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Full Name of Author – Nom complet de l'auteur

RANDOLPH RICHARD WOJTOWICZ

Date of Birth – Date de naissance

06 APRIL 1952

Canadian Citizen – Citoyen canadien

[x] Yes / Oui [ ] No / Non

Country of Birth – Lieu de naissance

CANADA

Permanent Address – Résidence fixe

11119 73 AVE
EDMONTON ALBERTA
T6G 0C5

### THESIS – THÈSE

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DR. F. P. VAN DE PITTE

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# KANT: A DEFENCE OF SYNTHETIC JUDGMENTS *A PRIORI* IN MATHEMATICS

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For Judy who created,

Helgard who preserved,

and most of all, my parents and Sylvia
who were always there.

ABSTRACT

In this thesis I shall attempt to defend Kant's claim that the propositions of mathematics are synthetic and known *a priori*. Kant's basic problem is the following: if he can demonstrate that the propositions of mathematics are synthetic in nature, it seems difficult to provide for their apriority; if Kant can show how the propositions of mathematics are known *a priori*, it appears difficult to argue for their syntheticity. I argue that any successful defence of both the apriority and syntheticity of mathematical propositions or judgments must take into account how Kant construes the analytic/synthetic distinction, how Kant explains his notions of intuition and pure intuition, and will eventually turn upon how mathematical concepts are constructed.

Kant maintains that mathematical concepts must be treated *in concreto*. This means that, for Kant, the propositions of mathematics have content or existential import. And it is precisely in virtue of this that Kant is able to provide for the possibility of distinguishing between analytic and synthetic judgments. Synthetic judgments have content; analytic judgments are invariant with respect to content. In order for Kant to make a formal distinction between analytic and synthetic judgments, however, I argue that this must turn upon the activity of the understanding somehow functioning at the basal level of intuition. In Chapter II, I endeavour to make clear how Kant construes the analytic/synthetic distinction; and since the success of this enterprise requires an adequate account of intuition and figurative synthesis, in Chapter III, I address the problem of intuition, and the connection between intuition and concepts, directly. Chapter IV

continues my exposition of intuition, but from the point of view of pure sensibility. In particular, I examine the Metaphysical Expositions of Space and Time and cull from them those arguments that I take are valid and of import in demonstrating the apriority and syntheticity of mathematics. With these pieces in place, in Chapter V, I direct my attention specifically to the issue of concept construction. Herein I first explore the two main lines of argumentation that purport to defend the Kantian thesis of the synthetic and *a priori* nature of the propositions of mathematics. I find in favour of neither line, and turn to a consideration of the differences between mathematics and logic, in what way a reduction of mathematics to logic is possible, and how many of the logicist's objections can be undercut or rendered otiose. I also examine the close relationship, that Kant asserts holds, between necessity and apriority. This examination naturally lends itself to an account of the Kantian sense of the transcendental conditions of necessity. And this, in turn, leads to a consideration of "real" possibility and objective validity. In drawing from the intuitionists, I argue that it is possible to identify the notion of constructive proof with what would count as a "really" possible construction in mathematics. Armed, thus, with the intuitionist's tensed notion of truth and the Kantian sense of form of intuition and formal intuition, I argue that the construction of mathematical concepts defines its own limits of possibility in virtue of the method of contruction, and its own content in virtue of the proof process in the making of mathematical judgments.

## ACKNOWLEDGEMENT

I should like to express my gratitude to the members of my committee for their assistance and fruitful comments. In particular, I am indebted to Professor F.P. Van de Pitte without whose many constructive criticisms and infinite patience the number of faults that this thesis may contain would be considerably greater.

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# CHAPTER I

# INTRODUCTION

It would be difficult to gainsay the influence of Kant's philosophy on those who succeeded him; it would be even more difficult to gainsay the great impact Kant had on philosophy in general. Whether in criticism or defence, virtually every major philosophical figure since Kant has referred to, or had something to say about, various aspects of Kant's philosophy. And this is no less true in the area of his philosophy of mathematics. Kant has been subject to the criticisms of those explorers into the foundations of mathematics such as Russell and Frege; on the other hand, whole schools of thought, such as the intuitionists, praise the intent, if not the content, of the Kantian enterprise. The intent behind Kant's philosophy of mathematics is, at least, clear. He maintains that the propositions of mathematics are both synthetic in nature and known *a priori*. Unfortunately Kant did not produce a work directed particularly to mathematics and hence the brevity and, perhaps, seeming inconsistency of some of his arguments in this area, that are to be found scattered throughout his writings, have often been misconstrued or misinterpreted. Yet to hold to a thesis that simultaneously asserts both the syntheticity and apriority of the propositions of mathematics is clearly a position that demands close argumentation. The task I set before me is a defence of just this position.

Now Kant is a systemic philosopher. Hence any reasonable attempt to defend his position on mathematics, and also maintain the spirit of Kantianism, must be approached from such a perspective. This fact, I believe, has not been given its proper due in the literature on the subject. Kant's philosophy of mathematics is embedded in a greater epistemological structure--critical idealism. Thus, if one were to separate from Kant's critical idealism only those arguments that refer specifically to mathematics, one, perhaps, makes the mistake of removing the arguments from the context in which they were embedded, and hence leaving them more readily open to misinterpretation or misrepresentation. On the other hand, by drawing upon too much of Kant's "system" in order to explain or defend apparently weak or inconsistent arguments, one runs the risk of producing a tome the size of, say, the First Critique, and losing sight of the position Kant assumes with respect to mathematics. The problem is thus twofold. One must first carefully identify those aspects of Kant's critical idealism that will provide for the most faithful rendering of his philosophy of mathematics; and then one must thoroughly examine those aspects to see if an adequate defence of Kant's position is to be found.

Under the premise, then, that Kant's philosophy of mathematics should not be separated entirely from his critical idealism, my attempt at a defence will proceed in four distinct stages. Clearly, prior to making any assertions concerning whether the propositions or judgments of mathematics are both synthetic and known *a priori*, one must come to some conclusions regarding what is to count, in any given instance, as a synthetic or analytic judgment. The first stage will, therefore, comprise an examination of how Kant construes the analytic/synthetic distinction and upon what criteria he grounds the possibility of a distinction. Although Kant proffers various arguments regarding the differences between analytic and synthetic judgments, in the final analysis I conclude that the distinction rests upon a question of content. A judgment is synthetic if it has content; mathematical judgments are synthetic because one constructs mathematical concepts in intuition. And it will turn out it is the process of construction, itself, that somehow supplies the content.

If it can be shown that the propositions of mathematics have content (and the route to cogently argue for this is tortuous, indeed), then intuition, in Kantian terms, would arguably be a necessary condition under which mathematical knowledge is possible. To protect against the possibility of the propositions of mathematics being thus synthetic and knowable only *a posteriori*, the intuition in question must be pure. Pure intuition, therefore, would arguably be a necessary condition under which the propositions of mathematics are synthetic and knowable *a priori*. Hence stages two and three of my defence of the Kantian program will comprise an account of intuition, in general, and pure intuition, in particular. Of critical importance in my survey of intuition will be a consideration of sensibility and particularity. I hold that one cannot divorce intuition from sensibility. This is not without little import for those philosophers (such as Hintikka) whose reconstruction of Kant's philosophy of mathematics mitigates or denies such a connection between sensibility and intuition. But if the objects of mathematics are, like other objects, objects of possible experience, the question remains: what status is one to grant the individual object in experience? Now Kant would argue against any philosophical position (such a Platonism) that would confer some sort of ontological status upon mathematical objects. Objects of experience and mathematical objects alike are, in the end, constituted or constructed by the understanding. Yet if the ground of unity of the object is to be located in the understanding, how then is one to come to terms with the appearance as phenomenal and the thing-in-itself as noumenal? It can be argued, I believe, that by giving intuition a more constitutive role and providing for the functioning of the understanding at the level of intuition in terms of the notion of figurative synthesis, the construction of the object, for Kant, becomes less mysterious.

Stage three of my defence will continue the examination of intuition --this time from the perspective of pure intuition. Few notions in Kant's philosophy have been subject to as much criticism as that of pure intuition. In particular, the arguments of the Metaphysical and Transcendental Expositions of space and time, it is claimed, have been conclusively refuted. And, perhaps, much of the criticism has not been unwarranted. However, many of the more common objections (such as those of Strawson, for example) I believe can be found wanting. There exists a kernel of solid, valid argumentation to be found in the Metaphysical Expositions. This kernel I hope to identify and expand upon. Clearly, the Metaphysical and Transcendental Expositions do not succeed in proving what it appears Kant wished to prove: the transcendental ideality of space and time. But, then, it would be questionable to assume that Kant's full blown transcendental idealism rests solely upon the arguments of the Expositions. What the Metaphysical Expositions do show, however, is that our representation of space and time is intuitive and singular and not merely conceptual or abstracted from experience. These notions, coupled with Kant's arguments that space and time are necessary conditions or pre-conditions for the possibility of experience, give some sense to what Kant might mean by a form of intuition. Moreover, Kant is careful to note the distinction between space and time as forms of intuition and of our formal intuition of space and time. And, armed with this distinction, in addition to the arguments of the Metaphysical Expositions that I take to be valid, Kant, I maintain, has sufficient resources to provide for the sense of concept construction in mathematics that he requires.

The fourth, and crucial, stage will therefore concern the construction of concepts in mathematics. I shall first discuss the two main lines of argumentation that purport to defend the Kantian claim that the propositions of mathematics are synthetic and known *a priori*. Brittan, whom I take as paradigmatic of one line, maintains that syntheticity accrues in virtue of the axioms upon which the various branches of mathematics are based. These axioms are synthetic, and hence any theorems derived therefrom are similarly synthetic. Hintikka, whom I take as paradigmatic of the other major line, asserts that mathematics is synthetic in virtue of the use of the logical notion of existential instantiation. I shall find in favour of neither of these lines, but will draw from some of the salient points of both that I feel are valid. I shall then readdress the issue of content. If, as I have argued, Kant maintains that the propositions of mathematics are synthetic because they have content, one must be careful in making arbitrary distinctions between mathematics and logic. When viewed from a correct perspective (in particular, by paying close attention to the status of logic as it existed in Kant's time), some sort of reduction of mathematics to logic is possible--even within the scope of Kant's philosophy. This, of course, immediately renders otiose many of the logicist's objections; but problems however, still exist. The question of apriority must also be addressed. And to this end an examination of the transcendental sense of necessity that Kant attaches to the propositions of mathematics seems called for.

Although Brittan's reconstruction, here, appears preferable, neither line is successful in fully capturing the sense of necessity Kant requires.

From an examination of the satisfaction of Kant's transcendental conditions of necessity, it is natural to turn to a consideration of "real" possibility and "really" possible construction. And this, in turn, leads to a re-examination of what it means for mathematics to have content, to treat of mathematical concepts *in concreto*. I argue that it is possible to identify the notion of constructive proof with "really" possible constructions; possessing a constructive proof of a mathematical proposition is, therefore, a necessary condition under which a mathematical concept can be represented *in concreto*. What remains, then, is to demonstrate how this representation *in concreto* necessitates an appeal to the forms of intuition. To this end both the notion of formal intuition developed earlier and the intuitionist theory of truth and meaning is of assistance. What I suggest Kant hopes to demonstrate is that the construction of mathematical concepts defines its own limits of possibility in virtue of the method of construction, and its own content in virtue of the proof process in making mathematical judgments. If this holds, then the construction of mathematical concepts is a synthetic process which adheres to a transcendental sense of necessity. Hence it becomes possible to maintain that the propositions of mathematics are synthetic and known *a priori*.

# CHAPTER II

## THE ANALYTIC/SYNTHETIC DISTINCTION

### i.

One of the major themes, perhaps arguably the central theme, in the Kantian enterprise is the possibility of synthetic judgments that are known *a priori*; my undertaking will be of a somewhat more modest scope; *i.e.* a defence of synthetic judgments known *a priori* in mathematics. In general, then, Frege enumerates four types of judgments or propositions that are logically possible:[1] (1) analytic judgments *a priori*, (2) analytic judgments *a posteriori*, (3) synthetic judgments *a priori*, and (4) synthetic judgments *a posteriori*. I take it as uncontroversial that few would dispute the existence of instances of (4); I also take it that most would admit that set of propositions or sentences that comprise (2) to be empty.[2] On the other hand, while the judgments of (1) are a matter of some contention (usually turning on what one means by the terms analytic and *a priori*), whether (3) has any instances has been the focus of a great deal of controversy. There are numerous lines of attack that can be made against the claim that (3) is a non-empty set. One can, for example, question the adequacy of Kant's distinction between analytic and synthetic judgments or propositions. One may claim that Kant is vague or confused, and that, were he not, the analyticity of mathematics, for instance, could not be gainsaid. Moreover, one could, perhaps, claim the distinction cannot be coherently made at all. It is my intent herein to argue that Kant is anything but vague. What he does assert with respect to the analytic/synthetic distinction, although not voluminous, and scattered throughout several of his works, is reasonably precise, focusing on several somewhat different sets of criteria (some or all of which can be effectively used to answer the queries of his detractors). That Kant, himself, found the distinction to be important cannot be doubted. Indeed, at times, Kant writes as if he were its originator. Perhaps even, he says, 'the distinction between analytic and synthetic judgments, has never previously been considered (B19).'[3] In any event, before the real possibility of synthetic judgments known *a priori* can be established, a critical survey of precisely how Kant construed the analytic/synthetic distinction, and his responses to various objections (or possible responses to more contemporary objections) seems in order. And again, perhaps, such a survey would prove useful as a prolegomenon to the crucial problem of the synthetic *a priori* in general.

ii.

The most oft quoted formulation of Kant's analytic/synthetic distinction appears in the Introduction to the First Critique.

> [E]ither the predicate B belongs to the subject A, as something which is (covertly) contained [*enthalten*] in the concept A; or lies outside the concept A, although it does indeed stand in connection with it. In the one case I entitle the judgment analytic, in the other synthetic. Analytic judgments (affirmative) are therefore those in which the connection of the predicate with the subject is thought through identity; those in which the connection is thought without identity should be entitled synthetic. The former, as adding nothing through the predicate to the concept of the subject, but merely breaking it up into those constituent concepts that have all along been thought in it, although confusedly, can also be entitled explicative. The latter, on the other hand, add to the concept of the subject a predicate which has not in any wise been thought in it, and which no analysis could possibly extract from it; and they may therefore be entitled ampliative (A6-7/B10-11).

Kant can herein be seen to be making at least three separable points: the first, wherein the predicate of a judgment is somehow contained in the subject concept, I shall refer to (from Quine) as the containment metaphor; the second, wherein the analyticity of judgments is somehow demonstrated by identifying an identity relation between subject and predicate, I shall entitle the identity argument; and the third I shall entitle the ampliative/explicative distinction since, here, analytic judgments can be broken down by analysis alone and are thus explicative, while synthetic judgments add to the subject concept that which cannot be provided by mere analysis, and are thus ampliative. Quine, for one, has two objections: the first refers to the passage in general, and the second to the containment metaphor in particular. Kant's proposal, Quine says, 'limits itself to statements of subject-predicate form, and it appeals to a notion of containment which is left at the metaphorical level.'[4] Now Quine's points certainly seem forceful, and if this were all Kant had to say on the matter, would, I suspect, be quite damaging to the Kantian program at a rather fundamental level. In the first place, Kant does appear to restrict himself unnecessarily to judgments of subject-predicate form. Thus, for example, while the categorial statement "All Bs are Es" (where B represents bodies and E represents extension), can be considered an analytic judgment for Kant, it seems that the hypothetical judgment, "If *a* is greater than *b*, and *b* is greater than *c*, then *a* is greater than *c*," cannot. Are we to agree with, say, Garver[5] that truth-functional tautologies (being usually either hypothetical or disjunctive in form) and the valid formulae of modern logic are to be excluded from the distinction Kant draws between analytic and synthetic judgments? Frege makes a similar point.[6] He claims that Kant "underestimated" the value of analytic judgments as a result of defining them too narrowly. With respect to the subject-predicate relation, Kant's definition of analytic and synthetic judgments is simply not exhaustive. Kant, in speaking of categorial judgments, is, in fact, referring only to Frege's universal affirmative judgments. What of, for instance, single or particular judgments--or existential judgments?[7] Now, although Kant does, in the Introduction, speak in terms of subject and predicate, he elsewhere has some rather explicit things to say about analyticity and logical truth. For Kant, the highest principle of all analytical knowledge is the principle of contradiction. The truth of any analytic judgment can only be known in accordance with this principle, and thus any proposition, the negation of which is self-contradictory [*selbst widersprechend*], is entitled to be called analytic. Indeed, 'the principle of contradiction must therefore be recognized as being the universal and completely sufficient principle of all analytic knowledge' (A151/B191). Analytic propositions, from a purely formal perspective, are those which can be designated as logical truths, *i.e.* those which, when any substitution instances are negated, are inconsistent.[8] In this vein, then, precisely those propositions which are not analytic are synthetic.

Secondly, the containment metaphor seems inadequate for the reasons both Quine and Frege bring to the fore. As Frege notes, if the analytic/synthetic distinction is considered in terms of only the subject-predicate relation, then it is not exhaustive. Now the containment metaphor surely entails such a formulation, and hence is also not exhaustive. However, as I have indicated, I do not believe that Kant is necessarily restricted to this formulation. Even so, it is difficult to make clear exactly what Kant means by "containment". As Garver points out, concepts cannot literally contain one another, and thus Quine's objection hits home: "containment" is a metaphorical notion, and, although it may be instructive or heuristically helpful, it does not make precise what one means when one asserts certain judgments are analytic or synthetic. Perhaps the identity argument is of some assistance here. Kant maintains that analytic judgments are those wherein the connection between predicate and subject is "thought through identity"; synthetic judgments are those wherein the connection is "thought without identity". If the identity argument, then, is to be of any assistance, it would seem we would be best served by attempting to make clear what Kant means by "thinking through identity". In the first place, Kant makes a distinction between judgments of identity and identical judgments. Identical judgments are, for Kant, tautologies. Kant says they are empty or void of consequence, as in the proposition: "Man is man".[9] Here the identity of subject concept with predicate concept is explicit. A more interesting case occurs when the identity of subject and predicate concepts is not explicit, yet analytical. The set of these sentences Kant would place under the rubric of judgments of identity. Now this notion of identity of concepts, Kant claims, is a formal notion (although not, perhaps, what the modern logician means by identity). Take, for example, the judgment or proposition: "All bodies are extended". Kant maintains this is an analytic proposition since 'to every $X$ to which appertains the concept of body $(a + b)$ appertains also extension $(b)$.'[10] Propositions such as, "All bodies attract", however, are synthetic since 'to every $X$ to which appertains the concept of body $(a + b)$ appertains also attraction $(c)$.'[11] Beck[12] asserts that by this Kant means that $a$ and $b$ are representations or marks of the object, $X$, under consideration. And if "$X$ is $a$" logically implies "$X$ is $b$", then the judgment is analytic and reference to $X$ is otiose. Hence to say, "$X$ is a body" logically implies "X is extended", and thus the proposition "All bodies are extended" counts as analytic. It is in this sense, then, that the judgment is a judgment of identity. When thinking of the subject concept "body", one is thinking of a conceptual complex ($a + b$); in thinking of the predicate concept "extension", one is thinking of a concept ($b$) which is identical to at least part of the subject conceptual complex.

It seems to me, however, that the identity argument still leaves problems involving the ability to distinguish between analytic and synthetic judgments unresolved. Firstly, the objections of Quine and Frege regarding Kant's being committed only to a subject-predicate relation still appear to hold. And secondly, it appears to simply shift the focus of the problem of the nebulous notion of containment from the predicate concept within the subject concept to a notion of containment within the conceptual complex. If Beck's analysis is correct, aided by the Kantian sense of identity, one can, indeed, formally distinguish between what is to count as an analytic or synthetic judgment. But how can one identify what is to count, in any given instance, as the constituent parts of the appropriate conceptual complex which comprises the complex subject concept?

Kant's third attempt to make clear the analytic/synthetic distinction offers some promise. Here Kant distinguishes between explicative and ampliative [*erweiternd*] judgments--the former being that class of judgments which is analytic, the latter being that class which is synthetic. So, for example, when one asserts that the proposition, "All bodies are extended", is analytic, what one really means is that $(x)((Bx \,\&\, Ex) \supset Ex)$ rather than $(x)(Bx \supset Ex)$. This is to say that the predicate concept in an analytic judgment can be "analyzed out" of the subject concept; and analysis of an analytic judgment serves only to make clear or explicate that which was already known, albeit perhaps confusedly. In an ampliative or synthetic judgment the predicate concept cannot be so simply "analyzed out" of the subject concept (since it is not a part of the complex of which the subject concept consists), and thus truly counts as an extension of knowledge.

It would, however, still appear that some criterion must be given in order to decide what is to count as explication or amplification, and hence as an analytic or synthetic judgment. To say that ampliative judgments extend one's knowledge by adding to the content of knowledge does not provide any substantial clues as to how one is to identify or recognize what is to count as an extension of content. Consider, for example, the mathematical proposition, 7+5=12. Now Kant would say that the concept one possesses of twelve in no way "contains" or reduces to the concepts of seven and five; and hence the judgment, 7+5=12, is synthetic and ampliative. Frege, on the other hand, would have us reduce such mathematical formulae to logic or general laws of logic.

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgments and consequently *a priori*. Arithmetic becomes simply a development of logic and every proposition of arithmetic a law of logic, albeit a derivative one ... they are laws of the laws of nature. They assert not connexions between phenomena, but connexions between judgments.[13]

Even Kant admits that logic (general, not transcendental) proceeds analytically.[14] Kant's problem is thus twofold. Firstly, he must still explain what is to count as an explicative or ampliative judgment in mathematics. Secondly, he must either block the reduction of mathematics to logic or explain how such a reduction does not count against judgments in this area being simply explicative. And it is not so obvious how, purely in terms of the arguments of the Critique of Pure Reason, he can set about demonstrating that.

Kant, however, does have an argument (one that does not appear explicitly in the Introduction, and, perhaps arguably, only implicitly in the First Critique at all) that addresses most of the objections previously raised. This argument I shall entitle the argument from content. For a precise statement we must turn to the Prolegomena; for a supporting argument we must turn to On a Discovery. In the Prolegomena Kant asserts that,

> ... whatever be their origin or logical form, there is a distinction in judgments, as to their content, according to which they are either merely explicative, adding nothing to the content of knowledge, or expansive [*erweiternd*], increasing the given knowledge. The former may be called analytical, the latter synthetical judgments.[15]

Hence whether a particular judgment is analytic and explicative or synthetic and expansive (ampliative) depends not one whit upon the origin or logical form of the judgment. It is the content (or lack thereof) of the judgment that is the determining factor. The principle of contradiction belongs to general logic insofar as it is a necessary and sufficient criterion of analytic truth, and 'holds of knowledge, merely as knowledge in general, irrespective of content' (A151/B190). Analytic judgments are thus abstract from and are invariant with respect to content. Synthetic judgments, on the other hand, are judgments made of possible experience and are therefore dependent upon the conditions of possible experience. Whereas analytic relations concern the logical relation between concept terms apart from the particular content of such concepts, synthetic judgments concern the relation of a predicate concept to the content of the subject concept, *i.e.* to an object of possible experience.[16]

Of the four preceding Kantian claims, *i.e.* the containment metaphor, the identity argument, the explication/amplification distinction and the argument from content, I take the last to be the critical one in distinguishing between analytic and synthetic judgments. And this I do for two reasons. Firstly, the argument from content most clearly addresses the objections that can be raised concerning the distinction (those of Eberhard and Maass in particular, and Frege in general), and it actually forms the basis of Kant's only extended discussion of the distinction in his On a Discovery. Secondly, it seems to me that, in setting out the criteria under which a proposition is judged to be synthetic, the first three claims can be said to be dependent, in at least some ways, upon the fourth. For example, it is the content of judgment that in fact justifies the meaningfulness of asserting that any particular proposition amplifies or merely explicates the subject concept.

If my interpretation is correct, then I suggest that most commentators or critics either misrepresent the Kantian position or attack little more than a straw man; by taking the few comments Kant proffers about the analytic/synthetic distinction in the Introduction at face value, they undertake an undermining of the possibility of synthetic judgments *a priori* from an incomplete premise set. It is a great tribute to a philosopher such as Frege that he clearly recognized the problem, and thus sets about his own resolution from, at least, a correct determination of the issue at hand.

> Now these distinctions between *a priori* and *a posteriori*, synthetic and analytic, concern, as I see it, not the content of the judgment but the justification for making the judgment. Where there is no such justification, the possibility of drawing the distinctions vanishes ... When a proposition is called *a posteriori* or analytic, in my sense, this is not a judgment about conditions, psychological, physiological and physical, which have made it possible to form the content of the proposition

> in our consciousness ... rather, it is a judgment about the ultimate ground upon which rests the justification for holding it to be true.[17]

Kant's concern is the content of a judgment; Frege's concern is with its justification or formal grounds. Yet the two positions are not that dissimilar. Kant, too, regards justification as critical, but the justification, for Kant, will reside in the process of concept construction. Before examining this aspect of Kant's philosophy of mathematics, however, I think we would be best served by examining some of the objections raised to his construal of the analytic/synthetic distinction (particularly objections raised by his contemporaries since Kant, then, at least, had the possibility of responding) and exploring the relevance of Kant's notion of intuition. To this end, then, I shall turn to the arguments Kant offers in On a Discovery and the Kant-Eberhard controversy.

iii.

As adumbrated earlier, there are several lines of attack wherein one can criticize Kant regarding his thesis of the synthetic *a priori*. I take one method of assault to be an attempt to undermine the distinction Kant makes between analytic and synthetic judgments. I also take as typical the objections raised by Eberhard and Maass (particularly those objections that appeared in the volumes of the *Philosophisches Magazin* of 1788-92 and the *Philosophisches Archiv* of 1793-94). The Eberhard-Maass attack is two-pronged. On the one hand, Eberhard questions the adequacy of Kant's distinction on the grounds of lack of precision in formulation. He then goes on to offer his own construal of what kinds of judgments are analytic or synthetic. Maass, on the other hand, poses a different problem. He asks what decision procedure is available to

Kant in order to determine, in any particular instance, whether a proposition is analytic or synthetic. There are several related issues involved in Maass' objection, not the least of which is the real vs. relative distinction concerning what is to count as an analytic or synthetic judgment, or the problem of definition in analyticity and syntheticity.

Briefly, then, Eberhard maintains that the distinction between analytic and synthetic judgments was not precisely formulated by Kant and, in fact, the distinction lies in a more fundamental difference--that between essence and attribute. There are two classes of analytic judgments: those wherein the predicate is the essence of the subject itself, and those wherein the predicate is part of the essence of the subject. For example, the judgment, "All triangles are triangles", or "All triangles are three-sided figures", would be instances of the first class (Eberhard calls then entirely identical judgments); the judgment, "All triangles are figures" or "All bodies are extended", would be instances of the second class (Eberhard calls these partially identical judgments). Moreover, one may have synthetic judgments such that the predicate is non-essential, *i.e.* Kant's synthetic judgments *a priori*, where the predicate is an attribute of the subject. And finally one has judgments wherein the predicate is known through experience only, *i.e.* synthetic judgments *a posteriori*.[18] For Eberhard, then, metaphysics contains the Kantian counterpart of synthetic judgments *a priori* where the analyticity of a judgment rests upon the principle of contradiction, and the syntheticity of a judgment rests upon the principle of sufficient reason.

Kant, naturally, retorts that the essence/attribute distinction that

Eberhard draws does not address the issue at all.

> ... if one has not already given a criterion for a synthetic *a priori* proposition, the statement that its predicate is an attribute in no way illumines its distinction from an analytic proposition. For by naming it an attribute nothing more is said than it can be derived as a necessary consequence from the essence. Whether it is derived analytically according to the principle of contradiction, or synthetically according to some other principle [sufficient reason], remains thereby undetermined.[19]

It is still the case that we must be able to distinguish whether an attribute is analytic or synthetic. To take Kant's example, in the proposition "Every body is divisible", the predicate is an attribute because it can be derived as a necessary consequence from the subject concept. Yet it is an attribute which belongs to the subject concept according to the principle of contradiction and hence is entitled analytic. Now if one asserts, "Every substance is permanent", again the predicate is an attribute since permanance is a necessary predicate of substance. Yet it is not "contained" in the subject concept and so cannot be in any way derived by analysis. The proposition is therefore synthetic. Clearly then, with regard to the possibility of synthetic judgments *a priori*, whether limited, indeed determined, by the realm of possible experience, or actually having validity in the province of metaphysics, any 'hope of explaining synthetic propositions *a priori* which have attributes of the subject as a predicate is destroyed, unless one adds to this that they are synthetic, and thus perpetrates an obvious tautology.'[20]

The fact that the analytic/synthetic distinction cannot be reduced to an essence/attribute dinstinction, however, does not absolve Kant from the responsibility of making his position clear. As it turns out, in fact, there is a much larger issue at stake here--an issue that is brought out in Kant's criticism of the role that the principle of sufficient reason plays in Eberhard's arguments. Kant notes that Eberhard seems to want the principle of sufficient reason to be taken in two ways: as a logical or formal principle and as a transcendental or material principle of knowledge. Kant asserts that, for example, "Every proposition must have a reason" is a formal principle of knowledge and is thus subordinate to the principle of contradiction. When one claims, on the other hand, that "Everything must have its reason", one is asserting a transcendental principle of knowledge, 'which no one has ever proven, or will ever prove by means of the principle of contradiction.'[21] To affirm the contrary is to "smuggle" in a principle of causality (which is actually a material or transcendental principle) in the guise of the principle of contradiction. The assertion, however, is in fact synthetic, *i.e.* is itself a synthetic judgment and thus requires an adequate ground in its own right. And it is this point that is absolutely critical. Formally one can speak of the relation between concepts (in this case presumably between subject and predicate) in abstraction from content; materially the relation between concepts takes into account the content of a given judgment, the thing(s) to which the thought is directed. Nevertheless, Kant has all along in the Critique of Pure Reason and the Prolegomena been careful to distinguish the formal from the transcendental: general logic from transcendental logic. In his Lectures on Logic (para. 36) Kant separates the formal from the material extension of knowledge. This, and the various arguments Kant offers in criticism of Eberhard, leads Allison to conclude that the question of the analytic/synthetic distinction 'as developed in

On a Discovery does not at all concern the logical form, but rather the content of a judgment,'[22] *i.e.* it is a question only to be approached epistemologically from within the province of transcendental logic.

Now I agree that it is a question of content that determines whether a judgment is synthetic or analytic. Allison's contention is that once content is brought in as a justification for being able to make the distinction, then the distinction is no longer formally grounded. Analytic judgments still concern the relation between concepts--a logical or formal relation; synthetic judgments concern the predicating a concept of an intuition--a transcendental or material relation. If this were the case then it would seem that any distinction between analytic and synthetic judgments must therefore eventually be approached through transcendental logic rather than general logic. Again, I agree that the distinction can be made transcendentally. If this were the only manner in which it could be made, however, it might not be without consequences that would, perhaps, be troublesome to a purely formal account of analyticity.[23] Beck, for one, is troubled by Kant's failure to adequately distinguish the logical from, what Beck calls, the phenomenological aspects of thought:

> Where definitions or fairly complete analyses are available, he thinks of the distinction between analytic and synthetic judgments as logical; when they are not, but rather the objects of search, he has recourse to a phenomenological criterion by virtue of which he seeks definitions through analysis of what, in the plainest sense, is "actually thought" in a concept or even "contained in" a complex experience subject to subsequent analysis.[24]

Allison, meanwhile, appears perfectly satisfied with the shift in emphasis from the logical to the epistemological. It seems to me, however, that

Beck and Allison are really addressing two somewhat different issues. Beck is searching for a criterion under which judgments can be said to be analytic or synthetic; Allison is attempting to provide for some justification whereby one is able to make the distinction between analytic and synthetic judgments in the first place.

Nevertheless, perhaps the two approaches can be reconciled. Perhaps it is possible to provide a criterion that will permit the distinction to be made on formal grounds while justifying the ability to draw the distinction epistemologically. Allison, himself, provides a clue to a solution to this problem. He points out that Kant considers two different orders of concepts. On the one hand, concepts are the result of a synthetic activity of judgment; on the other hand, concepts function as the rules or forms of judgment. Kant, in the Critique of Pure Reason, clearly wishes to distinguish between analytic and synthetic judgments on purely formal grounds. This position is further reinforced in the Prolegomena where he speaks of the content of a judgment as though it consisted in the relation between concepts. Yet, as Allison indicated, when pressed on the issue, as Kant is in On a Discovery, he speaks of the content of a concept. And it certainly sounds as if this refers to the predicating of a concept of an intuition. He then reverts to epistemological justification (what Beck calls a phenomological criterion) which appears to be inconsistent with the logical distinction he has already drawn. I take it, therefore, that what Kant requires to salvage his position, is the functioning of the understanding, in some manner, at the level of intuition. If this were the case, then the subject of a judgment would have both an intuitive and conceptual component. Kant would, thus, still have at his disposal his notion of identity in order to identify the relationship between the predicate concept and the subject complex.[25] Hence, Allison is partly correct. There are (at least) two "orders" of concepts at work in Kant's epistemology: the order of concepts that arises through synthesis and judgment, and the order of concepts that not only prescribes rules of judgment but functions at the basal level of intuition. What remains, then, is to provide an adequate decision procedure whereby the relationship between concepts in a judgment can be said to be either analytic or synthetic. Since Kant's position on definition appears to most clearly address this issue, and since Maass' criticism of Kant is directed, in part, to this area, I shall now turn to Maass' objections proper.

iv.

Maass' basic objection, as I have already indicated, focuses upon what would count as a decision procedure whereby one could, in any particular instance, decide on the analyticity or syntheticity of a given judgment. Perhaps the same judgment would for one person be analytic, for another synthetic, depending upon background knowledge and context. Thus unless one is willing to accept a relativized distinction, some universal rule for deciding each case on its own merit seems warranted. The problem for Maass is figurative (for which we should probably read psychological). Now there are, I believe, at least three separate and critical problems that issue from the objections that Maass either raises or could have raised. Firstly, Maass offers the avenue of definition as a possible candidate for the universal rule he feels is required.

> Only by means of a definition ... can we determine with certainty what is contained in a concept. We can say that a judgment is analytic if the predicate B either

> gives the definition of the subject A or some characteristic found in that definition. If it does not do so then it is synthetic.[26]

Maass then closes this avenue for Kant in his discussion of real and nominal definitions. But as can be gleaned from various of Kant's writings (and as expressed explicitly by Schulze) definitions have a rôle to play in concept construction--particularly the construction of mathematical concepts. Secondly, Maass maintains that the subject concept is variable in meaning and thus if we even entertain the possibility of a relativized analytic/synthetic distinction, it surely makes sense to ask if it is possible, by adding to the concept of the subject appropriate data or definitions, to arbitrarily create analytic judgments from those previously determined synthetic. Thirdly, there is the variability problem in general--the meta-problem: there simply is neither a criterion nor a justification that could count against a relativized distinction. Since no conceptual scheme could possibly have any propositions that could not be subject to revision, including those propositions that we call analytic truths, it is meaningless to hold on to any such hard and fast distinctions.

If one takes the approach from definition one notes immediately that there are two different kinds or classes, depending upon the content of that which is to be defined, to be considered: nominal definitions and real definitions. Now nominal definitions state the logical essence of the concept of the thing in question and are thus explanations that arbitrarily assign a particular meaning to a certain name.[27] Real definitions 'contain a clear property by which the defined object can always be known with certainty, and which makes the explained concept serviceable in application' (A241n.), and hence states the real essence of the subject that is constitued by real, and not logical, predicates (and is therefore conditioned by the limits of possible experience). What is important here, I presume, is that a real definition is not a mere substitution of names for words or descriptions, but it actually reflects a real property, a real distinguishing feature of the object which belongs to the referent class of the concept to be defined (a position not far removed from the contemporary notion of "rigid designator"). I shall not belabour the distinction between real and nominal definitions[28] other than to point out one critical issue it brings to the fore: that of real vs. logical essence. One can get a logical essence by simply reflecting upon the concepts involved in abstraction from actual content; real essences must be arrived at through experience, through that which is really possible. Beck maintains that those very features of an object that make that object an object of possible experience, are entitled to be called a real essence. But how can this be? Kant has repeatedly argued that intuition cannot reveal to us the thing-in-itself. Yet if we are dealing within an experiential framework, such intuition, as a mode of knowledge, seems required. And that is a situation which Kant could not countenance. However, if one grants Kant his Transcendental Deduction, the conditions for the possibility of experience are precisely those conditions for the possibility of the objects of experience. So if intuition is pure (thus one must, I suppose, also tentatively grant Kant at least some of the results of the Transcendental Aesthetic),[29] the real essence is indeed revealed insofar as it is an intuition of the conditions under which that thing, whose essence we are seeking, is an object for us at all. And this, in turn, actually provides for the possibility of synthetic judgments *a priori*.

> Judgments can be both synthetic and known *a priori*, therefore, only if there is something *a priori* which is not logically necessary. This is the condition of pure sensible intuitability. Without pure sensible intuition, all judgments are either analytic and *a priori* or synthetic and *a posteriori*.[30]

The preceding discussion of real and nominal definitions provides us with a springboard from which to approach the more serious difficulty of variability. How do we, indeed, decide whether a given proposition is analytic or synthetic? If it were simply a matter of definitions it would seem that the choice is arbitrary and would depend upon how much one knows about the proposition in question, how much relevant background knowledge one possesses, in what context the proposition appears, and therefore how much one can "pack into" the subject concept of the judgment. Beck notes, however, that definability is, for Kant, a much stricter notion than analyzability. To define 'means only to present the complete, original concept of a thing within the limits of its concept' (A727/B755). Completeness here refers to the clarity of the predicate; limitation means that only those predicates are applicable; originality asserts that those predicates cannot be derived from anything else. Kant maintains that one can always define an invented concept since it involves neither the nature of the understanding nor experience. But this would not be a concept of a true object and does not guarantee the possibility of the object so defined. 'There remains, therefore, no concept which allows of definition, except only those which contain an arbitrary synthesis that admits of *a priori* construction' (A729/B757), *i.e.* the concepts of mathematics. This appears to permit Beck to correctly affirm that, in general, 'it is, in fact, through organizing analytic judgments that we gradually approach definition, which is the end, not the beginning, of knowledge.'[31]

Now clearly Kant would not accept any kind of variability or arbitrariness regarding judgments that are analytic or synthetic. And the method of "packing in" definitions into a subject concept does not alter his stance one bit. The analyticity or syntheticity of a particular proposition, once correctly determined, cannot change in virtue of definitions. So, for example, "packing in" with respect to nominal definitions would have no effect on the syntheticity or analyticity of a judgment. Nominal definitions are, as Werkmeister points out,[32] essentially grammatical; and since they prescind, in general, from the content of any given judgment, they cannot alter the criterion under which the judgment was seen to be analytic or synthetic in the first place. With respect to real definitions, "packing in" merely begs the question, since a real definition defines the objects of the referent class of the concept essentially, not the mere name. Beck frames it nicely:

> A definition which will change a synthetic into an analytic judgment must either be nominal or real. If nominal it does not affect the cognitive status of the original judgment; while it may make the original sentence formally analytic, it does not give to the knowledge it expresses any logical or epistemic necessity it previously lacked. If real we must know the necessary conjunction of independent, coordinate attributes in order to make it; and this conjunction is precisely what was stated in the synthetic judgment whose status is now being disputed. All that is effected by such a procedure, we might say, is that the locus of *a priori* synthesis is shifted.[33]

Let us take a mathematical example, namely, Euler's conjecture. The conjecture, itself, has a somewhat controversial career. Can Euler's conjecture be proven? Perhaps Kant would ask if Euler's conjecture has objective validity? Originally it was thought that Euler's conjecture was valid for all polyhedra, then only convex polyhedra, then only convex polyhedra without tunnels, and so on. At one time geometricians took the conjecture to be analytically true (at least those who held geometry to be analytic--a position to which Kant obviously does not adhere). Now it would seem that Euler's conjecture holds only of certain specific types of polyhedra; and those types have been appropriately dubbed Eulerian polyhedra. Perhaps all one can effectively claim is the rather uninformative: "Euler's conjecture is valid with respect to Eulerian polyhedra." Although Lakatos, for one, tells a much longer story, in this case, at least, we have packed into the concept of polyhedron an Eulerianess that would make the assertion analytic (were it synthetic to begin with; and even Kant would accept this formulation--but notice how the locus of alleged syntheticity has shifted). Lakatos questions how precise definitions in mathematics can be made. It would seem, Lakatos argues, that if definitions are arbitrary one can never be certain that one has achieved a level of precision sufficient to guarantee that the object of a mathematical investigation is identical with the stipulative definition. Definitions in mathematics, for Kant, however, actually prescribe rules for the construction of mathematical concepts--and as such are not entirely arbitrary for they must conform to the limits of real possibility. And it is this construction, according to rules and not far removed from the notion of constructive proof for the intuitionist, that preserves syntheticity.

There are objections that can be made--many objections; I shall, at this time however, only deal with those that refer to the establishment and demonstration of the analytic/synthetic distinction itself. Frege complains, and perhaps justifiably so, that Kant 'seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful.'[34] Fruitful definitions, according to Frege, such as the continuity of a function, are those wherein all the elements are "intimately" and "organically" interconnected. Kant, however, makes his case clear in the Transcendental Doctrine of Method. Empirical concepts are made explicit, <u>not</u> defined, by producing a list of characteristics. But this 'is more properly to be regarded as merely a description than as a concept of the thing, the so-called definition is nothing more than a determining of the word' (A728/B756). Nor can concepts given *a priori* (such as cause, substance, *etc.*) be so defined since the completeness of any analysis is always in doubt. What remains, then, are concepts arbitrarily defined, but arbitrarily defined with respect to their exhibition or verification in intuition (and this means subject to the conditions of possible experience). Such definitions contain nothing more than we put there. In this sense, then, the concept is original and complete.[35] Now, mathematics, allowing of definition, is thus subject to these parameters. This is not, though, to say that Kant is a conventionalist in his approach to mathematics.[36] While Kant's position certainly permits any number of competing consistent axiomatic systems (say, for instance, Euclidean, Lobachewskian and Riemannian geometries), the problem is to ascertain which system is really possible (that is to say, which system provides the necessary and sufficient conditions for the possibility of the objects of experience). So, for example, Kant could quite easily accept Peano's postulates of arithmetic and possibly, non-standard arithmetic, as well, in terms of pure logical possibility; but he could find no possibility of application for the latter, *i.e.* non-standard arithmetic does not describe a "really possible world". Moreover, problems such as Cantor's Continuum Hypothesis would probably remain alien to Kant as they lack significant meaning for the intuitionist.

In the non-mathematical realm many other examples present themselves (examples that are, perhaps, even more clear since one, presumably, is more readily prejudiced towards associating analyticity with mathematics than with vague notions of synonymy). For instance, given the synthetic judgment, "All bodies are heavy", there is no way in which one can build the concept of heaviness into the subject concept (thus attempting to create an analytic judgment) without obviously shifting the locus of syntheticity and therefore leaving the problem unresolved. Schulze, probably in cooperation with Kant, formulates a solution in terms of synthetic judgments *a priori* which is not without value to the general problem at hand.

> Let one place just so many marks in the concept of the subject that the predicate, which he wishes to prove of the subject, can be derived from its concept through the mere principle of contradiction. This trick does not help him at all. For the Critique grants to him without dispute this kind of analytic judgment. Then it takes the concept of the subject into consideration, and it asks: how did it come about that you have placed so many different marks in this concept that already contain synthetic propositions? First prove the objective reality of your concept ... then ... prove that the other marks belong to the same thing that the

> first one belongs to without themselves belonging to the first mark. The entire dispute ... belongs merely in the logical theory of definition.[37]

V.

We have thus far seen that Kant can adequately deal with the typical, if not classical, objections of Eberhard and Maass, leaving us with a residue of positive statements regarding his formulation of the analytic/synthetic distinction, *i.e.* the containment metaphor, the identity argument, the explicative/ampliative distinction and, most important of all, the argument from content. All the objections raised so far, however, have been more or less within the context of the Kantian enterprise. What has not been considered is the meta-problem alluded to earlier. I have argued (with some promissory notes) that if one can grant that an analytic/synthetic distinction can be made at all, Kant can do so with reasonable precision, and can provide reasonable justification for such a distinction with the tools and methods his philosophy places at his disposal. But what if one denies that the distinction can be made at all, asserts that the distinction is without meaning, or that the distinction remains relative to a conceptual scheme that is, itself, subject to change? To even attempt to answer this type of objection permit me to reformulate the problem in terms of Kripkean possible world semantics.[38] Analytic propositions would then be propositions that are true of all possible worlds; synthetic propositions would be propositions true of the actual world and possibly some others as well. Propositions that are both synthetic and *a posteriori* would be true of the actual world, but not of all really possible worlds; propositions that are both synthetic and *a priori* would be true of all really possible worlds but not all possible worlds. Now the meta-critic (Quine, for example) would deny that one could coherently assert what would constitute a really possible world. What one construes as really possible is subject to revision (indeed from inside the scheme what one construes as possible at all is subject to revision --for example, the fundamental laws of logic). Brittan maintains that Kant need not give up the analytic/synthetic distinction on these grounds.

> Kant appears to have accepted the implication. That there are certain necessary propositions that can be known *a priori* is coupled by him with the claim that these propositions are not revisable. But one could surely separate the two claims ... Kant's main claim is that certain judgments are synthetic *a priori* with respect to a given body of knowledge. He has no separate argument ... for claiming that these same judgments are not revisable ... [M]oreover, even if the judgments he lists as synthetic *a priori* have since been revised or rejected, all that follows is that he was wrong about these particular judgments, and not that there are not judgments that play the role he assigns to them. Thus, Quine's argument applies only partially to Kant, in that, even if we grant Quine's point, it does not show that certain principles cannot play the kind of role Kant assigns them.[39]

I agree, in general, with Brittan's intent, but not with his formulation. It seems to me that if Kant were wrong about particular synthetic judgments *a priori*, there is no longer any guarantee that he could not be wrong about all of them--and Quine's point hits home. What is required is an adequate ground and the application will take care of itself. But elsewhere in the Critique of Pure Reason Kant does supply an argument that only a particular kind of world within a particular range of experience is really possible. If one grants Kant the Transcendental

Deduction then the conditions for the possibility of experience are simultaneously the conditions for the possibility of the objects of experience. Hence what is a really possible object for us is also what counts as a really possible world. One could, I suppose, deny these conclusions (as I suspect Quine would), but fortunately the validity of the Deduction is not at issue here. One thing that might be said in passing, however, is that the range of Kant's conclusions in the Transcendental Deduction may be interpreted in one of three ways (Kant sometimes seems to oscillate between the three): (i) over all possible experience, (ii) over all possible human experience, or (iii) over all possible human experience as we now know it. I suggest the third claim, the weakest, would be sufficient, though not without argument, to substantiate an adequate and workable formulation of the analytic/synthetic distinction.

# CHAPTER III

## INTUITION

i.

Any account of Kant's theory of mathematics would be incomplete without a concomitant account of intuition. That intuition plays a key role in the construction of mathematical concepts, for Kant, cannot be denied; what the scope of that role is remains, yet, to be decided. In this chapter I shall endeavour to make clear what, precisely, Kant meant by intuition in terms of his general epistemology. I have also been charged with the added task of providing an explanation of how the understanding can function, in at least some manner, at the level of intuition. Now I shall not pretend to solve all of the problems inherent in Kant's epistemology; my intention is to offer a reasonable explanation of Kant's notion of intuition in order to allow smooth passage to his theories of pure intuition and construction of mathematical concepts.

According to Kant, all human knowledge involves two different faculties: sensibility and understanding. Thus, for Kant, human knowledge springs from two different, yet fundamental, sources: the capacity for receiving impressions or representations, realized in intuition; and the spontaneous power of knowing the object through these representations, realized in the production of a concept. Intuitions and concepts, therefore, become two modes of knowledge or cognition [*Erkenntnisse*]. Now all knowledge intends or requires an object of which it can be called knowledge. An object cannot, however, be known by intuition. If this were the case then what need have we of the understanding? Intellectual intuition, for Kant, is perhaps possible for God (and is arguably, thus, a paradigm for any knowledge claim), but not characteristic of what one would construe as finite human knowledge. On the other hand, the understanding, by itself, cannot wholly constitute the object. For, then, what need have we of sensibility? If the sense data proffered by the object cannot in any way affect the knowledge one may claim one has of that object, then the object will be constituted solely in virtue of the powers of the understanding. And such transcendent use of concepts is forbidden by Kant. These assertions inevitably lead to Kant's famous formulation: 'Thoughts without content are empty, intuitions without concepts are blind' (A51/B75). This does leave him, however, in a somewhat peculiar, if not unenviable, situation. On the one side, one has the noumenal realm, the province of the thing-in-itself, a province of which one can never have any theoretical knowledge; on the other, one has the phenomenal realm, the province of the appearance, and one in which knowledge of the object as a representation or appearance is possible. And so the problem of the Two Worlds Dichotomy, and its concomitant dilemma of the object as thing-in-itself and the object as appearance, arises. On a cursory inspection, then, it would appear that if there is to be any knowledge at all,[1] either Kant must accept that one can have sensible knowledge of the object as it is (as a thing-in-itself) at the level of intuition, or the understanding must wholly constitute the undetermined object of experience, in virtue of its own formal rules, at the level of concept formation. It is my purpose to demonstrate the dilemma to be a false one. I hope to show that Kant has adequate resources at his disposal to allow sensibility to meaningfully affect the determination of an object of experience by the understanding. The Kantian solution revolves around what it means to be an object of experience, indeed what it means for us to know an object at all; and this, in turn, depends upon how the sensible manifold is unified by the understanding. If an object of knowledge is to have a determinate unity, this must clearly be provided in virtue of formal rules of the understanding. Yet this object of knowledge must, at the same time, have some ground, something to limit the arbitrariness of the conceptual process. And since the touchstone for sensibility is, for Kant, intuition, it is here one must turn, I think, to find the necessary direction for the resolution of the problem.

ii.

With respect to intuition [*Anschauung*] Kant, I believe, makes a five-fold distinction. Aside from it being a mode of knowledge or cognition [*eine Erkenntnis*] (A19/B33, A68/B93), intuition is also sensible (A51/B75), particular or of individual objects (A320/B377), immediate (A19/B33, A320/B377), and that to which thought as a means is directed (A19/B33). This, however, merely lists a set of conditions that must be fulfilled in order for one to have an intuition; if does not tell one how intuitions arise, or what "causes" one to have an intuition in the first place. It is sensations, Kant says, that yield intuitions. A sensation [*Empfindung*] is 'the effect of an object upon the faculty of representation, so far as we are affected by it' (A19-20/

B34). The intuition which relates to the object through sensation Kant calls empirical. Kant, moreover, designates the undetermined object of an empirical intuition as appearance [*Erscheinung*]. It is, rather, the representation that is determinate and this requires the process of synthesis provided by the faculty of judgment. Hence a somewhat crude and preliminary formulation of how knowledge can come about is proffered. The object somehow "causes" an appearance to arise by means of sensation; and the intuition which we have of this appearance, in turn, somehow serves for the predicating of appropriate concepts in judgment.

Now sensation is not, in itself, a mode of knowledge or cognition. It is simply the matter of experience. Knowledge requires judgment; it is only through judgment that any assertions concerning the truth or falsity of any particular propositions may be made.[2] 'It is therefore correct to say that the senses do not err--not because they always judge rightly but because they do not judge at all' (A293/B350). Hence that in appearance which corresponds to sensation is its matter; that in appearance which determines the manifold is its form. But a determinate unity is not, in itself, present in the manifold; it is rather prescribed by the understanding (and so, for Kant, is ideal). The sensible manifold, as such, is quite indeterminate and undifferentiated. So, it would seem, it does not make sense to speak of the unity of the object (and hence, perhaps, of an object at all) prior to categorization or judgment. And thus another account of knowledge presents itself: the union of intuitions and concepts in judgments gives rise to representations that refer to a constituted object when the sensible manifold, in all its undifferentiatedness, is thought, and made determinate according to the formal rules of the understanding. But this is as little gratifying as is the first account. Representations must be understood as having content; the content of a representation lies in sensation; and sensation comes from outside consciousness, from outside the understanding. So how is it Kant can maintain, as he does in the A-edition Deduction of the Categories, that each representation 'insofar as it is contained in a single moment, can never be anything but absolute unity' (A99)?

One story that could be told, I suppose, is that of affection. The content of a representation is a result of sensory affection. Affection is pure passivity, *i.e.* the passive reception of sensations or sense data. This being the case, it seems reasonable to ask, then, about the cause of sensations. On this issue, obviously, one can say little in a positive vein for, if one accepts the general thrust of Kant's arguments, it is that of which one, virtually by definition, cannot have knowledge. Pippin suggests[3] that any line of argumentation turning upon the cause of sensation should be ignored. The problem, according to Pippin, rather calls for an account of the nature of the effects of sensations in experience and their role in empirical knowledge, regardless of the original cause. And this, I must admit, is a position with which I find much sympathy. Kant has certainly limited any knowledge claims to the realm of phenomena and forbids any causal link between the noumenal thing-in-itself and the object of experience. It seems to me that it is to Kant's credit, perhaps one of his great achievements, that whatever characteristics a sensible manifold may intrinsically possess (I admit that even this could only be intelligently stated after much argumentation), any meaning or significance that may be attached to the unification of the manifold is in virtue of rules of the understanding known *a priori*, rules contributed by the subject which set the limits and conditions of objectivity. However, it also seems to me that to speak only of effects, to speak, as Kant does, only of a ground rather than a cause, does not make the "causal" problem disappear. And, indeed, I believe Kant does provide the framework of a reasonable account of the problem in both the Transcendental Deduction of the Categories (in particular, the solution lies in the difference between the A and B editions, and the role of figurative synthesis in B) and the Schematism of the Categories (although I admit that the locus of the account seems to reside in both cases in the somewhat mysterious activities of the productive imagination).

How, then, can one favourably interpret the problem of affection? It would appear, irrespective of whether one construes the problem as causal, that one avenue might be the differing perspectives one could take when viewing an empirical object.[4] Kant, in his Observations on the Transcendental Aesthetic, says that one must be careful to note the merely empirical distinction between the object in itself and its appearances, and the transcendental distinction wherein the empirical object is taken in its general character as appearance and entirely distinct from the thing-in-itself. This sort of assertion is also found in the Fourth Paralogism where Kant maintains that an object can be thought from two separate perspectives (and the thought, itself, can be considered in two different senses). Thus the claim is that the status of an object can be examined empirically or transcendentally. If viewed empirically, the question of the thing-in-itself does not meaningfully arise. One can speak of sensations causing our intuitions; appearances are just appearances of an external object; one can have empirical knowledge in virtue of a correspondence theory of truth as long as one recognizes that this is subject to the usual empirical constraints and open to appropriate sceptical criticism. If, on the other hand, one takes a transcendental perspective the spectre of the thing-in-itself arises. One can no longer meaningfully speak of the cause of appearance or sensation; appearance is seen for what it is--phenomenal; and one cannot have any knowledge of an object independent of the rules of the understanding that prescribe the conditions under which an object of experience is possible; knowledge, and any criterion thereof, shifts from the empirical to the transcendental realm where a coherence theory of truth subsumes a correspondence theory.[5]

Many other passages in the First Critique also seem to lend credence to this "perspectival argument". For example, in the Preface to the Second Edition, Kant supposes that his critical philosophy has shown necessary the distinction 'between things as object of experience and those same things as things in themselves' (Bxxvii, my emphasis). Even more explicitly Kant maintains that the validity of the Transcendental Deduction depends upon the teaching that the 'object is to be taken in a twofold sense, namely as appearance and as thing in itself' (Bxxvii).

Now I do not wish to gainsay the importance of the thing-in-itself in Kant's philosophy. Nor do I wish to deny that the perspectival argument is valuable in describing how one can think of a thing-in-itself. The thing-in-itself is, clearly, for Kant a necessary condition of appearance. After all, if he is to argue for his transcendental idealism and empirical realism *vis a vis* transcendental realism and empirical idealism he requires that there cannot be an appearance without anything that appears. Westphal makes the point quite forcefully.

> But the thing in itself lies at the heart of Kant's great achievement. Without it the distinction between transcendental ideality and empirical reality is vacuous. Without it the antinomies are unresolved. Without it the purported originality of the Copernican Revolution is reduced to the giving of fancy names to familiar distinctions.[6]

What I wish to say is that the perspectival argument is not sufficient. That there exists empirical knowledge seems to be undeniable except to the staunchest sceptic; but when one makes the transcendental turn and raises the meta-question of guarantees or grounds, then to simply argue that there are two perspectives one can take when viewing the same object misses the point of raising the meta-question in the first place. Surely, on a transcendental level, we are permitted to think of the thing-in-itself as the cause of sensation; but then we can think of many things. What we need, since we cannot have knowledge in this area, is some justification for what constrains us to think in this manner. And the answer to this just as surely cannot emerge from within the compass of empirical knowledge. Let us, for a moment, grant that Kant has demonstrated that the unity of the manifold is determined by the formal rules of the faculty of judgment. If the Transcendental Deduction goes through, then the necessary conditions for the possibility of experience are precisely the necessary conditions for the possibility of the objects of experience. Hence, whatever it is one can truly or falsely assert of the object must conform to how the sensible manifold is thought together according to a rule. So the correspondence theory that functions on the empirical level is replaced by the coherence theory that is manifest on the transcendental level, *i.e.* the necessary conditions for the possibility of empirical truth are found to be just those conditions for the possibility of transcendental truth. But in virtue of what can one provide a transcendental criterion of truth?[7] It cannot be simple consistency, for that would leave Kant in a position not significantly superior to, say, Leibniz. It seems to me that, when considering the sensible manifold, in order to know what general concepts to apply to particular intuitions, we must already possess some awareness or understanding of what this manifold is presenting. Thus, an account of the determination of the manifold by the understanding is only half an answer. A ground of unity from the side of sensibility must also be provided.[8]

To come to terms with the Two Worlds Dichotomy, then, is to come to terms with how the relation or connection between a manifold as an amorphous, indeterminate and undifferentiated object of thought, and the manifold as a structured, determinate and unified object of knowledge, can be explained. And this, in turn, means to come to terms with the manifold ways in which a unified, individual object can be constructed by, or presented to, consciousness. As a preliminary discussion to a possible solution of the problem, I take to be critical an examination of the *transzendentale Gegenstand*. The *Gegenstand* can be viewed as roughly equivalent with the object as it is in itself, but with positive nuances that the thing-in-itself does not carry. What more the *transzendentale Gegenstand* offers other than a mere indeterminate thought of something in general, or other than the ground of the synthetic unity of appearance that will give rise to the fully constituted empirical object, however, remains to be seen. At this stage, clearly, the manner in which sensation guides or affects the object of appearance remains undecided, but it is at least conceivable that such direction can emerge from a consideration of the regulative ideas of reason. I take this as the next step in a possible solution. The regulative ideas will provide the necessary contextual and background information to seek unity; but being regulative, and not determinative, they do not guarantee a unique interpretation of the manifold. They, however, lay the foundations for the final stage wherein a layered synthesis and interpretation of experience takes place. The categories will be found to be operating, in some fashion, at each layer of synthesis; and thus by providing for a more constitutive role at the level of intuition, the demands upon the synthesis of the manifold in imagination will not be as great.

iii.

The Transcendental Deduction of the Categories, on a minimal interpretation, purports to establish the conditions of objectivity for human knowledge. Now, Kant maintains that wherever one finds unity one finds it as a result of some synthetic process. Such synthetic unity is a necessary condition for the possibility of experience and is the product of the understanding, the source of the categories. Therefore, the general rules which will account for the unity of various sense data must be provided for, at some point, by the understanding. There is a problem, though, as Pippin notes.

> It is clear that Kant meant only for the categories to be the necessary conditions of experience, that without which no experience would be possible for us, and not that they were sufficient conditions of experience. He did not intend that our conceptual structure was a priori sufficiently complex to prescribe the "discrimination rules" by virtue of which any conceivable empirical manifold could be apprehended.[9]

Pippin's observations have some merit. It seems correct that our conceptual structures cannot be the sole arbiters of "discrimination rules". Yet the answer, to me, does appear to lie within the compass of the Deduction. I firmly believe that Kant thought he had proved much more than just the necessary conditions for the possibility of experience. Kant wished to demonstrate, as is clear, I think, in any reading, that not only are the categories the necessary conditions for the possibility of (human)[10] experience, but that they are the only conditions, that is, they are uniquely so. Hence, I suggest, the groundwork for such sufficiency conditions are implicit (if not explicit) in the text. In this view, however, it must be carefully noted that the Deduction does not stand alone in the Kantian corpus. Kant, for example, in sec. 26 draws on both the results of the Transcendental Aesthetic and the Metaphysical Deduction.[11] Moreover, I think that any reading of the Deduction is incomplete without a means of applying the categories to experience. Hence, I take it, it is incomplete without the Schematism chapter. If this is granted, then all the tools required are already present in the Deduction.

Now, at the transcendental level (or meta-level, if you like) a conundrum arises with respect to sensation. Empirically, one might say that sensation causes the appearance in intuition. Transcendentally, the appearance is seen as the undetermined object of intuition and it is the representation, through judgment, that is determinate. Hence one may well ask about the nature of the object of representation, an object that obviously cannot arise through intuition--for that would merely be yet another representation of which we could ask the same question. In the A-edition version of the Deduction Kant supplies just such an object, 'which may, therefore, be named the non-empirical, that is, transcendental object [*Gegenstand*] = *X*' (A109). The cause of sensation, though unknown in the negative realm of the noumena, has thus been assigned a positive value in terms of both the ground of objective validity and the necessary relation between the representations themselves. Hence, any unifying of the manifold must conform to that ground. And that is to say that the synthetic unity, which is a product of the categories through judgment, must conform to the conditions which make it possible for that unity to be grounded in the *transzendentale Gegenstand*. As Allison points out, this leads to the very idea or formula of objectivity,

> ... since this necessary synthetic unity of representations has already been identified with the unity of consciousness, and since this unity is itself grounded in the unity of the rule or function whereby the manifold was synthesized, we can see that the proclaimed necessity of the relation to a transcendental object is simply another way of stating, this time from the side of the object, what must be regarded as the fundamental thesis of the Transcendental Deduction.[12]

The thesis to which Allison is referring is the condition for the possibility of knowledge. This is to say that appearance in experience must stand under those *a priori* rules of synthetical unity whereby the relation of appearances in empirical intuition is possible under the conditions of the necessary unity of apperception.

This brief examination now permits one to draw some prefatory conclusions. When Kant speaks of an *Objekt* he is referring to the logical conditions to which an object of experience must conform. *Gegenstand,* on the other hand, refers to a possible object of experience which is the ground of that which appears. The *Objekt* must conform to what is known, and thus objectively valid, for the *Gegenstand.* On a meta-level, however, what do we know about the *Gegenstand überhaupt* other than that the *transzendentale Gegenstand* is the ground of the necessary unity of appearances? Surely nothing but this necessary synthetic unity; for at this level we are forbidden to draw any correspondence between the *transzendentale Gegenstand* and the thing-in-itself.

Thus far we have been considering only the A-edition version of the Transcendental Deduction. In the B-edition Kant shifts his perspective in three important areas: the capacity of synthesis is located entirely in the understanding, the notion of figurative synthesis is brought to the fore, and the concept of a *transzendentale Gegenstand* dissappears. Drawing from the previous discussion it would appear that in regard to the third area Kant requires a more complete explanation of what can affect sensibility--and Kant realized this was so. The three-fold synthesis and the *transzendentale Gegenstand,* in the A-edition, alone do not provide for such affection.

> ... according to the second edition, the object which is given to the senses [is] the object qua object

> instead of some indeterminate X. In fact, the text of this second version attests to the disappearance of the transcendental object X in Kant's argumentation. Kant speaks indistinctly of the object, thought and known (sec. 22), as if the object of knowledge and the object of thought were the same. Consequently, the object is said to be given, whereas in the first edition, the given object was a mere X.[13]

However, that the object is now given, and not a mere X, is in itself of little explicative value. A more active role of the understanding in providing for the synthetic process of unification (and thus, in passing, a more constitutive role at the level of intuition in virtue of the activity of the understanding at its level), must be possible. And all this seems to hinge on the notions of figurative and intellectual synthesis (B151-52). What Kant appears to want is for the understanding to actively function within intuition. Intellectual synthesis is of no assistance here since it is merely that activity of the understanding which relates, or brings, the manifold of intuition to the unity of apperception. But figurative synthesis makes possible the determinate representation of space and time, and thus, through imagination, synthesizes and unifies the manifold. And space and time, for Kant, belong in the province of intuition. Hence it seems that it is the figurative synthesis which unifies or determines an intuition, *i.e.* makes possible a determinate intuition (B154), by organizing the spatio-temporal manifold given by the imagination. The rub here is, of course, the role of the imagination. How much or how little does one require of a faculty whose mechanism is 'an art concealed in the depths of the human soul, whose real modes of activity nature is hardly likely ever to allow us to discover, and to have open to our eyes' (B180-81).

Permit me, then, to approach the problem somewhat indirectly. What is given, is always given in context, a successful interpretation of which can only be made with reference to that context and with regard to suitable background information. For Kant, understanding has appearance for its object; reason has the understanding and its concept for its object. Thus when one is given a manifold one must unify it according to the *a priori* rules of the understanding. There exists, then, a subjective necessity for the ordering of objects given (A305-06/B362-63). But what of objective necessity and objective ordering? Consistency, surely, is not a sufficient test for there may be an infinite number of possible consistent interpretations for a given proposition or set of propositions. Yet, Kant maintains the ordering of objects of experience must still meet the systematic demands of reason.

> The law of reason which requires us to seek for this unity, is a necessary law, since without it we should have no reason at all, and without reason no coherent employment of the understanding, and in the absence of this no sufficient criterion of empirical truth. In order, therefore, to secure an empirical criterion we have no option save to presuppose the systematic unity of nature as objectively valid and necessary (A651/B679).

The claim, I take it, is this: the necessary connection of one's representations, be it through the *transzendentale Gegenstand* or some activity of the productive imagination, must cohere within the context in which it arises, and the background system against which it is in relief. Kant repeatedly makes this claim. For instance, in the Analogies, he maintains that an order and regularity must be presupposed for nature (an order of which we can have no direct knowledge) in order that any systematic knowledge be at all possible.

Yet I am somewhat troubled by the use of the regulative ideas. Kant says they 'are not arbitrarily invented; they are imposed by the very nature of reason itself and therefore stand in necessary relation to the whole employment of the understanding' (A327/B384). But when queried as to the ground for this activity of reason, one wonders whether he is trying to have reason pull itself up by its own bootstraps.

> Reason does not here follow the order of things as they present themselves in appearance, but frames for itself with perfect spontaneiry an order of its own according to ideas, to which it adapts the empirical conditions and according to which it declares actions to be necessary, even although they have never taken place, and perhaps never will take place (A548/B576).

Reason, Kant asserts, must exhibit an empirical character since every cause presupposes a rule according to which, as an effect, certain appearances follow. Still it remains the case that these dictates of human reason are self-referentially imposed by reas[illegible]nt nowhere in the First Critique supplies, to my knowledge, any [illegible]empirical grounds.

iv.

In any event, what we determine we must determine contextually. The regulative ideas of reason provide for such a contextual determination but attach no necessity thereupon. Only if we assume an order, a preliminary meaning, can we meaningfully make judgments in experience. And without a certain amount of background already painted in, so to speak, that of which we make judgments would have nothing at all with which to cohere. There is much to be said of the unity of the object and the unity of consciousness being reciprocally related in a system-a[illegible] whole. Thus we are, indeed, further ahead than before. Appearances are not only part of a systematic unity imposed for whatever reason, by reason itself; it would also seem that there is an order to appearances. At the empirical level, appearances would be uninterpreted; at the transcendental level appearances become interpreted contextually for "goodness of fit". What I am concerned with is a transcendental explication of first order appearances. Now something like phenomenological object profiles are represented.[14] Recall that in the first edition Deduction each representation in the manifold of appearance, as it is contained in a single moment, cannot be anything but an absolute unity (A99). According to the second edition, that unity must be provided solely by the understanding in terms of a figurative synthesis. Indeed, intuition is only possible through this figurative synthesis. Now in the section concerning the Synthesis of Recognition in a Concept Kant states: 'It is only when we have thus produced synthetic unity in the manifold of intuition that we are in a position to say that we know the object' (A105). Rolf George suggests that Kemp Smith's translation is incorrect.[15] For "manifold of intuition" one should, perhaps, read "manifold of an intuition" or "manifold of the intuition". If this were the case the amorphous manifold of intuition acquires some characteristics. The addition of a definite or indefinite article in front of "intuition" seems to indicate that it is some set of sensations that makes up a particular intuition. Now whether Kemp Smith's translation is infelicitous or George's suggested rendering simply unfounded, it does make sense to say that it is some set of sensations that provide the material for the appearance of the object profile. This must, itself, in each successive temporal profile, be an absolute unity, the representation of which is generated as a unity by the figurative synthesis. Being an activity of the understanding, figurative synthesis, as such, is subject to the constraints of reason and produces unity from the manifold of an intuition in terms of contextual warrant dictated by the demands of the regulative function of reason. And the Schematism provides for the application of the categories in their temporal aspect since it specifies how the pure concepts of the understanding are to be construed as conditions of human experience, *i.e.* by being considered as modes of our time consciousness.

If this interpretation is correct, it seems to square well with the conclusions of the B-edition Deduction. The givenness of the object is explicated in terms of figurative synthesis; but the object is still given in terms of appearance. The material of appearance, sense data, now reflects a specifiable set of data particular to the manifold of an intuition. Of course, this appearance must still be interpreted, even at the level of intuition, by the understanding before any claims regarding knowledge can be made of the constituted object. However, it now makes sense to assert that the categories no longer wholly determine what an object of sense, in fact, is (this is guaranteed by the givenness of a set of sense data in the manifold of an intuition), although they still set the conditions for what it is to be a possible object of experience.

# CHAPTER IV

# PURE INTUITION

## i.

Many commentators have criticized Kant on the grounds of inconsistency regarding his presentation of space and time in the Transcendental Aesthetic and the Axioms and Anticipations. That one can so criticize Kant is, I suppose, possible, but probably not terribly profitable.[1] However, the differences in presentation are, in themselves, instructive. Kant maintains that knowledge, in general, requires two necessary ingredients: intuitions and concepts. Now one of the pervading problems and recurring techniques in Kant's philosophy is the search for the necessary conditions for the possibility of (human) experience. From the side of intuition, it would seem, in the Transcendental Aesthetic, that Kant argues for space and time being such necessary conditions; from the side of concepts it would appear that, in the Analytic of Concepts, Kant argues for the necessity of the Categories. Clearly, one cannot deny that the Analytic of Concepts at times assumes the successful demonstration of various claims of the Aesthetic (the latter parts of the B-edition Transcendental Deduction, in particular, seem to draw heavily upon these claims). Moreover, I have previously argued that figurative synthesis requires the operation of the understanding, in some manner, at the level of intuition. Indeed it seems to me that the problem of the separation of sense data and appearances, noumena and phenomena, must be continually reappraised as the <u>Critique</u> progresses. Kant's philosophy is aptly described as an architectonic. All the pieces are there, but their coherence depends, at least in part, upon a systematic exposition, upon the careful erection of self-supporting arguments. From a reading of the Transcendental Aesthetic several puzzles present themselves, puzzles involving intuition, pure intuition *a priori, etc.* The process of figurative synthesis resolves, I believe, some of these problems; others involving pure intuition and space and time remain. But just as it would have been inappropriate to bring in the notion of figurative synthesis at the level of the Transcendental Aesthetic (that is, before the necessary apparatus of the understanding had been established in the Analytic of Concepts),[2] it seems to me that a successful interpretation of pure intuition must take into account the implications of the Axioms and Anticipations of the Analytic of Principles. And it is, thus, from this perspective that I shall approach the issue of Kant's much criticized, often maligned, and undoubtedly peculiar notion of space and time.[3]

ii.

Kant claims there are four sets of principles of the understanding: the Axioms of Intuition, Anticipations of Perception, Analogies of Experience and Postulates of Empirical Thought. The former two he asserts are mathematical principles and are constitutive, the latter two are dynamical and regulative. The principles, and their concomitant proofs, purport, in the former case, to deal with the conditions which must be met within experience, if experience is to be possible and the principles to have application. The Axioms of Intuition, therefore, are not axioms in the mathematical sense of the word, but rather principles that ground the applicability of mathematical judgments within a possible experiential framework:

> While, therefore, I leave aside the principles of mathematics, I shall none the less include these [more fundamental] principles upon which the possibility and *a priori* objective validity of mathematics are grounded. These latter must be regarded as the foundation of all mathematical principles. They proceed from concepts to intuition, not from intuition to concepts (A160/B199).

The proof of the Axioms of Intuition thus 'serves only to specify the principle of the possibility of axioms in general' (A733/B761). The possibility of mathematics, in general, then is to be demonstrated by transcendental philosophy.

The principle of the Axioms of Intuition is: All intuitions are extensive magnitudes (A162/B202; in the A-edition: All appearances are, in their intuition, extensive magnitudes). A magnitude is defined as extensive, Kant says, 'when the representation of the parts makes possible, and therefore necessarily precedes, the representation of the whole [*das Ganze*]' (A162/B203). It is this formulation, and others like it or derived from it, that has been the cause of some concern by Kant's critics. In the Transcendental Aesthetic Kant holds to the doctrines of the intuitive nature of space as a whole, as essentially one [*wesentlich einig*], which makes possible our apprehension of parts of space; in the Axioms he seems to say that our representation of the parts of space is a necessary preliminary for the apprehension of space as a whole. Now, to begin with, one could question to what extent "*das Ganze*" means whole as in the sense of unity or oneness. If it does not, then much of the criticism to which the Axioms has been subjected seems amiss. The translation remains unclear. But, perhaps little is lost if we accept Kemp Smith's rendering, since a closer examination discloses that what is actually at issue here is the problem of determination or determinateness. Intuitions are, for Kant, sensible; and it can be maintained that what is sensible must clearly have some spatial and/or temporal properties in order for it to be sensible at all. The question that thus arises is twofold: how are these sensible spatial and temporal properties made determinate and under what conditions is this determination possible?. Simply put, then, these spatial (and temporal) properties, being spatial (or temporal), must admit of being additive (continuous). Being additive (continuous), they are thus extensive (intensive). And so, if intuitions are sensible, and that which is sensible is extensive (intensive), all intuitions are therefore extensive (intensive) magnitudes. But this says little about the conditions under which the determination takes place. For Kant, such a determination must be necessary; but here we are speaking of empirical intuitions and all the concomitant contingency they imply. Empirical intuition is possible, according to Kant, only on the ground of pure intuition *a priori*. Hence appearances must formally contain an intuition in space and time which conditions them *a priori*; and so the Axioms and Anticipations must address this problem directly. Nor does our simple exposition resolve the apparent inconsistency between the notions of space and time in the Aesthetic and the Analytic of Principles, for we are still at a loss to explain how this determination in space and time takes place. Merely noting that objects present themselves spatially and temporally does not reconcile the subordination of part to whole in the Aesthetic with the creation of a whole from parts in the Analytic.

Recall, for a moment, the location of the Axioms with respect to the entire First Critique. It occurs after Kant has laid out the Metaphysical and Transcendental Deductions of the Categories. Recall also that the Transcendental Deduction purports to demonstrate that the conditions for the possibility of experience are the conditions for the possibility of the objects of experience (this claim is reiterated at A158/B197 of the Analytic of Principles). Brittan points out[4] that the Axioms are necessary conditions of the unity of consciousness insofar as they make possible the concept of a public objective world in which individuals can determinately locate themselves. Hence, objects are objects of experience only if they are determinate. Brittan refers to this as a fundamental condition of objectivity and asserts that the Axioms thus deliver us "objectivity concepts".

Brittan maintains, and rightly so, that there are two distinct arguments present in the Axioms--both of which involve the notion of what he calls "objectivity concepts". From A162/B202 to A163/B204 Kant argues progressively from the unity of consciousness to the Axioms as necessary conditions for the possibility of this unity; from A163/B204 to A166/B207 Kant argues regressively from an established body of mathematical truths to the Axioms as an *a priori* condition for the possibility of knowledge of these truths. It is the former argument that I take to be critical, although the latter, I believe, carries with it some interesting nuances.

Now appearances serve as the undetermined object of an empirical intuition. Appearances have two aspects: the material or that which corresponds to sensation, and the formal or that which determines the manifold. The formal aspect contains an intuition in space and time which determines the manifold *a priori* and which can neither be apprehended nor,

> ... taken up into empirical consciousness, save through that synthesis of the manifold whereby the representations of a determinate space or time are generated, that is, through combination of the homogeneous manifold and consciousness of its synthetic unity. Consciousness of the synthetic unity of the manifold [and] homogeneous in intuition in general, in so far as the representation of an object first becomes possible by means of it, is, however, the concept of a magnitude (quantum) (A162/B202-03).

What Kant wants to argue is this: the Transcendental Deduction has demonstrated that the perception of an object as appearance is only possible through a synthesis, or series of syntheses, whereby the object is constituted as an object of consciousness;[5] the manner in which an object as appearance is constituted as an extensive magnitude is possible only through a successive synthesis whereby the unity of the combination of the manifold is thought; and these respective syntheses are one and the same:

> ... appearances are all without exception magnitudes, indeed extensive magnitudes. As intuitions in space or time, they must be represented through the same synthesis whereby space and time in general are determined (A162/B203).

One result of the Transcendental Deduction is to lay out an account of what is to count as an object of human experience, to make clear the conditions of objectivity. The Axioms purport to fulfill what Brittan would term the application of "objectivity concepts". And the condition under which this obtains is through a determination of the object in a successive synthesis. Moreover, the Deduction has putatively already shown that a unity of intuition can arise out of the manifold only through an act of synthesis which must be exercised *a priori*. If this were not the case then one could never have *a priori* the representation of either space or time (A99).[6] But this, I take it, is precisely the synthesis required for the generation of extensive (intensive) magnitudes. Kant's example of the representation of a line of thought, I think, demonstrates just this. The line is generated 'from a point all its parts one after another' (A163/B203). That a line must be so generated is moot. Take, for instance, the intersection of two planes --but Kant's point remains clear, I believe, in any event, for then might one not speak of the generation of the plane from lines, one after another, and so on? To represent the line as an extensive magnitude simply is to represent it as an object of experience.

Kant's argument from an established body of synthetic mathematical propositions known *a priori* I take to be less successful, but rather more instructive. Leaving aside for a moment, the obvious fact that the position that mathematics as a discipline is, in general, a body of knowledge the nature of the propositions of which is synthetic and known *a priori*, is surely a thesis that must be argued for and not given, the arguments here touch the heart of Kant's critical idealism: the notion of the transcendental ideality of space and time. It seems to me that the previous claims made in regard to the Axioms do not necessitate adherence to a transcendental idealist picture of space and time. Such a picture might prove sufficient to substantiate these claims, but other stories, perhaps many other stories, could be told.[7] The argument from mathematics, however, supplies a different twist. By definition, measurement of any sort presupposes an extensive magnitude; hence measureability presupposes the Axioms. But for Kant a metric is brought to space and time by the understanding.

> Unless we assume satisfaction of the appropriate topological and metrical conditions, measurements cannot be carried out. In other words, statements assigning numbers to objects, for instance, that projectile *a* has terminal velocity *ϕ*, have no truth value. But the "possibility" of a mathematical physics, of which Newtonian physics is the paradigm for Kant, depends on such measurements.[8]

Yet on what basis does the understanding determine the nature of the metric which it brings to space and time? Clearly it may provide any sort of metric and may do so arbitrarily. But this is not what Kant wishes to argue. On the other hand, Kant does not want to assert that space and time have an intrinsic metric that is merely "read off" by the understanding. If Kant's argument so far is sound, then he has shown that the synthesis whereby appearances as extensive magnitudes are generated is precisely that same synthesis whereby space and time, in general, are determined. So the mathematics of space, *i.e.* geometry, is grounded upon this successive synthesis of the productive imagination in the generation of figures. 'This is the basis of the axioms which formulate the conditions of sensible *a priori* intuition under which alone the schema of a pure concept of outer appearance can arise' (A163/B204). Hence, if it is the same synthesis that constitutes the object as appearance in the Transcendental Deduction that also provides for the possibility of a metric, then what geometry asserts of pure intuition would similarly be valid for empirical intuition.

> The synthesis of spaces and times, being a synthesis of the essential forms of all intuition, is what makes possible the apprehension of appearance, and consequently every outer appearance and all knowledge of the objects of such experience. Whatever pure mathematics establishes in regard to the synthesis of the form of apprehension is also necessarily valid of the objects apprehended (A165-66/B206).

Against the Newtonian Kant would argue that the metric cannot be intrinsic since space is not a possible object of experience; against, say, Poincare, Kant would argue that there is a sense in which the choice of geometries is not conventional; against Reichenbach, he would assert that although our choice of congruence standards may be conventional, this does not necessarily mean that the metric, itself, is conventional. These contrasting positions, and many others (relativity, Neo-Newtonian spacetime, quantum physics, *etc.*), are certainly quite relevant to the Kantian enterprise and demand some sort of response. However, it seems that Kant's position is somehow more fundamental (in the sense that it would ground the possibility or non-possibility of alternative metrics and geometries). Although it appears he would wish to affirm the veracity of Euclidean geometry, Newtonian mechanics, and the like, it is not necessary that he be read as such. What he does maintain, in fact, depends upon his rather peculiar notion of pure intuition. In light of this, then, I turn to the Transcendental Aesthetic and various other accounts where some of the objections that may be made may be met directly.

iii.

From the perspective of the Axioms, if Kant can show the objective validity of space he will have the genesis of an answer to the relationist and conventionalist. To this end, Kant will attempt to argue for a notion of space that is dependent upon a further notion of pure intuition. That is to say, he must demonstrate that space is neither a concept nor an abstraction of empirical intuition and thus, perhaps, a convenient fiction. Against the realist, on the other hand, Kant will argue that our knowledge of space is *a priori* and that we cannot have any direct intuition of space as an absolute entity.[9] Kant's investigation will give rise to what he will maintain is the only correct position: the transcendental ideality of space and time.

As I indicated earlier, although my concern will be with Kant's theory of space and time, the arguments of the Metaphysical Expositions of Time closely parallel those arguments with respect to space. I shall thus content myself with Kant's exposition of space, while merely highlighting those differences that exist in his exposition of time. In this vein, then, Kant presents several arguments that he asserts, taken together, will vindicate his transcendental idealism. The arguments that I shall consider are the four Metaphysical Expositions (ME) as enumerated in the B-edition, the Transcendental Exposition (TE), and the argument of Incongruent Counterparts (ICs).[10] ME-1 and ME-2 purport to show that our knowledge of space, our manner of cognizing space, is *a priori* and not *a posteriori* or empirical; ME-3 and ME-4 purport to show that our knowledge of space is not discursive and lies in intuition;[11] the TE presupposes that space has some determinate structure and purports to show that our knowledge of this very structure must be *a priori* and intuitive; the ICs argument appears to specifically attack the relationist position and is therefore of value in a more or less negative sense. What I hope to accomplish is this: after examining these arguments for validity and soundness, I wish to cull from them those claims that I take can be adequately substantiated; I shall then develop what I take would be a minimally acceptable position from which Kant can argue, within the context of critical philosophy, for the synthetic and *a priori* nature of the propositions of mathematics.

The Metaphysical Expositions, as Pippin notes 'are best viewed as establishing what must be the case if we are able to represent numerically distinct objects or regions of space'.[12] ME-1 seems to underscore this thesis. Space, Kant asserts, is not an empirical concept derived from outer experience. In some respect, however, one's notion of space is acquired empirically. One has perceptual experiences insofar as sensation affects the faculty of representation and yields intuition. Nevetheless, as I have elsewhere argued, for Kant the unity of the sensible manifold in terms of which one could speak of an object of experience is not provided by the manifold itself, but rather by the subject. To be sure, there must be something in the various sets of sense data that affects the pure passivity of reception, but the unity is possible only by a synthesizing activity provided by the subject (an examination of figurative synthesis was crucial in this regard). Now it is not my intent to embroil myself in the pros and cons of the empiricists's position--~~and I shall~~ not do so. I simply take it that lurking in the background of any empiricism is some sort of transcendental mechanism that makes sense of, or gives meaning to, the empirical framework. The interesting task, then, is to uncover or discover the structure of this mechanism in providing a coherent account of experience. And the entire Kantian enterprise, it seems to me, is directed in one way or another to this problem. Thus what does it mean to say that I refer to something, other than myself, outside of myself?[13] Perception in this sense (not necessarily the strong Kantian sense) merely means that I am immediately aware of something other outside of me. Any meaning that might accrue from this otherness seems to presuppose an activity of the subject that provides for the possibility of meaning. To say that the subject interprets the sense data does not gainsay that there might be something in the sense data that might make the difference in interpretation; but the interpretation is, essentially, a product of the subject. The question, then, is this: is the notion of space (and time) presupposed by the ability to represent the otherness (or coexistence and succession) that is perceived?

Suppose, for example, one maintains that one's notion of space is obtained by abstraction from the perceived object(s). Kant would probably retort that the object, as an object of experience, has already gone through successive syntheses, syntheses provided by the subject. Thus the "object of experience" is already a constituted "object". What of the sense data then? The apprehension of mere sensation, for Kant, takes place in an instant and thus 'sensation is not in itself an objective representation, and since neither the intuition of space nor that of time is to be met within it, its magnitude is not extensive but intensive' (A166/B208). Given this, it seems to me, Kant's point is incontrovertible. If there is nothing in sensation which admits of spatiality (as we represent it), there is nothing in sensation from which spatiality can be abstracted. However, this is surely not a position which is indubitably true. The arguments of the Axioms and Anticipations are directed towards providing a metric for space and time on the basis of the manner in which an object is constituted and thus is objectively valid; the object is not so constructed as a consequence of the imposition of a metric alone (although the imposition of a metric may be a necessary condition for construction) since the metric, itself, depends in some way upon our conceptualizing process. You will recall that from the Axioms the same synthesis that constitutes the object as appearance also provides for the possibility of a metric. Thus the construction of the object and the imposition of a metric go hand in hand. This line of refutation, thus, appears merely to beg the question.[14] Consider then, how this reflective, abstractive process of generating notions of space and time out of empirical intuition would work. Kant could say that this very process presupposes some faculty or "innate" capacity of making spatial (or temporal) that which is intuited in sensation. Such a claim may seem trivial, but it says more than, for instance, Strawson claims.

> It is difficult to extract anything from it [ME-1] remotely to the purpose except the tautology that we could not become aware of objects as spatially related unless we had the capacity to do so. If the "pre-supposing" of the "representation of space" means more than this, the argument, by itself, sheds no light on what more it means.[15]

Kant is careful to speak of the representation of space in ME-1, not of space in itself (whatever that might mean). He therefore is not making any claims about the nature of space, *prima facia*, in this passage: rather his arguments pertain to our cognitive awareness of how certain sensations may be referred to something outside of us, and how they may be represented as numerically distinct (*i.e. als ausser- und neben einander*) by our cognitive faculties. Hence, the argument, itself, says nothing about whether that which is in sensation possesses spatial qualities from which a notion of space can be abstracted. This may, or may not, yet be the case. His arguments do assert that any representation of an object in space presupposes a representation of space, that is not empirically derived, such that it is possible to represent numerically distinct objects.[16]

ME-2 is a peculiar argument. In ME-1 Kant wanted to prove that one's representation of space could not be derived empirically; in ME-2 he wishes to demonstrate that space is a necessary *a priori* representation which underlies all outer intuition. The proof, however, leaves something to be desired. There are, as I see it, two possible interpretations. Firstly, one could read the argument as a thought experiment. This is suggested by the manner in which Kant formulates the first premise of the argument: 'We can never represent to ourselves the absence of space, though we can quite well think it empty of objects' (A24/B38-39). Broad indicates that if by "represent" Kant means "conceive" then the mere possibility of there being extended objects involves the actuality of space. This interpretation seems to me to be too strong and really involves a notion of absolute space (a position Kant could scarcely have held at the time of the writing of the First Critique). Therefore, Kant simply could not mean by the phrase "we can quite well think it [space] empty of objects" that space is some sort of container without which objects would not be possible, *i.e.* the actuality of space is possible without objects. What he could mean leads us to the second interpretation: objects could not be represented except as in some determinate space. This claim plays on what Kant means by "represent" and "think". One can "think" many things insofar as they are not inconsistent. "Representation", in general, has subordinate to it representation within consciousness (*Perceptio*). 'A perception which relates solely to the subject as the modification of its state is sensation (*sensatio*), an objective perception is knowledge (*cognitio*)' (A320/B376-77). From our discussions of the Axioms we found that Kant was searching for a guarantee of the application of "objectivity concepts". In ME-1, I think "representation" refers to real possibility. One can, thus, very well think an object in and out of existence without thinking out of existence the volume of space it determinately occupies. Nowhere does Kant argue for the logical necessity of space. To the contrary, if space were logically necessary, then Kant would have a difficult time, indeed, arguing for its intuitive nature; rather it is the case of this space in which one thinks things in and out of existence being already a determinate space. When one speaks of the real possibility of "representing" the absence of space, one is confronted with the problem of how one can represent anything at all except as, in some sense, spatial.[17] Moreover, Kant is quite clear, and repeatedly states, that space (and time) is not, itself, a possible object of experience. Experience (in the strong Kantian sense of the word) simply demands the ability to represent spatially (or temporally) and thus the ability to represent that thing or object determinately as <u>in</u> space (or time).

If this interpretation is correct, then it would appear that one can take ME-2 as an extension of ME-1. In some fundamental sense, therefore, space <u>can</u> be regarded as a condition of the possibility of appearance (the "can" could be replaced by "must", I believe, only with further argumentation--but this Kant provides in ME-3 and ME-4). Hence, insofar as space is a condition of the possibility of appearance, it is *a priori*. This, I believe, one could grant in at least the Strawsonian austere sense of the word. Allison makes a similar point.

> ... the extension and figure of body is the primary content of the representation of space. Since this content remains when one abstracts from the other properties and relations thought in connection with the representation of a body, whereas these do not remain when one abstracts from it, it cannot be viewed as derived from these other properties and relations. In a word, it is a priori.[18]

So, when Kant maintains that space is a necessary *a priori* representation that underlies all outer intuition, in a minimal sense one can take him as asserting that all objects of outer sense encountered in intuition are so encountered in virtue of a representation of space that simultaneously arises with (and is not subordinate to) the determination of said object. Now Kant will, of course, go on to argue that since the representation of space *a priori* is found to be a necessary condition for the possibility of human experience, then it must be subjectively contributed and its transcendental ideality follows as a direct consequence. Just how far I wish to pursue this tack I shall leave open for the moment; but surely Kant owes us some explanation of the nature of this representation of space that underlies all outer intuition. To say, for instance, that space is not a discursive or general concept generated out of the relations that hold between objects that are putatively "in" space is to deny certain reductionist or relationist theses on the nature of space; to affirm that space is an intuition *a priori* is to assert something positive about our manner of cognizing space. And ME-3 argues both these points.

The crucial premise upon which ME-3 hinges is to be found in the claim that space is unique: 'we can represent to ourselves only one space; and if we speak of diverse spaces, we mean thereby only parts of one and the same unique space' (A25/B39). This claim, in itself, says nothing, of course, as to whether this one unique space is Euclidean, Riemannian, Lobachewskian, Minkowskian spacetime, *etc.*. At this stage, the geometrization of space would still be an empirical issue. Now, if one grants this premise, [illegible] many of Kant's conclusions follow. For example, if one could represent to oneself only one space, within the context of Kant's critical philosophy, that representation must be grounded in intuition.[19] And since the representation of space underlies all outer intuition it belongs to the conditions of experience in even Strawson's austere sense of *a priori*; hence, for Kant, our intuition of space must be pure. But what does it mean to say that we can represent only one space? Borad suggests that Space (with a capital "s") serves as the grammatical subject of sentences 'which cannot be replaced by statements about bodies, actual or possible, and their actual or possible spatial relations'.[20] The term "Space", then, in this sense serves as a sort of proper name. Yet surely this cannot be the sense in which Kant construes space. As I have previously indicated, space is not a possible object of experience for Kant; it is a form of sensibility underlying *a priori* any representation one might have of objects of outer sense. It is true that Kant would not aver that space, as such, is reducible to descriptions that relate objects of experience in spatial terms; it is false to claim that Kant would assert the kind of substantival space that Broad suggests.

Kant says that the parts of space cannot precede (representatively) the representation of the "one all-embracing space". To conceive of distinct, diverse spaces depends upon a notion of limitation of the "one space". And there is a very real difference between space as a whole and parts or sub-regions of space, and the properties of things in general and their instantiations. One can, perhaps, generate a general concept of, say, book or dog. When one represents to oneself the concept of book or dog, one recognizes that there are an infinite number of possible representations which could instantiate that concept. When one "forms" a representation of space, however, one does not similarly recognize the possibility of instantiating an infinite number of possible parts of space. One "thinks" of all of the parts or sub-regions of space as coexisting; one does not "think" of all of the possible instantiations of book or dog, *i.e.* all particular books or dogs, as coexisting (if they exist at all--but I shan't dwell on what it would mean for fictional objects or entities to be instantiated). This line of argumentation is, I think, the force behind ME-4, the force behind what Kant intends when he refers to the ability of a concept to contain an infinite number of representations under itself but not within itself (B40). One perceives objects and events (and does so in virtue of their taking place in space and time); one does not perceive space itself or time itself. The concept one has of individual, distinct or diverse spaces is arrived at by a process of limitation of space as a whole. For example, the space of the book before me is not an instance of Space (as this particular book is an instance of some general concept of Bookness), but rather a limited part of, say, the space in this room. A general concept of space 'is found alike in a foot and in an ell' (A25). Spatiality is a general concept and can be instantiated in objects, but one cannot speak of parts of spatiality. A concept of spatiality, then, would range over sets of objects (potentially infinite) that have in common some range of characteristics or coordinates. The concept can thus be limited or expanded by varying restrictions on the set over which it ranges. One can, however, speak of parts of space; and parts of space are not instances of space, but sub-regions, limitations of a whole. One cannot, for example, restrict a volume of space in the same manner in which one restricts the members of a set. Hence, to represent numerically distinct objects or sub-regions of space presupposes a representation of one space, of space as a whole.[21] And if one adds the further premise that it is only in intuition that an individual is given, space being one, a whole, an individual, one's representation of space, then, must be intuitive.

iv.

Let us, for a moment, examine what Kant hoped to conclude from the MEs. He wished to demonstrate that 'space does not represent any determination that attaches to the objects themselves, and which remains even when abstraction has been made of all the subjective conditions of intuition' (A26/B42). In this, I believe, he was partly successful. He also hoped to conclude that space 'is nothing but the form of all appearances of outer sense' (A26/B42). This, I take it, he certainly did not prove in the four MEs (or in the corresponding arguments with respect to time). Now it is true that at A28/B44 Kant draws a correlation between that which underlies things-in-themselves and a transcendental idealist notion of space; and I would even go so far as to grant Kant that he was successful in demonstrating that some notion of space as a whole must underlie all representations of outer appearance; but it is not until Kant has gone through the TE that he speaks of space as being "the form of outer sense in general". And as I have already indicated, the premise from which Kant argues in the TE, *i.e.* 'Geometry is a science which determines the properties of space synthetically, and yet *a priori*' (B40), is a claim to be argued for, not from. It seems to me that Kant's arguments for the intuitive character of our representation of space (or time) is a strong one; and I have also argued that there is a clear sense in which space (and time) is a representation which underlies all outer intuition (it can thus be considered *a priori* in at least this manner). But does it follow that space and time are nothing but the forms of sensibility? The form of appearance is that which determines and orders the manifold *a priori*. The program of the Transcendental Aesthetic, Kant says, is twofold. Firstly, one must isolate sensibility by separating from it the conceptual process provided by the understanding and leaving as a residue only empirical intuition. Secondly, one must 'separate off from [that] everything which belongs to sensation, so that nothing may remain save pure intuition and the mere form of appearances, which is all that sensibility can supply *a priori*' (A22/B36). Now Kant explicitly proffers two distinct interpretations of pure intuition, and, I suggest, it is possible to construe the form of sensibility in a corresponding manner.

Before I endeavour to undertake a further examination of pure intuition, I think it would be best if I made may position clear on a point or two. Obviously the thesis of the transcendental ideality of space and time is at stake here, a cornerstone of Kant's critical philosophy. The four MEs purport to demonstrate that our representation of space is both intuitive and *a priori*. This means, among other things, that the notion of space cannot be reduced to relations obtaining between objects of experience; it means that a notion of space cannot be empirically derived, but must underlie and be presupposed by objects of experience. But if what is presupposed by experience cannot be derived from it, it must, for Kant, be contributed by the subject. And thus, concomitant with the TE, space becomes a form of sensibility and its transcendental ideality follows as a matter of course. As I have already argued, I agree, for the most part, with what Kant has asserted concerning the necessity to conceive of a representation of space as underlying our intuition of outer sense (at least insofar as this representation must arise simultaneously with a perception of an extended object), and that the concept of space is irreducible, *i.e.* that our representation of space lies in intuition. Hence, space and time are not things that appear to us, but rather how things appear to us. But if space and time are forms of sensibility, Kant owes us an explanation of how this ordering process, as a subjective contribution, occurs. There must be some connection between the structuring process and that which is structured. Kant says,

> [O]ur exposition therefore establishes the reality, that is objective validity, of space in respect of whatever can be presented to us outwardly as object, but also at the same time the ideality of space in respect of things when they are considered in themselves through reason, that is, without regard to the constitution of our sensibility (A28/B44).

One of Kant's great achievements is the laying out and determining of the limits of knowledge; one of Kant's great difficulties is reconciling the various problems that arise through the phenomenon/noumenon distinction. Kant is an empirical realist as well as a transcendental idealist. I have earlier attempted to show that one can hold these two positions without contradiction as long as one is careful to appropriately limit the knowledge claims applicable in each. It seems to me, therefore, there is no manifest reason why one could not, for example, on an empirical realist's level hold some sort of substantivalist position regarding space (or spacetime) while concomitantly holding space as a form of sensibility when considering the issue from a transcendental idealist's perspective. To accomplish this, however, one requires a notion of space as a form of sensibility that is, perhaps, somewhat un-Kantian. One can then, I suspect, extract a notion of pure intuition that is both consonant with substantival space and sufficient to provide for the synthetic and *a priori* nature of mathematics.

v.

The argument from ICs, in their various forms, provides an excellent starting point for the issues I raised in section (iv). One thing all the ICs arguments have in common is their attack on the reductionist approach to the mathematics of space and the relationist approach to the concept of space. Another common denominator is the avail made of the explanatory powers of taking space as a form of sensibility. Broad summarizes the central ideas of the arguments from ICs as follows:

> [They] are taken to show that spatial characteristics do not belong to things as they are in themselves, but are only ways in which such things appear when perceived by observers whose minds are provided with a certain innate form of sensibility.[22]

The argument of the Inaugural Dissertation (para. 17c) discusses the well known examples of the left and right hand and spherical triangles; the Prolegomena (para. 13) reiterates the spherical triangle example and uses a variant of the "hand" argument. I take it, in both texts, Kant is trying to maintain that the difference between two incongruent counterparts must be some internal difference, for, from a purely relational view of space, there simply is no difference. There are several objections one can raise, most of which, however, miss the point Kant is attempting to make. For instance, one could object (with Broad) that although one could not obtain the idea of incongruent counterparts unless one has perceived instances of such pairs of objects, this does not mean that one could not form a general concept of "incongruent counterpartness", generalize it, and apply it to possible instances which have not yet even been perceived. This, I agree, may be true; but it does not address the problem Kant is posing. He is asking what makes the difference, not how we can identify such differences. It is also true that the matter can be treated mathematically in virtue of algebraic transformations; but these equations must be interpreted in terms of perceived instances. Kant never denies the possibility of, say, pure uninterpreted mathematical systems (actually the question of the possibility of uninterpreted mathematical systems, for Kant, does not even arise). Yet what we require is a mathematical system that will explain the putative intrinsic difference, and this is just what mathematics cannot do. Broad also points out that the argument from ICs is bound up with the number of dimensions that we assign to space. By some manipulation, then, one can resolve the problem of spherical triangles by reflecting them through a third dimension. The left and right hand examples, thus, require a similar maneuver through 4-space. Again mathematical transformations can accomplish this feat (topology might even offer inversion). What one needs to consider, though, is the insistence upon real possibility in Kant's critical philosophy. And unless the real possibility of n-space can be demonstrated (Kant is, of course, wont to argue for three spatial and one temporal dimension) this mathematical ploy will not work.[23]

So how does Kant explain what it is that intrinsically makes the difference?

> These objects are not representations of things as they are in themselves and as some mere understanding would know them, but sensuous intuitions, that is, appearances whose possibility rests upon the relation of certain things unknown in themselves to something else, namely,

> to our sensibility. Space is the form of external intuition of this sensibility, and the internal determination of every space is possible only by the determination of its external relation to the whole of space of which it is a part.[24]

Kant's explanation makes two distinct points. Firstly, there is the usual assertion that intuition is required. Secondly, space as a form of intuition provides for the internal determination of any given subregion of space as a whole. But notice well that there is nothing in the argument from ICs that mandates against a substantival view of space. Moreover, not only is a substantival theory of space consonant with Kant's arguments so far (except, perhaps, the TE), it, in fact, at the empirical level seems implied.[25]

Bearing this in mind, let us return to pure intuition and the forms or sensibility. Early in the Aesthetic Kant appears to equate these two notions.

> I term all representations pure (in the transcendental sense) in which there is nothing that belongs to sensation. The pure form of sensible intuitions, in general, in which all the manifold of intuition is intuited in certain relations, must be found in the mind *a priori*. This pure form of sensibility may also itself be called pure intuition (A20/B34-35).

Extension and figure, for example, would belong to pure intuition. Moreover, since Kant has argued that a representation of space must underlie all outer intuitions, if space is a pure form of sensibility, then the notions of extension and figure would remain in the mind, *a priori*, even without the representation of any actual object of the senses. Understood in this manner, pure intuition would seem to be taken in a purely passive sense. That is to say that a pure intuition of space is merely a capacity to be affected in a particular way by something other and external. But this interpretation gives rise to a familiar problem. Just as there must be something in the set of sense data that "makes the difference" between my taking this object of perception as a piece of paper and my taking that object of perception as a cup, there must similarly be something that "makes the difference" between my taking this piece of paper as rectangular and that cup as cylindrical. But if space (or time) is construed merely as a form of intuition in the sense so far considered, then that cannot be it-- pure passivity is surely not constitutive. Part of the solution to this conundrum lies in the fact that Kant not only considers space and time as the pure forms of intuition, but as themselves, intuitions. In a passage from the Transcendental Deduction (which takes place after the structure of the understanding has been laid out and the machinery of concept formation has been put in place), Kant returns to some of the issues brought to light in the Aesthetic.

> In the representation of space and time we have *a priori* forms of outer and inner sensible intuition; and to these the synthesis of apprehension of the manifold of appearance must always conform, because in no other way can the synthesis take place at all. But space and time are represented *a priori* not merely as forms of sensible intuition, but as themselves intuitions which contain a manifold, and therefore are represented with the determination of the unity of this manifold (*vide* the Transcendental Aesthetic) (B160).

So space cannot only be construed as a pure form of intuition. But this should not be particularly surprising. Clearly there is *a* sense in which the space of the geometrician of the Axioms is not the "same" space as that of the philosopher of the Aesthetic.[26]

> Space, represented as object (as we are required to do in geometry), contains more than mere form of intuition; it also contains combination of the manifold, given according to the form of sensibility, in an intuitive representation, so that the form of

> intuition gives only a manifold, the formal intuition gives unity if representation (B160n.).

Allison notes that pure intuition, taken under these conditions, appears contradictory. On the one hand, pure intuition as a form of sensibility is both universal and necessary; on the other hand, taken as a formal intuition it is particular and determinate. I, however, do not see it as such. In the case of formal intuition we are speaking of the space of the geometrician, a space of determinate content, a space wherein a metric has already been imposed. The space of the philosopher provides for an account of how the geometrician can determine the metric of space as a whole. And so we have returned to the applicability problem of the Axioms; yet our position has been improved considerably. From the Transcendental Deduction Kant has purported to show the necessary conditions for the possibility of experience are the necessary conditions for the possibility of the objects of experience. The Axioms have been shown to be necessary conditions for the unity of consciousness, and objects are considered objects of experience only if they are determinate. But the same synthesis in which an object as appearance is constituted as an extensive magnitude is that which the unity of the manifold is thought as intuition in space and time. And this unity of the manifold is now grounded in formal intuition. From the results of the MEs, space is an individual and thus a representation of space must be intuitive. Moreover, this representation of one space, as given, must underlie any external intuition and hence any determinations made thereupon. In this sense, then, space does have a particular "form". Any determination made by the geometrician upon the representation of space he thus takes for his object must conform to what would count as a "really" possible object or sub-region of the one, underlying representation of the substantival space of the philosopher. It is in this sense of pure intuition, as a form of a really possible object, that the crucial part of the Kantian enterprise, thus far unexplained, becomes clear. Brittan makes essentially the same point[27] (though in a different context). He asserts,

> [t]he only guarantee of "real possibility" in the case of pure concepts such as those of a mathematical geometry is an *a priori* construction.[28]

This, I believe, is quite correct; but Brittan goes on to argue that the only mathematical concepts capable of such construction with regard to space, for Kant, are those of Euclidean geometry. If my arguments, so far, are sound, then this is not necessary (it is in the construction, itself, that the apriority is revealed). The notion of space gleaned from the MEs leaves it quite open as to what metric is applicable; it simply guarantees that a metric will be applicable. The one, substantival, space does not, in itself as an object, appear to us; rather it is how objects (that are really possible) of a one, substantival, space appear to us.[29] For all we know, the metric may be decided upon from empirical considerations. But it does guarantee the possibility of mathematizing substantival space considered as an object insofar as that mathematization must conform to the aforementioned notion of pure intuition.

Finally, Kant maintains that pure intuition must underlie pure concepts in synthetic judgments *a priori*. The very possibility of making synthetic judgments *a priori* in, say, mathematics, depends then upon an account of pure intuition. Such a notion as I have developed here (*i.e.* the minimal notion of pure intuition generated from Kant's account of formal intuition), I suggest is sufficient to provide for this possibility.

## CHAPTER V

## CONCEPT CONSTRUCTION

i.

According to Kant, the propositions of mathematics are both synthetic and known *a priori*. For Kant this means that the theorems or truths of mathematics are arrived at through some process that is in some way synthetic and yet is, at the same time, necessary and independent with respect to any possible experience. Against this thesis there are several competing schemes, one of which is the logicist program of Frege and Russell. And to this program I shall refer periodically since the vivid contrast serves well in underscoring those areas wherein Kant actually locates the syntheticity and apriority of mathematics. The logicists argue that mathematics, as a discipline, is both analytic and *a priori*. They maintain that mathematics is reducible to logical notions, and because logic is analytic, then so, too, is mathematics.[1] Now one thing, at least, is clear. Kant would agree that logic (or what he calls general logic) is both analytic and known *a priori*. General logic abstracts from all content of knowledge; it concerns itself only with logical form in the relation of knowledge to knowledge, not knowledge to its object; 'it treats of the form of thought in general' (A55/B79). Hence if mathematics is, as Kant claims, synthetic and *a priori*, there must be something about either mathematical propositions or mathematical reasoning, or both, that is quite different from logic.

Recent literature has proffered two main lines wherein the syntheticity and apriority of mathematics can be cogently argued. Line

(1), in general, grants the *a priori* nature of mathematics, permits mathematical reasoning to proceed analytically (via the principle of contradiction) and locates syntheticity in the axioms or primitive popositions of the discipline; line (2) locates syntheticity and apriority in the reasoning process itself. Several issues are involved in the exposition of these lines, three of which are crucial: (i) the distinction between logic and mathematics, and the concomitant distinction between what is merely logically possible and what is "really" possible; (ii) the relationship between apriority and necessity and what it means for mathematical propositions to be true or follow from previous propositions necessarily; and (iii) precisely what role intuition plays in the generation of mathematical concepts. Now, as I hope to make clear, although both lines certainly address these issues, neither are entirely successful in interpreting or arguing for, or against, the issues. Hintikka's enterprise, for example, which I take as paradigmatic of line (2), involves the divorcing of intuition from sensibility and thus leads to a quite un-Kantian reconstruction. Or again, Brittan's proposal, which I take to be paradigmatic of line (1), requires such a strong position with respect to Kantian transcendental idealism that I believe it not to be defensible. What I shall attempt, rather, is a defence of the synthetic *a priori* nature of mathematics that is both consistent with Kant's philosophy as a whole and adequate in its own right.

Now, much of the argumentation in the earlier chapters has been directed to the thesis that the solution to the problem of the possibility of making synthetic judgments *a priori* in mathematics lies in how one constructs, or what it means to construct, mathematical concepts. Mathematics, Kant maintains, as opposed to, say, philosophy, presents the most splendid example of the successful extension of pure reason. Why is this so? How is it that our mathematical knowledge has grown so, proven so effective in application, and so on, while in metaphysics one is constantly retracing one's steps and arguing again and again on old battle grounds. Kant has an answer, an answer that is the key to understanding how synthetic judgments are possible *a priori* in mathematics.

> Philosophical knowledge is the knowledge gained by reason from concepts; mathematical knowledge is the knowledge gained by reason from the construction of concepts. To construct a concept means to exhibit *a priori* the intuition which corresponds to the concept (A713/B741).[2]

What one needs is an account of what would count as a really possible construction, of the necessity in such a construction and the relation of intuition to the construction. To this end I shall first draw from both lines (1) and (2) those arguments that are consistent with and faithful to Kant's program in general, and the construction of concepts in particular, while avoiding the pitfalls and shortcomings and infelicitous or incorrect interpretations of Kant that I believe are characteristic of these lines considered by themselves. I shall then formulate, on the basis of the construction of mathematical concepts, a defence of the syntheticity and apriority of all mathematical judgments.

ii.

As adumbrated earlier, Frege and Russell object to Kant's use of intuition in mathematical inferences. The reduction of arithmetic to logic, they assert, renders any reliance on intuition otiose, or even a liability. There are those who would admit syntheticity in some parts of mathematics but staunchly affirm the analyticity of other parts. Frege, for instance, in maintaining that arithmetic is reducible to logic, says this lends credence to what he feels is the analytic nature of arithmetic (Russell is much more forceful in his analysis); geometry, on the other hand, Frege avers admits of synthetic construction. Dryer, a Kantian scholar, to give yet another example, makes short work of the possible syntheticity of arithmetic and then tells a very long story, a large part of which is devoted to showing the syntheticity of geometry.[3] Another putative line is that pure mathematics is analytic, while applied mathematics is synthetic. Presumably arithmetic, algebra and geometry as axiomatized and uninterpreted qualify as the former; axiomatized geometry given an interpretation would be applied, and hence qualify as the latter. Regardless of these possible construals, or variations thereupon, Kant is quite clear on the matter: 'all mathematical judgments, without exception, are synthetic; (B14). As far as syntheticity is concerned, there is no distinction to be made between pure and applied mathematics,[4] between geometry and arithmetic, etc.

So where is syntheticity located in mathematics? It certainly seems that intuition, whatever that may be at this stage, must play a crucial role. In geometry it appears, at least superficially, that it is in virtue of the construction of actual figures (either on paper or in the imagination) that syntheticity obtains. At A716/B744 Kant proffers a specific example. Suppose, he says, one considers the relation between the sum of the angles in a triangle and a right angle. Kant asserts that by mere analysis of the concept triangle alone, all that one can derive is the concept of a figure enclosed by three lines and containing three angles. "Philosophical analysis", he maintains, cannot get beyond this. The geometrician, however, 'at once begins by constructing a triangle' (A716/B744).[5] Presumably such construction can either be imaginative or actual--the point being that one starts with an arbitrarily constructed figure and performs further constructions upon it. Eventually, and 'in this fashion, through a chain of inferences guided throughout by intuition, he [the geometer] arrives at a fully evident and universally valid solution of the problem' (A716-17/B744-45).

Now, at first glance, it appears that this ostensive connection between geometry and sensible construction warrants some plausibility that is not so obviously attainable in, say, arithmetic. When one constructs a geometrical figure, one constructs it in space with certain spatial characteristics or determinations (be it Euclidean, Riemannian, Lobachewskian, or whatever--the number of different axiomatic systems seems to lend support to the synthetiticy of geometry, though not necessarily the apriority). But what sensible characteristics do arithmetic objects, numbers, possess? Frege, here, draws the line. For him arithmetical propositions can be defined in terms of logical notions. Given "0", "is a number", a successor operation, a few quantificational and propositional operators and connectives, and the set theoretic notion of "$\epsilon$", all propositions of arithmetic can be generated. What need has one, then, of sensible intuition? Kant has a somewhat different approach. At B15-16 he asks us to consider the proposition "7+5=12". Kant maintains that the concept of the sum of 7 and 5 "contains" nothing but the union of two numbers. The concept of 12 in no way already "contains" this union and no amount of analysis could possibly generate the concept of such a union. Kant says one must "go outside" [*hinausgehen*] and enlist the aid of intuition. For example, one starts with the number 7 and with the assistance of fingers or points, successively counts off 5 units and somehow "sees the number 12 come into being" [*sehe so die Zahl 12 entspringen*]. Now, of course, construed in this manner, the arithmetic operation of addition appeals to images, to intuition at a sensible level. And if addition were merely the counting off of strokes in this rather primitive fashion, then not only does Frege's program seem far superior, but even Dryer's Leibnizian formulation is infinitely preferable. But Kant has much more to say. He concludes:

> Arithmetical propositions are therefore always synthetic. This is still more evident if we take larger numbers. For it is then obvious that, however we might turn or twist our concepts, we could never, by mere analysis of them, and without the aid of intuition, discover what [the number is that] is the sum (B16).

Clearly, for large numbers, the ability to use one's fingers, or tick off points or strokes, breaks down. I, thus, take Kant's example to be merely heuristic, indicative that it is the general process of counting by successive addition of units that is important insofar as synthesis is concerned. It is the possibility of providing the construction of a magnitude in geometry and arithmetic that is relevant.[6] Yet has Kant actually provided anything of real import? Frege and Russell would hardly answer in the positive. If all Kant means by addition is the successive iteration of units, and if arithmetic can be viewed as including all that can strictly be called pure in mathematics, then for all algebra and analysis and even geometry (uninterpreted) 'it is unnecessary to assume any material beyond the integers, which, as we have seen, can themselves be defined in logical terms. It is this science, far more than non-Euclidean Geometry, that is really fatal to the Kantian theory of *a priori* intuitions as the basis of mathematics'.[7] Again it seems to come down to the problem of the containment metaphor outlined in Chapter II. The metaphor is simply too vague to support a precise interpretation. As a mere thought experiment it appears impossible to determine whether one can actually think the union of 7+5 in thinking the concept 12. Clearly a much more precise account of how a concept is constructed is required.

One such account is offered by Beck and Brittan and constitutes, by and large, what I have presented as line (1). In the Introduction to the First Critique one passage that Brittan, in particular, picks up on, seems to permit an answer, of sorts, to the Frege-Russell program.

> For as it was found that all mathematical inferences proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires), it was supposed that the fundamental propositions of the science can themselves be known to be true through that principle. This is an erroneous view. For though a synthetic proposition can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from this other proposition; it can never be so discerned in and by itself (B14).[8]

Here Kant is probably responding to Leibniz; but the passage can be read, also, as a response to the logicist. Even though the inferences made in a mathematical proof can be rendered analytic by the logicist enterprise, the axioms or premises upon [illegible] are, on the other hand, synthetic. According to Brittan, Kant contends that although in a mathematical proof the steps follow one another analytically, that is, in accordance with the principle of contradiction, the initial premises of the proof and the conclusion are themselves synthetic'.[10] Thus, according to Brittan, the synthetic character of the propositions of mathematics is 'a function of some feature of the propositions themselves and not the way in which they come to be established'.[11] And this does, indeed, seem to be a response to the logicist. A common criticism of, say, Frege and Russell, is that the most their arguments demonstrate is that mathematics is reducible to logic and some set theory. Moreover, this set theory involves an appeal to existence axioms; and existence axioms are certainly synthetic in nature since they can always be falsified in some possible world.[12]

Hintikka, however, has two quarrels with Brittan's line.[13] Firstly, Hintikka points out that Kant does not say that mathematical truths *always* get their syntheticity from earlier synthetic theorems, and from which it *always* is the case they can be derived by the principle of contradiction, *i.e.* analytically. What Kant, in fact, says is that this *can* happen and is, rather, the only way in which one synthetic proposition can be analytically derived from another synthetic proposition. This, I believe, is correct, and is borne out by the use of "*kann*" in both the *Critique of Pure Reason* and the corresponding passage of the *Prolegomena*. If one reads these passages as a criticism of Leibniz, what Kant is putatively offering is one way in which Leibniz *can* go awry, and not that Leibniz is always wrong for those reasons; but one way, of course, is enough to topple this part of the Leibnizian position.[14]

Secondly, Hintikka objects that Brittan maintains that the syntheticity of a given mathematical proposition can always be traced back to the syntheticity of its axoms. Now, although geometry can, and has been, axiomatized, arithmetic certainly was not--at least in Kant's day. And though Brittan alludes to axiomatizations he was just as aware as was Hintikka of the passage at A164/B204 where Kant holds that in arithmetic 'there are no axioms in the strict meaning of the term, although there are a number of propositions which are synthetic and immediately certain'. Frege states the problem quite clearly.

> We must distinguish numerical formulae, such as 2+3=5, which deal with particular numbers, from general laws, which hold good for all whole numbers. The former are held by some philosophers to be unprovable and immediately self-evident like axioms. Kant declares them to be unprovable and synthetic, but hesitates to call them axioms because they are not general and because the number of them is infinite.[15]

Peano's postulates aside, then, syntheticity according to Brittan is essentially derivable from some unprovable, yet self-evident, primitive propositions of numerical relations. But if, as Brittan maintains, mathematical proofs always proceed through analytic inferences, then the onus is upon the premise set, and thus eventually these primitive propositions of numerical relations, to provide for both syntheticity and apriority. Now, as I have already pointed out, premises based on existence assumptions can always be falsified in some possible world. Hence, if the propositions of mathematics are to be necessary (and for Kant apriority and necessity and universality are more or less synonymous terms), then Kant is committed to showing how only certain possible worlds are "really" possible. Now that, alone, is not a position with which Kant would disagree. Indeed, a great part of the Transcendental Analytic is devoted to precisely this task. The problem, I believe, is to identify and appropriately limit the domain over which these existence assumptions range. For example, to maintain that the axioms of, say, (Euclidean) geometry are, in fact, the self-evident primitive propositions upon which geometry, as a science, is founded requires a position akin to the full-blown transcendental idealism of the Aesthetic. And that is a position that requires much more argumentation; surely much more than Kant provides. It is a position for which I know no adequate defence. This is not, however, to gainsay the merit of Brittan's reconstruction; there is much of value to be garnered from his arguments (in particular, his position on real possibility, objective validity and meaning). But since I wish to appraise his position with respect to that of Hintikka, it is to Hintikka's reconstruction that I now turn.

iii.

One interesting interpretation regarding Kant's theory of mathematics that has surfaced recently is due to Jaakko Hintikka (an interpretation that I regard as paradigmatic of line (2)). Hintikka would not block any Fregean reduction of mathematics to logic. To the contrary, when mathematical arguments are translated into arguments of the quantificational calculus, it is precisely this mode of logical argumentation that provides for syntheticity. Hintikka claims that what Kant meant by construction in mathematics (in the late eighteenth century) would now be more properly rendered as quantification theory and quantificational proofs.[16] He notes that many of these proofs of quantificational logic (particularly those that are transformations of constructive proofs in geometry) involve the natural deduction rule of existential instantiation--and that is a synthetic process.

Now the first thing that is of concern to us is the permission, granted by Hintikka, of the Fregean program. Just because mathematics could be reduced to logic does not preclude, Hintikka claims, its being synthetic. Yet, as I have previously argued, Kant would agree with Frege that logic is both analytic and *a priori*. There must therefore be something about mathematical reasoning, or mathematical propositions, or the discipline of mathematics itself wherein syntheticity arises. In defence of Hintikka, however, one must be aware of both the state of logic, in general, at the time Kant formulated his critical philosophy, and Kant's own views on the subject. Formal logic, for Kant, was more or less restricted to the syllogism (and at that, basically, in categorial form) and the various inferences of Aristotelian logic. For Kant, the sphere of logic was precisely delimited and 'since Aristotle it has not been required to retrace a single step, unless, indeed we care to count as improvements the removal of needles subtleties which concern the elegance rather than the certainty of the science' (Bviii). Kant further remarks that logic, in his era, to all intents and purposes, appeared to be a 'closed and completed' science. Clearly he had no conception of the possibility of what would become, for example, modal or deontic logic, or even simple quantification theory. It is not even entirely certain that he would consider the developments, developments of logic. In any event, were he to so consider modern logic as an advance of Aristotelian syllogistic logic, given nearly two centuries of hindsight, it seems reasonable to suppose that, given also the extension of logic into the field of quantification theory (with appropriate existence assumptions), Kant would no longer assert the analyticity of logic.

The separation between logic and mathematics, for Kant, seemed quite precise; now the distinction is not so clear. But what Kant took to be logic, and what we now know it as, are obviously quite different. As Brittan points out,[17] Hintikka draws the line at monadic predicate calculus; it also seems reasonable to assume Kant would have done likewise. In Hintikka's terms, then,

> [W]hat Kant says of mathematics pertains more to contemporary first-order logic, whose mainstays are precisely instantiation rules, than to what we twentieth-century philosophers would classify as specifically mathematical modes of reasoning.[18]

Let us, thus, provisionally grant this part of Hintikka's reconstruction, see how it applies to Kant's program, and what of value can be culled from it. For Hintikka, the ground of mathematical syntheticity is to be found in existential instantiation rules. Mathematical proofs that introduce new free individual symbols are synthetic in nature; proofs that do not introduce such symbols are analytic.[19] Now, much of what Kant has to say about syntheticity and content seems to lend credence to this view. Moreover, Kant would also appear to assert that analytic statements could never support the existence claims represented in existential instantiation. For example, in the Analogies Kant maintains that 'through the mere concept of these things, analyse them as we may, we can never advance from one object and its existence to the existence of another or to its mode of existence' (A217/B264). Kant is, at least, clear that the existential import of a statement is sufficient for its syntheticity: 'all existential propositions are synthetic, [for] how can we maintain that the predicate of existence can be negated without contradiction? This is a feature which is found only in analytic propositions, and is indeed precisely what constitutes their analytic character' (A598/B626).

Now much of Hintikka's analysis appears to be tied to the translation of constructive geometric proofs into quantificational form. Indeed, he claims that Kant's theory of Mathematics is actually based on a generalization of the constructive types of proofs seen in geometry.[20]

> What made mathematics synthetic was the introduction of such singular terms to represent the individuals to which certain general concepts apply, in the same way that geometrical entities are introduced by geometrical "construction" into the figure by means of which we are illustrating the interrelations of the geometrical concepts involved in the proof. Kant's wider notion of a construction is thus nothing but a generalization from the constructions which make geometrical arguments synthetic.[21]

Hintikka notes that Euclid's method of proof provides a clear model of Kant's mathematical theory,[22] and that considering the historical perspective, it is only natural that Kant would have adopted such a model. From a general enunciation or *protasis*, Euclid then proceeds to apply the enunciation to a particular case. So, for instance, if one were to endeavour to prove Pythagoras' theorem, one generally starts with the phrase: "Let ABC be a triangle ... and so on". This is called the setting-out or *ekthesis*, and closely corresponds with the manner in which Kant approaches a geometric proof; recall that even Brittan notes that, for Kant, the geometrician 'at once begins by constructing a triangle' (A716/B744). The *ekthesis* is followed by auxiliary construction, or *kataskeue*, and the proof proper, or *apodeixis*. Both the *ekthesis* and *kataskeue* introduce new individual terms and, as such, represent the synthetic element of the proof; the *apodeixis*, being based essentially on Euclid's Common Notions, theorems proved earlier, *etc.*, represents the analytic part of the proof.[23]

One question that immediately presents itself is how closely Hintikka's analysis parallels the Kantian program. On the surface, at least from the perspective of geometry, Hintikka's case appears quite forceful. For example, take a simple constructive proof of Pythagoras' theorem (Euclid's I, 47). One first satisfies the *ekthesis* by letting ABC be a right-angled triangle. As Kant points out, 'philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance'(A714/B742). On the basis of a given right-angled triangle, then, one will proceed to prove that the sum of the squares of the sides containing the right angle will be equal to the square of the side subtending the right angle--and do so universally and necessarily. Notice that the given triangle, ABC, is given arbitrarily (at least under the constraints of "right-angleness" and Euclideaness), and as such, any proofs established on the basis can be generalized and are valid universally of all right-angled (Euclidean) triangles. In the next step in the proof, one is asked to construct squares on each of the sides of the triangle. The altitude of triangle ABC is constructed from the vertex of the right angle and extended to the opposite side of the square constructed on the hypotenuse. Finally two other line segments are constructed connecting an appropriate vertex of a square with an appropriate vertex of the triangle. This completes the *kataskeue* part of the proof. The construction of the squares and line segments clearly represent what Hintikka would call the introduction of new free terms into the proof. As these constructions are what so obviously give rise to the synthetic nature of the proof, it is the translation into a quantificational form of argumentation that introduces new terms, in virtue of the natural deduction rule of existential instantiation, that would provide for syntheticity. For the geometrician, as Kant puts it,

> [T]he true method, so he found, was not to inspect what he discerned either in the figure, or in the bare concept of it, and from this, as it were, to read off its properties; but to bring out what was necessarily implied in the concepts that he had himself found *a priori*, and had put into the figure in the construction by which he presented it to himself. If he is to know anything with *a priori* certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept (Bxii).

One can now, perhaps, attach some meaning to Kant's notoriously vague containment metaphor: How is it that from the concept of a triangle one can analytically derive Pythagoras' theorem? Analyze as one may, in the proof outlined above certain auxiliary constructions are required, new line segments and figures were drawn that were not present in the *ekthesis*. In this case, at least, it would be difficult to find any determinate sense in which Pythagoras' theorem is "contained" in the concept of right-angled triangle. Once some auxiliary constructions have been performed, however, the proof becomes trivial, and the conclusion can be "read off" or "analyzed out" of the concept of the right -angled triangle plus the extra construction. And this is precisely Hintikka's point, I take it, concerning the *apodeixis*. The *apodeixis* proceeds analytically; and in our example, the proof proper is, indeed, trivial. All one needs is a previous notion of congruence and the proof of Pythagoras' theorem readily follows. Hence, both Hintikka and Brittan would both be in agreement concerning this part of the proof. For Brittan the inferences are made analytically; syntheticity is found in the axioms from which the proof could eventually be traced. For Hintikka the inferences are, in general, similarly analytic (at least in the *apodeixis*); syntheticity is located in the auxiliary constructions without which the proof (or this type of proof) could not be established.

It, therefore, appears that Hintikka's program provides some measure of progress from that of Brittan. As I have previously argued, Brittan's reconstruction asserts that syntheticity obtains from the axioms or premises upon which a mathematical argument is based, and any necessity arises from following, precisely, various rules of inference. But what is it that makes the premises necessary? In the quotation from Bxxi above, Kant is wont to argue for the *a priori* certainty of mathematical theorems. And for him this apriority means more than mere logical necessity. Logical necessity is involved, surely, but for Kant the possibility of mathematics, itself, must be demonstrated--and that is a problem for transcendental philosophy. I shall have much more to say on this issue later, but suffice it for now to show how Hintikka's position differs from that of Brittan. Hintikka locates syntheticity in the *kataskeue* and, perhaps, the *ekthesis* parts of a geometrical proof. This does not deny the possible syntheticity of the axioms as well; it is just that it is not absolutely necessary to trace syntheticity back to axioms or premises. And it is here, by locating syntheticity in the proof process itself, that I find Hintikka's program superior to Brittan's enterprise.

Several objections can, however, be raised regarding Hintikka's reconstruction. For one, it is unclear how well his reconstruction works when considering arithmetic rather than geometry. The analogy between auxiliary constructions in geometrical proofs and the logical notion of existential instantiation and introducing new free terms into the course of a proof seems reasonably pellucid. An extension into the area of arithmetic, however, is not so clear. Hintikka, himself, says that not all proofs will be synthetic--only those that avail themselves of the introduction of new terms or variables; Kant, to the contrary, holds that all mathematical judgments are synthetic. In Hintikka's defence, though, it would appear that all constructive proofs (and this applies to arithmetic as well as geometry), and perhaps some non-constructive proofs as well, would be synthetic under his criteria. And it is quite uncertain as [illegible] Kant would accept a non-constructive proof as a proof at all.[24]

A much more serious objection, however, can be put forth. How would Hintikka interpret what Kant means by concept construction in mathematics? For Hintikka, it would seem to follow that what it means to construct a mathematical concept, what Kant meant by exhibiting *a priori* the intuition which corresponds to the concept, is to introduce new individual objects or representation, new free terms, into a mathematical proof. And <u>that</u> can be done by existential instantiation in the corresponding argument in quantificational form of, say, a geometrical proof. Hence, for Hintikka, any new free terms introduced in a proof find their counterpart in Kant's notion of intuition. But this intuition need not be sensible in nature. Hintikka declares that when Kant defines intuition as a particular *Vorstellung*, what he has in mind is what Euclid took to be particular cases. Mathematical construction, therefore, is simply the introduction of particular entities as representations used to instantiate general concepts. 'For him [Kant], an intuition is almost like a "proper name" in Frege's unnaturally wide sense of the term, except that it did not have to be a linguistic entity, but could also be anything in the human mind which "stands for" an individual'.[25] Hintikka asks us to take intuition as a singular term, as representing an individual object. He further supports his position by providing a plethora of examples from the Kantian corpus wherein Kant mentions intuitions without any reference to sensibility. Now I shan't deny that particularity is certainly part of the Kantian definition of intuition; nor shall I enumerate the examples Hintikka proffers.[26] Rather, I hold that the connection between intuition and sensibility in Kant is so fundamental that any attempt to sever or divorce the two would be intolerable (the position being so fundamental, in fact, that Kant need not refer to sensibility every time he mentions intuition--it must always be assumed).

The strongest argument against Hintikka's position comes from Kant, himself, when he denies the possibility of a non-sensible intuition or the possibility of intellectual intuition (this is particularly true of the second edition of the First Critique). Examples are too numerous to list,[27] but several commentators, such as Mitscherling, do a credible job of attacking Hintikka on all fronts, and provide an adequate supply of counterexamples to his position.[28] Moreover, in addition to forbidding any non-sensible intuition, Kant makes several forceful positive remarks concerning intuition and sensibility. To take just one example, Hintikka holds that the construction of mathematical concepts depends only upon the representation of individuals by instantiation of free symbols in modern logic. That this representation of individuals by instantiation might be the case I do not deny (in fact, I believe Hintikka is quite correct so far as he goes--it is just not the whole story); but to say that mathematical construction depends only upon this sort of instantiation is, I think, quite wrong. When Kant talks about the construction of concepts at A713/B741, he says that this means 'to exhibit *a priori* the intuition which corresponds to the concept'; and at A720/B748 he states 'the <u>only</u> intuition that is given *a priori* is that of the mere form of appearances, space and time' (my emphasis). Leaving aside, for a moment, what Kant "really" means by *a priori* here, I cannot imagine a more clear statement of the issue.

Finally, I shall countenance one further objection, an objection that is directed not only to Hintikka's notion of intuition, but to his reconstruction of Kant's theory of mathematics as a whole. Pippin asserts,

> ... in discussing Hintikka, I also claimed that a reconstruction which eliminates all such appeals to sensibility eliminates too much. This is so not only because it eliminates a possible explanation of the applicability of mathematics, but because Kant's whole idealism argument depends on his being able to show that the only way space could be representable a priori is if it were a form of sensibility. This claim is supposed to account for what we are able to do in mathematics, and thus is involved in a strategy which must be accepted or rejected as a whole. And if we reject the idealism argument (which I believe must be done if we are to reconstruct the *Methodenlehre* in the way Hintikka suggests), much of Kant's case for his whole formal idealism collapses.[29]

One could, perhaps, argue that Pippin states his case too strongly, but I think his point is essentially correct. Consider Hintikka's reconstruction, in itself, and what he has putatively shown. Kant had hoped to prove that mathematics, as a body of necessary truths, was both synthetic in nature and known *a priori*. Hintikka has demonstrated that mathematics, even reduced to first-order logic, does not lose its synthetic character. And, moreover, the proposition of mathematics are, if true, necessarily true. But what guarantees the truth of the antecedent of this conditional (which I shall refer to as the (4) axiom)? Two questions at once present themselves: is this a mathematical problem or a meta-mathematical problem; and if the latter, can any resources be found in the former that would provide some ground for an adequate answer to the latter? I shall not, yet, hasten towards any definitive answer, but it is surely a question Kant would have wanted resolved; it is surely a question which would, for Kant, require embedding in a larger epistemological framework; yet it is not a question that, to my knowledge, Hintikka addresses at all. In addition, by severing the ties between intuition and sensibility, Hintikka also severs the ties between intuition and the forms of sensibility; and that is a position that not only Kant could not countenance, but it seems to render any attempt by Kant to guarantee the truth of the antecedent of the (4) axiom, with respect to mathematical judgments, otiose.

How, then, can Kant provide us with an account of the syntheticity and apriority of mathematics that he desires? Hintikka has shown us that syntheticity can be a part of the mathematical process itself and, although it can also be derived from the axioms or premise set of an argument, it need not. Now coupled with an account of what is to count as "really" possible experience, or a "really" possible object of experience, the next step in the program may be taken. We need to know not only how mathematical concepts are constructed, but how only certain constructions are possible.

iv.

As I have already indicated Kant felt that philosophical knowledge is obtained by reason from concepts; mathematical knowledge is obtained by reason from the construction of concepts. Moreover, to construct a concept means to exhibit *a priori* the intuition which corresponds to the concept. Now if the exhibition is to be *a priori*, the intuition which corresponds to the concept must be non-empirical in nature. And that means that the intuition must be formal, that is, of space or time. Yet what, exactly, this means is not entirely perspicuous. For example, is the intuition in question to be considered merely as a form of intuition, or, rather as a formal intuition? Is the intuition to be an intuition in space or time, or is it to be of our representation of space or time, or both? One can even ask (as would presumably Frege and Russell) about the necessity of intuition, in any form, at all. Now since the success or failure of the Kantian program of demonstrating the syntheticity and apriority of mathematics hinges, I believe, upon how one construes the construction of concepts, these problems must be addressed. It is my intent, therefore, in this section, to lay the foundation for a resolution to these problems.

It is clear that Kant thought mathematics and logic to be different in kind. Though judgments of both are known *a priori*, judgments of the former are synthetic, while judgments of the latter are analytic. In this area, then, Kant is quite opposed to the logicist program. Both Hintikka and Brittan, however, offer reconstructions that, although not blocking the reductionist move of the logicists, permit of the syntheticity of mathematics. Hintikka locates syntheticy in the natural deduction rule of existential instantiation, which could then be generalized over all logical proofs utilizing this natural deduction rule. Brittan finds syntheticity in the axioms or premises upon which a mathematical proof is based. This reconstruction gains support from the fact that mathematics, at best, seems reducible to logic plus some set theory, a set theory that contains existence axioms. And regardless of whether set theory is considered extra-logical, it remains that existence axioms must be synthetic, if anything is to be synthetic at all. The programs of both Brittan and Hintikka have, thus, at least one point in common: they agree that mathematical propositions have (or can have) existential import, and it is by virtue of this that mathematics is synthetic. Brittan makes this clear.

> Mathematics, in contrast to logic, is not "empty" or "merely formal". It has "content", and for this reason is synthetic rather than analytic. I have tried to indicate one way in which "having content" can be understood: the propositions of mathematics have existential import, whereas the propositions of logic do not. This fact has a further consequence that the propositions of mathematics, unlike those of logic, do not hold in or of all possible worlds.[30]

There is also a sense in which Hintikka's and Brittan's reconstructions fail for the same reason: they fail to provide for the sense of necessity Kant attached to mathematics. If Hintikka or Brittan are correct, then mathematics is, indeed, synthetic and known *a priori*. But one should recall that, for Kant, apriority is virtually synonymous with necessity. For both Hintikka and Brittan necessity obtains from the correct employment of logical (or mathematical) rules applied to the steps of a given argument. Hence, mathematics is synthetic and known *a priori*--but in an unedifying fashion. To be sure, logical necessity is involved here. Mathematical propositions are certain or necessary propositions insofar as they follow other certain or necessary propositions according to rules (of inference or construction). So, for example, the propositions of Euclidean geometry are necessary with respect to a particular set of axioms; one can say that the theorems of Euclidean geometry are known *a priori* with respect to its axioms. But this is not the only sense of apriority or necessity to which Kant is referring. If it were, it seems to me it would make Kant's theory of mathematics almost a trivial extenstion of the conventionalists.[31] Kant says that 'all necessity, without exception, is grounded in a transcendental condition'(A106). It is this epistemological twist that separates Kant from the logicist or conventionalist. Brittan is quite correct when he maintains that, for Kant, mathematics is not empty or merely formal and that the propositions of mathematics do not hold in all possible worlds, but only in those "really" possible worlds. Yet how do the reconstructions of either Brittan or Hintikka permit such a transcendental condition of necessity?

For Kant, if geometry (and presumably arithmetic) is to be possible (and here one should read "really" possible) certain spatial and temporal conditions must obtain. And these conditions will specify what is "really" constructible in mathematics.

> Geometry is a science which determines the properties of space synthetically, and yet *a priori*. What then, must be our representation of space, in order that such knowledge of it may be possible? It must in its origin be intuition ... Further, this intuition must be *a priori* (B40-41).

Although Kant has no comparable argument with respect to arithmetic and our representation of time[32] it is not impossible to supply such. But what really concerns us here is how Brittan and Hintikka can account for the transcendental sense of necessity that Kant's position entails. It seems to me, that, insofar as Hintikka's position is concerned, the situation is hopeless. In severing intuition from sensibility Hintikka, in fact, loses any contact with what would count as a "really" possible construction. For Hintikka, necessity meets the demands of logical consistency, and only the demands of logical consistency. In Kantian terms it is true that there is no more in a mathematical proposition than what we put there; but in terms of construction there is no guarantee that what we put there is "really" possible. In this sense, then, Brittan's reconstruction has a better chance of working than Hintikka's does. If the axioms or premises of a mathematical argument conform to our forms of sensibility, then Kant's transcendental condition of necessity is satisfied. But to substantiate this thesis seems a difficult enterprise indeed. With the advent of consistent (and constructible) non-Euclidean geometries, and the uncertainty regarding which geometry "really" describes space,[33] this avenue seems closed with respect to the pure form of space. And it is unclear, even had Kant the benefit of Peano's postulates, whether he would have accepted the axiomatization of arithmetic as founded on the pure form of time.

Brittan's analysis of what it means to be "really" possible, however, possesses great merit. If one can delimit what constructions are "really" possible when one constructs one's concepts in mathematics, then one has taken a large step in any defence of Kant's theory of mathematics. Now Brittan equates real possibility with objective reality; and various passages in the First Critique lend support to this claim. For instance, Kant points out, in the Postulates of Empirical Thought, that logical consistency alone is not sufficient to determine the objective validity of a concept.

> Thus there is no contradiction in the concept of a figure which is enclosed within two lines, since the concepts of two straight lines and of their coming together contains no negation of a figure. The impossibility arises not from the concept in itself, but in connection with its construction in space, that is, from the conditions of space and of its determination.

> And since these contain *a priori* in themselves the form of experience in general, they have objective reality; that is, they apply to possible things (A210-11/B268).[34]

In addition, Brittan connects objective reality with meaningfulness. This position, although somewhat more controversial, also seems to find support from the First Critique.[35] One consequence of this, for Brittan, is that to say a proposition of mathematics is meaningful is also to say that it has a truth value. The controversy arises over Brittan's contention that although synthetic propositions will thus have truth values, analytic propositions need not.[36] For example, the judgment, "the round square=the round square", as a substitution instance of the analytic identity, a=a, is neither true nor false--but rather meaningless. After all, at B16, after admitting that some propositions of mathematics may be analytic, Kant goes on to say that even though these propositions are valid according to pure concepts, they are 'only admitted in mathematics because they can be exhibited in intuition'. Presumably, then, there is little chance of a singular term such as "round square" being exhibited in intuition. Now, although I shall not enter into this issue here, Brittan does, it seems to me, raise an extremely important point with respect to existential judgments.

> That is to say, I think, that if the existential proposition is not satisfied, then the judgment does not have a truth value. Kant puts his point in a misleading way by saying that if you reject the subject you reject the contradiction, for he has already asserted that any judgment of the form a=a is analytic, hence any judgment of the form a≠a is self-contradictory. What he wants to say is that if you reject the subject, then the principle of contradiction cannot be used to establish the truth of the judgment. Thus, once a triangle is posited, it is true, analytically, that it have three angles. But if a triangle is not posited, then no judgment about "it" is either true or false.[37]

Kant has said that 'to posit a triangle, and yet to reject its three angles, is self-contradictory; but there is no contradiction in rejecting the triangle together with its three angles' (A594/B622). Hence, before any meaningful assertion of judgment may be made at all concerning, say, a triangle, it (the triangle) must first be posited or given.[38] Brittan is onto something here; it is a pity he does not carry this line of argumentation further. Once a triangle is given, certain auxiliary constructions are performed and certain theorems can be derived by following pre-established rules of inference or rules of construction. Geometrically, then, one begins with a "really" possible construction and performs further "really" possible constructions on that basis. But a "really" possible construction here need not be actually a construction of a figure in space. It suffices that the concept constructed be "really" constructible. And what is it that determines this? It is simply the providing of a proof or providing for, in principle, a proof of the object in question. The proof must be constructive; that is to say, it must not only establish the existence of something, but offer an adequate means of finding it or demonstrating it (thus any non-constructive proofs are ruled out). This, I take it, is the force behind the passage in the first chapter of the Transcendental Doctrine of Method.

> Philosophy confines itself to universal concepts; mathematics can achieve nothing by concepts alone but hastens at once to intuition, in which it considers the concept *in concreto*, though not empirically, but only in an intuition which it presents *a priori*, that is, which it has constructed, and in which whatever follows from the universal conditions of the construction must be universally valid of the object of the concept thus constructed (A716/B744).

Many arguments may treat logically of concepts *in abstracto;* mathematics proper, for Kant, treats of concepts *in concreto*. This I take to mean, in agreement with Hintikka and Brittan, that mathematics has content, has existential import. But it also means that the content of the concept must be "really" possible (it must conform to the pure forms of intuition *a priori*). Kant states that philosophical knowledge is obtained by reason from concepts, whereas mathematical knowledge is obtained by reason from the construction of concepts. Considering a mathematical concept *in concreto* then, for Kant, means exhibiting *a priori* the intuition which corresponds to the concept one has constructed. And this I now take to mean that, in mathematics, the knowledge obtained through judgments, is obtained in virtue of constructive proofs, *i.e.* "really" possible constructions.[39]

Now this will immediately restrict mathematics to a province quite familiar to the intuitionists. But there is nothing, in particular, inconsistent with an interpretation of mathematics in this manner with respect of the rest of Kant's philosophy. Kant is quite firm about the restriction of any knowledge claims to the realm of possible experience, and about prescinding from any metaphysical knowledge claims made beyond this realm. I strongly suspect that Kant would treat attempts to prove Cantor's Continuum Hypothesis in much the same manner as he would the ontological proof of God. Having a constructive proof of a proposition is thus a necessary condition for the possibility of representing the proposition, and its constituent elements, the concepts that comprise the proposition, *in concreto*. Intuitionistically, mathematics is actually limited to the realm of the finite or denumerably infinite. But again, this is not inconsistent with Kant's philosophy in general.

The Antinomies of Pure Reason, for example, purport to demonstrate the problems and contradictions involved in any metaphysical consideration of the infinite.[40] Moreover, in what sense could one, say, represent the concept of aleph-one *in concreto*? If the question of the construction of concepts in mathematics turns upon a question of meaning, as I think it must, then an intuitionistic interpretation of Kant seems warranted. For instance, what meaning could Kant attach to the realm of transfinite numbers? Very little I suspect. The denumerably infinite ordinal number $\omega$, the cardinality of which is aleph-null, is constructible in an intuitionistic sense and, hence, has meaning. On the other hand, the set of all denumerably infinite ordinal numbers is not intuitionistically constructible, and so lacks meaning. Brouwer makes this point clear.

> Because it is possible to argue to the satisfaction of both formalist and intuitionist, first, that denumerably infinite sets of denumerably infinite ordinal numbers can be built up in various ways, and second, that for every such set it is possible to assign a denumerably infinite ordinal number, not belonging to this set, the formalist concludes: "aleph-one is greater than aleph-null", a proposition that has no meaning for the intuitionist. Because it is possible to argue to the satisfaction of both formalist and intuitionist that it is impossible to construct a set of denumerably infinite ordinal numbers, which could be proved to have a power less than that of aleph-one, but greater than that of aleph-null, the formalist concludes: "aleph-one is the second smallest infinite number", a proposition that has no meaning for the intuitionist.[41]

We are now in a position where the notions of constructive proof and intuitionism may be able to shed some light on a few of the problems posed earlier. The fact that a proposition of mathematics for which a constructive proof cannot be given is thus devoid of meaning, engenders a notion of truth and real possibility not dissimilar to that proffered by Brittan. Clearly for a proposition to have a truth value, it must in some sense be meaningful. The statement "7+5=12" is decidable and hence is both meaningful and has a truth value. But what of propositions such as "$1+\omega=\omega$"? At B16-17 Kant asserts that certain mathematical propositions such as a=a and a+b a (where $b>0$) are analytic. Moreover, at A151/B190 he appears to affirm that these propositions have truth values. They must thus refer only to those decidable statements for which a constructive proof can be provided; and this reliance on constructive proof will prescind from statements of transfinite arithmetic which, for Kant, as well as the intuitionist, lack meaning and whose truth values are hence indeterminate. An intuitionistic interpretation of Kant will therefore replace a notion of mathematical truth as central to a theory of meaning by a notion of proof. What we must consider now is how such an interpretation is of assistance in demonstrating the syntheticity and apriority of mathematics.

By restricting knowledge claims to the realm of possible experience Kant can be seen to be, in some sense, a verificationist, *i.e.* we can only know what we can or could, in principle, verify as possible experientially. To be sure, problems or paradoxes arise when considering any metaphysical notion of the infinite by the finite understanding; Kant recognized that, and appropriately limited the domain over which knowledge claims by the understanding range. Similarly in mathematics there exists a plethora of problems and paradoxes concerning infinite sets and the infinite in general. Intuitionists resolve the problem by attaching meaningfulness only to decidable statements (statements of a finite or denumerably infinite nature) for which one can, at least in principle, provide a proof. Now Parsons points out that 'it would be a plausible interpretation of Kant to say that the form of intuition must be appealed to in order to verify the existence assumptions of mathematics'.[42] A bold claim, but to pursue this verification perspective in mathematics seems profitable. Dummett, in attempting to provide a philosophical foundation for the intuitionist theories of mathematics and logic, maintains that this foundation could be provided by replacing the notion of truth with the notion of proof, and then this could be generalized over arbitrary statements within a greater epistemological framework.

> Since we were concerned with mathematical statements, which we recognize as true by means of a proof (or, in simple cases, a computation), this meant replacing the notion of truth by that of proof: evidently, the appropriate generalization of this, for statements of an arbitrary kind, would be the replacement of the notion of truth, as the central notion of the theory of meaning, but that of verification: to know the meaning of a statement is, on such a view, to be capable of recognizing whatever counts as verifying the statement, i.e. as conclusively establishing it as true.[43]

In terms of geometrical proofs the appeal to a form of intuition as verificatory of mathematical propositions is not difficult to visualize. One sets out with a figure, performs some construction and derives various conclusions. The problem with this conclusion is just as easy to visualize. There exists the familiar difficulty of selecting the correct interpretation, and there seems to be no way that, from a consideration of the pure form of space alone, one can make a choice. As I have earlier indicated, this very difficulty tends to support the claim that geometry is a synthetic discipline. And perhaps the fact that the theorems of geometry are, if true, necessarily true, influenced logicians such as Frege, and Kant scholars such as Dryer, in declaring that, although arithmetic is analytic and known *a priori*, geometry is both synthetic and *a priori* in nature. But as we have already seen, this conception of geometry turns upon a notion of logical necessity with respect to a chosen axiomatic scheme. To be sure, Kant would ask this of any branch of mathematics, but he would also require a transcendental condition of necessity--an epistemological ground, if you like. Hence, for Kant, to say that geometry is *a priori* is to say it is descriptive of what would constitute a "really" possible world, and it is at least arguable that Kant would hold that this geometry must be Euclidean.

Clearly, though, alternative geometries exist--geometries that are internally consistent, but inconsistent with each other (many contain, for example, the negation of Euclid's famous fifth postulate). And even if Kant were not tied to one particular interpretation, that is, Euclidean geometry, how is one to account for his transcendental sense of necessity? Formal axiomatization, even for all of geometry, would not be sufficient. Hilbert, for example, would take Euclid's constructive postulates, replace their non-logical terms, such as line, point, circle, centre, *etc.*, with sentence letters and then consider the whole as a relational structure. This structure could then be "reinterpreted" in virtue of points, lines, *etc.*; but such an interpretation is not necessarily unique. Many commentators would assert that Kant's position demands a unique interpretation. I do not hold to such a strong line, but clearly for Kant the range of interpretation must be, at least, specifiable (that is, the interpretation must minimally range over that which is "really" possible); and even that is not implied by Hilbert's program.

Now, in Kant's defence, several points can be raised. To begin with, although Hilbert's program prescinds from the purely constructive nature of an axiom scheme, it still employs the existential quantifier. And if Hintikka is correct, then even purely deductive manipulation of symbols and sentences would not preclude syntheticity. Of greater importance, though, is the question of content. If no specific meaning is attached to the primitive terms of an axiomatic system, then any manipulation of the sentences in that system becomes little more than a logical exercise. I do not see Kant denying the possibility of investigations into geometrical systems as uninterpreted sets of sentences, any more than he would of investigations into transfinite arithmetic or even pure logic. What Kant would deny is the meaningfulness of such investigations. Mathematics, for Kant, has content, and in virtue of its constructive character, in fact, creates its own content; it is this that separates mathematics from logic. Having content means having existential import; and to make judgments upon statements having existential import involves us in a method of satisfying that judgment. Satisfaction of the judgment, in turn, is dependent upon the provisions of our notion of constructive proof. We are therefore led to a proof-theoretic notion of meaning which, in terms of the representation of mathematical concepts *in concreto*, leads us to the problem of objective validity and "real" possibility--the application of our mathematical concepts to what would count as "really" possible experience.

> Presumably Kant saw the "existence problem" and the "application problem" as two sides of the same coin. They come together in Kant's claim that the propositions of Euclidean geometry have "objective reality". On the one hand, to say that a proposition has objective reality is to say that it has a (knowable) truth value. On the other hand, to say that a

> proposition has objective reality is to say that it applies to objects. But how, on the one hand, do we guarantee truth values, and how, on the other hand, do we guarantee application of objects? The answer, once again, is to claim that Euclidean geometry is *a priori*.[44]

Brittan, I believe, is correct in his assessment of the "application problem" with respect to the "existence problem".[45] The point is this: to verify a proposition of geometry or arithmetic requires that some interpretation be given. One then constructs a proof of said proposition which must either conform to what would count as a "really" possible construction, or constitute what would count as a "really" possible construction. And this proof process would, in fact, take place in time. Hence, even if the formalist program of Hilbert or the reductionist program of the logicists is granted, the method of actually proving propositions under these constraints appeals to another pure form of intuition--time.

It seems to me, therefore, three issues remain to be considered in terms of a defence of Kant's philosophy of mathematics. Firstly, with respect to time and the pure form of time, we must note how the notion of constructive proof engenders a tensed notion of meaningfulness and, hence, truth. Secondly, we must note how this notion of meaningfulness is effected by the construction of concepts (the constituent elements of a judgment) in mathematics. And thirdly, we must note how this must conform to what would count as a "really" possible experience.

V.

Kant claims that geometry is a science which determines the properties of space synthetically and yet *a priori* (B40f.). It would seem to follow, then, that arithmetic is arguably a science which determines the properties of time synthetically and yet *a priori*. Kant then asks what must be our representation of space (or time) in order that geometric (or arithmetic) knowledge is possible. He concludes that the origin of the representation must be intuitive, and further that the intuition must be *a priori*. But to argue from the synthetic and *a priori* nature of mathematics to the intuitive and *a priori* nature of our representations of space and time has been justifiably subject to much criticism.[46] Kant, however, has other arguments (notably the Metaphysical Expositions) to support his claim that space and time are pure intuitions. Now, I have already considered what Kant feels he proved and what he, in fact, does demonstrate in the Metaphysical Expositions; what is of concern, at this point, is how what is representable in space and time conditions what is mathematically constructible.

> Of course it is one thing to speak of representations in space and time and another to speak of representations to the senses. What is represented to the senses is presumably represented in space and time, but maybe not vice versa. To establish a link of these two Kant would appeal to his theory of space and time as forms of sensibility. The relevant part of this theory is that the structures which can be represented in space and time are structures of possible objects of perception. The kind of possibility at stake here must be essentially mathematical and go beyond 'practical' or physical possibility.[47]

Now the possibility that Parsons mentions here is equated, by Brittan, with objective reality. Brittan further connects objective reality with meaningfulness. This is to say that only those sentences whose subject terms are "really" possible, *i.e.* have objective validity, have truth values. And those sentences for which it is not possible to assign a truth value are simply not meaningful.[48] Hence, for a mathematical proposition to be meaningful and thus have a truth value, it must have objective validity. And that, as Brittan points out, means its "existential presuppositions" must be satisfied.[49] But the only method of satisfying these "existential presuppositions" is the "real" possibility of giving an object (at least a mathematical object) in intuition *a priori*. Therefore, it is the 'possibility of providing *a priori* intuitions for Euclidean geometry [that] guarantees satisfaction of its presuppositions'.[50]

Although I agree with the thrust of Brittan's argument, in general, my position is somewhat weaker. For a mathematical proposition to be meaningful, and hence have a truth value, one must be able to give, or provide in principle, a proof (constructive) of said proposition. In Kantian terms, then, the proposition or judgment must allow of verification. Now, the construction of such a proof must not only conform to what is "really" possible, it will, in fact, turn out that the construction, itself, will determine the limits of "real" possibility. Therefore, if the notion of a constructive proof not only appeals to a notion of intuition (as do, for example, most proofs in Euclidean geometry), and satisfied Brittan's "existential presupposition" (by providing a method whereby one can actually construct a specific instance of a closed statement of the form $(\exists x)Fx$), but is grounded in the same conditions under which any experience, at all, is possible, then the transcendental conditions of necessity Kant requires in mathematics can be established.

Before examining precisely how the construction of concepts in mathematics will provide for this transcendental condition of necessity, let me recapitulate some of the problems Kant faces. The issue at stake is the synthetic and *a priori* nature of the propositions of mathematics;

and, on the surface, it would appaear that since analyticity and apriority usually go hand in hand, if one were to argue for syntheticity, it would also seem to follow that one should argue for the *a posteriori* nature of mathematical knowledge. Thus if Kant can show how it is that mathematics is synthetic, his major stumbling block will be to demonstrate its apriority or its transcendental conditions of necessity. Kant argues that mathematics has content, has existential import, and in virtue of this is synthetic. Both Hintikka and Brittan have proffered lines of argument in support of this thesis. Brittan traces syntheticity back to synthetic axioms from which mathematical propositions can be derived; Hintikka locates syntheticity in the proof process itself in virtue of an appeal to the use of auxiliary constructions or existential instantiation. Note that these two lines are not mutually exclusive, and both have much of value to say. My complaint has been that neither line can successfully account for the sense of apriority that Kant requires. Brittan's program has the advantage that, indeed, it seems to be true that mathematics is reducible to logic plus some set theory; and this set theory contains existence axioms, which are synthetic, if anything is. Since the theorems of mathematics follow according to certain rules of construction or inference, they are, if true, necessarily true. But what guarantees the truth of the antecedent of this conditional (the (4) axiom)? Geometrically, there are many different systems from which to choose: Euclidean, spherical, hyperbolic, *etc.*; arithmetically there exist standard and non-standard arithmetics. The axioms of, for example, Euclidean, Absolute, Lobachewskian and Riemannian geometries are mutually inconsistent, yet the theorems of each are necessarily true, if true. Moreover, in some sense it seems to be a matter of empirical investigation which set of axioms correctly describes our representation of space.[51] This surely lends support to the thesis that geometry is synthetic in nature, but mitigates the position that geometry can be known *a priori*. One might respond that the theorems can be known *a priori* with respect to the respective axiom sets, but this is not a position acceptable to Kant --he would ask much more. The theorems or propositions must not only be logically necessary with respect to the particular system, but the system, itself, must be transcendentally necessary with respect to the conditions under which one can represent, in space, the mathematical objects of the system at all. Any attempt to argue for the apriority of mathematics must take into account the possible verification of a particular geometric system. Hence, it seems to me that Brittan's proposal is doomed from the onset. What is of import is how we construct our geometrical concepts, and how these concepts are descriptive of our representation of space; therein will be the sought after transcendental conditions of necessity.

For Kant, however, how we construct our concepts depends upon what is constructible; and what is constructible surely depends upon limitations set upon our conceptual process by intuition. For example, there is nothing inconsistent with conceiving two straight lines (taken in a Euclidean sense) enclosing a figure. Such a figure, however, cannot be represented in intuition. Herein lies the failure of Hintikka's reconstruction. By divorcing intuition from sensibility, Hintikka eschews any limitations by intuition on what is constructible. And, for Kant, the construction of mathematical concepts must appeal to intuition in this sense. The notion of constructive proof that I have developed, on the other hand, permits precisely this. To satisfy the proposition, "two straight lines enclose a figure (in Euclidean space)", one must be in possession of a constructive proof of the proposition. This is to say that one must not only be in a position to assert there exists a figure enclosed by two straight lines, but one must actually provide a method whereby such a figure can be constructed. In the absence of such a method, the proposition has no truth value and is meaningless.[52] Since a constructive proof of the proposition, in Euclidean geometry, cannot, indeed can never, be provided, the mathematical object (a figure enclosed by two straight lines) is not a possible object of experience; and since such an object could not be represented in space it, as Parsons puts it, is not a possible object of perception.

Naturally, this intuitionistic stance concerning constructive proof and meaningfulness gives rise to a tensed notion of truth. Now, although I shall not specifically enter into the controversy revolving around tensed vs. tenseless truth, I do have a few comments that are pertinent. Firstly, a tensed notion of truth seems to be consistent with Kant's philosophy of mathematics in particular, and his epistemology as a whole. The Platonist, whatever else he may assert, certainly makes an ontological commitment concerning mathematical objects. Kant, on the other hand, speaks repeatedly of the construction of mathematical concepts. This strongly implies that the objects of mathematics are similarly the product of a constructive process. Indeed, in general, any object of experience is the result of some constructive process, and as such cannot be said to acquire meaning without some contribution from the side of the subject.[53]

Secondly, this tensed notion of truth brings to the fore the crucial importance of time as a form of sensibility. Although Kant offers us no Transcendental Exposition of time with reference to arithmetic, I think Parson, for one, is quite correct in assuming 'the dependence of arithmetic on the form of our intuition is in the first instance only on time'.[54] Parsons points out that whenever Kant speaks about arithmetic, he claims that number always ...ves succession. Kant's characterization of number in the Schematism makes this explicit.

> The pure image of all magnitudes (*quantorum*) for outer sense is space; that of all objects of the senses in general is time. But the pure schema of magnitude (*quantitatis*), as a concept of the understanding, is number, a representation which comprises the successive addition of homogeneous units (A142/B182).

If Kant is correct, *i.e.* if number depends upon the temporal notion of the successive addition of homogeneous units, then it is not so long a step to asserting that arithmetical propositions, indeed arithmetic in general, are dependent upon the pure intuition of time developed in the Metaphysical Expositions. And if this is true, then it is a very short step to asserting the *a priori* and synthetic nature of mathematics.

> In this way the apriority of time does not only qualify the properties of arithmetic as synthetic *a priori* judgments, but it does the same for those of geometry, and not only elementary two- and three-dimensional geometry, but for non-euclidean and *n*-dimensional geometries as well. For since Descartes we have learned to reduce all these geometries to arithmetic by means of the calculus of coordinates.[55]

This line of argumentation clearly goes against the grain of both common Platonist practice and the logicists's program. For the Platonist, the truths of mathematics in general, and of number-theoretic statements in particular, are if true, necessarily true. The Platonist makes certain ontological commitments concerning mathematical objects; propositions of mathematics may be true even though one might not be in a position to affirm their truth. And although there is much to be said for and against the acceptance of one scheme or the other, I tend to agree with, say, Dummett, that such acceptance must be made from within a greater epistemological framework or theory of meaning. Kant proffers such a theory, a theory which is consistent with the intuitionist approach. The logicist program, on the other hand, poses, in some ways, a more difficult problem for Kant. For the logicist the arithmetic operation of, for instance, addition can be explained by the set-theoretic union of classes. And that can be further defined in terms of a few primitive ideas and the propositions of logic. The point to be stressed here is the putative tenseless relation between classes and the universality of the logical relation. There is no need to appeal to a temporal notion of successive addition of units that involves intuition or pure intuition when one can employ the atemporal notion of class and achieve the same result.

How, then, does Kant propose to introduce the idea of the pure intuition of time as conditioning what is arithmetically constructible? In the first place, nowhere does Kant assert that time, or a pure intuition of time, conditions the properties of the concept of number. Number, Kant has claimed, is a pure schema of magnitude, but the concept of number is arrived at through an intellectual synthesis that echos some of the assertions of the logicist.

> ... pure mathematics considers space in geometry and time in pure mechanics. To these is to be added a certain concept, intellectual to be sure in itself, but whose becoming actual in the concrete requires the auxiliary notions of time and space in the successive addition and simultaneous juxtaposition of separate units, which is the concept of number treated in arithmetic.[56]

As Kant notes, however, it is the concept being represented *in concreto* (becoming actual) that necessitates the appeal to the forms of intuition: space and time. Just as many arguments outside the realm of mathematics may confine themselves to universal concepts and treat logically of these concepts *in abstracto*, so one can also treat arithmetic in a similar manner. The whole arena of standard and non-standard arithmetic is open to such treatment. If one is to make any successful knowledge claims regarding these concepts in judgment, however, one must deal with them *in concreto*, and what was once an intellectual synthesis involving the concept of number is now replaced by a figurative synthesis involved in the application of the schema number. This is to say that one must supply content, provide for an interpretation. And that, I have argued, means to provide, at least in principle, for a constructive proof of the judgment of which the concept is predicated.[57] It is in terms of constructive proof, then, an area so critical to the construction of concepts in mathematics, that the connection between pure intuition and arithmetic constructibility will be found.

Pursuing the notion of intuition in arithmetic, consider, for example, the simple proposition: "7+5+12". There are several ways in which one could approach such a proposition. All of these ways, I claim, when viewed from the proper perspective, involve in some manner, sensible intuition or time as a form of intuition. Firstly, there is an immediate appeal to inspection. Given a group of seven objects, a group of five objects, and a group of twelve objects, one can draw a one-to-one correspondence between the number of objects contained in the first two groups and the number of objects contained in the third group (note that one need not even require the notion of counting in order to perform this task). This quite obviously appeals to intuition in its spatial and temporal aspects, but other alternatives are, just as obviously, present.[58] Secondly, one might try the method of direct computation. I take it that what one is doing when one performs a computation, is endeavouring to establish the truth (or falsity) of the proposition "7+5=12" by discovering whether, in fact, 7 and 5 equals 12. This, I suspect, is the solution Kant envisaged. Given the concept of the number 7, one enlists the aid of intuition in the form of fingers or points in counting off the number 5 and arrives at the sum 12. One, Kant says, <u>sees</u> the number 12 come into being, and it is in virtue of this that he claims the propositions of mathematics are synthetic. The rather crude example Kant offers clearly refers to sensible intuition; just as clearly, on the other hand, this method is cumbersome, if not effectively impossible to employ, when considering very great numbers. Kant goes on to maintain, however, that when considering large numbers the synthetic nature of the propositions of mathematics is even more evident. Obviously it is not the use of fingers or points as visual aids, then, that is critical in the counting process. What Kant must mean is that it is the method, itself, that is important, and not necessarily the realization of the method. And what could this mean except the successive addition of units in time? For example, consider how the natural numbers are generated from the primitives, "0", and a successor operation. The concept of the number 5 would be represented by SSSSSO; the concept of the number 246532 would be represented by an appropriate number of S's successively attached in front of the 0. Hence, we have a method by which the natural numbers can be generated (Peano's postulates serve nicely) and the ability to recognize that any given numeral is the name for a number that can be generated in that fashion. Examine, then, the notion of constructive proof I have offered. A proof is constructive just in case it, in fact, proves a specific instance of the proposition which one sets out to prove, or permits, in principle, an adequate means of providing a proof of that instance. Thus, by analogy, although one might not in every case count off S's in order to solve a numerical problem, one possesses, with the assistance of Peano's postulates, an effective means of doing so.[59] But the actual employment of the successor operation takes place in time. One successively attaches S's; and just as successive times are different, so are successive numbers. If the appeal to intuition is not already clear, one should note that each S one attaches is, in itself, a sensibly perceptible object. Note also, that whether one starts with "0", "1", "101", or any other given number, each construction is isomorphic. As a matter of fact, any generation of a series of natural numbers utilizing Peano's postulates turns out to be isomorphic, no matter what numeral names or tokens are used. In this sense, then, perhaps Kant has a right to regard the appeal to intuition as formal.

> The basis for the use of a concrete perception of a sequence of *n* terms in verifying general propositions is that, since it serves as a representation of a structure, the same purpose could be served by any other instance of the same structure, that is any other perceptible sequence which can be placed in a one-one correspondence with the given one so as to preserve the successor relation. This might justify us in calling such a perception a "formal intuition". We might note that the physical existence of objects is not directly necessary, so that we can abstract also from that "material" factor.[60]

We have all along been considering the propositions that comprise mathematics as a body of knowledge. Owing, in large part, to the greater epistemological framework within which Kant embeds his theory of mathematics (a framework that contains both a theory of meaning and a theory of truth), the question we have been examining has, therefore, not been simply: are mathematics and the propositions of mathematics both synthetic and *a priori* in nature? Rather, I would frame the question more carefully: are the proofs or verifications of the propositions of mathematics both synthetic and known *a priori*? Kant has claimed that the propositions of mathematics have content and existential import, and thus any proofs of these propositions must be synthetic. The problem has, therefore, become a problem of demonstrating how the relevant sense of apriority is consistent with the established syntheticity of mathematics. But what of the logicist? Certainly there exists a further manner in which the proposition, "7+5=12", can be construed. The logicist might offer a way which not only restricts the use of synthetic set-theoretic notions, but provides for a solution to the problem by employing only formal, first-order predicate calculus. Take, for instance, the following schema:

(1) $((\exists x)_7 Fx \ \& \ (\exists x)_5 Gx \ \& \ (x)\sim(Fx \ \& \ Gx)) \supset (\exists x)_{12}(Fx \vee Gx)$

where $(\exists x)_n Fx$ is expanded as follows,

(1.1) '$(\exists x)_0 Fx$' for '$\sim(\exists x)Fx$', and
(1.2) '$(\exists x)_{n+1} Fx$' for '$(\exists x)(Fx \ \& \ (\exists y)_n(Fy \ \& \ y \neq x))$'.[61]

Here, then, is a formulation of the proposition, "7+5=12", that is not only purely formal, but does not even require the postulation of any existence axioms. Yet as Parsons points out, 'although the schema [1] does not imply that the universe contains any elements or that any constructions can be carried out, the proof of it involves writing down a group of [seven] symbols representing the Fs, another such group [of five symbols] representing the Gs, and putting them together to get [twelve] symbols'.[62] Firstly, it is not trite to assert that when carrying out the proof, one must carry it out in time, *i.e.* when one constructs (on paper or in the imagination) the respective groups, one carries out the operation successively, unit by unit. Whatever else the symbols may be they are representative of something with respect to the proof process, and again we return to the grounding notion of the form of inner intuition, *i.e.* time. Secondly, we encounter the notion of symbolic construction. In the Transcendental Doctrine of Method Kant avers that mathematics not only constructs magnitudes in geometry, but in algebra as well, Algebraic symbolic construction, Kant maintains, abstracts completely from the properties of the object that is to be thought in terms of magnitude, and then chooses some sort of notation to represent the magnitudes to be related.

> Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the various operations through which the magnitudes are produced and modified. When, for instance, one magnitude is to be divided by another, their symbols are placed together, in accordance with the sign for division, and similarly in the other processes; and thus in algebra by means of symbolic construction, just as in geometry by means of ostensive construction ... we succeed in arriving at results which discursive knowledge could never have reached by means of mere concepts (A717/B745).

Kant's peculiar phrasing at B15-16 of "seeing" the number 12 come into being can now be more easily explained. When one adopts a notation, one abbreviates or represents certain concepts with symbols, and the constructions performed are symbolic constructions. By manipulating the symbols, according to certain pre-established rules, one can "see" with certainty securing 'all inferences against error by setting each one before ones eyes' (A734/B762). The point Kant is making is this: the difference between geometric construction and algebraic construction is not a difference in kind. If one accepts that ostensive geometric proofs are synthetic in character, as I think we must, then one must accept the synthetic nature of symbolic algebraic proofs for the same reason. Just as after certain constructions are performed on the given triangle used in the proof of Pythagoras' theorem we can "see" the steps that would count as a proof of that theorem, after a choice of notation we can "see" what would count as a proof or solution to an algebraic equation. Schema (1), then, considered formally, would describe an analytic relation, but would be contentless and devoid of determinate meaning. Once, however, one takes it to represent a relation among magnitudes (and the concept of number is, for Kant, inevitably tied to that of magnitude, it is the pure schema of magnitude), it acquires a determinate sense and is subject to the same conditions as symbolic algebraic construction.[63]

If one puts these claims in perspective, Kant's meaning becomes more clear. For Kant, all meaning originates on the side of the subject. The question of a proposition, be it mathematical or otherwise, possessing a truth value without it having been supplied by the subject, simply does not arise. The truth value of a proposition is determined by a proof, construction, computation or verification regarding its constituent concepts. So how do we know we have correctly performed a computation, construction, or proof? In arithmetic, geometry, *etc.*, we "see" it to be true. In other words, we must already possess the ability to recognize what would count as a proof, a valid construction of a correct computation. In constructing a concept we must already know what would count as a proof of that concept. The connection to sensibility in construction is clear in virtue of the "setting out before one's eyes"; the connection with the pure forms of sensibility is clear in virtue of the ability to recognize what would count as a proof or verification of what we are, in fact, "seeing". Note that we do not require of space that it be a particular, determinate, space (such as Euclidean 3-space); it suffices that our representation of space be intuitive and *a priori*. And a similar claim may be made about time; but a consideration of time provides us with an added bonus, since time, as a pure form, conditions the successive nature of our representations provides a model for the arithmetic notion of a successor operation. And <u>that</u> notion is determinate, insofar as any interpretation would be isomorphic, and has a claim on the sense of necessity Kant requires.

vi.

Thus far I have argued that the construction of mathematical concepts must appeal to pure intuition; I have not discussed at any length the mechanism by which concepts are connected with pure intuition, and thus how such an appeal can be successfully made. Now I earlier alluded to a distinction that must be drawn when considering pure intuition: one must be careful to keep separate the notions of form of intuition and formal intuition. Since I have already laid out much of the foundation for this discussion in Chapter IV, my remarks will be brief. A form of intuition can be regarded as either a form or manner of intuiting, a disposition, if you like, or as a form or essential structure of that which is intuited.[64] In either case, however, being a pure form, it is quite indeterminate. Nevertheless, I have argued that, for Kant, mathematics must always consider its concepts *in concreto* in order to successfully make any knowledge claims. In other words, the objects of any mathematical inquiry must have a determinate structure. In a note to B160 Kant proposes the idea of a formal intuition, an idea that is able to connect the pure forms of intuition *a priori* with the determinate structure of space and time demanded by the construction of concepts in mathematics.

> Although this note is attached to a discussion of the synthesis of apprehension, that is, the empirical synthesis which Kant contends is involved in sense perception, it is intended to explicate the claim made in the text that space and time are not only *a priori* forms of intuition but are themselves *a priori* intuitions with a manifold, or content, of their own.[65]

Allison calls this formal intuition a "hybrid" which requires both the form of intuition and a concept by means of which this form is determined in a particular fashion. Now, if the forms of intuition must somehow condition or model what is "really" possible, and therefore what is mathematically constructible, in order for Kant's arguments regarding the syntheticity and apriority of mathematics to go through, then the mathematician must have some way of representing this form in a determinate sense such that the concepts he constructs will correspond to what is "really" possible. But if a formal intuition is involved whenever a mathematical concept is constructed, then the concept must "conform" to what is "really" possible since the construction, itself, will in effect determine the limits of "real" possibility. There remains, then, only an examination of how this is the case.

The claim is this: the constructions of mathematical concepts define their own limit of possibility and supply their own content in virtue of the construction process, itself. A mathematical judgment or proposition has meaning if one can, through a proof process, supply a truth value for the proposition. But the proof process, itself, involves a mathematical construction without which the proposition and its constituent concepts would have no content (and thus no meaning). Consider, for example, the construction of the concept of triangle. I suggest that when one constructs the concept one does so in a two-fold manner. The preliminary construction gives the concept of triangle a meaning with a sort of quasi-content. This is to say that one can analytically or discursively derive certain things immediately from the construction (such as three angles, the fact that the triangle is a figure, *etc.*). The concept triangle "contains" within it at least this much information. A concept, Kant says, is 'always as regards its form, something universal which serves as a rule' (A106). Now, if a concept functions as a rule for the understanding, one should be able to "read" something out of the rule discursively. But this requires a judgment; and all judgments, by definition, require some sort of content--even analytic judgments. The content required in order to make an analytic judgment, I call quasi-content. In terms of mathematics, one might also hold that this is the formation of a preliminary definition of the object of which the concept is predicated. To make determinate judgments, however, is to give the concept a fully constituted content. By this, I take it, the concept serves as a rule for a determinate representation in space and time. And this, in the final analysis, will provide for a complete definition. For example, the judgment or proposition that the interior angles of a triangle equal two right angles, or have a defect greater than 0°, or have an excess greater than 0°, requires the construction of a proof in order to give it truth and meaning. But this, in turn, requires that the initial concept of triangle be given a determinate structure. Hence the very act of judging gives the concept its content, To even make the judgment, "This is a triangle", selects, because of the demonstrative, a particular triangle, with a particular spatial structure from which various other propositions can be synthetically derived and proven *a priori*. To take an arithmetic example makes the point even more perspicuous. The construction of the concept of "one" is, in fact, the basal intuition of mathematics without which the number series could not be generated at all.[66] To construct a concept of, say, the number five means to actually generate the number five. In this case the very meaning of the concept is given by its method of construction. The construction of the concept of the number five, constructs, at the same time, its own content. The construction is thus synthetic, and in virtue of the necessity involved in the method of construction, is also *a priori*. The construction of any mathematical concept, then, defines its own limits of possibility in virtue of its method of construction, and supplies its own content in virtue of the process of proof it must undergo in making any determinate judgments. This entire process is a synthetic and necessary one. Hence it is possible to demonstrate that judgments in mathematics are synthetic and known *a priori*.

# CHAPTER VI

## CONCLUSION

I have herein endeavoured to defend Kant's position that the propositions of mathematics are synthetic and known *a priori*. To this end I have, in the first place, attempted to select those parts of Kant's general epistemology that pertain specifically to the issue of the synthetic *a priori* in mathematics. I then offer, in the form of a four stage argument, my defence proper. I have, in the course of my defence, tried, in most cases, to maintain the spirit of Kantianism as much as possible. This seems to me to be of crucial importance, and is perhaps a point not well observed by many of Kant's commentators or critics. Kant's philosophy, Kant's system, is an architectonic. And his thesis concerning the synthetic *a priori* is inextricably embedded within his epistemology, his critical idealism. Hence, to defend Kant, while not maintaining the spirit of his philosophy, runs the great risk of seriously affecting other parts of his system, perhaps with disasterous consequences. And that runs the risk of bringing the whole edifice down, and taking with it the thesis of the synthetic *a priori*. This is nowhere more evident than in attempting to defend Kant against the more contemporary philosophical objections of, say, the logicists (Kant is more than capable of handling the objections of his own contemporaries One must be careful here. The advances in logic, mathematics, science and, indeed philosophy, since the time of the writing of the First Critique (in particular, since the time of Frege and Russell in whom, quite arguably, one can find the genesis of the logicist enterprise), demand responses to questions Kant could not have foreseen. Some reconstructions, such as that of Hintikka, respond to objections by divorcing those elements of contention in Kant's philosophy from his theory of mathematics in order to salvage his position. I am thinking, here, of Hintikka's attempt to sever intuition from sensibility. This attempt, however interesting it may be in its own right, is a mistake with respect to Kant's philosophy as a whole. I have, therefore, attempted a reconstruction that is faithful to, or compatible with, the major tenets of Kant's general epistemology and critical idealism.

The first issue that is of concern is the possibility of drawing the distinction between analytic and synthetic judgments to begin with. Kant's remarks on this matter, in the Critique of Pure Reason, are brief, somewhat metaphorical and open to interpretation (or misinterpretation). I feel, however, in response to various critics, such as Eberhard and Maass, Kant makes his position clear. The determination of the syntheticity of a judgment revolves around the question of content. Synthetic judgments have content; they have existential import and, as such, are at least subject to, and perhaps themselves establish, the limits of "real" possibility. Analytic judgments, on the other hand, are invariant with respect to content. *Prima facia*, this seems to undercut the logicist's objections by rendering the determination of a distinction between analytic and synthetic judgments subject to the rules of transcendental, rather than general, logic. This therefore makes the distinction an epistemological or phenomenological one, rather than purely logical. But this seems to imply that there is nothing to distinguish, or that analytic judgments are, in some manner, a mere subset of synthetic judgments. Kant, however, wishes the distinction to be formal. I have argued that, in order to accomplish this, Kant must allow for the functioning of the understanding at the level of intuition. He will then be able to permit the predicating of concepts in mathematical judgments of a subject conceptual complex containing both intuitive and conceptual components. The intuitive component would provide for the content of judgment which determines syntheticity; the conceptual component permits the relationship of identity (in the Kantian sense) to hold between the subject and predicate concepts in determining analyticity. That, however, can only come about if a notion of intuition adequate to the task is provided.

In the second stage of my defence, I thus address the Kantian notion of intuition directly. I examine how one's knowledge claims depend upon two fundamental sources: that of receiving impression or representations realized in intuition, and that of the spontaneous power of knowing the object through these representations realized in the predicating of concepts. After laying out some of the problems that arise in Kant's theory of knowledge when considering the Two World's Dichotomy, I turn to the notion of figurative synthesis. Here the faculty of the understanding can be seen to be operating, in some fashion, at the level of intuition. My intention, at this stage, is not to solve all of the problems inherent, or claimed to be inherent, in Kant's epistemology, but to provide a coherent and adequate account of how knowledge of mathematical objects is possible. Now, although the figurative synthesis will resolve some of the puzzles implicit in the Two World's Dichotomy, of critical importance is the permission it grants to the possibility of drawing a formal distinction between analytic and synthetic judgments.

Moreover, the account of intuition, in general, will provide for a sense of "content" in judgments while reinforcing the connection between intuition and sensibility.

Kant asserts that the propositions of mathematics are synthetic in nature and known *a priori*. Although more must be said about syntheticity, and what it means for a judgment or proposition to have content, one should also have a sense of what it means for a judgment or proposition to be known *a priori*. For Kant, apriority is roughly equivalent to universality and necessity. For mathematical judgments to be known *a priori* thus means something much stronger than their being only logically necessary. Pure intuition, Kant claims, provides for this sense of apriority and guarantees the objective validity of mathematical judgments in their application. Stage three of my defence thus comprises an account of pure intuition and the pure forms of sensibility: space and time. In the Axioms of Intuitions and Anticipations of Perception, Kant argues that the synthesis whereby extensive and intensive magnitudes are generated is the very same synthesis whereby space and time, in general, are determined. Hence, to represent something as an extensive or intensive magnitude is to represent it as an object of experience. Now measurement presupposes an extensive magnitude which can be measured. Thus measurability presupposes the Axioms. And the Axioms are the principles that ground the possibility of mathematical judgments within a possible experiential framework. But the metric, Kant argues, is brought to space and time by the understanding. The question, here, is upon what basis he can ground this argument? One method of approach is to demonstrate the objective validity of space and time. And Kant has an argument that putatively establishes precisely this: the Metaphysical Expositions of Space and Time. I thus undertake an examination of the Metaphysical Expositions, Expositions which, it has been claimed, have been decisively refuted. I hold, however, that they do contain a kernel of solid, valid argumentation. I argue that not only does Kant have a strong case for taking our representations of space and time as intuitive in character, but that there is a clear sense in which space and time are representations which underlie our representations of outer sense (of numerically distinct objects) and inner sense (of successive or simultaneous events). I further argue that the Metaphysical Expositions and the argument from Incongruent Counterparts are consonant with at least some notion of substantival space and time (or spacetime). This lays open a path to an account of formal intuition, an account which permits the particularity and determinateness of space and time at one level, which is, in turn, grounded in the universality and necessity of the forms of intuition at a transcendental level.

The final stage of my defence pulls together all the pieces brought to the fore in the previous chapters when I deal with the problem of concept construction in mathematics, itself. I first explore the two main lines of argumentation that purport to defend Kant's assertion that the propositions of mathematics are synthetic and known *a priori*. Brittan, whom I take as paradigmatic of one line, maintains that mathematics can be reduced to logic plus some set theory. This set theory contains existence axioms which are, indeed, synthetic. Hence, mathematics proceeds analytically (and thus necessarily), but is synthetic in virtue of its axioms or primitive propositions. Hintikka, whom I take as paradigmatic of the other major line of argumentation, permits a reduction of mathematics to logic, but argues that syntheticity accrues in virtue of the logical rule of existential instantiation. I find both lines to fail for various reasons. Brittan claims that the syntheticity of the propositions of mathematics is a feature of the propositions, themselves, and not the way in which they come to be established. I argue, rather, that syntheticity is obtained in and through the proof process itself. Hintikka maintains that the notion of intuition should be separated from syntheticity. I argue that that cannot be the case when viewing Kant's epistemology in general. In addition, for Hintikka, only those mathematical proofs that avail themselves of the natural deduction rule of existential instantiation are synthetic. Kant, I believe, is quite clear on the matter. Any mathematical proposition, derived through some proof process, is synthetic. Moreover, the reconstructions of both Brittan and Hintikka fail for a similar reason: neither can successfully account for the sense of necessity, and thus apriority, that Kant requires, *i.e.* neither provides for the transcendental conditions of necessity that Kant claims must hold. There is, however, much merit to be garnered from both reconstructions. Synthetic propositions have content, existential import. This means that the propositions of mathematics must conform to the limits of "real" possibility (if the proof process, itself, does not prescribe these limits). Now it is arguable that something is "really" possible just in case it is objectively real. It is also quite arguable that objective reality can be connected with a notion of meaningfulness. But one way to say a proposition of mathematics is meaningful is to say it has a truth value. So, if something is meaningful just in case it is "really" possible, what is it that determines the limits of "real" possibility? In mathematics, I take it, that to show something is "really" possible is to provide for the possibility of a "really" possible construction of that object. And this is simply the providing of a proof, or providing for, in principle, a proof of the object in question. (For that which is not "really" possible, one would provide a refutation.) The question remains, though: what kind of proof? I argue that, for Kant, the proof must be constructive. Hence, if Kant can be considered, in some ways, a verificationist, and if mathematical propositions must contain an intuitive component (insofar as mathematical propositions must be represented *in concreto* in order for any successful knowledge claims to be made), the notion of truth as central to a theory of meaning is supplanted by the, somewhat verificationist, notion of proof. This, in turn, implies a tensed notion of truth that is consonant with Kant's philosophy as a whole, and brings into relief the importance of time as a form of sensibility. I argue that time, as a pure form, conditions the successive nature of our representations and provides a model for the logicist's arithmetic conception of a successor operation. Thus, also, do the forms of intuition condition what is "really" possible. And if a formal intuition is involved whenever a mathematical concept is constructed, then the concept must conform to what is "really" possible. But the proof process, itself, involves a mathematical construction without which the proposition, and its constituent concepts, would have no content. Thus the very act of proving a mathematical proposition gives content to the constituent concepts in the judgment, and so supplies syntheticity. But in virtue of the method of construction, the necessity involved in the rules of construction and the appeal to pure intuition, apriority is also supplied. It is in this sense, then, that I offer my defence of the syntheticity and apriority of mathematical judgments in Kant's theory of mathematics.

Notes to Chapter II:

1. Frege, *Grundlagen*, p. 17.

2. Although Kripke, for example, has argued that there are propositions that, even though known *a posteriori*, are yet necessary. The Kripkean proposal, however, is somewhat different than that of Kant. In the first place, he distinguishes between the epistemological notions of that which is known *a priori* or *a posteriori* and the metaphysical notions of necessity and contingency. And it is not at all clear whether Kant would have countenanced such a distinction. Secondly, Kripke, himself, phrases the problem as a problem involving how contingent identity statements are possible. He then argues for the existence of a set of sentences which are identity statements that are true necessarily but only known to be true *a posteriori*. Kant, on the other hand, is more inclined to identify necessity with *a priori* knowledge. Hence the sense of necessity to which the two philosophers refer is quite different. Kant, in fact, refers to the transcendental conditions of necessity. These are conditions that, though they include logical necessity, are much wider in scope (a position I hope to make clear in Chapter V). Thus, although I do not wish to denigrate, in any way, the arguments Kripke proffers in the area, the issues Kripke addresses are not the same issues Kant envisages. Hence, however interesting Kripke's analysis of contingent identity statements may be, it is not particularly pertinent to Kant's philosophy of mathematics.

3. I shall follow common practice in identifying quotations from the Critique of Pure Reason by referring to the standard First and Second Edition pagination. All references to the Critique of Pure Reason are included in the text and follow Norman Kemp Smith's translation.

4. Quine, "Two Dogmas of Empiricism," From a Logical Point of View, p. 21. There also exists in the literature on the subject some discussion of the correct translation of "*enthalten*", a word which Kemp Smith renders as "contains" (for example, see Lucey and Palmer in the *Akten des 4. Internationalen Kant-Kongresses, 1974*; the major point raised here seems to concern the spatial or non-spatial senses of containment). These dicussions, however, do not, I believe, significantly alter the thrust of the argument to follow.

5. Garver, "Analyticity and Grammar," Kant Studies Today, pp. 247-48.

6. Frege, *Grundlagen*, p. 99ff.

7. Existential judgments are not a problem for Kant; they are synthetic. I suppose existential judgments of the sort, "Unicorns do not exist" or "Horses exist", superficially pose a puzzle in virtue of the

subject-predicate distinction. Kant certainly holds that "being" is not a real predicate; for Frege existence is not an ordinary predicate, but rather a concept of the second order. But as I shall argue, Kant is not locked into the subject-predicate form in distinguishing between analytic and synthetic judgments.

8. Thus any alleged logical truth, such as "p v~p", can be said to be analytic. Moreover, sentences such as "No unlearned man is learned", represent an instance of the schema (x)(~Fx⊃~Fx) which is, of course, truth-functionally true. This I suppose, though, still does not gainsay Quine--for he would demand clarification of the Kantian notion of self-contradictoriness. But it does extend the scope of what is to count as analytic from Kant's initial formulation in the Introduction. Note also that I use the terms "proposition" and "judgment" as virtually synonomous. I recognize that Kant, in general, speaks of "judgments" rather than "propositions", and that there exists some literature (Garver, for example) on the subject that advocates the necessity of keeping the distinction clear. However, for Kant judgments do not actually represent the <u>act</u> of judging but rather that which is judged (Kant's account of judgment at A68-69/B93-94 makes this quite clear), and hence, I believe, little is lost with respect to my exposition in using the terms more or less interchangeably.

9. Kant, <u>Logic</u>, para. 37, p. 118.

10. Kant, <u>Logic</u>, para. 36. pp. 117-18.

11. Kant, <u>Logic</u>, para. 37, p. 118.

12. Beck, "Can Kant's Synthetic Judgments Be Made Analytic?" <u>Studies in the Philosophy of Kant</u>, pp. 74-76.

13. Frege, *Grundlagen*, p. 99.

14. Russell points out that Kant '... never doubted for a moment that the propositions of logic are analytic, whereas he rightly perceived that those of mathematics are synthetic. It has since appeared that logic is just as synthetic as all other kinds of truth' (<u>Principles of Mathematics</u>, p. 457).

15. Kant, <u>Prolegomena</u>, para. 2, p. 14.

16. The notion of "content" I shall leave at this point, somewhat vague. Some philosophers, such as Hintikka, would simply identify the content of mathematical judgments with the introduction of new free variables (which would seem to closely parallel the natural deduction rule of existential instantiation); I, however, feel that any explanation of "content" must involve, in a crucial way, some notion of intuition. The critical point to note is that analytic judgments are invariant of possible experience, are true across all possible worlds, if you like; synthetic judgments are subject to the constraints of possible experience, are true in only "really" possible

worlds. I shall later argue (Chapter V) that the content of a mathematical concept, and thus its meaning, is inexorably tied to the process by which the concept is constructed. And there I shall endeavour to show how this constructive process in mathematics describes those worlds that are really possible.

17. Frege, *Grundlagen*, p. 3.

18. On this basis, Eberhard goes on to criticize Kant's critical idealism insofar as the very possibility of what Kant construes to be synthetic judgments *a priori* or *a posteriori* in fact presupposes a reference to things as they are in themselves (a point which Kant would, of course, strongly deny).

19. Kant, On a Discovery, [229] (in Allison's The Kant-Eberhard Controversy, p. 141).

20. Kant, On a Discovery, [231] (in Allison's The Kant-Eberhard Controversy, p. 143).

21. Kant, On a Discovery, [194] (in Allison's The Kant-Eberhard Controversy, p. 113).

22. Allison, The Kant-Eberhard Controversy, p. 56.

23. There is a sense, for example, in which one might maintain that, for Kant, one must be able to decide upon a criterion of syntheticity prior to making any claims about analyticity. All knowlege must begin with a synthesis; and this synthesis is a necessary condition for the possibility of experience. It is that prior synthesis which confers meaning--even with respect to analytic judgments. Kant, himself, says that with regard to content no concepts can first arise by way of analysis. 'Synthesis of a manifold (be it given empirically or *a priori*) is what first gives rise to knowledge' (A77/B103). Though this knowledge may be crude and confused (and so may require analysis) it is in virtue of synthesis that the elements necessary for knowledge are first combined to produce "a certain content." In any event, to emphasize this aspect of synthesis (as Allison does) seems quite inconsistent with the position Kant lays out earlier in the First Critique with respect to the analytic/synthetic distinction. I suggest, rather, one should view this issue from the perspective of epistemological justification, while attempting to keep the ability to draw the distinction at a purely formal level.

24. Beck, "Can Kant's Synthetic Judgments Be Made Analytic," Studies in the Philosophy of Kant, p. 78.

25. And I shall argue, in Chapter III, that the notion of figurative synthesis provides for this very activity, *i.e.* the functioning of the understanding at the level of intuition.

26. Allison, The Kant-Eberhard Controversy, p. 44.

27. Kant, Logic, para. 106, p. 144.

28. For an extended discussion and a reasonably complete categorization of analytic, synthetic, real and nominal definitions, see Beck's "Kant's Theory of Definition," anthologized in Studies in the Philosophy of Kant. Beck also argues that nowhere does Kant base even analytic judgments upon definitions.

29. Frege was quite concerned that the notion of intuition be kept separate from any mathematical inquiry grounded on pure logic. He maintained that Kant had 'no alternative but to invoke a pure intuition as the ultimate ground' (*Grundlagen*, p. 18). Russell's view, at one time, was that the reduction of mathematics to logic rendered any notion of intuition in mathematics otiose: 'thanks to the progress of Symbolic Logic, especially as treated by Professor Peano, this part of the Kantian philosophy is now capable of a final and irrevocable refutation' (Principles of Mathematics, p. 4). If we take the argument from content seriously, however, then mathematical reasoning is not merely formal. One constructs mathematical concepts (in intuition); mathematics has content, existential import. It seems to me that some notion of pure intuition is necessary; and insofar as this is the case I agree with Kant; but I do not feel that the full-blown critical-idealism of the Transcendental Aesthetic is either warranted or necessary (and so, in some sense, I sympathize with Frege). Nevertheless, I shall let the issue remain open until Chapter IV.

30. Frege, *Grundlagen*, p. 100.

31. Beck, "Can Kant's Synthetic Judgments Be Made Analytic," Studies in the Philosophy of Kant, p. 77.

32. Werkmeister, Kant, The Architectonic and Development of His Philosophy, pp. 19-20.

33. Beck, "Can Kant's Synthetic Judgments Be Made Analytic," Studies in the Philosophy of Kant, p. 84.

34. Frege, *Grundlagen*, p. 100.

35. There does exist a question of how complete the concept actually is. Lakatos, for example, would undoubtedly object that one cannot always be certain one has captured what one wishes to capture in any given definition (his presentation of the different positions held with respect to the proofs of Euler's conjecture bear this out); but even he, I suspect, would admit that once a set of sentences or propositions has been axiomatized, then the given concept with respect to that set of sentences or propositions is pellucid.

36. To the contrary, Kant's program has all along been to delimit the realm wherein knowledge claims can be justifiably asserted (in this area Kant's enterprise is intimately involved with justification and Frege's comments seem amiss). Frege argues (*Grundlagen*, p.

101 ff.) that, since knowledge claims for Kant require concepts and intuition as necessary ingredients, no intuition would give us nought, or one, or infinitely many objects (to be fair to both Frege and Kant, Frege realizes that Kant assuredly uses the word "object" in a different sense than he). But of course Kant will deny the possibility of any meaningful sentences being asserted about completed infinite totalities (the Antinomies of Pure Reason attest to this fact). Kant would argue that such considerations regarding infinities would lead to contradiction; the least one could argue would be that these infinities would lack significant meaning. As far as zero and one are concerned, the latter can come about through the concept of unity described in the Analytic of concepts as a necessary precondition for the possibility of experience. The former presents a more difficult problem and is not specifically addressed, to my knowledge, by Kant. In the Anticipations, and in a different context, Kant speaks in a rather unedifying fashion of zero being the negation of reality. The problem though, I suspect, is not insuperable. In the Anticipations he also speaks of "absence" and, perhaps equating nought with the experiential perception of "lack" represents a possible solution. In any event, if Kant's program goes through, both the conventionalist and logicist positions will be undercut. Hilbert, even though it turned out his enterprise failed, frames it well.

> [Material logical deduction] deceives us only when we form arbitrary abstract definitions, especially from those which involve infinitely many objects. In such cases we have illegitimately used material logical deduction, i.e. we have not paid sufficient attention to the preconditions necessary for its valid use. In recognizing that there are such preconditions that must be taken into account we find ourselves in agreement with the philosophers, notably with Kant. Kant taught--and it is an integral part of his doctrine--that mathematics treats a subject matter which is given independently of logic. Mathematics, therefore, can never be grounded solely on logic. Consequently, Frege's and Dedekind's attempts to so ground it were doomed to failure ("On the Infinite." Philosophy of Mathematics, p. 142).

37. Schulze, "Schulze's Review," [409] (in Allison's The Kant-Eberhard Controversy, p. 175).

38. As, for example, does Brittan in his Kant's Theory of Science. I am aware, as is Brittan, that this places the emphasis on ontology rather than epistemology. Like Brittan, I feel that little is lost in such a reformulation (indeed, much is gained in both ease of explication and clarity) as long as one does not lose sight of the fact that Kant's program is fundamentally epistemological; unlike Brittan I prefer the epistemological approach whenever possible since the onotological approach tends to stress somewhat different problems than Kant was addressing.

39. Brittan, Kant's Theory of Science, pp. 26-27.

## Notes to Chapter III:

1. Although the sceptical problem of the possibility of any knowledge at all is an issue that warrants some response, I shall not do so directly in the course of my arguments. In any case, if one can resolve some of the problems that plague the Transcendental Deduction of the Categories, then the Deduction does, I believe, go through. And that, of course, would itself be a refutation of most of the sceptic's arguments.

2. More precisely, judgment is actually the mediate knowledge of an object; it is the representation of a representation of an object (A68/B93).

3. Pippin, Kant's Theory of Form, p. 47ff.

4. This avenue seems to be a favourite of many commentators. It is, however, I think at best inadequate. Nevertheless, the "perspectival argument" is quite instructive and makes clear several distinctions required for the layered synthesis I propose.

5. Allison, in "Kant's Concept of the Transcendental Object," quite correctly maintains that the transcendental distinction between appearances and things-in-themselves is the result of second-order analysis of the necessary conditions of their constitution. However, he goes on to assert that since appearances are empirically real, one can affirm that one is immediately aware of real spatio-temporal objects. It is only when viewed transcendentally that these empirically real things are seen solely as representations that, themselves, must have an object. And this seems to lie in the main avenue of approach which I earlier adumbrated. Nevertheless, the strength of Allison's exposition resides in his account of the *transzendentale Genenstand* and its relation to appearance and the thing-in-itself--a relation that is critical to any attempt to resolve the problem of Kant's Two Worlds Dichotomy.

6. Westphal, "In Defense of the Thing In Itself," *Kant-Studien* 59, p. 119.

7. This I also take to be a species of the Hegelian criticism (a sort of meta-meta-criticism) of how, or under what conditions, Kant's critical philosophy is possible. Thus, it is at least logically possible that any resolution obtained might serve as an answer to the Hegelian problem as well.

8. Perhaps this can be interpreted too stongly. Intellectual intuition, of course, must still be ruled out; but there has to be some coherent way of speaking of the unity of the manifold prior to the constitution of the empirical object, prior even to the synthesis of recognition in a concept; the manifold must have, at least, the potential

to be unified in a manner that is not entirely arbitrary. Kant, himself, offers the vague formulation of the affinity of the manifold.

9. Pippin, Kant's Theory of Form, p. 46.

10. It is arguable that the adjective "human" unnecessarily limits what Kant thought he had shown--though whether he actually demonstrated more is moot.

11. With respect to the Transcendental Aesthetic, it seems to me that without the pure forms of intuition of space and time, Kant cannot hope to uphold his uniqueness thesis for human experience. It also seems to me that Kant repeatedly assumes that the conclusions of the Metaphysical Deduction have already been adequately demonstrated (this is explicit at B159, for instance). Now I shall not herein argue for the necessity of the Metaphysical Deduction in the construction of the Transcendental Deduction (Horstmann's article, "The Metaphysical Deduction in Kant's Critique of Pure Reason," makes some interesting points); but by analogy I ask: how much does Kant really prove in the Transcendental Exposition of space and time without granting the validity of the Metaphysical Expositions?

12. Allison, "Kant's Concept of the Transcendental Object," *Kant-Studien* 59, p. 178.

13. Meyer, "Why Did Kant Write Two Versions of the Transcendental Deduction of the Categories?" Synthese 47, pp. 373-74.

14. Perhaps "presented" is more correct here, but the there exists some controversy as to the correct English rendering of "*Vorstellung*". In any case, I do not believe that the controversy affects my argument to any significant extent.

15. George, "Kant: Sensationism," Synthese 47, makes this point. George's suggestion does not appear to have much textual support. However, it does seem that his interpretation of "manifold of intuition" is of great merit. Whether Kant really intended "manifold of an intuition" or "manifold of the intuition", he clearly should have.

Notes to Chapter IV:

1. There are innumerable works by commentators such as Brittan, Broad, Cassirer and Walsh who argue there is not such inconsistency present. There do exist many lines of defence. In the Aesthetic, for example, Kant asserts that since intuitions are sensible it makes sense to ask for their spatial and temporal properties. In the Axioms, at least, Kant speaks of the imposition of a metric upon space and time. Thus it is no wonder that one can pick up on isolated phrases here and there and claim inconsistency. Kant's intent, what he purports to demonstrate, in the Aesthetic and Analytic of principles is quite different; and this is reflected in his manner of argumentation and approach. To argue for inconsistency on such a basis is, minimally, to commit a category mistake.

2. Moreover, it would appear more faithful to interpret the notion of figurative synthesis, itself, with what Kant would call a formal intuition.

3. In point of fact, the bulk of my initial discussion will be directed towards Kant's arguments regarding space and not those, except cursorily, regarding time. It would, I suppose, be somewhat trite, though not perhaps incorrect, to maintain that Kant's arguments for the intuitive and *a priori* nature of space are the same with respect to time (especially in the Metaphysical Expositions) and are but mere variations of a theme. Hence what one could confidently assert of space, one could also confidently assert of time (taking into account, of course, their respective realms). On the other hand, there is a sense in which time can be considered the more primitive of the pair, and thus, perhaps should be treated with more respect. However, since many of the arguments concerning time are almost word for word transposed, I shall content myself with highlighting important points or differences.

4. In Brittan's Kant's Theory of Science, ch. 4, pp. 90-116.

5. It is clear that if the object to which we are referring is the *Objekt*, then the object is a logical construct conforming to the requirements of thought. If, following the A-edition Deduction, the object is the *transzendental Gegenstand*, then one is referring to a possible object of experience which is the ground of that which appears. If, following the B-edition Deduction, the object is spoken of as merely given, then I have argued that the role of figurative synthesis provides for the functioning of the understanding at the intuitive level.

6. In the A-edition this synthesis is inseparably bound to that synthesis of representation in imagination, and, in terms of knowledge claims, to the synthesis of recognition in a concept. It is the first synthesis, however, that is crucial here since it is

there that the spatio-temporal field is provided.

7. I recognize that this might depend, in turn, upon how much one thinks the Transcendental Deduction of the Categories has proven, and how much the Deduction depends upon the results of the Transcendental Aesthetic. My own opinion is that Kant's only chance at establishing his uniqueness thesis is to draw heavily from the Aesthetic. This is not to gainsay the validity of the Deduction; if one rejects the Aesthetic, it can more or less stand alone, but the less said about the Aesthetic the more trivial, I think, the Deduction becomes. Hence I question the force of the Axioms without some adequate notion of space and time.

8. Brittan, Kant's Theory of Science, p. 113.

9. In particular, Kant directs his attack against Leibniz and only to a lesser extent against Newton; but taking them to be somewhat representative of the rationalist (dogmatist) and realist camps, his attack serves a more general purpose. Note that if Kant's arguments hold they will also hold, with perhaps some amelioration, against the more contemporary views of, say, Reichenbach or Einsteinian relativity. Only quantum physics seems to pose problems requiring an entirely different line of argumentation.

10. These arguments are variously found in perhaps somewhat different forms in the Critique of Pure Reason, the Inaugural Dissertation and the Prolegomena.

11. The corresponding argum[illegible]ith respect to time, MEt-1 and MEt-2, show that our manner [illegible]zing time is *a priori;* MEt-4 and MEt-5 show our knowl[illegible]time is not discursive and lies in intuition.

12. Pippin, Kant's Theory of Form, p. 60. Pippin's perspective seems to square well with the interpretation of the Axioms already considered. To represent numerically distinct regions of space is to determine space or objects with respect to space. In other words, it sets the foundations for the imposition of a metric that Kant will claim are mind imposed. At the onset, however, one point should be noted well. There appears to be some tradition that affirms Kant's program to be setting out a series of positions, say, $p$, $q$, $r$ (where $p$ represents Newtons's claims, $q$ represents Leibniz, and $r$ represents trancendental idealism), arguing for not-$p$ and not-$q$, and then by disjunctive syllogism asserting $r$. Clearly, part of the Kantian enterprise is to argue against Newton and Leibniz; but just as clearly he does not offer transcendental idealism as a "what else" solution. The transcendental ideality of space and time is a critical part of his program, and both negative and positive arguments are proffered. Kant, himself, maintains that an "exposition" is 'not necessarily an exhaustive account, but rather a clear representation of what belongs to a concept' (A23/B38).

13. In MEt-1 the question is how I refer to some thing(s) as existing simultaneously or successively.

14. Actually, I believe that one can argue for some sense of the spatiality and temporality of the manifold in terms of what one might call "spreadoutness", though this will be touched on again with reference to "formal intuition".

15. Strawson, The Bounds of Sense, p. 59. In regard to this capacity being innate, Kant maintains, in On a Discovery [221-22], that it is only the ground in the subject that makes possible the representation of space in this manner that is innate; of the representation, itself, it can only be spoken of in terms of being originally acquired (although, at times, I admit I am at a loss in adequately distinguishing between "innate" and "original acquisition").

16. In the case of time Kant maintains that the representation of an object in time presupposes a representation of time which is not empirically derived such that it is possible to represent object(s) as existing simultaneously or successively. Note that this interpretation is much closer to, say, Strawson's austere sense of *a priori* than the transcendental idealist sense Kant would hold. It would seem that, in an austere way, space and time are essential elements in any coherent explanation of experience that one could form. This, of course, is Strawson's line and does not necessitate that the presence of space and time is a function of experience that is attributed entirely to the subject.

17. A similar argument is offered with respect to time (A31/B46) which should take care of those cases where one is concerned with purely intensive magnitudes (such as pitch). Frankly, I cannot even "conceive" what it would be like to "represent" an empty space or time. How one thinks of space or time is fundamentally different in kind than how one can think of the objects of space or events in time (or even, in a weaker sense, spatial objects or temporal events).

18. Allison, Kant's Transcendental Idealism, p. 89.

19. If it were not thus, then one must be able to make sense of what it would mean to say that space is an individual concept. Kant, himself, seems unclear on this issue (for example, para. 12, Inaugural Dissertation), and I am at a loss to explain how a coherent account can be laid out. For an excellent exposition on the problem of individual concept and "*Anschauung*" in general, see Gram's "The Sense of a Kantian Intuition" anthologized in Interpreting Kant. Note also that the arguments corresponding to ME-3 and ME-4 with respect to time are quite similar, if somewhat abbreviated. I shall thus concern myself primarily with only the intuitive nature of space.

20. Broad, Kant: An Introduction, p. 32.

21. I shall have more to say about the shortcomings of the reductionist or relationist account of space when considering the ICs; suffice it to say now that the reductionist cannot account for the possibility of representing numerically distinct objects without invoking the somewhat circular notion that the object in question must already possess individuating spatial properties or positions that permit of such distinction.

22. Broad, Kant: An Introduction, pp. 37-38.

23. It seems to me that what it means to have human experience is to experience an external world in terms of three spatial and one temporal dimension. (In The Bound of Sense, Strawson argues essentially the same point, but I disagree with many of his conclusions). And even were a proof for, say, 11-space to be given tomorrow, a proof that would unify the four known forces, it would not alter, one iota, the manner in which we "experience" the external world.

24. Kant, Prolegomena, para. 13, pp. 33-34.

25. Newton, of course, was a substantivalist, but in an absolute sense, *i.e.* all positions, velocity and acceleration are determinate. One need not hold such a strong position, however, and yet remain a substantivalist.

26. This is made quite explicit by Schulze's comments (made in collaboration with Kant) on an essay by Kästner. Allison's The Kant-Eberhard Controversy contains the text to the comments.

27. In Brittan's, Kant's Theory of Science, chapter 3.

28. Brittan, Kant's Theory of Science, p. 82.

29. This formulation occurs in various forms throughout Kant's works. It is explicit, for example, in On a Discovery [241].

Notes to Chapter V:

1. This, I suppose, is not precisely correct. Frege, for example, held that while arithmetic was analytic, geometry was synthetic. Russell's views seemed to fluctuate, although early in his career he advocated most of the major tenets of the logicist position. Nevertheless, nothing is lost if, in general, we take the logicist program to be a reduction of mathematics (or even that part of mathematics we call arithmetic) to logic, and thus an analytic enterprise.

2. Note that this particular formulation appears throughout the Kantian corpus. For example, the Prolegomena (para. 2) contains such a restatement of the position held in the Critique of Pure Reason; in the Logic (Intro., sec. III) the quotation is virtually word for word identical with that in the First Critique. The problem in the formulation is how, precisely, to cash out what concept construction in mathematics has to do with exhibiting *a priori* the intuition which corresponds to the concept.

3. Dryer's argument for the analyticity of arithmetic, in Kant's Solution for Verification in Metaphysics (pp. 48-53), is curiously Leibnizian--and extremely brief. One cannot help but wonder if he has not only missed the nuances of the situation, but the point entirely.

4. There is a question whether a difference exists at all from any perspective. I suppose it could be argued that all mathematics, for Kant, requires an interpretation and is, hence, in this sense applied. And though I do not find this position as queer as it may sound, it is not one for which I shall argue; Kant's program can, I think, be put through without any such claims.

5. The interpretation to follow will parallel that of the line (1); but note that this very passage can be (and has been, by Hintikka) used as a jumping-off point for line (2).

6. Note also that at A717/B745 Kant asserts that magnitudes are constructible even in algebra. Algebra allegedly abstracts completely from the properties of the object as such, and a choice is made regarding notation whereby all construction of magnitudes, such as addition, subtraction, the extraction of roots, and so on, takes place. This Kant calls symbolic construction (in contrast to ostensive construction in geometry).

7. Russell, Principles of Mathematics, p. 158. Essentially, Russell claims that if all of mathematics can somehow be reduced to number theoretic statements, and if these can be generated from merely logical notions, then it follows that all of mathematics is analytic in character.

8. Kant restates this position in a similar, although somewhat briefer, fashion in the Prolegomena (para. 2, p. 15).

> For as it was found that the conclusions of mathematics all proceed according to the law of contradiction (as is demanded by all apodeitic certainty), men persuaded themselves that the fundamental principles were known from the same law. This was a great mistake, for a synthetical proposition can indeed be established by the law of contradiction, but only by presupposing another synthetical proposition from which it follows, but never by that law alone.

9. Recall, also, Beck's argument from Chapter II regarding the shifting of the locus of syntheticity in attempting to make synthetic judgments analytic.

10. Brittan, Kant's Theory of Science, p. 46.

11. Brittan, Kant's Theory of Science, pp. 55-56.

12. Russell, for example, was deeply troubled by the axiom of infinity. It is also worth mentioning that any system of logic that accepts as axiomatic existence claims such as, $(\exists x)x=x$, will be in a similar distressing situation.

13. Hintikka, "Kant's Theory of Mathematics Revisited", Essays on Kant's Critique of Pure Reason, pp. 208-09.

14. Note that in the following passage, when Kant argues 7+5=12 is a synthetic proposition, he does so not on the basis of the syntheticity of any prior mathematical proposition, but through reference to the oft disputed containment metaphor.

15. Frege, *Grundlagen*, p. 5.

16. As, for example, in "Kant's Theory of Mathematics Revisited," Essays on Kant's Critique of Pure Reason, p. 202, or "Kant and the Tradition of Analysis," Logic, Language Games and Information, p. 220.

17. Brittan, Kant's Theory of Science, p. 55f.

18. Hintikka, "Kant's Theory of Mathematics Revisited," Essays on Kant's Critique of Pure Reason, p. 202.

19. Hintikka argues, rather convincingly, for this thesis in "An Analysis of Analyticity" and "Kant Vindicated" (both anthologized in Logic, Language Games and Information). In the latter paper he further maintains that the "degree" of a quantificational sentence is the maximal number of individuals or free terms considered in their relation to one another. A step in a proof is synthetic if the "degree" of the sentence in question increases, a step in analytic if it does not.

20. Such a position has assets and liabilities. To its merit, the analogy with geometry permits an extremely clear presentation; the Euclidean method of proof highlights exactly those points Hintikka wishes to make. On the debit side, however, geometry and, say, arithmetic are only two aspects of mathematics. It is not clear that what is true of one is necessarily true of the other. Moreover, whereas Kant would appear to allow for axiomatization of geometry, he certainly would not permit anything of the sort in arithmetic. Thus unless one first argues for the foundational nature of geometry, or the foundational nature of constructive proofs in both geometry and arithmetic transformed into quantificational form, then the import of Hintikka's reconstruction is somewhat weakened.

21. Hintikka, "Kant and the Tradition of Analysis," Logic, Language Games and Information, p. 207.

22. Much of the following is drawn from Hintikka's "Kant's Theory of Mathematics Revisited," Essays on Kant's Critique of Pure Reason, pp. 203ff.

23. Although Hintikka does not explicitly say so, I think it reasonable to suppose that it is the *apodeixis* to which Kant is referring in B14 where he asserts that mathematical inferences proceed according to the principle of contradiction. Brittan, as I have already noted, uses this passage to claim that Kant must trace syntheticity back to axioms of the initial premises of the proof; Hintikka, on the other hand, would trace syntheticity back to the *kataskeue* and possibly *ekthesis*.

24. Intuitionists such as Brouwer or Heyting, for example, would advocate the use of only constructive proofs. And not only do they have much sympathy with Kant's theory of mathematics but allegedly ground their own position in parts of Kant's program. Brouwer, for instance, asserts:

> However weak the position of intuitionism seems to be after this period of mathematical development [formalism and non-Euclidean geometry], it has recovered by abandoning Kant's apriority of space but adhering more resolutely to the apriority of time ("Intuitionism and Formalism," Philosophy of Mathematics, p. 69).

25. Hintikka, "Kant and the Tradition of Analysis," Logic, Language Games and Information, p. 207. I am frankly puzzled by the amount of support this position has received. Even Werkmeister (in Kant, the Architectonic and Development of His Philosophy, p. 22) pleads his case.

26. Many are pre-critical and hence are somewhat suspect for that reason; but enough are drawn from critical writings to be cause for a response. Note also that Hintikka claims Kant provides two different approaches to his philosophy of mathematics. His

"preliminary theory" affirms no connection between intuition and sensibility, while his "full theory" does. Hintikka's reconstruction is, therefore, based on the "preliminary theory".

27. Here are a few: Bxln., B72, B159, and especially B307ff.

28. Mitscherling, "Kant's Notion of Intuition: in Response to Hintikka," *Kant-Studien*, 72, pp. 187-94.

29. Pippin, Kant's Theory of Form, p. 85.

30. Brittan, Kant's Theory of Science, p. 61.

31. At least with regard to apriority it is conventional. One could say that one has knowledge of the theorems of ... *a priori* (where one could fill in the blank with their favourite axiomatic scheme). Syntheticity is a different problem, but I doubt if the conventionalist would argue as long as it was synthetic and *a priori* with respect to ... , and there was nothing upon which to base a choice.

32. That is to say that the Transcendental Exposition of Time proffers no comparable argument. At B15-16, though, Kant seems to indicate that the solution of arithmetical propositions depends upon the concept of succession, and thus time. And in the Prolegomena (sec. 10) Kant explicitly says that arithmetic comes by the concept of number through successive addition of units in time.

33. Current thought tends more towards Riemannian geometry. In any event, it would now seem clear that Euclidean geometry does not truly represent physical, substantival, space. Hopkins (in "Visual Geometry," Kant on Pure Reason, pp. 41-65) argues convincingly that Euclidean geometry does not even truly represent what Strawson would call phenomenal figures in space.

34. This would, I suppose, depend upon a Euclidean notion of straightness. The notion of two intersecting geodesics, however, is not only logically possible, but is also really possible. I have, though, argued earlier that Kant's notion of space is not tied to a Euclidean interpretation; and the argument can, in any case, be suitably rearranged to encompass non-Euclidean geometries as well.

35. For example, at A155/B194-95, Kant asserts,

> [I]f knowledge is to have objective reality, that is, to relate to an object, and is to acquire meaning and significance in respect to it, the object must be capable of being in some manner given, otherwise the concepts are empty .... That an object be given ... means simply that the representation through which the object is thought relates to actual or possible experience.

Even space and time, as pure forms of intuition, would have no objective reality and hence would be senseless and meaningless were their application to objects of experience not established through the schematism of the categories.

36. Actually, it would appear that Kant would admit that analytic judgments have truth values. At A151/B190 Kant asserts that the truth of analytic propositions can always be known in accordance with the principle of contradiction; and hence any substitution instance of a=a would be formally true. In a mathematical proof, however, the extension of "round square" would be the null set. And since the singular term, "round square", would not, indeed could not, refer to any possible object of experience, it would not be mathematically constructible. To say the proposition is meaningless seems to me to be overstating the case--unless by meaningless one means lack of constructibility.

37. Brittan, Kant's Theory of Science, p. 63.

38. Recall that in any proof the geometrician starts at once by a construction (A716/B744). Note that his squares well with Hintikka's interpretation; after the *protasis*, the *ekthesis* is the next step in any Euclidean proof. Note also that at A155/B194 Kant holds that if an object is to have any significance or meaning it must be capable of being given. In terms of mathematics, I read this as meaning being capable of being constructed.

39. Three questions immediately surface here. The first is the familiar question of apriority and pure intuition and how it functions in the construction of mathematical concepts. Secondly, we have a new problem area: the content of a concept and what that would entail. And thirdly, there is the question of meaningfulness, of precisely how mathematical judgments are meaningful. These are, I believe, three closely related issues, and I think we should be best served by approaching the last first, since it, after a fashion, serves as a basis for an explanation of the first two.

40. The thesis side of the First Antinomy, for instance, bears some resemblance to the Burali-Forti paradox (if one stretches a bit what Kant might mean by the completion of an infinite series). In any event, however weak one may take Kant's arguments in the Antinomies to be, it is clear that there remain problems and paradoxes having to do with the infinite that are not easily resolved, if they can be resolved at all. Russell, for one, took great pains to effect such a resolution with his theory of types; it is moot as to whether his resolution did not generate more problems than it putatively solved.

41. Brouwer, "Intuitionism and Foramlism," Philosophy of Mathematics, p. 74.

42. Parsons, "Kant's Philosophy of Arithmetic," Kant on Pure Reason, pp. 32-33.

43. Dummett, "The Philosophical Basis of Intuitionistic Logic," Proceedings of the Logic Colloquium, 1973, p. 17. To be fair Dummett offers other possible methods of defence for intuitionistic logic and mathematics. At the conclusion of the above article (p. 40) he notes:

> Anyone who can hang on to a view as hard-headed as this [that there is no notion of truth applicable even to numerical equations save that in which a statement is true when we have actually performed a computation or effected a proof] has no temptation at all to accept a platonist view of number-theoretic statements involving unbounded quanitification: he has a rationale for an intuitionistic interpretation ... But, for anyone who is not prepared to be quite as hard-headed as that, the route to a defence of an intuitionistic interpretation of mathematical statements which begins from the ontological status of mathematical objects is closed; the only path that he can take to this goal is that which I sketched at the outset: one turning on the answers given to general questions in the theory of meaning.

And, although hard-headed, I shall, in general, concern myself with the issues revolving around the theory of meaning.

44. Brittan, Kant's Theory of Science, pp. 83-84.

45. I disagree that one must claim the transcendental sense of necessity or apriority with respect only to Euclidean geometry. But this will become clear later.

46. Strawson, for example, deals with the Transcendental Exposition of space in terms of a more comprehensive critique of transcendental idealism in his Bounds of Sense, Pt. 2, Ch. 1 (particularly pp. 67ff.).

47. Parsons, "Kant's Philosophy of Arithmetic," Kant on Pure Reason, pp. 36-37.

48. Brittan also asserts that analytic propositions do not have truth values. This would, presumably, include most of the truths of logic. I have suggested that, perhaps, this is overstating the case; but one must be careful here. Truths of logic, being true in all possible worlds, are, in fact, also true of all "really" possible worlds. One must, however, be cautious about what one accepts as a truth of logic. For example, one cannot necessarily assert that p v~p is truth-functionally true. It is true that p v~p might hold, but only in the sense that one can actually construct a proof of p, or construct a refutation of p. Whether this would subordinate logic to mathematics, as Heyting, for example, would have us believe, I leave open.

49. This, I take it, is why Brittan maintains that the "existence

problem" and the "application problem" are two sides of the same coin.

50. Brittan, Kant's Theory of Science, p. 82.

51. This may not be precisely correct. Empirical evidence could, I suppose, once and for all "prove" the axioms of spherical or hyperbolic geometry; but I know of no evidence that would find conclusively in favour of Euclidean or Absolute geometry. Here, perhaps, the best for which one could hope is the lack of evidence for alternative systems.

52. In fact the assignment of the truth value 'false' would only be applicable if one could prove the proposition could never be satisfied. Notice also, that by replacing "straight line" by "geodesic" in spherical geometry, a constructive proof could be provided; and hence with respect to spherical geometry the proposition would have a determinate meaning. But then what are we to make of Kant's claim that geometry is a science that determines the properties of space synthetically and yet *a priori*? There is one sense in which it will have no effect. I have all along been arguing that it is the propositions of mathematics to which syntheticity and apriority attach. Geometry as a science, then, could be considered as comprising all those propositions of Euclidean, Riemannian, Lobachewskian, etc., geometries. Syntheticity would still accrue according to the proof process; a priority would remain relative to the interpretation selected. If geometry can be further reduced to number-theoretic statements, however, the transcendental sense of necessity, and hence apriority Kant demands can then be supplied, if one can demonstrate how such number-theoretic statements are conditioned by the pure form of time. I do not, however, assert that this is a position that Kant would have wished to hold.

53. This is not surprising for those of an idealist persuasion. Nevertheless, for Kant, the notion of figurative synthesis, outlined in Chapter III, makes possible the determinate representation of objects in space and time by permitting the understanding to actively function within intuition. This is to say that, although the pure concepts of the understanding do not completely determine an object as an object of experience, they do provide the conditions under which an object is a possible object of experience.

54. Parsons, "Kant's Philosophy of Arithmetic," p. 31. The reason, I suspect, that Kant does not offer a Transcendental Exposition of time drawing upon arithmetic, rather than pure mechanics, is that arithmetic does not admit of axiomatization. Arithmetical propositions are numerical formulae which are synthetic and immediately certain (A164/B204). I frankly do not know what Kant's response might have been had he the benefit of the advances made by Peano and Dedekind. Certainly, he would have still maintained that arithmetic, as part of mathematics, has content and is not reducible to

purely logical notions. The "arithmetization" of mathematics would not have troubled Kant; the "logicizing" of arithmetic would have been of some concern. In any event, as shall become clear, I feel Kant could have turned Peano's accomplishment to his advantage.

55. Brouwer, "Intuitionism and Formalism," Philosophy of Mathematics, pp. 69-70.

56. Kant, Inaugural Dissertation, para. 12, p. 52.

57. Parsons complains that the object to which a concept refers is quite different in mathematics than in every day, mundane, experience. Mathematical objects, Parsons claims, are abstract entities. In geometry it is arguable that these entities could be considered in space and time; in arithmetic, Parsons claims this is not so. This objection, I believe, does not hold. One should recall that, for Kant, any object, if not a fully constituted object of consciousness, is at least partially constituted by the understanding. There may be a difference between the objects of reference of empirical concepts and mathematical concepts, but the difference is not necessarily a difference in kind. Even Frege recognizes that Kant uses the word "object" in a special sense (*Grundlagen*, p. 101), and one must be aware of this.

58. This example is not as trite as it sounds. If one is to explain the arithmetic operation of addition by the notion of class union, one can ask how one initially develops the notion of what it is to be a class. One answer, although I admit it is not the only one, is that one must begin with a group of perceptible or sensible objects. If this holds, the notion of class is not as alien to sensibility as some logicists would have it.

59. At least as long as the numeral token involved does not extend into the realm of the transfinite.

60. Parsons, "Kant's philosophy of Arithmetic," Kant on Pure Reason, p. 33. Parsons is onto something here. There is a crucial distinction to be made, though, between formal intuition and form of intuition. And this is a distinction that Parsons does not draw. I shall return to this issue later; in the meantime, it is quite important to note that all interpretations of the generation of the natural numbers are isomorphic. Recall that Brittan was searching for a way in which Kant could demonstrate the objective validity of Euclidean geometry. This task turns out to be virtually impossible; but if all interpretations of Peano's postulates (in the realm of the finite or denumerably infinite) are, in fact, isomorphic, what more of a claim to objective validity could one request? What might not have been possible with respect of geometry is possible with respect to arithmetic.

61. This formulation is drawn from the Encyclopedia of Philosophy, Vol. 5, p. 197.

62. Parsons, "Kant's Philosophy of Arithmetic," Kant on Pure Reason, p. 35.

63. Hence, schema (1) can be satisfactorily explained by Kant under the rubric of symbolic construction. I do not agree with Parsons, however, that this blurs the distinction between logic and mathematics. The explanation of (1), for Kant, requires that it be given an interpretation. In doing so it is given content and falls under the compass of mathematics proper. One may wish to make a distinction between pure logic and what one might call mathematical logic; but then Kant has already made other distinctions (such as that between transcendental and general logic). If any distinction occurs, it is between mathematical and transcendental logic; and that, at least has little impact on the separation of analytic from synthetic judgments and the possibility of drawing synthetic judgments *a priori*.

64. Allison, in his Kant's Transcendental Idealism (p. 96ff.) provides an excellent account of the ways in which a form of intuition may be taken.

65. Allison, Kant's Transcendental Idealism, p. 96.

66. Brouwer (in "Intuitionism and Formalism," Philosophy of Mathematics, p. 69) calls two-oneness the basal intuition of mathematics. Here the reference is possibly more closely related to a Hegelian type of unity in diversity and diversity in unity, than any particular Kantian notion. However, little is lost, I believe, whether one holds one to be the basal intuition or holds to Brouwer's thesis concerning two-oneness.

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