University of Alberta

COVARIANCE ANALYSIS, CONTROL AND FAULT DETECTION

by

Xin Huang ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

in

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Covariance Analysis, Control and Fault Detection** submitted by Xin Huang in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in *Process Control.*

Steven Ding

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"A journey of a thousand miles must begin with a single step"

-Lao-Tzu

To

My Family

Abstract

Chemical processes are often multivariate and stochastic due to interaction of variables and randomness of disturbances. Increasing global competition of the modern chemical industry demands better control of process variability in order to increase product quality consistency and to reduce energy consumption.

The optimal linear quadratic control subject to the generalized covariance constraints (*LQGCC*) for both discrete-time systems and continuous-time systems is first considered in the thesis. It is shown that LQGCC for both cases can be solved via semi-definite programming methods. Complete LQGCC is typically high order and its implementation is nontrivial. On the other hand, PID controllers have simple structure and they are widely used in chemical industry. Controlling the process variation through PID controllers is practically attractive. Multiloop PID controller design methods are proposed to control the process variation for both discrete-time systems and continuous-time systems. The necessary and sufficient conditions for the existence of such a multiloop PID controller are derived and computational algorithms are proposed.

In addition to process variability control, fault detection and isolation (*FDI*) is a key to maintaining process operation and sustaining benefit of advanced control. A novel observer based fault detection and isolation scheme, in which the trace of the covariance matrix of the estimation error is minimized to enhance the sensitivity of the detection scheme, is proposed for discrete-time stochastic systems. As an alternative to the observer based fault detection method, a new recursive PCA based process monitoring approach is also explored and applied in industry.

A filter design method for sampled-data systems is proposed to restrict the covariance matrix of the estimation error within sampling intervals. A necessary and sufficient condition

is derived and a computational algorithm is proposed. The advantage of this filter design method is that the inter-sample behavior is considered while most of other schemes only focus on the sampling instant behavior.

Descriptor systems are more suitable to represent large scale systems than regular systems. Certain control problems of descriptor systems are similar to covariance control problems of regular systems. The control of descriptor systems is therefore also explored.

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NOMENCLATURE

Notation	Description
R	Real number
R^n	n dimensional vector
$Re(s)$	real part of complex number s
$x \in A$	x belongs to set A
A^{ij}	the element at the $i-th$ row and $j-th$ column of matrix A
$\|A\|_F$	the Frobenius norm of matrix A
A^T	transpose of matrix A
A^{-1}	inverse of matrix A
A^+	Penrose-Moore inverse of matrix A
$A^{\perp}$	left inverse of matrix A satisfying $A^{\perp}A^{\perp T} > 0$
$A^{\frac{1}{2}}$	square root of semidefinite positive matrix A
$tr(A)$	trace of matrix A
$A > B$	$A - B$ is positive definite
$A \geq B$	$A - B$ is positive semidefinite
$diag(A_1, \cdots, A_m)$	a block diagonal matrix with diagonal term A_i
$\Lambda(E, A)$	generalized eigenvalue set of matrix pair (E, A)
$deg(sE - A)$	order of $\|sE - A\|$
$E(x)$	expectation value of random variable x
$\sigma(x)$	variance of random variable x
$x \sim \mathcal{N}(\mu, \Sigma)$	Gaussian distributed vector x with mean μ and covariance Σ
discrete-time $\delta(t)$	$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases}$
continuous-time $\delta(t)$	$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$ and $\delta(t) = 0$ for any $t \neq 0$
null(A)	null space of matrix A
$dim(\star)$	the dimension of linear space $\star$
$\|T_{rf}\|_2$	H_2 norm of the transfer function T_{rf}

Part I

Introduction

1
Introduction

The covariance matrix of the process variables is the core of the thesis and several different areas related to it are explored:

1. the output covariance control and optimization.
2. output covariance control subject to controller structure constraints.
3. fault detection and isolation (*FDI*).
4. process performance monitoring based on covariance analysis.
5. filter design for sampled-data systems.
6. static output feedback control of descriptor systems.

The structure of the thesis is shown in Figure 1.1.

1.1 Covariance control

1.1.1 Variance control - scalar control objective

Variance control is one of the most important subjects for process industries as variance typically represents product quality consistency (Shunta 1995). In any manufacturing process,

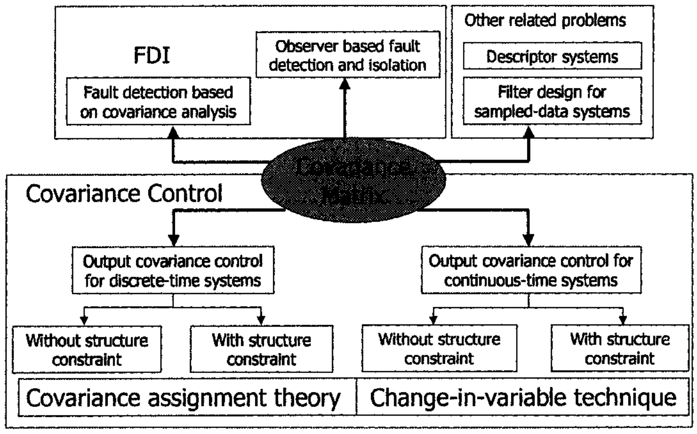

Figure 1.1: Research structure

there exist some variables that characterize the product quality. Reducing the variances of these variables means improving product quality consistency. Many different variance control strategies have been proposed in the literature. The minimum variance control (*MVC*) law, which minimizes the output variance, was derived by Ästrom (1970) for single-input-single-output (*SISO*) systems. MVC provides an absolute lower bound of variance that can be used to measure the controller performance (Harris 1989, Huang *et al.* 1997). However, MVC is rarely implemented in industry due to its aggressive action, which is caused by no penalty on control action in the cost function for the control design. Linear quadratic gaussian (*LQG*) control overcomes the drawback of MVC by considering the variations of input and output simultaneously.

1.1.2 State covariance control

Both MVC and LQG control have a scalar objective function. Modern process control often involves multiple objectives that are difficult to be represented by one objective function. Skelton and co-workers pioneered the research in controlling the covariance matrix of the process variables, which is the direct extension of variance control from SISO systems to multi-input-multi-output (*MIMO*) systems. The beauty of covariance control lies in the following facts: (1) If a system holds certain covariance structure, every state is well behaved in the sense of variance. Other techniques, such as LQG control, H_2 or H_∞ control, can only guarantee that a scalar index is well behaved. Covariance control takes the correlation among state variables into consideration as well. (2) The covariance controller automatically guarantees the stability of the closed-loop system because the control law is associated with a

Lyapunov function. (3) Covariance control theory provides parameterization of all fixed order controllers, which can stabilize the system and assign the pre-specified covariance matrix to the closed-loop state variables. This property of covariance control presents the possibility of solving multi-objective controller design problem.

The state covariance assignment problem was first introduced by Hotz and Skelton (1987) for continuous linear time-invariant (*LTI*) systems and Collins and Skelton (1985) for discrete-time LTI systems. The main idea of state covariance control (*SCC*) is to select a positive definite matrix, which is obtained according to requirements on the system performance, and then to design a controller such that the specified positive definite matrix is assigned to the closed-loop state vector. The basic procedure of (*SCC*) via the state feedback is briefly discussed in the rest of this section.

For the following continuous-time LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) + G\upsilon(t) \tag{1.1}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the input vector and $\upsilon \in R^d$ is the zero mean white disturbance with covariance Ω. The matrices A, B and G have appropriate dimensions. The objective is to design a state feedback controller $u = Kx$ such that the closed-loop is asymptotically stable and the state x has pre-specified covariance Σ. It is well known (Bryson and Ho 1975) that the closed-loop state covariance Σ should satisfy:

$$(A+BK)\Sigma + \Sigma(A+BK)^T + G\Omega G^T = 0 \tag{1.2}$$

If Σ is known, Equation (1.2) is linear in K so that the solution K can be easily obtained through linear algebra manipulations. However, there may not exist a solution K for every positive definite Σ. Rearranging Equation (1.2), we can obtain

$$BK\Sigma + \Sigma K^T B^T = -G\Omega G^T - A\Sigma - \Sigma A^T \tag{1.3}$$

Equation (1.3) can be regarded as an equation with unknown K. Equation (1.3) is solvable for the unknown K if and only if Σ satisfies (Hotz and Skelton 1987):

$$\begin{aligned} &(I - BB^+)\left(A\Sigma + \Sigma A + G\Omega G^T\right)(I - BB^+) = 0 \\ &\Sigma > 0 \end{aligned} \tag{1.4}$$

Equation (1.4) characterizes the condition for a positive definite matrix to be assigned to the state $x(t)$ via state feedback.

After the pioneering work on covariance assignment via state feedback, Skelton and his co-workers have done extensive research on state covariance control problem. Assigning a covariance matrix to the closed-loop state variables through dynamic output feedback was studied by Hsieh and Skelton (1990), but the measurement noise was not considered. An improved result was given by Xu and Skelton (1992). However, measurement noise was again ignored. Covariance control with measurement noises was considered by Skelton and Iwasaki (1993) and necessary and sufficient conditions for a covariance matrix to be assignable were proposed there.

1.1.3 Output covariance control

As we can see from the covariance assignment via state feedback, not every positive definite matrix is an assignable covariance matrix. This is because matrix $K \in R^{m \times n}$ has $n \times m$ free parameters, but there are $\frac{n(n+3)}{2}$ constraints (Equation (1.2) has $\frac{n(n+1)}{2}$ constraints and $\Sigma > 0$ has n constraints). Since the constraint number is more than that of free parameters, it is very possible that the pre-specified positive definite matrix is not an assignable covariance matrix. Two different methods were proposed to solve this problem.

The first method is to find the assignable positive definite matrix that is "closest" to the unassignable pre-specified positive definite matrix. For example, Grigoriadis and Skelton (1994) proposed the alternating convex projection methods to calculate the closest (measured in Frobenius norm) assignable covariance matrix to the unassignable pre-specified matrix. Actually, the problem of calculating the assignable Σ, which is closest (in Frobenius norm) to the unassignable pre-specified Σ_0, can be casted to a semi-definite programming (*SDP*) problem. This can be shown by the equivalence of the following two optimization problems:

1.

$$\min \|\Sigma - \Sigma_0\|_F \tag{1.5}$$

2.

$$\min \sum_{i,j} \gamma_{ij} \tag{1.6}$$

s. t.

$$\begin{bmatrix} \gamma_{ij} & \Sigma^{ij} - \Sigma_0^{ij} \\ \Sigma^{ij} - \Sigma_0^{ij} & 1 \end{bmatrix} \geq 0 \tag{1.7}$$

The second method is to reduce the number of constraints by only specifying the meaningful covariance constraints. State covariance may not be always meaningful in industry because in some industrial applications the state variables do not have physical meanings. Instead, controlling the covariance of the output variables is more attractive. The output covariance constrained problem was studied by Skelton *et al.* (1998) for one output vector using the well known variable linearization technique (Masubuchi *et al.* 1995, Scherer *et al.* 1997), which was originally proposed to solve the multiple-objective controller design problem. Since one Lyapunov function is used for each control performance specification, the Lyapunov paradigm (Masubuchi *et al.* 1995, Scherer *et al.* 1997) is conservative. In this thesis, the variable linearization approach is used to solve the generalized covariance control problem and the optimal LQ control subject to generalized covariance constraints for both discrete-time systems and continuous-time systems. The covariance constrained problem subject to controller structure constraints (*multi-loop PID*) is also considered.

A good design of controllers by itself does not constitute a complete picture of the modern control system. Maintenance of the controllers through routine process monitoring has increasingly become inseparable part of the control system. Thus, in this thesis we will also investigate process monitoring and fault detection problems using the same criterion as that of the controller design discussed in this thesis, the covariance criterion.

1.2 Fault detection and isolation

FDI is important to ensure reliable and safe operations of the modern process industry. Two basic methods, model-based approach and data driven approach, are widely used to detect process abnormality. Both of these fault detection methods are considered in this thesis.

1.2.1 FDI and covariance constraints

It was shown (Chow and Willsky 1984) that the parity space method was a powerful tool to detect and isolate faults. Gertler and Singer (1990) proved that the parity space approach was equivalent to the observer-based fault detection scheme. One of the advantages of the observer-based fault detection is that the well-studied control theory can be transferred to the observer design due to duality. As a result, the observer-based fault detection scheme has been widely studied (Commault *et al.* 2000, Ding and Guo 1998, Frank 1994, Wünnenberg 1990).

In Chapter 6, a novel observer-based FDI scheme is proposed. The FDI scheme involves two steps: 1) the signature of the target fault on the primary residual is magnified and the trace of the estimation error covariance is minimized; 2) a quadratic evaluation function is constructed to isolate the target fault from the nuisance faults, to constrain the mean value of the evaluation function and to maximize the effect of target fault on the evaluation function.

1.2.2 Performance monitoring based on covariance information

Statistical Process Control (*SPC*), which uses statistical methods for process monitoring to improve process quality and productivity, has received substantial interest in the chemical process industries. Various control charts, such as Shewhart charts, cumulative sum (*CUSUM*) charts, and exponentially weighted moving average (*EWMA*) charts, are widely used to monitor single variables. It has been realized that applying univariate control charts to multivariate system is not appropriate (Jackson 1959). Multivariate control chart, plotting one common statistic from a multivariate vector, should be used instead. Principal Component Analysis (*PCA*) is one of the most widely used data-driven performance monitoring tools for multivariate systems (Jackson 1991). In the conventional PCA method it is often necessary to scale the data according to the standard deviation of each variable, *i.e.*, the correlation matrix of the process variables is used instead of the covariance matrix to extract the linear relations among variables. Two statistics are generated in the PCA approach: *Hotelling* T^2 and square prediction error(*SPE*).

Two assumptions are made when PCA is used to monitor process performance: (1) the process is operated at steady state; (2) the data is independent, which means that the data should be uncorrelated in time. The validity of the assumptions determines the validity of the monitoring result. Unfortunately, these two assumptions do not always hold for many chemical plants.

First, industrial processes are usually subject to some time-varying factors, such as slow environment changes, process aging, *etc.* These factors usually make the mean values of process variables change with time. In the conventional PCA methods, the training data determines the PCA model so that the PCA model does not change with time. To increase the robustness of the PCA monitoring scheme, it is preferable to let the PCA monitoring scheme have adaptive feature. For example, an adaptive PCA method was proposed by Li *et al.* (2000). In Chapter 7, a new adaptive PCA method, called recursive moving window PCA (*RMWPCA*), is proposed. The RMWPCA is used to detect the sanding phenomena in the tailing system of Syncrude Canada Ltd.

Second, the dynamics of the process can not be neglected for many systems. There must be an interesting relation between the covariance control theory and PCA based process monitoring. This relation has been posted as an open problem for future research. A brief discussion can be found in the future work section in order to shed some light on this topic.

1.3 Other related problems

Two related problems are also studied in the thesis: the first one is the filter design problem for sampled-data systems and the second is the control of descriptor systems.

1.3.1 Filter design for the sampled-data systems with covariance constraints

For some control design problems, sometimes one needs to estimate the unmeasurable state variables. The filter design problems for discrete-time systems and continuous-time systems are dual to the controller design problems so that it is a straightforward extension of the results presented in Part 2 and is therefore omitted. Instead, the filter design for sampled-data systems is studied.

A special filter, which assigned a covariance matrix to the estimation error, was first presented by Yaz and Skelton (1991). The error covariance assignment (*ECA*) theory for continuous-time systems or discrete-time systems is dual to the covariance assignment via state feedback. The ECA theory was extended to sampled-data systems by Wang *et al.* (2001). In Chapter 8, we give an improved result by specifying an estimation error covariance upper bound.

1.3.2 Static output feedback control of descriptor systems

Descriptor systems are natural in modelling large scale processes because they capture not only the dynamics of the systems but also the static constraints. Descriptor systems have the capacity of describing impulsive behaviors and non-causal behaviors, which the regular systems are lack of. Due to the similarity between descriptor systems and regular systems, some technique developed in this thesis can be extended to descriptor systems as well.

1.4 Linear matrix inequalities, convexity and semi-definite programming

The linear matrix inequality (*LMI*) technique is used as a basic tool to solve many of the problems proposed in the thesis so that it is worth a brief discussion. A general form of LMI is given as:

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i \geq 0 \tag{1.8}$$

where $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T \in R^m$ is the decision variable and $F_i = F_i^T \in R^{n\times n}$, $i = 0, 1, \cdots, m$ are given. The inequality (1.8) is equivalent to a set of polynomial inequalities in x, *i.e.*, the leading principal minors of $F(x)$ must be positive semi-definite.

One of the distinguishing features of LMI is that Inequality (1.8) is a convex constraint, *i.e.*, the set, determined by (1.8),

$$\Gamma = \{x \,|\, F(x) \geq 0\} \tag{1.9}$$

is a convex set. This can be verified by the definition of convexity and the definition of LMI.

Definition 1.4.1 *A set $\mathscr{C}$ is called convex if for any x_1, $x_2 \in \mathscr{C}$ and any $0 \leq \alpha \leq 1$ such that $\alpha x_1 + (1-\alpha)x_2 \in \mathscr{C}$.*

For any $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}^T \in \Gamma$ and $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}^T \in \Gamma$, we have

$$\begin{aligned} F & \;\; (\alpha x + (1-\alpha)y) \\ = & \;\; F_0 + \sum_{i=1}^{m} (\alpha x_i + (1-\alpha)y_i) F_i \\ = & \;\; \alpha \left(F_0 + \sum_{i=1}^{m} x_i F_i \right) + (1-\alpha) \left(F_0 + \sum_{i=1}^{m} y_i F_i \right) \\ = & \;\; \alpha F(x) + (1-\alpha) F(y) \geq 0 \end{aligned} \tag{1.10}$$

The convexity of LMIs plays a crucial role in optimization because it is well known that a convex function has a global optimum over a convex set. In particular, many of the problems studied in the thesis are transferred to semi-definite programming (*SDP*) problems. Here, a SDP problem (Vandenberghe and Boyd 1996) is referred to a problem of minimizing a linear function of the decision variable $x \in R^m$ subject to LMI (1.8). SDP problems are special forms of cone programming. For interested readers are referred to (Boyd and Barratt 1991). In addition, robust and efficient computational algorithms, such as the ellipsoid algorithm and the interior-point methods, are available to solve LMIs and SDP problems. These algorithms have proven to be very efficient in both theory and practice. Commercial software packages for solving LMIs are also available, such as the MATLAB LMI toolbox and SeDuMi (This is a free software package that you can download from this website: http://fewcal.kub.nl/sturm/software/sedumi.html).

1.5 Contributions of the thesis

The main contributions of the thesis are as follows:

- Necessary and sufficient conditions for the feasibility of the generalized covariance constrained control (*GCC*) are given in terms of LMIs for both discrete-time systems and continuous-time systems.
- It is shown that the optimal LQ control subject to the generalized covariance constraints for both discrete-time and continuous-time systems can be obtained via SDP so that global optimality is guaranteed.
- A necessary and sufficient condition for the feasibility of the GCC problem through a multi-loop PID controller is derived for discrete-time systems.
- A convergent computational algorithm to calculate the multi-loop PID controller parameters is given for discrete-time systems in order to satisfy the generalized covariance constraints.
- A necessary and sufficient condition for the feasibility of covariance constrained problem through a multi-loop PID controller is given for continuous-time systems.
- A convergent computational algorithm to calculate the multi-loop PID controller parameters is given for continuous-time systems in order to satisfy the covariance constraints.
- A novel observer-based scheme is proposed to detect and isolate multiple faults.
- A necessary and sufficient condition for the H_2 norm of a transfer function to be larger than a positive constant is given in terms of LMIs.
- A necessary and sufficient condition for the sampling-interval estimation error covariance constrained problem is provided.
- A computational algorithm is presented to calculate the filter parameters for the sampled-data system to satisfy the sampling-interval estimation error covariance constraints.
- A recursive moving window PCA monitoring scheme is presented and is applied to sanding detection for the tailing system of *Syncrude Canada Ltd.*

1.6 Outline of the thesis

To present the results of the thesis in a straightforward manner, each chapter in the thesis is self-contained: starting with the introduction, problem formulation and then presenting the new results and certain illustrating examples.

In Chapter 2, the LQ control subject to the generalized covariance constraint (*GCC*) problem for discrete-time linear time- invariant systems is considered. It is shown that the feasibility of the GCC problem is equivalent to the feasibility of several LMIs. Furthermore, if the LMIs are feasible, the admissible controller set can be parameterized by the solutions to the LMIs. In addition, LQ performance can be optimized over the admissible controller set, and the global optimal solution can be obtained by solving a SDP problem.

In Chapter 3, the optimal LQ control with generalized covariance constraints (*LQGCC*) for the continuous linear time-invariant systems is studied. This problem consists of two aspects: (1) the feasibility of the generalized covariance constrained control problem, which is to make the covariances of different controlled variables satisfy certain pre-specified covariance constraints; (2) the optimization of LQ performance over the feasible controller set. It is shown that the feasibility of the GCC problem is equivalent to the feasibility of several linear matrix inequalities (LMIs). Furthermore, if the LMIs are feasible, the controller set can be parameterized by the solutions of the LMIs. If the GCC is feasible, then the minimization of the LQ performance is equivalent to solving a SDP problem and our approach ensures the global optimality.

In Chapter 4, the design of multi-loop PID controllers for discrete-time systems such that the process variables satisfy the generalized covariance constraints is studied. A convergent computational algorithm is proposed to calculate the multi-loop PID controller parameters for a process with stable disturbances. This algorithm is then extended to a process with integrated disturbances. The feasibility of the proposed algorithm is verified by several simulation examples.

In Chapter 5, designing multi-loop PID controllers for continuous-time systems in order to constrain the state covariance is studied. A convergent iterative computational algorithm, in which a sequence of SDP problems are solved, is presented to calculate the multi-loop PID controller parameters. A numerical example is used to demonstrate the efficiency of the algorithm.

In Chapter 6, we consider the fault detection and isolation (*FDI*) problem for discrete-time stochastic systems. A novel two-step FDI scheme is proposed. In the first step, an observer is designed so that the H_2 norm of the transfer function from the target fault to the residual is larger than a pre-specified value and the trace of the estimation error covariance is minimized. In the second step, an evaluation function is constructed for three purposes: (1) to isolate the nuisance fault from the target fault; (2) to constrain the disturbance effect on evaluation function; (3) to maximize the effect of the target fault on the evaluation function. One of the main contributions of the chapter is that the necessary and sufficient condition for the H_2 norm of the transfer function from the target fault to the residual to be larger than some pre-specified β is given in LMI form. The observer gain is parameterized in the solution to the LMIs. The relationship between the constructed evaluation function and the structured residual vector (*SRV*) is also investigated. A threshold for the evaluation function is generated based on the distribution of the quadratic form in random variables. The simulation results show the effectiveness of the proposed method.

In Chapter 7, a new adaptive PCA is implemented to detect sanding in tailing systems of Syncrude Canada Ltd. Due to slow time variation of the process, the traditional PCA can not work properly. RMWPCA that is efficient in online and off-line calculation is introduced to enhance the robustness of the detection algorithm. The comparison between traditional PCA and RMWPCA shows that RMWPCA is more robust to normal process operation changes without compromising its ability to detect sanding in pipe lines.

In Chapter 8, the filter design problem for the sampled-data system subject to the estimation error variance constraints is studied. A necessary and sufficient condition for the existence of a discrete filter such that the variances of the estimated error have specified upper bounds is given in terms of linear matrix inequalities LMIs. If the LMIs are feasible, then the filter is parameterized by the solution to the LMIs. An illustrative example is used to demonstrate the efficiency of the proposed algorithm.

In Chapter 9, the problem of admissible stabilization of continuous-time descriptor systems and simultaneous stabilization of several descriptor systems via static output-feedback is addressed. Necessary and sufficient conditions for a descriptor system to be admissibly stabilized via static output feedback are given in a bilinear matrix inequality form. An iterative optimization algorithm is proposed to numerically calculate the static feedback gain. The results are then extended to the simultaneous admissible output feedback stabilization.

This thesis has been written in a paper-format in accordance with the rules and regulations of the Faculty of Graduate Studies and Research, University of Alberta. Many of the chapters have appeared or are to appear in archival journals and conference proceedings. In order to link the different chapters, there is some overlap and redundancy of material. This has been done to ensure completeness and cohesiveness of the thesis material and help the reader understand the material easily.

Part II

Covariance control

2

LQ control with Generalized Covariance Constraint for Discrete-time Systems

2.1 Introduction

Variance control plays an important role in stochastic control theory and a number of controller design methods have been proposed and are well-known. Minimization of variances of certain process variables (Äström 1970) is well justified in process control because the reduction of the variances of quality variables not only means improved product quality but also makes it possible to increase throughput, reduce energy consumption and save raw materials. However, it is well known that minimizing the variances of some process variables alone may result in unacceptable control input variance (MacGregor 1973). Minimum variance control can be treated as a singular solution to LQG controller design (Brockett 1970, Kalman 1968). The latter is to minimize the scalar sum of the variances of the output variables and the input variables based on a known stochastic noise disturbance model:

$$J(Q,R) = \lim_{k\to\infty} E\left(x_k^T Q x_k + u_k^T R u_k\right) \tag{2.1}$$

In LQG controller design the weighting matrices Q and R are usually considered to be free design parameters, and properly choosing the weighting matrices Q and R is important to the design of a controller such that the closed-loop system has satisfactory performance. Even

[1]Some versions of this chapter was presented in CSCHE (Huang *et al.* 2002) or accepted by *Dynamics of continuous, discrete, and impulse systems* (Huang *et al.* 2003*b*)

though some rules of thumb to choose the weighting matrices were given by Athans (1971), there is no systematic methods of selecting the weighting matrices to ensure the overall closed-loop performance.

It is convenient and often necessary to set the variances or covariance constraints on the input and output variables directly in many industrial applications. But the traditional LQG controller design can not achieve this straightforward goal. Several modified LQG controller design methods (Mäkilä 1982, Skelton and Delorenzo 1985, Toivonen 1983) were proposed to solve the variance constraints problem. The main idea of the modified LQG controller design is to iterate the weighting matrices until the variance constraints are satisfied. Mäkilä *et al.* (1982) presented an iterative algorithm to minimize the quadratic loss function (2.1) subject to constraints on the variances of some variables specified by the designer, but the global convergence property of the algorithm is not known. Some self-tuning controllers (Toivonen 1983, Mäkilä 1982) were also proposed to explicitly restrict the variance of the state variables and the input variables based on adaptively adjusting the Lagrange multiplier of the variance constrained control problem.

It is natural to describe the variation of a random vector by its covariance; however the constraints considered in the modified LQG control problem are only the variance constraints for scalar random variables. The minimum energy output covariance control theory (Hsieh *et al.* 1989, Zhu *et al.* 1997) considered the covariance restriction on the output vector, and at the same time minimized the input energy. The output covariance control problem was based on the state covariance assignment theory (Hsieh and Skelton 1990, Collins and Skelton 1985, Skelton and Iwasaki 1993), first introduced by Collins and Skelton (1985). The basic idea of the state covariance assignment theory is to design a controller such that the specified state covariance, which is obtained according to system performance requirements and assignable condition, is assigned to the closed-loop system. One advantage of the covariance control theory is that it provides a parameterization of all controllers that achieve the specified covariance and additional cost functions can then be optimized over such a controller set. Therefore, covariance control gives a multiple objective flavor to the controller design. Assigning a covariance matrix to the closed-loop state variables through a fixed-order dynamic output feedback has been solved by Hsieh and Skelton (1990), and the closed form of such a controller is parameterized by two orthogonal matrices. Grigoriadis and Skelton (1997) found the analytical solution to the minimum energy control subject to the known state covariance matrix. Some necessary or sufficient conditions were discussed, and an algorithm that iteratively selected a weighting matrix was given by Hsieh *et al.* (1989), but the convergence of the algorithm was not guaranteed. Another iterative algorithm was also proposed by Zhu *et al.* (1997), and this algorithm was proved to converge to an optimal solution provided that user specified parameters in the algorithm were properly chosen. A Lyapunov equation and a Riccati equation are required to be solved in each iteration, and also global optimality is not guaranteed.

In this chapter LQ control subject to the generalized covariance constraint (*GCC*) is considered. The generalized covariance constrained problem is to design a controller such that the covariance matrices of both inputs and outputs or their subset are constrained by certain user-specified upper bounds. The feasibility of GCC is equivalent to the solvability of several linear matrix inequalities (*LMIs*). If GCC is feasible, all the controller set can be parameterized by the solutions to LMIs. Furthermore, one can optimize the LQ performance (2.1) over the feasible controller set. The achievable optimal LQ performance can be found by solving a semi-definite programming problem. This method guarantees the global optimality, and robust computational tool is available (Gahinet *et al.* 1995). The rest of this chapter is organized as follows: In Section 2, the problem statements and the preliminary results are presented. The main results are stated in Section 3, which contains two parts: in Section 3.1 the necessary and sufficient conditions for the feasibility of the GCC problem are stated in term of several LMIs for both the state feedback and the dynamic output feedback; in Section 3.2 it is also shown that one can optimize the LQ performance by semi-definite programming approach. Numerical examples are given in Section 4 and concluding remarks are presented in Section 5.

2.2 Problem statement

Consider the finite-dimensional discrete-time linear system $\mathscr{P}$ given as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G\varsigma_k \\ y_k &= Cx_k + F\varsigma_k \\ z_k^i &= C_i x_k + D_i u_k, \quad i = 1 \ldots t \end{aligned} \tag{2.2}$$

where $x_k \in R^n$ is the system state vector, $u_k \in R^m$ is the system input, $y_k \in R^p$ is the system measurement, $\varsigma_k \in R^g$ is the external disturbance and measurement noise, and $z_k^i \in R^{p_i}$ is the i-th controlled variable. It is assumed that the external disturbance ς_k is white noise, satisfying:

$$E(\varsigma_k) = 0 \tag{2.3}$$

$$E\left(\varsigma_i \varsigma_j^T\right) = \Omega \delta_{ij} \tag{2.4}$$

where $\Omega > 0$.

Both the state feedback controller and the dynamic output feedback controller are considered here. A state feedback controller is given as follows:

$$u_k = Kx_k \tag{2.5}$$

and a full order dynamic output feedback controller is described as:

$$\begin{aligned} x_{k+1}^C &= A_c x_k^C + B_c y_k \\ u_k &= C_c x_k^C + D_c y_k \end{aligned} \tag{2.6}$$

where K, A_c, B_c, C_c and D_c are unknown matrices with appropriate dimensions. With the state feedback control law (2.5) or the full order dynamic control law (2.6), the closed-loop system can be written as

$$\begin{aligned} X_{k+1} &= A_{cl} X_k + G_{cl} \varsigma_k \\ Z_k^i &= \bar{C}_i X_k + \bar{D}_i \varsigma_k,\ i = 1 \ldots t \end{aligned} \tag{2.7}$$

where $X_k = x_k$, $A_{cl} = A + BK$, $G_{cl} = G$, $\bar{C}_i = C_i + D_i K$ and $\bar{D}_i = 0$ for the state feedback case; $X_k = \begin{bmatrix} x_k \\ x_k^C \end{bmatrix}$, $A_{cl} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$, $G_{cl} = \begin{bmatrix} G + BD_cF \\ B_cF \end{bmatrix}$, $\bar{C}_i = \begin{bmatrix} C_i + D_iD_cC & D_iC_c \end{bmatrix}$ and $\bar{D}_i = D_iD_cF$ for the dynamic output feedback case.

The covariance of the state is defined as:

$$\Sigma_{cl} = \lim_{k \to \infty} E\left(X_k X_k^T\right) \tag{2.8}$$

If the closed-loop system is asymptotically stable, it is well known that the covariance of the state vector should satisfy the following equation:

$$\Sigma_{cl} = A_{cl} \Sigma_{cl} A_{cl}^T + G_{cl} \Omega G_{cl}^T \tag{2.9}$$

Skelton and his co-workers studied the state covariance control problem, obtained the necessary and sufficient conditions for a positive matrix to be assignable to the state vector and parameterized all controllers that assign such a covariance to the state vector (Hsieh and Skelton 1990, Hsieh *et al.* 1989, Collins and Skelton 1985, Grigoriadis and Skelton 1997, Skelton and Delorenzo 1985, Zhu *et al.* 1997). In many control applications assigning a covariance to the state vector is not necessary. Instead, specifying certain bounds on the covariance for the input and output vectors is often required. For example, the variance and covariance bounds for the controlled variables, which are only dependent on the state variables, were considered by Hsieh *et al.* (1989) and Zhu *et al.* (1997) respectively. This concept is extended to including the input in the controlled variables. We state the generalized covariance constrained problem as follows:

Problem 2.2.1 *Generalized covariance constrained (GCC) problem: For the discrete-time linear time-invariant system (2.3), find a state feedback controller (2.5) or a dynamic feedback controller (2.6) such that the closed-loop system is asymptotically stable and the covariance of the controlled variable* Z_k^i $(i = 1...t)$ *satisfies*

$$\Phi_i = \lim_{k\to\infty} E\left(Z_k^i Z_k^{iT}\right) < \bar{\Phi}_i \tag{2.10}$$

where $\bar{\Phi}_i$ $(i = 1...t)$ *is a pre-specified positive matrix.*

If there is a controller such that the closed-loop system is asymptotically stable and the covariance constraints (2.10) are satisfied, then we call the GCC problem feasible. In the next section, it is shown that feasibility of GCC is equivalent to feasibility conditions of some linear matrix inequalities (LMIs). Since LMIs are linear convex constraints, we can always find a solution, if there is one.

Suppose that the GCC problem is feasible via state feedback or dynamic output feedback. The feasible state feedback controller set is denoted by $\mathscr{C}_s$ and the feasible dynamic output controller set is denoted by $\mathscr{C}_d$. We are interested in optimizing certain performance indices, which are functionals of the controller $\mathscr{C} \in \mathscr{C}_s\,(\mathscr{C}_d)$. Several widely-used performance indices, including the LQ index and input energy, are investigated in this chapter in particular. The LQ objective is a quadratic function:

$$\begin{aligned} J &= \lim_{k\to\infty} E\left(x_k^T Q x_k + u_k^T R u_k\right) \\ &= tr\left(Q\Sigma_x\right) + tr\left(R\Sigma_u\right) \end{aligned}$$

where Σ_x is the stationary covariance matrix for the plant state vector x_k, Σ_u is the covariance matrix of the input vector u_k, and Q and R are positive semi-definite weighting matrices. If $R = 0$, the LQ index is the input vector's energy, denoted by J^e. The above stated problems can be formulated as the following optimization problem, called as LQ control problem subject to the generalized covariance constraints:

Problem 2.2.2

$$\begin{aligned} J_{\mathscr{C}_s} &= \inf_{\mathscr{C}\in\mathscr{C}_s} J \\ J_{\mathscr{C}_d} &= \inf_{\mathscr{C}\in\mathscr{C}_d} J \end{aligned} \tag{2.11}$$

Remark 2.2.1 *The strict inequality sign is used in GCC so that the minimum of the cost function may not be reachable. This is why infimum instead of minimum is used in the above formulation.*

The well-known Schur complement Lemma (Boyd and Barratt 1991) will be frequently used in the proofs, and it is stated as follows:

Lemma 2.2.1 *(Schur complement lemma) The following statements are equivalent:*
1)

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} > 0 \tag{2.12}$$

2) $C > 0$ *and* $A - BC^{-1}B^T > 0$.

2.3 Main results

One needs to make sure that the generalized covariance constraints (*GCC*) are feasible before optimizing the LQ performance (2.11). The GCC problem itself is a nonlinear multi-objective controller design problem. Generally speaking multi-objective control design is a challenging problem (Masubuchi *et al.* 1995, Scherer *et al.* 1997). In the next subsection it is shown that the GCC problem is feasible via the state feedback or dynamic output feedback if and only if certain LMIs are feasible. To state the main results, we need the following lemma:

Lemma 2.3.1 *The closed-loop system (2.7) is asymptotically stable and satisfies the constraint (2.10) if and only if there exists a matrix* $\Sigma > 0$ *such that*

$$A_{cl}\Sigma A_{cl}^T - \Sigma + G_{cl}\Omega G_{cl}^T < 0 \tag{2.13}$$

$$\bar{C}_i\Sigma\bar{C}_i^T + \bar{D}_i\Omega\bar{D}_i^T < \bar{\Phi}_i \tag{2.14}$$

where $i = 1, ..., t$.

Proof:
Necessity: If the closed-loop system is asymptotically stable, the unique solution to (2.9), denoted by Σ_{cl}, is the covariance matrix. Furthermore, it can also be shown that

$$\bar{C}_i\Sigma_{cl}\bar{C}_i^T + \bar{D}_i\Omega\bar{D}_i^T = \lim_{k\to\infty} E\left(Z_k^i Z_k^{i\,T}\right) < \bar{\Phi}_i \tag{2.15}$$

because ς_k is uncorrelated with x_k. There is a unique solution Σ_ε to the following Lyapunov equation ($\varepsilon > 0$):

$$A_{cl}\Sigma A_{cl}^T - \Sigma + G_{cl}WG_{cl}^T + \varepsilon I = 0 \tag{2.16}$$

For sufficiently small ε, Σ_ε satisfies constraint (2.14) because $\Sigma_\varepsilon \to \Sigma_{cl}$. Equation (16) implies that Σ_ε also satisfies (2.13).
Sufficiency: (2.13) implies that the closed-loop system (2.7) is asymptotically stable. (2.13) and (2.14) imply that the state covariance matrix Σ_{cl} satisfies constraint (2.10) because $\Sigma > \Sigma_{cl}$.
∇∇∇

Remark 2.3.1 *If the nonstrict inequality sign is used in GCC, the necessity of GCC requires that the nonstrict inequality sign should be used in Inequality (2.13) as well. However, to prove the asymptotically stability of the closed-loop system through Inequality (2.13) (with nonstrict inequality sign), we need to assume that* (A_{cl}, G_{cl}) *is a controllable pair. Since matrices* A_{cl} *and* G_{cl} *can only be determined after the controller is given, we have to check if* (A_{cl}, G_{cl}) *is a controllable pair after the controller is obtained. Some related discussion, for the state feedback case, can be found in (Collins and Skelton 1985) (Hotz and Skelton 1987), where* (A, G) *was assumed to be a controllable pair.*

2.3.1 Feasibility of the GCC problem

Theorem 2.3.1 *The GCC problem is feasible for the discrete-time system (2.3) via state feedback if and only if there exist matrices* $\Sigma > 0$ *and* L *such that*

$$\begin{bmatrix} G\Omega G^T - \Sigma & A\Sigma + BL \\ \Sigma A^T + L^T B^T & -\Sigma \end{bmatrix} < 0 \tag{2.17}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_i L \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{2.18}$$

Proof:

Necessity: Since the GCC is feasible, according to Schur complement Lemma, inequality (2.13) implies:

$$\begin{bmatrix} G\Omega G^T - \Sigma & A\Sigma + BK\Sigma \\ \Sigma A^T + \Sigma B^T & -\Sigma \end{bmatrix} < 0 \tag{2.19}$$

and (2.14) implies:

$$\begin{bmatrix} \bar{\Phi} & C_i\Sigma + D_i K\Sigma \\ \Sigma C_i^T + \Sigma K^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{2.20}$$

Inequalities (2.17) and (2.18) are easy to derive by letting $L = K\Sigma$ in inequalities (2.19) and (2.20).

Sufficiency: Let $K = L\Sigma^{-1}$, it is easy to show that (2.13) and (2.14) hold. ∇∇∇

For the state feedback case, we can transfer the nonlinear matrix inequalities to linear matrix inequalities by a simple variable transformation. It is long believed that such kind of transformation does not exist for dynamic output feedback. However, the recent work (Scherer *et al.* 1997, Masubuchi *et al.* 1995) showed that we can also linearize the matrix variables by some nonlinear matrix transformation. We will extend the technique used for continuous-time LTI systems (Scherer *et al.* 1997) to discrete-time LTI systems to solve the GCC problem for the dynamic output feedback.

Theorem 2.3.2 *The GCC problem is feasible via a dynamic controller for the discrete-time LTI system (2.3) if and only if there exist matrices $\Sigma_1 > 0$, $M_1 > 0$, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$, $\hat{D}_c$ such that*

$$\begin{bmatrix} -\Sigma_1 & -I & A+B\hat{D}_cC & A\Sigma_1+B\hat{C}_c & G+B\hat{D}_cF \\ -I & -M_1 & M_1A+\hat{B}_cC & \hat{A}_c & M_1G+\hat{B}_cF \\ A^T+C^T\hat{D}_c^TB^T & A^TM_1+C^T\hat{B}_c^T & -M_1 & -I & 0 \\ \Sigma_1A^T+\hat{C}_c^TB^T & \hat{A}_c^T & -I & -\Sigma_1 & 0 \\ G^T+C^T\hat{D}_c^TB^T & G^TM_1+F^T\hat{B}_c^T & 0 & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{2.21}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i+D_i\hat{D}_cC & C_i\Sigma_1+D_i\hat{C}_c & D_i\hat{D}_cF \\ C_i^T+C^T\hat{D}_c^TD_i^T & M_1 & I & 0 \\ \Sigma_1C_i^T+\hat{C}_c^TD_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TD_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.22}$$

Furthermore, if LMIs (2.21) and (2.22) are feasible, the full-order dynamic controller can be parameterized as:

$$\begin{aligned} D_c &= \hat{D}_c \\ C_c &= \left(\hat{C}_c - \hat{D}_cC\Sigma_1\right)\Sigma_2^{-T} \\ B_c &= M_2^{-1}\left(\hat{B}_c - M_1B\hat{D}_c\right) \\ A_c &= M_2^{-1}\left(\hat{A}_c - M_1A\Sigma_1 - M_1BD_cC\Sigma_1 - M_2B_cC\Sigma_1 - M_1BC_c\Sigma_2^T\right)\Sigma_2^{-T} \end{aligned} \tag{2.23}$$

where $M_2 \in R^{n\times n}$ and $\Sigma_2 \in R^{n\times n}$ are any matrices satisfying $\Sigma_2M_2^T = I - \Sigma_1M_1$.

Proof:

Necessity: If the GCC problem is feasible via full-order dynamic output feedback, there exists a controller with a form (2.6) and $\Sigma > 0$ such that inequalities (2.13) and (2.14) are satisfied. (2.13) is equivalent to the following LMI:

$$\begin{bmatrix} -\Sigma & A_{cl} & G_{cl} \\ A_{cl}^T & -\Sigma^{-1} & 0 \\ G_{cl}^T & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{2.24}$$

and (2.14) is equivalent to the following LMI:

$$\begin{bmatrix} \bar{\Phi}_i & \bar{C}_i & \bar{D}_i \\ \bar{C}_i^T & \Sigma^{-1} & 0 \\ \bar{D}_i^T & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.25}$$

Partition Σ as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix} \tag{2.26}$$

Let $M = \Sigma^{-1} = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$. It can be verified that

$$\Sigma_1M_1 + \Sigma_2M_2^T = I \tag{2.27}$$

$$\Sigma_1M_2 + \Sigma_2M_3 = 0 \tag{2.28}$$

$$M_1\Sigma_2 + M_2\Sigma_3 = 0 \tag{2.29}$$

Without losing generality, we can assume that Σ_2 and M_2 are both full rank matrices. Let $T_1 = \begin{bmatrix} I & 0 \\ M_1 & M_2 \end{bmatrix}$, and $T_2 = \begin{bmatrix} I & 0 \\ \Sigma_1 & \Sigma_2 \end{bmatrix}$. The following equations can be verified:

$$T_1 \Sigma T_1^T = \begin{bmatrix} \Sigma_1 & I \\ I & M_1 \end{bmatrix} \tag{2.30}$$

$$T_2 \Sigma^{-1} T_2^T = \begin{bmatrix} M_1 & I \\ I & \Sigma_1 \end{bmatrix} \tag{2.31}$$

$$T_1 A_{cl} T_2^T = \begin{bmatrix} A + B\hat{D}_c C & A\Sigma_1 + B\hat{C}_c \\ M_1 A + \hat{B}_c C & \hat{A}_c \end{bmatrix} \tag{2.32}$$

$$T_1 G_{cl} = \begin{bmatrix} G + B\hat{D}_c F \\ M_1 G + \hat{B}_c F \end{bmatrix} \tag{2.33}$$

$$T_2 \bar{C}^T = \begin{bmatrix} C_i^T + C^T \hat{D}_c^T D_i^T \\ \Sigma_1 C_i^T + \hat{C}_c^T \hat{D}_i^T \end{bmatrix} \tag{2.34}$$

where

$$\hat{D}_c = D_c \tag{2.35}$$

$$\hat{C}_c = D_c C \Sigma_1 + C_c \Sigma_2^T \tag{2.36}$$

$$\hat{B}_c = M_1 B D_c + M_2 B_c \tag{2.37}$$

$$\hat{A}_c = M_1 A \Sigma_1 + M_1 B D_c C \Sigma_1 + M_2 B_c C \Sigma_1 + M_1 B C_c \Sigma_2^T + M_2 A_c \Sigma_2^T \tag{2.38}$$

Pre-multiplying and post-multiplying both sides of (2.24) with $\begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I \end{bmatrix}$ and $\begin{bmatrix} T_1^T & 0 & 0 \\ 0 & T_2^T & 0 \\ 0 & 0 & I \end{bmatrix}$ respectively, we obtain (2.21). Pre-multiplying and post-multiplying both sides of (2.25) with $\begin{bmatrix} I_n & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I_m \end{bmatrix}$ and $\begin{bmatrix} I_n & 0 & 0 \\ 0 & T_2^T & 0 \\ 0 & 0 & I_m \end{bmatrix}$ respectively, we obtain (2.22).

Sufficiency:

Both inequalities (2.21) and (2.22) imply that $\begin{bmatrix} M_1 & I \\ I & \Sigma_1 \end{bmatrix} > 0$, which in turn implies that

$$\Sigma_1 - M_1^{-1} > 0 \tag{2.39}$$

Suppose that $\Sigma_2 M_2^T = I - \Sigma_1 M_1$. We can show that matrices Σ_2 and M_2 are both full rank; otherwise $\Sigma_1 - M_1^{-1} = -\Sigma_2 M_2^T M_1^{-1}$ will be rank deficient, which is against (2.39).

Let the controller be defined as (2.23) and

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_2^T \left(\Sigma_1 - M_1^{-1}\right)^{-1} \Sigma_2 \end{bmatrix} \tag{2.40}$$

We can obtain (2.13) and (2.14) with some matrix manipulations. $\nabla\nabla\nabla$

2.3.2 Optimization with the generalized covariance constraints

If the GCC problem is feasible, then we can optimize certain performance indices over the feasible controller set $\mathscr{C}_s$ or $\mathscr{C}_d$. LQ performance index is considered in this chapter, and this kind of problem is called optimal LQ control subject to generalized covariance constraints. A similar problem, the constrained LQG problem, was studied by Mäkilä *et al.* (1984). The constrained LQG problem is to minimize the cost function J subject to the scalar variance constraints for the state variables and input variables. Complex numerical algorithms for solving the constrained LQG problem were given, but the global convergency of the algorithms has not been proved in the literature. If let $Q = 0$ in (2.11), the constrained LQG control problem then reduces to the minimum energy output covariance control problem (Hsieh *et al.* 1989, Zhu *et al.* 1997). The dual problem to the minimum energy control problem, called the input variance constraint controller design, was also studied by Hsieh *et al.* (1989). Several necessary or sufficient conditions for minimum energy output covariance control were presented and an algorithm that iteratively selected a diagonal output weighting matrix was given by Hsieh *et al.* (1989). An iterative algorithm was presented by Zhu *et al.* (1997) to solve the minimum energy output covariance control, and this algorithm was proved to be convergent provided that a parameter was properly chosen. Strictly proper dynamic controllers were assumed in both (Hsieh *et al.* 1989) and (Zhu *et al.* 1997), and the covariance constraints for the input vectors were not considered either.

The constrained LQG control or the minimum energy output covariance control problem is a nonlinear optimization problem, and the algorithms available so far cannot guarantee global convergence. In this section we will show that the LQ problem subject to the generalized covariance constraints can be solved by semi-definite programming, by which we mean minimizing a linear objective function subject to some LMI constraints, for both the state feedback and dynamic feedback. Due to the convexity of LMI constraints, the global optimality is guaranteed. Similarly, the minimum energy output covariance control and the input variance controller design can also be solved by semi-definite programming. The convexity property of semi-definite programming ensures global convergence. Also the feasible controller that is considered here is not limited to strictly proper controllers. The results are presented in the following theorems:

Theorem 2.3.3 *The generalized constrained LQ control via state feedback law (2.5) can be solved by semi-definite programming:*

$$J_{\mathscr{C}_s} = min \quad tr(P_1) + tr(P_2)$$

subject to:

$$\begin{bmatrix} P_1 & Q^{\frac{1}{2}}\Sigma \\ \Sigma Q^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{2.41}$$

$$\begin{bmatrix} P_2 & R^{\frac{1}{2}}L \\ L^T R^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{2.42}$$

$$\begin{bmatrix} G\Omega G^T - \Sigma & A\Sigma + BL \\ \Sigma A^T + L^T B^T & -\Sigma \end{bmatrix} < 0 \tag{2.43}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_i L \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{2.44}$$

Proof: The proof is similar to that of Theorem 2.3.1 and 2.3.2. To make the GCC feasible, LMIs (2.43) and (2.44) are required. One only need to pay attention to the LQ term:

$$\begin{aligned} J &= tr(Q\Sigma_{cl}) + tr(R\Sigma_u) \\ &< tr\left(Q^{\frac{1}{2}}\Sigma Q^{\frac{1}{2}} + R^{\frac{1}{2}}K\Sigma K R^{\frac{1}{2}}\right) \\ &= tr\left(Q^{\frac{1}{2}}\Sigma\Sigma^{-1}\Sigma Q^{\frac{1}{2}} + R^{\frac{1}{2}}L\Sigma^{-1}LR^{\frac{1}{2}}\right) \\ &< tr(P_1) + tr(P_2) \end{aligned}$$

where P_1 and P_2 satisfy constraints (2.41) and (2.42). In the proof of Lemma 2.3.1 it is noted that we can let $\Sigma \rightarrow \Sigma_{cl}$, and the inequalities (2.41) and (2.42) allow the free parameters $P_1 \rightarrow R^{\frac{1}{2}}K\Sigma K R^{\frac{1}{2}}$ and $P_2 \rightarrow Q^{\frac{1}{2}}\Sigma Q^{\frac{1}{2}}$ so that $tr(P_1) + tr(P_2) \rightarrow J_{\mathscr{C}_s}$. ∇∇∇ Similarly we have the following theorem for the dynamic output feedback control.

Theorem 2.3.4 *The optimal LQ control with the generalized covariance constraints via dynamic output feedback law (2.6) can be solved by the following semi-definite programming:*

$$J_{\mathscr{C}_d} = min \quad tr(P_1) + tr(P_2)$$

subject to:

$$\begin{bmatrix} P_1 & R^{1/2}\hat{D}_cC & R^{1/2}\hat{C}_c & R^{1/2}\hat{D}_cF \\ C^T\hat{D}_cR^{1/2} & M_1 & I & 0 \\ \hat{C}_c^TR^{1/2} & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TR^{1/2} & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.45}$$

$$\begin{bmatrix} P_2 & Q^{1/2} & Q^{1/2}\Sigma_1 \\ Q^{1/2} & M_1 & I \\ \Sigma_1Q^{1/2} & I & \Sigma_1 \end{bmatrix} > 0 \tag{2.46}$$

$$\begin{bmatrix} -\Sigma_1 & -I & A+B\hat{D}_cC & A\Sigma_1+B\hat{C}_c & G+B\hat{D}_cF \\ -I & -M_1 & M_1A+\hat{B}_cC & \hat{A}_c & M_1G+\hat{B}_cF \\ A^T+C^T\hat{D}_c^TB^T & A^TM_1+C^T\hat{B}_c^T & -M_1 & -I & 0 \\ \Sigma_1A^T+\hat{C}_c^TB^T & \hat{A}_c^T & -I & -\Sigma_1 & 0 \\ G^T+C^T\hat{D}_c^TB^T & G^TM_1+F^T\hat{B}_c^T & 0 & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{2.47}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i+D_i\hat{D}_cC & C_i\Sigma_1+D_i\hat{C}_c & D_i\hat{D}_cF \\ C_i^T+C^T\hat{D}_c^TD_i^T & M_1 & I & 0 \\ \Sigma_1C_i^T+\hat{C}_c^TD_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TD_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.48}$$

Proof:

For the dynamic feedback case:

$$\begin{aligned} J &= tr(Q\Sigma_x)+tr(R\Sigma_u) \\ &= tr\left(\bar{Q}\Sigma_{cl}\right)+tr(R\Sigma_u) \\ &= tr\left(\bar{Q}\Sigma_{cl}\right)+tr\left(R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma_{cl}\begin{bmatrix} D_cC & C_c \end{bmatrix}^T+RD_cF\Omega F^TD_c^T\right) \\ &< tr\left(\bar{Q}\Sigma\right)+tr\left(R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma\begin{bmatrix} D_cC & C_c \end{bmatrix}^T+RD_cF\Omega F^TD_c^T\right) \\ &< tr(P_1)+tr(P_2) \end{aligned} \tag{2.49}$$

where $\bar{Q}=\begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$, and P_1 and P_2 are free parameters that satisfy the following inequalities respectively:

$$\begin{bmatrix} P_1 & R^{1/2}\begin{bmatrix} D_cC & C_c \end{bmatrix} & R^{1/2}D_cF \\ \begin{bmatrix} C^TD_c^T \\ C_c^T \end{bmatrix}R^{1/2} & \Sigma^{-1} & 0 \\ R^{1/2}F^TD_c^T & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.50}$$

$$\begin{bmatrix} P_2 & \bar{Q}^{1/2} \\ \bar{Q}^{1/2} & \Sigma^{-1} \end{bmatrix} > 0 \tag{2.51}$$

It can be verified that

$$\begin{bmatrix} Q^{1/2} & 0 \\ 0 & 0 \end{bmatrix}\cdot T_2^T=\begin{bmatrix} Q^{1/2} & Q^{1/2}\Sigma_1 \\ 0 & 0 \end{bmatrix} \tag{2.52}$$

Pre-multiplying the left side of (2.50) by $\begin{bmatrix} I & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I \end{bmatrix}$ and post-multiplying the right side of (2.50) by $\begin{bmatrix} I & 0 & 0 \\ 0 & T_2^T & 0 \\ 0 & 0 & I \end{bmatrix}$, (2.45) is obtained. Pre-multiplying the left side of (2.51) by $\begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix}$ and post-multiplying the right side of (2.51) by $\begin{bmatrix} I & 0 \\ 0 & T_2^T \end{bmatrix}$, (2.46) is obtained. In the proof of the Lemma 2.3.1 it is noted that we can let $\Sigma \to \Sigma_{cl}$. Also, we can let the free parameters $P_1 \to R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma\begin{bmatrix} D_cC & C_c \end{bmatrix}^T + RD_cF\Omega F^T D_c^T$ and $P_2 \to \bar{Q}\Sigma$. So $tr(P_1) + tr(P_2) \to J_{\mathscr{C}_s}$. ∇∇∇

Specially we can optimize the energy of the input signal: $J^{in} = \lim_{k\to\infty} E\left(u_k^T R u_k\right)$ subject to the generalized covariance constraints (2.10). The results can be obtained by letting $Q = 0$ in Theorem 2.3.3 and 2.3.4, and are presented as follows:

Corollary 2.3.1 *The minimized energy control via state feedback law (2.5) can be solved by semi-definite programming:*

$$J^e_{\mathscr{C}_s} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & R^{\frac{1}{2}}L \\ L^T R^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{2.53}$$

$$\begin{bmatrix} G\Omega G^T - \Sigma & A\Sigma + BL \\ \Sigma A^T + L^T B^T & -\Sigma \end{bmatrix} < 0 \tag{2.54}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_iL \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{2.55}$$

Corollary 2.3.2 *The minimized energy control via dynamic output feedback law (2.6) can be solved by the following semi-definite programming:*

$$J_{\mathscr{C}_d} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & R^{1/2}\hat{D}_cC & R^{1/2}\hat{C}_c & R^{1/2}\hat{D}_cF \\ C^T\hat{D}_cR^{1/2} & M_1 & I & 0 \\ \hat{C}_c^TR^{1/2} & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TR^{1/2} & 0 & 0 & I \end{bmatrix} \geq 0 \tag{2.56}$$

$$\begin{bmatrix} -\Sigma_1 & -I & A+B\hat{D}_cC & A\Sigma_1+B\hat{C}_c & G+B\hat{D}_cC \\ -I & -M_1 & M_1A+\hat{B}_cC & \hat{A}_c & M_1G+\hat{B}_cF \\ A^T+C^T\hat{D}_c^TB^T & A^TM_1+C^T\hat{B}_c^T & -M_1 & -I & 0 \\ \Sigma_1A^T+\hat{C}_c^TB^T & \hat{A}_c^T & -I & -\Sigma_1 & 0 \\ G^T+C^T\hat{D}_c^TB^T & G^TM_1+F^T\hat{B}_c^T & 0 & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{2.57}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i+D_i\hat{D}_cC & C_i\Sigma_1+D_i\hat{C}_c & D_i\hat{D}_cF \\ C_i^T+C^T\hat{D}_c^TD_i^T & M_1 & I & 0 \\ \Sigma_1C_i^T+\hat{C}_c^TD_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TD_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.58}$$

Similarly we can minimize the output variance $J^{out} = \lim_{k\to\infty} E\left(x_k^TQx_k\right)$ subject to the generalized covariance constraints (2.10). A similar problem, called the input variance control problem, which is to minimize the output variance J^{out} subject to the scalar input variance constraints, was considered for the continuous LTI system (Hsieh *et al.* 1989). Now we extend the scalar constraints to the covariance constraints of the controlled variables. This problem can also be solved by semi-definite programming, and the proof can be done by letting $R = 0$ in Theorem 2.3.3 and 2.3.4. The results are listed as follows:

Corollary 2.3.3 *The input variance constraint control via state feedback can be solved by semi-definite programming:*

$$J_{\mathscr{C}_s} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & Q^{\frac{1}{2}}\Sigma \\ \Sigma Q^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{2.59}$$

$$\begin{bmatrix} G\Omega G^T-\Sigma & A\Sigma+BL \\ \Sigma A^T+L^TB^T & -\Sigma \end{bmatrix} < 0 \tag{2.60}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma+D_iL \\ \Sigma C_i^T+L^TD_i^T & \Sigma \end{bmatrix} > 0 \tag{2.61}$$

Corollary 2.3.4 *The input variance control subject to the generalized covariance constraints (9) can be solved by the following semi-definite programming for the dynamic output feedback:*

$$J_{\mathscr{C}_d} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & Q^{1/2} & Q^{1/2}\Sigma_1 \\ Q^{1/2} & M_1 & I \\ \Sigma_1 Q^{1/2} & I & \Sigma_1 \end{bmatrix} \geq 0 \tag{2.62}$$

$$\begin{bmatrix} -\Sigma_1 & -I & A+B\hat{D}_cC & A\Sigma_1+B\hat{C}_c & G+B\hat{D}_cC \\ -I & -M_1 & M_1A+\hat{B}_cC & \hat{A}_c & M_1G+\hat{B}_cF \\ A^T+C^T\hat{D}_c^TB^T & A^TM_1+C^T\hat{B}_c^T & -M_1 & -I & 0 \\ \Sigma_1A^T+\hat{C}_c^TB^T & \hat{A}_c^T & -I & -\Sigma_1 & 0 \\ G^T+C^T\hat{D}_c^TB^T & G^TM_1+F^T\hat{B}_c^T & 0 & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{2.63}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i+D_i\hat{D}_cC & C_i\Sigma_1+D_i\hat{C}_c & D_i\hat{D}_cF \\ C_i^T+C^T\hat{D}_c^TD_i^T & M_1 & I & 0 \\ \Sigma_1C_i^T+\hat{C}_c^TD_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TD_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{2.64}$$

2.4 Examples

Several simulation examples are used to demonstrate the results presented in the last section. MATLAB LMI toolbox (Gahinet *et al.* 1995) is used, and the MATLAB code is available from the author.

2.4.1 Example 1

This example is taken from (Westerlund 1981, Mäkilä *et al.* 1984). A stochastic model of a dry process cement kiln with a capacity of 1000t of clinker a day is given as follows:

$$y_{k+1}+\begin{bmatrix} -0.914 & -0.08 \\ 0.126 & -0.917 \end{bmatrix} y_k = \begin{bmatrix} 2.091 & -0.0744 \\ -0.211 & -0.0156 \end{bmatrix} u_k + e_{k+1} + \begin{bmatrix} 0 & 0 \\ 0 & 0.715 \end{bmatrix} e_k$$

A state space realization of the above model is:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.9140 & 0.08 \\ -0.1260 & 0.9170 \end{bmatrix} x_k + \begin{bmatrix} 2.0910 & -0.0744 \\ -0.2110 & -0.0156 \end{bmatrix} u_k + \begin{bmatrix} 0.9140 & 0.08 \\ -0.126 & 1.6320 \end{bmatrix} e_k \\ y_k &= x_k + e_k \end{aligned}$$

The minimum variance control strategy gives the output variances $E\left(y_1^2\right) = 0.0644, E\left(y_2^2\right) = 0.0214$; but the variances for the control signal $E\left(u_1^2\right) = 0.0148, E\left(u_2^2\right) = 108$ are not acceptable (Westerlund 1981, Mäkilä *et al.* 1984). It is required that the input variances satisfy $E\left(u_1^2\right) < 0.004, E\left(u_2^2\right) < 1.5$. With the algorithm described in last section, a full order controller that minimizes the output variances ($Q = I_2$) is obtained:

$$\begin{aligned} x_{k+1}^C &= \begin{bmatrix} -0.0001 & -0.1165 \\ 0.0000 & -0.6754 \end{bmatrix} x_k^C + \begin{bmatrix} -0.3948 & -0.3023 \\ 0.1232 & -1.5412 \end{bmatrix} y_k \\ u_k &= \begin{bmatrix} 0.0000 & 0.0986 \\ -0.0001 & 1.2065 \end{bmatrix} x_k^C + \begin{bmatrix} -0.1719 & 0.2117 \\ 2.1475 & 2.9609 \end{bmatrix} y_k \end{aligned} \tag{2.65}$$

It can be verified that with the above controller implemented, the output covariance matrix is $\begin{bmatrix} 0.0938 & 0.0480 \\ 0.0480 & 0.1890 \end{bmatrix}$ and the input covariance is $\begin{bmatrix} 0.0040 & 0.0092 \\ 0.0092 & 1.4992 \end{bmatrix}$.

2.4.2 Example 2

The second example is taken from the thickness control of plastic film in blown film line (Mäkilä *et al.* 1984). The variation of the film should be controlled in a very tight thickness limit. The model from an identification experiment was used to control the thickness of film:

$$y_{k+1} - 0.986y_k = -2.87u_{k-1} - 1.86u_{k-2} + e_{k+1} - 0.912e_k$$

It is assumed that $E\left(e_i e_j^T\right) = R_e \delta (i-j)$. One of the state space realization of the above system is:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 2.0 & 0.986 \end{bmatrix} x_k + \begin{bmatrix} -0.93 \\ -0.7175 \\ 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 0 \\ 0.037 \end{bmatrix} e_k \\ y_k &= \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} x_k + e_k \end{aligned} \tag{2.66}$$

The minimum variance control law will make the process output variance $E\left(y^2\right) = 1.0054R_e$ and the input variance $E\left(u^2\right) = 1.1 \times 10^{-3} R_e$. It was found (Mäkilä *et al.* 1984) that if the output variance was enlarged to $E\left(y^2\right) = 1.0115R_e$, then the input variance could be reduced significantly. Setting the output variance constraint $E\left(y^2\right) < 1.0115R_e$ and using the algorithm presented in section 3 with $Q = 0$ and $R = 1$, we can obtain the following controller:

$$\begin{aligned} x_{k+1}^C &= \begin{bmatrix} -0.1048 & -0.1229 & -0.0561 \\ 0.7743 & -0.2009 & -0.0919 \\ -0.0438 & 1.9923 & 0.9084 \end{bmatrix} x_k^C + \begin{bmatrix} 0.0024 \\ 0.0037 \\ -0.0370 \end{bmatrix} y_k \\ u_k &= \begin{bmatrix} -0.3133 & -0.2796 & -0.1279 \end{bmatrix} x_k^C + 0.0051 y_k \end{aligned} \tag{2.67}$$

With this controller the control energy is $4.5736 \times 10^{-5} R_e$ and the output variance is $1.0115R_e$.

2.4.3 Example 3

The third example is from (Grigoriadis and Skelton 1997). The state space model is as follows:

$$
\begin{aligned}
x_{k+1} &= \begin{bmatrix} 0.8778 & 0.4782 & 0 & 0 \\ -0.4782 & 0.8730 & 0 & 0 \\ 0 & 0 & -0.4075 & 0.2251 \\ 0 & 0 & -3.6011 & -0.4165 \end{bmatrix} x_k \\
&+ \begin{bmatrix} 0.0718 & -0.1222 \\ 0.2811 & -0.4782 \\ 0.0837 & 0.1759 \\ 0.2141 & 0.4501 \end{bmatrix} u_k \\
&+ \begin{bmatrix} 0.0214 & 0.0055 & 0 & 0 \\ -0.0055 & 0.0214 & 0 & 0 \\ 0 & 0 & 0.0102 & 0.0039 \\ 0 & 0 & -0.0629 & 0.0101 \end{bmatrix} \varsigma_k
\end{aligned}
$$

where the covariance of ς_k is $100I_4$. With the state feedback controller:

$$
K = \begin{bmatrix} -0.7604 & -3.4667 & 9.5586 & 0.2955 \\ -0.0010 & 1.2221 & 3.3200 & -0.2642 \end{bmatrix}
$$

the covariance for the state is:

$$
\Sigma = \begin{bmatrix} 0.1378 & -0.0509 & 0.0003 & -0.0115 \\ -0.0509 & 0.1256 & 0.0176 & 0.0008 \\ 0.0003 & 0.0176 & 0.0301 & -0.0769 \\ -0.0115 & 0.0008 & -0.0769 & 0.5218 \end{bmatrix}
$$

The control input energy $E\left(u^2\right) = 3.3546$. Let the variance of each state variable satisfy the corresponding upper bound. Minimize the control input energy by using the proposed algorithm and we can obtain the following state feedback controller:

$$
K = \begin{bmatrix} -0.0645 & -0.6584 & 1.6018 & -0.0641 \\ -0.210 & 0.6362 & 1.7428 & -0.1799 \end{bmatrix}
$$

With this controller the covariance for the state variables is:

$$
\Sigma = \begin{bmatrix} 0.1377 & -0.0509 & 0.0003 & -0.0115 \\ -0.0509 & 0.1255 & 0.0175 & 0.0006 \\ 0.0003 & 0.0175 & 0.0300 & -0.0769 \\ -0.0115 & 0.0006 & -0.0769 & 0.5217 \end{bmatrix}
$$

The control energy is 0.3557.

2.5 Conclusion

The generalized covariance control has been considered in this chapter. It is proved that the feasibility of the GCC problem is equivalent to the feasibility of several LMIs for both state feedback and dynamic output feedback. The controller can also be parameterized by the solutions to the LMIs. If the GCC problem is feasible, i.e. the feasible controller set is not empty, we can optimize some quadratic loss functions, including the LQ index, the input energy and/or the output variance. It is shown that all these optimizations over the feasible controller set can be solved by semi-definite programming methods that guarantee the global convergence. Several simulation examples have been used to validate the results.

3

LQ Control with Generalized Covariance Constraint for Continuous-time Systems

3.1 Introduction

In this chapter, the optimal LQ controller design subject to the generalized covariance constraints (*LQGCC*) for continuous-time linear time-invariant systems is studied. This work is a theoretical extension of our previous work on the discrete-time system and the new results are applied to two simulated chemical engineering processes. In particular, the feasibility of the generalized covariance constrained problem is first addressed and then it is shown that the global optimal LQGCC controller can be obtained by a semi-definite programming approach. The rest of the chapter is organized as follows: In Section 2, the problem statements and the preliminary results are presented. The main results are stated in Section 3. In Section 3.1, the necessary and sufficient conditions for the feasibility of the generalized covariance constrained problem are stated in terms of several LMIs for both the state feedback and the dynamic output feedback. The feasible controller set can be parameterized by the solutions to the LMIs. In Section 3.2, it is shown that the LQGCC controller can be obtained by solving a semi-definite programming problem. Numerical examples are given in Section 4 and concluding remarks are presented in Section 5. Considering some mathematical proofs are similar to those of the previous chapter, we put all mathematical proofs in appendices.

[1]Some versions of this chapter was presented in *American Control Conf.* (Huang *et al.* 2003*a*) or accepted by *Dynamics of continuous, discrete, and impulse systems* (Huang *et al.* 2003*b*)

3.2 Problem statement

A finite-dimensional continuous-time linear system $\mathscr{P}$ is given as follows:

$$\begin{array}{rcl} \dot{x} & = & Ax(t)+Bu(t)+G\varsigma(t) \\ y(t) & = & Cx(t)+F\varsigma(t) \\ z_i(t) & = & C_i x(t)+D_i u(t), i=1...l \end{array} \tag{3.1}$$

where $x(t) \in R^n$ is the system state vector, $u(t) \in R^m$ is the system input, $y(t) \in R^p$ is the system measurement, $\varsigma(t) \in R^g$ is the external disturbance and measurement noise, and $z_i(t) \in R^{p_i}$ is the i-th controlled vector. It is assumed that the external disturbance $\varsigma(t)$ is white noise satisfying:

$$\begin{array}{rcl} E(\varsigma(t)) & = & 0 \\ E\left(\varsigma(t)\varsigma(s)^T\right) & = & \Omega\delta(t-s) \end{array} \tag{3.2}$$

Both the state feedback controller and the dynamic output feedback controller are considered in this chapter. A state feedback controller is given as follows:

$$u(t) = Kx(t) \tag{3.3}$$

and a full order dynamic output feedback controller is described as:

$$\begin{array}{rcl} \dot{x}_c(t) & = & A_c x_c(t)+B_c y(t) \\ u(t) & = & C_c x_c(t)+D_c y(t) \end{array} \tag{3.4}$$

where K, A_c, B_c, C_c and D_c are unknown matrices with proper dimensions. With the state feedback control law (3.3) or the full order dynamic control law (3.4), the closed-loop system is

$$\begin{array}{rcl} \dot{X}(t) & = & A_{cl}X(t)+G_{cl}\varsigma(t) \\ Z_i(t) & = & \bar{C}_i X(t)+\bar{D}_i\varsigma(t), i=1...l \end{array} \tag{3.5}$$

where $X(t) = x(t)$, $A_{cl} = A+BK$, $G_{cl} = G$, $\bar{C}_i = C_i + D_i K$ and $\bar{D}_i = 0$ for the state feedback case; $X(t) = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}$, $A_{cl} = \begin{bmatrix} A+BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$, $G_{cl} = \begin{bmatrix} G+BD_cF \\ B_cF \end{bmatrix}$, $\bar{C}_i = \begin{bmatrix} C_i+D_iD_cC & D_iC_c \end{bmatrix}$ and $\bar{D}_i = D_iD_cF$ for the dynamic output feedback case.

The covariance of the closed-loop state vector is defined as:

$$\Sigma_{cl} = \lim_{t\to\infty} E\left(X(t)X(t)^T\right) \tag{3.6}$$

If the closed-loop system (3.5) is asymptotically stable, it is well known that the closed-loop state covariance should satisfy the following equation:

$$A_{cl}\Sigma_{cl}+\Sigma_{cl}A_{cl}^T+G_{cl}\Omega G_{cl}^T = 0 \tag{3.7}$$

The state covariance assignment problem for continuous-time LTI systems has been studied extensively (Hotz and Skelton 1987, Skelton and Delorenzo 1985, Yasuda *et al.* 1993) and several necessary and sufficient conditions for a positive definite matrix to be assigned to the state vector have been presented there. One important feature of the covariance assignment theory is that the controller can be parameterized in time domain and additional performance can be optimized over the controller set. For example, minimizing the input energy subject to a specified closed-loop state covariance was studied by Grigoriadis and Skelton (1997) , and the analytical solutions to this problem were given for both discrete-time and continuous-time LTI systems. Since specifying the covariance upper bounds on the controlled variables gives the control engineers more freedom than strictly assigning a covariance to the state vector, the variance or covariance constrained problem for continuous-time LTI systems was studied by Hsieh *et al.* (1989) and Zhu *et al.* (1997). The covariance constrained problem is extended here by introducing the input term into the controlled variables. The generalized covariance constrained problem is stated as follows:

Problem 3.2.1 *Generalized covariance constrained (GCC) problem: For the continuous-time LTI system (3.1), find a state feedback controller as given by (3.3) or a dynamic feedback controller as given by (3.4) such that the closed-loop system is asymptotically stable and the covariance of the controlled variable* $Z_k^i (i = 1 \ldots l)$ *satisfies*

$$\Phi_i = \lim_{t \longrightarrow \infty} E\left(Z_i(t)) Z_i(t)^T\right) < \bar{\Phi}_i \tag{3.8}$$

where $\bar{\Phi}_i \, (i = 1 \ldots l)$ *is some pre-specified positive definite matrix.*

If there exists a controller such that the closed-loop system given by (3.5) is asymptotically stable and the covariance constraints given by (3.8) are satisfied, then we call that the GCC problem is feasible. In the next section, it will be shown that the feasibility of GCC is equivalent to the feasibility of certain linear matrix inequalities (*LMIs*). It is worthwhile to point out that a numerically robust algorithm (Gahinet *et al.* 1995) is available to solve LMIs, and the feasibility of GCC is easily verified.

Suppose that the GCC problem is feasible via state feedback or dynamic output feedback, the feasible state feedback controller set is denoted by $\mathscr{C}_s$ and the feasible dynamic output controller set is denoted by $\mathscr{C}_d$. Our final goal is to optimize the LQ performance over the controller set $\mathscr{C}_s$ or $\mathscr{C}_d$. The LQ performance can also be written as:

$$\begin{aligned} J &= \lim_{t \longrightarrow \infty} E\left(x(t)^T Q x(t) + u(t)^T R u(t)\right) \\ &= tr(Q\Sigma_x) + tr(R\Sigma_u) \end{aligned}$$

where Σ_x is the stationary covariance matrix for the plant state vector $x(t)$, Σ_u is the stationary covariance matrix of the input vector $u(t)$. Now what we want to solve is the following optimization problem, called the optimal LQ control with the generalized covariance constraints:

Problem 3.2.2

$$\begin{aligned} J_{\mathscr{C}_s} &= \inf_{\mathscr{C}\in\mathscr{C}_s} J \\ J_{\mathscr{C}_d} &= \inf_{\mathscr{C}\in\mathscr{C}_d} J \end{aligned} \tag{3.9}$$

3.3 Main results

To state the main results, we need the following lemma:

Lemma 3.3.1 *The closed-loop system (3.5) is asymptotically stable and satisfies the constraints (3.8) if and only if there exists a matrix $\Sigma > 0$ such that*

$$A_{cl}\Sigma + \Sigma A_{cl}^T + G_{cl}\Omega G_{cl}^T \;<\; 0 \tag{3.10}$$

$$\bar{C}_i\Sigma\bar{C}_i^T + \bar{D}_i\Omega\bar{D}_i^T \;<\; \bar{\Phi}_i \tag{3.11}$$

where $i = 1, \ldots, l$.

The proof is similar to the proof of Lemma 2.3.1 and it is omitted here for brevity.

Before we start considering the LQGCC problem, we need to ensure that the specified generalized covariance constraints are feasible.

3.3.1 Feasibility of the GCC problem

The GCC problem itself is a nonlinear multi-objective controller design problem. Generally speaking, multi-objective control design is difficult and still remains open, especially when several criteria, such as $\mathscr{H}_2$, $\mathscr{H}_\infty$ and pole placement are considered. For example, the multiple-objective controller design problem was studied by Masubuchi *et al.* (1995) and Scherer *et al.* (1997) with the Lyapunov paradigm method, in which one Lyapunov function was used for all the constraints. In this section it is shown that for both the state feedback and dynamic output feedback when only the covariance bound criterion is considered all the control objectives can be nicely unified by one Lyapunov function.

Proposition 3.3.1 *The GCC problem is feasible for the continuous-time system (3.1) via state* *feedback if and only if there exist matrices* $\Sigma > 0$ *and* L *such that*

$$A\Sigma + \Sigma A + BL + L^T B^T + G\Omega G^T \; < \; 0 \tag{3.12}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_i L \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} \; > \; 0 \tag{3.13}$$

The proof follows from Lemma 3.3.1 and the well known matrix transformation (Boyd and Barratt 1991) directly so that it is omitted here. For the state feedback case, we can transfer the nonlinear matrix inequalities to linear matrix inequalities by variable transformation (Boyd and Barratt 1991). It is long believed that such kind of transformation does not exist for dynamic output feedback. However, the recent work (Masubuchi *et al.* 1995, Scherer *et al.* 1997) shows that we can also linearize the matrix variables by some nonlinear matrix transformation. We will extend this technique (Scherer *et al.* 1997) to solve the GCC problem for the dynamic output feedback. The result is stated as follows:

Proposition 3.3.2 *The GCC problem is feasible via a dynamic controller for the continuous-time LTI system (3.1) if and only if there exist matrices* $\Sigma_1 > 0$, $M_1 > 0$, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$, $\hat{D}_c$ *such that*

$$\begin{bmatrix} A\Sigma_1 + \Sigma_1 A^T + B\hat{C}_c + \hat{C}_c^T B^T & A + B\hat{D}C + \hat{A}_c^T & G + B\hat{D}_c F \\ \hat{A}_c + A^T + C^T\hat{D}_c^T B^T & M_1 A + A^T M_1 + \hat{B}_c C + C^T \hat{B}_c^T & M_1 G + \hat{B}_c F \\ G^T + F^T \hat{D}_c^T B^T & G^T M_1 + F^T \hat{B}_c^T & -\Omega^{-1} \end{bmatrix} \; < \; 0 \tag{3.14}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i + D_i\hat{D}_c C & C_i\Sigma_1 + D_i\hat{C}_c & D_i\hat{D}_c F \\ C_i^T + C^T\hat{D}_c^T D_i^T & M_1 & I & 0 \\ \Sigma_1 C_i^T + \hat{C}_c^T D_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^T D_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} \; > \; 0 \tag{3.15}$$

Furthermore if the LMIs (3.14) and (3.15) are feasible, the dynamic controller can be parameterized as

$$D_c = \hat{D}_c \tag{3.16}$$

$$C_c = \left(\hat{C}_c - \hat{D}_c C\Sigma_1\right)\Sigma_2^{-T} \tag{3.17}$$

$$B_c = M_2^{-1}\left(\hat{B}_c - M_1 B\hat{D}_c\right) \tag{3.18}$$

$$A_c = M_2^{-1}\left(\hat{A}_c - M_1 A\Sigma_1 - M_1 BD_c C\Sigma_1 - M_2 B_c C\Sigma_1 - M_1 BC_c\Sigma_2^T\right)\Sigma_2^{-T} \tag{3.19}$$

where $M_2 \in R^{n\times n}$ *and* $\Sigma_2 \in R^{n\times n}$ *are any matrices satisfying* $\Sigma_2 M_2^T = I - \Sigma_1 M_1$.

The proof is given in the appendix A. The results given in this subsection are stated in terms of LMIs, which can be easily verified by using MATLAB LMI toolbox (Gahinet *et al.* 1995).

3.3.2 Optimization with the generalized covariance constraints

If the generalized covariance constraints (3.8) are feasible by the state feedback or the dynamic output feedback, then there exists an infimum of the LQ performance over the non-empty state feedback controller set $\mathscr{C}_s$ or the non-empty dynamic output feedback controller set $\mathscr{C}_d$. Minimizing the LQ performance over $\mathscr{C}_s$ or $\mathscr{C}_d$ is considered in this subsection. If let $Q = 0$ in (3.9) and $D_i = 0$, the LQGCC is then reduced to the minimum energy covariance control problem (Hsieh *et al.* 1989, Zhu *et al.* 1997) subject to covariance constraints (3.8); if let $R = 0$ in (3.9) and $D_i = 0$, the LQGCC is then reduced to input variance constraints problem, which has been considered by Hsieh *et al.* (1989) and is dual to the minimum energy covariance control. Generally speaking, all the problems mentioned above (the constrained LQ control, the minimum energy output covariance control problem and the input variance constrained problem) are nonlinear optimization problem, and the algorithms available so far (Hsieh *et al.* 1989, Mäkilä *et al.* 1984, Zhu *et al.* 1997) could not guarantee the global optimality. In this subsection it will be shown that all these problems can be solved by a semi-definite programming approach, by which we mean minimizing a linear objective function subject to some LMI constraints, for both the state feedback and dynamic feedback. The convexity of semi-definite programming ensures the global optimality. The results for the LQGCC are presented in the following Propositions.

Proposition 3.3.3 *The optimal state feedback LQ control subject to the generalized covariance constraints can be obtained by solving the following semi-definite programming problem:*

$$J_{\mathscr{C}_s} = min \quad tr(P_1) + tr(P_2)$$

subject to:

$$\begin{bmatrix} P_1 & R^{\frac{1}{2}}L \\ L^T R^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{3.20}$$

$$\begin{bmatrix} P_2 & Q^{\frac{1}{2}}\Sigma \\ \Sigma Q^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{3.21}$$

$$A\Sigma + \Sigma A + BL + L^T B^T + G\Omega G^T < 0 \tag{3.22}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_i L \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{3.23}$$

The proof is similar to that for the discrete-time system given in last chapter and is stated in Appendix B for the sake of completeness. A similar result for the dynamic output feedback control is presented as follows:

Proposition 3.3.4 *The optimal LQ control subject to the generalized covariance constraints via dynamic output feedback can be obtained by solving the following semi-definite programming problem:*

$$J_{\mathscr{C}_d} = min \quad tr(P_1) + tr(P_2)$$

subject to:

$$\begin{bmatrix} P_1 & R^{1/2}\hat{D}_c C & R^{1/2}\hat{C}_c & R^{1/2}\hat{D}_c F \\ C^T\hat{D}_c R^{1/2} & M_1 & I & 0 \\ \hat{C}_c^T R^{1/2} & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^T R^{1/2} & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{3.24}$$

$$\begin{bmatrix} P_2 & Q^{1/2} & Q^{1/2}\Sigma_1 \\ Q^{1/2} & M_1 & I \\ \Sigma_1 Q^{1/2} & I & \Sigma_1 \end{bmatrix} > 0 \tag{3.25}$$

$$\begin{bmatrix} A\Sigma_1 + \Sigma_1 A^T + B\hat{C}_c + \hat{C}_c^T B^T & A + B\hat{D}C + \hat{A}_c^T & G + B\hat{D}_c F \\ \hat{A}_c + A^T + C^T\hat{D}_c^T B^T & M_1 A + A^T M_1 + \hat{B}_c C + C^T\hat{B}_c^T & M_1 G + \hat{B}_c F \\ G^T + F^T\hat{D}_c^T B^T & G^T M_1 + F^T\hat{B}_c^T & -\Omega^{-1} \end{bmatrix} < 0 \tag{3.26}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i + D_i\hat{D}_c C & C_i\Sigma_1 + D_i\hat{C}_c & D_i\hat{D}_c F \\ C_i^T + C^T\hat{D}_c^T D_i^T & M_1 & I & 0 \\ \Sigma_1 C_i + \hat{C}_c^T D_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^T D_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{3.27}$$

The optimal controller has the same form as that in (3.3.2).

The proof is similar to that for the discrete-time system given in last chapter and is stated in Appendix B for the sake of completeness.

On the special case, one can minimize the input energy $J^e = \lim_{k\to\infty} E\left(u_k^T R u_k\right)$ subject to the generalized covariance constraints (3.8). The solution can be obtained through solving a semi-definite programming problem by letting $Q = 0$ in Propositions 3.3.3 and 3.3.4:

Corollary 3.3.1 *The minimized energy control subject to the generalized covariance constraints via state feedback can be obtained by solving the following semi-definite programming problem:*

$$J^e_{\mathscr{C}_s} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & R^{\frac{1}{2}}L \\ L^T R^{\frac{1}{2}} & \Sigma \end{bmatrix} > 0 \tag{3.28}$$

$$A\Sigma + \Sigma A + BL + L^T B^T + G\Omega G^T < 0 \tag{3.29}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i\Sigma + D_i L \\ \Sigma C_i^T + L^T D_i^T & \Sigma \end{bmatrix} > 0 \tag{3.30}$$

Corollary 3.3.2 *The minimized energy control subject to the generalized covariance control via dynamic feedback can be obtained by solving the following semi-definite programming problem:*

$$J^e_{\mathscr{C}_d} = min \quad tr(P)$$

subject to:

$$\begin{bmatrix} P & R^{1/2}\hat{D}_cC & R^{1/2}\hat{C}_c & R^{1/2}\hat{D}_cF \\ C^T\hat{D}_cR^{1/2} & M_1 & I & 0 \\ \hat{C}_c^TR^{1/2} & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TR^{1/2} & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{3.31}$$

$$\begin{bmatrix} A\Sigma_1+\Sigma_1A^T+B\hat{C}_c+\hat{C}_c^TB^T & A+B\hat{D}C+\hat{A}_c^T & G+B\hat{D}_cF \\ \hat{A}_c+A^T+C^T\hat{D}_c^TB^T & M_1A+A^TM_1+\hat{B}_cC+C^T\hat{B}_c^T & M_1G+\hat{B}_cF \\ G^T+F^T\hat{D}_c^TB^T & G^TM_1+F^T\hat{B}_c^T & -\Omega^{-1} \end{bmatrix} < 0 \tag{3.32}$$

$$\begin{bmatrix} \bar{\Phi}_i & C_i+D_i\hat{D}_cC & C_i\Sigma_1+D_i\hat{C}_c & D_i\hat{D}_cF \\ C_i^T+C^T\hat{D}_c^TD_i^T & M_1 & I & 0 \\ \Sigma_1C_i^T+\hat{C}_c^TD_i^T & I & \Sigma_1 & 0 \\ F^T\hat{D}_c^TD_i^T & 0 & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{3.33}$$

Semi-definite programming can be solved by available numerically robust interior point algorithms (Boyd and Barratt 1991, Gahinet *et al.* 1995). Compared with the available algorithms (Hsieh *et al.* 1989, Mäkilä *et al.* 1984, Zhu *et al.* 1997) semi-definite programming approach not only guarantees the global optimality but is also numerically efficient. For example, a Lyapunov equation and a Riccati equation are required to solve at each iteration according to the algorithm by Zhu *et al.* (1997) .

In the same way, one can obtain the solution to the input variance constraints problem through solving a semi-definite programming problem by letting $R=0$ in Propositions 3.3.3 and 3.3.4.

3.4 Simulation results

3.4.1 Example 1

The first example is a stirred tank reactor (Wu 2001), and in this example it is illustrated how to make the LQGCC controller track the setpoint change in addition to regulatory control. Stirred tank reactor is a common operating unit in process industry. The reaction discussed here is a first order, irreversible, exothermic kinetic reaction ($A \rightarrow B$). The mass and energy balances of the reaction are described as following nonlinear differential equations:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{Q_f}{V}\left(C_{Af}-C_A\right)-k_0C_Ae^{-\frac{E_a}{R_IT}} \\ \frac{dT}{dt} &= \frac{Q_f}{V}\left(T_f-T\right)+\frac{k_0C_A}{C_p}\Delta He^{-\frac{E_a}{R_IT}}-\frac{UA_h}{VC_p}\left(T-T_c\right) \end{aligned} \tag{3.34}$$

where C_A is the reactant concentration, T the reactor temperature, A_h the reaction area, C_{Af} the feed concentration, C_p the heat capacity, E_a the activation energy, ΔH the heat of the reaction, k_0 the reaction rate constant, Q_f the feed flow rate, R_I the ideal gas temperature, U the overall heat transfer coefficient, V the reactor volume, and the manipulated variable is the feed temperature T_f. The nonlinear CSTR model (3.34) can be simplified by the normalization method:

$$\begin{aligned} \dot{x}_1 &= -\alpha x_1 + D_a(1-x_1)e^{\frac{\gamma x_2}{\gamma+x_2}} \\ \dot{x}_2 &= -(\alpha+\beta)x_2 + BD_a(1-x_1)e^{\frac{\gamma x_2}{\gamma+x_2}} + \alpha d + \beta u \end{aligned} \tag{3.35}$$

where $x_1 = \frac{C_{Af}-C_A}{C_{Af}}$, $x_2 = \gamma\frac{T-T_{f0}}{T_{f0}}$, $u = \gamma\frac{T_c-T_{f0}}{T_{f0}}$, $x_2 = \gamma\frac{T_f-T_{f0}}{T_{f0}}$, $B = -\frac{\gamma\Delta H C_{Af}}{C_p T_{f0}}$, $D_a = k_0 e^{-\gamma}$, $\beta = \frac{UA_h}{VC_p}$, $\gamma = \frac{E_a}{R_I T_{f0}}$, $\alpha = \frac{V}{Q_f}$, $d = \frac{\gamma(T_f - T_{f0})}{T_{f0}}$ and T_{f0} is the nominal feed temperature.

This nonlinear model (3.35) is linearized at the equilibrium point $(x_1, x_2, u) = (0.3, 1.96, 7.5)$, and the linear model obtained is as follows:

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ B(C_{11}+\alpha) & BC_{12}-\alpha-\beta \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u + \begin{bmatrix} 0 \\ \beta \end{bmatrix} d \tag{3.36}$$

For this linear model (3.36) a dynamic controller is designed to minimize the LQ performance with $Q = I_2$ and $R = 1$ subject to the variance constraints: $var(x_1) < 0.001$ and $var(x_2) < 0.005$. According to the algorithm in section 3 the controller is calculated as follows:

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} -2.1 & 0.2 \\ -115.3 & -1917.1 \end{bmatrix} x_c + \begin{bmatrix} -0.1 & -0.2 \\ -45.2 & -1923.9 \end{bmatrix} y \\ u &= 10^{-5}\begin{bmatrix} 0.4642 & 0.00559 \end{bmatrix} x_c + \begin{bmatrix} 0.0036 & -0.0141 \end{bmatrix} y \end{aligned} \tag{3.37}$$

The closed-loop responses of the system are shown in Figure 1 and Figure 2. The initial point of the CSTR is at $(0.4, 2.4)$, and the LQGCC controller brings the state to the equilibrium point. The calculated variance of the normalized concentration x_1 is 0.0001 and the calculated variance of the normalized reactor temperature x_2 is 0.0044.

In practice, it is desired for the normalized concentration x_1 to track the set-point. However, the controller (3.37) cannot eliminate the tracking error. To make the concentration track the set point change, another differential equation is introduced to the normalized CSTR model(3.35):

$$\dot{x}_3 = x_1 - r \tag{3.38}$$

where r is the set-point for the concentration. Based on the same performance specification as

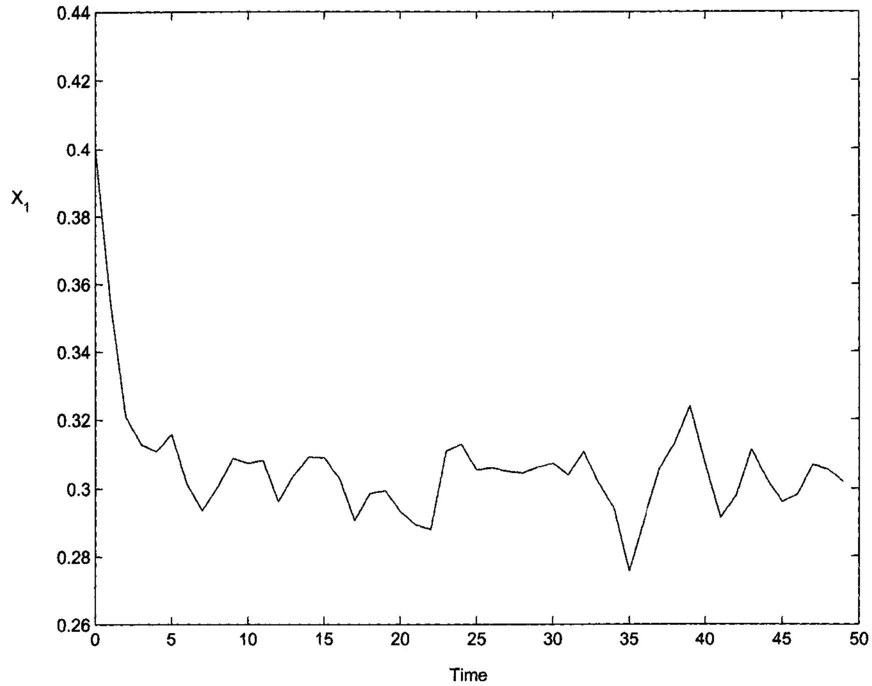

Figure 3.1: Closed-loop response of normalized concentration x_1

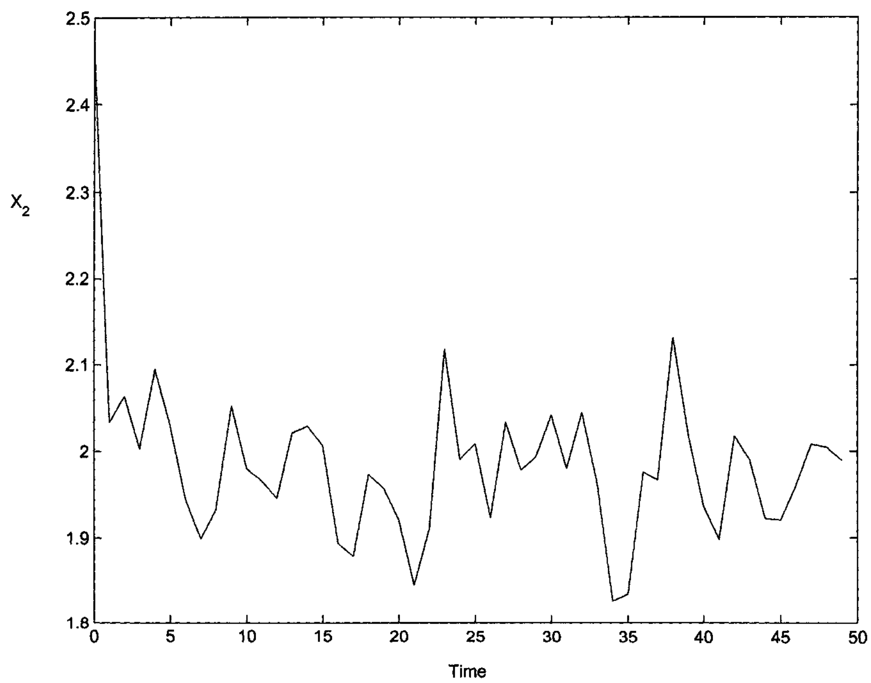

Figure 3.2: Closed-loop response of normalized reactor temperature x_2

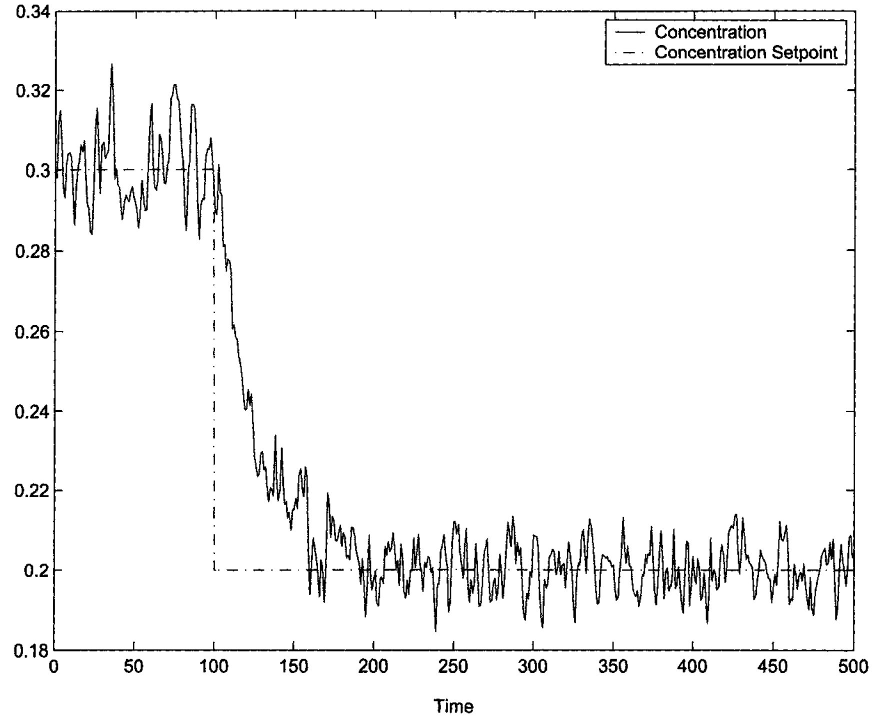

Figure 3.3: Normalized concentration tracks the setpoint changes

mentioned above the LQGCC controller is calculated for the augmented system:

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} -2.1 & 0.2 & -0.1 \\ -116.6 & -1736.9 & -21.6 \\ 0.9 & 0 & -0.6 \end{bmatrix} x_c + \begin{bmatrix} -0.1 & -0.2 & -0.1 \\ -44.7 & -1743.7 & -22.1 \\ -0.1 & 0 & -0.6 \end{bmatrix} e \\ u &= \begin{bmatrix} -0.0009223 & 0.0000566 & -0.0002579 \end{bmatrix} x_c + \begin{bmatrix} -0.3978 & -0.2748 & -0.9952 \end{bmatrix} e \end{aligned} \tag{3.39}$$

where $e = \begin{bmatrix} x_1 - r \\ x_2 - 1.96 \\ x_3 \end{bmatrix}$. Figure 3 shows that when the setpoint of the normalized concentration changes from 0.3 to 0.2 the controller makes the concentration track the setpoint change. The calculated variance of x_1 is 5×10^{-5}, the calculated variance of x_2 is 0.004, and they both satisfy the variance constraints.

3.4.2 Example 2

This example is a quadruple-tank process (Johansson 2000) with unstable random walk disturbance. The quadruple-tank process consists of four interconnected water tanks and two pumps. One interesting feature of the quadruple-tank process is that it has a multivariable zero, which can be located in either the left or the right half-plane by simply changing the position of a valve. The block diagram of the quadruple-tank is shown in Figure 4:

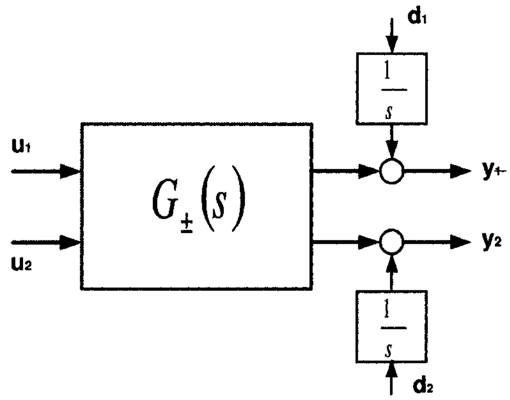

Figure 3.4: The block diagram of the quadruple-tank process with unstable disturbance

The process runs at two different operating points. The transfer function matrices of the process running at these two operating points are given below:

$$G_-(s) = \begin{bmatrix} \frac{2.6}{1+62s} & \frac{1.5}{(1+23s)(1+62s)} \\ \frac{1.4}{(1+30s)(1+90s)} & \frac{2.8}{1+90s} \end{bmatrix} \tag{3.40}$$

and

$$G_+(s) = \begin{bmatrix} \frac{1.5}{1+63s} & \frac{2.5}{(1+39s)(1+63s)} \\ \frac{2.5}{(1+56s)(1+91s)} & \frac{1.6}{1+91s} \end{bmatrix} \tag{3.41}$$

It can be verified that $G_-(s)$ has two left plane zeros: one is located at -0.0595, and the other is located at -0.0173; $G_+(s)$ has two zeros, one is located at -0.0565, and the other is located at 0.0130. To stabilize the quadruple-tank with the random walk disturbance the controller needs to have an integrator built-in. Since there is an integrator in the disturbance channel and there is an integrator in the controller, we can relocate the integrator as shown below and design a controller for this reconfigured system:

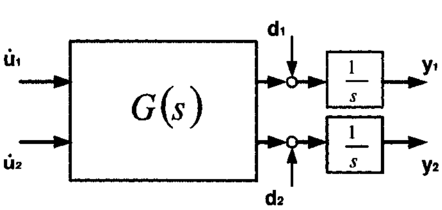

Figure 3.5: The block diagram of the quadruple-tank process after relocating the integrator

Suppose the controller obtained for the reconfigured system is denoted as C_s. The actual controller, implemented for the quadruple-tank process, is shown in the following block diagram:

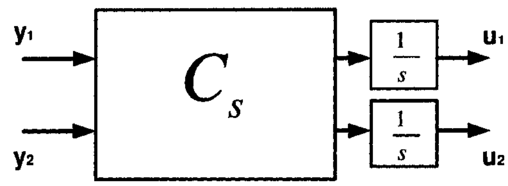

Figure 3.6: The actual implemented controller

With the following variance constraints: $var(y_1) < 0.1$, $var(y_2) < 0.2$, $var(\dot{u}_1) < 0.2$ and $var(\dot{u}_2) < 0.2$ the minimum LQ performance ($Q = I_6$ and $R = I_2$) for the minimum-phase system (3.40) is 0.1512. However, the same variance constraints are not feasible for the nonminimum-phase system (3.41). For the nonminimum-phase system (3.41) when the variance constraints are enlarged to $var(y_1) < 0.8$, $var(y_2) < 0.8$, $var(\dot{u}_1) < 1$ and $var(\dot{u}_2) < 1$, the GCC controller has a feasible solution, and the minimum LQ index ($Q = I_6$ and $R = I_2$) is 3.5726; but for the minimum-phase system (3.40) with the same variance constraints the minimum LQ index ($Q = I_6$ and $R = I_2$) is 0.1512. It is well known that the nonminimum-phase imposes limitation on achievable control performance and it is shown that with the same variance constraints the lower bound of the LQ performance for the nonminimum-phase system is much larger than that of the minimum-phase system.

3.5 Conclusion

The optimal LQ control with the generalized covariance constraints for continuous-time systems has been considered in this chapter. It is proved that the feasibility of the GCC problem is equivalent to the feasibility of several LMIs for both state feedback and dynamic output feedback. The controller can also be parameterized by the solutions of the LMIs. If the GCC problem is feasible, we can optimize some quadratic loss functions, including the LQ index, the input energy and/or the output variance, and it is shown that all these optimization problems can be solved by a semi-definite programming approach. Our approach ensures the global optimality. Several simulation examples have been used to verify our results.

3.6 Appendix A : proof of Proposition 3.3.2

Proof:

Necessity: If the GCC problem is feasible via full-order dynamic output feedback, there exists a controller with a form of (3.4) and $\Sigma > 0$ such that inequalities (3.10) and (3.11) are satisfied. According to Schur complement Lemma, inequality (3.11) is equivalent to the following LMI:

$$\begin{bmatrix} \bar{\Phi}_i & \bar{C}_i & \bar{D}_i \\ \bar{C}_i^T & \Sigma^{-1} & 0 \\ \bar{D}_i^T & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{3.42}$$

Partition Σ as

$$\Sigma = \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix} \tag{3.43}$$

Let $M = \Sigma^{-1} = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_3 \end{bmatrix}$, and it is easy to verify that

$$\Sigma_1 M_1 + \Sigma_2 M_2^T = I \tag{3.44}$$

$$\Sigma_1 M_2 + \Sigma_2 M_3 = 0 \tag{3.45}$$

$$M_1 \Sigma_2 + M_2 \Sigma_3 = 0 \tag{3.46}$$

Without losing generality, we can assume that matrices Σ_2 and M_2 are both full rank. Let $T_1 = \begin{bmatrix} I & 0 \\ M_1 & M_2 \end{bmatrix}$ and $T_2 = \begin{bmatrix} I & 0 \\ \Sigma_1 & \Sigma_2 \end{bmatrix}$, and it can be verified that the following equations hold:

$$T_1 A_{cl} \Sigma T_1^T = \begin{bmatrix} A\Sigma_1 + B\hat{C}_c & A + B\hat{D}_c C \\ \hat{A}_c & M_1 A + \hat{B}_c C \end{bmatrix} \tag{3.47}$$

$$T_1 G_{cl} = \begin{bmatrix} G + B\hat{D}_c F \\ M_1 G + \hat{B}_c F \end{bmatrix} \tag{3.48}$$

$$T_2 \Sigma^{-1} T_2^T = \begin{bmatrix} M_1 & I \\ I & \Sigma_1 \end{bmatrix} \tag{3.49}$$

$$T_2 \bar{C}^T = \begin{bmatrix} C_i^T + C^T \hat{D}_c^T D_i^T \\ C_i^T \hat{D}_c^T \end{bmatrix} \tag{3.50}$$

where

$$\hat{D}_c = D_c \tag{3.51}$$

$$\hat{C}_c = D_c C \Sigma_1 + C_c \Sigma_2^T \tag{3.52}$$

$$\hat{B}_c = M_1 B D_c + M_2 B_c \tag{3.53}$$

$$\hat{A}_c = M_1 A \Sigma_1 + M_1 B D_c C \Sigma_1 + M_2 B_c C \Sigma_1 + M_1 B C_c \Sigma_2^T + M_2 A_c \Sigma_2^T \tag{3.54}$$

Pre-multiplying and post-multiplying both sides of (3.10) with T_1 and T_1^T, and then using Schur complement Lemma and equations (3.47) $\sim$ (3.50) we can obtain (3.14).

Pre-multiplying and post-multiplying both sides of (3.42) with $\begin{bmatrix} I_n & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I_m \end{bmatrix}$ and $\begin{bmatrix} I_n & 0 & 0 \\ 0 & T_2^T & 0 \\ 0 & 0 & I_m \end{bmatrix}$ we obtain (3.15).

Sufficiency:

Both inequalities (3.14) and (3.15) imply that $\begin{bmatrix} M_1 & I \\ I & \Sigma_1 \end{bmatrix} > 0$, which in turn implies that

$$M_1 - \Sigma_1^{-1} > 0 \tag{3.55}$$

Suppose that $\Sigma_2 M_2^T = I - \Sigma_1 M_1$. We can show that matrices Σ_2 and M_2 are both full rank; otherwise $\Sigma_1^{-1} - M_1 = \Sigma_1^{-1}\Sigma_2 M_2^T$ will be rank deficient, which is in contradiction with (3.55). Let

$$\begin{aligned}
D_c &= \hat{D}_c & (3.56)\\
C_c &= \left(\hat{C}_c - \hat{D}_c C \Sigma_1\right)\Sigma_2^{-T} & (3.57)\\
B_c &= M_2^{-1}\left(\hat{B}_c - M_1 B \hat{D}_c\right) & (3.58)\\
A_c &= M_2^{-1}\left(\hat{A}_c - M_1 A \Sigma_1 - M_1 B D_c \Sigma_1 - M_2 B_c C \Sigma_1 - M_1 B C_c \Sigma_2^T\right)\Sigma_2^{-T} & (3.59)\\
\Sigma &= \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2^T & -\Sigma_2^T\left(\Sigma_1 - M_1\right)^{-1}\Sigma_2 \end{bmatrix} & (3.60)
\end{aligned}$$

We can obtain (3.10) and (3.11) with some matrix manipulations. ∇∇∇

3.7 Appendix B : proof of Proposition 3.3.3

Proof: The proof is similar to that of Proposition 3.3.1 and Proposition 3.3.2. We only need to pay attention to the LQ index that we want to optimize

$$\begin{aligned}
J &= tr\left(Q\Sigma_x\right) + tr\left(R\Sigma_u\right)\\
&< tr\left(Q^{\frac{1}{2}}\Sigma Q^{\frac{1}{2}} + R^{\frac{1}{2}} K \Sigma K R^{\frac{1}{2}}\right)\\
&= tr\left(Q^{\frac{1}{2}}\Sigma\Sigma^{-1}\Sigma Q^{\frac{1}{2}} + R^{\frac{1}{2}} L \Sigma^{-1} L^T R^{\frac{1}{2}}\right)\\
&< tr\left(P_1\right) + tr\left(P_2\right)
\end{aligned}$$

where P_1 and P_2 satisfy (3.20) and (3.21) respectively. In the proof of Lemma 3.3.1 it is noted that we can make $\Sigma \to \Sigma_{cl}$ so that $tr\left(P_1\right) + tr\left(P_2\right) \to J_{\mathscr{C}_s}$. ∇∇∇

3.8 Appendix C : proof of Proposition 3.3.4

Proof: For the dynamic feedback case:

$$\begin{aligned}
J &= tr(Q\Sigma_x) + tr(R\Sigma_u) \\
&= tr(\bar{Q}\Sigma_{cl}) + tr(R\Sigma_u) \\
&= tr(\bar{Q}\Sigma_{cl}) + tr\left(R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma_{cl}\begin{bmatrix} D_cC & C_c \end{bmatrix}^T + RD_cF\Omega F^T D_c^T\right) \\
&< tr(\bar{Q}\Sigma) + tr\left(R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma\begin{bmatrix} D_cC & C_c \end{bmatrix}^T + RD_cF\Omega F^T D_c^T\right) \qquad (3.61) \\
&< tr(P_1) + tr(P_2) \qquad (3.62)
\end{aligned}$$

where $\bar{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix}$. P_1 and P_2 satisfy the following inequalities respectively:

$$\begin{bmatrix} P_1 & R^{1/2}\begin{bmatrix} D_cC & C_c \end{bmatrix} & R^{1/2}D_cF \\ \begin{bmatrix} C^TD_c^T \\ C_c^T \end{bmatrix} R^{1/2} & \Sigma^{-1} & 0 \\ R^{1/2}F^TD_c^T & 0 & \Omega^{-1} \end{bmatrix} > 0 \qquad (3.63)$$

$$\begin{bmatrix} P_2 & \bar{Q}^{1/2} \\ \bar{Q}^{1/2} & \Sigma^{-1} \end{bmatrix} > 0 \qquad (3.64)$$

It is easy to verify that

$$\begin{bmatrix} Q^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \cdot T_2^T = \begin{bmatrix} Q^{1/2} & Q^{1/2}\Sigma_1 \\ 0 & 0 \end{bmatrix} \qquad (3.65)$$

Pre-multiplying the left side of (3.63) by $\begin{bmatrix} I & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I \end{bmatrix}$ and post-multiplying the right side of (3.63) by $\begin{bmatrix} I & 0 & 0 \\ 0 & T_2^T & 0 \\ 0 & 0 & I \end{bmatrix}$, (3.24) is obtained; pre-multiplying the left side of (3.64) by $\begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix}$ and post-multiplying the right side of (3.64) by $\begin{bmatrix} I & 0 \\ 0 & T_2^T \end{bmatrix}$, (3.25) is obtained. In the proof of the Lemma 3.3.1 it is noted that we can let $\Sigma \to \Sigma_{cl}$. Also, we can let the free parameters $P_1 \to R\begin{bmatrix} D_cC & C_c \end{bmatrix}\Sigma\begin{bmatrix} D_cC & C_c \end{bmatrix}^T + RD_cF\Omega F^TD_c^T$ and $P_2 \to \bar{Q}\Sigma$. So $tr(P_1) + tr(P_2) \to J_{\mathscr{C}_s}$. ∇∇∇

4

Decentralized PID Tuning for Discrete-Time Systems Based on Covariance Criterion

4.1 Introduction

The proportional-integral-derivative (*PID*) controller is extensively used in industry and is well documented in the literature since the classic Ziegler-Nichols method (Ziegler and Nichols 1942) was proposed. This is because the PID controller is simple, robust and well understood. Optimization of PID parameters has been extensively studied to meet the high performance requirement of the modern industry. Many different tuning criteria and procedures have been proposed, for example, decay ratio method (Cohen and Coon 1953), gain and phase margin method (Ästrom and Hagglund 1995) and the internal model control based PID tuning method (Morari and Zafiriou 1989, Rivera *et al.* 1986). Recently, with the popularity of the interior point algorithm several PID design methods based on Linear Matrix Inequality (*LMI*) were proposed for the continuous-time systems (Bao *et al.* 1999, Feng *et al.* 2002, Ge *et al.* 2002). However, none of the above mentioned PID design methods is related to achieving variance specification on the outputs for multivariable systems.

The significance of reducing the process variation is well appreciated in the manufacture industry (Shunta 1995); however, there are only a few papers on optimizing PID parameters in order to reduce the process variances. Stochastic predictive PID controllers were proposed

[1] Some version of this chapter is probationally accepted by *ISA Trans.* (Huang and Huang 2003*a*)

by Kwok *et al.* (2000) and Miller *et al.* (1995) by equating a discrete PID control law with the linear form of the Generalized Predictive Control (*GPC*) with steady state weighting; several self-tuning PID controllers were proposed by Miura *et al.* (1998), Sato *et al.* (2002) and Yamamoto *et al.* (1999) for discrete-time Linear Time-invariant (*LTI*) systems by approximating the PID controller to the generalized minimum variance control (*GMVC*). The philosophy behind these methods (Kwok *et al.* 2000, Miller *et al.* 1995, Miura *et al.* 1998, Sato *et al.* 2002, Yamamoto *et al.* 1999) is to calculate the PID parameters by approximating the PID controller to other advanced controllers. In such a way, it is expected that the PID controller has similar property as the other advanced controllers. However, there is no theoretical guarantee about how "close" such approximation can be; moreover, these approximation methods are available only for the single-input-single-output (*SISO*) systems so far. The extension of the PID design from the SISO system to the multi-input-multi-output (*MIMO*) system is nontrivial.

In this chapter, a state space approach of designing multi-loop PID controllers is proposed such that the closed-loop system satisfies the generalized covariance constraints. One of the main advantages of the proposed method is that the controller parameters are calculated directly according to the covariance constraints on process variables instead of approximating other controllers. A convergent computational algorithm, in which a sequence of semi-definite programming problems are solved using the LMItool (Gahinet *et al.* 1995), is proposed to calculate the multi-loop PID controller parameters. The proposed algorithm initially intends for the process with stable disturbances. The algorithm is then extended to the process with unstable disturbance (random walk disturbance). The proposed multi-loop PID controller design method is for the purpose of controlling the variation of process variables. However, it can also be applied to design multi-loop PID controller for other performance indices, such as H_2 or H_∞.

The generalized covariance constraints problem and the preliminary results are stated in Section 2. The state space realization of the multi-loop PID controller is given in Section 3. An algorithm is presented to calculate the multi-loop PID controller parameters in Section 4, where the disturbance model is assumed to be stable. The multi-loop PID controller design for the random walk disturbance is addressed in Section 5. Numerical examples are presented in Section 6, followed by concluding remarks in Section 7.

4.2 Problem statement

Consider a finite-dimensional discrete-time LTI system $\mathscr{P}$, given as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + G\zeta_k \\ y_k &= Cx_k + F\zeta_k \\ z_k^i &= C_i x_k + D_i u_k, i = 1 \ldots l \end{aligned} \tag{4.1}$$

where $x_k \in R^n$ is the state vector, $u_k \in R^m$ is the input, $y_k \in R^m$ is the measurement, $\zeta_k \in R^g$ is the external disturbance and measurement noise, and $z_k^i \in R^{p_i}$ is the i-th controlled vector. A, B, G, C, F, C_i and D_i are matrices with appropriate dimensions. The disturbance ζ_k is unmeasurable, but it is assumed that some of its statistical properties are known:

$$\begin{aligned} &E(\zeta_k) = 0 \\ &E\left(\zeta_i \zeta_j^T\right) = \Omega\delta(i-j) \end{aligned} \tag{4.2}$$

It is assumed that the state space representation is a minimal realization.

To achieve better product quality, controlling the variation of the process variables is well accepted in industry (Shunta 1995), and different control strategies have been presented, such as minimum variance control (Ästrom 1970), the generalized minimum variance control (*GMVC*) (Yamamoto *et al.* 1999), linear quadratic Gaussian (*LQG*) control (Kalman 1968, MacGregor 1973, Mäkilä *et al.* 1984) and covariance control (Skelton and Iwasaki 1993, Skelton and Delorenzo 1985). For multivariable systems controlling the plant's covariance is one of the main objectives (Skelton and Iwasaki 1993, Skelton and Delorenzo 1985). The covariance of x_k is defined as:

$$\Sigma = \lim_{k \to \infty} E\left(x_k x_k^T\right) \tag{4.3}$$

It is well known that for the following stable LTI system driven by the white noise ζ_k:

$$X_{k+1} = A_{cl} X_k + G_{cl} \zeta_k \tag{4.4}$$

the covariance of the state vector X_k should satisfy the following equation:

$$A_{cl} \Sigma_{cl} A_{cl}^T + G_{cl} \Omega G_{cl}^T - \Sigma_{cl} = 0 \tag{4.5}$$

However, as it is pointed out in Chapter 2, there is often no physical interpretations of the covariance of the states. Therefore, it is more desirable to control covariance of process output rather than that of states. The generalized covariance constrained control (*GCC*) problem is stated as follows:

Problem 4.2.1 *For the continuous-time LTI system (4.1), find a controller such that the closed-loop system is asymptotically stable and the covariance of the controlled variable* $z_k^i (i = 1...l)$ *satisfies*

$$\Phi_i = \lim_{k\to\infty} E\left(z_k^i z_k^{i\,T}\right) < \bar{\Phi}_i \tag{4.6}$$

where $\bar{\Phi}_i\,(i = 1...l)$ *is some pre-specified positive definite matrix.*

If there exists a multi-loop PID controller such that the closed-loop system is asymptotically stable and the generalized covariance constraints (4.6) are satisfied, then the GCC problem is feasible via a multi-loop PID controller. It has been shown in Chapter 2 that the feasibility of GCC is equivalent to feasibility of some linear matrix inequalities (*LMIs*) if a full order dynamic controller is considered. Unfortunately, this conclusion does not hold for the fixed-order controller and decentralized controller. Interior point algorithm is very efficient in solving LMIs (Boyd and Barratt 1991, Gahinet *et al.* 1995), but it cannot be used here to solve the multi-loop PID design with the generalized covariance constraints. However, an iterative algorithm, in which a sequence of optimization problems are solved, is proposed in section 5 to calculate the multi-loop PID controller parameters.

4.3 Multi-loop PID controller

Whenever it is feasible, multi-loop PID controllers are preferred over the complex multivariable control systems because multi-loop PID controllers are easier to implement on DCS and requires less training compared to the multivariable controller. However, optimization of the PID parameters to reduce the effect of disturbance is not a trivial task due to the non-convex nature of the optimization problem. The presented approach is based on the state space representation and a state space realization of the multi-loop PID controller is given in this section.

A discrete-time single-loop PID controller can be described as:

$$u_k = k_1 e_k + k_2 \sum_{i=0}^{k} e_i + k_3 (e_k - e_{k-1}) \tag{4.7}$$

where u_k is the manipulated variable and $e_k = r_k - y_k$ is the error between setpoint r_k and the measurement y_k. The velocity form of the discrete-time PID controller can be obtained from equation (4.7):

$$\Delta u_k = (k_1 + k_2 + k_3) e_k + (-k_1 - 2k_3) e_{k-1} + k_3 e_{k-2} \tag{4.8}$$

The transfer function of the discrete-time PID controller (4.7) can be obtained easily from the velocity form:

$$C\left(z^{-1}\right)=\frac{(k_1+k_2+k_3)+(-k_1-2k_3)z^{-1}+k_3z^{-2}}{1-z^{-1}} \tag{4.9}$$

The controllable state space realization of the PID controller (4.7) is obtained from the transfer function (4.9):

$$\begin{aligned} x^s_{k+1} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x^s_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_k \\ u^s_k &= \begin{bmatrix} \tilde{k}_3 & \tilde{k}_2 \end{bmatrix} x^s_k + \tilde{k}_1 e_k \end{aligned} \tag{4.10}$$

where $\tilde{k}_1 = k_1+k_2+k_3$, $\tilde{k}_2 = k_2-k_3$, $\tilde{k}_3 = k_3$ and x^s_k represents the state vector of a single-loop PID. For the multivariable systems, the multi-loop PID controller $C_m\left(z^{-1}\right)$, consisted of a group of individual PID controllers, is given as:

$$C\left(z^{-1}\right)=diag\left(c_1\left(z^{-1}\right),c_2\left(z^{-1}\right),\cdots,c_m\left(z^{-1}\right)\right) \tag{4.11}$$

where $c_i\left(z^{-1}\right)$ is the *ith* single-loop PID controller with the same form as (4.9). With this multi-loop PID controller, the diagram of the closed-loop system is shown as:

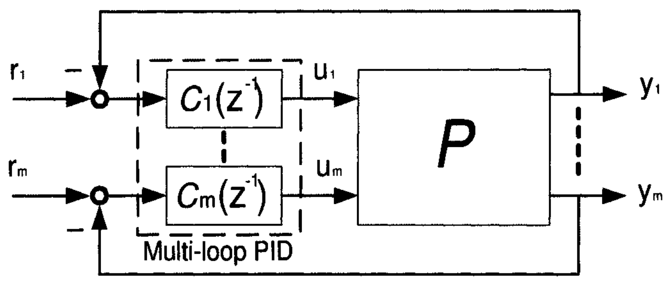

Figure 4.1: Closed-loop diagram

The state space representation of the multi-loop PID controller can be obtained by stacking the state space realization of each individual PID controller (where superscript m represents multi-loop):

$$\begin{aligned} x^m_{k+1} &= A_c x^m_k + B_c e^m_k \\ u^m_k &= C_c x^m_k + D_c e^m_k \end{aligned} \tag{4.12}$$

where matrices A_c, B_c, C_c and D_c are defined as:

$$\begin{aligned} A_c &= diag\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\cdots,\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) \in R^{2m\times 2m} \\ B_c &= diag\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix},\begin{bmatrix} 0 \\ 1 \end{bmatrix},\cdots,\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \in R^{2m\times m} \\ C_c &= diag\left(\begin{bmatrix} \tilde{k}^1_3, \tilde{k}^1_2 \end{bmatrix}, \begin{bmatrix} \tilde{k}^2_3 & \tilde{k}^2_2 \end{bmatrix},\cdots,\begin{bmatrix} \tilde{k}^m_3 & \tilde{k}^m_2 \end{bmatrix}\right) \in R^{m\times 2m} \\ D_c &= diag\left(\tilde{k}^1_1,\tilde{k}^2_1\cdots\tilde{k}^m_1\right) \in R^{m\times m} \end{aligned} \tag{4.13}$$

With the multi-loop PID controller (4.12) the closed-loop system can be written as (assuming $r = 0$):

$$\begin{array}{rcl} X_{k+1} & = & (A_0 + B_0KC_0)X_k + (G_0 + B_0KF_0)\zeta_k \\ z_k^i & = & (\bar{C}_i + \bar{D}_iKC_0)X_k + \bar{D}_iKF_0\zeta_k, i = 1...l \end{array} \tag{4.14}$$

where $X_k = \begin{bmatrix} x_k \\ x_k^m \end{bmatrix}$ and the matrices $A_0, B_0, C_0, G_0, F_0, K, \bar{C}_i$ and $\bar{D}_i$ are composed as follows:

$$\begin{array}{l} A_0 = \begin{bmatrix} A & 0 \\ -B_cC & A_c \end{bmatrix}, B_0 = \begin{bmatrix} B \\ 0 \end{bmatrix}, C_0 = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}, G_0 = \begin{bmatrix} G \\ -B_cF \end{bmatrix} \\ F_0 = \begin{bmatrix} F \\ 0 \end{bmatrix}, \bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}, \bar{D}_i = D_i, K = \begin{bmatrix} -D_c & C_c \end{bmatrix} \end{array} \tag{4.15}$$

To make the closed-loop system (4.14) satisfy the generalized covariance constraints we use the following lemma from Chapter 2:

Lemma 4.3.1 *The closed-loop system (4.14) is asymptotically stable and satisfies constraints (4.6) if and only if there exists a matrix $\Sigma > 0$ such that*

$$(A_0 + B_0KC_0)\Sigma(A_0 + B_0KC_0)^T - \Sigma + (G_0 + B_0KF_0)\Omega(G_0 + B_0KF_0)^T < 0 \tag{4.16}$$

$$(\bar{C}_i + \bar{D}_iKC_0)\Sigma(\bar{C}_i + \bar{D}_iKC_0)^T + \bar{D}_iKF_0\Omega F_0^TK^T\bar{D}_i^T < \bar{\Phi}_i \tag{4.17}$$

where $i = 1, ..., l$.

4.4 Computational algorithm

Inequalities (4.16) and (4.17) are difficult to solve and the difficulty lies in two facts: first, both Inequality (4.16) and Inequality (4.17) contain cubic terms; second, the decentralized control structure of the multi-loop PID controller makes the unknown matrix K a sparse matrix. To solve the nonlinear matrix inequalities (4.16) and (4.17), one need change them to some equivalent forms that can be solved. By applying the Schur complement Lemma, it can be obtained:

Proposition 4.4.1 *The discrete-time system (4.1) is stabilized by a multi-loop PID controller, defined in (4.12), and the constraints (4.6) are satisfied if and only if there exist matrices $X > 0$, $Y > 0$ and K (the decision variable K is composed as that in (4.15)) such that*

$$\begin{bmatrix} -X & A_0 + B_0KC_0 & G_0 + B_0KF_0 \\ (A_0 + B_0KC_0)^T & -Y & 0 \\ (G_0 + B_0KF_0)^T & 0 & -\Omega^{-1} \end{bmatrix} < 0 \tag{4.18}$$

$$\begin{bmatrix} \bar{\Phi}_i & (\bar{C}_i + \bar{D}_iKC_0) & \bar{D}_iKF_0 \\ (\bar{C}_i + \bar{D}_iKC_0)^T & Y & 0 \\ F_0^TK^T\bar{D}_i^T & 0 & \Omega^{-1} \end{bmatrix} > 0 \tag{4.19}$$

$$XY = I_{n+2m} \tag{4.20}$$

where $i = 1, ..., l$.

The proof is straightforward and it is omitted here. Obviously the condition in the proposition (4.4.1) is not convex because $X > 0$ and $Y > 0$ are inverse to each other. To find a feasible solution to (4.18) - (4.20) the idea of the cone complementary linearization method (Ghaoui *et al.* 1997) is adopted. The algebraic equation (4.20) is relaxed with the following LMI:

$$\begin{bmatrix} X & I_{n+2m} \\ I_{n+2m} & Y \end{bmatrix} \geq 0 \tag{4.21}$$

and the linearized version of $trace(XY)$ is minimized at each step.

The algorithm to calculate the multi-loop PID controller is summarized as follows:

Algorithm 4.4.1 *1. Set k=0. Find* $X_k > 0 \in R^{n+2m}$, $Y_k > 0 \in R^{n+2m}$ *that solve the following semi-definite programming problem:*

$$\min_{X,Y,K} \quad trace(X+Y) \quad subject\ to\ (4.18), (4.19), (4.21)$$

2. Find $X_{k+1} > 0 \in R^{n+2m}$, $Y_{k+1} > 0 \in R^{n+2m}$ *that solve the following semi-definite programming problem:*

$$\min_{X,Y,K} \quad trace(X_kY + Y_kX) \quad subject\ to\ (4.18), (4.19), (4.21)$$

Set $t_k = trace(X_kY_{k+1} + Y_kX_{k+1})$.
3. Set $k = k+1$. *If the decrease of* t_k *in last L steps is less than a small constant number* $\varepsilon_1 > 0$, *then the algorithm stops. If* $trace(X_kY_k) - n - 2m < \varepsilon_2$ *then go to step 4; otherwise, go to step 2.*
4. Find $\Sigma > 0$ *by solving LMI (4.16) and LMI (4.17) (where K is obtained from step 3). If there is a solution, then one feasible solution is found; otherwise, go to step 2.*

Remark 4.4.1 *The above computational algorithm is an extension of the one so called cone complementarity linearization algorithm presented in (Ghaoui* et al. *1997) by introducing the space matrix K into inequalities (4.18) and (4.19) as decision variable. Similar to the proof of Theorem (2.1) in (Ghaoui* et al. *1997) it can be shown that* t_k *decreases with each step so that the algorithm converges.*

Claim 4.4.1 *The Algorithm (4.4.1) is convergent.*

4.5 Process with random walk disturbance

In chemical industry random walk disturbance is often used to represent slow dynamic of disturbances. The algorithm presented in the last section can not deal with random walk disturbance. This is because the closed-loop system, composed by the multi-loop PID controller and the process, is not a minimal realization. To generate a minimal realization of the closed-loop system, one needs to change the process diagram, the procedure is illustrated by using a unit feedback example. A block diagram of a single-loop feedback system is shown in Figure (4.2), where $G(z)$ is the process model and $\frac{H(z)}{z-1}$ is the disturbance model.

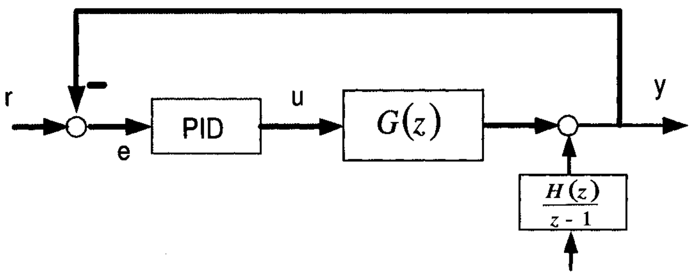

Figure 4.2: Unit feedback block diagram

Since the PID controller block and this disturbance block both contain a pole at 1, this unstable pole is moved out of these two blocks. The reconfigured block diagram is shown in Figure (4.3): where $G_1(z) = G(z)z$ and $C(z) = \frac{(k_1+k_2+k_3)z^2+(-k_1-2k_2)z+k_3}{z^2}$. The PID controller

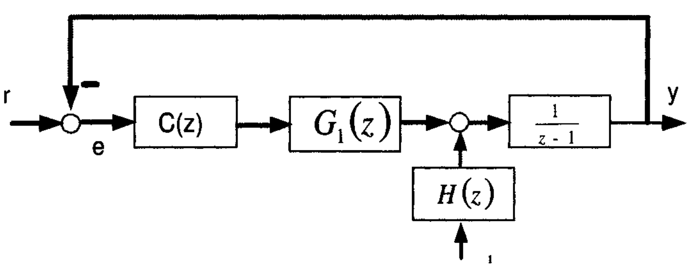

Figure 4.3: Unit feedback block diagram after reconfiguration

"borrows" a pole (located at the origin) from the process in order to preserve its properness. The zero order hold ensures that the process model $G(z)$ is strictly proper, in other words, $G(z)$ can always have an extra pole that is "lent" to the controller block. After this block diagram reconfiguration, a state space representation of the controller $C(z)$ is:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e_k \\ u_k &= \begin{bmatrix} k_3 & -k_1 - 2k_2 \end{bmatrix} x_k + (k_1 + k_2 + k_3) e_k \end{aligned} \tag{4.22}$$

Following the same procedure as in Section 3, we can build the controller and the closed-loop system in state space for the MIMO case, and then use the algorithm (4.4.1) to calculate the multi-loop PID controller parameters.

Remark 4.5.1 *Changing the block diagram preserves the transfer function from the disturbance to the output so that only the variances for the output can be specified. The variance for the manipulated variables can not be specified because the PID controller output is not a stationary signal.*

4.6 Simulation results

4.6.1 Example 1

The first example (Qin 1998) is to design a single-loop PID controller for a first order plus time-delay process subject to unstable disturbance containing an integrator. The process is as follows:

$$y_k = \frac{z^{-6}}{1-0.8z^{-1}} u_k + \frac{1+0.6z^{-1}}{(1-0.6z^{-1})(1+0.7z^{-1})(1-0.5z^{-1})\Delta} \zeta_k \tag{4.23}$$

where the series $\{\zeta_k\}$ is white noise with zero mean and unit variance. The known minimal output variance achieved by a PID controller is 123.54 (Qin 1998). However, our algorithm shows that the variance of the output can be further reduced by optimizing the PID parameters. Using the algorithm presented in the last section, we obtain the optimal PID controller as:

$$C\left(z^{-1}\right) = \frac{0.8021 - 1.3402z^{-1} + 0.5788z^{-2}}{1-z^{-1}} \tag{4.24}$$

The corresponding output variance is 87.85. Comparing with the known output variance 123.54, we can see that by using the PID controller (4.24) the variance of the output is reduced by 30%.

If let disturbance $\zeta_k = 0$, the process (4.23) is actually obtained by sampling a first order plus time-delay system with sampling rate 1:

$$G(s) = \frac{5}{4.48s+1} e^{-5s} \tag{4.25}$$

PID tuning for the first order plus time-delay systems is well studied and there are many tuning rules available. The PID controllers, obtained using these tuning algorithms, and the corresponding output variance are listed in Table 4.1. The PID controllers are calculated according to the Tables[1] 15.2, 15.3, 15.4 and 15.6 in (Ogunnaike and Ray 1994).

The PID design algorithm in this chapter is proposed for MIMO systems, but from this

Tuning methods	PID controller	output variance
Ziegler-Nichols method	$\frac{0.7742-1.2903z^{-1}+0.5376z^{-2}}{1-z^{-1}}$	105.87
Cohen-Coon	$\frac{0.759-1.163z^{-1}+0.437z^{-2}}{1-z^{-1}}$	112.22
IMC ($\lambda=0.2$)	$\frac{0.6393-0.9794z^{-1}+0.3734z^{-2}}{1-z^{-1}}$	93.93
IMC ($\lambda=0.4$)	$\frac{0.548-0.839z^{-1}+0.320z^{-2}}{1-z^{-1}}$	98.29
IMC ($\lambda=0.6$)	$\frac{0.478-0.735z^{-1}+0.280z^{-2}}{1-z^{-1}}$	107.31
Integral Time-weighted Absolute Error	$\frac{0.753-1.176z^{-1}+0.466z^{-2}}{1-z^{-1}}$	93.63
Our algorithm	$\frac{0.8021-1.3402z^{-1}+0.5788z^{-2}}{1-z^{-1}}$	87.85

Table 4.1: PID controllers and the corresponding output variances

example it can be used for SISO systems and the obtained PID controller (4.24) has the best performance compared with other PID controllers.

4.6.2 Example 2

The second example is from (Inoue *et al.* 2000). The process is described as follows:

$$y_k + A_1 y_{k-1} = B_0 u_{k-2} + \zeta_k \tag{4.26}$$

where $A_1 = \begin{bmatrix} -0.99101 & 8.80512\cdot 10^{-3} \\ -0.80610 & -0.77089 \end{bmatrix}$, $B_0 = \begin{bmatrix} 0.89889 & -0.409329 \\ -0.56 & 0.88052 \end{bmatrix}$ and the series $\{\zeta_k\}$ is assumed to be zero mean white noise and its covariance matrix: $E\left(\zeta_i\zeta_j^T\right) = \delta(i-j)\,0.01\times I_2$.

The process is controlled by the conventional GMVC (Inoue *et al.* 2000):

$$u_k = -\begin{bmatrix} 1.8989+1.9957z^{-1} & -0.4093-0.4134z^{-1} \\ -0.56+0.2929z^{-1} & 1.8805+1.4488z^{-1} \end{bmatrix} \begin{bmatrix} 0.975 & -0.0155 \\ 1.4203 & 0.5872 \end{bmatrix} y_k \tag{4.27}$$

With the controller (4.27) implemented on the process, the covariance for the output y_k is $\begin{bmatrix} 0.0621 & 0.0782 \\ 0.0782 & 0.1411 \end{bmatrix}$ and the covariance for the input u_k is $\begin{bmatrix} 0.0169 & 0.0255 \\ 0.0255 & 0.0413 \end{bmatrix}$.

[1]It is assumed that the ratio between time-delay and time constant is less than 1 in the tables. However, since here $\frac{5}{4.48} = 1.1$ is not far away from 1, from the engineering point of view, the tables could be used to calculate the PID parameters.

To use the algorithm in Section 4, the state space model is first generated from (4.26):

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} -A_1 & B_0 \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ I_2 \end{bmatrix} u_k + \begin{bmatrix} -A_1 \\ 0 \end{bmatrix} \zeta_k \\ y_k &= \begin{bmatrix} I_2 & 0 \end{bmatrix} x_k + \zeta_k \end{aligned} \tag{4.28}$$

where $x_k = \begin{bmatrix} y_k - \zeta_k \\ u_{k-1} \end{bmatrix}$.

The obtained multi-loop PID control is as follows:

$$C\left(z^{-1}\right) = \begin{bmatrix} \frac{0.5647-0.5387z^{-1}-0.02z^{-2}}{z^{-1}\left(1-z^{-1}\right)} & \\ & \frac{0.3815-0.5476z^{-1}+0.1678z^{-2}}{z^{-1}\left(1-z^{-1}\right)} \end{bmatrix} \tag{4.29}$$

With the multi-loop PID controller (4.29) implemented on the process, the covariance for the simulated y_k is $\begin{bmatrix} 0.0249 & 0.0224 \\ 0.0224 & 0.1158 \end{bmatrix}$ and the covariance for the simulated u_k is $\begin{bmatrix} 0.0084 & 0.0041 \\ 0.0041 & 0.0075 \end{bmatrix}$. It can be seen that by using the multi-loop PID controller the variances of the process input and output are smaller than those by using the GMVC. The performance comparison of multi-loop PID and GMVC is shown in Figure (4.4).

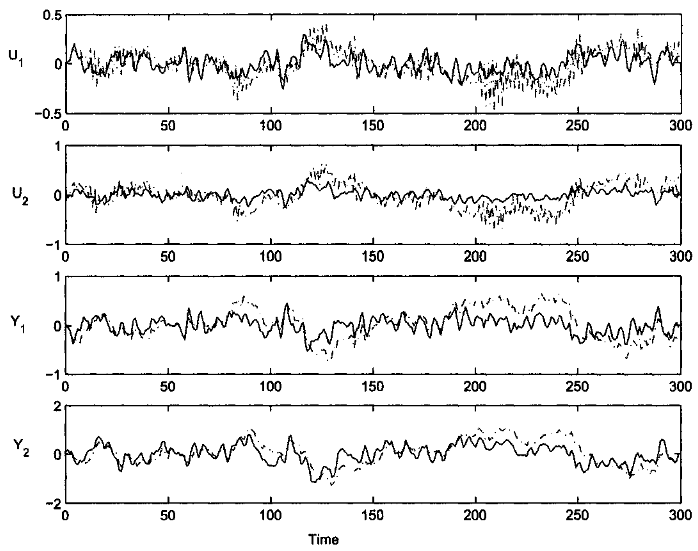

Figure 4.4: Simulation results: the dashed line is from the GMVC; the solid line is the from the multi-loop PID controller

4.6.3 Example 3

The third example is a dry rotary cement kiln with capacity 1000 tons/day (Mäkilä *et al.* 1984). The kiln is 105 meters long and 5 meters in diameter. After two pre-heaters, where the dry homogenized raw material is heated to 800^{o}C, then goes to the kiln. The dry homogenized raw material enters the kiln and then passes it. The final temperature of the material is around 1450^{o}C. Coal is burned in the lower front end of the kiln in order to produce the high temperature, which is required to start the chemical reactions taking place in the raw materials. The product of the reaction is called clinker that is cooled in a planetary cooler before it leaves the process. The kiln process has two controlled variables: the combustion gas temperature and the kiln drive power. The latter is chosen as a controlled variable because it correlates to the burning temperature and clinker quality and the clinker quality can only be analyzed every two hours. The two manipulated variables of the kiln process are the kiln exhaust fan speed and raw material feed rate. The process is exposed to random disturbance. The sampling rate is 5 minutes. The original process model is shown as follows:

$$y_{k+1} + A_0 y_k = B_0 u_k + \zeta_{k+1} + C_0 \zeta_k \tag{4.30}$$

where

$$\begin{aligned} A_0 &= \begin{bmatrix} -0.917 & -0.0846 \\ 0.132 & -0.915 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 2.06 & -0.0746 \\ -0.108 & -0.0192 \end{bmatrix} \\ C_0 &= \begin{bmatrix} -0.0449 & -0.216 \\ 0.0256 & 0.841 \end{bmatrix}, \quad E\left(\zeta_i \zeta_j^T\right) = \begin{bmatrix} 0.0639 & 0.00188 \\ 0.00188 & 0.0233 \end{bmatrix} \delta(i-j) \end{aligned}$$

One can obtain the following state space model from the kiln process (4.30):

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.917 & 0.0846 \\ -0.132 & 0.915 \end{bmatrix} x_k + \begin{bmatrix} 2.06 & -0.0746 \\ -0.108 & -0.0192 \end{bmatrix} u_k + \begin{bmatrix} 0.8721 & -0.1314 \\ -0.1064 & 1.7560 \end{bmatrix} \zeta_k \\ y_k &= x_k + \zeta_k \end{aligned} \tag{4.31}$$

It is desired to control the variances of both the output and input so that the controlled variables are chosen as:

$$\begin{aligned} z_k^1 &= \begin{bmatrix} 1 & 0 \end{bmatrix} y_k \\ z_k^2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} y_k \\ z_k^3 &= \begin{bmatrix} 1 & 0 \end{bmatrix} u_k \\ z_k^4 &= \begin{bmatrix} 0 & 1 \end{bmatrix} u_k \end{aligned} \tag{4.32}$$

It is shown in (Mäkilä *et al.* 1984) that a reasonable control criterion is to minimize the joint variation of the controlled variables:

$$J = \lim_{k \to \infty} E y_k^T y_k \tag{4.33}$$

However, the variances of the input variables become unacceptable if the minimum variance control law is implemented: $\lim\limits_{k\to\infty} E\left(z_k^3 z_k^3\right) = 0.148$ and $\lim\limits_{k\to\infty} E\left(z_k^4 z_k^4\right) = 108$. As it is pointed out in (Mäkilä *et al.* 1984) it is appropriate to restrict the variances of the input variables according to:

$$\begin{aligned} &\lim_{k\to\infty} E\left(z_k^3 z_k^3\right) < 0.004 \\ &\lim_{k\to\infty} E\left(z_k^4 z_k^4\right) < 1.5 \end{aligned} \tag{4.34}$$

Minimization of the (4.33) subject to the variance constraints (4.34) by using a full order dynamic controller leads to the output variances (Huang *et al.* 2002, Mäkilä *et al.* 1984):

$$\begin{aligned} &\lim_{k\to\infty} E\left(z_k^1 z_k^1\right) = 0.0939 \\ &\lim_{k\to\infty} E\left(z_k^2 z_k^2\right) = 0.189 \end{aligned} \tag{4.35}$$

The proposed algorithm can not find a multi-loop PID controller if the variance constraints for the input variables are chosen as (4.34) and the variance constraints for the output variables are specified as 0.939 and 0.189 respectively. This is not surprising because as it is known that decentralized controller structure adds performance limit compared to the full order centralized controller. If the variance constraint for z_k^2 is relaxed to 0.345, by using the algorithm in Section 4 a multi-loop PID controller can be obtained as follows:

$$C\left(z^{-1}\right) = \begin{bmatrix} \frac{0.1743-0.1612z^{-1}+0.0064z^{-2}}{\left(1-z^{-1}\right)z^{-1}} & \\ & \frac{-1.8252+1.8139z^{-1}-0.0207z^{-2}}{\left(1-z^{-1}\right)z^{-1}} \end{bmatrix} \tag{4.36}$$

The simulation results are shown in Figure (4.5):

With the multi-loop PID controller (4.36) implemented, the variances for the controlled variables, calculated from the simulated data, are:

$$\begin{cases} \lim\limits_{k\to\infty} E z_k^1 z_k^1 = 0.0923 \\ \lim\limits_{k\to\infty} E z_k^2 z_k^2 = 0.3419 \end{cases}, \quad \begin{cases} \lim\limits_{k\to\infty} E z_k^3 z_k^3 = 0.0034 \\ \lim\limits_{k\to\infty} E z_k^4 z_k^4 = 1.1198 \end{cases}$$

It is shown that the variance for the first output is even slightly better than that achieved by the constrained LQG controller in (Mäkilä *et al.* 1984); the variances of the input variables are better than those achieved by the constrained LQG controller. However, the variance of z_k^2 is larger than that achieved by the optimal constrained LQG controller in (Mäkilä *et al.* 1984) but satisfies the design specification. Simulation also shows that if the variance bound for the second output is smaller than 0.33, then the algorithm can not find a solution. This may be caused by two reasons: 1) the multi-loop-PID-controller structure is far simpler than that of the full-order multivariable controller, and this simplicity will reduce the closed-loop performance compared to the full-order optimal constrained LQG controller in (Mäkilä *et al.* 1984); 2) the

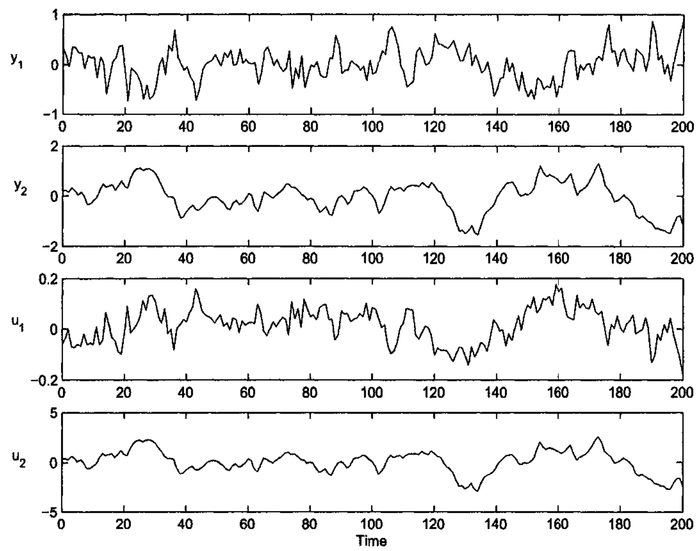

Figure 4.5: Simulation results for the kiln process

proposed algorithm is not globally convergent, which means that even though there maybe exists a multi-loop PID controller such that the variance z_k^2 is smaller than 0.33, however, the proposed algorithm can not find it. There is no strict proof of the global convergency of the proposed algorithm in Section 4, however, our simulation shows the algorithm always converges to one value no matter the initial condition. It is worth a investigation of the global convergent properties of the proposed algorithm in the future.

4.7 Conclusion

The multi-loop or decentralized PID controller design for discrete-time systems based on the generalized covariance constraints has been considered in this chapter and an iterative LMI approach is proposed to solve the problem. The algorithm is shown to be convergent. The algorithm is originally derived for the process with stable disturbances; after the reconfiguration of the process block diagram, it can also be applied to the process with unstable random-walk disturbances. Several simulation results are used to illustrate the effectiveness of the proposed method.

5

Decentralized Multi-loop PID Tuning for Continuous-time Systems Based on Covariance Criterion

5.1 Introduction

Proportional-integral-derivative (*PID*) controllers have been extensively used in chemical industry for several decades. This is because the PID controller has a simple structure and its parameters are well understood to control engineers as discussed in the last chapter. Numerous PID tuning methods have been documented in the literature. The classic Ziegler-Nichols method (Ziegler and Nichols 1942) is one of the very first presented PID tuning method, in which the controller gain is increased until sustained oscillation occurs and then the PID parameters are determined based on the gain and oscillation period. A PID design relation for first order plus time-delay systems was proposed by Cohen and Coon (1953) to provide closed loop responses with a decay ratio of $\frac{1}{4}$. However, both these two methods may make the closed-loop system oscillate more than desired. A PID tuning method was proposed based on moving one point on the system Nyquist curve to a desired point in order to make the closed-loop system have sufficient gain margin and phase margin (Äström and Hagglund 1984). The internal model control based PID tuning method (Morari and Zafiriou 1989, Rivera *et al.* 1986) considers model uncertainty by introducing the IMC filter and it allows the designer to trade off the closed-loop performance against the robustness.

To meet the high performance requirement of the modern industry optimization of the PID parameters is also extensively studied. However, due to the non-convexity nature of the optimization problem it is difficult to obtain the optimal solution, especially for the MIMO systems. Recently, PID controller design based on LMIs has been an active area with the popularity of the LMI tools (Boyd and Barratt 1991, Gahinet *et al.* 1995) and different computation algorithms (Bao *et al.* 1999, Feng *et al.* 2002, Ge *et al.* 2002) have been presented. One of the major advantages of these LMI based approaches is that they can be applied not only for the SISO systems but also for the MIMO systems. The algorithm proposed by Ge *et al.* (2002) is only for system modelled as first order or second order plus time-delay. Feng *et al.* (2002) assumed the centralized PID structure that might not be easily implemented on the distributed control system (*DCS*) because of extra programming effort. Moreover, one more LMI was added in order to ensure the well-posedness of the closed-loop system so that extra conservativeness was introduced. The algorithm presented by Bao *et al.* (1999) approximates the nonlinear constraints by LMIs in the neighborhood of the estimated solution and solves the linearized sub-problem by using semi-definite programming approach, but selection of the initial value is not discussed. None of these LMI-based algorithms is proposed to reduce the process variation directly.

As it is pointed out in previous chapters, the significance of reducing process variations is well appreciated in manufacture industry. In this chapter, a multi-loop PID controller design method is proposed for continuous-time systems in order for the closed-loop system to satisfy the covariance constraint. A necessary and sufficient condition for the covariance constraints to be satisfied via a multi-loop PID controller is derived in terms of nonlinear matrix inequalities. A convergent computational algorithm, in which a sequence of optimization problems are solved using the LMItool (Gahinet *et al.* 1995), is proposed to solve the nonlinear matrix inequalities.

The covariance constrained problem is formulated in Section 2. The state space realization of the multi-loop PID controller is given in Section 3. A computational algorithm is proposed in Section 4. Numerical examples are presented in Section 5 to illustrate the efficiency of the algorithm and concluding remarks are presented in Section 6.

5.2 Problem statement

A finite-dimensional continuous-time linear time-invariant (*LTI*) system $\mathscr{P}$ is given as follows:

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) + G\varsigma(t) \\ y(t) &= Cx(t) + F\varsigma(t) \end{aligned} \tag{5.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^m$ is the input, $y(t) \in R^m$ is the measurement and $\varsigma(t) \in R^g$ is the external disturbance and measurement noise. The external disturbance ς_k is not measurable, but its first and second order moments are known:

$$\begin{aligned} &E(\varsigma(t)) = 0 \\ &E\left(\varsigma(t)\varsigma(s)^T\right) = \Omega\delta(t-s) \end{aligned} \tag{5.2}$$

In some applications, controlling the state covariance is attractive (Skelton and Iwasaki 1993). The state covariance of $x(t)$ is defined as:

$$\Sigma = \lim_{t \longrightarrow \infty} E\left(x(t)x(t)^T\right) \tag{5.3}$$

It is well known that for the following stable LTI system driven by the white noise ς_k:

$$\dot{X}(t) = A_{cl}X(t) + G_{cl}\varsigma(t) \tag{5.4}$$

the covariance of the state vector should satisfy the following equation:

$$A_{cl}\Sigma_{cl} + \Sigma_{cl}A_{cl}^T + G_{cl}\Omega G_{cl}^T = 0 \tag{5.5}$$

However, as it is pointed in Chapter 3 specifying an upper bound for the closed-loop covariance is desirable and sometimes even necessary in process industry. In this chapter, the state covariance constrained multi-loop PID control (*SCCMPID*) problem is stated as follows:

Problem 5.2.1 *For the continuous-time LTI system (5.1), find a multi-loop PID controller such that the closed-loop system is asymptotically stable and the closed-loop state covariance satisfies*

$$\Sigma_{cl} < \Phi \tag{5.6}$$

where Φ is a pre-specified positive-definite matrix.

If there exists a multi-loop PID controller such that the closed-loop system is asymptotically stable and the covariance constraint (5.6) is satisfied, then we call the SCCMPID problem is feasible. It has been shown in Chapter 2 that the feasibility of the generalized covariance control problem (*GCC*) is equivalent to the feasibility of certain linear matrix inequalities (*LMIs*), which can be solved by the interior point algorithm quite efficiently (Boyd and Barratt 1991, Gahinet *et al.* 1995), if a full order dynamic controller is considered. Unfortunately, this conclusion does not hold for the SCCMPID because multi-loop PID controllers are not only the reduced order controller but also the decentralized controller. In other words, there is no polynomial time algorithm to compute the multi-loop PID controllers satisfying the covariance constraint (5.6). However, This does not mean that we can not find solutions to this problem. An iterative algorithm, in which a sequences of semi-definite programming problems are solved, is proposed in section 4 to calculate the multi-loop PID controller parameters.

5.3 Multi-loop PID controllers

Different from the centralized PID controller used in (Feng *et al.* 2002), a multi-loop PID controller structure is assumed because the decentralized controller is easier to be implemented at the DCS and it is preferable compared with the centralized controller. Our method is a state space approach so that the state space realization of the multi-loop PID controllers are given in this section.

An ideal continuous-time PID controller for the single-input single-output system is given as follows:

$$u(t) = k_1 e(t) + k_2 \int_0^t e(\tau) d\tau + k_3 \frac{de}{dt} \tag{5.7}$$

where $u(t)$ is the manipulated variable and $e(t) = r(t) - y(t)$ is the error between set-point $r(t)$ and the measurement $y(t)$. The ideal PID controller is not applicable because of its non-causality and usually a first order filter is added to make it physically realizable. The practical PID controller, described in the frequency domain, is as follows:

$$c(s) = \frac{u(s)}{e(s)} = \frac{k_2 + sk_1 + s^2 k_3}{s(k_4 s + 1)} \tag{5.8}$$

where the filter time constant k_4 is usually pre-chosen.

The controllable state space realization of the practical PID controller (5.8) is as follows:

$$\begin{aligned} \dot{x}_c^s(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{k_4} \end{bmatrix} x_c^s(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(k) \\ y(k) &= \begin{bmatrix} \tilde{k}_2 & \tilde{k}_1 \end{bmatrix} x_c^s(t) + \tilde{k}_3 e(k) \end{aligned} \tag{5.9}$$

where $\tilde{k}_1 = \frac{k_1}{k_4} - \frac{k_3}{k_4^2}$, $\tilde{k}_2 = \frac{k_2}{k_4}$ and $\tilde{k}_3 = \frac{k_3}{k_4}$. The multi-loop PID controllers, consisting of a group of individual PID controllers, have the following form:

$$C(s) = diag(c_1(s), c_2(s), \cdots, c_m(s)) \tag{5.10}$$

The state space realization of the multi-loop PID controllers can be obtained by stacking the state space realization of each individual PID controller:

$$\begin{aligned} x_c^m(t) &= A_c x_c^m(t) + B_c e(t) \\ u(t) &= C_c x_c^m(t) + D_c e(t) \end{aligned} \tag{5.11}$$

where

$$\begin{aligned} A_c &= diag\left(\begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{k_4^1} \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{k_4^2} \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{k_4^m} \end{bmatrix} \right) \\ B_c &= diag\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \cdots, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ C_c &= diag\left(\begin{bmatrix} \tilde{k}_2^1 & \tilde{k}_1^1 \end{bmatrix}, \begin{bmatrix} \tilde{k}_2^2 & \tilde{k}_1^2 \end{bmatrix}, \cdots, \begin{bmatrix} \tilde{k}_2^m & \tilde{k}_1^m \end{bmatrix} \right) \\ D_c &= diag\left(\tilde{k}_3^1, \tilde{k}_3^2 \cdots \tilde{k}_3^m \right) \end{aligned} \tag{5.12}$$

With the multi-loop PID controllers (5.11) the closed-loop system can be written as ($r = 0$):

$$\dot{X} = A_{cl}X(t) + G_{cl}\varsigma(t) \tag{5.13}$$

where the closed-loop state vector $X(t) = \begin{bmatrix} x(t) \\ x_c^m(t) \end{bmatrix}$ and the matrices $A_{cl}, G_{cl}, \bar{C}_i$ and $\bar{D}_i$ are composed as follows:

$$\begin{aligned} A_{cl} &= \begin{bmatrix} A - BD_cC & BC_c \\ -B_cC & A_c \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ -B_cC & A_c \end{bmatrix}}_{A_0} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_0} \underbrace{\begin{bmatrix} -D_c & C_c \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} C & 0 \\ 0 & I_{2m} \end{bmatrix}}_{C_0} \\ G_{cl} &= \begin{bmatrix} G - BD_cF \\ -B_cF \end{bmatrix} = \underbrace{\begin{bmatrix} G \\ -B_cF \end{bmatrix}}_{G_0} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} -D_c & C_c \end{bmatrix} \underbrace{\begin{bmatrix} F \\ 0 \end{bmatrix}}_{F_0} \end{aligned} \tag{5.14}$$

To make the closed-loop system (5.13) satisfy the generalized covariance constraints we have the following lemma from Chapter 3:

Lemma 5.3.1 *The SCCMPID problem is feasible if and only if there exists a matrix* $0 < \Sigma < \Phi$ *such that*

$$A_{cl}\Sigma + \Sigma A_{cl}^T + G_{cl}\Omega G_{cl}^T < 0 \tag{5.15}$$

where A_{cl} and G_{cl} are matrices shown in (5.14).

The proof is similar to Lemma 4.3.1 and it is omitted here.

5.4 Computational algorithm

Generally speaking it is difficult to solve the Matrix Inequality (5.15). The difficulties come from the fact that the inequality (5.15) is a bilinear matrix inequality (*BMI*) with respect to $\{A_c, C_c, D_c\}$ and Σ. BMI (5.15) is non-convex and known to be a NP-hard problem in contrast to LMI problems that can be solved by polynomial-time interior point methods. Although global optimization approaches using branch and bound methods for general BMIs have been proposed in (Goh *et al.* 1994*b*, Tuan *et al.* 1999), the necessary computational efforts would be prohibitive when their methods are used to solve our BMI for systems of high dimensions in unlimited regions of the variables $\{A_c, C_c, D_c\}$ and Σ. To solve the nonlinear matrix inequality (5.15), we need change them to some equivalent forms that are easier solved.

Proposition 5.4.1 *The SCCMPID problem is feasible if and only if there exist a scalar* $\varepsilon > 0$ *and matrices* $\Phi > \Sigma > 0$ *and* K *such that*

$$\begin{bmatrix} A_0\Sigma + \Sigma A_0^T + G_0\Omega G_0^T - \Gamma\Theta^{-1}\Gamma^T & B_0K + \Gamma\Theta^{-1} \\ K^TB_0^T + \Theta^{-T}\Gamma^T & -\Theta^{-1} \end{bmatrix} < 0 \tag{5.16}$$

where

$$\Gamma = \Sigma C_0^T + G_0 \Omega F_0^T \tag{5.17}$$

$$\Theta = F_0 \Omega F_0^T + \varepsilon I_{n+2m} \tag{5.18}$$

Proof:

Necessity: If the *SCCMPID* is feasible, then according to Lemma 5.3.1 there exists a matrix $\Phi > \Sigma > 0$ such that inequality (5.15) holds.

Substituting (5.14) and (5.17) to (5.15), one can obtain that

$$A_0\Sigma + \Sigma A_0^T + G_0\Omega G_0^T + B_0 K \Gamma^T + \Gamma K^T B_0^T + B_0 K F_0 \Omega F_0^T K^T B_0^T < 0 \tag{5.19}$$

The inequality (5.19) implies that

$$A_0\Sigma + \Sigma A_0^T + G_0\Omega G_0^T + B_0 K \Gamma^T + \Gamma K^T B_0^T + B_0 K \left(F_0 \Omega F_0^T + \varepsilon I_{n+2m}\right) K^T B_0^T < 0 \tag{5.20}$$

as long as $\varepsilon > 0$ is sufficiently small. Substitute (5.17) and (5.18) to inequality (5.20) and with the help of Schur Lemma, Inequality (5.16) can be obtained.

Sufficiency: Sufficiency is straightforward by reversing the procedure of proving the necessary condition and it is omitted here. ∇∇∇

Obviously the conditions in the Proposition 5.4.1 are not convex because of the quadratic term $-\Gamma\Theta^{-1}\Gamma^T$. Expand $-\Gamma\Theta^{-1}\Gamma^T$ and (5.16) is equivalent to

$$\Xi = \begin{bmatrix} \Xi_{11} & B_0 K + \Gamma\Theta^{-1} \\ K^T B_0^T + \Theta^{-1}\Gamma^T & -\Theta^{-1} \end{bmatrix} < 0 \tag{5.21}$$

where

$$\Xi_{11} = \left(A_0 - G_0\Omega F_0^T \Theta^{-1} C_0^T\right)\Sigma + \Sigma\left(A_0 - G_0\Omega F_0^T \Theta^{-1} C_0^T\right)^T + Q - \Sigma C_0 \Theta^{-1} C_0^T \Sigma \tag{5.22}$$

$$Q = G_0 \Omega G_0^T - G_0 \Omega F_0^T \Theta^{-1} F_0 \Omega G_0^T \tag{5.23}$$

To eliminate the quadratic term $-\Sigma C_0 \Theta^{-1} C_0^T \Sigma$, we need the following theorem:

Theorem 5.4.1 *The SCCMPID problem is feasible if and only if there exist a scalar $\varepsilon > 0$ and matrices $\Phi > \Sigma > 0$, K and T such that*

$$\Pi = \begin{bmatrix} \Pi_{11} & B_0 K + \Gamma\Theta \\ K^T B_0^T + \Theta^T \Gamma^T & -\Theta \end{bmatrix} < 0 \tag{5.24}$$

where Q, Θ are defined in (5.23) and (5.18) respectively and

$$\Pi_{11} = \left(A_0 - G_0\Omega F_0^T \Theta^{-1} C_0^T - T C_0 \Theta^{-1} C_0^T\right)\Sigma + \Sigma\left(A_0 - G_0\Omega F_0^T \Theta^{-1} C_0^T - C_0 \Theta^{-1} C_0 T^T\right)^T + Q \tag{5.25}$$

Proof:

Necessity: (5.21) implies that

$$\Xi+\begin{bmatrix} (\Sigma-T)C_0\Theta^{-1}C_0^T(\Sigma-T)^T & 0 \\ 0 & 0 \end{bmatrix}<0 \tag{5.26}$$

as long as $\|\Sigma-T\|_2$ is sufficiently small. The above inequality is equivalent to (5.24).

Sufficiency: Sufficiency is straightforward if it is noticed that

$$\begin{bmatrix} (\Sigma-T)C_0\Theta^{-1}C_0^T(\Sigma-T)^T & 0 \\ 0 & 0 \end{bmatrix}\geq 0 \tag{5.27}$$

$\nabla\nabla\nabla$

The matrix inequality (5.4.1) is not linear so that an iterative algorithm is used to calculate a feasible solution to (5.4.1).

Algorithm 5.4.1 *1. Select ε_0 and calculate Θ. Find the solution to the following Riccati equation:*

$$\begin{aligned}&\left(A_0-G_0\Omega F_0^T\Theta^{-1}C_0^T\right)\Sigma+\Sigma\left(A_0-G_0\Omega F_0^T\Theta^{-1}C_0^T\right)^T\\&+G_0\Omega G_0^T-G_0\Omega F_0^T\Theta^{-1}F_0\Omega G_0^T-\Sigma C_0\Theta^{-1}C_0^T\Sigma=0\end{aligned} \tag{5.28}$$

Let $T=\Sigma$.

2. Solve the following optimization problem:

$$\min_{\Sigma,K}\alpha$$

subject to:

$$\Pi+\alpha\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}<0 \tag{5.29}$$

If $\alpha\leq 0$, then one solution is obtained and the algorithm stops; otherwise, if α decreases slowly in the last L steps, then the algorithm may not find a solution for the specified ε and we need decrease ε and try again; or go to step 3.

3. Solve the following optimization problem:

$$\min_{\Sigma,K} tr(\Sigma) \quad subject\ to\ LMI\ (5.29)$$

Let $T=\Sigma$ and go to step 2.

Remark 5.4.1 *The algorithm was first used by Cao and Sun (1998) to calculate the static feedback controller and then was extended by Feng et al. (2002) to calculate the centralized PID controller parameters. It can be verified that Algorithm 5.4.1 is convergent. This is because α is decreased during iterations.*

5.5 Numerical example

An example is presented here to demonstrate the efficiency of the algorithm. This example is borrowed from (Zhai *et al.* 2001). The system (5.1) considered here is a 2-input-2-output system:

$$A = \begin{bmatrix} 1.0 & -1.0 & -2.2 & -1.0 & 2.0 & 2.0 & 0.2 & -2.0 \\ 2.1 & -5.1 & -1.2 & 0 & 1.1 & 1.0 & 0.1 & -0.7 \\ 2.1 & -1.0 & -3.2 & -0.9 & 2.0 & 2.0 & 0.2 & -2.0 \\ 8.3 & -10.4 & -7.4 & -1.0 & 7.4 & 7.0 & 0.1 & -6.5 \\ 2.2 & -4.0 & -1.3 & 0 & 0.2 & 1.1 & 0.1 & 0.2 \\ -2.2 & 7.8 & 3.2 & 0.3 & -7.2 & -2.3 & -0.9 & 1.3 \\ 2.4 & 5.1 & -0.2 & -0.9 & -4.0 & 2.0 & -2.8 & -2.0 \\ -1.2 & 6.0 & 2.2 & 0.2 & -6.2 & -0.2 & -1.0 & 0.2 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ -1 & 2 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 0 & -1 \\ -2 & 1 & 3 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

A decentralized controller is designed to attenuate the effect of disturbance on the controlled variables:

$$\begin{aligned} \dot{x}_c &= \begin{bmatrix} 0.17 & -0.47 & 0 & 0 & 0 \\ 2.45 & -1.71 & 0 & 0 & 0 \\ 0 & 0 & 2.37 & -11.78 & -16.81 \\ 0 & 0 & 2.72 & -3.58 & -3.63 \\ 0 & 0 & 0.96 & -5.44 & -9.13 \end{bmatrix} x_c + \begin{bmatrix} 0.54 & 0 \\ 1.84 & 0 \\ 0 & 0.36 \\ 0 & 0.12 \\ 0 & 0.10 \end{bmatrix} y \\ u &= \begin{bmatrix} 3.75 & -1.91 & 0 & 0 & 0 \\ 0 & 0 & -1.61 & -0.12 & 0.96 \end{bmatrix} x_c + \begin{bmatrix} 1.48 & 0 \\ 0 & -0.01 \end{bmatrix} y \end{aligned} \tag{5.30}$$

With the above controller, the process state covariance matrix is:

$$\Sigma = \begin{bmatrix} 2.0004 & -1.0984 & 2.0517 & 0.9132 & -1.1555 & 0.7340 & 2.2928 & 0.2454 \\ -1.0984 & 1.6590 & -1.1056 & 0.3947 & 1.7117 & -0.3442 & -1.1241 & -0.0208 \\ 2.0517 & -1.1056 & 2.5343 & 0.9874 & -1.1379 & 1.3303 & 2.5008 & 0.4197 \\ 0.9132 & 0.3947 & 0.9874 & 3.4752 & 0.5315 & 0.4098 & 1.0237 & 0.2946 \\ -1.1555 & 1.7117 & -1.1379 & 0.5315 & 1.8059 & -0.2624 & -1.1908 & 0.0537 \\ 0.7340 & -0.3442 & 1.3303 & 0.4098 & -0.2624 & 1.5081 & 1.1912 & 0.5998 \\ 2.2928 & -1.1241 & 2.5008 & 1.0237 & -1.1908 & 1.1912 & 2.8984 & 0.5154 \\ 0.2454 & -0.0208 & 0.4197 & 0.2946 & 0.0537 & 0.5998 & 0.5154 & 0.4729 \end{bmatrix}$$

Multi-loop PID controllers are designed so that the variance of each state will be smaller than 1.5 times of that achieved by the decentralized controller (5.30). The pre-specified filter time constant is chosen as $k_4 = 0.001$ for both the PID controllers. Using the algorithm in the last section, it can be obtained

$$K = \begin{bmatrix} 2.3287 & 0 & -6.8985 & 21.2553 & 0 & 0 \\ 0 & -0.0442 & 0 & 0 & 0.1977 & 0.8716 \end{bmatrix}$$

With the above multi-loop PID controller the process state covariance is:

$$\Sigma = \begin{bmatrix} 1.7913 & 0.4683 & 1.7303 & -1.6428 & 0.3366 & -0.2415 & 2.1093 & 0.0007 \\ 0.4683 & 2.0256 & 0.4365 & -1.4272 & 2.0473 & -0.5476 & 0.5347 & -0.1618 \\ 1.7303 & 0.4365 & 2.1795 & -1.5146 & 0.4213 & 0.3946 & 2.1297 & 0.2305 \\ -1.6428 & -1.4272 & -1.5146 & 4.1255 & -1.2442 & 0.8236 & -1.8175 & 0.4784 \\ 0.3366 & 2.0473 & 0.4213 & -1.2442 & 2.1773 & -0.4211 & 0.3612 & -0.0602 \\ -0.2415 & -0.5476 & 0.3946 & 0.8236 & -0.4211 & 1.5399 & 0.1484 & 0.6699 \\ 2.1093 & 0.5347 & 2.1297 & -1.8175 & 0.3612 & 0.1484 & 2.7711 & 0.2671 \\ 0.0007 & -0.1618 & 0.2305 & 0.4784 & -0.0602 & 0.6699 & 0.2671 & 0.5162 \end{bmatrix}$$

The corresponding frequency domain representative of the above PID controllers are:

$$C(s) = \begin{bmatrix} \frac{-4.3s^2-4345.6s-0.4}{s(s+1000)} & \\ & \frac{-0.001s^2-0.9865s+0.2119}{s(s+1000)} \end{bmatrix}$$

5.6 Conclusion

The multi-loop PID controller design for the continuous-time systems has been considered in this chapter based on the covariance constraint. An iterative LMI approach is proposed to solve the problem. A simulation example is used to illustrate the effectiveness of the proposed method.

Part III

Fault detection

6

A Novel LMI Approach towards Fault Detection and Isolation for Stochastic Discrete-time Systems

6.1 Introduction

Modern process industry has become increasingly complex. Fault Detection and Isolation (*FDI*) plays a crucial role in maintaining normal operation of the process. The term 'fault' refers to any kind of malfunction that can lead to unacceptable abnormality in the system performance. The early detection of process failure, identification of the sources of the faults and then timely action can avoid plant shutdown and catastrophic results. FDI usually consists of two steps: (1) residual generation and (2) decision making. Residual is usually the difference between various functions of the sensor output and the expected values of these functions under the normal operation conditions. The effect of the fault on the residual is called signature of the fault. Observer based fault detection scheme is one of the most widely studied residual generation schemes (Ding and Frank 1989, Frank 1990, Gertler and Kunwer 1995, Patton *et al.* 1989, Wünnenberg 1990). The residual generation step is crucial to the decision making step, in which whether or not faults present in the process is decided according to some logics. A simple and commonly applied logic rule is to compare the evaluation function, a function of the residual, with its threshold. An alarm is issued only

[1]Some version of this chapter is submitted to *Int. J. Control.* (Huang and Huang 2003*b*) for publication

when the evaluation function exceeds the threshold (Ding and Guo 1998, Zhong *et al.* 2003).

In practice, the difference between the observer state and the process state is always affected by the process disturbance, sensor and actuator noise, which impede the reliability of fault detection schemes. A lot of research has been done to enhance robustness of the detection observer in the presence of unknown inputs or disturbances (Frank 1994, Massoumnia 1986, Patton and Chen 1991, Wünnenberg 1990). The main idea of these approaches is to decouple the residual (or innovation) from the unknown disturbances. In other words, residual does not respond to the unknown input and disturbance. For example, the eigenstructure assignment method was used by Patton *et al.* (1989,1991) to decouple the residual from the disturbance; Wünnenberg (1990) applied a so-called unknown input observer to the disturbance decoupling problem; Nikoukhah (1994) proposed a method to generate innovations, which were decoupled from the unknown input and disturbance. These methods may be restrictive because the conditions for the perfect disturbance decoupling from residual may not be satisfied. Two approximate decoupling methods were introduced by Gertler and Kunwer (1995) if the unknown disturbance can not be totally decoupled from the residual. Ding and Guo (1998) presented a time domain optimization approach, in which a quadratic function was used to evaluate if fault existed or not. A threshold was generated for the evaluation function to enhance the robustness of the detection algorithm.

In this chapter, we propose a two-step approach towards robust fault detection and isolation for stochastic discrete-time systems, where the unknown disturbance is treated as a stochastic process and its statistical information is assumed known. In the first step, the H_2 norm of the transfer function from a single fault, called target fault, to the residual is magnified to some level and the effect of the unknown disturbance on the residual is minimized. The effect of unknown disturbance on the residual is measured by the covariance matrix of the residual. In the second step, a general quadratic evaluation function is optimized to isolate the nuisance faults from the target fault. A threshold is determined by the statistical distribution of the evaluation function. The contributions of this chapter are three fold: (1) a necessary and sufficient condition for the H_2 norm of a transfer function to be larger than a pre-specified value is derived in terms of LMIs; (2) if such a condition holds, the observer gain is parameterized in the solution to the LMIs; (3) the quadratic evaluation function, used by Ding and Guo (1998), is extended to a general quadratic form in order to isolate the target fault. The evaluation function method by nature is closely related to the structure residual generation approach (Li and Shah 2002, Gertler and Singer 1990).

The problem is formulated in section 2. The main results are given in Section 3, which contains two parts: (1) the residual generation method is proposed in the first subsection;

(2) the residual evaluation function and its threshold are derived in the second subsection. Simulated results are presented in Section 4, which is followed by the conclusion section.

6.2 Problem statement

The linear discrete-time system under consideration is described by the following state space equation:

$$x_{k+1} = Ax_k + Bu_k + Dd_k + \sum_{i=1}^{f} F_i f_k^i \tag{6.1}$$

where $x_k \in R^n$ is the state vector, $u_k \in R^m$ is the input vector, $d_k \in R^d$ is the disturbance vector, and the system is subject to multiple faults: f_k^i is a scalar representing the magnitude of $i-th$ fault and F_i is a vector representing the direction, on which the $i-th$ fault affects the system. We are considering the error in variable (*EIV*) case, *i.e.* both the measurement y_k and input u_k are corrupted with measurement noise or the actuator noise:

$$\begin{aligned} y_k &= Cx_k + G\varpi_k \\ u_k &= \hat{u}_k + H\varsigma_k \end{aligned} \tag{6.2}$$

where $\hat{u}_k \in R^m$ is the controller output, $\varpi_k \in R^t$ is the measurement noise vector and $\varsigma_k \in R^a$ is the actuator noise. A, B, C, D, G, H and F_i are known constant matrices with appropriate dimensions. Without losing generality, it is assumed that (A, C) is observable and the matrix C has the full row rank. The disturbance signal d_k and noise signal ϖ_k and ς_k are Gaussian distributed white noise. Let

$$v_k = \left[d_k^T \ \varsigma_k^T \ \varpi_k^T\right]^T \tag{6.3}$$

It is assumed that:

$$\begin{aligned} &E(v_k) = 0 \\ &E\left(v_i v_j^T\right) = \Phi\delta(i-j) \end{aligned} \tag{6.4}$$

where $\Phi = \begin{bmatrix} U & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & W \end{bmatrix}, U > 0, V > 0, W > 0.$

The main objective of the chapter is to detect and isolate the faults by designing a sequences of FDI subsystems. Each FDI subsystem shares the same structure as shown in Figure 6.1.

The observer, the evaluation function and the threshold for the evaluation function will be determined in the design procedure of the FDI subsystem.

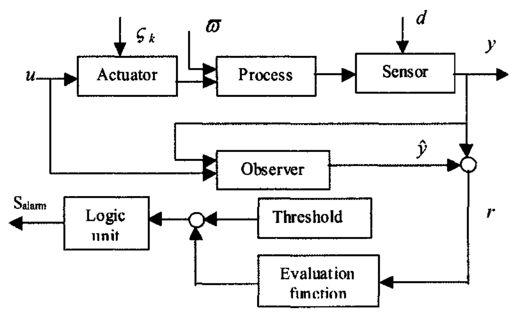

Figure 6.1: A general configuration of observer

6.2.1 Residual generation

The observer in the $i-th$ subsystem for the linear discrete-time system (6.1) is given as:

$$z_{k+1} = Az_k + B\hat{u}_k + L_i(Cz_k - y_k) \tag{6.5}$$

where z_k is the observer state and L_i is the observer gain matrix with appropriate dimension.

With the state observer described in Equation (6.5), the state estimation error $(e_k = x_k - z_k)$ dynamics is as follows:

$$e_{k+1} = (A + L_iC)e_k + Mv_k + \sum_{i=1}^{f} F_i f_k^i \tag{6.6}$$

where $M = \begin{bmatrix} D & BH & L_iG \end{bmatrix}$.

The residual of the observer is given by:

$$r_k = y_k - Cz_k = Ce_k + G\varpi_k \tag{6.7}$$

The state estimation error can be partitioned into two parts: the first part, denoted as e_k^U, is the response to the unmeasured disturbance and noise; the second part, denoted as e_k^F, is the response to the fault:

$$\begin{aligned} e_k &= e_k^U + e_k^F \\ e_{k+1}^U &= (A + L_iC)e_k^U + Mv_k \\ e_{k+1}^F &= (A + L_iC)e_k^F + \bar{F}_i\bar{f}_k^i + F_i f_k^i \end{aligned} \tag{6.8}$$

where

$$\bar{F}_i = \begin{bmatrix} F_1 & \cdots & F_{i-1} & F_{i+1} & \cdots & F_f \end{bmatrix}, \bar{f}_k^i = \begin{bmatrix} f_k^1 & \cdots & f_k^{i-1} & f_k^{i+1} & \cdots & f_k^f \end{bmatrix}^T$$

Correspondingly, the residual can be written as:

$$r_k = Ce_k^U + Ce_k^F + \bar{G}v_k \tag{6.9}$$

where $\bar{G} = \begin{bmatrix} 0 & 0 & G \end{bmatrix}$.

If unknown disturbances cannot be perfectly decoupled from the residual, the observer gain L_i should be designed in such a way that the faults can still "distinguish" itself from the disturbance even though the disturbance effect still appears on the residual. To achieve this goal, one needs to design an asymptotically stable observer such that the effect of the fault on the residual is magnified. In other words, it is desired that f_k^i has "significant" effect on r_k. Different norms have been applied to characterize the significance of the fault on the residual. H_∞ norm was used by Chen and Patton (1999) to measure the worst unknown disturbance effect on the residual and the minimum singular value of the transfer function from the fault to the residual was used to measure the effect of fault on the residual; H_2 norm was used to measure the effect of both disturbance and fault on the residual by Ding *et al.* (1989,1998). In this chapter, the effect of the fault f^i on r_k is measured by the H_2 norm of the corresponding transfer function:

Condition 6.2.1

$$\left\| T_{rf^i} \right\|_2^2 > \beta_i \tag{6.10}$$

where T_{rf^i} is the transfer function from f_k^i to r_k and β_i is a pre-specified constant number.

If there exists an asymptotically stable observer such that condition (6.2.1) is satisfied, then the observer gain is not unique and it can be parameterized with an additional variable. The trace of the covariance matrix of the estimated error e_k is minimized on this additional variable in order to reliably detect the target fault.

If there is no fault, the state estimation error is a stochastic process driven by the unknown disturbance and noise only. The state covariance of r_k can be calculated as follows:

$$\lim_{k \to \infty} \sigma(r_k) = C\Sigma C^T + GWG^T \tag{6.11}$$

where Σ is the covariance of e_k^U and is defined as:

$$\Sigma = \lim_{k \to \infty} E\left(e_k^U e_k^{U^T} \right) \tag{6.12}$$

It is well known (Bryson and Ho 1975) that Σ is the unique positive-definite solution to the following equation:

$$\Sigma = (A + L_i C)\Sigma (A + L_i C)^T + M\Phi M^T \tag{6.13}$$

6.2.2 Residual evaluation

After the observer gain L_i is calculated, another important task for designing the FDI subsystem is to evaluate the generated residual. A widely applied approach to enhance the robustness is to compare the evaluation function with a threshold (Ding and Guo 1998, Zhong *et al.* 2003). The residual evaluation function $J\left(S_{r,k}\right)$ is defined in a quadratic form:

$$J\left(S_{r,k}\right) = S_{r,k}^T Q S_{r,k} \tag{6.14}$$

where Q is a semi-definite positive matrix that will be determined later and $S_{r,k}$ is defined as:

$$S_{r,k} = \left[\begin{array}{cccc} r_{k-s}^T & r_{k-s+1}^T & \cdots & r_k^T \end{array}\right]^T \tag{6.15}$$

For the choice of the window length, $s+1$, it is suggested in (Ding and Guo 1998) that the window length s should be much larger than n.

Remark 6.2.1 *The evaluation function $J\left(S_{r,k}\right)$ can be interpreted as the weighted energy of the residual r in the interval $[k-s,k]$. In (Ding and Guo 1998, Zhong* et al. *2003), the Q matrix in the evaluation function is defined as a unitary matrix; however, the faults may not be isolated if Q is defined in that way. To isolate the faults, Q should satisfy specific conditions as shown in the next section.*

6.3 Main results

6.3.1 Observer design

There are numerous references on restricting the H_2 norm of a transfer function through *LMIs*. On the contrary, there are few results on how to magnify the H_2 norm of a transfer function as shown in inequality (6.10). Some sufficient conditions were given by Chen and Patton (1999), Ding *et al.* (2001). In this subsection a necessary and sufficient condition for the existence of observer gain L_i satisfying condition (6.10) is given. To prove the result, several useful lemmas are given as follows:

Lemma 6.3.1 *(Hsieh and Skelton 1990) Let $T_1, T_2 \in R^{n\times m}$. Then $T_1T_1^T = T_2T_2^T$ if and only if $T_1 = T_2V$, where $VV^T = I_m$.*

Lemma 6.3.2 *(Hsieh and Skelton 1990) The linear matrix equation $AXC = B$ has solution if and only if*

$$AA^+BC^+C = B \tag{6.16}$$

If the above equality holds, the general solution to the linear matrix equation is given as:

$$X = A^{+}BC^{+} + A^{+}AZCC^{+} - Z \tag{6.17}$$

where Z is an arbitrary matrix with appropriate dimension.

Lemma 6.3.3 *For the following discrete-time system:*

$$\begin{aligned} x_{k+1} &= \mathscr{A}x_k + \mathscr{B}u_k \\ y_k &= \mathscr{C}x_k \end{aligned} \tag{6.18}$$

It is assumed that $(\mathscr{A}, \mathscr{B})$ is controllable. The following statements are equivalent:

[1] *The matrix $\mathscr{A}$ is asymptotically stable and $\|T_{yu}\|_2^2 > \beta$.*

[2] *There exist a matrix $\mathscr{P} > 0$ and a scalar $0 \leq \gamma < 1$ such that*

$$\gamma\mathscr{B}\mathscr{B}^{\mathscr{T}} \geq \mathscr{A}\mathscr{P}\mathscr{A}^{\mathscr{T}} - \mathscr{P} + \mathscr{B}\mathscr{B}^{\mathscr{T}} \geq 0 \tag{6.19}$$

$$tr\left(\mathscr{C}\mathscr{P}\mathscr{C}^{\mathscr{T}}\right) > \beta \tag{6.20}$$

Proof:

$[1] \rightarrow [2]$:

If the system (6.18) is asymptotically stable, the H_2 norm of the transfer function T_{yu} can be calculated as follows:

$$\|T_{yu}\|_2^2 = tr\left(\mathscr{C}\mathscr{X}_0\mathscr{C}^{\mathscr{T}}\right) > \beta$$

where $\mathscr{X}_0 > 0$ is the unique solution to the following Lyapunov function:

$$\mathscr{A}\mathscr{X}\mathscr{A}^{\mathscr{T}} - \mathscr{X} + \mathscr{B}\mathscr{B}^{\mathscr{T}} = 0 \tag{6.21}$$

Clearly, inequalities (6.19) and (6.20) hold for any $1 > \gamma \geq 0$ and the matrix $\mathscr{X}_0$.

$[2] \rightarrow [1]$:

For any $1 > \gamma \geq 0$, if $(\mathscr{A}, \mathscr{B})$ is controllable, then $(\mathscr{A}, \sqrt{1-\gamma}\mathscr{B})$ is controllable as well. Inequalities (6.19) imply that

$$\mathscr{A}\mathscr{P}\mathscr{A}^{\mathscr{T}} - \mathscr{P} + (1-\gamma)\mathscr{B}\mathscr{B}^{\mathscr{T}} \leq 0 \tag{6.22}$$

The above inequality indicates that $\mathscr{A}$ is an asymptotically stable matrix according to the well known Lyapunov theorem. Monotonicity of solution to Lyapunov equation (6.21) implies that $\mathscr{P} \leq \mathscr{X}_0$. $\|T_{yu}\|_2^2 = tr\left(\mathscr{C}\mathscr{X}_0\mathscr{C}^{\mathscr{T}}\right) \geq tr\left(\mathscr{C}\mathscr{P}\mathscr{C}^{\mathscr{T}}\right) > \beta$. ∇∇∇

We can give the following Corollary without proof because of the duality of linear systems:

Corollary 6.3.1 *For the discrete-time system (6.18), if $(\mathscr{A},\mathscr{C})$ is observable, the following statements are equivalent:*

- *The matrix $\mathscr{A}$ is asymptotically stable and $\|T_{yu}\|_2^2 > \beta$.*
- *There exist a matrix $\mathscr{Q} > 0$ and a scalar $0 \leq \gamma < 1$ such that*

$$\gamma \mathscr{C}^{\mathscr{T}} \mathscr{C} \geq \mathscr{A}^{\mathscr{T}} \mathscr{Q} \mathscr{A} - \mathscr{Q} + \mathscr{C}^{\mathscr{T}} \mathscr{C} \geq 0 \tag{6.23}$$

$$tr\left(\mathscr{B}^{\mathscr{T}} \mathscr{Q} \mathscr{B}\right) > \beta \tag{6.24}$$

Remark 6.3.1 *In Lemma 6.3.3 and Corollary 6.3.1 matrix inequalities are derived to characterize the lower bound of the H_2 norm of a transfer function matrix. Similarly, matrix inequalities can also be obtained for H_∞ norm of a transfer function matrix.*

Lemma 6.3.4 *For any $C \in R^{p\times n}$ and $1 > \gamma \geq 0$, $\gamma C^T C \geq T \geq 0$ if and only if there exists $T_1 \in R^{p\times p}$ such that $I_p > T_1 \geq 0$ and $T = C^T T_1 C$.*

Lemma 6.3.4 can be easily proved by using singular value decomposition and it is omitted here.

Theorem 6.3.1 *There exists an observer gain L_i such that* (6.10) *holds if and only if there exist matrices $Y > 0$ and $I_p \geq T > 0$ such that*

$$\left(I_n - C^T C^{T+}\right)\left(A^T Y A - Y + C^T T C\right)\left(I_n - C^T C^{T+}\right) = 0 \tag{6.25}$$

$$Y \geq C^T T C \tag{6.26}$$

$$F_i^T Y F_i > \beta \tag{6.27}$$

and the observer gain is given by:

$$L_i = \left(Y^{-\frac{1}{2}} V^T \left(Y - C^T T C\right)^{\frac{1}{2}} - A\right) C^+ + Z C C^+ - Z \tag{6.28}$$

where $V \in R^{n\times n}$ is an orthogonal matrix satisfying (6.33) and $Z \in R^{n\times p}$ is an arbitrary matrix.

Proof:

Necessity: If there exists an observer such that (6.10) holds, Corollary 6.3.1 and Lemma 6.3.4 imply that there exist matrices $Y > 0$ and $I_p > T_1 \geq 0$ such that

$$(A + L_i C)^T Y (A + L_i C) = Y - C^T (I_p - T_1) C \tag{6.29}$$

$$F_i^T Y F_i > \beta \tag{6.30}$$

Let $T = I_p - T_1 > 0$ and it can be verified that $I_p \geq T > 0$. Inequality (6.29) implies (6.26). According to Lemma 6.3.1, there exists an orthogonal matrix V such that

$$C^T L_i^T Y^{\frac{1}{2}} = \left(Y - C^T TC\right)^{\frac{1}{2}} V - A^T Y^{\frac{1}{2}} \tag{6.31}$$

The equation above has at least one solution: L_i^T. According to Lemma 6.3.2, it is implied that

$$C^T C^{T+}\left(\left(Y - C^T TC\right)^{\frac{1}{2}} V - A^T Y^{\frac{1}{2}}\right) = \left(Y - C^T TC\right)^{\frac{1}{2}} V - A^T Y^{\frac{1}{2}} \tag{6.32}$$

Equation (6.32) implies that

$$\left(I_n - C^T C^{T+}\right) A^T Y^{\frac{1}{2}} = \left(I_n - C^T C^{T+}\right)\left(Y - C^T TC\right)^{\frac{1}{2}} V \tag{6.33}$$

Multiplying both sides of the above equation with their transpose matrices, equation (6.25) is obtained.

Sufficiency can be proved by reversing the necessity part and it is omitted here. $\nabla\nabla\nabla$

If the condition (6.2.1) holds, the observer gain is not unique and an extra performance index can be optimized over the admissible observer gain set. To enhance the sensitivity of the fault detection scheme, the trace of the covariance matrix of the estimated error e_k is minimized over the admissible observer gain set. This optimization problem can be equivalently transferred to a semi-definite programming problem as listed in the second step of the following algorithm. The algorithm to calculate the observer gain is given as:

Algorithm 6.3.1 *1.Find $Y > 0 \in R^n$ that solves the following semi-definite programming problem:*

$$\min_{Y,T} \quad trace(Y) \quad subject\ to\ (6.25), (6.26), (6.27)$$

The solution is denoted as Y^*, T^* and V^*.
Let $L_0 = \left(Y^{*-\frac{1}{2}} V^{*T}\left(Y^* - C^T T^* C\right)^{\frac{1}{2}} - A\right) C^+$.

2. Find $Z \in R^{n \times p}$ that solves the following semi-definite programming problem:

$$\min_{Z} trace(\Sigma) \quad s.t.$$
$$\begin{bmatrix} (A + L_0 C)\Sigma(A + L_0 C)^T - \Sigma + DUD^T + BHVH^T B^T & L_0 G + ZCC^+ G - ZG \\ G^T L_0^T + G^T (C^+)^T C^T Z^T - G^T Z^T & -W^{-1} \end{bmatrix} < 0$$

The solution is denoted by Z^*.

3. The observer L_i is then calculated as:

$$L_i = L_0 + Z^* CC^+ - Z^* \tag{6.34}$$

Remark 6.3.2 *In step (1) the trace of Y is minimized. This is inspired by the idea of Lyapunov diagram shaping method (Scherer* et al. *1997). As a matter of fact, besides minimizing the trace of Y, one can minimize other index as well, such as the condition number of Y.*

6.3.2 Evaluation function and the threshold calculation

Suppose that the observer gain L_i is known. $S_{r,k}$ can be written as:

$$S_{r,k} = \Psi_1 e_{k-s} + \Psi_2 V_k + \Psi_3 \tilde{F}_k^o + \Psi_4 \tilde{F}_k^i \tag{6.35}$$

where

$$V_k = \begin{bmatrix} v_{k-s} \\ v_{k-s+1} \\ v_{k-s+2} \\ \vdots \\ v_k \end{bmatrix}, \quad \tilde{F}_k^o = \begin{bmatrix} \bar{f}^i_{k-s} \\ \bar{f}^i_{k-s+1} \\ \bar{f}^i_{k-s+2} \\ \vdots \\ \bar{f}^i_k \end{bmatrix}, \quad \tilde{F}_k^i = \begin{bmatrix} f^i_{k-s} \\ f^i_{k-s+1} \\ f^i_{k-s+2} \\ \vdots \\ f^i_k \end{bmatrix}, \Psi_1 = \begin{bmatrix} C \\ C\bar{A} \\ C\bar{A}^2 \\ \vdots \\ C\bar{A}^s \end{bmatrix}$$

$$\Psi_2 = \begin{bmatrix} \bar{G} & 0 & 0 & \cdots & 0 \\ CM & \bar{G} & 0 & \cdots & 0 \\ C\bar{A}M & CM & \bar{G} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C\bar{A}^{s-1}M & C\bar{A}^{s-2}M & \cdots & CM & \bar{G} \end{bmatrix},$$
$$\Psi_3 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ C\bar{F}_i & 0 & 0 & \cdots & 0 \\ C\bar{A}\bar{F}_i & C\bar{F}_i & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C\bar{A}^{s-1}\bar{F}_i & C\bar{A}^{s-2}\bar{F}_i & C\bar{A}^{s-3}\bar{F}_i & \cdots & 0 \end{bmatrix}, \tag{6.36}$$
$$\Psi_4 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ CF_i & 0 & 0 & \cdots & 0 \\ C\bar{A}F_i & CF_i & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C\bar{A}^{s-1}F_i & C\bar{A}^{s-2}F_i & C\bar{A}^{s-3}F_i & \cdots & 0 \end{bmatrix}$$

To isolate the fault, ideally, the evaluation function should be designed such that it does not depend on $\tilde{F}_k^o$. Since e_{k-s} may be affected by the faults that occur at time before $k-s$, the evaluation function should not depend on e_{k-s} either. Q is symmetric and semi-definite positive so that Q can be decomposed as: $Q = \Gamma\Gamma^T$, where $\Gamma \in R^{(s+1)p\times\gamma}$ is a matrix with full column rank and its column number γ will be determined later. It can be verified that if and only if

$$\begin{bmatrix} \Psi_3^T \\ \Psi_1^T \end{bmatrix} \Gamma = 0 \tag{6.37}$$

then $J(S_{r,k})$ is not a function on $\tilde{F}_k^o$ and e_{k-s}. Let $\gamma = dim\left(null\left(\begin{bmatrix} \Psi_3^T \\ \Psi_1^T \end{bmatrix}\right)\right)$. Equation (6.37) defines a subspace Λ, from which each column of the candidate Γ is chosen:

$$\Lambda = \left\{ \Gamma_i \middle| \Gamma_i = \sum_{j=1}^{\gamma} \theta_{ij}\bar{\Gamma}_j \right\} \tag{6.38}$$

where $\bar{\Gamma}_j$, $j = 1, 2, \cdots, \gamma$ is an orthogonal base of the subspace defined by (6.37) and Γ_i is the $i-th$ column of matrix Γ.

Remark 6.3.3 *The applicability of this evaluation function construction method is the nonemptiness of Λ. The necessary and sufficient condition to ensure the nonemptiness of Λ is the column rank of $\begin{bmatrix} \Psi_3^T \\ \Psi_1^T \end{bmatrix}$ is less than its column number. Apparently, a sufficient condition for the set Λ to be nonempty is that the row number of $\begin{bmatrix} \Psi_3^T \\ \Psi_1^T \end{bmatrix}$ is less than its column number, i.e., $p(s+1) > n + (f-1)(s+1)$. This inequality holds only if $p > f-1$. In other words, the output number is larger or equal to the number of the faults. Suppose $p = f$, the window length $s+1$ should satisfy $s+1 > n$.*

Let $\bar{\Gamma} = \begin{bmatrix} \bar{\Gamma}_1 & \bar{\Gamma}_2 & \cdots & \bar{\Gamma}_\gamma \end{bmatrix}$ and $\Theta = [\alpha_{ij}]_{\gamma\times\gamma}$, Γ can be written as:

$$\Gamma = \bar{\Gamma}\Theta \tag{6.39}$$

Remark 6.3.4 *Actually, if Γ satisfies (6.37), then $\prod = \Gamma^T S_{r,k} \in R^{\gamma\times 1}$ is the so called structured residual vector (SRV) (Gertler and Kunwer 1995, Gertler and Singer 1990, Li and Shah 2002). The evaluation function, defined in (6.15), is nothing but the 2-norm of the SRV, and it is different from the one used in (Ding and Guo 1998, Zhong* et al. *2003), where the 2-norm of $S_{r,k}$ is used.*

The similar evaluation function is introduced in (Li and Shah 2002). However, the square weighted residual *SWR* is used as an index for fault detection so that the detection power of the evaluation function is not fully optimized. In constructing the evaluation function $J\left(S_{r,k}\right)$, Q should be designed such that the fault signature of the $i-th$ fault is maximized. To increase the sensitivity of the fault detection subsystem to the $i-th$ fault, it is desired to reduce the threshold of the evaluation function. However, the threshold can only be determined after Q is fixed. Instead, it is the mean of $J\left(S_{r,k}\right)$ that is constrained here because the mean value of J relates to the threshold.

Fault detection is to detect the abnormal behavior of the dynamic system. From statistical point of view, it is the following hypothesis test.

- H_0: no $i-th$ fault present in the process
- H_1: $i-th$ fault present in the process

It can be verified that if there is no fault,

$$\begin{aligned}
\lim_{k\to\infty} E(J) &= \lim_{k\to\infty} E\left(V_k^T\Psi_2^T\Gamma\Gamma^T\Psi_2V_k\right) \\
&= \lim_{k\to\infty} E\left(tr\left(V_k^T\Psi_2^T\Gamma\Gamma^T\Psi_2V_k\right)\right) \\
&= tr\left(\lim_{k\to\infty} E\left(\Psi_2^T\Gamma\Gamma^T\Psi_2V_kV_k^T\right)\right) \\
&= tr\left(\Psi_2^T\Gamma\Gamma^T\Psi_2\Omega\right) = tr\left(\Omega^{\frac{1}{2}}\Psi_2^T\bar{\Gamma}\Theta\Theta^T\bar{\Gamma}^T\Psi_2\Omega^{\frac{1}{2}}\right) \le \mu_i
\end{aligned} \tag{6.40}$$

where $\Omega = diag\underbrace{\left(\begin{array}{cccc}\Phi & \Phi & \cdots & \Phi\end{array}\right)}_{s+1}$ and μ_i is an appropriately selected upper bound for the mean value of the evaluation function. In deriving the equality above, $tr(AB) = tr(BA)$ is used, where A and B are matrices with appropriate dimensions.

Let $P = \Theta\Theta^T$. The constraint (6.40) can be rewritten as:

$$\begin{aligned}
\lim_{k\to\infty} E(J) &= tr\left(\Omega^{\frac{1}{2}}\Psi_2^T\bar{\Gamma}P\bar{\Gamma}^T\Psi_2\Omega^{\frac{1}{2}}\right) \\
&= \sum_{j=1}^{(d+t+a)\times(s+1)} b_jPb_j^T \le \mu_i
\end{aligned} \tag{6.41}$$

where b_j is the $j-th$ row of $\Omega^{\frac{1}{2}}\Psi_2^T\bar{\Gamma}$.

Usually, it is desired to select Γ such that the worst case is maximized:

$$\max_{\Gamma}\min_{\tilde{F}_k^i}\frac{\tilde{F}_k^{iT}\Psi_4^T\Gamma\Gamma^T\Psi_4\tilde{F}_k^i}{\tilde{F}_k^{iT}\tilde{F}_k^i}$$

However, because of the rank deficiency of matrix $\Psi_4^T \Gamma \Gamma^T \Psi_4$ the above optimization always ends up with 0. The step type fault is one of the most occurred faults in process so that it is reasonable to assume that $\tilde{F}_k^i = a_{s+1}$, where $a_{s+1} \in R^{s+1}$ is a constant vector and each of its element is 1. As a consequence of the step fault assumption, the signature of the $i-th$ fault, measured by $a_{s+1}^T \Psi_4^T \Gamma \Gamma^T \Psi_4 a_{s+1}$, is maximized subject to the constraint (6.41).

$$\begin{aligned} \Xi &= a_{s+1}^T \Psi_4^T \Gamma \Gamma^T \Psi_4 a_{s+1} \\ &= a_{s+1}^T \Psi_4^T \bar{\Gamma} \Theta \Theta^T \bar{\Gamma}^T \Psi_4 a_{s+1} \\ &= a_{s+1}^T \Psi_4^T \bar{\Gamma} P \bar{\Gamma}^T \Psi_4 a_{s+1} \end{aligned} \tag{6.42}$$

Both the objective function (6.42) and the constraint (6.41) are linear to the positive definite decision variable P so that this problem can be solved by semi-definite programming. The solution is denoted by P^*, which can then be fractionized as: $P^* = \Theta^* \Theta^{*T}$. It should be pointed out that Θ^* is not unique because $\bar{\Theta}\bar{\Theta}^T = P^*$, where $\bar{\Theta} = \Theta^* X$ and X is any orthogonal matrix with appropriate dimension.

Under the H_0 hypothesis, $\prod = \Gamma^T S_{r,k} \sim \mathcal{N}\left(0, \Gamma^T \Phi_2 \Omega \Phi_2^T \Gamma\right)$. The evaluation function $J = \prod^T \prod$ is referred to as the central quadratic form (Mathai and Provost 1992). The threshold of the evaluation function J can be determined from the distribution function of the quadratic forms. However, The density function and the distribution function of J are complicated and their computations are prohibitive. The interested readers are referred to Section 2 of Chapter 4 in (Mathai and Provost 1992). Several approximation methods were also covered in (Mathai and Provost 1992). For example, J can be approximated by $A\omega^r$, where ω is the chi-square distribution with p degrees of freedom. A, r and p are determined by comparing the known first three moments of J with those of $A\omega^r$.

A simple threshold of the evaluation function J can be approximately calculated according to the following Corollary:

Corollary 6.3.2

$$Pr\{J \geq \delta_1 | H_0\} \leq \frac{E(J)}{\delta_1} \tag{6.43}$$

This corollary is the direct result of the Theorem 4.8.2 in (Mathai and Provost 1992) and the proof is omitted for the brevity reason.

Corollary 6.3.2 implies that the probability of *type I error* is less than $1 - \frac{E(J)}{\delta}$ if the threshold is chosen as δ. For example, suppose that the significance level is chosen as 10%, the threshold of J can be chosen as $10E(J)$.

Remark 6.3.5 *The threshold calculated based on Corollary 6.3.2 is conservative because the knowledge on the distribution of J is not used. To get more accurate threshold, we should use the the approximation methods that are mentioned above. Corollary 6.3.2 is presented here in order to demonstrate the motivation of constraining E (J) in constructing the evaluation function. It should also be pointed out that Corollary 6.3.2 can apply not only for the Gaussian distributed noise but also for other distributed noise.*

6.4 Simulation results

6.4.1 Case study 1

The first example is taken from (Ding and Guo 1998) in order to demonstrate the detection capacity of the fault observer that is calculated using the proposed method.

The process model is as follows:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.5 & -0.7 & 0.7 & 0 \\ 0 & 0.8 & 0 & 0 \\ -1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0.4 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 0.6 \\ 0 \\ 0 \end{bmatrix} d_k + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} f_k \\ y_k &= \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} x_k \end{aligned}$$

The disturbance d_k is white noise bounded by

$$\sum_{i=k}^{k+25} d(i)^2 \leq 0.25$$

In other words, d_k is 0 mean white noise with variance 0.01. With the algorithm presented in the last section (the condition number instead of the trace of Y is minimized) and $\beta = 30$, the obtained observer is:

$$L = \begin{bmatrix} -0.8147 \\ 1.3912 \\ 1.5521 \\ -2.5286 \end{bmatrix}$$

There is only one fault so that it is not necessary to construct the evaluation function as described in the last section. Simply we set $Q = I_8$. The evaluation function (window length is 8) and its threshold are shown in Figure (6.2). In the simulation, a step fault occurs at 50 second (sampling interval is 0.01 s) and $f_k = 0.2$.

It can be shown that fault can be clearly detected after it occurs. This is because the observer designed separates the effect of the disturbance from the effect of the fault, which can be shown clearly in Figure (6.3).

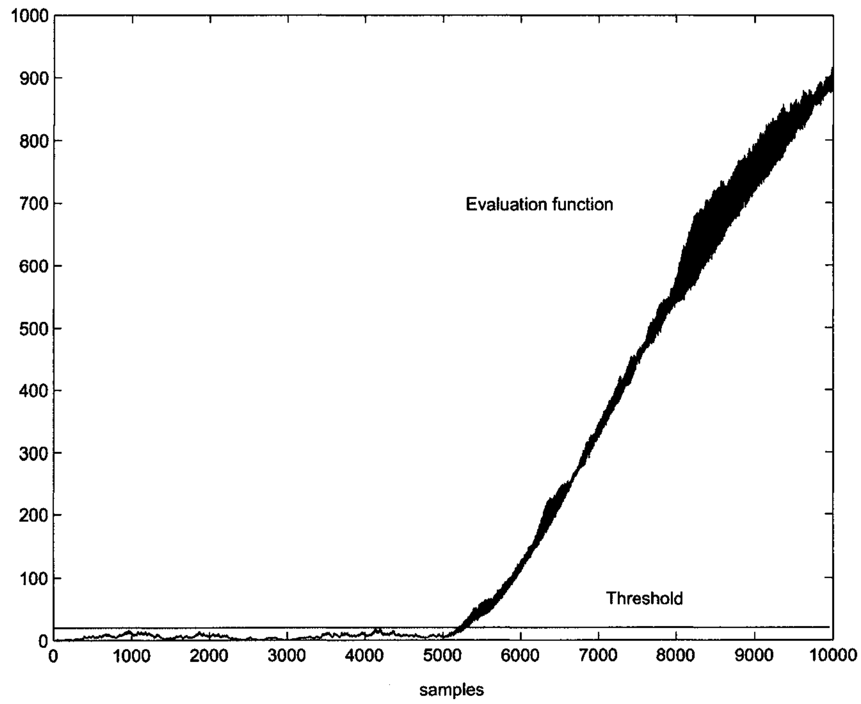

Figure 6.2: Evaluation function and its threshold

6.4.2 Case study 2

The second example is taken from (Keller 1999) to show the detection and isolation capacity of the proposed method. The stochastic discrete-time system is as follows:

$$A = \begin{bmatrix} 0.2 & 1 & 0 & 0.2 \\ 0 & 0.5 & 1 & 0.4 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0 & 0 & 0.3 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, F = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, G = I_3, U = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, W = I_3, H = 0$$

Design of the 1st FDI subsystem:

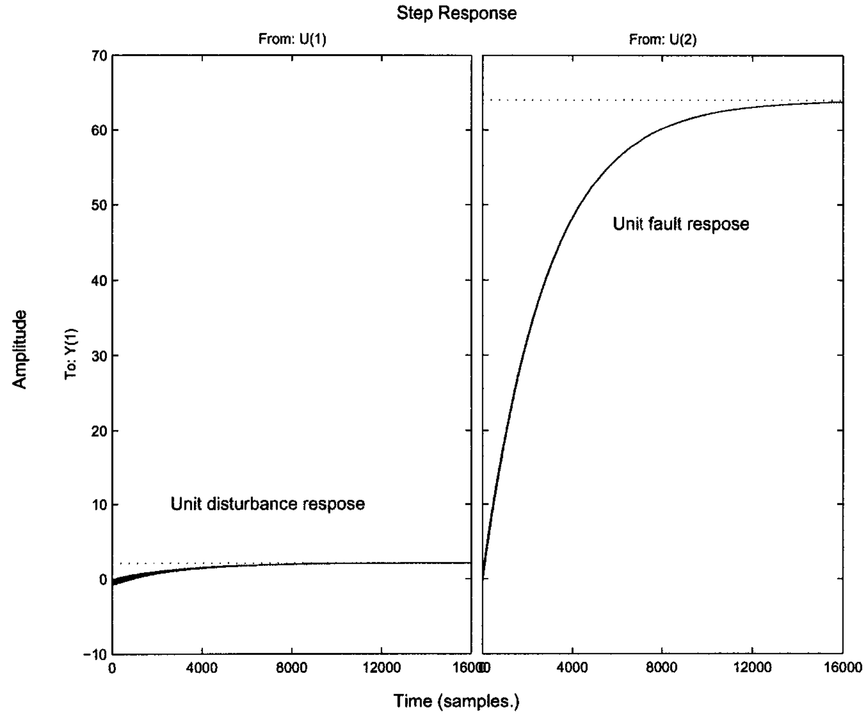

Figure 6.3: Step responses from disturbance or fault to the residual

Let $\beta = 50$. Applying Algorithm (6.3.1) and the obtained observer gain is

$$L_1 = \begin{bmatrix} -1.0127 & 0 & -0.2127 \\ 0.2561 & -1 & -0.1662 \\ 0 & -0.2781 & -1 \\ 0.2338 & 0 & 0.4561 \end{bmatrix}$$

Let the window length for the evaluation function $s+1 = 12$. After calculation, it turns out that $\gamma = 22$ and the threshold for the evaluation function is 69.

The 2nd FDI subsystem can be designed following the same procedure. The observer gain

$$L_2 = \begin{bmatrix} -1 & -0.0127 & -0.1873 \\ 0.0219 & -1 & -0.4 \\ 0 & -0.0439 & -1.2338 \\ 0 & -0.2338 & 0.4561 \end{bmatrix}$$

Let the window length for the evaluation function $s+1 = 12$. After calculation, it turns out that $\gamma = 22$ and the threshold for the evaluation function is 53.

The evaluation functions and their thresholds for both FDI subsystems are shown in Figure (6.4), where the first fault occurs at 400 and its magnitude is 1. The evaluation functions and their threshold for both FDI subsystems are shown in Figure (6.5), where the second fault occurs at 400 with magnitude 1.

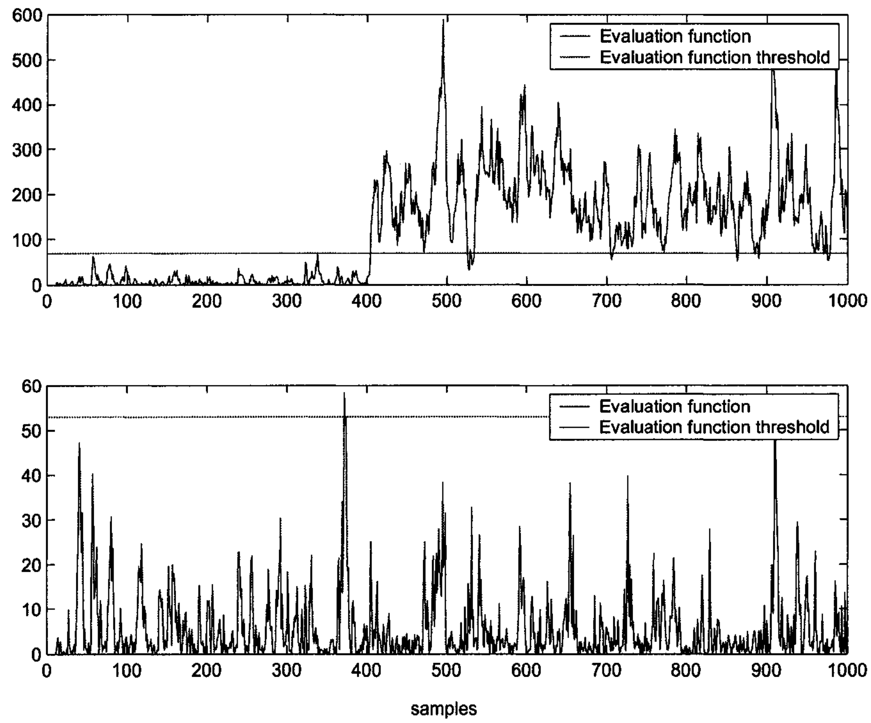

Figure 6.4: Evaluation function and its threshold for the first FDI subsystem

6.5 Conclusion

We have considered the fault detection and isolation problem (FDI) for stochastic discrete-time systems. A novel two-step FDI scheme is proposed. A necessary and sufficient condition for the H_2 norm of the transfer function from the fault to the residual to be larger than some pre-specified β is given in LMI forms. The relationship between the constructed evaluation function and the structured residual vector is also discussed. The simulation results show the effectiveness of the proposed method.

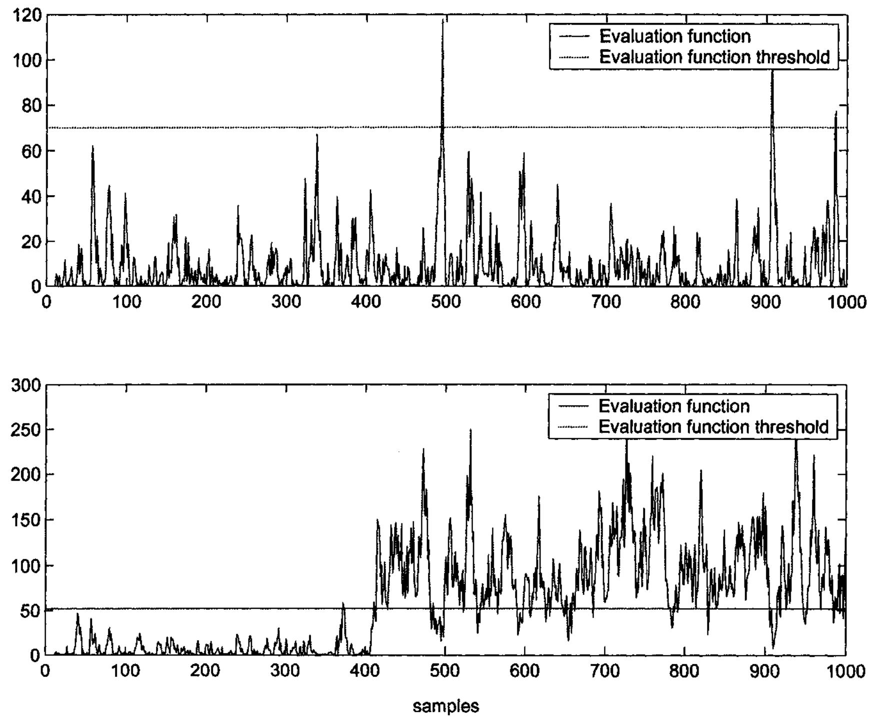

Figure 6.5: Evaluation function and its threshold for the second FDI subsystem

7

Covariance Based Fault Detection - an Alternative Approach through PCA

7.1 Introduction

Fault detection is a key to the reliable operation for any Modern process. Generally speaking, fault detection algorithms can be categorized in two groups: model based methods, such as diagnostic observers, Kalman filters (or extended Kalman filters) and parity space methods; and data driven methods, such as PCA, PLS and Neural Networks. It is desirable to use the data driven methods to monitor the process, when it is difficult to obtain a model. The data based methods use some "implicit" model, for example the weightings in Neural Networks and the loading factors in PCA. The basic idea of data based methods is to extract an "implicit" model from the data directly and then check if the new observations fit the "implicit" model or not.

In contrast to the last chapter, this chapter considers data based approach for fault detection through a practical example and, in particular, is concerned with the tailing lines' sanding detection using principal component analysis (*PCA*). PCA is one of the most widely used data based methods, and it has wide applications in chemical industry, such as chemical process monitoring (Wise and Ricker 1991, MacGregor 1994, MacGregor and Kourti 1995); sensor fault identification and reconstruction (Dunia *et al.* 1996); process fault identification and reconstruction (Dunia and Qin 1998). Formulated as a multivariate statistical process control

(*MSPC*) task, PCA is used to extract a few independent variables, called latent variables, from a large amount of process variables, thus reducing the dimensionality of monitoring problem. Typically, process monitoring via PCA involves two indices: the squared prediction error (*SPE*) and the Hotelling T^2. An abnormal situation will usually cause at least one of the two indices to exceed the control limits.

However, the implementation of the traditional PCA in the tailing line system shows that the traditional PCA cannot work properly. The reason is that traditional PCA based monitoring scheme cannot deal with the normal process variation, including mean drift, variance change and correlation structure change, because once the data set, with which the PCA model is built, is selected the PCA model is fixed and can not adapt to the normal process changes. The performance of PCA-based process monitoring scheme degrades when processes demonstrate timevarying behaviors. To enhance the robustness of the PCA methods, several modified PCA methods were proposed. Wold (1994) discussed the use of the exponentially weighted moving average (*EWMA*) filters in conjunction with PCA and PLS. A recursive PCA algorithm was presented by Li *et al.* (2000), with which the mean and (covariance) correlation are recursively updated, to adapt the process changes. This recursive PCA algorithm keeps all the historical information when it updates the mean and covariance (correlation) matrix. However, it is desirable to discard the most historical data because they do not represent the current process behavior. To overcome this the forgetting factor was introduced in (Li *et al.* 2000). Based on the work of Li *et al.* (2000), we present a recursive moving window PCA algorithm here and implement it in the tailing line's sanding detection problem. The results showed that RMWPCA was efficient in calculation, robust to process operation changes and the false alarm rate was significantly reduced.

The tailing line system is briefly described in Section 2 and the root cause of the sanding is also briefly discussed; in Section 3 the solid-liquid mixture model is introduced; in Section 4, the traditional PCA is discussed briefly; the RMWPCA is proposed in the Section 5; the sanding detection comparison results are shown in Section 6, which is followed by the conclusion section.

7.2 The tailing system

The tailing system is used to transport tailings, which contains sands (60% $\sim$ 70%), water and very little of bitumen, to tailings pond. The tailing system starts with a tailing distributor, which distributes the waste to several tailing lines. At the beginning of each line there is a pumpbox, followed by three tailing pumps driven by varied Speed Drivers (*VSD*). These three tailings pumps can not generate sufficient head to pump the tailings through the required distance to the tailing pond so that there are two or three (according to the length of the pipeline) pumping stations at strategic locations along the pipeline. Each pumping station contains one surge tank and two or three pumps (according to pumping requirement). The surge tanks are open to atmosphere and cannot transfer any unused pump head from one pump station to the next pump station. Some pipes use standard carbon steel pipe with the high wear areas using chromium carbide overlaid pipe and other tailings lines use induction hardened pipe. The beginning tailing station uses pump suction pressure to vary the speed of strategically placed variable speed driven (*VSD*) pumps. The other tailing stations utilize level, which is measured by ultrasonic level sensors, in the surge tank to vary the speed of all the tailings pumps, which are driven by induction motors controlled by a single variable frequency drive (*VFD*) in each station simultaneously. Figure (7.1) shows a simplified diagram of a tailing line. For each surge tank, a level controller is used to adjust the VFD speed in order

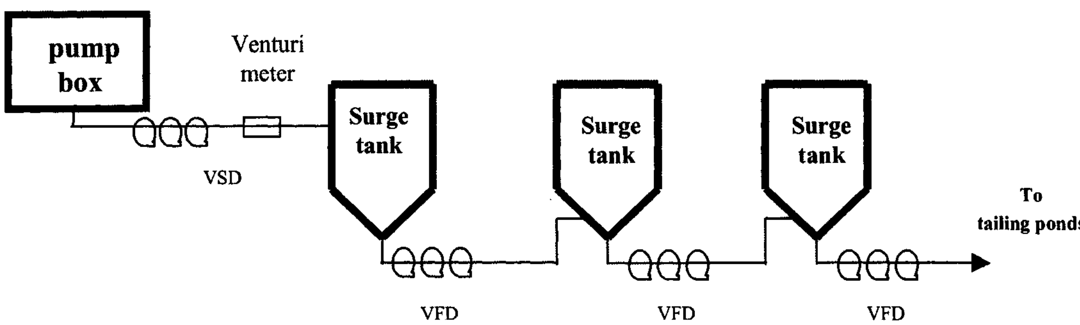

Figure 7.1: The simplified diagram of the tailing pipe line

to control the surge tank level. It is well known that the tailing line velocity should be above a critical value (for the detailed analysis, please refer to the next section). However, in the existing control strategy, the pump speed is adjusted by the surge tank level controller and the velocity constraint is not considered. For the continuous production, the feed to a tailing line may be varied, and it will cause the level of the surge tank varied. Correspondingly, the VFDs will speed up or slow down to maintain the surge tank level. Sometimes the slurry speed is even smaller than the critical velocity so that it is potential to sand in these cases.

When the volume flow rate is reduced to certain critical value, a stationary sand bed is formed on the bottom of the tailing lines. More reduction of the flow rate, more deposition occurs. When sand starts to block a tailing line, it may cause two consequences: (1) the pressure in the pump increases dramatically so that it may damage the tailing lines or even explode the pumps; (2) the slurry overflows the surge tank. Therefore, monitoring tools are required to detect sanding at the early stage so that the operators have enough time to prevent sand from blocking the pipe line.

7.3 Solid-liquid mixture model

The slurry in the pipe line mainly consists of sand (solid part) and water (liquid part) so that it can be viewed as a two phase flow. It is well known that particles immersed in the two-phase flow have a tendency to rise or sink according to the relative densities of the solid and fluid phase. Since we only deal with the slurry in which the density of sand is larger than that of the water, the background knowledge of solid-liquid mixtures is covered only for the two-phase flow in which the solid is denser than the liquid.

The behavior of two-phase flow in a pipe is governed by a number of parameters, which can be classified as follow:

- Pipe line parameters:
  - Diameter (assuming section to be circular)
  - Inclined or the horizontal (in this study, it is assumed that the pipeline is in the horizontal position)
- Liquid parameters:
  - density
  - viscosity
- Solid particle parameters:
  - density
  - size distribution
  - shape
- System parameters:
  - velocity of the flow
  - solid-liquid ratio of mixture flowing

The transport regime in the pipeline can be partitioned to four different flow categories according to the flow velocity and sand grades (Bain and Bonnington 1970): homogenous flow, heterogeneous flow, flow wit saltation or moving bed and flow with stationary bed. The transport regime for the slurry in a 6" diameter pipe is shown in Figure 7.2.

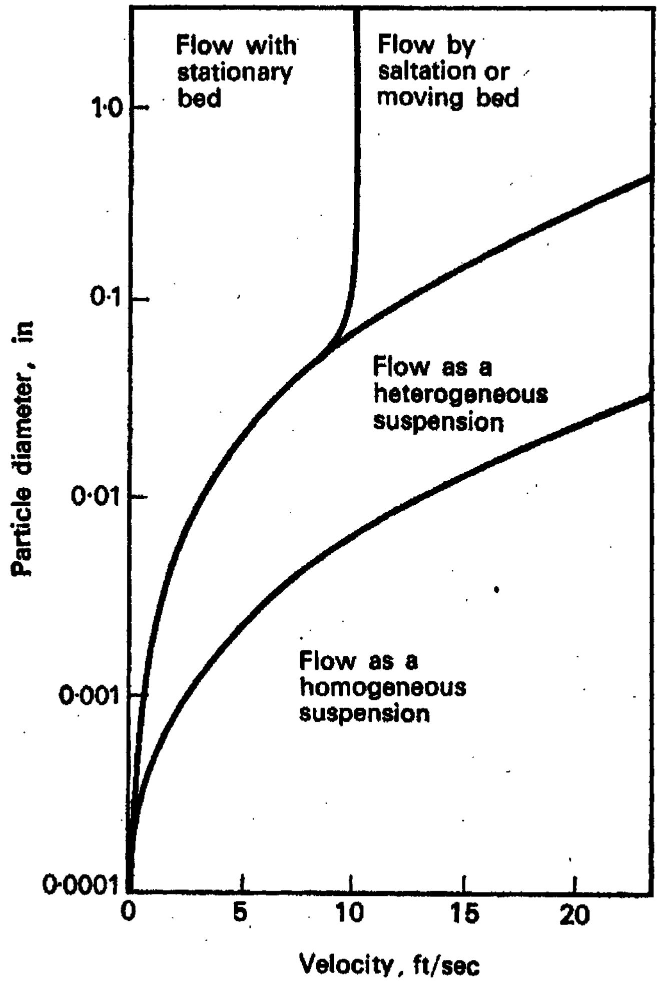

Figure 7.2: The flow pattern of solid-liquid two phase flow

The flow regimes given in this classification are obtained at the lowest velocities consistent with the absence of a bed of solids and related to material having a specific gravity of 2.65. In the tailing system of Syncrude Canada Ltd., the sand specific gravity is 2.65, but the diameters of the tailing pipelines are usually 24". Figure 7.2 may not be accurate, but it nevertheless characterizes the basic relationship between the particle size and the slurry velocity.

It can be seen in the Figure (7.2) that flow pattern is mainly determined by the average grade of the solid particle and the velocity of the slurry for a given pipe line. For the tailing line 3, the sand density is about 2.65 sg. If the tailing line density is 1.4 sg, the volume concentration of the sand is about 24%. If the tailing line density is 1.6 sg, the volume concentration of the sand is about 37%. The volume flow rate of tailing line 3 is about $1.69m^3/s$ and the mean velocity is about 18 ft/s (corresponding to 24 inches pipeline). According to the laboratory data, the particle size usually ranges from $10\mu \sim 0.2$ mm for the normal operation and in the sanding case the sand particle ranges from 0.1 mm $\sim$ 0.5 mm. The slurry is normally operated as heterogeneous flow or saltation flow or flow with stationary bed. The worst scenario is that stationary sand bed occurs in the pipe line. For a given concentration and particle size, the deposition happens at a constant Froude number (Bain and Bonnington 1970), which is defined by

$$F = \frac{V_L}{\sqrt{2gD(s-1)}} \tag{7.1}$$

where V_L is the critical velocity, s is the specific gravity of sand and D is the pipeline diameter. The critical velocity V_L is defined as the minimal velocity, at which sand starts to accumulate at the bottom of the pipe line. The critical velocity is slightly increased when the sand concentration increases (up to 15%). The critical velocity is determined by the same set of parameters that governs the increase in pipeline pressure drop when solid is introduced into the flow. The increase in the pipe line pressure drop (Bain and Bonnington 1970), caused by solid introduced into the flow, is given by

$$\frac{P-P_w}{PC_v} = 81\left(\frac{\sqrt{gD}}{V}\right)^3 (C_d)^{-\frac{3}{4}}(s-1)^{\frac{3}{2}} \tag{7.2}$$

where P is the pressure drop for the slurry; P_w is the pressure drop for the water within the same condition; C_v is the volume concentration of the solid; C_d is the drag coefficient; V is the velocity; D is the pipeline diameter and s is the relative gravity of solid. Equation (7.2) is not universal, as a matter of fact, it is summarized from data covering the following range: $D \in [40mm, 580mm]$, $C_v \in [2\%, 22.5\%]$. Equation (7.2) sheds some light on pipe line monitoring by monitoring relationship between the pressure drop and velocity because, roughly speaking, Equation (7.2) does not hold when sand bed appears at the bottom of the pipe line. However, directly using Equation (7.2) is not possible for the on line tailing line monitoring because of two reasons: (1) Equation (7.2) may not fit the tailing pipe line that is under monitoring; (2) there is lack of most sanding-related on-line measurements: the average particle size, viscosity and drag coefficient. Since the empirical fluid-mechanical model can not be used, the data-driven PCA method is used to monitor the tailing system instead. However, knowing the above discussed information will help select the appropriate variables for PCA.

7.4 Traditional PCA

Consider a data matrix X of n samples (*rows*) and m variables (*columns*). PCA decomposition can be applied to the covariance matrix and also can be applied to the correlation matrix. Here it is assumed that the original data matrix X has already been normalized. X can be decomposed as follows:

$$X = \hat{X} + E = TP^T + T_e P_e^T = \sum_{i=1}^{p} t_i p_i^T + \sum_{i=p+1}^{m} t_i p_i^T \tag{7.3}$$

where the matrices $\hat{X}$ and E represent the modelled and unmolded variations of X respectively, and p represents the number of principal components, the matrices T and P are the score and loading matrices respectively, t_i is the i_th column of $\begin{bmatrix} T & T_e \end{bmatrix}$ and p_i is the i_th column of $\begin{bmatrix} P & P_e \end{bmatrix}$. The decomposition of X satisfies that the matrices $\begin{bmatrix} T & T_e \end{bmatrix}$ and $\begin{bmatrix} P & P_e \end{bmatrix}$ are orthogonal, and also the cross covariance between $\hat{X}$ and E is zero. The number of principal components, p, is chosen such that most of the variation of the original data matrix can be explained by $\hat{X}$.

It is easy to show that

$$\Sigma = \frac{1}{n-1} X^T X = \begin{bmatrix} P & P_e \end{bmatrix} \Lambda \begin{bmatrix} P & P_e \end{bmatrix}^T \tag{7.4}$$

where $\Lambda = \frac{1}{n-1} \begin{bmatrix} T & T_e \end{bmatrix}^T \begin{bmatrix} T & T_e \end{bmatrix} = diag\begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \end{pmatrix}$

$$\lambda_i = \frac{1}{n-1} t_i^T t_i = \text{var}(t_i) \tag{7.5}$$

λ_i is the i_th largest eigenvalues of the correlation matrix S.

In this traditional PCA model, the first p principal components capture the most process variation and the last principal components capture the redundancy information among variables. There are many methods (Dunia *et al.* 1996) to decide how many PCs are required at time $k+1$, we use the cumulative percent variance (CPV) to decide how many PCs.

A new observation x_i (column vector) can be projected to the space spanned by the column vector of P

$$\hat{x}_i = PP^T x_i \tag{7.6}$$

The projection error vector is given by

$$e_i = x_i - \hat{x}_i = (I - PP^T) x_i \tag{7.7}$$

There are usually two indices related to PCA test, Hotelling T^2 test and squared prediction error (SPE) test. Hotelling T^2 test is to check if the underlying variation structure changes or not.The Hotelling T^2 statistics based on the first p PCs is given by

$$T^2 = \sum_{i=1}^{p} \frac{\hat{x}_i^2}{\lambda_i} \tag{7.8}$$

Confidence limit for T^2 at confidence level $(1-\alpha)\%$ is given by the following F-test:

$$T^2_{n,p} = \frac{(n^2-1)\,p}{(n-p)\,n} F_{p,n-p} \tag{7.9}$$

where $F_{p,n-p}$ is the upper $100\alpha\%$ critical point of the $F-$distribution with p and $n-p$ degree of freedom. It is possible that the correlation structure among variables is varied but the Hotelling T^2 statistics still remain the confidence region. To overcome this, SPE is used together with Hotelling T^2 charts. The SPE is given as

$$SPE = e_i^T e_i = x_i^T \left(I - PP^T\right) x_i \tag{7.10}$$

And the confidence limit for SPE is given by (Jackson 1991):

$$Q_\alpha = \Theta_1 \left(1 + \frac{c_\alpha h_0 \sqrt{2\Theta_2}}{\Theta_1} + \frac{\Theta_2 h_0 \left(h_0 - 1\right)}{\Theta_1^2}\right)^{\frac{1}{h_0}} \tag{7.11}$$

where $\Theta_i = \sum_{j=p+1}^{m} \lambda_j^i$ $(i = 1,2,3)$, $h_0 = 1 - \frac{2\Theta_1\Theta_3}{3\Theta_2^2}$ and c_α is the confidence limit for $1-\alpha$ percentile for a normal distribution.

7.5 Recursive moving window PCA

When the traditional PCA method is first implemented, it is shown that the traditional PCA not only detects the changes caused by sanding but also picks up the process operation changes. To reduce the false alarm rate caused by slow process operation change, it is desired to modify the PCA model such that it only detects the fast changes (sanding) and it is not sensitive to the slow changes (process operation changes). To achieve this goal, a recursive moving window PCA (RMWPCA) method is proposed. The idea is to update the PCA model using the data within a sliding window.

For the RMWPCA, it is important to update the mean and correlation matrix efficiently. For simplicity reason, the updating formulas are derived assuming that at each time we only have one observation. The general case can be derived similarly.

Suppose at current time k, we keep a window length $L+1$. The data set is denoted as: $X_k^T = \left[\ x_{k-L}, ..., x_{k-1}, x_k\ \right]$, where x_i is the observation sampled at time i. The mean value and the covariance matrix estimated from the data matrix X_k are μ_k and Σ_k respectively. At time $k+1$, the data matrix is updated to $X_{k+1}^T = \left[\ x_{k-L+1}, ..., x_k, x_{k+1}\ \right]$, and the mean value and the covariance matrix estimated from the data matrix X_{k+1} are u_{k+1} and Σ_{k+1} respectively.

Let $\Delta_1 = x_{k+1} - x_{k-L}$ and $\Delta_2 = x_{k-L} - u_k$, it is easy to show that the mean value μ_{k+1} and the covariance matrix Σ_{k+1}, estimated from X_{k+1}, are

$$\begin{aligned} \mu_{k+1} &= \mu_k + \frac{1}{L+1}(x_{k+1} - x_{k-L}) = \mu_k + \frac{\Delta_1}{L+1} && (7.12) \\ \Sigma_{k+1} &= \frac{1}{L}\left(X_{k+1}^T - \mu_{k+1}1_{L+1}^T\right)\cdot\left(X_{k+1} - 1_{L+1}\mu_{k+1}^T\right) \\ &= \frac{1}{L}\sum_{i=k-L+1}^{k+1}(x_i - \mu_{k+1})\cdot(x_i - \mu_{k+1})^T && (7.13) \end{aligned}$$

Substitute Equation (7.12) to Equitation (7.13) and it can be obtained

$$\Sigma_{k+1} = \Sigma_k + \frac{1}{L+1}\Delta_1\Delta_1^T + \frac{1}{L}\Delta_1\Delta_2^T + \frac{1}{L}\Delta_2\Delta_1^T \tag{7.14}$$

Let $\Psi_{k+1} = diag\left(\ \Sigma_{k+1}^{11} \quad \Sigma_{k+1}^{22} \quad \cdots \quad \Sigma_{k+1}^{mm}\ \right)$, where Σ_{k+1}^{ii} is the i^{th} diagonal term in matrix Σ_{k+1}. It is easy to know that the correlation matrix Ξ_{k+1} at time $k+1$ is

$$\begin{aligned} \Xi_{k+1} &= \Psi_{k+1}^{\frac{1}{2}}\Sigma_{k+1}\Psi_{k+1}^{\frac{1}{2}} && (7.15) \\ &= \Psi_{k+1}^{\frac{1}{2}}\Sigma_k\Psi_{k+1}^{\frac{1}{2}} + \frac{1}{L+1}\Psi_{k+1}^{\frac{1}{2}}\Delta_1\Delta_1^T\Psi_{k+1}^{\frac{1}{2}} + \frac{1}{L}\Psi_{k+1}^{\frac{1}{2}}\left(\Delta_1\Delta_2^T + \Delta_2\Delta_1^T\right)\Psi_{k+1}^{\frac{1}{2}} \\ &\approx \Xi_k + \frac{1}{L+1}\Psi_{k+1}^{\frac{1}{2}}\Delta_1\Delta_1^T\Psi_{k+1}^{\frac{1}{2}} + \frac{1}{L}\Psi_{k+1}^{\frac{1}{2}}\left(\Delta_1\Delta_2^T + \Delta_2\Delta_1^T\right)\Psi_{k+1}^{\frac{1}{2}} && (7.16) \end{aligned}$$

From Equations (7.14) and (7.16), we can see that the covariance and correlation matrix updating is determined by a *rank* 2 modification, and efficient algorithm was proposed in (Xu and Kailath 1994).

It may require a lot of computation effort to update if the mean value and the covariance matrix are updated at every sample. To reduce computation effort, it is practical to update them when D new observations are available, *i.e.*, the first D observations, $x_{k-L}^1, ..., x_{k-L}^D$, will be discarded and D new observations, $x_{k+1}^1, ..., x_{k+1}^D$, will be appended to the data matrix X_{k+1}. Let

$$\begin{aligned} \Delta_1 &= \left[\ x_{k+1}^1 \quad \cdots \quad x_{k+1}^D\ \right] - \left[\ x_{k-L}^1 \quad \cdots \quad x_{k-L}^D\ \right] \\ \Delta_2 &= \left[\ x_{k-L}^1 \quad \cdots \quad x_{k-L}^D\ \right] - \left[\ u_k \quad \cdots \quad u_k\ \right] \end{aligned}$$

It can be shown that

$$
\begin{aligned}
u_{k+1} &= u_k + \frac{\Delta_1 1_D}{L} && (7.17)\\
\Sigma_{k+1} &= \Sigma_k + \frac{1}{L+1}\Delta_1 1_D 1_D^T \Delta_1^T + \frac{1}{L}\Delta_1\Delta_2^T + \frac{1}{L}\Delta_2\Delta_1^T && (7.18)
\end{aligned}
$$

where $1_D = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T \in R^{D\times 1}$.

Remark 7.5.1 *The window length is a parameter that decides how adaptive the algorithm is: the shorter the window length is, the more robust of RMWPCA is to fast changes. It can be determined by the off-line test.*

Remark 7.5.2 *With on-line implementation of RMWPCA, the updating of data matrix X_{k+1} can be simplified with a pointer. When the data matrix is updated, the new observation needs not to be appended to the end of X_{k+1}, and it can be filled in the position of x_{k-L} in the array with a pointer pointing to x_{k-L+1} such that the program knows that the next time x_{k-L+1} will be updated.*

7.6 Comparison of PCA and RMWPCA

In the application, it was found that if a fixed PCA model was used, then many false alarms were reported after some time even though the tailing line was running under the normal condition. On the contrary, the false alarm is greatly reduced if RMWPCA is applied. The comparison between the conventional PCA and RMWPCA is shown in Figure 7.3.

RMWPCA can adapt the slow operation changes. However, it is necessary for RMWPCA not to adapt the changes caused by the sanding bed, *i.e.*, RMWPCA should still detect changes when sand bed appears. The simulation of RMWPCA on one set of sanding data shows that RMWPCA has the capacity detecting deposit of sand long before sand blocks the pipe line. The simulation results are shown in Figure 7.4.

7.7 Conclusion

In this chapter, sanding detection using RMWPCA is reported for the tailing lines system in Syncrude Canada Ltd. The detection results of traditional PCA and RMWPCA are compared. The comparison shows that RMWPCA is robust to normal process changes (slow changes) and at the same time it can also be used to detect sanding (fast changes) at early stage. Thus, RMWPCA significantly reduces the false alarm rate caused by slow process changes. This

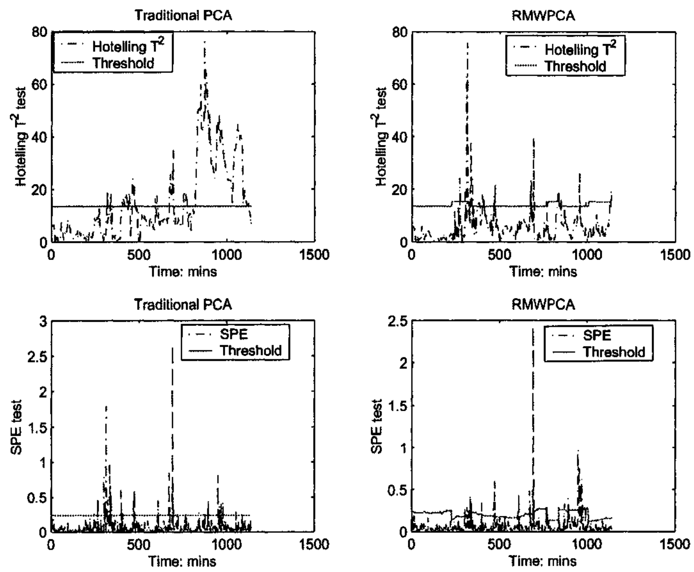

Figure 7.3: Comparison of the detection results between conventional PCA and RMWPCA

chapter thus illustrates an alternative approach to model based fault detection as discussed in the last chapter.

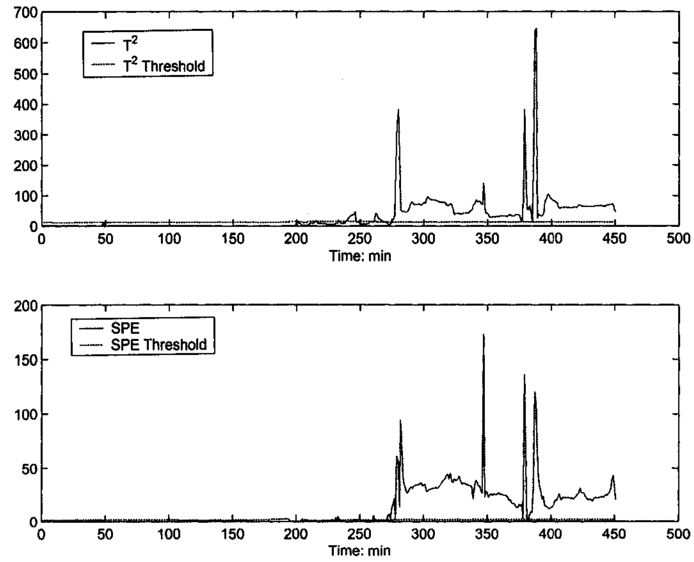

Figure 7.4: RMWPCA detect sanding long before pipeline was blocked

Part IV

Other related topics

8

Variance Constrained Filter Design for Stochastic Sampled-data Systems

8.1 Introduction

It is common that not all process states are measurable in process industry so that optimally estimating the process states based on the process input and output has been widely studied, *e.g.* (Anderson and Moore 1979), and the references therein. It is well known that the discrete-time Kalman filter (Kalman 1968) minimizes the trace of the estimation error covariance matrix based on the discrete-time equivalent model (*DTEM*) of a continuous-time process. However, it is more practical to consider the filtering problem for the sampled-data systems directly due to the popularity of digital computer control in industry. Sampled-data systems are referred to systems composed of a continuous-time system and a discrete-time system that are connected to each other using sample and hold devices. Optimal filtering problem for sampled-data systems has drawn a lot of attention and interested readers are referred to (de Souza and Trofino 1999, Haddad and Bernstein 1992, Nagpal and Khargonekar 1991, Shaked 1990, Shaked and Theodor 1992, Wang *et al.* 2001) and references therein.

Generally, the objective of filter design for the sampled-data system is to minimize certain estimation criterion. For example, the Kalman filter (Kalman 1968) utilizes trace of the estimation error covariance matrix as a performance index; H_∞ filtering design (Haddad and

Bernstein 1992, Nagpal and Khargonekar 1991, Shaked 1990, Shaked and Theodor 1992, Xu and Chen 2002) minimizes or constrain the H_∞ norm of the transfer function from the disturbance to the estimation error; H_2 robust filtering design (de Souza and Trofino 1999) is to minimize the upper bound of the 2-norm of the estimation error for the admissible uncertainties. The filter design methods mentioned above only involve one performance objective. This may not fit in the industrial applications that multi-objectives are explicitly specified. Recently, a novel filtering design approach for the sampled-data systems, so called sampling interval covariance assignment theory (*SCC*), was presented by Wang *et al.* (2001). SCC is a natural extension of the error covariance assignment (*ECA*) theory, first proposed by Yaz and Skelton (1991) and then explored further by Wang *et al.* (2001). The SCC theory for the sampled-data systems is based on the feedback controller design for the sampled-data systems (Fujioka and Hara 1995), which is the dual to the filtering problem.

Since a covariance matrix is considered as the objective, the filter design method itself is a multi-objective oriented approach. Furthermore, SCC or the ECA theory can parameterize all filters that assign a pre-specified covariance to the estimation error vector so that the resulting filter is not unique and other performance can be optimized over the parameterized filter set. Another distinguishing feature of this filtering design approach is that the inter-sample behavior is considered in addition to the sampling instant behavior. However, for sampled-data systems most other filter design methods, such as H_2 filtering design approach and H_∞ filtering design approach, only consider the signal at the sampling instant. A lot of performance requirements, such as disturbance attenuation, transient properties, need us to have a closer look at the inter sample behavior.

SCC theory assigns a specific positive definite matrix to the estimation error. However, it is often not necessary to assign a covariance matrix to estimation error; instead, it is more realistic to specify the upper bounds on the variances of the estimation errors. In this chapter, we extend the results of Wang *et al.* (2001) by specifying the variances upper bounds on estimation errors in the sampling interval so that some design freedom can be utilized to achieve other performance. Another improvement is that our result is given in linear matrix inequalities (*LMIs*) that can be solved by numerically robust tools (Gahinet *et al.* 1995). The rest of this chapter is organized as follows: the sampling-interval covariance constrained problem (*SCC*) is formulated in Section 2; the solution of the SCC is presented in Section 3, where the necessary and sufficient condition for the solvability of SCC is given in terms of linear matrix inequalities (*LMIs*) and a computational algorithm is presented; a simulation result is used to demonstrate the effectiveness of the proposed algorithm in Section 4 and concluding remark is given in Section 5.

8.2 Problem statement

Consider the finite-dimensional continuous-time linear system $\mathscr{P}$, which is given by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + \varsigma(t) \\ y(t) &= Cx(t) + \upsilon(t) \end{aligned} \tag{8.1}$$

where $x(t) \in R^n$ is the state, $y(t) \in R^p$ is the measurement, $\varsigma(t) \in R^g$ is the external disturbance and $\upsilon(t)$ is measurement noise. It is assumed that A is stable. Usually the bandwidths of disturbance and noise are much higher than that of the process band so that it is reasonable to model them as white noise signals. In other words, the external disturbance $\varsigma(t)$ and measurement noise $\upsilon(t)$ are white noise. Let $\omega = \begin{bmatrix} \varsigma \\ \upsilon \end{bmatrix}$ and it is assumed that

$$\begin{aligned} &E(\omega(t)) = 0 \\ &E\left(\omega(t)\,\omega(s)^T\right) = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-s) \end{aligned} \tag{8.2}$$

where $W > 0$ and $V > 0$. The objective of filter design is to reconstruct the state $x(t)$ using a discrete filter that is defined as:

$$\hat{x}_{k+1} = G\hat{x}_k + Hy_k \tag{8.3}$$

where G, H are matrices that need to be determined and y_k is the sampler output. Because the direct sampling of white noise is not permitted, which will lead to unbounded variances of the discrete-time variables (Åstrom 1970), an average type *A/D* device is employed:

$$y_k = \frac{1}{T}\int_{(k-1)T}^{kT} y(t)dt, \quad (k-1)T \le t < kT \tag{8.4}$$

Since the connection between the continuous-time system (8.1) and the discrete-time filter (8.3) is through an average sampling device, the system composed of (8.1) and (8.3) is a sampled-data system. Usually, the filter (8.3) is designed in discrete-time domain based on the discrete-time model of (8.1) and then applied to the original continuous-time system (8.1). Therefore, one of the most critical tasks in filter design is to select a proper DTEM for system (8.1). According to (Haddad and Bernstein 1992, Shaked and Theodor 1992) a present-state-dependent DTEM for (8.1) is given as follows:

$$\begin{aligned} x_{k+1} &= A_T x_k + \varsigma_k \\ y_k &= C_T x_k + \upsilon_k \end{aligned} \tag{8.5}$$

where

$$\begin{aligned} &A_T = e^{AT} \\ &C_T = \tfrac{1}{T} C \textstyle\int_0^T e^{A(T-\tau)} d\tau \\ &\varsigma_k = \tfrac{1}{T} \textstyle\int_0^T e^{A(T-\tau)} \varsigma(kT+\tau)\, d\tau \\ &\upsilon_k = \tfrac{1}{T} \textstyle\int_0^T \upsilon((k-1)T+\tau) d\tau - \tfrac{1}{T} \int_0^T \int_\xi^T e^{A(\xi-T)} \varsigma((k-1)T+\tau) d\tau d\xi \end{aligned} \tag{8.6}$$

Let $\varepsilon_k = \begin{bmatrix} \varsigma_k \\ \upsilon_k \end{bmatrix}$ and it can be verified that ε_k is a zero mean white noise sequence satisfying:

$$E\left(\varepsilon\varepsilon^T\right) = \Omega = \begin{bmatrix} \Omega_1 & \Omega_{12} \\ \Omega_{12}^T & \Omega_2 \end{bmatrix} \tag{8.7}$$

where

$$\begin{array}{rcl} \Omega_1 & = & \int_0^T e^{A\tau} W e^{A^T\tau} d\tau \\ \Omega_2 & = & \frac{1}{T}V + \frac{1}{T^2}\int_0^T F(\tau) W F(\tau)^T d\tau \\ \Omega_{12} & = & -\frac{1}{T}\int_0^T e^{A(T-\tau)} W F(\tau)^T d\tau \\ F(\tau) & = & C\int_0^\tau e^{A(\eta-\tau)} d\eta \end{array}$$

Let $e_k = x_k - \hat{x}_k$ and it can be obtained that

$$e_{k+1} = Ge_k + (A_T - G - HC_T)x_k + \varsigma_k - H\upsilon_k \tag{8.8}$$

Rearranging equations (8.5) and (8.8), it can be obtained that

$$X_{k+1} = \left(\bar{A} + \bar{B}M\bar{C}\right)X_k + \left(\bar{G} + \bar{B}M\bar{D}\right)\varepsilon_k \tag{8.9}$$

where

$$X_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix}, \bar{A} = \begin{bmatrix} A_T & 0 \\ A_T & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, M = \begin{bmatrix} G & H \end{bmatrix}$$
$$\bar{C} = \begin{bmatrix} I & -I \\ C_T & 0 \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} I & 0 \\ I & 0 \end{bmatrix}, \quad \bar{D} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

Remark 8.2.1 *A is stable so that it can be verified that A_T is stable as well. From (8.9) it can be seen that the stability of filter (8.3) ensures the stability of (8.9) and vice versa.*

For the sampled-data system, defined by (8.1), (8.3) and (8.5), two types of covariance matrix are of interest: one is called sampling-instant covariance and the other is called sampling-interval covariance. The sampling-instant covariance matrix is defined as:

$$\Sigma_d = \lim_{k\to\infty} E\left(X_k X_k^T\right) \tag{8.10}$$

It is well known (Bryson and Ho 1975) that if (8.9) is stable, Σ_d exists and it satisfies:

$$\left(\bar{A} + \bar{B}M\bar{C}\right)\Sigma_d\left(\bar{A} + \bar{B}M\bar{C}\right)^T - \Sigma_d + \left(\bar{G} + \bar{B}M\bar{D}\right)\Omega\left(\bar{G} + \bar{B}M\bar{D}\right)^T = 0 \tag{8.11}$$

The sampling-instant covariance describes the variation of the sampled-data system at the sampling instant. In addition to variation at the sampling instant, the variation between sampling interval is also of concern. Let $X_s(kT+t) = \begin{bmatrix} x(kT+t) \\ e(kT+t) \end{bmatrix}$ for any $t \in (0, T]$, where

$e(kT+t) = x(kT+t) - \hat{x}_k 1_k(t)$ and $1_k(t) = \begin{cases} 1, & t \in [kT,\ (k+1)T) \\ 0, & otherwise \end{cases}$. The sampling-interval covariance of the sampled-data systems is defined as:

$$\Sigma_s = \lim_{k\to\infty} \frac{1}{T} \int_0^T E\left(X_s(kT+t) X_s(kT+t)^T\right) dt = \begin{bmatrix} \Sigma_{s11} & \Sigma_{s12} \\ \Sigma_{s12}^T & \Sigma_{s22} \end{bmatrix} \tag{8.12}$$

An algorithm was developed by Wang *et al.* (2001) that determined whether a matrix Σ_s was assignable to $X_s(t)$. First, the corresponding sampling-instant covariance Σ_d is calculated based on the algorithm presented by Fujioka and Hara (1995) and then it is checked if Σ_d is assignable to the discrete-time system (8.9). If Σ_d is assignable, then the matrix pair (G,H) can be calculated accordingly. In most applications it is not necessary to assign a sampling-interval covariance to the vector $X_s(t)$. On the contrary, it is more relevant to assign Σ_{s22} that describes variation of the estimation error $e(kT+t)$ during the sampling interval. In this chapter, we will specify the variance upper bounds for the continuous-time estimation error $e(kT+t)$:

Problem 8.2.1 *The sampling-interval covariance constrained (SCC) problem for the sampled-data system is to design a stable filter (8.3) such that the covariance in the sampling interval satisfies:*

$$\Sigma_{s22}(i,i) < \Phi_i, \quad i = 1,2,\cdots,n \tag{8.13}$$

If there exists a matrix pair (G,H) such that the sampling-interval covariance satisfies (8.13), then we say that the sampling-interval covariance constrained problem is feasible and the constraints given in (8.13) are called admissible constraints.

8.3 Solutions to SCC

A necessary and sufficient condition for a matrix to be assigned to $X_s(t)$ is given by Wang *et al.* (2001):

Lemma 8.3.1 *A matrix Σ_s is assignable to the sampled-data system defined by (8.1), (8.3) and (8.5), if and only if there exist matrices G, H and Σ_d such that in addition to (8.11) the following equation also holds:*

$$\Sigma_s = \frac{1}{T} \int_0^T \left[C_s(\tau) \Sigma_d C_s(\tau)^T + W_s(\tau)\right] d\tau \tag{8.14}$$

where Σ_d is the covariance matrix at the sampling instant and

$$
\begin{aligned}
C_s(\tau) &= \begin{bmatrix} e^{A\tau} & 0 \\ e^{A\tau}-I & I \end{bmatrix} \\
W_s(\tau) &= \begin{bmatrix} W_1(\tau) & W_1(\tau) \\ W_1(\tau) & W_1(\tau) \end{bmatrix} \\
W_1(\tau) &= \int_0^T e^{A\xi} W e^{A^T\xi} d\xi
\end{aligned} \tag{8.15}
$$

Remark 8.3.1 *It is well known (Skelton and Iwasaki 1993, Yaz and Skelton 1991) that (8.11) is the necessary and sufficient condition for Σ_d to be assigned to the discrete-time system (8.9), i.e., Σ_d is the sampling-instant covariance matrix. Lemma (8.3.1) actually characterizes the relationship between sampling-instant covariance and sampling-interval covariance through the integration constraint (8.14).*

Based on Lemma 8.3.1, a necessary and sufficient condition for the feasibility of the SCC problem is given in matrix inequalities as follows:

Theorem 8.3.1 *The sampling-interval covariance constrained problem is feasible if and only if there exist matrices $M \in R^{n\times(n+p)}$ and $\Sigma > 0$ such that*

$$(\bar{A}+\bar{B}M\bar{C})\Sigma(\bar{A}+\bar{B}M\bar{C})^T - \Sigma + (\bar{G}+\bar{B}M\bar{D})\Omega(\bar{G}+\bar{B}M\bar{D})^T < 0 \tag{8.16}$$

$$\frac{1}{T}\int_0^T \left[e_i^T C_s(\tau)\Sigma C_s(\tau)^T e_i + e_i^T W_s(\tau) e_i\right] d\tau < \Phi_i \quad i = 1,2,\ldots,n \tag{8.17}$$

where $e_i \in R^{2n\times 1}$ is a vector with all elements 0 except the one at $n+i$ position, $i = 1,2,\ldots,n$.

Proof:

Necessity: If there exists a stable discrete-time filter (8.3) such that the SCC problem is feasible, then according to Lemma (8.3.1) the instant-sampling covariance matrix Σ_d satisfies (8.11) and the inter-sampling covariance matrix Σ_s, calculated according to (8.14), satisfies (8.13). The stability of the filter ensures that for any sufficiently small $\varepsilon > 0$ there is a unique solution $\Sigma_\varepsilon > 0$ to the discrete Lyapunov function:

$$(\bar{A}+\bar{B}M\bar{C})\Sigma(\bar{A}+\bar{B}M\bar{C})^T - \Sigma + (\bar{G}+\bar{B}M\bar{D})\Omega(\bar{G}+\bar{B}M\bar{D})^T + \varepsilon I = 0 \tag{8.18}$$

Let $\Sigma_{s,\varepsilon} = \frac{1}{T}\int_0^T \left[C_s(\tau)\Sigma_\varepsilon C_s(\tau)^T + W_s(\tau)\right] d\tau$. If ε is sufficiently small, then $\Sigma_{s,\varepsilon}$ also satisfies (8.13) because $\Sigma_{s,\varepsilon} \to \Sigma_s$.

Sufficiency: Let M be partitioned to two parts: $M = [M_1 \quad M_2]$, where $M_1 \in R^{n\times n}$ and $M_2 \in R^{n\times p}$. Inequality (8.16) implies that the discrete-time system (8.9) is stable if we construct a filter (8.3) with $G = M_1$ and $H = M_2$. So, the filter is also stable. $\Sigma_d \leq \Sigma$ because of the monotonicity of Lyapunov function. It is straightforward to show that

$$\begin{aligned}\Sigma_s &= \frac{1}{T}\int_0^T \left[C_s(\tau)\Sigma_d C_s(\tau)^T + W_s(\tau)\right]d\tau \\ &\leq \frac{1}{T}\int_0^T \left[C_s(\tau)\Sigma C_s(\tau)^T + W_s(\tau)\right]d\tau\end{aligned}$$

The inequality above implies that the SCC problem is feasible with the constructed filter. ∇∇∇

The theorem above is given in matrix inequalities that are difficult to solve because of the nonlinearity of (8.18). To obtain a tractable procedure to calculate the filter (8.3) the matrix inequality (8.18) should be transformed to a form that is easier to solve. The following lemmas are useful in the transformation.

Lemma 8.3.2 *(Skelton and Iwasaki 1993) Let matrices B, C, P and R be given. Suppose $P = P^T$, $R = R^T > 0$ and $CC^T > 0$. Then the following statements are equivalent.*

- *There exists a matrix G such that*

$$(B+GC)^T R (B+GC) < P \tag{8.19}$$

- *$P > 0$ and*

$$C^{T\perp}\left(P - B^T RB\right)C^{T\perp T} > 0 \quad or \quad C^T C > 0 \tag{8.20}$$

If (8.3.2) holds, then the matrix G is given by:

$$G = -BP^{-1}C^T\left(CP^{-1}C^T\right)^{-1} + S^{\frac{1}{2}}L\left(CP^{-1}C^T\right)^{-\frac{1}{2}}, \quad \|L\| < 1 \tag{8.21}$$

where $S = R^{-1} - B\left[P^{-1} - P^{-1}C^T\left(CP^{-1}C^T\right)^{-1}CP^{-1}\right]B^T > 0.$

Lemma 8.3.3

$$\left(\Sigma_1 + \Sigma_2\Sigma_3\Sigma_4\right)^{-1} = \Sigma_1^{-1} - \Sigma_1^{-1}\Sigma_2\left(\Sigma_3^{-1} + \Sigma_4\Sigma_1^{-1}\Sigma_2\right)^{-1}\Sigma_4\Sigma_1^{-1} \tag{8.22}$$

With the help of Lemmas stated above, we can obtain the following Theorem:

Theorem 8.3.2 *The SCC problem is feasible if and only if there exist a scalar $\varepsilon > 0$ and matrices $\Sigma > 0$ and $\Gamma > 0$ such that*

$$\bar{B}^{\perp}\left(\Sigma - \bar{A}\Sigma\bar{A}^T - \bar{G}\Omega\bar{G}^T\right)\bar{B}^{\perp T} > 0 \tag{8.23}$$

$$\begin{bmatrix} \Gamma & \Gamma\tilde{A} & \Gamma Q^{\frac{1}{2}} \\ \tilde{A}^T\Gamma & \Gamma + \bar{C}^T R^{-1}\bar{C} & 0 \\ Q^{\frac{1}{2}}\Gamma & 0 & I \end{bmatrix} > 0 \tag{8.24}$$

$$\Sigma\Gamma = I \tag{8.25}$$

$$\frac{1}{T}\int_0^T \left[e_i^T C_s(\tau)\Sigma C_s(\tau)^T e_i + e_i^T W_s(\tau) e_i\right] d\tau < \Phi_i \tag{8.26}$$

where

$$\begin{aligned} R &= \bar{D}\Omega\bar{D}^T + \varepsilon I \\ Q &= \bar{G}\Omega\bar{G}^T - \bar{G}\Omega\bar{D}^T R^{-1}\bar{D}\Omega\bar{G}^T \\ \tilde{A} &= \bar{A} - \bar{G}\Omega\bar{D}^T R^{-1}\bar{C} \end{aligned} \tag{8.27}$$

Proof:

Since (8.26) appears in both Theorem (8.3.1) and Theorem (8.3.2) we only need to show the equivalence between Inequalities (8.16) and (8.23) $\sim$ (8.25).

Necessity: Matrix inequality (8.16) implies that for sufficiently small $\varepsilon > 0$ the following inequality holds:

$$\left(\bar{A} + \bar{B}M\bar{C}\right)\Sigma\left(\bar{A} + \bar{B}M\bar{C}\right)^T - \Sigma + \left(\bar{G} + \bar{B}M\bar{D}\right)\Omega\left(\bar{G} + \bar{B}M\bar{D}\right)^T + \varepsilon\bar{B}MM^T\bar{B}^T < 0 \tag{8.28}$$

Rearrange (8.28) and it can be shown that

$$\left(\bar{B}M + \Theta\Xi^{-1}\right)\Xi\left(\bar{B}M + \Theta\Xi^{-1}\right)^T < \Sigma - \bar{A}\Sigma\bar{A}^T - \bar{G}\Omega\bar{G}^T + \Theta\Xi^{-1}\Theta^T \tag{8.29}$$

Generally $\bar{B}$ is not a full dimension matrix so that $\bar{B}\bar{B}^T$ is not a full rank matrix either. According to Lemma (8.3.2), (8.29) is solvable if and only if

$$\Sigma - \bar{A}\Sigma\bar{A}^T - \bar{G}\Omega\bar{G}^T + \Theta\Xi^{-1}\Theta > 0 \tag{8.30}$$

$$\bar{B}^{\perp}\left(\Sigma - \bar{A}\Sigma\bar{A}^T - \bar{G}\Omega\bar{G}^T\right)\bar{B}^{\perp T} > 0 \tag{8.31}$$

where

$$\begin{aligned} \Theta &= \bar{A}\Sigma\bar{C}^T + \bar{G}\Omega\bar{D}^T \\ \Xi &= \bar{C}\Sigma\bar{C} + \bar{D}\Omega\bar{D}^T + \varepsilon I \end{aligned}$$

Substitute (8.27) to (8.30) and it can be verified that (8.30) is equivalent to

$$\Sigma - \tilde{A}\Sigma\tilde{A}^T + \tilde{A}\Sigma\bar{C}^T\left(\bar{C}\Sigma\bar{C}^T + R\right)^{-1}\bar{C}\Sigma\tilde{A}^T - Q > 0 \tag{8.32}$$

Substituting $\Gamma = \Sigma^{-1}$ to (8.32) and then with the help of (8.22) it can be shown that (8.32) is equivalent to:

$$-\tilde{A}\left(\Gamma+\tilde{C}^T R^{-1}\tilde{C}\right)^{-1}\tilde{A}^T+\Gamma^{-1}-Q>0 \tag{8.33}$$

By using Lemma 8.22, it can be shown that

$$\begin{aligned} Q &= \bar{G}\left(\Omega-\Omega\bar{D}^T\left(\bar{D}\Omega\bar{D}^T+\varepsilon I\right)^{-1}\bar{D}\Omega\right)\bar{G}^T \\ &= \bar{G}\left(\Omega^{-1}+\varepsilon^{-1}\bar{D}^T\bar{D}\right)^{-1}\bar{G} \end{aligned} \tag{8.34}$$

Equation (8.34) shows that $Q \geq 0$. Pre-multiplying and post-multiplying inequality (8.33) by Γ, it can be shown that (8.33) is equivalent to (8.24) with the well-known Schur Complement Lemma.

Sufficiency: Reverse the necessity part and it can be easily shown the equivalence between (8.23) $\sim$ (8.26) and (8.28), which implies that (8.16) holds. $\nabla\nabla\nabla$

The inequality (8.26) is an integration constraint that is difficult to be incorporated in LMI framework even though it is linear in the decision variable Σ. To obtain a numerically tractable algorithm we need change the integration constraint to a non-integrated form. As we know that

$$e^{A\tau}=I+A\tau+\frac{1}{2!}A^2\tau^2+\frac{1}{3!}A^3\tau^3+\cdots \tag{8.35}$$

Substituting (8.35) to $C_s(\tau)$ defined in (8.3.1), we can obtain that

$$\begin{aligned} C_s(\tau) &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}+\tau\begin{bmatrix} A & 0 \\ A & 0 \end{bmatrix}+\frac{\tau^2}{2!}\begin{bmatrix} A^2 & 0 \\ A^2 & 0 \end{bmatrix}+\cdots \\ &= C_0+\tau C_1+\tau^2 C_2+\tau^3 C_3+\cdots \end{aligned} \tag{8.36}$$

Substitute (8.36) to (8.26) and it is obtained that

$$\begin{aligned} &\int_0^T \tilde{C}_i C_s(\tau)\Sigma C_s(\tau)^T\tilde{C}_i^T d\tau \\ &= \int_0^T \sum_{j=0}^{\infty}\left(\tau^j\sum_{k=0}^{i}\tilde{C}_i C_k\Sigma C_{j-k}^T\tilde{C}_i^T\right)d\tau \\ &= \sum_{j=0}^{\infty}\left(\frac{T^{j+1}}{j+1}\sum_{k=0}^{i}\tilde{C}_i C_k\Sigma C_{j-k}^T\tilde{C}_i^T\right) \end{aligned} \tag{8.37}$$

To get a tractable algorithm, we need truncate the equation above at N, *i.e.*, the integrated constraint (8.26) is replaced by the following LMI:

$$\sum_{j=0}^{N}\left(\frac{T^{j+1}}{j+1}\sum_{k=0}^{i}e_i^T C_k\Sigma C_{j-k}^T e_i\right)<\Phi_i \tag{8.38}$$

In Theorem (8.3.2) it is required that Σ and Γ are inverse to each other. This requirement describes the linearity and convexity of the matrix inequalities (8.23) $\sim$ (8.26). The cone complementary linearization algorithm is adopted here to find a feasible solution to sampling-interval covariance constrained problem. The idea of the cone complementary linearization method is to relax the matrix inverse condition (8.25) with the following LMI:

$$\begin{bmatrix} \Sigma & I \\ I & \Gamma \end{bmatrix} \geq 0 \tag{8.39}$$

and the linearized version of $trace\,(\Sigma\Gamma)$ is minimized at each step to saturate (8.39).

The algorithm to check the feasibility of the sampling-interval covariance constrained problem (8.2.1) is summarized as follows:

Algorithm 8.3.1 *1. Set k=0. Find $\Sigma_k > 0$, $\Gamma_k > 0$ that solve the following semi-definite programming problem:*

$$\min_{\Sigma,\Gamma} trace\,(\Sigma+\Gamma) \quad subject\ to\ (8.23),(8.24),(8.38),(8.39)$$

2. Find $\Sigma_{k+1} > 0$, $\Gamma_{k+1} > 0$ that solve the following semi-definite programming problem:

$$\min_{\Sigma,\Gamma} trace\,(\Sigma_k\Gamma+\Gamma_k\Sigma) \quad subject\ to\ (8.23),(8.24),(8.38),(8.39)$$

Set $t_k = trace\,(\Sigma_k\Gamma_{k+1}+\Gamma_k\Sigma_{k+1})$.
3. Set $k = k+1$. If the decrease of t_k in last L steps is less than a small constant number $\varepsilon_1 > 0$, then the algorithm stops. If $trace\,(\Sigma_{k+1}\Gamma_{k+1}) - 2n < \varepsilon_2$ then go to step 4; otherwise, go to step 2.

8.4 Illustrative example

A first order example is used to illustrate the proposed algorithm. The example is taken from (Wang *et al.* 2001):

$$\begin{aligned} \dot{x}(t) &= -1.7329x(t)+\varsigma(t) \\ y(t) &= 0.8x(t)+\upsilon(t) \end{aligned}$$

where $W = 2$ and $V = 0.64$. The sampling interval is 0.4 second. The DTEM is as follows:

$$x_{k+1} = 0.5x_k+\varsigma_k$$
$$y_k = 1.1542x_k+\upsilon_k$$

where $\Omega = \begin{bmatrix} 0.4328 & -0.8325 \\ -0.8325 & 1.8970 \end{bmatrix}$. It is desired to design a discrete filter such that the sampling-interval variance for the estimation error is smaller than 1.15. Let $\varepsilon = 1e^{-6}$ and the truncation order $N = 10$. Apply the algorithm presented in the last section, the following discrete-time filter is obtained:

$$x_{k+1} = -0.8053x_k + 0.3104y_k \tag{8.40}$$

With the above filter, the sampling-instant covariance matrix is:

$$\begin{bmatrix} 5.9690 & -0.6191 \\ -0.6191 & 0.5253 \end{bmatrix}$$

and the sampling-interval covariance matrix is:

$$\begin{bmatrix} 7.6037 & 0.3106 \\ 0.3106 & 0.7500 \end{bmatrix}$$

8.5 Conclusion

The filter design for the sampled-data system subject to variance constraints for the estimation error is studied. A necessary and sufficient condition for the existence of a discrete-time filter such that the variances of the estimated error have specified upper bounds is given in terms of LMIs. If the LMIs are feasible, then the filter is parameterized by the solution to the LMIs. An illustrative example is used to demonstrate the feasibility of the proposed algorithm.

9

A BMI Approach towards Single and Simultaneous Admissible Stabilization of Continuous-time Descriptor Systems via Static Output Feedback

9.1 Introduction

Descriptor systems are also referred to as differential-algebraic systems, generalized state space systems, semi-state systems, or constrained systems. Study of descriptor systems has attracted significant attentions recently due to its flexibility in modelling physical systems. Descriptor systems have a great significance for process control because they capture not only the dynamics of the systems but also the static constrains. In chemical process this type of algebraic constraints accounts for equilibrium relations, thermodynamic relations and empirical correlations (Kumar and Daoutidis 1998). There is a considerable similarity in the methodologies between control of descriptor systems and covariance control theory. Thus we also extend some of our previous results to descriptor systems in this chapter.

The studies of descriptor systems can be traced back to 1970s (Singh and Liu 1973, Luenberger 1978). After their pioneering work, extensive studies have been done on descriptor

[1]Some version of this chapter was published in *Dynamics of continuous, discrete, and impulse systems* (Huang and Huang 2002)

systems, such as (Bender and Laub 1987, Campbell 1980, Chu and Ho 1999, Dai 1988, Duan 1999, Fletcher 1988, Lewis 1986, Lin *et al.* 1999, Masubuchi *et al.* 1997, Rehm and Allgower 2000, Verghese *et al.* 1981), just to list a few of them.

Stabilization of linear systems via output feedback is theoretically appealing because of its simple control structure. Static output regulation has been extensively studied for regular systems (Cao and Sun 1998, J. C. Geromel and Skelton 1998, Iwasaki and Skelton 1995, Syrmos *et al.* 1997). However, the literature has been relatively sparse on static feedback control for descriptor systems. Some representative work can be found in (Bunse-Gerstner *et al.* 1994, Lovass-Nagy *et al.* 1994, Chu and Ho 1999). Their results are obtained by transferring the original descriptor system to a standard form, such as SVD form or Weierstrass form. Unlike the results mentioned above, in this chapter the necessary and sufficient conditions for output feedback stabilization of a descriptor system, based on generalized Lyapunov Inequalities (Masubuchi *et al.* 1997), are presented in a Bilinear Matrix Inequalities (*BMI*) form. BMIs are then solved by an iterative algorithm. The proposed algorithm can be applied to simultaneous stabilization as well.

Simultaneous stabilization is to stabilize a finite collection of plants simultaneously by using one controller. This problem has been intensively studied for the regular systems (Vidyasagar and Viswanadham 1982, Chow 1990, Chen *et al.* 1995, Cao and Sun 1998). However, to the best of the our knowledge there is only one paper (Liu *et al.* 1996) available on the simultaneous admissible stabilization of descriptor systems in the literature. In this chapter, necessary and sufficient conditions for the existence of static output feedback controllers that admissibly stabilize the given collection of descriptor systems are presented based on the result of static output feedback stabilization. Then the algorithm for the static output stabilization is extended to simultaneous stabilization of a groups of descriptor systems.

The rest of this chapter is organized as follows: In Section 2, several definitions and preliminary results are presented. The main results are stated in Section 3, which contains two parts: in Section 3.1 the necessary and sufficient conditions of admissibly stabilizing a descriptor system via static output feedback are given and an algorithm to calculate the static feedback gain is presented; in Section 3.2 the results obtained in Section 3.1 are extended to simultaneous admissible stabilization via static output feedback; numerical examples are given in Section 4 and concluding remarks are presented in Section 5.

9.2 Formulation of the problem

Consider a time-invariant continuous-time descriptor system described as follows:

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{9.1}$$

where $x \in R^n$ is the descriptor vector, $u \in R^m$ is the control vector, $y \in R^p$ is the output vector, E, A, B and C are constant matrices with appropriate dimensions, and $rank(E) = r < n$. r is called generalized order in (Verghese *et al.* 1981).

If a static output feedback control law $u = My$ is implemented, then the closed-loop state space representation is given by:

$$E\dot{x} = (A + BMC)x \tag{9.2}$$

The following are some definitions and known results for descriptor systems:

Definition 9.2.1 *A pencil (E,A) is called regular if $|sE - A|$ is not identically zero.*

Definition 9.2.2 *A pencil (E,A) is called impulse-free if $\deg(|sE - A|) = r$.*

A regular pencil can be transferred to a special form, called Weierstrass form:

Lemma 9.2.1 *(Dai 1988) For any regular pencil (E,A), there exist nonsingular matrices M and N such that*

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & J \end{bmatrix}, MAN = \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{9.3}$$

where J is a nilpotent matrix.

If all finite eigenvalues of pencil (E,A) are stable, we say that (E,A) is stable. Lemma 9.3 shows that $\Lambda(E,A) = \Lambda(I_r, A_{11})$, *i.e.*, the stability of the regular system (E,A) is completely determined by that of the system (I_r, A_{11}). Note that (E,A) is impulse-free if and only if $J = 0$.

Definition 9.2.3 *The closed-loop descriptor system (9.2) is asymptotically stable if it is regular and its finite eigenvalues are all located in the left half of the complex plane.*

Definition 9.2.4 *The closed-loop descriptor system (9.2) is called admissible if it is regular, impulse-free and asymptotically stable.*

The following lemma that relates the admissibility of a pencil (E,A) with linear matrix inequalities (*LMIs*) is from (Masubuchi *et al.* 1997):

Lemma 9.2.2 *Pencil (E,A) is admissible if and only if there exists a nonsingular matrix X such that*

$$\begin{aligned} E^T X = X^T E &\geq 0 \\ A^T X + X^T A &< 0 \end{aligned} \tag{9.4}$$

For a descriptor system (9.1) if there exists a controller $u = My$ such that the closed-loop system (9.2) is admissible, we say that the descriptor system can be admissibly stabilized by a static output feedback control law. Our first goal is to derive necessary and conditions for a descriptor system to be admissibly stabilized by a static controller. Second, we want to build an algorithm to calculate the static output feedback gain that can admissibly stabilize the descriptor system (9.1). Our last goal is to derive necessary and sufficient conditions for simultaneous stabilization of a group of descriptor systems. The following well known lemma is required in the proof:

Lemma 9.2.3 *The linear system $\dot{x} = Ax$ is asymptotically stable and all its eigenvalues are located to the left of* $\mathrm{Re}(s) = -\frac{\alpha}{2}$ *($\alpha > 0$), if and only if there exists a positive definite matrix P such that*

$$A^T P + PA^T + \alpha P < 0 \tag{9.5}$$

9.3 Main results

9.3.1 Static output feedback admissible stabilization

The following lemma is a corollary of Lemma 9.4, and it is the counterpart of Lemma 9.2.3 for the regular systems.

Lemma 9.3.1 *Pencil (E,A) is admissible and its every finite eigenvalue λ_i satisfies* $\mathrm{Re}(\lambda_i) < -\frac{\alpha}{2}$ $(\alpha > 0)$ *if and only if there exists a nonsingular matrix X such that*

$$\begin{aligned} E^T X = X^T E &\geq 0 \\ A^T X + X^T A + \alpha E^T X &< 0 \end{aligned} \tag{9.6}$$

Proof: :

Sufficiency: If there exists a nonsingular matrix X such that inequalities (9.6) are satisfied, the pencil (E,A) is admissible according to Lemma 9.2.2. We only need to prove that all its finite eigenvalues are located to the left of $\mathrm{Re}\,(s) = -\frac{\alpha}{2}$.

Since the pencil (E,A) is admissible, according to lemma 9.2.1 there exist nonsingular matrices M and N satisfying (9.3). Then we can see that the nonsingular matrix X has a special structure:

$$X = M^T \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} N \tag{9.7}$$

where $X_1 > 0$ and X_3 is nonsingular.

$$A^T X + X^T A + \alpha E^T X < 0$$

$$\Leftrightarrow N^T \left(A^T X + X^T A + \alpha E^T X\right) N < 0$$

$$\Leftrightarrow N^T A^T M^T M^{-T} X N + N^T X^T M^{-1} M A N + \alpha N^T E^T M^T M^{-T} X N < 0$$

$$\Rightarrow A_{11}^T X_1 + X_1 A_{11} + \alpha X_{11} < 0$$

From Lemma 9.2.3, we can conclude that all the finite eigenvalues are located to the left of $\mathrm{Re}\,(s) = -\frac{\alpha}{2}$.

Necessity: Since the pencil is admissible, there exist nonsingular matrices M and N such that (9.3) are satisfied. The finite eigenvalues of the pencil (E,A) are the same as the eigenvalues of A_{11}. According to Lemma 9.2.3 there exists a positive definite matrix $X_{11} \in R^{r \times r}$ such that:

$$A_{11}^T X_{11} + X_{11} A_{11} + \alpha X_{11} < 0 \tag{9.8}$$

Choosing $X = M^T \begin{bmatrix} X_1 & 0 \\ 0 & -I_{n-r} \end{bmatrix} N$, then it is easy to verify that Inequalities (9.6) are satisfied. ∇∇∇

Theorem 9.3.1 *The descriptor system (9.1) is admissibly stabilized by a static controller and its every finite eigenvalue has a real part smaller than $-\frac{\alpha}{2}$ if and only if there exist a nonsingular matrix X and a matrix F such that*

$$\begin{aligned} A^T X + X^T A - X^T B B^T X + \left(B^T X + FC\right)^T \left(B^T X + FC\right) + \alpha E^T X < 0 \\ E^T X = X^T E \geq 0 \end{aligned} \tag{9.9}$$

Furthermore, $u = Fy$ is a controller that can admissibly stabilize the descriptor system (9.1).

Proof:
Sufficiency: Suppose that there exist nonsingular matrix X and matrix F such that inequalities (9.9) are satisfied,

$$\begin{aligned}
&(A+BFC)^T X+X^T (A+BFC)+\alpha E^T X \\
&\leq A^T X+X^T A+X^T BFC+C^T F^T B^T X+C^T F^T FC+\alpha E^T X \\
&= A^T X+X^T A-X^T BB^T X+\left(B^T X+FC\right)^T \left(B^T X+FC\right)+\alpha E^T X \\
&< 0
\end{aligned}$$

From Lemma 9.3.1, it is easy to see that the closed-loop descriptor system (9.2) is admissible and all its finite eigenvalues have a real part smaller than $-\frac{\alpha}{2}$ when the controller $u = Fy$ is implemented.

Necessity: If a static output feedback control law $u = My$ makes the closed-loop system (9.2) admissible and its every finite eigenvalue has a real part smaller than $-\frac{\alpha}{2}$, then according to Lemma 9.3.1 there exists a nonsingular matrix X such that

$$\begin{aligned}
(A+BMC)^T X+X^T (A+BMC)+\alpha E^T X < 0 \\
E^T X = X^T E \geq 0
\end{aligned} \tag{9.10}$$

$$\begin{aligned}
&\exists \varepsilon > 0 \\
&0 > (A+BMC)^T X+X^T (A+BMC)+\varepsilon^2 C^T M^T MC+\alpha E^T X \\
&= A^T X+X^T A-\varepsilon^{-2} X^T BB^T X+\alpha E^T X \\
&+\left(\varepsilon^{-1} B^T X+\varepsilon MC\right)^T \left(\varepsilon^{-1} B^T X+\varepsilon MC\right)
\end{aligned} \tag{9.11}$$

Inequality (9.11) is equivalent to

$$\begin{aligned}
&A^T \varepsilon^{-2} X+\varepsilon^{-2} X^T A-\varepsilon^{-2} X^T BB^T X\varepsilon^{-2}+\frac{\alpha}{\varepsilon^2} E^T X \\
&\quad +\left(\varepsilon^{-2} B^T X+MC\right)^T \left(\varepsilon^{-2} B^T X+MC\right) < 0
\end{aligned} \tag{9.12}$$

The inequality (9.12) shows that the nonsingular matrix $\varepsilon^{-2} X$ and the matrix M satisfy (9.9). $\nabla\nabla\nabla$

Corollary 9.3.1 *For the descriptor system (9.1) to be admissibly stabilized by a static controller and its every finite eigenvalue has a real part smaller than $-\frac{\alpha}{2}$ if and only if there exist a nonsingular matrix Y and matrices Y_1 and F such that*

$$\begin{aligned}
\begin{bmatrix} Y^T A^T+AY-BB^T+\alpha Y^T E^T & B+Y_1^T \\ B^T+Y_1 & -I_m \end{bmatrix} < 0 \\
EY = Y^T E^T \geq 0 \\
Y_1 = FCY
\end{aligned} \tag{9.13}$$

Furthermore, $u = Fy$ is a controller that can admissibly stabilize the descriptor system (9.1).

This corollary is the direct result of Theorem 9.3.1 by choosing $Y = X^{-1}$, and the proof is omitted. The condition in Theorem 9.3.1 is stated in Generalized Algebraic Riccati Inequalities (*GARI*) and the condition in Corollary 9.3.1 is LMIs with algebraic constraint. Similar results for the regular system can be found in the paper by Kucera and de Souza (1995), and also in the paper by Cao and Sun (1998). Since the conditions presented in Theorem 9.3.1 is in a Quadratic Matrix Inequalities (*QMI*) form, there is no efficient algorithm available to solve the QMI problem yet. It is the quadratic term X^TBB^TX that prevents us from using Schur complement lemma to transfer inequalities (9.9) to LMIs. To accommodate this quadratic term, an extra matrix variable is introduced. This technique, first introduced by Shiau and Chow (1996) and then used by Cao and Sun (1998) to solve the static output feedback control problem for the regular systems, leads to a bilinear matrix inequality (*BMI*) forms.

Theorem 9.3.2 *The descriptor system (9.1) can be admissibly stabilized by a static output feedback controller and all the finite eigenvalues of the closed-loop descriptor system are located to the left of $R_e(s) = -\frac{\alpha}{2}$ if and only if there exist a nonsingular matrix X, a matrix T and a matrix F such that*

$$\begin{bmatrix} A^TX+X^TA-T^TBB^TX-X^TBB^TT+\alpha E^TX & (B^TX+FC)^T & T^TB \\ (B^TX+FC) & -I_m & 0 \\ B^TT & 0 & -I_m \end{bmatrix} < 0$$
$$E^TX = X^TE \geq 0 \tag{9.14}$$

Proof: First, it is easy to see that the first inequality in (9.14) is equivalent to the following inequality:

$$\begin{aligned} A^TX+X^TA-X^TBB^TX+\left(B^TX+FC\right)^T\left(B^TX+FC\right) \\ +(T-X)^TBB^T(T-X)+\alpha E^TX<0 \end{aligned} \tag{9.15}$$

Inequality (9.15) ensures that the first inequality in (9.9) is satisfied because of the fact $(T-X)^TBB^T(T-X) \geq 0$.

Necessity: If there exists a static output feedback control law $u = My$ such that the closed-loop descriptor system (9.2) is admissible, then there exist a nonsingular matrix X and a matrix F such that inequalities (9.9) are true. We can always choose a matrix T that is close enough to the matrix X such that (9.14) is satisfied. ∇∇∇

The necessary and sufficient condition stated in Theorem 9.3.2 is in a BMI form. BMI was popularized by Safonov and co-researchers in a series of papers (Goh *et al.* 1994*a*, Goh *et* *al.* 1994*b*). Since a BMI is not convex, there is no guarantee of always finding a solution even if feasible solutions exist. There are several algorithms, proposed by Hassibi *et al.* (1999), Goh *et al.* (1994a,1994b), to solve the BMI problem. The most outstanding feature of BMI is that if we fix one variable, BMI becomes an LMI. For example, fixing the matrix T in inequalities (9.14), inequalities (9.14) become LMIs on X. Since there exist powerful numerical tools solving the LMI problems, an iterative algorithm is presented here to solve the static output feedback controller design problem. The algorithm is based on following observation: The descriptor system (9.1) is stabilizable via static output feedback if and only if there exist a nonsingular matrix X, matrices T, F and $\alpha \geq 0$ such that inequalities (9.14) are satisfied. It is easy to see that if (α^*, X^*, T^* ,F^*) satisfy (9.13), so do (α , X^*, T^*,F^*), where $\alpha < \alpha^*$. In other words, if the maximum value of α , to which correspondingly there exist matrices X, T and F satisfying (9.14), is larger than zero, then the descriptor system (9.1) is static output feedback stabilizable. This idea leads to the following algorithm:

Algorithm 9.3.1 *(1) Fix T_i, and find X_i by solving the following optimization problem:*

$$\alpha_i^1 = \max_{X,F} \alpha$$

subject to:

$$\begin{bmatrix} A^TX+X^TA-T_i^TBB^TX-X^TBB^TT_i+\alpha E^TX & (B^TX+FC)^T & T_i^TB \\ (B^TX+FC) & -I_m & 0 \\ B^TT_i & 0 & -I_m \end{bmatrix} < 0$$

$$E^TX = X^TE \geq 0$$

(2) Fix X_i, and find T_{i+1} by solving the following optimization problem (SP Optimization):

$$\alpha_i^2 = \max_{T,F,\alpha} \alpha$$

subject to:

$$\begin{bmatrix} A^TX_i+X_i^TA-T^TBB^TX_i-X_i^TBB^TT+\alpha E^TX & (B^TX_i+FC)^T & T^TB \\ (B^TX_i+FC) & -I_m & 0 \\ B^TT & 0 & -I_m \end{bmatrix} < 0$$

(3) At step (1) or (2), if $\alpha > 0$ then the descriptor system (1) can be admissibly stabilized by the static output feedback control law $u = Fy$ and the algorithm stops; if $\alpha_i^1 < 0, \alpha_i^2 < 0$ and $|\alpha_i^1 - \alpha_i^2| < \varepsilon$, the system may not be stabilized by output feedback and the algorithm stops; otherwise, go to step (1).

Remark 9.3.1 *BMIs do not have the global convergence property, and thus the result may depend critically on the initial point. There is no unique way of selecting the initial point. We want the initial point T_0 to be close to a globally optimal solution so that we suggest that T_0 may be given by solving the following optimization problem:*

$$\min_{T_0} \gamma$$

subject to

$$A^T T_0 + T_0^T A - \gamma E^T T_0 < 0$$

Remark 9.3.2 *To simplify the computation, we may transfer the descriptor system (9.1) to a SVD form (Dai 1988) before implement Algorithm 9.3.1.* $UEV = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, UAV = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, UB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. *Then, the nonsingular matrix X, which satisfies the second inequality in (9.14), must have the following form:*

$$X = \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix}$$

where $0 < X_1 \in R^{r \times r}$, $X_3 \in R^{(n-r)\times(n-r)}$ *is nonsingular,* $X_2 \in R^{(n-r)\times r}$.

This algorithm ensures that α monotonically increases. If at some step α is larger than zero, it means that all the finite eigenvalues of the closed-loop system (9.2) have been moved to the left-half plane. However, if the algorithm stops and α is still less than zero, then the descriptor system (9.1) may not be stabilized by static output feedback or the algorithm converges to an local optimal point even though the original descriptor system (9.1) can be stabilized by static output feedback.

9.3.2 Simultaneous admissible stabilization via output feedback

Consider a collection of m linear time-invariant continuous descriptor systems:

$$\begin{aligned} E_i \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x \end{aligned} \tag{9.16}$$

where $x_i \in R^n$ is the descriptor vector, $u_i \in R^m$ is the control vector, $y_i \in R^p$ is the output vector, E_i, A_i,B_i and C_i are constant matrices with appropriate dimensions, and $rank(E_i) = r_i < n$, $i = 1...m$.

From the result obtained in the last section, it is easy to show the following theorem:

Theorem 9.3.3 *There exists a static output feedback control law that admissibly simultaneously stabilizes the descriptor systems (9.16) if and only if there exist some matrices X_i (nonsingular), T_i and F such that $(i = 1 \ldots m)$*

$$\begin{bmatrix} A_i^T X_i + X_i^T A_i - T_i B_i B_i^T X_i - X_i^T B_i B_i^T T_i + \alpha E_i^T X_i & (B_i^T X_i + FC_i)^T & T_i^T B_i \\ (B_i^T X_i + FC_i) & -I_m & 0 \\ B_i^T T_i & 0 & -I_m \end{bmatrix} < 0$$

$$E_i^T X_i = X_i^T E_i \geq 0$$

Proof: Sufficiency is the direct result of Theorem 9.3.2, and the proof is omitted here.

Necessary: According to Theorem 9.3.2, the collection of descriptor systems (9.16) are simultaneously stabilizable if and only if there exist matrices X_i, T_i and F and $\alpha_i \geq 0$ such that inequalities (9.14) are satisfied. Let $\alpha = \min\{\alpha_i\}$, it can be verified that matrices X_i, T_i and F and $\alpha \geq 0$ satisfy (9.17). ∇∇∇

Algorithm 9.3.2 *1. Fix T_i^j $(i = 1, \ldots, m)$, and find X_i^j $(i = 1, \ldots, m)$ by solving the following optimization problem:*

$$\alpha_j^1 = \max_{X_i, F} \alpha$$

subject to

$$\begin{bmatrix} A_i^T X_i + X_i^T A_i - T_i^j B_i B_i^T X_i - X_i^T B_i B_i^T T_i^j + \alpha E_i^T X_i & (B_i^T X_i + FC_i)^T & (T_i^j)^T B_i \\ (B_i^T X_i + FC_i) & -I_m & 0 \\ B_i^T T_i^j & 0 & -I_m \end{bmatrix} < 0 \qquad (9.17)$$

$$E_i X_i = X_i^T E_i \geq 0 \qquad (9.18)$$

where $i = 1, \ldots, m$.

2. Fix X_i^j (i=1,,m), and find T_i^{j+1} $(i = 1, , m)$ by solving the following optimization problem (SP Optimization):

$$\alpha_j^2 = \max_{T_i} \alpha_i$$

subject to

$$\begin{bmatrix} A_i^T X_i^j + (X_i^j)^T A_i - T_i B_i B_i^T X_i^j - (X_i^j)^T B_i B_i^T T_i + \alpha E_i^T X_i^j & \left(B_i^T X_i^j + FC_i\right)^T & T_i^T B_i \\ \left(B_i^T X_i^j + FC_i\right) & -I_m & 0 \\ B_i^T T_i & 0 & -I_m \end{bmatrix} < 0$$

where $i = 1, ..., m$.

3.At step (1) or (2), if $\alpha > 0$ *then the descriptor system (1) can be admissibly stabilized by the static output feedback control law* $u = Fy$ *and the algorithm stops; if* $\alpha_j^1 < 0$, $\alpha_j^2 < 0$, *and* $\left|\alpha_j^1 - \alpha_j^2\right| < \varepsilon$ *the system may not be stabilized by output feedback, and the algorithm stops; otherwise, go to step 1.*

9.4 Numerical examples

9.4.1 Example 1

This example is from (Fletcher 1988, Duan 1999):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t)$$

Using Algorithm 9.3.1, the solutions are given by

$$X = \begin{bmatrix} 32.8679 & -5.3982 & 0.3267 & 0 \\ -5.3982 & 184.9933 & 12.2591 & 0 \\ -0.3267 & 12.2591 & 62.3613 & 0 \\ -138.0631 & 10.3193 & -472.9530 & 104.5537 \end{bmatrix}$$

$$F = \begin{bmatrix} -10.6798 & -5.4373 \\ 42.3220 & -10.1914 \\ -0.9303 & -1.4774 \end{bmatrix}$$

$$\alpha = 0.612$$

The finite eigenvalues of the closed-loop system are

$$\{-0.6987,\ -4.4253,\ -57.0703\}$$

9.4.2 Example 2

This example is taken from (Liu and Si 1997). Consider the following three systems:
System 1 is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -5 & 2 \\ 0 & -2 & 4 & 8 \\ 2 & 4 & -8 & 1 \\ 1 & 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} x(t)$$

System 2 is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -4 & 0 \\ 1 & -2 & -4 & 0 \\ 0 & 3 & -8 & 5 \\ 0 & 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

System 3 is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

With the implementation of the Algorithm 9.3.2, it is obtained that the finite eigenvalues for the three closed-loop systems are

$$\{-4.0358, -108.9870, -258.1996\}$$

$$\{-2.9663, -2.3554, -18.5898\}$$

$$\{-5.1132, -62.2481\}$$

and the static feedback controller gain is

$$F = \begin{bmatrix} -0.2496 & 0.0346 & -0.2440 \\ 0.3279 & 0.5202 & -0.6737 \end{bmatrix}$$

9.5 Conclusion

The static output feedback control of descriptor systems and simultaneous stabilization of a collection of descriptor systems via a static output feedback controller are studied. A sufficient and necessary condition of stabilizing a descriptor system via static output feedback is presented in a GARI form, and then the GARI is transferred to a BMI form, which leads to the algorithm 1, an iterative algorithm to calculate the static output feedback gain. The results for the static output feedback control of descriptor systems are then extended to simultaneous stabilization of a group of descriptor systems via a static output feedback controller. Numerical results are given to illustrate the proposed algorithms.

Part V

Future work

10

Future work

10.1 Covariance control and PCA

As mentioned in the introduction section, it is assumed that the data is not autocorrelated in PCA theory. However, this assumption is valid only if the process dynamics are fast enough compared with the sampling period. Unfortunately, this assumption does not hold for most of the process because process control requires that the sampling period should be at least twice smaller than the process time constant. If the sampling period is not long enough, the process data may exhibit some degree of autocorrelation. A modified PCA method, called dynamic PCA, tries to incorporate the dynamic relations among variables by augmenting the random vector. However, these methods do not reveal the relationship between covariance of the process variable and dynamics of the process.

Take the following discrete-time stochastic process as an example to illustrate the relationship:

$$\begin{aligned} x_{k+1} &= Ax_k + w_k \\ y_k &= Cx_k + v_k \end{aligned} \tag{10.1}$$

where $x_k \in R^n$ is the process state, $y_k \in R^p$ is the process measurement, $w_k \in R^n$ is the disturbance and $v_k \in R^p$ is measurement noise. It is assumed that $\zeta_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix}$ is zero mean white sequences satisfying

$$E\left(\zeta_i \zeta_j\right) = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta\left(i-j\right) \tag{10.2}$$

As we know from the covariance control theory, Σ_y, the sampling covariance of the output y_k, satisfies:

$$\begin{aligned} \Sigma_y &= C\Sigma C^T + V \\ \Sigma &= A\Sigma A^T + W \end{aligned} \tag{10.3}$$

Usually in chemical process p is much larger than n, *i.e.*, the rows of matrix C are not independent of each other and this is the reason why the last several eigenvalues of Σ_y are close to 0. It is worth further studies on how Equations (10.3) affect performance of PCA monitor scheme in the future.

10.2 Control structure selection based on covariance performance

Control structure selection is one of the fundamental problems in control system design. The control structure selection can be divided into two steps: input/output selection and control configuration selection. Input/output selection involves selecting an appropriate number, location, and type of actuators and sensors in order to achieve the overall system performance requirement. Control configure selection is about how to pair the selected input and output variables and it is only for the decentralized control. Appropriate selected input and output variables can help control engineers achieve the desired system performance. On the other hand, a poor selection of input and output variables might make it difficult for a controller to achieve the preferred performance.

Two basic approaches can be found in literatures on the control structure selection: open-loop and closed-loop methods. The open loop approach does not involve the controller design so that there is no guarantee about the closed-loop performance. The closed-loop approach is jointly selecting the input/output variables and designing the controller according to some performance criteria.

The linear time-invariant (*LTI*) system considered is given by:

$$\dot{x}(t) = Ax(t) + Bu(t) + G\omega(t) \tag{10.4}$$

$$y(t) = Cx(t) + \upsilon(t) \tag{10.5}$$

$$z_i(t) = C_i x(t) + D_i(t) \tag{10.6}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input candidate vector, $y \in R^p$ is the output, $z_i \in R^{l_i}$ $(i = 1, 2, \cdots, L)$ is the controlled variable, $\omega \in R^d$ is the external disturbance and $\upsilon \in R^p$ is the measurement noises. A, B, C and G are matrices with appropriate dimensions.

The control performance is defined as the generalized covariance constraints (*GCC*). The objective of the controller structure selection is to jointly select the input and output such that the generalized covariance constraints are satisfied and at the same time to minimize the economic cost associated with the selected sensors and actuators.

Bibliography

Anderson, B. D. O. and J. B. Moore (1979). *Optimal Filtering*. NJ: Prentice-Hall.

Ästrom, K. J. (1970). *Introduction to stochastic control theory*. Academic Press.

Ästrom, K. J. and T. Hagglund (1984). Automatic tuning of simple regulators with specification on phase and amplitude margins. *Automatica*.

Ästrom, K. J. and T. Hagglund (1995). PID controller. *Instrument of Society of America*.

Athans, M. (1971). The role and use of the stochastic linear quadratic gaussian problem in control system design. *IEEE Trans. A. C.* **16**, 529–552.

Bain, A. G. and S. T. Bonnington (1970). The hydraulic transport of solids by pipeline. *Pergmon press*.

Bao, J., F. Forbes and P. J. McLellan (1999). Robust multi-loop PID controller design - a successive semi-definite programming approach. *Ind. Eng. Chem. Res.* **38**, 3407–3419.

Bender, D. and A. Laub (1987). The linear quadratic optimal regulator for descriptor systems. *IEEE Trans. A. C.* **32**, 672–688.

Boyd, S. and C. H. Barratt (1991). *Linear control design*. Prentice Hall.

Brockett, R. W. (1970). *Finite dimensional linear systems*. New York: Wiley.

Bryson, A. E. and Y. C. Ho (1975). *Applied Optimal Control*. Hemisphere Publishing.

Bunse-Gerstner, A., V. Mehrmann and N.K. Nichols (1994). Regulation of descriptor systems by output feedback. *IEEE Trans. A. C.* **39**, 1742–1748.

Campbell, S. L. (1980). *Singular systems of differential equations*. Pitman Publishing Co.

Cao, Y. and Y. Sun (1998). Static output feedback simultaneous stabilization: ILMI approach. *Int. J. Control* **70**, 803–814.

Chen, H., J. H. Chow, M. A. Kale and K. D. Minto (1995). Simultaneous stabilization using stable system inversion. *Automatica* **31**, 531–542.

Chen, J. and R. J. Patton (1999). H_∞ formulation and solution for robust fault diagnosis. *Proceedings of 14th World Congress of IFAC* pp. 127–132.

Chow, E. Y. and A. S. Willsky (1984). Analytical redundancy and the design of robust failure detection systems. *IEEE Trans. A. C.* **29**, 603–614.

Chow, J. H. (1990). A pole-placement design approach for systems with multiple operating conditions. *IEEE Trans. A. C.* **35**, 278–288.

Chu, D. L. and D. W. Ho (1999). Necessary and sufficient conditions for the output feedback regulation of descriptor systems. *IEEE Trans. A. C.* **44**, 405–412.

Cohen, G. H. and G. A. Coon (1953). Theoretical considerations of retarded control. *Transactions of ASME* **75**, 827.

Collins, E. G. and R. E. Skelton (1985). Covariance control of discrete systems. *Proceedings of the 24th Conf. On Decision and Control* pp. 542–547.

Commault, C., J. M. Dion, O. Sename and R. Moteyian (2000). Observer-based fault diagnosis for structured systems. *Proceedings of the 39th IEEE Conf. on Decision and Control* pp. 4138–4143.

Dai, L. (1988). Observer for discrete singular systems. *IEEE Trans. A. C.* **33**, 187–191.

de Souza, C. E. and A. Trofino (1999). An LMI aproach to the design of robust H_2 filters. *Chapter in Recent advances on linear matrix inequality methods in control.*

Ding, X. and L. Guo (1998). An approach to time domain optimization of observer-based fault detection systems. *Int. J. Control.* **69**, 419–442.

Ding, X. and P. M. Frank (1989). Fault detection via optimally robust detection filters. *Proceedings of the 28th IEEE Conf. on Decision and Control* pp. 1767–1772.

Ding, X., M. Zhong, B. Tang and P. Zhang (2001). An LMI approach to the design of fault detection filter for time-delay LTI systems with unknown inputs. *Proceedings of American Control Conf.* pp. 2137–2142.

Duan, G. (1999). Eigenstructure assignment in descriptor systems via feedback: a new complete parametric approach. *Int. J. Control* **72**, 345–364.

Dunia, R. and J. Qin (1998). A unified geometric approach to process and sensor fault identification. *Comput. Chem.* **22**, 927–943.

Dunia, R., J. Qin, T. F. Edgar and T. J. McAvoy (1996). Sensor fault identification and reconstruction using principal component analysis. *Proceedings of 13th World Congress of IFAC* pp. 259–264.

Feng, Z., Q. G. Wang and T. H. Lee (2002). On the design of multivariable PID controllers via LMI approach. *Automatica* **38**, 517–526.

Fletcher, L. R. (1988). Eigenstructure assignment by output feedback in descriptor systems. *IEE Proceeding Control Theory and Applications* pp. 302–308.

Frank, P. M. (1990). Fault diagnosis in dynamic systems using analytical and knowledge based redundancy, a survey of some new results. *Automatica* **26**, 459–474.

Frank, P. M. (1994). Enhancement of robustness in observer based fault detection. *Int. J. Control* **59**, 955–981.

Fujioka, H. and S. Hara (1995). State covariance assignment for sampled-data feedback control. *Int. J. Control* **61**, 719–737.

Gahinet, P., A. Nemirovski, A. Laub and M. Chilali (1995). *LMI Control toolbox*. The Mathworks.

Ge, M., M. Chiu and Q. G. Wang (2002). Robust PID controller design via LMI approach. *J. Process. Control.* **12**, 3–13.

Gertler, J. and D. Singer (1990). A new structural framework for parity equation based failure detection and isolation. *Automatica* **26**, 381388.

Gertler, J. and M. Kunwer (1995). Optimal residual decoupling for robust fault diagonosis. *Int. J. Control* **61**, 395–421.

Ghaoui, L. El, F. Oustry and M. AitRami (1997). A cone complementarity linearization algorithm for static output feedback and related problem. *IEEE. Trans. A. C.* **42**, 1171–1176.

Goh, K. C., L. Turan, M. G. Safonov, G. P. Papvassilopoulos and J. H. Ly (1994*a*). Biaffine matrix inequality properties and computational methods. *Proceedings of America Control Conf.* pp. 850–855.

Goh, K. C., M. G. Safonov and G. P. Papavassilopoulos (1994*b*). A global optimization approach for the bmi problem. *Proceedings of the 33rd IEEE Conf. on Decision and Control* pp. 2009–2014.

Grigoriadis, K. and R. Skelton (1994). Alternating convex projection methods for covariance control design. *Int. J. Control* **60**, 1083–1106.

Grigoriadis, K. and R. Skelton (1997). Minimum-energy covariance controllers. *Automatica* **33**, 569–578.

Haddad, W. and D. S. Bernstein (1992). Fixed-order sampled-data estimation. *Int. J. Control* **55**, 129–139.

Harris, T. J. (1989). Assessment of closed loop performance. *Can. J. Chem. Eng.* **67**, 856–861.

Hotz, A. and R. Skelton (1987). A covariance control theory. *Int. J. Control* pp. 13–32.

How, A. Hassibi J. and S. Boyd (1999). A path-following method for solving bmi problems in control. *Proceedings of American Control Conf.* pp. 1385–1389.

Hsieh, C. and R. E. Skelton (1990). All covariance controllers for linear discrete-time systems. *IEEE Trans. A. C.* **35**, 908–915.

Hsieh, C., R. E. Skelton and F. M. Damra (1989). Minimum energy controllers with inequality constraints on output variances. *Optimal Control Application & methods* **10**, 347–366.

Huang, B., S.L. Shah and K.Y. Kwok (1997). Good, bad or optimal? performance assessment of mimo processes. *Automatica* **33**, 1175–1183.

Huang, X. and B. Huang (2002). A bmi approach towards single and simultaneous admissible stabilization of continuous-time descriptor systems via static output feedback. *Dynamics of continuous, discrete, and impulse systems* pp. 71–84.

Huang, X. and B. Huang (2003*a*). Multi-loop decentralized PID control based on covariance criterion: an LMI approach. *ISA Trans.*

Huang, X. and B. Huang (2003*b*). A novel LMI approach towards fault detection and isolation for discrete-time stochastic systems. *submitted to Inter. J. Control.*

Huang, X., L. Q. Zhang and B. Huang (2002). LQ control with generalized covariance constraints for discrete-time systems. *CSChE.*

Huang, X., L. Q. Zhang and B. Huang (2003*a*). Covariance constrained LQ control and applications. *American Control Conf.* pp. 3392–3397.

Huang, X., L. Q. Zhang and B. Huang (2003*b*). Stochastic LQ control with the covariance constraints. *Dynamics of continuous, discrete, and impulse systems.*

Inoue, A., A. Yanou, T. Sato and Y. Hirashima (2000). An extension of generalized minimum variance control for multi-input multi-output system using coprime factorization approach. *Proceedings of American Control Conf.* pp. 4184–4188.

Iwasaki, T. and R. E. Skelton (1995). The XY-centering algorithm for the dual LMI problem: a new approach to fixed-order control desogn. *Int. J. Control* pp. 1257–1272.

J. C. Geromel, C. C. de Souza and R. E. Skelton (1998). Static output feedback controllers: stability and convexity. *IEEE Trans. A. C.* **43**, 120–125.

Jackson, J. E. (1959). Quality control methods for several related variables. *Technometrics* pp. 359–377.

Jackson, J. E. (1991). *A User's Guide to Principal Components*. WileyInterscience, New York.

Johansson, K. H. (2000). The quadruple-tank process: a multivariable laboratory process with an adjustable zero. *IEEE Trans. Control. Sys. Tech.* **8**, 456–465.

Kalman, R. E. (1968). Contribution to the theory of optimal control. *Bio. Soc. Mat. Mex.* **5**, 102–199.

Keller, J. Y. (1999). Fault isolation filter design for linear stochastic systems. *Automatica* **35**, 1701–1706.

Kucera, V. and C. E. de Souza (1995). A necessary and sufficient condition for the output feedback stability. *Automatica* **31**, 1357–1359.

Kumar, A. and P. Daoutidis (1998). Control of nonlinear differential algebraic equation systems: an overview. *Nato ASI Series E, Kluwer Acadymics* **353**, 311–344.

Kwok, K. E., M. C. Ping and P. Li (2000). A model based augmented PID algorithm. *J. Process. Control.* **10**, 9–18.

Lewis, F. L. (1986). A survey of linear singular systems. *IEEE Trans. Circuit, Sys. Signal Process* **43**, 3–36.

Li, W. and S. Shah (2002). Structured residual vector-based approach to sensor fault detection and isolation. *J. Process Control* **12**, 429–443.

Li, W., H. Yue, S. Vale-Cervantes and S. Qin (2000). Recursive PCA for adaptive process monitoring. *J. Process Control* **10**, 471–486.

Lin, C., J. Wang, G. H. Yang and C. B. Soh (1999). Robust controllability and robust closed-loop stability with static output feedback for a class of uncertain descriptor systems. *Linear Algebra Appl.* **43**, 133–155.

Liu, B. and J. Si (1997). Fault isolation filter design for time-invariant systems. *IEEE Trans. A. C.* **42**, 704–707.

Liu, W. Q., K. L. Teo and W. Y. Yan (1996). Optimal simultaneous stabilization of descriptor systems via output feedback. *Int. J. Control* **64**, 595–613.

Lovass-Nagy, V., D. L. Powers and R. J. Schilling (1994). On regularizing of descriptor systems by output feedback. *IEEE Trans. A. C.* **39**, 1507–1509.

Luenberger, D. G. (1978). Time-invariant descriptor systems. *Automatica* **14**, 473–480.

MacGregor, J. F. (1973). Optimal discrete stochastic control theory for process application. *Can. J. Chem. Eng.* **51**, 468–478.

MacGregor, J. F. (1994). Statistical process control of multivariate processes. *Proceedings of IFAC ADCHEM.*

MacGregor, J. F. and T. Kourti (1995). Statistical process control of multivariate processes. *Control Eng. Practice* pp. 403–414.

Mäkilä, P. M. (1982). Self-tuning control with linear input constraints. *Optim. Control. Appl. Methods* **3**, 337–353.

Mäkilä, P. M., T. Westerlund and H. T. Toivonen (1984). Constrained linear quadratic gaussian control with process applications. *Automatica* **20**, 15–29.

Massoumnia, M. (1986). A geometric approach to synthesis of failure detection filters. *IEEE Trans. A. C.* **31**, 839–846.

Masubuchi, I., A. Ohara and N. Suda (1995). LMI-based output feedback controller design. *Proceedings of American Control Conf.* pp. 3473–3477.

Masubuchi, I., Y. Kamitane, A. Oharas and N. Suda (1997). H_∞ control for descriptor systems: a matrix inequalities approach. *Automatica* **33**, 669–673.

Mathai, A. M. and S. B. Provost (1992). Quadratic forms in random variables: theory and applications. *Marcel Dekker, Inc.*

Miller, R. M., K. E. Kwok, S. L. Shah and R. K. Wood (1995). Development of a stochastic predictive PID controller. *Proceedings of American Control Conf.* pp. 4204–4208.

Miura, N., M. Imaeda, K. Hashimoto, R. K. Wood, H. Hattori and M. Onishi (1998). Auto-tuning PID controller based on generalized minimum variance control for a pvc reactor. *J. Chem. Eng. of Japan* **31**, 626–632.

Morari, M. and E. Zafiriou (1989). *Robust process control.* Prentice Hall.

Nagpal, K. M. and P. P. Khargonekar (1991). Filtering and smoothing in an H_∞ setting. *IEEE Trans. A. C.* **36**, 151–166.

Nikoukhah, R. (1994). Innovations generation in the presence of unknown inputs: application to robust failure detection. *Automatica* **30**, 1851–1867.

Ogunnaike, B. A. and W. Harmon Ray (1994). *Process dynamics, modeling and control.* Oxford University Press.

Patton, R. J. and J. Chen (1991). Robust fault detection using eigenstructure assignment: a tutorial consideration and some new results. *Proceedings of the 30th IEEE Conf. on Decision and Control* pp. 2242–2247.

Patton, R. J., P. M. Frank and R. N. Clark (1989). *Fault diagnosis in dynamic systems: theory and application*. Prentice Hall.

Qin, J. (1998). Control performance monitoring - a review and assessment. *Comput. and Chem. Eng.* **23**, 178–186.

Rehm, A. and F. Allgower (2000). H_∞ control for descriptor systems: a matrix inequalities approach. *Proceeding of IFAC* pp. 57–62.

Rivera, D. E., M. Morari and S. Skogestad (1986). Internal model control - PID control design. *Ind. Eng. Chem. Process Des. Dev.* **25**, 252–265.

Sato, T., A. Inoue and Y. Hirashima (2002). Self-tuning two-degree-of freedom PID controller reducing the effect of disturbance. *Am. Control. Conf.* pp. 3997–4002.

Scherer, C., P. Gahinet and M. Chilali (1997). Multiobjective output-feedback control via LMI optimization. *IEEE Trans. A. C.* **42**, 896–911.

Shaked, U. (1990). H_∞ minimum error state estimation of linear stationary processes. *IEEE Trans. A. C.* **35**, 554558.

Shaked, U. and Y. Theodor (1992). H_∞-optimal estimation: A tutorial. *Proceedings of the 31st IEEE Conf. on Decision and Control* pp. 2278–2286.

Shiau, J. K. and J. H. Chow (1996). Robust decentralized state feedback control design using an iterative linear matrix inequality algorithm. *Proceedings of 13th IFAC* pp. 203–208.

Shunta, J. P. (1995). *Achieving world class manufacturing through process control.* Prentice Hall PTR.

Singh, S. P. and R. W. Liu (1973). Existence of state equation representation of linear large-scale dynamic systems. *IEEE Trans. A. C.* **20**, 239–246.

Skelton, R. and M. Delorenzo (1985). Space structure control design by variance assignment. *J. Guidance* **8**, 454–462.

Skelton, R. E. and T. Iwasaki (1993). Liapunov and covariance controller. *Int. J. Control* **57**, 519–536.

Skelton, R.E., T. Iwasaki and K. Grigoriadis (1998). *A Unified Algebraic Approach to Linear Control Design*. Taylor & Francis.

Syrmos, V. L., C. Abdallah, P. Dorato and K. Grigoriadis (1997). Static output feedback: A survey. *Automatica* **33**, 125–137.

Toivonen, H. T. (1983). Self-tuning with explicit variance restriction. *Int. J. Control.* **38**, 229–235.

Tuan, H. D., P. Apkarian and Y. Nakashima (1999). A new lagrangian dual global optimization algorithm for solving bilinear matrix inequalities. *Am. Control. Conf.* pp. 1851–1855.

Vandenberghe, L. and S. Boyd (1996). Semi-definite programming. *SIAM Review* **38**, 49–95.

Verghese, G. C., B. C. Levy and T. Kailath (1981). A generalized state space for singular systems. *IEEE Trans. A. C.* **26**, 811–831.

Vidyasagar, M. and N. Viswanadham (1982). Algebraic design techniques for reliable stabilization. *IEEE Trans. A. C.* **27**, 1085–1095.

Wang, Z. and B. Huang (2000). Robust H_2 H_∞ filtering for linear systems with error variance constraints. *IEEE Trans. A. C.* **48**, 2463–2467.

Wang, Z., B. Huang and P. Huo (2001). Sampled-data filtering with error covariance assignment. *IEEE Trans. Signal Processing* **49**, 666–670.

Westerlund, T. (1981). A digital quality control system for an industrial dry process. *IEEE Trans. A. C.* **26**, 885–890.

Wise, B. M. and N. L. Ricker (1991). Recent advances in multivariate statistical process control: Improving robustness and sensitivity. *Proceedings of IFAC ADCHEM* pp. 125–130.

Wold, S. (1994). Exponentially weighted moving principal component analysis and projection to latent structures. *Chemometrics and Intelligent Laboratory Systems* **23**, 149–161.

Wu, F. (2001). LMI-based robust model predictive control and its application to an industrial cstr problem. *J. of Process Control* **11**, 649–659.

Wünnenberg, J. (1990). Observer-based fault detection in dynamic systems. *Ph.D Thesis.*

Xu, G. and T. Kailath (1994). Fast estimation of principal eigenspace using lanczos algorithm. *SIAM J. Matrix Anal. Appl.* **15**, 974–994.

Xu, J. and R. Skelton (1992). An improved covariance assignment theory for discrete sytems. *IEEE Trans. A. C.* **37**, 1588–1591.

Xu, S. and T. Chen (2002). Reduced-order H_∞ filtering for stochastic systems. *IEEE Trans. Signal Processing* **50**, 2998–3007.

Yamamoto, T., A. Inoue and S. L. Shah (1999). Generalized minimum variance self-tuning pole-assignment controller with a PID structure. *Proc. IEEE Int. Conf. on Control Appl.* pp. 125–130.

Yasuda, K., R. E. Skelton and K. M. Grigoriadis (1993). Covariance controller: a new parameterization of the class of all stabilizing controllers. *Automatica* **29**, 785–788.

Yaz, E. and R. E. Skelton (1991). Continuous and discrete state estimation with error covariance assignment. *Proceedings of the 30th IEEE Conf. on Decision and Control* pp. 3091–3092.

Zhai, G., M. Ikeda and Y. Fujisaki (2001). Decentralized H_∞ controller design: a matrix inequality approach using a homotopy method. *Automatica* **37**, 565–572.

Zhong, M., X. Ding, J. Lam and H. Wang (2003). An LMI approach to design robust fault detection filter for uncertain LTI systems. *Automatica* **39**, 543–550.

Zhu, G., M. A. Rotea and R. E. Skelton (1997). A convergent algorithm for the output covariance constraint control problem. *SIAM J. Control Optim.* **35**, 341–361.

Ziegler, J. G. and N. B. Nichols (1942). Optimum settings for automatic controllers. *Trans. A.S.M.E.* **62**, 759–768.