University of Alberta

EQUIVARIANT DEGREE METHOD AND ITS APPLICATIONS

TO

SYMMETRIC SYSTEMS OF VAN DER POL EQUATIONS

AND

HOPF BIFURCATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SYMMETRIES

by

Meymanat Farzamirad ©

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DEDICATION

To memory of my father
To my mother for her love, encouragement and support
To Arezoo my precious daughter

ABSTRACT

In my thesis, I apply the recently developed equivariant degree theory to study the properties of solutions in symmetric systems of van der Pol equations and to classify the Hopf bifurcations in symmetric systems of functional differential equations. The existence of periodic solutions or the occurence of Hopf bifurcation in symmetric dynamical systems can be determined by means of the so-called primary equivariant degree associated with the considered problem, which provides with an algebraic invariant containing equivariant topological information about the existence of solutions and their symmetric properties. The equivariant degree computational techniques are based on the reduction to the so-called basic maps (on irreducible representations) and the usage of the splitting lemma and multiplicativity property. In my work, I establish the algebraic basis for the computations of the equivariant degree for the groups $G = \Gamma \times S^1$, with Γ being dihedral, tetrahedral, octahedral and icosahedral groups. Based on the obtained tables the existence and bifurcation results are formulated in terms of the equivariant degree. Multiplicativity of solutions is also discussed.

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KEY WORDS AND PHRASES

Irreducible representations, twisted subgroups, Burnside ring, basic maps, normal maps, regular normal maps, equivariant degree, primary degree, basic degree, dihedral, tetrahedral, octahedral and icosahedral symmetry group, van der Pol systems of non-linear oscillators, periodic solutions, topological classification of symmetric periodic solutions, Hopf bifurcation, functional differential equations, crossing numbers, symmetric bifurcation, and orbit types.

Contents

INTRODUCTION

Classical degree theory became an important tool of nonlinear analysis for the detection of single and multiple solutions of nonlinear equations. Its effectiveness and universality rely on its standard properties, which can be considered as the axioms of the degree theory: existence, additivity, homotopy, suspension and normalization properties. In simple words, the degree of a map f means an "algebraic count" of solutions in a set Ω to the equation $f(x) = 0$ (additivity), which does not depend on perturbations of f (homotopy). There are numerous applications of the degree theory in differential equations and functional differential equations (cf. [36]) to establish the existence and multiplicity results for nonlinear equations.

For a long time, there were many attempts made in order to create a similar degree theory to study nonlinear equations for equivariant maps. These efforts were motivated by symmetries in dynamical systems. Many such systems were studied in physics, chemistry, biology, engineering, etc.

The impact of symmetry can result in large varieties of different solutions with various symmetric properties, complicated bifurcations and pattern formations, requiring the usage of sophisticated mathematical tools.

The idea of the equivariant degree is based on a desire to "count algebaically" the orbits of solutions to $f(x) = 0$ for an equivariant map f.

Several names should be mentioned in this place; In 1976 R. Rubinsztein published a paper on S^1-equivariant degree (cf. [44]), and E. N. Dancer (cf. [14]) in 1980's studied S^1 equivariant variational problems using a specific variant of S^1-degree. A more general degree theory (without free parameter) was introduced in 1988 by H. Ulrich (cf [48]), who also generalized the notion of Burnside ring used as the range for this degree. An important contribution was made by a group of Polish mathematicians, Dylawerski, Gęba, Jodel, and Marzantowicz (cf. [15]), who published in 1991 a paper introducing a one-parameter S^1-degree. This degree theory turned out to be fundamental for the computations of the general G-degree (with one free parameter).

The general G-degree was introduced by J. Ize, Massabó, and Vignoli (cf. [30]), and this definition is the basis for the equivariant degree theory. Although the work of Ize and his collaborator, was concentrated on the abelian group actions, for which they established computational formulae, this framework can also be used for general non-abelian group actions. Independently of Ize, in 1990's, following a different construction (using normal approximations) another degree theory (for non-abelian actions) was introduced by Geba, Krawcewicz, and Wu (cf. [21]). This degree theory, turned out to be a part of the G-degree which was introduced by Ize. I will refer in my work to this degree as the so-called *primary degree.* In the recent years, there was a large progress made in the development of the equivariant degree theory for non-abelian groups.

The main advantage of using the primary degree for non-abelian group actions lies in the fact that computations related to such situations can be standarized. More precisely, it is possible to establish the so-called *multiplicativity property* for the primary degree (with one parameter) for the twisted orbit types (with $G = \Gamma \times S^1$), and use the reduction technique to basic maps, in order to establish computational tables for the G-degree.

Since in the case of a non-abelian group G, the G-equivariant degree theory requires very particular computational techniques, depending on the specific properties of the group G (lattice of conjugacy classes of subgroups in G, irreducible representations of G, and other requirements), there is a need for creating a standard approach to the computation of G-degree.

For this purpose, in my work, I have explored the idea of basic maps (the most elementary equivariant maps) and the algebraic properties of the primary equivariant degree in order to establish standard tables needed for these computations. These tables can be used practically, even without knowledge of the definition of the G-degree, to compute the degree of equivariant maps associated with the studied applied problems.

Chapter 1

Preliminaries

1.1 Topological Degree

It has been about one hundred years since the topological methods, such as degree theory, were introduced to study nonlinear equations, mainly to prove the existence and multiplicity of solutions.

Consider an open bounded set $\Omega \subset \mathbb{R}^N$ and a continuous map $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $f(x) \neq 0$ for $x \in \partial\Omega$ (such a map is called Ω-*admissible*). We will call the pair (f, Ω) *admissible*. It is easy to observe that knowing the values of the map f on the boundary $\partial\Omega$ may be sufficient to predict the existence of a solution $x \in \Omega$ of the equation $f(x) = 0$. Indeed, for instance if $f : [a, b] \to \mathbb{R}$ is a continuous map such that $f(a)f(b) < 0$, then by the Intermediate Value Theorem there exists a point $x_1 \in (a, b)$ such that $f(x_1) = 0$ (see Figure 1.1).

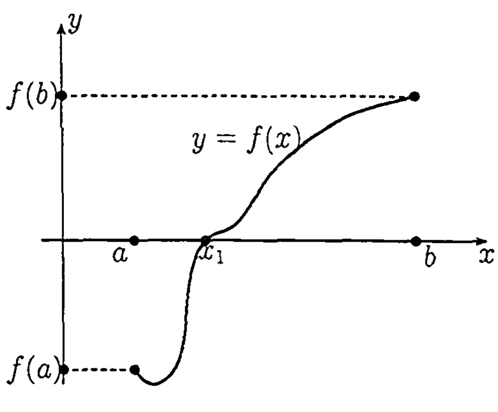


Figure 1.1: Existence of a solution to $f(x) = 0$.

We denote by $\mathcal{M}$ the set of all admissible pairs (f,Ω). There is an integer-valued function $\deg : \mathcal{M} \to \mathbb{Z}$, called the *local Brouwer degree*, satisfying the following properties:

P1: (EXISTENCE[1]) If $\deg(f,\Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$.

P2: (HOMOTOPY[2]) If $h : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous map such that $h(t,x) \neq 0$ for all $(t,x) \in [0,1] \times \partial\Omega$ (we will call such h an *Ω-admissible homotopy* or *deformation*) then $\deg(h(t,\cdot),\Omega) = \text{constant}$.

P3: (EXCISION[3]) Let Ω_o be an open subset of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_o$, then $\deg(f,\Omega) = \deg(f,\Omega_o)$.

P4: (ADDITIVITY[4]) Let Ω_1 and Ω_2 be two disjoint open subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then

$$\deg(f,\Omega) = \deg(f,\Omega_1) + \deg(f,\Omega_2).$$

P5: (NORMALIZATION[5]) Let $f(x) = x - a$, where $a \notin \partial\Omega$. Then

$$\deg(f,\Omega) = \begin{cases} 1 & \text{if } a \in \Omega, \\ 0 & \text{if } a \notin \Omega. \end{cases}$$

It can be proved (cf. [36]) that there exists only one such function $(f,\Omega) \mapsto \deg(f,\Omega) \in \mathbb{Z}$ satisfying the properties (P1)–(P5). There are many ways to construct the degree $\deg(f,\Omega)$. For example, if f is an Ω-admissible C^1 map such that zero is a regular value of $f|_\Omega$. Then

$$\deg(f,\Omega) = \sum_{x_k \in f^{-1}(0)\cap\Omega} \text{sign}\det Df(x_k)$$

where $\sum$ over an empty set is defined an zero.

We recall that zero is a regular value of a smooth map $f|_\Omega$, if either $f^{-1}(0) \cap \Omega = \emptyset$, or for each $x \in f^{-1}(0) \cap \Omega$ we have $\det Df(x) \neq 0$.

[1]The Existence Property is used to establish the existence of a solution to the equation $f(x) = 0$.

[2]The Homotopy Property is used to deform a complicated map $f(x) = h(0,x)$ to a rather simpler map $\tilde{f}(x) = h(1,x)$, for which the computation of degree is possible, and if $\deg(\tilde{f},\Omega) \neq 0$, then $\deg(f,\Omega) = \deg(\tilde{f},\Omega) \neq 0$, and the existence of a solution for $f(x) = 0$ can be established.

[3]The Excision Property means simply that the degree of f depends on the location of zeros of f and not the set Ω.

[4]The Additivity Property is useful to establish the existence of multiple solutions of the equation $f(x) = 0$. It also expresses the fact that the degree is a kind of an algebraic count of zeros of f.

[5]The Normalization Property guarantees the non-triviality of the degree (and its uniqueness).

Example 1.1.1. *As an illustration, we can give a degree-theoretical proof of the fundamental theorem of algebra: Every polynomial*

$$P(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0.$$

has at least one complex zero if $n \geq 1$, where a_i, $0 \leq i \leq n-1$, are complex numbers. To prove this, we first identify the complex plane with $\mathbb{R}^2$, then P defines a mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$. Let $\epsilon > 0$ be given and define

$$H(t, z) = z^n - \epsilon + t(a_{n-1}z^{n-1} + \dots + a_1 z + a_0 + \epsilon)$$

for $t \in [0, 1]$. It is easy to show that if ϵ is sufficiently small and $R > 0$ is sufficiently large, then H is Ω-admissible homotopy with $\Omega = B_R(0)$. Consequently,

$$\deg(P, \Omega) = \deg(H(1, \cdot), \Omega) = \deg(H(0, \cdot), \Omega)$$

Clearly, $H(0, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ is a C_1-map and

$$H(0, \cdot)^{-1}(0) = \{\epsilon^{\frac{1}{n}} \cdot e^{i(2\pi/n)j} \mid 0 \leq j \leq n-1\}$$

and $detDH(0, z)|_{z=\epsilon^{\frac{1}{n}} \cdot e^{i(2\pi/n)j}} > 0$. Therefore, zero is a regular value of $H(0, \cdot)|_\Omega$ and

$$\deg(H(0, \cdot), \Omega) = \sum_{j=0}^{n-1} \text{sign} \;\; \det DH(0, z)|_{z=\epsilon^{\frac{1}{n}} \cdot e^{i(2\pi/n)j}} = n$$

Consequently, $\deg(P, \Omega) = n > 0$ and the conclusion follows from the existence property of the degree.

1.2 G-actions and G-spaces

Definition 1.2.1. *A Lie group is a smooth manifold G, which is also a group such that the group multiplication $\cdot : G \times G \to G$ and the inverse map $\nu : G \to G$, $\nu(g) = g^{-1}$, are smooth.*

Example 1.2.2. *The unit circle $S^1 = \{z \in \mathbb{C} | \;\; |z| = 1\}$, viewed as a multiplicative subgroup of $\mathbb{C}$, is a compact Lie group.*

Throughout this section, we assume that G is a compact Lie group.

Definition 1.2.3. *Let X be a Hausdorff topological space. By a topological transformation group we mean a triple (G, X, φ), where $\varphi : G \times X \to X$ is a continuous map such that* *(i)* $\varphi(g, \varphi(h, x)) = \varphi(gh, x)$ *for all* $g, h \in G$ *and* $x \in X$,

(ii) $\varphi(1, x) = x$ *for all* $x \in X$, *where* 1 *is the identity of* G.

For a toplogical transformation group (G, X, φ), *we call the map* φ *an action of* G *on* X *and the space* X, *under the action of* G, *a* G*-space. The action* φ *is called free if* $\varphi(g, x) \neq x$ *for all* $g \neq 1 \in G$ *and* $x \in X$. *For* $A \subset X$, *we put* $G(A) = \{gx| \ g \in G, x \in A\}$. *A set* $A \subset X$ *is said to be* G*-invariant, or simply invariant, if* $G(A) = A$.

We will denote by $\mathbb{Z}_k \subset S^1$ *the cyclic subgroups of* S^1 *of order* k, $k = 1, 2, \cdots$, *i.e.* $\mathbb{Z}_k := \{z \in \mathbb{C}| \ z^k = 1\}$, *i.e.* $\mathbb{Z}_k = \{1, \gamma, \gamma^2, \cdots, \gamma^{k-1}\}$ *where* $\gamma = e^{\frac{i2\pi}{k}}$. *Recall that two closed subgroups* H *and* K *of* G *are said to be conjugate subgroups in* G, *denoted by* $H \sim K$, *if* $H = gKg^{-1}$ *for some* $g \in G$. *The relation* $\sim$ *is an equivalence relation and the equivalence class of* H, *denoted by* (H), *is called the conjugacy class of* H *in* G. *We denote by* $\Phi(G)$ *the set of all conjugacy classes. The set* $\Phi(G)$ *is partially ordred by the relation* $\leq$ *defined as follows*

$$(H) \leq (K) \overset{Def}{\Longleftrightarrow} \exists g \in G \ \ gHg^{-1} \subset K.$$

1.3 Elements of Representation Theory

1.3.1 Definitions

Let V be a vector space over the field $\mathbb{C}$ of complex numbers (respectively, over the field $\mathbb{R}$ of real numbers) and let $GL(V)$ be the group of isomorphisms of V onto itself. An element A of $GL(V)$ is a linear mapping of V into V which has an inverse A^{-1}. Assume that $\dim V = n$, by choosing a basis (e_i), each linear map $A : V \to V$ can be represented by a square matrix (a_{ij}) of order n. In such a case saying that A is an isomorphism is equivalent to the fact that the determinant $\det(A)$ of A is not zero. In this way the group $GL(V)$ can be identified with the group of invertible square matrices of order n.

Suppose G is a compact lie group, with identity element 1 and with compositon $(g, h) \to gh$. A *complex (*respectively, *real) linear representation* of G in *complex (*respectively, *real)* vector space V is a continuous homomorphism T from the group G into the group $GL(V)$. In other words, we associate with each element $g \in G$ an element T_g of $GL(V)$, and we have the equalities

$$T_{g_1 g_2} = T_{g_1} \circ T_{g_2} \qquad \text{for} \ \ g_1, g_2 \in G,$$

and

$$T_1 = Id.$$

For a given homomorphism T, we say that V is the *complex* (repectively, *real*) *representation space* of G. Suppose $\dim V = n$, then n is called the *dimension* of the complex (respectively, real) representation V.

Let T and T' be two complex (respectively, real) representations of the same group G in the complex (respectively, real) vector spaces V_1 and V_2. These two representations are said to be *equivalent* if there exists a linear isomorphism $A : V_1 \to V_2$ that satisfies the identity

$$A \circ T_g = T'_g \circ A \qquad \text{for all } g \in G.$$

1.3.2 Basic Examples

(a) A complex representation of dimension 1 of a group G is given by the representation space $V = \mathbb{C}$ and a homomorphism $T : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of nonzero complex numbers. If we take $T(g) = 1$ for all $g \in G$, then the obtained representation of G is called *trivial (unit) representation* of G.

(b) Assume that G is finite and let n be the order of G. Let V be a complex (respectively, real) vector space of dimension n, with a basis $(e_g)_{g\in G}$ indexed by the elements g of G. For $g \in G$, let $T_g : V \to V$ be a linear map of V into V which sends e_h to e_{gh}, for all $h \in V$. This defines a complex (respectively, real) linear representation, which is called *the complex* (respectively, *real*) *regular representation* of G.

(c) More generally, suppose that G acts on a finite set X, which means that for each $g \in G$ there is a permutation $X \to X$ which takes x to gx, satisfying the identities

$$1x = x, \quad g(hx) = (gh)x, \qquad g, h \in G,\ x \in X.$$

Let V be a complex (respectively, real) vector space having a basis $(e_x)_{x\in X}$ indexed by the elements of X. For $g \in G$ let T_g be a linear map of V into V which sends e_x to e_{gx} for all $x \in X$. The obtained in this way linear representation of G is called *the complex* (respectively, *real*) *permutation representation* associated with X.

1.3.3 Subrepresentation

Let $T : G \to GL(V)$ be a complex (respectively, real) linear representation and let W be a complex (respectively, real) vector subspace of V. Suppose that W is invariant under the action of G, or in other words, $w \in W$ implies that $T_g(w) \in W$ for all $g \in G$, $w \in W$. The restriction T_g^W of T_g to W is an isomorphism of W onto itself, and we have $T_{gh}^W = T_g^W \cdot T_h^W$ thus $T^W : G \to GL(W)$ is also a complex (respectively, real) linear representation of G in W. In such a case W is said to be a *complex (*repectively, *real) subrepresentation* of V.

Example 1.3.1. *Let V be the complex (respectively, real) regular representation of a finite group G and let W be the subspace of dimension 1 of V spanned by the element $x = \sum_{g \in G} e_g$. It is clear that we have $T_g(x) = x$ for all $g \in G$. Consequently W is a complex (respectively, real) subrepresentation of V, isomorphic to the complex (respectively, real) trivial (unit) representation.*

Theorem 1.3.2. *(cf. [36]) Let $T : G \to GL(V)$ be a complex (respectively, real) linear representation of G in V and let W be a complex (respectively, real) vector subspace of V invariant under G. Then there exists an algebraic complement W^c of W in V which is invariant under G and $V = W \oplus W^c$.*

1.3.4 Irreducible Representations

Let $T : G \to GL(V)$ be a complex (respectively, real) linear representation of G, we say that it is *irreducible* or *simple* if no vector subspace of V is invariant under G, except of $\{0\}$ and V. In other words V is irreducible if it is not the direct sum of two proper subrepresentations. Note that any complex (respectively, real) representation of dimension 1 is evidently irreducible.

Example 1.3.3. *Every irreducible complex representation of an abelian group is one dimensional.* [1]

Theorem 1.3.4. *(Complete Reducibility Theorem) (cf. [36])*

Every representation is a direct sum of irreducible representations.

Definition 1.3.5. *Let V be a finite dimensional complex (respectively, real) representation of G. An Hermitian inner product (respectively, inner product), $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ (respectively, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$), is called G-invariant if $\langle gu, gv \rangle = \langle u, v \rangle$ for all $g \in G, u, v \in V$. A representation together with a G-invariant inner product is called an unitary (respectively, orthogonal) representation.*

[1]This fact results directly from the Schur's lemma

1.3.5 Character Theory

Let V be a vector space having a basis (e_i) of n elements, and let A be a linear map of V into itself, with matrix (a_{ij}). By the trace of A we mean the scalar $Tr(A) = \sum_{i=1}^{n} a_{ii}$.

Let $T : G \to GL(V)$ be a linear complex (respectively, real) representation of a finite group G in the complex (respectively, real) vector space V. We define the function $\chi_T : G \to \mathbb{C}$ such that $\chi_T(g) = Tr(T_g)$ for all $g \in G$. This complex (respectively, real) valued function χ_T on G is called the *character* of the representation T. The importance of this function comes primarily from the fact that it allows to characterize the irreducible representations T of G.

Proposition 1.3.6. *(cf. [46]) If χ is the character of a unitary representation T of dimension n, then we have;*

(i) $\chi(1) = n,$

(ii) $\chi(g^{-1}) = \chi(g)^*,$ *for* $g \in G$

(iii) $\chi(ghg^{-1}) = \chi(h),$ *for* $g, h \in G.$

Definition 1.3.7. *Let V_1 and V_2 be two complex representations of the group G. A morphism $A : V_1 \to V_2$ is a linear map that is equivariant, i.e. $A(gv) = gA(v)$ for $g \in G$, $v \in V_1$. We denote by $L_G(V_1, V_2)$ the set of all equivariant morphisms from V_1 to V_2.*

Proposition 1.3.8. *(cf. [46]) Let $T_1 : G \to GL(V_1)$ and $T_2 : G \to GL(V_2)$ be two linear complex representations of G, and let χ_1 and χ_2 be their characters respectively. Then the character χ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$.*

Proposition 1.3.9. Schur's lemma *(cf. [36]) Let V_1 and V_2 be two irreducible complex representations of G. Then we have the following:*

(i) A morphism $A : V_1 \to V_2$ is either zero or an isomorphism,

(ii) Let $A : V_1 \to V_1$ be a morphism. Then there exists $\lambda \in \mathbb{C}$ such that $A(v) = \lambda v$ for every $v \in V_1$.

Assume that G is a finite group and Φ and Ψ are two complex-valued functions on G, we put

$$(\Phi|\Psi) = \frac{1}{n}\sum_{g\in G}\Phi(g)\Psi(g)^*,$$

n being the order of G. This function satisfies the properties of an Hermitian inner product in the vector space of all complex-valued characters from G to $\mathbb{C}$.

Theorem 1.3.10. *(cf. [46])*

(i) If χ is the character of an irreducible complex representation of G, we have $(\chi|\chi)$=1 (i.e. χ is of norm 1),

(ii) If χ and χ' are the characters of two non-isomorphic irreducible complex representations, then we have $(\chi|\chi') = 0$ (i.e. χ and χ' are orthogonal).

Theorem 1.3.11. *(cf. [46]) Let V be a linear complex representation of G, with character Φ, and suppose V can be decomposed into a direct sum of irreducible complex subrepresentations $W_1, W_2, \cdots, W_k$, i.e.*

$$V = W_1 \oplus \cdots \oplus W_k.$$

Then if W is an irreducibe complex representation with character χ, the number of W_i isomorphic to W is equal to the inner product $(\Phi|\chi)$.

Corollary 1.3.12. *Suppose $\chi_1, \cdots, \chi_N$ are the distinct characters of N irreducible complex representations $W_1, \cdots, W_N$, then $V = m_1W_1 \oplus \cdots \oplus m_NW_N$ for any complex representation V of G, where $mW = \underbrace{W \oplus \cdots \oplus W}_{m}$, and the character of V,*

$\Phi = m_1\chi_1 + \cdots + m_h\chi_N$, $m_i = (\Phi|\chi_i)$, and $(\Phi|\Phi) = \sum_{i=1}^{i=N} m_i{}^2$.

Corollary 1.3.13. *Let T be a linear complex representation with the character χ. Then the number of trivial (unit) subrepresentations contained in T is equal to $(\chi|I) = \frac{1}{n}\sum_{g\in G}\chi(g)$, $n = |G|$.*

1.3.6 Decomposition of Regular Complex Representations

Proposition 1.3.14. *(cf.[46]) The character χ of the regular complex representation V of G is given by the formulas:*

(i) $\chi(g) = 0, \qquad g \neq 1$

(ii) $\chi(1) = n = |G| = \dim V$.

Corollary 1.3.15. *Suppose that* $W_1, W_2, \cdots, W_k$ *are all irreducible complex representations of* G *and* $\dim W_i = n_i$ *and let* $n = |G|$. *Then*

(a) Every irreducible complex representation W_i *is contained in the regular complex representation of* G *with multiplicity equal to its dimension* n_i,

(b) The dimensions n_i *of the complex representation* W_i *satisfy the relation* $\sum_{i=1}^{N} n_i^2 = n$,

(c) For $g \in G$, $g \neq 1$, *we have* $\sum_{i=1}^{N} n_i \chi_i(g) = 0$.

1.3.7 Number of Irreducible Complex Representations

Theorem 1.3.16. *(cf.[46]) The number of irreducible complex representations of* G *is equal to the number of the conjugacy classes in* G.

Proposition 1.3.17. *(cf. [46]) Let* $g \in G$, *and let* $C(g)$ *be the number of elements in the conjugacy class of* g. *Then*

(a) $\sum_{i=1}^{N} \chi_i(g)^* \chi_i(g) = \frac{n}{C(g)}$,.

(b) For $h \in G$ *not conjugate to* g, *we have* $\sum_{i=1}^{N} \chi_i(g)^* \chi_i(h) = 0$.

1.3.8 Isotypical Decomposition of Complex Representations

Let $T : G \to GL(V)$ be a linear complex representation of G. We are going to define a unique direct sum of certain representations of V.

Let $\chi_1, \cdots, \chi_N$ be distinct characters of the irreducible complex subrepresentations $W_1, \cdots, W_N$ of G and $n_1, \cdots, n_N$ their dimensions. Let $V = U_1 \oplus \cdots \oplus U_m$ be a decomposition of V into a direct sum of irreducible complex representations. For $i = 1, \cdots, N$ denote by V_i the direct sum of those U_l which are isomorphic to W_i. We put $V_l = \{0\}$ if there is no such component U_l. Clearly we have

$$V = V_1 \oplus \cdots \oplus V_N, \tag{1.1}$$

the composition (1.1) (which is unique) is called the *isotypical decomposition* of V and V_l is called the *isotypical component* of V associated with the irreducible representation W_l.

Remark 1.3.18. *Here we are discussing the case of complex representations, but in the real case, of course, a similar construction leads to the isotypical decomposition of real representations.*

1.3.9 Examples

Dihedral group D_N: The dihedral group D_N of order $2N$, is the symmetry group of N-sided regular polygon for $N > 1$. We consider the elements of D_N to be real 2×2-matrices (e.g. the element $\gamma = a + ib$ represents the matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$). The dihedral group D_N is generated by $\mathbb{Z}_N$, the cyclic group of order N, which the generators being the rotation $\gamma = e^{\frac{2\pi i}{N}}$, together with the reflection $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let's classify the irreducible representation for some dihedral groups D_N, and use them to describe the isotypical decomposition of some representations of D_N.

(i) **D_3**

The dihedral group $D_3 = \{1, \gamma, \gamma^2, \kappa, \kappa\gamma, \kappa\gamma^2\}$ has three conjugacy classes: $x_1 = \{1\}$, $x_2 = \{\gamma, \gamma^2\}$, and $x_3 = \{\kappa, \kappa\gamma, \kappa\gamma^3\}$. The characters of D_3 are presented in Table 1.1.

χ_i	x_1	x_2	x_3
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

Table 1.1: Representations of D_3.

There exist three non-isomorphic complex irreducible representations V_1, V_2 with dimension 1, and V_3 with dimension 2. The isotypical decomposition of the regular representation V of D_3 is:

$$V = V_1 \oplus V_2 \oplus V_3.$$

(ii) **D_4**

The dihedral group $D_4 = \{1, \gamma, \gamma^2, \gamma^3, \kappa, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3\}$ has 5 conjugacy classes: $x_1 = \{1\}$, $x_2 = \{\gamma, \gamma^3\}$, $x_3 = \{\gamma^2\}$, $x_4 = \{\kappa, \kappa\gamma^2\}$, and $x_5 = \{\kappa\gamma, \kappa\gamma^3\}$. The characters of D_4 are presented in Table 1.2.

χ_i	x_1	x_2	x_3	x_4	x_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	-1	1	1	-1
χ_4	1	-1	1	-1	1
χ_5	2	0	-2	0	0

Table 1.2: Representations of D_4.

There exist five non-isomorphic irreducible complex representations V_1, V_2, V_3, V_4 with dimension 1 and V_5 with dimension 2. The isotypical decomposition of the regular representation V of D_4 is

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5.$$

(iii) **D_5**

The dihedral group $D_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \kappa, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4\}$ has 4 conjugacy classes: $x_1 = \{1\}$, $x_2 = \{\gamma, \gamma^4\}$, $x_3 = \{\gamma^2, \gamma^3\}$, $x_4 = \{\kappa, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4\}$. The characters of D_5 are presented in Table 1.3.

There exist four non-isomorphic irreducible complex representations V_1, V_2 with dimension 1 and V_3, V_4 with dimension 2. The isotypical decomposition of the regular representation V of D_5 is

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

χ_i	x_1	x_2	x_3	x_4
χ_1	1	1	1	1
χ_2	1	1	1	-1
χ_3	2	$\frac{\sqrt{5}-1}{4}$	$-\frac{\sqrt{5}+1}{4}$	0
χ_4	2	$-\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{5}-1}{4}$	0

Table 1.3: Representations of D_5.

(iv) **A$_4$**

The alternating group A_4 is the group of even permutations of four symbols $\{1, 2, 3, 4\}$. This group is also called the *tetrahedral* group, because it is isomorphic to the group of symmetries (preserving orientation) of a regular tetrahedron. Every permutation $\sigma \in A_4$ can be written as a composition of cycles $(n_1 n_2 \cdots n_k)$, $(k = 1, 2, 3)$. By using this notation, we divide the elements of A_4 into four conjugacy classes:

$$x_1 = \{(1)\},\ x_2 = \{(12)(34), (13)(24), (14)(23)\},$$

$$x_3 = \{(123), (142), (134), (243)\},\ x_4 = \{(132), (124), (143), (234).$$

The following is the character table of A_4:

χ_i	x_1	x_2	x_3	x_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

Table 1.4: Representations of A_4

where $\omega = e^{\frac{2\pi i}{3}}$. There exist four non-isomorphic irreducible complex representations V_1, V_2, V_3 with dimension 1 and V_4 with dimension 3. The isotypical decomposition of the permutation representation V of A_4 is

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

(v) S_4

The permutation group S_4 of four symbols $\{1, 2, 3, 4\}$ is isomorphic to the *octahedral* group of symmetries (preserving orientation) of a regular cube. By using the notation of elements in S_4 as compositions of cycles, we can list the following five conjugacy classes of elements in S_4:
$x_1 = \{(1)\}$, $x_2 = \{(12), (13), (14), (23), (24), (34)\}$,
$x_3 = \{(12)(34), (13)(24), (14)(23)\}$,
$x_4 = \{(123), (142), (134), (243), (132), (124), (143), (234)\}$, and
$x_5 = \{(1234), (1342), (1324), (1243), (1423), (1432)\}$.

The following is the character table of S_4:

χ_i	x_1	x_2	x_3	x_4	x_5
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

Table 1.5: Representations of S_4

There exist five non-isomorphic irreducible complex representations V_1, V_2 with dimension 1, V_3 with dimension 2 and V_4, V_5 with dimension 3. The isotypical decomposition of the permutation representation V of S_4 is

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5.$$

(vi) **A_5**

The alternation group A_5 is the group of even permutations of five symbols $\{1,2,3,4,5\}$, which is isomorphic to the icosahedral group $\mathbb{I}$ consisting of the symmetries (preserving orientation) of a regular icosahedron. The group A_5 has 60 elements. There are five conjugacy classes of the elements in A_5, which are listed explicitely in Table 1.6.

#1	#2	#3	#4	#5
(1)	(12)(35), (13)(25)	(123), (125)	(12345)	(21345)
	(12)(35), (13)(45)	(132), (152)	(12354)	(21354)
	(12)(45), (14)(23)	(124), (234)	(12435)	(21435)
	(13)(24), (14)(25)	(142), (243)	(12453)	(21453)
	(14)(35), (15)(23)	(235), (253)	(12534)	(21534)
	(14)(25), (14)(35)	(345), (354)	(12543)	(21543)
	(15)(23), (15)(34)	(134), (143)	(13425)	(31425)
	(15)(24), (25)(34)	(135), (153)	(13524)	(31524)
	(24)(35), (23)(45)	(145), (154)	(14325)	(41325)
		(245), (254)	(14523)	(41523)
			(15324)	(51324)
			(15423)	(51423)

Table 1.6: Conjugacy classes of elements in A_5.

The following is the character table of A_5:

χ_i	x_1	x_2	x_3	x_4	x_5
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

Table 1.7: Representations of A_5.

There exist five non-isomorphic irreducible complex representations V_1 with dimension 1, V_2 with dimension 4, V_3 with dimension 5 and V_4, V_5 with dimension 3. The isotypical decomposition of the permutation representation V of A_5 is

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus V_5.$$

1.3.10 Irreducible Real Representations of $\Gamma \times S^1$

Let Γ be a finite group and V_k be a complex irreducible representation of Γ. Since the group $S^1 \subset \mathbb{C}$ acts on V_k by complex multiplication, the irreducible complex representation V_k of Γ leads to a real representation of $\Gamma \times S^1$, with the action of $\Gamma \times S^1$ defined by $(\gamma, z)v = \gamma(z^j \cdot v)$ for $j \geq 1$, where $(\gamma, z) \in \Gamma \times S^1$, $v \in V_k$. The space V_k equipped with this action of $\Gamma \times S^1$, will be denoted by $V_{k,j}$.

Chapter 2

Equivariant Degree Theory: Construction and Basic Properties

In this chapter we present a construction of the G-equivariant degree theory with one free parameter for $G = \Gamma \times S^1$, where Γ is a finite group. This construction can be generalized to the case of any compact lie group G (cf. [28]) but we do not need such generality in this thesis. We also introduce the definition of the Burnside ring $A(G)$, and provide several examples.

2.1 Construction of Equivariant Degree

Let U be an Euclidean space and X a subset in U. We will use the following notations throughout this section:

$$B(X) = \{x \in X \mid \|x\| < 1\},$$
$$\overline{B}(X) = \{x \in X \mid \|x\| \leq 1\},$$
$$S(X) = \{x \in X \mid \|x\| = 1\}.$$

Assume that V is an orthogonal G-representation, and consider the space $W := \mathbb{R} \oplus V$, where G acts trivially on $\mathbb{R}$. Let $\Omega \subset W$ be an open bounded G-invariant set.

Let $f : W \to V$ be an Ω-admissible G-equivariant map, i.e. $f(gx) = gf(x)$ for $x \in W$ and $g \in G$, then there exists a G-invariant neighborhood $\mathcal{N}$ of $\partial\Omega$ such that $f(x) \neq 0$ for all $x \in \mathcal{N}$. We put $\Omega_{\mathcal{N}} := \Omega \cup \mathcal{N}$ and suppose that $R > 0$ is a real number such that $\overline{\Omega_{\mathcal{N}}} \subset B_R(0) := \{x \in W \mid \|x\| < R\}$. Assume that $\eta : \overline{B_R(0)} \to \mathbb{R}$

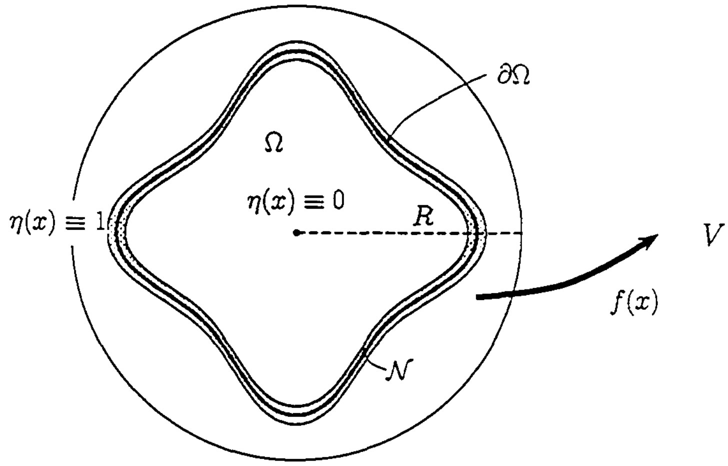


Figure 2.1: Invariant Set Ω and Equivariant Map f

is an G-invariant Urysohn function such that (cf. Figure 2.1)

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \notin \Omega_{\mathcal{N}}. \end{cases} \tag{2.1}$$

We define $F : [-1, 1] \times \overline{B_R(0)} \to \mathbb{R} \oplus V$ by

$$F(t, x) = (t + 2\eta(x), f(x)), \quad (t, x) \in [-1, 1] \times \overline{B_R(0)}. \tag{2.2}$$

Since $F^{-1}(0, 0) = \{0\} \times (f^{-1}(0) \cap \Omega)$, by identifying $[-1, 1] \times \overline{B_R(0)}$ with $\overline{B}(\mathbb{R}^2 \oplus V)$, we get the equivariant map

$$F : (\overline{B}(\mathbb{R}^2 \oplus V), S(\mathbb{R}^2 \oplus V)) \to (\mathbb{R} \oplus V, \mathbb{R} \oplus V \setminus \{0\}).$$

Now by taking the equivariant homotopy class of F (denoted by $[F]$), we obtain the element

$$[F] \in [\overline{B}(\mathbb{R}^2 \oplus V), S(\mathbb{R}^2 \oplus V); \mathbb{R} \oplus V, \mathbb{R} \oplus V \setminus \{0\}]^G =: \Pi_1^G,$$

where by $[X, A; Y, B]^G$ we denote the set of all G-equivariant homotopy classes of maps from (X, A) to (Y, B). It is well known (cf. [29]) that the suspension homomorphism ξ from $\Pi_N^G := \left[\overline{B}(\mathbb{R}^{N+1} \oplus V), S(\mathbb{R}^{N+1} \oplus V); \mathbb{R}^N \oplus V, \mathbb{R}^N \oplus V \setminus \{0\}\right]^G$ to $\Pi_{N+1}^G := \left[\overline{B}(\mathbb{R}^{N+2} \oplus V), S(\mathbb{R}^{N+2} \oplus V); \mathbb{R}^{N+1} \oplus V, \mathbb{R}^{N+1} \oplus V \setminus \{0\}\right]^G$ defined by $\xi([F]) = [\mathrm{Id}_{\mathbb{R}} \times F]$, is an isomorphism (by Freudenthal Theorem (cf. [32])) for N sufficiently large. We put $\Pi^G := \Pi_N^G$. The group Π^G, which is the *stable equivariant homotopy group of sphere*, is the range for the equivariant degree.

By taking the class of $N-1$-suspension of F, we define

$$\deg_G(f,\Omega) := [\mathrm{Id}_{\mathbb{R}^{N-1}} \times F], \tag{2.3}$$

and we call it *G-equivariant degree* of f in Ω. The equivariant degree introduced above satisfies all the properties expected from any reasonable degree theory. In particular, we have the following result:

Theorem 2.1.1. (cf. [32]) *The G-equivariant degree $\deg_G(f,\Omega)$ has the following properties:*

(i) EXISTENCE: *If $\deg_G(f,\Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$,*

(ii) ADDITIVITY: *If $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two disjoint open invariant subsets of Ω then*

$$\deg_G(f,\Omega) = \deg_G(f,\Omega_1) + \deg_G(f,\Omega_2),$$

(iii) HOMOTOPY: *If $f_t : [0,1] \times V \to W$ is an equivariant homotopy of Ω-admissible maps then $\deg_G(f_t,\Omega) =$constant,*

(iv) SUSPENSION: $\deg_G(\mathrm{Id} \times f, (-1,1) \times \Omega) = \deg_G(f,\Omega)$,

(v) EXCISION: *If $f^{-1}(0) \cap \Omega \subset \Omega_o$, where $\Omega_o \subset \Omega$ is an invariant open subset of Ω, then*

$$\deg_G(f,\Omega) = \deg_G(f,\Omega_o),$$

(vi) HOPF PROPERTY: *Assume $\Omega = B(V)$ is the unit ball in V and f_1, f_2 are two $B(V)$-admissible G-equivariant maps with $\deg_G(f_1, B(V)) = \deg_G(f_2, B(V))$. Then for N big enough, $Id_{R^N} \times F_1$ and $Id_{R^N} \times F_2$ are G-equivariantly homotopic by a $B(\mathbb{R}^N) \oplus B(V)$-admissible homotopy.*

2.2 Regular Normal Approximation of Equivariant Mappings

2.2.1 Normal Maps: Definition and Examples

Assume that V is a real, finite-dimensional, orthogonal G-representation and $W = \mathbb{R} \oplus V$, where G acts on $\mathbb{R}$ trivially. For $x \in W$, we denote by G_x the subgroup of G defined by $G_x := \{g \in G|\ gx = x\}$. The subgroup G_x is a closed subgroup of G and is called the *isotropy group* of x. The set $G(x) := \{gx|g \in G\}$ is called the *orbit* of x. The orbit $G(x)$ is homeomorphic to G/G_x. Since $G_{gx} = g^{-1}G_x g$ it follows that the conjugacy class (G_x) of G_x describes the orbit $G(x)$.

In what follows, for a given $x \in W$, we will call (G_x) the *orbit type* of x. It is easy to verify that $V^G := \{x \in V; gx = x \text{ for all } g \in G\}$ is a linear subspace of V, called the subspace of *G-fixed points.*

For a G-invariant set $\Omega \subset W$ and a closed subgroup H of G, we put

$$\begin{aligned} \Omega^H &:= \{x \in \Omega |\ hx = x \text{ for all } h \in H\}, \\ \Omega_H &:= \{x \in \Omega |\ G_x = H\}, \\ \Omega^{(H)} &:= \{x \in \Omega |\ (G_x) \geq (H)\}, \\ \Omega_{(H)} &:= \{x \in \Omega |\ (G_x) = (H)\}, \\ \mathcal{J}(\Omega) &:= \{(G_x) |\ x \in \Omega\} \subseteq \Phi(G). \end{aligned}$$

The set $\mathcal{J}(\Omega)$ is called the set of the *orbit types* in Ω, and (H) for a subgroup $H \subseteq G$ such that $H = G_x$, for some $x \in \Omega$ is called an *orbit type* in Ω. It is well known that Ω_H is open and dense in Ω^H (cf. [9]). Moreover, $\mathcal{J}(\Omega)$ is a finite set, partially ordered by the relation $\leq$ defined for the set $\Phi(G)$ (see section 1.2). It is also well known (see [9]) that the set $\Omega_{(H)}$ is a submanifold of W. We will denote by $\tau(\Omega_{(H)})$ the *tangent bundle* to $\Omega_{(H)}$ and by $\nu(\Omega_{(H)})$ the *normal bundle* to $\Omega_{(H)}$ in W.

Definition 2.2.1. Let $\Omega \subset W$ be an open bounded G-invariant set and $f : W \to V$ an Ω-admissible G-equivariant map. We say that f satisfies the *normality* condition at $x \in \Omega$ if there exists $\delta_x > 0$ such that for all $v \in \nu_x(\Omega_{(H)})$ i.e. $v \perp \tau_x(\Omega_{(H)})$, where $H = G_x$, with $\|v\| < \delta_x$ and $x + v \in \Omega$, we have

$$f(x+v) = f(x) + v. \tag{2.4}$$

Definition 2.2.2. *Let $\Omega \subset W$ be an open, bounded, G-invariant set and $f : W \to V$ an Ω-admissible G-map, We say that f is normal if for every $\alpha = (H) \in \mathcal{J}(\Omega)$ and every $x \in f^{-1}(0) \cap \Omega_{(H)}$, the α-normality condition at x is satisfied (see Figure 2.2); i.e. there exists $\delta_x > 0$ such that for all $v \in \nu_x(\Omega_{(H)})$, $\|v\| < \delta_x$, then*

$$f(x+v) = f(x) + v = v. \tag{2.5}$$

Similarly, an Ω-admissible G-homotopy $h : [0,1] \times W \to V$ is called a normal homotopy in Ω, if for every $(H) \in \mathcal{J}(\Omega)$ and for every $(t,x) \in h^{-1}(0) \cap ([0,1] \times \Omega_{(H)})$, the following α-normality condition at (t,x) is satisfied, i.e. there exists $\delta_{(t,x)} > 0$ such that for all $v \in \nu_{(t,x)}([0,1] \times \Omega_{(H)})$ with $\|v\| < \delta_{(t,x)}$,

$$h(t, x+v) = h(t,x) + v = v. \tag{2.6}$$

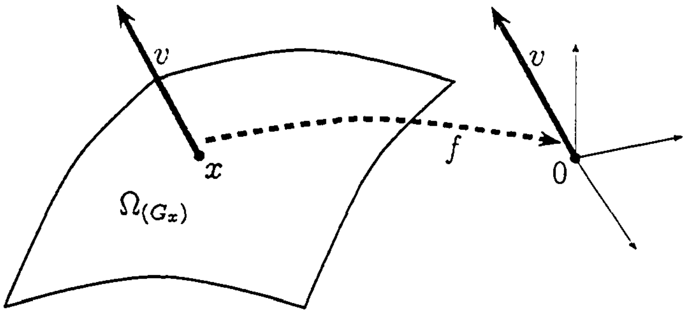


Figure 2.2: Normal Map

Example 2.2.3. Consider the antipodal action of $G = \mathbb{Z}_2$ on the space $V = \mathbb{R}$. Then the map $f : V \to V$ defined by

$$f(x) = \begin{cases} x & \text{if } |x| < 1, \\ 2 - x & \text{if } x \geq 1, \\ -(2 + x) & \text{if } x \leq -1, \end{cases}$$

is a G-equivariant normal map. Indeed, there are only two orbit types in V; $(H), (K)$ with $H = \mathbb{Z}_2$ and $K = \{1\}$. Since $(V^H)^\perp = \mathbb{R}$ and for $|x| < 1$ we have that $f(x) = x$, it follows that f satisfies $(\mathbb{Z}_2)$-normality condition. On the other hand $(V^K)^\perp = \{0\}$, thus it also satisfies the (K)-normality condition.

In the following, we consider an orthogonal representation V of G and its set of orbit types

$$\mathcal{J}(V) = \{(H_1), (H_2), \dots, (H_l)\}.$$

To simplify the notation, we put $(H_k) = \alpha_k$ for $k = 1, 2, \dots, l$.

Theorem 2.2.4. (Normal Approximation Theorem) *(cf. [36]) Assume that $\Omega \subset W$ is a bounded open G-invariant subset and let $f : W \to V$ be an Ω-admissible G-equivariant map. Then for every $\eta > 0$ there exists a normal (in Ω) G-equivariant map $\widetilde{f} : W \to V$ such that*

$$\sup_{x \in \Omega} \left\| \widetilde{f}(x) - f(x) \right\| < \eta.$$

Theorem 2.2.4 can be extended to the case of Ω-admissible homotopies.

Theorem 2.2.5. (Normal Approximation Theorem for Homotopies) *(cf. [36]) Let $\Omega \subset W$ be an open bounded invariant set and $h : [0, 1] \times W \to V$ be an Ω-admissible homotopy. Then for every $\eta > 0$ there exists a normal G-homotopy*

$\tilde{h} : [0,1] \times W \to V$ *in* Ω *such that* $\sup_{(t,x)\in[0,1]\times\Omega} \|\tilde{h}(t,x) - h(t,x)\| < \eta$. *In addition, if* $h_0 := h(0,\cdot)$ *and* $h_1 := h(1,\cdot)$ *are normal in* Ω, *then* $\tilde{h}_0 = h_0$ *and* $\tilde{h}_1 = h_1$.

2.2.2 Regular Normal Approximations

Definition 2.2.6. Let $\Omega \subset W$ be an open bounded G-invariant set and let $f : W \to V$ be an Ω-admissible G-equivariant map. We say that f is a *regular normal* map in Ω if:

(i) f is of class C^1,

(ii) f is normal in Ω,

(iii) for every $\alpha \in \mathcal{J}(f^{-1}(0) \cap \Omega)$, $\alpha = (H)$, zero is a regular value of

$$f_H := f_{|\Omega_H} : \Omega_H \to V^H.$$

where $\Omega_H = \Omega \cap W_H$. Notice that Ω_H is not G-invariant, in general, but it is $N(H)$-invariant. Therefore, it is $W(H)$-invariant, where $W(H) := \frac{N(H)}{H}$ is called the *Weyl group* of H in G.

Similarly, an Ω-admissible G-equivariant homotopy $h : [0,1] \times W \to V$ is called *regular normal* homotopy in Ω if:

(i) h is of class C^1,

(ii) h is a normal homotopy in Ω,

(iii) for every $\alpha \in \mathcal{J}(h^{-1}(0) \cap [0,1] \times \Omega)$, $\alpha = (H)$, zero is a regular value of the maps h_H, $(h_0)_H$ and $(h_1)_H$, where

$$h_H := h_{|[0,1]\times\Omega_H} : [0,1] \times \Omega_H \to V^H,$$
$$(h_0)_H := h_{0|\Omega_H} : \Omega_H \to V^H,$$
$$(h_1)_H := h_{1|\Omega_H} : \Omega_H \to V^H.$$

Theorem 2.2.7. (Regular Normal Approximation Theorem) *(cf. [36]) Let* $\Omega \subset W$ *be an open bounded* G*-invariant set and* $f : W \to V$ *an* Ω*-admissible* G*-equivariant map. Then for every* $\eta > 0$ *there exists a regular normal (in* Ω*)* G*-equivariant map* $\tilde{f} : W \to V$ *such that* $\sup_{x\in\Omega} \|\tilde{f}(x) - f(x)\| < \eta$. *Similarly, if* $h : [0,1] \times W \to V$ *is an* Ω*-admissible* G*-equivariant homotopy, then for every* $\eta > 0$ *there exists a regular* *normal (in Ω) G-homotopy $\tilde{h} : [0,1] \times W \to V$ such that* $\sup_{(t,x)\in[0,1]\times\Omega} \|h(x,t) - \tilde{h}(x,t)\| < \eta$. *In addition, if h_0 and h_1 are regular normal in Ω, then $\widetilde{h_0} = h_0$ and $\widetilde{h_1} = h_1$.*

2.3 Computation and Decomposition of the Group Π^G

The goal of this section is to describe the group Π^G. We will look at the group Π^G as the set of values for the G-equivariant degree, so we can describe the equivariant homotopy classes in Π^G, by studying the G-equivariant degree of regular normal representatives of these classes. Notice that, by the definition of Π^G, we are, in fact, dealing with free equivariant homotopy classes, which are very awkward for studying the group structure of Π^G. However, the equivariant degree, and in particular, its additivity property, provides us with a geometric method for the computation of the (abstract) group structure of Π^G.

2.3.1 Equivariant Degree Techniques

A: Let $a \in \Pi^G$ and $f : \mathbb{R}^N \oplus W \to \mathbb{R}^N \oplus V$ be a $B(\mathbb{R}^N \oplus W)$-admissible map such that the equivariant homotopy class of f is exactly a. In what follows we will always assume that N is large enough. Then obviously, from the construction of the G-equivariant degree, the class a is equal to $\deg_G(f, B(\mathbb{R}^N \oplus W))$. Now we can take advantage of the equivariant degree properties. Suppose that $Z := f^{-1}(0)$ and let K be a compact invariant subset of Z such that there exists an open invariant subset Ω of $B(\mathbb{R}^N \oplus W)$ satisfying $\Omega \cap Z = K$. Since the map f is Ω-admissible, we can apply the construction described in section 2.1 as follows:

(i) Choose an invariant neighborhood $\mathcal{N}$ of $\partial\Omega$ in $B(\mathbb{R}^N \oplus W)$ such that $f(x) \neq 0$ for $x \in \mathcal{N}$.

(ii) Find an invariant Urysohn function $\eta : \overline{B(\mathbb{R}^N \oplus W)} \to [0,1]$ satisfying

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \notin \mathcal{N} \cup \Omega. \end{cases}$$

(iii) We can identify $[-1,1] \times \overline{B(\mathbb{R}^N \oplus W)}$ with $\overline{B(\mathbb{R}^{N+1} \oplus W)}$ and define the G-equivariant map $F : [-1,1] \times \overline{B(\mathbb{R}^N \oplus W)} \to \mathbb{R} \oplus (\mathbb{R}^N \oplus V)$, by

$$F(t,x) = (t + 2\eta(x), f(x)), \tag{2.7}$$

where $(t,x) \in [-1,1] \times \overline{B(\mathbb{R}^N \oplus W)}$.

Notice that

(a) $F(t,x)=0 \Longleftrightarrow t=0$ and $x \in K$.

(b) If f is regular normal in Ω, then F is regular normal in $B(\mathbb{R}^{N+1}\oplus W)$.

(c) If $K=Z$ then $\deg_G(F, B(\mathbb{R}^{N+1}\oplus W)) = \deg_G(f, B(\mathbb{R}^{N}\oplus W)) = a$.

We will call the map F a *localization* of f about the set Ω.

B: Notice that for two elements a and $b \in \Pi^G$, it is always possible to find two representatives f_a, $f_b : \overline{B(\mathbb{R}^N \oplus W)} \to \mathbb{R}^N \oplus V$ such that $f_a^{-1}(0) \cap f_b^{-1}(0) = \emptyset$. Indeed, suppose that f_a', $f_b' : \overline{B(\mathbb{R}^{N'} \oplus W)} \to \mathbb{R}^{N'} \oplus V$ be two representatives of a and b, then we can define $f_a, f_b : [-1,1] \times \overline{B(\mathbb{R}^{N'} \oplus W)} \to \mathbb{R} \oplus \left(\mathbb{R}^{N'} \oplus V\right)$ by

$$f_a(t,x) = (t - \frac{1}{2}, f_a'(x)), \qquad f_b(t,x) = (t + \frac{1}{2}, f_b'(x)),$$

where $(t,x) \in [-1,1] \times \overline{B(\mathbb{R}^{N'} \oplus W)}$. Clearly, the zeros of f_a and f_b are separated. We will call this procedure a *separation of zeros.* Notice that if f_a' and f_b' are regular normal, then f_a and f_b are also regular normal.

C: If f_a, $f_b : \overline{B(\mathbb{R}^N \oplus W)} \to \mathbb{R}^N \oplus V$ are such that $f_a^{-1}(0) \cap f_b^{-1}(0) = \emptyset$ then we can find two open invariant sets Ω_a and Ω_b such that $f_a^{-1}(0) \subset \Omega_a$, $f_b^{-1}(0) \subset \Omega_b$ and $\overline{\Omega_a} \cap \overline{\Omega_b} = \emptyset$. Then we put $\Omega = \Omega_a \cup \Omega_b$ and define the map $f : \overline{\Omega} \to \mathbb{R}^N \oplus V$ by

$$f(x) = \begin{cases} f_a(x) & \text{if } x \in \overline{\Omega_a}, \\ f_b(x) & \text{if } x \in \overline{\Omega_b}. \end{cases}$$

The map f can be extended equivariantly to $\overline{B(\mathbb{R}^N \oplus W)}$. Let F be a localization of f about Ω, then clearly, by the additivity and suspension properties,

$$\begin{aligned} \deg_G(F, B(\mathbb{R}^{N+1} \oplus W)) &= \deg_G(F, (-1,1) \times \Omega) = \deg_G(f, \Omega) \\ &= \deg_G(f_a, \Omega_a) + \deg_G(f_b, \Omega_b) = a + b. \end{aligned}$$

D: Suppose that $h : [0,1] \times \overline{B(\mathbb{R}^N \oplus W)} \to \mathbb{R}^N \oplus V$ is a regular normal homotopy of $B(\mathbb{R}^N \oplus W)$-admissible maps. Since the set $Z = \{(t,x) : h(t,x) = 0\} \subset [0,1] \times B(\mathbb{R}^N \oplus W)$ is compact and invariant, it follows that the set $K = \pi(Z)$, where $\pi : [0,1] \times B(\mathbb{R}^N \oplus W) \to B(\mathbb{R}^N \oplus W)$ is the natural projection, is also compact and invariant. Let $\mathcal{J}(Z) = \{(H_1), (H_2), \ldots, (H_k)\}$. Since h is regular normal, the sets $Z_{(H_l)}$ are compact, so the set K can be represented as a union $K = \bigcup_{l=1}^k K_{(H_l)}$, where each of the sets $K_{(H_l)}$ is compact and invariant. Therefore, there exist invariant disjoint open neighborhoods Ω_l of the sets $K_{(H_l)}$. Let $\Omega := \bigcup_{l=1}^{k} \Omega_l$. Notice that $Z \subset [-1,1] \times \Omega$, and $Z_{(H_l)} \subset [-1,1] \times \Omega_l$ for $l = 1, \dots, k$, so h is also an Ω-admissible regular normal homotopy and Ω_l-admissible homotopy for each $l = 1, \dots, k$. We will call this procedure *restricting a regular normal homotopy to an orbit type* (H_l).

2.3.2 Decomposition of the Group Π^G

Definition 2.3.1. For every orbit type (H) in W we define the subset $\Pi(H)$ of Π^G which consists of all elements $a \in \Pi^G$ such that there exists a regular normal map $f : \mathbb{R}^N \oplus W \to \mathbb{R}^N \oplus V$, with the following properties:

(i) f is $B(\mathbb{R}^N \oplus W)$-admissible,

(ii) $f^{-1}(0) \cap B(\mathbb{R}^N \oplus W) = \left(f^{-1}(0) \cap B(\mathbb{R}^N \oplus W)\right)_{(H)}$,

(iii) $\deg_G(f, B(\mathbb{R}^N \oplus W)) = a$.

Since a constant non-zero equivariant map clearly satisfies the conditions (i)–(iii) of Definition 2.3.1, the element 0 belongs to $\Pi(H)$ for every orbit type (H) in W. We have the following:

Theorem 2.3.2. *(cf. [5]) For every orbit type $\alpha = (H)$ in W:*

(a) *the set $\Pi(H)$ is a subgroup of Π^G,*

(b) $\Pi(H) = \{0\}$ *if* $\dim W(H) > 1$,

(c) $\Pi^G = \bigoplus_{\dim W(H) \leq 1} \Pi(H)$.

In what follows, we will denote by $a_{(H)}$ the $\Pi(H)$-component of $a \in \Pi^G$. We will also write

$$\deg_G(f, \Omega) = \sum_{(H)} a_{(H)} \in \bigoplus_{(H)} \Pi(H). \tag{2.8}$$

Proposition 2.3.3. *(cf. [5]) Let $f : W \to V$ be an Ω-admissible map such that $\deg_G(f, \Omega) = a \neq 0$, i.e. $a_{(H)} \neq 0$ for some (H). Then there exists $x \in \Omega^H$ such that $f(x) = 0$. In other words, the equation $f(x) = 0$ has a solution x in Ω with symmetries at least H, i.e. $G_x \supset H$.*

2.4 Burnside Ring A(G)

In this section we introduce the definition of the *Burnside ring* $A(G)$, and provide several examples. Let $\Phi(G)$ denote the set of conjugacy classes (H) such that the Weyl group $W(H)$ is finite. We denote by $A(G)$ the free abelian group generated by $(H) \in \Phi(G)$. There is a *multiplication* operation on $A(G)$ defining in $A(G)$ a structure of a ring with identity. The multiplication is given by

$$(H) \cdot (K) = \sum_{(L)\in\Phi(G)} n_L(L),$$

where

$$n_L = \left[n(L,H) \cdot n(L,K)|W(H)| \cdot |W(K)| - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L})n_{\widetilde{L}}|W(\widetilde{L})|\right]/|W(L)|,$$

and the integer $n(L,H)$ denotes the number of conjugate copies of H containing the subgroup L.

Example 2.4.1. Dihedral group D_n

(a) $n = 3$

In this case, the subgroups of the group

$$D_3 = \{1, \gamma, \gamma^2, \kappa, \kappa\gamma, \kappa\gamma^2\},$$

where $\gamma = e^{\frac{2\pi i}{3}} \in \mathbb{C}$, are represented by

$$\mathbb{Z}_1 = \{1\}, \quad \mathbb{Z}_3 = \{1, \gamma, \gamma^2\}, \quad D_1 = \{1, \kappa\} \sim \{1, \kappa\gamma\} \sim \{1, \kappa\gamma^2\}.$$

It is easy to notice that in this case $N(D_1) = D_1$ and $N(\mathbb{Z}_3) = N(\mathbb{Z}_1) = D_3$, which implies that $W(D_1) = \mathbb{Z}_1$, $W(\mathbb{Z}_3) = \mathbb{Z}_2$, and $W(\mathbb{Z}_1) = D_3$.

Following is the lattice of conjugacy classes in D_3:

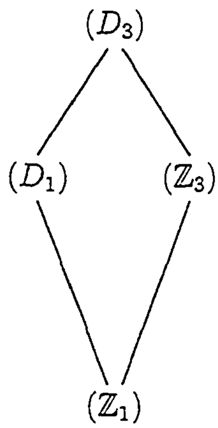


Figure 2.3: Lattice of Conjugacy Classes in D_3.

L	H	n(L,H)
$\mathbb{Z}_1$	D_3	1
D_1	D_3	1
$\mathbb{Z}_3$	D_3	1
$\mathbb{Z}_1$	$\mathbb{Z}_3$	1
$\mathbb{Z}_1$	D_1	3

Table 2.1: Numbers n(L,H) for subgroups of D_3.

	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
(D_3)	(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$
(D_1)	(D_1)	$(D_1)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$

Table 2.2: Multiplication Table for the Burnside Ring $A(D_3)$.

(b) $n = 4$

In this case, The subgroups of

$$D_4 = \{1, i, -1, -i, \kappa, \kappa i, -\kappa, -\kappa i\},$$

are exactly:
$\mathbb{Z}_4 = \{1, i, -1, -i\}$,
$\mathbb{Z}_2 = \{1, -1\}$,
$\mathbb{Z}_1 = \{1\}$,
$D_2 = \{1, -1, \kappa, -\kappa\}$,
$\tilde{D}_2 = \{1, -1, \kappa i, -\kappa i\}$,
$D_1 = \{1, \kappa\} \sim \{1, -\kappa\}$,
$\tilde{D}_1 = \{1, \kappa i\} \sim \{1, -\kappa i\}$.

Notice that, $N(D_k) = D_{2k}$ and $N(\tilde{D}_k) = \tilde{D}_{2k}$, where $k = 1, 2$ and $N(\mathbb{Z}_k) = D_4$ for $k = 1, 2, 4$, which implies that $W(\mathbb{Z}_k) = D_{\frac{4}{k}}$ and $W(D_k) = W(\tilde{D}_k) = \mathbb{Z}_2$. We have the following lattice of the conjugacy classes of subgroups in D_4:

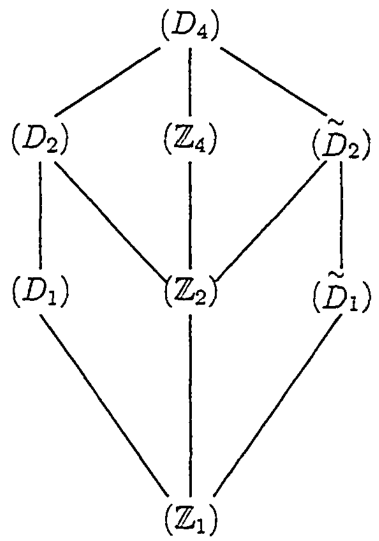


Figure 2.4: Lattice of Conjugacy Classes in D_4.

L	H	$n(L,H)$	L	H	$n(L,H)$
$\mathbb{Z}_1$	D_4	1	$\mathbb{Z}_1$	$\tilde{D}_2$	1
D_1	D_4	1	$\mathbb{Z}_2$	$\mathbb{Z}_4$	1
$\mathbb{Z}_2$	D_4	1	$\mathbb{Z}_1$	$\mathbb{Z}_4$	1
$\tilde{D}_1$	D_4	1	$\mathbb{Z}_1$	D_2	1
D_2	D_4	1	$\mathbb{Z}_2$	D_2	1
$\mathbb{Z}_4$	D_4	1	D_1	D_2	1
$\tilde{D}_2$	D_4	1	$\mathbb{Z}_1$	D_1	2
$\tilde{D}_1$	$\tilde{D}_2$	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	1
$\mathbb{Z}_2$	$\tilde{D}_2$	1	$\mathbb{Z}_1$	$\tilde{D}_1$	2

Table 2.3: Numbers n(L,H) for subgroups of D_4.

	(D_4)	(D_2)	$(\mathbb{Z}_4)$	$(\widetilde{D}_2)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_4)	(D_4)	(D_2)	$(\mathbb{Z}_4)$	$(\widetilde{D}_2)$	(D_1)	$(\widetilde{D}_1)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(D_2)	(D_2)	$2(D_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(D_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
$(\widetilde{D}_2)$	$(\widetilde{D}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\widetilde{D}_2)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
(D_1)	(D_1)	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\widetilde{D}_1)$	$(\widetilde{D}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\widetilde{D}_1)$	$2(\mathbb{Z}_1)$	$2(\widetilde{D}_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$

Table 2.4: Multiplication Table for the $A(D_4)$.

(c) $n = 5$
We have the following subgroups of

$$D_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4\},$$

where $\gamma = e^{\frac{2\pi i}{5}}$:

$$\mathbb{Z}_1 = \{1\}, \quad D_1 = \{1, \kappa\} \sim \{1, \kappa\gamma\} \sim \{1, \kappa\gamma^2\} \sim \{1, \kappa\gamma^3\} \sim \{1, \kappa\gamma^4\},$$

$$\mathbb{Z}_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4\}.$$

In this case, $N(D_1) = D_1$ and $N(\mathbb{Z}_5) = N(\mathbb{Z}_1) = D_5$, which implies that $W(D_1) = \mathbb{Z}_1$, $W(\mathbb{Z}_1) = D_5$ and $W(\mathbb{Z}_5) = \mathbb{Z}_2$. The lattice of subgroups in D_5 is shown on Figure 2.5.

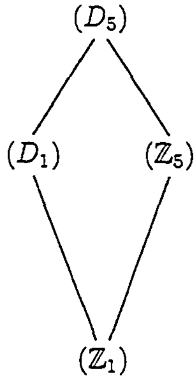


Figure 2.5: Lattice of Conjugacy Classes in D_5.

L	H	$n(L,H)$
$\mathbb{Z}_1$	D_5	1
D_1	D_5	1
$\mathbb{Z}_3$	D_5	1
$\mathbb{Z}_1$	$\mathbb{Z}_5$	1
$\mathbb{Z}_1$	D_1	5

Table 2.5: Numbers $n(L,H)$ for subgroups of D_5.

	(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$
(D_5)	(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$
(D_1)	(D_1)	$(D_1)+2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$
$(\mathbb{Z}_5)$	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5)$	$2(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$

Table 2.6: Multiplication Table for the Burnside Ring $A(D_5)$.

Example 2.4.2. Tetrahedral Group A_4

We have the following representatives for the conjugacy classes of the subgroups H in A_4:(see Section 1.3.9)

H	$\lvert(H)\rvert$
$\mathbb{Z}_1 = \{(1)\}$	1
$\mathbb{Z}_2 = \{(1), (12)(34)\}$	3
$\mathbb{Z}_3 = \{(1), (123), (132)\}$	4
$V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$	1

Table 2.7: Subgroups of A_4 and the order of each conjugacy class.

The lattice of subgroups in A_4 is shown on Figure 2.6.

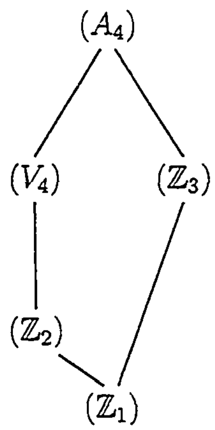


Figure 2.6: lattice of conjugacy classes in A_4.

In addition, $N(V_4) = A_4$, $N(\mathbb{Z}_3) = \mathbb{Z}_3$, $N(\mathbb{Z}_2) = V_4$, and $N(\mathbb{Z}_1) = A_4$, so $W(V_4) = \mathbb{Z}_3$, $W(\mathbb{Z}_3) = \mathbb{Z}_1, W(\mathbb{Z}_2) = \mathbb{Z}_2$, and $W(\mathbb{Z}_1) = A_4$. Now we can find the numbers of $n(L, H)$.

L	H	$n(L,H)$
$\mathbb{Z}_1$	A_4	1
$\mathbb{Z}_2$	A_4	1
$\mathbb{Z}_3$	A_4	1
V_4	A_4	1
$\mathbb{Z}_1$	V_4	1
$\mathbb{Z}_2$	V_4	1
$\mathbb{Z}_1$	$\mathbb{Z}_3$	4
$\mathbb{Z}_1$	$\mathbb{Z}_2$	3

Table 2.8: Numbers $n(L,H)$ for subgroups of A_4.

	(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_4)	(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(V_4)	(V_4)	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$

Table 2.9: Multiplication table for the Burnside ring $A(A_4)$.

Example 2.4.3. Octahedral Group S_4

Since A_4 is a subgroup of S_4, it is clear that all the subgroups of A_4, including V_4, $\mathbb{Z}_3$, $\mathbb{Z}_2$, and $\mathbb{Z}_1$ are also subgroups of S_4. In addition, there are the following subgroups in S_4 (up to the conjugacy class)-(see Section 1.3.9)

$$
\begin{aligned}
D_4 &= \{(1), (1324), (12)(34), (1423), (34), (14)(23), (12), (13)(24)\},\\
\mathbb{Z}_4 &= \{(1), (1324), (12)(34), (1423)\},\\
D_3 &= \{(1), (123), (132), (12), (12)(23), (13)\},\\
D_2 &= \{(1), (12)(34), (12), (34)\},\\
D_1 &= \{(1), (12)\}.
\end{aligned}
$$

These subgroups are shown in Figure 2.7.

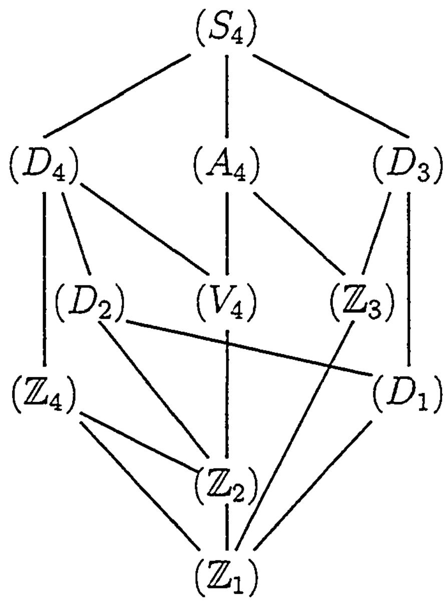


Figure 2.7: Lattice of Conjugacy Classes in S_4.

Let us notice that the subgroup D_4, which is composed of the elements $\{(1)$, (1324), $(12)(34)$, (1423), (34), $(14)(23)$, (12), $(13)(24)\}$ has the normalizer $N(D_4) = D_4$. The conjugacy class (D_4) contains three elements, corresponding to the symmetry subgroups of the three pairs of parallel faces of the cube. The normalizer of the subgroup A_4 is $N(A_4) = S_4$. The group D_3 consists of the elements $\{(1)$, (123),

$(132), (12), (23), (13)\}$. The conjugacy class (D_3) contains four subgroups corresponding to the symmetries of the cube around each of four pairs of opposite vertices of the cube. In addition $N(D_3) = D_3$. The subgroup $\mathbb{Z}_4$ consisting of the rotations belonging to D_4, has the normalizer $N(\mathbb{Z}_4) = D_4$. There are three subgroups in the conjugacy class $(\mathbb{Z}_4)$, which correspond to the rotations of the three pairs of the parallel faces of the cube. The normalizer of V_4 is $N(V_4) = S_4$, the subgroup D_2 has the normalizer $N(D_2) = D_4$, and the subgroup D_1 has the normalizer $N(D_1) = D_2$. Finally, the subgroup $\mathbb{Z}_2$ has the normalizer $N(\mathbb{Z}_2) = D_4$.

The following table shows the numbers of $n(L,H)$.

H	L	$n(L,H)$	L	K	$n(L,H)$
A_4	S_4	1	D_2	D_4	1
V_4	S_4	1	D_1	D_4	1
$\mathbb{Z}_3$	S_4	1	$\mathbb{Z}_1$	D_4	3
$\mathbb{Z}_2$	S_4	1	D_1	D_3	2
$\mathbb{Z}_1$	S_4	1	$\mathbb{Z}_3$	D_3	1
D_4	S_4	1	$\mathbb{Z}_1$	D_3	4
$\mathbb{Z}_4^-$	S_4	1	$\mathbb{Z}_2$	V_4	1
D_3	S_4	1	$\mathbb{Z}_1$	V_4	3
D_2	S_4	1	D_1	D_2	1
D_1	S_4	1	$\mathbb{Z}_2$	D_2	1
V_4	A_4	1	$\mathbb{Z}_1$	D_2	1
$\mathbb{Z}_3$	A_4	1	$\mathbb{Z}_1$	D_1	6
$\mathbb{Z}_2$	A_4	1	$\mathbb{Z}_2$	$\mathbb{Z}_4$	1
$\mathbb{Z}_1$	A_4	1	$\mathbb{Z}_1$	$\mathbb{Z}_4$	3
V_4	D_4	3	$\mathbb{Z}_1$	$\mathbb{Z}_3$	4
$\mathbb{Z}_4$	D_4	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	3
$\mathbb{Z}_2$	D_4	3			

Table 2.10: Numbers $n(L,H)$ for Subgroups of S_4.

	(S_4)	(A_4)	(D_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(S_4)	(S_4)	(A_4)	(D_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	(D_1)	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_4)	(A_4)	$2(A_4)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$
(D_4)	(D_4)	(V_4)	$(D_4)+(V_4)$	(D_1)	$(D_2)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$
(D_3)	(D_3)	$(\mathbb{Z}_3)$	(D_1)	$(D_3)+(D_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$
(D_2)	(D_2)	$(\mathbb{Z}_2)$	$(D_2)+(\mathbb{Z}_2)$	$2(D_1)$	$2(D_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
(V_4)	(V_4)	$2(V_4)$	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$6(V_4)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$
(D_1)	(D_1)	$(\mathbb{Z}_1)$	$(D_1)+(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$2(D_1)+5(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$24(\mathbb{Z}_1)$

Table 2.11: Multiplication Table for the Burnside ring $A(S_4)$.

Example 2.4.4. Alternating Group A_5

Let us list the representatives of the conjugacy classes of the subgroups in A_5:

$\mathbb{Z}_2 = \{(1),\ (12)(34)\}$,
$\mathbb{Z}_3 = \{(1),(123),\ (132)\}$,
$V_4 = \{(1),\ (12)(34),\ (13)(24),\ (23)(14)\}$,
$\mathbb{Z}_5 = \{(1),\ (12345),\ (13524),\ (14253),\ (15324)\}$,
$D_3 = \{(1),\ (123),\ (132),\ (12)(45),\ (13)(45),\ (23)(45)\}$,
$A_4 = \{(1),\ (12)(34),\ (123),\ (132),\ (13)(24),\ (14)(23),\ (124),\ (142),\ (134),$
$(143),\ (234),\ (243)\}$,
$D_5 = \{(1),\ (12345),\ (13524),\ (15432),\ (14253),\ (12)(35),\ (13)(54),\ (14)(23),$
$(15)(24),\ (25)(34)\}$.

The conjugacy classes of the subgroups of A_5 can be classified as follows: there are 15 elements in the conjugacy class of the subgroup $\mathbb{Z}_2$, 10 elements in the conjugacy class of the subgroup $\mathbb{Z}_3$, 5 elements in the conjugacy class of the subgroup V_4, 6 elements in the conjugacy class of $\mathbb{Z}_5$, 10 elements in the conjugacy class of D_3, 5 elements in the conjugacy class of A_4, and 6 elements in the conjugacy class of the subgroup D_5.

The lattice of the conjugacy subgroups in A_5 is shown in Figure 2.8.

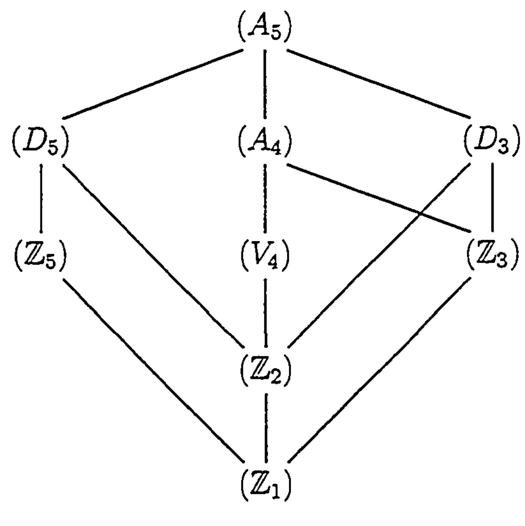


Figure 2.8: Lattice of the Conjugacy Classes for A_5.

L	H	$n(L,H)$	L	H	$n(L,H)$
$\mathbb{Z}_3$	A_5	1	$\mathbb{Z}_2$	A_4	1
$\mathbb{Z}_2$	A_5	1	$\mathbb{Z}_1$	A_4	1
$\mathbb{Z}_1$	A_5	1	$\mathbb{Z}_1$	$\mathbb{Z}_3$	10
A_4	A_5	1	$\mathbb{Z}_5$	D_5	1
$\mathbb{Z}_5$	A_5	1	$\mathbb{Z}_2$	D_5	2
V_4	A_5	1	$\mathbb{Z}_1$	D_5	6
D_3	A_5	1	$\mathbb{Z}_3$	D_3	1
D_5	A_5	1	$\mathbb{Z}_2$	D_3	2
V_4	A_4	1	$\mathbb{Z}_1$	D_3	10
$\mathbb{Z}_3$	A_4	2	$\mathbb{Z}_2$	V_4	1
$\mathbb{Z}_1$	$\mathbb{Z}_2$	15	$\mathbb{Z}_1$	V_4	5
$\mathbb{Z}_1$	$\mathbb{Z}_5$	6			

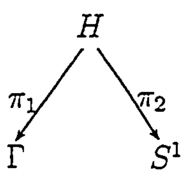
Table 2.12: Numbers $n(L,H)$ for Subgroups of A_5.

In addition $N(D_3) = N(\mathbb{Z}_3) = D_3$, $N(\mathbb{Z}_5) = N(D_5) = D_5$, and $N(A_4) = A_4$. the subgroup V_4 has the normalizer $N(V_4) = A_4$. Finally, the subgroup $\mathbb{Z}_2$ has the normalizer $N(\mathbb{Z}_2) = V_4$.

2.5 Subgroups of $\Gamma \times S^1$ and Twisted Subgroups

In this section we introduce the definition of twisted subgroups of $\Gamma \times S^1$, and provide several examples.

Let Γ be a finite group and consider the group $G = \Gamma \times S^1$. In order to classify the subgroups H of $\Gamma \times S^1$ we consider the following diagram



where π_1 and π_2 are projections (homomorphisms) on Γ and S^1 respectively. Let $K := \pi_1(H)$ and consider $\ker \pi_1 = H \cap \{e\} \times S^1$, where e denotes the neutral element of Γ.

	(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_5)	(A_5)	(A_4)	(D_5)	(D_3)	$(\mathbb{Z}_5)$	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$
(A_4)	(A_4)	$(A_4)+(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_3)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(V_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$
(D_5)	(D_5)	$(\mathbb{Z}_2)$	$(D_5)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_5)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$
(D_3)	(D_3)	$(\mathbb{Z}_3)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(D_3)+(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+3(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$
$(\mathbb{Z}_5)$	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_5)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_5)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$
(V_4)	(V_4)	$(V_4)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(V_4)+3(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+6(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_3)$	$2(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$
$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)+6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+14(\mathbb{Z}_1)$	$30(\mathbb{Z}_1)$
$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$15(\mathbb{Z}_1)$	$20(\mathbb{Z}_1)$	$30(\mathbb{Z}_1)$	$60(\mathbb{Z}_1)$

Table 2.13: Multiplication Table for the Burnside ring $A(A_5)$.

If ker $\pi_1 = \{e\} \times S^1$, then simply $H = K \times S^1$,i.e. H is a"product" subgroup and

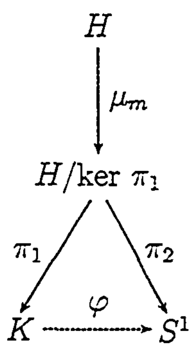
$$N(H) = N(K \times S^1) = N(K) \times S^1,$$

therefore

$$W(H) = \frac{N(H)}{H} = \frac{N(K) \times S^1}{K \times S^1} = \frac{N(K)}{K} = W(K).$$

Consequently, we obtain that $\dim W(H) = \dim W(K) = 0$ since $K \subset \Gamma$ and Γ is finite.

If ker $\pi_1 = \{e\} \times \mathbb{Z}_m$, for some $m \geq 1$, then we are dealing with"twisted" subgroup and in this case still we have $H/\ker \pi_1 \subset \Gamma \times S^1$, thus we can consider the diagram below.



Since $\pi_1 : H/\ker \pi_1 \to K$ is one-to-one and onto, we can define the homomorphism $\varphi := \pi_2 \circ \pi_1^{-1} : K \to S^1$, and consequently the subgroup $H/\ker \pi_1$ is the graph of φ, i.e.

$$H/\ker \pi_1 := \{(\gamma, z) \in \Gamma \times S^1 | \varphi(\gamma) = z\},$$

and since the subgroup H is the inverse image $\mu_m^{-1}(H/\ker \pi_1)$, we obtain:

$$H = \{(\gamma, z) \in \Gamma \times S^1 | \varphi(\gamma) = z^m\}.$$

In this case we will call the subgroup H a *twisted* (by φ) *m-folded* subgroup which will be denoted by $K^{\varphi,m}$. Let us describe the normalizer $N(H)$ of the group H. Notice that

$$N(H) = N(K^{\varphi,m}) = \{(\gamma, z) \in \Gamma \times S^1 : (\gamma, z)K^{\varphi,m}(\gamma^{-1}, z^{-1}) = K^{\varphi,m}\}$$

$$= \{(\gamma, z) \in \Gamma \times S^1 : \varphi(\gamma k \gamma^{-1}) = \varphi(k) \ \ \forall k \in K\}$$

$$= \{\gamma \in N(K) : \varphi(\gamma k \gamma^{-1}) = \varphi(k) \ \ \forall k \in K\} \times S^1 := N \times S^1.$$

That means N is finite and $W(H) = W(K^{\varphi,m}) = \frac{N \times S^1}{K^{\varphi,m}}$. Consequently, we obtain that $\dim W(H) = \dim W(K^{\varphi,m}) = 1$.

2.5.1 Examples and Computations

In this section we consider several examples of groups Γ and the product groups $G = \Gamma \times S^1$, for which we classify their conjugacy classes of twisted subgroups and determine the values of the numbers $n(L, H)$ and their Weyl groups.

Example 2.5.1. $\mathbf{G = D_3 \times S^1}$

By Example 2.4.1, we know the subgroups of D_3 are represented by

$$\mathbb{Z}_1 = \{1\}, \ \ \mathbb{Z}_3 = \{1, \gamma, \gamma^2\}, \ \ D_1 = \{1, \kappa\} \sim \{1, \kappa\gamma\} \sim \{1, \kappa\gamma^2\}.$$

On the other hand, besides all the listed above subgroups of D_3 (up to conjugacy class), which are clearly also the subgroups of $D_3 \times S^1$, we have the following twisted (one-folded) subgroups of $G = D_3 \times S^1$

$$\mathbb{Z}_3^t = \{(1,1), (\gamma, \gamma), (\gamma^2, \gamma^2)\} \sim \{(1,1), (\gamma, \gamma^2), (\gamma^2, \gamma)\},$$

$$D_1^{\tilde{z}} = \{(1,1), (\kappa, -1)\} \sim \{(1,1), (\kappa\gamma, -1)\} \sim \{(1,1), (\kappa\gamma^2, -1)\},$$

$$D_3^{\tilde{z}} = \{(1,1), (\gamma, 1), (\gamma^2, 1), (\kappa, -1), (\kappa\gamma, -1), (\kappa\gamma^2, -1)\}.$$

Additional properties of these subgroups are listed in Tables 2.14 and 2.15. The lattice of the conjugacy classes of subgroups in $D_3 \times S^1$ is shown on Figure 2.9.

$H = K^{\varphi,1}$	K	$\varphi(K)$	$\operatorname{Ker}\varphi$	$N(H)$	$W(H)$
D_3	D_3	$\mathbb{Z}_1$	D_3	$D_3 \times S^1$	S^1
D_1	D_1	$\mathbb{Z}_1$	D_1	$D_1 \times S^1$	S^1
$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3$	$D_3 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$D_3 \times S^1$	$D_3 \times S^1$
$D_3^{\tilde{z}}$	D_3	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$D_3 \times S^1$	S^1
$\mathbb{Z}_3^t$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3 \times S^1$	S^1
$D_1^{\tilde{z}}$	D_1	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$D_1 \times S^1$	S^1

Table 2.14: Twisted Subgroups of $D_3 \times S^1$.

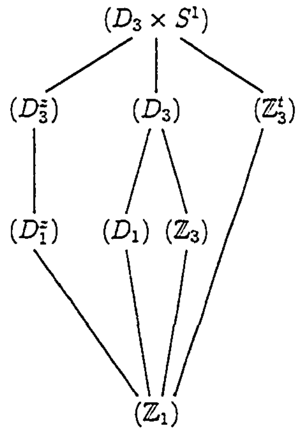


Figure 2.9: Lattice of the conjugacy classes of twisted subgroups in $D_3 \times S^1$.

L	H	$n(L,H)$	L	H	$n(L,H)$
$\mathbb{Z}_1$	D_3	1	$D_1^{\tilde{z}}$	$D_3^{\tilde{z}}$	1
D_1	D_3	1	$\mathbb{Z}_1$	$\mathbb{Z}_3^t$	2
$\mathbb{Z}_3$	D_3	1	$\mathbb{Z}_1$	$D_1^{\tilde{z}}$	3
$\mathbb{Z}_1$	$\mathbb{Z}_3$	1	$\mathbb{Z}_1$	D_1	3
$\mathbb{Z}_1$	$D_3^{\tilde{z}}$	1	$\mathbb{Z}_1$	$\mathbb{Z}_3$	1

Table 2.15: Numbers $n(L,H)$ for Twisted Subgroups in $D_3 \times S^1$.

The complete list of numbers $n(L,H)$ for the twisted subgroups of $D_3 \times S^1$ is given.

Example 2.5.2. $\mathbf{D_4 \times S^1}$

As we know the subgroups of

$$D_4 = \{1, i, -1, -i, \kappa, \kappa i, -\kappa, -\kappa i\},$$

classifying the conjugacy classes (H) are exactly
$\mathbb{Z}_4 = \{1, i, -1, -i\}$,
$\mathbb{Z}_2 = \{1, -1\}$,
$\mathbb{Z}_1 = \{1\}$,
$D_2 = \{1, -1, \kappa, -\kappa\}$,
$\tilde{D}_2 = \{1, -1, \kappa i, -\kappa i\}$,
$D_1 = \{1, \kappa\} \sim \{1, -\kappa\}$,
$\tilde{D}_1 = \{1, \kappa i\} \sim \{1, -\kappa i\}$.

Then, in addition to the above subgroups of $G = D_4 \times S^1$, there are the following twisted (one-folded) subgroups of G:

$\mathbb{Z}_2^- = \{(1,1), (-1,-1)\}$,
$\mathbb{Z}_4^t = \{(1,1), (i,i), (-1,-1), (-i,-i)\} \sim \{(1,1), (i,-i), (-1,-1), (-i,i)\}$,
$\mathbb{Z}_4^d = \{(1,1), (i,-1), (-1,1), (-i,-1)\}$,
$D_1^z = \{(1,1), (\kappa,-1)\} \sim \{(1,1), (-\kappa,-1)\}$,
$\widetilde{D}_1^z = \{(1,1), (\kappa i,-1)\} \sim \{(1,1), (-\kappa i,-1)\}$,
$D_2^z = \{(1,1), (-1,1), (\kappa,-1), (-\kappa,-1)\}$,
$\widetilde{D}_2^z = \{(1,1), (-1,1), (\kappa i,-1), (-\kappa i,-1)\}$,
$D_2^d = \{(1,1), (-1,-1), (\kappa,1), (-\kappa,-1)\}$,
$D_2^{\hat{d}} = \{(1,1), (-1,-1), (\kappa,-1), (-\kappa,1)\}$,
$\widetilde{D}_2^d = \{(1,1), (-1,-1), (\kappa i,1), (-\kappa i,-1)\}$,
$\widetilde{D}_2^{\hat{d}} = \{(1,1), (-1,-1), (\kappa i,-1), (-\kappa i,1)\}$,
$D_4^z = \{(1,1), (i,1), (-1,1), (-i,1), (\kappa,-1), (\kappa i,-1), (-\kappa,-1), (-\kappa i,-1)\}$,
$D_4^d = \{(1,1), (i,-1), (-1,1), (-i,-1), (\kappa,1), (\kappa i,-1), (-\kappa,1), (-\kappa i,-1)\}$,
$D_4^{\hat{d}} = \{(1,1), (i,-1), (-1,1), (-i,-1), (\kappa,-1), (\kappa i,1), (-\kappa,-1), (-\kappa i,1)\}$.

All the twisted subgroups of $D_4 \times S^1$ (up to their conjugacy class), their normalizers and Weyl groups, are listed in Table 2.16. The lattice of the conjugacy classes of the twisted subgroups in $D_4 \times S^1$ is shown in Figure 2.10. The numbers $n(L,H)$ for twisted subgroups in $D_4 \times S^1$ are listed in Table 2.17.

Example 2.5.3. $D_5 \times S^1$

We have the following subgroups (up to conjugacy class) of

$$D_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4\},$$

where $\gamma = e^{\frac{2\pi i}{5}}$:

$$\mathbb{Z}_1 = \{1\}, \quad D_1 = \{1, \kappa\} \sim \{1, \kappa\gamma\} \sim \{1, \kappa\gamma^2\} \sim \{1, \kappa\gamma^3\} \sim \{1, \kappa\gamma^4\},$$
$$\mathbb{Z}_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4\},$$
$$D_5 = \{1, \gamma, \gamma^2, \gamma^3, \gamma^4, \kappa\gamma, \kappa\gamma^2, \kappa\gamma^3, \kappa\gamma^4\}.$$

$H = K^{\varphi,1}$	K	$\varphi(K)$	$\mathrm{Ker}\,\varphi$	$N(H)$	$W(H)$
D_4	D_4	$\mathbb{Z}_1$	D_4	$D_4 \times S^1$	S^1
D_2	D_2	$\mathbb{Z}_1$	D_2	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$D_4 \times S^1$	$D_1 \times S^1$
$\widetilde{D}_2$	$\widetilde{D}_2$	$\mathbb{Z}_1$	$\widetilde{D}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_1	D_1	$\mathbb{Z}_1$	D_1	$D_2 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$D_4 \times S^1$	$D_2 \times S^1$
$\widetilde{D}_1$	$\widetilde{D}_1$	$\mathbb{Z}_1$	$\widetilde{D}_1$	$\widetilde{D}_2 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$D_4 \times S^1$	$D_4 \times S^1$
D_4^{z}	D_4	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$D_4 \times S^1$	S^1
D_4^{d}	D_4	$\mathbb{Z}_2$	D_2	$D_4 \times S^1$	S^1
$D_4^{\tilde{d}}$	D_4	$\mathbb{Z}_2$	$\widetilde{D}_2$	$D_4 \times S^1$	S^1
D_2^{z}	D_2	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\widetilde{D}_2^{z}$	$\widetilde{D}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\widetilde{D}_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_2^{d}	D_2	$\mathbb{Z}_2$	D_1	$D_2 \times S^1$	S^1
$\widetilde{D}_2^{d}$	$\widetilde{D}_2$	$\mathbb{Z}_2$	$\widetilde{D}_1$	$\widetilde{D}_2 \times S^1$	S^1
D_1^{z}	D_1	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$D_2 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\widetilde{D}_1^{z}$	$\widetilde{D}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\widetilde{D}_2 \times S^1$	$\mathbb{Z}_2 \times S^1$
$\mathbb{Z}_4^{d}$	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4 \times S^1$	$D_1 \times S^1$
$\mathbb{Z}_4^{t}$	$\mathbb{Z}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_1$	$\mathbb{Z}_4 \times S^1$	S^1
$\mathbb{Z}_2^{-}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$D_4 \times S^1$	$D_2 \times S^1$

Table 2.16: Twisted Subgroups of $D_4 \times S^1$.

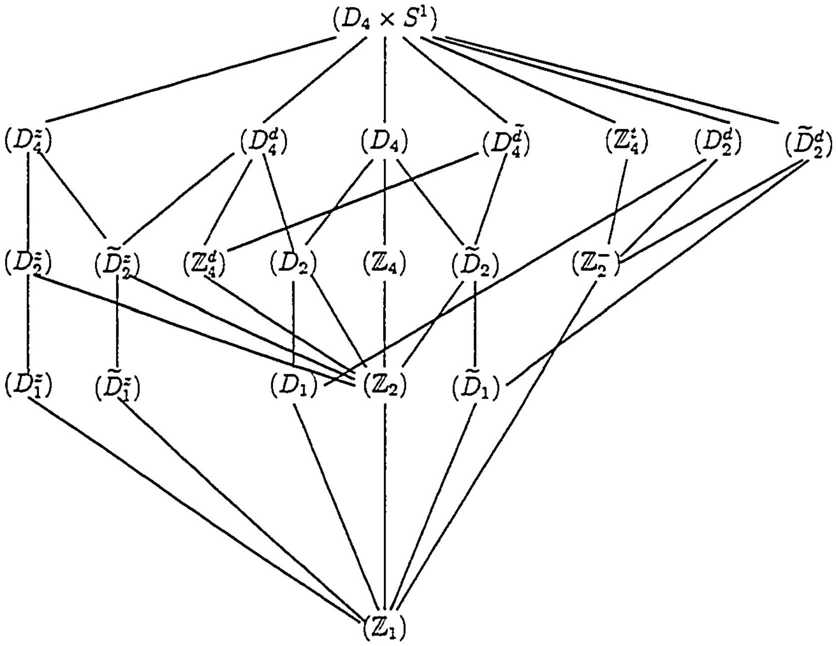


Figure 2.10: Conjugacy Classes of Twisted Subgroups in $D_4 \times S^1$.

L	H	$n(L,H)$	L	H	$n(L,H)$	L	H	$n(L,H)$	L	H	$n(L,H)$
$\mathbb{Z}_1$	D_4	1	$\widetilde{D}_2^{\tilde{z}}$	D_4^d	1	D_1	D_2^d	1	$\mathbb{Z}_1$	$\mathbb{Z}_4^d$	1
D_1	D_4	1	$\mathbb{Z}_4^d$	D_4^d	1	$\mathbb{Z}_2^-$	D_2^d	2	D_1	D_2	1
$\mathbb{Z}_2$	D_4	1	D_2	D_4^d	1	$\mathbb{Z}_1$	D_2^d	2	$\mathbb{Z}_2$	D_2	1
$\widetilde{D}_1$	D_4	1	$\mathbb{Z}_2$	D_4^d	1	$\widetilde{D}_1$	$\widetilde{D}_2^d$	1	$\mathbb{Z}_1$	D_2	1
D_2	D_4	1	D_1	D_4^d	1	$\mathbb{Z}_2^-$	$\widetilde{D}_2^d$	2	$\mathbb{Z}_2$	$\mathbb{Z}_4$	1
$\mathbb{Z}_4$	D_4	1	$\mathbb{Z}_1$	D_4^d	1	$\mathbb{Z}_1$	$\widetilde{D}_2^d$	2	$\mathbb{Z}_1$	$\mathbb{Z}_4$	1
$\widetilde{D}_2$	D_4	1	$\mathbb{Z}_4^d$	$D_4^{\tilde{d}}$	1	$D_1^{\tilde{z}}$	$D_2^{\tilde{z}}$	1	$\widetilde{D}_1$	$\widetilde{D}_2$	1
$D_2^{\tilde{z}}$	$D_4^{\tilde{z}}$	1	$\widetilde{D}_2$	$D_4^{\tilde{d}}$	1	$\mathbb{Z}_2$	$D_2^{\tilde{z}}$	1	$\mathbb{Z}_2$	$\widetilde{D}_2$	1
$\widetilde{D}_2$	$D_4^{\tilde{z}}$	1	$\mathbb{Z}_2$	$D_4^{\tilde{d}}$	1	$\mathbb{Z}_1$	$D_2^{\tilde{z}}$	1	$\mathbb{Z}_1$	$\widetilde{D}_2$	1
$D_1^{\tilde{z}}$	$D_4^{\tilde{z}}$	1	$\widetilde{D}_1$	$D_4^{\tilde{d}}$	1	$\widetilde{D}_1^{\tilde{z}}$	$\widetilde{D}_2^{\tilde{z}}$	1	$\mathbb{Z}_1$	$D_1^{\tilde{z}}$	2
$\widetilde{D}_1^{\tilde{z}}$	$D_4^{\tilde{z}}$	1	$\mathbb{Z}_1$	$D_4^{\tilde{d}}$	1	$\mathbb{Z}_2$	$\widetilde{D}_2^{\tilde{z}}$	1	$\mathbb{Z}_1$	$\widetilde{D}_1^{\tilde{z}}$	2
Z_2	$D_4^{\tilde{z}}$	1	$\mathbb{Z}_2^-$	$\mathbb{Z}_4^t$	2	$\mathbb{Z}_1$	$\widetilde{D}_2^{\tilde{z}}$	1	$\mathbb{Z}_1$	D_1	2
$\mathbb{Z}_1$	$D_4^{\tilde{z}}$	1	$\mathbb{Z}_1$	$\mathbb{Z}_4^t$	2	$\mathbb{Z}_2$	$\mathbb{Z}_4^d$	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	1
$\mathbb{Z}_1$	$\widetilde{D}_1$	2	$\mathbb{Z}_1$	$\mathbb{Z}_2^-$	1						

Table 2.17: Numbers $n(L,H)$ for Twisted Subgroups in $D_4 \times S^1$.

Then, in addition to the above, the twisted (one-folded) subgroups of $D_5 \times S^1$ are:

$$
\begin{aligned}
D_1^{\tilde{z}} &= \{(1,1),(\kappa,-1)\} \sim \{(1,1),(\kappa\gamma,-1)\} \sim \{(1,1),(\kappa\gamma^2,-1)\},\\
&\sim \{(1,1),(\kappa\gamma^3,-1)\} \sim \{(1,1),(\kappa\gamma^4,-1)\},\\
\mathbb{Z}_5^{t_1} &= \{(1,1),(\gamma,\gamma),(\gamma^2,\gamma^2),(\gamma^3,\gamma^3),(\gamma^4,\gamma^4)\}\\
&\sim \{(1,1),(\gamma,\gamma^4),(\gamma^2,\gamma^3),(\gamma^3,\gamma),(\gamma^4,\gamma^2)\},\\
\mathbb{Z}_5^{t_2} &= \{(1,1),(\gamma,\gamma^2),(\gamma^2,\gamma^4),(\gamma^3,\gamma),(\gamma^4,\gamma^3)\}\\
&\sim \{(1,1),(\gamma,\gamma^3),(\gamma^2,\gamma^4),(\gamma^3,\gamma^3),(\gamma^4,\gamma)\},\\
D_5^{\tilde{z}} &= \{(1,1),(\gamma,1),(\gamma^2,1),(\gamma^3,1),(\gamma^4,1),(\kappa,-1),(\kappa\gamma,-1),\\
&\quad (\kappa\gamma^2,-1),(\kappa\gamma^3,-1),(\kappa\gamma^4,-1)\}.
\end{aligned}
$$

All the twisted subgroups of $D_5 \times S^1$ (up to conjugacy class), their normalizers and Weyl groups, are listed in Table 2.18. The lattice of the conjugacy classes of the twisted subgroups in $D_5 \times S^1$ is shown in Figure 2.11. The numbers $n(L,H)$ for twisted subgroups in $D_5 \times S^1$ are listed in Table 2.19.

$H = K^{\varphi,1}$	K	$\varphi(K)$	$\operatorname{Ker}\varphi$	$N(H)$	$W(H)$
D_5	D_5	$\mathbb{Z}_1$	D_5	$D_5 \times S^1$	S^1
$\mathbb{Z}_5$	$\mathbb{Z}_5$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$D_5 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_1	D_1	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$D_1 \times S^1$	S^1
$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$D_5 \times S^1$	$D_5 \times S^1$
$D_5^{\tilde{z}}$	D_5	$\mathbb{Z}_2$	$\mathbb{Z}_5$	$D_5 \times S^1$	S^1
$\mathbb{Z}_5^{t_1}$	$\mathbb{Z}_5$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_5 \times S^1$	S^1
$D_1^{\tilde{z}}$	D_1	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$D_1 \times S^1$	S^1
$\mathbb{Z}_5^{t_2}$	$\mathbb{Z}_5$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_5 \times S^1$	S^1

Table 2.18: Twisted Subgroups of $D_5 \times S^1$.

L	H	$n(L,H)$
$\mathbb{Z}_1$	D_5	1
$\mathbb{Z}_5$	D_5	1
D_1	D_5	1
$\mathbb{Z}_1$	D_5	1
$\mathbb{Z}_1$	$D_5^{\tilde{z}}$	1
$\mathbb{Z}_1$	$\mathbb{Z}_5^{t_1}$	2
$\mathbb{Z}_1$	$D_1^{\tilde{z}}$	5
$\mathbb{Z}_1$	$\mathbb{Z}_5^{t_2}$	2

Table 2.19: Numbers $n(L,H)$ for Twisted Subgroups in $D_5 \times S^1$.

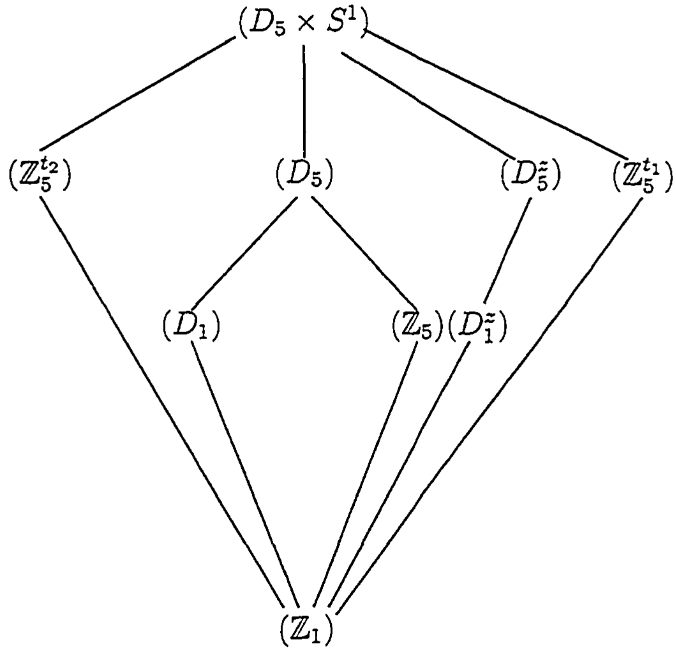


Figure 2.11: Conjugacy Classes of Twisted Subgroups in $D_5 \times S^1$.

Example 2.5.4. $\mathbf{A_4 \times S^1}$

Let us consider the group A_4. We already know the subgroups H of A_4 (up to conjugacy class):

$$\begin{aligned}
&\mathbb{Z}_1 = \{(1)\}, \quad \mathbb{Z}_2 = \{(1), (12)(34)\}, \\
&\mathbb{Z}_3 = \{(1), (123), (132)\}, \quad V_4 = \{(1), (12)(34), (13)(24), (14)(23)\}, \\
&A_4 = \{(1), (12)(34), (123), (132), (13)(24), (142), (124), (14)(23), (134), (143), \\
&\qquad (243), (234)\}.
\end{aligned}$$

The additional twisted (one-folded) subgroups of $A_4 \times S^1$ are

$$\begin{aligned}
\mathbb{Z}_2^- &= \{((1),1),((12)(34),-1)\},\\
\mathbb{Z}_3^{t_1} &= \{((1),1),((123),\gamma),((132),\gamma^2)\},\\
\mathbb{Z}_3^{t_2} &= \{((1),1),((123),\gamma^2),((132),\gamma)\},\\
V_4^- &= \{((1),1),((12)(34),1),((13)(24),-1),((14)(23),-1)\},\\
A_4^{t_1} &= \{((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma),\\
&\quad ((132),\gamma^2),((142),\gamma),((124),\gamma^2),((134),\gamma),((143),\gamma^2),\\
&\quad ((243),\gamma),((234),\gamma^2)\},\\
A_4^{t_2} &= \{((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma^2),\\
&\quad ((132),\gamma),((142),\gamma^2),((124),\gamma),((134),\gamma^2),((143),\gamma),\\
&\quad ((243),\gamma^2),((234),\gamma)\},
\end{aligned}$$

where $\gamma = e^{\frac{2\pi i}{3}}$. Additional properties of these groups are listed in Tables 2.20 and 2.21. The lattice of the conjugacy classes of subgroups in $A_4 \times S^1$ is shown on Figure 2.12.

$H = K^{(\varphi,1)}$	K	$\varphi(K)$	Ker φ	$N(H)$	$W(H)$	Comments
A_4	A_4	$\mathbb{Z}_1$	A_4	$A_4 \times S^1$	S^1	
V_4	V_4	$\mathbb{Z}_1$	V_4	$A_4 \times S^1$	$\mathbb{Z}_3 \times S^1$	
$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times S^1$	S^1	
$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$V_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$A_4 \times S^1$	$A_4 \times S^1$	
$A_4^{t_k}$ $k = 1,2$	A_4	$\mathbb{Z}_3$	V_4	$A_4 \times S^1$	S^1	
V_4^-	V_4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$V_4 \times S^1$	S^1	
$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$V_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_3^{t_k}$, $k = 1,2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3 \times S^1$	S^1	$\varphi(g) = g^k$

Table 2.20: Twisted Subgroups of $A_4 \times S^1$.

The complete list of numbers $n(L, H)$ for the twisted subgroups of $A_4 \times S^1$ is given in Table 2.21.

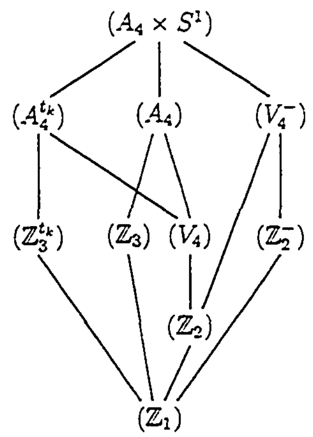


Figure 2.12: Lattice of the conjugacy classes of twisted subgroups in $A_4 \times S^1$.

H	L	$n(L,H)$	L	K	$n(L,H)$	H	L	$n(L,H)$	H	L	$n(L,H)$
$\mathbb{Z}_1$	A_4	1	$\mathbb{Z}_1$	V_4^-	1	$\mathbb{Z}_1$	$A_4^{t_k}$	1	$\mathbb{Z}_1$	V_4^-	3
$\mathbb{Z}_2$	A_4	1	$\mathbb{Z}_2$	V_4	1	V_4	$A_4^{t_k}$	1	$\mathbb{Z}_2$	V_4^-	1
$\mathbb{Z}_3$	A_4	1	$\mathbb{Z}_1$	$\mathbb{Z}_3$	4	$\mathbb{Z}_2$	$A_4^{t_k}$	1	$\mathbb{Z}_2^-$	V_4^-	2
V_4	A_4	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	3	$\mathbb{Z}_3^{t_k}$	$A_4^{t_k}$	1	$\mathbb{Z}_1$	$\mathbb{Z}_3^{t_k}$	4
$\mathbb{Z}_1$	$\mathbb{Z}_2^-$	3									

Table 2.21: Numbers $n(L,H)$ for Twisted Subgroups in $A_4 \times S^1$ $(k = 1, 2)$.

Example 2.5.5. $\mathbf{S_4 \times S^1}$

In addition to the listed subgroups in Example 5.2.3, there are the following twisted (one-folded) subgroups in $S_4 \times S^1$, representing conjugacy classes:

$$
\begin{aligned}
\mathbb{Z}_2^- &= \{((1),1),((12)(34),-1)\},\\
\mathbb{Z}_3^t &= \{((1),1),((123),\gamma),((132),\gamma^2)\},\\
\mathbb{Z}_4^c &= \{((1),1),((1324),i),((12)(34),-1),((1423),-i)\},\\
\mathbb{Z}_4^- &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1)\},\\
D_1^z &= \{((1),1),((12),-1)\},\\
V_4^- &= \{((1),1),((12)(34),1),((13)(24),-1),((14)(23),-1)\},\\
D_2^d &= \{((1),1),((12)(34),-1),((12),1),((34),-1)\},\\
D_2^z &= \{((1),1),((12)(34),1),((12),-1),((34),-1)\},\\
D_3^z &= \{((1),1),((123),1),((132),1),((12),-1),((12)(23),-1),((13),-1)\},\\
A_4^t &= \{((1),1),((12)(34),1),((13)(24),1),((14)(23),1),((123),\gamma),\\
&\quad ((132),\gamma^2),((142),\gamma),((124),\gamma^2),((134),\gamma),((143),\gamma^2),\\
&\quad ((243),\gamma),((234),\gamma^2)\},\\
D_4^d &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),1),\\
&\quad ((14)(23),-1),((12),1),((13)(24),-1)\},\\
D_4^{\tilde{d}} &= \{((1),1),((1324),-1),((12)(34),1),((1423),-1),((34),-1),\\
&\quad ((14)(23),1),((12),-1),((13)(24),1)\},\\
D_4^z &= \{((1),1),((1324),1),((12)(34),1),((1423),1),((34),-1),\\
&\quad ((14)(23),-1),((12),-1),((13)(24),-1)\},\\
S_4^- &= \{((1),1),((12),-1),((12)(34),1),((123),1),((1234),-1),((13),-1),\\
&\quad ((13)(24),1),((132),1),((1342),-1),((14),-1),((14)(23),1),((142),1),\\
&\quad ((1324),-1),((23),-1),((124),1),((1243),-1),((24),-1),((134),1),\\
&\quad ((1423),-1),((142),1),((34),-1),((143),1),((1432),-1),((243),1),\\
&\quad ((234),1)\}.
\end{aligned}
$$

The properties of the twisted subgroups in $S_4 \times S^1$ are listed in Table 2.22.

The numbers $n(L,H)$ for the twisted subgroups in $S_4 \times S^1$, which can be again established by inspection, are given in Table 2.23.

H^φ	$\mathrm{Ker}\,\varphi$	$\mathrm{Im}\,\varphi$	$N(H^\varphi)$	$W(H^\varphi)$	Comments
S_4^-	A_4	$\mathbb{Z}_2$	$S_4 \times S_4$	S^1	
D_4^z	$\mathbb{Z}_4$	$\mathbb{Z}_2$	$D_4 \times S^1$	S^1	
D_4^d	D_2	$\mathbb{Z}_2$	$D_4 \times S^1$	S^1	
$D_4^{\bar{d}}$	V_4	$\mathbb{Z}_2$	$D_4 \times S^1$	S^1	
A_4^t	V_4	$\mathbb{Z}_3$	$A_4 \times S^1$	S^1	
D_3^z	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$D_3 \times S^1$	S^1	
D_2^z	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	
D_2^d	D_1	$\mathbb{Z}_2$	$D_2 \times S^1$	S^1	
V_4^-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	
D_1^z	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$D_2 \times S^1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_4^-$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_4^c$	$\mathbb{Z}_1$	$\mathbb{Z}_4$	$\mathbb{Z}_4 \times S^1$	S^1	$\varphi(g) = g$
$\mathbb{Z}_3^t$	$\mathbb{Z}_1$	$\mathbb{Z}_3$	$\mathbb{Z}_3 \times S^1$	S^1	$\varphi(g) = g$
$\mathbb{Z}_2^-$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$D_4 \times S^1$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \times S^1$	

Table 2.22: Twisted Subgroups in $S_4 \times S^1$, where $\varphi : H \to S^1$ is a homomorphism.

H	L	$n(L,H)$	L	K	$n(L,H)$	H	L	$n(L,H)$	H	L	$n(L,H)$
A_4	S_4^-	1	$\mathbb{Z}_2^-$	$\mathbb{Z}_4^c$	2	D_2^z	$D_4^{\bar{d}}$	1	D_2^z	D_4^z	1
V_4	S_4^-	1	$\mathbb{Z}_1$	$\mathbb{Z}_4^c$	6	$\mathbb{Z}_4^-$	$D_4^{\bar{d}}$	1	D_2	D_4^z	1
$\mathbb{Z}_3$	S_4^-	1	$\mathbb{Z}_4^-$	D_4^d	1	V_4	$D_4^{\bar{d}}$	3	$\mathbb{Z}_4$	D_4^z	1
$\mathbb{Z}_2$	S_4^-	1	V_4^-	D_4^d	1	$\mathbb{Z}_2$	$D_4^{\bar{d}}$	3	V_4^-	D_4^z	1
$\mathbb{Z}_1$	S_4^-	1	$\mathbb{Z}_2^-$	D_4^d	2	D_1^z	$D_4^{\bar{d}}$	1	$\mathbb{Z}_2^-$	D_4^z	2
$D_4^{\bar{d}}$	S_4^-	1	D_2	D_4^d	1	$\mathbb{Z}_1$	$D_4^{\bar{d}}$	3	D_1^z	D_4^z	1
$\mathbb{Z}_4^-$	S_4^-	1	D_1	D_4^d	1	V_4	A_4^t	2	$\mathbb{Z}_1$	D_4^z	3
V_4^-	S_4^-	1	$\mathbb{Z}_2$	D_4^d	1	$\mathbb{Z}_3^t$	A_4^t	1	$\mathbb{Z}_2$	$\mathbb{Z}_4^-$	2
$\mathbb{Z}_2^-$	S_4^-	1	$\mathbb{Z}_1$	D_4^d	3	$\mathbb{Z}_2$	A_4^t	2	$\mathbb{Z}_1$	$\mathbb{Z}_4^-$	3
$\mathbb{Z}_2^-$	V_4^-	2	$\mathbb{Z}_3$	D_3^z	1	$\mathbb{Z}_1$	A_4^t	1	$\mathbb{Z}_2^-$	D_2^d	2
$\mathbb{Z}_2$	V_4^-	1	D_1^z	D_3^z	2	D_1^z	D_2^z	1	D_1^z	D_2^d	1
$\mathbb{Z}_1$	$\mathbb{Z}_3^t$	8	$\mathbb{Z}_1$	D_3^z	4	$\mathbb{Z}_2$	D_2^z	1	D_1	D_2^d	1
V_4	A_4	1	V_4	D_4	3	$\mathbb{Z}_3$	D_3	1	$\mathbb{Z}_2$	$\mathbb{Z}_4$	1
$\mathbb{Z}_3$	A_4	1	$\mathbb{Z}_4$	D_4	1	D_1	D_3	2	$\mathbb{Z}_1$	$\mathbb{Z}_4$	3
$\mathbb{Z}_2$	A_4	1	D_2	D_4	1	$\mathbb{Z}_1$	D_3	4	$\mathbb{Z}_1$	$\mathbb{Z}_3$	4
$\mathbb{Z}_1$	A_4	1	D_1	D_4	1	$\mathbb{Z}_2$	V_4	1	D_1	D_2	1
$\mathbb{Z}_1$	$\mathbb{Z}_3^t$	1	$\mathbb{Z}_2$	D_4	3	$\mathbb{Z}_1$	V_4	3	$\mathbb{Z}_2$	D_2	1
$\mathbb{Z}_1$	D_1^z	6	$\mathbb{Z}_1$	D_4	3	$\mathbb{Z}_1$	D_1	6	$\mathbb{Z}_1$	D_2	1
$\mathbb{Z}_1$	$\mathbb{Z}_2^-$	1	$\mathbb{Z}_1$	$\mathbb{Z}_2$	3						

Table 2.23: Numbers $n(L,H)$ for Twisted Subgroups in $S_4 \times S^1$.

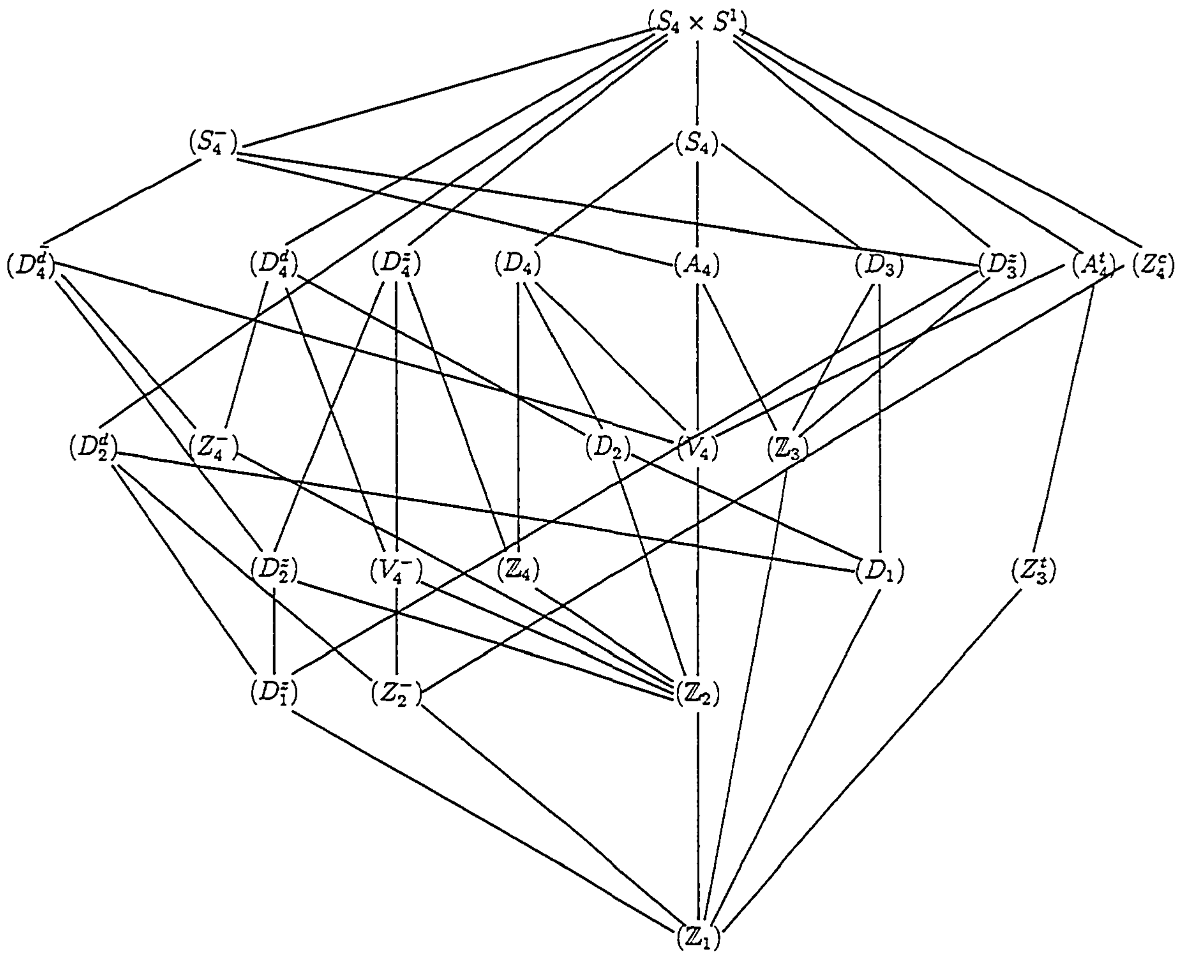


Figure 2.13: Lattice of Conjugacy Classes in $S_4 \times S^1$.

Example 2.5.6. $\mathbf{A_5 \times S^1}$ Let us list, up to conjugacy class, the twisted subgroups H^φ of $A_5 \times S^1$, where H is a subgroup of A_5, $\varphi : H \to S^1$ a group homomorphism, and $H^\varphi = \Big\{(h,z) \in H \times S^1 : \varphi(h) = z\Big\}$:

$$\mathbb{Z}_2^- = \Big\{((1),1),\ ((12)(34),-1)\Big\},$$
$$V_4^- = \Big\{((1),1),\ ((12)(34),-1),\ ((13)(24),-1),\ (23)(14),1)\Big\},$$
$$\mathbb{Z}_5^{t_k} = \Big\{((1),1),\ ((12345),\xi^{2k}),\ ((13524),\xi^{3k}),\ ((14253),\xi^{4k}),\ ((15324),\xi^{k})\Big\},$$
$$\mathbb{Z}_3^t = \Big\{((1),1),\ ((123),\gamma),\ ((132),\gamma^2)\Big\} \sim \Big\{((1),1),\ ((132),\gamma),\ ((123),\gamma^2)\Big\},$$
$$D_3^{\tilde z} = \Big\{((1),1),\ ((123),1),\ ((132),1),\ ((12)(45)),\ ((13)(45),-1),$$
$$((23)(45),-1)\Big\},$$
$$A_4^{t_1} = \Big\{((1),1),\ ((12)(34),1),\ ((123),\gamma),\ ((132),\gamma^2),\ ((13)(24),1),\ ((14)(23),1),$$
$$((124),\gamma^2),\ ((142),\gamma),\ ((134),\gamma),\ ((143),\gamma^2),\ ((234),\gamma^2),\ ((243),\gamma)\Big\}.$$

$$A_4^{t_2} = \Big\{((1),1),\ ((12)(34),1),\ ((123),\gamma^2),\ ((132),\gamma),\ ((13)(24),1),\ ((14)(23),1),$$
$$((124),\gamma),\ ((142),\gamma^2),\ ((134),\gamma^2),\ ((143),\gamma),\ ((234),\gamma),\ ((243),\gamma^2)\Big\},$$
$$D_5^{\tilde z} = \Big\{((1),1),\ ((12345),1),\ ((13524),1),\ ((154323),1),\ (14253),1),$$
$$((12)(35),-1),\ ((13)(54),-1),\ ((14)(23),-1),\ ((15)(24),-1),$$
$$((25)(34),-1)\Big\},$$

where $k = 1,2$, $\xi = e^{\frac{2\pi}{5}i}$, $\gamma = e^{\frac{2\pi i}{3}}$.

There are 15 elements in the conjugacy class of the subgroup $\mathbb{Z}_2^-$, 15 elements in the conjugacy class of V_4^-, 12 elements in the conjugacy class of $\mathbb{Z}_5^{t_k}$, $(k = 1,2)$, 20 elements in the conjugacy classes of $\mathbb{Z}_3^t$, 10 elements in the conjugacy class of $D_3^{\tilde z}$, 5 elements in the conjugacy classes of $A_4^{t_k}$, $(k = 1,2)$, 6 elements in the conjugacy class of $D_5^{\tilde z}$.

All the twisted subgroups of $A_5 \times S^1$ (up to their conjugacy class), their normalizers and Weyl groups, are listed in Table 2.24. The lattice of the conjugacy classes of the twisted subgroups in $A_5 \times S^1$ is shown in Figure 2.14. The numbers $n(L,H)$ for twisted subgroups in $A_5 \times S^1$ are listed in Table 2.25.

H^φ	$\mathrm{Im}\,\varphi$	$\mathrm{Ker}\,\varphi$	$N(H^\varphi)$	$W(H^\varphi)$	Comments
A_5	$\mathbb{Z}_1$	A_5	$A_5 \times S^1$	S^1	
D_5	$\mathbb{Z}_1$	D_5	$D_5 \times S^1$	S^1	
A_4	$\mathbb{Z}_1$	A_4	$A_4 \times S_1$	S^1	
D_3	$\mathbb{Z}_1$	D_3	$D_3 \times S_1$	S^1	
$\mathbb{Z}_5$	$\mathbb{Z}_1$	$\mathbb{Z}_5$	$D_5 \times S_1$	$\mathbb{Z}_2 \times S^1$	
V_4	$\mathbb{Z}_1$	V_4	$A_4 \times S_1$	$\mathbb{Z}_3 \times S^1$	
$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3$	$D_3 \times S_1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$V_4 \times S_1$	$\mathbb{Z}_2 \times S^1$	
$\mathbb{Z}_1$	$\mathbb{Z}_1$	$\mathbb{Z}_1$	$A_5 \times S_1$	$A_5 \times S^1$	
V_4^-	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$V_4 \times S^1$	S^1	
D_5^z	$\mathbb{Z}_2$	$\mathbb{Z}_5$	$D_5 \times S^1$	S^1	
$A_4^{t_k}$	$\mathbb{Z}_3$	V_4	$A_4 \times S^1$	S^1	$k = 1, 2$
D_3^z	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$D_3 \times S^1$	S^1	
$\mathbb{Z}_3^t$	$\mathbb{Z}_3$	$\mathbb{Z}_1$	$\mathbb{Z}_3 \times S^1$	S^1	
$\mathbb{Z}_5^k$	$\mathbb{Z}_5$	$\mathbb{Z}_1$	$\mathbb{Z}_5 \times S^1$	S^1	$k = 1, 2$
$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$V_4 \times S^1$	$\mathbb{Z}_2 \times S^1$	

Table 2.24: Twisted Subgroups H^φ in $A_5 \times S^1$, where $\varphi : H \to S^1$ is a homomorphism.

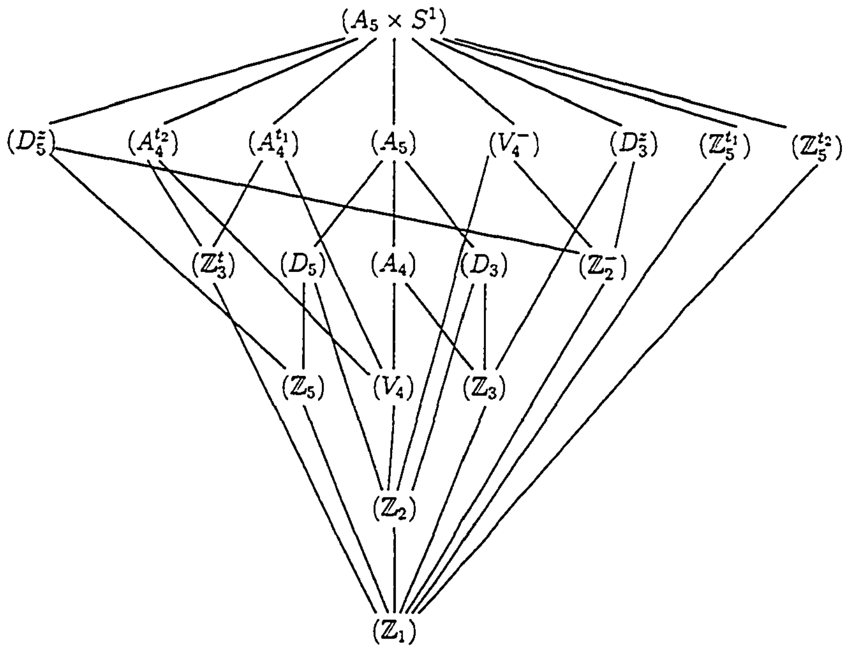


Figure 2.14: Conjugacy Classes of Twisted Subgroups in $A_5 \times S^1$.

L	H	$n(L,H)$	L	H	$n(L,H)$	L	H	$n(L,H)$	L	H	$n(L,H)$
$\mathbb{Z}_5$	$D_5^{\tilde{z}}$	1	$\mathbb{Z}_2$	$A_4^{t_2}$	1	$\mathbb{Z}_3$	A_5	1	$\mathbb{Z}_1$	$D_3^{\tilde{z}}$	10
$\mathbb{Z}_2^-$	$D_5^{\tilde{z}}$	2	$\mathbb{Z}_1$	$A_4^{t_2}$	5	$\mathbb{Z}_2$	A_5	1	$\mathbb{Z}_1$	$\mathbb{Z}_5^{t_k}$	12
$\mathbb{Z}_1$	$D_5^{\tilde{z}}$	6	$\mathbb{Z}_3^{t_1}$	$A_4^{t_1}$	2	$\mathbb{Z}_1$	A_5	1	$\mathbb{Z}_1$	$\mathbb{Z}_2^-$	15
$\mathbb{Z}_5$	D_5	1	V_4	$A_4^{t_1}$	1	$\mathbb{Z}_2^-$	V_4^-	2	V_4	A_4	1
$\mathbb{Z}_2$	D_5	2	$\mathbb{Z}_2$	$A_4^{t_1}$	1	$\mathbb{Z}_2$	V_4^-	1	$\mathbb{Z}_3$	A_4	2
$\mathbb{Z}_1$	D_5	6	$\mathbb{Z}_1$	$A_4^{t_1}$	5	$\mathbb{Z}_1$	V_4^-	15	$\mathbb{Z}_2$	A_4	1
$\mathbb{Z}_3^{t_2}$	$A_4^{t_2}$	2	A_4	A_5	1	$\mathbb{Z}_3$	D_3	1	$\mathbb{Z}_1$	A_4	5
V_4	$A_4^{t_2}$	1	V_4	A_5	1	$\mathbb{Z}_2$	D_3	2	$\mathbb{Z}_1$	$\mathbb{Z}_3^t$	20
$\mathbb{Z}_1$	D_3	10	$\mathbb{Z}_1$	$\mathbb{Z}_5$	6	$\mathbb{Z}_3$	$D_3^{\tilde{z}}$	1	$\mathbb{Z}_1$	$\mathbb{Z}_3$	10
$\mathbb{Z}_2^-$	$D_3^{\tilde{z}}$	2	$\mathbb{Z}_2$	V_4	1	$\mathbb{Z}_1$	V_4	5	$\mathbb{Z}_1$	$\mathbb{Z}_2$	15

Table 2.25: Numbers $n(L,H)$ for Twisted Subgroups in $A_5 \times S^1$.

Chapter 3

Computation of Equivariant Degree

3.1 Equivariant Degree: Definitions and Properties

We vary the groups $\Pi(H)$ defined in the previous section according to the values of dim $W(H)$. More precisely,

Definition 3.1.1. Let $\Pi(H)$ be a subgroup described in Definition 2.3.1. $\Pi(H)$ is called *primary* (resp. *secondary*) group if dim $W(H) = 1$ (resp. dim $W(H) = 0$).

Remark 3.1.2. *Notice that if* (H) *is an orbit type in* $\mathbb{R} \oplus V$ *such that* $\dim W(H) > 1$, *then by Theorem 2.3.2, we obtain that* $\Pi(H) = \{0\}$. *In other words*

$$\Pi^G = \bigoplus_{\dim W(H) \leq 1} \Pi(H),$$

where the summation is taken over the orbit types (H) *in* $\mathbb{R} \oplus V$.

In what follows, we will assume that $G = \Gamma \times S^1$, *where* Γ *is a finite group, and denote by* $\Phi_1(G)$ *the set of all the conjugacy classes* (H) *of subgroups* H *such that* $\dim W(H) = 1$. *Then we have the following theorem.*

Theorem 3.1.3. *(cf. [9]) If* (H) *is an orbit type in* $\mathbb{R} \oplus V$ *such that* $(H) \in \Phi_1(G)$, *then* $\Pi(H) \cong \mathbb{Z}$.

It will be convenient to write the elements in $\bigoplus_{\dim W(H)=1} \Pi(H)$ in a form of finite sums, indexed by the orbit types $(H) \in \Phi_1(G)$. i.e. $\alpha \in \bigoplus_{\dim W(H)=1} \Pi(H)$ can be written as $\alpha = \sum_{(H)} n_H \cdot (H)$, $n_H \in \mathbb{Z}$.

Let us denote by π the natural projection of Π^G onto $\bigoplus_{\dim W(H)=1} \Pi(H)$.

Definition 3.1.4. If $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible, G-equivariant map and $\Omega \subset \mathbb{R} \oplus V$ an open, bounded and G-invariant set, we put

$$G\text{-Deg}\,(f, \Omega) := \pi(\deg_G(f, \Omega)),$$

and write

$$G\text{-Deg}\,(f, \Omega) := \sum_{\dim W(H)=1} n_H \cdot (H) \in \bigoplus_{\dim W(H)=1} \Pi(H) \subset \Pi^G. \tag{3.1}$$

The G-Deg (f, Ω) is called *primary degree* of f in Ω.

Clearly, since G-Deg (f, Ω) is a "projection" of the G-equivariant degree, it satisfies the existence, additivity, homotopy and suspension properties.

Remark 3.1.5. Let us point out that $(H) \in \Phi_1(G)$ if and only if H is a twisted m-folded subgroup, i.e. there exists $K \subset \Gamma$ and a homomorphism $\varphi : K \to S^1$ such that

$$H = K^{\varphi,m} = \{(\gamma, z) \in K \times S^1 \,|\, \varphi(\gamma) = z^m\}.$$

One of the advantages of using the primary degree is that it is possible to consider, as a range of this degree, the free $\mathbb{Z}$-module $A_1(G)$ generated by the orbit types $(H) \in \Phi_1(G)$, i.e. $A_1(G) = \mathbb{Z}[\Phi_1(G)]$. Thus $\mathbb{Z}$-module does not depend on the representation V, which will allow us to explore additional properties of the primary degree, such as multiplicativity property.

Before we establish the computational formula for the primary degree, we need to discuss the equivariant degree introduced by H. Ulrich (cf. [48]) in the case without free parameter.

3.2 Γ-Equivariant Degree without Free Parameter

The equivariant degree Γ-Deg(f, Ω) can be computed using appropriate recurrence formula. Since in this case $\Pi(H) = \mathbb{Z}$ for any orbit (H) in V such that $\dim W(H) =$

0, so we have $\Pi^\Gamma \subset A(\Gamma)$, where $A(\Gamma)$ denotes the Burnside ring of Γ. The fact that $A(\Gamma)$ is a ring with identity is very important for this degree. The $\Gamma\text{-Deg}\,(f, \Omega)$ has additional property, called the *multiplicativity property*, which is formulated below.

Theorem 3.2.1. (Multiplicativity Property) *(cf. [35]) Let V and W be two Γ-representations. Suppose $\Omega_1 \subset V$, $\Omega_2 \subset W$ be two Γ-invariant open bounded sets, $f: V \to V$ an Ω_1-admissible map and $g: W \to W$ an Ω_2-admissible map. Then*

$$\Gamma\text{-Deg}\,(f \times g, \Omega_1 \times \Omega_2) = \Gamma\text{-Deg}\,(f, \Omega_1) \cdot \Gamma\text{-Deg}\,(g, \Omega_2),$$

where the multiplication is taken in the Burnside ring $A(\Gamma)$.

Suppose $f : V \to V$ is an Ω-admissible, Γ-equivariant map, where $\Omega \subset V$ is bounded, open and Γ-invariant, Γ is a finite group. Recall that $\Pi^\Gamma = \bigoplus_{\dim W(H)=0} \Pi(H)$.

We will discuss the computational technique for $\Gamma\text{-Deg}\,(f, \Omega)$ for two cases.

Case 1: **f is a regular normal map:** Since f is a regular normal map, for every orbit type (K) in Ω, we have that $f^K : V^K \to V^K$ is Ω_K-admissible and $W(K)$-equivariant. Since $f^K|_{\Omega_k}$ has zero as a regular value and

$$(f^K)^{-1}(0) \cap \Omega_K = W(K)x_1 \cup \cdots \cup W(K)x_m.$$

It is clear that $\forall \gamma \in W(K)$ we have

$$\operatorname{sign}\det Df^K(x_i) = \operatorname{sign}\det Df^K(\gamma x_i).$$

In this case we obtain:

Theorem 3.2.2. *If f is a regular normal, Γ-invariant, Ω-admissible map from V to V, then*

$$\Gamma\text{-Deg}\,(f, \Omega) = \sum_{\dim W(K)=0} n_K(K),$$

where $n_K = \sum_{i=1}^{m} \operatorname{sign}\,\det Df^K(x_i)$ and $(f^K)^{-1}(0) \cap \Omega_K = W(K)x_1 \cup \cdots \cup W(K)x_m$.

Notice that

$$n_K = \frac{\sum_{x \in (f^K)^{-1}(0) \cap \Omega_K} \operatorname{sign}\det Df^K(x)}{|W(K)|} = \frac{\deg(f^K, \Omega_K)}{|W(K)|},$$

where $\deg(f^K, \Omega_K)$ is the local Brower degree.

Case 2: f **is an arbitrary** Γ**-maps:** (not necessary a regular normal map)
Let

$$\mathcal{J}(\Omega) = \{(K) \mid K = \Gamma_x \text{ for some } x \in \Omega\}.$$

Let us begin with an explanation of the computational method applied to a simplified situation, where the only orbit types in Ω are $(K_0), (K_1), (K_2)$, and (K_3), with the lattice of the orbit types illustrated in the diagram below.

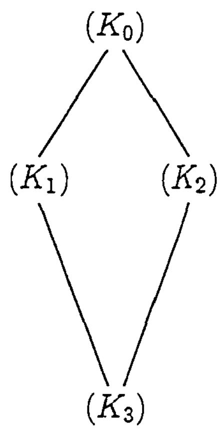


Figure 3.1: Lattice of isotropies.

Then the maximal orbit type in Ω is K_0 and

$$\Omega^{K_0} = \{x \in \Omega \mid \Gamma_x \supset K_0\} = \Omega_{K_0} = \{x \in \Omega \mid \Gamma_x = K_0\},$$

We consider a regular normal approximation $\tilde{f}$ of f, which is Γ-homotopic to f (notation: $\tilde{f} \sim f$). By the regular normal approximation theorem for homotopies, we have that for all $(K) \in \mathcal{J}(\Omega)$, $\tilde{f}^K \sim f^K$in Ω^K, therefore

$$n_{K_0} = \frac{\deg(\tilde{f}^{K_0}, \Omega_{K_0})}{|W(K_0)|} = \frac{\deg(\tilde{f}^{K_0}, \Omega^{K_0})}{|W(K_0)|} = \frac{\deg(f^{K_0}, \Omega^{K_0})}{|W(K_0)|}. \tag{3.2}$$

Now, let us illustrate the computation of n_{K_1}. Since $K_1 \subset K_0$ then $\Omega^{K_1} \supset \Omega_{K_1}$ and $\Omega^{K_0} \subset \Omega^{K_1}$. Let $x \in (\tilde{f}^{K_1})^{-1}(0) \cap \Omega_{K_0}$ then by normality of $\tilde{f}$,

$$\det D\tilde{f}^{K_1}(x) = \det(D\tilde{f}^{K_0}(x) \times \mathrm{Id})(x) = \det D\tilde{f}^{K_0}(x),$$

so

$$n_{K_1} = \frac{\deg(\tilde{f}^{K_1}, \Omega_{K_1})}{|W(K_1)|} = \frac{\deg(\tilde{f}^{K_1}, \Omega^{K_1}) - n(K_1, K_0)\deg(\tilde{f}^{K_0}, \Omega^{K_0})}{|W(K_1)|},$$

and by homotopy

$$n_{K_1} = \frac{\deg(f^{K_1}, \Omega^{K_1}) - n(K_1, K_0) \deg(f^{K_0}, \Omega^{K_0})}{|W(K_1)|},$$

thus, by Equation (3.2)

$$n_{K_1} = \frac{\deg(f^{K_1}, \Omega^{K_1}) - n(K_1, K_0) n_{K_0} |W(K_0)|}{|W(K_1)|}. \tag{3.3}$$

The same idea can be applied to compute n_{K_2} and then n_{K_3}, i.e.

$$n_{K_3} = \frac{\deg(f^{K_3}, \Omega^{K_3}) - \sum_{k=0}^{2} n(K_3, K_k) n_{K_k} |W(K_k)|}{|W(K_3)|}.$$

In the general case, by applying a simple induction over the orbit types in Ω, we can easily prove the following recurrence formula:

$$n_L = \frac{\deg(f^L, \Omega^L) - \sum_{(K)>(L)} n(L, K) n_K |W(K)|}{|W(L)|}. \tag{3.4}$$

Let us illustrate these computations on several examples.

Example 3.2.3. $\Gamma = D_4$

Assume $V = \mathbb{C}$ is the orthogonal irreducible D_4-representation, with the action of D_4 on V, given by $\gamma z = \gamma \cdot z$ (complex multiplication) and $\kappa z = \bar{z}$. We define the map $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = -z$, which is clearly not normal. Suppose $\Omega = B$ is the unit ball. then

$$\mathcal{J}(B) = \{(D_4), (D_1), (\widetilde{D_1}), (\mathbb{Z}_1)\},$$

and the following is the lattice of isotropy groups:

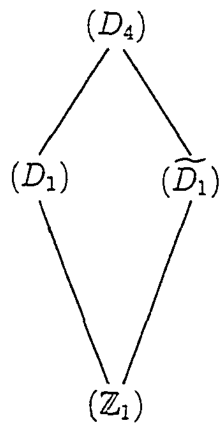


Figure 3.2: Lattice of isotropies.

$$D_4\text{-Deg}\,(f, B) = n_{D_4}(D_4) + n_{D_1}(D_1) + n_{\widetilde{D_1}}(\widetilde{D_1}) + n_{\mathbb{Z}_1}(\mathbb{Z}_1).$$

Clearly, $V^{D_4} = \{(0,0)\}$, $V^{\mathbb{Z}_1} = \mathbb{C}$, $V^{D_1} = \mathbb{R}$, and $V^{\widetilde{D_1}} = \mathbb{R}$. By using tabels 2.16 and 2.17 we have

$$n_{D_4} = \frac{\deg(f, B\cap\{(0,0)\})}{|W(D_4)|} = \frac{(-1)^0}{1} = 1,$$

$$n_{D_1} = \frac{deg(f, B\cap\mathbb{R}) - n(D_1, D_4)n_{D_4}|W(D_4)|}{|W(D_1)|} = \frac{(-1)^1 - 1(1)(1)}{2} = -1,$$

$$n_{\widetilde{D_1}} = \frac{deg(f, B\cap\mathbb{R}) - n(\widetilde{D_1}, D_4)n_{D_4}|W(D_4)|}{|W(\widetilde{D_1})|} = \frac{(-1)^1 - 1(1)(1)}{2} = -1,$$

$$\begin{aligned} n_{\mathbb{Z}_1} &= \frac{deg(f, B\cap\mathbb{R}^2) - n(\mathbb{Z}_1, D_4)n_{D_4}|W(D_4)| - n(\mathbb{Z}_1, D_1)n_{D_1}|W(D_1)| - n(\mathbb{Z}_1, \widetilde{D_1})n_{\widetilde{D_1}}|W(\widetilde{D_1})|}{|W(\mathbb{Z}_1)|} \\ &= \frac{(-1)^2 - 1(1)(1) - 2(-1)(2) - 2(-1)(2)}{8} = 1. \end{aligned}$$

Then

$$D_4\text{-Deg}\,(f, B) = (D_4) - (D_1) - (\widetilde{D_1}) + (\mathbb{Z}_1).$$

Example 3.2.4. $\Gamma = \mathbf{A_4}$

Let us consider $V = \mathbb{R}^4$ to be the permutation representation of A_4, where A_4 acts on $\mathbb{R}^4$ by permuting the coordinates of vectors. Then $V = V^{A_4} \oplus V_0$ where V^{A_4} is the fixed point space of A_4, which is spanned by $(1, 1, 1, 1)$, and $V_0 = (V^{A_4})^{\perp} = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 + x_4 = 0\}$. The orbit types in V are $(A_4), (\mathbb{Z}_2), (\mathbb{Z}_3)$, and $(\mathbb{Z}_1)$, which can be illustrated by the following lattice:

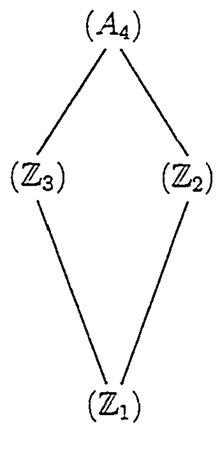


Figure 3.3: Lattice of isotropies.

Clearly, $\dim V^{A_4} = 1$, $\dim V^{\mathbb{Z}_2} = \dim V^{\mathbb{Z}_3} = 2$, and $\dim V^{\mathbb{Z}_1} = 4$. Supposed B is the unit ball and let f be $-\mathrm{Id}$. Then

$$A_4\text{-Deg}\,(-\mathrm{Id}\,, B) = n_{A_4}(A_4) + n_{\mathbb{Z}_3}(\mathbb{Z}_3) + n_{\mathbb{Z}_2}(\mathbb{Z}_2) + n_{\mathbb{Z}_1}(\mathbb{Z}_1),$$

then by using Tables 2.20 and 2.21 we obtain:

$$n_{A_4} = \frac{\deg(f, B\cap\mathbb{R})}{|W(A_4)|} = \frac{(-1)^1}{1} = -1,$$

$$n_{\mathbb{Z}_3} = \frac{deg(f, B\cap\mathbb{R}^2) - n(\mathbb{Z}_3, A_4)n_{A_4}|W(A_4)|}{|W(\mathbb{Z}_3)|} = \frac{(-1)^2 - 1(-1)(1)}{1} = 2,$$

$$n_{\mathbb{Z}_2} = \frac{deg(f, B\cap\mathbb{R}^2) - n(\mathbb{Z}_2, A_4)n_{A_4}|W(A_4)|}{|W(\mathbb{Z}_2)|} = \frac{(-1)^2 - 1(-1)(1)}{2} = 1,$$

$$\begin{aligned} n_{\mathbb{Z}_1} &= \frac{deg(f, B\cap\mathbb{R}^4) - n(\mathbb{Z}_1, A_4)n_{A_4}|W(A_4)| - n(\mathbb{Z}_1, \mathbb{Z}_3)n_{\mathbb{Z}_3}|W(\mathbb{Z}_3)| - n(\mathbb{Z}_1, \mathbb{Z}_2)n_{\mathbb{Z}_2}|W(\mathbb{Z}_2)|}{|W(\mathbb{Z}_1)|} \\ &= \frac{(-1)^4 - (-1)(1)(1) - 2(4)(1) - 1(3)(2)}{12} = -1. \end{aligned}$$

Thus:

$$A_4\text{-Deg}\,(f, B) = -(A_4) + 2(\mathbb{Z}_3) + (\mathbb{Z}_2) - (\mathbb{Z}_1).$$

3.3 S^1-Equivariant Degree

We start with a particular case of the primary degree for the group $G = \Gamma \times S^1$, where $\Gamma = \{1\}$ i.e. $G = S^1$. We denote by $A_1(S^1)$, the free $\mathbb{Z}$-module generated by the symbols $(\mathbb{Z}_k)$, $k = 1, 2, 3, \ldots$. Consider an orthogonal S^1-representation V, an open S^1-invariant bounded set $\Omega \subset \mathbb{R} \oplus V$, and an Ω-admissible S^1-equivariant map $f : \mathbb{R} \oplus V \to V$. Then the primary degree S^1-Deg (f, Ω), which we will simply call the *S^1-equivariant degree*, is an element in $A_1(S^1)$ which can be written as

$$S^1\text{-Deg}\,(f, \Omega) = \sum_k n_k(\mathbb{Z}_k) = n_{k_1}(\mathbb{Z}_{k_1}) + n_{k_2}(\mathbb{Z}_{k_2}) + \cdots + n_{k_r}(\mathbb{Z}_{k_r}), \tag{3.5}$$

where $n_k \in \mathbb{Z}$.

The S^1-degree can be introduced axiomatically, based on its fundamental properties only, (its existence follows from the general construction) and we can use the axioms for its computations.

3.3.1 Basic and m-Folding Maps

We denote by $\mathcal{V}_k$, $k = 1, 2, 3, \dots$, the (non-trivial) k-th real irreducible representation of the group S^1, i.e. $\mathcal{V}_k$ is the space $\mathbb{R}^2 = \mathbb{C}$ with the S^1-action given by $\gamma z := \gamma^k \cdot z$, $\gamma \in S^1$, $z \in \mathbb{C}$, and define the set

$${}^k\Omega := \left\{ (t, z) \in \mathbb{R} \oplus \mathcal{V}_k \middle| \; |t| < 1, \; \frac{1}{2} < |z| < 2 \right\}, \tag{3.6}$$

and $\mathfrak{b} : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ by

$$\mathfrak{b}(t, z) := \big(1 - |z| + it\big) \cdot z, \quad (t, z) \in \mathbb{R} \times \mathcal{V}_k, \tag{3.7}$$

where '$\cdot$' denotes the complex multiplication in $\mathcal{V}_k = \mathbb{C}$. It is clear that the map $\mathfrak{b}$ is S^1-equivariant and ${}^k\Omega$-admissible. We will call the map $\mathfrak{b}$ the *S^1-basic map* on ${}^k\Omega$ (or simply *basic map* if it will be clear from the context what representation is involved).

Further, for every integer $m = 1, 2, 3, \dots$, we define the homomorphism $\theta_m : S^1 \to S^1$ (called *m-folding*), by $\theta_m(\gamma) = \gamma^m$, $\gamma \in S^1$, and define the induced by θ_m homomorphism $\Theta_m : A_1(S^1) \to A_1(S^1)$, by

$$\Theta_m(\mathbb{Z}_k) := (\mathbb{Z}_{km}), \quad k = 1, 2, 3, \dots, \tag{3.8}$$

i.e. $\Theta_m(\mathbb{Z}_k) = (\theta_m^{-1}(\mathbb{Z}_k))$, where $(\mathbb{Z}_k)$ are the free generators of $A_1(S^1)$.

Notice that if $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible map for a certain open bounded S^1-invariant subset $\Omega \subset \mathbb{R} \oplus V$, then for every integer $m = 1, 2, 3, \dots$, we can, first, define the *associated m-folded* S^1-representation ${}^mV$, which is the same vector space V with the S^1-action "$\cdot$" given by

$$\gamma \cdot v := \theta_m(\gamma)v = \gamma^m v, \quad \gamma \in S^1, \;\; v \in V. \tag{3.9}$$

Next, the map f considered from $\mathbb{R} \oplus {}^mV$ to ${}^mV$, is S^1-equivariant as well. The set Ω considered as an S^1-subset of $\mathbb{R} \oplus {}^mV$ will be denoted by ${}^m\Omega$. In what follows, we will say that the pair $(f, {}^m\Omega)$ is the *m-folded* admissible pair *associated* with (f, Ω).

3.3.2 Axiomatic Definition of S^1-Degree

Theorem 3.3.1. *(cf.[4]) There exists a unique function, denoted by S^1-Deg, assigning to each admissible pair (f, Ω) an element $S^1\text{-}Deg(f, \Omega) \in A_1(S^1)$ satisfying the following properties:*

(P1) (Existence) *If $S^1\text{-}Deg(f, \Omega) = \sum_k n_k(\mathbb{Z}_k)$ is such that $n_{k_o} \neq 0$ for some $k_o = 1, 2, \ldots,$ then there exists $x \in \Omega$ with $f(x) = 0$ and $G_x \supset \mathbb{Z}_{k_o}$.*

(P2) (Additivity) *Assume that Ω_1 and Ω_2 are two S^1-invariant open disjoint subsets of Ω such that $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$. Then*

$$S^1\text{-}Deg(f, \Omega) = S^1\text{-}Deg(f, \Omega_1) + S^1\text{-}Deg(f, \Omega_2).$$

(P3) (Homotopy) *Suppose that $h : [0, 1] \times \mathbb{R} \times V \to V$ is an Ω-admissible S^1-equivariant homotopy (i.e. $h_\lambda := h(\lambda, \cdot, \cdot)$ is Ω-admissible for all $\lambda \in [0, 1]$). Then*

$$S^1\text{-}Deg(h_\lambda, \Omega) = \text{constant}.$$

(P4) (Suspension) *Suppose that W is another orthogonal S^1-representation and let U be an open, bounded S^1-invariant neighborhood of origin in W. Then*

$$S^1\text{-}Deg(f \times \mathrm{Id}, \Omega \times U) = S^1\text{-}Deg(f, \Omega).$$

(P5) (Normalization) *For the basic map $\mathfrak{b} : \mathbb{R} \oplus \mathcal{V}_1 \to \mathcal{V}_1$, we have*

$$S^1\text{-}Deg(\mathfrak{b}, {}^1\Omega) = (\mathbb{Z}_1),$$

and if V is a trivial S^1-representation, then

$$S^1\text{-}Deg(f, \Omega) = 0.$$

(P6) (Folding) *Let ${}^mV$ be the m-folded representation associated with V, and $(f, {}^m\Omega)$ the m-folded admissible pair associated with (f, Ω). Then*

$$S^1\text{-}Deg(f, {}^m\Omega) = \Theta_m\big[S^1\text{-}Deg(f, \Omega)\big].$$

3.3.3 Computation of S^1-Degree via Reduction to Basic and $\mathbb{C}$-Complementing Maps

In order to use the primary degree without referring to its topological construction, we need the definition of $\mathbb{C}$-complementing maps and splitting lemma.

3.3.4 $\mathbb{C}$-Complementing Maps

Let $V = \mathcal{V}_k$ be a k-th irreducible S^1-representation. We define on $\mathcal{V}_k$ a complex structure "sensitive" to the S^1-action as follows: for $z \in \mathbb{C}$ we put $z = \gamma|z|$, where $\gamma = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. The complex multiplication of $v \in \mathcal{V}_k$ by the number z is defined by

$$z \cdot v := |z|e^{\frac{i\theta}{k}}v. \tag{3.10}$$

We call this complex structure on $\mathcal{V}_k$ by a *natural complex structure.*

Definition 3.3.2. Let $\mathfrak{b} : \mathbb{R} \oplus \mathcal{V}_k \to \mathcal{V}_k$ be the k-th basic map and let ${}^k\Omega$ be a subset of $\mathbb{R} \oplus \mathcal{V}_k$. Assume, further, that $\mathcal{V}_k$ is equipped with the natural complex structure and $\mathcal{O}$ is given by (3.11). Suppose, finally, that $f : \mathbb{C} \oplus \mathcal{V}_k \to \mathbb{R} \oplus \mathcal{V}_k$ is defined by $f(\lambda, v) = \Big(|\lambda|(\|v\| - 1) + \|v\| + 1, \lambda \cdot v\Big)$, where $\lambda \in \mathbb{C}$, $v \in \mathcal{V}_k$. Then the pair $(f, \mathcal{O})$ is called a $\mathbb{C}$-*complementing pair* to $(\mathfrak{b}, {}^k\Omega)$.

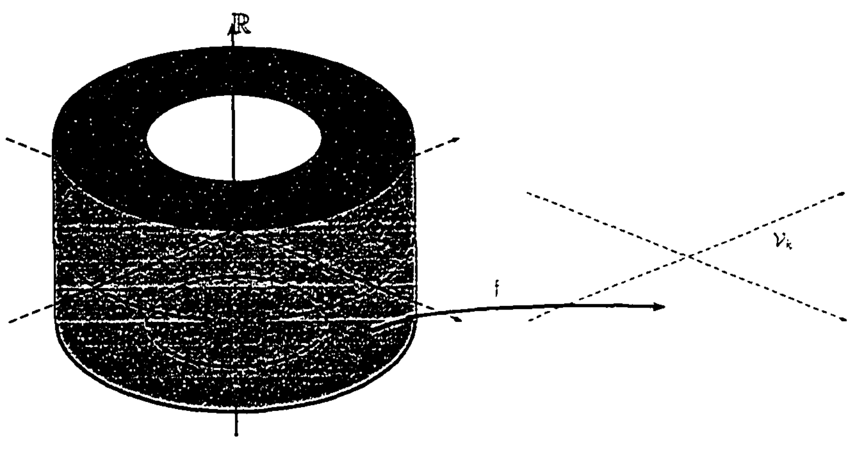


Figure 3.4: The admissible pair $(\mathfrak{b}, {}^k\Omega)$.

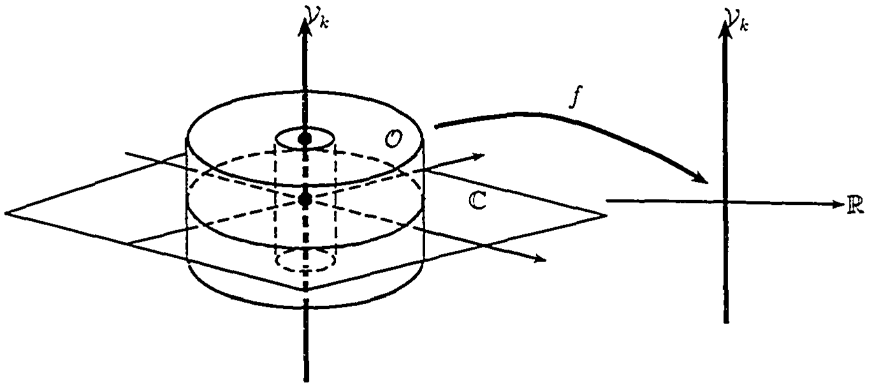


Figure 3.5: The admissible pair $(f, \mathcal{O})$.

It is clear that $(f, \mathcal{O})$ and $(\mathfrak{b}, {}^k\Omega)$ are admissible pairs (see Figures 3.4 and 3.5).

Lemma 3.3.3. **Splitting Lemma***(cf. [4])*

Let G be a compact Lie group, V_1 and V_2 orthogonal G-representations, $V = V_1 \oplus V_2$. Assume that the isotypical decomposition of V contains only components modeled on irreducible G-representations of complex type. Suppose that $a_j : S^1 \to GL^G(V_j)$, $j = 1, 2$, are two continuous maps and $a : S^1 \to GL^G(V)$ is given by

$$a(\lambda) = a_1(\lambda) \oplus a_2(\lambda), \quad \lambda \in S^1.$$

Let

$$\mathcal{O}_j := \left\{(\lambda, v_j) \in \mathbb{C} \oplus V_j \middle| \;\|v_j\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\},$$

$$\mathcal{O} := \left\{(\lambda, v) \in \mathbb{C} \oplus V \middle| \;\|v\| < 2,\ \frac{1}{2} < |\lambda| < 4\right\}, \tag{3.11}$$

Define the maps $f_{a_j} : \overline{\mathcal{O}}_j \to \mathbb{R} \oplus V_j$, $j = 1, 2$, $f_a : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ by

$$f_{a_j}(\lambda, v_j) = \left(|\lambda|(\|v_j\| - 1) + \|v_j\| + 1, a_j\left(\frac{\lambda}{|\lambda|}\right) v_j\right), \quad j = 1, 2,$$
$$f_a(\lambda, v) = \left(|\lambda|(\|v\| - 1) + \|v\| + 1, a\left(\frac{\lambda}{|\lambda|}\right) v\right),$$

where $v_j \in V_j$, $j = 1, 2$, $v \in V$ and $\frac{1}{2} \le |\lambda| \le 4$. Then

$$G\text{-Deg}\,(f_a, \mathcal{O})) = G\text{-Deg}\,(f_{a_1}, \mathcal{O}_1) + G\text{-Deg}\,(f_{a_2}, \mathcal{O}_2). \tag{3.12}$$

By using the splitting lemma we can provide the following proposition.

Proposition 3.3.4. *(cf.[4]) Let $(f, \mathcal{O})$ be a $\mathbb{C}$-complementing pair to $(\mathfrak{b}, {}^k\Omega)$. Then*

$$S^1\text{-}Deg(f, \mathcal{O}) = S^1\text{-}Deg(\mathfrak{b}, {}^k\Omega) = (\mathbb{Z}_k).$$

3.4 Basic Maps for Irreducible Representations of $G = \Gamma \times S^1$

Assume that Γ is a finite group and $\mathcal{V}_k$ is a complex irreducible representation of Γ. We define an action of $z \in S^1$ on $\mathcal{V}_k$ by formula

$$zv := z^j \cdot v, \quad z \in S^1, \ \ v \in \mathcal{V}_k, \ \ j \in \mathbb{N},$$

where '$\cdot$' denotes the usual complex multiplication in $\mathcal{V}_k$. The obtained real irreducible representation of the group $G = \Gamma \times S^1$ we will denote by $\mathcal{V}_{k,j}$. The following definition provides two examples of the simplest possible G-equivariant maps with non-zero primary degree.

Definition 3.4.1.

(a) Let $\mathcal{O} \subset \mathbb{R} \oplus \mathcal{V}_{j,k}$ be the set

$$\mathcal{O} = \{(t, v) \in \mathbb{R} \oplus \mathcal{V}_{j,k} \,|\, \frac{1}{2} < \|v\| < 2, \ \ -1 < t < 1\},$$

and $\mathfrak{b} : \overline{\mathcal{O}} \to \mathcal{V}_{j,k}$ be defined by

$$\mathfrak{b}(t, v) = (1 - \|v\| + it) \cdot v, \quad (t, v) \in \overline{\mathcal{O}}. \tag{3.13}$$

Then the map $\mathfrak{b}$ is called a *basic map* on $\mathcal{O}$, and the pair $(\mathfrak{b}, \mathcal{O})$ is called a *basic pair* for the irreducible G-representation $\mathcal{V}_{j,k}$;

(b) Let $\Omega = \{(\lambda, v) \in \mathbb{C} \oplus \mathcal{V}_{j,k} \,|\, \|v\| < 2, \ \frac{1}{2} < |\lambda| < 4\}$ and $f : \overline{\Omega} \to \mathbb{R} \oplus \mathcal{V}_{j,k}$ be defined as

$$f(\lambda, v) = \Big(|\lambda|(\|v\| - 1) + \|v\| + 1, \lambda \cdot v\Big), \quad (\lambda, v) \in \overline{\Omega}, \tag{3.14}$$

where $\lambda \cdot v$ denotes the usual complex multiplication of v by λ. Then the map f is called a $\mathbb{C}$-*complementing map* on Ω, and the pair (f, Ω) is called a $\mathbb{C}$-*complementing pair* for the irreducible G-representation $\mathcal{V}_{j,k}$.

Remark 3.4.2. Notice that the basic map $\mathfrak{b}$ and the $\mathbb{C}$-complementing map f (see section 3.3) are G-equivariant. Moreover, $\mathfrak{b}$ is $\mathcal{O}$-admissible. Indeed, since for all $(t, v) \in \mathcal{O}$, $v \neq 0$, it follows from the equality

$$0 = \mathfrak{b}(t, v) = (1 - \|v\| + it) \cdot v,$$

that $1 - \|v\| + it = 0$, which is equivalent to $\|v\| = 1$ and $t = 0$. Consequently, we have

$$\mathfrak{b}^{-1}(0) = \Big\{(0, v) \big| \; \|v\| = 1\Big\} \subset \mathcal{O}.$$

The map $\mathfrak{b}$ is illustrated below on Figure 3.6.

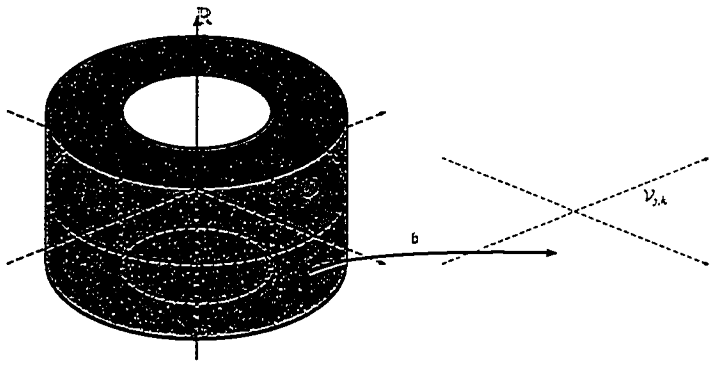


Figure 3.6: Basic Map $\mathfrak{b}$.

Similarly, the map f is Ω-admissible. Indeed, since $\lambda \neq 0$ we have for $(\lambda, v) \in \overline{\Omega}$ that the equality

$$0 = f(\lambda, v) = \Big(|\lambda|(\|v\| - 1) + \|v\| + 1, \lambda \cdot v\Big),$$

implies that $v = 0$ and $|\lambda| = 1$. Consequently,

$$f^{-1}(0) = \Big\{(\lambda, 0) \big| \; |\lambda| = 1\Big\} \subset \Omega.$$

The basic map $\mathfrak{b}$ seems to be the simplest G-equivariant map for which the primary G-degree on $\mathcal{O}$ may be non-zero. On the other hand, the definition of f was motivated by the applications to the bifurcation theory. Although, these two maps are essentially different, their primary G-equivariant degrees are equal. (cf. [4]))

3.5 Computational Formula for Basic Maps without Parameter and Examples

Let $\mathcal{V}_k$ be an irreducible Γ-representation, and $\mathfrak{b} : \mathcal{V}_k \to \mathcal{V}_k$ be simply the map $\mathfrak{b} = -\mathrm{Id}$. The map $\mathfrak{b}$ will be called *basic* for the equivariant degree without parameter.

Consider the set $\Phi_0(\Gamma, \mathcal{V}_k)$ of all orbit types (L) in $\mathcal{V}_k$ such that $\dim W(L) = 0$. We can assume that the partial order in $\Phi_0(\Gamma, \mathcal{V}_k)$ is extended to the total order, i.e. $\Phi_0(\Gamma, \mathcal{V}_k) = \{(L_1), (L_2), \cdots, (L_n)\}$, $(L_1) < (L_2) < \cdots < (L_n)$. Then the Γ-equivariant degree

$$\Gamma\text{-Deg}\,(-\mathrm{Id}, B_k) = \sum_{(L_j)\in\Phi_0(\Gamma,\mathcal{V}_k)} n_{L_j} \cdot (L_j),$$

where B_k denotes the unit ball in $\mathcal{V}_k$, can be computed from the formula

$$n_{L_j} = \frac{(-1)^{n_j} - \sum_{k=1}^{j-1} n(L_j, L_k) \cdot n_{L_k} \cdot |W(L_k)|}{|W(L_j)|}, \tag{3.15}$$

where $n_j = \dim V^{L_j}$.

In this section we present several examples of computations of the equivariant degree for the basic maps in the case of various groups Γ. These results will be used later to concrete applied problems. Notice that to compute the equivariant degree we need the isotropy lattice for each irreducible representation and the dimension of the fixed point space for each subgroup K of Γ. For this purpose we can use the following theorem.

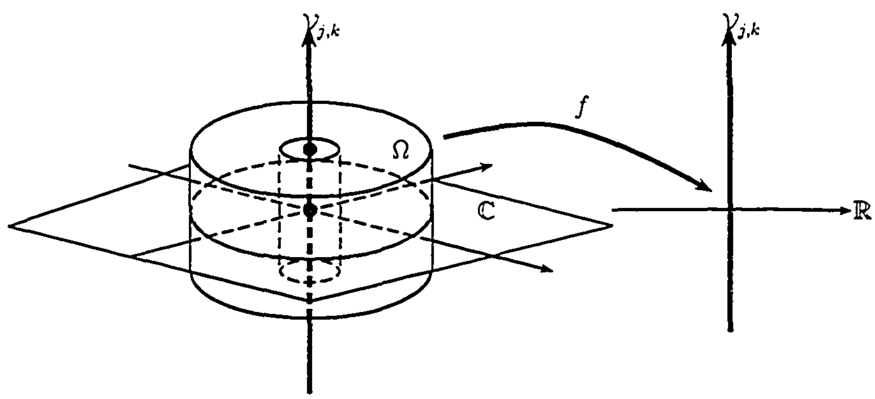


Figure 3.7: $\mathbb{C}$-complementing Map f.

Theorem 3.5.1. *(cf. [18]) Let T be a complex (resp. real) representation of the finite group Γ in a space V. then*

$$\dim V^H = \frac{1}{|H|}\sum_{h\in H}\chi_T(h),$$

for any subgroup $H \subseteq \Gamma$.

3.5.1 Degrees of Basic Maps for the Dihedral Group D_3

(i) There is a one-dimensional trivial representation $\mathcal{V}_0$. In this case we have

$$\deg_{\mathcal{V}_0} = -(D_3).$$

(ii) There is a one-dimensional representation $\mathcal{V}_1$, given by the homomorphism $c : D_3 \to \mathbb{Z}_2$ such that $\ker c = \mathbb{Z}_3$. Let us obtain the corresponding degree of the basic map by using Theorem 3.5.1, Table 1.1, and Formula 3.15.

$$\dim V_1^{D_3} = \tfrac{1}{6}(\chi_2(1) + 2\chi_2(\gamma) + 3\chi_2(\kappa)) = \tfrac{1}{6}(1 + 2(1) + 3(-1)) = 0,$$
$$\dim V_1^{\mathbb{Z}_3} = \tfrac{1}{3}(\chi_2(1) + 2\chi_2(\gamma)) = \tfrac{1}{3}(1 + 2(1)) = 1.$$

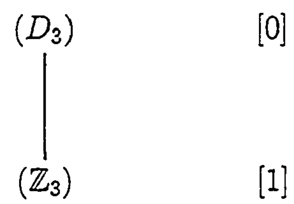


Figure 3.8: Isotropy lattice for $\mathcal{V}_1$.

And

$$\deg_{\mathcal{V}_1} = (D_3) - (\mathbb{Z}_3).$$

(iii) There is one two dimensional representation $\mathcal{V}_2$ of D_3 on $\mathbb{C}$ given by

$$\gamma z := \gamma \cdot z, \text{ for } \gamma \in \mathbb{Z}_3 \text{ and } z \in \mathbb{C},$$

$$\kappa z := \overline{z},$$

where $\gamma \cdot z$ denotes the usual complex multiplication. Then by applying Theorem 3.5.1, and Table 1.1 we obtain

$$\begin{aligned}
\dim V_2^{D_3} &= \tfrac{1}{6}(\chi_3(1) + 2\chi_3(\gamma) + 3\chi_3(\kappa)) = \tfrac{1}{6}(2 + 2(-1) + 3(0)) = 0,\\
\dim V_2^{\mathbb{Z}_3} &= \tfrac{1}{3}(\chi_3(1) + 2\chi_3(\gamma)) = \tfrac{1}{3}(2 + 2(-1)) = 0,\\
\dim V_2^{D_1} &= \tfrac{1}{2}(\chi_3(1) + \chi_3(\kappa)) = \tfrac{1}{2}(2 + 0) = 1,\\
\dim V_2^{\mathbb{Z}_1} &= \tfrac{1}{1}(\chi_3(1)) = 2.
\end{aligned}$$

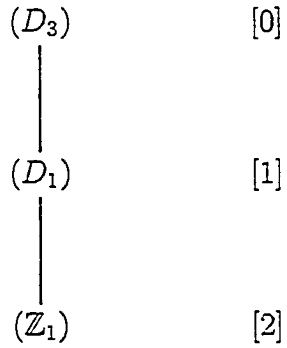


Figure 3.9: Isotropy lattice for $\mathcal{V}_2$.

By Formula 3.15 we have the following degree of the basic map:

$$\deg_{\mathcal{V}_2} = (D_3) - 2(D_1) + (\mathbb{Z}_1).$$

3.5.2 Degrees of Basic Maps for the Dihedral Group D_4

(i) There is a one-dimensional trivial representation $\mathcal{V}_0$. In this case we have

$$\deg_{\mathcal{V}_0} = -(D_4).$$

(ii) There is a one-dimensional representation $\mathcal{V}_1$, given by the homomorphism $c : D_4 \to \mathbb{Z}_2$ such that $\ker c = \mathbb{Z}_4$. Then by applying Theorem 3.5.1, and Table 1.2 we obtain

$$\begin{aligned}
\dim V_1^{D_4} &= \tfrac{1}{8}(\chi_2(1) + 2\chi_2(\gamma) + \chi_2(\gamma^2) + 2\chi_2(\kappa) + 2\chi_2(\kappa\gamma))\\
&= \tfrac{1}{8}(1 + 2(1) + 1 + 2(-1) + 2(-1)) = 0,
\end{aligned}$$

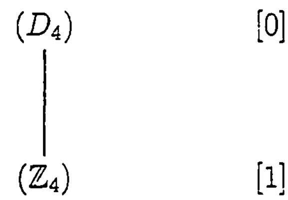


Figure 3.10: Isotropy lattice for $\mathcal{V}_1$.

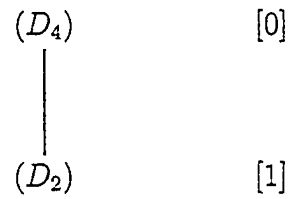


Figure 3.11: Isotropy lattice for $\mathcal{V}_2$.

$$\dim V_1^{\mathbb{Z}_4} = \tfrac{1}{4}(\chi_2(1) + 2\chi_2(\gamma) + \chi_2(\gamma^2)) = \tfrac{1}{4}(1 + 2(1) + 1) = 1.$$

And

$$\deg_{\mathcal{V}_1} = (D_4) - (\mathbb{Z}_4).$$

(iii) There is an irreducible representation $\mathcal{V}_2$, given by the homomorphism $d : D_4 \to \mathbb{Z}_2$ such that $\ker d = D_2$. Then by applying Theorem 3.5.1, and Table 1.2 we obtain

$$\begin{aligned}
\dim V_2^{D_4} &= \tfrac{1}{8}(\chi_3(1) + 2\chi_3(\gamma) + \chi_3(\gamma^2) + 2\chi_3(\kappa) + 2\chi_3(\kappa\gamma)) \\
&= \tfrac{1}{8}(1 + 2(-1) + 1 + 2(1) + 2(-1)) = 0, \\
\dim V_2^{\mathbb{Z}_4} &= \tfrac{1}{4}(\chi_3(1) + 2\chi_3(\gamma) + \chi_3(\gamma^2)) = \tfrac{1}{4}(1 + 2(-1) + 1) = 0, \\
\dim V_2^{D_2} &= \tfrac{1}{4}(\chi_3(1) + \chi_3(\gamma^2) + 2\chi_3(\kappa)) = \tfrac{1}{4}(1 + 1 + 2(1)) = 1,
\end{aligned}$$

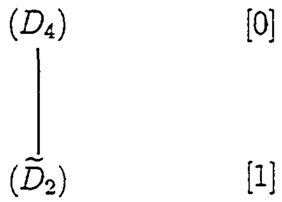


Figure 3.12: Isotropy lattice for $\mathcal{V}_3$.

And

$$\deg_{\mathcal{V}_2} = (D_4) - (D_2).$$

(iv) There is an irreducible representation $\mathcal{V}_2$, given by the homomorphism $\widetilde{d} : D_4 \to \mathbb{Z}_2$ such that $\ker \widetilde{d} = \widetilde{D}_2$. Then we obtain

$$\begin{aligned}
\dim V_3^{D_4} &= \tfrac{1}{8}(\chi_4(1) + 2\chi_4(\gamma) + \chi_4(\gamma^2) + 2\chi_4(\kappa) + 2\chi_4(\kappa\gamma)) \\
&= \tfrac{1}{8}(1 + 2(-1) + 1 + 2(-1) + 2(1)) = 0, \\
\dim V_3^{\mathbb{Z}_4} &= \tfrac{1}{4}(\chi_4(1) + 2\chi_4(\gamma) + \chi_4(\gamma^2)) = \tfrac{1}{4}(1 + 2(-1) + 1) = 0, \\
\dim V_3^{D_2} &= \tfrac{1}{4}(\chi_4(1) + \chi_4(\gamma^2) + 2\chi_4(\kappa)) = \tfrac{1}{4}(1 + 1 + 2(-1)) = 0, \\
\dim V_3^{\widetilde{D}_2} &= \tfrac{1}{4}(\chi_4(1) + \chi_4(\gamma^2) + 2\chi_4(\kappa)) = \tfrac{1}{4}(1 + 1 + 2(1)) = 1.
\end{aligned}$$

And

$$\deg_{\mathcal{V}_3} = (D_4) - (\widetilde{D}_2).$$

(v) There is an orthogonal two dimensional representation $\mathcal{V}_2$ of D_4 on $\mathbb{C}$ given by

$$\gamma z := \gamma \cdot z, \text{ for } \gamma \in \mathbb{Z}_4 \text{ and } z \in \mathbb{C},$$

$$\kappa z := \overline{z},$$

where $\gamma \cdot z$ denotes the usual complex multiplication.Then

$$\begin{aligned}
\dim V_4^{D_4} &= \tfrac{1}{8}(\chi_5(1) + 2\chi_5(\gamma) + \chi_5(\gamma^2) + 2\chi_5(\kappa) + 2\chi_5(\kappa\gamma)) \\
&= \tfrac{1}{8}(2 + 2(0) + (-2) + 2(0) + 2(0)) = 0, \\
\dim V_4^{\mathbb{Z}_4} &= \tfrac{1}{4}(\chi_5(1) + 2\chi_5(\gamma) + \chi_5(\gamma^2)) = \tfrac{1}{4}(2 + 2(0) + (-2)) = 0, \\
\dim V_4^{D_2} &= \tfrac{1}{4}(\chi_5(1) + \chi_5(\gamma^2) + 2\chi_5(\kappa)) = \tfrac{1}{4}(2 + (-2) + 2(0)) = 0,
\end{aligned}$$

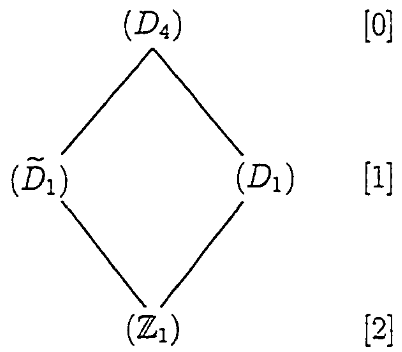


Figure 3.13: Isotropy lattice for $\mathcal{V}_4$.

$$\begin{aligned}
\dim V_4^{\widetilde{D}_2} &= \tfrac{1}{4}(\chi_5(1) + \chi_5(\gamma^2) + 2\chi_5(\kappa\gamma)) = \tfrac{1}{4}(2 + (-2) + 2(0)) = 0,\\
\dim V_4^{\mathbb{Z}_2} &= \tfrac{1}{2}(\chi_5(1) + \chi_5(\gamma^2)) = \tfrac{1}{2}(2 + (-2)) = 0,\\
\dim V_4^{D_1} &= \tfrac{1}{2}(\chi_5(1) + \chi_5(\kappa)) = \tfrac{1}{2}(2 + 0) = 1,\\
\dim V_4^{\widetilde{D}_1} &= \tfrac{1}{2}(\chi_5(1) + \chi_5(\kappa\gamma)) = \tfrac{1}{2}(2 + 0) = 1,\\
\dim V_4^{\mathbb{Z}_1} &= \tfrac{1}{1}(\chi_5(1)) = 2.
\end{aligned}$$

By Formula (3.15) we have the following degree of the basic map:

$$\deg_{\mathcal{V}_4} = (D_4) - (D_1) - (\widetilde{D}_1) + (\mathbb{Z}_1).$$

3.5.3 Degrees of Basic Maps for the Dihedral Group D_5

(i) There is a one-dimensional trivial representation $\mathcal{V}_0$. In this case we have

$$\deg_{\mathcal{V}_0} = -(D_5).$$

(ii) There is a one-dimensional representation $\mathcal{V}_1$, given by the homomorphism $c : D_5 \to \mathbb{Z}_2$ such that $\ker c = \mathbb{Z}_5$. Let us obtain the corresponding degree of the basic map by applying Theorem 3.5.1, Table 1.3, and Formula (3.15).

$$\begin{aligned}
\dim V_1^{D_5} &= \tfrac{1}{10}(\chi_2(1) + 2\chi_2(\gamma) + 2\chi_2(\gamma^2) + 5\chi_2(\kappa)),\\
&= \tfrac{1}{10}(1 + 2(1) + 2(1) + 5(-1)) = 0\\
\dim V_1^{\mathbb{Z}_5} &= \tfrac{1}{5}(\chi_2(1) + 2\chi_2(\gamma) + 2\chi_2(\gamma^2)) = \tfrac{1}{5}(1 + 2(1) + 2(1)) = 1.
\end{aligned}$$

And

$$\deg_{\mathcal{V}_1} = (D_5) - (\mathbb{Z}_5).$$

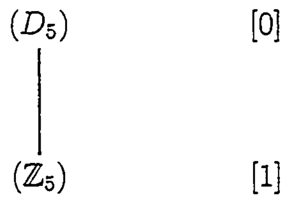


Figure 3.14: Isotropy lattice for $\mathcal{V}_1$.

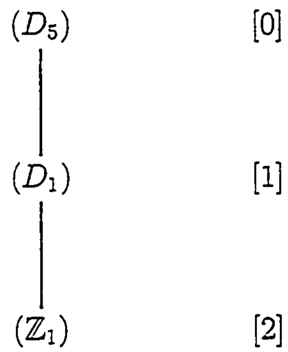


Figure 3.15: Isotropy lattice for $\mathcal{V}_2$.

(iii) There is one two dimensional representation $\mathcal{V}_2$ of D_5 on $\mathbb{C}$ given by

$$\gamma z := \gamma \cdot z, \text{ for } \gamma \in \mathbb{Z}_5 \text{ and } z \in \mathbb{C},$$

$$\kappa z := \overline{z},$$

where $\gamma \cdot z$ denotes the usual complex multiplication. Then we obtain

$$\begin{aligned}
\dim V_2^{D_5} &= \tfrac{1}{10}(\chi_3(1) + 2\chi_3(\gamma) + 2\chi_3(\gamma^2) + 5\chi_3(\kappa)) \\
&= \tfrac{1}{10}(2 + 2(\tfrac{\sqrt{5}-1}{4}) + 2(-\tfrac{\sqrt{5}+1}{4})) + 5(0)) = 0, \\
\dim V_2^{\mathbb{Z}_5} &= \tfrac{1}{5}(\chi_3(1) + 2\chi_3(\gamma) + 2\chi_3(\gamma^2)) = \tfrac{1}{5}(2 + 2(\tfrac{\sqrt{5}-1}{4}) + 2(-\tfrac{\sqrt{5}+1}{4})) = 0, \\
\dim V_2^{D_1} &= \tfrac{1}{2}(\chi_3(1) + \chi_3(\kappa)) = \tfrac{1}{2}(2+0) = 1, \\
\dim V_2^{\mathbb{Z}_1} &= \tfrac{1}{1}(\chi_3(1)) = 2.
\end{aligned}$$

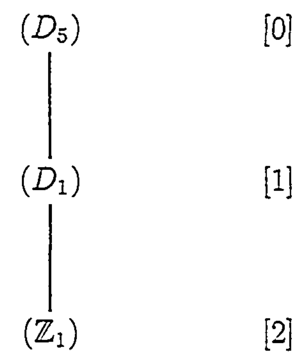


Figure 3.16: Isotropy lattice for $\mathcal{V}_3$.

$$\deg_{\mathcal{V}_2} = (D_5) - 2(D_1) + (\mathbb{Z}_1).$$

(iv) There is one two dimensional representation $\mathcal{V}_3$ of D_5 on $\mathbb{C}$ given by

$$\gamma z := \gamma^2 \cdot z, \text{ for } \gamma \in \mathbb{Z}_5 \text{ and } z \in \mathbb{C},$$
$$\kappa z := \overline{z},$$

where $\gamma^2 \cdot z$ denotes the usual complex multiplication. Then

$$\begin{aligned}
\dim V_3^{D_5} &= \tfrac{1}{10}(\chi_3(1) + 2\chi_3(\gamma) + 2\chi_3(\gamma^2) + 5\chi_3(\kappa)) \\
&= \tfrac{1}{10}(2 + 2(-\tfrac{\sqrt{5}+1}{4}) + 2(\tfrac{\sqrt{5}-1}{4}) + 5(0)) = 0, \\
\dim V_3^{\mathbb{Z}_5} &= \tfrac{1}{5}(\chi_3(1) + 2\chi_3(\gamma) + 2\chi_3(\gamma^2)) = \tfrac{1}{5}(2 + 2(-\tfrac{\sqrt{5}+1}{4}) + 2(\tfrac{\sqrt{5}-1}{4})) = 0, \\
\dim V_3^{D_1} &= \tfrac{1}{2}(\chi_3(1) + \chi_3(\kappa)) = \tfrac{1}{2}(2+0) = 1, \\
\dim V_3^{\mathbb{Z}_1} &= \tfrac{1}{1}(\chi_3(1)) = 2.
\end{aligned}$$

By Formula 3.15 we have the following degree of the basic map:

$$\deg_{\mathcal{V}_3} = (D_5) - 2(D_1) + (\mathbb{Z}_1).$$

3.5.4 Degrees of Basic Maps for the Alternating Group A_4

To compute basic degree, we describe real irreducible A_4-representations. Using the homomorphism $\varphi : A_4 \to \frac{A_4}{V_4} \simeq \mathbb{Z}_3$, we obtain the one-dimensional trivial representation $\mathcal{V}_0$ and the two-dimensional $\mathcal{V}_1$, $\mathcal{V}_2$, which are associated with the

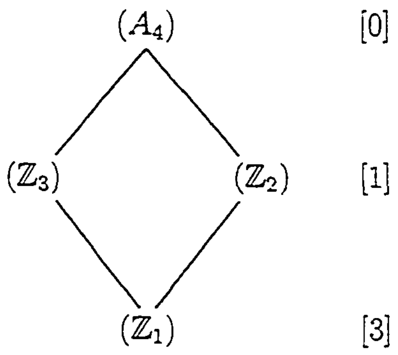


Figure 3.17: Isotropy lattice for $\mathcal{V}_3$.

$\mathbb{Z}_3$-actions are on $\mathbb{R}^2 \simeq \mathbb{C}$ given by $\gamma\mathbb{Z} = \gamma^k \cdot z$, $k = 1, 2$, respectively. There is also one three-dimensional natural representation $\mathcal{V}_3$ of A_4.

The computation of the basic degrees, related to the representations $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ is a strightforward application of Theorem 3.5.1, and formula (3.15). (as we did for the dihedral groups)

$$\deg_{\mathcal{V}_0} = -(A_4), \qquad \deg_{\mathcal{V}_1} = \deg_{\mathcal{V}_2} = (A_4).$$

Following is the isotropy lattice for $\mathcal{V}_3$:

$$\deg_{\mathcal{V}_3} = (A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1).$$

3.5.5 Degrees of Basic Maps for the Permutation Group S_4

There are exactly five real (and also complex) irreducible representations of S_4: The trivial representation $\mathcal{V}_0$, the one-dimensional representation $\mathcal{V}_1$ corresponding to the homomorphism $\varphi : S_4 \to \mathbb{Z}_2$, where $\ker \varphi = A_4$, the two dimensional representation $\mathcal{V}_2$ corresponding to the homomorphism $\psi : S_4 \to S_4/V_4 = S_3 \simeq D_3$, and two different three-dimensional representations of S_4, one of them being the natural representation $\mathcal{V}_3$ of S_4 , while the other $\mathcal{V}_4$ being the tensor product $\mathcal{V}_1 \otimes \mathcal{V}_3$ of the natural three-dimensional representation with the non-trivial one-dimensional representation. Following we obtain the isotropy lattice of each representation and the corresponding basic degree.
$\deg_{\mathcal{V}_0} = -(S_4)$,
$\deg_{\mathcal{V}_1} = (S_4) - 2(D_4)$,
$\deg_{\mathcal{V}_2} = (S_4) - 2(D_4) + (V_4)$,

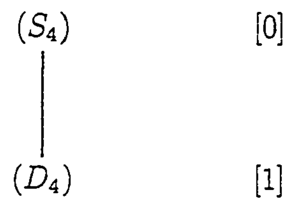


Figure 3.18: Isotropy lattice for $\mathcal{V}_1$.

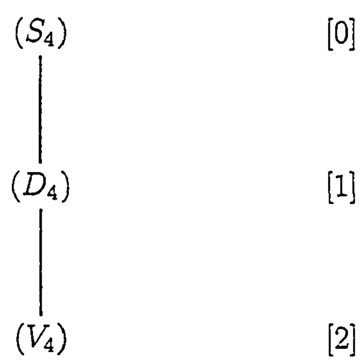


Figure 3.19: Isotropy Lattice for $\mathcal{V}_2$.

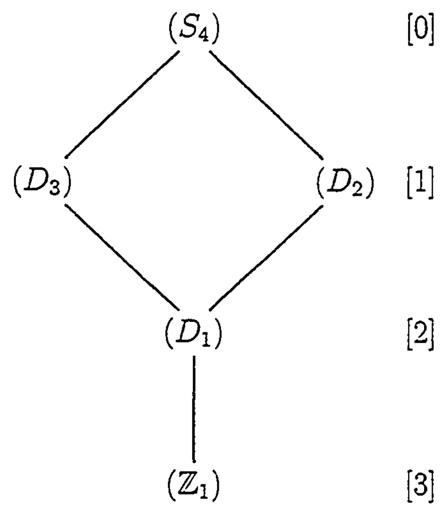


Figure 3.20: Isotropy Lattice for $\mathcal{V}_3$.

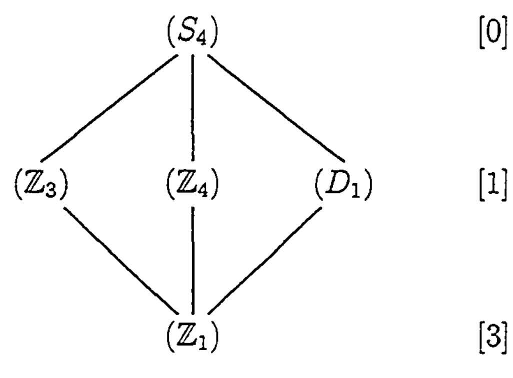


Figure 3.21: Isotropy Lattice for $\mathcal{V}_4$.

$\deg_{\mathcal{V}_3} = (S_4) - 2(D_3) - (D_2) + 3(D_1) - (\mathbb{Z}_1),$
$\deg_{\mathcal{V}_4} = (S_4) - (\mathbb{Z}_4) - (D_1) - (\mathbb{Z}_3) + (\mathbb{Z}_1).$

3.5.6 Degrees of Basic Maps for the Alternating Group A_5

There are exactly 5 irreducible representations of A_5: $\mathcal{V}_0$ – the trivial representation, $\mathcal{V}_1$ – the natural 4-dimensional representation of A_5, $\mathcal{V}_2$ – 5-dimensional representation of A_5, and two 3-dimensional representations $\mathcal{V}_3$ and $\mathcal{V}_4$.

Clearly, for the representation $\mathcal{V}_0$ we have the basic degree $\deg_0 = -(A_5)$. We have the following isotropy lattice for the representation $\mathcal{V}_2$ (the dimension of the fixed point spaces is marked on the left of each row) and the basic degree $\deg_2$:

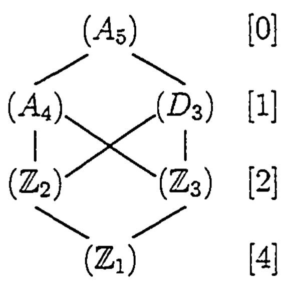


Figure 3.22: Isotropy Lattice for $\mathcal{V}_1$.

$$\deg_{\mathcal{V}_1} = (A_5) - 2(A_4) - 2(D_3) + 3(\mathbb{Z}_2) + 3(\mathbb{Z}_3) - 2(\mathbb{Z}_1).$$

We have the following lattice of isotropies and the basic degree for the 5-dimensional representation $\mathcal{V}_2$:

$$\deg_{\mathcal{V}_2} = (A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1).$$

We have the following lattice of isotropies and the basic degree for the 3-dimensional representations $\mathcal{V}_3$ and $\mathcal{V}_4$:

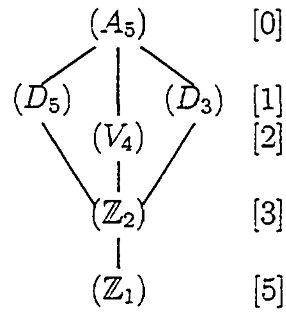


Figure 3.23: Isotropy Lattice for $\mathcal{V}_2$.

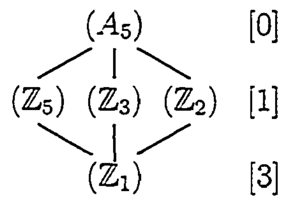


Figure 3.24: Isotropy Lattice for $\mathcal{V}_3$ and $\mathcal{V}_4$.

$$\deg_{\mathcal{V}_3} = \deg_4 = (A_5) - (\mathbb{Z}_5) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1).$$

3.6 Computational Formula for Basic Maps with One Parameter and Examples

The equivariant degree $\Gamma \times S^1$-Deg(f, Ω), where $f : \mathbb{R} \oplus V \to V$ is an Ω-admissible map, can be also computed by applying a recurrence formula. Let us discuss the following cases.

Suppose f is a regular normal map (otherwise, we consider a regular normal approximation map $\widetilde{f}$ of f), then for every orbit type (H) in Ω (we assume $\dim W(H) = 1$), the map $f^H|_{\Omega_H}$ has zero as a regular value, thus

$$(f^H)^{-1}(0) \cap \Omega_H = W(H)x_1 \cup \cdots \cup W(H)x_m,$$

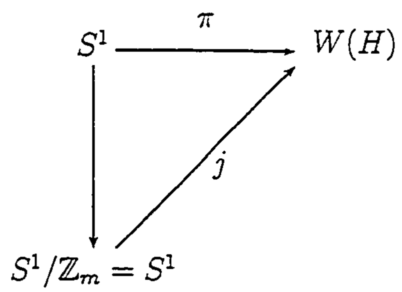
on the other hand, since H is a twisted subgroup, i.e. $H = K^{\varphi,m} = \{(\gamma, z) \in K \times S^1 |\ \varphi(\gamma) = z^m\}$ for some $K \subset \Gamma$ and $\varphi : K \to S^1$ being a homomorphism.

Then there exists a natural homomorphism $S^1 \to W(H)$, defined as follows:

We have $N(H) = N_0 \times S^1$, where $N_0 = \{g \in N(K) | \varphi(g\gamma g^{-1}) = \varphi(\gamma) \text{ for all } \gamma \in K\}$, so we have the composition

$$S^1 \to N_0 \times S^1 = N(H) \xrightarrow{p} W(H),$$

denoted by p which has a kernel $\mathbb{Z}_m$. We define this homomorphism by j,



and $j : S^1 \to W(H)$ is an injective homomorphism.

So, we can consider S^1 as a subgroup of $W(H)$. Since f^H is $W(H)$-equivariant, it is also S^1-equivariant. Then we get

$$n_K = \frac{\deg_1(f^K, \Omega^K)}{\left|\frac{W(H)}{S^1}\right|},$$

where $\deg_1(f^K, \Omega_K)$ is the first coefficient of S^1-Deg (f^K, Ω_K) corresponding to the orbit type $(\mathbb{Z}_1)$.

By applying the induction over orbit types we get the recurrence formula

$$n_K = \frac{\deg_1(f^K, \Omega_K) - \sum\limits_{(K)<(L)} n(K,L) \cdot n_L \cdot \left|\frac{W(L)}{S^1}\right|}{\left|\frac{W(K)}{S^1}\right|}. \tag{3.16}$$

Now, consider the representation $\mathcal{V}_{k,1}$ (we consider here $\mathcal{V}_{k,1}$for simplicity but the general case $\mathcal{V}_{k,j}$ for $j > 1$ can also be analyzed in the same way). We consider a basic map $\mathfrak{b} : \mathbb{R} \times \mathcal{V}_k \to \mathcal{V}_k$, and we put G-Deg $(\mathfrak{b}, \mathcal{O}) = \deg_{\mathcal{V}_{k,1}}$. We will show how to compute these degrees.

By recurrence formula $\deg_{\mathcal{V}_{k,1}} = \sum_{(L)} n_L \cdot (L)$, where

$$n_L = \frac{1}{|\frac{W(L)}{S^1}|}\left[\frac{M_L}{2} - \sum_{(\widetilde{L})>(L)} n(L,\widetilde{L}) \cdot n_{\widetilde{L}} \cdot |\frac{W(\widetilde{L})}{S^1}|\right], \tag{3.17}$$

and $M_L = \dim \mathcal{V}^L_{j,1}$.

3.6.1 Degrees of Basic Maps for Group $D_3 \times S^1$

There are three complex irreducible D_3-representations

(i) The representation V_0^c defined on $\mathbb{C}$.

(ii) The representation V_1^c defined on $\mathbb{C} \oplus \mathbb{C}$ by

$$\gamma(z_1, z_2) := (\gamma \cdot z_1, \gamma^{-1} \cdot z_2), \quad \text{for } \gamma \in \mathbb{Z}_3, \text{ and } z_1, z_2 \in \mathbb{C},$$
$$\kappa(z_1, z_2) := (z_2, z_1).$$

(iii) The representation $\mathcal{V}_2^c$ defined by $c : D_3 \to \mathbb{Z}_2$, such that $\ker c = \mathbb{Z}_3$.

For j=0,1,2, we define the action of S^1 on $\mathcal{V}_j^c$ by $zv = z \cdot v$, for $z \in S^1$ and $v \in \mathcal{V}_j^c$, where the product '$\cdot$' is the usual complex multiplication. In this way we obtain a real irreducible representation for $D_3 \times S^1$, which we denote by $\mathcal{V}_{j,1}$. For each of the representations $\mathcal{V}_{j,1}$ of $D_3 \times S^1$, we can compute the $\deg_{\mathcal{V}_{j,1}}$of the associated basic maps on $\mathcal{V}_{j,1}$, by using the isotropy lattice for $\mathcal{V}_{j,1}$ and the Formula (3.17). (the numbers located on the right side of the isotropy lattice denote the real dimension of the fixed-point space).

Clearly, $\deg_{\mathcal{V}_{0,1}} = (D_3)$ and $\deg_{\mathcal{V}_{2,1}} = (D_3^{\mathbb{Z}})$.

$\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_3^t) + (D_1) + (D_1^{\mathbb{Z}}) - (\mathbb{Z}_l)$.

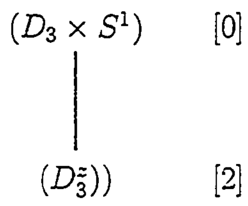


Figure 3.25: Isotropy Lattice for $\mathcal{V}_{2,1}$.

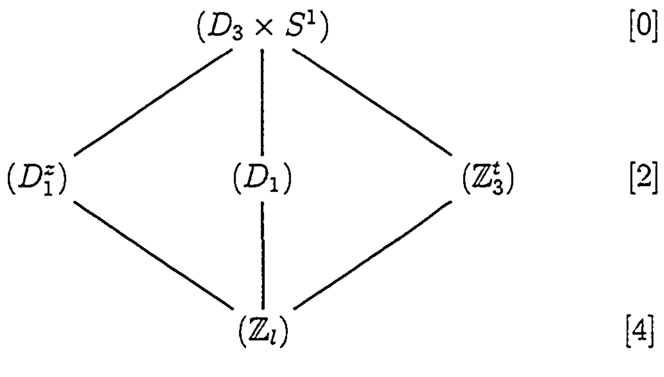


Figure 3.26: Isotropy Lattice for $\mathcal{V}_{1,1}$.

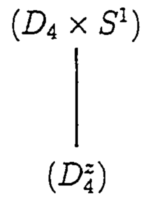


Figure 3.27: Isotrpy lattice for $\mathcal{V}_{2,1}$.

3.6.2 Degrees of Basic Maps for Group $D_4 \times S^1$

There are five complex irreducible D_4-representations

(i) The representation V_0^c defined on $\mathbb{C}$.

(ii) The representation V_1^c defined on $\mathbb{C} \oplus \mathbb{C}$ by

$$\begin{aligned} &\gamma(z_1, z_2) := (\gamma \cdot z_1, \gamma^{-1} \cdot z_2), \quad \text{for } \gamma \in \mathbb{Z}_4, \text{ and } z_1, z_2 \in \mathbb{C}, \\ &\kappa(z_1, z_2) := (z_2, z_1). \end{aligned}$$

(iii) The representation $\mathcal{V}_2^c$ defined by $c : D_4 \to \mathbb{Z}_2$, such that $\ker c = \mathbb{Z}_4$.

(iv) The representation $\mathcal{V}_3^c$ defined by $d : D_4 \to \mathbb{Z}_2$, such that $\ker d = D_2$.

(v) The representation $\mathcal{V}_4^c$ defined by $\tilde{d} : D_4 \to \mathbb{Z}_2$, such that $\ker \tilde{d} = \tilde{D}_2$.

The same as the first example we obtain five real irreducible representation for $D_4 \times S^1$ which we denote by $\mathcal{V}_{j,1}$, $j = 0, \cdots, 4$. For each of the representations $\mathcal{V}_{j,1}$ of $D_4 \times S^1$, we can compute the $\deg_{\mathcal{V}_{j,1}}$ of the associated basic maps on $\mathcal{V}_{j,1}$, by using the isotropy lattice for $\mathcal{V}_{j,1}$ and the equation (3.17).

It is clear $\deg_{\mathcal{V}_{0,1}} = (D_4)$ and

$$\deg_{\mathcal{V}_{2,1}} = (D_4^{\tilde{z}}), \quad \deg_{\mathcal{V}_{3,1}} = (D_4^{d}), \quad \deg_{\mathcal{V}_{4,1}} = (D_4^{\tilde{d}}).$$

$$\deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_4^t) + (D_2) + (D_2^{\tilde{d}}) - (\mathbb{Z}_l).$$

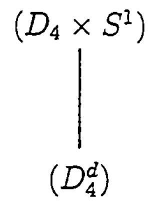


Figure 3.28: Isotrpy lattice for $\mathcal{V}_{3,1}$.

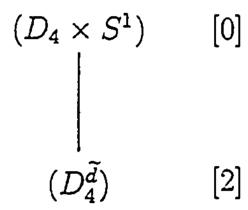


Figure 3.29: Isotrpy lattice for $\mathcal{V}_{4,1}$.

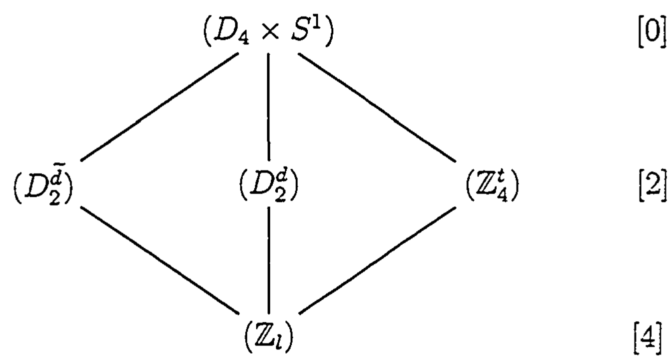


Figure 3.30: Isotropy Lattice for $\mathcal{V}_{1,1}$.

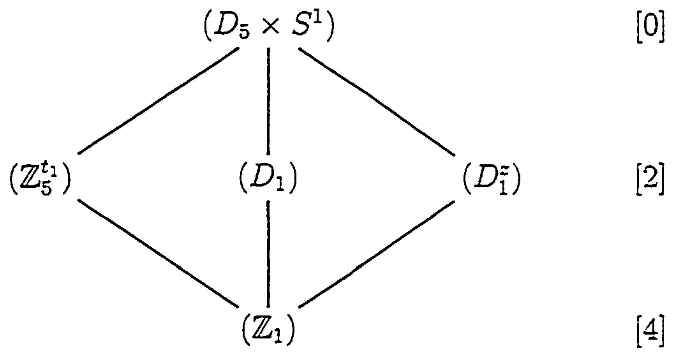


Figure 3.31: Isotropy Lattice for $\mathcal{V}_{2,1}$.

3.6.3 Degrees of Basic Maps for Group $D_5 \times S^1$

There are four complex irreducible D_5-representations

(i) The representation V_0^c defined on $\mathbb{C}$.

(ii) The representation V_1^c defined on $\mathbb{C} \oplus \mathbb{C}$ by

$$\gamma(z_1, z_2) := (\gamma \cdot z_1, \gamma^{-1} \cdot z_2), \quad \text{for } \gamma \in \mathbb{Z}_5, \text{ and } z_1, z_2 \in \mathbb{C},$$
$$\kappa(z_1, z_2) := (z_2, z_1).$$

(iii) The representation V_2^c defined on $\mathbb{C} \oplus \mathbb{C}$ by

$$\gamma(z_1, z_2) := (\gamma^2 \cdot z_1, \gamma^{-2} \cdot z_2), \quad \text{for } \gamma \in \mathbb{Z}_5, \text{ and } z_1, z_2 \in \mathbb{C},$$
$$\kappa(z_1, z_2) := (z_2, z_1).$$

(iv) The representation $\mathcal{V}_3^c$ defined by $c : D_5 \to \mathbb{Z}_2$, such that $\ker c = \mathbb{Z}_5$.

We obtain four real irreducible representations for $D_5 \times S^1$ which we denote by $\mathcal{V}_{j,1}$, $j = 0, \cdots, 3$. For each of the representations $\mathcal{V}_{j,1}$ of $D_5 \times S^1$, we can compute the $\deg_{\mathcal{V}_{j,1}}$ of the associated basic maps on $\mathcal{V}_{j,1}$, by using the isotropy lattice for $\mathcal{V}_{j,1}$ and the equation (3.17).

It is clear $\deg_{\mathcal{V}_{0,1}} = (D_5)$ and $\deg_{\mathcal{V}_{1,1}} = (D_5^{\tilde{z}})$.

$$\deg_{\mathcal{V}_{2,1}} = (\mathbb{Z}_5^{t_1}) + (D_1) + (D_1^{\tilde{z}}) - (\mathbb{Z}_1).$$

$$\deg_{\mathcal{V}_{3,1}} = (\mathbb{Z}_5^{t_2}) + (D_1) + (D_1^{\tilde{z}}) - (\mathbb{Z}_1).$$

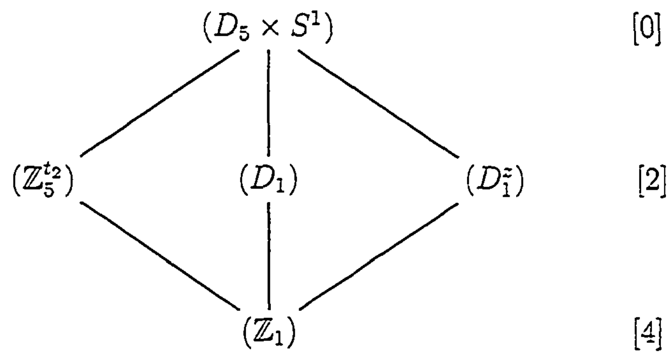


Figure 3.32: Isotropy Lattice for $\mathcal{V}_{3,1}$.

3.6.4 Degrees of Basic Maps for Group $A_4 \times S^1$

There are four irreducible real representation $\mathcal{V}_{j,1}$ of $A_4 \times S^1$. Let us discuss the isotropy lattices for the representations $\mathcal{V}_{j,1}$, $j = 0, 1, 2, 3$. Of course, the only twisted orbit type for $\mathcal{V}_{0,1}$ is (A_4). For $j = 1$, or $j = 2$, there is also only one twisted isotropy class $(A_4^{t_j})$ in $\mathcal{V}_{j,1}$ determined by the homomorphism $\varphi_j : A_4 \xrightarrow{\varphi} A_4/V_4 \simeq \mathbb{Z}_3 \xrightarrow{\gamma \to \gamma^j} \mathbb{Z}_3$. To obtain the lattice for $\mathcal{V}_{3,1}$, consider the action of A_4 on $\mathbb{C}^4$ permuting the coordinates of the vectors $\vec{z} = \langle z_1, z_2, z_3, z_4 \rangle$ and let S^1 act by the complex multiplication. The subspace $\{\langle z, z, z, z \rangle : z \in \mathbb{C}\}$ is the fixed-point subspace for the action of A_4, and its complement is equivalent to the representation $\mathcal{V}_3^c$. Let us choose the following basis in this subspace: $\vec{v}_1 = \langle 1, -1, 1, -1 \rangle$, $\vec{v}_2 = \langle 1, 1, -1, -1 \rangle$, and $\vec{v}_3 = \langle -1, 1, 1, -1 \rangle$. Notice that the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ have the isotropy groups (with respect to $G = A_4 \times S^1$) belonging to the class (V_4^-).

Indeed:

$$G_{\vec{v}_1} = \Big\{ ((1), 1), ((13)(24), 1), ((12)(34), -1), ((14)(23), -1) \Big\},$$

$$G_{\vec{v}_2} = \Big\{ ((1), 1), ((12)(34), 1), ((13)(24), -1), ((14)(23), -1) \Big\},$$

$$G_{\vec{v}_3} = \Big\{ ((1), 1), ((14)(23), 1), ((12)(34), -1), ((13)(24), -1) \Big\}.$$

Next, notice that the vectors $\vec{x} = \vec{v}_1 + \vec{v}_2 = \langle 0, 2, 0, -2 \rangle$ and $\vec{y} = v_1 - \vec{v}_2 = \langle 2, 0, -2, 0 \rangle$ have the isotropy group $H = \Big\{ ((1), 1), ((13)(24), -1) \Big\}$, which belongs to the class

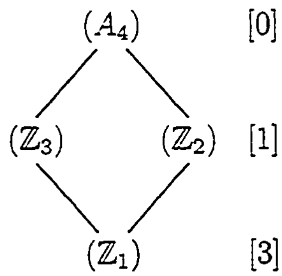


Figure 3.33: Isotropy lattice for $\mathcal{V}_3$.

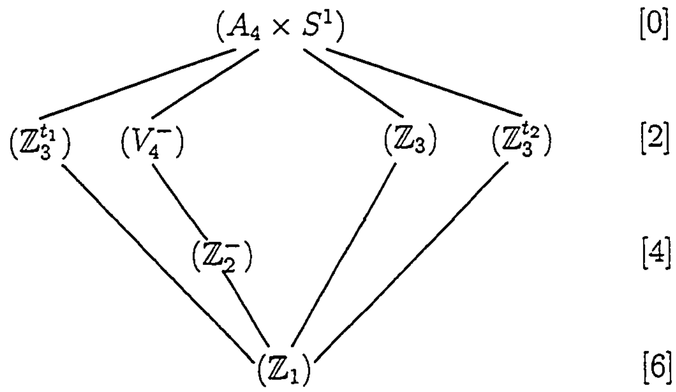


Figure 3.34: Isotropy lattice for $\mathcal{V}_{3,1}$.

$(\mathbb{Z}_2^-)$. The elements $\vec{v}_1+\vec{v}_2+\vec{v}_3$, $\vec{v}_1+\vec{v}_2-\vec{v}_3$, $\vec{v}_1-\vec{v}_2-\vec{v}_3$, and $-\vec{v}_1+\vec{v}_2-\vec{v}_3$ have the isotropy group belonging to the class of the subgroup $(\mathbb{Z}_3)$. Let $\omega = e^{\frac{2\pi i}{3}}$. Then the elements $\vec{w}_1^1 = \langle 1,\omega,\omega^2,0\rangle$, $\vec{w}_2^1 = \langle 1,\omega,0,\omega^2\rangle$ $\vec{w}_3^1 = \langle 1,0,\omega,\omega^2\rangle$, and $\vec{w}_4^1 = \langle 0,1,\omega,\omega^2\rangle$ have the isotropy groups belonging to the class $(\mathbb{Z}_3^{t_1})$, and $\vec{w}_1^2 = \langle 1,\omega^2,\omega,0\rangle$, $\vec{w}_2^2 = \langle 1,\omega^2,0,\omega\rangle$, $\vec{w}_3^2 = \langle 1,0,\omega^2,\omega\rangle$, and $\vec{w}_4^2 = \langle 0,1,\omega^2,\omega\rangle$ have the isotropy groups belonging to the class $(\mathbb{Z}_3^{t_2})$. The isotropy lattices for $\mathcal{V}_3$ (as the representation of A_4) and $\mathcal{V}_{3,1}$ (as the representation of $A_4\times S^1$) are shown on the diagram above.

Finally, we can list all the $A_4\times S^1$-degrees of the basic mappings associated with these representations:

$$\deg_{\mathcal{V}_{0,1}} = (A_4), \qquad \deg_{\mathcal{V}_{1,1}} = (A_4^{t_1}),$$
$$\deg_{\mathcal{V}_{2,1}} = (A_4^{t_2}), \qquad \deg_{\mathcal{V}_{3,1}} = (\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (V_4^-) + (\mathbb{Z}_3) - (\mathbb{Z}_1).$$

3.6.5 Degrees of Basic Maps for Group $S_4\times S^1$

There are exactly five real irreducible representations of S_4, which described in section 3.5.5. We consider the complexifications $\mathcal{V}_j^c$ of the representations $\mathcal{V}_j$, $j=0$, 1, 2, 3, 4, and define the S^1-action on $\mathcal{V}_j^c$ by $\gamma\vec{v} = \gamma^l\cdot\vec{v}$, where $l=0,1,2,\dots$; $\gamma\in S^1$, $\vec{v}\in\mathcal{V}_j^c$. We will denote the obtained irreducible $S_4\times S^1$-representations by $\mathcal{V}_{j,l}$, $j=0$, 1, 2, 3, 4 and $l=0,1,2,3,\dots$.

The representation $\mathcal{V}_{0,1}$ contains two orbit types: $(S_4\times S^1)$ and (S_4), so we have $\deg_{\mathcal{V}_{0,1}} = (S_4)$. For the representations $\mathcal{V}_{1,1}$ there are also two classes of the isotropy groups: $(S_4\times S^1)$ and (S_4^-), so we have $\deg_{\mathcal{V}_{1,1}} = (S_4^-)$. In the case of the representations $\mathcal{V}_{2,1}$, we have the following lattice of the isotropy groups:

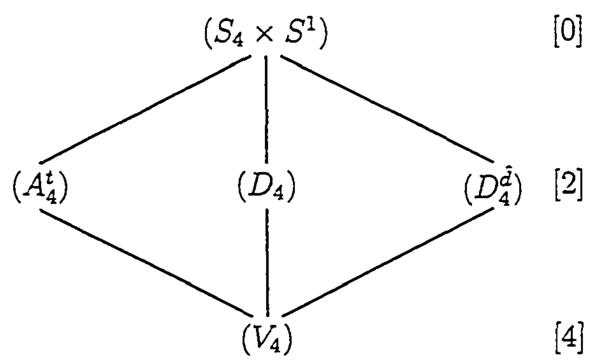


Figure 3.35: Isotropy Lattice for $\mathcal{V}_{2,1}$.

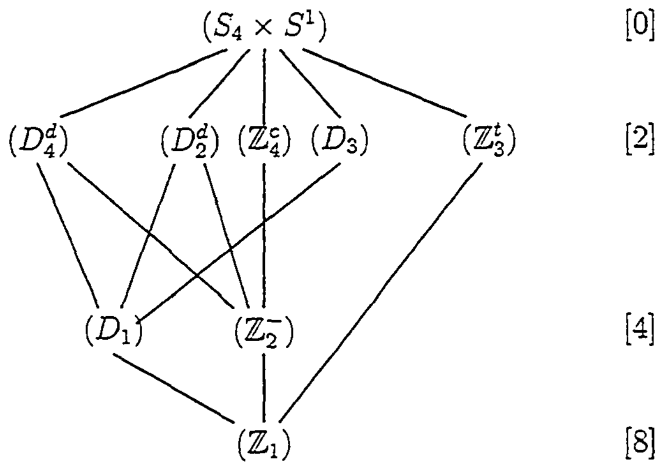


Figure 3.36: Isotropy Lattice for $\mathcal{V}_{3,1}$.

For the representation $\mathcal{V}_{2,1}$ we obtain that the corresponding $S_4 \times S^1$-degree of the basic map is

$$\deg_{\mathcal{V}_{2,1}} = (A_4^t) + (D_4) + (D_4^{\tilde{d}}) - (V_4).$$

Now, let us consider the representation $\mathcal{V}_{3,1}$ of $S_4 \times S^1$, which is obtained by taking the complexification of $\mathcal{V}_3$ and defining the action of S^1 by complex multiplication. The isotropy lattice for the natural representation $\mathcal{V}_{3,1}$ of $S_4 \times S^1$ is shown in the following diagram. Notice that the isotropy group G_x of $x = \vec{v}_1$ is represented by D_4^d, for $x = \vec{v}_1 + \vec{v}_2$ it is D_2^d, for $x = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ it is D_3, for $x = \vec{v}_1 + 2\vec{v}_2$ it is $\mathbb{Z}_2^-$, for $x = \vec{v}_1 + \gamma\vec{v}_2 + \gamma^2\vec{v}_3$, where $\gamma \in S^1$ is the third root of 1, it is $\mathbb{Z}_3^t$, and finally for $x = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$, it is $\mathbb{Z}_1$.

By applying the standard computational formulae, we obtain the following value of the G-equivariant degree for the basic map on the representation $\mathcal{V}_{3,1}$:

$$\deg_{\mathcal{V}_{3,1}} = (D_4^d) + (D_2^d) + (D_3) + (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1).$$

By taking the complexification of $\mathcal{V}_4$ and defining the action of $z \in S^1$ by the complex multiplication, we obtain the irreducible representation $\mathcal{V}_{4,1}$ of the group $S_4 \times S^1$.

Notice that the isotropy group G_x of $x = \vec{v}_1$ is represented by $D_4^{\tilde{z}}$, for $x = \vec{v}_1 + \vec{v}_2$ it is D_2^d, for $x = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$ it is $D_3^{\tilde{z}}$, for $x = \vec{v}_1 + 2\vec{v}_2$ it is $\mathbb{Z}_2^-$, for $x = \vec{v}_1 + \gamma\vec{v}_2 + \gamma^2\vec{v}_3$, where $\gamma \in S^1$ is the third root of 1, it is $\mathbb{Z}_3^t$, and finally for $x = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$, it is $\mathbb{Z}_1$. By applying the standard computational formulas, we obtain the following value of the G-equivariant degree for the basic map on the representation $\mathcal{V}_{4,1}$

$$\deg_{\mathcal{V}_{4,1}} = (D_4^{\tilde{z}}) + (D_2^d) + (\mathbb{Z}_4^c) + (D_3^{\tilde{z}}) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1^{\tilde{z}}).$$

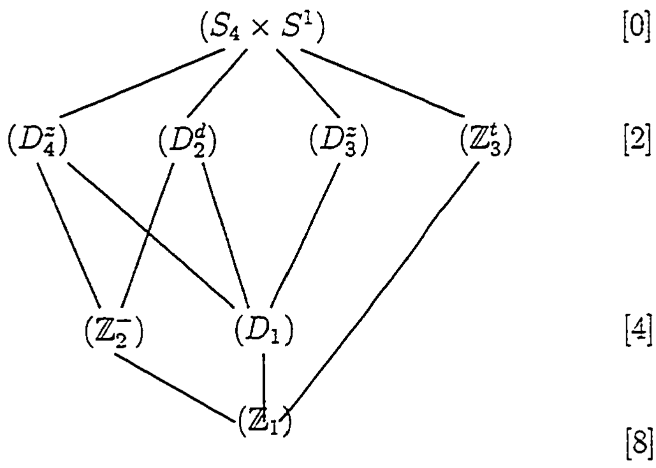


Figure 3.37: Isotropy Lattice for $\mathcal{V}_{4,1}$.

3.6.6 Degrees of Basic Maps for Group $A_5 \times S^1$

Let us present the computations of the basic $A_5 \times S^1$-degrees for the representations $\mathcal{V}_{1,1}$, $\mathcal{V}_{2,1}$, $\mathcal{V}_{3,1}$ and $\mathcal{V}_{4,1}$. In the case of the representations $\mathcal{V}_{1,1}$ and $\mathcal{V}_{2,1}$ we have the following lattice of twisted subgroups:

The basic degree for this representation is given by:

$$\begin{aligned}\deg_{\mathcal{V}_{1,1}} = {} & (A_4) + (D_3) + (D_3^{z}) + (V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) \\ & -(\mathbb{Z}_2) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-),\end{aligned}$$

The basic degree for this representation is given by:

$$\deg_{\mathcal{V}_{2,1}} = (D_5) + (D_3) + (A_4^{t_1}) + (A_4^{t_2}) + (V_4^-) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_2).$$

For the representations $\mathcal{V}_{3,1}$ and $\mathcal{V}_{4,1}$, we have the isotropy lattice of twisted subgroups:

The basic degrees for these two representations are equal to:

$$\begin{aligned}\deg_{\mathcal{V}_{3,1}} &= (D_5^{z}) + (V_4^-) + (D_3^{z}) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-), \\ \deg_{\mathcal{V}_{4,1}} &= (D_5^{z}) + (V_4^-) + (D_3^{z}) + (\mathbb{Z}_5^{t_2}) + (\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-).\end{aligned}$$

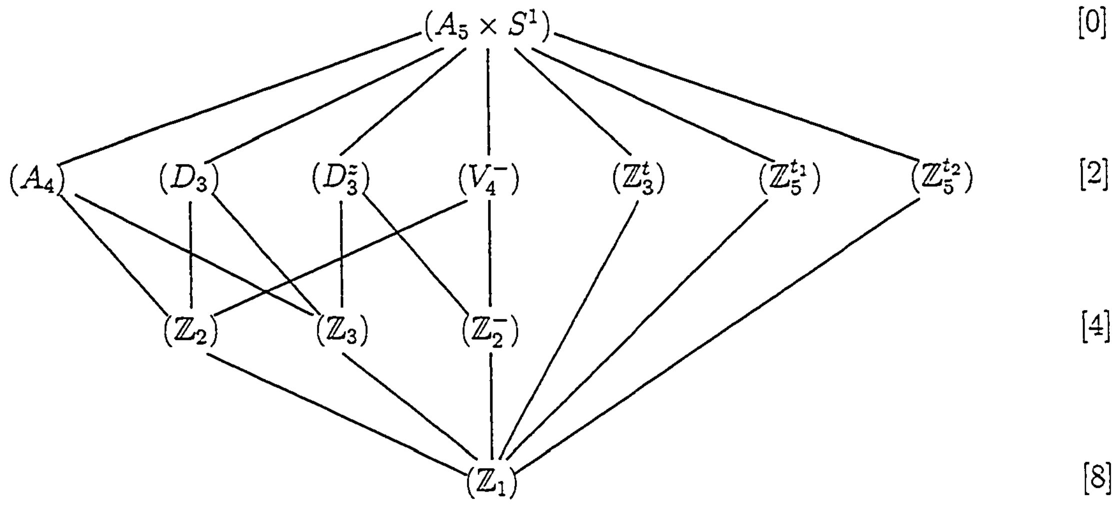


Figure 3.38: Isotropy lattice for $\mathcal{V}_{1,1}$.

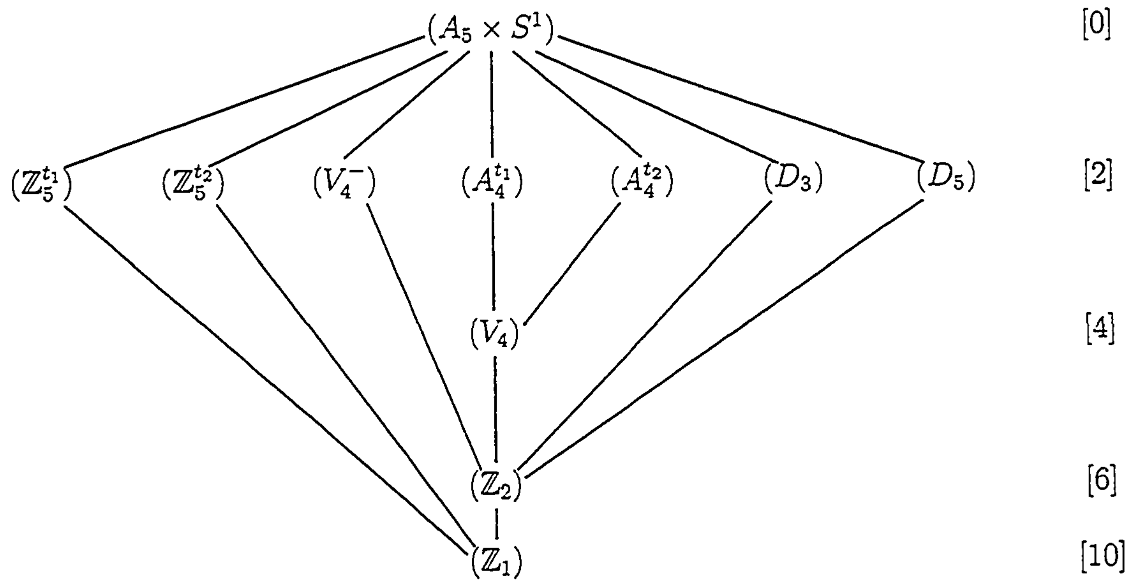


Figure 3.39: Isotropy lattice for $\mathcal{V}_{2,1}$.

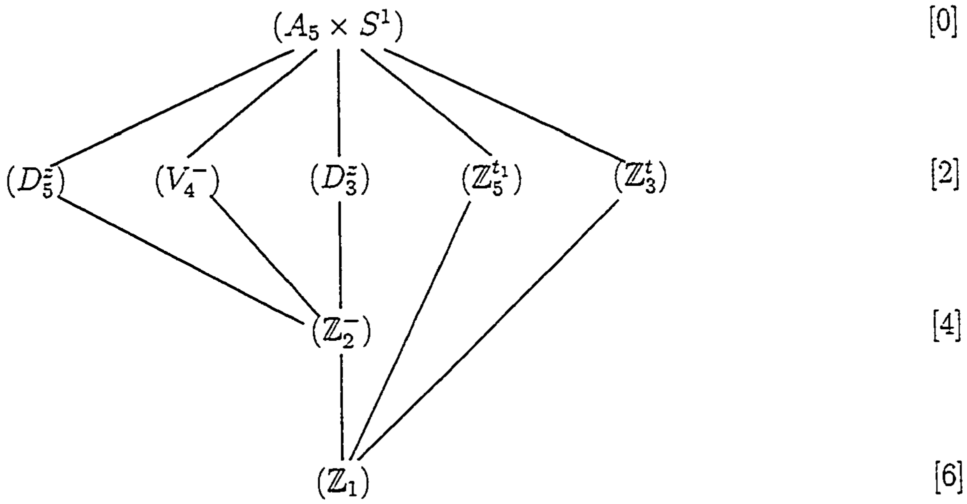


Figure 3.40: Isotropy lattice for $\mathcal{V}_{3,1}$.

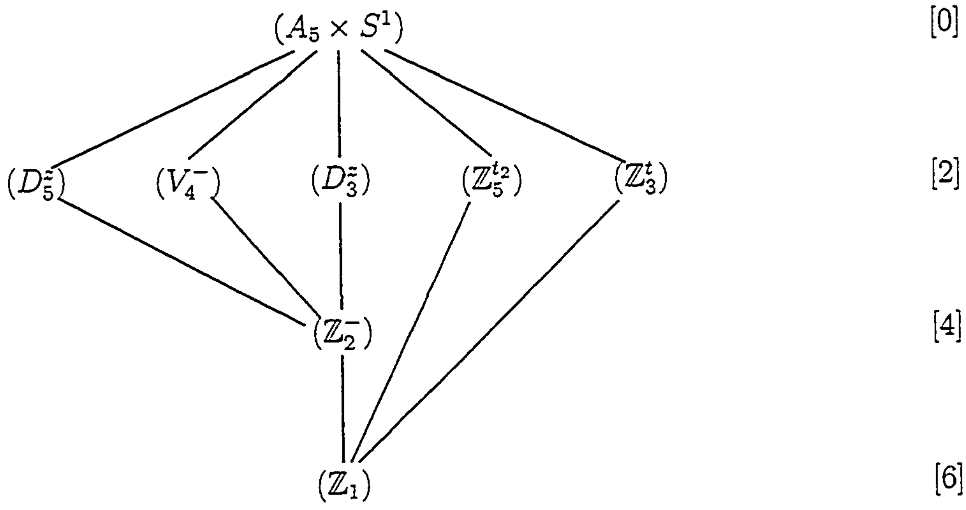


Figure 3.41: Isotropy lattice for $\mathcal{V}_{4,1}$.

Chapter 4

$A(\Gamma)$-Module Structure on $A_1(\Gamma \times S^1)$ and the Multiplication Tables

In this section we consider several paticular cases of the group Γ, for which we present the computations of the $A(\Gamma)$-module structure of $A_1(\Gamma \times S^1)$ the multiplication tables. Recall that the $\mathbb{Z}$-module $A_1(\Gamma\times S^1)$ is generated by all the conjugacy classes of twisted subgroups $(H^{\varphi,l})$ in $\Gamma \times S^1$. The $A(\Gamma)$-multiplication on the generators $(K) \in A(\Gamma)$ and $(H^{\varphi,l}) \in A_1(\Gamma \times S^1)$, is defined by the formula

$$(K).(H^{\varphi,l}) = \sum_{(L)} n_L \cdot (L^{\varphi,l}),$$

where the number n_L are computed using the recurrence formula

$$n_L = \frac{\left[n(L,K)|W(K)|n(L^{\varphi,l},H^{\varphi,l})|\frac{W(H^{\varphi,l})}{S^1}| - \sum_{(\tilde{L})>(L)} n(L^{\varphi,l},\tilde{L}^{\varphi,l})n_{\tilde{L}}|\frac{W(\tilde{L}^{\varphi,l})}{S^1}|\right]}{|\frac{W(L^{\varphi,l})}{S^1}|}, \tag{4.1}$$

where for a set Y, we denote by $|Y|$ the number of elements in Y.

4.1 Examples of $A(\Gamma)$-Modules $A_1(\Gamma \times S^1)$

We devote this section to several examples of the multiplication tables for the $A(\Gamma)$-Modules $A_1(\Gamma \times S^1)$.

4.1.1 $A(D_3)$-Module $A_1(D_3 \times S^1)$

Let us consider the $A(D_3)$-module $A_1(D_3 \times S^1)$. We will use the notation and the computations of the numbers $n(L, H)$ that were presenteed in section 2.4. By applying formula (4.1), it is an easy task to derive the multiplication table for the $A(D_3)$-module $A(D_3 \times S^1)$, which is shown in Table 4.1.

(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	
(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	(D_3)
(D_1)	$(D_1)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
(D_3^z)	(D_1^z)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	(D_3^z)
(D_1^z)	(D_1^z)	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_1^z)

Table 4.1: Multiplication Table for $A(D_3)$-module $A_1(D_3 \times S^1)$.

4.1.2 $A(D_4)$-Module $A_1(D_4 \times S^1)$

(D_4)	(D_2)	$(\tilde{D}_2)$	(D_1)	$(\tilde{D}_1)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	
(D_4)	(D_2)	$(\tilde{D}_2)$	(D_1)	$(\tilde{D}_1)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4)
(D_2)	$2(D_2)$	$(\mathbb{Z}_2)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	(D_2)
$(\tilde{D}_2)$	$(\mathbb{Z}_2)$	$2(\tilde{D}_2)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_2)$
(D_1)	$2(D_1)$	$(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_1)
$(\tilde{D}_1)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1)$	$2(\mathbb{Z}_1)$	$2(\tilde{D}_1)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\tilde{D}_1)$
$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4)$
$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)$	$4(\mathbb{Z}_2)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$
(D_4^z)	(D_2^z)	$(\tilde{D}_2^z)$	(D_1^z)	$(\tilde{D}_1^z)$	$(\mathbb{Z}_4)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4^z)
$(D_4^{\hat{d}})$	(D_2^z)	$(\tilde{D}_2)$	(D_1^z)	$(\tilde{D}_1)$	$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(D_4^{\hat{d}})$
(D_4^d)	(D_2)	$(\tilde{D}_2^z)$	(D_1)	$(\tilde{D}_1^z)$	$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(D_4^d)
(D_2^d)	$2(D_2^d)$	$(\mathbb{Z}_2^-)$	$(D_1^z)+(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	(D_2^d)
$(\tilde{D}_2^d)$	$(\mathbb{Z}_2^-)$	$2(\tilde{D}_2^d)$	$(\mathbb{Z}_1)$	$(\tilde{D}_1^z)+(\tilde{D}_1)$	$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_2^d)$
(D_2^z)	$2(D_2^z)$	$(\mathbb{Z}_2)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	(D_2^z)
$(\tilde{D}_2^z)$	$(\mathbb{Z}_2)$	$2(\tilde{D}_2^z)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\tilde{D}_2^z)$
(D_1^z)	$2(D_1^z)$	$(\mathbb{Z}_1)$	$2(D_1^z)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_1^z)
$(\tilde{D}_1^z)$	$(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)$	$2(\mathbb{Z}_1)$	$2(\tilde{D}_1^z)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\tilde{D}_1^z)$
$(\mathbb{Z}_4^t)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4^t)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^t)$
$(\mathbb{Z}_4^d)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_4^d)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_4^d)$
$(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_2^-)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table 4.2: Multiplication Table for the $A(D_4)$-module $A_1(D_4 \times S^1)$.

4.1.3 $A(D_5)$-Module $A_1(D_5 \times S^1)$

(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	
(D_5)	(D_1)	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	(D_5)
(D_1)	$(D_1) + 2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5)$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5)$
$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$10(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$
$(\mathbb{Z}_5^{t_1})$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_1})$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_1})$
$(\mathbb{Z}_5^{t_2})$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_5^{t_2})$	$2(\mathbb{Z}_1)$	$(\mathbb{Z}_5^{t_2})$
(D_5^{z})	(D_1^{z})	$(\mathbb{Z}_5)$	$(\mathbb{Z}_1)$	(D_5^{z})
(D_1^{z})	$(D_1^{z}) + 2(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$5(\mathbb{Z}_1)$	(D_1^{z})

Table 4.3: Multiplication Table for $A(D_5)$-module $A_1(D_5 \times S^1)$.

4.1.4 $A(A_4)$-Module $A_1(A_4 \times S^1)$

By applying the standard reccurence formula (4.1) one can easily establish the $A(A_4)$-multiplication table for the generators of $A_1(A_4 \times S^1)$, which is shown in Table 4.4.

(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	
(A_4)	(V_4)	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	(A_4)
(V_4)	$3(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	(V_4)
$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$
$(A_4^{t_k})$	(V_4)	$(\mathbb{Z}_3^{t_k})$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(A_4^{t_k})$
(V_4^-)	$3(V_4^-)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-) + (\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	(V_4^-)
$(\mathbb{Z}_3^{t_k})$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3^{t_k}) + (\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$(\mathbb{Z}_3^{t_k})$
$(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$

Table 4.4: Multiplication Table for $A(A_4)$-module $A_1(A_4 \times S^1)$.

4.1.5 $A(S_4)$-Module $A_1(S_4 \times S^1)$

The multiplication table for the $A(S_4)$-module $A_1(S_4 \times S^1)$ (and implicitly for the Burnside ring $A(S_4)$) is given in Table 4.5.

(S_4)	(D_4)	(A_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1)	$(\mathbb{Z}_1)$	
(S_4)	(D_4)	(A_4)	(D_3)	(D_2)	(V_4)	$(\mathbb{Z}_4)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1)	$(\mathbb{Z}_1)$	(S_4)
(D_4)	$(D_4)+(V_4)$	(V_4)	(D_1)	$(D_2)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4)
(A_4)	(V_4)	$2(A_4)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(A_4)
(D_3)	(D_1)	$(\mathbb{Z}_3)$	$(D_3)+(D_1)$	$2(D_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_3)
(D_2)	$(D_2)+(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(D_1)$	$2(D_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2)
(V_4)	$3(V_4)$	$2(V_4)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$6(V_4)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(V_4)
$(\mathbb{Z}_4)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4)$
$(\mathbb{Z}_3)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_3)$
$(\mathbb{Z}_2)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_2)$
(D_1)	$(D_1)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(D_1)+(\mathbb{Z}_1)$	$2(D_1)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$2(D_1)+5(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	(D_1)
$(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$24(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$
(D_4^z)	$(D_4^z)+(V_4^-)$	(V_4^-)	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4^z)
(D_4^d)	$(D_4^d)+(V_4^-)$	(V_4^-)	(D_1)	$(D_2)+(\mathbb{Z}_2^-)$	$3(V_4^-)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$(D_1)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	(D_4^d)
$(D_4^{\hat{d}})$	$(D_4^{\hat{d}})+(V_4)$	(V_4)	(D_1^z)	$(D_2^z)+(\mathbb{Z}_2)$	$3(V_4)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(D_1^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$(D_4^{\hat{d}})$
(A_4^t)	(V_4)	$2(A_4^t)$	$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_2)$	$2(V_4)$	$(\mathbb{Z}_2)$	$2(\mathbb{Z}_3)$	$2(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	(A_4^t)
(D_3^z)	(D_1^z)	$(\mathbb{Z}_3)$	$(D_3^z)+(D_1^z)$	$2(D_1^z)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(D_1^z)+(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	(D_3^z)
(D_2^z)	$(D_2^z)+(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$2(D_1^z)$	$2(D_2^z)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2^z)
(D_2^d)	$(D_2^d)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(D_1)+(D_1^z)$	$2(D_2^d)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$(D_1)+(D_1^z)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(D_2^d)
(V_4^-)	$3(V_4^-)$	$2(V_4^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$6(V_4^-)$	$(\mathbb{Z}_2)+2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+4(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	(V_4^-)
$(\mathbb{Z}_4^c)$	$(\mathbb{Z}_4^c)+(\mathbb{Z}_2^-)$	$(\mathbb{Z}_2^-)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_4^c)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4^c)$
$(\mathbb{Z}_3^t)$	$(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)$	$(\mathbb{Z}_3^t)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_3^t)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$8(\mathbb{Z}_1)$	$(\mathbb{Z}_3^t)$
$(\mathbb{Z}_2^-)$	$3(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$6(\mathbb{Z}_2^-)$	$2(\mathbb{Z}_2^-)+2(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$4(\mathbb{Z}_2^-)+4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	$(\mathbb{Z}_2^-)$
(S_4^-)	$(\tilde{D}_4^d)$	(A_4)	(D_3^z)	(D_2^z)	(V_4)	$(\mathbb{Z}_4^-)$	$(\mathbb{Z}_3)$	$(\mathbb{Z}_2)$	(D_1^z)	$(\mathbb{Z}_1)$	(S_4^-)
(D_1^z)	$(D_1^z)+(\mathbb{Z}_1)$	$(\mathbb{Z}_1)$	$2(D_1^z)+(\mathbb{Z}_1)$	$2(D_1^z)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$4(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$2(D_1^z)+5(\mathbb{Z}_1)$	$12(\mathbb{Z}_1)$	(D_1^z)
$(\mathbb{Z}_4^-)$	$(\mathbb{Z}_4^-)+(\mathbb{Z}_2)$	$(\mathbb{Z}_2)$	$(\mathbb{Z}_1)$	$(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$3(\mathbb{Z}_2)$	$2(\mathbb{Z}_4^-)+(\mathbb{Z}_1)$	$2(\mathbb{Z}_1)$	$2(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	$3(\mathbb{Z}_1)$	$6(\mathbb{Z}_1)$	$(\mathbb{Z}_4^-)$

Table 4.5: Multiplication Table for $A(S_4)$-module $A_1(S_4 \times S^1)$.

4.1.6 $A(A_5)$-Module $A_1(A_5 \times S^1)$

In the case of the Icosahedral group A_5, the structure of the $A(A_5)$-module on $A_1(A_5 \times S^1)$, which contains (implicitly) the multiplication formulae for the Burnside ring $A(A_5)$, is described in Table 4.6. Notice that this table contains (implicitly) also the multiplication formulae for the Burnside ring $A(A_5)$.

(A_5)	(D_5)	(A_4)	(D_3)	(Z_5)	(V_4)	(Z_3)	(Z_2)	(Z_1)	
(A_5)	(D_5)	(A_4)	(D_3)	(Z_5)	(V_4)	(Z_3)	(Z_2)	(Z_1)	(A_5)
(D_5)	$(D_5)+(Z_1)$	(Z_2)	$2(Z_2)$	$(Z_5)+(Z_1)$	$3(Z_2)$	$2(Z_1)$	$2(Z_2)+2(Z_1)$	$6(Z_1)$	(D_5)
(A_4)	(Z_2)	$(A_4)+(Z_3)$	$(Z_3)+(Z_2)$	(Z_1)	$(V_4)+(Z_1)$	$2(Z_3)+(Z_1)$	$(Z_2)+2(Z_1)$	$5(Z_1)$	(A_4)
(D_3)	$2(Z_2)$	$(Z_3)+(Z_2)$	$(D_3)+(Z_2)+(Z_1)$	$2(Z_1)$	$3(Z_2)+(Z_1)$	$(Z_3)+3(Z_1)$	$2(Z_2)+4(Z_1)$	$10(Z_1)$	(D_3)
(Z_5)	$(Z_5)+(Z_1)$	(Z_1)	$2(Z_1)$	$2(Z_5)+2(Z_1)$	$3(Z_1)$	$4(Z_1)$	$6(Z_1)$	$12(Z_1)$	(Z_5)
(V_4)	$3(Z_2)$	$(V_4)+(Z_1)$	$3(Z_2)+(Z_1)$	$3(Z_1)$	$3(V_4)+3(Z_1)$	$5(Z_1)$	$3(Z_2)+6(Z_1)$	$15(Z_1)$	(V_4)
(Z_3)	$2(Z_1)$	$2(Z_3)+(Z_1)$	$(Z_3)+3(Z_1)$	$4(Z_1)$	$5(Z_1)$	$2(Z_3)+6(Z_1)$	$10(Z_1)$	$20(Z_1)$	(Z_3)
(Z_2)	$2(Z_2)+2(Z_1)$	$(Z_2)+2(Z_1)$	$2(Z_2)+4(Z_1)$	$6(Z_1)$	$3(Z_2)+6(Z_1)$	$10(Z_1)$	$2(Z_2)+14(Z_1)$	$30(Z_1)$	(Z_2)
(Z_1)	$6(Z_1)$	$5(Z_1)$	$10(Z_1)$	$12(Z_1)$	$15(Z_1)$	$20(Z_1)$	$30(Z_1)$	$60(Z_1)$	(Z_1)
$(A_4^{t_1})$	(Z_2)	$(A_4^{t_1})+(Z_3^{t_1})$	$(Z_2)+(Z_3^{t_1})$	(Z_1)	$(V_4)+(Z_1)$	$2(Z_3^{t_1})+(Z_1)$	$(Z_2)+2(Z_1)$	$5(Z_1)$	$(A_4^{t_1})$
$(A_4^{t_2})$	(Z_2)	$(A_4^{t_2})+(Z_3^{t_2})$	$(Z_2)+(Z_3^{t_2})$	(Z_1)	$(V_4)+(Z_1)$	$2(Z_3^{t_2})+(Z_1)$	$(Z_2)+2(Z_1)$	$5(Z_1)$	$(A_4^{t_2})$
(D_5^z)	$(D_5^z)+(Z_2^-)$	(Z_2^-)	$2(Z_2^-)$	$(Z_5)+(Z_1)$	$3(Z_2^-)$	$2(Z_1)$	$2(Z_2^-)+2(Z_1)$	$6(Z_1)$	(D_5^z)
(V_4^-)	$2(Z_2^-)+(Z_2)$	$(V_4^-)+(Z_1)$	$2(Z_2^-)+(Z_2)+(Z_1)$	$3(Z_1)$	$3(V_4^-)+3(Z_1)$	$5(Z_1)$	$2(Z_2^-)+(Z_2)+6(Z_1)$	$15(Z_1)$	(V_4^-)
(D_3^z)	$2(Z_2^-)$	$(Z_3)+(Z_2^-)$	$(D_3^z)+(Z_2^-)+(Z_1)$	$2(Z_1)$	$3(Z_2^-)+(Z_1)$	$(Z_3)+3(Z_1)$	$2(Z_2^-)+4(Z_1)$	$10(Z_1)$	(D_3^z)
$(Z_5^{t_1})$	$(Z_5^{t_1})+(Z_1)$	(Z_1)	$2(Z_1)$	$2(Z_5^{t_1})+2(Z_1)$	$3(Z_1)$	$4(Z_1)$	$6(Z_1)$	$12(Z_1)$	$(Z_5^{t_1})$
$(Z_5^{t_2})$	$(Z_5^{t_2})+(Z_1)$	(Z_1)	$2(Z_1)$	$2(Z_5^{t_2})+2(Z_1)$	$3(Z_1)$	$4(Z_1)$	$6(Z_1)$	$12(Z_1)$	$(Z_5^{t_2})$
$(Z_3^{t_1})$	$2(Z_1)$	$2(Z_3^{t_1})+(Z_1)$	$(Z_3^{t_1})+3(Z_1)$	$4(Z_1)$	$5(Z_1)$	$2(Z_3^{t_1})+6(Z_1)$	$10(Z_1)$	$20(Z_1)$	$(Z_3^{t_1})$
$(Z_3^{t_2})$	$2(Z_1)$	$2(Z_3^{t_2})+(Z_1)$	$(Z_3^{t_2})+3(Z_1)$	$4(Z_1)$	$5(Z_1)$	$2(Z_3^{t_2})+6(Z_1)$	$10(Z_1)$	$20(Z_1)$	$(Z_3^{t_2})$
(Z_2^-)	$2(Z_2^-)+2(Z_1)$	$(Z_2^-)+2(Z_1)$	$2(Z_2^-)+4(Z_1)$	$6(Z_1)$	$3(Z_2^-)+6(Z_1)$	$10(Z_1)$	$2(Z_2^-)+14(Z_1)$	$30(Z_1)$	(Z_2^-)

Table 4.6: Multiplication Table for $A(A_5)$-module $A_1(A_5 \times S^1)$. The upper half of the table describes the multiplication in the Burnside Ring $A(A_5)$.

4.2 Multiplicativity Property for the Primary Equivariant Degree

The following multiplicativity property of the primary degree plays an important role in practical computations of the primary degree.

Let Γ be a compact Lie group, V (resp. W) be an orthogonal representation of $G := \Gamma \times S^1$ (resp. of Γ), $\Omega \subset \mathbb{R} \oplus V$ (resp. $\mathcal{U} \subset W$) an invariant open bounded set and $f : \mathbb{R} \oplus V \to V$ (resp. $g : W \to W$) an Ω-admissible (resp. $\mathcal{U}$-admissible) equivariant map. Then

$$G\text{-Deg}\,(f \times g, \Omega \times \mathcal{U}) = \Gamma\text{-Deg}\,(g, \mathcal{U}) \cdot G\text{-Deg}\,(f, \Omega),$$

where the multiplication is taken in the $A(\Gamma)$-module $A_1(\Gamma \times S^1)$. (See[3])

Chapter 5

Application of Equivariant Degree to Symmetric Systems of Van Der Pol Equations

In this chapter we set up a standard framework for analyzing the van der Pol systems with symmetries and introduce several examples of symmetric van der Pol systems based on the geometric symmetries of regular polygons, tetrahedron, octahedron and dodecahedron. Later on, we present a series of results for these systems, describing and classifying the symmetry types of different periodic solutions occuring in these systems, which can be deduced from properties of the linearized systems. Finally, the necessary algebraic computations for the considered groups D_3, D_4, D_5, A_4, S_4 and A_5 are presented.

5.1 Definitions and Some Basic Facts

Definition 5.1.1. Let V be a real (resp. complex) Banach space, and G be a compact Lie group. We say that V is a real (resp. complex) *Banach representation* of G or *Banach G-representation* if the space V is a G-space such that the translation map $T_g : V \to V$, defined by $T_g(v) = gv$ for $v \in V$, is a *bounded* $\mathbb{R}$-linear (resp. $\mathbb{C}$-linear) operator for every $g \in G$.

Clearly, every finite-dimensional G-representation is a Banach G-representation.

We say that a Banach G-representation V is *isometric* if for each $g \in G$, the translation operator $T_g : V \to V$ is an isometry, i.e. $\|T_g v\| = \|v\|$ for all $v \in V$ and we call the norm $\|\cdot\|$ a *G-invariant* norm.

Theorem 5.1.2. *(cf. [36]) Let V be a Banach G-representation. Then for every irreducible G-representation $\mathcal{V}_j$, there exists a closed subspace $V_j \subset V$ satisfying:*

(i) If $W_0 \subset V$ is an irreducible subrepresentation of V equivalent to $\mathcal{V}_j$, then $W_0 \subset V_j$.

(ii) If $W_0^1 \subset V$ is an irreducible subrepresentation of V not equivalent to $\mathcal{V}_j$, then $W_0^1 \cap V_j = \{0\}$.

(iii) The set $V_\infty = \oplus_j V_j$ is dense in V i.e. $V = \overline{\oplus_j V_j}$.

(iv) There exists G-equivariant, projection $P_j : V \to V_j$.

Definition 5.1.3. The *Sobolev space* of order 1 on $[a,b]$ is defined by

$$H^1([a,b];\mathbb{R}^n) = \{u \in L^2([a,b];\mathbb{R}^n) \mid u' \in L^2([a,b];\mathbb{R}^n)\}.$$

The space $H^1([a,b];\mathbb{R}^n)$ can be equipped with scalar product

$$u \bullet v = \int_a^b u(t) \cdot v(t) dt + \int_a^b u'(t) \cdot v'(t) dt, \tag{5.1}$$

and $\|u\|_{1,2} = \sqrt{u \bullet u}$. Notice that $H^1([a,b];\mathbb{R}^n)$ is a Hilbert space for the inner product defined by (5.1).

Definition 5.1.4. Let X and Y be two Banach spaces. A linear operator $A : X \to Y$ is called *compact*, if $A(B) \subset Y$ is relatively compact, (B is the unit ball in X).

Theorem 5.1.5. *(cf. [10]) The natural imbedding*

$$j : H^1([a,b];\mathbb{R}^n) \hookrightarrow C([a,b];\mathbb{R}^n),$$

defined by $j(u) = u$ is a compact operator.

Corollary 5.1.6. *The natural imbedding*

$$j : H^1([a,b];\mathbb{R}^n) \hookrightarrow L^2([a,b];\mathbb{R}^n),$$

is a compact operator.

Definition 5.1.7. Let $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map. We define the map

$$N_f : C([a,b];\mathbb{R}^n) \to C([a,b];\mathbb{R}^n),$$

which is called *Nemysky Operator* associated with f, by $N_f(u)(t) = f(t, u(t))$.

Notice that N_f is a continuous map.

5.2 Systems of van der Pol Equations with Symmetries

The van der Pol equations are related to the so-called self-excited dynamical systems arising in many models of mechanics, electronics and biology. For more information on van der Pol oscillators and related results, we refer to [3].

We are interested in systems of coupled identical van der Pol equations of the type

$$\begin{cases} \ddot{u}_1 + \epsilon(u_1^2 - a)\dot{u}_1 + c_{11}u_1 + c_{12}u_2 + \cdots + c_{1n}u_n & = 0 \\ \ddot{u}_2 + \epsilon(u_2^2 - a)\dot{u}_2 + c_{21}u_1 + c_{22}u_2 + \cdots + c_{2n}u_n & = 0 \\ \vdots \quad \vdots \qquad\qquad \vdots \qquad\quad \vdots \quad \cdots \quad \vdots & \vdots \\ \ddot{u}_n + \epsilon(u_n^2 - a)\dot{u}_n + c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{nn}u_n & = 0 \end{cases} \tag{5.2}$$

where $a > 0$, $\epsilon > 0$, admitting certain "spatial" symmetries. The system (5.2) can be reformulated using the vector "multiplication":

$$uv = \begin{bmatrix} u_1v_1 \\ u_2v_2 \\ \vdots \\ u_nv_n \end{bmatrix}, \quad \text{where } u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

in the following form

$$\ddot{u} + \varepsilon(u^2 - \vec{a})\dot{u} + Cu = 0 \tag{5.3}$$

where

$$\varepsilon = \begin{bmatrix} \epsilon \\ \epsilon \\ \vdots \\ \epsilon \end{bmatrix}, \quad \vec{a} = \begin{bmatrix} a \\ a \\ \vdots \\ a \end{bmatrix}, \quad u^2 = \begin{bmatrix} u_1^2 \\ u_2^2 \\ \vdots \\ u_n^2 \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix}.$$

There are many possible examples of symmetric van der Pol systems of the type (5.3), where the matrix C is equvariant with respect to a certain group Γ acting on $u = (u_1, u_2, \cdots, u_n)$ by permuting its coordinates. Let us discuss some of them.

Example 5.2.1. We consider a ring of n identical van der Pol oscillators where the interaction takes place only between the neighboring oscillators (see Figure 5.1), i.e. in this case the matrix C is of the type

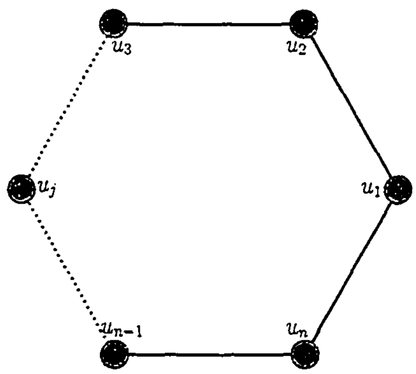


Figure 5.1: System with dihedral symmetries.

$$C = \begin{bmatrix} c & d & 0 & \dots & 0 & d \\ d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \dots & d & c \end{bmatrix}.$$

It is clear that the system (5.3) has the dihedral group D_n of symmetries.

In the subsequent examples, we present the concrete systems of van der Pol equations modelled on three regular polyhedrons: *tetrahedron, octahedron* and *dodecahedron*. In each case, the symmetry group Γ of the system is composed of the orthogonal symmetries of the corresponding polyhedron. To simplify the presentation, we have considered only those orthogonal symmetries T for which $\det T = 1$. This assumption is not essential, and in the general case, similar results can be easily derived based on the already obtained computations.

Example 5.2.2. Let us consider four identical inter-connected van der Pol oscillators having exactly the same linear interaction with all the other oscillators. In this case, the matrix C in the system (5.3) can be written as:

$$C = \begin{bmatrix} c & d & d & d \\ d & c & d & d \\ d & d & c & d \\ d & d & d & c \end{bmatrix}. \tag{5.4}$$

The situation is illustrated on Figure 5.2, where the vertices of the tetrahedron symbolize the oscillators and its edges correspond to the connections between the

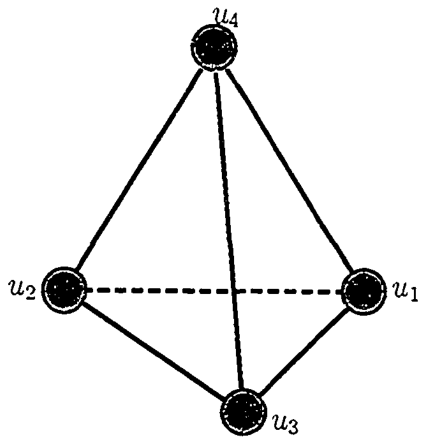


Figure 5.2: System with tetrahedral symmetries.

oscillators, indicating the interaction between them. It is clear that this system of differential equations is symmetric with respect to the tetrahedral group $\mathbb{T} = A_4$.

Example 5.2.3. Suppose that the van der Pol oscillators are arranged in a configuration corresponding to the vertices of a cube. We assume that the interaction takes place between those oscillators that are connected by an edge of the cube (see Figure 5.3). We assume that all the oscillators are identical.

The eight identical van der Pol oscillators, which are inter-connected, illustrated on Figure 5.3, lead to the system of equations with the matrix C of the following type:

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}. \tag{5.5}$$

It is clear that the system of van der Pol equations (5.3) is symmetric with respect to the octahedral symmetry group $\mathbb{O}$ which is isomorphic to the symmetric group S_4.

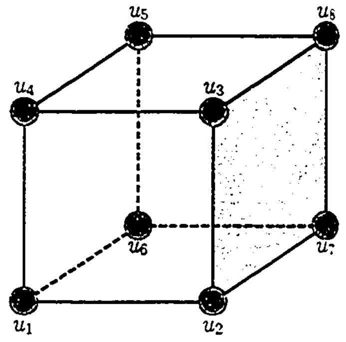


Figure 5.3: System with octahedral symmetries.

Example 5.2.4. Let us consider an arrangement of van der Pol oscillators based on the inter-connections given by the edges of a dodecahedron (see Figure 5.4). It is clear that the group of symmetries of the dodecahedron, which is the icosahedral group $\mathbb{I}$, is the symmetry group of the system (5.3). Let us point out that the icosahedral group $\mathbb{I}$ is isomorphic to the alternating group A_5. In this case we have the system (5.3) composed of 20 equations, where the matrix C is given by:

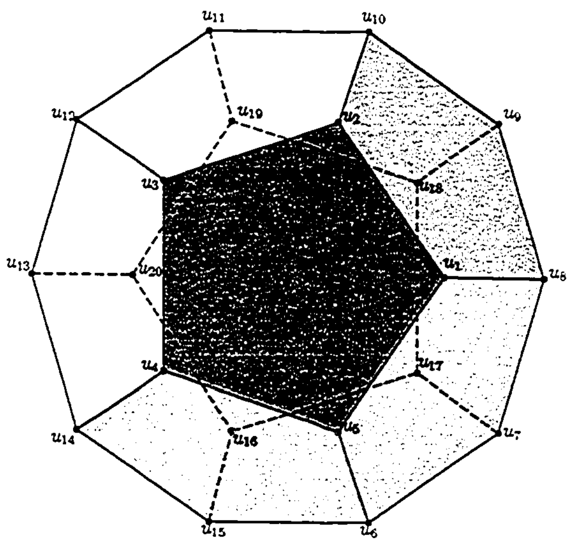


Figure 5.4: System with icosahedral symmetries.

$$C = \begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}. \tag{5.6}$$

5.3 Reformulation of the Problem as an Equivariant Fixed-Point Problem with One Parameter

In this section we discuss a general strategy based on the application of the equivariant degree allowing us to study symmetric periodic solutions for (5.3).

5.3.1 Preliminaries

Notice that in the all examples discussed above, the space $V := \mathbb{R}^n$ was an orthogonal representation of a certain finite group Γ, acting on vectors $u \in \mathbb{R}^n$ by permuting their components, the matrix C commuted with the action of Γ on V, and $\det(C) \neq 0$. In addition C was symmetric, i.e.

$$Cu \bullet v = u \bullet Cv, \quad u,\ v \in \mathbb{R}^n,$$

where $u \bullet v$ denotes the usual inner product in $\mathbb{R}^n$.

By replacing the independent variable t by $\frac{p}{2\pi}\tau$, where $p > 0$, the equation (5.3) can be rewritten as

$$\ddot{u} + \frac{p}{2\pi}\varepsilon(u^2 - \vec{a})\dot{u} + \frac{p^2}{4\pi^2}Cu = 0. \tag{5.7}$$

Since, we are looking for a 2π-periodic solution, the boundary conditions for the system (5.7) are

$$u(0) = u(2\pi) \quad \text{and} \quad \dot{u}(0) = \dot{u}(2\pi).$$

Let us put $\alpha := \frac{p}{2\pi}$, so the equation (5.7) can be rewritten as

$$\ddot{u} + \alpha\varepsilon(u^2 - \vec{a})\dot{u} + \alpha^2 Cu = 0. \tag{5.8}$$

Set

$$F(u) = \left(\frac{1}{3}u^3 - \vec{a}u\right). \tag{5.9}$$

Then the equation (5.8) becomes

$$\ddot{u} + \alpha\varepsilon\frac{d}{dt}F(u) + \alpha^2 Cu = 0. \tag{5.10}$$

The equation (5.10), together with the periodic boundary conditions, can be reformulated as a non-linear operator equation in an appropriate Hilbert representation of the group $G = \Gamma \times S^1$, where Γ denotes the symmetry group of the system (5.8).

We will need another technical assumption, which is used later to establish *a priori* bounds for the periodic solutions. We will restrict our analysis to the solutions u of (5.3) satisfying the following additional condition:

$$u(t+\pi) = -u(t), \quad \text{for all} \quad t \in \mathbb{R}.$$

In this way, we transform (5.8) into the following system

$$\begin{cases} -\ddot{u} = \alpha\varepsilon(u^2 - \bar{a})\dot{u} + \alpha^2 Cu, & u(t) \in V \\ u(t+\pi) = -u(t). \end{cases} \tag{5.11}$$

5.3.2 Setting in Functional Spaces

Let us introduce the functional spaces, which are appropriate for studying (5.11). First we define the subspace $\mathbb{H}_o$ of the Sobolev space $H^2_{2\pi}(\mathbb{R}, V)$ of 2π-periodic, twice-differentiable, V-valued functions such that $\dot{u}$ and $\ddot{u} \in L^2(\mathbb{R}, V)$, defined as

$$\mathbb{H}_o = \{u \in H^2_{2\pi}(\mathbb{R}, V) \mid u(t+\pi) = -u(t), \ \forall\, t \in \mathbb{R}\}.$$

We will also identify V with the space of all constant V-valued functions. The space $\mathbb{H}_o$ can be equipped with an inner product, given by

$$\langle u, v\rangle_{\mathbb{H}_o} = \int_0^{2\pi} u(t) \bullet v(t)dt + \int_0^{2\pi} \dot{u}(t) \bullet \dot{v}(t)dt + \int_0^{2\pi} \ddot{u}(t) \bullet \ddot{v}(t)dt.$$

In addition, we define the subspace $\mathbb{L}_o \subset L^2([0, 2\pi]; V)$ by $\mathbb{L}_o := L(\mathbb{H}_o)$, where $Lu = -\ddot{u}$. It is clear that $L : \mathbb{H}_o \to \mathbb{L}_o$ is an isomorphism. Let us define

$$\mathbb{H} := V \oplus \mathbb{H}_o, \quad \mathbb{L} := V \oplus \mathbb{L}_o.$$

We put

$$K : \mathbb{H} \to \mathbb{L}, \quad Ku = \frac{1}{2\pi}\int_0^{2\pi} u(t)dt.$$

It is clear that the operator K is an orthogonal projection on the subspace V of constant functions and $L + K : \mathbb{H} \to \mathbb{L}$ is an isomorphism such that $(L+K)_{|V} = \mathrm{Id}$ and $(L+K)_{|\mathbb{H}_o} = L_{|\mathbb{H}_o}$. Given $u \in \mathbb{H}$, denote by $\overline{u}$ (resp. u_o) its orthogonal projection of u on V (resp. $\mathbb{H}_o$).

The space $H^2_{2\pi}(\mathbb{R}; V)$ is a Hilbert representation of the group $\Gamma \times S^1$, where the element $(\gamma, e^{i\tau}) \in \Gamma \times S^1$ acts on a function $u \in H^2_{2\pi}(\mathbb{R}; V)$ by the formula

$$(\gamma, e^{i\tau})u(t) = \gamma(u(t+\tau)), \quad \text{for all} \quad t \in \mathbb{R}, \ \gamma \in \Gamma, \ e^{i\tau} \in S^1. \tag{5.12}$$

The S^1-isotypical components of the space $H^2_{2\pi}(\mathbb{R}; V)$ are the subspaces V_l^c, $l = 1, 2, \dots$, and the subspace of constant functions V (which is the S^1-fixed-point subspace), where

$$V_l^c = \{a_l \cos(lt) + b_l \sin(lt) |\ a_l, b_l \in V\}.$$

A function $u \in V_l^c$, $u(t) = a_l \cos(lt) + b_l \sin(lt)$, can be identified with

$$u(t) = e^{ilt}\Big(x_l + i\, y_l\Big),$$

where $x_l = \frac{a_l + b_l}{2}$ and $y_l = \frac{a_l - b_l}{2}$, so the action of $e^{il\tau} \in S^1$ on $u(t)$ is simply the complex multiplication by $e^{il\tau}$, i.e. $e^{il\tau} \cdot u(t) = e^{il(t+\tau)}(x_l + i\, y_l)$. It is clear that V_l^c are G-invariant subspaces of $H^2_{2\pi}(\mathbb{R}; V)$; in addition, V_1^c (considered as a complex linear space) is S^1-isomorphic to the complexification of V. Let $D(u)(t) = \dot{u}(t)$, then for $u(t) = e^{ilt}\big(x_l + i\, y_l\big)$ we have

$$D(u) = il\, u, \quad \text{and} \quad L(u) = l^2 u, \tag{5.13}$$

so L and D preserve V_l^c, $l = 1, 2, 3, \dots$.

Notice that V_l^c, $l = 1, 3, 5, \dots$, are the S^1-isotypical components of $\mathbb{H}_o$.

5.3.3 Operator Reformulation and Setting for the Equivariant Degree Treatment

Let us now reformulate the problem (5.11) as a parameterized G-equivariant fixed point problem in the space $\mathbb{H}_0$, where $G = \Gamma \times S^1$.

We consider the following (infinite dimensional) representation of the group G:

$$C_o := \{u : \mathbb{R} \to V;\ u_{|[0,2\pi]} \in C^1([0, 2\pi], V),\ \ u(t + \pi) = -u(t)\ \forall\, t \in \mathbb{R}\},$$

where $u(t+2\pi) = u(t)$, $\dot{u}(t+2\pi) = \dot{u}(t)$. Let us define $C := V \oplus C_o$, notice that $\dot{u}$ is a continuous periodic function for every function $u \in C$, in particular $\dot{u}(t+\pi) = -\dot{u}(t)$ for all $t \in \mathbb{R}$, therefore the map $N : C \to L^2([0, 2\pi]; V)$ defined by

$$N(u)(t) = (u^2(t) - \vec{a})\dot{u}(t), \quad t \in \mathbb{R},$$

satisfies

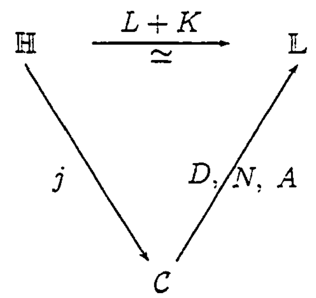
$$N(u)(t + \pi) = -N(u)(t),$$

thus $N : C \to \mathbb{L}$. It is clear that N is a continuous map.

We also define the operators:

$$\begin{aligned} j :& \mathbb{H} \hookrightarrow \mathcal{C}, & j(u) &= u, \\ A :& \mathcal{C} \to \mathbb{L}, & (Au)(t) &= C(u(t)), \\ D :& \mathcal{C} \to \mathbb{L}, & (Du)(t) &= \dot{u}(t). \end{aligned}$$

The relations between the operators L, j, D, N and A are illustrated in the following diagram.



Notice that the linear operator j is compact, and A is a bounded linear operator. In addition, all the above operators are G-equivariant, where the G-action on all the above functional spaces is defined by (5.12). The equation $Lu = -\ddot{u}$ can be written in the following operator form:

$$L(u) = \alpha\varepsilon N(j(u)) + \alpha^2 A(j(u)), \quad u \in \mathbb{H}. \tag{5.14}$$

5.3.4 Computations of the Equivariant Degree

We are going to introduce additional parameters to the original system of differential equations to allow its deformation to a "linear" system. Then, by applying *a priori* bounds to parameterized systems, the existence result can be obtained by using the *homotopy property* of the degree.

Let us introduce additional parameters $\delta \in [0, 1]$ and $\lambda \in \mathbb{R}$ to the equation (5.11):

$$\begin{cases} -\ddot{u} = \delta\alpha\varepsilon(u^2 - \vec{a})\dot{u} + \alpha^2 Cu - \lambda\alpha\rho\dot{u}, & u(t) \in V \\ u(t+\pi) = -u(t) \end{cases} \tag{5.15}$$

where $\rho = \varepsilon\vec{a}$ (see [27]).

Assume for a moment that there exists an increasing positive function $\mathfrak{m}(\cdot)$ such that every solution u_o of the system (5.15) for $\lambda = 0$, which by the imposed conditions belongs to $\mathbb{H}_o$, satisfies the inequality (cf. Lemma 5.5.1)

$$\|u_o\|_{\mathbb{H}_o} \leq \mathfrak{m}(\alpha).$$

Given $\alpha > 0$, take $M > \mathfrak{m}(\alpha)$, and choose $m < \mathfrak{m}(\alpha)$ to be small enough. We define $\eta : \mathbb{R} \to [0,1]$ by

$$\eta(t) = \begin{cases} 0 & \text{if } t < m, \\ \frac{t-m}{M-m} & \text{if } m \leq t \leq M, \\ 1 & \text{if } t > M, \end{cases}$$

and $\theta : \mathbb{H}_0 \to [0,1]$ by $\theta(u_o) = \eta\Big(\|u_o\|_{\mathbb{H}_o}\Big)$, where $u_o \in \mathbb{H}_o$. We modify the problem (5.15) as follows

$$\begin{cases} -\ddot{u} = \delta\alpha\varepsilon(u^2 - \bar{a})\dot{u} + \alpha^2\theta(u_o)Cu - \lambda\alpha\rho\dot{u}, \quad u(t) \in V \\ u(t+\pi) = -u(t). \end{cases} \tag{5.16}$$

The problem (5.16) can be reformulated as the following parameterized equation in the functional space $\mathbb{H} = V \oplus \mathbb{H}_o$

$$\begin{cases} Lu_o = \delta\alpha\varepsilon N(j(u_o)) + \alpha^2\theta(u_o)A(j(u_o)) - \lambda\alpha\rho D(j(u_o)), \\ 0 = \alpha^2\theta(u_o)A\overline{u}. \end{cases} \tag{5.17}$$

Notice that the equation (5.17) can be written as

$$\begin{aligned} (L+K)u \ &= \delta\alpha\varepsilon N(j(u_o)) + \alpha^2\theta(u_o)A(j(u_o)) + \alpha^2\theta(u_o)A\overline{u} \\ &\quad -\lambda\alpha\rho D(j(u_o)) + K(u), \end{aligned} \tag{5.18}$$

and since $L+K$ is a G-equivariant isomorphism, (5.18) is equivalent to

$$\begin{aligned} u \ &= (L+K)^{-1}\Big[\delta\alpha\varepsilon N(j(u_o)) + \alpha^2\theta(u_o)A(j(u_o)) + \alpha^2\theta(u_o)A\overline{u} \\ &\quad -\lambda\alpha\rho D(j(u_o)) + K(u)\Big]. \end{aligned} \tag{5.19}$$

Consequently, the equation (5.19) can be represented as the system of equations

$$\begin{cases} u_o = \delta\alpha\varepsilon L^{-1}N(j(u_o)) + \alpha^2\theta(u_o)L^{-1}A(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)) \\ \overline{u} = \alpha^2\theta(u_o)A\overline{u} + \overline{u}. \end{cases} \tag{5.20}$$

We define $\widetilde{G}(\alpha, \delta, \cdot, \cdot) : \mathbb{R} \times \mathbb{H}_o \to \mathbb{H}_o$, by

$$\widetilde{G}(\alpha, \delta, \lambda, u_o) := \delta\alpha\varepsilon L^{-1}N(j(u_o)) + \alpha^2\theta(u_o)L^{-1}A(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \tag{5.21}$$

and $G(\alpha, \delta, \cdot, \cdot) : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, by

$$G(\alpha, \delta, \lambda, u) = \Big(\overline{u} + \alpha^2\theta(u_o)A(\overline{u}), \widetilde{G}(\alpha, \delta, \lambda, u_o)\Big), \quad u = \overline{u} + u_o, \quad \overline{u} \in V, \ u_o \in \mathbb{H}_o. \tag{5.22}$$

Clearly, $G(\alpha, \delta, \lambda, u)$ is a completely continuous G-equivariant map.

Remark 5.3.1. Notice that, the original van der Pol equation (5.11) corresponds to the case $\lambda = 0$ and $\delta = 1$, except for the nonlinear factor $\theta(u_o)$ in (5.17). However, if $||u_o||_{\mathbb{H}_o} \geq M$, then $\theta(u_o) = 1$ so the solution u_o of (5.17) is also a solution of (5.11).

Remark 5.3.2. In the case of one free parameter, the simplest equivariant maps (needed for the computations of the equivariant degree) turn out to be the so-called basic maps, which on the isotypical components have a form

$$\mathfrak{b}(\lambda, v) = \Big(1 - ||v|| + i\beta\lambda\Big)v, \quad \lambda \in \mathbb{R}, \ \beta > 0. \tag{5.23}$$

In section 5.4, we will show that the term $-\lambda\alpha\rho L^{-1}D(j(u_o))$ in the system (5.20) corresponds to the term $i\beta\lambda v$ in (5.23), while $(1 - ||v||)v$ corresponds to $u_o - \alpha^2\theta(u_o)L^{-1}A(j(u_o))$, i.e. the basic maps (5.23) "emerge" from the "linearized system". However, the "linearized system" can not be connected by an admissible homotopy to the original van der Pol system! The "breaking" of the homotopy occurs for those solutions u_o with $||u_o||_{\mathbb{H}_o} = M$, which are in fact the solutions of the original van der Pol system, so the existence results still can be obtained (see section 5.5).

We define

$$\Omega := \{(\lambda, u) \in \mathbb{R} \times \mathbb{H} \mid \lambda \in (-\lambda_o, \lambda_o), \ m < ||u_o||_{\mathbb{H}_o} < M, \ ||\overline{u}|| < 1\},$$

$$\Omega_o := \{(\lambda, u_o) \in \mathbb{R} \times \mathbb{H}_o \mid \lambda \in (-\lambda_o, \lambda_o), \ m < ||u_o||_{\mathbb{H}_o} < M\},$$

$$B(0,1) := \{v \in V \mid ||v|| < 1\},$$

where $u = \overline{u} + u_o$, $u_o \in \mathbb{H}_o$, $\overline{u} \in V$ and the constant $\lambda_o > 0$ is a fixed number, which will be specified later. Notice that the set Ω is a product of $\Omega_o \subset \mathbb{R} \times \mathbb{H}_o$ and $B(0,1) \subset V$. The boundary $\partial\Omega_o$ is composed of three parts

$$\partial_m := \{(\lambda, u_o) \in \overline{\Omega_0} \mid ||u_o||_{\mathbb{H}_o} = m\},$$

$$\partial_M := \{(\lambda, u_o) \in \overline{\Omega_0} \mid \|u_o\|_{\mathbb{H}_o} = M\},$$
$$\partial_o := \{(\lambda, u_o) \in \overline{\Omega_0} \mid |\lambda| = \lambda_o\}.$$

It is possible to show that for appropriate values of α and M, the homotopy $\widetilde{G}(\alpha, \delta, \lambda, u_o)$ with respect to $\delta \in [0, \delta_o]$ (where δ_o will be chosen to be large enough) has no fixed points in $\partial_m \cup \partial_o$. Notice that for $\delta = 0$ the equation (5.20) can be written as

$$u_o = \alpha^2\theta(u_o)L^{-1}A(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \quad u_o \in \mathbb{H}_o. \tag{5.24}$$

In addition, the equation (5.24) has no solutions in ∂_M. Let us put

$$\mathcal{F}(\lambda, \overline{u}, u_o) = (\overline{u} + \alpha^2\theta(u_o)A(\overline{u}), \alpha^2\theta(u_o)L^{-1}A(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o))) \in V \times \mathbb{H}_o. \tag{5.25}$$

It is possible to show that the primary equivariant degree

$$G\text{-Deg}\,(\mathrm{Id} - \mathcal{F}, \Omega) = \sum_{(H)} n_H\,(H), \tag{5.26}$$

is different from zero. In the next section we will reduce the computations of (5.26) to studying the equivariant degrees of the *basic maps* related to irreducible Γ- and G-representations.

On the other hand, it is possible to apply a G-equivariant homotopy $\mathrm{Id} - \Psi(s, \lambda, u)$, $s \in [0, 1]$, to the map $\mathrm{Id} - G(\alpha, \delta_o, \lambda, u)$, where

$$u - \Psi(0, \lambda, u) = u - G(\alpha, \delta_o, \lambda, u) \quad \text{for } (\lambda, u) \in \overline{\Omega},$$
$$u - \Psi(s, \lambda, u) \neq 0, \quad \text{for } (\lambda, u) \in \partial\Omega,$$

and the map $\mathrm{Id} - \Psi(1, \cdot, \cdot)$ satisfies

$$G\text{-Deg}\,(\mathrm{Id} - \Psi(1, \cdot, \cdot), \Omega) = 0.$$

By using the standard argument, it will follow that for every orbit type (H_o) in Ω for which n_{H_o} is different from zero, there exist $\delta > 0$ and $u_0 \in \partial_M$ satisfying

$$\begin{cases} -\ddot{u}_o = \delta\alpha\varepsilon(u_o^2 - \vec{a})\dot{u}_o + \alpha^2\theta(u_o)Cu_o - \lambda\alpha\rho\dot{u}_o, & u_o(t) \in V \\ u_o(t+\pi) = -u_o(t), \end{cases}$$

and having a symmetry at least H_o. Since $u_o \in \partial_M$, we have $\theta(u_o) = 1$, so u_o is a solution of the equation

$$\begin{cases} -\ddot{u}_o = \delta\alpha\varepsilon(u_o^2 - \vec{a})\dot{u}_o + \alpha^2 Cu_o - \lambda\alpha\rho\dot{u}_o, & u_o(t) \in V \\ u_0(t+\pi) = -u_0(t). \end{cases}$$

Definition 5.3.3. Let V be a finite-dimensional or Banach representation of $\Gamma \times S^1$ and let V_0 be the G-invariant complement of V^{S^1} in V. We will call an orbit type (H) in V to be *dominating*, if (H) is a maximal isotropy type in $V_0 \setminus \{0\}$ with respect to the usual order relation.

5.4 Computations of the Equivariant Degree: Reduction to Basic Maps

In this section we reduce the computation of the G-equivariant degree $G\text{-Deg}(\mathrm{Id} - \mathcal{F}, \Omega)$, where $\mathcal{F}$ is defined by (5.25), to the computation of the degrees of basic maps.

5.4.1 Finite-Dimensional Reduction

We, first, study the solution set for the equation

$$u_o = \mathcal{F}_o(\lambda, u_o) \iff u_o = \alpha^2\theta(u_o)L^{-1}A(j(u_o)) - \lambda\alpha\rho L^{-1}D(j(u_o)), \quad u_o \in \mathbb{H}_o \tag{5.27}$$

(in particular, we will show that the solution set is finite-dimensional). The above equation (5.27) can be rewritten as follows:

$$\begin{cases} \ddot{u}_o + \alpha^2\theta(u_o)Cu_o - \lambda\alpha\rho\dot{u}_o = 0, & u_o(t) \in V \\ u_o(t+\pi) = -u_o(t). \end{cases} \tag{5.28}$$

Since the matrix C is nonsingular, symmetric and Γ-equivariant, it is diagonalizable and every eigenspace is a Γ-invariant subspace. Let $\sigma(C) = \{\mu_s\}$ denote the spectrum of C and assume that for every $v \in V$ we have a decomposition $v = \sum_s v_s$, where v_s is an eigenvector corresponding to the eigenvalue μ_s. Then, we can split the system (5.28) into

$$\begin{cases} \ddot{u}_s + \alpha^2\theta(u_o)\mu_s u_s - \lambda\alpha\rho\dot{u}_s = 0, & u_o = \sum_s u_s \\ u_s(t+\pi) = -u_s(t). \end{cases} \tag{5.29}$$

Since (5.29) is a system with constant coefficients, it follows that (5.29) has 2π-periodic solutions u_s satisfying $u_s(t+\pi) = -u_s(t)$ if

$$\alpha^2\theta(u_o)\mu_s = (2r-1)^2 \quad \text{and} \quad \lambda = 0 \tag{5.30}$$

for some $r = 1, 2, 3, \dots$. By construction, the function u_o lives in Ω_o, therefore $\theta(u_o) \in (0,1)$ (see the definition of $\theta(\cdot)$ and requirements on Ω_o). From this it follows that (5.30) can be satisfied only if

$$\mu_s > \frac{1}{\alpha^2} > 0. \tag{5.31}$$

By the same argument, the requirement for possible values of α should be

$$\alpha \neq \frac{(2r-1)}{\sqrt{\mu_s}}, \quad \text{for all } \mu_s,\ r = 1, 2, 3, \ldots. \tag{5.32}$$

Bearing in mind the isotypical decomposition of $\mathbb{H}_o$, formulae (5.13) and the second inequality from (5.31), the solution set to (5.29) satisfies the following equations in $\mathbb{H}_o$:

$$u_{s,r} - \frac{\theta(u_o)\mu_s\alpha^2}{(2r-1)^2} u_{s,r} = 0, \tag{5.33}$$

where $r = 1, 2, 3, \ldots$ and $u_{s,r}(t) = e^{(2r-1)it}(x_r + iy_r)$, for some μ_s-eigenvectors x_r and y_r of C. Thus (5.30) and (5.33) give rise to a non-zero solution for (5.29) if (5.31) and (5.32) are satisfied. In particular, since there are only finitely many such $r > 0$, the solution set to (5.33) (and respectively, to (5.29)), is finite-dimensional. Combining the above argument with the suspension property of the equivariant degree, one obtains that G-Deg$\,(\text{Id} - \mathcal{F}_o, \Omega_o) = G$-Deg$\,(\text{Id} - F_o, \Omega_1)$, where

$$F_o(\lambda, v) = \alpha^2\theta(\|v\|)Av + \lambda Tv, \quad (\lambda, v) \in \mathbb{R} \times U,$$

U is a *finite-dimensional* G-representation such that $U^{S^1} = \{0\}$, $\Omega_1 = \Omega_0 \cap (\mathbb{R} \times U)$ (see Figure 5.5), $A : U \to U$ is a G-equivariant nonsingular linear operator with spectrum

$$\sigma(A) = \left\{ \frac{\mu_s}{(2r-1)^2};\ r = 1, 2, \ldots, k,\ \mu_s > \frac{1}{\alpha^2} \right\}. \tag{5.34}$$

The linear operator $T : U \to U$ is diagonal with respect to the eigenvectors of A, with all its diagonal entries being positive multiples of i. Notice that since A is G-equivariant, one may consider A as a complex linear operator. In particular, the set $(\text{Id} - F_o)^{-1}(0) \cap \Omega_1$ is composed of finitely many S^1-orbits $S^1(v_0), \ldots, S^1(v_R)$.

5.4.2 Isotypical Decomposition and Basic Maps

In order to compute the G-degree G-Deg$\,(\text{Id} - F_o, \Omega_1)$, we need to consider the following S^1-isotypical decomposition of the space U:

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_k,$$

where U_l denotes the isotypical S^1-component of U with the S^1-action given by the complex multiplication $(\gamma, v) \mapsto \gamma^l \cdot v$, $(\gamma, v) \in S^1 \times U_l$, and the product '$\cdot$' denotes a complex multiplication. Every subspace U_l is invariant with respect to the Γ-action. We can consider the Γ-isotypical decomposition of U_l, which we denote by $U_l = U_{0,l} \oplus U_{1,l} \oplus \cdots \oplus U_{k,l}$, where each of the components $U_{j,l}$, $j = 0, \ldots, k$,

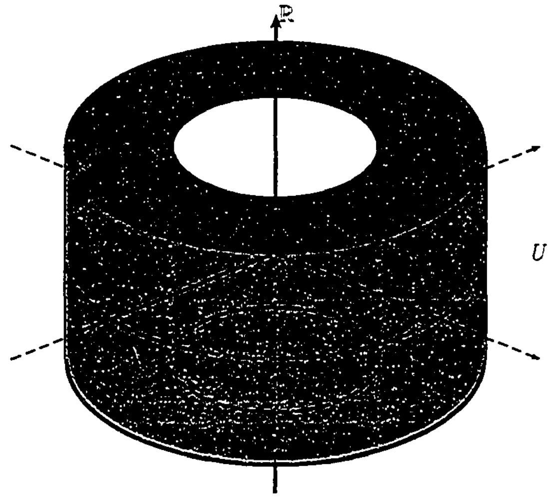


Figure 5.5: The set Ω_1

is modelled on the complex irreducible Γ-representation $\mathcal{V}_j^c$, $j = 1, \dots, k$, and $\mathcal{V}_0^c$ being the trivial representation of Γ. It is clear that the space $\mathcal{V}_j^c$ equipped with the above $\Gamma \times S^1$-action is a real irreducible representation of $G = \Gamma \times S^1$, which we will denote by $\mathcal{V}_{j,l}$. Consequently, we obtain the following isotypical decomposition of the space U with respect to the G-action:

$$U = \bigoplus_{j,l} U_{j,l}, \quad U_{j,l} \text{ modelled on } \mathcal{V}_{j,l}.$$

For an orthogonal irreducible representation $\mathcal{V}_{j,l}$ of $G = \Gamma \times S^1$ such that $\mathcal{V}_{j,l}{}^{S^1} = \{0\}$, we put

$$\mathcal{O} = \left\{(\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l} : \frac{1}{2} < \|v\| < 2, \ -1 < \lambda < 1\right\}.$$

The simplest $\mathcal{O}$-admissible map is $\mathfrak{b} : \overline{\mathcal{O}} \to \mathcal{V}_{j,l}$, by

$$\mathfrak{b}(\lambda, v) = (1 - \|v\| + i\lambda) \cdot v,$$

where $(\lambda, v) \in \mathbb{R} \times \mathcal{V}_{j,l}$. Notice that $\mathfrak{b}(\lambda, v) = 0$ if and only if $1 - \|v\| + i\lambda = 0$, i.e. $\lambda = 0$, $\|v\| = 1$. In what follows, for every G-irreducible representation $\mathcal{V}_{j,l}$, on which S^1 acts non-trivially, we denote by $(\mathfrak{b}, \mathcal{O})$ the so-called $\mathcal{V}_{j,l}$*-basic pair*, and we define

$$\deg_{\mathcal{V}_{j,l}} = G\text{-Deg}\,(\mathfrak{b}, \mathcal{O}) \in A_1(\Gamma \times S^1).$$

Similarly, let $\mathcal{V}_j$ be an irreducible representation of Γ and $\mathcal{B}_j$ be the unit ball in $\mathcal{V}_j$. The simplest (in some sense non-trivial) $\mathcal{B}_j$-admissible map is $-\text{Id} : \mathcal{V}_j \to \mathcal{V}_j$, which we denoted by $(-\text{Id}\,, \mathcal{B}_j)$ the so-called $\mathcal{V}_j$*-basic pair*. We put

$$\deg_{\mathcal{V}_j} := \Gamma\text{-Deg}\,(-\text{Id}\,, \mathcal{B}_j) \in A(\Gamma).$$

5.4.3 Product Formula

Return to the computations of G-Deg (Id $- F_o, \Omega_1$) and, respectively, G-Deg (Id $- \mathcal{F}, \Omega$) (see (5.25)). Let $\xi \in \sigma(A)$ be an eigenvalue of the G-equivariant linear operator $A : U \to U$. Then the eigenspace $E(\xi) = \{v \in U : Av = \xi v\}$ is a G-invariant subspace of U. Clearly, the subspace $E(\xi)$ can be represented as the direct sum of its G-isotypical components

$$E(\xi) = \bigoplus_{j,l} E_{j,l}(\xi), \quad E_{j,l}(\xi) \text{ modelled on } \mathcal{V}_{j,l}.$$

We will call the number

$$n_{j,l}(\xi) := \dim E_{j,l}(\xi)/\dim \mathcal{V}_{j,l},$$

the $\mathcal{V}_{j,l}$-*multiplicity* of the eigenvalue ξ. Consider the Γ-equivariant map Id $-\overline{F} : V \to V$, $(\mathrm{Id} - \overline{F})(\overline{v}) = -\alpha^2\theta(v_o)C(\overline{v})$, $\overline{v} \in V$ (cf. the second equation in system (5.20)). Let $\mathcal{B} = B(0,1)$ in V. Then the Γ-equivariant degree Γ-deg (Id $- \overline{F}, \mathcal{B}) \in A(\Gamma)$ can be computed as follows: for every eigenvalue $\mu_o \in \sigma(C)$ such that $\mu_o > \frac{1}{\alpha^2}$, we consider the Γ-isotypical decomposition of the associated with μ_o eigenspace $E(\mu_o) = \bigoplus_j E_j(\mu_o)$. We put

$$n_j(\mu_o) = \dim\, E_j(\mu_o)/\dim\, \mathcal{V}_j.$$

Then we have

$$\Gamma\text{-deg}\,(\mathrm{Id} - \overline{F}, \mathcal{B}) = \prod_{j,s} \left(\deg_{\mathcal{V}_j}\right)^{n_j(\mu_s)},$$

where the product is taken in the Burnside ring $A(\Gamma)$ and we assume that $\left(\deg_{\mathcal{V}_j}\right)^0 = (\Gamma)$[1].

Define Id $- F : \mathbb{R} \times V \times U \to V \times U$ by

$$(\mathrm{Id} - F)(\lambda, \overline{v}, v) = \Big(-\alpha^2\theta(v_o)C(\overline{v}), v - \alpha^2\theta(v)Av - \lambda Tv\Big), \quad (\lambda, \overline{v}, v) \in \mathbb{R} \times V \times U,$$

and put $\Omega_2 = \Omega \cap \Big(\mathbb{R} \times V \times U\Big)$. By the argument given in subsection 5.4.1, G-Deg (Id $- \mathcal{F}, \Omega$) $=$ G-Deg (Id $- F, \Omega_2$). In the statement following below, we present the result for the computation of G-Deg (Id $- F, \Omega_2$)

[1]Notice that we always have $\left(\deg_{\mathcal{V}_j}\right)^2 = (\Gamma)$

Proposition 5.4.1. *Under the notations of previous subsections we have*

$$G\text{-Deg}\,(\mathrm{Id}-F,\Omega_2)=\prod_{j,s}\left(\deg_{\mathcal{V}_j}\right)^{n_j(\mu_s)}\cdot\left[\sum_{\xi\in\sigma(A)}\sum_{j,l}n_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}}\right] \tag{5.35}$$

where the product '$\cdot$' denotes the $A(\Gamma)$ multiplication on the $\mathbb{Z}$-module $A_1(\Gamma\times S^1)$ generated by the twisted orbit types (H) in $\Gamma\times S^1$.

Proof: By using the homotopy invariance, we can modify the operator A (using a small perturbation) in such a way that each eigenvalue $\xi\in\sigma(A)$ is "simple", i.e. there exists exactly one (j,l) such that $n_{j,l}(\xi)=1$. Let us consider an eigenvalue $\xi\in\sigma(A)$ and suppose that $E(\xi)=E_{j,l}(\xi)$ for some isotypical component $U_{j,l}$. By (5.33), for every eigenvector $v\in E_{j,l}(\xi)$ we have that $(\mathrm{Id}-F_o)(\lambda,v)=0$ if $\lambda=0$ and $\alpha^2\theta(v)\xi=1$. Put $K_{j,l}(\xi)=(\mathrm{Id}-F_o)^{-1}(0)\cap\Omega_1\cap E_{j,l}(\xi)$. The sets $K_{j,l}(\xi)$ are compact and it is possible to separate them by choosing small open G-invariant neighborhoods $\Omega_{j,l}(\xi)$ in $\mathbb{R}\times U$. Notice that for every neighborhood $\Omega_{j,l}(\xi)$ the map $\mathrm{Id}-F_o$ is G-homotopic to a map, which is normal to $E_{j,l}(\xi)$. Consequently, by the additivity and suspension properties of the G-equivariant degree, we obtain

$$\begin{aligned}G\text{-Deg}\,(\mathrm{Id}-F_o,\Omega_1)&=\sum_{\xi,j,l}G\text{-Deg}\,(\mathrm{Id}-F_o,\Omega_{j,l}(\xi))\\&=\sum_{\xi,j,l}G\text{-Deg}\left((\mathrm{Id}-F_o)_{|E_{j,l}(\xi)},\Omega_{j,l}(\xi)\cap E_{j,l}(\xi)\right).\end{aligned}$$

On the other hand, it can be easily verified that the map $(\mathrm{Id}-F_o)_{|E_{j,l}(\xi)\cap\Omega_{j,l}(\xi)}$ is G-homotopic to a basic map on $\mathcal{V}_{j,l}$. This reduction to basic maps is fundamental for the computations of the primary degree. Consequently,

$$G\text{-Deg}\left((\mathrm{Id}-F_o)_{|E_{j,l}(\xi)},\Omega_{j,l}(\xi)\cap E_{j,l}(\xi)\right)=\deg_{\mathcal{V}_{j,l}}.$$

Therefore, by applying the homotopy and additivity properties again, we get

$$G\text{-Deg}\,(\mathrm{Id}-F_o,\Omega_1)=\sum_{\xi\in\sigma(A)}\sum_{j,l}n_{j,l}(\xi)\deg_{\mathcal{V}_{j,l}}.$$

On the other hand, since $\mathrm{Id}-F$ is a product of two maps $\mathrm{Id}-\overline{F}:V\to V$ (Γ-equivariant) and $\mathrm{Id}-F_o:\mathbb{R}\times U\to U$ (G-equivariant), it follows from the multiplicativity property (section 4.2) that

$$G\text{-Deg}\,(\mathrm{Id}-F,\Omega_2)=\Gamma\text{-deg}\,(\mathrm{Id}-\overline{F},\mathcal{B})\cdot G\text{-Deg}\,(\mathrm{Id}-F_o,\Omega_1).$$

Finally, since the map $\mathrm{Id} - \overline{F} = -\alpha^2\theta(v_o)C : V \to V$ (notice that since $\theta(v_o) > 0$, we can simply consider it to be equal to 1) can be represented by a diagonal-block matrix on the eigenspaces of C, one has

$$\Gamma\text{-deg}\,(\overline{F}, \mathcal{B}) = \prod_{j,s} \left(\deg_{\mathcal{V}_j} \right)^{n_j(\mu_s)},$$

and the result follows. □

5.5 Existence of Symmetric Periodic Solutions in Van Der Pol Systems

Let us recall that we consider the space $V = \mathbb{R}^n$, a group $\Gamma \subset S_n$, and an $n \times n$ non-singular matrix C commuting with the Γ-action on V. Throughout this section we continue to keep the same notations as in section 5.3. We consider a solution to (5.10) as a function living in the G-space $\mathbb{H} = V \times \mathbb{H}_o$, where $G = \Gamma \times S^1$, V is identified with the Γ-space of constant functions.

As it was indicated in section 5.3, in order to provide the equivariant degree treatment to system (5.10) (see also (5.11)), one needs to obtain *a priori* estimates for the solutions.

5.5.1 A Priori Estimate

The required *a priori* estimates are provided by the two lemmas following below.

Lemma 5.5.1. *There exists an increasing function* $\mathfrak{m} : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for each* $\delta \in (0, 1]$, $\alpha \in \mathbb{R}_+$ *and for each solution* $u \in \mathbb{H}_o$ *of the system*

$$\ddot{u} + \delta\alpha\varepsilon\frac{d}{dt}F(u) + \alpha^2 Cu = 0, \tag{5.36}$$

where F *is given by (5.9), we have*

$$\|u\|_{\mathbb{H}_o} \le \mathfrak{m}(\alpha).$$

Proof: Let us fix $\alpha \in \mathbb{R}_+$ and $\delta \in (0, 1]$ and assume that u is a solution of (5.36). Bearing in mind that C is symmetric and using integration by parts we have

$$\int_0^{2\pi} \ddot{u}(t) \bullet \dot{u}(t)dt = 0, \quad \text{and} \quad \int_0^{2\pi} Cu(t) \bullet \dot{u}(t)dt = 0. \tag{5.37}$$

Thus, by multiplying (5.36) by $\dot{u}$ and integrating over $[0, 2\pi]$ we obtain that

$$0 = \int_0^{2\pi} \left(\ddot{u} + \delta\alpha\varepsilon \frac{d}{dt} F(u) + \alpha^2 C u\right) \bullet \dot{u}\, dt$$

$$= \delta\alpha\varepsilon \int_0^{2\pi} (u^2 - \vec{a})\dot{u} \bullet \dot{u}\, dt$$

$$= \delta\alpha\varepsilon \int_0^{2\pi} \left(u^2 \bullet \dot{u}^2 - a\dot{u} \bullet \dot{u}\right) dt.$$

Therefore,

$$\int_0^{2\pi} u^2 \bullet \dot{u}^2\, dt = \int_0^{2\pi} a\dot{u} \bullet \dot{u}\, dt = a\|\dot{u}\|_2^2. \tag{5.38}$$

Since for $u \in \mathbb{H}_o$, $u(t) = -u(\pi + t)$, for each component $u_k(t)$ of $u(t)$, there exists $s_k \in [0, 2\pi]$ such that $u_k(s_k) = 0$. Consequently, using integration by parts one easily obtains for every $t \in [0, 2\pi]$ satisfying $t > s_k$:

$$u_k^2(t) \le 2 \int_{s_k}^{t} |u_k \dot{u}_k|\, dt \le 2 \int_0^{2\pi} |u_k \dot{u}_k|\, dt. \tag{5.39}$$

Using (5.38) and (5.39) one obtains (by the Cauchy Schwartz inequality)

$$\begin{aligned} \|u\|_2^2 = \int_0^{2\pi} u \bullet u\, dt \le 4\pi \sum_{k=1}^{n} \int_0^{2\pi} |u_k \dot{u}_k|\, dt \le 4\pi\sqrt{2n\pi} \left(\int_0^{2\pi} u^2 \bullet \dot{u}^2\right)^{\frac{1}{2}} dt \\ = 2^{\frac{5}{2}} \pi^{\frac{3}{2}} \sqrt{na} \|\dot{u}\|_2 \end{aligned} \tag{5.40}$$

On the other hand, if we multiply (5.36) by u and again integrate over $[0, 2\pi]$, we get

$$0 = \int_0^{2\pi} \left(\ddot{u} + \delta\alpha\varepsilon \frac{d}{dt} F(u) + \alpha^2 C u\right) \bullet u\, dt$$

$$= -\int_0^{2\pi} \dot{u} \bullet \dot{u}\, dt - \delta\alpha\varepsilon \int_0^{2\pi} \left(\frac{1}{3}u^3 - \bar{a}u\right) \bullet \dot{u}\, dt + \alpha^2 \int_0^{2\pi} Cu \bullet u\, dt$$

$$\le -\|\dot{u}\|_2^2 + \alpha^2 \|C\|\, \|u\|_2^2,$$

where $\|C\|$ denotes the operator norm of C. So, we obtain

$$\|\dot{u}\|_2^2 \le \alpha^2 \|C\|\, \|u\|_2^2. \tag{5.41}$$

Then, by (5.40) and (5.41), we get

$$\|u\|_2^2 \le 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\sqrt{an}\|\dot{u}\|_2 \le 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\sqrt{an}\alpha\sqrt{\|C\|}\,\|u\|_2$$

so

$$\|u\|_2 \le 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\alpha\sqrt{na\|C\|} \quad \text{and} \quad \|\dot{u}\|_2 \le 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\alpha^2\|C\|\sqrt{na}. \tag{5.42}$$

Notice that if $u \in \mathbb{H}_o$ is a solution to (5.36), then it is clearly of class C^2. By multiplying the equation (5.36) by $\ddot{u}$ and integrating it from 0 to 2π we obtain by (5.37) and (5.39)

$$\begin{aligned}\|\ddot{u}\|_2^2 &\le \delta\alpha\varepsilon\int_0^{2\pi} u^2(t)\dot{u}(t)\bullet\ddot{u}(t)dt + \alpha^2\,\|C\|\,\|\dot{u}\|_2^2 \\ &\le 2\delta\alpha\varepsilon\|u\|_2\|\dot{u}\|_2^2\|\ddot{u}\|_2 + \alpha^2\|C\|\,\|\dot{u}\|_2^2,\end{aligned} \tag{5.43}$$

so

$$\begin{aligned}\|\ddot{u}\|_2 &\le \delta\alpha\varepsilon\|u\|_2\|\dot{u}\|_2^2 + \sqrt{(\delta\alpha\varepsilon\|u\|_2\|\dot{u}\|_2^2)^2 + \alpha^2\|C\|\|\dot{u}\|_2^2} \\ &\le 2\delta\alpha\varepsilon\|u\|_2\|\dot{u}\|_2^2 + \alpha\sqrt{\|C\|}\|\dot{u}\|_2.\end{aligned}$$

Since the norms $\|u\|_2$ and $\|\dot{u}\|_2$ are bounded, it follows from (5.42) that

$$\|\ddot{u}\|_2 \le 2^{\frac{17}{2}}\pi^{\frac{9}{2}}\delta\alpha^6\varepsilon\|C\|^{\frac{5}{2}}(na)^{\frac{3}{2}} + 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\alpha^3\sqrt{na}\|C\|^{\frac{3}{2}}.$$

Therefore, it is to observe that

$$\|u\|_{\mathbb{E}_o} \le \mathfrak{m}(\alpha),$$

where

$$\mathfrak{m}(\alpha) := 2^{\frac{5}{2}}\pi^{\frac{3}{2}}\alpha\sqrt{na\|C\|}(1 + \alpha\|C\|^{\frac{1}{2}} + 2^6\pi^3\delta\alpha^5\varepsilon\|C\|^2 na + \alpha^2\|C\|).$$

Notice that $\mathfrak{m}(\alpha)$ is clearly increasing. □

Lemma 5.5.2. *For every $\tilde{\alpha} > 0$ there exists $\delta_1(\tilde{\alpha}) > 0$ such that the equation (5.36) has no non-zero solution in $\mathbb{H}_o$ for all $\alpha \in (0, \tilde{\alpha})$ and $\delta > \delta_1(\tilde{\alpha})$.*

Proof: Fix $\tilde{\alpha} > 0$ and take $\alpha \in (0, \tilde{\alpha})$. Let $\mathfrak{m}(\cdot)$ be a function provided by Lemma 5.5.1. Let $u \in \mathbb{H}_o$ be a solution to (5.36). By multiplying (5.36) by $\dot{u}$ and integrating it over $[0, 2\pi]$, we get

$$0 = \delta\alpha\varepsilon\int_0^{2\pi}(u^2 - \bar{a})\dot{u}\bullet\dot{u}\,dt. \tag{5.44}$$

Combining (5.44) with the condition $u(t+\pi) = -u(t)$ for all t, and using the standard continuity argument, one can find $t_0 \in [0, 2\pi]$ and $k \in \{1, 2, \ldots, n\}$ such that $u_k(t_0) = \sqrt{a}$ and $\dot{u}_k(t_0) \leq 0$. Notice, in particular, that $\|u\|_\infty \geq \sqrt{a}$. Since $u(t) = -u(t-\pi)$, $\dot{u}(t) = -\dot{u}(t-\pi)$, and $F(u)$ is an odd function, we have

$$\begin{aligned}
0 &= \int_{t_0-\pi}^{t_0} \left(\ddot{u} + \delta\alpha\varepsilon\frac{d}{dt}F(u) + \alpha^2 Cu\right)dt \\
&= \dot{u}(t_0) - \dot{u}(t_0-\pi) + \delta\alpha\varepsilon F(u)\Big|_{t_0-\pi}^{t_0} + \alpha^2\int_{t_0-\pi}^{t_0} Cu\,dt \\
&= 2\dot{u}(t_0) + 2\delta\alpha\varepsilon F(u(t_0)) + \alpha^2\int_{t_0-\pi}^{t_0} Cu\,dt.
\end{aligned}$$

Consequently,

$$\begin{aligned}
0 &\geq \dot{u}_k(t_0) \\
&= -\delta\alpha\varepsilon\left(\frac{a^{\frac{3}{2}}}{3} - a^{\frac{3}{2}}\right) - \frac{\alpha^2}{2}\int_{t_0-\pi}^{t_0}(Cu)_k\,dt \\
&\geq \alpha\left[\frac{2}{3}\delta\varepsilon a^{\frac{2}{3}} - \frac{\widetilde{\alpha}}{2}\int_{t_0-\pi}^{t_0}\|C\|\|u(t)\|_\infty\,dt\right] \\
&\geq \alpha\left[\frac{2}{3}\delta\varepsilon a^{\frac{3}{2}} - \frac{\widetilde{\alpha}}{2}\sqrt{\pi}\|C\|\|u\|_{\mathbb{H}_o}\right] \geq \alpha\left[\frac{2}{3}\delta\varepsilon a^{\frac{3}{2}} - \frac{\widetilde{\alpha}}{2}\sqrt{\pi}\|C\|\mathfrak{m}(\widetilde{\alpha})\right],
\end{aligned}$$

where $\|u(t)\|_\infty$ stands for $\max\{|u_1(t)|,\dots,|u_n(t)|\}$. Therefore, it is sufficient to take

$$\delta_1 = \frac{3\widetilde{\alpha}\sqrt{\pi}\|C\|\mathfrak{m}(\widetilde{\alpha})}{4\varepsilon a^{\frac{3}{2}}}.$$

□

5.5.2 Existence Result: Formulation

Take a function $\mathfrak{m} : \mathbb{R}_+ \to \mathbb{R}_+$ provided by Lemma 5.5.1 and let $\alpha > 0$ be a fixed number such that $\alpha^2 \neq \frac{(2r-1)^2}{\mu}$ for all $\mu \in \sigma(C)$ and $\Sigma(\alpha) := \left\{\mu \in \sigma(C) \mid \mu > \frac{1}{\alpha^2}\right\} \neq \emptyset$. Let $J : \mathbb{H}_o \to C(S^1; V)$ be the natural injection. We choose $m > 0$ such that $m < \frac{\sqrt{a}}{\|J\|}$. Then for every $u \in \mathbb{H}_o$ such that $\|u\|_{\mathbb{H}_o} \leq m$ we have

$$\|u\|_\infty = \|J(u)\|_\infty \leq \|J\|\,\|u\|_{\mathbb{H}_o} \leq \|J\|m < \sqrt{a}.$$

Notice that for any solution u of the equation (5.11) we have $\|u\|_\infty > \sqrt{a}$ (see the proof of Lemma 5.5.2), thus there is no solution u such that $\|u\|_{\mathbb{H}_o} = m$. Next, we choose $M > \mathfrak{m}(\alpha)$ and the numbers λ_o and δ_o to be large enough in order to have

$$\lambda_o - \delta_o > \delta_1(\alpha), \quad [0,\lambda_o] \subset \{\lambda : \lambda \geq \delta_1(\alpha)\} \cup \left\{\lambda : 0 \leq \lambda \leq \frac{\delta_o m^2}{\mathfrak{m}(\alpha)^2}\right\}.$$

Next, we define Ω, Ω_o, ∂_m, ∂_M and ∂_o according to formulas in subsection 5.3.4.

We are now in a position to formulate the existence theorem providing a general framework for the classification of periodic solutions to (5.2) according to their symmetries.

Theorem 5.5.3. *Let $\alpha > 0$ be such that $\alpha^2 \neq \frac{(2r-1)^2}{\mu}$ for all $\mu \in \sigma(C)$ and $r = 1, 2, 3, \dots$ and assume that $\Sigma(\alpha) := \left\{\mu \in \sigma(C) \mid \mu > \frac{1}{\alpha^2}\right\}$ is not empty.*

(i) Suppose that for a certain orbit type (H_o) in Ω, the coefficient n_{H_o} of the equivariant degree (5.26) is non-zero. Then the van der Pol system of equations (5.10) has a $2\pi\alpha$-periodic solution u such that $G_u \supset H_o$.

(ii) If in addition, the orbit type (H_o) is dominating, then the system (5.10) has at least $|G/H_o|_{S^1}$ different $2\pi\alpha$-periodic solutions, where $|X|_{S^1}$ stands for the number of different S^1-orbits in X.

Proof: (i) The idea of the proof of the first part of Theorem 5.5.3 is based on the following fact: *Let* $\mathrm{Id} - \mathcal{F}^t$ *be a homotopy of two equivariant maps* $\mathrm{Id} - \mathcal{F}^0$ *and* $\mathrm{Id} - \mathcal{F}^1$ *such that* $G\text{-}\mathrm{Deg}(\mathrm{Id} - \mathcal{F}^j, \Omega) = \sum n_H^j(H)$, $j = 0, 1$. *If* $n_{H_o}^0 \neq n_{H_o}^1$, *then there exists* $t_0 \in (0,1)$ *such that the map* $\mathcal{F}^{t_0}$ *has a fixed point in* $\partial\Omega^{H_o}$. We present only a sketch of the proof. For more details see [27].

Let
$$\mathcal{S}_o := \left\{(\lambda, u) : u = \widetilde{G}(\alpha, \delta, \lambda, u) \quad \text{for some } \delta \in [0, \delta_o]\right\},$$
where $\widetilde{G}(\alpha, \delta, \cdot, \cdot) : \mathbb{R} \times \mathbb{H}_o \to \mathbb{H}_o$ is defined by (5.21). We can show that
$$\mathcal{S}_o \cap \left(\partial_o \cup \partial_m\right) = \emptyset.$$
Notice that if $(\lambda, u) \in \mathcal{S}_o \cap \partial_M$, then $\theta(u) = 1$ and $\delta + \lambda > 0$ and (by Lemma 3.5 in [27]) the function $w = \sqrt{\frac{\delta}{\delta+\lambda}}u$ satisfies the equation
$$\ddot{w} + (\delta + \lambda)\alpha\varepsilon(w^2 - \bar{a})\dot{w} + \alpha^2 C w = 0. \tag{5.45}$$
In particular, that means the function w is a 2π-periodic solution of equation (5.11) with ε replaced by $(\delta + \lambda)\varepsilon$.

Following [27], define the parameterized nonlinear operators $F_s : V \to V$ by
$$F_s(u) := \frac{1}{3}u^3 - (1 - s)\bar{a}u, \qquad s \in [0, 1],$$

and consider the following family of parameterized differential equations

$$\begin{cases} -\ddot{u} = \delta_o \alpha\varepsilon \frac{d}{dt} F_s(u) + \alpha^2\theta(u_o) Cu - \lambda\alpha\varepsilon a\dot{u}, & u(t) \in V, \\ u(t+\pi) = -u(t). \end{cases} \tag{5.46}$$

Again, we can reformulate the above system using the setting in the functional space $\mathbb{H}_o$. We define $\widetilde{H}(\alpha, s, \cdot, \cdot) : \mathbb{R} \times \mathbb{H}_o \to \mathbb{H}_o$ by

$$\widetilde{H}(\alpha, s, \lambda, u_0) := \delta_o \alpha\varepsilon L^{-1} N_s(j(u_0)) + \alpha^2 \theta(u_o) L^{-1} A(j(u_0)) - \lambda\alpha\varepsilon a L^{-1} D(j(u_0)),$$

and $H(\alpha, s, \cdot, \cdot) : \mathbb{R} \times \mathbb{H} \to \mathbb{H}$, by

$$H(\alpha, s, \lambda, u) = \left(\overline{u} + \alpha^2\theta(u_o) A(\overline{u}), \widetilde{H}(\alpha, s, \lambda, u_o)\right), \quad u = \overline{u} + u_o, \quad \overline{u} \in V, \ u_o \in \mathbb{H}_o,$$

where

$$N_s(u) = \frac{d}{dt} F_s(u) = \frac{d}{dt}\left(\frac{1}{3}u^3 - (1-s)\vec{a}u\right) = (u^2 - (1-s)\vec{a})\dot{u}, \quad u \in \mathbb{H}_o.$$

The map $\mathrm{Id} - H(\alpha, s, \cdot, \cdot)$ is a G-equivariant homotopy. Using Lemmas 5.5.1 and 5.5.2, one can show that:

(a) the homotopy $\mathrm{Id} - \widetilde{H}(\alpha, s, \cdot, \cdot)$ has no zeros in the set $\partial_o \cup \partial_m$;

(b) the map $\mathrm{Id} - \widetilde{H}(\alpha, 1, \cdot, \cdot)$ has no zeros in $\overline{\Omega_o}$ (in particular, by the existence property of the G-equivariant degree, $G\text{-Deg}\,(\mathrm{Id} - H(\alpha, 1, \cdot, \cdot), \Omega) = 0$).

Next, we can define the following G-equivariant homotopy $\mathrm{Id} - \Psi(\tau, \cdot, \cdot)$ by

$$\Psi(\tau, \lambda, u) := \begin{cases} G(\alpha, 2\tau\delta_o, \lambda, u), & \text{for } (\lambda, u) \in \overline{\Omega}, \ \tau \in \left[0, \frac{1}{2}\right], \\ H(\alpha, 2\tau - 1, \lambda, u), & \text{for } (\lambda, u) \in \overline{\Omega}, \ \tau \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

As it was explained in section 5.4, the solution set to the equation (5.33) is non-empty only if conditions (5.31) and (5.32) are satisfied. Therefore, these conditions are necessary for the equivariant degree to be different from zero. Assume (according to the Theorem 5.5.3 conditions) that $n_{H_0} \neq 0$. Suppose that $u - \Psi(\tau, \lambda, u) \neq 0$ for all $(\lambda, u) \in \partial\Omega$. Then, by the homotopy property of the G-equivariant degree, we would also have that the (H_o)-coefficient of $G\text{-Deg}\,(\mathrm{Id} - H(\alpha, 1, \cdot, \cdot), \Omega_o)$ is non-zero, what is impossible. Since for $u = \overline{u} + u_o \in V \times \mathbb{H}_o$, the equation $u = \Psi(\tau, \lambda, u)$ implies that $-\alpha^2\theta(u_o)C(\overline{u}) = 0$, thus $\overline{u} = 0$, therefore, there exists $(\lambda, u) = (\lambda, u_o) \in \partial\Omega_o$ such that $u = \Psi(\tau, \lambda, u)$ for some $\tau \in [0, 1]$. However, the equation $u - \Psi(\tau, \lambda, u) = 0$ has no solution u in $\partial_o \cup \partial_m$. Consequently, it has a solution u in ∂_M. By applying a standard transformation, we obtain a solution for the equation (5.11), for the value of ε replaced by another (appropriate) value, with the period equal to 2π.

(ii) Assume now that (H_o) is a dominating orbit type. Then, we obtain the existence of at least $|G/H_o|_{S^1}$ different 2π-periodic solutions. □

5.6 Conclusions and Applications

In this section we will show how the existence Theorem 5.5.3 in combination with the computations related to the basic maps and the equivariant degree product formula provided by Proposition 5.4.1, allow us to study symmetric periodic solutions to concrete van der Pol equations. In particular, we will discuss the examples 5.2.1, 5.2.2, 5.2.3 and 5.2.4

Let us recall that for each of the discussed systems of van der Pol equations, we have the following associated G-equivariant degree ($G = \Gamma \times S^1$):

$$G\text{-Deg}\,(\mathrm{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega) = \sum_{(L^{\varphi,l})} n_{L^{\varphi,l}}(L^{\varphi,l}),$$

where $(L^{\varphi,l})$ are the generators of $A_1(\Gamma \times S^1)$, $L \subset \Gamma$, and $\mathcal{F}$ is given in (5.25). Although the entire value of the degree G-Deg $(\mathrm{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)$ should be considered as the *equivariant invariant* classifying the solutions of the corresponding equations, in order to simplify our exposition, we will restrict our computations to the coefficients $n_{L^{\varphi,1}}$, which will be called *first coefficients* (the corresponding part of the eqivariant degree will be denoted by G-Deg $(\mathrm{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1$). As it follows from Theorem 5.5.3(i), if $n_{L^{\varphi,1}} \neq 0$, then system (5.11) has *at least one* periodic solution u with symmetry $G_u \supset L^{\varphi,1}$. However (see theorem 5.5.3(ii)), only dominating orbit types occuring in eigenspaces, relevant to suitable eigenvalus of C, give a possibility to estimate a *precise* number of periodic solutions with the corresponding symmetry.

In addition, we will assume here, that the value of the parameter α was always chosen in the most favorable way, i.e. the set $\Sigma(\alpha)$ contains all the positive eigenvalues of the matrix C.

5.6.1 Conclusions for the Dihedral Group D_N

Let us consider again the system describing the ring of identical van der Pol oscillators, which was discussed in Example 5.2.1. This system has the group of symmetries $\Gamma = D_N$. Let us describe explicitly the D_N-action on $V = \mathbb{R}^N$, its isotypical decomposition and the spectrum of the linear operator C. We denote by $\xi := e^{\frac{2\pi}{N}i}$ the generator of $\mathbb{Z}_N$. Notice that ξ acts on a vector $\vec{x} = (x_0, x_1, \dots, x_{N-1})$ by sending the k-th coordinate of $\vec{x}$ to the $k+1$ (mod N) coordinate. It is convenient to consider this D_N-action on the complex space $U := \mathbb{C}^N$. Notice that we have the following $\mathbb{Z}_N$-isotypical decomposition of U

$$U = \widetilde{U}_0 \oplus \widetilde{U}_1 \oplus \cdots \oplus \widetilde{U}_{N-1},$$

$\lambda_j > 0$	$\deg_{\mathcal{V}_j}$	$\deg_{\mathcal{V}_{j,1}}$
$j = 0$	$-(D_N)$	(D_N)
$0 < j < N/2$ m is odd	$(D_N) - 2(D_h) + (\mathbb{Z}_h)$	$(\mathbb{Z}_N^{t_j}) + (D_h) + (D_h^{\bar{z}}) - (\mathbb{Z}_h)$
$0 < j < N/2$ N is even and $m \equiv 2 \pmod 4$	$(D_N) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h)$	$(\mathbb{Z}_N^{t_j}) + (D_{2h}^{\hat{d}}) - (\mathbb{Z}_{2h})$
$0 < j < N/2$ N is even and $m \equiv 0 \pmod 4$	$(D_N) - (D_h) - (\tilde{D}_h) + (\mathbb{Z}_h)$	$(\mathbb{Z}_N^{t_j}) + (\tilde{D}_{2h}^d) - (\mathbb{Z}_{2h})$
$j = j_N + 1$ N is even	$(D_N) - (D_{\frac{N}{2}})$	(D_N^d)

Table 5.1: Values of $\deg_{\mathcal{V}_j}$ and $\deg_{\mathcal{V}_{j,1}}$ corresponding to $\lambda_j > 0$, where $j_N = [(N+1)/2]$, $h = gcd(N, j)$ and $m = N/h$.

where $\widetilde{U}_j = \text{span}\,(\langle 1, \xi^j, \xi^{2j}, \dots, \xi^{(N-1)j}\rangle)$. Since κ sends $\widetilde{U}_j$ onto $\widetilde{U}_{-j}$ (where $-j$ is taken (mod N)), thus the D_N-isotopical components of U are

$$U_0 = \widetilde{U}_0, \quad U_j := \widetilde{U}_j \oplus \widetilde{U}_{-j}, \;\; 0 < j < N/2,$$

and, in addition, if N is even there is also the component

$$U_{\frac{N}{2}} := \widetilde{U}_{\frac{N}{2}}.$$

It is easy to check that the isotypical component U_j, $0 \le j < N/2$, is equivalent with the irreducible representation $\mathcal{V}_j^c$ of D_N, and $U_{\frac{N}{2}}$ (for N even) is equivalent to $\mathcal{V}_{j_N+1}^c$. The subspace U_j is also an eigenspace of the matrix C corresponding to the eigenvalue $\lambda_j := c + 2d\cos\frac{2\pi j}{N}$. We put $\Sigma(C) := \Big\{\lambda_j \mid \lambda_j > 0\Big\}$. Then by Proposition 5.4.1 we have:

$$G\text{-Deg}\,(\text{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1 = \prod_{\lambda_j \in \Sigma(C)} \deg_{\mathcal{V}_j} \cdot \left[\sum_{\lambda_j \in \Sigma(C)} \deg_{\mathcal{V}_{j,1}}\right] \tag{5.47}$$

Moreover, for an eigenvalue $\lambda_j > 0$ the values of $\deg_{\mathcal{V}_j}$ and $\deg_{\mathcal{V}_{j,1}}$ are listed in Table 5.1, where $h = gcd(j, N)$.

Let us illustrate these results for the particular cases $N = 3$, 4 and 5.

In the case $N = 3$, the spectrum $\sigma(C)$ of the matrix C is $\Big\{\lambda_0 = c + 2d, \lambda_1 = c - d\Big\}$ and the dominating orbit types (occuring in V^c) are $(\mathbb{Z}_3^t)$, (D_3) and $(D_1^{\bar{z}})$. If a coefficient n_L is standing by a dominating orbit type, then there is an orbit of periodic solutions of the system (5.3) composed of exactly $|G/L|_{S^1}$ periodic solutions. In particular, for the orbit type $(\mathbb{Z}_3^t)$ there are 2 distinct periodic solutions, for $(D_1^{\tilde{z}})$ there are 3 periodic solutions, and 1 periodic solution for (D_3). If $n_{D_3} = 0$, then still one more periodic solution can be detected as long as $n_L \neq 0$ for some $(L) < (D_3)$. The number of periodic solutions for the equation (5.3) in the case $N = 3$ is summarized in the following table:

Possible Cases for $N = 3$

$\Sigma(C)$	G-Deg (Id $- \mathcal{F}(0,\cdot,\cdot),\Omega)_1$	Minimal # of Solutions
$\emptyset$	0	0
$\{c-d\}$	$(\mathbb{Z}_3^t) - (D_1^{\tilde{z}}) - (D_1) + 3(\mathbb{Z}_1)$	6
$\{c+2d\}$	$-(D_3)$	1
$\{c+2d, c-d\}$	$-(\mathbb{Z}_3^t) + (D_1^{\tilde{z}}) - (D_3) + 3(D_1) - 2(\mathbb{Z}_1)$	6

In the case $N = 4$, the spectrum $\sigma(C)$ of the matrix C is $\Big\{\lambda_0 = c + 2d, \lambda_1 = c, \lambda_2 = c - 2d\Big\}$ and the dominating orbit types (occuring in V^c) are $(\mathbb{Z}_4^t)$, (D_4^d), (D_2^d), $(\widetilde{D}_2^d)$ and (D_4). For the orbit type $(\mathbb{Z}_4^t)$ there are 2 distinct periodic solutions, for (D_4^d) there is 1 periodic solutions, for (D_2^d) and $(\widetilde{D}_2^d)$ there are 2 periodic solutions, and there is 1 periodic solution for (D_4). We also have[*2]

$$\deg_{\mathcal{V}_0} = -(D_4), \quad \deg_{\mathcal{V}_1} = (D_4) - (D_1) - (\widetilde{D}_1) + (\mathbb{Z}_1), \quad \deg_{\mathcal{V}_3} = (D_4) - (D_2)$$
$$\deg_{\mathcal{V}_{0,1}} = (D_4), \quad \deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_4^t) + (D_2^d) + (\widetilde{D}_2^d) - (\mathbb{Z}_2^-), \quad \deg_{\mathcal{V}_{3,1}} = (D_4^d).$$

The number of periodic solutions for the equation (5.3) in the case $N = 4$ is summarized in the following table:

[2*] Notice that for $N = 4$ we have $\widetilde{D}_2^d = D_2^{\tilde{d}}$ (cf. Table 5.1).

Possible Cases for $N = 4$

$\Sigma(C)$	G-Deg $(\mathrm{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1$	Minimal # of Solutions
$\emptyset$	0	0
$\{c+2d\}$	$-(D_4)$	1
$\{c-2d\}$	(D_4^d)	1
$\{c+2d, c\}$	$-(\mathbb{Z}_4^t) - (\tilde{D}_2^d) - (D_2^d) - (D_4) + (\mathbb{Z}_{\bar{2}}^-) + (D_{\bar{1}}^z) + (\widetilde{D_1}^z)$ $+2(D_1) + 2(\tilde{D}_1) - 3(\mathbb{Z}_1)$	7
$\{c-2d, c\}$	$(D_4^d) + (D_2^d) + (\tilde{D}_2^d) + (\mathbb{Z}_4^t) + (\mathbb{Z}_{\bar{2}}^-) - (D_{\bar{1}}^z) - 2(\widetilde{D_1}^z)$ $-2(D_1) - (\tilde{D}_1) - (\tilde{D}_{\bar{1}}^z) - 2(\mathbb{Z}_{\bar{2}}^-) + 3(\mathbb{Z}_1)$	8
$\{c+2d, c-2d, c\}$	$-(D_4) - (D_4^d) - (D_2^d) - (\tilde{D}_2^d) - (\mathbb{Z}_4^t) + 3(D_1)$ $+(D_{\bar{1}}^z) + 2(\tilde{D}_{\bar{1}}^z) + 2(\tilde{D}_1) + (\mathbb{Z}_{\bar{2}}^-) - 4(\mathbb{Z}_1)$	8

In the case $N = 5$, the spectrum $\sigma(C)$ of the matrix C is $\left\{\lambda_0 = c + 2d, \lambda_1 = c + 2d\frac{\sqrt{5}-1}{4}, \lambda_2 = c - 2d\frac{\sqrt{5}+1}{4}\right\}$ and the dominating orbit types are $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, (D_5) and $(D_{\bar{1}}^z)$. We have the following equivariant degrees of the basic maps related to the eigenspaces of C

$$\deg_{\mathcal{V}_0} = -(D_5),\quad \deg_{\mathcal{V}_1} = (D_5) - 2(D_1) + (\mathbb{Z}_1),\quad \deg_{\mathcal{V}_2} = (D_5) - 2(D_1) + (\mathbb{Z}_1)$$
$$\deg_{\mathcal{V}_{0,1}} = (D_5),\quad \deg_{\mathcal{V}_{1,1}} = (\mathbb{Z}_5^{t_1}) + (D_{\bar{1}}^z) + (D_1) - (\mathbb{Z}_1),$$
$$\deg_{\mathcal{V}_{2,1}} = (\mathbb{Z}_5^{t_2}) + (D_{\bar{1}}^z) + (D_1) - (\mathbb{Z}_1).$$

For the orbit types $(\mathbb{Z}_5^{t_1})$ and $(\mathbb{Z}_5^{t_2})$ there are 2 distinct periodic solutions, for $(D_{\bar{1}}^z)$ there are 5 periodic solutions, and 1 periodic solution for (D_5). The number of periodic solutions for the equation (5.3) in the case $N = 5$ is summarized in the following table:

Possible Cases for $N = 5$

$\Sigma(C)$	G-Deg $(\mathrm{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1$	Minimal # of Solutions
$\emptyset$	0	0
$\{c+2d\}$	$-(D_5)$	1
$\left\{c - 2d\frac{\sqrt{5}+1}{4}\right\}$	$(\mathbb{Z}_5^{t_2}) - (D_{\bar{1}}^z) - (D_1) + (\mathbb{Z}_1)$	8
$\left\{c + 2d\frac{\sqrt{5}-1}{4}, c - 2d\frac{\sqrt{5}+1}{4}\right\}$	$(\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + 2(D_{\bar{1}}^z) + 2(D_1) - 2(\mathbb{Z}_1)$	10
$\left\{c + 2d\frac{\sqrt{5}-1}{4}, c + 2d\right\}$	$-(\mathbb{Z}_5^{t_1}) + (D_{\bar{1}}^z) - (D_5) + 3(D_1) - 6(\mathbb{Z}_1)$	8
$\left\{c+2d, c + 2d\frac{\sqrt{5}-1}{4}, c - 2d\frac{\sqrt{5}+1}{4}\right\}$	$-(\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (D_5) - 2(D_{\bar{1}}^z) - 2(D_1) + 2(\mathbb{Z}_1)$	10

5.6.2 Conclusions for the Tetrahedral Group $\mathbb{T}$

Let us consider the system of van der Pol oscillators with the tetrahedral symmetry group, which was studied in Example 2.2. Here, the tetrahedral group A_4 acts on the space $V = \mathbb{R}^4$ by permuting the coordinates of vectors. The subspace V_0 of the fixed-points of this action is spanned by the vector $\langle 1,1,1,1\rangle$, and its orthogonal complement V_3 is the natural three-dimensional representation of A_4, which was in section 6 denoted by $\mathcal{V}_3$. These two subspaces are the eigenspaces of the matrix C: the subspace V_0 corresponds to the eigenvalue $c+3d$ and V_3 to the eigenvalue $c-d$. The dominating orbit types in V^c are (A_4), $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$, and (V_4^-). For non-zero first coefficient corresponding to the orbit type (A_4) there is at least one periodic solution, for $(\mathbb{Z}_3^{t_1})$ or $(\mathbb{Z}_3^{t_2})$ — at least 4 periodic solutions, and for (V_4^-) there exist at least 3 periodic solutions.

In order to compute the equivariant degree $A_4 \times S^1\text{-Deg}(\text{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1$, we apply the computational formula (5.47). Depending on the set $\Sigma(C)$, we need the basic degrees: $\deg_{\mathcal{V}_0} \in A(A_4)$ (if $c+3d>0$), degree $\deg_{\mathcal{V}_3} \in A(A_4)$ (if $c-d>0$), $\deg_{\mathcal{V}_{0,1}} \in A_1(A_4\times S^1)$ (if $c+3d>0$), $\deg_{\mathcal{V}_{3,1}} \in A_1(A_4\times S^1)$ (if $c-d>0$). The related to this formula basic degrees are presented below:

Rep.	Basic Degrees $\deg_{\mathcal{V}_j}$ or $\deg_{\mathcal{V}_{j,1}}$	Eigenvalue of C
$\mathcal{V}_0$	$-(A_4)$	$c+3d>0$
$\mathcal{V}_3$	$(A_4)-2(\mathbb{Z}_3)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$c-d>0$
$\mathcal{V}_{0,1}$	(A_4)	$c+3d>0$
$\mathcal{V}_{3,1}$	$(\mathbb{Z}_3^{t_1})+(\mathbb{Z}_3^{t_2})+(V_4^-)+(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$c-d>0$

By using the established multiplication tables for the $A(A_4)$-module $A_1(A_4\times S^1)$, and applying the formula (5.47), we obtain the following first coefficients of the equivariant degrees $A_4 \times S^1\text{-Deg}(\text{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)$:

$\Sigma(C)$	$A_4\times S^1\text{-Deg}(\text{Id}-\Psi(0,\cdot,\cdot),\Omega)_1$	# Solutions
$c+3d$	$-(A_4)$	1
$c-d$	$-(\mathbb{Z}_3^{t_1})-(\mathbb{Z}_3^{t_2})+(V_4^-)-(\mathbb{Z}_3)-(\mathbb{Z}_2)+2(\mathbb{Z}_1)$	12
$c+3d, c-d$	$(\mathbb{Z}_3^{t_1})+(\mathbb{Z}_3^{t_2})-(A_4)-(V_4^-)+3(\mathbb{Z}_3)+2(\mathbb{Z}_2)-3(\mathbb{Z}_1)$	12

5.6.3 Conclusions for the Octahedral Group $\mathbb{O}$

Let us discuss the system of van der Pol equations described in Example 2.3. Here we have the group S_4 is acting on the eight-dimensional space $V := \mathbb{R}^8$ by permuting the coordinates of the vectors in the same way as the symmetries of the cube in $\mathbb{R}^3$ permutes the eight vertices of the cube. It can be easily verified, that the representation V can be decomposed into a direct sum of four subspaces:

$$V = V_0 \oplus V_1 \oplus V_3^1 \oplus V_3^2,$$

where

$$\begin{aligned}
V_0 &= \text{span}\Big\{\langle 1,1,1,1,1,1,1,1\rangle\Big\},\\
V_1 &= \text{span}\Big\{\langle 1,-1,1,-1,1,-1,1,-1\rangle,\rangle\Big\},\\
V_3^1 &= \text{span}\Big\{\langle 1,1,-1,-1,1,-1,-1,1\rangle,\ \langle 1,-1,1,-1,-1,1,-1,1\rangle,\\
&\qquad \langle -1,1,1,-1,1,1,-1,-1\rangle\Big\},\\
V_3^2 &= \text{span}\Big\{\langle 1,-1,-1,1,1,1,-1,-1\rangle,\ \langle 1,1,1,1,-1,-1,-1,-1\rangle,\\
&\qquad \langle -1,-1,1,1,1,-1,-1,1\rangle\Big\}.
\end{aligned}$$

Notice that these subspaces are irreducible representations of S_4, where V_3^1 is equivalent to the natural three-dimensional representation $\mathcal{V}_3$ of S_4, and V_3^2 is equivalent to the another three-dimensional irreducible representation $\mathcal{V}_4$ of S_4. The subspace V_0 is the fixed-point space of the action of S_4. The subspaces V_0, V_1, V_3^1 and V_3^2 are eigenspaces for the matrix C. Indeed, it is easy to check that:

Subspace	Eigenvalue of C	Type of Representation	Dimension
V_0	$c+3d$	Trivial	1
V_1	$c-3d$	Representation $\mathcal{V}_1$	1
V_3^1	$c+d$	Natural $\mathcal{V}_3$	3
V_3^2	$c-d$	Representation $\mathcal{V}_4$	3

In order to compute the equivariant degree $S_4 \times S^1\text{-Deg}(\text{Id} - \Psi(0,\cdot,\cdot),\Omega)$, we will apply the computational formula (5.47). All the related to this formula degrees of the basic maps are presented in the following table:

Rep.	Basic Degree $\deg_{\mathcal{V}_j}$ or $\deg_{\mathcal{V}_{j,1}}$	Eigenvalue of C
$\mathcal{V}_0$	$-(S_4)$	$c+3d>0$
$\mathcal{V}_1$	$(S_4)-2(D_4)$	$c-3d>0$
$\mathcal{V}_3$	$(S_4)-2(D_3)-(D_2)+3(D_1)-(\mathbb{Z}_1)$	$c+d>0$
$\mathcal{V}_4$	$(S_4)-(\mathbb{Z}_4)-(D_1)-(\mathbb{Z}_3)+(\mathbb{Z}_1)$	$c-d>0$
$\mathcal{V}_{0,1}$	(S_4)	$c+3d>0$
$\mathcal{V}_{1,1}$	(S_4^-)	$c-3d>0$
$\mathcal{V}_{3,1}$	$(D_4^d)+(D_2^d)+(D_3)+(\mathbb{Z}_3^t)-(\mathbb{Z}_2^-)-(D_1)+(\mathbb{Z}_4^c)$	$c+d>0$
$\mathcal{V}_{4,1}$	$(D_4^z)+(D_2^d)+(D_3^z)+(\mathbb{Z}_3^t)-(\mathbb{Z}_2^-)-(D_1)+(\mathbb{Z}_4^c)$	$c-d>0$

Let us list the dominating orbit types: (S_4) (orbit contains one periodic solution), (S_4^-) (orbit contains one periodic solution), (D_4^d) (orbit contains 3 periodic solutions), $(D_4^{\hat{d}})$ (orbit contains 3 periodic solutions),(D_2^d) (orbit contains 6 periodic solutions), $(\mathbb{Z}_4^c)$ (orbit contains 6 periodic solutions), $((\mathbb{Z}_3^t)$ (orbit contains 8 periodic solutions),$(\mathbb{Z}_4^-)$ (orbit contains 6 periodic solutions) and $(D_4^{\tilde{z}})$ (orbit contains 3 periodic solutions).

By using the above equivariant degrees of the basic maps, as well as the multiplications table for the $A(S_4)$-module $A_1(S_4 \times S^1)$ we obtain the following values of $S_4 \times S^1$-Deg(Id $- \Psi(0,\cdot,\cdot),\Omega)_1$, for all the possible distributions of the eigenvalues of the matrix C.

$\Sigma(C)$	$S_4 \times S^1$-Deg(Id $- \mathcal{F}(0,\cdot,\cdot),\Omega)_1$	# Sol.
$c+3d$	$-(S_4)$	1
$c-3d$	$(S_4^-) - 2(D_4^{\hat{d}})$	4
$c+d, c+3d$	$-4(D_4^d - 4(D_2^d) - 4(\mathbb{Z}_4^c) - 4(\mathbb{Z}_3^t) - (S_4)$ $-4(D_3) + 4(D_1) + 4(\mathbb{Z}_2^-)$	18
$c-d, c-3d$	$-2(D_4^{\hat{d}}) - (D_4^{\tilde{z}}) + (D_3^{\tilde{z}}) - (D_2^d) - 2(V_4^-)$ $-(\mathbb{Z}_3^t) + (S_4^-) + 2(D_1^{\tilde{z}})$ $+(\mathbb{Z}_4^-) - (A_4) + (\mathbb{Z}_4) - 2(\mathbb{Z}_3) + 4(\mathbb{Z}_2)$ $+(D_1) - 4(\mathbb{Z}_1) + 5(\mathbb{Z}_2^-)$	34
$c-d, c+d, c+3d$	$-(D_4^d) - (D_4^{\tilde{z}}) + (D_3^{\tilde{z}}) + (D_2^{\tilde{z}})$ $+2(D_2^d) - 2(\mathbb{Z}_3^t) - (D_1^{\tilde{z}})$ $+(\mathbb{Z}_4^-) - (S_4) + 3(D_3) + 2(D_2) + 2(\mathbb{Z}_4^c) + 2(\mathbb{Z}_4)$ $-3(\mathbb{Z}_3) - 3(\mathbb{Z}_2) - 3(D_1)$ $+3(\mathbb{Z}_1) - 2(\mathbb{Z}_2^-)$	33
$c+d, c-d, c-3d$	$-2(D_4^{\hat{d}}) - (D_4^{\tilde{z}}) - (D_4^d) - 3(D_3^{\tilde{z}}) + 2(D_2^{\tilde{z}})$ $+2(D_2^d) - 4(V_4^-) + 2(\mathbb{Z}_3^t)$ $-2(\mathbb{Z}_2^-) + (S_4^-) + 3(D_1^{\tilde{z}}) + 2(\mathbb{Z}_4^-) - (D_3)$ $+2(\mathbb{Z}_4^c) + (D_2) + (\mathbb{Z}_4)$ $+3(\mathbb{Z}_3) - 3(\mathbb{Z}_2) + (D_1) - 3(\mathbb{Z}_1)$	36
$c-3d, c-d, c+d,$ $c+3d$	$2(D_4^{\hat{d}}) + (D_4^d) + (D_4^{\tilde{z}}) + 2(D_4) + 3(D_3^{\tilde{z}}) - 2(D_2^{\tilde{z}})$ $-2(D_2^d) + 4(V_4^-) - 2(\mathbb{Z}_3^t)$ $+2(\mathbb{Z}_2^-) - (S_4^-) - 3(D_1^{\tilde{z}}) - 2(\mathbb{Z}_4^-)$ $-(S_4) + 3(D_3) - 2(D_2) - 2(\mathbb{Z}_4^c)$ $-2(\mathbb{Z}_4) - 4(\mathbb{Z}_3) + 4(\mathbb{Z}_2) - 3(D_1) + 4(\mathbb{Z}_1)$	37

5.6.4 Conclusions for the Icosahedral Group $\mathbb{I}$

Finally we consider the system of van der Pol equations with icosahedral symmetry group described in Example 5.2.4. Here we have the group A_5 acting on the twenty-dimensional space $V := \mathbb{R}^{20}$ by permuting the coordinates of the vectors in the same way as the symmetries in $\mathbb{R}^3$ permutes the vertices of the dodecahedron. It can be verified, that the matrix C, defined by (5.6) in Example 5.2.4 has the following eigenvalues:

$$\sigma(C) := \left\{\lambda_0 = c + 3d, \lambda_1 = c - 2d, \lambda_2 = c + d, \lambda_3 = c + \sqrt{5}d\right\}$$

and there is the following decomposition of V into the eigenspaces of C:

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3,$$

where V_0 is a one dimensional subspace of V, with a trivial action of A_5 (i.e. $V_0 = V^{A_5}$), and

$$V_1 \simeq \mathcal{V}_1 \oplus \mathcal{V}_1, \quad V_2 \simeq \mathcal{V}_2, \quad V_3 \simeq \mathcal{V}_3 \oplus \mathcal{V}_3,$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are irreducible representations of A_5.

In order to compute the equivariant degree $A_5 \times S^1\text{-Deg}(\text{Id} - \Psi(0,\cdot,\cdot),\Omega)_1$, we will apply the computational formula:

$$G\text{-Deg}\left(\text{Id} - \Psi(0,\cdot,\cdot),\Omega\right)_1 = \prod_{\lambda_j \in \Sigma(C)} \deg_{\mathcal{V}_j}^{m_j} \cdot \left[\sum_{\lambda_j \in \Sigma(C)} m_j \deg_{\mathcal{V}_{j,1}}\right], \tag{5.48}$$

where m_j denotes the "multiplicity" of the eigenvalue λ_j, which is 2 in the case of λ_1 and λ_3.

We need the basic degrees $\deg_{\mathcal{V}_j} \in A(A_5)$ and $\deg_{\mathcal{V}_{j,1}} \in A_1(A_5 \times S^1)$ (in the case the eigenvalue corresponding to the irreducible representation $\mathcal{V}_j$ is positive). All the related to this formula degrees of the basic maps are presented in the following table:

Rep.	Basic Degree $\deg_{\mathcal{V}_j}$ or $\deg_{\mathcal{V}_{j,1}}$	Eigenvalue of C
$\mathcal{V}_0$	$-(A_5)$	$c+3d>0$
$\mathcal{V}_1$	$(A_5)-2(A_4)-2(D_3)+3(\mathbb{Z}_2)+3(\mathbb{Z}_3)-2(\mathbb{Z}_1)$	$c-2d>0$
$\mathcal{V}_2$	$(A_5)-2(D_5)-2(D_3)+3(\mathbb{Z}_2)-(\mathbb{Z}_1)$	$c+d>0$
$\mathcal{V}_3$	$(A_5)-(\mathbb{Z}_5)-(\mathbb{Z}_3)-(\mathbb{Z}_2)+(\mathbb{Z}_1)$	$c+\sqrt{5}d>0$
$\mathcal{V}_{0,1}$	(A_5)	$c+3d>0$
$\mathcal{V}_{1,1}$	$(A_4)+(D_3)+(D_3^{z})+(V_4^-)+(\mathbb{Z}_3^t)+(\mathbb{Z}_5^{t_1})$	$c-2d>0$
	$+(\mathbb{Z}_5^{t_2})-(\mathbb{Z}_2)-(\mathbb{Z}_3)-(\mathbb{Z}_2^-)$	
$\mathcal{V}_{2,1}$	$(D_5)+(D_3)+(A_4^{t_1})+(A_4^{t_2})+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_5^{t_2})-2(\mathbb{Z}_2)+(V_4^-)$	$c+d>0$
$\mathcal{V}_{3,1}$	$(D_5^{z})+(V_4^-)+(D_3^{z})+(\mathbb{Z}_5^{t_1})+(\mathbb{Z}_3^t)-2(\mathbb{Z}_2^-)$	$c+\sqrt{5}d>0$

Let us list the dominating orbit types: $(A_4^{t_1})$ and $(A_4^{t_2})$ (orbit contains 5 periodic solutions), (A_5) (orbit contains 1 periodic solution), (V_4^-) (orbit contains 15 periodic solutions), (D_5^{z}) (orbit contains 6 periodic solutions), (D_3^{z}) (orbit contains 10 periodic solutions), $(\mathbb{Z}_5^{t_1})$ and $(\mathbb{Z}_5^{t_2})$ (orbit contains 12 periodic solutions).

By using the above equivariant degrees of the basic maps, as well as the multiplications table for the $A(A_5)$-module $A_1(A_5\times S^1)$ we obtain the following values of $A_5\times S^1$-Deg(Id $-\mathcal{F}(0,\cdot,\cdot),\Omega)_1$, for the possible distributions of the eigenvalues of the matrix C.

$\Sigma(C)$	$A_5 \times S^1$-Deg$(\text{Id} - \mathcal{F}(0,\cdot,\cdot),\Omega)_1$	# Sol.
$c+3d$	$-(A_5)$	1
$c-2d$	$2(A_4)+2(V_4^-)+2(D_3^{\tilde{z}})+2(\mathbb{Z}_5^{t_1})$ $+2(\mathbb{Z}_5^{t_2})-2(\mathbb{Z}_2^-)+2(\mathbb{Z}_3^t)$ $+2(D_3)-2(\mathbb{Z}_3)-2(\mathbb{Z}_2)$	45
$c+3d, c+\sqrt{5}d$	$-2(D_5^{\tilde{z}})-2(V_4^-)-2(D_3^{\tilde{z}})-2(\mathbb{Z}_5^{t_1})$ $-2(\mathbb{Z}_3^t)+4(\mathbb{Z}_2^-)-(A_5)$	29
$c-2d, c+d$	$(A_4^{t_1})+(A_4^{t_2})+3(V_4^-)-2(D_3^{\tilde{z}})$ $-3(\mathbb{Z}_5^{t_1})-3(\mathbb{Z}_5^{t_2})$ $-6(\mathbb{Z}_3^t)-4(\mathbb{Z}_2^-)+2(A_4)-3(D_3)-(D_5)-2(\mathbb{Z}_3)$ $-3(\mathbb{Z}_2)+7(\mathbb{Z}_1)$	45
$c+\sqrt{5}d, c+3d, c+d$	$-(A_4^{t_1})-(A_4^{t_2})+2(D_5^{\tilde{z}})$ $-3(V_4^-)+2(D_3^{\tilde{z}})+(\mathbb{Z}_5^{t_2})+3(\mathbb{Z}_5^{t_1})$ $+6(\mathbb{Z}_3^t)-(A_5)+3(D_5)+3(D_3)+2(\mathbb{Z}_2^-)-4(\mathbb{Z}_1)$	51
$c+d, c-2d, c+\sqrt{5}d$	$-(D_5)+(A_4^{t_1})+(A_4^{t_2})-(D_5^{\tilde{z}})$ $+4(V_4^-)-3(D_3^{\tilde{z}})+3(\mathbb{Z}_5^{t_2})$ $+4(\mathbb{Z}_5^{t_1})+3(\mathbb{Z}_3^t)+2(A_4)-3(D_3)+4(\mathbb{Z}_3)$ $+4(\mathbb{Z}_2^-)+4(\mathbb{Z}_2)-8(\mathbb{Z}_1)+2(\mathbb{Z}_5)$	51
$c+3d, c-2d, c+\sqrt{5}d, c+d$	$-(A_4^{t_1})-(A_4^{t_2})+2(D_5^{\tilde{z}})-5(V_4^-)$ $+4(D_3^{\tilde{z}})+5(\mathbb{Z}_5^{t_1})+3(\mathbb{Z}_5^{t_2})$ $+8(\mathbb{Z}_3^t)+4(\mathbb{Z}_2^-)-(A_5)+3(D_5)-2(A_4)$ $+5(D_3)+2(\mathbb{Z}_3)+2(\mathbb{Z}_2)-8(\mathbb{Z}_1)$	51

Chapter 6

Symmetric Hopf Bifurcation for Functional Differential Equations

The equivariant degree theory provides the most effective and complete method for studying the symmetric Hopf bifurcation problems. It allows to directly translate the equivariant spectral properties of the characteristic operator (associated with the system) into a topological invariant containing the information related to the occurences of the Hopf bifurcation, the symmetric structure of the bifurcating branches of non-constant periodic solutions, and their multiplicity.

In this chapter we will apply the equivariant degree method to a Hopf bifurcation problem for a system of symmetric functional differential equations. As examples, symmetric configurations of identical oscillators, with dihedral, tetrahedral, octahedral, and icosahedral symmetries, are analyzed.

6.1 Symmetric Hopf Bifurcation for Delayed Functional Differential Equations

Let us discuss a general setting for studying symmetric Hopf bifurcation problems for delayed functional differential equations, with a finite group Γ of symmetries.

6.1.1 Idea of Bifurcation

By a *nonlinear eigenvalue problem* means the problem of finding appropriate solutions of a nonlinear equation of the form

$$F(u, \lambda) = 0. \tag{6.1}$$

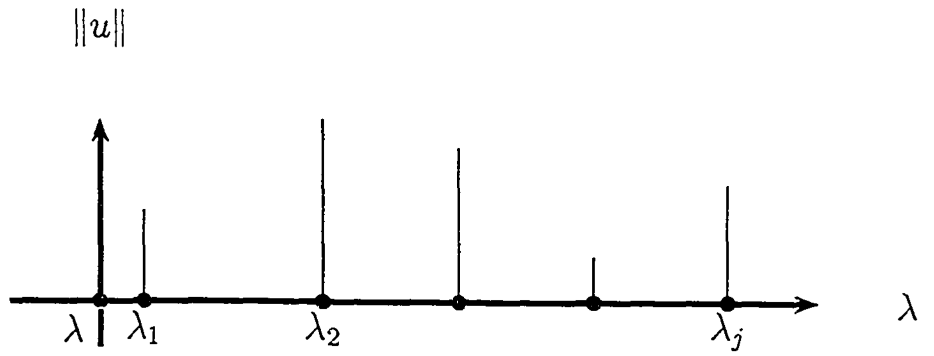


Figure 6.1:

F is a nonlinear operator, depending upon the parameter λ, which operates on the unknown function or vector u. The questions are:

(i) Whether or not the equation (6.1) has any solution u for a given values of λ?

(ii) If it does, how many solutions?

(iii) How this numbers varies with λ?

Of interest is the process of bifurcation whereby a given solution of (6.1) splits into two or more solutions as λ passes throught a critical value λ_0, called a *bifurcation point*.

To illustrate bifurcation, let us consider the linear eigenvalue problem

$$Lu = \lambda u. \tag{6.2}$$

L is a linear operator acting on vector u in some normed linear space and λ is a real number. For any value of λ a solution of (6.2) is $u = 0$. Suppose that there is a sequence of eigenvalues $\lambda_1 < \lambda_2 < \cdots$ and corresponding normalized eigenvectors $u_1, u_2, \cdots$ such that

$$Lu_j = \lambda_j u_j \quad ||u_j|| = 1, \quad j = 1, 2, \cdots$$

Then if c is any real number, other solutions of (6.2) are given by $u = cu_j \quad j = 1, 2, \cdots$ with $||u_j|| = |c|$. A graph of these solutions is shown in Figure 6.1.

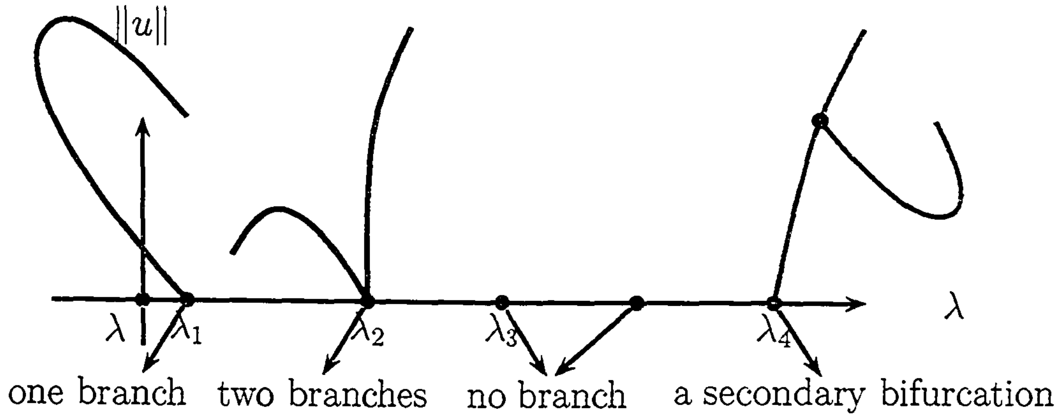


Figure 6.2:

As the figure shows, the solution $u = 0$ splits into two branches at each of eigenvalues λ_j, therefore the points $u = 0,\ \lambda = \lambda_j$ are bifurcation points of (6.2).

We now consider the nonlinear eigenvalue problem (6.1) which has equation (6.2) as its linearization. An illustrative plot of $||u||$ versus λ, called the *responses diagram*, is shown in Figure 6.2.

6.1.2 Bifurcation Setting for Delayed Functional Differential Equation

Let U be an orthogonal representation of the group Γ, and let $\tau \geq 0$ be a given constant. We denote by $C_{U,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ into U equipped with the usual supremum norm

$$||\varphi|| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|, \qquad \varphi \in C_{U,\tau}.$$

In what follows, if $x : [-\tau, A] \to U$ is a continuous function with $A > 0$ and if $t \in [0, A]$, then $x_t \in C_{U,\tau}$ is defined by

$$x_t(\theta) = x(t + \theta), \quad \theta \in [-\tau, 0].$$

Also, for any $x \in U$ we denote by $\bar{x}$ the constant mapping from $[-\tau, 0]$ into U with the value $x \in U$.

Consider the following one parameter family of delayed functional differential equations

$$\dot{x} = f(\alpha, x_t), \tag{6.3}$$

where $x(t) \in U$, $\alpha \in \mathbb{R}$, $f : \mathbb{R} \times C_{U,\tau} \to U$ is a continuously differentiable and completely continuous mapping. Clearly, the Γ-action on U induces a natural isometric Banach representation of Γ on the space $C_{U,\tau}$ with the action $\cdot : \Gamma \times C_{U,\tau} \to C_{U,\tau}$ given by:

$$(\gamma\varphi)(\theta) := \gamma(\varphi(\theta)), \qquad \gamma \in \Gamma, \ \theta \in [-\tau, 0].$$

We make the following assumptions

(A1) The mapping f is Γ-equivariant, i.e.

$$f(\alpha, \gamma\varphi) = \gamma f(\alpha, \varphi), \quad \varphi \in C_{U,\tau}, \ \alpha \in \mathbb{R}, \ \gamma \in \Gamma.$$

(A2) $f(\alpha, 0) = 0$ for all $\alpha \in \mathbb{R}$, i.e. $(\alpha, 0)$ is a *stationary solution* of (6.3) for every $\alpha \in \mathbb{R}$.

Consider the Γ-isotypical decomposition of U

$$U = U_0 \oplus U_1 \oplus \cdots \oplus U_r, \tag{6.4}$$

where $U_0 = U^\Gamma$ and U_j is modeled on the irreducible Γ-representation $\mathcal{V}_j$ and $\mathcal{V}_0$ stands for the trivial irreducible Γ-representation.

6.1.3 Characteristic Equation

An element $(\alpha_o, x_o) \in \mathbb{R} \times U$ is called a stationary solution of (6.3) if $f(\alpha_o, \bar{x}_o) = 0$. A complex number $\lambda \in \mathbb{C}$ is said to be a *characteristic value* of the stationary solution (α_o, x_o) if it is a root of the following *characteristic equation*

$$\det_{\mathbb{C}} \triangle_{(\alpha_o, x_o)}(\lambda) = 0, \tag{6.5}$$

where

$$\triangle_{(\alpha_o, x_o)}(\lambda) := \lambda \mathrm{Id} - D_x f(\alpha_o, x_o)(e^{\lambda \cdot}\mathrm{Id}).$$

A stationary solution (α_o, x_o) is called *nonsingular* if $\lambda = 0$ is not a characteristic value of (α_o, x_o), and a nonsingular stationary point (α_o, x_o) is called a *center* if it has a purely imaginary characteristic value. We will call (α_o, x_o) an *isolated center* if it is the only center in some neighborhood of (α_o, x_o) in $\mathbb{R} \times U$.

We now make the following assumption:

(A3) There is a nonsingular stationary solution $(\alpha_o, 0)$ which is an isolated center such that $\lambda = i\beta_o$, $\beta_o > 0$, is a characteristic value of $(\alpha_o, 0)$.

Let $B := (0, \delta_1) \times (\beta_o - \delta_2, \beta_o + \delta_2) \subset \mathbb{C}$. Under assumption (A3), the constants $\delta_1 > 0$, $\delta_2 > 0$ and $\varepsilon > 0$ can be chosen such that the following condition is satisfied:

(A4) For every $\alpha \neq \alpha_o \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$, the characteristic value of $(\alpha, 0)$ does not belong to ∂B.

Note that $\triangle_\alpha(\lambda) := \triangle_{(\alpha,0)}(\lambda)$ is analytic in $\lambda \in \mathbb{C}$ and continuous (see [25]) in $\alpha \in [\alpha_o - \varepsilon, \alpha_o + \varepsilon]$. It follows that $\det_{\mathbb{C}} \triangle_{\alpha_o \pm \varepsilon}(\lambda) \neq 0$ for $\lambda \in \partial B$.

Since the mapping f is Γ-equivariant, for every $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{C}$ the operator $\triangle_\alpha(\lambda) : U^c \to U^c$ is Γ-equivariant and consequently for every isotypical component U_j^c of U^c we have $\triangle_\alpha(\lambda)(U_j^c) \subseteq U_j^c$ for $j = 0, 1, \dots, r$. Thus, we can put

$$\triangle_{\alpha;j}(\lambda) := \triangle_\alpha(\lambda)|_{U_j^c}.$$

Let λ be a complex root of the characteristic equation $\det_{\mathbb{C}} \triangle_{(\alpha_o,0)}(\lambda) = 0$. In what follows, we will use the following notations

$$\begin{aligned} E(\lambda) &:= \ker \triangle_{(\alpha_o,0)} \subset U^c, \\ E_j(\lambda) &:= E(\lambda) \cap U_j^c, \\ m_j(\lambda) &:= \dim_{\mathbb{C}} E_j(\lambda) / \dim_{\mathbb{C}} \mathcal{V}_j. \end{aligned}$$

The integer $m_j(\lambda)$ will be called the $\mathcal{V}_j$*-multiplicity* of the characteristic root λ.

6.1.4 Crossing Numbers

We define

$$\mathfrak{t}_{j,1}(\alpha_o, \beta_o) := \deg(\det_{\mathbb{C}} \triangle_{\alpha_0-\varepsilon;j}(\cdot), B) - \deg(\det_{\mathbb{C}} \triangle_{\alpha_0+\varepsilon;j}(\cdot), B) \qquad (6.6)$$

for $0 \leq j \leq r$. The number $\mathfrak{t}_{j,1}(\alpha_o, \beta_o)$ is called the first $\mathcal{V}_j$*-isotypical crossing number* for the isolated center $(\alpha_o, 0)$ corresponding to the characteristic value $i\beta_o$, where $\mathcal{V}_j$ is the Γ-irreducible representation on which is modeled the isotypical component U_j. The crossing number $\mathfrak{t}_{j,1}$ has a very simple interpretation. In the case $\det_{\mathbb{C}}(\triangle_{\alpha_o;j}(i\beta_o)) = 0$ (i.e. $i\beta_o$ is a U_j^c*-characteristic value*), the number $\mathfrak{t}_{j,1}^- := \deg(\det_{\mathbb{C}} \triangle_{\alpha_0-\varepsilon;j}(\cdot), B)$ counts in the set B all the U_j^c-characteristic values (with multiplicity) before α crosses the value α_o, and the number $\mathfrak{t}_{j,1}^+ := \deg(\det_{\mathbb{C}} \triangle_{\alpha_0+\varepsilon;j}(\cdot), B)$

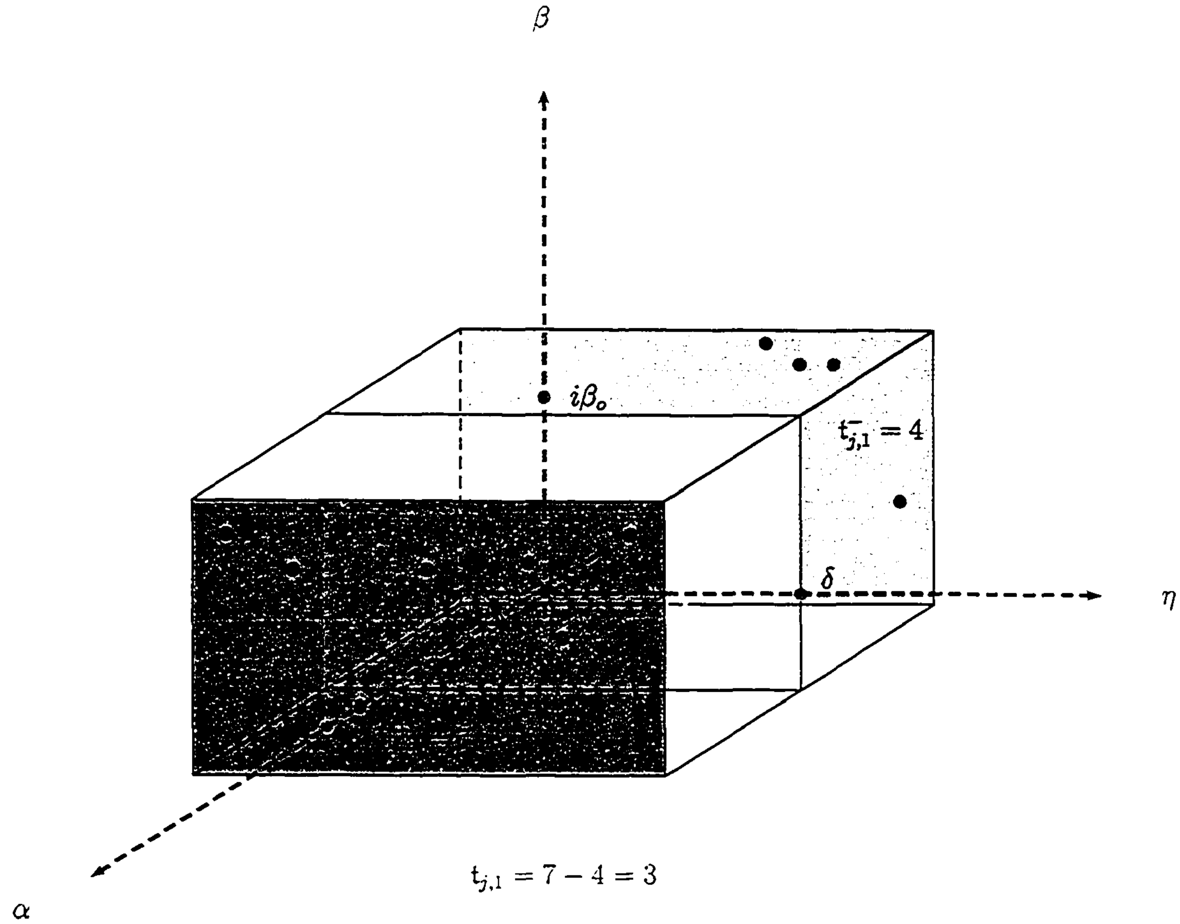


Figure 6.3: The $\mathcal{V}_j$-isotypical crossing number.

counts the U_j^c-characteristic values in B after α crosses α_o. The difference, which is exactly the number $\mathfrak{t}_{j,1}$ represents the net number of the U_j^c-characteristic values which 'escaped' (if $\mathfrak{t}_{j,1}$ is positive) or 'entered' (if $\mathfrak{t}_{j,1}$ is negative) the set B when α was crossing α_o. This situation is illustrated on Figure 6.3.

Since $l\beta_o$, where $l > 1$ is an integer, may also be an j-th isotypical characteristic value of $(\alpha_o, 0)$, we put

$$\mathfrak{t}_{j,l}(\alpha_o, \beta_o) := \mathfrak{t}_{j,1}(\alpha_o, l\beta_o).$$

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(\alpha_o, 0)$, i.e. the existence of Hopf bifurcations at the stationary point $(\alpha_o, 0)$, and to associate with $(\alpha_o, 0)$ a *local bifurcation invariant*, we apply the standard steps for the degree-theoretical approach.

6.1.5 Normalization of the Period

Normalization of the period is obtained by making the following change of variable $u(t) = x(\frac{p}{2\pi}t)$ for $t \in \mathbb{R}$. We obtain the following, equivalent to (6.3), equation

$$\dot{u}(t) = \frac{p}{2\pi} f(\alpha, u_{t,\frac{2\pi}{p}}), \tag{6.7}$$

where $u_{t,\frac{2\pi}{p}} \in C_{U,\tau}$ is defined by

$$u_{t,\frac{2\pi}{p}}(\theta) = u(t + \tfrac{2\pi}{p}\theta), \quad \theta \in [-\tau, 0].$$

Evidently, $u(t)$ is an 2π-periodic solution of (6.7) if and only if $x(t)$ is a p-periodic solution of (6.3). We can also introduce $\beta := \frac{2\pi}{p}$ into the equation (6.7) to get

$$\dot{u}(t) = \frac{1}{\beta} f(\alpha, u_{t,\beta}), \tag{6.8}$$

6.1.6 $\Gamma \times S^1$-Equivariant Setting in Functional Spaces

We use the standard identification $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and we introduce the operators $L : H^1(S^1; U) \to L^2(S^1; U)$, $Lu(t) = \dot{u}(t)$, and $K : H^1(S^1; U) \to L^2(S^1; U)$ by $Ku = \frac{1}{2\pi}\int_0^{2\pi} u(s)\, ds$, $u \in H^1(S^1; U)$, $t \in \mathbb{R}$, where $H^1(S^1; U)$ denotes the first Sobolev space of 2π-periodic U-valued functions. Put $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$. It can be easily shown that $(L + K)^{-1} : L^2(S^1; U) \to H^1(S^1; U)$ exists and the map $\mathcal{F} : \mathbb{R}^2_+ \times H^1(S^1; U) \to H^1(S^1; U)$ defined by

$$\mathcal{F}(\alpha, \beta, u) = (L + K)^{-1}\Big[Ku + \tfrac{1}{\beta} N_f(\alpha, \beta, u)\Big] \tag{6.9}$$

is completely continuous, where $N_f : \mathbb{R}^2_+ \times C(S^1; U) \to L^2(S^1; U)$ is defined by

$$N_f(\alpha, \beta, u)(t) = f(\alpha, u_{t,\beta}),$$

where $e^{it} \in S^1$, $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times H^1(S^1; U)$.

We put $W := H^1(S^1; U)$. The space W is an isometric Hilbert representation of the group $G = \Gamma \times S^1$ with the action being given by

$$(\gamma, \theta)u(t) = \gamma(u(t + \theta)), \quad e^{i\theta}, e^{it} \in S^1, \ \gamma \in \Gamma, \ u \in W.$$

The nonlinear operator $\mathcal{F}$ is clearly G-equivariant.

Notice that, $(\alpha, \beta, u) \in \mathbb{R}^2_+ \times W$ is a 2π-periodic solution of (6.8) if and only if $u = \mathcal{F}(\alpha, \beta, u)$. Consequently, the occurences of a Hopf bifurcation at $(\alpha_o, 0)$ for the equation (6.3) is equivalent to a bifurcation of 2π-periodic solutions for (6.8) from $(\alpha_o, \beta_o, 0)$ for some $\beta_o > 0$. On the other hand, the *necessary condition* for the occurences of a bifurcation at $(\alpha_o, \beta_o, 0) \in \mathbb{R}^2_+ \times W$ for (6.8) implies that, in such a case, the operator $\text{Id} - D_u\mathcal{F}(\alpha_o, \beta_o, 0) : W \to W$ is not an isomorphism, or equivalently, $i\beta_o$ is a purely imaginary characteristic value of $(\alpha_o, 0)$, i.e. $\det_{\mathbb{C}} \Delta_{\alpha_o}(i\beta_o) = 0$.

6.1.7 Sufficient Conditions for Hopf Bifurcation

It is convenient to identify $\mathbb{R}^2_+$ with a subset of $\mathbb{C}$, i.e. an element $(\alpha, \beta) \in \mathbb{R}^2_+$ will be written as $\lambda = \alpha + i\beta$, and we put $\lambda_o = \alpha_o + i\beta_o$. By assumption (A3), $(\alpha_o, 0)$ is an isolated center, thus there exists $\eta > 0$ such that

$$a(\lambda) := \text{Id} - D_u\mathcal{F}(\lambda, 0) : W \to W \tag{6.10}$$

is an isomorphism for $0 < |\lambda - \lambda_o| \leq \eta$. Consequently, by Implicit Function Theorem, there exists r, $\min\{1, \eta\} > r > 0$, such that for (λ, u) satisfying $|\lambda - \lambda_o| = \eta$ and $0 < \|u\| \leq r$, we have $u - \mathcal{F}(\lambda, u) \neq 0$. We define the subset $\Omega \subset \mathbb{R}^2_+ \times W$ by

$$\Omega := \left\{(\lambda, u) \in \mathbb{R}^2_+ \times W : |\lambda - \lambda_o| < \eta,\ \|u\| < r\right\} \tag{6.11}$$

and put

$$\partial_0 := \overline{\Omega} \cap \left(\mathbb{R}^2_+ \times \{0\}\right) \quad \text{and} \quad \partial_r := \{(\lambda, u) \in \overline{\Omega} : \|u\| = r\}.$$

We introduce an *auxiliary* function $\theta : \overline{\Omega} \to \mathbb{R}$, which is a G-invariant function satisfying the conditions

$$\begin{cases} \theta(\lambda, u) > 0 & \text{for } (\lambda, u) \in \partial_r, \\ \theta(\lambda, u) < 0 & \text{for } (\lambda, u) \in \partial_0. \end{cases}$$

Such a function θ can be easily constructed, (see Figure 6.4), for example by

$$\theta(\lambda, u) = |\lambda - \lambda_o|(\|u\| - r) + \|u\| - \frac{r}{2}; \quad (\lambda, u) \in \overline{\Omega}.$$

We define the map $\mathfrak{F}_\theta := \overline{\Omega} \to \mathbb{R} \oplus W$, by

$$\mathfrak{F}_\theta(\lambda, u) = \Big(\theta(\lambda, u), u - \mathcal{F}(\lambda, u)\Big), \quad (\lambda, u) \in \overline{\Omega}, \tag{6.12}$$

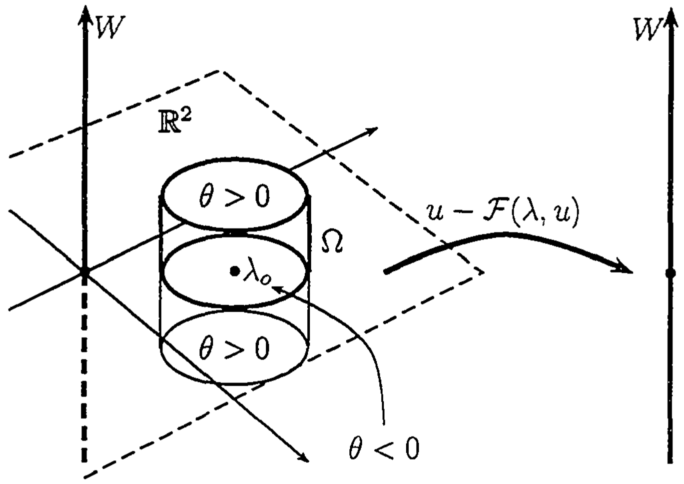


Figure 6.4: Auxiliary Function for Hopf Bifurcation.

Since $\mathfrak{F}_\theta(\lambda, u) \neq 0$ for $(\lambda, u) \in \partial\Omega$, thus $\mathfrak{F}_\theta$ is an Ω-admissible G-equivariant compact field. Therefore, the standard Leray-Schauder extension of the primary G-equivariant degree can be applied to the admissible pair $(\mathfrak{F}_\theta, \Omega)$. We put

$$\omega(\lambda_o) := G\text{-Deg}\,(\mathfrak{F}_\theta, \Omega) \in A_1(G), \tag{6.13}$$

and we will call $\omega(\lambda_o)$ the *local* $\Gamma \times S^1$*-invariant* for the Γ-symmetric Hopf bifurcation at the point $(\lambda_o, 0)$.

We have the following

Theorem 6.1.1. (Sufficient Condition for Hopf Bifurcation) *Suppose that* $(\alpha_o, 0)$ *is an isolated center for (6.3) satisfying the above assumptions. If*

$$\omega(\lambda_o) = \sum_{(H)} n_H\,(H),$$

and $n_{H_o} \neq 0$ *for some* $(H_o) \in \Phi_1(G)$*, then there exists a branch of non-constant periodic solutions* (λ, u) *of (6.7) bifurcating from* $(\lambda_o, 0)$ *such that* $G_u \supset H_o$.

6.1.8 Linearization of the Problem

We define

$$\widetilde{\theta}(\lambda, u) = |\lambda - \lambda_o|(\|u\| - r) + \|u\| + \frac{\eta r}{2}, \quad (\lambda, u) \in \overline{\Omega},$$

and the map

$$\mathfrak{F}_{\widetilde{\theta}}(\lambda,u) = \Big(\widetilde{\theta}(\lambda,u), u - \mathcal{F}(\lambda,u)\Big), \quad (\lambda,u) \in \overline{\Omega}. \tag{6.14}$$

It is clear that $\mathfrak{F}_\theta$ and $\mathfrak{F}_{\widetilde{\theta}}$ are homotopic by an Ω-admissible (linear) homotopy, thus

$$G\text{-Deg}\,(\mathfrak{F}_\theta, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\theta}}, \Omega). \tag{6.15}$$

Notice that for $|\lambda - \lambda_o| \le \frac{\eta}{4}$ and $\|u\| \le r$ we have

$$\begin{aligned}\widetilde{\theta}(\lambda,u) &= \|u\| + \frac{\eta r}{2} - |\lambda-\lambda_o|(r - \|u\|)\\ &\ge \frac{\eta r}{2} - \frac{\eta r}{4} \ge \frac{\eta r}{2} - \frac{\eta r}{4} = \frac{\eta r}{4} > 0.\end{aligned}$$

Put

$$\Omega_1 := \Big\{(\lambda,u) \in \mathbb{R}^2_+ \times W : \|u\| < r,\ \frac{\eta}{4} < |\lambda - \lambda_o| < \eta\Big\}.$$

By excision property of the G-equivariant degree, we obtain

$$G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\theta}}, \Omega) = G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\theta}}, \Omega_1). \tag{6.16}$$

We define $\widetilde{\mathfrak{F}} := \overline{\Omega}_1 \to \mathbb{R} \oplus W$ by

$$\widetilde{\mathfrak{F}}(\lambda,u) := (\widetilde{\theta}(\lambda,u), u - D_u\mathcal{F}(\lambda,0)u), \quad (\lambda,u) \in \overline{\Omega}_1. \tag{6.17}$$

By standard linearization argument, it is easy to show that

$$G\text{-Deg}\,(\mathfrak{F}_{\widetilde{\theta}}, \Omega_1) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega_1) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}, \Omega). \tag{6.18}$$

6.1.9 Representation of $\widetilde{\mathfrak{F}}$ on $\Gamma \times S^1$-Isotypical Components

With respect to the restricted $\Gamma \times S^1$-action on W, we have the usual isotypical decomposition of the space W

$$W = U_0 \oplus U_1 \oplus \cdots \oplus U_r \oplus \overline{\bigoplus_{j,l} U_{j,l}},$$

where U_j is the Γ-isotypical component of U modeled on $\mathcal{V}_j$, and $U_{j,l}$, $j = 0,1,\dots,r$, $l = 0,1,2,\dots$, are $\Gamma \times S^1$-isotypical components modeled on $\mathcal{V}_{j,l}$ (see [3] for more details).

For every $j = 0,1,\dots,r$ and $l = 0,1,2,\dots$ we define

$$a_{j,l}(\lambda) := \text{Id} - (L+K)^{-1}\Big[K + \tfrac{1}{\beta}D_uN_f(\alpha,\beta,0)\Big]\Big|_{U_{j,l}}, \quad \lambda = \alpha + i\beta,$$

where $|\lambda - \lambda_o| \leq \eta$.

We observe that, for $l > 0$,

$$(L+K)^{-1}\Big(e^{il\cdot}(x+iy)\Big) = \tfrac{1}{il}e^{il\cdot}(x+iy), \quad x, y \in U, \tag{6.19}$$

and since

$$a_{j,l}(\lambda) = \mathrm{Id} - (L+K)^{-1}\left[\tfrac{1}{\beta}D_u N_f(\alpha,\beta,0)\right]|_{U_{j,l}},$$

we obtain

$$\begin{aligned} a_{j,l}(\lambda)e^{il\cdot}u &= \mathrm{Id} - (L+K)^{-1}\left[\tfrac{1}{\beta}e^{il\cdot}D_u f(\alpha,0)(e^{il\beta\cdot})u\right] \\ &= e^{il\cdot}\tfrac{1}{il\beta}\,\triangle_{(\alpha,0)}\,(il\beta)(u). \end{aligned}$$

Consequently,

$$a_{j,l}(\lambda) = \tfrac{1}{il\beta}\,\triangle_{(\alpha,0)}\,(il\beta).$$

Notice in addition that

$$a_{j,0}(\lambda) = -\frac{1}{\beta}D_u f(\alpha,0)|_{U_j}.$$

6.1.10 Computation of G-Deg$\,(\widetilde{\mathfrak{F}},\Omega)$

By applying the standard finite dimensional reduction, we can assume without loss of generality that W is finite dimensional. We represent the set Ω as a product $\mathcal{B}\times\Omega_o \subset \mathbb{R}^2\oplus W$, where $\mathcal{B}$ is the unit ball in $W^{S^1}\cong U$.

The computation of G-Deg$\,(\mathfrak{F}_\theta,\Omega) = G$-Deg$\,(\widetilde{\mathfrak{F}},\Omega)$ can be reduced to computation of primary degrees of the basic maps, associated with the problem (6.3) (see Chapter 3).

More precisely, we consider the operator $\overline{\mathfrak{F}} := -\frac{1}{\beta_o}D_x f(\alpha_o,0) : U \to U$, which is clearly Γ-equivariant. Put $W_o := \bigoplus_{n>0} U_n^c$ and $\Omega_o = \Omega \cap \mathbb{R}^2\oplus W_o$ and define $\mathfrak{F}_o : \overline{\Omega}_o \to \mathbb{R}\oplus W_o$ by

$$\mathfrak{F}_o(\lambda,u_o) = (\widetilde{\theta}(\lambda,u_o), u_o - D_u\mathcal{F}(\lambda,0)u_o), \quad (\lambda,u_o)\in\overline{\Omega}_o.$$

It is easy to verify that the product map $\overline{\mathfrak{F}}\times\mathfrak{F}_o$ is homotopic to $\widetilde{\mathfrak{F}}$, therefore we have

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}},\Omega) = G\text{-Deg}\,(\overline{\mathfrak{F}}\times\mathfrak{F}_o,\mathcal{B}\times\Omega_o).$$

By applying the multiplicativity property of the equivariant degree (see section 4.2) we obtain that

$$G\text{-Deg}\,(\overline{\mathfrak{F}}\times\mathfrak{F}_o,\mathcal{B}\times\Omega_o) = \Gamma\text{-Deg}\,(\overline{\mathfrak{F}},\mathcal{B})\cdot G\text{-Deg}\,(\mathfrak{F}_o,\Omega_o). \tag{6.20}$$

Computation of Γ-Deg $(\overline{\mathfrak{F}}, \mathcal{B})$. For every negative eigenvalue μ of the linear Γ-equivariant operator $\overline{\mathfrak{F}}$, which is clearly a positive eigenvalue of the operator $A := D_x f(\alpha_o, 0)$, we consider the Γ-isotypical decomposition of the eigenspace $E(\mu)$

$$E(\mu) = E_0(\mu) \oplus E_1(\mu) \oplus \cdots \oplus E_r(\mu),$$

where the component $E_j(\mu)$ is modeled on the irreducible Γ-representation $\mathcal{V}_j$.

Let $\sigma_- = \{\mu_1, \mu_2, \ldots, \mu_k\}$ denote the set of all negative eigenvalues of $\overline{\mathfrak{F}}$. Then, we define the element

$$\prod_{l=1}^{k}\prod_{j=0}^{r}\left(\deg_{\mathcal{V}_j}\right)^{m_j(\mu_l)},$$

where $\deg_{\mathcal{V}_j}$ stands for the Γ-equivariant degree of the operator $-\mathrm{Id} : \mathcal{V}_j \to \mathcal{V}_j$ on the unit ball in $\mathcal{V}_j$, and $m_j(\mu_l)$ is the $\mathcal{V}_j$-multiplicity of μ_l. Then, by the multiplicativity property of the Γ-equivariant degree (in the case without parameter) we obtain

$$\Gamma\text{-Deg}\,(\overline{\mathfrak{F}}, \mathcal{B}) = \prod_{l=1}^{k}\prod_{j=0}^{r}\left(\deg_{\mathcal{V}_j}\right)^{m_j(\mu_l)}. \tag{6.21}$$

Computation of G-Deg $(\mathfrak{F}_o, \Omega_o)$. We put

$$\Omega_{o1} := \left\{(\lambda, u_o) \in \mathbb{R}^2_+ \times W_o : \|u_o\| < r,\ \frac{\eta}{4} < |\lambda - \lambda_o| < \eta\right\},$$

and define $\widetilde{\mathfrak{F}}_o : \overline{\Omega}_o \to \mathbb{R} \oplus W_o$ by

$$\widetilde{\mathfrak{F}}_o(\lambda, u_o) := \left(|\lambda - \lambda_0|(\|u_o\| - r) + \|u_o\| + \frac{\eta r}{2}, \mathfrak{a}(\lambda)u_o\right), \quad (\lambda, u_o) \in \overline{\Omega}_{o1},$$

where $\mathfrak{a}(\lambda) := a\left(\lambda_o + \frac{2(\lambda-\lambda_o)\eta}{3|\lambda-\lambda_o|}\right)$ and $a(\lambda) : W_o \to W_o$ is given by (6.10). By the excision and homotopy properties of the equivariant degree, we have

$$G\text{-Deg}\,(\mathfrak{F}_o, \Omega_o) = G\text{-Deg}\,(\mathfrak{F}_o, \Omega_{o1}) = G\text{-Deg}\,(\widetilde{\mathfrak{F}}_o, \Omega_{o1}).$$

Let us consider the isotypical decomposition

$$W_o = \bigoplus_{j,l} U_{j,l}. \tag{6.22}$$

Then, we have the following representation of the map $\mathfrak{a}$ with respect to this representation

$$\mathfrak{a}(\lambda) = \bigoplus_{j,l} \mathfrak{a}_{j,l}(\lambda), \tag{6.23}$$

where $\mathfrak{a}_{j,l}(\lambda) := a_{j,l}\left(\lambda_o + \frac{2(\lambda-\lambda_o)\eta}{3|\lambda-\lambda_o|}\right)$. By applying the Splitting Lemma, we obtain that

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}}_o, \Omega_{o1}) = \sum_{j,l} G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l}) \tag{6.24}$$

where $\widetilde{\mathfrak{F}}_{j,l} : \overline{\Omega}_{j,l} \to \mathbb{R} \oplus U_{j,l}$, with

$$\Omega_{j,l} := \left\{(\lambda, v) \in \mathbb{R}^2_+ \times U_{j,l} : \|v\| < r,\ \frac{\eta}{4} < |\lambda - \lambda_o| < \eta\right\},$$
$$\widetilde{\mathfrak{F}}_{j,l}(\lambda, v) := \left(|\lambda - \lambda_0|(\|v\| - r) + \|v\| + \frac{\eta r}{2}, \mathfrak{a}_{j,l}(\lambda)v\right), \quad (\lambda, v) \in \overline{\Omega}_{j,l}.$$

The degrees $G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l})$ can be easily computed.

We consider the complexification

$$U^c = U_0^c \oplus U_1^c \oplus \cdots \oplus U_r^c \tag{6.25}$$

of the isotypical decomposition (6.4) and, for $j = 0, 1, 2, \dots, r$, the set $\{i\beta_o, i\beta_1, \dots, i\beta_m\}$ of all the purely imaginary roots λ of the equation

$$\det_{\mathbb{C}} \Delta_{\alpha_o;j}(\lambda) = 0,$$

and assume that

$$\left\{i\beta_o, il_1\beta_o, \dots, il_s\beta_o\right\} \subset \left\{i\beta_o, i\beta_1, \dots, i\beta_m\right\},$$

where $l_k > 1$ are integers, is the subset composed of all the integer multiples of $i\beta_o$. Then, for every j and $l = l_k$ we have that

$$G\text{-Deg}\,(\widetilde{\mathfrak{F}}_{j,l}, \Omega_{j,l}) = \mathfrak{t}_{j,l}(\alpha_o, \beta_o)\ \deg_{\mathcal{V}_{j,l}}$$

where the numbers $\mathfrak{t}_{j,l}(\alpha_o, \beta_o) = \mathfrak{t}_{j,1}(\alpha_o, l\beta_o)$ are the $\mathcal{V}_{j,l}$-isotypical crossing numbers at $(\alpha_o, \beta_0, 0)$, and $\deg_{\mathcal{V}_{j,l}}$ denotes the primary G-degree of the so-called *basic* map (see [3] for more details) on $\mathcal{V}_{j,l}$ given by

$$\mathfrak{f}_{j,l}(t, v) = (1 - \|v\| + it) \cdot v, \quad (t, v) \in \overline{\mathcal{O}}_{j,l}, \tag{6.26}$$

where $\mathcal{O}_{j,l} \subset \mathbb{R} \oplus \mathcal{V}_{j,l}$ is the set

$$\mathcal{O}_{j,l} = \{(t, v) : \frac{1}{2} < \|v\| < 2,\ -1 < t < 1\}.$$

(Notice that the space $\mathcal{V}_{j,l}$ admits a complex structure induced by the action of S^1.)

Consequently, we obtain that

$$G\text{-Deg}(\mathfrak{F}_o, \Omega_o) = \sum_{j,l} \mathrm{t}_{j,l}(\alpha_o, \beta_o) \deg_{\mathcal{V}_{j,l}}.$$

In this way, we have obtained that

$$G\text{-Deg}(\mathfrak{F}, \Omega) = \prod_{l=1}^{k} \prod_{j=0}^{r} \left(\deg_{\mathcal{V}_j}\right)^{m_j(\mu_l)} \cdot \sum_{j,l} \mathrm{t}_{j,l}(\alpha_o, \beta_o) \deg_{\mathcal{V}_{j,l}}. \tag{6.27}$$

6.2 Hopf Bifurcation in Symmetric Configurations of Identical Oscillators

Let us begin with several examples. Consider n identical cells coupled symmetrically by diffusion between certain selected couples of the cells. We will denote by $x^j(t)$, say for example, the concentration of the chemical species in the j-th cell. We assume that the coupling is taking place between adjacent cells connected by the edges of a regular polygon or polyhedron, symbolizing geometrically this configuration of the cells. More precisely, the coupling strength between cells is represented by the mapping K from $\mathbb{R}$ to the space of bounded linear operators $L\Big(C([-\tau, 0]; \mathbb{R}), \mathbb{R}\Big)$, which is continuously differentiable. In our case, the linear operator $K(\alpha)$ will represent the coupling strength $K(\alpha)(x_t^{j-1} - x_t^j)$ between the adjacent cells x^{j-1} and x^j. This term is supported by the ordinary law of diffusion, which simply means that the chemical substance moves from a region of greater concentration to a region of less concentration, at the rate proportional to the gradient of the concentration. Since, in general, the coupling strength between the $j-1$-th and j-th cell may be nonlinear and depend on the concentration x^j, we will assume that it is of the form

$$h(x(t))\big(g(\alpha, x_t^j) - g(\alpha, x_t^{j-1})\big),$$

where $h : \mathbb{R} \to \mathbb{R}$ is continuously differentiable function, $h(t) \neq 0$ for all $t \in \mathbb{R}$, and $g : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$, $\tau > 0$, a continuously differentiable map, $g(\alpha, 0) = 0$. We will also suppose that the kinetic law obeyed by the concentration x^j in every cell is described by a certain function $f : \mathbb{R} \times C([-\tau, 0]; \mathbb{R}) \to \mathbb{R}$, $\tau > 0$, which is continuously differentiable.

As it will be shown on several examples below, a dynamical system describing such a configuration of cells, is of the type

$$\frac{d}{dt}x(t) = F(\alpha, x_t) + H(x(t)) \cdot C(G(\alpha, x_t)), \tag{6.28}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad x_t = \begin{bmatrix} x_t^1 \\ x_t^2 \\ \vdots \\ x_t^n \end{bmatrix}, \quad F(\alpha, x_t) = \begin{bmatrix} f(\alpha, x_t^1) \\ f(\alpha, x_t^2) \\ \vdots \\ f(\alpha, x_t^n) \end{bmatrix}, \quad G(\alpha, x_t) = \begin{bmatrix} g(\alpha, x_t^1) \\ g(\alpha, x_t^2) \\ \vdots \\ g(\alpha, x_t^n) \end{bmatrix},$$

$$H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^n) \end{bmatrix}, \quad x \cdot y = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} x^1 y^1 \\ x^2 y^2 \\ \vdots \\ x^n y^n \end{bmatrix},$$

and C is a symmetric non-singular $n \times n$-matrix.

Suppose, in addition, that the geometrical configuration of these cells has a symmetry group Γ. The group Γ permutes the vertices of the related polygon or polyhedron, which means it acts on $\mathbb{R}^n$ by permuting the coordinates of the vectors $x \in \mathbb{R}^n$. Clearly, the system (6.28) is symmetric with respect to this action of the group Γ on $U := \mathbb{R}^n$. In this way, we are dealing here with a Γ-symmetric system of FDEs.

We will assume that

(H1) $f(\alpha, 0) = 0$ for $\alpha \in \mathbb{R}$, i.e. $(\alpha, 0)$ is a stationary solution of (6.28).

In the subsequent examples, we present concrete configurations of such identical cells coupled symmetrically by diffusion between adjacent cells, modeled on the regular n-gon, tetrahedron, octahedron, and dodecahedron. In each case, the symmetry group Γ of the system is composed of the orthogonal symmetries corresponding to the given n-gon or polyhedron. To simplify the presentation, in the case of a symmetry group modeled on the above polyhedrons, we consider only those orthogonal symmetries T for which $\det T = 1$. This assumption is not essential, and in the general case, similar results can be easily derived based on the already obtained computations.

Dihedral Configurations of Identical Oscillators. We consider a ring of n identical oscillators where the interaction takes place only between the neighboring oscillators. In this case the matrix C is of the type

$$C = \begin{bmatrix} c & d & 0 & \dots & 0 & d \\ d & c & d & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d & 0 & 0 & \dots & d & c \end{bmatrix}. \tag{6.29}$$

It is easy to check that the system (6.28) is symmetric with respect to the action of dihedral group D_n.

Tetrahedral Configuration of Identical Oscillators We consider four identical inter-connected oscillators having exactly the same linear interaction between all the other oscillators. In this case, the matrix C is of the type

$$C = \begin{bmatrix} c & d & d & d \\ d & c & d & d \\ d & d & c & d \\ d & d & d & c \end{bmatrix} \tag{6.30}$$

It is also clear that this system of differential equations (6.28) is symmetric with respect to the tetrahedral group $\mathbb{T} = A_4$.

Octahedral Configuration of Identical Oscillators Suppose that the identical cells (oscillators) are arranged in a configuration corresponding to the vertices of a cube. We assume that the interaction takes place between those oscillators that are connected by an edge of the cube. We assume that all the oscillators are identical. These identical oscillators, with such inter-connections lead to the system of eight equations with the matrix C of the type

$$C = \begin{bmatrix} c & d & 0 & d & 0 & d & 0 & 0 \\ d & c & d & 0 & 0 & 0 & d & 0 \\ 0 & d & c & d & 0 & 0 & 0 & d \\ d & 0 & d & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & d & c & d & 0 & d \\ d & 0 & 0 & 0 & d & c & d & 0 \\ 0 & d & 0 & 0 & 0 & d & c & d \\ 0 & 0 & d & 0 & d & 0 & d & c \end{bmatrix}. \tag{6.31}$$

It is clear that the system of equations (6.28) is symmetric with respect to the octahedral symmetry group $\mathbb{O}$ which is isomorphic to the symmetric group S_4.

Icosahedral Configuration of Identical Oscillators Let us consider an arrangement of identical oscillators based on the inter-connections given by the edges of a dodecahedron. It is clear that the group of symmetries of the dodecahedron, which is the icosahedral group $\mathbb{I}$, is the symmetry group of the system (6.28). Let us point out that the icosahedral group $\mathbb{I}$ is isomorphic to the alternating group A_5. In this case we have the system (6.28) composed of 20 equations, where the matrix C is of the type

$$C = \begin{bmatrix}
c & d & 0 & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 \\
d & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 \\
d & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 \\
0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 \\
0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 & d \\
0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & d & c & d \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & 0 & d & c
\end{bmatrix}. \tag{6.32}$$

Of course, other configurations of identical oscillators could also be considered, for example based on octahedron, icosahedron or other higher dimensional polyhedra.

6.2.1 Characteristic Equation for a Symmetric Configuration of Identical Oscillators

The linearization of the system (6.28) at $(\alpha, 0)$ is simply the system

$$\frac{d}{dt}x(t) = D_xF(\alpha, 0)x_t + h(0)C(D_xG(\alpha, 0)x_t). \tag{6.33}$$

Since $D_xG(\alpha, 0)$ is diagonal and C has constant coefficients, $CD_xG(\alpha, 0) = D_xG(\alpha, 0)C$. We put $K(\alpha) := h(0)D_xG(\alpha, 0)$, i.e. the linearized system can be written as

$$\frac{d}{dt}x(t) = D_xF(\alpha, 0)x_t + K(\alpha)\,C(x_t), \tag{6.34}$$

therefore, a number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution $(\alpha, 0) \in \mathbb{R} \oplus U$ if there exists a nonzero vector $z \in U^c$ such that

$$\triangle_\alpha(\lambda)z := \lambda z - D_xF(\alpha, 0)(e^{\lambda \cdot})z - K(\alpha)C(e^{\lambda \cdot}z) = 0. \tag{6.35}$$

Therefore, we have the following characteristic equation for the bifurcation problem (6.28)

$$\det{}_{\mathbb{C}} \triangle_\alpha(\lambda) = 0. \tag{6.36}$$

Since the matrix C is symmetric (and non-singular), it is completely diagonalizable by using a basis composed of its eigenvectors. Thus, suppose that $\sigma(C) = \{\xi_1, \xi_2, \dots, \xi_n\}$ is the set of all eigenvalues of C and let $z_1, z_2, \dots, z_n \in U^c$ denote the corresponding eigenvectors. Then we have the following formula:

Proposition 6.2.1. *A number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution $(\alpha, 0)$ for the system (6.28) if and only if*

$$\det{}_{\mathbb{C}} \triangle_\alpha(\lambda) = \prod_{l=1}^{n} \left[\lambda - D_xf(\alpha, 0)e^{\lambda \cdot} - \xi_l K(\alpha)e^{\lambda \cdot}\right], \tag{6.37}$$

where $\xi_1, \xi_2, \dots, \xi_n$ are the eigenvalues of the matrix C.

Of course, the characteristic operator $\triangle_\alpha(\lambda) : U^c \to U^c$ is Γ-equivariant, so its eigenspaces are Γ-invariant.

In order to satisfy the necessary condition for the occurences of Hopf bifurcation, we need to make the following assumption

(H2) *There exists $\alpha_o \in \mathbb{R}$ such that $(\alpha_o, 0) \in \mathbb{R} \oplus U$ is an isolated center of (6.28) such that* $\det_{\mathbb{C}} \triangle_{\alpha_o}(i\beta_o) = 0$ *for some $\beta_o > 0$.*

(H3) The system (6.28) has no constant periodic solution.

6.2.2 Application of the Equivariant Degree Method

By following the steps, which were explained in section 6.1, we associate with the point (α_o, β_o) a local bifurcation invariant $\omega(\alpha_o, \beta_o, 0) := G\text{-Deg}(\mathcal{F}_\theta, \Omega)$, where $G = \Gamma \times S^1$, Ω (defined by (6.11)) is an open neighborhood of $(\alpha_o, \beta_o, 0)$ in the space

$\mathbb{R}^2_+ \times W$, and $\mathcal{F}_\theta : \mathbb{R}^2_+ \times W \to \mathbb{R} \oplus W$ (defined by a similar formula to (6.9)), $W := H^1(S^1; U)$, is the mapping associated with the bifurcation problem (6.28). This bifurcation invariant can be evaluated by applying the standard steps, which were explained in subsection 6.1.10.

Put $A := D_x F(\alpha_o, 0) + K(\alpha_o)C : U \to U$, and consider the real spectrum $\sigma(A)$ of the operator A. It is easy to check that a number μ belongs to $\sigma(A)$ if and only if for some eigenvalue ξ of the matrix C we have

$$\mu - k(\alpha_o) - \xi f'_x(\alpha_o, 0) = 0, \quad \text{i.e.} \quad \mu = k(\alpha_o) + \xi f'_x(\alpha_o, 0),$$

where $k(\alpha_o) = K(\alpha_o)(1)$ and $f'_x(\alpha_o, 0) = D_x f(\alpha_o, 0)(1)$ are constants. Consequently, we obtain

$$\sigma(A) = \Big\{ \mu_l : \mu_l := k(\alpha_o) + \xi_l\, f'_x(\alpha_o, 0),\ l = 1, 2, \dots, n \Big\}.$$

To every eigenvalue $\mu \in \sigma(A)$ ($\mu = \mu_l$, for some $l = 1, 2, \dots, n$) corresponds an eigenvector $x_l \in U$ of the matrix C associated with $\xi_l \in \sigma(C)$.

Let $\sigma_+(A) = \{\mu_1, \dots, \mu_k\}$ denote all the positive eigenvalues of A. Then we put

$$\deg_0(\alpha_o, \beta_o) := \prod_{s=1}^{k} \prod_{j=0}^{r} \Big(\deg_{\mathcal{V}_j} \Big)^{m_j(\mu_s)}.$$

We consider the isotypical decomposition (6.4) and the set $\{i\beta_o, i\beta_1, \dots, i\beta_m\}$ of all the purely imaginary roots λ of the equation

$$\det_{\mathbb{C}} \triangle_{\alpha_o, j}(\lambda) = 0, \quad j = 0, 1, 2, \dots, r.$$

We assume that

$$\Big\{ i\beta_o, ik_1\beta_o, \dots, ik_s\beta_o \Big\} \subset \Big\{ i\beta_o, i\beta_1, \dots, i\beta_m \Big\},$$

where $k_l > 1$ are integers, is the subset composed of all the integer multiples of $i\beta_o$. The element $\deg_1(\alpha_o, \beta_o) \in A_1(G)$ is given by

$$\deg_1(\alpha_o, \beta_o) = \deg_{1,k_0}(\alpha_o, \beta_o) + \deg_{1,k_1}(\alpha_o, \beta_o) + \dots + \deg_{1,k_s}(\alpha_o, \beta_o),$$

where $k_0 = 1$ and

$$\deg_{1,k_l}(\alpha_o, \beta_o) = \sum_{j=0}^{r} \mathfrak{t}_{j,1}(\alpha_o, k_l\beta_o)\, \deg_{\mathcal{V}_{j,k_l}}$$

where the numbers $\mathfrak{t}_{j,1}(\alpha_o, k_l\beta_o) = \mathfrak{t}_{j,k_l}(\alpha_o, \beta_o)$ are the $\mathcal{V}_{j,k_l}$-isotypical crossing numbers at (α_o, β_o).

Then, under the above assumptions we obtain

$$G\text{-Deg}\,(\mathcal{F}_\theta, \Omega) = \deg_0(\alpha_o, \beta_o)\, \deg_1(\alpha_o, \beta_o). \tag{6.38}$$

6.3 Hopf Bifurcation Results for Configurations of Identical Oscillators

In this section and its subsections, we will present the computations of the degree $G\text{-Deg}\,(\mathcal{F}_\theta, \Omega)$ for concrete models of symmetric configurations of identical oscillators.

More precisely, we consider the following system of delayed functional differential equations

$$\frac{d}{dt}x(t) = -\alpha x(t) - \alpha H(x(t)) \cdot C(G(x(t-1))), \tag{6.39}$$

where

$$x = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix}, \quad H(x) = \begin{bmatrix} h(x^1) \\ h(x^2) \\ \vdots \\ h(x^n) \end{bmatrix}, \quad G(x) = \begin{bmatrix} g(x^1) \\ g(x^2) \\ \vdots \\ g(x^n) \end{bmatrix},$$

and the product '$\cdot$' is defined on the vector by component-wise multiplication, i.e.

$$x \cdot y = \begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = \begin{bmatrix} x^1y^1 \\ x^2y^2 \\ \vdots \\ x^ny^n \end{bmatrix},$$

the functions $h, g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, $h(t) \neq 0$ and $g(0) = 0$, $g'(0) > 0$. and C is a non-singular symmetric $n \times n$-matrix. Such a system can be obtained from (6.28) by rescaling the time and making an appropriate change of variable.

The linearization of the system (6.39) at $(\alpha, 0)$ is

$$\frac{d}{dt}x(t) = -\alpha x(t) - \alpha h(0)g'(0)C(G(x(t-1))), \tag{6.40}$$

and

$$\Delta_\alpha(\lambda) = (\lambda + \alpha)\text{Id} + \alpha h(0)g'(0)e^{-\lambda}C.$$

Therefore, by Proposition 6.2.1, a number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution if and only if

$$\det{}_{\mathbb{C}} \Delta_\alpha(\lambda) = \prod_{l=1}^{n} \left[\lambda + \alpha + \alpha h(0)g'(0)\xi_l e^{-\lambda}\right] = 0, \tag{6.41}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are the eigenvalues of the matrix C.

We put $\eta := h(0)g'(0)$. In order to find the characteristic values $\lambda \in \mathbb{C}$ for the stationary point $(\alpha, 0)$ of the system (6.39) we need to solve the following equation

$$\lambda + \alpha + \alpha\eta\xi_o e^{-\lambda} = 0, \tag{6.42}$$

where ξ_o is an eigenvalue of C. The equation (6.42) can be written as the system

$$\begin{cases} u + \alpha + \alpha\eta\xi_o e^{-u}\cos v = 0 \\ v - \alpha\eta\xi_o e^{-u}\sin v = 0, \end{cases} \tag{6.43}$$

where $\lambda = u + iv$.

Since we are interested in purely imaginary eigenvalues $\lambda = i\beta_o$, by substituting $u = 0$ and $v = \beta$ into the system (6.43), we obtain

$$\begin{cases} \alpha + \alpha\eta\xi_o\cos\beta = 0 \\ \beta - \alpha\eta\xi_o\sin\beta = 0, \end{cases} \tag{6.44}$$

which can be easily transformed to

$$\begin{cases} \cos\beta = -\frac{1}{\eta\xi_o} \\ \frac{1}{\alpha\eta\xi_o}\beta = \sin\beta. \end{cases} \tag{6.45}$$

If $\left|\frac{1}{\xi_o\eta}\right| < 1$, then there exists $\beta_o \in (0, \pi)$ such that $\cos\beta_o = -\frac{1}{\eta\xi_o}$, and, in addition, it is possible to find a unique $\alpha_o \neq 0$ such that $\alpha_o = \frac{\beta_o}{\eta\xi_o\sin\beta_o}$. Therefore, we obtain a pair of solutions (α_o, β_o).

In order to determine the value of the crossing number associated with this purely imaginary characteristic value $\lambda_o = i\beta_o$, we will compute (by implicit differentiation) $\frac{d}{d\alpha}u(\alpha)|_{\alpha=\alpha_o}$. By differentiating the system (6.43) with respect to α we obtain

$$\begin{cases} u'(1 - \alpha\eta\xi_o e^{-u}\cos v) - v'(\alpha\eta\xi_o e^{-u}\sin v) = -\eta\xi_o e^{-u}\cos v - 1 \\ u'(\alpha\eta\xi_o e^{-u}\sin v) + v'(1 - \alpha\eta\xi_o e^{-u}\cos v) = \eta\xi_o e^{-u}\sin v, \end{cases} \tag{6.46}$$

which, by (6.43), leads to

$$\begin{cases} u'(1 + u + \alpha) - v'v = \frac{u+\alpha}{\alpha} - 1 \\ u'v + v'(1 + u + \alpha) = \frac{v}{\alpha}. \end{cases} \tag{6.47}$$

By substituting $\alpha = \alpha_o$, $u = 0$ and $v = \beta_o$ into the system (6.47), we get

$$\begin{cases} u'(1+\alpha_o) - v'\beta_o = 0 \\ u'\beta_o + v'(1+\alpha_o) = \frac{\beta_o}{\alpha_o}. \end{cases} \tag{6.48}$$

The system (6.48) yields

$$\frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \frac{\beta_o^2}{\alpha_o(\alpha_o^2 + (1+\beta_o)^2}. \tag{6.49}$$

$$\text{sign } \frac{d}{d\alpha}u|_{\alpha=\alpha_o} = \text{sign } \alpha_o. \tag{6.50}$$

Remark 6.3.1. The equivariant degree $\Gamma \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)$ provides a complete description of the symmetric Hopf bifurcation at $(\alpha_o, 0)$, i.e. every non zero coefficient n_{H_o} in

$$\Gamma \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega) = \sum_{(H)} n_H\,(H), \tag{6.51}$$

indicates a "topological obstruction" resulting in the existence of a branch of non-trivial periodic solutions to (6.39) of the orbit type at least (H_o) (with respect to the indicated partial order). Although, the entire value of the degree $G\text{-Deg}(\text{Id} - \mathcal{F}_\theta, \Omega)$ should be considered as the *equivariant invariant* classifying the symmetric Hopf bifurcation, in order to simplify the exposition (by reducing the number of additional cases) we will restrict our computations to the coefficients $n_{H_o} = n_{L^{\varphi,1}}$, which will be called *first coefficients*, and we will denote the corresponding part of the eqivariant degree (6.51) by $G\text{-Deg}(\text{Id} - \mathcal{F}_\theta)_1$.

6.3.1 Positive Eigenvalues

We will use the same notation as in section 6.2. We have $A = -\alpha\text{Id} - \alpha h(0)g'(0)C = -\alpha\text{Id} - \alpha\eta C$, so

$$\sigma(A) = \Big\{\mu_j : \mu_j = -\alpha - \alpha\eta\xi_j,\ \xi_j \in \sigma(C)\Big\}.$$

Let us consider an eigenvalue $\xi_l \in \sigma(C)$ such that $\left|\frac{1}{\xi_l\eta}\right| < 1$. Then, there exists a purely imaginary characteristic root $i\beta_l$, $\beta_l > 0$, of the characteristic equation (6.41) for $\alpha = \alpha_l$, where

$$\cos\beta_l = \frac{1}{\eta\xi_l}, \quad \alpha_l = \frac{\beta_l}{\eta\xi_l \sin\beta_l}.$$

We always assume in what follows that $h(0) > 0$ and, consequently, $\eta > 0$. Therefore, sign α_l = sign ξ_l. Therefore, we have

$$\begin{array}{lll} \text{if} \;\; \alpha_l > 0 & \text{then} & \mathfrak{t}_{j,1}(\alpha_l, \beta_l) = -m_j(i\beta_l) \\ \text{if} \;\; \alpha_l < 0 & \text{then} & \mathfrak{t}_{j,1}(\alpha_l, \beta_l) = m_j(i\beta_l) \end{array}$$

In order to determine all the positive eigenvalues of the operator A, we divide the spectrum $\sigma(C)$ into two parts $\sigma_a(C)$ and $\sigma_b(C)$

$$\sigma_a(C) = \{\xi_j \in \sigma(C) : -1 < \eta\xi_j\}$$
$$\sigma_b(C) = \{\xi_j \in \sigma(C) : \eta\xi_j < -1\}$$

Since we assume here that A is an isomorphism, the condition $1+\eta\xi_j \neq 0$ is satisfied by all the eigenvalues ξ_j of C, thus $\sigma(C) = \sigma_a(C) \cup \sigma_b(C)$. Therefore, if we put

$$\Sigma(C) := \begin{cases} \sigma_a(C) & \text{if } \alpha < 0 \\ \sigma_b(C) & \text{if } \alpha > 0 \end{cases}$$

then the set $\sigma_+(A)$ of all positive eigenvalues of A can be identified as

$$\sigma_+(A) = \Big\{\mu_j : \mu_j = -\alpha(1+\eta\xi_j,\ \xi_j \in \Sigma(C)\Big\}.$$

Consequently, we can apply the equivariant degree method described in subsection 6.2.2 with $(\alpha_o, \beta_o) = (\alpha_l, \beta_l)$, and we obtain

$$\Gamma \times S^1\text{-Deg}\,(\mathfrak{F}_\theta, \Omega)_1 = \prod_{\mu_j \in \sigma_+(A)} \prod_{i=0}^{r} \Big(\deg_{\mathcal{V}_i}\Big)^{m_i(\mu_j)} \cdot \sum_{j=0}^{s} \Big(\mathfrak{t}_{j,1}(\alpha_j, \beta_l)\ \deg_{\mathcal{V}_{j,1}}\Big). \quad (6.52)$$

Now we are in the position to discuss the concrete examples of the system (6.39), where the matrix C is symmetric with respect to a certain finite group of symmetries Γ.

6.3.2 Hopf Bifurcation in a System with Dihedral Symmetries

We consider here the system of equations (6.28) with the matric C of the type (6.29). This system is symmetric with respect to the dihedral group $\Gamma = D_n$ action on $U = \mathbb{R}^n$. We denote by $\rho := e^{\frac{2\pi}{n}i}$ the generator of $\mathbb{Z}_n$. Notice that ρ acts on a vector $x = (x^0, x^1, \dots, x^{n-1})$ by sending the k-th coordinate of x to the $k+1$ (mod n) coordinate. It is convenient to consider this D_n-action on the complex space $U^c := \mathbb{C}^n$. We have the following $\mathbb{Z}_n$-isotypical decomposition of U^c

$$U^c = \widetilde{U}_0 \oplus \widetilde{U}_1 \oplus \cdots \oplus \widetilde{U}_{n-1},$$

where $\widetilde{U}_j = \operatorname{span}(\langle 1, \rho^j, \rho^{2j}, \dots, \rho^{(n-1)j} \rangle)$. Since κ sends $\widetilde{U}_j$ onto $\widetilde{U}_{-j}$ (where $-j$ is taken (mod n)), thus the D_n-isotypical components of U are

$$U_0 = \widetilde{U}_0, \quad U_j := \widetilde{U}_j \oplus \widetilde{U}_{-j}, \ \ 0 < j < n/2,$$

and, in addition, if n is even, there is also the component

$$U_{\frac{n}{2}} := \widetilde{U}_{\frac{n}{2}}.$$

It is easy to check that the isotypical component U_j, $0 \le j < n/2$, is equivalent to the irreducible representation $\mathcal{V}_j^c$ of D_n, and $U_{\frac{n}{2}}$ (for n even) is equivalent to $\mathcal{V}_{j_n+1}^c$ (see [3] for more details). The subspace U_j is also an eigenspace of the matrix C corresponding to the eigenvalue $\xi_j := c + 2d\cos\frac{2\pi j}{n}$. We put $\sigma(C) := \Big\{\xi_j : 0 < j \le [n/2]\Big\}$.

It seems to be a difficult task to completely evaluate the $D_n \times S^1$-equivariant degree $D_n \times S^1\text{-Deg}\,(\mathfrak{F}_\theta, \Omega)_1$ for an arbitrary n. However, in the case of the Hopf bifurcation with the symmetry group D_n, it is possible to determine the coefficients n_{H_o} of the degree $D_n \times S^1\text{-Deg}\,(\mathfrak{F}_\theta, \Omega)_1 = \sum_{(H)} n_H (H)$ corresponding to the dominating orbit types (H_o). For this purpose, we need the following

Lemma 6.3.2. *Under the above assumptions, if (H_o) is a dominating orbit type in U^c (see [3]), then for all j such that $0 \le j \le n/2$ the coefficient of (H_o) in $\deg_{\mathcal{V}_j} \cdot (H_o)$ is non-zero.*

Proof: Suppose that $H_o = K_o^\theta$ and consider $\deg_{\mathcal{V}_j} = (D_n) + (K) + \dots$. It is clear that the product $(K)(K_o^\theta)$ may contain a term $a(H_o)$, with $a \neq 0$, only if $K_o \subset K$. This is exactly the case when a cancelation of the coefficients of (H_o) can take place. For example $(\mathbb{Z}_n)(\mathbb{Z}_n^{t_j}) = 2(\mathbb{Z}_n^{t_j})$. We put $h = gcd(j,n)$ and $m = n/h$, according to the list of basic degrees $\deg_{\mathcal{V}_j}$ (given in [3]) we have the following cases:

The case m is odd, $(H_o) = (D_h^{\tilde{z}})$ or $(H_o) = (\mathbb{Z}_n^{t_j})$, and $\deg_{\mathcal{V}_j} = (D_n) - 2(D_h) +$ other terms. We have

$$\deg_{\mathcal{V}_j} \cdot (D_h^{\tilde{z}}) = (D_h^{\tilde{z}}) - 2(D_h^{\tilde{z}}) + \text{ other terms} = -(D_h^{\tilde{z}}) + \text{ other terms},$$

thus, in this case, the coefficients of $(D_h^{\tilde{z}})$ in the above product do not reduce to zero. Of course, we also have

$$\deg_{\mathcal{V}_j} \cdot (\mathbb{Z}_n^{t_j}) = (\mathbb{Z}_n^{t_j}) + \text{ other terms.}$$

There are no other possibilities for the cancelation.

The case m is even. The following orbit types (H_o) should be considered in this case: (D_{2h}^d), $(\widetilde{D}_{2h}^d)$, $(D_{2h}^{\hat{d}})$ and $(\mathbb{Z}_n^{t_j})$. The same argument, as in the previous case, can be used for the orbit type $(\mathbb{Z}_n^{t_j})$. Assume that $H_o = D_{2h}^d$ (for the subgroups $\widetilde{D}_{2h}^d$ and $D_{2h}^{\hat{d}}$ the same argument can be applied). If there exists another j' $(0 < j' < n/2)$ such that $h' = gcd(j', n)$ and $h' = 2h$, then n/h' must be odd (or otherwise (H_o) could not be dominating), and in this case we have that $\deg_{\mathcal{V}_{j'}} = (D_n) - 2(D_{2h}) +$ other terms. Therefore,

$$\deg_{\mathcal{V}_{j'}} \cdot (D_{2h}^d) = (D_{2h}^d) - 2(D_{2h}^d) + \text{ other terms} = -(D_{2h}^d) + \text{ other terms,}$$

and again we obtain that the cancelation is not possible.

The case $j = n/2$. Assume that $(H_o) = (D_n^d)$. In this case, the cancelation is clearly not possible. □

As an immediate consequence of Lemma 6.3.2 we have

Proposition 6.3.3. *Let $\Gamma = D_n$. Under the above assumptions, if the crossing number $t_{j,1} \neq 0$, standing in (6.52), then every dominating orbit type appearing in $\deg_{\mathcal{V}_{j,1}}$ with non-zero coefficient, will also appear with non-zero coefficient in $D_n \times S^1$-Deg$(\mathfrak{F}_\theta, \Omega)$*

Remark 6.3.4. Let us point out that Proposition 6.3.3 is not true for an arbitrary group Γ. In fact, it is shown in the subsequent examples, cancelation of coefficients standing by dominating orbit types is possible. Therefore, it is necessary to use the complete degree $\Gamma \times S^1$-Deg$(\mathfrak{F}_\theta, \Omega)_1$ to detect branches of solutions and classify their symmetries.

We are now in a position to present the following general result (cf. [35])

Theorem 6.3.5. *Suppose that $\xi_j \in \sigma(C)$ is such that $\left|\frac{1}{\xi_j \eta}\right| < 1$. Then, there exists $\beta_o > 0$ and α_o satisfying*

$$\cos \beta_o = \frac{1}{\eta \xi_j}, \quad \alpha_o = \frac{\beta_o}{\eta \xi_j \sin \beta_o}, \tag{6.53}$$

such that the equation (6.39) has a Hopf bifurcation at $(\alpha_o, 0)$. *More precisely, there exists a branch of non-constant* $\frac{2\pi}{\beta}$*-periodic solutions* ($\beta \to \beta_o$ *as* $\alpha \to \alpha_o$) *of (6.39) bifurcating from* $(\alpha_o, 0)$. *In addition*

(i) *If* $j = 0$, *then there exists at least one branch of non-constant periodic solutions of the orbit type* (D_n),

(ii) *If* $0 < j < \frac{n}{2}$, *and* n *is odd then there exists at least: 2 branches of non-constant periodic solutions of the orbit type* $(\mathbb{Z}_n^{t_j})$, $\frac{n}{h}$ *branches of the orbit type* $(D_{\frac{n}{h}}^{z})$ *(where* $h = gcd(n, j)$*), and one branch of the orbit type larger or equal than* (D_n),

(iii) *If* $0 < j < \frac{n}{h}$, n *is even and* $m \equiv 2$ *(mod 4) (where* $m = \frac{n}{h}$*), then there exists at least: 2 branches of non-constant periodic solutions of the orbit type* $(\mathbb{Z}_n^{t_j})$, $\frac{n}{2h}$ *branches of the orbit type* (D_{2h}^{d}), *and* $\frac{n}{2h}$ *branches of the orbit type* $(\widetilde{D}_{2h}^{d})$,

(iv) *If* $0 < j < \frac{n}{2}$, n *is even and* $m \equiv 0$ *(mod 4), then there exists at least: 2 branches of the orbit type* $(\mathbb{Z}_n^{t_j})$, $\frac{n}{2h}$ *branches of the orbit type* (D_{2h}^{d}), *and* $\frac{n}{2h}$ *branches of the orbit type* $(\widehat{D}_{2h}^{d})$,

(v) *If* $j = \frac{n}{2}$ *(for* n *even) then there exists at least one branch of non-constant periodic solutions of the orbit type* (D_n^d).

Let us discuss several examples of dihedral groups, for which we obtain a complete classification of the symmetric Hopf bifurcation, in terms of the equivariant degree.

6.3.3 Hopf Bifurcation with D_3 Symmetries

In this case we have $\sigma(C) = \{\xi_0 = c + 2d,\ \xi_1 = c - d\}$. To each of the eigenvalues ξ_l, $l = 0, 1$, corresponds the pair (α_l, β_l) such that $i\beta_l$ is a purely imaginary characteristic value for $(\alpha_l, 0)$. Then we can apply the equivariant degree $D_3 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)$ to classify the Hopf bifurcation at the point $(\alpha_l, 0)$. We summarize in Table 6.1 the topological invariants $D_3 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$ corresponding to elements in $\Sigma(C)$, and under the condition that $\alpha_l < 0$ (for $\alpha_l > 0$, one should simply reverse the signum of the listed degree for $-\alpha_l$). The dominating orbit types in this case are (D_3), $(\mathbb{Z}_3^t)$ (we write here simply t instead t_1), and (D_1^z).

α_l	$\Sigma(C)$	$D_3 \times S^1$-Deg $(\mathfrak{F}_\theta, \Omega)_1$	# Branches
α_0	$\emptyset$	(D_3)	1
α_0	$\{\xi_0\}$	$-(D_3)$	1
α_0	$\{\xi_1\}$	$(D_3) - 2(D_1) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_1\}$	$-(D_3) + 2(D_1) - (\mathbb{Z}_1)$	1
α_1	$\emptyset$	$(\mathbb{Z}_3^t) + (D_1) + (D_1^z) - (\mathbb{Z}_1)$	6
α_1	$\{\xi_0\}$	$-(\mathbb{Z}_3^t) - (D_1) - (D_1^z) + (\mathbb{Z}_1)$	6
α_1	$\{\xi_1\}$	$(\mathbb{Z}_3^t) - (D_1) - (D_1^z) + (\mathbb{Z}_1)$	6
α_1	$\{\xi_0, \xi_1\}$	$-(\mathbb{Z}_3^t) + (D_1) + (D_1^z) - (\mathbb{Z}_1)$	6

Table 6.1: Equivariant classification of the Hopf bifurcation with D_3 symmetries.

6.3.4 Hopf Bifurcation with D_4 Symmetries

We summarize in Table 6.2 the corresponding results for D_4-symmetric Hopf bifurcation. Here we have $\sigma(C) = \left\{\xi_0 = c + 2d, \xi_1 = c, \xi_2 = c - 2d\right\}$ and the dominating orbit types in this case are: $(\mathbb{Z}_4^t)$ (here we simply write t instead of t_1), (D_4^d), (D_2^d), $(\tilde{D}_2^d)$, and (D_4).

α_l	$\Sigma(C)$	$D_4 \times S^1$-Deg $(\mathfrak{F}_\theta, \Omega)_1$	# Br.
α_0	$\{\emptyset\}$	(D_4)	1
α_0	$\{\xi_0\}$	$-(D_4)$	1
α_0	$\{\xi_2\}$	(D_4)	1
α_0	$\{\xi_0, \xi_1\}$	$-(D_4) + (D_1) + (\tilde{D}_1) - (Z_1)$	1
α_0	$\{\xi_1, \xi_2\}$	$(D_4) - (D_1) - (\tilde{D}_1) + (Z_1)$	1
α_0	$\{\xi_0, \xi_1, \xi_2\}$	$-(D_4) + (D_1) + (\tilde{D}_1) - (Z_1)$	1
α_1	$\{\emptyset\}$	$(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-)$	6
α_1	$\{\xi_0\}$	$-(\mathbb{Z}_4^t) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^-)$	6
α_1	$\{\xi_2\}$	$(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (\mathbb{Z}_2^-)$	6
α_1	$\{\xi_0, \xi_1\}$	$-(\mathbb{Z}_4^t) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\tilde{D}_1^z) - 2(\mathbb{Z}_1) + (D_1) + (\tilde{D}_1)$	6
α_1	$\{\xi_1, \xi_2\}$	$(\mathbb{Z}_4^t) + (D_2^d) + (\tilde{D}_2^d) - (D_1) - (\tilde{D}_1) - (\mathbb{Z}_2^-) + 2(\mathbb{Z}_1) - (D_1^z) - (\tilde{D}_1^z)$	6
α_1	$\{\xi_0, \xi_1, \xi_2\}$	$-(\mathbb{Z}_4^t) - (D_2^d) - (\tilde{D}_2^d) + (\mathbb{Z}_2^-) + (D_1^z) + (\tilde{D}_1^z) - 2(\mathbb{Z}_1) + (D_1) + (\tilde{D}_1)$	6
α_2	$\{\emptyset\}$	(D_4^d)	1
α_2	$\{\xi_0\}$	$-(D_4^d)$	1
α_2	$\{\xi_2\}$	(D_4^d)	1
α_2	$\{\xi_0, \xi_1\}$	$-(D_4^d) + (D_1) + (\tilde{D}_1^z) - (Z_1)$	1
α_2	$\{\xi_1, \xi_2\}$	$(D_4^d) - (D_1) - (\tilde{D}_1^z) + (Z_1)$	1
α_2	$\{\xi_0, \xi_1, \xi_2\}$	$-(D_4^d) + (D_1) + (\tilde{D}_1^z) - (Z_1)$	1

Table 6.2: Equivariant classification of the Hopf bifurcation with D_4 symmetries.

6.3.5 Hopf Bifurcation with D_5 Symmetries

The topological invariants $D_5 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$ corresponding to elements in $\Sigma(C)$, and $\alpha_l < 0$, are summarized in Table 6.3. Here we have $\sigma(C) = \left\{\xi_0 = c + 2d, \xi_1 = c + 2d\frac{\sqrt{5}-1}{4}, \xi_2 = c - 2d\frac{\sqrt{5}+1}{4}\right\}$ and the dominating orbit types in this case are: $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$, (D_5), and (D_1^z).

α_l	$\Sigma(C)$	$D_5 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$	# Branches
α_0	$\{\emptyset\}$	(D_5)	1
α_0	$\{\xi_0\}$	$-(D_5)$	1
α_0	$\{\xi_2\}$	$(D_5) - 2(D_1) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_1\}$	$-(D_5) + 2(D_1) - (\mathbb{Z}_1)$	1
α_0	$\{\xi_1, \xi_2\}$	(D_5)	1
α_0	$\{\xi_0, \xi_1, \xi_2\}$	$-(D_5)$	1
α_1	$\{\emptyset\}$	$(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
α_1	$\{\xi_0\}$	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_1	$\{\xi_2\}$	$(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_1	$\{\xi_0, \xi_1\}$	$-(\mathbb{Z}_5^{t_1}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
α_1	$\{\xi_1, \xi_2\}$	$(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_1	$\{\xi_0, \xi_1, \xi_2\}$	$-(\mathbb{Z}_5^{t_1}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_2	$\{\emptyset\}$	$(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
α_2	$\{\xi_0\}$	$-(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_2	$\{\xi_2\}$	$(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8
α_2	$\{\xi_0, \xi_1\}$	$-(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
α_2	$\{\xi_1, \xi_2\}$	$(\mathbb{Z}_5^{t_2}) + (D_1^z) + (D_1) - (\mathbb{Z}_1)$	8
α_2	$\{\xi_0, \xi_1, \xi_2\}$	$-(\mathbb{Z}_5^{t_2}) - (D_1^z) - (D_1) + (\mathbb{Z}_1)$	8

Table 6.3: Equivariant classification of the Hopf bifurcation with D_5 symmetries.

6.3.6 Hopf Bifurcation in a System with Tetrahedral Symmetries

We consider here the system of equations (6.28) with the matrix C of the type (6.30). This system is symmetric with respect to the tetrahedral group $\Gamma = A_4$ action on $U = \mathbb{R}^n$, which acts on the space $V = \mathbb{R}^4$ by permuting the coordinates of vectors. We have $\sigma(C) = \{\xi_0 = c+3d,\ \xi_3 = c-d\}$. The subspace V_0 of the fixed-points of this action is spanned by the vector $\langle 1, 1, 1, 1\rangle$, and its orthogonal complement V_3 is the natural three-dimensional representation of A_4, which was denoted by $\mathcal{V}_3$. These two subspaces are the eigenspaces of the matrix C: the subspace V_0 corresponds to the eigenvalue $\xi_0 = c + 3d$ and V_3 to the eigenvalue $\xi_3 = c - d$. In addition $V_0 \oplus V_3$ is the isotypical decomposition of V, where V_0 is modeled on the trivial A_4-representation $\mathcal{V}_0$ and V_3 is modeled on the natural A_4-representation $\mathcal{V}_3$. The dominating orbit types are (A_4) (orbit contains 1 periodic solution), $(\mathbb{Z}_3^{t_1})$ (orbit contains 2 periodic solutions), $(\mathbb{Z}_3^{t_2})$ (orbit contains 4 periodic solutions), and (V_4^-) (orbit contains 3 periodic solutions). The equivariant degree $A_4 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$, can be evaluated using the computational formula (6.52).

We summarize in Table 6.4 the topological invariants $A_4 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$ corresponding to elements in $\Sigma(C)$, and $\alpha_l < 0$. Table 6.4 provides us with an example of a situation where a non-zero coefficient in $\deg_{\mathcal{V}_3}$, corresponding to a dominating orbit type, gets canceled after multiplication.

α_l	$\Sigma(C)$	$A_4 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$	# Branches
α_0	$\{\emptyset\}$	(A_4)	1
α_0	$\{\xi_0\}$	$-A_4)$	1
α_0	$\{\xi_3\}$	$(A_4) - 2(\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_3\}$	$-(A_4) + 2(\mathbb{Z}_3) + (\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
α_3	$\{\emptyset\}$	$(\mathbb{Z}_3^{t_1}) + (\mathbb{Z}_3^{t_2}) + (V_4^-) + (\mathbb{Z}_3) - (\mathbb{Z}_1)$	12
α_3	$\{\xi_0\}$	$-(\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (V_4^-) - (\mathbb{Z}_3) + (\mathbb{Z}_1)$	12
α_3	$\{\xi_3\}$	$-(\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) + (V_4^-) - (\mathbb{Z}_3) - 2(\mathbb{Z}_2) + (\mathbb{Z}_1)$	12
α_3	$\{\xi_0, \xi_3\}$	$(\mathbb{Z}_3^{t_1}) + \mathbb{Z}_3^{t_2}) - (V_4^-) + (\mathbb{Z}_3) + 2(\mathbb{Z}_2) - (\mathbb{Z}_1)$	12

Table 6.4: Equivariant classification of the Hopf bifurcation with A_4 symmetries.

6.3.7 Hopf Bifurcation in a System with Octahedral Symmetries

Here we have the group S_4 is acting on the eight-dimensional space $V := \mathbb{R}^8$ by permuting the coordinates of the vectors in the same way as the symmetries of the cube in $\mathbb{R}^3$ permutes the eight vertices of the cube. It can be easily verified, that the representation V can be decomposed into a direct sum of four subspaces:

$$V = V_0 \oplus V_1 \oplus V_3^1 \oplus V_3^2,$$

where

$$V_0 = \text{span}\left\{\langle 1,1,1,1,1,1,1,1\rangle\right\}$$
$$V_1 = \text{span}\left\{\langle 1,-1,1,-1,1,-1,1,-1\rangle,\rangle\right\}$$
$$V_3^1 = \text{span}\left\{\langle 1,1,-1,-1,1,-1,-1,1\rangle,\ \langle 1,-1,1,-1,-1,1,-1,1\rangle,\ \langle -1,1,1,-1,1,1,-1,-1\rangle\right\}$$
$$V_3^2 = \text{span}\left\{\langle 1,-1,-1,1,1,1,-1,-1\rangle,\ \langle 1,1,1,1,-1,-1,-1,-1\rangle,\ \langle -1,-1,1,1,1,-1,-1,1\rangle\right\}$$

Notice that these subspaces are irreducible representations of S_4, where V_3^1 is equivalent to the natural three-dimensional representation $\mathcal{V}_3$ of S_4, and V_3^2 is equivalent to the another three-dimensional irreducible representation $\mathcal{V}_4$ of S_4. The subspace V_0 is the fixed-point space of the action of S_4. The subspaces V_0, V_1, V_3^1 and V_3^2, which are the isotypical component of V, are eigenspaces for the matrix C. We have

Subspace	Eigenvalue of C	Type of Representation	Dimension
V_0	$c+3d$	Trivial	1
V_1	$c-3d$	Representation $\mathcal{V}_1$	1
V_3^1	$c+d$	Natural $\mathcal{V}_3$	3
V_3^2	$c-d$	Representation $\mathcal{V}_4$	3

Let us list the dominating orbit types: (S_4) (orbit contains one periodic solution), (S_4^-) (orbit contains one periodic solution), (D_4^d) (orbit contains 3 periodic solutions), (D_2^d) (orbit contains 6 periodic solutions), $(\mathbb{Z}_3^t)$ (orbit contains 8 periodic solutions), $(D_4^{\hat{z}})$ (orbit contains 3 periodic solutions), $(D_3^{\hat{z}})$ (orbit contains 4 periodic solutions), and (Z_4^c).

We summarize in Table 6.5 the topological invariants $S_4 \times S^1$-Deg$(\mathfrak{F}_\theta, \Omega)_1$ corresponding to elements in $\Sigma(C)$, and $\alpha_l < 0$.

α_l	$\Sigma(C)$	$S_4 \times S^1\text{-Deg}\,(\mathfrak{F}_\theta, \Omega)_1$	# Branches
α_0	$\{\emptyset\}$	(S_4)	1
α_0	$\{\xi_0\}$	$-(S_4)$	1
α_0	$\{\xi_1\}$	$(S_4) - 2(D_4)$	1
α_0	$\{\xi_0, \xi_3\}$	$-(S_4) + 2(D_2) - 3(D_1) - 2(D_3) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_1, \xi_4\}$	$(S_4) - 2(D_4) + (\mathbb{Z}_4) - (\mathbb{Z}_3) + 2(\mathbb{Z}_2) + (D_1) - (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_3, \xi_4\}$	$-(S_4) + 2(D_3) + (D_2) + (\mathbb{Z}_4) - (\mathbb{Z}_3) - (\mathbb{Z}_2) - 2(D_1) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_1, \xi_3, \xi_4\}$	$-(S_4) - 2(D_4) - 2(D_3) + (D_2) + (\mathbb{Z}_4) + (\mathbb{Z}_3) - (\mathbb{Z}_2) + 2(D_1) - (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_1, \xi_3, \xi_4\}$	$(S_4) + 2(D_4) + 2(D_3) - (D_2) - (\mathbb{Z}_4) - (\mathbb{Z}_3) + (\mathbb{Z}_2) - 2(D_1) + (\mathbb{Z}_1)$	1
α_1	$\{\emptyset\}$	(S_4^-)	1
α_1	$\{\xi_0\}$	$-(S_4^-)$	1
α_1	$\{\xi_1\}$	$(S_4^-) - 2(D_4^{\hat{d}})$	2
α_1	$\{\xi_0, \xi_3\}$	$-(S_4^-) + 2(D_3^z) + (D_2^z) - 3(D_1^z) + (\mathbb{Z}_1)$	5
α_1	$\{\xi_1, \xi_4\}$	$(S_4^-) - 2(D_4^{\hat{d}}) + (\mathbb{Z}_4^-) + (D_1^z) - (\mathbb{Z}_3) + (D_1^z) + 2(\mathbb{Z}_2) - (Z_1)$	2
α_1	$\{\xi_0, \xi_3, \xi_4\}$	$-(S_4^-) + 2(D_3^z) + (D_2^z) + (\mathbb{Z}_4^-) - (\mathbb{Z}_3) - 2(D_1^z) - (\mathbb{Z}_2) + (Z_1)$	2
α_1	$\{\xi_1, \xi_3, \xi_4\}$	$-2(D_3^z) + (D_2^z) + (S_4^-) - 2(D_4^{\hat{d}}) + 2(D_1^z) + (\mathbb{Z}_4^-) + (\mathbb{Z}_3) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	6
α_1	$\{\xi_0, \xi_1, \xi_3, \xi_4\}$	$2(D_3^z) - (D_2^z) - (S_4^-) + 2(D_4^{\hat{d}}) - 2(D_1^z) - (\mathbb{Z}_4^-) - (\mathbb{Z}_3) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	6
α_3	$\{\emptyset\}$	$(D_4^d) + (D_2^d) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) + (\mathbb{Z}_4^c) + (D_3) - (D_1)$	24
α_3	$\{\xi_0\}$	$-(D_4^d) - (D_2^d) - (\mathbb{Z}_4^c) - (\mathbb{Z}_3^t) + (\mathbb{Z}_2^-) - (D_3) + (D_1)$	24
α_3	$\{\xi_1\}$	$-(D_4^d) - (D_2^d) + (\mathbb{Z}_3^t) - (\mathbb{Z}_4^c) - 2(V_4^-) + (D_3) + (\mathbb{Z}_2^-) - (D_1)$	24
α_3	$\{\xi_0, \xi_3\}$	$-(D_4^d) + (D_2^d) - (\mathbb{Z}_4^c) + (\mathbb{Z}_3^t) - (D_1^z) + (D_3) + (D_2) - 3(D_1) + (\mathbb{Z}_2^-) + (\mathbb{Z}_1)$	24
α_3	$\{\xi_1, \xi_4\}$	$-(D_4^d) - (D_2^d) + (\mathbb{Z}_4^c) + \mathbb{Z}_4^-) - 2(V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) + (D_1^z) + 5(\mathbb{Z}_2^-) + 2(\mathbb{Z}_2) + (D_3) - 3(\mathbb{Z}_1)$	24
α_3	$\{\xi_0, \xi_3, \xi_4\}$	$-(D_4^d) + (D_2^d) + (\mathbb{Z}_4^c) + (\mathbb{Z}_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) + (D_3) + (D_2) - (\mathbb{Z}_3) - (\mathbb{Z}_2) + (\mathbb{Z}_1) - (D_1)$	24
α_3	$\{\xi_1, \xi_3, \xi_4\}$	$-(D_4^d) + (D_2^d) - 2(V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) - (\mathbb{Z}_2^-) + (\mathbb{Z}_4^-) + (\mathbb{Z}_4^c) - (D_3) + (D_2) - (\mathbb{Z}_2) + (D_1) - (\mathbb{Z}_1)$	24
α_3	$\{\xi_0, \xi_1, \xi_3, \xi_4\}$	$-(D_4^d) - (D_2^d) + 2(V_4^-) + (\mathbb{Z}_3^t) - (\mathbb{Z}_3) + (\mathbb{Z}_2^-) - (\mathbb{Z}_4^-) - (\mathbb{Z}_4^c) + (D_3) - (D_2) + (\mathbb{Z}_2) - (D_1) + (\mathbb{Z}_1)$	24
α_4	$\{\emptyset\}$	$(D_4^z) + (D_3^z) + (D_2^d) - (\mathbb{Z}_3^t) - (\mathbb{Z}_2^-) - (D_1^z) + (\mathbb{Z}_4^c)$	27
α_4	$\{\xi_0\}$	$(D_4^z) - (D_3^z) - (D_2^d) - (\mathbb{Z}_4^c) - (\mathbb{Z}_3^t) + (\mathbb{Z}_2^-) - (D_1^z)$	27
α_4	$\{\xi_1\}$	$-(D_4^z) + (D_3^z) - (\mathbb{Z}_4^c) - 2(V_4^-) - (D_2^d) + (\mathbb{Z}_3^t) - (D_1^z) + (\mathbb{Z}_2^-)$	27
α_4	$\{\xi_0, \xi_3\}$	$-(D_4^z) + (D_3^z) + (D_2^d) + (D_2^z) + (\mathbb{Z}_3^t) - 3(D_1^z) - (\mathbb{Z}_4^c) + (\mathbb{Z}_2^-) - (D_1) + (Z_1)$	27
α_4	$\{\xi_1, \xi_4\}$	$-(D_4^z) + (D_3^z) - (D_2^d) + (\mathbb{Z}_4^c) - 2(V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) + (D_1^z) + (\mathbb{Z}_4) + 2(\mathbb{Z}_2) + 5(\mathbb{Z}_2^-) + (D_1) - 3(Z_1)$	27
α_4	$\{\xi_0, \xi_3, \xi_4\}$	$-(D_4^z) + (D_3^z) + (D_2^z) + (D_2^d) + (\mathbb{Z}_4^c) - (\mathbb{Z}_3^t) - (D_1^z) + (\mathbb{Z}_4) - (\mathbb{Z}_3) - (\mathbb{Z}_2^-) + (\mathbb{Z}_1)$	27
α_4	$\{\xi_1, \xi_3, \xi_4\}$	$-(D_4^z) - (D_3^z) + (D_2^z) + (D_2^d) - 2(V_4^-) + (\mathbb{Z}_3^t) + (\mathbb{Z}_3) - (\mathbb{Z}_2^-) + (D_1^z) + (\mathbb{Z}_4) + (\mathbb{Z}_4^c) - (\mathbb{Z}_2) - (\mathbb{Z}_1)$	27
α_4	$\{\xi_0, \xi_1, \xi_3, \xi_4\}$	$(D_4^z) + (D_3^z) - (D_2^z) - (D_2^d) + 2(V_4^-) - (\mathbb{Z}_3^t) - (\mathbb{Z}_3) + (\mathbb{Z}_2^-) - (D_1^z) - (\mathbb{Z}_4) - (\mathbb{Z}_4^c) + (\mathbb{Z}_2) + (\mathbb{Z}_1)$	27

Table 6.5: Equivariant classification of the Hopf bifurcation with S_4 symmetries.5

6.3.8 Hopf Bifurcation in a System with Icosahedral Symmetries

Finally we consider the system (6.28) with icosahedral symmetry group. Here we have the group A_5 acting on the twenty-dimensional space $V := \mathbb{R}^{20}$ by permuting the coordinates of the vectors in the same way as the symmetries in $\mathbb{R}^3$ permutes the vertices of the dodecahedron. It can be verified, that the matrix C, defined by (6.32) has the following eigenvalues:

$$\sigma(C) := \left\{\xi_0 = c + 3d, \xi_1 = c - 2d, \xi_2 = c + d, \xi_3 = c + \sqrt{5}d\right\}$$

and there is the following decomposition of V into the eigenspaces of C:

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3,$$

where V_0 is a one dimensional subspace of V, with a trivial action of A_5 (i.e. $V_0 = V^{A_5}$), and

$$V_1 \simeq \mathcal{V}_1 \oplus \mathcal{V}_1, \quad V_2 \simeq \mathcal{V}_2, \quad V_3 \simeq \mathcal{V}_3 \oplus \mathcal{V}_3,$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are irreducible representations of A_5 (see [3]).

Let us list the dominating orbit types: $(A_4^{t_1})$ and $(A_4^{t_2})$ (orbit contains 5 periodic solutions), (A_5) (orbit contains 1 periodic solution), (V_4^{-}) (orbit contains 15 periodic solutions), (D_5^{z}) (orbit contains 6 periodic solutions), (D_3^{z}) (orbit contains 10 periodic solutions), $(\mathbb{Z}_5^{t_1})$, $(\mathbb{Z}_5^{t_2})$ (orbit contains 12 periodic solutions), (D_3), and $(\mathbb{Z}_3^{t})$.

We summarize in Table 6.6 and the topological invariants $A_5 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$ corresponding to elements in $\Sigma(C)$, and $\alpha_l < 0$.

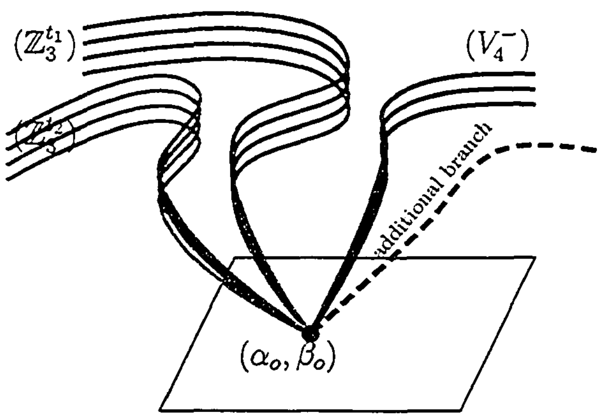
α_l	$\Sigma(C)$	$A_5 \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$	# Br.
α_0	$\{\emptyset\}$	(A_5)	1
α_0	$\{\xi_0\}$	$-(A_5)$	1
α_0	$\{\xi_1\}$	(A_5)	1
α_0	$\{\xi_0, \xi_3\}$	$-(A_5)$	1
α_0	$\{\xi_1, \xi_2\}$	$(A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_2, \xi_3\}$	$-(A_5) + 2(D_5) + 2(D_3) - 3(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
α_0	$\{\xi_1, \xi_2, \xi_3\}$	$(A_5) - 2(D_5) - 2(D_3) + 3(\mathbb{Z}_2) - (\mathbb{Z}_1)$	1
α_0	$\{\xi_0, \xi_1, \xi_2, \xi_3\}$	$-(A_5) + 2(D_5) + 2(D_3) - 3(\mathbb{Z}_2) + (\mathbb{Z}_1)$	1
α_1	$\{\emptyset\}$	$2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_5^{t_2}) + 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) + 2(A_4) + 2(D_3) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2)$	52
α_1	$\{\xi_1\}$	$-2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_5^{t_2})) - 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-) - 2(A_4) - 2(D_3) + 2(\mathbb{Z}_3) + 2(\mathbb{Z}_2)$	52
α_1	$\{\xi_0, \xi_3\}$	$-2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-) - 2(A_4) - 2(D_3) + 2(\mathbb{Z}_3) + 2(\mathbb{Z}_2)$	52
α_1	$\{\xi_1, \xi_2\}$	$2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) + 2(A_4) - 2(D_3) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	52
α_1	$\{\xi_0, \xi_2, \xi_3\}$	$-2(A_4) + 2(D_3) + 2(D_3^z) - 2(V_4^-) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_5^{t_2})) + 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-) + 2(\mathbb{Z}_3) + 2(\mathbb{Z}_2) - 4(\mathbb{Z}_1)$	52
α_1	$\{\xi_1, \xi_2, \xi_3\}$	$2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_5^{t_2}) - 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_2^-) + 2(A_4) - 2(D_3) - 2(\mathbb{Z}_3) - 2(\mathbb{Z}_2) + 4(\mathbb{Z}_1)$	52
α_1	$\{\xi_0, \xi_1, \xi_2, \xi_3\}$	$-2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_5^{t_2}) + 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2^-) - 2(A_4) + 2(D_3) + 2(\mathbb{Z}_3) + 2(\mathbb{Z}_2) - 4(\mathbb{Z}_1)$	52
α_2	$\{\emptyset\}$	$(A_4^{t_1}) + (A_4^{t_2}) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (D_5) + (D_3) - 2(\mathbb{Z}_2)$	51
α_2	$\{\xi_0\}$	$-(A_4^{t_1}) - (A_4^{t_2}) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (V_4^-) - (D_5) - (D_3) + 2(\mathbb{Z}_2)$	51
α_2	$\{\xi_1\}$	$(A_4^{t_1}) + (A_4^{t_2}) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) + (V_4^-) + (D_5) + (D_3) - 2(\mathbb{Z}_1)$	51
α_2	$\{\xi_0, \xi_3\}$	$-(A_4^{t_1}) - (A_4^{t_2}) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - (D_5) - (D_3) - (V_4^-) + 2(\mathbb{Z}_2)$	51
α_2	$\{\xi_1, \xi_2\}$	$(A_4^{t_1}) + (A_4^{t_2}) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) - 4(\mathbb{Z}_3^t) - (D_5) - (D_3) + (V_4^-) - 2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	51
α_2	$\{\xi_0, \xi_2, \xi_3\}$	$-(A_4^{t_1}) - (A_4^{t_2}) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + 4(\mathbb{Z}_3^t) + (D_5) + (D_3) + 2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	51
α_2	$\{\xi_1, \xi_2, \xi_3\}$	$(A_4^{t_1}) + (A_4^{t_2}) - (\mathbb{Z}_5^{t_1}) - (\mathbb{Z}_5^{t_2}) + (V_4^-) - 4(\mathbb{Z}_3^t) - (D_5) - (D_3) - 2(\mathbb{Z}_2^-) - (\mathbb{Z}_2) + 3(\mathbb{Z}_1)$	51
α_2	$\{\xi_0, \xi_1, \xi_2, \xi_3\}$	$-(A_4^{t_1}) - (A_4^{t_2}) + (\mathbb{Z}_5^{t_1}) + (\mathbb{Z}_5^{t_2}) - (V_4^-) + 4(\mathbb{Z}_3^t) + (D_5) + (D_3) + 2(\mathbb{Z}_2^-) + (\mathbb{Z}_2) - 3(\mathbb{Z}_1)$	51
α_3	$\{\emptyset\}$	$2(D_5^z) + 2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_3^t) - 4(\mathbb{Z}_2^-)$	44
α_3	$\{\xi_0\}$	$-2(D_5^z) - 2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_3^t) + 4(\mathbb{Z}_2^-)$	44
α_3	$\{\xi_1\}$	$2(D_5^z) + 2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_3^{t_1}) - 4(\mathbb{Z}_2^-)$	44
α_3	$\{\xi_0, \xi_3\}$	$-2(D_5^z) - 2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_3^{t_1}) + 4(\mathbb{Z}_2^-)$	44
α_3	$\{\xi_1, \xi_2\}$	$-2(D_5^z) + 2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	44
α_3	$\{\xi_0, \xi_2, \xi_3\}$	$2(D_5^z) - 2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	44
α_3	$\{\xi_1, \xi_2, \xi_3\}$	$-2(D_5^z) + 2(V_4^-) - 2(D_3^z) - 2(\mathbb{Z}_5^{t_1}) - 2(\mathbb{Z}_3^t) - 2(\mathbb{Z}_2) + 2(\mathbb{Z}_1)$	44
α_3	$\{\xi_0, \xi_1, \xi_2, \xi_3\}$	$2(D_5^z) - 2(V_4^-) + 2(D_3^z) + 2(\mathbb{Z}_5^{t_1}) + 2(\mathbb{Z}_3^t) + 2(\mathbb{Z}_2) - 2(\mathbb{Z}_1)$	44

Table 6.6: Equivariant classification of the Hopf bifurcation with A_5 symmetries.

Remark 6.3.6. Let us explain briefly how to read the information provided by the equivariant degree $\omega(\alpha_o, \beta_o) := \Gamma \times S^1\text{-Deg}(\mathfrak{F}_\theta, \Omega)_1$. Consider for example the case $\lambda_l = \lambda_3$, $\Sigma(C) = \{\xi_0\}$, and $\alpha_0 < 0$, listed in Table 6.4 for $\Gamma = A_4$. In this case we have

$$\omega(\alpha_o, \beta_o) = -(\mathbb{Z}_3^{t_1}) - (\mathbb{Z}_3^{t_2}) - (V_4^-) - (\mathbb{Z}_3) + (\mathbb{Z}_1).$$

The dominating orbit types in $\omega(\alpha_o, \beta_o)$ with non-zero coefficients are $(\mathbb{Z}_3^{t_1})$, $(\mathbb{Z}_3^{t_2})$, and (V_4^-). Therefore, there is a Hopf bifurcation taking place with non-constant branches of periodic solutions with exactly these orbit type. That means we can expect the occurences of at least four branches of periodic solutions with the orbit type $(\mathbb{Z}_3^{t_1})$, four branches with the orbit type $(\mathbb{Z}_3^{t_2})$, and three branches of the orbit type (V_4^-). Since $\omega(\alpha_o\beta_o)$ has also non-zero coefficients corresponding to $(\mathbb{Z}_3)$ and $(\mathbb{Z}_1)$, there must be also another branch of non-trivial solutions with the isotropy group larger or equal than $\mathbb{Z}_3$. In this way we can predict the existence of at least 12 branches of non-trivial periodic solutions. We illustrate this situation on a diagram below.



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