,

$$A_3 = \hat{\Pi}(1-\tau_D) := \frac{\pi_u u + \pi_d u^{-1}}{1 - \pi_u u - \pi_d u^{-1}}(1-\tau_D).$$

We have $n > 0$ when $r > 0$, and $n < 2$ when $q < (1+r)/(1+u^2) = \breve{R}/(1+u^2)$. For the numerical examples of Dammon and Spatt (1996), both conditions are satisfied, so $0 < n < 2$.

The paper of Dammon and Spatt (1996) concentrates exclusively on the case $n > 1$ (see equation (A19)), but this condition is not always satisfied.

Example 8.1. *a) For* $u = 1.046012$, $d = u^{-1} = 0.956012$, *and* $r = 0.001835$ *they obtain in their Section 4.1*

$$n = 1.383026 > 1.$$

b) If we halve the weekly interest rate of their bond: $r = 0.001835/2 = 0.0009175$ *(from 10% per annum to 4.88% per annum) and keep all other parameters unchanged, we obtain (see Appendix C)*

$$n = 0.988968 < 1.$$

This reduced interest rate, which is also the discount factor of the intertemporal utility, satisfies the no arbitrage condition (see Prop. 2.3 of Björk (1998) or Section 3.5 of Pliska (1997)) for the binomial model of Dammon and Spatt (1996):

$$d < 1 + r < u \iff 0.956 < 1.000917 < 1.046.$$

Before investigating their solution, let us analyze the condition $n > 1$.

Proposition 8.1. *The condition $n > 1$ holds if and only if the mean growth rate in the stock price is smaller than the interest rate of the riskless bond.*

Proof. Condition $n > 1$ is equivalent to

$$\frac{1+\sqrt{1-4\pi_u\pi_d}}{2u\pi_u} > 1 \iff \sqrt{1-4\pi_u\pi_d} > 2u\pi_u - 1$$

$$\iff 1-4\pi_u\pi_d > 4u^2\pi_u^2 - 4u\pi_u + 1 \iff u^2\pi_u^2 - u\pi_u + \pi_u\pi_d < 0$$

$$\iff u^2\frac{q^2}{\breve{R}^2} - u\frac{q}{\breve{R}} + \frac{q}{\breve{R}}\frac{1-q}{\breve{R}} < 0 \iff u^2q^2 - uq\breve{R} + q(1-q) < 0$$

$$\iff (u^2-1)q^2 - (u\breve{R}-1)q < 0 \iff q < \frac{u\breve{R}-1}{u^2-1}.$$

But, since $\breve{\mu} = E(\breve{X}_{t+1}|\breve{X}_t)$ (see their equation (22))

$$q = \frac{\breve{\mu}-u^{-1}}{u-u^{-1}} = \frac{\breve{\mu}u-1}{u^2-1},$$

where $\breve{\mu}-1$ is the mean of the binomial process of dividends. Therefore

$$q = \frac{\breve{\mu}u-1}{u^2-1} < \frac{u\breve{R}-1}{u^2-1} \iff \mu < \breve{R} \iff \breve{\mu}-1 < \breve{R}-1 = r.$$

The condition $n > 1$ is satisfied if and only if the mean growth rate in dividends $\breve{\mu}-1$ is smaller than the interest rate $r = \breve{R}-1$ of the riskless bond. Recalling that $\breve{P}_t = \Pi\breve{X}_t$, we get that the stock price follows the same binomial process as the dividends, with the same mean growth rate.

□

Remark 8.6. *If $n > 1$ means that the mean growth rate of the price of one share of the stock is smaller than the interest rate of the bond, why not invest all the money in the bond?*

To compare this with the model of Cadenillas and Pliska (1999) where there is only one stock, amounts to having the discount rate $r = 0$. By Proposition 8.1 this means that the mean growth rate of the stock is negative. In Cadenillas and Pliska (1999) this was positive (see their condition (2.2)). Therefore the two models consider mutually exclusive cases.

Remark 8.7. *In particular, for $r = 0$ a stock with negative mean growth rate will indeed result in an optimal strategy that cuts short only the losses.*

Using the same technique as in Williams (1985), Dammon and Spatt (1996) prove in the Lemma of their Appendix that there can be only one optimal cut-off point, corresponding to cutting short losses. The proof of the Lemma makes the same mistake as mentioned in the previous section; specifically, the inequality $x_3^n - x_4^n < 0$ following equation (A21) should be $x_3^{n-2} - x_4^{n-2} > 0$ since $n - 2 < 0$. This way no contradiction results. A different proof can be given along the lines of the proof of Lemma 8.1.

This results in a cut-off point for losses

$$x_L = \tau_L m / [(1-c)(1-\tau_L)\Pi(1-m)],$$

and a value function

$$v(x) = \frac{\tau_L}{1-m}\left(\frac{x}{x_L}\right)^m + (1-\tau_D)\hat{\Pi}x.$$

The optimal strategy is to make a transaction whenever the losses go below the cut-off point x_L.

To compare this with our model that has no dividends, we consider symmetric taxation $\tau_s = \tau_L$, and $\tau_D = 1$. This way the dividends remain in the model, but are not cashed by the investor. Replacing $\tau_D = 1$ in the function $v(x)$ gives

$$v(x) = \frac{\tau_L}{1-m}\left(\frac{x}{x_L}\right)^m + (1-\tau_D)\hat{\Pi}x = \frac{\tau_L}{1-m}\left(\frac{x}{x_L}\right)^m.$$

The constant Π is obtained from the market clearing condition $v(1/\Pi) = 1$, thus

$$\Pi x_L = \left(\frac{1-m}{\tau_L}\right)^{-\frac{1}{m}}.$$

On the other hand, x_L was derived as $\tau_L m/[(1-c)(1-\tau_L)\Pi(1-m)]$, so

$$\Pi x_L = \frac{\tau_L m}{(1-c)(1-\tau_L)(1-m)}.$$

We obtain that the two expressions of Πx_L are equal

$$\left(\frac{1-m}{\tau_L}\right)^{-\frac{1}{m}} = \frac{\tau_L m}{(1-c)(1-\tau_L)(1-m)}.$$

Since the transaction costs c can be varied, we obtain a contradiction. Therefore the two models are not compatible.

Even the optimality criteria are different: we maximize the long-run growth rate, which induces a continuation fee, while the model of Dammon and Spatt (1996) maximizes the discounted return, so there is no continuation fee.

Our criterion is consistent with logarithmic utility, which is a HARA type utility function for risk-averse investors, while Dammon and Spatt (1996) use a risk-neutral utility (identity function).

We conclude with the following remark.

Remark 8.8. *The model of Dammon and Spatt (1996) is less realistic than our model for three main reasons.*

All the money is invested in the stock, and none in the bond, even though a bond is available (but used only for discounting).

The mean growth rate of the stock is poor in comparison to the discount rate. In our model this results in all the money being invested in the bond.

Finally, it considers risk-neutral utility instead of risk-averse utility.

Dammon, Spatt and Zhang (2001) explain the optimality of the capital gains deferral by the forgiveness of capital gains taxes at death under the U.S. tax code, i.e. due to the role of the mortality risk. Their optimal strategy is motivated by that of Constantinides (1983), which realizes losses and defers taxes while allowing short-selling. However, they note that "while these results are appealing from a theoretical perspective, they seem to be inconsistent with the observed realization behaviour of investors. Poterba (1987), using U.S. tax return data, and Odean (1998), using brokerage account data, document that investors realize substantial gains". They conclude that "investors [...] may optimally sell assets with embedded capital gains", particularly young and middle-aged investors who wish to finance consumption.

Recently, Dammon, Spatt and Zhang (2003) consider realizing gains in the optimal strategy, in addition to realizing losses: "If the investor is overexposed to equity, the investor will trade off the tax cost of selling some equity with the diversification benefit of the reduced exposure to the risky asset."

The results of this dissertation are consistent with the empirical findings of Odean (1998) and Poterba (1987), in that it is optimal to make a transaction not only when there is a loss, but also when there is a gain. This was first observed by Cadenillas and Pliska (1999).

Chapter 9

Conclusions

9.1 Results and Contributions

The problem of managing a portfolio with one stock and one bond leads to interesting results when taxes are considered in addition to transaction costs. In the following we summarize these results and present new questions that arise out of them.

9.1.1 Mathematical Results and Contributions

1. Two mathematical models are unified and generalized: the one of a portfolio of one stock under taxes of Cadenillas and Pliska (1999), and the one of Morton and Pliska (1995) where the portfolio has one stock and one bond, but there are no taxes (see Remark 3.3 and Remark 4.2).

2. The approach of Cadenillas and Pliska (1999) is generalized to solve analytically an optimal stopping problem (see Theorem 5.1).

3. A numerical algorithm is derived and implemented in *C++*, combining the

Markov chain approximation technique with an iterative method that identifies the long-run growth rate (see Figure 6.1). This allows us to solve numerically a problem of optimal stopping.

4. The mathematical models of Williams (1985) and Dammon and Spatt (1996) are discussed in Chapter 8. In particular, we show that the results of Dammon and Spatt (1996) which are qualitatively different from ours are based on assumptions which are less realistic than those of our model.

9.1.2 Financial Results and Contributions

1. The optimal strategy is determined explicitly for the case when the bond has a zero interest rate, and numerically when the interest rate is positive (see Theorem 5.1 and Example 6.5).

2. It is proved that it is optimal to make a transaction not only when the losses reach a lower boundary, but also when the gains reach an upper boundary, hence generalizing Cadenillas and Pliska (1999) and Morton and Pliska (1995) (see Example 5.1).

3. The influences that the tax rate and the interest rate of the bond have on the return of the portfolio interact with each other, and affect the optimal strategy that the investor employs in order to manage the risk.

a) When the interest rate is small or zero, the investor prefers a positive tax rate (see Example 6.2 and Remark 5.2).

b) A significant interest rate determines the investor to prefer the safety of the bond to the cushion of tax credits that a positive tax rate would offer (see Example 6.6). As a consequence, when the interest rate is large, the investor prefers a tax rate equal to zero.

4. The end of Section 5.1 contains an analysis of the influence of the volatility of the stock on the tax rate that would be the best for the investor. When the interest rate of the bond is zero, an increasing volatility of the stock determines the investor to seek an increased tax rate. This is consistent with the result of Cadenillas and Pliska (1999), which showed, for a portfolio of one stock in the absence of transaction costs and with continuous trading, that an increased tax rate reduces the volatility of the return. We generalized their result in Section 5.3 for a portfolio having a zero-interest bond in addition to the stock.

9.2 Directions for Future Research

9.2.1 The Case of Multiple Stocks and One Bond

One direction for further research that has already been mentioned in the conclusion of Chapter 7 is the numerical solution in the case when the portfolio has two stocks and a zero-interest bond. This case has been solved in the absence of taxes by Morton and Pliska (1995).

We would like to solve the more general problem when the financial market consists of many stocks and one bond with positive interest rate, complete with numerical examples and economic interpretation of the results.

9.2.2 Using a More General Optimality Criterion

A more general optimality criterion than the current Kelly criterion is the risk-sensitive criterion, which takes into account the exposure to risk the investor is willing to assume. This criterion has been recently introduced in Finance (see Bielecki and Pliska (1999) and Bielecki and Pliska (2000)).

The portfolio management problem can then be written in the following way (see also Whittle (1990)).

Problem 9.1. *Let $\Psi > -2$, $\Psi \neq 0$. In the context of Chapter 2, the investor wants to maximize*

$$J_\theta := \liminf_{t\to\infty} \frac{1}{t}\left(-\frac{2}{\Psi}\right)\log E\left[\exp\left(-\frac{\Psi}{2}\log V_t\right)\right], \tag{9.1}$$

where V_t is the value of the investment at time t.

A Taylor expansion around $\Psi = 0$ yields

$$-\frac{2}{\Psi}\log E\left[\exp\left(-\frac{\Psi}{2}\log V_t\right)\right] = E[\log V_t] - \frac{\Psi}{4}\mathrm{VAR}[\log V_t] + O(\Psi^2). \tag{9.2}$$

Therefore J_Ψ can be interpreted as the long-run expected growth rate minus a penalty term, with an error that is proportional to Ψ^2. The penalty term is proportional with Ψ, so Ψ is interpreted as a *risk sensitivity parameter*, with $\Psi > 0$ and $\Psi < 0$ corresponding to risk averse and risk seeking investors, respectively. The particular case when $\Psi = 0$ is the risk-neutral case, and the criterion J_0 is the Kelly criterion.

Bielecki and Pliska (2000) obtained some results for Problem 9.1 in the case in which there are transaction costs ($\alpha > 0$) but no taxes ($\beta = 0$), using a model that also accounts for some economical factors. The optimal strategy is described in terms of a risk-sensitive quasi-variational inequality, which is solved explicitly only for the case of Morton and Pliska (1995).

A natural generalization would be to add taxes to their model and see how that affects the optimal strategy.

9.2.3 Adding Consumption to The Model

It would be interesting to investigate if these results can be generalized by incorporating consumption in the model. Øksendal and Sulem (2002) use impulse control and quasi-variational inequalities to treat the case when there are transaction costs ($\alpha > 0$), but no taxes ($\beta = 0$).

A nice endeavor would then combine and unify the directions of research mentioned above, resulting in a very general model, better suited for the complex reality of today's financial market.

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Appendix A. *C/C++* code and output for Chapter 5

```
/////////////////////////////////////////////////////////
//      ZERO INTEREST RATE                            //
//  This code is based on the algorithm of            //
//  Gaussian elimination with backward substitution   //
//  and uses the numerical values of Example 4.3      //
//  (was also used for Example 5.1)                   //
/////////////////////////////////////////////////////////

/* Libraries */

#include <stdio.h>
#include <stdlib.h>
#include <time.h>
#include <math.h>

/* Define constants */

#define STEP   10
#define alpha  0.001
#define be     0.0
#define mu     0.112
#define sigma  0.4
#define sigmas 0.16              //sigmas=sigma*sigma
#define lambda mu-sigma*sigma/2
#define INDEX  3
#define gamma  2*mu/sigmas
#define delta  1/(mu-sigmas/2)
#define TOL    0.00001

/* Function prototypes */

double r    (double,double);
double r1   (double,double);
double fone (double,double,double,double);
double ftwo (double,double,double,double,double);
double j00  (double,double,double,double);
double j02  (double,double,double);
double j20  (double,double,double,double);
double j22  (double,double,double,double);

int main()
{
       /* Variables */
FILE *fp;
int flag,flag1;
double coef[INDEX];          //  -F(x)
```

```
double sol [INDEX];          //  initial x
double jac [INDEX][INDEX];   //  Jacobian J(x)
double bias[INDEX];          //  vector y
double temp[INDEX];          //  temporary storing vector
double tem,bone,prop,oare,bet,beta,beta_best=-1.0;
double pi_beta=-1.0,pi_best=-1.0,R_beta_pi=0.0,R_beta=0.0;
int i,j,k,p,r,notfound;      //  various indices

fp=fopen("Example4.3.dat","w");

fprintf(fp,"\n beta       pi            R\n");
fprintf(fp,"\n----------------------------------------\n");

for(bet=0;bet<=0;bet++)     //this loop identifies the
{                           //best tax rate (beta_hat)
 beta=bet/1000;

  sol[0]=.65;            // input initial values for
  sol[1]=1.84;           // a=sol[0], b=sol[1], theta=sol[3]
  sol[2]=.038;
  oare=0.0;

for (prop=705;prop>=695;prop--) //loop for best proportion
{
  printf("\n%3.15f-%3.15f",oare,sol[2]);
  if(oare>sol[2])
      break;
  printf("\n CODE RUNNING \n");
  printf("\n Proportion= %3.15f",prop/1000);
  bone=prop/1000;
  flag=0;
  flag1=0;
  i=0;
  j=0;
  p=0;
  r=0;
  printf("\n TEST \n");
  printf("%3.15f",fone(sol[0],sol[2],bone,beta));
  printf("\n %3.15f",fone(sol[1],sol[2],bone,beta));
  printf("\n %3.15f",ftwo(sol[0],sol[1],sol[2],bone,beta));

   // main loop
                              // STEP 10.1
  k=0;
                              // STEP 10.2
  while ((k<STEP)&&(flag==0))
  {
                              // STEP 10.3
       // vector of -F(x)
```

```
coef[0]=-fone(sol[0],sol[2],bone,beta);
coef[1]=-fone(sol[1],sol[2],bone,beta);
coef[2]=-ftwo(sol[0],sol[1],sol[2],bone,beta);

// Jacobian  J(x)
jac[0][0]=j00(sol[0],sol[2],bone,beta);
jac[0][1]=0.0;
jac[0][2]=j02(sol[0],bone,beta);
jac[1][0]=0.0;
jac[1][1]=j00(sol[1],sol[2],bone,beta);
jac[1][2]=j02(sol[1],bone,beta);
jac[2][0]=j20(sol[0],sol[2],bone,beta);
jac[2][1]=-j20(sol[1],sol[2],bone,beta);
jac[2][2]=j22(sol[0],sol[1],bone,beta);


                        // STEP 10.4

 // Call the Gaussian elimination with
 //backward substitution

 // Initialize the bias
for (j=0;j<INDEX;j++)
     bias[j]=0.0;

                        // STEP 6.1
for (i=0;i<INDEX-1;i++)
{
                        // STEP 6.2
     notfound=1;
     p=i;

     while ((notfound==1) && (p<INDEX))
     {
         if (jac[p][i]==0)
              p+=1;
         else
              notfound=0;
     }
     if (notfound==1)
         flag=1;             // forces exit

                        // STEP 6.3
     if (p!=i)
     {
         for (j=0;j<INDEX;j++)
         {
           temp[j]=jac[p][j];
           jac[p][j]=jac[i][j];
```

```
            jac[i][j]=temp[j];
          }
          tem=coef[p];
          coef[p]=coef[i];
          coef[i]=tem;
     }
                                  // STEP 6.4
     for (j=i+1;j<INDEX;j++)
     {
                                  // STEPS 6.5 and 6.6
          tem=jac[j][i]/jac[i][i];

          for (r=0;r<INDEX;r++)
                jac[j][r]-=(tem*jac[i][r]);
          coef[j]-=tem*coef[i];
     }
}                                 // endfor ENDSTEP 6.1

                                  // STEP 6.7
if (jac[2][2]==0)
    flag=1;

                                  // STEP 6.8
bias[2]=(coef[2])/(jac[2][2]);

                                  // STEP 6.9
for (r=INDEX-2;r>-1;r--)
{
    for (j=r+1;j<INDEX;j++)
          bias[r]+=jac[r][j]*bias[j];
    bias[r]*=-1;
    bias[r]+=coef[r];
    bias[r]/=jac[r][r];
}

                                  // STEP 10.5
for (r=0;r<INDEX;r++)
    sol[r]+=bias[r];
if ((sol[0]>1)||(sol[1]<1))
    printf("Error: a>1 or b<1 !");

for (r=0;r<INDEX;r++)
{
    if (sol[r]<0)
          printf("Error: NEGATIVE solution !");
}
                                  // STEP 10.6
 // Check the stopping condition
```

```
        if (pow(bias[0],2)+pow(bias[1],2)+pow(bias[2],2)<TOL)
        {
             printf("\n Stopping Condition Met!\n");
             for (i=0;i<INDEX;i++)
                  printf("\n Sol= %3.15f",sol[i]);

             fprintf(fp," %3.3f",beta);
             fprintf(fp,"     %3.3f",bone);
             fprintf(fp,"   %3.10f",sol[2]);
             fprintf(fp,"      ( a=%3.5f",sol[0]);
             fprintf(fp," , b=%3.5f)   \n",sol[1]);
             flag=1;
             flag1=1;
        }

                                   // STEP 10.7
        k+=1;
  }                       // endwhile STEP 10.2

  if(flag1==0.0)
  {
        printf("\n #################################### \n");
        printf("\n #####                          ##### \n");
        printf("\n #####     CHOOSE DIFFERENT     ##### \n");
        printf("\n #####     INITIAL VALUES       ##### \n");
        printf("\n #####                          ##### \n");
        printf("\n #################################### \n");
        prop-=1.0;
        printf("\n temporary solution \n ");
        for (i=0;i<INDEX;i++)
             printf("\n Temp= %3.15f",sol[i]);
  }

  oare=sol[2];
  if(oare>R_beta_pi)
  {
        R_beta_pi=oare;
        pi_beta=bone;
  }
                                   // STEP 10.8
 }  //Algorithm Completed

  if(R_beta_pi>R_beta)
  {
       R_beta=R_beta_pi;
       pi_best=pi_beta;
       beta_best=beta;
  }
 }
```

```
fprintf(fp,"\nbeta*=%1.3f",beta_best);
fprintf(fp," pi*=%1.3f",pi_best);
fprintf(fp," R=%1.13f",R_beta);
fclose(fp);
return(0);
}   // end main

/**********************************************/
/* procedure and function definitions*/

double r (double m,double beta)
 {
     double l;
     l=(1-alpha)*(1-beta)/(beta+(1-alpha)*(1-beta)*m);
     return(l);
 }

double r1(double m,double beta)
{
     double l;
     l=-pow(r(m,beta),2);
     return(l);
}

double ftwo(double m, double n, double q,double bone,
            double beta)
{
    double l;

    l=r(m,beta)*pow(m-1+bone,gamma)
      -delta*q*pow(m-1+bone,gamma-1);
    l-=r(n,beta)*pow(n-1+bone,gamma);
    l+=delta*q*pow(n-1+bone,gamma-1);
    return(l);
}

double fone(double m, double q,double bone,double beta)
{
     double l;

     l=(r(m,beta)-delta*q/(m-1+bone))*(m-1+bone-
        pow(bone,1-gamma)*pow(m-1+bone,gamma));
     l/=(1-gamma);
     l+=delta*q*log((m-1+bone)/bone)
        -log(beta+(1-alpha)*(1-beta)*m);
     return(l);
}
```

```
double j20(double m, double q,double bone,double beta)
 {
     double l;

     l=r1(m,beta)*pow(m-1+bone,gamma)
       +gamma*r(m,beta)*pow(m-1+bone,gamma-1)
       -delta*(gamma-1)*q*pow(m-1+bone,gamma-2);
     return(l);
}

double j22 (double m, double n, double bone,double beta)
{
     double l;

     l=delta*(pow(n-1+bone,gamma-1)-pow(m-1+bone,gamma-1));
     return(l);
}

double j00 (double m, double q,double bone,double beta)
{
     double l;

     l=(r1(m,beta)+delta*q/pow(m-1+bone,2))*(m-1+bone-
       pow(bone,1-gamma)*pow(m-1+bone,gamma));
     l+=(r(m,beta)-delta*q/(m-1+bone))*(1-pow(bone,1-gamma)
        *gamma*pow(m-1+bone,gamma-1));
     l/=(1-gamma);
     l+=delta*q/(m-1+bone)-r(m,beta);
     return(l);
}

double j02 (double m, double bone,double beta)
{
     double l;

     l=delta*(1-pow(bone,1-gamma)*pow(m-1+bone,gamma-1));
     l/=-(1-gamma);
     l+=delta*log((m-1+bone)/bone);
     return(l);
}
```

Example4.3.dat

```
 beta        pi          R
-----------------------------------------
 0.000      0.705    0.0385972718       ( a=0.64131 ,b=1.80990)
 0.000      0.704    0.0385974243       ( a=0.64351 ,b=1.81586)
 0.000      0.703    0.0385975257       ( a=0.64570 ,b=1.82188)
 0.000      0.702    0.0385975762       ( a=0.64789 ,b=1.82796)
 0.000      0.701    0.0385975758       ( a=0.65006 ,b=1.83409)
 0.000      0.700    0.0385975248       ( a=0.65223 ,b=1.84028)
 0.000      0.699    0.0385974233       ( a=0.65438 ,b=1.84653)
 0.000      0.698    0.0385972715       ( a=0.65652 ,b=1.85284)
 0.000      0.697    0.0385970695       ( a=0.65866 ,b=1.85921)
 0.000      0.696    0.0385968174       ( a=0.66078 ,b=1.86564)
 0.000      0.695    0.0385965154       ( a=0.66289 ,b=1.87213)

SOLUTION: beta*=0.000, pi*=0.702, R=0.0385975762309.
```

```
/////////////////////////////////////////////////////////////
//    POSITIVE INTEREST RATE                               //
//  This code is based on the algorithm of Figure 6.1,     //
//  and uses the numerical values of Example 6.4.          //
//    (Y(t)= [ t, lambda t+sigma W(t)])                    //
//                                                         //
/////////////////////////////////////////////////////////////


#include<stdio.h>
#include<stdlib.h>
#include<math.h>

#define INDEX1 160  //for large number of points use
#define INDEX2 160  //dynamic memory allocation

//define mesh size h=(h1,h2)=(dt,dx),
//and input the numerical values of the parameters
//(lam=lambda, the rest are self-explanatory)

const double dt=0.247, dx=0.05, lam=0.02,sigma=0.3,r=0.015;
const double alpha=0.02,beta=0.3, f1=1.0;
const double epsilon_u=0.00000001,epsilon_tau=0.01;
const double epsilon_R=0.00001,a11=0.0,a22=sigma*sigma;
const int    PROPS=100;
//take PROPS values for the proportion pi in (0,1];
//here: 0.01, 0.02 and so on up to 1.00

int main()
{
 FILE *fp;
 double pr,pu,pd,q,tn,sj,update=1.0,q1,q2,q3;
 double wait,trade,u1,u2,b2,R,update1=-10000.0;
 double prop,pie_hat=-1.0,update2=-10.0;
 int    krun=1,iloop,i,n,j,propo,flag=1;
 double u_old[INDEX1][INDEX2], u_new[INDEX1][INDEX2];
 double etau_old[INDEX1][INDEX2], etau_new[INDEX1][INDEX2];
 double R_pi[PROPS+1];
 int    inC[INDEX1][INDEX2];

 fp=fopen("Example6.4.dat","w");

 fprintf(fp,"\n        IMPLICIT SCHEME:");
 fprintf(fp,"  Y(t)={");  fprintf(fp,f1<1.0?"ln B(t)":"t");
 fprintf(fp,",ln S(t)}\n");
 fprintf(fp,"\n    lambda=%2.3f ,sigma=%2.3f, r=%2.3f,",lam,
       sigma,r);
```

```
fprintf(fp," alpha=%2.3f, beta=%2.2f\n",alpha,beta);
fprintf(fp,"\n     Grid: %3dx%3d points,",INDEX1,INDEX2);
fprintf(fp," mesh: (dt=%1.4f,dx=%1.3f)",dt,dx);
fprintf(fp,"\n     Range  Y(t): [0,%2.2f)X",
       (double)INDEX1*dt);
fprintf(fp,"(-%2.2f,%2.2f),",(double)(INDEX2*0.5)*dx,
       (double)(INDEX2*0.5)*dx);
fprintf(fp," S(t):(%2.3f,%2.2f)",exp((double)(-INDEX2*0.5)
       *dx),exp((double)(INDEX2*0.5)*dx));

tn=0.0; //running variables denoting the
sj=0.0; //coordinates of the current point

q1=1.0+lam*dt/dx+sigma*sigma*dt/(dx*dx);
q=q1;
q2=lam*dt/dx+sigma*sigma*0.5*dt/(dx*dx);
q3=sigma*sigma*0.5*dt/(dx*dx);
pr=1.0/q1; //transition probabilities are
pu=q2/q1;  //independent of the state
pd=q3/q1;

fprintf(fp,"\n\n      Prob:(pr=%2.3f,pu=%2.3f,pd=%2.3f),",
       pr,pu,pd);
fprintf(fp," total prob=%3.2f,  2003 iters\n",pu+pd+pr);
fprintf(fp,"\n          ->(t+dt,x+dx) with pu=%2.3f",pu);
fprintf(fp,"\n     (t,x)->(t+dt, x)}  with pr =%2.3f",pr);
fprintf(fp,"\n          ->(t+dt,x-dx) with pd=%2.3f",pd);
fprintf(fp,"\n  ____________________________________\n");

for(propo=0;propo<=PROPS;propo++) R_pi[propo]=-1.0;
//initialize the vector of values R_pi

for(propo=1;propo<=PROPS;propo++) //for each pi solve pb.
{
  prop=(double) propo/PROPS;
  fprintf(fp,"\n Prop=%1.2f",prop);
  for(n=0;n<INDEX1;n++)
     for(j=0;j<INDEX2;j++)
     {
            u_old[n][j]=-9999999.0;//old value function
            u_new[n][j]=-9999999.0;//updated value function
            etau_old[n][j]=0.0;     //old E(time)
            etau_new[n][j]=0.0;     //new E(time)
            inC[n][j]=-11;  //in_continuation_region
     }
  R=lam;       //start with theta_0=lambda
  krun=1;      //number of updates of theta
  flag=2;      //flag<=1 when theta converged
  fprintf(fp,"\n     R_0=%3.8f",R);
```

```
while(flag>1)
{
   update1=-111.0;
   update2=0.0;
   for(iloop=1;iloop<=2003;iloop++) //2003 Jacobi
   {                                //iterations

          for(n=0;n<INDEX1-1;n++)
          {
//this paragraph updates the points [t,log(s_min)]
//for all t (s_min=smallest value of S(t) on grid)

                 tn=(double) n*dt;
                 sj=(double)-INDEX2*0.5;
                 trade=log(beta+(1.0-alpha)*(1.0-beta)
                 *((1.0-prop)*exp(r*tn)+prop*exp(sj)));
//trade=reward for stopping & making a transaction

                 u_new[n][0]=(u_old[n][0]>trade)
                             ?u_old[n][0]:trade;

//this for loop updates the points [t,log(s)]
//for s_min<s<s_max
                 for(j=1;j<INDEX2-1;j++)
                 {
                      tn=(double) n*dt;
                      sj=(double)(j-INDEX2*0.5)*dx;
                      wait=pr*u_old[n+1][j]
                          +pu*u_old[n][j+1]
                          +pd*u_old[n][j-1]-R*dt/q;
//wait=reward-for-continuing (no trade)
                      trade=log(beta+(1.0-alpha)
                                  *(1.0-beta)
                            *((1.0-prop)*exp(r*tn)
                            +prop*exp(sj)));

                      if(trade>wait)
                      {
                            u_new[n][j]=trade;
                            inC[n][j]=0;
                            etau_new[n][j]=0.0;
                      }
                      else
                      {
                            u_new[n][j]=wait;
                            inC[n][j]=1;
                            etau_new[n][j]=
                              pr*etau_old[n+1][j]
                             +pu*etau_old[n][j+1]
```

```
                                +pd*etau_old[n][j-1]+dt/q;
                        }
                        if((tn==0.0)&&(sj==0.0))
                        {
                            update1=u_new[n][j];
                            update2=etau_new[n][j];
                        }
                        u_old[n][j]=u_new[n][j];
                        etau_old[n][j]=etau_new[n][j];
                    }//end_for j loop

//this paragraph updates points (t,s_max) for
//all t (s_max=largest value of S(t) on grid)
                    tn=(double) n*dt;
                    sj=(double)(INDEX2-1-INDEX2*0.5)*dx;
                    trade=log(beta+(1.0-alpha)*(1.0-beta)
                      *((1.0-prop)*exp(r*tn)+prop*exp(sj)));
                    u_new[n][INDEX2-1]=(u_old[n][INDEX2-1]
                          >trade)?u_old[n][INDEX2-1]:trade;
            }//end_for n loop

            for(j=0;j<INDEX2;j++)
                    u_new[INDEX1-1][j]=u_old[INDEX1-1][j];
    }//end_for iloop

    //update theta
    if(update2!=0.0)
    {
            update=update1/update2;
            R+=update;
            fprintf(fp,"   update=%2.8f/%2.2f",
                       update1,update2);
            R_pi[propo]=R;
            fprintf(fp,"\n     R_%d=%3.8f",krun++,R);

            if((fabs(update1)<epsilon_u)
                          &&(fabs(update)<epsilon_R))
                    flag=0;
    }
    else
    {
            update=0.0;
            fprintf(fp,"   no update -  E(tau)=0 !");
            flag=0;
    }
  }//end_while loop
}//end_for propo

//identify R=max{R_pi}
```

```
R=0.0;
for(propo=1;propo<=PROPS;propo++)
{
  if(R_pi[propo]>R)
  {
     R=R_pi[propo];
     pie_hat=(double) propo/PROPS;
  }
}
fprintf(fp,"\n\n\n SOL> pi=%1.2f(+/-%1.2f), R=%1.8f",
pie_hat, (double)1.0/PROPS,R);

//for the optimal pi_hat determine the continuation region
prop=pie_hat;

for(n=0;n<INDEX1;n++)
  for(j=0;j<INDEX2;j++)
  {
     u_old[n][j]=-9999999.0;
     u_new[n][j]=-9999999.0;
     etau_old[n][j]=0.0;
     etau_new[n][j]=0.0;
     inC[n][j]=-11;
  }
  R=lam;
  krun=1;
  flag=2;
  fprintf(fp,"\n     R_0=%3.8f",R);
  while(flag>1)
  {
     update1=-111.0;
     update2=0.0;
     for(iloop=1;iloop<=2003;iloop++)
     {
        for(n=0;n<INDEX1-1;n++)
        {
              tn=(double) n*dt;
              sj=(double)-INDEX2*0.5;
              trade=log(beta+(1.0-alpha)*(1.0-beta)
              *((1.0-prop)*exp(r*tn)+prop*exp(sj)));
              u_new[n][0]=(u_old[n][0]>trade)
                         ?u_old[n][0]:trade;

              for(j=1;j<INDEX2-1;j++)
              {
                   tn=(double) n*dt;
                   sj=(double)(j-INDEX2*0.5)*dx;
                   wait= pr*u_old[n+1][j]+pu*u_old[n][j+1]
                        +pd*u_old[n][j-1]-R*dt/q;
```

```
            trade=log(beta+(1.0-alpha)*(1.0-beta)
               *((1.0-prop)*exp(r*tn)+prop*exp(sj)));

            if(trade>wait)
            {
                   u_new[n][j]=trade;
                   inC[n][j]=0;
                   etau_new[n][j]=0.0;
            }
            else
            {
                   u_new[n][j]=wait;
                   inC[n][j]=1;
                   etau_new[n][j]=pr*etau_old[n+1][j]
                                 +pu*etau_old[n][j+1]
                           +pd*etau_old[n][j-1]+dt/q;
            }
            if((tn==0.0)&&(sj==0.0))
            {
                   update1=u_new[n][j];
                   update2=etau_new[n][j];
            }
            u_old[n][j]=u_new[n][j];
            etau_old[n][j]=etau_new[n][j];
       }//end_for j loop

       tn=(double) n*dt;
       sj=(double)(INDEX2-1-INDEX2*0.5)*dx;
       trade=log(beta+(1.0-alpha)*(1.0-beta)
         *((1.0-prop)*exp(r*tn)+prop*exp(sj)));
       u_new[n][INDEX2-1]=(u_old[n][INDEX2-1]>trade)
                          ?u_old[n][INDEX2-1]:trade;
   }//end_for n loop

   for(j=0;j<INDEX2;j++)
       u_new[INDEX1-1][j]=u_old[INDEX1-1][j];
} //end_for iloop

if(update2!=0.0)
{
   update=update1/update2;
   R+=update;
   fprintf(fp,"   update=%2.8f/%2.2f",
            update1,update2);
   fprintf(fp,"\n      R_%d=%3.8f",krun++,R);
   if((fabs(update1)<epsilon_u)&&
              (fabs(update)<epsilon_R))
         flag=0;
```

```
    }
    else
    {
       update=0.0;
       fprintf(fp,"   no update -  E(tau)=0 !");
       flag=0;
    }

 }//end_while loop

 fprintf(fp,"\n Boundaries at time 0");
 //determine the continuation region when t=0

 for(j=INDEX2-3;j>0;j--)
 {
    if(inC[0][j]!=inC[0][j+1])
    {
       sj=(double) (j-INDEX2*0.5)*dx+dx*0.5;
       fprintf(fp,"\n         S(0)=%1.3f",exp(sj));
       fprintf(fp,"   in (%1.3f,%1.3f)",exp(sj-dx*0.5),
                  exp(sj+dx*0.5));
       fprintf(fp," (ln S(0)=%2.3f",sj);
       fprintf(fp,"  in (%1.2f,%1.2f))",sj-dx*0.5,
                  sj+dx*0.5);
    }
 }

//output final result
fprintf(fp,"\n\n\n SOLUTION: pi*=%3.2f(+/-%1.2f),",
            pie_hat,(double) 1.0/PROPS);
fprintf(fp," R=%3.9f,  E(tau)=%3.2f\n",R,update2);


///////PLOT Continuation Region in RAW FORM  ////////////

fprintf(fp,"\nPLOT ln S(t) vs. t\n\n");

for(j=INDEX2-3;j>1;j--)
{
   for(n=0;n<INDEX1-2;n++)
      if((fmod(n,4)==0)&&(fmod(j,4)==0))
      {
            if((n==0)&&(j==(int)INDEX2*0.5))
                  fprintf(fp,"0");
            else
                if(n==0)
                  fprintf(fp,"|");
                else
                  fprintf(fp,(inC[n][j]==1)?"*":"-");
```

```
        }//end for n
        if(fmod(j,4)==0)
              fprintf(fp,"\n");
  }//end for j

 //OUTPUT  Continuation Region's Boundaries a(t), b(t)


  fprintf(fp,"\n OUTPUT time t \n");

  for(i=0;i<INDEX1-1;i++)
     for(j=1;j<INDEX2-1;j++)
        if(inC[i][j]==1)
        {
             tn=dt*((double)i);
             u1=tn/f1;
             fprintf(fp,"\n  %2.3f     ",u1);
        }

  fprintf(fp,"\n OUTPUT u2=S(t) \n");
  for(i=0;i<INDEX1-1;i++)
     for(j=1;j<INDEX2-1;j++)
        if(inC[i][j]==1)
        {
             sj=dx*((double)(j-INDEX2/2));
             u2=exp(sj);
             fprintf(fp,"\n  %2.3f",u2);
        }

 return(0);
}//end main
```

Example6.4.dat

```
        IMPLICIT SCHEME:  Y(t)={ t,ln {S(t)/S(0)} }

 lambda=0.020 ,sigma=0.300, r=0.015, alpha=0.020, beta=0.30

 Grid: 160x160 points, mesh: (dt=0.247,dx=0.05).
 Range  Y(t): [0,39.52)X(-4.00,4.00), S(t):(0.018,54.60).
 Prob:(pr=0.100,pu=0.455,pd=0.445), total prob=1.00.
 No. of iterations: 2003.

             ->(t+dt,x+dx) with pu=0.455
        (t,x)->(t+dt, x)}  with pr=0.100
             ->(t+dt,x-dx) with pd=0.445
 -------------------------------------------------------

Prop=0.63
    R_0=0.02000000   update=0.10838321/32.10
    R_1=0.02337628   update=0.01244922/22.22
    R_2=0.02393662   update=0.00097700/18.44
    R_3=0.02398959   update=0.00001323/18.07
    R_4=0.02399032   update=0.00000000/18.06
    R_5=0.02399032
Prop=0.64
    R_0=0.02000000   update=0.10892117/32.21
    R_1=0.02338164   update=0.01240631/22.33
    R_2=0.02393725   update=0.00097905/18.69
    R_3=0.02398962   update=0.00001818/18.03
    R_4=0.02399063   update=0.00000000/18.03
    R_5=0.02399063
Prop=0.65
    R_0=0.02000000   update=0.10937700/32.33
    R_1=0.02338313   update=0.01239731/22.48
    R_2=0.02393469   update=0.00099845/18.78
    R_3=0.02398787   update=0.00001050/18.25
    R_4=0.02398845   update=0.00000000/18.24
    R_5=0.02398845


SOL> pi=0.64(+/-0.01), R=0.02399063
    R_0=0.02000000   update=0.10892117/32.21
    R_1=0.02338164   update=0.01240631/22.33
    R_2=0.02393725   update=0.00097905/18.69
    R_3=0.02398962   update=0.00001818/18.03
    R_4=0.02399063   update=0.00000000/18.03
    R_5=0.02399063

SOLUTION: pi*=0.64(+/-0.01), R=0.023990626,  E(tau*)=18.03.
```

```
> restart;mu:=1.0009387:sigma:=0.045:R:=1+.0018346/1:
> printf("Input values of Dammon and Spatt, 1996
> (Section 4.1):");
> printf(" mu = %f, sigma = %f, R = %f.",mu,sigma,R);
> printf(" (in percent per annum:");
> printf(" mu = %f, sigma = %f, interest rate = %f) ",
> 100*(evalf(power(mu,52))-1),
> 100*sqrt(evalf(power(1+sigma^2,52))-1),
> 100*(evalf(power(R,52))-1));
> u:=(1+mu*mu+sigma*sigma+
> sqrt((1+mu*mu+sigma*sigma)*(1+mu*mu+sigma*sigma)-4*mu*mu))
> /(2*mu):
> u_inv:=1/u:q:=(mu-1/u)/(u-1/u):pi_u:=q/R:pi_d:=(1-q)/R:
> pi_hat:=(pi_u*u+pi_d/u)/(1-pi_u*u-pi_d/u):
> m:=(log(1-sqrt(1-4*pi_u*pi_d))-log(pi_u)-log(2.0))/log(u):
> n:=(log(1+sqrt(1-4*pi_u*pi_d))-log(pi_u)-log(2.0))/log(u):
> printf(" From (21) we get the up factor:  u =%f,",u);
> printf(" and down factor:  1/u =%f.  ",1/u);
> printf(" From (22) the up probability is:  q =%f,",q);
> printf(" and down probability is:  1-q =%f.",1-q);
> printf(" Equilibrium state prices are obtained
> from (23)-(24):  pi_u = %f, pi_d = %f.  ",pi_u, pi_d);
> printf(" The pricing operator of (3) is pi_hat = %f.",
> pi_hat);
> printf(" Replacing these values in (A8) and (A9) gives:");
> printf(" m = %f < 0,", m);
> printf(" n = %f > 1.",n);
```

```
> printf(" Now HALVE the weekly interest rate:");
> mu:=1.0009387:sigma:=0.045:R:=1+.0018346/2:
> printf(" mu = %f, sigma = %f, R = %f.",mu,sigma,R);
> printf(" (in percent per annum:");
> printf(" mu = %f, sigma = %f, interest rate = %f) ",
> 100*(evalf(power(mu,52))-1),
> 100*sqrt(evalf(power(1+sigma^2,52))-1),
> 100*(evalf(power(R,52))-1));
> u:=(1+mu*mu+sigma*sigma+
> sqrt((1+mu*mu+sigma*sigma)*(1+mu*mu+sigma*sigma)-4*mu*mu))
> /(2*mu):
> u_inv:=1/u:q:=(mu-1/u)/(u-1/u):pi_u:=q/R:pi_d:=(1-q)/R:
> pi_hat:=(pi_u*u+pi_d/u)/(1-pi_u*u-pi_d/u):
> m:=(log(1-sqrt(1-4*pi_u*pi_d))-log(pi_u)-log(2.0))/log(u):
> n:=(log(1+sqrt(1-4*pi_u*pi_d))-log(pi_u)-log(2.0))/log(u):
> printf(" From (21) we get the up factor:  u =%f,",u);
> printf(" and down factor:  1/u =%f.",1/u);
> printf(" From (22) the up probability is:  q =%f,",q);
> printf(" and down probability is:  1-q =%f.",1-q);
> printf(" Equilibrium state prices are obtained
> from (23)-(24):  pi_u = %f, pi_d = %f.  ",pi_u, pi_d);
> printf(" The pricing operator of (3) is pi_hat = %f.
> (<0!)", pi_hat);
> printf(" Replacing these values in (A8) and (A9) gives:");
> printf(" m = %f < 0,", m);
> printf(" n = %f.  (SHOULD BE GREATER THAN 1!)",n);
```

```
Input values of Dammon and Spatt, 1996 (Section 4.1):

    mu = 1.000939, sigma = .045000, R = 1.001835.

 (in percent per annum:

    mu = 4.999931, sigma = 33.305493, interest rate = 10.000178)

 From (21) we get the up factor:   u  =1.046012,

                 and down factor:  1/u =.956012.

 From (22) the up probability is:   q  =.499186,

         and down probability is:  1-q =.500814.

 Equilibrium state prices are obtained from (23)-(24):

    pi_u = .498272, pi_d = .499897.

 The pricing operator of (3) is pi_hat = 1117.243642.

   Replacing these values in (A8) and (A9) gives:

                 m = -1.310616 < 0,

                 n = 1.383026 > 1.

  Now HALVE the weekly interest rate:

    mu = 1.000939, sigma = .045000, R = 1.000917.

 (in percent per annum:

    mu = 4.999931, sigma = 33.305493, interest rate = 4.883260)

 From (21) we get the up factor:   u  =1.046012,
```

and down factor: 1/u =.956012.

From (22) the up probability is: q =.499186,

and down probability is: 1-q =.500814.

Equilibrium state prices are obtained from (23)-(24):

pi_u = .498728, pi_d = .500355.

The pricing operator of (3) is pi_hat = -46773.028440. (<0!)

Replacing these values in (A8) and (A9) gives:

m = -.916558 < 0,

n = .988968. (SHOULD BE GREATER THAN 1!)