UNIVERSITY OF ALBERTA

AMERICAN PUT OPTIONS

BY

DONNA MARY SALOPEK

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled American Put Options submitted by Donna Mary Salopek in partial fulfillment of the requirements for the degree of Master of Science.

R. J. Elliott

G. Barone-Adesi

B. Schmuland

August 9, 1994

Abstract. The Black-Scholes model is described, and the differential equations for the American put, and its generalizations, are derived. S. D. Jacka's results on optimal stopping and the free boundary formulation are presented. Several analytical approximations to the American put value, and numerical methods for its calculation, are also given and analyzed.

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Contents

1 Background

The theory of option pricing began in 1900 when Bachelier [2] deduced an option pricing formula based on the assumption that stock prices follow a Brownian motion with zero drift.

Bachelier assumed that the stock price S is a random variable, that price changes are independent and identically distributed, and that

$$\Pr(S_{t+T} \leq S^* \mid S_t = x) = F(S^* - x, T), \tag{1}$$

where F is the distribution function of the stock price changes.

Bachelier also incorrectly deduced that (1) implies the following,

$$F(S^* - x, T) = \Phi\left(\frac{S^* - (x + \mu T)}{\sigma\sqrt{T}}\right), \tag{2}$$

where μ is the mean expected price change per time period, σ^2 is the variance per time period, and Φ is the standard normal distribution. However, (1) is not sufficient to deduce (2), since it is known that any stable Paretian family of distributions satisfies (1). To deduce (2) from (1), one must assume that the variance is finite. In addition, as T tends to infinity, $\Pr(S_T < S^*)$ tends to 1/2 for every S^*. Since nothing in this formulation restricts S^* to the positive numbers, there is a positive probability of negative stock prices, which is a violation of the property of limited liability.

Bachelier also assumed that the mean expected price change per unit time μ

is zero, since he viewed stock market transaction as a gamble, and felt that competition would reduce the expected return to zero. This is unsuitable as an equilibrium specification, since it seems to deny both positive interest rates and risk aversion.

Applying the same logic to call option pricing, Bachelier claimed that the current call price, c, was equal to the expected final call price, c^*. Since $c^* = c(S^*, 0; X) = (S^* - X)^+$ (where $f^+ = max(f, 0)$), the model implies that

$$c = E(c^*) \equiv \int_X^\infty (S^* - X)\Phi'(S^*)dS^*, \tag{3}$$

where Φ' is the normal density function. Changing variables,

$$c = \int_{(X-S)/\sigma\sqrt{T}}^\infty (W\sigma\sqrt{T} + S - X)\Phi'(W)dW, \tag{4}$$

where $W \equiv (S^* - S)/\sigma\sqrt{T}$, and so

$$c = S\cdot\Phi\left(\frac{S-X}{\sigma\sqrt{T}}\right) - X\cdot\Phi\left(\frac{S-X}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}\Phi'\left(\frac{S-X}{\sigma\sqrt{T}}\right), \tag{5}$$

Note that as the time to expiration is increased, the call price increases without bound. This implication seems to violate the restriction that the maximum value which the call price can assume be equal to the stock price. This result arises because Bachelier (implicitly) did not assume that stock possesses limited liability.

The major problems with Bachelier's model are

1. The assumption of Gaussian Brownian Motion in the description of expected price movement implies both a nonzero probability of negative prices for the security, and option prices greater than their respective security prices for large T.

2. The assumption that the mean expected price change is zero, which suggests no time preference or risk neutrality.

3. The implicit assumption that the variance is finite, thereby ruling out other members of the stable Paretian family except the normal.

1.1 Why study options?

Since the option is a particularly simple type of the contingent claim asset, a theory of option pricing may lead to a general theory of contingent claims pricing. The development of an option pricing theory is also an intermediate step toward a unified theory about the pricing of a firm's liabilities, the term and risk structure of interest rates, and the theory of speculative markets.

DEFINITION. An *American warrant* is a security, issued by a company, giving its owner the right to purchase a share of stock at a given exercise price on or before a given date. An *American call option* has the same terms as the warrant except that it is issued by an individual instead of a company. An *American put option* gives its owner the right to sell a share of stock at a given exercise price on or before a given date. A *European option* has the same terms as its American counterpart except that it cannot be exercised before the last date of the contract.

Samuelson [45] showed that the American and the European options may not have the same value. The contracts may differ with respect to other provisions such as antidilution clauses, exercise price changes, etc.

The following notation will be used throughout the thesis:

- t = current time.
- t^* = expiration date.
- $T = t^* - t$ = time to expiration.
- r = risk-free interest rate.
- X = exercise (strike) price.
- S = stock price at time t.
- S_c = the critical stock price = the price at which the put should be exercised.
- σ = standard deviation of the stock price processes.
- P = price of the American put at time t.
- C = price of the American call at time t.
- p = price of the European put at time t.

- c = price of European call at time t.
- $M = 2r/\sigma^2$.

Merton [42] deduces a set of restrictions on option pricing formulae from the the assumption that investors prefer more to less (a necessary condition for the formula to be consistent with a rational pricing theory).

DEFINITION. Security A is *dominant* over security B, if, the return on A is never less than the return on B, and is greater than that on B at some known date in the future.

Note that in perfect markets with no transaction costs and the ability to borrow and short-sell without restriction, the existence of dominated securities would be equivalent to the existence of an arbitrage situation. Thus, assuming that investors prefer more wealth to less, any investor willing to purchase B would prefer A.

Assumption 1 *A necessary condition for a rational option pricing theory is that the option be priced such that it is neither a dominant nor a dominated security.*

THE EUROPEAN PUT OPTION PRICE

The put option has received relatively little analysis in the literature because it is a less popular option than the call and, because it is commonly believed (e.g., Black-Scholes [5]) that given the price of a call option and the common stock, the value of a put is uniquely determined. This belief is false for American put options, and the mathematics of put option pricing is more difficult than that of the corresponding call option.

Let $B(t^*)$ be the price of a riskless, discounted loan (or bond), which pays \$1, t^* years from now, and $B'(t^*)$ the current value of \$1 payable t^* years from now at the borrowing rate of a t^*-year, noncallable, discounted loan. Since it is assumed that current and future interest rates are positive, it follows that $1 = B(0) > B(t_1^*) > B(t_2^*) > \ldots > B(t_n^*)$ whenever $0 < t_1^* < t_2^* < \ldots < t_n^*$. To avoid arbitrage, $B'(t^*) \leq B(t^*)$.

At the expiration of a put option, we have

$$P(S, 0; X) = p(S, 0; X) = (X - S)^+. \tag{6}$$

To determine the rational European put option price, consider two portfolio positions. First, a long position in the common stock at S dollars, a long position in a t^*-year European put at $p(S, t^*; X)$ dollars, and borrowing $XB'(t^*)$. The value of the portfolio t^* years from now with the stock price at S^* will be: $S^* + (X - S^*) - X = 0$, if $S^* \leq X$, and $S^* + 0 - X = S^* - X$ if $S^* > X$. The pay-off structure is identical in every respect to a European call option with the same exercise price and duration. Hence to stop the call option from being a dominated security, the put and the call must be priced so that

$$p(S, t^*; X) + S - XB'(t^*) \geq c(S, t^*; X). \tag{7}$$

The values of the portfolio prior to expiration are not computed because the call option is European and cannot be exercised prematurely .

Next, consider taking a long position in a t^*-year European call, a short position in the common stock price at price S, and lending $XB(t^*)$ dollars. The value of the portfolio t^* years from now with the stock price at S^* will be: $0 - S^* + X = X - S^*$, if $S^* \leq X$, and $(S^* - X) - S^* + X = 0$ if $S^* > X$. The pay-off structure is identical in every respect to a European put option with the same exercise price and duration. If the put is not be a dominated security, then

$$c(S, t^*; X) - S + XB(t^*) \geq p(S, t^*; X), \tag{8}$$

must hold.

Theorem 1 *If Assumption 1 holds and if the borrowing and lending rates are equal (i.e., $B(t^*) = B'(t^*)$), then*

$$p(S, t^*; X) = c(S, t^*; X) - S + XB(t^*).$$

PROOF. The proof follows directly from the simultaneous application of (7) and (8), using $B(t^*) = B'(t^*)$. □

Thus the value of a rationally priced European put option is determined once one has a rational theory of the call option value. Note that no distributional assumptions about the stock price or future interest rates are required to prove Theorem 1.

Two corollaries to Theorem 1 follow directly from the above analysis.

Corollary 1 $p(S,t^*;X) \leq XB(t^*)$.

PROOF. Since $S \geq C(S,t^*;X) \geq c(S,t^*;X)$, we have $c - S \leq 0$, so the result follows from (8). □

Corollary 2 *The value of a perpetual (i.e., $t^* = \infty$) European put option is zero.*

PROOF. The put is a limited liability security ($p(S,t^*;X) \geq 0$). From Corollary 1 and the condition that $B(\infty) = 0$, one obtains $0 \geq p(S,\infty;X)$. □

Since the American put option can be exercised at any time, its price must satisfy the arbitrage condition

$$P \geq (X - S)^+ \tag{9}$$

and it can be shown that

$$P(S,t^*;X) \geq p(S,t^*;X) \tag{10}$$

where the strict inequality holds only if there is a positive probability of exercising early.

Unlike a European option, the value of an American option is always a nondecreasing function of its expiration date. If there is no possibility of exercising prematurely, then the value of the American option will be equal to its European counterpart. Also, if the strike price is constant and no dividends are paid, then the European and American call options will have the same value. Even under these assumptions, there is usually a positive probability of prematurely exercising an American put, so that the American put will sell for more than its European counterpart.

Theorem 2 *If, for some $t < t^*$, there is positive probability that $c(S, t; X) < X(1 - B(t))$, then there is a postive probability that a t^*-year American put option will be exercised prematurely, and the value of the American put will strictly exceed the value of its European counterpart.*

PROOF. The only reason that an American put will sell for a premium over its European counterpart is because there is a positive probability of exercising prior to expiration. Hence, it is sufficient to prove that $p(S, t^*; X) < P(S, t^*; X)$. By Assumption 1, if for some $t \le t^*$, $p(S^*, t; X) < P(S^*, t; X)$ for some value of S^*, then $p(S, t^*; X) < P(S, t^*; X)$. By assumption, there exists S^* such that $c(S^*, t; X) < X(1 - B(t))$. Then Theorem 1 implies that $p(S^*, t^*; X) = c(S^*, t; X) - S^* + XB(t) < X - XB(t) - S^* + XB(t) = X - S^* \le (X - S^*)^+ \le P(S^*, t; X)$. □

Since there is almost always a chance of exercising early, the formula in Theorem 1 will not lead to a correct valuation of an American put since the valuation of such options is analytically more difficult than its European counterpart.

1.2 The Black-Scholes Model

A number of option pricing theories satisfy the general restrictions on a rational theory. One such theory developed by Black and Scholes [5] is particularly attractive because it is a complete general equilibrium formulation of the problem and because the final formula is a function of observable variables, making the model subject to direct empirical tests.

The assumptions of Black-Scholes model are as follows:

1. The risk-free interest rate, r, is known and constant through time.

2. The stock price follows a random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is lognormal. The variance rate of the return on the stock is contant.

3. No dividends are paid, and the exercise price does not change over the life of the contract.

4. The option is European so that it can only be exercised at maturity.

5. There are no transaction costs in buying or selling the stock or the option and no taxes.

6. It is possible to borrow any fraction of the price of a security to buy it or to hold it, at the risk-free interest rate.

7. There are no penalties to short selling. A seller who does not own a security will simply accept the price of the security from a buyer, and will agree to settle with the buyer on some future date by paying her an amount equal to the price of the security on that date.

Black and Scholes used Samuelson's theory of warrant pricing [45] to express the expected return on the option in terms of the option price function and its partial derivatives (possibly since the option price is a function of the common stock price). From the equilibrium conditions on option yield, a partial differential equation (see (23)) for the option price is derived. The solution of this equation for the European call option is

$$c(S, t^*; X) = S\Phi(d_1) - Xe^{-rt^*}\Phi(d_2), \tag{11}$$

where

$$d_1 = [ln(S/X) + (r + \sigma^2/2)T]/(\sigma\sqrt{T}) \tag{12}$$

and $d_2 = d_1 - (\sigma\sqrt{T})$.

An exact formula for an asset price, based on observable variables only, is a rare finding from a general equilibrium model.

The most noticeable property of (11) is the variables that it does not depend on. The option price does not depend on the expected return of the common stock, risk preferences of investors, or on the aggregate supply of assets. It does depend on the rate of interest and the total variance of the return on the common stock (which is often a stable number, so accurate estimates are possible from time series data).

The Samuelson and Merton [47] model is a complete, although very simple (three assets and one investor), general equilibrium formulation. Their formula is

$$c(S, t^*; X) = e^{-rt^*} \int_{X/S}^{\infty} (ZS - X) dQ(Z; t^*), \tag{13}$$

where dQ is a probability density function with $E_Q(Z) = e^{rt^*}$. Equations (11) and (13) will be the same only when dQ is a lognormal density and $Var(\log(Z)) = \sigma^2 t^*$. Note that this occurs only if

1. the objective returns on the stock are lognormally distributed.
2. the investor's utility function is iso-elastic.

However, dQ is a risk-adjusted distribution, dependent on both risk preferences and aggregate supplies, while the distribution in (11) is the objective distribution of returns on the common stock. Black and Scholes claim that one reason that Samuelson and Merton [47] did not arrive at formula (11) was because they did not consider other assets, although a result which does not obtain for a simple, three asset, case is unlikely to hold in a more general example. In fact, it is only necessary to consider three assets to derive the Black-Scholes formula. (Black and Scholes also provide an alternative derivation of their equation using the capital asset pricing model.)

The Black and Scholes derivation of (11) is intuitively appealing, but a rigorous derivation can also be provided, particularly to gain an insight into sufficient conditions for the formula to obtain. Note that because Black and Scholes consider only terminal boundary conditions, their analysis is only applicable to European options, even though the European valuation is often equal to the American option. Finally, although their model is based on a different economic structure, the formal analytical content is identical to Samuelson's [45] linear $\alpha = \beta$ model when the returns on the common stock are lognormal. Hence, with a different interpretation of the parameters, theorems proved in Samuelson [45] and McKean [39] are directly applicable to the Black-Scholes models and vice versa.

1.3 Alternative derivation of the Black-Scholes model

Initially, we consider the case of a European option where no payouts are made to the common stock over the life of the contract. We make the following further assumptions.

Assumption 2 *"Frictionless" market: There are no transaction costs or differential taxes. Trading takes place continuously and borrowing and short-selling are allowed without restriction, and the borrowing rate equals the lending rate.*

Assumption 3 *Stock price dynamics: The instantaneous return on the common stock is described by the stochastic differential equation*

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{14}$$

where μ is the instantaneous expected return on the common stock, σ^2 is the instantanous variance of the return and dW is a standard Gauss-Wiener process. μ is a stochastic variable which may even be dependent on the level of the stock price or other assets' returns. σ is a non-stochastic, known function of time.

No assumption is made that dS/S be an independent increment process or stationary, although dW clearly is.

Assumption 4 *Bond price dynamics: With $B(t^*)$ as before,*

$$\frac{dB}{B} = \alpha(t^*)dt + \delta(t^*)dq(t,t^*) \tag{15}$$

where α is the instantaneous expected return, δ^2 is the instantaneous variance, and $dq(t,t^*)$ is a standard Gauss-Wiener process for maturity t^*. It is not assumed that dq for one maturity is perfectly correlated with the dq for another, i.e.,

$$dq(t;t^*)dq(t;u) = \rho_{t^*u}dt, \tag{16}$$

where ρ_{t^*u} may be less than 1 for $t^* \neq u$. However, it is assumed that there is no serial correlation (Cootner, [8]) among the (unanticipated) returns on the assets, i.e.,

$$dq(s;t^*)dq(t;u) = 0 \quad \text{for } s \neq t$$
$$dq(s;t^*)dW(t) = 0 \quad \text{for } s \neq t, \tag{17}$$

(consistent with the general efficient market hypothesis of Fama [14] and Samuelson [46]). $\alpha(t^*)$ may be stochastic, and different for different maturities. Because $B(t^*)$ is the price of a discounted loan with no risk of default, $B(0) = 1$ with certainty, and $\delta(0) = 0$. However, δ is non-stochastic and independent of B. In the special case when the interest rate is constant over time, $\delta \equiv 0$, $\alpha = r$, and $B(t^*) = e^{-rt^*}$.

Assumption 5 *Investor preference and expectations: Apart from Assumption 1, it is assumed that all investors agree on the values of σ and δ, and on the distributions of dW and dq. It is not assumed that they agree on either μ or α.*

It is reasonable to assume that the option price is a function of the stock price, the riskless bond price, and the length of time to expiration. If $H(S, B, t^*; X)$ is the option price function, then, given the distributions of S and B, we have, by Itô's lemma, that the change in the option price over time satisfies the stochastic differential equation

$$\begin{aligned} dH &= \frac{\partial H}{\partial S}dS + \frac{\partial H}{\partial B}dB + \frac{\partial H}{\partial t^*}dt^* \\ &\quad + \frac{1}{2}\left[\frac{\partial^2 H}{\partial S^2}(dS)^2 + 2\frac{\partial^2 H}{\partial S \partial B}(dSdB) + \frac{\partial^2 H}{\partial B^2}(dB)^2\right], \end{aligned} \tag{18}$$

where $(dS)^2 \equiv \sigma^2 S^2 dt$, $(dB)^2 \equiv \delta^2 B^2 dt$, $dt^* = -dt$, and $(dSdB) \equiv \rho\sigma\delta SBdt$ with ρ the instantaneous correlation coefficient between the (unanticipated) return on the stock and on the bond. Substituting (14) and (15) and rearranging terms, (18) becomes

$$dH = \beta H dt + \gamma H dW + \eta H dq, \tag{19}$$

where $\beta = [\sigma^2 S^2(\partial^2 H/\partial S^2)/2 + \rho\sigma\delta SB(\partial^2 H/\partial S\partial B) + \delta^2 B^2(\partial^2 H/\partial B^2)/2 + \alpha B(\partial H/\partial B) + \mu S(\partial H/\partial S) - (\partial H/\partial t^*)]/H$, $\gamma = \sigma S(\partial H/\partial S)/H$, and $\eta = \delta B(\partial H/\partial B)/H$. Denoting by u_1(resp. u_2, u_3) the amounts invested in the common stock (resp., option, bonds), it can be shown (Merton [40] or [41]) that the instantaneous return on investment is

$$[(\mu - \alpha)u_1 + (\beta - \alpha)u_2]dt + [\sigma u_1 + \gamma u_2]dW + [\eta u_2 - \delta(u_1 + u_2)]dq. \tag{20}$$

To make the above return non-stochastic, the coefficients of dW and dq in (20) must be zero. To avoid arbitrage, the coefficient of dt must then be zero. This gives a system of three equations in u_1 and u_2, which will have a non-zero solution if and only if

$$\begin{aligned} \sigma(\beta - \alpha) &= \gamma(\mu - \alpha) \\ \delta\gamma &= \sigma(\delta - \eta). \end{aligned} \tag{21}$$

When r is constant, we have $\delta = 0$, $\alpha = r$, and $B(t^*) = e^{-rt^*}$. Thus (21) reduces to

$$\beta - r = \frac{\gamma}{\sigma}(\mu - r). \tag{22}$$

Making use of (22) and the fact $B(t^*)$ is non-stochastic for this model of the American put, we obtain

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP - \frac{\partial P}{\partial T} = 0, \tag{23}$$

subject to $P(\infty, t^*, X) = 0$, $P(S, 0, X) = (X - S)^+$, and $P(S, t^*, X) \geq (X - S)^+$.

For $t^* = \infty$, equation (23) is an ordinary differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 P}{dS^2} + rS\frac{dP}{dS} - rP = 0, \tag{24}$$

valid for all $S \geq S_c$, with the boundary conditions

$$P(\infty, \infty, X) = 0,$$

$$P(S_c, \infty, X) = X - S_c,$$

and S_c chosen so as to maximize the return. It is elementary that the solution to (24), subject to the above boundary conditions, is

$$P(S, \infty, X) = (X - S_c)(S/S_c)^{-M}. \tag{25}$$

The value of S_c which maximizes (25) is easily found to be $S_c = MX/(1+M)$, whence (25) becomes

$$P(S, \infty, X) = \frac{X}{1+M}\left(\frac{(1+M)S}{MX}\right)^{-M}. \tag{26}$$

2 Optimal Stopping and the American Put

An American put option provides *the right, but not the obligation, to sell for price X a unit of stock at any time up to the time horizon T*, the original expiry date of the option. To simplify the problem, we follow the Black-Scholes model by assuming that the stock pays no dividends during the lifetime of the option and that the stock price S_t at time t is an exponential Brownian motion:

$$S_t = S_0 e^{(\sigma W_t + (\mu - \sigma^2/2)t)} \tag{27}$$

where $(W_t)_{t \geq 0}$ is a canonical Brownian motion, S_0 $(= x)$ is the fixed initial value, μ is the expected rate of return on the stock, and σ is the variance of the return of the stock. This price S_t is the unique (strong) solution to the stochastic differential equation

$$dS_t = S_t(\sigma dW_t + \mu dt). \tag{28}$$

We also assume that interest on cash is earned at a fixed rate $r > 0$.

In 1991, Jacka established the following basic results: (i) The fair price for an option is a function of the present stock price and the time horizon only, and is the unique solution to a parabolic free boundary problem. (ii) The option price is the solution to an optimal stopping problem which is a generalization of the Black-Scholes option pricing formula. McKean [39] and Van Moerbeke [49] had earlier derived a result equivalent to (i), and parallel results were established by Kim [31], Carr, Jarrow and Myneni [7], and Jacka and Lynn [23].

2.1 The Optimal Stopping Problem

We consider the problem of optimally stopping $e^{-rt}S_t$ before time T, that is, we find

$$f(x,t) \stackrel{\text{def}}{=} \sup_{\tau \leq t} E_x[e^{-r\tau}(X - S_\tau)^+] \tag{29}$$

with τ ranging over stopping times. Let P be the (martingale) measure with respect to which the discounted stock price $e^{-rt}S_t$ is a martingale. It will be shown (Theorem 8) that there is an exercise boundary $b(t) = S_c$ and that we exercise the option the first time that S_s, the price of the underlying stock, falls below the (lower) boundary $b(t)$ at $t = T - s$.

It is well known that under P, S_t satisfies the stochastic differential equation

$$S_t(x) = x + \int_0^t \sigma S_s(x) dW_s + \int_0^t r S_s(x) ds \tag{30}$$

where W is the P-Brownian motion such that

$$S_t = x e^{(\sigma W_t + (r - \sigma^2/2)t)}. \tag{31}$$

Let $g(x) = (X - x)^+$. Then one has the following:

Theorem 3 *The payoff function V_T for the stopping problem is a function of the present price of the stock and of the time to the expiry of the option only. That is,*

$$V_T = f(S_0, T). \tag{32}$$

The function f is continuous, $f(x,t) \geq g(x)$, and the optimal stopping time τ^* is

$$\tau^* = \inf\ \{s : f(S_s, T - s) = g(x)\} = \inf\ \{t \geq 0 : (S_t, T - t) \notin D\} \tag{33}$$

where

$$D = \{(x,t) \in \Re^+ \times \Re^+ : f(x,t) > g(x)\}$$

is the continuation region for the stopping problem.

PROOF. The results follow from Krylov's theorems (Appendix, Theorems 9 and 10) upon letting $w(s,x) = f(x,t)$ in Krylov's notation. □

Further, one knows the following about the structure of the region D:

Proposition 1 *For each $t > 0$, the t-section of D has the form*

$$D_t \equiv \{x : (x,t) \in D\} = (b(t), \infty)$$

for some $b(t)$ satisfying $X > b(t) > 0$.

PROOF. We know that zero is not in D_t, so we only need to show that for any $y > x$, $(x \in D_t) \implies (y \in D_t)$. We show this by using a pathwise comparison based on $S_t = xe^{(\sigma W_t + (r - \sigma^2/2)t)}$. Suppose $x \in D_t$, $y > x$, and let

$\tau = \inf(s \geq 0 : (S_s(x), T-s) \in D)$. (That is, τ is a stopping time for $S_s(x)$.) Note that τ is bounded. Then

$$\begin{aligned}
&f(y,t) - f(x,t) \\
&\quad \equiv f(y,t) - E[e^{-r\tau}(X - S_\tau(x))^+] \\
&\quad \geq E[e^{-r\tau}((X - S_\tau(y))^+ - (X - S_\tau(x))^+)] \\
&\quad = Ee^{-r\tau}[(X - S_\tau(y)) - (X - S_\tau(x)) \\
&\quad\quad + (X - S_\tau(y))^- - (X - S_\tau(x))^-]. \qquad (34)
\end{aligned}$$

Now $S_\tau(y) > S_\tau(x)$ since $y > x$, so the last term of (34) is non-negative. Thus

$$\begin{aligned}
f(y,t) - f(x,t) &\geq E[e^{-r\tau}(S_\tau(x) - S_\tau(y))] \\
&= (x-y)E[e^{(\sigma W_\tau - \sigma^2\tau/2)}] = (x-y),
\end{aligned}$$

since $e^{(\sigma W_t - \sigma^2 t/2)}$ is a martingale starting at 1 and τ is a bounded stopping time. Therefore,

$$f(y,t) \geq f(x,t) + (x-y).$$

If $f(x,t) > (X-x)^+$, then

$$f(y,t) > (X-x)^+ + (x-y) \geq (X-x) + (x-y) = (X-y).$$

Since $f(y,t) > 0$, for all $t > 0$ and all $y \geq 0$ (Appendix, Lemma 3), we have $f(y,t) > g(y)$, so $(y,t) \in D$, as required. □

Note that $b(t) \leq X$ for all $t \geq 0$, since if $x > X$, $(X-x)^+ = 0$ but $f(x,t) > 0$. It can be shown (see Appendix, Propositions 3 and 4) that b is a continuous, decreasing function of t.

The limiting behaviour of b and f is given by:

Proposition 2 *Let (F,b) be the unique solution, with $F \in C^1$, to*

$$\begin{aligned} \sigma^2 x^2 F''(x)/2 + rxF'(x) - rF = & \quad 0 \quad \text{if } x \geq b \\ F(x) = & \quad X - x \quad \text{if } x < b \end{aligned}$$

Then

(i) $b = [MX/(1+M)]$,

$$F(x) = \begin{cases} (X-b)(b/x)^M & \text{if } x \geq b \\ (X-x) & \text{if } x < b \end{cases}$$

where $M = 2r/\sigma^2$ *and* $F(x) = \sup_\tau \ E_x[e^{-r\tau} g(S_\tau)]$.

(ii) $\lim_{t\to\infty} f(x,t) = F(x)$.

(iii) $\lim_{t\to\infty} b(t) = b$.

PROOF. To begin with, observe that f is decreasing in x, as is immediate from its definition. Also, f is increasing in t, since if τ is admissible for the optimal stopping problem with horizon t, it is also admissible for the problem with horizon s, for any $s \geq t$.

(i) Trivial (see, e.g., Merton [42] or Karatzas and Shreve, [30]).

(ii) It clear that $f(x,t) \leq F(x)\ \forall t$. Conversely, let $\tau = \inf\{t : S_t \leq b\}$. Then $F(x) = E[e^{-r\tau} g(S_\tau)]$. If we let $\tau_t = \tau \wedge t$, then for $x > b$,

$$
\begin{aligned}
f(x,t) &\geq E_x\left[e^{-r\tau_t}g(S_{\tau_t})\right] \\
&\geq (X-b)E_x\left[e^{-r\tau_t}\mathbf{1}_{\tau\leq t}\right] \\
&\longrightarrow (X-b)E_x[e^{-r\tau}] \equiv F(x)
\end{aligned}
$$

as $t \to \infty$, since $e^{-r\tau}$ is a bounded continuous function on $\Re^+$, and $\tau_t \to \tau$. On the other hand, for $x \leq b$, we have $f(x,t) \geq g(x) = F(x)$, so $\lim_{t\to\infty} f(x,t) = F(x)$.

(iii) Given $x > b$, we have $F(x) > g(x)$. Let $\varepsilon = F(x) - g(x)$, and take $t(\varepsilon)$ such that

$$F(x) - f(x,t) \leq \varepsilon/2$$

for all $t \geq t(\varepsilon)$. Then $f(x,t) > g(x)$ for $t \geq t(\varepsilon)$, so $(x,t) \in D$, whence $b(t) \leq x$. But this is for all $x > b$, so $\limsup_{t\to\infty} b(t) \leq b$. Similarly, if $x < b$, then $g(x) = F(x) \geq f(x,t)$, so $x \leq b(t)$ for all t, and so $b \leq \liminf_{t\to\infty} b(t)$. The result follows. □

Jacka's main theorem is the following.

Theorem 4 *The pair (f,b) is a solution pair (φ,h) to the free boundary problem*

$L\varphi = 0,\ x > h(t)$

$\varphi(h(t),t) = X - h(t)$

$\frac{\partial\varphi}{\partial x}(h(t),t) = -1$

$\varphi(x,0) \to g(x),\ x \geq h(0)$

$\varphi(x,t) \to 0$ uniformly on compact sets as $x \to \infty$

and f is maximal.

Before proceeding with the proof of the above theorem, we need several lemmas. Recall that a function on a metric space X is Hölder continuous if, $\forall x \in X$, there exist constants $C_x > 0$ and $0 < \gamma_x \leq 1$ and an open neighbourhood $\mathcal{V}$ of x such that $\forall x_1, x_2 \in \mathcal{V}$, $d(f(x_1), f(x_2)) \leq C_x d(x_1, x_2)^{\gamma_x}$.

Lemma 1 *f is C^1 in x.*

PROOF. Define

$$M_s = M_s^{t,f} = e^{-rs} f(S_s, t-s) + rX \int_0^s e^{-ru} \mathbf{1}_{S_u \leq b(t-u)} du. \tag{35}$$

Since $\partial D = \{b(s,s); s \geq 0\}$ has zero measure, and the decreasing component of the semimartingale decomposition of $(e^{-rt} g(S_t))$ is absolutely continuous with respect to the Lebesgue measure, it follows from Theorem 12 and Corollary 3 of the Appendix, that $M_s^{t,f}$ is a martingale. Therefore, we can write

$$f(x,t) = M_0 = EM_t = p(x,t) + rX \textstyle\int_0^t e^{-ru} \psi(x, b(t-u), u) du,$$

where $p(x,t) = E(e^{-rt}(X - S_t)^+)$ (price of an European put option) and $\psi(x,y,t) = \Pr(S_t(x) \leq y)$. Then we can rewrite $\partial f / \partial x = f_x(x,t)$ as

$$f_x(x,t) = rX \int_0^t e^{-rs} \psi_x(x, b(t-s), s) ds + p_x(x,t)$$

(Lebesgue a.e.).

Define $u(x,t) = \int_0^t e^{-rs}\psi_x(x, b(t-s), s)ds$. We need to show that $u(x,t)$ is continuous. Since b is bounded and Borel measurable, it is sufficient to show that for any Borel measurable function k, the function

$$d(x,t) = \int_0^t e^{-rs} n(x - k(s), \sigma^2 s)ds$$

where $n(x,v)$ is the normal density at x with mean zero and variance v, is continuous in x. Clearly, for any $0 \le \delta \le t$, we have $| n_x | \le (e/2\pi)^{1/2}/(\sigma^2 t)$ and $0 < n < 1/(2\pi\sigma^2 t)^{1/2}$, so

$$\begin{aligned}
&| d(x,t) - d(y,t) | \\
&= \left| \int_0^t e^{-rs} n(x - k(s), \sigma^2 s)ds - \int_0^t e^{-rs} n(y - k(s), \sigma^2 s)ds \right| \\
&\le \int_0^t | e^{-rs} n(x - k(s), \sigma^2 s)ds | + \int_0^t | e^{-rs} n(y - k(s), \sigma^2 s)ds | \\
&\le \int_0^\delta ds/\sigma(2\pi s)^{1/2} + \int_\delta^t | x - y | (e/2\pi)^{1/2}/\sigma^2 t ds \\
&= (2\sqrt{\delta})/(\sigma\sqrt{2\pi}) + | x - y | (e/2\pi)^{1/2} \ln(t/\delta)/\sigma^2.
\end{aligned}$$

In particular, letting $\delta = (x-y)^2$, we get

$$\begin{aligned}
&| d(x,t) - d(y,t) | \\
&\le 2 | x - y | /(\sigma\sqrt{2\pi}) + | x - y | (e/2\pi)^{1/2}/\sigma^2(\ln(t/ | x - y |^2)) \\
&= | x - y | [(2/\sigma^2)(e/2\pi)^{1/2}\{\sigma e^{-1/2} + (1/2)\ln t - \ln | x - y |\}] \\
&\le K_t | x - y | (1 + | \ln | x - y | |).
\end{aligned}$$

Therefore, $| d(x,t) - d(y,t) | \leq K_t | x - y | (1 + | \ln | x - y | |)$, so d is locally Hölder continuous. Thus, f is C^1. □

Lemma 2 *Let $\mathcal{L} = \sigma^2\ x^2/2\ \partial^2/\partial x^2 + rx\partial/\partial x - r - \partial/\partial t$ (a parabolic operator). Then f is the unique solution φ of the initial value problem*

$\mathcal{L}\varphi = 0,\ x > b(t)$

$\varphi(b(t), t) = X - b(t)$, *where* $t \geq 0$

$\varphi(x, 0) = g(x)$

$\varphi(x,t) \to 0$ *as* $x \to \infty$, *uniformly on compact sets.*

PROOF. Take $(x,t) \in D$. D is open so we can take an open rectangle $R = (x_1, x_2) \times (t_1, t_2)$ with $(x,t) \in R \subseteq D$. Consider the initial value problem $\mathcal{L}\varphi = 0$ in R and $\varphi(x,t) = f(x,t)$ on $\partial R \setminus (x_1, x_2) \times \{t_2\}$. The existence and uniqueness of a solution φ to this problem follows from Theorem 11 of the Appendix. Next, let

$$N_s^t = e^{-rs}\varphi(S_s(x), t - s).$$

It follows from Itô's formula that $N_{s\wedge\tau}$, where $\tau = \inf\{s \geq 0 : (S_s, t-s) \notin R\}$, is a bounded martingale, so that,

$$\varphi(x,t) = N_0^t = EN_\tau^t = Ee^{-r\tau} f(S_\tau, \tau - t) = f(x,t),$$

since τ is bounded from above by the first exit time of $(S_s(x), t - s)$ from D. Therefore f satisfies $\mathcal{L}f = 0$ in D.

The boundary conditions are also satisfied by f. The fact that $f(x,t) \to 0$ as $x \to \infty$ uniformly on compact sets follows, since $f(x,t) \to 0$ as $x \to \infty$

for each t ($0 < f(x,t) \leq X \Pr(\tau_x S.(x) \leq t)$ for $x > X$, and $\tau_X S.(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ a.s.) and f is increasing in t and decreasing in x, as noted at the beginning of the proof of Proposition 2. As for uniqueness, for any T, φ is bounded on $\Re^+ \times [0,T]$ and $\varphi(x,t) = (X - x)$ for $x \leq b(t)$. Define

$$M_t^T = e^{-r(t\wedge\tau)}\varphi(S_{t\wedge\tau}, T - (t \wedge \tau))$$

where $\tau = \inf\{s \geq 0 : S_s = b(s)\}$ and $S_0 = x \geq b(T)$. By Itô's formula, M_t^T is a bounded martingale, so we have

$$M_0^T \equiv \varphi(x,T) = EM_T^T = Ee^{-r(T\wedge\tau)}g(S_{T\wedge\tau}) = f(x,T),$$

for all $T \in \Re^+$, since we know $T \wedge \tau$ is optimal for the T-horizon problem, as required. □

Proof of Theorem 4. That (f, b) satisfies the conditions follows immediately from Lemmata 1 and 2. It remains to show that f is maximal. Suppose (φ, h) is another solution pair. By the same argument as in Lemma 2, we get $\varphi(x,t) \leq E(e^{-r(t\wedge\tau_h)}g(S_{t\wedge\tau_h})$ since $g(x) \geq (X - x)$ and $\tau_h = \inf\{s \geq 0 : S_s = h(t - s)\}$. Thus $\varphi(x,t) \leq f(x,t) = \sup_{\tau \leq t} Ee^{-r\tau}g(S_\tau)$, as required. □

2.2 Pricing the Option

The problem is to find a fair price for an American put option.

Theorem 5 *Let* $dS'_t = \sigma S'_t dW_t + rS'_t dt$ *with* $S'_0 = S_0$ *and let* $f(x,t) = \sup_{\tau \le t} E_x[e^{-r\tau}(X - S'_\tau)^+]$ *be the optimal payoff from optimally stopping the process* $e^{-rs}(X - S_s)^+$. *Then there is a self-financing portfolio of stock and cash,* $((Y_s, Z_s); 0 \le s \le t)$ *with* Y *and* Z *adapted, and whose value* V_s *at time* s *satisfies* $V_s \equiv (S_s Y_s + Z_s) \ge (X - S_s)^+$ *and such that if* τ^* *is the optimal stopping time corresponding to the payoff function* $f(S_0, t)$, *then* $V_{\tau^*} = (X - S_{\tau^*})^+$. *Therefore* $S_0 Y_0 + Z_0 \equiv f(S_0, t)$ *is the fair price for the option.*

That is, if one writes the option and sells it for V_0 and then purchase the portfolio (S_0, Y_0) and manages it in the prescribed manner, then one will not lose money, and will make money unless the option is not rationally used.

PROOF. In the previous section it was shown that the optimal stopping time t is to exercise the option when S'_s falls below $b(t-s)$. Moreover f is piecewise C^2 in x, C^1 in t with

$$d(e^{-rs} f(S'_s, t-s)) = e^{-rs} f_S(S'_s, t-s)\sigma S'_s dW_s - rXe^{-rs}\mathbf{1}_{S'_s \le b(t-s)} ds.$$

So

$$\begin{aligned} df(S'_s, t-s) &= rf(S'_s, t-s)ds + f_S(S'_s, t-s)\sigma S'_s dW_s \\ &\quad - rX\mathbf{1}_{S'_s \le b(t-s)} ds. \end{aligned}$$

Set

$$Y_s = f_S(S'_s, t-s),$$

$$
\begin{aligned}
Z_s &= f(S_s, t-s) - S_s f_S(S_s, t-s) + rXe^{rs}\int_0^s e^{-ru}\mathbf{1}_{S_u \le b(t-u)}du, \\
V_s &= S_s Y_s + Z_s \\
&= S_s f_S(S_s, t-s) + f(S_s, t-s) - S_s f_S(S_s, t-s) \\
&\quad + rXe^{rs}\int_0^s e^{-ru}\mathbf{1}_{S_u \le b(t-u)}du \\
&= f(S_s, t-s) + rXe^{rs}\int_0^s e^{-ru}\mathbf{1}_{S_u \le b(t-u)}du.
\end{aligned}
$$

Then

$$
\begin{aligned}
dV_s &= rf(S_s, t-s)ds + \sigma f_S(S_s, t-s)dW_s \\
&\quad + (\mu - r)f(S_s, t-s)S_s ds + r(e^{rs}rX\int_0^s e^{-ru}\mathbf{1}_{S_u \le b(t-u)}du)ds \\
&= Y_s dS_s + rZ_s ds.
\end{aligned}
$$

Since Y_s, S_s, and Z_s are continuous, V is the value of a self-financing portfolio, so $dV_s = Y_{S^-}dS_s + rZ_{S^-}dS_s$ is the instantaneous return of a portfolio of Y_{S^-} shares and Z_{S^-} cash. By $V_s \equiv S_s Y_s + Z_s$, we know that $V_t \ge (X - S_t)^+$ since $f(x,t) \ge g(x)$. □

2.3 An Integral Transform of the Stopping Time

Theorem 6 *Let* (φ, h) *be a solution pair to the free boundary problem in Theorem 4. Then*

$$
M_s \equiv M_s^{t,\varphi} = e^{-rs}\varphi(S_s, t-s) + rX\int_0^t e^{-ru}\mathbf{1}_{S_u \le h(t-u)}du, \qquad (36)
$$

is a martingale.

PROOF. By Itô's formula,

$$\begin{aligned} dM_s &= e^{-rs}(\mathcal{L}\varphi)(S_s, t-s)ds + rXe^{-rs}\mathbf{1}_{S_s \le h(t-s)}ds \\ &\quad + e^{-rs}\varphi(S_s, t-s)\sigma S_s dW_s \end{aligned}$$

since φ is C^1, piecewise C^2 in x and piecewise C^1 in t. But

$$\mathcal{L}\varphi(x, s) = \begin{cases} 0 & \text{if } x > h(s) \\ -rX & \text{if } x < h(s) \end{cases}$$

so M is a local martingale. Since $\varphi(x, u) \to 0$ as $x \to \infty$ uniformly for u in a compact set, and φ is C^1 in x, φ is bounded on $\Re^+ \times [0, t]$, so M is a martingale. □

Theorem 7 *The pair (f, b) is the unique solution pair (φ, h) to the conditions of Theorem 4 (with h Lebesgue measurable), satisfying $h(t) > 0$ infinitely often.*

PROOF. We know that (f, b) satisfies the conditions of Theorem 4. Now suppose (φ, h) is another solution pair to with $h(t) > 0$ infinitely often. Define

$$M_s^0 = e^{-rs}f(S_s(x), t-s) + rX \int_0^s e^{-ru}\mathbf{1}_{S_u \le b(t-u)}du$$

$$M_s^1 = e^{-rs}\varphi(S_s(x), t-s) + rX \int_0^s e^{-ru}\mathbf{1}_{S_u \le h(t-u)}du.$$

For a fixed $t \ge 0$, consider $x < b(\infty) \wedge h(t) \le b(t) \wedge h(t)$. The last inequality holds since b is bounded by X and b is decreasing in t (see Appendix, Proposition 3). Thus, $x < b(t) \wedge h(t)$, $f(x,t) = \varphi(x,t) = g(x)$. Define $M = M^0 - M^1$ with $M_0 = 0$. Define $\tau = \inf\{s \geq 0 : S_u > b(t-u)\} \wedge t$. Then $EM_{\tau\wedge s} = M_0 = 0$, but $M_{\tau\wedge s} = e^{-r(\tau\wedge s)}[f(S_{\tau\wedge s}(x), t-(\tau\wedge s)) - \varphi(S_{\tau\wedge s}(x), t-(\tau\wedge s))] + rX\int_0^{\tau\wedge s} e^{-ru}\mathbf{1}_{S_u(x)>h(t-u)}du$. Then the right hand side is positive because f is maximal and $S_u(x) \leq b(t-u)$ on $[0,\tau]$. Therefore, $S_{\cdot\wedge\tau\wedge s}(x) \leq h(t-(\cdot\wedge\tau))$ (*Lebesgue* $\times$ *P*) a.e. and $\varphi(S_{s\wedge\tau}(x), t-s) = f(S_{s\wedge\tau}(x), t-s) = g(S_{s\wedge\tau})$ (P a.s.). Note that $h(s) \leq b(s)$ for $s \leq t$.

Next let

$$M_t = rX\int_0^t e^{-rs}\{\mathbf{1}_{S_s\leq b(t-s)} - \mathbf{1}_{S_s\leq h(t-s)}\}ds,$$

and let $W = \mathbf{1}_{S_s\leq b(t-s)} - \mathbf{1}_{S_s\leq h(t-s)}$. Then W is negative since $\varphi(x,0) = f(x,0) = g(x)$. But $EM_t = M_0 = 0$, so W is zero (*Lebesgue* $\times$ *P*) a.e. . Therefore, $h = b$ a.e. on $[0,t]$. For arbitrary x and $s \leq t$,

$$\varphi(x,s) = M_0^{s,1} = EM_s^{s,1} = M_0^{s,0} = f(x,s),$$

so $\varphi = f$ and hence $h \leq b$ everywhere else on $\{s \in [0,t]\}$ since $f > g$ on D. This implies that $h = b$ since $\mathcal{L}f \neq 0$ on D^c. □

Note that the argument fails if there exist a t such that $h(s) \equiv 0$ $\forall s \geq t$. Also, the theorem indicates that prohibited stopping before the horizon t corresponds to the case where $h(s) = 0$ for all $s > t$.

One also has the following result which was also noted by El Karoui and Karatzas [13].

Theorem 8 *The boundary $b(\cdot)$ is the unique left continuous solution $h(\cdot)$, satisfying $X > h > 0$ for all $t > 0$, of the integral equation*

$$X - x = p(x,t) + rX \int_0^t \psi(x, h(t-s); s)ds \tag{37}$$

for all $x \leq h(t)$, where $p(x,t) = E[e^{-rt}(X - S_t(x))^+]$ is the price of the European option, and $\psi(x,y;t) = \Pr(S_t(x) \leq y)$.

PROOF. Since

$$(M_s^0 : s \leq t) \equiv e^{-rs} f(S_s(x), t-s) + rX \int_0^s e^{-ru} \mathbf{1}_{S_u(x) \leq b(t-u)} du$$

is a martingale, with initial value $(X - x)$ if $x \leq b(t)$, the boundary b satisfies the conditions. Suppose h also satisfies the conditions. Define

$$\varphi(x,t) = p(x,t) + rX \int_0^t e^{-rs} \psi(x, h(t-s), s)ds.$$

Then we can write

$$M_s^1 \equiv e^{-rs} \varphi(S_s(x), t-s) + rX \int_0^s e^{-ru} \mathbf{1}_{S_u(x) \leq h(t-u)} du,$$

which is a $(\sigma W_u : u \leq s)$ martingale for $(x,t) \in \Re_+^2$. Define

$$\tau = \inf\{s \geq 0 : S_s(x) \leq h(t-s)\} \wedge t.$$

For $x \geq h(t)$,

$$\varphi(x,t) \equiv M_0^1 = Ee^{-r\tau} \varphi(S_\tau, t - \tau) = Ee^{-r\tau} g(S_\tau).$$

Then $\varphi \leq f$ if $x \geq h(t)$ and $\varphi = g \leq f$ if $x \leq h(t)$. Now take $0 < x \leq h(t) \wedge b(t)$ and set

$$\theta = \inf\{s \geq 0 : S_s(x) \geq b(t-s)\} \wedge t.$$

Then

$$\begin{aligned}0 &= f(x,t) - \varphi(x,t) \\ &= Ee^{-r\theta}[f(S_\theta, t-\theta) - \varphi(S_\theta, t-\theta)] \\ &+ ErX \int_0^\theta e^{-ru} 1_{S_u > h(t-s)} du.\end{aligned}$$

Both terms on the right-hand side are nonnegative (since f is maximal), so $h \geq b$ (Lebesgue a.e). However this implies that $\varphi \geq f$. Therefore $\varphi = f$ and so (by the left continuity of b) one obtains $h = b$. □

2.4 Appendix

In this section, we have collected some background information necessary for the previous sections.

1. Krylov [33]

Let A be a separable metric space (a set of admissible controls); $\Re^d$ the Euclidean space of dimension d; T a nonnegative number. Consider a controlled process in the space $\Re^d$ in a time interval $[0,T]$. Let $(W_t, \mathcal{F}_t)$ be a d_1-dimensional Wiener process for some integer d_1. Suppose $\alpha \in A$, $t \geq 0$, $x \in \Re^d$, and suppose we are given $\sigma(\alpha, t, x)$ which characterizes the diffusion component of the process as a matrix of dimension $d \times d_1$ and $b(\alpha, t, x)$ which characterizes the deterministic component as a d-dimensional vector. We are also given real-valued functions $C^\alpha(t,x) \geq 0$, $f^\alpha(t,x)$, and $G(x)$, where $f^{\alpha}(t,x)$ is the payoff during the time interval $[t, \Delta t + t]$, if the controlled process is near a point x at time t, and if a control α is used, $G(x)$ is the gain at time T, and $C^{\alpha}(t,x)$ is the measure of discounting.

Assume that σ, b, C, f, G are continuous with respect to (α, x) and continuous with respect to x uniformly over α for each t. Further, assume that the above functions are Borel measurable with respect to (α, t, x).

For some constants m, $k \geq 0$, all $x, y \in \Re^d$, $t \geq 0$, $\alpha \in A$, suppose

$$\| \sigma(\alpha,t,x) - \sigma(\alpha,t,y) \| + | b(\alpha,t,x) + b(\alpha,t,y) | \leq k \mid x - y \mid \tag{38}$$

$$\| \sigma(\alpha,t,x) \| + | b(\alpha,t,x) | \leq k(1 + | x |) \tag{39}$$

$$| C^{\alpha}(t,x) | + | f^{\alpha}(t,x) | + | G(x) | \leq k(1 + | x |)^m. \tag{40}$$

DEFINITION. A *strategy* means a process $\alpha_t(w)$ which is progressively measurable with respect to a system $\mathcal{F}_t$ of σ-algebras having values in A. Denote by $\mathcal{U}$ the set of all strategies.

For $s \leq T$, let

$$\varphi_t^{\alpha,s,x} = \int_0^t C^{\alpha_r}(s + r, x_r^{\alpha,x,r})dr \tag{41}$$

$$v^{\alpha}(s,x) = m[\int_0^{T-s} f^{\alpha_t}(s + t, x_t^{\alpha,s,x})e^{-\varphi_t^{\alpha,s,x}} dt + G(x_{T-s}^{\alpha,s,x})e^{-\varphi_{T-s}}] \tag{42}$$

$$v(s,x) = \sup_{\alpha \in \mathcal{U}} v^{\alpha}(s,x) \tag{43}$$

Let $C([0,\infty),\Re^d)$ be the space of all continuous functions x_t with values in $\Re^d$ defined on $[0,\infty)$, and N_t the smallest σ-algebra of subsets of $C([0,\infty),\Re^d)$ which contains all sets of the form $(x_{[0,\infty)} : x_r \in \Gamma)$ for $r \in [0,T]$ and Borel $\Gamma \subset \Re^d$.

DEFINITION. For $t \in [0,\infty)$ and $x_{[0,\infty)} \in C([0,\infty),\Re^d))$, an A-valued function α_t, with $\alpha_t(x_{[0,\infty)}) = \alpha_t(x_{[0,t)})$ is said to be a *natural strategy admissible at the point* (s,x) if α_t is progressively measurable with respect to (N_t) and if, in addition, there exist at least one solution of the stochastic equation

$$x_t = x + \int_0^t \sigma(\alpha_r(x_{[0,r)}), s+r, x_r)dW_r + \int_0^t b(\alpha_r(x_{[0,r)}), s+r, x_r)dr, \quad (44)$$

which is progressively measurable with respect to $\mathcal{F}_t$. Denote by $\mathcal{U}_E(s,x)$ the set of all natural strategies admissible at the point (s,x).

For each strategy $\alpha \in \mathcal{U}_E(s,x)$, choose one (fixed) solution $x_t^{\alpha,s,x}$ of (44).

DEFINITION. The natural strategy $\alpha_t(x_{[0,t)})$ is said to be a *(nonstationary) Markov strategy* if $\alpha_t(x_{[0,t)}) = \alpha_t(x_t)$ for some Borel function $\alpha_t(x_t)$. Denote by $\mathcal{U}_M$ the set of all Markov strategies admissible at the point (s,x). Note that $\mathcal{U}_M(s,x) \subset \mathcal{U}_E(s,x)$, and $\mathcal{U}_M(s,x) \neq \emptyset$.

Let $H(t,x)$ be a continuous function of (t,x), $(x \in \Re^d, t \geq 0)$ such that

$$| H(t,x) | \leq k(1+ | x |)^m \quad (45)$$

for all t and x.

For $s \in [0,T]$, denote by $\mathcal{M}(T-s)$ the set of all Markov times (with respect to $\mathcal{F}_t$) not exceeding $(T-s)$. For $\alpha \in \mathcal{U}$, $\tau \in \mathcal{M}(T-s)$, let

$$v^{\alpha,\tau}(s,x) = M^{\alpha}_{s,x}[\int_0^{\tau} f^{\alpha_t}(s+t, x_t^{\alpha,s,x})e^{-\varphi_t}dt + H(s+\tau, x_\tau)e^{-\varphi_\tau}] \tag{46}$$

$$w(s,x) = \sup_{\alpha\in\mathcal{U}} \sup_{\tau\in\mathcal{M}(T-s)} v^{\alpha,\tau}(s,x). \tag{47}$$

For $s \leq T$, let

$$w_E(s,x) = \sup_{\alpha\in\mathcal{U}_E(s,x)} v^{\alpha,\tau}(s,x)$$

and

$$w_M(s,x) = \sup_{\alpha\in\mathcal{U}_M(s,x)} v^{\alpha,\tau}(s,x).$$

Theorem 9 *The function $w(s,x)$ is continuous with respect to (s,x) on $[0,T]\times\Re^d$, $w(s,x) \geq G(x)$, $w(T,x) = H(T,x)$. Also, there exists a constant $N = N(m,k,T)$ such that for all $s \in [0,T]$, and $x \in \Re^d$,*

$$| w(s,x) | \leq N(1+ | x |)^m.$$

□

Let $\varepsilon > 0$, and let

$$\tau_\varepsilon^{\alpha,s,x} = \inf(t \geq 0 : w(s+t, x_t^{\alpha,s,x})) \leq H(s+t, x_t^{\alpha,s,x}) + \varepsilon.$$

Note that $\tau_\varepsilon^{\alpha,s,x}$ is the time of the first exit of the process $(s+t, x_t^{\alpha,s,x})$ from the open set $Q_\varepsilon = \{(s,x) \mid w(s,x) > H(s,x)+\varepsilon\}$. It is clear that $\tau_\varepsilon^{\alpha,s,x} \in \mathcal{M}(T-s)$.

Theorem 10 *$w(s,x) = w_E(s,x)$. Furthermore, for $s \in [0,T]$, $x \in \Re^d$, the following inequality holds:*

$$w(s,x) \leq \sup_{\alpha \in \mathcal{U}_E(s,x)} M^{\alpha}_{s,x}\left[\int_0^{\tau_\varepsilon} f^{\alpha_t}(s+t,x_t)e^{-\varphi_t}dt + H(s+\tau_\varepsilon, x_{\tau_\varepsilon})e^{-\varphi_{\tau_\varepsilon}}\right] + \varepsilon.$$

□

This theorem shows that $\tau_\varepsilon^{\alpha,s,x}$ is an ε-optimal stopping time of the controlled process.

If A consists of a single point, the last inequality becomes an equality for $\varepsilon = 0$.

2. Friedman [15]

Let Q be a bounded domain in the $(n+1)$-dimensional space of variables (x,t). Assume that Q lies in the strip $0 < t \leq T$ and that $\tilde{B} = \bar{Q} \cap \{t = 0\}$,$\tilde{B}_T = \bar{Q} \cap \{t = T\}$ are nonempty. Let B_T be the interior of $\tilde{B}_T$, B be the interior of $\tilde{B}$. Denote by S_0 the boundary of Q lying in the strip $0 < t \leq T$, and let $S = S_0 \setminus B_T$. The set $\partial_0 Q = B \cup S$ is called the normal (or parabolic) boundary of Q. Let

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t) - \frac{\partial}{\partial t}$$

be a uniformly parabolic operator in Q. Consider the initial boundary value problem of finding a solution u of

$$\mathcal{L}u(x,t) = f(x,t) \text{ in } Q \cup B_T$$

$u(x,0) = \varphi(x)$ on B

$u(x,t) = g(x,t)$ on S

where f, φ, g are given functions. If $g = \varphi$ on $\bar{B} \cap \bar{S}$, then the solution u is always understood to be continuous in $\bar{Q}$.

Theorem 11 *Assume that $\mathcal{L}$ is uniformly parabolic in Q, that a_{ij}, b_j, c, f are uniformly Hölder continuous in $\bar{Q}$, and that g,φ are continuous functions on $\bar{B}$ and $\bar{S}$ respectively, and $g = \varphi$ on $\bar{B} \cap \bar{S}$. Then there exists a unique solution u of the initial boundary value problem above.*

□

3. Jacka [22]

Let $(\xi_t)_{t \geq 0}$ be a diffusion in $\Re^d$ (with respect to $(\mathcal{F}_t)$), and $X_t = e^{-\alpha t} g(\xi_t, T - t)$, where g is a continuous function. If

$$S_t = \text{ess sup}_{t \leq \tau} E\left[X_\tau \mid \mathcal{F}_t\right],$$

then clearly $S_t = e^{-\alpha t} f(\xi_t, T-t)$, where f is a continuous function and $f \geq g$ (Krylov [33]). Let $D := \{(x,s) : f(x,s) > g(x,s)\}$. Because f and g are continuous, it is apparent that the local time $L^0(S - X)$ will only increase for $\xi_t \in \partial D$.

Next, let $X = M + A$ be the canonical decomposition of X, where M is a martingale and A is a continuous, predictable process of integrable variation with $A_0 = 0$. Denote by A^- the decreasing component of A. Then we have the following theorem and corollary:

Theorem 12 *If ξ, f, g are defined as above, and there exist (deterministic) measures m_1, m_2 such that*

(i) ξ has density ρ with respect to m_1,

(ii) $dA^- \ll dm_2$,

(iii) $m_1 \otimes m_2(\partial D) = 0$,

then $L^0(S - X)$ is indistinguishable from 0.

PROOF. Let $\mu_t := (1/2)dL^0_t/dA^-_t$. It follows from the strong Markov property that μ and $K \overset{\text{def}}{=} dA^-_s/dm_2(s)$ are of the form $\mu_s = \mu(\xi_s, T - s)$ and $K_s = e^{-\alpha t}K(\xi_s, T - s)$ respectively. Denoting the t-section of ∂D by $(\partial D)_t$, we have

$$\begin{aligned}
&EL^0(S - X) \\
&\quad = E\int_0^t \mu_s \mathbf{1}_{(S-X)_s>0} dA^- \\
&\quad = E\int_0^t \mu_s \mathbf{1}_{(\xi_s, T-s)\in\partial D} dA^-_s \\
&\quad = \int_0^t \int_{\partial D_s} e^{-\alpha s}\mu(a, T - s)\rho(\xi_0, a; s)K(a, T - s)dm_1(a)dm_2(s) \\
&\quad = \int_{\partial D\cap(\Re^d\times[0,t])} e^{-\alpha s}\mu(a, T - s)\rho(\xi_0, a; s)K(a, T - s)d(m_1 \otimes m_2)(a, s) \\
&\quad = 0
\end{aligned}$$

by (iii). □

REMARK. Under the conditions of the above theorem, we can represent f as

$$\begin{aligned}
&f(\xi_0, T) \\
&= A_0 + E\int_0^T \mathbf{1}_{(\xi_s, T-s)\in D} dA_s \\
&= E\{A_T - \int_0^T \mathbf{1}_{(\xi_s, T-s)\in D^c} dA_s\} \\
&= E\{e^{-\alpha T} g(S_T, 0) + \int_0^T e^{-\alpha s} K(\xi_s, T-s) ds\} \\
&= \int_{\Re^d} e^{-\alpha T} \rho(\xi_0, a; T) g(a, 0) dm_1(a) \\
&+ \int_{D^c \cap (\Re^d \times [0,T])} e^{-\alpha s} \rho(\xi_0, a; s) K(a, T-s) d(m_1 \otimes m_2)(a, s).
\end{aligned}$$

Corollary 3 *Suppose the conditions of the above theorem are satisfied. If ρ is C^1 in ξ with derivatives which are uniformly continuous in $\Re^d \times [t_0, t_1]$ for any $0 < t_0 < t_1 < \infty$, then f' is C^1 in ξ for all $t > 0$.*

□

4. Jacka [21]

Let f be as in (29).

Lemma 3 *$f(x,t) > 0$ for all $x \geq 0, t > 0$.*

PROOF. Note that $g(x) > 0 \; \forall x < X$. Then for $x \geq X$ and $t > 0$,

$$f(x,t) \geq (X/2)(E_x e^{-r\tau_{X/2}} \mathbf{1}_{\tau_{X/2} < t}) > 0$$

where $\tau_{X/2} = \inf\{s \geq 0 : S_s \leq X/2\} \wedge t$. □

Proposition 3 *The boundary b is decreasing in t and bounded above by X.*

PROOF. We know that $f(x,\cdot)$ is increasing for each x, and $f(\cdot,t)$ is decreasing for each t, by the proof of Proposition 2 , so that for any $t>0$, $s>0$, $\varepsilon>0$

$$\begin{aligned} f(b(t)+\varepsilon,t+s) &\geq f(b(t)+\varepsilon,t) \\ &> g(b(t)+\varepsilon), \end{aligned}$$

since $b(t)+\varepsilon \in D_t$, and so $b(t)+\varepsilon \in D_{t+s}$. Thus $b(t+s) \leq b(t)$, so b is decreasing in t. Finally, note that g vanishes on $[X,\infty)$, whereas if $x \geq X$, $t>0$, then by Lemma 3, $f(x,t)>0$ so $b(t)<X$. □

Proposition 4 *The boundary b is continuous.*

PROOF. Both f and g are continuous , so since $D=\{(x,t): f-g>0\}$ is open, D^c is closed. If $t_n \uparrow t$, then $(b(t_n),t_n) \in D^c$ for all n. Thus $b(t^-) \leq b(t)$. This gives us left continuity of b, since b is decreasing in t. Recall that $\mathcal{L}f=0$ in D by Lemma 2, and

$$\frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} \geq rf$$

in D, since $\partial f/\partial t \geq 0$ and $\partial f/\partial x \leq 0$. Thus by letting

$$D^n \stackrel{\text{def}}{=} D \cap [0,X] \times [1/n,n]$$

we obtain

$$\inf_{(x,t)\in D^n} \frac{\sigma^2 x^2}{2}\frac{\partial^2 f}{\partial x^2} \geq \varepsilon_n > 0$$

for some ε_n, since $f > 0$ in the closure of D^n and f is continuous while the closure of D^n is compact. Given $t > 0$, take N such that $t \geq 1/N$. Then for any $x \in [b(t^+) + \eta, X]$ we can write

$$\begin{aligned} & f(x,s) - g(x,s) \\ = & \int_{b(s)}^{x} \int_{b(s)}^{y} \left[\frac{\partial^2 f}{\partial x^2}(u,s) - \frac{\partial^2 g}{\partial x^2}(u,s) \right] du dy \\ > & \frac{\eta^2}{\sigma^2 X} \varepsilon_n \end{aligned}$$

since $(\sigma^2 x^2/2)(\partial^2 g/\partial x^2)$ vanishes on $[0, X] \times \Re^+$, and f, g agree on b up to first derivatives and b is decreasing.

Take a sequence $(s_n) \downarrow t$, and use the continuity of $f - g$ to deduce that

$$f(b(t^+) + \eta, t) - g(b(t^+) + \eta, t) \geq \frac{\eta^2}{\sigma^2 X} \varepsilon_n > 0.$$

Thus $(b(t^+) + \eta) \in D_t$ for all $\eta > 0$. This implies that $b(t^+) \geq b(t)$. Since b is also decreasing, it follows that b is right continuous. The result follows. □

3 Analytic Approximation of the American Put

As stated earlier, Black and Scholes [5] derived the pricing equation for an European put option when the stock price follows a geometric Brownian motion. For the same model, Merton [42] derived the pricing equation for an American put with infinite time to maturity (see Chapter 1). In this chapter, several analytic approximations to the American put price will be reviewed. Numerical solutions, which will be discussed in the next chapter, are generally expensive and do not provide much insight. By studying analytic approximations, we hope to gain insight into the problem of pricing.

No one has yet been able to find the general solution to the American put price, P, although P is known for several special situations (Johnson, [26]):

- $rT = 0$, $P = p$, i.e. the American and European puts are equivalent.
- $\sigma^2 T = 0$, $P = (X - S)^+$, i.e. the put is either worthless or exercised immediately.
- $X = 0$, $P = 0$.
- $S = 0$, $P = X$.
- $S = \infty$, $P = 0$.
- $X = \infty$, $P = \infty$.

and Merton [42]:

- $T = \infty$,

$$P = \begin{cases} X/(1+M)(S_c/S)^M & \text{if } S \geq S_c = MX/(1+M) \\ X - S & \text{if } S \leq S_c \end{cases}$$

notation being as defined in Chapter 1. We write Φ for the distribution function of a standard normal variate (the number of arguments will be clear from the notation). Again, S_c, the critical stock price (i.e., the price below which early exercise occurs) is, in general, an unknown function of X, M, and $\sigma^2 T$.

In this chapter, the works of Johnson [26], Geske and Johnson [16], MacMillan [36], Barone-Adesi and Whaley [3], and Lamberton [35] will be reviewed.

3.1 Johnson

Using the fact that an American put is more valuable than a European put, but less valuable than a European put with an exercise price which is constant in present value terms, Johnson develops an approximate analytic expression for the price of an American put on a stock which is not paying dividends. The expression is very accurate for all values of the risk-free rate so far observed in the United States. More precisely, Johnson derives a general inequality by noting that an investor would be willing to trade an American put with exercise price X and time to maturity T for an American put with the same terms, but with a step-up of $X(e^{rT} - 1)$ at maturity. This latter put is equivalent to another European put (Margrabe [37]); hence, one can write

$$p(X) \leq P(X) \leq p(Xe^{rT}). \tag{48}$$

Note that any European put is equivalent to an American put with exercise price rising at the risk-free rate.

It might appear that inequality (48) provides an analytic expression for P, since we must have

$$P(X) = \alpha p(Xe^{rT}) + (1-\alpha)p(X) \tag{49}$$

with $0 \leq \alpha \leq 1$. Unfortunately, α is not a constant but a rather complicated function of S/X, rT, and $\sigma^2 T$. Johnson estimates a functional form of α by observing from Parkinson's tables [44] that α is between 0.2 and 0.25. A first attempt is made with a function of the form

$$\alpha = \frac{rT}{a_0 rT + a_1} \tag{50}$$

with $a_0 \approx 4$ and $a_1 \ll 1$, since then (50) gives values of α in the correct range, except when rT is small. Parkinson's tables were used to find α, by regressing rT/α on rT. This yields $a_0 = 3.9649$ and $a_1 = 0.03325$ with $R^2 = 0.9998$. For in-the-money puts, expression (50) gives values which are too large; the values are too small for out-of-the-money puts. Therefore, (50) was modified to permit α to become larger as S becomes smaller, while giving the same values as before when $S = X$. The following functional form was used:

$$\alpha = \left(\frac{rT}{a_0 rT + a_1}\right)^u \tag{51}$$

where

$$u = \frac{\log(S/S_c)}{\log(X/S_c)}. \qquad (52)$$

Finding S_c , in general, requires the solution of

$$X - S_c = P(X, S_c, rT, \sigma^2 T). \qquad (53)$$

Since the Amercian put pricing equation is not known, the equation for S_c must be estimated. For $\sigma^2 T = 0$, we must have $S_c = X$, while for $\sigma^2 T = \infty$ (Merton, [42]), we must have $S_c = XM/(1 + M)$. Thus, it seems natural to approximate the functional form for S_c as

$$S_c = X \left(\frac{M}{1 + M}\right)^m \qquad (54)$$

for some function m.

Note that for $r = 0$ there is no reason to exercise early, and, hence, S_c must be zero; (54) satisfies this requirement. In order to have $m = 0$ for $\sigma^2 T = 0$ and $m = 1$ for $\sigma^2 T = \infty$, one could try

$$m = \sigma^2 T/(b_0 \sigma^2 T + b_1) \qquad (55)$$

where b_0 should be one. In order to estimate b_0 and b_1, rewrite (54) and (55) as

$$m = \frac{\sigma^2 T}{b_0 \sigma^2 T + b_1} = \frac{\log(S_c/X)}{\log(M/(1 + M))} \qquad (56)$$

and (52) as

$$u = \frac{\log((S/X)(X/S_c))}{\log(X/S_c)} = \frac{\log \alpha}{\log(rT/(a_0 rT + a_1))}. \qquad (57)$$

Now, regress

$$\frac{\sigma^2 T}{\log(S/X)}\{1 - \log(\alpha - rT/(a_0 rT + a_1))\}\log(M/(1+M)) \qquad (58)$$

on $\sigma^2 T$. Since no listed puts have ever had $T > 0.12r^{-1}$, cases of large rT were excluded from the regression. Again, values for α were obtained using Parkinson's tables; a_0 and a_1 were found in the previous regression. The second regression yielded $b_0 = 1.04083$ and $b_1 = 0.00963$, with $R^2 = .9975$.

The formula developed by Johnson is valid only for American puts on stocks which do not pay dividends. The approximation is not very accurate for puts on stocks which pay large dividends.

To find the hedge ratio, rewrite (49) as

$$\begin{aligned} P &= \alpha[X\Phi(-d_{02}) - S\Phi(-d_{01})] \\ &\quad + (1-\alpha)[Xe^{-rT}\Phi(-d_2) - S\Phi(-d_1)] \end{aligned} \qquad (59)$$

where d_1 and d_2 are as in (12),

$$d_{01} = \frac{\log(S/X) + \sigma^2 T/2}{\sigma\sqrt{T}} \qquad (60)$$

$$d_{02} = d_{01} - \sigma\sqrt{T}. \qquad (61)$$

In this representation, one can view the American put as a weighted average of two European puts, one using a zero interest rate and one using the true interest rate. From (59) one obtains

$$\begin{aligned}\frac{\partial P}{\partial S} &= -\alpha\Phi(-d_{01}) - (1-\alpha)\Phi(-d_1) \\ &\quad + \frac{\alpha \log(rT)/(a_0 rT + a_1)}{S\log(X/S_c)}[p(Xe^{rT}) - p(X)]. \end{aligned} \tag{62}$$

In general, of course, the derivative of an approximation formula is less accurate than the formula itself. Using (62) to set a hedge will not necessarily produce an entirely risk-free hedge.

3.2 Geske and Johnson

Geske and Johnson present an alternative analytic formula which allegedly satisfies the partial differential equation and boundary conditions that characterize the American put valuation problem. As usual, the Black-Scholes hypotheses of perfect markets, constant r and σ, no dividends, and geometric Brownian motion for the stock price are assumed. The current time is set at zero.

The key to their solution is the assumption that each exercise decision is a discrete event, whence a formula is derived as the discounted expected value of all future cash flows. The cash flows arise because the put can be exercised at one of the future times t_1, t_2, ... The infinite series thus derived should be a continuous time solution to the free boundary value problem.

Since the assumption of geometric Brownian motion implies that the stock price at any future date is a lognormally distributed random variable, the correlation coefficient between the overlapping Brownian increments at close times t_1 and t_2 ($t_2 > t_1$) is given by $\rho_{12} = (t_1/t_2)^{1/2}$. At each instant, one will exercise the put if (a) the put has not already been exercised and (b) the payoff from exercising the put equals or exceeds the value of the put if it is not exercised. The critical stock price is independent of the current stock price and is determined from the free boundary whenever $X - S_c = P(S_c, T)$, for some $S = S_c$ and any T. Note that this can also be expressed as $\partial P/\partial S = -1$ when $S = S_c$. At the first instant, there is no probability that the put will already have been exercised, so one just integrates the exercise price less the future stock price at this date, and then discounts to the present. This yields two terms, one being the discounted exercise price times the probability that the stock price will be below $S_c^{(t_1)}$. At the next instant, one performs a similar integration up to $S_c^{(t_2)}$, the new critical stock price, but one must exclude all those cases where the put has been exercised at the first date. Again one obtains two terms, one being the discounted exercise price times the probability that the stock price at the first instant will be above the first critical stock price $S_c^{(t_1)}$, and the stock price at the second instant will be below the second critical stock price $S_c^{(t_2)}$, and so on. The correlation coefficient is negative between the argument for the last instant and the arguments of the previous ones, but positive between the arguments for the previous times. Intuitively, the put will be exercised at this instant if the stock price is below the critical stock price for this instant, given that it was not exercised at all previous instants because the stock price was always above the critical stock price. Proceeding in this way, one obtains the following formulae: Set

$$x_i = d_1(S_c^{(t_i)}, t_i), \tag{63}$$

$$y_i = d_2(S_c^{(t_i)}, t_i). \tag{64}$$

Then one has

$$P = Xw_2 - Sw_1 \tag{65}$$

where the weights w_1 and w_2 are given by

$$\begin{aligned} w_1 &= \Phi(-x_1) + \Phi(x_1, -x_2; -\rho_{12}) \\ &\quad + \Phi(x_1, x_2, -x_3; \rho_{12}, -\rho_{13}, -\rho_{13}) \\ &\quad + \cdots \end{aligned}$$

$$\begin{aligned} w_2 &= e^{-rt_1}\Phi(-y_1) + e^{-rt_2}\Phi(y_1, -y_2; -\rho_{12}) \\ &\quad + e^{-rt_3}\Phi(y_1, y_2, -y_3; \rho_{12}, -\rho_{13}, -\rho_{13}) \\ &\quad + \cdots \end{aligned}$$

and the correlation coefficients are $\rho_{ij} = i/\sqrt{j}$ for all $i, j \geq 1$. The series are both infinite.

Equation (65) cannot be used to compute actual numbers for the American put values, but it offers some insight into the portfolio which duplicates the

American put payoffs. Note that if we differentiate this formula, we also gain some information regarding the sensitivity to all the specified parameters, and obtain some simplification in computing the hedge ratio. Since one assumes that the risk-free rate is known and constant, the portfolio of bonds represented by Xw_2 is equivalent to investing the same amount in a risk-free bond of whatever maturity. However, if one were to introduce uncertainty about future interest rates, then term structure effects could be important. Note that the formula implies that the duplicating portfolio for the out-of-the-money puts is skewed toward longer maturity bonds, while for the in-the-money puts it is skewed toward shorter maturities.

Recall that the European put option is simply a special case of an American put option with only one exercise boundary. Then, (65) reduces to the European put formula when $P(S,T) = (X - S)^+$ only holds at $T = 0$. The critical stock price is a time-dependent path of stock prices that separates the exercise from the no exercise region in such a way as to maximize the value of the American put. Just at the point where the stock price is equal to the critical price, the put value would decrease one dollar for a one dollar increase in the stock price (i.e., $\partial P/\partial S(S = S_c) = -1$). As $S \rightarrow S_c$, the sensitivity of the American put, $\partial P/\partial T$, tends to zero. Also, at this exercise point, the interest rate effect on the American put exactly offsets the variance effects.

Note that because the option is linearly homogeneous with respect to the stock price and exercise price, the existence of either partial derivative implies the other. Since the time to maturity only appears in the formula multiplied by the interest rate or variance rate, the existence of any two of the partial derivatives with respect to r, σ and T implies that of the third.

The partial derivative of the valuation formula (65)with respect to the stock price is the hedge ratio (the number of shares of stock to options) in the portfolio:

$$\frac{\partial P}{\partial S} = -w_1 < 0. \tag{66}$$

The negative sign indicates that as the stock price rises, the put price falls. The hedge ratio can be thought of as either the negative amount of stock (sold short) to which the put is equivalent, or under risk neutrality, as the discounted expected cash outflow (divided by the stock price) that the put holder will experience.

As the exercise price rises, the put value rises:

$$\frac{\partial P}{\partial X} = w_2 > 0. \tag{67}$$

This can be considered as the expected cash inflow (divided by the exercise price).

As the interest rate rises, the American put value falls:

$$\frac{\partial P}{\partial r} = -Xdt[e^{-rt_1}\Phi(') + 2e^{-rt_2}\Phi(') + \ldots] \leq 0. \tag{68}$$

Since the present value of the bonds in the duplicating portfolio decreases as the interest rate increases.

As the variance rate rises, the put value rises:

$$\frac{\partial P}{\partial \sigma^2} = X\frac{\sqrt{T}}{2\sigma}w_2' \geq 0. \tag{69}$$

An increase in volatility increases the probability of both high and low stock prices, and with the asymmetry of an option's contingent payoffs, increases the option value.

As the time to expiration increases, the American put value rises:

$$\frac{\partial P}{\partial T} = Xr[e^{-rt_1}\Phi(') + 2e^{-rt_2}\Phi(') + \ldots] + X\frac{\sigma}{2\sqrt{T}} \geq 0. \tag{70}$$

The first four partial derivatives are functionally different from their corresponding European counterparts, but they do have the same sign and similar interpretation. The partial derivative with respect to time to expiration is strictly positive for the American put (provided $S > S_c$), while its sign is ambiguous for the European put. This ambiguity for the European put is obvious because more time helps if the put is out-of-the-money, but hurts if the holder wants to exercise immediately. The strictly positive sign for the American put is intuitively plausible because extending the life gives the holder more choices.

Geske and Johnson also show how to evaluate the American put formula (65) with a polynomial expression based upon an extrapolation from only a small number of exercise points to the infinite limit. The evaluation is very efficient because one is approximating an exact solution rather than the partial differential equation or the stock price process itself. Arbitrary accuracy can be obtained by adding exercise points. However, they showed that only a few (three) critical stock prices need to be computed in order to obtain penny accuracy. Unlike Brennan and Schwartz [6] or Parkinson [44], they do not approximate the partial differential equation, or the stock price process (Cox and Rubinstein, [9]). Instead, they evaluate their formula as an exact solution to the partial differential equation subject to the (discrete) free exercise boundary.

Let P_1 be the price of a put that can only be exercised at time T (i.e., at expiration); this option is just the European put, and we can write $P_1 = p$, the European put value. Let P_2 be the value of a put that can only be exercised at time $T/2$ or time T. Then

$$\begin{aligned} P_2 &= Xe^{-rT/2}\Phi(-d_2(S_c^{(T/2)}, T/2) - S\Phi(-d_1(S_c^{(T/2)}, T/2)) \\ &\quad + Xe^{-rT}\Phi[d_2(S_c^{(T/2)}, T/2), -d_2(S_c^X, T); -1/\sqrt{2}] \\ &\quad - S\Phi[d_1(S_c^{(T/2)}, T/2), -d_1(X, T); -1/\sqrt{2}]. \end{aligned} \tag{71}$$

The critical stock price, $S_c^{(T/2)}$, solves

$$S = X - p(S, X, T/2, r, \sigma) = S_c^{(T/2)}. \tag{72}$$

Similarly, let P_3 be the value of a put that can only be exercised at times $T/3$, $2T/3$, or T. Then

$$P_3 = Xe^{-rT/3}\Phi[-d_2(S_c^{(T/3)}, T/3)] - S\Phi[-d_1(S_c^{(T/3)}, T/3)]$$

$$+ Xe^{-2rT/3}\Phi[d_2(S_c^{(T/3)}, T/3), -d_2(S_c^{(2T/3)}, 2T/3); -1/\sqrt{2}]$$
$$- S\Phi[d_1(S_c^{(T/3)}, T/3), -d_1(S_c^{(2T/3)}, 2T/3); -1/\sqrt{2}]$$
$$+ Xe^{-rT}\Phi[d_1(S_c^{(T/3)}, T/3), -d_1(S_c^{(2T/3)}, 2T/3), -d_1(X, T);$$
$$1/\sqrt{2}, -1/\sqrt{3}, -2/\sqrt{3}]$$
$$- S\Phi[d_2(S_c^{(T/3)}, T/3), d_2(S_c^{(2T/3)}, 2T/3), -d_2(X, T);$$
$$1/\sqrt{2}, -1/\sqrt{3}, -2/\sqrt{3}] \quad (73)$$

and the critical stock prices $S_c^{(T/3)}$ and $S_c^{(2T/3)}$ solve

$$S = X - P_2(S, X, 2T/3, r, \sigma) = S_c^{(T/3)} \quad (74)$$

and

$$S = X - p(S, X, T/3, r, \sigma) = S_c^{(2T/3)} \quad (75)$$

respectively.

The values P_1, P_2, P_3, $\cdots$ define a sequence whose limit should be the American put value. Many techniques are available for computing such limits. One method is Richardson extrapolation (Dahlquist and Bjorck [11], p.269). In this case, the quantity to be determined is the American put price for a particular set of values S, X, T, r, and σ. The step length is the time between points at which exercise is permitted. The version of Richardson extrapolation (Appendix, equation (111)) leads to the following equation:

$$P = P_3 + 7/2(P_3 - P_2) - 1/2(P_2 - P_1). \quad (76)$$

This polynomial can be used to determine American put values and hedge ratios.

Preliminary evidence indicates that the analytic formula evaluation tabulated is faster to compute, by a factor of 10, than numerical methods. This is because the binomial and finite difference methods compute n critical stock prices ($n = 150$ in Cox and Rubinstein) while here only three points are used. (Even the tabulated four point method computes six.) Note that the three point extrapolation is about twice as fast as the four point.

Recall that the hedge ratio for the American put is given approximately by $\partial P/\partial S = -w_1$. In the absence of an analytic formula the hedge ratio is numerically approximated by computing two put values for two different stock prices and then using a difference equation to approximate the partial derivative at an intermediate stock price.

Geske and Johnson attribute their solution to an economic interpretation. First,the risk free hedge allows economists to avoid the transformations required for solutions to partial differential equations. Second, compound option theory provided a straightforward method for interpreting the infinite series of interrelated probability integrals arising from the free boundary condition. A key to the solution is that each exercise decision is considered as a discrete event. Thus, the formula derived is a continuous time solution to the parital differential equation instants. The formula adds to our intuition because it implies an exact duplicating portfolio for the American put, consisting of specific positions in discount bonds and stock sold short.

The evaluation of the formula is a separate problem. At first blush the American put formula might be considered intractable due to the infinite series of integrals. However, since the formula is exact in the limit, arbitrary accuracy can be obtained by extrapolating from a small sequence of terms to the actual solution containing an infinite series. This formula evaluation procedure leads to a polynomial expression similar to that used to evaluate the integral terms in the Black-Scholes European put option formula.

3.3 MacMillan

A third analytic approximation for the value of the American put option is given by MacMillan. The approximation is claimed to give accurate values and is easy to implement on a computer.
The value of the American put option satisfies the following partial differential equation (23)

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP - \frac{\partial P}{\partial T} = 0,$$

subject to the boundary conditions

$$P(S,t^*) = (X - S)^+; \tag{77}$$

$$P(S,t) \geq (X - S)^+; \tag{78}$$

$$P(S,t) \leq X; \tag{79}$$

$$\lim_{S \to \infty} P(S,t) = 0, \tag{80}$$

where t^* is the expiration date. In addition, we require equilibrium option prices to be continuous functions. If discontinuities arise, for example as S varies, then arbitrageurs would seek to buy on the low side of the discontinuity and sell on the high side; the resulting market forces would work to eliminate the discontinuity, and so at equilibrium no discontinuities could survive.

Recall that S_c is the price at which the put should rationally be exercised, defined by $P(S_c, t) = X - S_c$. Equation (23) is only applicable for $S \geq S_c$. For $S \leq S_c$, $P(S,t) = X - S$.

MacMillan finds an approximate solution to Merton's differential equation and its boundary conditions, in the following form: write

$$P = p + e, \tag{81}$$

so e is the *early exercise premium.* Since the European put satisfies an equation similar to the American put, e satisfies

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 e}{\partial S^2} + rS\frac{\partial e}{\partial S} - re - \frac{\partial e}{\partial T} = 0, \tag{82}$$

However the boundary conditions on e are substantially simpler than those on either P or p, since both P and p satisfy the same boundary condition:

$$P(S,t^*) = p(S,t^*) = (X - S)^+. \tag{83}$$

We are thus left with the simpler problem of solving (82) subject to the conditions $e(S,t^*) = 0$, $e(S,t) \leq X$, and $\lim_{S\to\infty} e(S,t) = 0$. Replacing T by a function $K(T)$, transforms (82) into

$$S^2 \frac{\partial^2 e}{\partial S^2} + MS\frac{\partial e}{\partial S} - Me - \frac{M}{r}\frac{dK}{dT}\frac{\partial e}{\partial K} = 0, \tag{84}$$

where $M = 2r/\sigma^2$. Now write $e(S,K)$ in the form:

$$e(S,K) = K(T)f(S,K). \tag{85}$$

One then has

$$\begin{aligned} \frac{\partial e}{\partial S} &= K\frac{\partial f}{\partial S}; \\ \frac{\partial^2 e}{\partial S^2} &= K\frac{\partial^2 f}{\partial S^2}; \\ \frac{\partial e}{\partial T} &= \frac{dK}{dT}f + K\frac{dK}{dT}\frac{\partial f}{\partial K}. \end{aligned}$$

Equation (82) then becomes

$$S^2 K\frac{\partial^2 f}{\partial S^2} + MSK\frac{\partial f}{\partial S} - MKf - \frac{M}{r}[\frac{dK}{dT}f + K\frac{dK}{dT}\frac{\partial f}{\partial K}] = 0. \tag{86}$$

This can be rewritten as:

$$S^2\frac{\partial^2 f}{\partial S^2} + MS\frac{\partial f}{\partial S} - M[1 + \frac{dK/dT}{rK}(1 + K\frac{\partial f/\partial K}{f})]f = 0. \tag{87}$$

The function $K(T)$ is now chosen so that the time-dependence of $e = Kf$ is contained in the factor $K(T)$. A simple and useful choice is

$$K(T) = 1 - e^{-rT}. \tag{88}$$

(86) the becomes:

$$S^2 \frac{\partial^2 f}{\partial S^2} + MS \frac{\partial f}{\partial S} - \frac{M}{K}[1 + (1 - K)K \frac{\partial f / \partial K}{f}]f = 0.$$

The term $(1 - K)K$ is zero at $T = 0$ (i.e. $K = 0$) and at $T = \infty$ (i.e. $K = 1$), and has a maximum value of $1/4$ at $K = 1/2$. Neglecting the term involving $(1 - K)K$ should produce a useful approximation for small T and for large T, with error at intermediate values of T well controlled by the term $(1 - K)K$, so consider

$$S^2 \frac{\partial^2 f}{\partial S^2} + MS \frac{\partial f}{\partial S} - \frac{M}{K} f \approx 0. \tag{89}$$

One can assume that K is not equal to zero; the case $K = 0$ is treated in the Appendix. The second-order ordinary differential equation (89) has two linearly independent solutions of the form aS^q. By substituting $f = aS^q$ into (89), one gets a quadratic equation in q:

$$q^2 + (M - 1)q - \frac{M}{K} = 0. \tag{90}$$

This is easily solved for q to give two roots q_1 and q_2, with $q_1 < 0 < q_2$, say. The general solution to (89) is

$$f(S) = a_1 S^{q_1} + a_2 S^{q_2}. \tag{91}$$

Necessarily $a_2 = 0$, since $S^{q_2} \to \infty$ with S, which is not allowed by condition (80). Writing $a = a_1$ and $q = q_1$, we obtain

$$P(S,T) = p + Kf = p(S,T) + (1 - e^{-rT})aS^q.$$

Let $S = S_c$ be the value of S at which the curve $P = p + Kf$ touches the line $Y = X - S$, so

$$P(S_c) = p(S_c) + K(T)f(S_c) = X - S_c. \tag{92}$$

Note that at $S = S_c$, the slope of the curve $P = p + Kf$ must equal the slope of the line $Y = X - S$, which is -1. That is,

$$\frac{d}{dS}P(S_c) = \frac{d}{dS}p(S_c) + K(T)\frac{d}{dS}f(S_c) = \frac{d}{dS}(X - S_c) = -1, \tag{93}$$

where S_c in parentheses indicates evaluation at S_c after all other operations have been performed. Finally, recall that for $S \le S_c$, one has $P = X - S$.

Use (92) and (93) to determine S_c and a. First of all note that

$$\frac{d}{dS}p(S) = \frac{d}{dS}c(S) - 1 = \Phi(d_1(S)) - 1$$

(see, Eq(45) in Smith [48]). Then from (92) and (93) obtain

$$\frac{Kf(S_c)}{K\frac{d}{dS}f(S_c)} = \frac{X - S_c - p(S_c)}{-\Phi(d_1(S_c))}.$$

But $f(S) = aS^q$, so that $\frac{d}{dS}f(S) = aqS^{q-1}$. Thus,

$$\frac{f(S_c)}{\frac{d}{dS}f(S_c)} = \frac{aS_c^q}{qaS_c^{q-1}} = \frac{S_c}{q}.$$

Hence

$$S_c = \frac{-q(X - p(S_c))}{\Phi(d_1(S_c)) - q}. \tag{94}$$

Thus S_c is the value of S which satisfies the equation $S = G(S)$, where

$$G(S) = \frac{-q(X - p(S))}{\Phi(d_1(S)) - q}.$$

This equation will not yield an explicit formula for S_c but can be solved to any desired degree of accuracy using standard iterative techniques. Having obtained S_c, use (93) to obtain

$$a = \frac{\Phi(d_1(S_c))S_c^{1-q}}{-qK}.$$

Now let

$$A = aKS_c^q = \frac{S_c\Phi(d_1(S_c))}{-q}. \tag{95}$$

Then an approximate solution of (23) subject to the boundary conditions (77)-(80) is

$$P(S,T) = p(S,T) + A(S/S_c)^q. \tag{96}$$

The limit $T = \infty$ is exact, and the solution in this case has been obtained by Merton [42]. For $T = \infty$, one has $K = 1 - e^{-rT} = 1$. Thus, the quadratic equation, $q^2 + (M - 1)q - M = 0$, has solutions $q = 1$ and $q = -M$. Select the negative root $q = -M$ as before. Obtain S_c from the equation

$$S_c = \frac{-q(X - p(S_c))}{\Phi(d_1(S_c)) - q}.$$

At $T = \infty$, $p(S) = 0$ and $\Phi(d_1(S)) = 1$, so that this equation becomes $S_c = MX/(1 + M)$, whence $A = S_c\Phi(d_1(S_c))/(-q) = X/(1 + M)$, and so finally

$$P = A\left(\frac{S}{S_c}\right)^q = \frac{X}{1 + M}\left(\frac{S}{S_c}\right)^{-M},$$

for $S \geq S_c$, while $P = X - S$ for $S \leq S_c$. These are precisely the results obtained by Merton.

3.4 Barone-Adesi and Whaley

Barone-Adesi and Whaley extended MacMillan's work to the theory of pricing commodity option contracts. A commodity option represents the right to buy or sell a specific commodity at a specified price within a specified period of time. The exact nature of the underlying commodity varies and may be anything from a precious metal such as gold or silver to a financial instrument such as a Treasury bond or a foreign currency. Usually the commodity option is labelled by the nature of the underlying commodity. For example, if the commodity option is written on a common stock, it is referred to as a stock option, and if the commodity option is written on a foreign currency, it is referred as a foreign currency option.

Consider a general commodity option-pricing model. The assumptions used in Barone-Adesi and Whaley's analysis are consistent with Black-Scholes and Merton. First, the short-term interest rate, r, and the cost of carry for the commodity, b, are assumed to be constant, proportional rates. For a non-dividend paying stock, the cost of carry is equal to the riskless rate of interest (i.e. $b = r$), but, for the most other commodities, this is not the case. For the traditional agricultural commodities such as grain and livestock, the cost of carry exceeds the riskless rate by the cost of storage, insurance, deterioration,etc.

A second common assumption in the option pricing literature is that the underlying commodity price change movements follows the stochastic differential equation,

$$\frac{dS}{S} = \mu dt + \sigma dW, \tag{97}$$

where μ is the expected instantaneous relative price change of the commodity, σ is the instantaneous standard deviation, and W is a Wiener process.

Finally, assuming a riskless hedge between the option and the underlying commodity may be formed, the partial differential equation governing the movements of the commodity option price V through time is

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + bS\frac{\partial V}{\partial S} - rV - \frac{\partial V}{\partial T} = 0. \tag{98}$$

This equation, which first appeared in Merton, is the heart of the commodity option pricing discussion contained herein. Note that, when the cost of carry rate b is equal to the riskles rate of interest, the above differential equation reduces to that of Black and Scholes. The non-dividend stock option pricing is a special case of the more general commodity option pricing problem.

Barone-Adesi and Whaley derive a quadratic approximation to the American put value. The key insight into this approximation is that, if the above partial differential equation applies to both the American and European options, it also applies to the early exercise premium of the American option. Recall that the early exercise premium is defined by (81).

Letting $M = 2r/\sigma^2$, and $N = 2b/\sigma^2$, the differential equation satisfied by e is

$$S^2\frac{\partial^2 e}{\partial S^2} + NS\frac{\partial e}{\partial S} - Me - \frac{M}{r}\frac{\partial e}{\partial T} = 0. \tag{99}$$

Next, let $e(S,K) = K(T)f(S,K)$ as in (85). It follows that

$$\begin{aligned}
\frac{\partial e}{\partial S} &= K\frac{\partial f}{\partial S};\\
\frac{\partial^2 e}{\partial S^2} &= K\frac{\partial^2 f}{\partial S^2};\\
\frac{\partial e}{\partial T} &= \frac{dK}{dT}f + K\frac{dK}{dT}\frac{\partial f}{\partial K}.
\end{aligned}$$

Substituting the above partial derivatives into (99) yields

$$S^2\frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - M[1 + \frac{dK/dT}{rK}(1 + (K/f)\frac{\partial f/\partial K}{f}))]f = 0. \tag{100}$$

Choose $K(T) = 1 - e^{-rT}$ as in (88), and substitute to obtain

$$S^2 \frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - \frac{M}{K}f - (1-K)M\frac{\partial f}{\partial K} = 0. \tag{101}$$

Now, assume that the last term on the left-hand side is negligible. For commodity options with either very short or very long times to expiration, this assumption is reasonable since, as $T \to 0$ or $T \to \infty$, $\partial f/\partial K \to 0$, and the term $(1-K)M(\partial f/\partial K)$ disappears. Therefore, one has

$$S^2 \frac{\partial^2 f}{\partial S^2} + NS\frac{\partial f}{\partial S} - \frac{M}{K}f \approx 0. \tag{102}$$

The characteristic equation for (102) is

$$q^2 + (N-1)q - \frac{M}{K} = 0, \tag{103}$$

with a unique negative root q_1. As in MacMillan, the only feasible solution is $f(S) = a_1 S^{q_1}$, whence

$$P(S,T) = p(S,T) + K a_1 S^{q_1}. \tag{104}$$

The value of a_1 is

$$a_1 = -\frac{[1 - e^{(b-r)T}\Phi(d_1(S_c))]}{K q_1 S_c^{q_1 - 1}}, \tag{105}$$

where $-e^{(b-r)T}\Phi(d_1(S_c))$ is the partial derivative of $p(S_c, T)$ with respect to S_c, $a_1 > 0$ since $q_1 < 0$, and all other terms are positive (for the definition of d_1 see (12)). The critical commodity price S_c is determined from

$$X - S_c = p(S_c, T) - [1 - e^{(b-r)T}\Phi(d_1(S_c))]S_c/q_1. \tag{106}$$

To find S_c, observe that $S_c(0) = X$, and

$$S_c(\infty) = \frac{X}{1 - (1/q_1(\infty))}, \tag{107}$$

where $q_1(\infty) = [(1 - N) - \sqrt{(N-1)^2 + 4M}]/2$. It is worthwhile to point out that when the cost of carry b is equal to the riskless rate of interest r, this result is exactly the same as Merton. In equation (106), S_c is a decreasing function of the time to expiration, and has the following approximate analytic expression

$$S_c \approx S_c(\infty) + [X - S_c(\infty)]e^{h_1}, \tag{108}$$

where $h_1 = (bT - 2\sigma\sqrt{T})X/(X - S_c(\infty))$.

With S_c known, the approximate value of an American put option (104) written on a commodity becomes

$$\begin{aligned} P(S,T) &= p(S,T) + A_1(S/S_c)^{q_1} \quad \text{when } S > S_c \\ P(S,T) &= X - S \quad \text{when } S \leq S_c, \end{aligned} \tag{109}$$

where $A_1 = -(S_c/q_1)[1 - e^{(b-r)T}\Phi(d_1(S_c))]$. Note that $A_1 > 0$ since $q_1 < 0$, $S_c > 0$, and $\Phi(d_1(S_c)) < e^{-bT}$.

3.5 Lamberton

For a probabilitistic approach, consider an American put option on one share, with exercise price X and maturity T. Following Karatzas [27] and Myneni [43], write the fair price of this option at time $t \in [0, T]$ as

$$V_t = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} E(e^{-r(\tau-t)}(X - S_\tau)^+ \mid \mathcal{F}_t),$$

where $\mathcal{T}_{t,T}$ is the family of stopping times $\tau \in [t, T]$. Due to the Markov property of the model, we have $V_t = u(t, S_t)$, where

$$\begin{aligned} u(t,x) &= \sup_{\tau \in \mathcal{T}_{t,T}} E(e^{-r(\tau-t)}(X - xe^{(r-\sigma^2/2)(\tau-t)+\sigma(W_\tau - W_t}))^+ \\ &= \sup_{\tau \in \mathcal{T}_{0,T-t}} Ee^{-r\tau}(X - xe^{(r-\sigma^2/2)\tau+\sigma W_\tau})^+. \end{aligned}$$

For $t \in [0, T]$, define the *critical price* at time t as

$$S_c(t) = \sup\{x \geq 0 \mid u(t,x) = (X - x)^+\}.$$

The function $t \mapsto S_c(t)$ is C^∞ on the interval $[0, T)$ (see Van Moerbeke [49] and Friedman [15]), and $\lim_{t \to T} S_c(t) = X$. Since $t \mapsto u(t, x)$ is nonincreasing, $t \mapsto S_c(t)$ must be nondecreasing.

From the theory of optimal stopping, we know that

$$V_0 = sup_{\tau \in \mathcal{T}_{0,T}} Ee^{-r\tau}(X - S_\tau)^+ = Ee^{-r\tau^*}(X - S_{\tau^*})^+$$

with $\tau^* = inf\{t \in [0,T] \mid u(t,S_t) = (X - S_t)^+\}$ being the smallest optimal stopping time (El Karoui, [12], 72-238). In fact, the following result of Jaillet et al. [25] (Remark 3.12) shows that τ^* is the only optimal stopping time.

Lemma 4 *If τ is a stopping time in $\mathcal{T}_{0,T}$ and satisfies*

$$V_0 = Ee^{-r\tau}(X - S_\tau)^+$$

then $\tau = \tau^*$ almost surely.

□

Note that on $\{\tau^* < T\}$, we have $\tau^* = inf\{t \in [0,T] \mid S_t \leq S_c(t)\}$ and that, if $S_0 \leq S_c(0)$, then $\tau^* = 0$ almost surely.

Lemma 5 *Assume $S_0 = x > S_c(0)$. Then, the support of the distribution of τ^* is equal to $[0,T]$.*

PROOF. Since $S_t = xe^{(r-\sigma^2/2)t+\sigma W_t}$, we have, with the convention that $\inf \emptyset = T$,

$$\begin{aligned} \tau^* &= inf\{t \in [0,T] \mid \log(x) + (r - \sigma^2/2)t + \sigma W_t = \log S_c(t)\} \\ &= inf\{t \in [0,T] \mid (r - \sigma^2/2)t - \log S_c(t) + \sigma W_t = -\log(x)\}. \end{aligned}$$

Now, for any $\varepsilon \in (0,T)$ the distribution of the process

$$(W_t + \frac{1}{\sigma}((r - \sigma^2/2)t - \log S_c(t)))_{0 \leq t \leq T-\varepsilon}$$

is equivalent to that of the standard Brownian motion on the interval $[0, T - \varepsilon]$, since S_c is smooth on $[0, T)$. Hence, the result follows from the well-known properties of the hitting times of Brownian motion. □

A natural way to approximate the function u is to introduce a random walk approximation of the driving Brownian motion $(W_t)_{0 \leq t \leq T}$. Let $(\varepsilon_k)_{k \geq 1}$ be a sequence of i.i.d. square-integrable random variables, with $E(\varepsilon_1) = 0$ and $E(\varepsilon_1^2) = T$, and let

$$W_t^{(n)} = \frac{\sum_{k=1}^{[nt/T]} \varepsilon_k}{\sqrt{n}}.$$

For $t = kT/n$, $k = 1, \ldots, n$, let

$$u^{(n)}(t, x) = sup_{\tau \in \mathcal{T}_{0,T-t}^{(n)}} E(e^{-r\tau} X - x e^{\sigma W_\tau^{(n)} - \sigma^2/2\tau})$$

where $\mathcal{T}_{0,T-t}^{(n)}$ is the set of stopping times (with respect to the natural filtration of $W^{(n)}$) with values in

$$[0, T - t] \cap \{0, T/n, 2T/n, \ldots, (n-1)T/n, T\}.$$

From the definition of $u^{(n)}$, it is clear that $x \mapsto u^{(n)}(t, x)$ is a nonincreasing, convex function, satisfying $u^{(n)}(t, x) \geq (X - x)^+$.

We extend the definition of $u^{(n)}$ to all of $[0, T]$ by setting

$$u^{(n)}(t, x) = u^{(n)}([nt/T]T/n, x),$$

where $[a]$ denotes the integer part of a. With this definition, $u^{(n)}$ is a nonincreasing function of t, whose values may be computed (by dynamic programming), by solving the following recursive equations:

$$u^{(n)}(T,x) = g(x)$$

$$u^{(n)}\left(\frac{kT}{n},x\right) = max(g(x), e^{-rT/n}Eu^{(n)}\left(\frac{(k+1)T}{n}, xe^{\frac{(r-\sigma^2/2)T}{n}+\frac{\sigma\varepsilon_1}{\sqrt{n}}})\right)$$

for all integers k between 0 and $n-1$.

(Note that when the ε_k are Bernoulli random variables, the algorithm we have just described is a variant of the Cox-Ross-Rubinstein method [10] for valuing American put options. The methods of Parkinson [44] and Geske and Johnson [16] are also relevant to this approach.)

The main properties of the approximate critical price associated with the $u^{(n)}$ are as follows.

Proposition 5 *Let* $\rho_n = \{0, T/n, 2T/n, \ldots, (n-1)T/n\}$. *For all large enough* n *and all* $t \in \rho_n$, *there is a number* $s_n(t) \in [0, X)$ *such that* $u^{(n)}(t,x) = (X-x)$ *for all* $x \in [0, s_n(t)]$, *and* $u^{(n)}(t,x) > (X-x)$ *for all* $x \in (s_n(t), X]$. *Moreover, letting* $S_t^{(n)} = xe^{(r-\sigma^2/2)t+\sigma W_t^{(n)}}$, *the stopping time defined by*

$$\tau_n^* = \inf\{t \in \rho_n \mid S_t^{(n)} \le s_n(t)\}$$

satisfies

$$u^{(n)}(0,x) = E(e^{-r\tau_n^*}X - xe^{\sigma W_{\tau_n^*}^{(n)} - \sigma^2/2\tau_n^*})^+.$$

PROOF. First observe that for $x \geq 0$,

$$u^{(n)}\left(\frac{(n-1)T}{n},x\right) = max(g(x), e^{-rT/n}E(X - xe^{(r-\sigma^2/2)T/n+\sigma(\varepsilon_1/\sqrt{n})})^+).$$

Therefore, for all $\alpha > 0$,

$$\begin{aligned} u^{(n)}((n-1)T/n, X) &= Xe^{-rT/n}E(1 - e^{(r-\sigma^2/2)T/n + \sigma\varepsilon_1/\sqrt{n}})^+ \\ &\geq Xe^{-rT/n}(1 - e^{(r-\sigma^2/2)T/n - \sigma\alpha/\sqrt{n}})\Pr(\varepsilon_1 < -\alpha) \\ &= Xe^{-rT/n}\Pr(\varepsilon_1 < -\alpha)\left(\frac{\sigma\alpha}{\sqrt{n}} + o(\frac{1}{\sqrt{n}})\right). \end{aligned}$$

Since $E(\varepsilon_1) = 0$ and $E(\varepsilon_1^2) = T$, one can find $\alpha > 0$ such that $P(\varepsilon_1 < -\alpha) > 0$, so that, for n large enough,

$$u^{(n)}((n-1)T/n, X) > 0.$$

Since $t \mapsto u^{(n)}(t, X)$ is nonincreasing, we have $u^{(n)}(t, X) > 0$ for all $t \in \rho_n$. The existence of $s_n(t)$ now follows from the convexity of the function $x \mapsto u^{(n)}(t, x)$.

To see that τ_n^* is an optimal stopping time, recall that the smallest optimal stopping time is given by

$$\overline{\tau}_n = \inf\{t \in \rho_n \mid u^{(n)}(t, S_t^{(n)}) = (X - S_t^{(n)})^+\}.$$

Obviously, $\tau_n^* \geq \overline{\tau}_n$. The inequality may be strict, since $u^{(n)}(t, x)$ may vanish for large values of x (especially when the distribution of ε_1 has finite support). But on $\{\tau_n^* > \overline{\tau}_n\}$, we have $(X - S_{\overline{\tau}_n}^{(n)})^+ = 0$, and therefore

$$e^{-r\tau_n^*}(X - S_{\tau_n^*}^{(n)})^+ \geq e^{-r\overline{\tau}_n}(X - S_{\overline{\tau}_n}^{(n)})^+,$$

which implies that τ_n^* is optimal too. □

Define s_n for all $t \in [0, T]$ by setting $s_n(t) = s_n([nt/T]T/n)$. The main result is the following.

Theorem 13 *The sequence of functions s_n converges uniformly to S_c as n tends to infinity.*

PROOF. Since T is arbitrary, it suffices to show uniform convergence on any interval $[\varepsilon, T)$ where $0 < \varepsilon < T$, because $u^{(n)}$ and u depend on T only through $T - t$. Next, since s is continuous and nondecreasing and each s_n is nondecreasing and bounded by $X = \lim_{t \to T} S_c(t)$, it suffices to prove pointwise convergence on $(0, T)$.

Let $t_0 \in (0, T)$. Then we know that

$$\lim_{n \to \infty} u^{(n)}(t_0, x) = u(t_0, x)$$

(cf Kushner [34], esp. Section 8.2). Thus

$$\limsup_{n \to \infty} s_n(t_0) \leq S_c(t_0).$$

Indeed, if $x < \limsup_{n \to \infty} s_n(t_0)$, then we have $u^{(n)}(t_0, x) = X - x$ for infinitely many values of n, whence $u(t_0, x) = X - x$.

In order to prove that $\liminf_{n \to \infty} s_n(t_0) \geq S_c(t_0)$, fix $x > S_c(0)$ and define τ_n^* as in Proposition 5. Since τ_n^* is optimal for the discretized problem, any weak limit of the law of $(\tau_n^*, W^{(n)})$ can be intrepreted as a law of (τ, W), where W is a standard Wiener process with respect to some filtration $\mathcal{F}$ and τ is an $\mathcal{F}$-stopping time which maximizes $E(Xe^{-r\tau} - xe^{\sigma W_\tau - \sigma^2 \tau/2})^+$. Using

Lemma 4, we know that the optimal stopping time for the limit is unique (note that the proof of Lemma 4 does not use the fact the filtration is the natural filtration of W, but only the fact that W is an $\mathcal{F}$-Brownian motion). Therefore, $(\tau_n^*, S_{\tau_n^*}^{(n)})$ converges in law to (τ^*, S_{τ^*}).

Now assume that $\liminf_{n\to\infty} s_n(t_0) < S_c(t_0)$, and take $\varepsilon > 0$ such that $\liminf_{n\to\infty} s_n(t_0) < S_c(t_0) - \varepsilon$. Since S_c is continuous, there exists $\eta > 0$ such that for all $t \in [t_0 - \eta, t_0 + \eta]$,

$$\liminf_{n\to\infty} s_n(t_0) < S_c(t) - \varepsilon.$$

Choose a subsequence $(s_{n_k}(t_0))_{k\geq 0}$ such that for all k

$$s_{n_k}(t_0) < S_c(t_0 - \eta) - \varepsilon. \tag{110}$$

Since s_{n_k} and S_c are nondecreasing, we have

$$s_{n_k}(t) < S_c(t) - \varepsilon$$

for all k and all $t \in (t_0 - \eta, t_0)$. By Lemma 5, we have $\Pr(\tau^* \in (t_0 - \eta, t_0)) > 0$ since $x > S_c(0)$. Let

$$A = \{(t, y) \in \Re^2 \mid t_0 - \eta < t < t_0 \quad \text{and} \quad y > S_c(t) - \varepsilon\}.$$

The set A is an open subset of $\Re^2$, since S_c is continuous, and

$$\Pr((\tau^*, S_{\tau^*}) \in A) = \Pr(\tau^* \in (t_0 - \eta, t_0)),$$

since on $\{\tau^* < T\}$ we have $S_{\tau^*} = S_c(\tau^*)$. As $(\tau_n^*, S_{\tau_n^*}^{(n)})$ converges in law to (τ^*, S_{τ^*}), we have

$$\liminf_{n\to\infty} \Pr((\tau_n^*, S_{\tau_n^*}^{(n)}) \in A) \geq \Pr((\tau^*, S_{\tau^*}) \in A).$$

Therefore, for n large enough,

$$\Pr\left[\tau_n^* \in (t_0 - \eta, t_0), S_{\tau_n^*}^{(n)} > S_c(\tau_n^*) - \varepsilon\right] > 0.$$

But on $\{\tau_n^* < T\}$ we have $s_n(\tau_n^*) \geq S_{\tau_n^*}^{(n)}$, so the above inequality contradicts (110). □

3.6 Appendix

Let $F(h)$ be the value of the function of interest when a step size of h is used. We wish to find $F(0)$. Suppose $F(h)$ takes the form

$$F(h) = F(0) + a_1 h^p + a_2 h^r + O(h^s)$$

where $s > r > p$. Then we can also write

$$F(kh) = F(0) + a_1(kh)^p + a_2(kh)^r + O(h^s)$$

and

$$F(qh) = F(0) + a_1(qh)^p + a_2(qh)^r + O(h^s)$$

where $q > k > 1$. Substituting for a_1 and a_2 and solving for $F(0)$ yields

$$F(0) = F(h) + \frac{A}{C}[F(h) - F(kh)] - \frac{B}{C}[F(kh) - F(qh)] \qquad (111)$$

where

$$A = q^r - q^p + k^p - k^r$$

$$B = k^r - k^p$$

$$C = q^r(k^p - 1) - q^p(k^r - 1) + k^r - k^p.$$

Using $P_1 = F(qh)$, $P_2 = F(kh)$, and $P_3 = F(h)$, we have $q = 3$, $k = 3/2$. If we expand $F(h)$ in a Taylor series around $F(0)$ and drop terms of the third order or higher, we obtain $p = 1$ and $r = 2$. Substitution into (111) gives (76). There is some error in (76) from dropping the higher order terms.

The Case $K = 0$

We are concerned here with the case $K = 1 - e^{rT} = 0$, that is, $rT = 0$. We assume $T \neq 0$, so that we deal only with the case $r = 0$. In the quadratic equation (90), the ratio M/K is then undefined. To deal with his case, suppose first that $r \neq 0$, and put $U = M/2 = r/\sigma^2$ and $V = \sigma^2 T$. Then $UV = rT$. For small rT expand K as

$$K = 1 - e^{rT} = 1 - (1 - rT + (rT)^2 + \cdots) = rT - (rT)^2 + \cdots .$$

Thus for r near zero K is approximately equal to rT, and thus M/K is approximately $M/rT = 2/V$, which is independent of r. Thus, for small r, equation (89) becomes

$$S^2 \frac{\partial^2 f}{\partial S^2} + MS\frac{\partial f}{\partial S} - \frac{2}{V} f = 0.$$

Substituting $f = aS^q$ gives the quadratic equation

$$q^2 + (M-1)q - \frac{2}{V} = 0.$$

The negative root must be selected. The case $r = 0$ is obtained by setting $M = 0$. Since $r = 0$ implies $K = 0$, the solution in this case is exact.

4 Numerical Methods

The Black-Scholes partial differential equation is relevant for many significant valuation problems where no analytic solutions have been found. This has led to considerable research employing numerical methods to find approximate solutions.

Efficient markets imply the rapid reflection of information in asset prices. Thus, the arrival of new information often will be accompanied by price changes. If the underlying asset is assumed to follow a diffusion process, then price changes are continuous. Alternatively, if the underlying asset is assumed to follow a jump process, then price changes are discontinuous. In the diffusion case, information is thought to arrive in a smooth, continuous fashion, and price changes can have either a constant or a changing variance, but with either a normal or a lognormal distribution. By contrast, a jump process signifies that the information arrival is discontinuous, and that the price changes have a Poisson distribution. In practice, it is suspected that a combined diffusion-jump process is generating the data.

The no-arbitrage partial equilibrium conditions have been derived for the pure diffusion, pure jump, and combined processes, and some analytic solutions have been found for each case. However, in many complex but realistic problems, numerical methods, including finite differences (Brennan and Schwartz) or numerical integration (Parkinson) must be employed to approximate the value of the assets.

The constant-variance diffusion approach to asset price changes leads to the partial differential equation (23). Despite the presence of variable coefficients, the Black-Scholes equation remains parabolic (this is essentially due to limited liability), although the approximation equations and the stability conditions become more complex with S. Fortunately, a change of variables (cf [42]) can transform (23) into the following equation:

$$a\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) \tag{112}$$

where a and k are C^2 functions of x and t. This may be a non- linear equation if a and k are allowed to vary with u as well as x and t. To use numerical schemes of high accuracy, equation (112) is transformed from variable to constant coefficients by making the transformation

$$y = \int \frac{dx}{k(x)}, \tag{113}$$

whereupon (112) becomes

$$\frac{\partial u}{\partial t} = \frac{1}{ak}\frac{\partial^2 u}{\partial y^2}. \tag{114}$$

In terms of the original variable x, this transformation scheme in y will have unequal spacing of the mesh points. Brennan and Schwartz [6] and Mason [38] have used a version of this transformation by substituting $y - ln(S)$ into equation (112).

A discussion of the works of Parkinson [44], Jaillet, Lamberton, and Lepeyre [25], Lamberton [35], and Allegretto, Barone-Adesi, and Elliott [1] follows.

4.1 Parkinson

Assume that one has an option which can only be exercised at definite times t_m, $m = 1, \ldots, N$. In the limit $N \longrightarrow \infty$, we obtain the true American option. In the time interval $t_{N-1} \leq t \leq t_N$, solve for the value $w(S, t_{N-1})$ of the option using the following equation

$$w(y,t) = e^{-r(T-t)} E_q[w(y',T) \mid y, T-t] \tag{115}$$

where q is the probability distribution defined, in terms of the density function $p(y,t) = exp(f(x)t)$ of the stock price, by

$$q(y'-y,t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} e^{ik(y'-y)+t(f(k)+ik(f(i)-r))} dk. \tag{116}$$

$w(S, t_{N-1})$ is then used as the boundary condition when (115) is applied to the computation of $w(S, t_{N-2})$, and so on, until the solution is obtained.

However, the computation done in this way will yield some put values which are less than $(X - S)^+$. Such values actually occur for European puts, since the option cannot be exercised until expiration. But for an American put, they cannot be allowed. In general, for the above procedure to be satisfactory, we must set

$$P(S, t_m) = max\left(P(S,T), e^{-r(t_{m+1}-t_m)} E_q[P(S, t_{m+1} \mid S, t_{m+1} - t_m]\right). \tag{117}$$

American puts will be more valuable than European puts because of the maximum taken in (117), whose effect is propagated back in time with each successive stage of the calculation; the longer the time until expiration, the greater the difference between them.

The above calculation is easily performed with the discrete forms of the relevant equations. For example, suppose the stock distribution is approximately lognormal, so $f(k) \approx -ika - k^2\sigma^2/2$, and

$$q(y,t) \approx \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{\frac{-(y-(r-\sigma^2/2)t)^2}{2\sigma^2 t}}. \tag{118}$$

Now proceed as follows. Define $y_m = mdy$ for all integer values of m, $b_m = (1-e^{-y_m})(-y_m)^+$, $t_j = T - jdt$ for $j \geq 0$, $P(m,j) = P(y_m, t_j)$ for $j \geq 1$, and $P(m,0) = b_m$. Then the discrete version of (117) is

$$P(n,j+1) = max(b_n, \sum_{m=-\infty}^{+\infty} q(y_n - y_m, dt)P(m,j)). \tag{119}$$

One may use an approximation to (118) for substitution into (119). Define $u = (2r/\sigma^2 - 1)dy$, $a' = 0.25u/(1+\sqrt{1+0.25u^2})$, $a = a' + 0.5$, $dt = 2a(1-a)(dy)^2/\sigma^2$, and define $c(m)$ by

$$\begin{cases} 0 & m \leq -2 \\ e^{-rdt}(1-a)^2 & m = -1 \\ e^{-rdt}2a(1-a) & m = 0 \\ e^{-rdt}a^2 & m = 1 \\ 0 & m \geq 2. \end{cases}$$

Then we may use

$$P(m,j+1) = max(b_m, c(-1)P(m-1,j) + c(0)P(m,j) + c(1)P(m+1,j)) \tag{120}$$

for $j \geq 0$ and all integer m. In the limit $dy \longrightarrow 0$, (120) converges to an exact equation for an American put, because the binomial distribution given by $c(m)$ produces the normal distribution given by (118).

For the actual numerical solution to (120), $dy \approx 0.032$ was used and $P(m, j)$ was set to 1 (respectively 0) for $m < -158$ (respectively $m > 32$), to concentrate on a practical region of interest. The numerical check performed by setting $b_m = 0$ agreed with the Black-Scholes value for the European put to three decimal places.

As expected, as r increases, a put option decreases in value. Parkinson includes a numerical comparision between formula (120) and the advertised put prices in the New York Times. On average, the percentage difference $(predicted/actual) - 1$ is -25 ± 8. Naturally, one would expect the advertised prices to be higher than the true values. From this numerical comparison, Parkinson also concludes that most of the advertised puts are significantly overpriced. In addition, the values computed with $r = 0$ are in much better agreement with the data than those with $r = 7\%$.

4.2 Jaillet, Lamberton, and Lapeyre

A detailed discussion of the Brennan and Schwartz algorithm for the valuation of the American put options (except for the logarithmic change of variable) is provided. Although the formulation of the boundary condition in Brennan and Schwartz's paper was mathematically incorrect, the algorithm is completely justified. The main advantage to this approach is that the techniques of variational inequalities provide an adequate framework for the study of numerical methods.

We begin by studying the pricing formulae of Bensoussan [4] and Karatzas [27] from the point of view of diffusion models. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $(W_t)_{t\geq 0}$ a standard Brownian motion with values in $\Re^n$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the $\mathbf{P}$-completion of the natural filtration of $(W_t)_{t\geq 0}$.

Consider a financial market with n risky assets, with prices $S_t^1, \ldots, S_t^n$ at time t, and let Y_t be the n-dimensional vector with components $Y_t^j = \log(S_t^j)$ for $j = 1, \ldots, n$. We will assume that Y_t satisfies the following stochastic differential equation:

$$dY_t = \beta(t, Y_t)dt + \sigma(t, Y_t)dW_t \tag{121}$$

on a finite interval $[0, T]$, where T is the horizon (namely, the date of maturity of the option).

We impose the following conditions on β and σ, and on the interest rate:

(H1) $\beta(t, x)$ is a bounded C^1 function from $[0, T] \times \Re^n$ into $\Re^n$, with bounded derivatives.

(H2) $\sigma(t, x)$ is a bounded C^1 function from $[0, T] \times \Re^n$ into the space of $n \times n$ matrices, with bounded derivatives. Also, σ admits bounded continuous second derivatives $\partial^2 \sigma_{i,j} / \partial x_i \partial x_j$, satisfying a Hölder condition in x, uniformly with respect to (t, x) in $[0, T] \times \Re^n$.

(H3) The entries $a_{i,j}$ of $\sigma(t, x)\sigma^*(t, x)/2$ (where $*$ denotes transposition) satisfy the following coercivity property: $\exists\eta > 0$ such that $\forall(t,x) \in [0,T]\times\Re^n$ and $\forall\xi \in \Re^n$ we have

$$\sum_{1\leq i,j\leq n} a_{i,j}(t,x)\xi_i\xi_j \geq \eta\sum_{i=1}^{n}\xi_i^2$$

(H4) The instantaneous rate of interest $r(t)$ is a C^1 function from $[0,T]$ into $[0,\infty)$.

REMARK. Condition (H3) can be interpreted in terms of the completeness of the market (Bensoussan [4], Karatzas [29, 28], Harrison and Pliska [19]).

An American contingent claim is defined by an adapted process $(h(t))_{0\leq t\leq T}$, where $h(t)$ is the payoff of the claim if exercised at time t. The typical example is an American put option: for an option on one unit of asset i, with exercise price X, one has $h(t) = (X - S_t^i)^+$. Note that, for simplicity, we do not consider contingent claims allowing a payoff rate per unit of time as in Karatzas [27]. Assume that $h(t)$ depends only on the prices of risky assets at time t, so that $h(t) = \psi(Y_t)$, where ψ is a continuous function from $\Re^n$ into $\Re$. Note that in the case of a put on asset i, one has $\psi(x_i) = (X - e^{x_i})^+$.

We will denote by $(Y_s^{t,x})_{s\geq t}$ a continuous version of the flow of the stochastic differential equation (121). Therefore, $(s,t,x) \mapsto Y_s^{t,x}(\omega)$ is continuous for almost all ω, $Y_t^{t,x} = x$, and $Y^{t,x}$ satisfies (121) on $[t,T]$.

Proposition 6 *Assume ψ is continuous and satisfies $\mid \psi(x) \mid \leq Me^{M|x|}$ for some $M > 0$, and define a function u^* on $[0,T] \times \Re^n$, by*

$$u^*(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} E(e^{-\int_t^\tau r(s)ds}\psi(Y_\tau^{t,x})), \tag{122}$$

where $\mathcal{T}_{t,T}$ is the set of all stopping times with values in $[t,T]$. Then u^* is continuous and, for any solution (Y_t) of (121)

$$u^*(t,Y_t) = ess\ \sup_{\tau \in \mathcal{T}_{t,T}} E\left(e^{-\int_t^\tau r(s)ds}\psi(Y_\tau) \mid \mathcal{F}_t\right). \tag{123}$$

Note that if, under probability $\mathbf{P}$, the discounted vector price is a martingale, then (123) means that $u^*(t,Y_t)$ is the 'fair price' of the American option defined by ψ at time t (cf. Karatzas [27], [29], [28], Bensoussan [4]).

PROOF. Using the equality

$$Y_s^{t,x} = x + \int_t^s \beta(v, Y_v^{t,x})dv + \int_t^s \sigma(v, Y_v^{t,x})dW_v \tag{124}$$

it is easy to prove that

$$E\left(\sup_{t\le s\le T} e^{M|Y_s^{t,x}|}\right) \le Ce^{M|x|}, \tag{125}$$

where C depends only on T, M, and on the bounds on β , σ. Now, observe that if $(t_1,x_1),(t_2,x_2) \in [0,T]\times\Re^n$, with $t_1 < t_2$, then

$$\begin{aligned}
&u^*(t_2,x_2) - u^*(t_1,x_1) = \\
&\sup_{\tau \in \mathcal{T}_{t_2,T}} E(e^{-\int_{t_2}^\tau r(s)ds}\psi(Y_\tau^{t_2,x_2})) - \sup_{\tau \in \mathcal{T}_{t_2,T}} E(e^{-\int_{t_1}^\tau r(s)ds}\psi(Y_\tau^{t_1,x_1})) \\
&+ \sup_{\tau \in \mathcal{T}_{t_2,T}} E(e^{-\int_{t_1}^\tau r(s)ds}\psi(Y_\tau^{t_1,x_1})) - \sup_{\tau \in \mathcal{T}_{t_1,T}} E(e^{-\int_{t_1}^\tau r(s)ds}\psi(Y_\tau^{t_1,x_1})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&| u^*(t_2, x_2) - u^*(t_1, x_1) | \\
&\quad \leq E(\sup_{t_2 \leq s \leq T} | e^{-\int_{t_2}^{s} r(v)dv} \psi(Y_s^{t_2,x_2}) - e^{-\int_{t_1}^{s} r(v)dv} \psi(Y_s^{t_1,x_1}) |) + \\
&\quad + E(\sup_{t_1 \leq s \leq t_2} | e^{-\int_{t_1}^{s} r(v)dv} \psi(Y_s^{t_1,x_1}) - e^{-\int_{t_1}^{s} r(v)dv} \psi(Y_{t_2}^{t_1,x_1}) |),
\end{aligned}$$

and the continuity of u^* follows from the continuity of ψ and of the flow and from (125) (which, applied with $2M$ instead of M, ensures uniform integrability).

The equality (123) can be proved directly, by arguing that the supremum in (122) and the essential supremum in (123) are the same when $\mathcal{T}_{t,T}$ is replaced by the set of stopping times with respect to the filtration $(\mathcal{G}_{t,s})_{s \geq t}$ of the increments $W_s - W_t$ for $s \geq t$ (cf. [24] for details). □

Next, we turn to the algorithm of Brennan and Schwartz. The inner product of two vectors $u, v \in \Re^n$ will be denoted by (u, v). Let $| u |^2 = (u, u)$, and write $u \geq v$ if $u_i \geq v_i$ for all i.

Proposition 7 *Let A be an $n \times n$ matrix and $u, \varphi, \theta \in \Re^n$. The following two systems are equivalent:*

$$Au \geq \theta, u \geq \varphi, (Au - \theta, \varphi - u) = 0, \tag{126}$$

$$u \geq \varphi, (Au - \theta, v - u) = 0 \tag{127}$$

for all $v \geq \varphi$.

□

This proposition, which expresses the linear complementarity problem as a variational inequality is well known (cf Kinderlehrer and Stampacchia, [32]). In *loc cit*, the following is also proved.

Theorem 14 *The linear complementarity problem* (127) *has a unique solution for all $0, \varphi$ if and only if A has positive principal minors.*

□

Note that this property is satisfied if A is *coercive* in the following sense: There exists $C > 0$ such that for all $x \in \Re^n$

$$(Ax, x) \geq C \mid x \mid^2 .$$

The following is a characterization of the solution.

Proposition 8 *Assume the nondiagonal coefficients of a coercive matrix are nonpositive. Then the solution of* (126) *is the smallest vector u satisfying $Au \geq 0$ and $u \geq \varphi$.*

□

In the next two propositions, we assume that A is a tridiagonal matrix of the form

$$\begin{pmatrix} b_1 & c_1 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & a_{n-1} & b_{n-1} & c_{n-1} \\ 0 & \dots & 0 & a_n & b_n \end{pmatrix} .$$

Proposition 9 *If A is a tridiagonal matrix as above with $c_i = 0$ for all $i \in \{1, \ldots, n-1\}$, and $b_i > 0$ for all $i \in \{1, \ldots, n\}$, then, for all $\theta, \varphi \in \Re^n$, the unique solution of the system* (126) *can be computed by solving the following recursive relations*

$$u_1 = (\theta_1/b_1) \vee \varphi_1$$

$$u_j = [(\theta_j - a_j u_{j-1})/b_j] \vee \varphi_j$$

for $2 \leq j \leq n$.

PROOF. This is an immediate consequence of the equalities

$$(Au)_1 = b_1 u_1$$

$$(Au)_j = b_j u_j + a_j u_{j-1}$$

for $2 \leq j \leq n$. □

Given any coercive tridiagonal matrix A as above, consider the following lower triangular matrix $\tilde{A}$

$$\begin{pmatrix} \tilde{b}_1 & 0 & 0 & \ldots & 0 \\ a_2 & \tilde{b}_2 & 0 & \ldots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & \ldots & 0 & a_n & \tilde{b}_n \end{pmatrix}$$

where $\tilde{b_n} = b_n$ and $\tilde{b_i} = b_i - [c_i/\tilde{b_{i+1}}]a_{i+1}$ for all $1 \leq i \leq n-1$.

Lemma 6 *If A is coercive then $\tilde{b}_i > 0$ for all $1 \leq i \leq n$.*

PROOF. Note that, for $1 \leq i \leq n$, $\prod_{j=i}^{n} \tilde{b}_j$ is the determinant of the matrix derived from A by deleting the first $(i-1)$ rows and columns. This matrix, being a submatrix of A, is coercive and therefore, its determinant is positive. Hence $\tilde{b}_i > 0$ for all i. □

If $\theta \in \Re^n$, denote by $\tilde{\theta}$ the vector with coordinates $\tilde{\theta}_n = \theta_n$ and $\tilde{\theta}_i = \theta_i - c_i\tilde{\theta}_{i+1}/\tilde{b}_{i+1}$ for $i < n$. Clearly, the two systems $Ax = \theta$ and $\tilde{A}x = \tilde{\theta}$ are equivalent.

Proposition 10 *Assume that A is a coercive tridiagonal matrix with $c_j \leq 0$ for all $1 \leq j \leq n-1$. If the solution u of system (126) satisfies $u_i = \varphi_i$ whenever $1 \leq i \leq k$, and $u_i > \varphi_i$ whenever $k < i$ for some $1 \leq k \leq n$, then u is also a solution to the following system:*

$$\tilde{A}u \geq \tilde{\theta}, u \geq \varphi, (\tilde{A}u - \tilde{\theta}, \varphi - u) = 0. \tag{128}$$

PROOF. Since $c_{i-1} \leq 0$ and $\tilde{b}_i > 0$ (by Lemma 6), we have $-c_{i-1}/\tilde{b}_i \geq 0$ for all $i \geq 2$. It follows that, for all $i \leq n-1$, there exist nonnegative numbers $\lambda_{i,i+1}, \ldots, \lambda_{i,n}$ such that

$$\tilde{\theta}_i = \theta_i + \lambda_{i,i+1}\theta_{i+1} + \ldots + \lambda_{i,n}\theta_n$$

and

$$(\tilde{A}u)_i = (Au)_i + \lambda_{i,i+1}(Au)_{i+1} + \ldots + \lambda_{i,n}(Au)_n.$$

Therefore, if u solves (126), we have $(\tilde{A}u)_i \geq \tilde{\theta}_i$. Moreover, by assumption, equality holds whenever $i > k$, since $(Au)_j$ is then equal to θ_j for all $j \geq i$. Hence, u solves (128). □

THE VALIDITY OF THE BRENNAN-SCHWARTZ ALGORITHM

Recall that for the classical models, the calculation of the price of the American put option reduces to solving the following system, in which we have made the change of variable $t \mapsto (T - t)$:

$$u \geq \psi,$$

$$\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial x} + ru \geq 0,$$

$$(\frac{\partial u}{\partial t} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \beta\frac{\partial u}{\partial x} + ru)(\psi - u) = 0,$$

$$u(0) = \psi,$$

with $\psi(x) = (X - e^x)^+$. The implicit discretization scheme approximates the vector

$$(u(i\Delta t, j\Delta x))_{0 \leq i \leq N, n_1 \leq j \leq n_2}$$

by

$$(\bar{u}_j^i)_{0 \leq i \leq N, n_1 \leq j \leq n_2}$$

obtained by solving the following set of equations:

$$\bar{u}_j^0 = \psi(j\Delta x) \tag{129}$$

for $n_1 + 1 \leq j \leq n_2 + 1$,

$$a\bar{u}^i_{j-1} + b\bar{u}^i_j + c\bar{u}^i_{j+1} \geq \bar{u}^{i-1}_j \quad (130)$$

$$\bar{u}^i_j \geq \psi(j\Delta x) \quad (131)$$

$$(a\bar{u}^i_{j-1} + b\bar{u}^i_j + c\bar{u}^i_{j+1} - \bar{u}^{i-1}_j)(\bar{u}^i_j - \psi(j\Delta x)) = 0 \quad (132)$$

$$\bar{u}^i_{n_1} = \psi(n_1\Delta x) \quad (133)$$

$$\bar{u}^i_{n_2} = 0 \quad (134)$$

for $1 \leq i \leq N$ and $n_1 + 1 \leq j \leq n_2 + 1$, where

$$a = -\frac{\Delta t\sigma^2}{2(\Delta x)^2} + \beta\frac{\Delta t}{2\Delta x},$$

$$b = 1 + \frac{\Delta t\sigma^2}{(\Delta x)^2} + r\Delta t,$$

$$c = -\frac{\Delta t\sigma^2}{2(\Delta x)^2} - \beta\frac{\Delta t}{2\Delta x},$$

and $N\Delta t = T$, with $[n_1\Delta x, n_2\Delta x]$ being a sufficiently large interval about $\log(X)$.

Note that (133) and (134) correspond to Dirichlet boundary conditions, whereas Brennan and Schwartz use mixed conditions, Dirichlet in $n_1\Delta x$ and Neumann in $n_2\Delta x$. The Brennan-Schwartz algorithm, as applied to the system (129)- (134), goes as follows:

(1) From (129) get $\bar{u}^0_j$ for $n_1 \leq j \leq n_2$.

(2) For $n_1 \leq j \leq n_2$, derive $(\bar{u}^i_j)$ from $(\bar{u}^{i-1}_j)$ from the following equations:

$$\bar{u}^i_{n_1} = \psi(n_1\Delta x),$$

$$(\bar{u}_j^i) = \left(\frac{1}{\tilde{b}_j}(\tilde{u}_j^{i-1} - a\bar{u}_{j-1}^{i-1})\right) \vee \psi(j\Delta x),$$

for $n_1 + 1 \le j \le n_2 - 1$,

$$\bar{u}_{n_2}^i = \psi(n_2\Delta x) = 0$$

where

$$\tilde{b}_{n_2-1} = b,$$

$$\tilde{b}_{j-1} = b_{j-1} - ca/\tilde{b}_j,$$

$$\tilde{u}_{n_2-1}^{i-1} = \bar{u}_{n_2-1}^{i-1}$$

$$\tilde{u}_{j-1}^{i-1} = \bar{u}_{j-1}^{i-1} - c\tilde{u}_j^{i-1}/\tilde{b}_j,$$

for $n_1 + 2 \le j \le n_2 - 1$.

To justify the algorithm, first restate (129)-(134) as a set of variational inequalities. Let $n = n_2 - n_1 - 1$, and let u^i be the vector in $\Re^n$ with components $u_j^i = \bar{u}_{j+n_1}^i$ for all j. Given any vector $v \in \Re^n$, denote by $\dot{v}$ the vector with components $\dot{v}_1 = v_1 - a\psi(n_1\Delta x)$ and $\dot{v}_j = v_j$ for $j > 1$, and let A be the following $n \times n$ matrix:

$$\begin{pmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ 0 & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & c \\ 0 & \dots & 0 & a & b \end{pmatrix}.$$

The system (129)-(134) is equivalent to the following set of variational inequalities:

$$\bar{u}^i_{n_1} = \psi(n_1\Delta x)$$

$$\bar{u}^i_{n_2} = \psi(n_2\Delta x)$$

for $0 \leq i \leq N$,

$$u^0 = \varphi$$

$$Au^i \geq \dot{u}^{i-1}$$

$$u^i \geq \varphi$$

$$(Au^i - \dot{u}^{i-1}, u^i - \varphi) = 0,$$

for $1 \leq i \leq N$, where φ is the vector with components $\varphi_j = \psi((n_1 + j)\Delta x)$ for all j.

It is clear that in order to justify the Brennan-Schwartz algorithm, it is sufficient to prove that the assumptions of Proposition 10 hold for the above variational inequalities. To this end, let us suppose that

$$\beta < \sigma^2/\Delta x. \tag{135}$$

The above, for example, holds when Δx is sufficiently small, and it implies that $a < 0$.

Lemma 7 *If* (135) *is satisfied, then A is a coercive matrix with the following property: if $x \neq 0$ and $Ax \geq 0$, then $x_i > 0$ for all i.*

PROOF. Using the fact that a and c are negative, we have

$$\begin{aligned}(Ax,x) &= \sum_{j=2}^{n} a x_{j-1}x_j + \sum_{j=1}^{n} b x_j^2 + \sum_{j=1}^{n-1} c x_j x_{j+1} \\ &\geq \frac{a}{2}\sum_{j=2}^{n}(x_{j-1}^2 + x_j^2) + b\sum_{j=1}^{n} x_j^2 + \frac{c}{2}\sum_{j=1}^{n-1}(x_j^2 + x_{j+1}^2) \\ &\geq (a+b+c)\sum_{j=1}^{n} x_j^2 = (1 + r\Delta t)\sum_{j=1}^{n} x_j^2.\end{aligned}$$

This proves coercivity. Next, suppose $Ax \geq 0$. Then $\tilde{A}x \geq 0$. Indeed, the numbers $\lambda_{i,j}$ introduced in the proof of Proposition 10 are positive since $a < 0$ and $c < 0$. If, in addition, $x \neq 0$, then $(\tilde{A}x)_1 > 0$, and using the negativity of a again, we find that all the components of x are positive. □

Lemma 8 *Let $\Gamma(\psi)$ be the set of all integers j such that*

$$a\psi((j-1)\Delta x) + b\psi(j\Delta x) + c\psi((j+1)\Delta x) \geq \psi(j\Delta x) > 0.$$

Then, $j \leq j_0 \in \Gamma(\psi)$ implies that $j \in \Gamma(\psi)$.

PROOF. Let $f(x) = X - e^x$ (so $\psi = f^+$). Using $c < 0$, it is easy to see that $\Gamma(\psi) \subseteq \Gamma(f)$. On the other hand, $\{j \mid \psi((j+1)\Delta x) \neq 0\} \subseteq \Gamma(\psi)$. Now for any integer j we have

$$\begin{aligned}&af((j-1)\Delta x) + bf(j\Delta x) + cf((j+1)\Delta x) - f(j\Delta x) \\ &\quad = (a+b+c-1)X - e^{j\Delta x}(ae^{-\Delta x} + b + ce^{\Delta x} - 1).\end{aligned}$$

In the above equation, the right-hand side is a monotone function of j, whose limit, as j goes to $-\infty$, is $(a+b+c-1)X = Xrh \geq 0$. It follows that $j \leq j_0 \in \Gamma(f)$ implies that $j \in \Gamma(f)$, and the result follows. □

The above results, together with the following proposition, complete the justification of the Brennan-Schwartz algorithm.

Proposition 11 *Let $u^0, u^1, \ldots, u^N$ be the solution of the above variational inequalities. Then the following hold:*
1. $u^i \geq u^{i-1}$ for all $i \in \{1, \ldots, N\}$.
2. For each i, there exists k_i such that $u^i_j = \varphi_j$ for all $j \leq k_i$, and $u^i_j > \varphi_j$ for all $j > k_i$.

PROOF. Clearly, $u^1 \geq u^0 = \varphi$. If $u^i \geq u^{i-1}$, then $Au^{i+1} \geq \dot{u}^{i-1}$, and $u^{i+1} \geq \varphi$, whence $u^{i+1} \geq u^i$ by Proposition 8.

As for (2), suppose first that $i = 1$. Sine $\varphi \neq 0$, it follows from the previous lemma that $u^1_j > 0$ for all j. Suppose there exist integers ℓ_1, ℓ_2 with $1 \leq \ell_1 \leq \ell_2 \leq n$ such that $u^1_{\ell_2} = \varphi_{\ell_2}$, and $u^1_j > \varphi_j$ for all integers $j \in [\ell_1, \ell_2 - 1]$. By reducing ℓ_1 we may assume, if necessary, that either $\ell_1 = 1$ or $u_{\ell_1 - 1} = \varphi_{\ell_1 - 1}$. From $(Au^1)_{\ell_2} \geq \varphi_{\ell_2}$ and $u^1_{\ell_2} = \varphi_{\ell_2}$, it follows that

$$au^1_{\ell_2 - 1} + b\varphi_{\ell_2} + cu^1_{\ell_2 + 1} \geq \varphi_{\ell_2},$$

and so, since a and c are negative,

$$a\varphi_{\ell_2 - 1} + b\varphi_{\ell_2} + c\varphi_{\ell_2 + 1} \geq \varphi_{\ell_2}.$$

Therefore, $n_1 + \ell_2 \in \Gamma(\varphi)$, and so by the previous lemma again, $[n_1 + 1, n_1 + \ell_2] \subset \Gamma(\varphi)$, whence $(A\varphi)_j \geq \varphi_j$ for $j \in [1, \ell_2]$. Since for all j in $[\ell_1, \ell_2)$, one has $(Au^1)_j = \varphi_j$, it follows that u^1 satisfies $(Au^1)_j \leq (A\varphi)_j$ for all j in $[\ell_1, \ell_2)$. Therefore, $v = \varphi - u^1$ satisfies the conditions $(Av)_j \geq 0$ for

$\ell_1 \leq j < \ell_2$, $v_{\ell_2} = 0$, and $v_{\ell_1} = 0$ if $\ell_1 > 1$. Applying Lemma 7 to a suitable submatrix of A, we find that $v_j \geq 0$ for $\ell_1 \leq j < \ell_2$. Hence, $u_j^1 \leq \varphi_j$ for the same range of values of j. But this is a contradiction, so (2) holds for $i = 1$. Finally, if (2) is true for some i, and $[\ell_1, \ell_2]$ is chosen such that $u_{\ell_2}^{i+1} = \varphi_{\ell_2}$ and $u_j^{i+1} > \varphi_j$ for all $j \in [\ell_1, \ell_2)$, then, using (1), we obtain $u_{\ell_2}^i = \varphi_{\ell_2}$, and so $u_j^i = \varphi_j$ for all $j \leq \ell_2$, and the same reasoning as above leads to a contradiction. □

REMARK. In the case of call options, a modified version of the Brennan-Schwartz algorithm can be justified, whereas the algorithm itself (as presented above) leads to a false solution. (The property to be used is $j \geq j_0 \in \Gamma(\psi)$ implies that $j \in \Gamma(\psi)$.)

For some functions ψ, such as $\psi(x) = |e^x - X|$, the Brennan-Schwartz algorithm fails to solve the variational inequality.

4.3 Lamberton

Lamberton aim here is to prove the convergence of the critical price in the finite difference method of Brennan and Schwartz.

For $x \in \Re$, let $\psi(x) = (X - e^x)^+$ and $\nu(t,x) = u(t, e^x)$ for $t \in [0,T]$. For $(t,x) \in [0,T] \times \Re$, introduce the stochastic processes $(Y_s^{t,x})_{t \leq s \leq T}$, defined for $s \in [t,T]$, by

$$Y_s^{t,x} = x + (r - \sigma^2/2)(s-t) + \sigma(W_s - W_t)$$

where W is a Wiener process. With these notations, we have

$$\nu(t,x) = u(t,e^x) = \sup_{\tau \in \mathcal{T}_{t,T}} E(e^{-r(\tau - t)}\psi(Y_\tau^{t,x})),$$

and it follows from the definition of $S_c(t)$ that for all $t \in [0,T)$, we have

$$\log(S_c(t)) = sup\{x \in \Re \mid \nu(t,x) = \psi(x)\}.$$

Before applying the finite difference method, we need to localize the problem in a bounded interval (see Jaillet et al [25], Section 4). Let M be a positive real number such that $e^{-M} < X < e^M$. For $(t,x) \in [0,T] \times \Re$, define

$$T_M^{t,x} = \inf\{s \in [t,T] \mid \mid Y_s^{t,x} \mid \geq M\},$$

with the convention $\inf \emptyset = T$, and define

$$\nu_M(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} E(e^{-r(\tau \wedge T_M^{t,x} - t)}\psi(Y_{\tau \wedge T_M^{t,x}}^{t,x})).$$

Obviously, we have

$$\psi(x) \leq \nu_M(t,x) \leq \nu(t,x)$$

and

$$\lim_{M \to \infty} \nu_M(t,x) = \nu(t,x).$$

Proposition 12 *1. If $(t,x) \in [0,T) \times (-M,M)$, then $\nu_M(t,x) > 0$.*

2. For $t \in [0,T)$ define $s_M(t)$ by

$$\log(s_M(t)) = sup\{x \in [-M,M) \mid \nu_M(t,x) = \psi(x)\}.$$

Then, $\log(s_M(t)) \in [-M, \log(X))$, and, for all $x \in [-M, \log(s_M(t))]$, we have $\nu_M(t,x) = \psi(x)$.

3. *The function $t \mapsto s_M(t)$ is continuous and nondecreasing on $[0,T)$ and, s_M converges uniformly to S_c as M tends to ∞.*

PROOF. To prove (1), observe that

$$\nu_M(t,x) \geq E(e^{-r(T_M^{t,x}-t)}\psi(Y_{T_M^{t,x}}^{t,x})),$$

and that $\Pr(Y_{T_M^{t,x}}^{t,x} = -M) > 0$ if $x \in (-M,M)$, and $\psi(-M) = X - e^{-M} > 0$. In order to prove (2), let $t \in [0,T)$, and

$$I_t = \{x \in [-M,M) \mid \nu_M(t,x) = \psi(x)\}.$$

It follows from (1) that $I_t \subset [-M, \log(X))$, since $\psi(x) = 0$ for $x \geq \log(X)$ and (2) will follow if we can prove that I_t is an interval. Now, let $x_1 < x_2$ belong to I_t. If $x \in [x_1, x_2]$, and we let

$$\rho_M^{t,x} = \inf\{s \in [t,T] \mid \nu_M(s, Y_s^{t,x}) = \psi(Y_s^{t,x})\},$$

then

$$\nu_M(t,x) = E(e^{-r(\rho_M^{t,x}-t)}\psi(Y_{\rho_M^{t,x}}^{t,x})).$$

By our choice of x, x_1, and x_2, we have $Y_s^{t,x} \in [x_1,x_2]$ on the set $\{s < \rho_M^{t,x}\}$, and therefore $\psi(Y_s^{t,x}) = X - e^{Y_s^{t,x}}$. But it is easy to check that the process $(e^{-r(s-t)}(X - e^{Y_s^{t,x}}))_{t \leq s \leq T}$ is a supermartingale. Hence $\nu_M(t,x) = \psi(x)$ and so $x \in I_t$.

The fact that s_M is nondecreasing follows from the fact that $t \mapsto \nu_M(t,x)$ is nonincreasing. The continuity of s_M can be proved by the same argument that establishes the continuity of S_c (cf Friedman [15], proof of Theorem 3.1).

The convergence of s_M to S_c follows easily from the fact that $\nu_M(t,x) \uparrow \nu(t,x)$ as $M \uparrow \infty$.

Now let $H_M = L^2(-M,M)$ and $V_M = H^1(-M,M) = \{f \in H_M \mid f' \in H_M\}$. For $f, g \in V_M$, let

$$\begin{aligned} a_M(f,g) &= \frac{\sigma^2}{2}\int_{-M}^{M} f'(x)g'(x)dx \\ &\quad - (r-\sigma^2/2)\int_{-M}^{M} f'(x)g(x)dx \\ &\quad + r\int_{-M}^{M} f(x)g(x)dx. \end{aligned}$$

Denote by $(.,.)_M$ the usual inner product on H_M. Now define $C_M = \{f \in V_M \mid f \geq g$ on $(-M,M)$ and $f = g$ if $x = \pm M\}$. We know that ν_M satisfies the following

$$\nu_M \in L^2([0,T], V_M)$$

$$\frac{\partial \nu_M}{\partial t} \in L^2([0,T], H_M)$$

$$\nu_M(t,.) \in C_M$$

$$-\left(\frac{\partial \nu_M}{\partial t}, w - \nu_M\right)_M + a_M(\nu_M, w - \nu_M) \geq 0$$

for all $w \in C_M$

$$\nu_M(T,.) = \psi.$$

For $h > 0$ let

$$R_h = \{m \in \mathcal{Z} \mid -M \leq (m - \frac{1}{2})h < (m + \frac{1}{2})h \leq M\},$$

$$m_1 = \min(R_h),$$

$$m_2 = \max(R_h).$$

Denote by $\chi_m^{(h)}$ the indicator function of the interval $((m - \frac{1}{2})h, (m + \frac{1}{2})h]$, and let H_h be the vector space spanned by the functions $\chi_m^{(h)}$ with $m \in R_h$. On H_h, the finite difference approximation of the operator

$$A = (\frac{\sigma^2}{2})\frac{\partial^2}{\partial x^2} + (r - \sigma^2/2)\frac{\partial}{\partial x} - r$$

is the operator A_h defined as follows: if $u = \sum_{m=m_1}^{m_2} u_m \chi_m^{(h)}$, then

$$A_h u = \sum_{m=m_1+1}^{m_2-1} (A_h u)_m \chi_m^{(h)},$$

with

$$(A_h u)_m = \frac{\sigma^2}{2h^2}(u_{m+1} - 2u_m + u_{m-1}) + \frac{r - \sigma^2/2}{2h}(u_{m+1} - u_{m-1}) - ru_m$$

for $m_1 \leq m \leq m_2$. The function ψ will be approximated by

$$\psi_h = \sum_{m \in R_h} \psi(mh)\chi_m^{(h)}.$$

Now let k be a positive number of the form $k = T/N$, with $N \in \mathcal{N}$. The finite difference approximation of ν_M is the function $\nu^{h,k}(t, x)$ defined by

$$\nu^{h,k}(t, x) = \sum_{i=1}^{N} \nu_h^i(x)\mathbf{1}_{((i-1)k, ik]}(t),$$

where $\nu_h^1, \nu_h^2, \ldots, \nu_h^N$ are in H_h and satisfy the following implicit scheme:

$$\nu_h^N = \psi_h$$

$$\nu_h^i(m_1 h) = \psi(m_1 h),$$

$$\nu_h^i(m_2h) = \psi(m_2h)$$

and for all $x \in ((m_1 + \frac{1}{2})h, (m_2 - \frac{1}{2})h)$

$$max((\nu_h^{i+1}(x) - \nu_h^i(x))/k + A_h\nu^i(x), \psi_h(x) - \nu_h^i(x)) = 0$$

for $i = 1, \ldots, N-1$.

The above system can be solved using backward regression and the Brennan-Schwartz algorithm. We know that as $h, k \to 0$, the functions $\nu^{h,k}$ converge strongly to ν_M in L^2 (Jaillet et al [25]).

By Proposition 11, for h small enough and all $t \in (0, T-k)$, there exists a number $\rho^{h,k}(t) \in [(m_1 + \frac{1}{2})h, (m_2 - \frac{1}{2})h)$ such that

$$\nu^{h,k}(t,x) = \psi_h(x)$$

for all $x \in (m_1 h, \rho^{h,k}(t))$, and

$$\nu^{h,k}(t,x) > \psi_h(x)$$

for all $x \in (\rho^{h,k}(t), (m_2 - \frac{1}{2})h)$. Since $\nu^{h,k}(t,x)$ is a nonincreasing function of t, $\rho^{h,k}$ is nonincreasing. From the proof of Proposition 11 one also knows that

$$(t,x) \in (0, T-k] \times ((m_1 + \frac{1}{2})h, (m_2 - \frac{1}{2})h) \Rightarrow \nu^{h,k}(t,x) > 0.$$

Therefore $\rho^{h,k}(t) \leq \log(X) + h$. Now define the approximate critical price

$$s^{h,k}(t) = e^{\rho^{h,k}(t)}.$$

Theorem 15 *As h and k go to zero, $s^{h,k}$ converges uniformly to s_M.*

PROOF. Due to the properties of $\rho^{h,k}$ and s_M, it suffices to show pointwise convergence on $(0,T)$. Let $t_0 \in (0,T)$. Assume that $x < \limsup s^{h,k}(t_0)$. Then, for a suitable subsequence, we have

$$\nu^{h,k}(t_0, \log(x)) = \psi_h(\log(x)).$$

Therefore, for all $(t,y) \in [t_0,T] \times (-M, \log(x)]$ one has $\nu^{h,k}(t_0,y) = \psi_h(y)$. Hence, letting $h,k \to 0$, one obtains $\nu_M(t,y) = \psi(y)$ almost everywhere on $[t_0,T] \times (-M, \log(x)]$, and so $x \leq s_M(t_0)$.

It remains to prove that $\liminf s^{h,k}(t_0) \geq s_M(t_0)$. Suppose not, and take $\varepsilon > 0$ such that $\liminf s^{h,k}(t_0) < s_M(t_0) - \varepsilon$. Since s_M is continuous, there exists $\eta > 0$ such that for all $t \in [t_0 - \eta, t_0 + \eta]$ we have

$$\liminf s^{h,k}(t_0) < s_M(t) - \varepsilon.$$

Passing to a subsequence, we may assume that

$$s^{h,k}(t_0) \leq s_M(t_0 - \eta) - \varepsilon.$$

Now if $t_0 - \eta < t < t_0$ and $\log(s_M(t_0 - \eta) - \varepsilon) < x < \log(s_M(t_0 - \eta))$, then

$$x > \log(s^{h,k}(t_0)) \geq \log(s^{h,k}(t)),$$

whence $\nu^{h,k}(t,x) > \psi_h(x)$ and, by the definition of the $\nu^{h,k}$

$$\frac{\nu^{h,k}(t+k,x) - \nu^{h,k}(t,x)}{k} + A_h \nu^{h,k}(t,x) = 0.$$

Now let

$$U = (t_0 - \eta, t_0) \times (\log(s_M(t_0 - \eta) - \varepsilon), \log(s_M(t_0 - \eta)))$$

and let ϕ be a C^∞ function with support in U. It is easy to check (using the fact that $\nu^{h,k} \to \nu_M$ in L^2) that

$$\lim_{h,k\to 0} \int (\frac{\nu^{h,k}(t+k,x) - \nu^{h,k}(t,x)}{k} + A_h \nu^{h,k}(t,x))\phi(t,x)dtdx$$
$$= \int \nu_M(-\frac{\partial\phi}{\partial t} + \frac{\sigma^2}{2}\frac{\partial\phi^2}{\partial x^2} - (r - \frac{\sigma^2}{2})\frac{\partial\phi}{\partial x} - r\phi)dtdx.$$

Hence

$$\frac{\partial\nu_M}{\partial t} + \frac{\sigma^2}{2}\frac{\partial\nu_M^2}{\partial x^2} - (r - \frac{\sigma^2}{2})\frac{\partial\nu_M}{\partial x} - r\nu_M = 0$$

in the sense of distributions on U. But on this set, $\nu_M(t,x) = \psi(x) = X - e^x$, since $s_M(t) \geq s_M(t_0 - \eta)$ if $t \geq t_0 - \eta$. Hence

$$\frac{\partial\nu_M}{\partial t} + \frac{\sigma^2}{2}\frac{\partial\nu_M^2}{\partial x^2} - (r - \frac{\sigma^2}{2})\frac{\partial\nu_M}{\partial x} - r\nu_M = -rX$$

and we have reached a contradiction. □

4.4 Allegretto, Barone-Adesi, and Elliott

As usual, $\epsilon(S,t)$ denotes the early exercise premium. From work of Carr, Jarrow, and Myneni [7], it is known that $\epsilon(S,t)$ has the following integral representation

$$\epsilon(S,t) = \int_t^T (rXe^{-r(s-t)}\Phi(-h_2) - (r-b)Se^{-(r-b)(s-t)}\Phi(-h_1))ds \qquad (136)$$

where $h_1 = (\log(S/S_c) + (b + \frac{\sigma^2}{2})(s-t))/\sigma\sqrt{s-t}$ and $h_2 = h_1 - \sigma\sqrt{s-t}$. Further, the boundary conditions imply

$$p(S_c, t) + e(S_c, t) = X - S_c, \tag{137}$$

$$\partial p/\partial S + \partial e/\partial S = -1 \text{ when } S = S_c\,. \tag{138}$$

Begin by trying a function of the form $A(t)(S/S_c)^{q(t)} = e'(S,t)$, say, for e, where $A(t)$ and $q(t)$ are functions to be determined. Substitution of e' in (137), (138), together with (99) leads to the following equation for q:

$$\frac{\sigma^2}{2}q(q-1) - r + bq + (\frac{dA/dt}{A} - \frac{q}{S_c}\frac{dS_c}{dt}) + \frac{dq}{dt}\log(S/S_c) = 0. \tag{139}$$

This equation implies that q is not independent of S, so $e(S,t)$ is not of the same form as e'. One may, however, regard e' as an approximation to e. Further, a useful approximation can be obtained by neglecting the last term of (139). That is, consider

$$\frac{\sigma^2}{2}q(q-1) - r + bq + (\frac{dA/dt}{A} - \frac{q}{S_c}\frac{dS_c}{dt}) \approx 0. \tag{140}$$

This approximation is reasonable when $\log(S/S_c)(dq/dt)$ is small, which happens when either (dq/dt) is small (at long maturities), or in a neighbourhood of S_c. Further manipulation of the equations eventually leads to

$$q^2 + (N-1)q - (M+G) \approx 0, \tag{141}$$

where $M = 2r/\sigma^2$, $N = 2b/\sigma^2$, and $G(t) = 2\sigma^{-2}A\partial p/\partial t$. As usual, the boundary conditions at infinity rule out one of the roots, leaving us with

$$q(t) \approx (1 - N - \sqrt{(1-N)^2 + 4(M + G(t))})/2. \tag{142}$$

A numerical procedure for finding S_c now suggests itself. Consider the equations

$$A(t) \approx -p(S_c, t) - S_c + X, \tag{143}$$

$$q^2 + (N-1)q - (M+G) \approx 0, \tag{144}$$

$$S_c \approx \frac{q(t)(X - p(S_c, t))}{-1 + q(t) + e^{(b-r)(T-t)}\Phi(-d_1(S_c, t))}. \tag{145}$$

For a fixed value of t, begin with an initial guess for S_c, and successively find new estimates for A, q, and S_c from the above equations. The cycle is to be repeated until the total difference between the successive approximations is less than some pre-assigned value (e.g., 10^{-4}), at which time one has hopefully obtained a good approximation to S_c.

As for the accuracy of the estimates, the integral expression (136) for e was evaluated by Bode's rule ([18], pp 649-652), and $(e(S_c, t) - A(t))/A(t)$ calculated. This percentage error was typically of the order of 2%.

To improve the accuracy, the equation (141) was modified by the introduction of a relaxation constant λ to yield

$$q^2 + (N-1)q - (M + \lambda G) = 0. \quad (146)$$

The same iterative scheme as above was followed, but now for different values of $\lambda \in [0,2]$, chosen so that $e(S_c, t) = A(t)$.

For example, with $\sigma \in [0.2, 0.4]$ and $r \in [0.04, 0.20]$, various ratios b/r were considered. For $r = b$, the following empirical formula was determined $\lambda = 1.2952 + 4.3338 \times 10^{-2} M - 4.6591 \times 10^{-3} M^2 + 2.1452 \times 10^{-4} M^3$, (recall that $M = 2r/\sigma^2$), and the error $(e(S_c, t) - A(t))/A(t)$ was well below 1% (except for very short times, where problems of computational accuracy arose). For $r = b/2$, and $\lambda = 1.227 + 0.12066M - 4.2737 \times 10^{-2} M^2 + 5.453 \times 10^{-3} M^3$, good results were again obtained. Finally, for $b = 0$, the expression $\lambda = 1.2495 - 4.15 \times 10^{-2} \sigma$ gave satisfaction. Generally, the error was of the order of 0.1% for periods longer than two years. The authors conjecture that the best value of λ is the smallest parameter value for which S_c is monotone.

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