On Anabelian Geometry of Mixed Characteristic Local Fields

by

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Abstract

The aim of this thesis is to provide an exposition to Mochizuki and Hoshi's approach to birational anabelian geometry of mixed characteristic local fields. In the introductory chapter, we begin by recalling the relevant backgrounds on the Grothendieck conjectures on the étale fundamental groups and their morphisms. Next, we review mixed characteristic local fields and their local class field theory. This leads us to derive that many invariants of a mixed characteristic local field can be reconstructed from its Galois group, but not the field itself. We then set out to demonstrate that isomorphisms between mixed characteristic local fields, which preserve certain arithmetic structures such as the ramification filtration or the class of Hodge-Tate representation, are induced by isomorphisms between the fields. We provide technical facts on Galois representations in support of this argument.

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Chapter 1

Introduction

Anabelian geometry is vaguely the study of the extend to which an algebro-geometric object can be determined by its homotopic data, and in particular, its étale fundamental group. The anabelian conjectures arose from Grothendieck's letter to Faltings.[1] We begin by discussing some of the backgrounds and motivations, and outlining major results.

1.1 Basics on Étale Fundamental Groups

Let X be a connected scheme. Denote $\mathrm{Fét}_X$ the category of finite étale covers of X, with objects finite étale morphisms of schemes $Y \to X$ and morphisms over X. Choose a geometric point $\overline{x} \to X$, it determines a fiber functor

$$\mathrm{Fib}_{\overline{x}} : \mathrm{Fét}_X \to \mathbf{Set}$$

defined by mapping a cover $Y \to X$ to the set $\mathrm{Fib}_{\overline{x}}(Y)$ of fibers over $\overline{x}$. We define the étale fundamental group $\pi_1(X, \overline{x})$ to be the automorphism group of the functor $\mathrm{Fib}_{\overline{x}}$.[2] Then the étale fundamental group is naturally a profinite group seen from the embedding

$$\pi_1(X, \overline{x}) \subset \prod_{Y \to X} \mathrm{Aut}\left(\mathrm{Fib}_{\overline{x}}(Y)\right).$$

[1]See an English translation in [SL97].

[2]We shall use π_1 to denote the étale fundamental group throughout, not to be confused with the topological fundamental group.

Moreover, the functor $\mathrm{Fib}_{\overline{x}}$ factorizes through the category $\pi_1(X, \overline{x}) - \mathbf{FSet}$ of finite sets equipped with an $\pi_1(X, \overline{x})$ action, which in fact yields an equivalence of categories.

Functoriality

Consider the category of schemes **Scheme** and the category of schemes with chosen geometric point $\mathbf{Scheme}_*$. For an étale cover $Y \to X$ and a morphism of scheme $X' \to X$ that preserves geometric base points $\overline{x'}$ and $\overline{x}$, one has

$$\mathrm{Fib}_{\overline{x'}}(Y \times_{X'} X) \longrightarrow \mathrm{Fib}_{\overline{x}}(X)$$

is a bijection. Then an automorphism of the functor $\mathrm{Fib}_{\overline{x'}}$ induces an automorphism of the functor $\mathrm{Fib}_{\overline{x}}$, and thus yielding a map

$$\pi_1(X', \overline{x'}) \longrightarrow \pi_1(X, \overline{x}).$$

Thus π_1 defines a covariant functor from $\mathbf{Scheme}_*$ to the category **Prof** of profinite groups.

The Role of Base points

Now consider the category $\mathbf{Scheme}_*^\circ$ of connected and pointed schemes. Let $\overline{x}$ and $\overline{y}$ be two different choices of geometric point on connected X, it is shown that they yield isomorphic fiber functors $\mathrm{Fib}_{\overline{x}}$ and $\mathrm{Fib}_{\overline{y}}$, and thus the fundamental groups $\pi_1(X, \overline{x})$ and $\pi_1(X, \overline{y})$ are isomorphic. Even though such isomorphisms are not canonical, they differ exactly by inner automorphisms of $\pi_1(X, \overline{y})$, essentially due to the fiber functor being pro-represented. One can thus coarsen the morphisms in the category by modding out such inner isomorphisms. More percisely, consider the category $\mathbf{Prof}^{\mathrm{out}}$, with objects profinite groups, and morphisms between profinite groups G and H given by

$$\mathrm{Hom}^{\mathrm{out}}(G, H) = \mathrm{Hom}(G, H) / \mathrm{Inn}(H).$$

We check that composition works corectly in this category. Given maps of groups $A \xrightarrow{f} B \xrightarrow{g} C$ and $a \in A, b \in B, c \in C$. One has

$$g^c \circ f^b(a) = c \cdot g(b \cdot f(a) \cdot b^{-1}) \cdot c^{-1} = (g \circ f)^{c \cdot g(b)}(a).$$

Remark 1. In fact, $\mathrm{Inn}(G)$ also acts on $\mathrm{Hom}(G, H)$ by conjugating the input. But conjugating $f \in \mathrm{Hom}(G, H)$ on the input by $g \in G$ coincide with conjugating on the output by $f(g) \in H$. This is in analogue to the topological situation where a loop is mapped to a loop.

Then in particular, $\pi_1(X, \overline{x})$ and $\pi_1(X, \overline{y})$ are canonically isomorphic in this category. Fixing a geometric point $\overline{x}$ for each $X \in \mathbf{Scheme}^\circ$, and define a functor $\pi_1 : \mathbf{Scheme}^\circ \to \mathbf{Prof}^{\mathrm{out}}$ by

$$\pi_1(X) = \pi_1(X, \overline{x}).$$

Then different choices of base points yield canonically isomorphic functors, which we denote by abusing notations as π_1.

The Case of Fields

In the situation where $X = \mathrm{Spec}(k)$ for some field k, choosing an seperable closure $k \hookrightarrow \overline{k}$ amounts to choosing a geometric point $\overline{x} : \mathrm{Spec}(\overline{k}) \to X$. Finite étale covers of X are exactly the spectra of étale algebras over k. Such algebras are exactly finite products of finite seperable extensions of k, and the fiber functor $\mathrm{Fib}_{\overline{x}}$ is seen to be represented by $\mathrm{Spec}(\overline{k})$. It follows that canonically

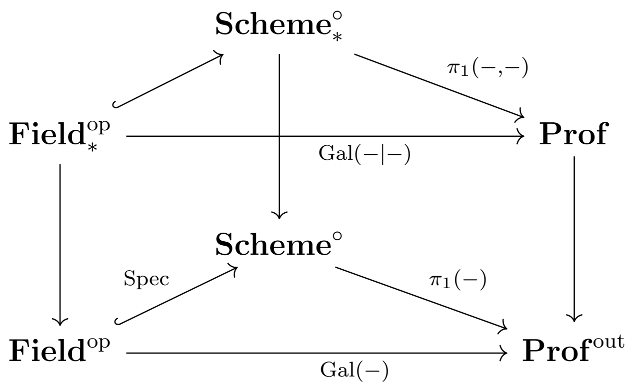
$$\pi_1(X, \overline{x}) = \mathrm{Gal}\left(\overline{k}|k\right)$$

From the above discussion, one has

$$\mathrm{Hom}^{\mathrm{out}}(\mathrm{Gal}\left(\overline{k_2}|k_2\right), \mathrm{Gal}\left(\overline{k_1}|k_1\right)) = \mathrm{Hom}(\mathrm{Gal}\left(\overline{k_2}|k_2\right), \mathrm{Gal}\left(\overline{k_1}|k_1\right))/\,\mathrm{Gal}\left(\overline{k_1}|k_1\right)$$

and the absolute Galois group of a field can be defined as an object in the category $\mathbf{Prof}^{\mathrm{out}}$ without an explicit choice of seperable closure.

We then have a commutative (up to natural isomorphism) diagram of categories:



Homotopy Exact Sequence and the Outer Galois Representation

Now fixing some field k together with an seperable closure $k \hookrightarrow \overline{k}$. Consider the category $\mathbf{Scheme}_k$ of schemes over $\mathrm{Spec}(k)$, together with the pointed category $\mathbf{Scheme}_{k,*}$.

There is a natural sequence of schemes

$$\overline{X} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k}) \longrightarrow X \longrightarrow \mathrm{Spec}(k).$$

It is then a fundemental theorem that the following induced sequence on fundemental groups is exact:

$$1 \to \pi_1(\overline{X}, \overline{x}) \to \pi_1(X, \overline{x}) \to \mathrm{Gal}\left(\overline{k}|k\right) \to 1.$$

In particular, $\pi_1(X, \overline{x})$ acts on its normal subgroup $\pi_1(\overline{X}, \overline{x})$ by conjugation. We then have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{X}, \overline{x}) & \longrightarrow & \pi_1(X, \overline{x}) & \longrightarrow & \mathrm{Gal}(\overline{k}|k) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho} & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & \mathrm{Aut}(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & \mathrm{Out}(\pi_1(\overline{X}, \overline{x})) & \longrightarrow & 1
\end{array}$$

When $\pi_1(\overline{X}, \overline{x})$ is center-free, namely when the left vertical arrow is isomorphic, the group $\pi_1(X, \overline{x})$ is the pullback

$$\mathrm{Aut}(\pi_1(\overline{X}, \overline{x})) \times_{\mathrm{Out}(\pi_1(\overline{X}, \overline{x}))} \mathrm{Gal}(\overline{k}|k)$$

determined by the geometric Galois group $\pi_1(\overline{X}, \overline{x})$ and the outer Galois representation ρ.

Geometric Fundamental Groups in characteristic 0

For a scheme X of finite type over the complex numbers $\mathbb{C}$, the étale fundamental group of X is known to be the profinite completion of the topological fundamental group of the complex analytic space of X. In particular, the topological fundamental group of smooth curves over $\mathbb{C}$ is well understood.

Theorem 2 ([Sza09])**.** *Let X be an integral proper normal curve over $\mathbb{C}$, and let $U \subset X$ be an open subset. The étale fundamental group $\pi_1(U)$ is isomorphic to the profinite completion of the group with given generators and relations*

$$\langle a_1, b_1, \ldots, a_g, b_g, \gamma_1, \ldots, \gamma_n \mid [a_1, b_1] \ldots [a_g, b_g]\, \gamma_1 \cdots \gamma_n = 1\rangle$$

where n is the number of points of X lying outside U, and g is the genus of X.

The complex analytic spaces of such U with $2g-2+n > 0$ are exactly the hyperbolic Riemann surfaces.

It is also a fact that base changing from one algebraically closed field of characteristic 0 to an algebraically closed extension preserves the étale fundemental group, the above discription applies for all algebraically closed fields of characteristic 0.

In general, hyperbolic k-curves are the smooth k-curves of genus g with at least $2 - 2g + 1$ geometric points at infinity. These are precisely the smooth curves with non-trivial center-free geometric étale fundemental groups. In this sense their étale fundemental groups are thought of as “far from abelian”, and Grothendieck considered hyperbolic curves as achitypical “anabelian” objects.

1.2 Anabelian Grothendieck Conjectures

Grothendieck proposed a number of conjectures for conjectural families of “anabelian” variety, namely varieties where the outer Galois representation sufficiently controls the geometry of the vairiety.

1.2.1 Bi-anabelian Conjectures and Results

A bi-anabelian conjecture studies the fully faithfulness of the functor π_1. In particular there are common variation to the forms of the conjecture.

Isom and Hom Form Conjectures

Conjecture 3 (Isom Form Conjecture). *For "anabelian" X_1 and X_2 over k, the canonical map*

$$\mathrm{Isom}_k\left(X_1, X_2\right) \to \mathrm{Isom}^{\mathrm{out}}_{G_k}\left(\pi_1(X_1), \pi_1(X_2)\right)$$

is a bijection.

Conjecture 4 (Hom Form Conjecture). *For "anabelian" X_1 and X_2 over k, the canonical map*

$$\mathrm{Hom}^{\mathrm{dom}}_k\left(X_1, X_2\right) \to \mathrm{Hom}^{\mathrm{out,open}}_{G_k}\left(\pi_1(X_1), \pi_1(X_2)\right)$$

is a bijection.

Evidently Hom form conjectures are stronger than Isom form.

Absolute and Relative Forms

It certain situations, it is posible to consider morphisms between fundamental groups themselves instead of over the absolute Galois group of the base field.

Example of bi-anabelian Results

An early result on hyperbolic curves by Tamagawa says

Theorem 5 (Tamagawa, absolute Isom form [Tam97]). *Let k be a finite field, and let X_1, X_2 be affine hyperbolic curves over k. Then*

$$\mathrm{Isom}\left(X_1, X_2\right) \to \mathrm{Isom}\left(\pi_1(X_1), \pi_1(X_2)\right)$$

is a bijection.

Mochizuki refined the result and eventually proved using p-adic Hodge theory that

Theorem 6 (Mochizuki, relative Hom form [Moc99]). *Let k be a field that may be embedded in a finitely generated extension of $\mathbb{Q}_p$, and let X_1, X_2 be hyperbolic curves over k. Then*

$$\mathrm{Hom}_k^{\mathrm{dom}}(X_1, X_2) \to \mathrm{Hom}_{G_k}^{\mathrm{out,open}}(\pi_1(X_1), \pi_1(X_2))$$

is a bijection.

Remark 7. In fact, Mochizuki showed the theorem holds more generally for X_1 arbitrary smooth vairety over k, and π_1 replaced by its maximal pro-p quotient.

Mochizuki also showed an absolute version

Theorem 8 (Mochizuki, absolute Hom form [Moc99]). *Let X_1, X_2 be hyperbolic curves over (possiblely different) finitely generated extensions of $\mathbb{Q}$. Then*

$$\mathrm{Isom}(X_1, X_2) \to \mathrm{Hom}^{\mathrm{out}}(\pi_1(X_1), \pi_1(X_2))$$

is a bijection.

One can also take the birational perspective and consider the Grothendieck conjectures on function fields. It this direction Mochizuki also showed the following birational analogy to the above theorem

Theorem 9 (Mochizuki, birational relative Hom form [Moc99]). *Let k be a field that may be embedded in a finitely generated extension of $\mathbb{Q}_p$, and let K, L be function fields of arbitrary dimensions (≥ 1) over k. Then*

$$\mathrm{Hom}_k(L, K) \to \mathrm{Hom}_{G_k}^{\mathrm{out,open}}(\mathrm{Gal}(K), \mathrm{Gal}(L))$$

is a bijection.

Remark 10. This result also holds more generally when the Galois groups are replaced by their maximal pro-p quotients.

Mochizuki's birational result can be seen as a relative characteristic 0 generalization of the following result:

Theorem 11 (Neukirch, Uchida, Iwasawa, Pop [Neu69] [Uch76] [Pop90] [Pop94])**.** *Let K, L be finitely generated infinite fields over their prime fields. Then*

$$\mathrm{Isom}\,(L, K) \to \mathrm{Isom}^{\mathrm{out}}\,(\mathrm{Gal}\,(K)\,, \mathrm{Gal}\,(L))$$

is a bijection.

Remark 12. Using Mochizuki's result Corry and Pop showed in [CP09] that for function fields over k the relative pro-p hom form result holds.

In the case of mixed characteristic local fields however, it is known that the birational Isom form statement does not hold. Namely, there exist outer isomorphisms that do not arise from field isomorphisms. Nevertheless, using finer invariants of the fields the following results hold and are of particular interests to this thesis:

Theorem 13 (Mochizuki [Moc97])**.** *Let K, L be mixed characteristic local fields. Then*

$$\mathrm{Isom}\,(L, K) \to \mathrm{Isom}^{\mathrm{Fil,out}}\,(\mathrm{Gal}\,(K)\,, \mathrm{Gal}\,(L))$$

is an isomorphism, where $\mathrm{Isom}^{\mathrm{Fil}}$ *denote the set of isomorphisms that preserves the upper ramification filtration, namely mapping* $\mathrm{Gal}\,(K)^{(v)}$ *to* $\mathrm{Gal}\,(L)^{(v)}$*.*

Hoshi adapted Mochizuki's argument and proved an Hom form statement:

Theorem 14 (Hoshi [Hos13])**.** *Let K, L be mixed characteristic local fields. Then*

$$\mathrm{Hom}\,(L, K) \to \mathrm{Hom}^{\mathrm{HT,out,open}}\,(\mathrm{Gal}\,(K)\,, \mathrm{Gal}\,(L))$$

is an isomorphism, where $\mathrm{Hom}^{\mathrm{HT}}$ *denote the set of isomorphisms that pullback p-adic Hodge-Tate representations to Hodge-Tate ones.*

Remark 15. Abrashkin used different argument and generalized Mochizuki's Isom form result (theorem 13) to local fields (not necessarily of mixed characteristic) and their pro-p Galois groups with upper ramification filtrations in [Abr10].

1.2.2 Mono-anabelian Results

Given the étale fundamental group π_1 of a scheme X, one hope that using only data of of the fundamental group one can reconstruct the data of the scheme X in a fuctorial way. Invariants obtainable this way are called mono-anabelian.

In the ideal situation where the scheme X can be reconstructed, this would imply the bi-anabelian conjectures. In fact, Hoshi showed that such a stronger statement holds for number fields, strengthening the result of theorem 11 on number fields:

Theorem 16 (Hoshi [Hos19]). *Given the absolute Galois group of a number field there is a functorial way to reconstruct the algebraic closure of the given number field equipped with its natural Galois action that gave rise to the given absolute Galois group.*

We shall see many examples of mono-anabelian invariants of mixed characteristic local fields that plays important role in the proof of the main theorem. However, since the Isom form bi-anabelian statement does not hold for mixed characteristic local fields, the absolute Galois group is not mono-anabelian.

1.3 The Main Goal of the Thesis

The main goal of this thesis is to explain the key ingredients that go into the proofs Mochizuki and Hoshi's results, and show for the following statement:

Theorem 17 (Main Theorem). *Let K, L be mixed characteristic local fields and $\alpha : \mathrm{Gal}\,(L) \to \mathrm{Gal}\,(K)$ be an isomorphism between the Galois groups. The following are equivalent:*

1. *There exists an isomorphism of fields $\beta : K \to L$ such that $\alpha = \pi_1(\beta)$;*
2. *The isomorphism α preserves the ramification filtration;*
3. *The isomorphism α is HT-preserving.*

Namely,

$$\mathrm{Isom}\left(L,K\right)=\mathrm{Isom}^{\mathrm{Fil,out}}\left(\mathrm{Gal}\left(K\right),\mathrm{Gal}\left(L\right)\right)=\mathrm{Isom}^{\mathrm{HT,out}}\left(\mathrm{Gal}\left(K\right),\mathrm{Gal}\left(L\right)\right)$$

in the above notations.

In particular, the implications (1) $\implies$ (2) $\implies$ (3) are shown in theorem 58 of chapter 3 after we study the mono-anabelian invariants of mixed characteristic local fields, and (3) $\implies$ (1) in theorem 138 of chapter 5 after we review the technical results from p-adic representations and p-adic Hodge theory.

Chapter 2

Mixed Characteristic Local Fields and Their Class Field Theory

2.1 Basic Invariants of Mixed Characteristic Local Fields

Definition 18 (Mixed Characteristic Local Fields)**.** A mixed characteristic local field is a finite extension of $\mathbb{Q}_p$ for some prime p.

In particular, associated to K are the following objects:

- $\mathcal{O}_K \subseteq K$ denotes the ring of integers of K,
- $\mathfrak{m}_K \subseteq \mathcal{O}_K$ denotes the maximal ideal of $\mathcal{O}_K$,
- $k_K = \mathcal{O}_K/\mathfrak{m}_K$ denotes the residue field of $\mathcal{O}_K$,
- For $n \geq 1$, $U_K^{(n)} = 1 + \mathfrak{m}_k^n \subseteq \mathcal{O}_k^\times$ for the n-th unit group of K,
- $p_K = \mathrm{char}(k_K)$ for the residue characteristic of K,
- $d_K = [k_K : \mathbb{Q}_{p_K}]$ for the degree of K over $\mathbb{Q}_{p_K}$,
- $f_K = [k_K : \mathbb{F}_{p_K}]$ for the degree of the residue extension,
- $q_K = p_K^{f_K}$, the size of k_K,
- e_K for the ramification index of K over $\mathbb{Q}_{p_K}$.

Lemma 19. *There is a topological isomorphism*

$$K^\times \cong \left(\mathbb{Z}/\left(p_K^{f_K}-1\right)\mathbb{Z}\right) \oplus (\mathbb{Z}/p_K^a\mathbb{Z}) \oplus (\mathbb{Z}_{p_K})^{\oplus d_K} \oplus \mathbb{Z}$$

which satisfies the following properties:

1. *The projection onto the last component agrees with the normalized discrete valuation $v_K : K^\times \to \mathbb{Z}$ on K,*

2. *$\mathcal{O}_K^\times$ corresponds to the sub-group $\left(\mathbb{Z}/\left(p_K^{f_K}-1\right)\mathbb{Z}\right) \oplus (\mathbb{Z}/p_K^a\mathbb{Z}) \oplus (\mathbb{Z}_{p_K})^{\oplus d_K}$ under the isomorphism,*

3. *The principle unit group $U_K^{(1)}$ corresponds to $(\mathbb{Z}/p_K^a\mathbb{Z}) \oplus (\mathbb{Z}_{p_K})^{\oplus d_K}$ under the isomorphism, and in particular is the pro-p_K Sylow sub-group of $K^\times$,*

4. *The roots of unity $\mu_K \subset K^\times$ are the torsion elements corresponding to the sum $\left(\mathbb{Z}/\left(p_K^{f_K}-1\right)\mathbb{Z}\right) \oplus (\mathbb{Z}/p_K^a\mathbb{Z})$.*

Let K be a mixed characteristic local field, we mention definitions and properties of its logarithmic and exponential function for later applications.

Proposition 20 ([Neu99, Proposition 2.5.4, 2.5.5])**.** *The power series*

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

is convergent on $U_K^{(1)}$.

The power series

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is convergent on $\mathfrak{m}_K^n$ for $n > \frac{e}{p-1}$, with image in $U_K^{(n)}$.

Definition 21. Let $n > \frac{e}{p-1}$. Define the logarithmic function

$$\log : U_K^{(1)} \to K$$

and the exponential function

$$\exp : \mathfrak{m}_K^n \to U_K^{(n)}$$

by the above power series.

Proposition 22 ([Neu99, Proposition 2.5.5]). *The maps* log *and* exp *are both continuous group homomorphism. Further for $n > \frac{e}{p-1}$ they yield isomorphisms*

$$U_K^{(n)} \underset{\exp}{\overset{\log}{\rightleftarrows}} \mathfrak{m}_K^n$$

that are inverse to each other.

Finally, one can show that the kernel of the logarithmic function is finite, and thus exactly the roots of unity in $U_K^{(1)}$.

Theorem 23 (Strassman's Theorem [Mur02, Theorem 3.3.4]). *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be a non-zero power series with $a_n \in K$. Suppose $a_n \to 0$ as $n \to \infty$ so that $f(x)$ converges for $x \in \mathcal{O}_K$. Then, the function $f : \mathcal{O}_K \to K$ given by $x \mapsto f(x)$ has finite many zeros.

Let $\pi \in K^\times$ be an uniformizer and apply the theorem to

$$-\log(1 - \pi x) = \sum_{n=1}^{\infty} \frac{\pi^n}{n} x^n,$$

one has that indeed Ker(log) is finite.

It is natural to extend log to $\mathcal{O}_K$, taking value 0 for the prime to p_K roots of unity.

Lemma 24. *Consider the natural bilinear map*

$$\mathcal{O}_K^\times \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_+$$

where $\mathbb{Q}$ is endowed with the discrete topology. The map is a topological group isomorphism, where K_+ is the underlying topological additive group.

Proof. The map is injective, as it is shown that ker(log) are all torsions.

The map is surjective as the following: Let π be some uniformizer of K. Note that $(p_k) = (\pi)^{e_K}$. We know $\mathfrak{m}_K^n = (\pi)^n$ is contained in the image, where n can be assume to be prime to e_K. Then by dividing powers of p_K in $\mathfrak{m}_K^n$, we have the $\mathcal{O}_K$ module $\mathbb{Q} \cdot \mathfrak{m}_K^n$ containing elements of valuations j for all $j \in \mathbb{Z}$.

It then follows

$$\mathbb{Q} \cdot \mathfrak{m}_K^n = (\mathcal{O}_K)_\pi = K$$

and the map is indeed surjective.

The map is a homeomorphism essentially follows from log restricts to a homeomorphism. □

Finally we comment on the fuctoriality of the isomorphism. Note that the definition of log and exp does not depend on K. Thus for an extension L of K the following natural diagram is commutative.

$$\begin{array}{ccc} \mathcal{O}_K^\times \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathcal{O}_L^\times \otimes_{\mathbb{Z}} \mathbb{Q} \\ \sim\downarrow & & \downarrow\sim \\ K_+ & \hookrightarrow & L_+ \end{array}$$

2.2 Galois Groups of Mixed Characteristic Local Fields

Letting $\overline{K}$ be some fixed algebraic closure of K, we further have the following Galois theoretic objects:

- $G_K = \mathrm{Gal}(\overline{K}|K)$ for the absolute Galois group of tower $\overline{K}|K$,
- $I_K \subseteq G_K$ for the inertia subgroup, corresponding to maximal unramified extension K^{ur} under the Galois correspondence,
- $\mathrm{Frob}_K \in G_K/I_K$ for the Frobenius element, which under the canonical isomorphism $G_K/I_K \simeq G_k$ acts as the Frobenius map on the residue field,
- $P_K \subseteq I_K$ for the wild inertia subgroup of G_K, corresponding to maximal tamely ramified extension K^{tr} under the Galois correspondence,

- The roots of unity $\mu_{\overline{K}} \subset \overline{K}^{\times}$, considered as a multiplicative group with a G_K-action.

We have the following facts:

1. $G_K/I_K \simeq \mathrm{Gal}(K^{\mathrm{ur}}|K) \simeq \mathrm{Gal}(\overline{k_K}|k_K) \simeq \widehat{\mathrm{Frob}_K^{\mathbb{Z}}}$. In particular $K^{\mathrm{ur}}|K$ is abelian, and K^{ur} lies in the maximal abelian extension K^{ab}.

2. Since K^{ur} contains all the prime to p_K-th roots of unity lifted from the residue extension, and K^{tr} is the composite of all (necessarily totally ramified) finite extension of K^{ur} with degree prime to p_K, these are all Kummer extensions. Thus they are extensions adjoining e-th roots of some uniformizer for all e prime to p_K. Then there is an isomorphism

$$I_K/P_K \xrightarrow{\sim} \mathrm{Hom}((K^{\mathrm{tr}})^{\times}/(K^{\mathrm{ur}})^{\times}, \overline{k_K}^{\times})$$

given by

$$g \longmapsto (a \mapsto \frac{g(a)}{a}).$$

Identifying $\Delta = \bigcup_{p\nmid n} \frac{1}{n}\mathbb{Z}$ as the value group of K^{tr} and $\mathbb{Z}$ as the value group of K^{ur}, we have

$$(K^{\mathrm{tr}})^{\times}/(K^{\mathrm{ur}})^{\times} = \Delta/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^{(p')}$$

where $(\mathbb{Q}/\mathbb{Z})^{(p'_K)}$ is the quotient by the pro-p_K Sylow sub-group of $\mathbb{Q}/\mathbb{Z}$.

Further notice the isomorphism is G_K/I_K-equivariant:

$$\sigma g \sigma^{-1} \longmapsto \left(a \mapsto \frac{\sigma g \sigma^{-1}(a)}{a} = \sigma(\frac{g(\sigma^{-1}(a))}{\sigma^{-1}(a)}) = \sigma(\frac{g(a)}{a})\right)$$

(where we abuse notation and take σ to be an arbitrary element in its coset.) In particular, the action of Frob_K on I_K/P_K is exactly multiplication by $p_K^{f_K}$ (written additively as an abelian group.)

Remark 25. In particular, K^{tr} is already not abelian over K. As above all finite extension in $K^{\mathrm{tr}}|K^{\mathrm{ur}}$ is of the form adjoining e-th roots of some uniformizer for e

prime to p_K. But one see from local class field theory that only those e that divides $p_K^{f_K} - 1$ yields abelian and tamely ramified extension over K.

2.3 Main Results of Local Class Field Theory

Theorem 26 (Local Reciprocity Map)**.** *Given a local field K, there exists a unique homomorphism*

$$\phi_K : K^\times \longrightarrow \mathrm{Gal}\left(K^{\mathrm{ab}}|K\right)$$

such that

1. *for every uniformizer π of K and every finite unramified extension L of $K, \phi_K(\pi)$ acts on L as $\mathrm{Frob}_{L/K}$,*
2. *for every finite abelian extension L of K, the image of the norm map $\mathrm{N}_{L/K}\left(L^\times\right)$ is contained in the kernel of $a \mapsto \phi_K(a)|_L$, and ϕ_K induces an isomorphism*

$$\phi_{L/K} : K^\times/\mathrm{N}_{L/K}\left(L^\times\right) \to \mathrm{Gal}(L/K).$$

Theorem 27 (Local Existence Theorem [Mil20, Theorem 3.5.1])**.** *The norm groups in $K^\times$ are exactly the open subgroups of finite index.*

There are multiple approaches to arrive at the local reciprocity map, see [Mil20, Section 1.1] and [FV93, Section 4.7]. Following [Mil20, Chapter 3], we outline the cohomological approach to construct the reciprocity maps, and compare it to the Hilbert symbol, which is used for a proof of the local existence theorem for mixed characteristic local fields.

2.3.1 Tate's Theorem

Theorem 28 (Tate's Theorem)**.** *Let G be a finite group, A a G-module. Assume that for each subgroup $H \subseteq G$, we have the following properties on Tate cohomology groups:*

- $\hat{H}^1(H, A) = 0$,
- $\hat{H}^2(H, A)$ *is cyclic of order* $|H|$.

Then there is an isomorphism

$$\hat{H}^q(G, \mathbb{Z}) \longrightarrow \hat{H}^{q+2}(G, A)$$

induced by taking cup-product with a generator of the group $\hat{H}^2(G, A)$.

In particular, for K a mixed characteristic local field, the pair $G = G_K$ and $A = K^\times$ satisfy the above assumptions. Then at $q = -2$ Tate's Theorem yields a canonical isomorphism between the abelianization $G^{\text{ab}} \cong \hat{H}^{-2}(G, \mathbb{Z})$ of G and the norm residue group $A^G/N_G A = \hat{H}^0(G, A)$:

$$\operatorname{Gal}(L|K)^{\text{ab}} = K^\times/\mathrm{N}_{L/K}\left(L^\times\right).$$

Remark 29. This canonical isomorphism from Tate's Theorem can be seen as a statement on the G module A. One can thus axiomatize these cohomological condition in the so called abstract class field theory of class formations. For a reference, see [Neu99, Chapter 4].

More precisely, cup product from group cohomology yields a bilinear map for finite extension L/F

$$H^0\left(\operatorname{Gal}(L/F), L^\times\right) \times H^2(\operatorname{Gal}(L/F), \mathbb{Z}) \to H^2\left(\operatorname{Gal}(L/F), L^\times\right).$$

Further consider the boundary homomorphism

$$H^1(\operatorname{Gal}(L/F), \mathbb{Q}/\mathbb{Z}) \to H^2(\operatorname{Gal}(L/F), \mathbb{Z})$$

from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Taking appropriate limit we have a bilinear pairing of profinite cohomology groups:

$$H^0\left(G_F, \overline{F}^\times\right) \times H^1(G_F, \mathbb{Q}/\mathbb{Z}) \to H^2\left(G_F, \overline{F}^\times\right)$$

where canonically

$$H^0\left(G_F, \overline{F}^\times\right) = F^\times,$$

$$H^1(G_F, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z})$$

since the action is trivial, and

$$H^2\left(G_F, \overline{F}^\times\right) = \mathbb{Q}/\mathbb{Z}$$

by the calculation of Brauer group for local fields in reference to [NSW08, Corollary 7.1.9].

In summary, we have a bilinear pairing

$$F^\times \times \mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}.$$

Then one shows that the left kernel is zero, and arrives at an isomorphism from $F^\times$ to $\mathrm{Hom}(\mathrm{Hom}(G_F, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = G_F^{\mathrm{ab}}$ by Pontryagin duality. This is exactly the local reciprocity map ϕ_L.

2.3.2 Functorial Properties of the Local Reciprocity Map

The reciprocity map satisfies the following functorial properties:

Proposition 30 ([FV93, Lemma 4.4.2])**.** *Let $L|K$ be a finite extension of local fields, we have commuting diagram*

$$\begin{array}{ccc} K^\times & \hookrightarrow & L^\times \\ {\scriptstyle\phi_K}\downarrow & & \downarrow{\scriptstyle\phi_L} \\ G_K^{\mathrm{ab}} & \xrightarrow[\mathrm{Ver}_{L/K}]{} & G_L^{\mathrm{ab}} \end{array}$$

where $\mathrm{Ver}_{L/K}$ *is the transfer map, and*

$$\begin{array}{ccc} K^\times & \xleftarrow{\mathrm{N}_{L/K}} & L^\times \\ {\scriptstyle\phi_K}\downarrow & & \downarrow{\scriptstyle\phi_L} \\ G_K^{\mathrm{ab}} & \longleftarrow & G_L^{\mathrm{ab}} \end{array}$$

where $\mathrm{N}_{L/K}$ *is the norm map.*

Further, Let $\overline{K}$ be some algebraic closure of K, $\overline{K'}$ some algebraic closure of K', and σ is an isomorphism of tower as the following

$$\begin{array}{ccc} \overline{K} & \xrightarrow[\sigma]{\sim} & \overline{K'} \\ \Big| & & \Big| \\ K & \xrightarrow{\sim} & K' \end{array}$$

we have the following commuting diagram

$$\begin{array}{ccc} K^\times & \xrightarrow{\quad\quad\sigma\quad\quad} & K'^\times \\ {\scriptstyle \phi_K}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_{K'}} \\ \mathrm{Gal}(\overline{K}|K)^{\mathrm{ab}} & \xrightarrow[g\mapsto\sigma g\sigma^{-1}]{} & \mathrm{Gal}(\overline{K'}|K')^{\mathrm{ab}} \end{array}$$

In particular, let $L|K$ be a finite extension of local fields and $\sigma \in \mathrm{Gal}\left(\overline{K}|K\right)$, we have

$$\begin{array}{ccc} \overline{K} & \xrightarrow[\sigma]{\sim} & \overline{K} \\ \Big| & & \Big| \\ L & \xrightarrow{\sim} & \sigma(L) \\ & \diagdown \quad \diagup & \\ & K & \end{array}$$

$$\begin{array}{ccc} L^\times & \xrightarrow{\quad\quad\sigma|_L\quad\quad} & \sigma(L)^\times \\ {\scriptstyle \phi_L}\Big\downarrow & & \Big\downarrow{\scriptstyle \phi_{\sigma(L)}} \\ \mathrm{Gal}(\overline{K}|L)^{\mathrm{ab}} & \xrightarrow[g\mapsto\sigma g\sigma^{-1}]{} & \mathrm{Gal}(\overline{K}|\sigma(L))^{\mathrm{ab}} \end{array}$$

2.3.3 Hilbert Symbol

Let F be a local field. Consider the Kummer exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \overline{F}^\times \xrightarrow{x\mapsto x^n} \overline{F}^\times \longrightarrow 1.$$

Form the long exact sequence in cohomology we have

$$\mathrm{H}^0\left(G_F, \overline{F}^\times\right) \longrightarrow \mathrm{H}^0\left(G_F, \overline{F}^\times\right) \longrightarrow \mathrm{H}^1\left(G_F, \mu_n\right) \longrightarrow \mathrm{H}^1\left(G_F, \overline{F}^\times\right)$$

where $\mathrm{H}^0\left(G_F, \overline{F}^\times\right) = F^\times$, and $\mathrm{H}^1\left(G_F, \overline{F}^\times\right) = 0$ by Hilbert's 90. Thus canonically

$$\mathrm{H}^1\left(G_F, \mu_n\right) = F^\times / F^{\times n}.$$

Also we have

$$\mathrm{H}^1\left(G_F,\overline{F}^\times\right)\longrightarrow \mathrm{H}^2\left(G_F,\mu_n\right)\longrightarrow \mathrm{H}^2\left(G_F,\overline{F}^\times\right)\longrightarrow \mathrm{H}^2\left(G_F,\overline{F}^\times\right)$$

where again $\mathrm{H}^1\left(G_F,\overline{F}^\times\right)=0$, $\mathrm{H}^2\left(G_F,\overline{F}^\times\right)=\mathbb{Q}/\mathbb{Z}$. Thus canonically

$$\mathrm{H}^2\left(G_F,\mu_n\right)=(\mathbb{Q}/\mathbb{Z})[n].$$

Now assume $\mu_n\subset F$, that is F contains primitive n-th roots of unity. Then $F^\times/F^{\times n}=\mathrm{H}^1\left(G_F,\mu_n\right)=\mathrm{Hom}\left(G_F,\mu_n\right)=\mathrm{Hom}\left(G_F^{\mathrm{ab}}/(G_F^{\mathrm{ab}})^n,\mu_n\right)$ which yields a bilinear map

$$F^\times/F^{\times n}\times G_F^{\mathrm{ab}}/(G_F^{\mathrm{ab}})^n\longrightarrow\mu_n.$$

By tracing through the cohomological calculation, we see this is exactly the Kummer pairing for the maximal abelian exponent n extension, namely maximal n-Kummer extension.

Finally, from cup product

$$H^0\left(G_F,\mu_n\right)\times H^2\left(G_F,\mu_n\right)\longrightarrow H^2\left(G_F,\mu_n\otimes\mu_n\right)$$

there is a canonical isomorphism

$$\mu_n=H^0\left(G_F,\mu_n\right)\simeq H^2\left(G_F,\mu_n\otimes\mu_n\right).$$

Consider the cup product

$$H^1\left(G_F,\mu_n\right)\times H^1\left(G_F,\mu_n\right)\longrightarrow H^2\left(G_F,\mu_n\otimes\mu_n\right)$$

which yields a bilinear map under above canonical isomorphisms:

$$F^\times/F^{\times n}\times F^\times/F^{\times n}\longrightarrow\mu_n.$$

This is the so-called Hilbert symbol, which among its other properties, also describes the reciprocity map for the maximal n-Kummer extension. More precisely, by tracing through the cohomological calculations we have

$$\phi(a)\left(\sqrt[n]{b}\right)=(a,b)\sqrt[n]{b}.$$

The Hilbert symbol and its properties can be found in [Mil20, Section 3.4]. In particular, it plays a key role in the proof of the local existence theorem.

2.4 Explicit Local Class Field Theory via Lubin-Tate Formal Group Laws

For each uniformizer π of K, Lubin-Tate theory explicitly constructs a maximal totally ramified abelian extension $K_\pi \subset K^{\mathrm{ab}}$ that is the fixed subfield of $\phi_K(\pi)$, and explicitly describes the Galois group $\mathrm{Gal}\,(K_\pi|K)$.

First we define formal group laws as a formal analogue of one-dimensional algebraic groups.

Definition 31 (Formal Group Law)**.** A one-parameter commutative formal group law over $\mathcal{O}_K$ is a power series $F \in \mathcal{O}_K[[X,Y]]$ such that

1. $F(X,Y) = X + Y +$ terms of degree ≥ 2,
2. $F(X, F(Y,Z)) = F(F(X,Y), Z)$,
3. there exists a unique $i_F(X) \in XA[[X]]$ such that $F\left(X, i_F(X)\right) = 0$,
4. $F(X,Y) = F(Y,X)$.

Note that one can substitute a formal power series without constant term into another formal power series. Thus one can define

Definition 32 (Morphisms for Formal Group Laws)**.** Let $F(X,Y)$ and $G(X,Y)$ be formal group laws. A homomorphism $F \to G$ is a power series $h \in T \cdot \mathcal{O}_K[[T]]$ such that

$$h(F(X,Y)) = G(h(X), h(Y)).$$

When there exists a homomorphism $h' : G \to F$ such that

$$h \circ h' = T = h' \circ h,$$

then h' is said to be an inverse to h, and h is called an isomorphism. A homomorphism $h : F \to F$ is called an endomorphism of F.

Let F be a formal group law. For any $f, g \in T \cdot \mathcal{O}_K[[T]]$, one define

$$f +_F g = F(f(T), g(T))$$

Lemma 33 (Ring Structure on End(F) [Mil20, Lemma 2.8])**.** *Given a formal group law F, the set* End(F) *of endomorphisms of F can be endowed a (not necessarily commutative) ring structure with $+_F$ as addition and composition $\circ$ as multiplication.*

Next we choose a uniformizer $\pi \in K^\times$, from which we define a special class of Lubin-Tate power series $\mathcal{F}_\pi$. Any such Lubin-Tate power series can be realized as an endomorphism of uniquely associated formal group laws. Such a group law would then admit an action from $\mathcal{O}_K^\times$, with π acting exactly as the corresponding Lubin-Tate power series. Further, different choices of Lubin-Tate power series in $\mathcal{F}_\pi$ result in isomorphic formal group laws.

Definition 34 (Lubin-Tate Power Series)**.** Let $\mathcal{F}_\pi$ be the set of $f(X) \in \mathcal{O}_K[[X]]$ such that

- $f(X) = \pi X +$ terms of degree ≥ 2,
- $f(X) \equiv X^q \bmod \pi$.

We call $\mathcal{F}_\pi$ the Lubin-Tate power series for uniformizer π.

Theorem 35 (Lubin-Tate Formal Group Laws [Mil20, Proposition 1.2.12])**.** *For every $f \in \mathcal{F}_\pi$, there is a unique formal group law F_f with coefficients in $\mathcal{O}_K$ admitting f as an endomorphism.*

Definition 36. We call F_f the Lubin-Tate formal group law associated to the Lubin-Tate power series f.

Definition 37. We say a ring $\mathcal{O}_K$ acts faithfully on a formal group law F when there is an injective ring homomorphism

$$\rho_F : \mathcal{O}_K \hookrightarrow \text{End}(F).$$

An isomorphism h between two formal group laws F and G with $\mathcal{O}_K$ action is said to be $\mathcal{O}_K$ equivariant if

$$\rho_G(a) \circ h = h \circ \rho_F(a)$$

for all $a \in \mathcal{O}_K$.

Theorem 38 ([Mil20, Corollary 1.2.17]). *For each $a \in \mathcal{O}_K$ there exists a unique $[a]_f \in \mathrm{End}(F_f)$ such that*

1. *$[a]_f = aT+$ terms of degree ≥ 2,*
2. *$[a]_f$ commutes with f under composition.*

Moreover, the map

$$\mathcal{O}_K \hookrightarrow \mathrm{End}(F_f)$$
$$a \mapsto [a]_f$$

is a ring homomorphism and thus endows F_f a faithful $\mathcal{O}_K$ action.

Theorem 39 ([Mil20, Corollary 1.2.16, Remark 1.2.19(c)]). *For $f, g \in \mathcal{F}_\pi$, there exists isomorphisms between formal group laws F_f and F_g that is $\mathcal{O}_K$ equivariant.*

Consider some algebraic closure $\overline{K}$ of K. The discrete valuation on K extends to a non-discrete valuation on $\overline{K}$. From the formal group law one can define functorially a new additive group structure on the maximal ideal $\mathfrak{m}$ of $\overline{\mathcal{O}}_K$. Then the $\mathcal{O}_K$ action on the group law yields an $\mathcal{O}_K$ action on the newly defined abelian group.

Definition 40 (From Formal Group F_f to Group $(\mathfrak{m}, +_{F_f})$). Let $F_f = \sum_{i,j}^{\infty} a_{ij} X^i Y^j$ be the Lubin-Tate formal group law associated to f over $\mathcal{O}_K$. For every $x, y \in \mathfrak{m}$ so that they have positive valuations, $a_{ij}x^i y^j \to 0$ as $(i,j) \to \infty$, and so the series

$$F_f(x, y) = \sum a_{ij} x^i y^j$$

converges to an element $x +_F y$ of $\mathfrak{m}$. In this way, $\mathfrak{m}$ becomes a abelian group $(\mathfrak{m}_K, +_F)$. Further, this process is seen to be functorial in the formal group F. Thus $(\mathfrak{m}_K, +_F)$ is seen to be an $\mathcal{O}_K$ module, independent of f up to canonical isomorphism.

For $n \in \mathbb{N}$, the π^n-torsions $\mathrm{Ker}([\pi^n]_f)$ of this newly defined $\mathcal{O}_K$ module $\left(\mathfrak{m}, +_{F_f}\right)$ would then be exactly the zeros of n-fold composition $f^{(n)} := f \circ f \circ \cdots \circ f$ in $\overline{K}$.

Theorem 41 ([Mil20, Proposition 1.3.4])**.** *For $n \in \mathbb{N}$ the $\mathcal{O}_K$ module $\mathrm{Ker}([\pi^n]_f)$ is isomorphic to $\mathcal{O}_K/\left(\pi^n\right)$. Then $\mathrm{End}(\mathrm{Ker}([\pi^n]_f)) \simeq \mathcal{O}_K/\left(\pi^n\right)$ and $\mathrm{Aut}(\mathrm{Ker}([\pi^n]_f)) \simeq (\mathcal{O}_K/\left(\pi^n\right))^\times$.*

Definition 42 (Lubin-Tate Tower)**.** For $n \in \mathbb{N}$ let

$$K_{\pi,n} = K(\mathrm{Ker}([\pi^n]_f)).$$

Noting that f has no constant term, one has $K_{\pi,n} \subseteq K_{\pi,n+1}$. Let

$$K_\pi = \bigcup_{n\in\mathbb{N}} K_{\pi,n}.$$

Let f and g be two Lubin-Tate power series, by noting that the any isomorphism between $\left(\mathfrak{m}_K, +_{F_f}\right)$ and $\left(\mathfrak{m}_K, +_{F_g}\right)$ that comes from an isomorphism between formal group laws is given by $\mathcal{O}_K$ power series, we have

Theorem 43 ([Sch17, Remark 1.3.7])**.** *The fields $K_{\pi,n}$ and K_π are independent to the choice of Lubin-Tate power series.*

Since one is free to choose arbitrary Lubin-Tate power series, one can let $f = \pi T + T^q$ where q is the size of residue field of K. Noting that f is a Eisenstein polynomial, generate totally ramified Galois extensions of K, with Galois group identified by $(\mathcal{O}_K/\pi^n)^\times$.

Theorem 44 ([Mil20, Theorem 1.3.6])**.** *For $n \in \mathbb{N}$ one has*

1. *$K_{\pi,n}/K$ is totally ramified of degree $(q-1)q^{n-1}$, where q is the size of residue field of K.*

2. *The action of $\mathcal{O}_K$ on $\ker([\pi^n]_f)$ defines an isomorphism*

$$(\mathcal{O}_K/(\pi^n))^\times \longrightarrow \mathrm{Gal}\left(K_{\pi,n}/K\right)$$

that is independent of the choice of Lubin-Tate power series. In particular, $K_{\pi,n}/K$ is an abelian extension.

3. *The uniformizer π is a norm from $K_{\pi,n}$.*

By passing to the limit, one has an isomorphism

$$\psi_\pi : \mathcal{O}_K^\times \longrightarrow \mathrm{Gal}\,(K_\pi/K)$$

that is independent of the choice of Lubin-Tate power series.

One also has

Theorem 45 (Local Kronecker-Weber Theorem [Mil20, Theorem 1.4.8])**.** *For every uniformizer π of K,*

$$K_\pi \cdot K^{\mathrm{ur}} = K^{\mathrm{ab}}.$$

It follows that K_π is a maximal totally ramified abelian extension of K. (Note that totally ramified abelian extensions are not closed under composites, so such maximal ones need not be unique.)

Finally, one explicitly recovers the local reciprocity map, and reinterpret the $\mathcal{O}_K$ module as a Galois representation.

Noting that $K_\pi \cap K^{\mathrm{un}} = K$, one has $G_K^{\mathrm{ab}} = \mathrm{Gal}\,(K_\pi|K) \times \mathrm{Gal}\,(K^{\mathrm{un}}|K)$. We define a homomorphism

$$\begin{aligned} \phi_\pi : K^\times &\longrightarrow G_K^{\mathrm{ab}} = \mathrm{Gal}\,(K_\pi|K) \times \mathrm{Gal}\,(K^{\mathrm{un}}|K) \\ u \times \pi^n &\longmapsto \left(\psi_\pi(u^{-1}), \mathrm{Frob}^n\right). \end{aligned}$$

Theorem 46 ([Mil20, Theorem 1.3.9, Section 1.1.4])**.** *The map ϕ_π is independent to the choice of π. Further, ϕ_π agrees with the local reciprocity map.*

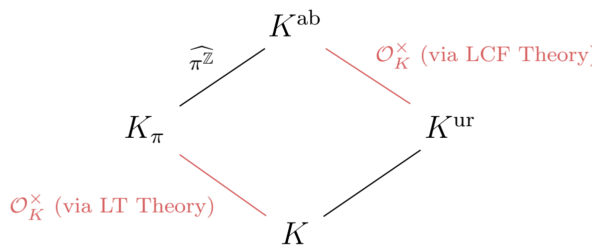
Remark 47. Let π be a uniformizer of K. With local class field theory in mind, one see that $\hat{K}^\times \simeq G_K^{\mathrm{ab}}$ is generated by $\pi^{\widehat{\mathbb{Z}}}$ and $\mathcal{O}_K^\times$. Their fixed subfields in K^{ab} are

$$(K^{\mathrm{ab}})^\pi = K_\pi$$

and

$$(K^{\mathrm{ab}})^{\widehat{\pi^{\mathbb{Z}}}} = K^{\mathrm{ur}}$$

respectively. We can identify the Galois group as in the diagram via the local reciprocity map.



Then the Galois group $\mathrm{Gal}\,(K_\pi|K)$ is abstractly isomorphic to $\mathcal{O}_K^{\times}$, as a quotient of G_K^{ab} depending on the choice of π. It is not a canonical object associated to K.

Further more, the map

$$\mathcal{O}_K^{\times} \hookrightarrow K^{\times} \twoheadrightarrow \mathrm{Gal}\,(K_\pi|K) \xrightarrow[\sim]{\psi_\pi^{-1}} \mathcal{O}_K^{\times}$$

is an isomorphism, and from the above discussion pertaining to local reciprocity map it is seen to be the map taking multiplicative inverse.

Chapter 3

Anabelian Geometry of Mixed Characteristic Local Fields

3.1 Mono-anabelian Results

Let K be a mixed characteristic local field with absolute Galois group $G = G_K$. Recall that

$$K^{\times} \simeq \left(\mathbb{Z}/\left(p_K^{f_K} - 1\right)\mathbb{Z}\right) \oplus (\mathbb{Z}/p_K^a\mathbb{Z}) \oplus (\mathbb{Z}_{p_K})^{\oplus d_K} \oplus \mathbb{Z}$$

and further via the local reciprocity map $\widehat{K}^{\times} \simeq G^{\mathrm{ab}}$.

In particular,

$$\mathrm{Tor}(G^{\mathrm{ab}}) \simeq \left(\mathbb{Z}/\left(p_K^{f_K} - 1\right)\mathbb{Z}\right) \oplus (\mathbb{Z}/p_K^a\mathbb{Z})$$

and

$$\frac{G^{\mathrm{ab}}}{\mathrm{Tor}(G^{\mathrm{ab}})} \simeq (\mathbb{Z}_{p_K})^{\oplus d_K} \oplus \widehat{\mathbb{Z}}$$

This allows us to recover certain invariants of K from its (abelianized) Galois group.

Proposition 48 ([Hos22]). *We have the following mono-anabelian invariants:*

1. *The residue characteristic $p(G)$ is the unique prime number such that*

$$\log_{p(G)}\left(\sharp\left(\frac{G^{\mathrm{ab}}}{\mathrm{Tor}(G^{\mathrm{ab}})}/p(G)\cdot\frac{G^{\mathrm{ab}}}{\mathrm{Tor}(G^{\mathrm{ab}})}\right)\right) \geq 2.$$

2. *The degree over $\mathbb{Q}_p$ is*

$$d(G) = \log_{p(G)}\left(\sharp\left(\frac{G^{\mathrm{ab}}}{\mathrm{Tor}(G^{\mathrm{ab}})}/p(G)\cdot\frac{G^{\mathrm{ab}}}{\mathrm{Tor}(G^{\mathrm{ab}})}\right)\right) - 1.$$

The residue degree is

$$f(G) = \log_{p(G)} \left(1 + \sharp\left(\left(\mathrm{Tor}(G^{\mathrm{ab}})\right)^{(p(G)')}\right)\right)$$

where we write $\left(\mathrm{Tor}(G^{\mathrm{ab}})\right)^{(p(G)')}$ *for the quotient of (finite abelian)* $\mathrm{Tor}(G^{\mathrm{ab}})$ *by the its $p(G)$-Sylow subgroup.*

The ramification index is

$$e(G) = d(G)/f(G).$$

3. *The inertia subgroup is*

$$I(G) = \bigcap_{N \subseteq G} N$$

where N ranges over the normal open subgroups of G such that $e(N) = e(G)$. Note it has the natural profinite topology.

4. *The wild inertia subgroup is*

$$P(G) = \bigcap_{N \subseteq G} N,$$

where N ranges over the normal open subgroups of G such that $e(N)/e(G)$ is prime to $p(G)$. Note it has the natural profinite topology.

Alternatively, $P(G)$ is the maximal pro-$p(G)$ submodule of $I(G)$.

5. *The Frobenius element*

$$\mathrm{Frob}(G) \in G/I(G) = G^{\mathrm{ab}}/I(G)^{\mathrm{ab}}$$

is the unique element of $G/I(G)$ such that the action of $\mathrm{Frob}(G)$ *on $I(G)/P(G)$ by conjugation is given by multiplication by $p(G)^{f(G)}$.*

6. *The set of uniformizers is*

$$\Pi(G) = \{g \in G^{\mathrm{ab}} \mid g \cdot I(G) = \mathrm{Frob}(G) \cdot I(G)\}.$$

7. *The topological multiplicative group of the ring of integers is*

$$\mathcal{O}^\times(G) = I(G)^{\mathrm{ab}} \subseteq G^{\mathrm{ab}}.$$

8. *The topological group of principal units is*

$$U^1(G) = P(G)^{\mathrm{ab}} \subseteq G^{\mathrm{ab}}.$$

Alternatively, $U^1(G)$ is the maximal pro-$p(G)$ submodule of $\mathcal{O}^\times(G)$.

9. *The multiplicative group of residue field is*

$$k^\times(G) = \mathcal{O}^\times(G)/U^1(G)$$

10. *The topological multiplicative group of the field is*

$$K^\times(G) = G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}} \subseteq G^{\mathrm{ab}},$$

that is, all $g \in G^{\mathrm{ab}}$ that acts on abelian group $G/I(G)$ by conjugation as multiplication by integer multiples of $p(G)^{f(G)}$.

Alternatively, $K^\times(G)$ can be taken as the subgroup of G^{ab} that is generated by either

- *an arbitrary uniformizer from $\Pi(G)$ together with $\mathcal{O}^\times(G)$, or*
- *the set of all uniformizers $\Pi(G)$.*

11. *The topological additive group of the field*

$$K_+(G) = \mathcal{O}^\times(G)^{\mathrm{pf}}$$

Using the functorial properties of the various map, many constructions can be naturally passed to limits.

Proposition 49 ([Hos22])**.** *We further have mono-anabelian invariants*

1. *We have topological G modules*

$$\overline{\mathcal{O}}^{\times}(G) = \varinjlim_{H\subseteq G} \mathcal{O}^{\times}(H)$$
$$\overline{U}^{1}(G) = \varinjlim_{H\subseteq G} U^{1}(H)$$
$$\overline{k}^{\times}(G) = \varinjlim_{H\subseteq G} k^{\times}(H)$$
$$\overline{K}^{\times}(G) = \varinjlim_{H\subseteq G} K^{\times}(H)$$
$$\overline{K}_{+}(G) = \varinjlim_{H\subseteq G} K_{+}(H)$$

where the limits are taken over open subgroups[1], *and transition maps for* $H \subseteq H' \subseteq G$ *induced from transfer maps* $\mathrm{Ver} : H' \longrightarrow H$ *and the functoriality of constructions* 7 *to* 11.

2.

$$\mu(G) = \mathrm{Tor}(\overline{K}^{\times}(G))$$

and in particular, we can construct the p-adic Tate module

$$T(G) = \varprojlim_{n} \mu(G)[p(G)^{n}]$$

Noting that as a $\mathbb{Z}_{p(G)}[G]$ *module* $T(G)$ *is (non-canonically) isomorphic to* $\mathbb{Z}_{p(G)}(1)$, *by choosing an arbitrary basis we also have the cyclotomic character*

$$\chi(G) : G \longrightarrow \mathbb{Z}_{p(G)}^{\times}.$$

Alternatively, Mochizuki has the following argument in [Moc97]: Suppose that M *is a* $\mathbb{Z}_p$*-module of finite length equipped with a continuous* G_K*-action. Then one has a natural isomorphism of Galois cohomology modules*

$$H^{i}(K, M) \cong H^{2-i}\left(K, M^{\vee}(1)\right)^{\vee}$$

for $i \geq 0$. *Here, with* (1) *denoting a Tate twist, and the duality is the "Pontrjagin dual"* $\mathrm{Hom}\,(-, \mathbb{Q}_p/\mathbb{Z}_p)$. *Suppose that* M *is isomorphic as a* $\mathbb{Z}_p$*-module to*

[1] namely subgroups of finite index

$\mathbb{Z}/p^n\mathbb{Z}$ for some $n \geq 1$. Then from the above isomorphism one has M is isomorphic as a G_K-module to $\mathbb{Z}/p^n\mathbb{Z}(1)$ if and only if $H^2(K, M) \cong \mathbb{Z}/p^n\mathbb{Z}$. This is a group-theoretic condition on M. Thus, one conclude that the isomorphism class of the Γ_K-module $\mathbb{Z}_p(1)$ can be recovered from G_K.

3.2 Bi-anabelian Results

In this section we study the birational anabelian results for mixed characteristic local fields. Motivated by the Neukirch–Uchida theorem on number fields, given local fields K and L one studys the following map induced by the absolute Galois group functor

$$\pi_1 : \mathrm{Isom}(L, K) \longrightarrow \mathrm{Isom}^{\mathrm{out}}(\mathrm{Gal}\,(K)\,, \mathrm{Gal}\,(L))$$

The following two theorems verify that mixed characteristic local fields indeed has certain anabelian features.

Theorem 50 ([Hos22])**.** *The absolute Galois group G_k is centre free.*

Theorem 51. *The map above map is injective, namely the absolute Galois group functor π_1 is faithful on isomorphisms.*

Proof. This follows from the functoriality of the local reciprocity isomorphism. In particular see the diagrams in proposition 30. □

In the last section, one see that from $\mathrm{Gal}\,(k)$ one can functorially recover the multiplicative and additive structure of k. However, one cannot realize the multiplicative and additive structure on the same set, so maps between the fundamental groups induces both multiplicative and additive maps that may not be consistent. In general this ramifies as π_1 fails being surjective on isomorphisms as seen in the following example.

Theorem 52. *The field $\mathbb{Q}_p$ is rigid, namely* $\mathrm{Aut}(\mathbb{Q}_p) = \{\mathrm{id}\}$

Proof. The following lemma says that $\mathbb{Z}_p^\times$ must be preserved by an field automorphism.

Lemma 53. *Let $x \in \mathbb{Q}_p^\times$. Then the following properties are equivalent:*

1. *x is a unit;*

2. *x^{p-1} possesses nth roots for infinitely many values of n.*

Noting that $\mathbb{Q}$ is rigid, it follows a field automorphism must preserve the valuation, and thus the p-adic topology. Then the field automorphism must be the identity by the density of $\mathbb{Q}$. □

Theorem 54 ([NSW08])**.** $\mathrm{Gal}(\mathbb{Q}_p)$ *has nontrivial outer automorphisms.*

Together we see that mixed characteristic local fields cannot induce every isomorphism between their Galois group, namely the naive bi-anabelian statement is false.

On the other hand, the Galois group of a local field has finer structures from the arithmetic of the field.

Definition 55 (Ramification Group)**.** Let k be a local field with normalized valuation v and Galois group $G = \mathrm{Gal}(k)$. Let i be an integer ≥ -1, then i-th ramification group of G is defined to be the group

$$G_i = \{g \in G \mid v(g(x) - x) \geq i + 1 \quad \forall x \in \overline{k}\}.$$

The characteristic ramification subgroups then yields a filtration on the Galois group.

Definition 56. Let K, L be mixed characteristic local fields, an isomorphism between the Galois groups $\alpha : \mathrm{Gal}(L) \to \mathrm{Gal}(K)$ is said to preserve the lower ramification filtration if

$$\alpha(\mathrm{Gal}(L)_i) = \mathrm{Gal}(K)_i$$

for all integer $i \geq -1$.

The lower numbering filtration is compatible with subgroups (extensions), and the data of the full filtration is sufficient to define an upper numbering filtration that is compatible with quotients.

The full knowledge of either the upper or the lower numbering filtration allows one to construct the Herbrand's function and its inverse, and thus yielding a description of the other filtration. In particular, an isomorphism between Galois group that preserves either filtration preserves the other. A reference can be found in [Ser79, Section 2.3].

There also exists a class of finite dimensional $\mathbb{Q}_p$ representation of the Galois groups that will be studied in the next chapter.

Definition 57. Let K, L be mixed characteristic local fields, an isomorphism between the Galois groups $\alpha : \mathrm{Gal}\,(L) \to \mathrm{Gal}\,(K)$ is said to be HT-preserving (Hodge-Tate preserving) if for any Hodge-Tate representation

$$\rho : \mathrm{Gal}\,(K) \longrightarrow \mathrm{GL}_{\mathbb{Q}_p}(V)$$

the composite

$$\rho \circ \alpha : \mathrm{Gal}\,(L) \longrightarrow \mathrm{GL}_{\mathbb{Q}_p}(V)$$

is also a Hodge-Tate representation.

The set of such isomorphisms is denoted $\mathrm{Isom}^{\mathrm{HT}}(\mathrm{Gal}\,(L)\,, \mathrm{Gal}\,(K))$.

Theorem 58. *Let K, L be mixed characteristic local fields and $\alpha : \mathrm{Gal}\,(L) \to \mathrm{Gal}\,(K)$ be an isomorphism between the Galois groups. The following are equivalent:*

1. *There exists an isomorphism of fields $\beta : K \to L$ such that $\alpha = \pi_1(\beta)$;*
2. *The isomorphism α preserves the ramification filtration;*
3. *The isomorphism α is HT-preserving.*

Proof. (1) $\implies$ (2) is trivial.

(2) $\implies$ (3): First we show that one can recover the additive module $\mathcal{O}_{K+}$.

With the full ramification filtration, one recover the ramification groups in upper numberings $\mathrm{Gal}\,(K)^{(v)}$, and thus the higher units $U_K^{(v)}$ via local reciprocity map. Also note that one can determine from the Galois group the ramification index e_K of K. Then the submodule

$$p^{-1} \cdot U_K^{e_K} \subseteq U_K \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

corresponds to the additive module $\mathcal{O}_{K+}$ of the ring of integers under the isomorphism induced by the p-adic logarithm.

Since the ramification groups are compatible with finite extensions by taking intersections, the above discussion also applies. One can thus recover the additive modules of rings of integers for finite extensions of K. The union is then the additive module $\mathcal{O}_{\overline{K}+}$.

Let $\widehat{\mathcal{O}_{\overline{K}+}}$ be the p-adic completion of $\mathcal{O}_{\overline{K}+}$, and

$$\widehat{\overline{K}}_+ = \widehat{\mathcal{O}_{\overline{K}+}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

In total, The $\mathrm{Gal}\,(K)$ module $\widehat{\overline{K}}_+$ is then re-constructed from the group-theoretic data of the Galois group and its full ramification filtration. And thus α induces exactly the pullback between the Galois modules $\widehat{\overline{K}}_+$ and $\widehat{\overline{L}}_+$.

Recall that one can also reconstruct the cyclotomic character from the Galois group. These ingredients together sufficiently characterize Hodge-Tate representation, as one shall see in the next chapter. Thus such α that preserves ramification filtration is necessarily HT-preserving.

(3) $\implies$ (1) will be proved as the main theorem in chapter 5 after relevant background on Galois representations is reviewed. □

3.3 Invariants Not Determined by the Galois Group

A mixed characteristic local field is not determined by their absolute Galois group. The following theorem characterize a certain case where isomorphic Galois groups arises.

Theorem 59 (Jarden, Ritter)**.** *Let K, L be mixed characteristic local fields. Assume K has residue characteristic p, and contains a primitive p-th root of unity ζ_p (and ζ_4 in the case where $p = 2$). Then K and L have isomorphic Galois group if and only if the following hold:*

1. *The maximal abelian subfield contained in K and L are isomorphic;*

2. *$[K : \mathbb{Q}_p] = [L : \mathbb{Q}_p]$.*

Let p be an odd prime. Let $K = \mathbb{Q}\left(\zeta_p, \sqrt[p]{p}\right)$ and $L = \mathbb{Q}\left(\zeta_p, \sqrt[p]{1+p}\right)$. Then by verifying the above conditions, one has K and L are non isomorphic local fields with isomorphic Galois groups.

From the above bi-anabelian results, we can further specified other invariants that are not mono-anabelian.

Theorem 60. *The following invariants of a mixed characteristic local field are not in general recoverable from its absolute Galois group:*

1. The ramification filtration on the absolute Galois group;

2. The additive group of the ring of integers $\mathcal{O}_+$;

3. The additive topological group $\overline{K}_+$ of the algebraic closure of the field together with p-adic topology.

Remark 61. For a mixed characteristic local field K and E a finite extension of K, we have seen that one can indeed reconstruct the additive topological group E_+ together with the p-adic topology. However, the topological limit

$$\varprojlim E_+$$

algebraically isomorphic to $\overline{K}$ is topologized much finer than the p-adic topology. The following example from Hoshi verifies this fact.

Consider the map

$$\mathrm{Tr} : \overline{K}_+ \longrightarrow K_+$$

well defined by mapping $a \in E_+ \subset \overline{K}_+$ to $\frac{1}{[E:K]} \cdot \mathrm{Tr}_{E/K}(a) \in K_+$ for E a finite extension of K contained in $\overline{K}$. Since the restriction of the map Tr to each finite extension of K contained in $\overline{K}$ is continuous with respect to the respective p-adic topologies, the map Tr is continuous with respect to the topology on the limit $\varprojlim E_+$ and the p-adic topology on K_+.

For each positive integer n write $K_n \subseteq \overline{K}$ for the (uniquely determined) unramified finite extension of K such that $[K_n : K] = p^n$. By the surjectivity of the trace map with respect to a finite extension of finite fields, there exists an element $a_n \in (\mathcal{O}_{K_n})_+$ such that $\mathrm{Tr}_{K_n/K}(a_n) \in \mathcal{O}_K^\times$. Then it is immediate that the sequence $(p^n \cdot a_n)_{n\geq 1}$ converges to $0 \in \overline{K}_+$ with respect to the p-adic topology on $\overline{K}_+$.

On the other hand,

$$\mathrm{Tr}\,(p^n \cdot a_n) = \frac{1}{p^n} \cdot \mathrm{Tr}_{K_n/K}\,(p^n \cdot a_n) = \mathrm{Tr}_{K_n/K}\,(a_n) \in \mathcal{O}_K^\times,$$

and thus the sequence $(\mathrm{Tr}\,(p^n \cdot a_n))_{n\geq 1}$ does not converge to $\mathrm{Tr}(0) = 0 \in K_+$. In particular, the map Tr is not continuous with respect to the p-adic topology on $\overline{K}_+$ and the p-adic topology on K_+.

Thus, we conclude that the p-adic topology on $\overline{K}_+$ does not coincide with the topology on the topological limit $\varprojlim E_+$.

For more detail on the normalized trace map and its significance see [Ban+22] and [And+19, Section 4.3.2]

Chapter 4

Galois Representations of Mixed Characteristic Local Fields

4.1 Galois Representations

Let K be a mixed characteristic local field with residue characteristic p. Let $\mathbb{C}_p$ be the completion of some fixed algebraic closure $\overline{K}$ of K. The Galois action of G_K on $\overline{K}$ extends continuously to an action on the completion $\mathbb{C}_p$. We have the following category:

Definition 62 ($\mathrm{Rep}_L(G_K)$)**.** Let L be a field such that $\mathbb{Q}_p \subseteq L \subseteq \mathbb{C}_p$, and further assume L is stable under the G_K action. Common choices for L would be finite extensions of $\mathbb{Q}_p$ in K, finite Galois extensions of K, finite Galois extensions of $\widehat{K^{\mathrm{ur}}}$ the completion of the maximal unramified extension of K, and $\mathbb{C}_p$.

A L-representation of G_K is a finite-dimensional L-vector space V with a continuous semilinear G_K-action, namely

$$G_K \times V \longrightarrow V$$

satisfies the following condition:

$$(g, a \cdot v) \longmapsto g(a) \cdot g(v) \quad \forall g \in G_K, a \in L, \text{and } v \in V.$$

Such objects together with L linear G_K-equivariant maps forms an abelian category denoted $\mathrm{Rep}_L(G_K)$.

In particular, when L is a subfield of K, the action of G_K is L linear.

One common way to construct G_K representations is by characters.

Definition 63 (Representation from Character)**.** Let $\phi : G_K \longrightarrow L^\times$ be a continuous homomorphism. Then one has $L(\phi) \in \mathrm{Rep}_L(G_K)$ which as a L module is identify by L, with the G_K action given by

$$\begin{aligned} G_K \times L &\longrightarrow L \\ (g, a) &\longmapsto \phi(g) \cdot a. \end{aligned}$$

Now if $L' \subseteq \mathbb{C}_p$ is some extension of L, ϕ can naturally be thought of as a character mapping to $L'^\times$ as well. Abusing notation and also denote it as ϕ. Then it follows that

$$L'(\phi) \simeq L' \otimes_L L(\phi)$$

in $\mathrm{Rep}_{L'}(G_K)$.

Definition 64 (Twist by Character)**.** Let $V \in \mathrm{Rep}_L(G_K)$, we define $V(\phi) \in \mathrm{Rep}_L(G_K)$, which is identified as a L vector space by V, but with the G_K semilinear action given by:

$$(g, v) \longmapsto v^g := \phi(g) \cdot g(v)$$

for $g \in G_K$ and $v \in V$.

Alternatively, $V(\phi)$ can be seen to be isomorphic to $V \otimes_L L(\phi) \in \mathrm{Rep}_L(G_K)$.

Definition 65. Given two characters $\phi, \phi' : G_K \longrightarrow \mathbb{C}_p$, we say that

$$\phi \sim \phi'$$

if there exists an $x \in \mathbb{C}_p$ such that for all $g \in G_K$,

$$g(x) = \frac{\phi(g)}{\phi'(g)} \cdot x.$$

Proposition 66 ([Ser97, Section 3.A.2])**.** *The characters satisfy $\phi \sim \phi'$ if and only if the $\mathbb{C}_p$ representations $\mathbb{C}_p(\phi)$ and $\mathbb{C}_p(\phi')$ are isomorphic.*

Proof. By noting $\phi \sim \phi'$ if and only if $\phi \cdot \phi'^{-1} \sim 1$ and $(\mathbb{C}_p(\phi))(\phi') = \mathbb{C}_p(\phi \cdot \phi')$, it suffices to show for $\phi' = 1$.

Assuming that $\phi \sim 1$ so that there is a $x \in \mathbb{C}_p$ such that

$$g(x) = \phi(g) \cdot x$$

Consider the $\mathbb{C}_p$ linear map

$$\begin{aligned} \gamma : \mathbb{C}_p(\phi) &\longrightarrow \mathbb{C}_p \\ c &\longmapsto c \cdot x. \end{aligned}$$

This map is G_K equivariant, since

$$\begin{aligned} \gamma(c)^g &= g(c \cdot x) \\ &= g(c) \cdot g(x), \end{aligned}$$

while

$$\begin{aligned} \gamma(c^g) &= \gamma(\phi(g) \cdot g(c)) \\ &= \phi(g) \cdot g(c) \cdot x \\ &= g(c) \cdot (\phi(g) \cdot x) \\ &= g(c) \cdot g(x). \end{aligned}$$

The converse is true by tracing back the argument. □

Definition 67 ($V\{\phi^n\}$)**.** Let $V \in \mathrm{Rep}_L(G_K)$. Given a character $\phi : G_K \to L^\times$, we can define the L^{G_K} sub-spaces of $V(\phi^n)$ where G_K acts as powers of ϕ. More precisely, we have a functor to L^{G_K} vector space

$$V \longmapsto V\{\phi^n\}_K = V(\phi^n)^{G_K} \simeq \left\{ v \in V \mid g(v) = \phi(g)^{-q} \cdot v \text{ for all } g \in G_K \right\}.$$

Note that when $K \subseteq L$, we have $L^{G_K} = K$.

Now when L is a finite extension of K (resp. L is a finite extension of $\widehat{K^{\text{ur}}}$), we have $G_L \subseteq G_K$ is a open subgroup of finite index (resp. $G_L \subseteq I_K$ is a open subgroup of finite index). Then there is a natural functor

$$\text{Rep}_{\mathbb{C}_p}(G_K) \longrightarrow \text{Rep}_{\mathbb{C}_p}(G_L)$$

and also natural K linear inclusions

$$V\{\phi^n\}_K = V(\phi^n)^{G_K} \hookrightarrow V(\phi^n)^{G_L} = V\{\phi^n\}_L.$$

It is not immediate that the K dimension of $V\{\phi^n\}_K$ is finite, but when it is (for example when $V = \mathbb{C}_p$ and ϕ is χ the cyclotomic character as will be shown soon) it has the following important descent property:

Theorem 68 (Descent [BC09])**.** *Let L be a finite extension of K or $\widehat{K^{\text{ur}}}$, and $W \in \text{Rep}_{\mathbb{C}_p}(G_K)$ such that W^{G_K} has finite K dimensions. The natural map*

$$W^{G_K} \otimes_K L \longrightarrow W^{G_L}$$

is an L linear isomorphism.

Then in particular, Assume $V\{\phi^n\}_K$ is a finite dimensional K vector space as above, and L is a finite extension of K or $\widehat{K^{\text{ur}}}$. The natural map

$$V\{\phi^n\}_K \otimes_K L \longrightarrow V\{\phi^n\}_L$$

is an isomorphism.

Next we discuss the two characters that are of importance to our application.

Definition 69 (Cyclotomic Character)**.** Let K be a mixed characteristic local field with residue characteristic p. The cyclotomic character encodes the action of G_K on the p power roots of unity. (Note that these come from a Lubin-Tate tower in the case that $K = \mathbb{Q}_p$.)

More precisely, let μ_{p^n} be the group of p^n-th roots of unity in $\overline{K}$, whereby non-cannically

$$\mu_{p^n}\left(\overline{K}\right) \simeq \mathbb{Z}/p^n\mathbb{Z}.$$

Such groups fit in an inverse system given by

$$\begin{aligned} \mu_{p^{n+1}} &\longrightarrow \mu_{p^n} \\ a &\longmapsto a^p. \end{aligned}$$

The inverse limit defines the so called Tate module of the multiplicative group $\mathbb{G}_m$ of $\overline{K}$

$$T_p\left(\mathbb{G}_m\right) = \varprojlim \mu_{p^n}.$$

One see that $T_p\left(\mathbb{G}_m\right)$ is isomorphic to a free $\mathbb{Z}_p$ module of rank 1. In particular, there is a canonical isomorphism

$$\mathrm{Aut}(T_p\left(\mathbb{G}_m\right)) = \mathbb{Z}_p^\times$$

and thus a canonical map which defines the so called p-adic cyclotomic character

$$\chi : G_K \longrightarrow \mathbb{Z}_p^\times.$$

Define $\mathbb{Z}_p(1)$ as the $\mathbb{Z}_p$ module identified by $\mathbb{Z}_p$, further with a G_K action

$$\begin{aligned} G_K \times \mathbb{Z}_p &\longrightarrow \mathbb{Z}_p \\ (g, a) &\longmapsto \chi(g) \cdot a. \end{aligned}$$

Then letting e be an arbitrary generator of $T_p\left(\mathbb{G}_m\right)$, there is a G_K-equivariant $\mathbb{Z}_p$ linear isomorphism

$$\begin{aligned} \mathbb{Z}_p(1) &\longrightarrow T_p\left(\mathbb{G}_m\right) \\ 1 &\longmapsto e \end{aligned}$$

which is not canonical, depending on the choice of generator e.

Remark 70. The significance of the above Tate module is that it is the first p-adic homology (or equivalently the étale fundamental group) of $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$, the multiplicative group scheme. It is free of rank 1, and the first p-adic cohomology group of $\mathbb{G}_m$ is the dual of $\mathbb{Z}_p(1)$, that is to say, $\mathbb{Z}_p(-1)$.

Definition 71 (Lubin-Tate Character)**.** Let E be a finite extensions of $\mathbb{Q}_p$, and let π be a uniformizer of E. One define

$$\chi^{\mathrm{LT}}_{E,\pi} : G_E \longrightarrow E^\times$$

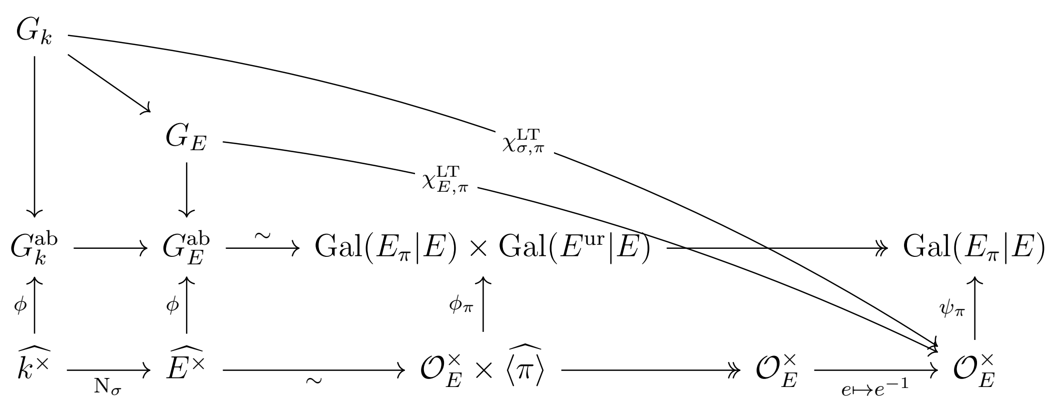
for the Lubin-Tate character discussed in section 2.4 on explicit class field theory .

Further let k be a finite extension of E. Let $\sigma : E \hookrightarrow k$ be an embedding. One define

$$\chi^{\mathrm{LT}}_{\sigma,\pi} : G_k \longrightarrow E^\times$$

for the restriction of a to the subgroup $G_k \subset G_E$.

In summary, the above continuous characters are induced from the following commutative diagram:



where ϕ is the local reciprocity map, N_σ is the norm with respect to the embedding σ.

Remark 72. In particular, the Lubin-Tate character $\chi^{\mathrm{LT}}_{\mathbb{Q}_p,p}$ is exactly the cyclotomic character χ.

We make the following definition:

Definition 73 (Inertial Equivalence)**.** Two representations ρ_1, ρ_2 of some fixed mixed characteristic local field are said to be inertially equivalent if they agree on some open subgroup of the inertia group. We denote the equivalence relation by

$$\rho_1 \sim \rho_2$$

Now note that in particular, the above composites are independent of the choice of π when restricted to the respective inertia subgroups. and thus the inertial equivalence classes of $\chi_{E,\pi}^{\mathrm{LT}}$ and $\chi_{\sigma,\pi}^{\mathrm{LT}}$ do not depend on the choice of $\pi \in \mathcal{O}_E$. Thus, we write χ_E^{LT} and $\chi_\sigma^{\mathrm{LT}}$ when in references to the equivalence classes.

Remark 74. Letting $\sigma : E \hookrightarrow k$ be as above, and $\mathcal{F}_\pi$ the Lubin-Tate formal group law for E. The absolute value on E induced by the valuation extends to $\overline{E}$. Then we have an $\mathcal{O}_E$ action on elements with absolute value strictly lesser than 1. In particular, the π^n (and respectively p^n) torsions, where π acts as some fixed Lubin-Tate polynomial, is a G_k module isomorphic to $\mathcal{O}_E/\mathfrak{m}_E^n$ (and respectively $\mathcal{O}_E/(p)^n$). The Tate module is then

$$T_p(\mathcal{F}_\pi) = \varprojlim \mathcal{O}_E/(p)^n = \varprojlim \mathcal{O}_E/\mathfrak{m}_E^n = \mathcal{O}_E$$

with automorphisms

$$\mathrm{Aut}\,(T_p(\mathcal{F}_\pi)) = \mathcal{O}_E^\times.$$

The G_k action is then given by the Lubin-Tate character $\chi_{\sigma,\pi}^{\mathrm{LT}}$. This picture is clarified when we study p-divisible group in a later section.

4.2 Group Cohomology and Representations

First we recall some facts on non-abelian group cohomology.

Definition 75. Let G be a topological group, and M a topological group that is not necessarily abelian, with group operation written multiplicatively. Assume M is equipped with a continuous G action $x \mapsto x^g$.

Defined the 0-th cohomology by taking invariants:

$$H^0(G,M) = M^G.$$

To define H^1, we first define the set of continuous 1-cocycles $Z^1(G,M)$ to be the set of continuous maps $f : G \longrightarrow M$ such that

$$f(gh) = f(g)f(h)^g$$

for all $g,h \in G$. Next we say that $f, f' \in Z^1(G,M)$ are cohomologous if there exists an element $x \in M$ such that

$$f'(g) = x^{-1}f(g)x^g$$

for all $g \in G$. This defines an equivelence relation on $Z^1(G,M)$ and we define $H^1(G,M)$ to be the quotient.

The non-abelian group cohomology is not defined for $n \geq 2$, but it does behave analogously for H^0 and H^1. In particular, it is functorial, induces a long exact sequence, and has a inflation-restriction sequence. For detail see [FY, Section 1.4].

Note that this definition agrees with the usual group cohomology when M is abelian, thus the notation is consistent.

One significant application of this definition is the following:

Proposition 76 ([FY, Proposition 3.7])**.** *Let $d \geq 1$, G some topological group and L some topological field with a continuous G action. There is a bijection between isomorphism classes of d dimensional G-semilinear representation over L, and $H^1(G, \mathrm{GL}_d(L))$. Moreover the representation is trivial if and only if the corresponding class of cocycle is trivial.*

Now we return to the case where K is a mixed characteristic local field, L some subfield of $\mathbb{C}_p$ stable under the action of Galois group G_K. Consider $\phi : G_K \longrightarrow \mathbb{C}_p^\times$ some character and $\mathbb{C}_p(\phi)$ the associated 1-dimensional representation in $\mathrm{Rep}_L(G_K)$.

After tracing through the theorem, one see that the isomorphism class of $\mathbb{C}_p(\phi)$ corresponds exactly to the element in $H^1(G_K, \mathrm{GL}_1(L)) = H^1(G_K, L^\times)$ represented by ϕ as a cocycle.

Furthermore, we shall see that $\phi \sim \phi'$ if and only if $\mathbb{C}_p(\phi) \simeq \mathbb{C}_p(\phi')$.

4.3 Motivating Fontaine's Approach

Fontaine's approch to study p-adic Galois representation can be motivated by the following geometric example.

Example 77 (De Rham Cohomology, De Rham Theorem and Hodge Decomposition)**.** Let X be a projective smooth scheme over $\mathbb{C}$. Then the associated complex analytic space X^{an} is a compact Kähler manifold.

Then X^{an} has the following cohomology theories:

- The singular cohomology $H^n_{\mathrm{top}}(X^{\mathrm{an}}, \mathbb{Z})$,
- The De Rham cohomology $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$.

They are classically related in the following ways:

Theorem 78 (De Rham Comparison Theorem)**.**

$$H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \simeq H^n_{\mathrm{top}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$$

This isomorphism can be obtained by considering a pairing obtained by integrating differential forms in the algebraic de Rham cohomology over cycles in the singular cohomology. Such an integration generally yields a complex number, and explains why the singular cohomology must be tensored to $\mathbb{C}$. Then $\mathbf{C}$ can be thought of as the so called period ring, containing all the periods necessary to express the isomorphism comparing algebraic de Rham cohomology with singular cohomology.

Letting Ω^q be the sheaves of complex differential q forms, we have

Theorem 79 (Hodge Decomposition)**.**

$$H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \simeq \bigoplus_{p+q=n} H^p\left(X^{\mathrm{an}}, \Omega^q\right)$$

where the right hand side are sheaf cohomology groups.

In particular, the complex de Rham cohomology $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ can be seen as the sheaf cohomology calculated from a resolution of the constant sheaf by the complex $\Omega^\bullet$. One can analogously define sheaves of algebraic differentials $\Omega^\bullet_{\mathrm{alg}}$ on X, which is shown in [Gro66] to yield the same sheaf cohomology and satisfy the corresponding decomposition.

In summary, we have

$$H^n_{\mathrm{top}}(X^{\mathrm{an}}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \bigoplus_{p+q=n} H^p\left(X, \Omega^q_{\mathrm{alg}}\right).$$

Now instead consider X a proper smooth scheme over a mixed characteristic local field K, we shall replace the singular cohomology by the ℓ-adic étale cohomology $H^n_{\mathrm{et}}\left(X_{\overline{K}}, \mathbb{Z}_p\right)$ Then there is the following theorem:

Theorem 80 (Faltings)**.**

$$\mathbb{C}_p \otimes_{\mathbb{Z}_p} H^n_{\mathrm{et}}\left(X_{\overline{K}}, \mathbb{Z}_p\right) \simeq \bigoplus_{p+q=n} \left(\mathbb{C}_p(-q) \otimes_K H^p\left(X, \Omega^q_{X/K}\right)\right).$$

This isomorphism can be interpreted as stating $H^n_{\mathrm{et}}\left(X_{\overline{K}}, \mathbb{Z}_p\right)$ is Hodge-Tate, and giving the explicit decomposition.

4.4 Fontaine's Formalism and Period Rings

We first discuss the general formalism.

Definition 81 (Period Rings)**.** Let F be a topological field, and G a topological group that acts trivially on F. Let B be a F algebra with a G action. One says that B is (F, G) regular if the following conditions hold:

1. B is a domain,

2. $(\operatorname{Frac} B)^G = B^G \supseteq F$,

3. if $b \in B, b \neq 0$ and the F-line $b \cdot F$ is stable under G, then $b \in B^\times$.

Lemma 82. *B^G is a field.*

Proof. For all $b \in B^G, b \neq 0$, the line $b \cdot F$ is clearly stable under G. Thus b has to be invertible in B, with an inverse that is also fixed by G. □

Definition 83. Let W be a B module with a semilinear G action. The comparison homomorphism is defined to be the natural map

$$\alpha_W : B \otimes_{B^G} W^G \to W.$$

Proposition 84 ([FY, Theorem 3.14])**.** *When W is free of finite rank over B, the comparison homomorphism α_W is injective.*

In particular, any W of the form $B \otimes_F V$ where V is a finite dimensional F representation of G satisfies the condition.

Definition 85 (B-Admissible Representations)**.** Let V be a d-dimensional F representation of G. Then V is said to be B-admissible if the B module $W = B \otimes_F V$ with semilinear G action is isomorphic to a finite product of B. We also say W is trivial in this case.

Proposition 86 ([**caruso_introduction_2019**] [FY, Theorem 3.6, 3.14] [Ber04, Section 1.2.3])**.** *The followings are equivalent:*

1. *V is B-admissible,*

2. *The comparison homomorphism α_W is an isomorphism,*

3. *$\dim_{B^G} W^G = \dim_F V = d$,*

4. *There is a G invariant basis of W,*

5. *The class $[W]$ in $H^1(G, \mathrm{GL}(d, B))$ that corresponds to the isomorphism class of W, which is the image of $[V]$ in $H^1(G_K, \mathrm{GL}(d, F))$ under the natural map, is trivial.*

Corollary 87. *If $B \subseteq B'$, then B-admissible representations are B'-admissible.*

In our application, G will be the Galois group of some mixed characteristic local field K, and F will be some subfield of K. We explore some classical choices of B.

4.4.1 The case for $B = \overline{K}$

Proposition 88 (Hilbert's Theorem 90 [FY, Theorem 1.114]). *Let L be a field and K be a Galois extension of L.* ***Assuming the field K is discrete****, and the Galois group has the natural profinite topology. Then for all $n \geq 1, H^1(\mathrm{Gal}(K|L), \mathrm{GL}_n(K))$ is trivial.*

Remark 89. This can be seen as the reformulation the classical Galois descent for vector space. See [Mil17, Section A.64].

Now for K an extension of mixed characteristic local field L, it comes equipped with a topology. Nevertheless one can apply the above result when the Galois group acts discretely, which recall is the following equivalent condition:

Proposition 90 ([NSW08, Proposition 1.1.8]). *Let G be a profinite group and let M be an abstract G-module. Then the following conditions are equivalent:*

1. *M is a discrete G-module, i.e. the action $G \times M \to M$ is continuous for the discrete topology on M.*

2. *For every $m \in M$ the stabilizer subgroup $\{g \in G \mid g(m) = m\}$ is open.*

Proposition 91 ([FY, Proposition 4.12]). *Let $L \subseteq K$ be mixed characteristic local fields and V a L representation for G_K. Let $\rho : G_K \to \mathrm{Aut}_L(V)$ the be corresponding representation map. Let $W = \overline{K} \otimes_L V$. The following are equivalent:*

1. *V is $\overline{K}$-admissible,*

2. *W is a discrete G_K vector space over $\overline{K}$,*

3. *$\mathrm{Ker}(\rho)$ is open in G_K,*

4. *V is a discrete G_K vector space over L.*

Proof. $1 \implies 2$: Given $V \otimes_L \overline{K}$ is trivial, it has a G_K invariant basis. Then the stabilizer for a vector is the finite intersection of the open Galois stabilizer of the coefficients.

$2 \implies 1$: Given the action is discrete, Hilbert's 90 applies and the representation is trivial.

1 and $2 \implies 3$: Let $\{v_i\}_i$ be a basis of V over L. Then $\{e_i = 1 \otimes v_i\}_i$ is a basis of W over $\overline{K}$. Then the stabilizers of the e_i's are open in G_K, with intersection being the kernel of ρ.

$3 \implies 4$: For any $g \in G_K$, $v \in V$ such that $g(v) = v$, there is an open neighbourhood $g \cdot \mathrm{Ker}(\rho) \times \{v\}$ that maps to v.

$4 \implies 2$: For any $a \otimes v \in W = \overline{L} \otimes_L V$, let $\mathrm{Stab}_V(v)$ be the open stabilizer in G_K for V, and $\mathrm{Stab}_{\overline{L}}(a)$ the stabilizer in G_K for $\overline{L}$. Then their open intersection is a neighbourhood for any stabilizer of $a \otimes v$. ☐

4.4.2 The case for $B = \overline{\widehat{K^{\mathrm{ur}}}}$

Now note that $\overline{\widehat{K^{\mathrm{ur}}}}$ is stable under G_K, with $G_{\widehat{K^{\mathrm{ur}}}} = \mathrm{Gal}\left(\overline{\widehat{K^{\mathrm{ur}}}} | \widehat{K^{\mathrm{ur}}}\right) = I_K$.

Proposition 92 ([FY, Proposition 4.14])**.** *Let $L \subseteq K$ be mixed characteristic local fields and V a L representation for G_K. Let $\rho : G_K \to \mathrm{Aut}_L(V)$ the be corresponding representation map. Let $W = \overline{\widehat{K^{\mathrm{ur}}}} \otimes_L V$. The following are equivalent:*

1. *V is $\overline{\widehat{K^{\mathrm{ur}}}}$-admissible,*

2. *W is a discrete I_K vector space over $\overline{\widehat{K^{\mathrm{ur}}}}$,*

3. $\mathrm{Ker}(\rho) \cap I_K$ *is open in* I_K*,*

4. V *is a discrete* I_K *vector space over* L*.*

Proof. The implications follow from exactly the same arguments as in the case $B = \overline{K}$, except for

$2 \implies 1$: Note that $\overline{\widehat{K^{\mathrm{ur}}}}$ is **not** a Galois extension of L! (There exists transcendentals in the competition.) So we can not directly apply the Hilbert's 90 argument. But it does follow that

$$\overline{\widehat{K^{\mathrm{ur}}}} \otimes_{\widehat{K^{\mathrm{ur}}}} W^{I_K} \longrightarrow W$$

is an isomorphism.

Consider the $\left(\overline{\widehat{K^{\mathrm{ur}}}}\right)^{I_K} = \widehat{K^{\mathrm{ur}}}$ vector space $W' = W^{I_K}$ and note that it is a natural $G_k = G_K/I_K$ representation. It suffices to prove that such $\widehat{K^{\mathrm{ur}}}$-representation W' of G_k is trivial.

Let $E = k$, the cohen ring of E is then the Witt vectors $\mathcal{O}_{\mathcal{E}} = W(k)$, with fraction field $\mathcal{E} = K_0$, the maximal unramified extension of $\mathbb{Q}_p$ in K. Taking limits of the above constructions over finite separable extensions of k, one has $\mathcal{E}^{\mathrm{ur}} = \bigcup_E \mathcal{E}$. Let $\widehat{\mathcal{E}^{\mathrm{ur}}}$ be its competition, one has $\widehat{\mathcal{E}^{\mathrm{ur}}} = K_0^{\mathrm{ur}}$.

It then follows from the theory of étale φ module that

$$\widehat{\mathcal{E}^{\mathrm{ur}}} \otimes_{\mathcal{E}} (W')^{G_k} = K_0^{\mathrm{ur}} \otimes_{K_0} (W')^{G_k} \longrightarrow W'$$

is an isomorphism, and thus W' is trivial over K_0^{ur}. It then follows W' is trivial over $\widehat{K^{\mathrm{ur}}}$ as claimed. □

4.4.3 The case for $B = \mathbb{C}_p$

Noting that $\overline{K} \subset \overline{\widehat{K^{\mathrm{ur}}}} \subset \mathbb{C}_p$, one has that two representation are isomorphic over $\mathbb{C}_p$ if they agree on some open subgroup of the inertia, namely they are inertially equivalent.

In fact, the converse also holds. Let K^{∞}/K be a totally ramified $\mathbb{Z}_p$-extension. Denote $\Gamma = \mathrm{Gal}\left(K^{\infty}|K\right)$ and $H = \mathrm{Gal}\left(\overline{K}|K^{\infty}\right)$. In this section we give an overview to Sen's method to study $\mathbb{C}_p$ representations without proof.

Firstly, by almost étale descent, one has

Theorem 93 ([FY, Proposition 4.18], [Ban+22, Proposition 4.2.8]). $H^1\left(H, \mathrm{GL}_d(C)\right) = 1$.

Then from the inflation-restriction sequence

$$1 \longrightarrow H^1\left(\Gamma, \mathrm{GL}_d\left(\mathbb{C}_p^H\right)\right) \longrightarrow H^1\left(G, \mathrm{GL}_d(\mathbb{C}_p)\right) \longrightarrow H^1\left(H, \mathrm{GL}_d(\mathbb{C}_p)\right)$$

it follows that

Theorem 94. *The inflation map yields a bijection*

$$H^1\left(\mathrm{Gal}\left(K\right), \mathrm{GL}_d(\mathbb{C}_p)\right) \longrightarrow H^1(\Gamma, \mathrm{GL}_d(\widehat{K^{\infty}})).$$

This reduces the study of $H^1\left(\mathrm{Gal}\left(K\right), \mathrm{GL}_d(\mathbb{C}_p)\right)$ to the study of $H^1(\Gamma, \mathrm{GL}_d(\widehat{K^{\infty}}))$.

Remark 95. Here more generally K^{∞}/K can be assumed to be a "deeply ramified" extension. For precise definition see [Ban+22].

Further by a decompletion technique, one has

Theorem 96 ([FY, Proposition 4.23]). *The inclusion* $\mathrm{GL}_d\left(K^{\infty}\right) \hookrightarrow \mathrm{GL}_d(\widehat{K^{\infty}})$ *induces a bijection*

$$H^1\left(\Gamma, \mathrm{GL}_d\left(K^{\infty}\right)\right) \longrightarrow H^1(\Gamma, \mathrm{GL}_d(\widehat{K^{\infty}})).$$

One can then show the fact that

Theorem 97 ([And+19]). *Let V be a $\mathbb{Q}_p$-linear finite dimensional representation of G_K. Then V is $\mathbb{C}_p$-admissible if and only if the inertia subgroup of G_K acts on V through a finite quotient.*

Remark 98. An alternative approach is by studying the so called Sen's operator Θ associated to the $\mathbb{C}_p$-representations, see [FY, Section 4.4].

We also record the following important cohomological fact:

Theorem 99 (Ax-Tate-Sen [FY, Proposition 4.46])**.** *One has*

1. $H^n(G_K, \mathbb{C}_p(i)) = 0$ *for* $i \neq 0$ *or* $n \geq 2$*;*

2. $H^0(G_K, \mathbb{C}_p) = K$*, and* $H^1(G_K, \mathbb{C}_p)$ *is a* 1*-dimensional* K*-vector space generated by* $\log \circ \chi = \left(G_K \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{\log} \mathbb{Z}_p \right) \in H^1(G_K, K_0)$.

4.4.4 The case for $B = B_{\mathrm{HT}}$

Definition 100. The Hodge-Tate period ring B_{HT} is defined to be

$$B_{\mathrm{HT}} = \bigoplus_{i \in \mathbb{Z}} \mathbb{C}_p(i) = \mathbb{C}_p\left[t, \frac{1}{t}\right]$$

where the element $c \otimes 1 \in \mathbb{C}_p(i) = \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ is identified by $ct^i \in \mathbb{C}_p\left[t, \frac{1}{t}\right]$.

One readily verifies that B_{HT} satisfies the conditions for a period ring.

Definition 101 (Hodge-Tate Representation)**.** A p-adic representation V of G_K is called Hodge-Tate if it is B_{HT}-admissible.

Let V be any p-adic representation, define

$$\mathbf{D}_{\mathrm{HT}}(V) := \left(B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V\right)^{G_K}.$$

The by the Fontain formalism, there is the canonical comparison map

$$\alpha_V : B_{\mathrm{HT}} \otimes_K \mathbf{D}_{\mathrm{HT}}(V) \longrightarrow B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$$

which is always injective and

$$\dim_K \mathbf{D}_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V.$$

Then V is Hodge-Tate exactly when the map is an isomorphism and the dimensions equal.

Denote $W = \mathbb{C}_p \otimes_{\mathbb{Q}_p} V$, one has

$$\begin{aligned} B_{\mathrm{HT}} \otimes_K \mathbf{D}_{\mathrm{HT}}(V) &= \left(\bigoplus_i \mathbb{C}_p(i) \right) \otimes_K \left(\bigoplus_j \mathbb{C}_p(j) \otimes_{\mathbb{Q}_p} V \right)^{G_K} \\ &= \bigoplus_{i,j} \mathbb{C}_p(i) \otimes_K \left(\mathbb{C}_p(j) \otimes_{\mathbb{Q}_p} V \right)^{G_K} \\ &= \bigoplus_{i,j} \mathbb{C}_p(i) \otimes_K \left(\mathbb{C}_p(j) \otimes_{\mathbb{C}_p} W \right)^{G_K} \end{aligned}$$

and

$$\begin{aligned} B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V &= \left(\bigoplus_i \mathbb{C}_p(i) \right) \otimes_{\mathbb{Q}_p} V \\ &= \bigoplus_i \mathbb{C}_p(i) \otimes_{\mathbb{Q}_p} V \\ &= \bigoplus_i \mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W. \end{aligned}$$

Now denote

$$W_i = \mathbb{C}_p(-i) \otimes_K \left(\mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W \right)^{G_K} \subset B_{\mathrm{HT}} \otimes_K \mathbf{D}_{\mathrm{HT}}(V)$$

Under such identifications the comparison map is given by

$$\begin{aligned} \alpha_V : \bigoplus_{i,j} \mathbb{C}_p(i) \otimes_K \left(\mathbb{C}_p(j) \otimes_{\mathbb{C}_p} W \right)^{G_K} &\longrightarrow \bigoplus_i \mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W \\ a \otimes (b \otimes w) &\longmapsto (a \cdot b) w \end{aligned}$$

where for $a \in \mathbb{C}_p(i), b \in \mathbb{C}_p(j)$ the group G_K acts on $a \cdot b$ as the element in $\mathbb{C}_p(i+j)$. Then we see that W_i's are exactly the summands that are mapped to $\mathbb{C}_p(0) \otimes_{\mathbb{C}_p} W$ on the right.

Noting that $\dim_{\mathbb{C}_p}(W_i) = \dim_K \left(\left(\mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W \right)^{G_K} \right)$, one has

$$\begin{aligned} \dim_{\mathbb{C}_p} \left(\bigoplus_i W_i \right) &= \sum_i \dim_{\mathbb{C}_p}(W_i) \\ &= \sum_i \dim_K \left(\left(\mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W \right)^{G_K} \right) \\ &= \dim_K \left(\left(\bigoplus_i \mathbb{C}_p(i) \otimes_{\mathbb{C}_p} W \right)^{G_K} \right) \\ &= \dim_K \mathbf{D}_{\mathrm{HT}}(V). \end{aligned}$$

The comparison map α_V restricts to an injective map

$$\xi_W : \left(\bigoplus_i W_i \right) \longrightarrow \mathbb{C}_p(0) \otimes_{\mathbb{C}_p} W = W$$

which when V is Hodge-Tate, the equality

$$\dim_K \mathbf{D}_{\mathrm{HT}}(V) = \dim_{\mathbb{Q}_p} V = \dim_{\mathbb{C}_p} W$$

yields that ξ_W is an isomorphism.

It is easy to verify that ξ_W being isomorphic implies V is conversely Hodge-Tate. Recall that

$$V\{\phi^n\}_K := V(\phi^n)^{G_K} \simeq \left\{ v \in V \mid g(v) = \phi(g)^{-n} v \text{ for all } g \in G_K \right\}.$$

This way one arrives at the following classical formulation of Hodge-Tate representation, which generalize to coefficients larger than $\mathbb{Q}_p$.

Definition 102. A representation W in $\mathrm{Rep}_{\mathbb{C}_p}(G_K)$ is said to be Hodge-Tate if

$$\xi_W : \bigoplus_i \left(\mathbb{C}_p(-i) \otimes_K W\{i\}_K \right) \to W$$

is an isomorphism.

Further V in $\mathrm{Rep}_L(G_K)$ is said to be Hodge-Tate if $\mathbb{C}_p \otimes_L V \in \mathrm{Rep}_{\mathbb{C}_p}(G_K)$ is Hodge-Tate. All the Hodge-Tate representations in $\mathrm{Rep}_L(G_K)$ form a full subcategory, denoted $\mathrm{Rep}_L^{\mathrm{HT}}(G_K)$.

Definition 103 (Hodge-Tate Weights and Multiplicities)**.** For any $W \in \mathrm{Rep}_{\mathbb{C}_p}^{\mathrm{HT}}(G_K)$ we define the Hodge-Tate weights of W to be those $i \in \mathbb{Z}$ such that $W\{\chi^i\}_K$ is nonzero. Also we call $h_i := \dim_K W\{i\}_K \geq 1$ the multiplicity of i as a Hodge-Tate weight of W.

One defines the Hodge-Tate weights and multiplicities for representations over other coefficients by base changing to $\mathbb{C}_p$ first.

Note that $i \in \mathbb{Z}$ is a Hodge-Tate weight of W precisely when there is an injection $\mathbb{C}_p(-i) \hookrightarrow W$ in $\mathrm{Rep}_{\mathbf{C}_p}(G_K)$. For example, $\mathbb{C}_p(i)$ has $-i$ as its unique Hodge-Tate weight.

Example 104. Let E be an elliptic curve over K, then $V_p(E) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(E)$ is a 2-dimensional Hodge-Tate representation, and

$$\dim_K \left(\mathbb{C}_p \otimes_{\mathbb{Q}_p} V_p(E)\right)^{G_K} = \dim_K \left(\mathbb{C}_p(-1) \otimes_{\mathbb{Q}_p} V_p(E)\right)^{G_K} = 1.$$

We shall give an overview for Hodge-Tate representations arising from étale cohomology of abelian varieties with good reduction in the next section.

Remark 105. The Hodge-Tate weights are the eigenvalues for Sen's operator Θ, for which being semi-simple and having such integral eigenvalues characterizes Hodge-Tate representations. See [FY, Proposition 6.5].

Remark 106. In particular, when W is Hodge-Tate $W\{\chi^i\}_K$ are all of finite K dimensions. One can apply a descent argument to show that for L a finite extension of K or $\widehat{K^{\mathrm{ur}}}$, it holds that $W \in \mathrm{Rep}_K^{\mathrm{HT}}(G_K)$ if and only if $W \in \mathrm{Rep}_L^{\mathrm{HT}}(G_K)$, and the natural functor

$$\mathrm{Rep}_{\mathbb{C}_p}(G_K) \longrightarrow \mathrm{Rep}_{\mathbb{C}_p}(G_L)$$

restricts to an equivalence between $\mathrm{Rep}_K^{\mathrm{HT}}(G_K)$ and $\mathrm{Rep}_L^{\mathrm{HT}}(G_K)$. For detail arguments see [BC09].

This again verifies the fact that $\mathbb{C}_p$-representation depends only on the Galois action restricted to open subgroup of the inertia.

4.5 p-divisible Groups and Their Hodge-Tate Representations

In this section we follow exposition from [Hon20] and [Sti12b] on the background and results of [Tat67]. We define p-divisible groups, show that they are categorically equivalent to certain Galois representation. Using structures of p-divisible groups one has that all such representation are Hodge-Tate. Further, p-divisible formal group laws induce many such representations, and provide rich applications in both arithmetic and geometry.

Definition 107 (p-divisible Group)**.** A p-divisible group G over R is an inductive system

$$G_1 \xrightarrow{i_1} \cdots \longrightarrow G_v \xrightarrow{i_v} \ldots$$

of finite flat group schemes over $\mathrm{Spec}(R)$ indexed by the natural numbers $v \in \mathbb{N}$ such that there is a natural number h, called the height such that

1. G_v has order p^{hv}, and

2. for each v we have an exact sequence

$$0 \to G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}$$

where $[p^v]$ is the multiplication by p^v map.

Morphisms between p-divisible group are morphisms between the level groups that are compatible with the inductive system.

Definition 108 (Cartier Duality)**.** Let $G = (G_v)$ be a p-divisible group over R. For each v, we have an exact sequence

$$G_{v+1} \xrightarrow{[p^v]} G_{v+1} \xrightarrow{j_v} G_v \longrightarrow 0$$

The Cartier duality yields maps

$$j_v^\vee : G_v^\vee \longrightarrow G_{v+1}^\vee$$

which can be shown to yield a p-divisible group $G^\vee = (G_v^\vee)$ over R of the same height.

Proposition 109 ([Hon20], [Sti12b])**.** *Let G be a p-divisible group of height h over R. Let d and $d^\vee$ denote the dimensions of G and $G^\vee$, respectively. Then $h = d + d^\vee$.*

Definition 110 (The p-adic Tate Module and Tate Comodule)**.** Let $G = (G_v)$ be a p-divisible group over $\mathcal{O}_K$. We define the Tate module of G by

$$T_p(G) := \varprojlim_v G_v(\overline{K}),$$

and the Tate comodule of G by

$$\Phi_p(G) := \varinjlim_v G_v(\overline{K}).$$

Proposition 111. *Let G be a p-divisible group G over $\mathcal{O}_K$, one has natural* $\mathrm{Gal}\,(K)$*-equivariant isomorphisms*

$$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^\vee\right), \mathbb{Z}_p(1)\right)$$

and

$$\Phi_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^\vee\right), \mu_{p^\infty}(\overline{K})\right)$$

by Cartier duality.

Theorem 112 ([Hon20], [Sti12b])**.** *A finite flat group scheme over R with order invertible in R is étale.*

Theorem 113 ([Hon20], [Sti12b])**.** *Assume that $R = K$ is a field with characteristic not equal to p. There is an equivalence between the categories of p-divisible groups over K and (finite free)* $\mathrm{Gal}\,(K)$ *representations over $\mathbb{Z}_p$ given by $G \rightsquigarrow T_p(G)$.*

A formal group law over R amounts to a underlying group structure on the formal scheme $\mathrm{Spf}(R\,[[X_1, \ldots, X_d]])$.

Definition 114 (p-divisible formal group laws)**.** We say that a formal group law is p-divisible if the multiplication by p map is a finite flat on the underlying formal scheme.

Definition 115. Let $G = (G_v)$ be a p-divisible group over R. We say that G is connected if each G_v is connected, and étale if each G_v is étale.

Theorem 116 (Serre-Tate [Hon20], [Sti12b])**.** *There exists an equivalence between the categories of p-divisible formal group laws over R and* ***connected*** *p-divisible groups over R.*

Definition 117 (Tangent Space)**.** Let $G = (G_v)$ be a (connected) p-divisible group over $\mathcal{O}_K$ and

$$\mathcal{O}_K\left[[X_1, \ldots, X_d]\right]$$

with augmentation ideal $I = (X_1, \ldots, X_d)$ be the $\mathcal{O}_K$-algebra representing the p-divisible formal group associated to (the connected part of) G.

Let L be a field extension of K. The tangent space of G at the unit section with values in a $\mathcal{O}_K$-algebra L is the $d = \dim(G)$ dimensional L vector space of continuous $\mathcal{O}_K$-derivations

$$\mathrm{t}_G(L) = \mathrm{Der}_{\mathcal{O}_K}\left(\mathcal{O}_K\left[[X_1, \ldots, X_d]\right], L\right) = \mathrm{Hom}_{\mathcal{O}_K}\left(I/I^2, L\right).$$

The cotangent space of G with values in the $\mathcal{O}_K$-algebra L is the $d = \dim(G)$ dimensional L vector space

$$\mathrm{t}_G^*(L) = I/I^2 \otimes_{\mathcal{O}_K} L = \mathrm{Hom}_L\left(\mathrm{t}_G(L), L\right).$$

Example 118. The p-divisible group

$$\mu_{p^\infty} = \mathbb{G}_{\mathrm{m}}\left[p^\infty\right]$$

has level groups $G_v = \mu_{p^v} = \mathbb{G}_{\mathrm{m}}\left[p^v\right]$ with transfer maps induced by the inclusions into $\mathbb{G}_{\mathrm{m}}$. The height of μ_{p^∞} is 1. The group has Cartier dual $\mathbb{Q}_p/\mathbb{Z}_p$ of height 1 and dimension 0 . The group corresponds to the p-adic cyclotomic character

$$\chi : \mathrm{Gal}\left(K\right) \longrightarrow \mathbb{Z}_p^\times.$$

Definition 119 (Formal Points on Formal Group)**.** Let $G = (G_v)$ be a p-divisible group over $\mathcal{O}_K$. For the p-adic completion L of an algebraic extension of K with p-adically completed ring of integers $\mathcal{O}_L$ the set of $\mathcal{O}_L$-valued points of G is defined to be the $\mathbb{Z}_p$-module

$$\begin{aligned} G\left(\mathcal{O}_L\right) &:= \varprojlim_i \varinjlim_v G_v\left(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L\right) \\ &= \operatorname{Hom}_{\mathcal{O}_K\text{-formal}}\left(\operatorname{Spf}\left(\mathcal{O}_L\right), \operatorname{Spf}\left(\varprojlim_v A_v\right)\right) \\ &= \operatorname{Hom}_{\mathcal{O}_K\text{-continuous}}\left(\varprojlim_v A_v, \mathcal{O}_L\right) \end{aligned}$$

where $G_v = \operatorname{Spec}\left(A_v\right)$.

Proposition 120 ([Hon20], [Sti12b])**.** *Let G be a p-divisible group over $\mathcal{O}_K$. We have an exact sequence*

$$0 \longrightarrow \Phi_p(G) \longrightarrow G\left(\mathcal{O}_{\mathbb{C}_p}\right) \xrightarrow{\log_G} \mathbb{C}_p \longrightarrow 0.$$

Example 121. For $G = \mu_{p^\infty}$ we recover the exact sequence for the p-adic logarithm.

$$0 \longrightarrow \mu_p(\overline{K}) \longrightarrow 1 + \mathfrak{m}_{\mathbb{C}_p} \xrightarrow{\log_{\mu_{p^\infty}}} \mathbb{C}_p \longrightarrow 0.$$

Proposition 122 ([Hon20], [Sti12b])**.** *Every p-divisible group G over $\mathcal{O}_K$ gives rise to a commutative diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_p}) & \xrightarrow{\log_G} & \mathrm{t}_G(\mathbb{C}_p) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}\left(T_p\left(G^\vee\right), \Phi_p\left(\mu_{p^\infty}\right)\right) & \longrightarrow & \operatorname{Hom}\left(T_p\left(G^\vee\right), \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}})\right) & \xrightarrow{\log_{\mu_{p^\infty}}} & \operatorname{Hom}\left(T_p\left(G^\vee\right), \mathrm{t}_{\mu_{p^\infty}}(\mathbb{C})\right) & \longrightarrow & 0 \end{array}$$

Proof. The first row is the lorgerithmetic exact sequence. The second row is from the lorgerithmetic exact sequence for the p-divisible group μ_{p^∞}, and noting that $T_p\left(G^\vee\right)$ is free.

Now the first vertical arrow is the aforementioned isomorphism.

To construct the last two vertical arrows, first we have that

$$\begin{aligned} T_p\left(G^\vee\right) &= \varprojlim_v G_v^\vee(\overline{K}) \\ &\cong \varprojlim_v G_v^\vee\left(\mathcal{O}_{\mathbb{C}_p}\right) \\ &= \varprojlim_v \operatorname{Hom}_{\mathcal{O}_{\mathbb{C}_p}\text{-group}}\left(\left(G_v\right)_{\mathcal{O}_{\mathbb{C}_p}}, \left(\mu_{p^v}\right)_{\mathcal{O}_{\mathbb{C}_p}}\right) \\ &= \operatorname{Hom}_{p\text{-divisible group}}\left(G \times_{\mathcal{O}_K} \mathcal{O}_{\mathbb{C}_p}, \left(\mu_{p^\infty}\right)_{\mathbb{C}_p}\right). \end{aligned}$$

The by applying the functor of $\mathcal{O}_{\mathbb{C}_p}$-valued formal points and the functor of tangent space with values in K, we have the maps

$$T_p\left(G^{\vee}\right) \longrightarrow \operatorname{Hom}\left(G\left(\mathcal{O}_{\mathbb{C}_p}\right), \mu_{p^{\infty}}\left(\mathcal{O}_{\mathbb{C}_p}\right)\right)$$

and

$$T_p\left(G^{\vee}\right) \longrightarrow \operatorname{Hom}\left(\mathrm{t}_G(\mathbb{C}_p), \mathrm{t}_{\mu_{p^{\infty}}}(\mathbb{C}_p)\right).$$

Then the two vertical arrows are maps induced by the above.

The left square is commutative by observing that the second vertical arrow restricts to the first. To see the right square is commutative it suffices to note that the logarithm map yields a natural transformation between the functor of $\mathcal{O}_{\mathbb{C}_p}$-valued formal points and the functor of tangent space with values in K. □

Theorem 123 ([Hon20], [Sti12b]). *Let G be a p-divisible group over $\mathcal{O}_K$. The restriction of the second and third vertical arrows to the* $\operatorname{Gal}(K)$*-invariant elements yield bijective maps*

$$G\left(\mathcal{O}_K\right) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_p[G_K]}\left(T_p\left(G^{\vee}\right), 1+\mathfrak{m}_{\mathbb{C}_p}\right)$$

and

$$\mathrm{t}_G(K) \longrightarrow \operatorname{Hom}_{\mathbb{Z}_p[G_K]}\left(T_p\left(G^{\vee}\right), \mathbb{C}_p\right).$$

It follows that

$$\begin{aligned}
d &= \dim_K\left(\operatorname{Hom}_{\mathbb{Z}_p[\operatorname{Gal}(K)]}\left(T_p\left(G^{\vee}\right), \mathbb{C}_p\right)\right) \\
&= \dim_K\left(\operatorname{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^{\vee}\right), \mathbb{C}_p\right)^{\operatorname{Gal}(K)}\right) \\
&= \dim_K\left(\left(\operatorname{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^{\vee}\right), \mathbb{Z}_p(1)\right) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(-1)\right)^{\operatorname{Gal}(K)}\right) \\
&= \dim_K\left(\left(T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_p(-1)\right)^{\operatorname{Gal}(K)}\right)
\end{aligned}$$

Theorem 124 ([Hon20], [Sti12b]). *Let G be a p-divisible group over $\mathcal{O}_K$. There is a canonical isomorphism of* $\operatorname{Gal}(K)$*-representations over* $\mathbb{C}_p$

$$\operatorname{Hom}_{\mathbb{Z}_p}\left(T_p(G), \mathbb{C}_p\right) \cong \mathrm{t}_{G^{\vee}}\left(\mathbb{C}_p\right) \oplus \mathrm{t}_G^*\left(\mathbb{C}_p\right)(-1).$$

Proof. From above we have natural isomorphisms

$$\mathrm{t}_G\left(\mathbb{C}_p\right)=\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^{\vee}\right),\mathbb{C}_p\right)^{\mathrm{Gal}(K)}\otimes_K\mathbb{C}_p$$

and

$$\mathrm{t}_{G^{\vee}}\left(\mathbb{C}_p\right)=\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(G\right),\mathbb{C}_p\right)^{\mathrm{Gal}(K)}\otimes_K\mathbb{C}_p.$$

By the Catier duality we have a $\mathrm{Gal}\,(K)$-equivariant perfect pairing of $\mathbb{Z}_p$-modules

$$T_p(G)\times T_p\left(G^{\vee}\right)\longrightarrow\mathbb{Z}_p(1)$$

which induces the perfect pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G),\mathbb{C}_p\right)\times\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(G^{\vee}\right),\mathbb{C}_p\right)\to\mathbb{C}_p(-1).$$

Taking $\mathrm{Gal}\,(K)$ invariants one has $\mathrm{t}_G\left(\mathbb{C}_p\right)$ and $\mathrm{t}_{G^{\vee}}\left(\mathbb{C}_p\right)$ are orthogonal complements for the pairing. Further more, one has dimension equality

$$\dim_{\mathbb{C}_p}\left(\mathrm{t}_G\left(\mathbb{C}_p\right)\right)+\dim_{\mathbb{C}_p}\left(\mathrm{t}_{G^{\vee}}\left(\mathbb{C}_p\right)\right)=\dim_{\mathbb{C}_p}\left(\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G),\mathbb{C}_p\right)\right).$$

Therefore one has an exact sequence

$$0\longrightarrow t_{G^{\vee}}\left(\mathbb{C}_p\right)\longrightarrow\mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G),\mathbb{C}_p\right)\longrightarrow t_G^*\left(\mathbb{C}_p\right)(-1)\longrightarrow 0$$

It follows from cohomological calculation that this sequence uniquely split. □

Theorem 125 ([Hon20], [Sti12b])**.** *For every p-divisible group G over $\mathcal{O}_K$, the rational Tate-module*

$$V_p(G):=T_p(G)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$$

is a Hodge-Tate p-adic representation of $\mathrm{Gal}\,(K)$.

Proof. As the $\mathbb{C}_p$-duals of $\mathrm{t}_{G^{\vee}}\left(\mathbb{C}_p\right)$ and $\mathrm{t}_G^*\left(\mathbb{C}_p\right)$ are respectively $\mathrm{t}_{G^{\vee}}^*\left(\mathbb{C}_p\right)$ and $\mathrm{t}_G\left(\mathbb{C}_p\right)$, the above isomorphism yields a decomposition

$$V_p(G)\otimes_{\mathbb{Q}_p}\mathbb{C}_p\cong\mathrm{t}_{G^{\vee}}^*\left(\mathbb{C}_p\right)\oplus\mathrm{t}_G\left(\mathbb{C}_p\right)(1)$$

Then for each n we have

$$\left(V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(n)\right)^{\mathrm{Gal}(K)} \cong \begin{cases} \mathrm{t}^*_{G^\vee}\left(\mathbb{C}_p\right) & \text{if } n = 0 \\ \mathrm{t}_G\left(\mathbb{C}_p\right) & \text{if } n = -1 \\ 0 & \text{otherwise} \end{cases}$$

□

Proposition 126 ([Hon20], [Sti12b]). *Let A be an abelian variety over K with good reduction. Then we have a canonical Γ_K-equivariant isomorphism*

$$H^n_{et}\left(A_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i\left(A, \Omega^j_{A/K}\right) \otimes_K \mathbb{C}_p(-j).$$

Proof. Let $A^\vee$ denote the dual abelian variety of A. Since A has good reduction, there exists an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ with $\mathcal{A}_K \cong A$. Then we have $T_p\left(\mathcal{A}\left[p^\infty\right]\right) = T_p\left(A\left[p^\infty\right]\right)$ by definition, and $\mathcal{A}^\vee\left[p^\infty\right] \cong \mathcal{A}\left[p^\infty\right]^\vee$. In addition, we have the following standard facts:

1. There is a canonical isomorphism

$$H^1_{\mathrm{et}}\left(A_{\overline{K}}, \mathbb{Q}_p\right) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p\left(A\left[p^\infty\right]\right), \mathbb{Z}_p\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

2. The formal completion of $\mathcal{A}$ along the unit section gives rise to the same formal group law as from the p-divisible group $\mathcal{A}\left[p^\infty\right]$

3. There are canonical isomorphisms

$$H^0\left(A, \Omega^1_{A/K}\right) \cong \mathrm{t}^*_e(A)$$

and

$$H^1\left(A, \mathcal{O}_A\right) \cong \mathrm{t}_e\left(A^\vee\right)$$

where $\mathrm{t}^*_e(A)$ and $\mathrm{t}_e(A)$ respectively denote the cotangent space of A and tangent space of $A^\vee$ (at the unit section).

4. We have identifications

$$H^n\left(A_{\overline{K}}, \mathbb{Q}_p\right) \cong \bigwedge^n H^1\left(A_{\overline{K}}, \mathbb{Q}_p\right)$$

and

$$H^i\left(A, \Omega^j_{A/K}\right) \cong \bigwedge^i H^1\left(A, \mathcal{O}_A\right) \otimes \bigwedge^j H^0\left(A, \Omega^1_{A/K}\right).$$

The statements (2) and (3) together yield identifications

$$H^0\left(A, \Omega^1_{A/K}\right) \cong \mathrm{t}^*_{\mathcal{A}[p^\infty]}(K)$$

and

$$H^1\left(A, \mathcal{O}_A\right) \cong \mathrm{t}_{\mathcal{A}^\vee[p^\infty]}(K).$$

Hence there is a canonical $\mathrm{Gal}\,(K)$-equivariant isomorphism

$$H^1_{\mathrm{et}}\left(A_{\overline{K}}, \mathbb{Q}_p\right) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \left(H^1\left(A, \mathcal{O}_A\right) \otimes_K \mathbb{C}_p\right) \oplus \left(H^0\left(A, \Omega^1_{A/K}\right) \otimes_K \mathbb{C}_p(-1)\right).$$

One then obtain the desired isomorphism. □

4.6 Lubin-Tate Character and its Hodge-Tate Representation

Let E, k be finite extensions of $\mathbb{Q}_p$ such that k contain the Galois closure of E. Let $\phi : G_k \longrightarrow E^\times$ be a character. In this section we study the representation $E(\phi) \in \mathrm{Rep}_{\mathbb{Q}_p}(G_k)$, and show that in the case where $\phi = \chi_E^{\mathrm{LT}}$, we have $E(\chi_E^{\mathrm{LT}}) \in \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_k)$.

Let $\mathrm{Hom}_{\mathbb{Q}_p}(E, k)$ be the set of field embeddings. In particular it is a finite set of size $n = [E : \mathbb{Q}_p]$. Then $E \otimes_{\mathbb{Q}_p} \mathbb{C}_p$, which is the underlying n dimensional $\mathbb{C}_p$ vector space of $E(\phi)_{\mathbb{Q}_p}\mathbb{C}_p \in \mathrm{Rep}_{\mathbb{C}_p}(G_k)$ is identified by the following isomorphism

$$\begin{aligned} E \otimes_{\mathbb{Q}_p} \mathbb{C}_p &\longrightarrow \bigoplus_{\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E,k)} \mathbb{C}_p. \\ e \otimes c &\longmapsto (c \cdot \sigma(e))_{\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E,k)} \end{aligned}$$

Now G_k acts on both of $E(\phi)$ and $\mathbb{C}_p$. One see that the above isomorphism is actually G_k equivariant, and thus in $\mathrm{Rep}_{\mathbb{C}_p}(G_k)$

$$E(\phi) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \longrightarrow \bigoplus_{\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E,k)} \mathbb{C}_p(\sigma \circ \phi)$$

is an isomorphism.

Further more, from the E action one can recover the summands as following: Let $W = E(\phi) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. (Note that E acts on the factor $E(\phi)$ on the left while $\mathbb{C}_p$ acts on the factor $\mathbb{C}_p$ on the right.)

For all $x \in E$ define

$$\begin{aligned} a_x : W &\longrightarrow W \\ e \otimes c &\longmapsto (x \cdot e) \otimes c \end{aligned}$$

to be the E action of x.

Then

$$W_\sigma = \{w \in W \mid a_x(w) = \sigma(x) \cdot w \text{ for all } x \in E\}$$

is exactly the summand $\mathbb{C}_p(\sigma \circ \phi)$ under the above isomorphism.

Proposition 127. *Let* $\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E, k)$

- *If* σ *is the identity embedding* id*, one has*

$$\mathrm{id} \circ \chi_E^{\mathrm{LT}} = \chi_E^{\mathrm{LT}} \sim \chi$$

- *On the other hand, if* σ *is not the identity embedding, one has*

$$\sigma \circ \chi_E^{\mathrm{LT}} \sim 1$$

Proof. Let π be a uniformizer of E. Let T be the Tate module associated to the Lubin-Tate p-divisible formal group for π. Recall that $T \simeq \mathcal{O}_E$ as a module over $\mathbb{Z}_p$ (and in fact is an $\mathcal{O}_E$ module). As a $\mathrm{Gal}(E)$ representation one has

$$T \simeq \mathcal{O}_E(\chi_{E,\pi}^{\mathrm{LT}}).$$

Now consider the $\mathbb{C}_p$ representation

$$W = T \otimes_{\mathbb{Z}_p} \mathbb{C}_p.$$

As is shown in the last section, W is Hodge-Tate, with decomposition

$$W = W_0 \oplus W_1.$$

Recall that the p-divisible Lubin-Tate formal group is of height n with dimension 1. It then implies as G_E representations over $\mathbb{C}_p$

$$W_0 \simeq \mathbb{C}_p^{n-1}$$

and

$$W_1 \simeq \mathbb{C}_p(1).$$

Now instead consider W as a $\mathrm{Gal}\,(k)$ representation over $\mathbb{C}_p$. By the above discussion we have

$$W \simeq \bigoplus_{\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E,k)} \mathbb{C}_p(\sigma \circ \chi_E^{\mathrm{LT}}) = \bigoplus_{\sigma \in \mathrm{Hom}_{\mathbb{Q}_p}(E,k)} W_\sigma$$

Note that by descent W is still Hodge-Tate with the same decomposition $W = W_0 \oplus W_1$. Now recall the E action on $W = E(\chi_{E,\pi}^{\mathrm{LT}}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ is visible from the Hodge-Tate decomposition

$$W = \left(\mathrm{t}_G^*(K) \otimes_K \mathbb{C}_p(-1)\right) \oplus \left(\mathrm{t}_{G^D}(K) \otimes_K \mathbb{C}_p\right).$$

In particular,

$$W_{\mathrm{id}} = \left(\mathrm{t}_G^*(K) \otimes_K \mathbb{C}_p(-1)\right) = W_1.$$

Thus

$$\mathbb{C}_p(\chi_E^{\mathrm{LT}}) \simeq \mathbb{C}_p(1)$$

and

$$\chi_E^{\mathrm{LT}} \sim \chi$$

as claimed.

Now for $\sigma \neq \mathrm{id}$, W_σ corresponds to the other summands contained in $W_0 \simeq \mathbb{C}_p^{n-1}$. Thus

$$W_\sigma \simeq \mathbb{C}_p$$

and

$$\sigma \circ \chi_E^{\mathrm{LT}} \sim 1$$

as claimed. □

Proposition 128. *Now assume E is Galois over $\mathbb{Q}_p$. Denote $\sigma_\circ : E \hookrightarrow k$ the embedding. Let $\phi : G_k \longrightarrow E^\times$ be a character. Assuming $E(\phi)$ is a Hodge-Tate G_k representation over $\mathbb{Q}_p$, one has that ϕ is inertially equivalent to the following character*

$$\prod_{\sigma \in \mathrm{Gal}(E/\mathbb{Q}_p)} \left(\chi_{\sigma_\circ \circ \sigma}^{\mathrm{LT}}\right)^{n_\sigma} : G_k \longrightarrow E^\times$$

for some $n_\sigma \in \mathbb{Z}$.

Proof. Let $W = E(\phi) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. For $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q}_p)$ define

$$W_\sigma = \{w \in W \mid a_x(w) = \sigma(x) \cdot w \text{ for all } x \in E\}$$

as above, whereby W_σ is exactly a summand $\mathbb{C}_p(\sigma \circ \phi)$ of W, which by Hodge-Tate assumption is isomorphic to $\mathbb{C}_p(n_\sigma)$ for some $n_\sigma \in \mathbb{Z}$. Namely, we have

$$\sigma \circ \phi \sim \chi^{n_\sigma}$$

for all $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q}_p)$.

On the other hand,

$$\sigma \circ \left(\prod_{\sigma' \in \mathrm{Gal}(E/\mathbb{Q}_p)} \left(\chi_{\sigma_\circ \circ \sigma'}^{\mathrm{LT}}\right)^{n_{\sigma'}} \right) = \prod_{\sigma' \in \mathrm{Gal}(E/\mathbb{Q}_p)} \left(\sigma \circ \chi_{\sigma_\circ \circ \sigma'}^{\mathrm{LT}}\right)^{n_{\sigma'}} \sim \chi^{n_\sigma}$$

for all $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q}_p)$ from discussion on χ_E^{LT}.

Then the proposition follows from the following cohomological calculation result using Ax-Sen-Tate theorem.

Lemma 129 ([Ser97, Porposition 3 on page III-38]). *The following are equivalent:*

1. $\sigma \circ \phi_1 \sim \sigma \circ \phi_2$ *for all* $\sigma \in \mathrm{Gal}\,(E/\mathbb{Q}_p)$;

2. $\phi_1 \sim \phi_2$.

$\square$

Chapter 5

Proof of the Main Theorem

Recall our main theorem 17:

Theorem 130 (Main Theorem)**.** *Let K, L be mixed characteristic local fields and $\alpha : \mathrm{Gal}(L) \to \mathrm{Gal}(K)$ be an isomorphism between the Galois groups. The following are equivalent:*

1. *There exists an isomorphism of fields $\beta : K \to L$ such that $\alpha = \pi_1(\beta)$;*

2. *The isomorphism α preserves the ramification filtration;*

3. *The isomorphism α is HT-preserving.*

We have shown the implication (1) $\implies$ (2) $\implies$ (3) in theorem 58.

For this chapter let $k_\circ$ and $k_\bullet$ be two mixed characteristic local fields. Denote $G_{k_\circ}$ and $G_{k_\bullet}$ their absolute Galois groups. We shall show the final implication (3) $\implies$ (1), namely any continuous group isomorphism $G_{k_\circ} \to G_{k_\bullet}$ that is Hodge-Tate preserving is induced by a field isomorphism $k_\bullet \to k_\circ$ in theorem 138.

Definition 131. (Inertial Compatibility) Let

$$\alpha : G_{k_\circ} \longrightarrow G_{k_\bullet}$$

be a continuous group isomorphism and

$$\beta : k_\bullet \longrightarrow k_\circ$$

a field isomorphism. We say that α and β are inertially compatible if there exists an open subgroup $S_\circ \subset G_{k_\circ}$ whose isomorphic images commute the square in the following diagram

$$\begin{array}{ccc} \widehat{k_\bullet^\times} & \xrightarrow{\widehat{\beta}} & \widehat{k_\circ^\times} \\ \wr\uparrow & & \wr\uparrow \\ G_{k_\bullet}^{\text{ab}} & \xleftarrow[\alpha^{\text{ab}}]{} & G_{k_\circ}^{\text{ab}} \end{array}$$

where α^{ab} is the isomorphism induce by α on the abelianizations, and $\widehat{\beta}$ is the isomorphism induce by β on the completions.

Lemma 132. *Let $S \subseteq \mathcal{O}_k^\times$ be an open subgroup, then the $\mathbf{Q}_p$-vector space generated by S in k is equal to k.*

Proof. Given $S \subseteq \mathcal{O}_k^\times$ is open, it contains some higher unit group $U_k^n = 1 + \mathfrak{m}_k^n$. Then the $\mathbf{Q}_p$-vector space generated by S contains $\mathfrak{m}_k^n$. Assume that $[k : \mathbb{Q}_p] = d$. Now $\mathfrak{m}_k^n$ is isomorphic to $\mathcal{O}_k$ as $\mathbb{Z}_p$ module with rank d. It follows that the $\mathbb{Q}_p$ subspace it generates has dimension d and thus is equal to k. ☐

It immediately follows that

Proposition 133. *Let α be as above, then there exists at most one such β such that they are inertially compatible. Further more, if such a β exists, it induces α by the faithfulness of the functor* $\operatorname{Gal}(-)$, *and the above isomorphism diagram commutes.*

Lemma 134. *Let $\alpha : G_{k_\circ} \to G_{k_\bullet}$ be an open continuous isomomorphism. The following are equivalent:*

1. *α is HT-preserving.*
2. *For every pair of respective finite extensions $k'_\circ \subset \overline{k_\circ}$, $k'_\bullet \subset \overline{k_\bullet}$ of $k_\circ$, $k_\bullet$ such that $\alpha(G_{k'_\circ}) \subset G_{k'_\bullet}$, the restriction $\alpha|_{G_{k'_\circ}} : G_{k'_\circ} \to G_{k'_\bullet}$ is HT-preserving.*

Proof. This essentially follows from the fact that Hodge-Tate representations are preserved when base changed to finite extensions. ☐

In particular, noting that Lubin-Tate characters are Hodge-Tate, the above lemma yields that

- For every pair of respective finite extensions $k'_\circ \subset \overline{k_\circ}$, $k'_\bullet \subset \overline{k_\bullet}$ of $k_\circ$, $k_\bullet$ such that $\alpha\left(G_{k'_\circ}\right) \subset G_{k'_\bullet}$,
- for every finite Galois extension E of $\mathbb{Q}_p$ that admits a pair of embeddings $\sigma_\circ : E \hookrightarrow k'_\circ, \sigma_\bullet : E \hookrightarrow k'_\bullet$,

the composite

$$G_{k'_\circ} \overset{\alpha|_{G_{k'_0}}}{\longrightarrow} G_{k'_\bullet} \overset{\chi^{\mathrm{LT}}_{\sigma_\bullet}}{\longrightarrow} E^\times$$

is Hodge-Tate.

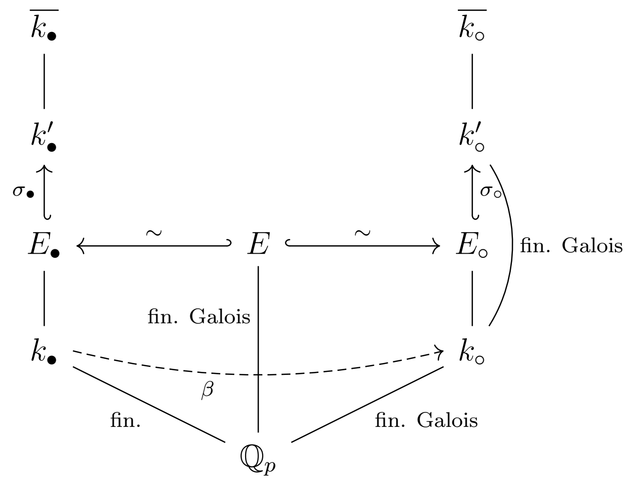
First we show for the special case:

Proposition 135. *Let $\alpha : G_{k_\circ} \to G_{k_\bullet}$ be an open continuous isomorphism. Let $k_\circ$ be Galois over $\mathbb{Q}_p$. Then there exists an isomorphism of fields $\beta : k_\bullet \to k_\circ$ that is inertially compatible with α.*

Proof. Let E be an finite Galois extension of $\mathbb{Q}_p$ that embeds into both $\overline{k_\bullet}$ and $\overline{k_\circ}$. Denote the images as $E_\bullet$ and $E_\circ$ respectively and further assume we have $k_\bullet \subset E_\bullet$ and $k_\circ \subset E_\circ$.

Let $k'_\circ$ be some finite Galois extension of $k_\circ$ that contains $E_\circ$. Then the corresponding Galois group $\mathrm{Gal}\left(\overline{k_\circ}|k'_\circ\right)$ is mapped to a open sub-group $\alpha(\mathrm{Gal}\left(\overline{k_\circ}|k'_\circ\right)) \subset \mathrm{Gal}\left(\overline{k_\bullet}|k_\bullet\right)$ and thus induces a sub-field $k'_\bullet \subset \overline{k_\bullet}$. by enlarging $k'_\circ$ if necessary, we can further assume that then $E_\bullet$ is a sub-field of $k'_\bullet$. Denote the embeddings as $\sigma_\circ : E_\circ \hookrightarrow k'_\circ$ and $\sigma_\bullet : E_\bullet \hookrightarrow k'_\bullet$.

In summary, we have the following diagram:



Now consider the equivalence class of characters from the following composite

$$G_{k'_\circ} \xrightarrow{\alpha|_{G_{k'_\circ}}} G_{k'_\bullet} \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} E_\bullet^\times \xleftarrow{\sim} E^\times \xrightarrow{\sim} E_\circ^\times.$$

Given α is HT-preserving, the composite is also Hodge-Tate. Further, since $E_\circ$ is Galois over $\mathbb{Q}_p$, we also have that the composite inertially equivalent to

$$\prod_{\sigma \in \mathrm{Gal}(E_\circ|\mathbf{Q}_p)} \left(\chi^{\mathrm{LT}}_{\sigma_\circ \circ \sigma}\right)^{n_\sigma} : G_{k'_\circ} \longrightarrow E_\circ^\times$$

for some integers n_σ by proposition 128.

To get inertial equivalence classes of characters on $G_{k_\bullet}$ and $G_{k_\circ}$, we need further pullback the characters by the functorial transfer maps $\mathrm{Ver}_{k'_\circ/k_\circ}$ and $\mathrm{Ver}_{k'_\bullet/k_\bullet}$ respectively. Note that the transfer map sends inertial subgroups to inertial subgroups, and thus it makes sense to pullback inertial equivalence class with them.

Explicitly then there is an open subgroup $S_\circ$ in the inertial subgroup $I_{k_\circ} \subset G_{k_\circ}$ on which our two representations of the same equivalent class agree. Denote its isomorphic image in the inertial subgroup $I_{k_\bullet} \subset G_{k_\bullet}$ under α by $S_\bullet$. We then have the following diagram where the outer square commutes:

$$\begin{array}{ccccccccc}
S_\circ & \hookrightarrow & I_{k_\circ} & \hookrightarrow & G_{k_\circ} & \xrightarrow{\mathrm{Ver}_{k'_\circ/k_\circ}} & G_{k'_\circ} & \xrightarrow{\prod_{\sigma\in\mathrm{Gal}(E_\circ|\mathbf{Q}_p)}\left(\chi^{\mathrm{LT}}_{\sigma_\circ\circ\sigma}\right)^{n_\sigma}} & E_\circ^\times \\
\downarrow{\scriptstyle \alpha|_{S_\circ}} & & \downarrow{\scriptstyle \alpha|_{I_{k_\circ}}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha|_{G_{k'_\circ}}} & & \uparrow\wr \\
 & & & & & & & & E^\times \\
 & & & & & & & & \downarrow\wr \\
S_\bullet & \hookrightarrow & I_{k_\bullet} & \hookrightarrow & G_{k_\bullet} & \xrightarrow{\mathrm{Ver}_{k'_\bullet/k_\bullet}} & G_{k'_\bullet} & \xrightarrow{\chi^{\mathrm{LT}}_{\sigma_\bullet}} & E_\bullet^\times
\end{array}$$

Now abelianize the diagram and identify the groups under the functorial reciprocity map. By abusing notations we denote also $S_\circ$ and $S_\bullet$ the open subgroups in $\mathcal{O}_\circ^\times$ and $\mathcal{O}_\bullet^\times$ the corresponding images of the reciprocity map. We have the following diagram where the outer square commutes:

$$\begin{array}{ccccccccccc}
S_\circ & \hookrightarrow & \mathcal{O}_\circ^\times & \hookrightarrow & \hat{k_\circ}^\times & \hookrightarrow & \hat{k'_\circ}^\times & \xrightarrow{\prod_{\sigma\in\mathrm{Gal}(E_\circ|\mathbf{Q}_p)}\left(\sigma^{-1}\circ\mathrm{N}_{k'_\circ/E_\circ}\right)^{n_\sigma}} & E_\circ^\times & \xrightarrow{\cdot\mapsto\cdot^{-1}} & E_\circ^\times \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \uparrow\wr \\
 & & & & & & & & & & E^\times \\
 & & & & & & & & & & \downarrow\wr \\
S_\bullet & \hookrightarrow & \mathcal{O}_\bullet^\times & \hookrightarrow & \hat{k_\bullet}^\times & \hookrightarrow & \hat{k'_\bullet}^\times & \xrightarrow{\mathrm{N}_{k'_\bullet/E_\bullet}} & E_\bullet^\times & \xrightarrow{\cdot\mapsto\cdot^{-1}} & E_\bullet^\times
\end{array}$$

Let $\mathrm{Im}(S_\circ)$ and $\mathrm{Im}(S_\bullet)$ denote the images in rightmost $E_\circ^\times$ and $E_\bullet^\times$ respectively.

Lemma 136. $\mathrm{Im}(S_\circ) \subset k_\circ^\times$

Proof. Let $s \in S_\circ$, we have

$$\begin{aligned}
\mathrm{Im}(s)^{-1} &= \prod_{\sigma\in\mathrm{Gal}(E_\circ|\mathbb{Q}_p)} \left(\sigma^{-1}\circ\mathrm{N}_{k'_\circ/E_\circ}\right)^{n_\sigma}(s) \\
&= \prod_{\sigma\in\mathrm{Gal}(E_\circ|\mathbb{Q}_p)} \sigma^{-1}\left(s^{n_\sigma\cdot[k'_\circ:E_\circ]}\right) \\
&\in k_\circ^\times
\end{aligned}$$

given $k_\circ$ is Galois over $\mathbb{Q}_p$ and thus every $\sigma \in \mathrm{Gal}\left(E_\circ|\mathbb{Q}_p\right)$ preserves $k_\circ$. $\square$

Lemma 137. $\mathrm{Im}(S_\bullet) \subset k_\bullet^\times$ *and it is open in* $\mathcal{O}_\bullet^\times$

Proof. Let $s \in S_\bullet$, we have

$$\begin{aligned} \operatorname{Im}(s)^{-1} &= \mathrm{N}_{k'_\bullet/E_\bullet}(s) \\ &= s^{[k'_\bullet:E_\bullet]}. \\ &\in k_\bullet^\times \end{aligned}$$

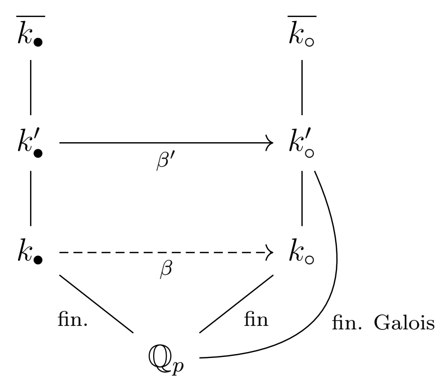
Thus $\operatorname{Im}(S_\bullet) = S_\bullet^{[k'_\bullet:E_\bullet]}$ is open. □

Now let β be the $\mathbb{Q}_p$ linear map induced by the above isomorphism between $\operatorname{Im}(S_\bullet)$ and $\operatorname{Im}(S_\circ)$. By the discussion at the beginning of the chapter we have that β is an isomorphism of fields $k_\bullet \longrightarrow k_\circ$ that is by definition inertially compatible with α. □

We can now drop the Galois condition and finally show for the final implication in our main theorem 17.

Theorem 138. *Let $\alpha : G_{k_\circ} \to G_{k_\bullet}$ be an open continuous isomorphism. Then there exists an isomorphism of fields $\beta : k_\bullet \to k_\circ$ that induces the isomorphism α on Galois groups.*

Proof.



Let $k'_\circ \subset \overline{k_\circ}$ be a finite extension of $k'_\circ$ that is Galois over $\mathbb{Q}_p$. Denote $k'_\bullet \subset \overline{k_\bullet}$ the finite Galois extension of $k'_\bullet$ corresponding to the open subgroup $\alpha\left(G_{k'_\circ}\right) \subset G_{k_\bullet}$. Then it follows from above that there exists an isomorphism of fields $\beta' : k'_\bullet \xrightarrow{\sim} k'_\circ$ that induces the map on Galois groups $\alpha|_{G_{k'_\circ}} : G_{k'_\circ} \to G_{k'_\bullet}$. Then β' restricts to an isomorphism

$$\beta : k_\bullet \longrightarrow k_\circ$$

that induces α and we are done. $\square$

Chapter 6

Future Work

The fields $\mathbb{Q}_p$ and $\mathbb{F}_p((T))$ shares striking similarities: they are both equipped with a discrete valuation, together with a multiplicative section from the multiplicative group of residue field to the ring of integers, namely the Teichmuller lifts. This allows one to write elements in their rings of integer as power series in the uniformizers p and T respectively, with coefficients from the group of Teichmuller lifts.

In the characteristic p case one immediately recovers $\mathbb{F}_p[[T]]$ from the power series representation, essentially due to the multiplicative Teichmuller lifts also respect the additive structure. On the other hand, the ring structure of $\mathbb{Z}_p$ is not obvious from the power series representation, as addition involves "carrying" when the coefficients of powers of p interacts.

Nevertheless, the coefficients of sums and products can be worked out explicitly. That is, when one represent elements of $\mathbb{Z}_p$ by a series of elements in the residue field via the power series representation and Teichmuller lifts, the arithmetic of $\mathbb{Z}_p$ itself can be determined entirely by the arithmetic of the residue field, together with a fixed set of inductive rules. This is in general the construction of p-typical Witt vectors, which in fact yields an equivalence of category between perfect $\mathbb{F}_p$-algebras and the so called strict $p$rings which are p-adically complete, p-torsionless with perfect $\mathbb{F}_p$-algebra as mod p quotient.

In particular, given a characteristic p perfect field k, the Witt vectors $W(k)$ is a characteristic 0 discretely valued complete ring with p as a uniformizer. By taking fraction field, one has an equivalence between characteristic p perfect fields and characteristic 0 discretely valued complete fields with p as uniformizers.

6.1 Fontaine-Wintenberger Theorem and the Fields of Norm

However, one shall note that the above algebraic comparison between $\mathbb{Q}_p$ and $\mathbb{F}_p((T))$ only goes so far. In particular, it can be shown that the cohomological dimension of the absolute Galois group of $\mathbb{Q}_p$ is 2, and that of $\mathbb{F}_p((T))$ is 1. Geometrically this suggests that the étale site of the spectrum of $\mathbb{Q}_p$ is bigger than that of $\mathbb{F}_p((T))$. Evidently the above equivalence using Witt vectors allows for only unramified extensions.

A deeper comparison is achieved by Fontaine and Wintenberger in the study of so called arithmetically profinite extensions, essentially extensions that have finite indexed ramification filtration.

Example 139. Typical examples of arithmetically profinite extensions are totally ramified $\mathbb{Z}_p$ extensions of mixed characteristic local fields. In particular, infinite arithmetically profinite extensions are deeply ramified.

Theorem 140 (Field of Norms)**.** *Let L/F be an infinite arithmetically profinite extension. Define*

$$X(L/K) = \varprojlim_{F \subset E \subset L} E^{\times} \cup \{0\}$$

where the limit is taken with respect to the norm maps. Let $\alpha = (\alpha_E)_E$ and $\beta = (\alpha_E)_E$ be elements in $X(L/K)$. Define addition and multiplication by the following:

$$(\alpha\beta)_E := \alpha_E \beta_E$$

$$(\alpha + \beta)_E := \lim_{E \subset E' \subset L} \mathrm{N}_{E'/E} \left(\alpha_{E'} + \beta_{E'}\right).$$

Then $\alpha\beta := ((\alpha\beta)_E)_E$ and $\alpha+\beta := ((\alpha+\beta)_E)_E$ are well-defined elements of $X(L/K)$. With addition and multiplication defined as above $X(L/K)$ is a field of characteristic p.

Let L_0 be the maximal unramified subextension of L/K. Define the valuation on $X(L/K)$ by

$$v(\alpha) = v_E(\alpha_E)$$

for any $L_0 \subset E \subset L$. Then $X(L/K)$ is furthermore a local field, with residue field canonically isomorphic to k_L, the residue field of L.

For any $\xi \in k_L$, let $[\xi]$ denote its Teichmuller lift. For each $L_0 \subset E \subset L$ set:

$$\xi_E := [\xi]^{1/[E:K_1]}$$

Then the map

$$k_L \longrightarrow X(L/K)$$

$$\xi \longmapsto (\xi_E)_E$$

is a canonical embedding.

In particular, by noting that the multiplicative groups of extensions of a fixed mixed characteristic local field are mono-anabelian, together with the norm map corresponding to the natural inclusion on subgroups, one has that the multiplicative group of the field of norm is mono-anabelian as well.

Theorem 141 (Fontaine-Wintenberger)**.** *Let L/K be an arithemtically profinite extension, and M/L be a finite extension. Then $X(M/K)/X(L/K)$ is a separable extension of degree $[M : L]$. If M/L is Galois, then the natural action of* $\mathrm{Gal}(M/L)$ *on $X(M/K)$ induces an isomorphism*

$$\mathrm{Gal}(M/L) \simeq \mathrm{Gal}(X(M/K)/X(L/K)).$$

Furthermore, taking field of norms establishes a canonical one-to-one correspondence between finite extensions of L and finite separable extensions of $X(L/K)$, which is compatible with the Galois correspondence. Namely, the spectra of L and $X(L/K)$ has isomorphic étale sites.

In particular, the extension $\mathbb{Q}_p(p^{1/p^\infty})/\mathbb{Q}_p$ is arithmetically profinite. The corresponding characteristic p field of norm is $X(\mathbb{Q}_p(p^{1/p^\infty})/\mathbb{Q}_p) = \mathbb{F}_p((t^{1/p^\infty}))$.

6.2 Perfectoid Fields and Their Tilts

Recall that the results of Fontaine-Wintenberger says that for any arithmetically profinite algebraic extension K of $\mathbb{Q}_p$, the Galois group of K and its norm field are isomorphic. The theory of perfectoid fields generalize and clarify the phenomena by defining a new class of so called perfectoid fields. Then given a perfectoid field of characteristic 0, tilting yields a correspondence between the étale site of K and the étale site of its characteristic p tilt $K^\flat$. For a arithmetically profinite extension K, the completion of K is perfectoid, with its tilt isomorphic to the completed perfect closure of its field of norm. This way one recovers the classical correspondence.

In particular, one has $\mathbb{Q}_p(\mu_{p^\infty})$ and its completion have the same Galois group; on the other hand, $\mathbb{F}_p((t))$, its perfect closure, and the completion of its perfect closure all have the same Galois group.

Furthermore, there exists a geometric space of moduli of untilts called the Fargue Fontaine curve, which has the same fundamental group as the spectra of $\mathbb{Q}_p$ and is shown to be $K(\pi, 1)$. It puts many interesting results in p-adic Hodge theory into a geometric context.

In light of the above modern developments, it would be natural to further study them from an anabelian perspective.

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