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Multiplicative Invariants

by

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Abstract

This thesis is devoted to the study of (twisted) multiplicative invariant rings and fields. Multiplicative invariant fields were first studied by Noether in her work on the inverse Galois problem. Since then, many others have continued her study of multiplicative invariant rings and fields. Endo-Miyata, Swan, Voskresenskii, Saltman and Farkas examined various rationality problems for (twisted) multiplicative invariant fields; Farkas and Lorenz examined the structure and divisor class groups of multiplicative invariant rings.

In this thesis, we first give a complete description of the divisor class group of a twisted multiplicative invariant ring in terms of cohomological data about the field, the lattice and the associated twist. Next we consider the rationality problem for a field extension given by a twisted multiplicative invariant field $K_\gamma(A)^G$ over a field K where here the group G acts by field automorphisms on the field K, faithfully on the lattice A and the action on $K_\gamma(A)$, the quotient field of the group algebra of A with base field K is determined by the 1-cocycle $\gamma : G \to \mathrm{Hom}(A, K^\times)$. The set of generators Γ of inertia subgroups of the extension $K_\gamma[A]/(K_\gamma[A])^G$ corresponding to height one primes consists of elements of G which act as reflections on A,

act trivially on K, and which satisfy a particular cocycle relation on γ. Γ generates a normal (reflection) subgroup R of G. We first show that $K_\gamma(A)^R$ is rational over K. Then under a hypothesis on the cocycle γ, are able to reduce the rationality problem for $K_\gamma(A)^G$ over $K_\gamma(A^R)^G$ to the rationality problem for $K_\gamma(A)^{\Omega_G}$ over $K_\gamma(A^R)^{\Omega_G}$ where Ω_G is a particular subgroup of G satisfying $G \cong R \rtimes \Omega_G$. We then applied this result in the untwisted case to show that $K(A)^G$ is rational over K where G is the automorphism group of a crystallographic root system Ψ acting faithfully on $V = \mathbf{Q}\Psi$ and trivially on the field K and A is any $\mathbf{Z}G$ lattice on V. Last, we considered the stable rationality problem for the centre of the generic ring of $n \times n$ matrices which Procesi and Formanek converted into a (stable) rationality problem for a multiplicative invariant field for the symmetric group S_n over $\mathbf{C}$. In the case when n is odd, we prove that this multiplicative invariant field is stably equivalent to a multiplicative invariant field for a smaller lattice which is the direct sum of a permutation lattice with an irreducible $\mathbf{Z}S_n$ lattice which corresponds over $\mathbf{Q}$ to the partition $(n-2, 1^2)$ of n.

For my sister Danielle who was unable to fulfil her many dreams and aspirations.

Kindred spirits exist but the proof is non-constructive.

I sincerely appreciate all the long discussions that I had with Dr. Weiss in the preparation of this document. His patience with me helped me complete this thesis successfully.

Contents

Chapter 1

Introduction

Classical invariant theory studies the linear action of groups on polynomial rings over fields. Given a group G, a field K, and a n-dimensional KG-module V, the action of G can be extended uniquely to an action on the symmetric algebra $K[V]$. Note that $K[V]$ is isomorphic to the polynomial algebra in n variables over K. The invariant ring of $K[V]$, denoted by $K[V]^G$, is called the *ring of polynomial invariants.* The action of G on $K[V]$ can be naturally extended to its quotient field $K(V)$. The invariant ring of $K(V)$ is denoted by $K(V)^G$ and is called the *field of polynomial invariants.* Note that $K(V)^G$ is also the quotient field of $K[V]^G$.

In this thesis, we study the analogous notion of (twisted) multiplicative invariant rings and fields. Let G be a finite group, A a $\mathbf{Z}G$ lattice of finite rank n, and K a field on which G acts as field automorphisms. Write elements of the group ring $K[A]$ as $\sum_{a\in A} c_a e(a)$ where $e : A \to K[A]^\times$ is the "exponential map" where $S^\times$ indicates the units of the ring S. Given a 1-cocycle $\gamma : G \to \mathrm{Hom}(A, K^\times)$, let G act on $K[A]$ by ring automorphisms as

$$g[ce(a)] = (gc)\gamma_g(ga)e(ga)$$

for all $g \in G$, $c \in K^\times$ and $a \in A$ and write $K_\gamma[A]$ for $K[A]$ with this G-action. Then under the isomorphism $\mathrm{Ext}^1_G(A, K^\times) \cong H^1(G, \mathrm{Hom}(A, K^\times))$ the extension class $[K^\times \rightarrowtail (K_\gamma[A])^\times \twoheadrightarrow A]$ corresponds to $[\gamma]$. The action of G can also be extended to the quotient field of $K_\gamma[A]$ which will be denoted by $K_\gamma(A)$. The ring of invariants of $K_\gamma[A]$ under the twisted action of G, $K_\gamma[A]^G$ is called a *twisted multiplicative invariant ring.* The ring of invariants of the quotient field $K_\gamma(A)^G$ which is also the quotient field of $K_\gamma[A]^G$ is denoted by

$K_\gamma(A)^G$ and is called a *twisted multiplicative invariant field.* If $[\gamma]$ represents the trivial extension $K^\times \oplus A$, then we will drop the subscript γ in reference to the ring of multiplicative invariants $K[A]^G$ and the field of multiplicative invariants $K(A)^G$. We will assume throughout that G acts faithfully on the group algebra and hence on its quotient field. In the first section of the first chapter, we review the construction of twisted multiplicative invariant fields and rings and prove a few useful technical facts about them.

A major theorem for rings of polynomial invariants is the Shephard-Todd-Chevalley Theorem. Let G be a finite subgroup of $GL(V)$ where V is a finite-dimensional KG module and K is a field with characteristic not dividing the group order. Recall that a pseudoreflection on V is an element s of $GL(V)$ with $\mathrm{Im}_V(1-s)$ of dimension 1. Then Shephard-Todd-Chevalley [6] states that $K[V]^G$ is a polynomial ring iff G is generated by pseudoreflections on V. The analogous question can also be posed for multiplicative invariants. This motivates a closer examination of the concept of a reflection acting on a lattice.

For a lattice A of $\mathbf{Z}$-rank n, a reflection $s \in GL(A)$ is an element of finite order with $\mathrm{Im}_A(s-1)$ cyclic. Note that s is a reflection on A iff s acts as a (pseudo)reflection on $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$. If s is a reflection on A, then $A \cong \mathbf{Z}^- \oplus \mathbf{Z}^{n-1}$ or $A \cong \mathbf{Z}\langle s\rangle \oplus \mathbf{Z}^{n-2}$ as $\mathbf{Z}\langle s\rangle$ lattices. The reflections corresponding to the first case are called *diagonalizable over* $\mathbf{Z}$.

A reflection space over $\mathbf{Q}$ for a finite group W is a finite dimensional $\mathbf{Q}$ space V such that W acts faithfully on V, $V^W = 0$, and such that W is generated by the set of reflections

$$\Gamma = \{s \in W \mid \dim_{\mathbf{Q}} \mathrm{Im}_V(s-1) = 1\}$$

Analogously, we may define a *reflection lattice* as a lattice A such that W acts faithfully on A, $A^W = 0$, and such that W is generated by the set of reflections on A in W or equivalently such that W is generated by the set of reflections on $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ in W. In Section 2.2, we classify the reflection lattices. Feit used a different method in [17] to classify the reflection lattices corresponding to irreducible root systems. Due to the classification of reflection spaces, it suffices to determine the $\mathbf{Z}W$ isomorphism classes of $\mathbf{Z}W$ lattices A such that $\mathbf{Q} \otimes_{\mathbf{Z}} A \cong V$ as $\mathbf{Q}W$ modules for a fixed reflection space V over $\mathbf{Q}$.

For a given reflection space V over $\mathbf{Q}$ with reflection set Γ generating W, and a crystallographic root system Φ, any lattice A lying between $\mathbf{Z}\Phi$

and the weight lattice $\Lambda(\Phi)$ is a reflection lattice with $\mathbf{Q} \otimes A \cong V$ as $\mathbf{Q}W$ modules. Conversely, we may associate to any reflection lattice A on V a crystallographic root system for V:

$$\Phi_A = \{\alpha \in V | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha \text{ for some } s \in \Gamma\}$$

with $\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$. $\mathrm{Aut}_{\mathbf{Q}W}(V)$ acts on the set of crystallographic root systems for V. To classify the $\mathbf{Z}W$ isomorphism classes of reflection lattices on V, we first identify the $\mathrm{Aut}_{\mathbf{Q}W}(V)$ orbit of a fixed crystallographic root system Φ and then determine the $\mathbf{Z}W$ isomorphism classes of $\mathbf{Z}W$-lattices lying between $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$. The precise statement is given by Theorem 2.2.20. For an irreducible reflection space V with root system Φ, we determine the number of isomorphism classes of $\mathbf{Z}W(\Phi)$ lattices. The crystallographic root system Φ_A associated to a reflection lattice A that is used in this classification determines the first cohomology group in Proposition 2.2.25

Proposition 1.0.1. *Let A be a reflection lattice on a reflection space V with reflection group W. Then*

$$H^1(W, A) \cong \Lambda(\Phi_A)/A$$

A classical result which begins the search for an analogue of Shephard-Todd-Chevalley for multiplicative invariants states that $\mathbf{C}[\Lambda]^W$ is a polynomial ring [6] where W is a Weyl group acting on the weight lattice Λ of its root system Φ. In order to state Farkas' theorem that addresses STC for multiplicative invariants we need two more definitions: Let G be a group acting faithfully on a lattice A and let R be the normal subgroup of G which is generated by the set of reflections Γ on $V = \mathbf{Q} \otimes A$ in G. Let $\pi : V \to V$ be the $\mathbf{Q}G$ map defined by $1 - \frac{1}{|R|}\sum_{r \in R} r$. A crystallographic root system Φ is called *suitable* for the $\mathbf{Z}G$ lattice A if Φ is G stable, $\Phi \subset A$, $\pi(A) \subset \Lambda(\Phi)$, and R is isomorphic to the Weyl group for Φ under the natural map $R \mapsto R|_{\mathbf{Q}\Phi}$. Φ_A is an example of a suitable root system for A. A *stretched weight lattice* for a root system Φ is a lattice $B = \oplus_{i=1}^n \mathbf{Z}m_i\omega_i$ where $m_i \in \mathbf{N}$ and $\{\omega_i\}_{i=1}^n$ is a set of fundamental dominant weights in $\Lambda(\Phi)$ corresponding to a base of Φ.

Theorem 1.0.2. *[16, 13, 14] Let A be a lattice, $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$, R be a finite subgroup of $GL(A)$ generated by reflections acting on A, π be the $\mathbf{Q}R$ map* *defined above, Φ be a suitable root system for the $\mathbf{Z}R$ lattice A and let K be a field with characteristic not dividing the order of G. Then the following statements are equivalent:*

(a) $\pi(A)$ *is a stretched weight lattice for* Φ.

(b) $K[A]^G$ *is a polynomial ring over* $K[A^G]$

(c) $K[A]^G$ *is a UFD.*

For a normal Noetherian domain S, the divisor class group of S is defined as

$$\mathrm{Cl}(S) = \frac{\bigoplus_{\mathfrak{p}\in \mathrm{Spec}(S), \mathrm{ht}(\mathfrak{p})=1} \mathbf{Z}\mathrm{div}(\mathfrak{p})}{\{\sum_{i=1}^{r} n_i \mathrm{div}(\mathfrak{p}_i) | \cap_{i=1}^{r} \mathfrak{p}_i^{n_i} = \langle x \rangle \text{ for some } x \in S\}}$$

where $\{\mathrm{div}(\mathfrak{p})|\mathfrak{p} \in \mathrm{Spec}(S), \mathrm{ht}(\mathfrak{p}) = 1\}$ is a $\mathbf{Z}$ linearly independent set in bijection with the set of height one primes of S. Since $\mathrm{Cl}(S) = 0$ [4, (3.5.1)] iff S is a unique factorization domain, the divisor class group of S measures in some sense how far S is from being a UFD.

Farkas' result determines precisely when a multiplicative invariant ring $K[A]^G$ is a UFD if G is a finite reflection group. In [15], Farkas expressed doubt that it would be possible to determine exactly when $K[A]^G$ is a UFD for an arbitrary finite group G and faithful $\mathbf{Z}G$ lattice A. In response to this remark , Lorenz posed the following more general question: Is it possible to describe the divisor class group of multiplicative invariants $\mathrm{Cl}(K[A]^G)$ for any finite group G and faithful $\mathbf{Z}G$ lattice A? For polynomial invariants, this question had already been answered. It had been shown that $\mathrm{Cl}(K[V]^G)$ is isomorphic to $\mathrm{Hom}(G/N, K^\times)$ where N is the subgroup of G generated by pseudoreflections on V. In fact, Lorenz proved the following theorem:

Theorem 1.0.3. *[24]*

$$\mathrm{Cl}(K[A]^G) \cong Hom(G/N, K^\times) \oplus H^1(G/D, A^D)$$

where A is a lattice, G is a finite subgroup of $GL(A)$ which acts trivially on the field K, N is the subgroup of G generated by reflections acting on A and D is the subgroup generated by reflections that are diagonalizable over $\mathbf{Z}$

In the third chapter of this thesis, we determine the divisor class group of the normal Noetherian domain $(K_\gamma[A])^G$ in terms of cohomological information about $K^\times$ and A. We used the following result of Lorenz based on work of Samuel [33]. Recall first that for a finite group G acting faithfully on a normal noetherian domain S, the inertia subgroup of a prime $\mathfrak{p}$ of S is given by

$$G^T(\mathfrak{p}) = \{g \in G | gs - s \in \mathfrak{p} \text{ for all } s \in S\}$$

Proposition 1.0.4. *[24] Let S be a unique factorization domain on which the finite group G acts faithfully. Let $\mathcal{H}$ be the collection of non-trivial inertia subgroups of G corresponding to height one primes. If all S^H for $H \in \mathcal{H}$ are UFDs then*

$$\mathrm{Cl}(S^G) \cong \bigcap_{H \in \mathcal{H}} \mathrm{Ker}(\mathrm{Res}^G_H)$$

where $\mathrm{Res}^G_H : H^1(G, S^\times) \to H^1(H, S^\times)$ is the restriction map.

To apply this result to the unique factorization domain S=$K_\gamma[A]$ with the given faithful action by the group G, we must first determine the non-trivial inertia groups of $K_\gamma[A]/K_\gamma[A]^G$ corresponding to height one primes and show that all the $K_\gamma[A]^H$ are UFDs. We do this in Section 3.1. Proposition 3.1.5 characterizes the height 1 primes:

Proposition 1.0.5. *The non-trivial inertia subgroups $G^T(\mathfrak{p})$ with $\mathfrak{p}$ a height one prime of $K_\gamma[A]$ are precisely the subgroups of G generated by an element s such that*

(a) *s acts as a reflection on A*

(b) *s acts trivially on K*

(c) $A^{(s)} \subset \mathrm{Ker}(\gamma_s)$.

and Lemma 3.1.6 shows that $K_\gamma[A]^H$ is a UFD for a non-trivial inertia subgroup H.

Let

$$\Gamma_\gamma = \{1 \neq g \in G | \langle g \rangle = G^T(\mathfrak{p}) \text{ for some height 1 prime } \mathfrak{p} \text{ of } K_\gamma[A]\}$$

Now by Proposition 1.0.4 we see that

$$\mathrm{Cl}(K_\gamma[A]^G) \cong \cap_{g \in \Gamma_\gamma} \mathrm{Ker}(\mathrm{Res}^G_{\langle g \rangle} : H^1(G, K_\gamma[A]^\times) \to H^1(\langle g \rangle, K_\gamma[A]^\times))$$

In Section 3.2, we relate $\mathrm{Cl}(K_\gamma[A]^G)$ to the $\mathbf{Z}G$ exact sequence $K^\times \rightarrowtail (K_\gamma[A])^\times \twoheadrightarrow A$ in terms of

$$\cap_{g\in\Gamma_\gamma}\mathrm{Ker}(\mathrm{Res}^G_{\langle g\rangle} : H^1(G, K^\times) \to H^1(\langle g\rangle, K^\times))$$

$$\cap_{g\in\Gamma_\gamma}\mathrm{Ker}(\mathrm{Res}^G_{\langle g\rangle} : H^1(G, A) \to H^1(\langle g\rangle, A))$$

Section 3.3 takes a closer look at the first of these and then Section 3.4 combines the previous results to obtain our description of the divisor class group of $K_\gamma[A]^G$ in Theorem 3.4.1. Here N_γ is the normal subgroup of G generated by Γ_γ and D_γ is the normal subgroup of G generated by the set of $s \in \Gamma_\gamma$ such that s is a diagonalizable reflection and $\gamma_s(A) \subset K^\times$.

Theorem 1.0.6. *Assume G acts faithfully on A. There is a short exact sequence*

$$H^1(G/N_\gamma, K^\times)/\partial'_G(A^G) \rightarrowtail \mathrm{Cl}((K_\gamma[A])^G) \twoheadrightarrow \mathrm{Ker}(\delta'_G) \cap H^1(G/D_\gamma, A^{D_\gamma})$$

where $\partial_G : A^G \to H^1(G, K^\times)$ and $\delta_G : H^1(G, A) \to H^2(G, K^\times)$ are the respective connecting homomorphisms associated with the short exact sequence

$$K^\times \rightarrowtail (K_\gamma[A])^\times \twoheadrightarrow A$$

and ∂'_G satisfies $\mathrm{inf}^G_{G/N_\gamma} \circ \partial'_G = \partial_G$, while $\delta'_G = \delta_G \circ \mathrm{inf}^G_{G/D_\gamma}$.

In Chapter 4, we turn to rationality problems for twisted multiplicative invariant fields. A large class of problems comes from the study of the rationality of field extensions. A field extension K/k is said to be *rational* if K is isomorphic to $k(X_1, \dots, X_n)$ for some algebraically independent variables $X_1, \dots, X_n$. K/k is said to be *stably rational* if there exists a field $L \supset K$ which is rational over both K and k. K/k is said to be *unirational* if K is a subfield of a field rational over k. Then rationality implies stable rationality which implies unirationality, but the converses [1] do not hold in general. Two fields K_1 and K_2 containing k are called *stably equivalent* over k if there is a k-isomorphism

$$K_1(X_1, \dots, X_l) \cong K_2(Y_1, \dots, Y_m).$$

In Noether's work on the inverse Galois problem, she showed that a finite group G could be realized as a Galois group over a given number field F if the invariant field $F(G) = F(X_g | g \in G)^G$ is rational over F where here G acts by permuting variables [25, 36]. In showing that the $F(G)'s$ were nonrational for certain G [28], Saltman first considered multiplicative invariant fields and showed in [29] that for certain lattices Q, $F(Q)^G$ was stably equivalent to $F(G')$ for G' a non-split extension of G with abelian kernel. He later expressed such a field $F(G')$ as a twisted multiplicative invariant field of G [32]. Saltman was also able to describe the invariant fields of reductive groups as twisted multiplicative invariant fields of their Weyl groups where the lattice is derived from the root lattice [30]. He used the concept of twisted multiplicative invariant fields to prove results on the existence of solutions to the embedding problem for Galois extensions[32] and as a tool to study retract rationality via the non-ramified Brauer group [31].

Farkas [16] proved the following theorem:

Theorem 1.0.7. *$\mathbf{C}(A)^R$ is rational over $\mathbf{C}$ if R is generated by reflections acting on the lattice A.*

Farkas' proof actually establishes the stronger result that $\mathbf{C}(A)^R$ is rational over $\mathbf{C}(A^R)$. A key tool in his proof is an embedding of the $\mathbf{Z}R$ lattice A into a $\mathbf{Z}R$ lattice B for which $\mathbf{C}[B]^R$ is a UFD so that it satisfies the equivalent conditions of Proposition 3.1.1. He relied heavily on this embedding to obtain a description of the structure of $\mathbf{C}[A]^R \subset \mathbf{C}[B]^R$ which later allows him to prove the rationality of $\mathbf{C}(A)^R$ over $\mathbf{C}$.

The 1-cocycle γ representing a class $[\gamma] \in H^1(R, \mathrm{Hom}(A, K^\times))$ turned out to be a complicating factor in the attempt to find a generalization of this result for our setting. Specializing to our case, Lemma 4.1.12 shows that

$$H^1_\Gamma(G, \mathrm{Hom}(A, T)) = \{[f] \in H^1(G, \mathrm{Hom}(A, T)) | A^{(s)} \subset \mathrm{Ker}(\gamma_s) \text{ for all } s \in \Gamma\}$$

is a group where A is a faithful $\mathbf{Z}G$ lattice, Γ is a G-stable set of reflections generating R and T is a G/R module.

Here are the required hypotheses to embed $K_\gamma[A]^R$ into a UFD of the form $K_{\tilde{\gamma}}[B]^R$:

(EMB) Let $\Gamma = \Gamma_\gamma$ and $R = \langle \Gamma_\gamma \rangle$. Let Φ be a suitable root system for A with weight lattice $\Lambda(\Phi)$. There exists a $\mathbf{Z}R$ lattice B on V which contains A and satisfies:

(E1) $[\gamma] \in \mathrm{Im}(H^1_\Gamma(R, \mathrm{Hom}(B, K^\times)) \to H^1_\Gamma(R, \mathrm{Hom}(A, K^\times)))$.

(E2) $\pi(B) \subset \Lambda(\Phi)$

(E3) $\pi(B) = \oplus_{i=1}^{n} \mathbf{Z} m_i \omega_i$ is a stretched weight lattice for Φ

With these hypotheses, we are able to prove Proposition 4.1.14, a generalization of Theorem 1.0.7.

Proposition 1.0.8. *Assume (EMB). Then $K_\gamma(A)^R$ is rational over K.*

In order to remove some of these hypotheses, we take another approach to the problem using the $\mathbf{Z}R$ short exact sequence of lattices

$$A^R \rightarrowtail A \twoheadrightarrow A/A^R$$

and the following isomorphism from Lemma 2.1.5: $K_\gamma(A)^R \cong L_\rho(X)^R$ where $X = A/A^R$, $L = K(A^R)$, $R = \langle \Gamma_\gamma \rangle$ and ρ is a 1-cocycle representing a class $[\rho] \in H^1(G, \mathrm{Hom}(X, L^\times))$ determined by γ and a $\mathbf{Z}$ splitting of the above exact sequence.

Here R acts effectively on X as a group of reflections. We find a suitable stretched weight lattice Y for which $X \subset Y \subset \Lambda$ and show that the induced map $H^1_\Gamma(R, \mathrm{Hom}(Y, L^\times)) \to H^1_\Gamma(R, \mathrm{Hom}(X, L^\times))$ is surjective. Hence we obtain Proposition 4.2.5.

Proposition 1.0.9. *If R is generated by $\Gamma_\gamma\rangle$, then $K_\gamma(A)^R$ is rational over $K(A^R)$.*

We next examine the rationality of $K_\gamma(A)^G$ over $K_\gamma(A^R)^G$ where G, acting faithfully on the lattice A, also acts on the field K, γ is a 1-cocycle representing $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$, and R is the group generated by Γ_γ. For a suitable root system Φ for A with weight lattice Λ and base Δ, we obtain a G-equivariant decomposition $V = V^R \oplus \mathbf{Q}\Phi$. using the idempotent $e_R = \frac{1}{|R|}\sum_{r \in R} r$ of $\mathbf{Q}G$. Setting $\Omega_G = \{g \in G | g\Delta = \Delta\}$, we show in Lemma 2.3.6 that $G = R \rtimes \Omega_G$ and we prove Proposition 4.3.2:

Proposition 1.0.10. *Suppose*

$$[\gamma] \in \mathrm{Im}(H^1_\Gamma(G, \mathrm{Hom}(B, K^\times) \to H^1_\Gamma(G, \mathrm{Hom}(A, K^\times)))$$

where $B = e_R A \oplus \Lambda$. Then the invariant fields $K_\gamma(A)^G$ and $K_\gamma(A)^{\Omega_G}$ are isomorphic under an isomorphism that is the identity on $K_\gamma(A^R)^G = K_\gamma(A)^{\Omega_G}$. In particular, $K_\gamma(A)^G$ is rational over $K_\gamma(A^R)^G$ if and only if $K_\gamma(A)^{\Omega_G}$ is rational over $K_\gamma(A^R)^{\Omega_G}$.

In the untwisted case, we obtain Corollary 4.3.3:

Corollary 1.0.11. *The invariant fields $K(A)^G$ and $K(A)^{\Omega_G}$ are isomorphic under an isomorphism that is the identity on $K(A^R)^G = K(A^R)^{\Omega_G}$. In particular, $K(A)^G$ is rational over $K(A^R)^G$ if and only if $K(A)^{\Omega_G}$ is rational over $K(A^R)^{\Omega_G}$.*

We are able to apply this Corollary to show in Theorem 4.4.9 that certain types of multiplicative invariant fields are rational:

Proposition 1.0.12. *Let Φ be a crystallographic root system for the $\mathbf{Q}$ space V. Then $G = \mathrm{Aut}(\Phi)$ acts faithfully on V. For any $\mathbf{Z}G$ lattice A on V, $K(A)^G$ is rational over K where G acts trivially on the field K.*

When G acts faithfully by automorphisms on the field K, the field of multiplicative invariants $K(A)^G$ is called the *field of tori invariants* of A under G. Rationality problems concerning the field of tori invariants seem to be the most tractable type of multiplicative invariant rationality problems. It is possible to relate questions on stable rationality and stable equivalence to problems on the structure of the $\mathbf{Z}G$-lattices involved: [9, 5]:

Proposition 1.0.13. *[12] (Endo-Miyata) Let P be a $\mathbf{Z}G$ permutation lattice. The field of tori invariants $K(P)^G$ is rational over K^G.*

Theorem 1.0.14. *[35, 38](Swan, Voskresenskii) The field of tori invariants $K(M)^G$ is stably rational over K^G iff there exists an exact sequence of $\mathbf{Z}G$ lattices*

$$0 \to M \to P \to Q \to 0$$

where P, Q are permutation lattices.

Theorem 1.0.15. *[35, 39, 11] Two fields of tori invariants $K(M)^G$ and $K(N)^G$ are stably equivalent over K^G iff there exist exact sequences of $\mathbf{Z}G$ lattices*

$$0 \to M \to E \to P \to 0$$

$$0 \to N \to E \to Q \to 0$$

where P, Q are $\mathbf{Z}G$ permutation lattices. The above condition defines an equivalence relation on $\mathbf{Z}G$ lattices.

It is sometimes possible to convert rationality problems of polynomial invariant fields into rationality problems of fields of tori invariants. This is the case in the following particularly prominent problem: Let $PGL_n(\mathbf{C})$ act on $M_n(\mathbf{C}) \oplus M_n(\mathbf{C})$ by simultaneous conjugation, and act on $\mathbf{C}$ trivially. Then one asks whether $C_n = \mathbf{C}(M_n(\mathbf{C}) \oplus M_n(\mathbf{C}))^{PGL_n(\mathbf{C})}$ is (stably) rational over $\mathbf{C}$. The fields C_n can also be described as the centre of the division ring of $n \times n$ generic matrices. Procesi [26, 27] and Formanek [18] were able to convert this problem into a tori invariant rationality problem over the Weyl group S_n. Let the permutation lattice $\mathbf{Z}[S_n/S_{n-2}]$ have basis $\{y_{ij} | i \neq j\}$ and let the permutation lattice $\mathbf{Z}[S_n/S_{n-1}]$ have basis $\{u_i | i = 1, \ldots, n\}$. Then S_n acts on $\mathbf{Z}[S_n/S_{n-2}]$ via $\sigma(y_{ij}) = y_{\sigma(i)\sigma(j)}$. Let

$$\rho_n : \mathbf{Z}[S_n/S_{n-2}] \to \mathbf{Z}[S_n/S_{n-1}]$$

be the $\mathbf{Z}S_n$ map sending y_{ij} to $u_i - u_j$. Then

$$0 \to G_n \to \mathbf{Z}[S_n/S_{n-2}] \overset{\rho_n}{\to} \mathbf{Z}[S_n/S_{n-1}] \overset{\varepsilon_n}{\to} \mathbf{Z} \to 0$$

is an exact sequence of $\mathbf{Z}S_n$-lattices where $G_n = \mathrm{Ker}(\rho_n)$ and $\varepsilon_n : \mathbf{Z}[S_n/S_{n-1}] \to \mathbf{Z}$ is the "augmentation" map sending u_i to 1. Procesi and Formanek were able to show that

$$\begin{aligned} \mathbf{C}(M_n(\mathbf{C}) \oplus M_n(\mathbf{C}))^{PGL_n(\mathbf{C})} &\cong \mathbf{C}(U_n \oplus U_n \oplus G_n)^{S_n} \\ &= \mathbf{C}(U_n)(U_n \oplus G_n)^{S_n}, \end{aligned}$$

where $U_n = \mathbf{Z}[S_n/S_{n-1}]$.

Progress on this rationality problem has a rather odd history [23]. C_2 was shown to be rational over $\mathbf{C}$ by Sylvester [37] at the end of the last century. Formanek proved that C_n is rational over $\mathbf{C}$ for $n = 3, 4$ [18, 19]. Bessenrodt and Lebruyn proved that C_n is stably rational over $\mathbf{C}$ for $n = 5, 7$ [5]. Beneish [3] later reproved the results of Bessenrodt and Lebruyn using a different approach and without the use of computers. Each of these cases was solved using the description in terms of multiplicative invariants given by Procesi and Formanek and the results on tori invariants and a wide range of representation theoretical results. However the proof of each new case was accompanied by a demonstration that the method used could not be extended to further cases. Using geometric techiques, Schofield proved a reduction result in [34]: For $n = a \cdot b$ where a and b are relatively prime, he showed that if C_a and C_b are both stably rational over $\mathbf{C}$, then C_n is stably rational over $\mathbf{C}$. Thus, so far, $\mathbf{C}(M_n(\mathbf{C}) \oplus M_n(\mathbf{C}))^{PGL_n(\mathbf{C})}$ has shown to be stably rational over $\mathbf{C}$ for any divisor of 420.

We make a small contribution to this problem. We first study the $\mathbf{Q}S_n$ structure of the $\mathbf{Q}S_n$ module $\mathbf{Q}G_n$ and determine its decomposition into irreducible $\mathbf{Q}S_n$ modules. It turns out that

$$\mathbf{Q}G_n = \begin{cases} S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,1^2)} \oplus S^{(n-2,2)} & \text{if } n \geq 4 \\ S^{(3)} \oplus S^{(2,1)} \oplus S^{(1^3)} & \text{if } n = 3 \end{cases}$$

where S^{λ} denotes the irreducible $\mathbf{Q}S_n$ module corresponding to the partition $\lambda \mapsto n$. We determine an explicit pure sublattice E_n of G_n satisfying

$$\mathbf{Q}G_n \cong \mathbf{Q}E_n \oplus \mathbf{Q}[S_n/S_{n-2} \times S_2]$$

over $\mathbf{Q}$. In Proposition 5.0.16 , we prove that $\mathbf{C}(U_n)(U_n \oplus G_n)^{S_n}$ is stably equivalent to $\mathbf{C}(U_n)(U_n \oplus E_n)^{S_n}$ for odd values of n. It would be interesting but undoubtably difficult to examine the stable rationality problem for this smaller lattice.

Chapter 2

Preliminaries

2.1 Twisted Multiplicative Invariant Rings and Fields

Twisted multiplicative invariant rings and fields were first introduced by Saltman in [32].

Let G act on the field K, let A be a $\mathbf{Z}G$-lattice of $\mathbf{Z}$ rank n, and let $K[A]$ be the group ring. We will write elements of $K[A]$ as $\sum_{a\in A} c_a e(a)$ where $e: A \to K[A]$ is the "exponential" map. Note that $K[A] \cong K[X_1^{\pm1}, \ldots, X_n^{\pm1}]$ is a domain and hence has a quotient field $K(A)$. Observe also that the unit group $K[A]^\times \cong K^\times \oplus A$. Using the following (standard) isomorphism as an identification, we will define a twisted action of G on $K[A]^\times$ corresponding to $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$ and extend it naturally to an action of G on $K[A]$ and $K(A)$.

Lemma 2.1.1. [10, (25.10)] $\mathrm{Ext}^1_G(A, N) \cong H^1(G, \mathrm{Hom}_{\mathbf{Z}}(A, N))$ *where A is a $\mathbf{Z}G$ lattice and N is any $\mathbf{Z}G$ module.*

Proof: Let $0 \to N \xrightarrow{i} X \xrightarrow{p} A \to 0$ be a short exact sequence of $\mathbf{Z}G$ modules. Replacing N by $i(N)$ if necessary, we may assume that i is an inclusion map.

Since A is a lattice, the sequence is $\mathbf{Z}$-split so that there exists $\tau \in \mathrm{Hom}_{\mathbf{Z}}(A, N)$ with $p\tau = 1_A$.

For each $g \in G$, let $f_g = g\tau - \tau \in \mathrm{Hom}_{\mathbf{Z}}(A, N)$.

Now $p(f_g(a)) = p(g\tau)(a) - p\tau(a) = p(g\tau(g^{-1}a)) - a = gp\tau(g^{-1}a) - a = g^{-1}ga - a = 0$ so that $f_g(a) \in \mathrm{Ker}(p) = N$.

The map defined by $f : G \to \mathrm{Hom}_{\mathbf{Z}}(A, N), g \mapsto f_g$ is a 1-cocycle: For $g, h \in G, a \in A$, we have

$$\begin{aligned}(f_g + gf_h)(a) &= (g\tau)(a) - \tau(a) + g[f_h(g^{-1}a)] \\ &= g(\tau(g^{-1}a)) - \tau(a) + g[(h\tau)(g^{-1}a) - \tau(g^{-1}a)] \\ &= g[\tau(g^{-1}a)] - \tau(a) + gh[\tau(h^{-1}g^{-1}a)] - g[\tau(g^{-1}a)] \\ &= (gh\tau)(a) - \tau(a) = f_{gh}(a)\end{aligned}$$

Note that the 1-cocycle f' for our given extension defined by a different $\mathbf{Z}$-splitting τ' also differs by a 1-coboundary since

$$f'_g - f_g = (g\tau' - \tau') - (g\tau - \tau) = g(\tau' - \tau) - (\tau' - \tau)$$

with $\tau' - \tau \in \mathrm{Hom}_{\mathbf{Z}}(A, N)$.

Now $X = N \oplus \tau(A)$. So each element x of X can be uniquely represented as $x = n + \tau(a)$ for some $n \in N, a \in A$.

$$gx = g(n + \tau(a)) = gn + g[\tau(a)] = gn + f_g(ga) + \tau(ga)$$

Using the isomorphism $X = N \oplus \tau(A) \stackrel{(id, \tau^{-1})}{\longrightarrow} N \oplus A$ as an identification, we have

$$g(n, a) = (gn + f_g(ga), ga) \tag{2.1.2}$$

Since for $g, h \in G,\ n \in N, a \in A$,

$$\begin{aligned}(gh)(n, a) &= ((gh)n + f_{gh}((gh)a), (gh)a) \\ g(h(n, a)) &= ((gh)n + f_g((gh)a) + (gf_h)(gha), gha)\end{aligned}$$

we see that $(gh)(n, a) = g(h(n, a))$ iff $f_{gh} = f_g + gf_h$ so that (2.1.2) defines an action of G on $X \cong N \oplus A$ iff f is a 1-cocycle.

Claim: The $\mathbf{Z}G$ extension $[0 \to N \to X' \to A \to 0]$ with 1-cocycle f' is equivalent to the $\mathbf{Z}G$ extension $[0 \to N \to X \to A \to 0]$ with 1-cocycle f iff $f' - f$ is a 1-coboundary.

The two extensions are equivalent iff there exists a $\mathbf{Z}G$ homomorphism $\varphi : X \to X'$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

If $\varphi : X \to X'$ is a **Z**-homomorphism, then it is a **Z**-isomorphism which makes the diagram commute iff $\varphi(n, a) = (n + \theta(a), a)$ for some $\theta \in \mathrm{Hom}_{\mathbf{Z}}(A, N)$. Now

$$g[\varphi(n, a))] = (gn + g[\theta(a)] + f'_g(ga), ga)$$
$$\varphi(g(n, a)) = \varphi(gn + f_g(ga), ga) = (gn + f_g(ga) + \theta(ga), ga)$$

Note that $\theta(ga) - g[\theta(a)] = [\theta - (g\theta)](ga)$. Then φ defined above is a $\mathbf{Z}G$ isomorphism iff $f' - f$ is a 1-coboundary.

So we have shown that $\mathrm{Ext}^1_G(A, N) \cong H^1(G, \mathrm{Hom}(A, N))$ as required. ■

Applying the lemma to our set-up, a 1-cocycle γ representing $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$ defines a short exact sequence of $\mathbf{Z}G$ modules

$$0 \to K^\times \to K_\gamma[A]^\times \to A \to 0$$

and hence an action of G on $K[A]$ and $K(A)$ such that

$$g \cdot [ce(a)] = (g \cdot c)\gamma_g(ga)e(ga), g \in G, c \in K^\times, a \in A$$

We will denote $K[A]$, respectively $K(A)$ with this G-action induced by γ as $K_\gamma[A]$ and $K_\gamma(A)$. The fixed ring $K_\gamma[A]^G$ is called a *twisted multiplicative invariant ring* and the fixed field $K_\gamma(A)^G$ is called a *twisted multiplicative invariant field*. Note that $K_\gamma(A)^G$ is the quotient field of $K_\gamma[A]^G$. If γ is the trivial 1-cocycle, then we write $K[A]^G$ and $K(A)^G$ for the multiplicative invariant ring and field respectively. The following lemma shows that $K_\gamma[A]^G$ and hence $K_\gamma(A)^G$ depend only on the class in $H^1(G, \mathrm{Hom}(A, K^\times))$ of the 1-cocycle γ.

Lemma 2.1.3. *If γ, γ' are 1-cocycles with $[\gamma] = [\gamma']$ in $H^1(G, \mathrm{Hom}(A, K^\times))$ then there exists a ring isomorphism $K_\gamma[A] \to K_{\gamma'}[A]$ compatible with G actions. In particular, $K_\gamma[A]^G \cong K_{\gamma'}[A]^G$.*

Proof: By hypothesis, there exists $\lambda \in \mathrm{Hom}(A, K^\times)$ such that $\gamma_g/\gamma'_g = g\lambda/\lambda$. Evaluating this at ga we get

$$\gamma_g(ga)/\gamma'_g(ga) = g(\lambda(a))/\lambda(ga)$$

Now define $\psi : K_\gamma[A] \to K_{\gamma'}[A]$ by $ce(a) \mapsto c\lambda(a)e(a)$. This is clearly a ring isomorphism. For $g \in G, c \in K^\times, a \in A$, we have

$$\psi(gce(a)) = \psi((gc)\gamma_g(ga)e(ga)) = (gc)\gamma_g(ga)\lambda(ga)e(ga)$$
$$= (gc)g(\lambda(a))\gamma'_g(ga)e(ga) = g[\psi(ce(a))]$$

which shows that ψ is compatible with the G action. ■

The following proposition shows that it suffices to consider twisted multiplicative invariant rings and fields in which G acts faithfully on the lattice.

Proposition 2.1.4. *Given G, K, A, γ as above, there exists a quotient $\overline{G}$ of G, a subfield $\overline{K}$ of K, a sublattice $\overline{A}$ of A and $\overline{\gamma} \in H^1(\overline{G}, \mathrm{Hom}(\overline{A}, \overline{K}^\times))$ such that $K_\gamma[A]^G \cong \overline{K}_{\overline{\gamma}}[\overline{A}]^{\overline{G}}$ and $\overline{G}$ acts faithfully on $\overline{A}$.*

Proof: This will be shown by two standard reductions, both of them constructive.

The first reduces the lattice. Let

$$N = \{g \in G | g \text{ acts trivially on } K \text{ and } A\}$$

and let

$$B = \{a \in A | \gamma_n(a) = 1 \text{ for all } n \in N\}$$

N is clearly a normal subgroup of G. Since $\theta : A \to \mathrm{Hom}(N, K^\times), a \mapsto [n \mapsto \gamma_n(a)]$ is a homomorphism with kernel B and with $\mathrm{Hom}(N, K^\times)$ finite, B is a sublattice of A. To show that B is G-stable it then suffices to check that θ is a G-homomorphism: Indeed, for $a \in A, n \in N, g \in G$, we have

$$\begin{aligned}
(g[\theta(a)])(n) &= g([\theta(a)](g^{-1}ng)) \\
&= g[\gamma_{g^{-1}ng}(a)] \\
&= g[\gamma_{g^{-1}}(a)g^{-1}(\gamma_n(ga))ng^{-1}(\gamma_g(n^{-1}ga)] \\
&= g[\gamma_{g^{-1}}(a)\gamma_g(ga)]\gamma_n(ga) \\
&= g[\gamma_{g^{-1}g}(a)]\gamma_n(ga) \\
&= g(1)\gamma_n(ga) \\
&= \gamma_n(ga) = \theta(ga)(n)
\end{aligned}$$

The inclusion $B \hookrightarrow A$ induces a G-homomorphism $\mathrm{Hom}(A, K^\times) \to \mathrm{Hom}(B, K^\times)$ which sends the 1-cocycle γ to γ', say. The G-action on the subring $K[B]$ of $K_\gamma[A]$ makes it into $K_{\gamma'}[B]$. By definition of B, N acts trivially on $K_{\gamma'}[B]$ and we claim that $K_\gamma[A]^N = K_{\gamma'}[B]$: for if $\sum_{a \in A} c_a e(a) \in K_\gamma[A]^N$ then for all $n \in N$, we have $\sum_{a \in A} c_a \gamma_n(a) e(a) = \sum_{a \in A} c_a e(a)$ so that for all $n \in N, a \in A$ we have $c_a(\gamma_n(a) - 1) = 0$. If $a \notin B$, then $\gamma_n(a) \neq 1$ for some $n \in N$ and thus $c_a = 0$. It follows that $\sum_{a \in A} c_a e(a) \in K_{\gamma'}[B]$.

If $b \in B$, then $\gamma_{gn}(b) = (g\gamma_n)(b)\gamma_g(b) = g(\gamma_n(g^{-1}b))\gamma_g(b) = g(1)\gamma_g(b) = \gamma_g(b)$. This means that γ' is inflated from a 1-cocycle $\overline{\gamma} : G/N \to \mathrm{Hom}(B, K^\times)$

so that G/N acts on $K_{\gamma'}[B]$ as $K_{\bar\gamma}[B]$. But then $K_\gamma[A]^G = (K_\gamma[A]^N)^{G/N} = K_{\bar\gamma}[B]^{G/N}$.

To reduce the field, our second step, we return to our original notation but can now assume that if $g \in G$ acts trivially on both K and A then $g = 1$. Let

$$N = \{g \in G | g \text{ acts trivially on } A\} \triangleleft G$$

and let F be the fixed field of N on K. By our assumption, N is then the Galois group of K/F.

Since N acts trivially on A, $\mathrm{Hom}(A, K^\times) \cong (K^\times)^{\mathrm{rank}(A)}$ so that $H^1(N, \mathrm{Hom}(A, K^\times)) = 0$ by Hilbert 90. By the inflation-restriction sequence, $[\gamma]$ is then the inflation of some $[\gamma'] \in H^1(G/N, \mathrm{Hom}(A, F^\times))$. Choose a 1-cocycle γ' representing $[\gamma']$. Then by Lemma 2.1.3, $K_\gamma[A]$ is G-isomorphic to $K_{\mathrm{inf}(\gamma')}[A]$. So we may assume that $\gamma = \mathrm{inf}(\gamma')$.

But γ is then F-valued hence $K_\gamma[A] = K \otimes_F F_\gamma[A]$ with the obvious G-actions. Since N acts trivially on $F_\gamma[A]$, it follows that $K_\gamma[A]^N = K^N \otimes_F F_\gamma[A] = F_\gamma[A]$ (a normal basis of K/F is an $F_\gamma[A]$-basis of $K_\gamma[A]$) and so $K_\gamma[A]^G = (K_\gamma[A]^N)^{G/N} = F_{\gamma'}[A]^{G/N}$ again as required. ■

The next lemma shows that one can decompose the group action on a twisted multiplicative invariant ring or field with $\mathbf{Z}G$ lattice A if there exists a short exact sequence of $\mathbf{Z}G$ lattices of the form

$$0 \to N \to A \to X \to 0$$

Lemma 2.1.5. *Let G be a finite group acting on the field K and the lattice A. Let $\gamma : G \to \mathrm{Hom}(A, K^\times)$ be a 1-cocycle representing the extension class of*

$$0 \to K^\times \to K_\gamma[A]^\times \to A \to 0$$

Let $0 \to N \to A \to X \to 0$ be a short exact sequence of $\mathbf{Z}G$ lattices with 1-cocycle $\beta : G \to \mathrm{Hom}(X, N)$ representing the extension given by $\beta_g = g\tau - \tau$ where $\tau : X \to A$ is a $\mathbf{Z}$-splitting of $A \to X$. Then there is an induced exact sequence

$$0 \to K_\gamma[N]^\times \to K_\gamma[A]^\times \to X \to 0$$

with extension class represented by the 1-cocycle $\rho : G \to \mathrm{Hom}(X, K_\gamma[N]^\times)$ given by

$$\rho_g(x) = \gamma_g(\tau(x))\gamma_g(\beta_g(x))e(\beta_g(x)), g \in G, x \in X$$

This allows the G-ring $K_\gamma[A]$ to be viewed as an iterated $(K_\gamma[N])_\rho[X]$. In particular, there is a ring isomorphism $K_\gamma[A]^G \cong (K_\gamma[N]_\rho[X])^G$ inducing $K_\gamma(A)^G \cong (K_\gamma(N)_\rho(X))^G$ where now ρ is also the image of the old ρ under

$$H^1(G, \operatorname{Hom}(X, K_\gamma[N]^\times)) \to H^1(G, \operatorname{Hom}(X, K_\gamma(N)^\times))$$

Proof: We build the following commutative diagram starting with the middle column and the bottom row:

$$\begin{array}{ccccc}
K^\times & \xrightarrow{id} & K^\times & & \\
\downarrow & & \downarrow & & \\
K_\gamma[N]^\times & \rightarrowtail & K_\gamma[A]^\times & \twoheadrightarrow & X \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle id} \\
N & \rightarrowtail & A & \twoheadrightarrow & X
\end{array}$$

To obtain the middle row, we define the map $K_\gamma[A]^\times \twoheadrightarrow X$ as the composite $K_\gamma[A]^\times \twoheadrightarrow A \twoheadrightarrow X$. The kernel of this composite is then $K_{\gamma'}[N]^\times$ where $\gamma'_g = \gamma_g|_N$ for all $g \in G$. Combining the $\mathbf{Z}$-splitting $\tau : X \to A$ of $A \to X$ with the usual $\mathbf{Z}$-splitting $e : A \to K_\gamma[A]^\times$ of $K_\gamma[A]^\times \to A$, which gives $\gamma_g = ge/e$ by construction, we get the $\mathbf{Z}$-splitting $e \circ \tau : X \to K_\gamma[A]^\times$ for the middle row from which its extension class is represented by $\rho_g = g(e \circ \tau)/(e \circ \tau)$. Since $\beta_g = g\tau - \tau$, we have $\beta_g(gx) = (g\tau)(gx) - \tau(gx) = g[\tau(x)] - \tau(gx)$ so that $g[\tau(x)] = \beta_g(gx) + \tau(gx)$. Then

$$\begin{aligned}
[g(e \circ \tau)](x) &= g[(e \circ \tau)(g^{-1}x)] \\
&= \gamma_g(g[\tau(g^{-1}x)])e(g[\tau(g^{-1}x)]) \\
&= \gamma_g(\tau(x) + \beta_g(x))e(\tau(x) + \beta_g(x)) \\
&= \gamma_g(\tau(x))\gamma_g(\beta_g(x))e(\beta_g(x))(e \circ \tau)(x)
\end{aligned}$$

shows that $\rho_g(x) = \gamma_g(\tau(x))\gamma_g(\beta_g(x))e(\beta_g(x))$

Let $f : X \to (K_\gamma[N])_\rho[X]^\times$ give a $\mathbf{Z}$-splitting of

$$0 \to K_\gamma[N]^\times \to K_\gamma[N]_\rho[X]^\times \to X \to 0,$$

so that $\rho_g = \frac{gf}{f}$. Let $\varphi : K_\gamma[N]_\rho[X] \to K_\gamma[A]$ be the ring homomorphism satisfying $\varphi(f(x)) = (e \circ \tau)(x)$ for all $x \in X$ and $\varphi|_{K_\gamma[N]}$ is the inclusion of

$K_\gamma[N]$ in $K_\gamma[A]$. Then φ is clearly a ring isomorphism. Let $g \in G$, $x \in X$. Then

$$\begin{aligned}\varphi(gf(x)) &= \varphi(\rho_g(gx)f(gx)) \\ &= \rho_g(gx)(e \circ \tau)(gx) \\ &= g[(e \circ \tau)(x)] \\ &= g[\varphi(f(x))]\end{aligned}$$

shows that φ commutes with the G-action and hence induces an isomorphism between $K_\gamma[N]_\rho[X]^G$ and $K_\gamma[A]^G$ and thus, by localization, an isomorphism between $K_\gamma(N)_\rho(X)^G$ and $K_\gamma(A)^G$. ■

2.2 Classification of Reflection Lattices

A reflection acting on a lattice A is an element s of $GL(A)$ of finite order with $\mathrm{Im}_A(s-1)$ cyclic. Let W be a finite subgroup of $GL(A)$ generated by reflections on A which acts effectively on A (that is, $A^W = 0$). We will call such a $\mathbf{Z}W$-lattice A satisfying the above hypotheses a *reflection lattice.* In [17], Feit classified the reflection lattices corresponding to irreducible root systems. We are interested in classifying all of the reflection lattices up to $\mathbf{Z}W$- isomorphism. The constructions used in this section will be useful in future sections in our study of divisor class groups of twisted multiplicative invariant rings and in our study of rationality problems for twisted multiplicative invariant fields. In order to make this classification, we will first show that for a reflection lattice A, the $\mathbf{Q}$- vector space $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ is a reflection space. Since the reflection spaces have been completely classified, it will suffice to find the isomorphism classes of full $\mathbf{Z}W$-lattices on a given reflection space.

Lemma 2.2.1. *A reflection $s \in GL(A)$ has order 2*

Proof: Let $\mathrm{Im}_A(s-1) = \mathbf{Z}\alpha$. Then $s(\alpha) = c\alpha$ for some $c \in \mathbf{Z}$. Let n be the order of s on A. Then $\alpha = s^n(\alpha) = c^n\alpha$ implies that c is a root of unity in $\mathbf{Z}$. So $c = \pm 1$.

Suppose $c = 1$. Let $x \in A$ be such that $s(x) = x + \alpha$. Then $x = s^n(x) = x + nx$. So $n = 0$ and $s = 1$. The contradiction implies that $c = -1$.

Let $a \in A$. Then $s(a) = a + n\alpha$ for some $n \in \mathbf{Z}$. Then $s^2(a) = a$. Since a was arbitrary, s has order 2. ■

The following result is standard, eg. [16, 2.2] but is often stated without proof. We include a proof for completeness.

Lemma 2.2.2. *For a reflection $s \in GL(A)$, $Ker_A(s+1)$ is cyclic and*

$$H^1(\langle s \rangle, A) \cong 0 \text{ or } \mathbf{Z}/2\mathbf{Z}.$$

In fact, the first cohomology group of A completely determines the isomorphism class of A as an $\mathbf{Z}\langle s \rangle$ lattice:

$$\begin{array}{ll} H^1(\langle s \rangle, A) \cong 0 & \Leftrightarrow A \cong \mathbf{Z}\langle s \rangle \oplus \mathbf{Z}^{n-2} \\ H^1(\langle s \rangle, A) \cong \mathbf{Z}/2\mathbf{Z} & \Leftrightarrow A \cong \mathbf{Z}^- \oplus \mathbf{Z}^{n-1} \end{array}$$

where s acts on $\mathbf{Z}^- = \mathbf{Z}x$ by $sx = -x$.

Proof

$$H^1(\langle s \rangle, A) \cong H^{-1}(\langle s \rangle, A) = \mathrm{Ker}_A(s+1)/\mathrm{Im}_A(s-1)$$

Since A has finite rank and s has order 2, $H^1(\langle s \rangle, A)$ is finite and has exponent dividing 2. In fact, since $\mathrm{Im}_A(s-1)$ is cyclic, so is $\mathrm{Ker}_A(s+1)$. This means that $H^1(\langle s \rangle, A) \cong 0$ or $\mathbf{Z}/2\mathbf{Z}$.

Let $\mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha$. Then $A/\mathbf{Z}\alpha = A/\mathrm{Ker}_A(s+1) \cong \mathrm{Im}_A(s+1)$ is torsion-free. So there exists a $\mathbf{Z}$-sublattice C of A such that $A = \mathbf{Z}\alpha \oplus C$. Note that any basis for C can be added to α to form a basis for A. Let $\{x_1, \ldots, x_{n-1}\}$ be an arbitrary basis for C. Then $s(x_i) = x_i + k_i\alpha$ for some $k_i \in \mathbf{Z}$.

Case 1: $H^1(\langle s \rangle, A) = 0$

Then $\mathrm{Im}_A(s-1) = \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha$. This implies that $\gcd(k_i) = 1$. Let $a_i \in \mathbf{Z}, i = 1, \ldots, n-1$ be such that $\sum_{i=1}^{n-1} a_i k_i = 1$. Set $x = \sum_{i=1}^{n-1} a_i x_i$. Then $C/\mathbf{Z}x$ is torsion-free since $\gcd(a_i) = 1$. Indeed if

$$m \sum_{i=1}^{n-1} b_i x_i \in \mathbf{Z}x$$

then there exists $r \in \mathbf{Z}$ such that $mb_i = ra_i$ for all $i = 1, \ldots, n-1$.

$$\sum_{i=1}^{n-1} mb_i k_i = \sum_{i=1}^{n-1} ra_i k_i = r$$

Then $\frac{r}{m} = \sum_{i=1}^{n-1} b_i k_i \in \mathbf{Z}$ so that

$$\sum_{i=1}^{n-1} b_i x_i = \frac{r}{m} x \in \mathbf{Z}x$$

We may create a new basis $\{x, y_2, \ldots, y_{n-1}\}$ for C since $C/\mathbf{Z}x$ is torsion-free. Then $P = \mathbf{Z}\alpha \oplus \mathbf{Z}x$ is a $\mathbf{Z}\langle s\rangle$ lattice on which s acts via $s(\alpha) = -\alpha, s(x) = x + \alpha$. A new basis for P is $\{x+\alpha, x\}$. Then $s(x+\alpha) = x$ and $s(x) = x + \alpha$. So $P \cong \mathbf{Z}\langle s\rangle$. We see that A/P is torsion-free and $s(a) - a \in \mathbf{Z}\alpha \subset P$ implies that A/P is a trivial $\mathbf{Z}\langle s\rangle$-lattice. Hence $0 \to P \to A \to A/P \to 0$ is an exact sequence of $\mathbf{Z}\langle s\rangle$ lattices. Since $\mathrm{Ext}^1_{\mathbf{Z}\langle s\rangle}(A/P, P) \cong \mathrm{Ext}^1_{\mathbf{Z}\langle s\rangle}(\mathbf{Z}^{n-2}, \mathbf{Z}\langle s\rangle) = 0$, we see that the sequence splits. Hence, $A \cong \mathbf{Z}\langle s\rangle \oplus \mathbf{Z}^{n-2}$ as a $\mathbf{Z}\langle s\rangle$-lattice.

Case 2: $H^1(\langle s\rangle, A) = \mathbf{Z}/2\mathbf{Z}$.
In this case, $\mathrm{Im}_A(s-1) = 2\mathbf{Z}\alpha$. This implies that for all i, k_i is even. A new basis for C is $\{x_i + \frac{k_i}{2}\alpha | i = 1, \ldots n-1\}$. Then

$$s(x_i + \frac{k_i}{2}\alpha) = x_i + k_i\alpha - \frac{k_i}{2}\alpha = x_i + \frac{k_i}{2}\alpha.$$

So the matrix of s with respect to

$$\{\alpha, x_1 + \frac{k_1}{2}\alpha, \ldots, x_{n-1} + \frac{k_{n-1}}{2}\alpha\}$$

is $\mathrm{diag}(-1, 1, \ldots, 1)$ and hence $A \cong \mathbf{Z}^- \oplus \mathbf{Z}^{n-1}$. ∎

Definition 2.2.3. A reflection s in $GL(A)$ is called *diagonalizable over* $\mathbf{Z}$ if $H^1(\langle s\rangle, A) \cong \mathbf{Z}/2\mathbf{Z}$.

Now consider $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$. V is a $\mathbf{Q}W$-space with $V^W = 0$. We will identify $1 \otimes s \in GL(V)$ with $s \in GL(A)$. Then $\mathrm{Im}_V(s-1) \cong \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Im}_A(s-1)$ is 1 dimensional and s is of order 2 on V. Note that $\mathrm{Ker}_V(s+1) = \mathrm{Im}_V(s-1)$.

Lemma 2.2.4. *If $s, t \in W$ are reflections with $\mathbf{Q}\alpha = \mathrm{Ker}_V(s+1) = \mathrm{Ker}_V(t+1)$ then $s = t$.*

Proof: $\mathbf{Q}\alpha$ is a $\mathbf{Q}\langle s, t\rangle$ submodule of V. By Maschke's theorem, $V = \mathbf{Q}\alpha \oplus U$, where U is a $\mathbf{Q}\langle s, t\rangle$ submodule of V. Now $s(\alpha) = -\alpha$ and $s(u) - u \in$

$U \cap \mathbf{Q}\alpha = 0$ for each $u \in U$. This implies that $s|_U$ is the identity. Similarly, $t(\alpha) = -\alpha$ and $t|_U$ is the identity. Then $t = s$.

Lemma 2.2.4 allows us to make the following definition:

Definition 2.2.5. Let $s_\alpha \in W$ be the unique reflection s on V in W with $\mathrm{Ker}_V(s+1) = \mathbf{Q}\alpha$. Now let

$$\langle \cdot, \alpha \rangle : V \mapsto \mathbf{Q}$$

be the $\mathbf{Q}$-linear map satisfying $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha$ for $x \in V$. $\Gamma = \{s \in W | \dim_{\mathbf{Q}}(\mathrm{Ker}_V(s+1)) = 1\}$ is the reflection set in W.

Lemma 2.2.6. *If A is a $\mathbf{Z}W$ lattice on the reflection space V,*

$$\Phi_A = \{\alpha | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha, s \in \Gamma\}$$

is a crystallographic root system for V with reflection set Γ. That is, Φ_A satisfies the following conditions:
(a) $\{s_\alpha | \alpha \in \Phi_A\} = \Gamma$
(b) Φ_A is a finite spanning set for V not containing 0.
(c) W stabilizes Φ_A.
(d) $\mathbf{Q}\alpha \cap \Phi_A = \{\pm\alpha\}$ for all $\alpha \in \Phi_A$.
(e) For all $\alpha, \beta \in \Phi_A, \langle \beta, \alpha \rangle \in \mathbf{Z}$.

Proof:
(a) Note that $\mathrm{Ker}_V(s+1) \cong \mathbf{Q} \otimes_{\mathbf{Z}} \mathrm{Ker}_A(s+1)$ implies that $\mathrm{Ker}_V(s+1)$ is one-dimensional if and only if $\mathrm{Ker}_A(s+1)$ is cyclic. If $s \in \Gamma$, $\mathrm{Ker}_V(s+1)$ is one-dimensional so that $\mathbf{Z}\alpha = \mathrm{Ker}_A(s+1)$ for some $\alpha \in A$ and hence $s = s_\alpha$ where $\alpha \in \Phi_A$. Conversely, if $\alpha \in \Phi_A$ then $\mathbf{Z}\alpha = \mathrm{Ker}_A(s+1)$ which implies that $\mathbf{Q}\alpha = \mathrm{Ker}_V(s+1)$ and hence that $s = s_\alpha \in \Gamma$.
(c) Let $w \in W$ and $s \in \Gamma$. Then w restricts to a $\mathbf{Z}$-module isomorphism between $\mathrm{Ker}_A(s+1)$ and $\mathrm{Ker}_A(wsw^{-1}+1)$. Since $\mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha$, we find that $\mathrm{Ker}_A(wsw^{-1}+1) = \mathbf{Z}w\alpha$ so that wsw^{-1} is a reflection. In fact $ws_\alpha w^{-1} = s_{w\alpha}$ implies that $w\alpha \in \Phi_A$.
(b) $0 \notin \Phi_A$ and Φ_A is finite since W is. By (c), $\mathrm{Span}_{\mathbf{Q}}(\Phi_A)$ is a $\mathbf{Q}W$-subspace of V. Then $V = \mathbf{Q}\Phi_A \oplus U$ for some $\mathbf{Q}W$-subspace U of V. Let $x \in U$ and let $\alpha \in \Phi_A$ be given. Then

$$x - s_\alpha(x) = \langle x, \alpha \rangle \alpha \in U \cap \mathbf{Q}\Phi_A = 0$$

This implies that $s_\alpha(x) = x$ for all $s_\alpha \in \Gamma$. But then $U \subset V^W = 0$. So $V = \text{Span}_{\mathbf{Q}}(\Phi_A)$

(d) Let $c\alpha \in \mathbf{Q}\alpha \cap \Phi_A$ for some $\alpha \in \Phi_A$. Then

$$\mathbf{Q}\alpha = \text{Ker}_V(s_{c\alpha} + 1) = \text{Ker}_V(s_\alpha + 1)$$

so that by Lemma 2.2.4, $s_{c\alpha} = s_\alpha$ which implies that

$$\mathbf{Z}c\alpha = \text{Ker}_A(s_{c\alpha} + 1) = \text{Ker}_A(s_\alpha + 1) = \mathbf{Z}\alpha$$

Hence $c = \pm 1$ as required.

(e) Let $\alpha, \beta \in \Phi_A$. then

$$(s_\alpha - 1)(\beta) = -\langle \beta, \alpha \rangle \alpha \in \text{Im}_A(s_\alpha - 1) \subset \text{Ker}_A(s_\alpha + 1) = \mathbf{Z}\alpha$$

So $\langle \beta, \alpha \rangle \in \mathbf{Z}$. ∎

Definition 2.2.7. Let Φ be a crystallographic root system in V. Then the *root lattice* $\mathbf{Z}\Phi$ is the $\mathbf{Z}$-span of Φ. The *weight lattice* is

$$\Lambda(\Phi) = \{v \in V | \langle v, \alpha \rangle \in \mathbf{Z} \text{ for all } \alpha \in \Phi\}$$

Since $V \to \mathbf{Q}^\Phi, v \mapsto [\alpha \mapsto \langle v, \alpha \rangle]$ is injective as $V^W = 0$, we see that $\Lambda(\Phi)$ embeds into $\mathbf{Z}^\Phi$ and hence is a lattice. Now $W = \langle s_\alpha | \alpha \in \Phi \rangle$. Since $\langle wv, w\alpha \rangle = \langle v, \alpha \rangle$ for all $w \in W$, and all $v \in V, \alpha \in \Phi$ and Φ is W-stable, $\Lambda(\Phi)$ is actually a $\mathbf{Z}W$ lattice. Finally $\mathbf{Z}\Phi \subset \Lambda(\Phi)$ implies that $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ both have $\mathbf{Z}$-rank equal to $\dim_{\mathbf{Q}}(V) = n$.

Lemma 2.2.8. $\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$.

Proof: The first inclusion is clear from the definition of Φ_A. To see that $A \subset \Lambda(\Phi_A)$, let $x \in A$ and $\alpha \in \Phi_A$. Then

$$s_\alpha x - x = -\langle x, \alpha \rangle \alpha \in \text{Im}_A(s_\alpha - 1) \subset \text{Ker}_A(s_\alpha + 1) = \mathbf{Z}\alpha$$

implies that $\langle x, \alpha \rangle \in \mathbf{Z}$. ∎

Remark 2.2.9. If we take an arbitrary positive definite symmetric bilinear form on V and average it over W, we would obtain a W-invariant positive definite symmetric bilinear form for V: $(\cdot,\cdot) : V \times V \to \mathbf{Q}$. Since for any $v \in V$,

$$(v,\alpha) = (s_\alpha v, s_\alpha \alpha) = (v - \langle v,\alpha\rangle\alpha, -\alpha) = -(v,\alpha) + \langle v,\alpha\rangle(\alpha,\alpha)$$

we find that $\langle v,\alpha\rangle = \frac{2(v,\alpha)}{(\alpha,\alpha)}$

So we have shown that our $\mathbf{Q}W$ space $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ is a reflection space with respect to a positive definite symmetric W-invariant bilinear form $(\cdot,\cdot)$ with root system Φ_A and Weyl group W. Conversely, given a finite dimensional $\mathbf{Q}$-space V, a positive definite symmetric bilinear form on V, and a root system Φ for V with Weyl group W, then the $\mathbf{Z}W$ lattices A such that $\mathbf{Q} \otimes_{\mathbf{Z}} A \cong V$ are reflection lattices since for any reflection $s \in W$ on V, $\mathrm{Im}_A(1-s) = A \cap \mathrm{Im}_V(1-s) \subset \mathrm{Im}_V(1-s)$ must be cyclic. We are now free to use our knowledge of the classification of reflection spaces to find a classification of the reflection lattices.

We would like to express an arbitrary root system for a reflection space V in terms of a fixed one.

Lemma 2.2.10. *Let V be a finite-dimensional reflection space with fixed root system Φ and set of reflections Γ. If Φ' is another root system with reflection set Γ, then Φ' must take the form $\Phi' = \{c_\alpha\alpha | \alpha \in \Phi\}, c_\alpha \in \mathbf{Q}^+$. The decomposition of Φ into irreducible root systems corresponds bijectively to the decomposition of Φ'. The corresponding decompositions of W and V are also the same for Φ and Φ'.*

Proof: Let $\beta \in \Phi'$. Then $s_\beta = s_\alpha$ for some $\alpha \in \Phi$ so that $\beta = c_\alpha\alpha$ for some $c_\alpha \in \mathbf{Q}^\times$. Since $\mathbf{Q}\beta \cap \Phi' = \{\pm\beta\}$ and $\mathbf{Q}\alpha \cap \Phi = \{\pm\alpha\}$, we may assume that $c_\alpha \in \mathbf{Q}^+$. Thus Φ' is as required.

Note that $\Phi' = \{c_\alpha\alpha | \alpha \in \Phi\}$ for an arbitrary choice of positive constants satisfies the first four axioms of Lemma 2.2.6. We will later determine the conditions on c_α so that Φ' satisfies the last axiom.

Our fixed root system Φ can be decomposed into a disjoint union of irreducible root systems:

$$\Phi = \Phi_1 \dot\cup \cdots \dot\cup \Phi_k$$

Take another root system Φ' with reflection set Γ. Then $\Phi' = \{c_\alpha\alpha | \alpha \in \Phi\}, c_\alpha \in \mathbf{Q}^+$. Its decomposition into irreducible root systems corresponds to that of the decomposition of Φ. That is

$$\Phi' = \Phi'_1 \cup \cdots \cup \Phi'_k$$

where $\Phi'_i = \{c_\alpha \alpha | \alpha \in \Phi_i\}$ since $\langle \alpha, \beta \rangle = 0$ if and only if $\langle c_\alpha \alpha, c_\beta \beta \rangle = 0$.

The decomposition of Φ into irreducible components gives a corresponding decomposition of the Weyl group

$$W = W_1 \times \cdots \times W_k$$

where W_i is the Weyl group of Φ_i and of the reflection space V:

$$V = V_1 \oplus \cdots \oplus V_k$$

where $V_i = \mathrm{Span}_{\mathbf{Q}}(\Phi_i)$ and the reflection set $\Gamma = \cup_{i=1}^k \Gamma_i$ where $\Gamma_i = \{s_\alpha | \alpha \in \Phi_i\}$ such that V_i is a $\mathbf{Q}W_i$ space with root system Φ_i and reflection set Γ_i. Now the correspondence of the decomposition of an arbitrary root system Φ' with reflection set Γ to that of Φ gives the same decomposition of W, V and Γ, proving the lemma. ∎

Notation: From this point on, we will fix an n-dimensional reflection space V, a crystallographic root system Φ, its reflection set Γ, its decomposition into irreducible root systems:

$$\Phi = \Phi_1 \cup \cdots \cup \Phi_k$$

and the corresponding decompositions of $V = V_1 \oplus \cdots \oplus V_k$ and $W = W_1 \times \cdots \times W_k$.

Lemma 2.2.11. $\mathrm{End}_{\mathbf{Q}W}(V) \cong \prod_{i=1}^k \mathbf{Q}$. *More precisely, a* $\mathbf{Q}W$*-automorphism of* V *is given by* $f|_{V_i} =$ *multiplication by* q_i *for some* $q_i \in \mathbf{Q}^\times$. *In particular, if* Φ *is an irreducible root system then* $\mathrm{Span}_{\mathbf{Q}}(\Phi)$ *is an absolutely irreducible* $\mathbf{Q}W$ *module.*

Proof: Each V_i is a $\mathbf{Q}W_i$ module. Since V_i is fixed pointwise by W_j for all $j \neq i$, and $W = W_1 \times \cdots \times W_k$, V_i is also a $\mathbf{Q}W$- module.

Let $f \in \mathrm{End}_{\mathbf{Q}W}(V)$. Since $\cap_{j \neq i} V^{W_i} = V_i$, we see that $f(V_i) \subset V_i$. So

$$\mathrm{End}_{\mathbf{Q}W}(V) \cong \prod_{i=1}^k \mathrm{End}_{\mathbf{Q}W}(V_i) \cong \prod_{i=1}^k \mathrm{End}_{\mathbf{Q}W_i}(V_i)$$

It now suffices to consider a $\mathbf{Q}W$-module V with irreducible root system Φ and to show that $\mathrm{End}_{\mathbf{Q}W}(V) \cong \mathbf{Q}$.

Let Φ be an arbitrary irreducible root system for V. Let $f \in \mathrm{Aut}_{\mathbf{Q}W}(V)$. Let $\alpha \in \Phi$. Then $s_\alpha f(\alpha) = f(s_\alpha \alpha) = -f(\alpha)$ so that $f(\alpha) \in \mathrm{Ker}_V(s_\alpha + 1) = \mathbf{Q}\alpha$. So $f(\alpha) = q_\alpha \alpha$ for some $q_\alpha \in \mathbf{Q}$. Moreover $\alpha, \beta \in \Phi$ imply

$$f(s_\beta(\alpha)) = f(\alpha - \langle \alpha, \beta \rangle \beta) = q_\alpha \alpha - \langle \alpha, \beta \rangle q_\beta \beta$$

$$s_\beta f(\alpha) = s_\beta(q_\alpha \alpha) = q_\alpha \alpha - \langle \alpha, \beta \rangle q_\alpha \beta$$

Since $f(s_\beta(\alpha)) = s_\beta f(\alpha)$, we see that $\langle \alpha, \beta \rangle q_\beta = \langle \alpha, \beta \rangle q_\alpha$. In particular, $\langle \alpha, \beta \rangle \neq 0$ implies that $q_\alpha = q_\beta$. Note that $\langle \alpha, \beta \rangle \neq 0 \Leftrightarrow \langle \beta, \alpha \rangle \neq 0$ by the description of $\langle \cdot, \cdot \rangle$ in terms of a positive definite symmetric bilinear form. So the relation $\alpha \sim \beta \Leftrightarrow$ there exists $\alpha_0, \ldots, \alpha_r$ so that $\alpha = \alpha_0, \alpha_r = \beta$ and $\langle \alpha_i, \alpha_{i+1} \rangle \neq 0$ for all i, is an equivalence relation whose equivalence classes are the connected components of the Coxeter graph of Φ [20, pp 56–57]. Thus Φ irreducible implies that all roots are equivalent and hence $q_\alpha = q$ for all $\alpha \in \Phi$. ■

Lemma 2.2.12. *[20, pp 44–45,53] Let α, β be roots in a crystallographic root system Φ. Define $p_{\alpha\beta} \equiv \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$. Then*
(a) $\alpha \neq \pm\beta$ implies that $p_{\alpha\beta} = 0, 1, 2$ or 3.
(b) $p_{\alpha\beta} = 1$ implies $W\alpha = W\beta$.
(c) $p_{\alpha\beta} \neq 0$ and $(\alpha, \alpha) = (\beta, \beta)$ implies that $p_{\alpha\beta} = 1$.
(d) If Φ is irreducible, then there are at most two W-orbits in Φ.

Proof: Choose a positive definite W-invariant symmetric bilinear form $(\cdot, \cdot)$ on V. Then, as before $\langle v, \alpha \rangle = 2(v, \alpha)/(\alpha, \alpha)$ from which trivially follows:
(i) $\langle \alpha, \beta \rangle = 0 \Leftrightarrow \langle \beta, \alpha \rangle = 0$
(ii) $p_{\alpha\beta} \neq 0$ implies $\langle \beta, \alpha \rangle / \langle \alpha, \beta \rangle = (\beta, \beta)/(\alpha, \alpha)$ and, in particular, $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ have the same sign.
(iii) If $\alpha, \beta \in \Phi$ irreducible then there exists $\beta' \in W\beta$ with $p_{\alpha\beta'} \neq 0$.

For (iii), note that $\langle w\beta, \alpha \rangle = 0$ for all $w \in W$ implies that $(w\beta, \alpha) = 0$ for all $w \in W$ implies by Lemma 2.2.11 that $(V, \alpha) = 0$ which implies that $\alpha = 0$. By contradiction, (iii) follows.

Moreover, expressing $p_{\alpha\beta}$ in terms of $(\cdot, \cdot)$ gives

$$p_{\alpha\beta} = 4(\alpha, \beta)^2 / (\alpha, \alpha)(\beta, \beta)$$

So (a) follows from the Cauchy Schwarz inequality and $p_{\alpha\beta} \in \mathbf{Z}$.

For (b), factoring $p_{\alpha\beta} = 1$ in $\mathbf{Z}$, in view of (ii), gives $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = \pm 1$. But then, $(s_\alpha s_\beta)\alpha = s_\alpha(\alpha - \langle\alpha,\beta\rangle\beta) = -\alpha - \langle\alpha,\beta\rangle(\beta - \langle\beta,\alpha\rangle\alpha) = -\langle\alpha,\beta\rangle\beta = \pm\beta$ so that either $(s_\alpha s_\beta)\alpha = \beta$ or $s_\beta s_\alpha s_\beta \alpha = \beta$.

For (c), note that (ii) gives $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle$. Looking at factorizations of $\langle\alpha,\beta\rangle = 1, 2, 3$, it follows that $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = \pm 1$ and so $p_{\alpha\beta} = 1$.

Finally for (d), suppose that $\alpha, \beta, \gamma \in \Phi$ are in three different W-orbits. The ratio $(\beta,\beta)/(\alpha,\alpha)$ is constant on W-orbits $W\beta$, $W\alpha$. By (iii), we can arrange that $p_{\alpha\beta} \neq 0$ and then by (ii), $(\beta,\beta)/(\alpha,\alpha) = \langle\beta,\alpha\rangle/\langle\alpha,\beta\rangle$. Now by (a) and (b), $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = p_{\alpha\beta} = 2$ or 3. Hence, looking at factorizations, $(\beta,\beta)/(\alpha,\alpha) \in \{2, 3, 1/2, 1/3\} = S$ in view of (ii). For the same reason. $(\gamma,\gamma)/(\beta,\beta)$ and $(\gamma,\gamma)/(\alpha,\alpha)$ are in S so

$$\frac{(\gamma,\gamma)}{(\alpha,\alpha)} = \frac{(\beta,\beta)}{(\alpha,\alpha)} \cdot \frac{(\gamma,\gamma)}{(\beta,\beta)}$$

means that two elements of S have product in S. But this is impossible.

Note that the choice of $(\cdot,\cdot)$ therefore does not affect the ratios of root lengths. ■

Definition 2.2.13. The dual root system $\Phi^\vee$ for a root system Φ is defined as:

$$\Phi^\vee = \{\frac{2\alpha}{(\alpha,\alpha)} | \alpha \in \Phi\}$$

for some choice of $(\cdot,\cdot)$. This satisfies the axioms of Lemma 2.2.6 with the same reflection set Γ on noting for the last axiom that $\langle\alpha^\vee, \beta^\vee\rangle = \langle\beta,\alpha\rangle$.

Lemma 2.2.14. *The* $\mathrm{Aut}_{\mathbf{Q}W}(V)$ *orbits of crystallographic root systems with fixed reflection set* Γ *for a* $\mathbf{Q}W$*-space* V *are in bijection with the* W*-orbits of roots in a fixed root system* Φ*. More precisely, a set of representatives of the* $\mathrm{Aut}_{\mathbf{Q}W}(V)$ *orbits of root systems for a* $\mathbf{Q}W$*-space* V *with fixed root system* $\Phi = \cup_{i=1}^k \Phi_i$ *is obtained by taking* $\Phi' = \cup_{i=1}^k \Phi_i'$ *where*

$$\Phi_i' = \begin{cases} \Phi_i & \text{if } \Phi_i \text{ has one } W\text{-orbit of roots} \\ \Phi_i \text{ or } \Phi_i^\vee & \text{if } \Phi_i \text{ has two } W\text{-orbits of roots} \end{cases}$$

Proof: Let Φ be a fixed root system for V and let $\Phi' = \{c_\alpha\alpha | \alpha \in \Phi\}$, $c_\alpha \in \mathbf{Q}^+$ be any other root system. Since, by Lemma 2.2.10, the decomposition of Φ' is compatible with the decomposition of Φ, we may reduce to the case when Φ is irreducible and must show for some constant c, $\Phi' = c\Phi$ if Φ has one root length and $\Phi' = c\Phi$ or $c\Phi^\vee$ otherwise where $\Phi^\vee$ is formed with respect to some form $(\cdot,\cdot)$ on V . Note that the $\mathrm{Aut}_W(V)$-orbit of $\Phi^\vee$ is not affected by the choice of form and that W is the Weyl group of Φ and $\Phi^\vee$.

If $\alpha \in \Phi$ and $w \in W$, then

$$w(c_\alpha \alpha) = c_\alpha w\alpha \in \mathbf{Q}c_{w\alpha} w\alpha \cap \Phi'$$

implies that $c_{w\alpha} = c_\alpha$ for all $w \in W$. In particular, if Φ has only one W orbit then $c_\alpha = c$ for all $\alpha \in \Phi$ and $\Phi' = c\Phi$.

Suppose that Φ has two W-orbits of roots. Note that

$$\langle \beta', \alpha' \rangle' c_\alpha \alpha = \langle \beta', \alpha' \rangle' \alpha' = \beta' - s_{\alpha'}\beta' = c_\beta(\beta - s_\alpha \beta) = c_\beta \langle \beta, \alpha \rangle \alpha$$

hence

$$\langle \beta', \alpha' \rangle' = \frac{c_\beta}{c_\alpha} \langle \beta, \alpha \rangle \tag{2.2.15}$$

and therefore that $p'_{\alpha'\beta'} = p_{\alpha\beta}$. Note that $\langle \beta', \alpha' \rangle'$ and $\langle \beta, \alpha \rangle$ are of the same sign since $c_\alpha, c_\beta \in \mathbf{Q}^+$. By Lemma 2.2.12(iii), we may choose $\alpha, \beta \in \Phi$ with $p_{\alpha\beta} \neq 0$ in different W-orbits. Then $p_{\alpha\beta} = 2$ or 3 by Lemma 2.2.12(a), (b).

Comparing factorizations of $p'_{\alpha'\beta'} = p_{\alpha\beta}$, we find that we have two cases to consider:

Case 1: $\langle \beta', \alpha' \rangle' = \langle \beta, \alpha \rangle$
Then by (2.2.15), $c_\beta = c_\alpha$ so that $c_\alpha = c_\beta \equiv c$ on all of Φ and $\Phi' = c\Phi$.

Case 2: $\langle \beta', \alpha' \rangle' = \langle \alpha, \beta \rangle$.
Then by Lemma 2.2.12(ii) and (2.2.15), we have

$$\frac{c_\beta}{c_\alpha} = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \alpha \rangle} = \frac{(\alpha, \alpha)}{(\beta, \beta)}$$

and so $\frac{1}{2}c_\alpha(\alpha, \alpha) = \frac{1}{2}c_\beta(\beta, \beta)$ is a constant c on all Φ and so for all $\alpha \in \Phi$, we get

$$c\alpha^\vee = \frac{1}{2}c_\alpha(\alpha, \alpha)\frac{2\alpha}{(\alpha, \alpha)} = c_\alpha \alpha = \alpha'$$

Hence $\Phi' = c\Phi^\vee$ as required. ∎

Lemma 2.2.16. *(a) W acts trivially on $\Lambda(\Phi)/\mathbf{Z}\Phi$.*
(b) Any $\mathbf{Z}$-lattice A lying between $\mathbf{Z}\Phi$ and $\Lambda(\Phi)$ is a $\mathbf{Z}W$-lattice with $\mathbf{Q}\otimes_{\mathbf{Z}}A \cong V$.
(c) There are only a finite number of such lattices.

Proof: $\Lambda(\Phi)/\mathbf{Z}\Phi$ is a W-trivial module since for all $\alpha \in \Phi$ and $\lambda \in \Lambda(\Phi)$,

$$s_\alpha\lambda - \lambda = -\langle\lambda, \alpha\rangle\alpha \in \mathbf{Z}\alpha \subset \mathbf{Z}\Phi$$

and $\langle s_\alpha | \alpha \in \Phi\rangle = W$.

So if $\mathbf{Z}\Phi \subset A \subset \Lambda(\Phi)$, $A/\mathbf{Z}\Phi$ is also W-trivial so that W stabilizes A. Note that $\mathbf{Z}\Phi \subset A \subset \Lambda(\Phi)$ are lattices of the same rank $n = \dim_{\mathbf{Q}}(V)$. So

$$V = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}\Phi = \mathbf{Q} \otimes_{\mathbf{Z}} A = \mathbf{Q} \otimes_{\mathbf{Z}} \Lambda(\Phi)$$

as required. The last statement follows immediately from the fact that $\Lambda(\Phi)/\mathbf{Z}\Phi$ is a finite group. ∎

Definition 2.2.17. Let Φ be any fixed root system in V with weight lattice Λ. A *forbidden* subgroup of $\Lambda(\Phi)/\mathbf{Z}\Phi$ is one of the form $\Lambda(\Phi) \cap \mathbf{Q}\alpha/\mathbf{Z}\Phi$ for $\alpha \in \Phi$. An *admissible* subgroup of $\Lambda(\Phi)/\mathbf{Z}\Phi$ is one which meets every forbidden subgroup trivially.

Remark 2.2.18. Let A be a $\mathbf{Z}W$ lattice with $\mathbf{Z}\Phi \subset A \subset \Lambda(\Phi)$. Since

$$\mathbf{Q}\alpha \cap A = \mathrm{Ker}_V(s_\alpha + 1) \cap A = \mathrm{Ker}_A(s_\alpha + 1),$$

it follows that $A/\mathbf{Z}\Phi$ is an admissible subgroup of $\Lambda/\mathbf{Z}\Phi$ iff $\mathbf{Q}\alpha \cap A = \mathbf{Z}\alpha$ for all $\alpha \in \Phi$ iff $\Phi_A = \Phi$. Note that, for any $\alpha \in \Phi$, $\mathbf{Q}\alpha \cap \mathbf{Z}\Phi = \mathbf{Z}\alpha$ so that the trivial subgroup of $\Lambda(\Phi)/\mathbf{Z}\Phi$ is always admissible.

We define the following subgroup of $\mathrm{Aut}_{\mathbf{Q}W}(V)$:

$$\Sigma = \{\sigma \in \mathrm{Aut}_{\mathbf{Q}W}(V) | \sigma|_{V_i} = \text{multiplication by } \epsilon_i, \epsilon_i \in \{\pm 1\}\}$$

Lemma 2.2.19. *Σ induces an action on $\Lambda(\Phi)/\mathbf{Z}\Phi$ which stabilizes the set of admissible subgroups.*

Proof: The decomposition of $\Phi = \cup_{i=1}^k \Phi_i$ leads to a decomposition of $\mathbf{Z}\Phi = \oplus_{i=1}^k \mathbf{Z}\Phi_i$ and of $\Lambda(\Phi) = \oplus_{i=1}^k \Lambda(\Phi_i)$. Let $\sigma \in \Sigma$ where $\sigma|_{V_i}$ = multiplication by ϵ_i. Then

$$\sigma(\Lambda) = \epsilon_1(\Lambda(\Phi_1)) \oplus \cdots \oplus \epsilon_k(\Lambda(\Phi_k)) = \Lambda(\Phi_1) \oplus \cdots \oplus \Lambda(\Phi_k) = \Lambda(\Phi).$$

Similarly $\sigma(\mathbf{Z}\Phi) = \mathbf{Z}\Phi$. So $\sigma(\sum_{i=1}^k \lambda_i + \mathbf{Z}\Phi) = \sum_{i=1}^k \epsilon_i\lambda_i + \mathbf{Z}\Phi$, where $\lambda_i \in \Lambda_i$, defines the action of Σ on $\Lambda/\mathbf{Z}\Phi$. Now if $A/\mathbf{Z}\Phi$ were an admissible subgroup,

$$\sigma(A) \cap \mathbf{Q}\alpha = \sigma(A \cap \mathbf{Q}\alpha) = \sigma(\mathbf{Z}\alpha) = \mathbf{Z}\alpha$$

for $\sigma \in \Sigma$ and $\alpha \in \Phi$ shows that $\sigma(A)/\mathbf{Z}\Phi$ is also an admissible subgroup of $\Lambda(\Phi)/\mathbf{Z}\Phi$. ■

Theorem 2.2.20. *Given a reflection space V with Weyl group W, the isomorphism class of a $\mathbf{Z}W$-lattice A on V is completely determined by the $\mathrm{Aut}_{\mathbf{Q}W}(V)$-orbit of the root system Φ_A and the Σ-orbit of admissible subgroups of $\Lambda(\Phi_A)/\mathbf{Z}\Phi_A$ determined by $A/\mathbf{Z}\Phi_A$.*

Proof: The reflection set Γ is the set of all reflections on V in W, by [21, p.24]

Suppose A and B are $\mathbf{Z}W$-lattices on V such that $f(\Phi_A) = \Phi_B$ for some $f \in \mathrm{Aut}_{\mathbf{Q}W}(V)$ and where $\sigma f(A)/\mathbf{Z}\Phi_B = B/\mathbf{Z}\Phi_B$ for some $\sigma \in \Sigma$. Note that $A/\mathbf{Z}\Phi_A$ and $B/\mathbf{Z}\Phi_B$ are both admissible and $\sigma \circ f : A \to B$ is a $\mathbf{Z}W$ isomorphism.

Conversely, let f be a $\mathbf{Z}W$-isomorphism mapping B onto A. Then f extends linearly to a $\mathbf{Q}W$-automorphism of V. So f maps $\mathrm{Ker}_B(s+1)$ isomorphically onto $\mathrm{Ker}_A(s+1)$ for each reflection s in W. This shows that $f(\Phi_B) = \Phi_A$. Thus $f \in \Sigma$ takes $B/\mathbf{Z}\Phi_B$ to $A/\mathbf{Z}\Phi_A$. ■

We will now apply this theorem to find the number of isomorphism classes of reflection lattices on an irreducible reflection space. We first recall the following:

Lemma 2.2.21. *[20, pp. 68,71]*

1. *If $\Phi = A_n$, then $\Lambda(\Phi)/\Phi \cong \mathbf{Z}/(n+1)\mathbf{Z}$.*
2. *If $\Phi = B_n, C_n, E_7$, then $\Lambda(\Phi)/\Phi \cong \mathbf{Z}/2\mathbf{Z}$.*

3. *If* $\Phi = D_{2n+1}$*, then* $\Lambda(\Phi)/\Phi \cong \mathbf{Z}/4\mathbf{Z}$.

4. *If* $\Phi = D_{2n}$*, then* $\Lambda(\Phi)/\Phi \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

5. *If* $\Phi = E_6$*, then* $\Lambda(\Phi)/\Phi \cong \mathbf{Z}/3\mathbf{Z}$.

6. *If* $\Phi = E_8, F_4, G_2$*, then* $\Lambda(\Phi)/\Phi = 1$.

Proposition 2.2.22. *Let* $c(\Phi)$ *denote the number of isomorphism classes of* $\mathbf{Z}W(\Phi)$ *lattices on the irreducible reflection space* $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}\Phi$*. Then*

- *For* $n \geq 2$*,* $c(A_n)$ *is the number of divisors of* $n+1$.
- $c(A_1) = c(E_8) = 1$.
- $c(B_2) = 2$.
- $c(B_n) = c(C_n) = 3, n > 2$.
- $c(D_{2n}) = 5$.
- $c(D_{2n+1}) = 3$.
- $c(E_6) = c(E_7) = 2$.
- $c(F_4) = c(G_2) = 2$.

Proof: In the irreducible case, we may ignore the Σ-orbit as for any $\mathbf{Z}W$-lattice A on V and any $\sigma \in \Sigma$, $\sigma(A) = \pm A = A$.

Since each root β of Φ is conjugate to a simple root α, we have $\Lambda(\Phi) \cap \mathbf{Q}\beta = w(\Lambda(\Phi) \cap \mathbf{Q}\alpha)$ where $w \in W$ is such that $w\alpha = \beta$. So to find the non-trivial forbidden subgroups it suffices to examine $\Lambda(\Phi) \cap \mathbf{Q}\alpha$ for simple roots α. Then $\frac{1}{n}\alpha \in \Lambda$ iff $\langle \frac{1}{n}\alpha, \beta\rangle \in \mathbf{Z}$ for all β simple iff n divides

$$n(\alpha) = \gcd\{\langle \alpha, \beta\rangle | \beta \text{ simple}\}$$

But since $\langle \alpha, \beta\rangle$ is the α, β entry of the Cartan matrix for Φ, we can examine the list of Cartan matrices ([20, p. 59]) to see that a forbidden subgroup occurs only for types A_1, B_2, C_n and that in each case, there is a unique forbidden subgroup of order 2.

For $\Phi = A_1$, there is only 1 root length and hence only 1 $\mathrm{Aut}_W(V)$-orbit of root system by Lemma 2.2.14. Moreover, $\Lambda(\Phi)/\mathbf{Z}\Phi \cong \mathbf{Z}/2\mathbf{Z}$ is not admissible. So the only isomorphism class is represented by $\mathbf{Z}\Phi$.

For $\Phi = A_n (n \geq 2), D_n, E_6, E_7, E_8$, there is only 1 root length and hence only 1 $\mathrm{Aut}_W(V)$-orbit of root system by Lemma 2.2.14. Since $\Lambda(\Phi)/\mathbf{Z}\Phi$ and hence all its subgroups are admissible, the isomorphism classes are in bijective correspondence with the subgroups of $\Lambda(\Phi)/\mathbf{Z}\Phi$. The result follows from an examination of the previous lemma.

Note that $W(B_n) \cong W(C_n)$ and that $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}B_n = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}C_n$. It then suffices to consider $\Phi = B_n$. There are two $\mathrm{Aut}_W(V)$-orbits of root systems, Φ and $\Phi^\vee = C_n$. For $n = 2$, $\Lambda(B_2)/\mathbf{Z}B_2$ and $\Lambda(C_2)/\mathbf{Z}C_2$ are both not admissible. So the representatives of the isomorphism classes are $\mathbf{Z}B_2$ and $\mathbf{Z}C_2$. For $n > 2$, $\Lambda(C_n)/\mathbf{Z}B_n \cong \mathbf{Z}/2\mathbf{Z}$ is not admissible but $\Lambda(B_n)/\mathbf{Z}B_n \cong \mathbf{Z}/2\mathbf{Z}$ is admissible. So the representatives of the 3 isomorphism classes are $\mathbf{Z}B_n, \mathbf{Z}C_n$, and $\Lambda(B_n)$.

For $\Phi = F_4$ or G_2, $\Lambda(\Phi)/\mathbf{Z}\Phi$ is trivial but there are two $\mathrm{Aut}_W(V)$-orbits of root systems. In each case, the representatives of the isomorphism classes are $\mathbf{Z}\Phi$ and $\mathbf{Z}\Phi^\vee$. ∎

We now wish to use our knowledge about the structure of reflection lattices to determine their first cohomology groups.

The following is an alternate proof of [24, 2.4]:

Lemma 2.2.23. *If $\Gamma_d = \{d_1, \ldots, d_r\}$ is a set of distinct diagonalizable reflections on A then the group D generated by Γ_d is an elementary abelian 2-group of rank r and*

$$H^1(D, A) \to \oplus_{i=1}^r H^1(\langle d_i \rangle, A)$$

is an isomorphism.

Proof: Since for each $i = 1, \ldots, r$, $A = \mathrm{Ker}_A(d_i + 1) \oplus A^{(d_i)}$, each d_i acts trivially on $A/2A$, hence so does D as it is generated by Γ_d. It follows that $d \in D$ implies that $d^2 = 1$ (hence D is an elementary abelian 2-group): for $d = 1 + 2x$ with $x \in E = \mathrm{End}_{\mathbf{Z}}(A)$ implies that $d^2 \in 1 + 4E$ so it suffices to prove that $1 + 4E$ contains no non-trivial element of finite order. If this is false, then there exists $m \geq 2, x \in E, x \notin 2E$ such that $1 + 2^m x$ has prime order p. Then $(1 + 2^m x)^p = 1$ shows that $\sum_{k=0}^p \binom{p}{k} (2^m x)^k = 1$ so that $x \in 2E$ which gives a contradiction.

Each character $\chi \in \mathrm{Hom}(D, \mathbf{Z}^\times)$ has an idempotent factorization $e_\chi = \prod_{i=1}^{r} \frac{1}{2}(1 + \chi(d_i)d_i)$. Note that $(1 \pm d_i)(A) \subset 2A$ since d_i acts trivially on $A/2A$. So the Wedderburn decomposition of $\mathbf{Q} \otimes_{\mathbf{Z}} A$ occurs on the $\mathbf{Z}$ level. That is e_χ acts on A and $A = \oplus_\chi e_\chi A$. Let $X = \{\chi \neq 1_D | e_\chi A \neq 0\}$. Note that $d \in D$ acts on $e_\chi A$ as $de_\chi = \chi(d)e_\chi$ and $\mathrm{Ker}_A(d+1) = \oplus_{\chi \in X, \chi(d)=-1} e_\chi A$. Since d_i is a reflection, there exists a unique $\chi_i \in X$ such that $\chi_i(d_i) \neq 1$ (ie with $\chi_i(d_i) = -1$). The χ_j are distinct since the d_i are and each $e_{\chi_i} A = \mathrm{Ker}_A(d_i + 1)$ has rank 1 since d_i is a reflection. It follows that D is a rank r elementary abelian 2-group.

Finallly $X = \{\chi_1, \ldots, \chi_r\}$: for if $\chi \notin X$, then $\chi \neq \chi_i$ implies that $\chi(d_i) = 1$ for all i so that $\chi = 1_D$. So $A = A^D \oplus \oplus_{i=1}^{r} e_{\chi_i} A$ where each $e_{\chi_i} A$ is a $\mathbf{Z}D$ lattice of $\mathbf{Z}$ rank 1 on which D acts with character χ_i. Since $H^1(D, A^D) = 0$, we see that $H^1(D, A) = \oplus_{i=1}^{r} H^1(D, e_{\chi_i} A)$. Since d_i acts trivially on $e_{\chi_j} A$ for $j \neq i$, we also see that $H^1(\langle d_i \rangle, A) = H^1(\langle d_i \rangle, e_{\chi_i} A)$ for all i. Note that $D/\langle d_i \rangle$ acts trivially on $H^1(\langle d_i \rangle, e_{\chi_i} A) \cong \mathbf{Z}/2\mathbf{Z}$ since $D/\langle d_i \rangle$ centralizes $\langle d_i \rangle$ and acts trivially on $e_{\chi_i} A$. Since $(e_{\chi_i} A)^{\langle d_i \rangle} = 0$ for all i, we see by the 5-term exact sequence that $\mathrm{Res}^D_{\langle d_i \rangle} : H^1(D, e_{\chi_i} A) \to H^1(\langle d_i \rangle, e_{\chi_i} A)$ is an isomorphism. Hence so is the map $\oplus_{i=1}^{r} \mathrm{Res}^D_{\langle d_i \rangle}$:

$$\begin{array}{ccc} H^1(D, A) & \longrightarrow & \oplus_{i=1}^{r} H^1(\langle d_i \rangle, A) \\ \downarrow \cong & & \downarrow \cong \\ \oplus_{i=1}^{r} H^1(\langle d_i \rangle, e_{\chi_i} A) & \xrightarrow{\cong} & \oplus_{i=1}^{r} H^1(\langle d_i \rangle, e_{\chi_i} A) \end{array}$$

■

Recall the following lemma due to Lorenz [24]:

Lemma 2.2.24. *Let G be a finite group acting on a lattice A. Let Γ be the set of reflections in G acting on A and let Γ_d be the subset of diagonalizable reflections. Let N be the (normal) subgroup of G generated by Γ and D be the (normal) subgroup of G generated by Γ_d. Then*

$$\mathrm{Ker}(\oplus_{s \in \Gamma} \mathrm{Res}^G_{\langle s \rangle} : H^1(G, A) \to \oplus_{s \in \Gamma} H^1(\langle s \rangle, A)) \cong H^1(G/D, A^D)$$

Proof:

$$\begin{aligned}
\mathrm{Ker}(\oplus_{s\in\Gamma}\mathrm{Res}^G_{\langle s\rangle}) &= \cap_{s\in\Gamma}\mathrm{Ker}(\mathrm{Res}^G_{\langle s\rangle}) \\
&= \cap_{s\in\Gamma_d}\mathrm{Ker}(\mathrm{Res}^G_{\langle s\rangle}) && \text{since } H^1(\langle s\rangle, A) = 0 \text{ if } s \notin \Gamma_d \\
&= \mathrm{Ker}(\oplus_{s\in\Gamma_d}\mathrm{Res}^G_{\langle s\rangle}) \\
&= \mathrm{Ker}[(\oplus_{s\in\Gamma_d}\mathrm{Res}^D_{\langle s\rangle}) \circ \mathrm{Res}^G_D] \\
&= \mathrm{Ker}(\mathrm{Res}^G_D) && \oplus_{s\in\Gamma_d}\mathrm{Res}^D_{\langle s\rangle} \text{ is an isomorphism by Lemma 2.2.23} \\
&\cong H^1(G/D, A^D) && \text{by inflation-restriction sequence}
\end{aligned}$$

Proposition 2.2.25. *Let A be a reflection lattice on a reflection space V with reflection set Γ generating W and set $\Phi = \Phi_A, \Lambda = \Lambda(\Phi_A)$. Then*

$$H^1(W, A) \cong \Lambda/A$$

Proof: Since $V^W = 0$, we note that $\Lambda^W = 0$. Also since $\Lambda/\mathbf{Z}\Phi$ is W-trivial and $\mathbf{Z}\Phi \subset A \subset \Lambda$, we see that Λ/A is W-trivial.

Applying cohomology to the exact sequence

$$0 \to A \to \Lambda \to \Lambda/A \to 0,$$

we obtain the exact sequence

$$0 \to \Lambda/A \xrightarrow{\partial} H^1(W, A) \xrightarrow{i_*} H^1(W, \Lambda).$$

To show that ∂ is an isomorphism, it suffices to prove that i_* is the zero map. Now

$$\begin{array}{ccc}
H^1(W,A) & \xrightarrow{\quad i_* \quad} & H^1(W,\Lambda) \\
\downarrow{\scriptstyle \oplus\mathrm{Res}^W_{\langle s_\alpha\rangle}} & & \downarrow{\scriptstyle \oplus\mathrm{Res}^W_{\langle s_\alpha\rangle}} \\
\oplus_{\alpha\in\Phi}H^1(\langle s_\alpha\rangle, A) & \xrightarrow{\oplus i_*} & \oplus_{\alpha\in\Phi}H^1(\langle s_\alpha\rangle, \Lambda)
\end{array}$$

is a commutative diagram. By the previous lemma applied to W and Λ, we find that

$$\mathrm{Ker}(\oplus_{\alpha\in\Phi}\mathrm{Res}^W_{\langle s_\alpha\rangle}) : H^1(W,\Lambda) \to \oplus_{\alpha\in\Phi}H^1(\langle s_\alpha\rangle, \Lambda)) \cong H^1(W/D, \Lambda^D)$$

But by Proposition 1.0.3 [24] , this is isomorphic to the divisor class group $\mathrm{Cl}(\mathbf{C}[\Lambda]^W)$. Since $\mathbf{C}[\Lambda]^W$ is a polynomial ring [6, Théorème 3, p. 188], we find that $\mathrm{Ker}(\oplus_{\alpha\in\Phi}\mathrm{Res}^W_{\langle s_\alpha\rangle}) = 0$. Thus, it suffices to show that for each $\alpha \in \Phi$, $i_* : H^1(\langle s_\alpha\rangle, A) \to H^1(\langle s_\alpha\rangle, \Lambda)$ is the zero map since then $i_*(H^1(W, A)) \subset \mathrm{Ker}(\oplus_{\alpha\in\Phi}\mathrm{Res}^W_{\langle s_\alpha\rangle}) = 0$. Viewing this map in H^{-1}, i_* is the map

$$\mathrm{Ker}_A(s_\alpha + 1)/\mathrm{Im}_A(s_\alpha - 1) \to \mathrm{Ker}_\Lambda(s_\alpha + 1)/\mathrm{Im}_\Lambda(s_\alpha - 1)$$

induced by the inclusion map

$$\mathbf{Z}\alpha = \mathrm{Ker}_A(s_\alpha + 1) \hookrightarrow \mathrm{Ker}_\Lambda(s_\alpha + 1)$$

To finish the proof, we need only show that $\mathrm{Im}_\Lambda(s_\alpha - 1) = \mathbf{Z}\alpha$.

By [21, p. 11], we may extend α to a base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ for the root system Φ with $\alpha_1 = \alpha$. Take $\{\omega_1, \ldots, \omega_n\}$ to be the $\mathbf{Z}$-basis for Λ of fundamental dominant weights corresponding to the base chosen for the root system [20, p. 67]. Then $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$. Since $(s_\alpha - 1)(\omega_1) = -\alpha$ and $(s_\alpha - 1)(\omega_j) = 0$ for $j \neq 1$, we see that $\mathrm{Im}_\Lambda(s_\alpha - 1) = \mathbf{Z}\alpha$ as required. ∎

2.3 Suitable Root Systems

This last preliminary section deals with non-effective reflection groups.

Let G be a finite group and let A be a lattice on which G acts faithfully. Set $V = \mathbf{Q}\otimes_{\mathbf{Z}} A$ and let Γ be a G-stable set of reflections on V (i.e. $g\Gamma g^{-1} = \Gamma$ for all $g \in G$.) Let R be the subgroup of G generated by Γ.

Definition 2.3.1.

$$e_R = \frac{1}{|R|}\sum_{r\in R} r \in \mathbf{Q}R \subset \mathbf{Q}G, \qquad \pi = 1 - e_R : V \to V$$

Lemma 2.3.2. *$V = \mathrm{Ker}_V(e_R) \oplus V^R$ is a $\mathbf{Q}G$ decomposition of V.*

Proof: For all $r \in R$, $re_R = e_R$. Hence e_R is an idempotent of $\mathbf{Q}R \subset \mathbf{Q}G$. Since R is a normal subgroup of G, we find that $ge_R = e_R g$ for all $g \in G$ and hence that e_R acts G-linearly on V. We obtain the following decomposition of V into $\mathbf{Q}G$ subspaces:

$$\begin{aligned} V &= \mathrm{Ker}_V(e_R) \oplus \mathrm{Im}_V(e_R) \\ \mathrm{Ker}_V(e_R) &= \mathrm{Im}_V(1 - e_R) \\ \mathrm{Im}_V(e_R) &= \mathrm{Ker}_V(1 - e_R) = V^R \end{aligned}$$

as required. ■

Definition 2.3.3. A crystallographic root system Φ for the $\mathbf{Q}$ space $\pi(V)$ with weight lattice $\Lambda(\Phi)$ is called *suitable* for the $\mathbf{Z}G$ lattice A on V and for Γ if
(i) $\Phi \subset A$
(ii) $\pi(A) \subset \Lambda(\Phi)$
(iii) R is isomorphic to the Weyl group $W(\Phi)$ under the natural map $R \to R|_{\mathbf{Q}\Phi}$
(iv) Φ is G stable.

Remark 2.3.4. If Φ is a suitable root system for A and Γ, then for $s \in \Gamma$, $\mathrm{Ker}_V(s+1) = \mathrm{Ker}_{\pi V}(s+1)$ is 1-dimensional so there exists $\alpha \in \Phi$ such that $\mathrm{Ker}_{\mathbf{Z}\Phi}(s+1) = \mathrm{Ker}_{\pi V}(s+1) \cap \mathbf{Z}\Phi = \mathbf{Z}\alpha$. We denote s by s_α and define $\langle \cdot, \alpha \rangle$ by $s_\alpha(v) = v - \langle v, \alpha \rangle \alpha$ for $v \in V$ and $\alpha \in \Phi$.

Lemma 2.3.5. *Let Φ be a suitable root system for A and Γ. Then*
(a) $V = \mathbf{Q}\Phi \oplus V^R$ is a $\mathbf{Q}G$ decomposition of V.
(b) For $g \in G$, $\alpha, \beta \in \Phi$, we have $gs_\alpha g^{-1} = s_{g\alpha}$ and $\langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle$.

Proof:
(a) Since Φ is a root system for $\pi(V) = \mathrm{Im}_V(1 - e_R) = \mathrm{Ker}_V(e_R)$, we see that $\mathbf{Q}\Phi = \mathrm{Ker}_V(e_R)$ and hence by Lemma 2.3.2, $V = \mathbf{Q}\Phi \oplus V^R$ is a $\mathbf{Q}G$ decomposition of V as required.
(b) For $g \in G, \alpha, \beta \in \Phi$, we have $s_\alpha \in \Gamma$, $gs_\alpha g^{-1} \in \Gamma$ and $g\alpha \in \Phi$. Since

$$\mathbf{Z}g\alpha = g\mathrm{Ker}_{\mathbf{Z}\Phi}(s_\alpha + 1) = \mathrm{Ker}_{\mathbf{Z}\Phi}(gs_\alpha g^{-1} + 1)$$

we see that $s_{g\alpha} = gs_\alpha g^{-1}$ Since

$$s_{g\beta}(g\alpha) = g\alpha - \langle g\alpha, g\beta \rangle g\beta$$
$$gs_\beta g^{-1}(g\alpha) = g\alpha - \langle \alpha, \beta \rangle g\beta$$

and $s_{g\beta} = gs_\beta g^{-1}$, we see that $\langle g\alpha, g\beta \rangle = \langle \alpha, \beta \rangle$ as required. ■

Fix a suitable crystallographic root system Φ for A and Γ. Then let Δ be a base for Φ and set

$$\Omega_G = \Omega_G(\Delta) = \{g \in G | g(\Delta) = \Delta\}$$

Lemma 2.3.6. *(a)* $G \cong R \rtimes \Omega_G$

(b) Let $H = \mathrm{Im}(G \to GL(\mathbf{Q}\Phi))$ *and let* $W = W(\Phi)$. *Then* $H \cong W \rtimes \Omega_H$ *is a subgroup of*

$$\mathrm{Aut}(\Phi) = \{g \in GL(\mathbf{Q}\Phi) | g\Phi = \Phi\}$$

Proof: (a) Let $\varphi : G \to GL(\mathbf{Q}\Phi)$. Note that R maps isomorphically to the Weyl group $W = W(\Phi)$ and that Ω_G maps surjectively onto $\Omega_{\varphi(G)}$. For each $g \in G$, $\varphi(g)(\Delta)$ is another base for the root system Φ. Since W acts simply transitively on the set of bases for the root system Φ [20, p. 51], we see that there exists a unique $w \in W$ with $w(\Delta) = \varphi(g)(\Delta)$ and $w = \varphi(r)$ for a unique $r \in R$. So $\varphi(g^{-1}r) \in \Omega_{\varphi(G)} = \varphi(\Omega_G)$ and since $\mathrm{Ker}(\varphi) \subset \Omega_G$, we have $g^{-1}r \in \Omega_G$ and hence $g \in R\Omega_G$. Let $r \in R \cap \Omega_G$. Then $\varphi(r) \in W \cap \Omega_{\varphi(G)} = 1$ so $r \in \mathrm{Ker}(\varphi) \cap R$. Since R acts faithfully on $\mathbf{Q}\Phi$, we see that $R \cap \Omega_G = 1$ as required.

(b) The isomorphism was proved in the course of the proof of (a). H is a subgroup of $\mathrm{Aut}(\Phi)$ since Φ is G-stable. ■

We need to show that suitable root systems for A and Γ exist. Let

$$\Phi_A = \{\alpha | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha \text{ for some } s \in \Gamma\}$$

Note that $\mathrm{Ker}_{\mathbf{Z}\Phi_A}(s+1) = \mathrm{Ker}_A(s+1)$ for all $s \in \Gamma$. The following Lemma is adapted from Farkas [16, Lemmas 1–3].

Lemma 2.3.7. Φ_A *is a suitable root system for* A *and* Γ.

Proof: Note that Φ_A is G-stable since Γ is G-stable and

$$\mathrm{Ker}_A(gs_\alpha g^{-1} + 1) = g\mathrm{Ker}_A(s_\alpha + 1) = \mathbf{Z}g\alpha$$

Applying e_R to $s_\alpha\alpha = -\alpha$, we get $e_R\alpha = -e_R\alpha$ and hence $e_R\alpha = 0$ so that $\mathbf{Q}\Phi_A \subset \mathrm{Ker}_V(e_R)$. Let $v \in \mathrm{Ker}_V(e_R)^R$. Then $rv = v$ for all $r \in R$ so that $v = e_Rv = 0$. We see $\mathrm{Ker}_V(e_R)$ is a $\mathbf{Q}R$ space containing $\mathbf{Q}\Phi_A$ with $\mathrm{Ker}_V(e_R)^R = 0$. By Lemma 2.2.6, this implies that

$$\mathrm{Ker}_V(e_R) = \mathbf{Q}\Phi_A$$

So Φ_A is a crystallographic root system for $\mathrm{Ker}_V(e_R)$. Thus R maps onto the Weyl group of Φ_A on $\pi(V) = \mathrm{Ker}_V(e_R)$ under the restriction map $R \to R|_{\pi(V)}$. Since R acts trivially on V^R and faithfully on V the decomposition $V = \mathbf{Q}\Phi_A \oplus V^R$ from Lemma 2.3.2 shows that the map is an isomorphism.

With our extended definition of $\langle \cdot, \alpha \rangle$ we find:

$$s_\alpha(e_R v) = e_R v - \langle e_R v, \alpha \rangle \alpha$$

which together with $s_\alpha e_R = e_R$ implies that $\langle e_R v, \alpha \rangle = 0$ and thus $\langle (1 - e_R)v, \alpha \rangle = \langle v, \alpha \rangle$ for all $v \in V$.

Now for $a \in A$, $\langle a, \alpha \rangle \alpha = (1 - s_\alpha)a \in \mathrm{Im}_A(1 - s_\alpha) \subset \mathrm{Ker}_A(s_\alpha + 1) = \mathbf{Z}\alpha$ implies $\langle a, \alpha \rangle \in \mathbf{Z}$. Thus $(1 - e_R)a \in \mathrm{Ker}(e_R) = \mathbf{Q}\Phi_A$ has $\langle (1 - e_R)a, \alpha \rangle = \langle a, \alpha \rangle \in \mathbf{Z}$ which proves $\pi(A) = (1 - e_R)A \subset \Lambda$. Since $\Phi_A \subset A$ by construction we see that Φ_A is a suitable root system for A. ■

Remark: For a suitable root system Φ for A and Γ, we have $\mathbf{Z}\Phi \subset \mathrm{Ker}_A(e_R) \subset \mathrm{Im}_A(1 - e_R) \subset \Lambda$ and clearly $\mathrm{Ker}_A(1 - e_R) \subset \mathrm{Im}_A(e_R)$. Moreover,

$$\mathrm{Ker}_A(e_R) \oplus \mathrm{Ker}_A(1 - e_R) \subset A \subset \mathrm{Im}_A(1 - e_R) \oplus \mathrm{Im}_A(e_R)$$

relative to $V = \mathbf{Q}\Phi \oplus V_R$. Acting on A by e_R respectively $1 - e_R$ gives exact sequences:

$$0 \to \mathrm{Ker}_A(e_R) \to A \to \mathrm{Im}_A(e_R) \to 0$$

$$0 \to \mathrm{Ker}_A(1 - e_R) \to A \to \mathrm{Im}_A(1 - e_R) \to 0$$

Chapter 3

Class Groups

3.1 Inertia Groups for $K_\gamma[A]/(K_\gamma[A])^G$

Lemma 3.1.1. *Let* $\mathfrak{p}$ *be a prime ideal of* $K_\gamma[A])$. *Let* $H = G^T(\mathfrak{p})$ *where*

$$G^T(\mathfrak{p}) = \{g \in G | gs - s \in \mathfrak{p}\}$$

is the inertia group of $\mathfrak{p}$ *and let* $B = I_H A$ *where* I_H *is the augmentation ideal of* H. *Then there is a unique group homomorphism* $\beta : B \to K^\times$ *so that*

$$e(b) \equiv \beta(b) \bmod \mathfrak{p}, b \in B$$

If $g \in G^T(\mathfrak{p})$, *then* g *acts trivially on the field* K *and* $\beta(ga - a) = \gamma_g(ga)^{-1}$

Proof: Since $g \in G^T(\mathfrak{p})$, then

$$\begin{aligned} g(ce(a)) - ce(a)) &\in \mathfrak{p} \text{ for all } a \in A, c \in K^\times \\ (gc)\gamma_g(ga)e(ga) &\equiv ce(a) \bmod \mathfrak{p} \\ e(ga - a) &\equiv \frac{c}{(gc)\gamma_g(ga)} \bmod \mathfrak{p} \end{aligned}$$

Given $b \in B$, suppose we have $e(b) \equiv k_1 \bmod \mathfrak{p}$ and $e(b) \equiv k_2 \bmod \mathfrak{p}$ where $k_1, k_2 \in K^\times$. Then if $k_1 \neq k_2$, $k_1 - k_2 \in (K_\gamma[A])^\times \cap \mathfrak{p}$ which is impossible. So, there is a unique element $\beta(b)$ of $K^\times$ such that $e(b) \equiv \beta(b) \bmod \mathfrak{p}$. It is easy to verify that β is a group homomorphism.

For any $c \in K^\times$ and $a \in A$, the above calculation shows that $\beta(ga-a) = \frac{c}{(gc)\gamma_g(ga)}$. But then $\frac{c}{(gc)\gamma_g(ga)} = \frac{1}{\gamma_g(ga)}$ so that $gc = c$ and $\beta(ga-a) = (\gamma_g(ga))^{-1}$ as required. ■

Definition: Let $H = G^T(\mathfrak{p})$, $B = I_H A$, and $\beta : B \mapsto K^\times$ be the group homomorphism defined in the previous lemma. Then we define J_β as the ideal of $K_\gamma[A]$ generated by the set $\{e(b) - \beta(b) \mid b \in B\}$. Note that $J_\beta \subset \mathfrak{p}$.

Lemma 3.1.2. *Let $\overline{K}$ denote an algebraic closure of K. Then*

$$\overline{K}[A]/\overline{K}J_\beta \cong \overline{K}[A/B]$$

Proof: Since $\overline{K}^\times$ is divisible as an abelian group, it is injective. So $\beta : B \mapsto K^\times$ may be extended to $\hat{\beta} : A \mapsto \overline{K}^\times$. Define

$$\begin{aligned} \rho : \overline{K}[A] &\to \overline{K}[A/B] \\ \sum_a c_a e(a) &\mapsto \sum_a c_a \hat{\beta}(a) e(a+B) \end{aligned}$$

It is clear that ρ is a ring epimorphism with $\overline{K}J_\beta \subset \mathrm{Ker}(\rho_{\hat{\beta}})$.

Conversely, let $\sum c_a e(a) \in \mathrm{Ker}(\rho_{\hat{\beta}})$. Then if T is a transversal for B in A, we find that

$$\sum_{b \in B} c_{a+b} \hat{\beta}(a+b) = 0 \text{ for all } a \in T$$

so that $\sum_b c_{a+b}\beta(b) = 0$ for all $a \in T$. Now

$$\begin{aligned} \sum_{a \in A} c_a e(a) &= \sum_{a \in T}\sum_{b \in B} c_{a+b} e(a+b) \\ &= \sum_{a \in T}\sum_{b \in B} c_{a+b}[e(b) - \beta(b)]e(a) + \sum_{a \in T}\sum_{b \in B} c_{a+b}\beta(b)e(a) \\ &= \sum_{a \in T}\sum_{b \in B} c_{a+b}[e(b) - \beta(b)]e(a) \\ &\in \overline{K}J_\beta \end{aligned}$$

as required. ■

Remark 3.1.3. Let $B^+ = \{a \in A \mid n \cdot a \in B \text{ for some } 0 \neq n \in \mathbf{Z}\}$. Then B^+ is a direct summand of A and B^+/B is the torsion part of A/B. Hence if $\beta : B \to K^\times$ could be extended to $\beta^+ : B^+ \to K^\times$, it can also be extended to $\hat{\beta} : A \mapsto K^\times$. In this case, we actually have $K[A]/J_{\beta^+} \cong K[A/B^+]$. In particular, if $B^+ = B$, we have $K[A]/J_\beta \cong K[A/B]$.

Definition 3.1.4. For a $\mathbf{Z}G$ lattice A, and a subgroup H,

$$A^H = \{a \in A | ha = a\}$$

is the $\mathbf{Z}$-sublattice of A fixed by H. The norm map is $N_H : A \to A, a \mapsto \sum_{h\in H} ha$. Note that N_H can be extended naturally to a $\mathbf{Q}$-linear map $N_H : V \to V$.

Now we can characterize the non-trivial inertia subgroups corresponding to height one primes:

Proposition 3.1.5. *The non-trivial inertia subgroups $G^T(\mathfrak{p})$ with $\mathfrak{p}$ a height one prime of $K_\gamma[A]$ are precisely the subgroups of G generated by an element s such that*

(a) *s acts as a reflection on A*

(b) *s acts trivially on K*

(c) $A^{(s)} \subset \mathrm{Ker}(\gamma_s)$.

Proof: Let $s \in G^T(\mathfrak{p})$, $\mathfrak{p}$ a height 1 prime. Then by Lemma 3.1.1, s acts trivially on K. Also by Lemma 3.1.1, for $a \in A^{(s)}$,

$$\gamma_s(a)^{-1} = \gamma_s(sa)^{-1} = \beta(sa - a) = 1$$

so that $\gamma_s|_{A^{(s)}} \equiv 1$.

Suppose that $s \neq 1$. If s acts trivially on A then $A^{(s)} = A$ and hence $\gamma_s \equiv 1$ by the last paragraph. So s acts trivially on $K_\gamma[A]$ and hence $s = 1$ since G acts faithfully on $K_\gamma[A]$. By contradiction, s must act non-trivially on A.

Let $H = G^T(\mathfrak{p}), B = I_H A$, and β be the homomorphism defined by Lemma 3.1.1. Take $J_\beta \triangleleft K_\gamma[A]$ as defined above. Since $\overline{K}$ is algebraic over K, we find $\overline{K}[A] \cong \overline{K} \otimes_K K[A]$ is integral over $K[A]$. Let $\mathfrak{P}$ be a prime of

$\overline{K}[A]$ lying over $\mathfrak{p}$. Then $\overline{K}[A]/\mathfrak{P}$ is integral over $K[A]/\mathfrak{P} \cap K[A] = K[A]/\mathfrak{p}$ so that $\dim \overline{K}[A]/\mathfrak{P} = \dim K[A]/\mathfrak{p}$. Hence $\mathfrak{P}$ is a height one prime of $\overline{K}[A]$ and $\overline{K}J_\beta \subset \overline{K}\mathfrak{p} \subset \mathfrak{P}$. By Lemma 3.1.2,

$$n - 1 = \dim \overline{K}[A]/\mathfrak{P} \leq \dim \overline{K}[A]/\overline{K}J_\beta = \dim \overline{K}[A/B] = n - \text{rank}(B)$$

which implies that the rank of B is precisely 1, as we have noted that B is non-trivial.

Since $\text{Im}_A(s-1) \subset B$, we find that s acts as a reflection on A and hence by Lemma 2.2.1, s has order 2 on A. Since $A/\text{Ker}_A(s+1)$ is torsion-free and $H^{-1}(\langle s\rangle, A) = \text{Ker}_A(s+1)/\text{Im}_A(s-1)$ is finite, the torsion subgroup of $A/\text{Im}_A(s-1)$ is $\text{Ker}_A(s+1)/\text{Im}_A(s-1)$. Since $B/\text{Im}_A(s-1)$ is finite, $B/\text{Im}_A(s-1) \subset \text{Ker}_A(s+1)/\text{Im}_A(s-1)$ so that $B \subset \text{Ker}_A(s+1)$. But then since $A/\text{Ker}_A(N_H)$ is torsion-free, $H^{-1}(H, A) = \text{Ker}_A(N_H)/I_H A$ is finite and $B = I_H A$, the torsion subgroup of A/B is $\text{Ker}_A(N_H)/B = \text{Ker}_A(s+1)/B$ so that $\text{Ker}_A(s+1) = \text{Ker}_A(N_H)$ for any $1 \neq s \in H$.

Let $s, t \in H$ be two non-trivial elements. Then s, t invert $\text{Ker}_A(N_H)$ and st fixes it. As $st \neq 1$ implies $\text{Ker}_A(st+1) = \text{Ker}_A(N_H)$, we must have $st = 1$ and so $t = s$. Hence $H = \langle s\rangle$ has order 2.

Conversely, let s be an element of G satisfying the three given hypotheses. We need to find a height 1 prime $\mathfrak{p}$ in $K_\gamma[A]$ with $G^T(\mathfrak{p}) = \langle s\rangle$. Since $G^T(\mathfrak{p})$ can only be cyclic of order 2, by above, we need only find $\mathfrak{p}$ with $s \in G^T(\mathfrak{p})$.

We first show s has order 2. If $c \in K, a \in A$ then by (a), (b),

$$\begin{aligned} s^2[ce(a)] &= s[c\gamma_s(sa)e(sa)] = c\gamma_s(sa)\gamma_s(s^2a)e(s^2a) \\ &= c\gamma_s(a+sa)e(a) = ce(a) \end{aligned}$$

where the last equality follows from the fact that $\text{Im}_A(s+1) \subset A^{\langle s\rangle}$ since s has order 2 on A. But then $s^2 = 1$ since s^2 acts trivially on $K_\gamma[A]$ and G acts faithfully on $K_\gamma[A]$.

Now $B = \text{Im}_A(s-1)$ is cyclic. Let $B = \mathbf{Z}b_0$ where $b_0 = sa_0 - a_0$. Define $\beta : B \to K^\times$ by $\beta(sa - a) = \gamma_s(a) = (\gamma_s(sa))^{-1}$. Since $\gamma_s : A \to K^\times$ is a homomorphism, and $A/A^{\langle s\rangle} \cong \text{Im}_A(s-1)$, we see that β is the homomorphism induced from γ_s. It is well-defined by property (c). But then $J_\beta = \langle e(y) - \beta(y) \mid y \in B\rangle$ is generated by $e(b_0) - \beta(b_0)$ since $e(nb_0) - \beta(nb_0) = e(b_0)^n - \beta(b_0)^n \in \langle e(b_0) - \beta(b_0)\rangle$. We wish to show that $sx - x \in J_\beta$ for all $x \in K_\gamma[A]$. It is sufficient to show this for $x = ce(a)$ for all $c \in K$ and $a \in A$.

In fact, since s acts trivially on $K[A^{\langle s\rangle}]$ and $A \cong A^{\langle s\rangle} \oplus \mathbf{Z}a_0$, it suffices to check that $se(a_0) - e(a_0) \in J_\beta$. Indeed,

$$\begin{aligned} se(a_0) - e(a_0) &= \gamma_s(sa_0)e(sa_0) - e(a_0) \\ &= \gamma_s(sa_0)e(a_0)[e(sa_0 - a_0) - (\gamma_s(sa_0))^{-1}] \\ &= \gamma_s(sa_0)e(a_0)[e(b_0) - \beta(b_0)] \in J_\beta \end{aligned}$$

Since J_β is generated by the non-zero non-unit element $e(b_0) - \beta(b_0)$ of the Noetherian domain $K_\gamma[A]$, we find that a minimal prime $\mathfrak{p}$ containing J_β must be of height 1 by Krull's Principal Ideal Theorem [2]. As $sx - x \in \mathfrak{p}$ for all $x \in K_\gamma[A]$, we have $\langle s\rangle = G^T(\mathfrak{p})$. ■

Remark: We can actually determine the height one primes $\mathfrak{p}$ containing J_β above.

For $B = \mathrm{Im}_A(s-1)$, we first claim that $B^+ = \mathrm{Ker}_A(s+1)$: Clearly $\mathrm{Im}_A(s-1) \subset \mathrm{Ker}_A(s+1)$ and $A/\mathrm{Ker}_A(s+1)$ is torsion-free. Since $H^{-1}(\langle s\rangle, A) = \mathrm{Ker}_A(s+1)/\mathrm{Im}_A(s-1)$ is finite, we may conclude that $B^+ = \mathrm{Ker}_A(s+1)$. So

$$B^+/B \cong H^1(\langle s\rangle, A) = \begin{cases} \mathbf{Z}/2\mathbf{Z} & \text{if } s \text{ is diagonalizable} \\ 0 & \text{else} \end{cases}$$

In particular, by the Remark after Lemma 3.1.2, we find that $K[A]/J_\beta \cong K[A/B]$ if s is a non-diagonalizable reflection and hence J_β is the required prime in this case.

Suppose that s is a diagonalizable reflection. Then $A = A^{\langle s\rangle} \oplus \mathrm{Ker}_A(s+1)$. Let $\mathbf{Z}a_0 = \mathrm{Ker}_A(s+1)$. Then $B = \mathbf{Z}(sa_0 - a_0) = \mathbf{Z}(-2a_0) = \mathbf{Z}(2a_0)$. So $B^+ = \mathbf{Z}a_0$. Let $b_0 = 2a_0$. Then J_β is generated by $e(b_0) - \beta(b_0) = e(a_0)^2 - \beta(b_0)$ and $T \mapsto e(a_0)$, $A^{\langle s\rangle} \hookrightarrow A$ induces an isomorphism

$$K[T]/(T^2 - \beta(b_0)) \otimes_K K[A^{\langle s\rangle}] \to K[A]/J_\beta$$

If $\beta(b_0) \notin (K^\times)^2$ then $T^2 - \beta(b_0)$ is irreducible over K so $K[T]/(T^2 - \beta(b_0))$ is a field which is algebraic over K. Hence the left side is a domain and again J_β is the required prime.

Suppose $\beta(b_0) \in (K^\times)^2$. Then if $\mathrm{char}(K) \neq 2$, $J_\beta = J_{\beta_1^+} J_{\beta_2^+}$ where $\beta_1^+(a_0) = \sqrt{\beta(b_0)}$ and $\beta_2^+(a_0) = -\sqrt{\beta(b_0)}$ define the two possible extensions of β to B^+. By the Remark after Lemma 3.1.2, $J_{\beta_1^+}$ and $J_{\beta_2^+}$ are the two possible minimal primes containing J_β. Finally if char$(K) = 2$, then $e(a_0)^2 - \beta(b_0) = (e(a_0) - \sqrt{\beta(b_0)})^2$ so the nil radical is $\sqrt{J_\beta} = J_{\beta^+}$ where $\beta^+(a_0) = \sqrt{\beta(b_0)}$ is the unique extension of β to B^+. As before, $\sqrt{J_\beta} = J_{\beta^+}$ is now the unique minimal prime containing J_β.

Lemma 3.1.6. *Let $G^T(\mathfrak{p}) = \langle s \rangle$ for some height 1 prime $\mathfrak{p}$. Then $(K_\gamma[A])^{\langle s \rangle}$ is a UFD.*

Proof: By Proposition 3.1.5, s acts as a reflection on A so that $A = A^{\langle s \rangle} \oplus \mathbf{Z} a_0$ where $sa_0 = -a_0 + x_0$ for some $x_0 \in A^{\langle s \rangle}$. It follows that $\{e(a_0)^n | n \in \mathbf{Z}\}$ is a $K[A^{\langle s \rangle}]$ basis of $K_\gamma[A]$ and since s acts trivially on K and $\gamma_s |_{A^{\langle s \rangle}} \equiv 1$, we see that s acts trivially on $K[A^{\langle s \rangle}]$. Here

$$se(a_0) = \gamma_s(sa_0)e(sa_0) = \gamma_s(sa_0)e(x_0)[e(a_0)]^{-1}$$

where $\gamma_s(sa_0)e(x_0) \in K[A^{\langle s \rangle}]$. Raising this identity to the kth power we see that

$$\{1\} \cup \{e(a_0)^k, (se(a_0))^k | k \geq 1\}$$

is also a $K[A^{\langle s \rangle}]$-basis for $K_\gamma[A]$ and therefore

$$\{1\} \cup \{e(a_0)^k + (se(a_0))^k | k \geq 1\}$$

is a $K[A^{\langle s \rangle}]$ basis of $K_\gamma[A]^{\langle s \rangle}$.

Note that $\tau = e(a_0)(se(a_0)) = \gamma_s(sa_0)e(x_0) \in K[A^{\langle s \rangle}]^\times$ and set $\omega_k = e(a_0)^k + (se(a_0))^k$ for $k \geq 0$ so that $\omega_0 = 2$. Put $\omega = \omega_1$. Now $se(a_0) = \omega - e(a_0)$ and $\tau = e(a_0)(\omega - e(a_0))$ so that for $k \geq 2$,

$$\begin{aligned}
&\omega\omega_{k-1} - \tau\omega_{k-2} \\
&= [e(a_0) + (\omega - e(a_0))][e(a_0)^{k-1} + (\omega - e(a_0))^{k-1}] \\
&- [e(a_0)(\omega - e(a_0))][e(a_0)^{k-2} + (\omega - e(a_0))^{k-2}] \\
&= e(a_0)^k + (\omega - e(a_0))e(a_0)^{k-1} + e(a_0)(\omega - e(a_0))^{k-1} + (\omega - e(a_0))^k \\
&- e(a_0)^{k-1}(\omega - e(a_0)) - e(a_0)[\omega - e(a_0)]^{k-1} \\
&= e(a_0)^k + (\omega - e(a_0))^k = \omega_k
\end{aligned}$$

so that

$$\omega_k = \omega\omega_{k-1} - \tau\omega_{k-2}, k \geq 2$$

Now induction on k implies that ω_k is a $K[A^{\langle s \rangle}]$- linear combination of powers of ω for all k. Hence it follows that $\{\omega^k | k \geq 0\}$ is a $K[A^{\langle s \rangle}]$-basis of $K_\gamma[A]^{\langle s \rangle}$ which means that $K_\gamma[A]^{\langle s \rangle}$ is a polynomial ring in one variable ω over $K[A^{\langle s \rangle}]$. Since $K[A^{\langle s \rangle}]$ is a UFD so is $K_\gamma[A]^{\langle s \rangle}$. ∎

3.2 Connecting Homomorphisms and Restriction Maps

Let

$$\Gamma_\gamma = \{1 \neq s \in G \mid \langle s \rangle = G^T(\mathfrak{p}) \text{ for some } \mathfrak{p} \text{ a height one prime of } K_\gamma[A]\}$$

In Section 3.1, we showed that the hypotheses of Lorenz' result 1.0.3 are satisfied for Γ_γ. So we now know that

$$\mathrm{Cl}(K_\gamma[A])^G) \cong \cap_{s \in \Gamma_\gamma} \mathrm{Ker}(\mathrm{Res}^G_{\langle s \rangle} : H^1(G, K_\gamma[A]^\times) \to H^1(\langle s \rangle, K_\gamma[A]^\times))$$

We now want to relate this class group to cohomology groups of $K^\times$ and A via the exact sequence of G-modules

$$0 \to K^\times \to (K_\gamma[A])^\times \to A \to 0$$

We first want to determine the connecting homomorphisms

$$\partial_G : A^G \mapsto H^1(G, K^\times) \qquad \delta_G : H^1(G, A) \mapsto H^2(G, K^\times)$$

arising from this sequence.

Lemma 3.2.1. *(a) The map* $\partial_G : A^G \to H^1(G, K^\times)$ *is given by*

$$\partial_G(a) = [g \mapsto \gamma_g(a)], a \in A^G$$

(b) The map $\delta_G : H^1(G, A) \to H^2(G, K^\times)$ *is given by*

$$\delta_G[f] = [(g, h) \mapsto \gamma_g(gf_h)], [f] \in H^1(G, A)$$

(c)Let $s \in \Gamma_\gamma$. *Then* $\partial_{\langle s \rangle}$ *is the trivial map.*

The homomorphism $\delta_{\langle s \rangle} : H^1(\langle s \rangle, A) \to H^2(\langle s \rangle, K^\times)$ *has a non-zero kernel iff s is a diagonalizable reflection with* $Res^G_{\langle s \rangle}[\gamma] = 0$ *iff s is a diagonalizable reflection with* $\gamma_g(A) \subset (K^\times)^2$. *Note that* $\mathrm{Ker}(\delta_{\langle s \rangle}) \neq 0$ *implies that* $\mathrm{Ker}(\delta_{\langle s \rangle}) = H^1(\langle s \rangle, A) \cong \mathbf{Z}/2\mathbf{Z}$.

Proof: (a) Lift $a \in A^G$ to $e(a) \in (K_\gamma[A])^\times$. Then $\partial_G(a)$ is represented by the 1-cocycle

$$g \mapsto (ge(a))/e(a) = \gamma_g(ga)e(ga)/e(a) = \gamma_g(a)$$

(b) Let $f : G \to A$ be a 1-cocycle. Lift each $f_g \in A$ to $e(f_g) \in (K_\gamma[A])^\times$. Then $[g \mapsto e(f_g)]$ maps to $[g \mapsto f_g]$ and $\delta[f]$ is represented by the 2-cocycle

$$(g,h) \mapsto ge(f_h)e(f_{gh})^{-1}e(f_g) = \gamma_g(gf_h)e(gf_h - f_{gh} + f_g) = \gamma_g(gf_h)$$

(c) Now let $s \in \Gamma_\gamma$. By condition (c) of Proposition 3.1.5, it is clear that $\partial_{\langle s\rangle}$ is trivial.

If $s \in \Gamma_\gamma$ is a non-diagonalizable reflection, then $H^1(\langle s\rangle, A) = 0$, so we may assume that s is diagonalizable and hence that $H^1(\langle s\rangle, A) \cong \mathbf{Z}/2\mathbf{Z}$. Then $A = A^{\langle s\rangle} \oplus \mathrm{Ker}_A(s+1)$ where $\mathrm{Ker}_A(s+1) = \langle a_0\rangle$. A 1-cocycle representing the generator of $H^1(\langle s\rangle, A) = H^1(\langle s\rangle, \mathrm{Ker}_A(s+1))$ is given by

$$f_g = \begin{cases} a_0 & , g = s \\ 0 & , g = 1 \end{cases}$$

By (b), $\delta_{\langle s\rangle}[f]$ is represented by the 2-cocycle r with $r(1,1) = r(s,1) = r(1,s) = 1$ and $r(s,s) = \gamma_s(a_0)^{-1}$. The cocycle r is a coboundary iff there exists a map $c : \langle s\rangle \to K^\times$ such that

$$r(g,h) = [gc(h)][c(gh)]^{-1}[c(g)]$$

Since $r(1,1) = r(s,1) = r(1,s) = 1$ we see that $c(1) = 1$. Also $r(s,s) = (c(s))^2$ since s acts trivially on K. So $\delta_{\langle s\rangle}$ is the trivial map iff $\gamma_s(a_0) \in (K^\times)^2$ iff $\gamma_s(A) \subset (K^\times)^2$.

Finally we must show, for diagonalizable s, that $\gamma_s(a) \in (K^\times)^2$ if and only if $\mathrm{Res}^G_{\langle s\rangle}[\gamma] = 0$ in $H^1(G, \mathrm{Hom}(A, K^\times))$. The map $\mathrm{Res}^G_{\langle s\rangle}\gamma : \langle s\rangle \to \mathrm{Hom}(A, K^\times)$ is a 1-coboundary iff $\gamma_s = \frac{s\varphi}{\varphi}$ for some $\varphi \in \mathrm{Hom}(A, K^\times)$. Since $A = A^{\langle s\rangle} \oplus \mathbf{Z}a_0$, we need only determine $s\varphi/\varphi$ for $x \in A^{\langle s\rangle}$ and for a_0. Since $s\varphi(x)/\varphi(x) = 1$ and $s\varphi(a_0)/\varphi(a_0) = (\varphi(a_0))^{-2}$, we find that $\mathrm{Res}^G_{\langle s\rangle}[\gamma] = 0$ iff $\gamma_s(A) \subset (K^\times)^2$. ■

Definition 3.2.2. Let N_γ be the subgroup of G generated by Γ_γ.

Definition 3.2.3. Let Γ^d_γ be the set consisting of $s \in \Gamma_\gamma$ such that s acts on A as a diagonalizable reflection and $\gamma_s(A) \subset (K^\times)^2$ and let D_γ be the subgroup of G generated by Γ^d_γ.

Notation: Let H be a subgroup of G, $\mathcal{H}$ a collection of subgroups of G, and M a $\mathbf{Z}G$ module. Then we will sometimes abbreviate the map

$\mathrm{Res}_H^G : H^1(G, M) \to H^1(H, M)$ by $\mathrm{Res}_H^G(M)$. We will also write $\oplus_{H\in\mathcal{H}}\mathrm{Res}_H^G$: $H^1(G, M) \to \oplus_{H\in\mathcal{H}} H^1(H, M)$ as $\oplus_{H\in\mathcal{H}}\mathrm{Res}_H^G(M)$.

Lemma 3.2.4. *(a) N_γ and D_γ are normal subgroups of G.*

(b) Let M be a G-module, H be a subgroup of G and $g \in G$. Then

$$\mathrm{Ker}(\mathrm{Res}_{gHg^{-1}}^G(M)) = \mathrm{Ker}(\mathrm{Res}_H^G(M))$$

(c)

$$\mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma^d/G}\mathrm{Res}_{\langle s\rangle}^G(A)\right) \cong H^1(G/D_\gamma, A^{D_\gamma})$$

$$\mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}_{\langle s\rangle}^G(K^\times)\right) \cong H^1(G/N_\gamma, K^\times)$$

Proof: (a) To show that N_γ and D_γ are normal in G, it suffices to show that Γ_γ and Γ_γ^d are stabilized by conjugation by $g \in G$. First note that for $g, s \in G$,

$$\mathrm{Im}_A(gsg^{-1} - 1) = g\mathrm{Im}_A(s-1) \cong \mathrm{Im}_A(s-1)$$
$$A^{\langle gsg^{-1}\rangle} = gA^{\langle s\rangle} \cong A^{\langle s\rangle}$$
$$\mathrm{Ker}_A(gsg^{-1} + 1) = g\mathrm{Ker}_A(s+1) \cong \mathrm{Ker}_A(s+1)$$

Now let $s \in \Gamma_\gamma$. Since s acts as a reflection on A, $\mathrm{Im}_A(s-1)$ is cyclic. But then $\mathrm{Im}_A(gsg^{-1} - 1) = g\mathrm{Im}_A(s-1)$ is cyclic so that gsg^{-1} is a reflection on A. Also

$$H^1(\langle gsg^{-1}\rangle, A) \cong H^{-1}(\langle gsg^{-1}\rangle, A) = g\mathrm{Ker}_A(s+1)/g\mathrm{Im}_A(s-1) \cong H^1(\langle s\rangle, A)$$

shows that gsg^{-1} is diagonalizable iff gsg^{-1} is. Since s acts trivially on K so does gsg^{-1}. Let $a \in A^{\langle gsg^{-1}\rangle}$. Then $g^{-1}a \in A^{\langle s\rangle}$ and $\gamma_s(A^{\langle s\rangle}) = 1$. So

$$\begin{aligned}\gamma_{gsg^{-1}}(a) &= \gamma_g(a)g(\gamma_s(g^{-1}a))gs(\gamma_{g^{-1}}(sg^{-1}a))\\ &= \gamma_g(a)g(1\cdot\gamma_{g^{-1}}(g^{-1}a))\\ &= \gamma_{gg^{-1}}(a) = \gamma_1(a) = 1\end{aligned}$$

If $s \in \Gamma_\gamma^d$, we already know that gsg^{-1} is diagonalizable so we need only show that $\gamma_{gsg^{-1}}(A) \subset (K^\times)^2$. But $A = A^{\langle s\rangle} \oplus \mathrm{Ker}_A(s+1)$ with $\mathrm{Ker}_A(s+1) =$

$\mathbf{Z}a_0$ and the above implies that $\text{Ker}_A(gsg^{-1}+1) = \mathbf{Z}ga_0$ so that it suffices to show $\gamma_{gsg^{-1}}(ga_0) \in (K^\times)^2$. But

$$\begin{aligned}\gamma_{gsg^{-1}}(ga_0) &= \gamma_g(ga_0)g(\gamma_s(a_0)s(\gamma_{g^{-1}}(sa_0))) \\ &= \gamma_g(ga_0)g(\gamma_{g^{-1}}(a_0))g(\gamma_{g^{-1}}(a_0))^{-1}g(\gamma_s(a_0))g(\gamma_{g^{-1}}(a_0))^{-1} \\ &= \gamma_{gg^{-1}}(ga_0)g(\gamma_s(a_0))g(\gamma_{g^{-1}}(a_0)^{-2}) \\ &= g(\gamma_s(a_0)\gamma_{g^{-1}}(a_0)^{-2}) \in (K^\times)^2\end{aligned}$$

since $\gamma_s(a_0) \in (K^\times)^2$. So we have shown that N_γ and D_γ are normal in G.

The following more conceptual proof of the normality of N_γ and D_γ was suggested by Lorenz: For $g \in G$, and any height one prime $\mathfrak{p}$ in $K_\gamma[A]$, $g \cdot \mathfrak{p}$ is also a height one prime in $K_\gamma[A]$ and $G^T(g\cdot\mathfrak{p}) = gG^T(\mathfrak{p})g^{-1}$. But

$$\Gamma_\gamma = \{s \in G | G^T(\mathfrak{p}) = \langle s\rangle \text{ for some height 1 prime of } K_\gamma[A]\}$$

So $s \in \Gamma_\gamma$ implies that $\langle gsg^{-1}\rangle = gG^T(\mathfrak{p})g^{-1} = G^T(g\cdot\mathfrak{p})$ so that $gsg^{-1} \in \Gamma_\gamma$. Hence Γ_γ is G-stable so that N_γ is a normal subgroup of G.

Recall from Lemma 3.2.1 that

$$\Gamma_\gamma^d = \{s \in \Gamma_\gamma | \text{Ker}(\delta_{\langle s\rangle}) \neq 0\}$$

Then the commutativity of the following diagram induced by the conjugation isomorphism $\langle s\rangle \to \langle gsg^{-1}\rangle$ [7, p. 80]

$$\begin{array}{ccc} H^1(\langle gsg^{-1}\rangle, A) & \xrightarrow{\cong} & H^1(\langle s\rangle, A) \\ \downarrow{\scriptstyle \delta_{\langle gsg^{-1}\rangle}} & & \downarrow{\scriptstyle \delta_{\langle s\rangle}} \\ H^2(\langle gsg^{-1}\rangle, K^\times) & \xrightarrow{\cong} & H^2(\langle s\rangle, K^\times) \end{array}$$

shows that Γ_γ^d is also G stable so that D_γ is normal in G.

(b) Now let M be a G-module, H a subgroup of G and $g \in G$. Let $[f] \in \text{Ker}(\text{Res}_H^G(M))$ and let f be a 1-cocycle representing $[f]$. Then there exists $m \in M$ such that $f_h = (h-1)m$ for all $h \in H$. So

$$\begin{aligned}f_{ghg^{-1}} &= f_g + gf_h + ghf_{g^{-1}} \\ &= f_g + g(h-1)m + ghg^{-1}gf_{g^{-1}} \\ &= (ghg^{-1}-1)(gm - f_g)\end{aligned}$$

shows that $f \in \mathrm{Ker}(\mathrm{Res}^G_{gHg^{-1}})$. The reverse inclusion is symmetric.

(c) By (b) applied to $M = A$ and $H = \langle s \rangle$ for some $s \in \Gamma_\gamma$, we find that

$$\begin{aligned}
&\mathrm{Ker}\left(\oplus_{s\in\Gamma^d_\gamma/G}\mathrm{Res}^G_{\langle s\rangle} : H^1(G, A) \to \oplus_{s\in\Gamma_\gamma/G} H^1(\langle s\rangle, A)\right) \\
&= \cap_{s\in\Gamma^d_\gamma/G}\mathrm{Ker}(\mathrm{Res}^G_{\langle s\rangle}(A)) \\
&= \mathrm{Ker}\left(\oplus_{s\in\Gamma^d_\gamma}\mathrm{Res}^G_{\langle s\rangle}(A)\right) \\
&= \mathrm{Ker}\left[\left(\oplus_{s\in\Gamma^d_\gamma}\mathrm{Res}^{D_\gamma}_{\langle s\rangle}(A)\right) \circ \mathrm{Res}^G_{D_\gamma}(A)\right]
\end{aligned}$$

We wish to show that $\oplus_{s\in\Gamma^d_\gamma}\mathrm{Res}^{D_\gamma}_{\langle s\rangle}(A)$ is injective. Let $\overline{D}_\gamma = \mathrm{Im}(D_\gamma \to \mathrm{Aut}(A))$ and let $\overline{\Gamma^d_\gamma}$ be the image of Γ^d_γ in $\overline{D}_\gamma$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
H^1(\overline{D}_\gamma, A) & \xrightarrow{\text{inf}} & H^1(D_\gamma, A) \\
\downarrow & & \downarrow \\
\oplus_{\overline{s}\in\overline{\Gamma^d_\gamma}} H^1(\langle \overline{s}\rangle, A) & \longrightarrow & \oplus_{s\in\Gamma^d_\gamma} H^1(\langle s\rangle, A)
\end{array}$$

Note from the inflation-restriction sequence, the top map is an isomorphism since $\mathrm{Ker}(D_\gamma \to \overline{D}_\gamma)$ acts trivially on A. The left vertical map is an isomorphism by [24, 2.4] or equivalently Proposition 2.2.23. Finally the bottom map is a diagonal map with $H^1(\langle\overline{s}\rangle, A)$ mapping diagonally into $\oplus_{\overline{t}=\overline{s}} H^1(\langle t\rangle, A)$ and hence is injective. We now see from the diagram that $\oplus_{s\in\Gamma^d_\gamma}\mathrm{Res}^{D_\gamma}_{\langle s\rangle}(A)$ is injective so that the above calculation shows that $\mathrm{Ker}(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}^G_{\langle s\rangle}(A)) = \mathrm{Ker}(\mathrm{Res}^G_{D_\gamma}(A))$ which is isomorphic by the inflation-restriction sequence to $H^1(G/D_\gamma, A^{D_\gamma})$ as required.

For the second isomorphism, apply (b) to $M = K^\times$ and $H = \langle s\rangle$ for some $s \in \Gamma_\gamma$. Note that

$$\oplus_{s\in\Gamma_\gamma}\mathrm{Res}^{N_\gamma}_{\langle s\rangle} : H^1(N_\gamma, K^\times) \to \oplus_{s\in\Gamma_\gamma} H^1(\langle s\rangle, K^\times)$$

is injective since Γ_γ generates N_γ and $\mathrm{Res}^{N_\gamma}_{\langle s\rangle}$ is just the restriction map on homomorphisms as N_γ acts trivially on K. Then

$$\begin{aligned}
&\text{Ker}\left(\ominus_{s\in\Gamma_\gamma/G}\text{Res}^G_{\langle s\rangle} : H^1(G,K^\times) \to \oplus_{s\in\Gamma_\gamma/G}H^1(\langle s\rangle, K^\times)\right) \\
&= \cap_{s\in\Gamma_\gamma/G}\text{Ker}(\text{Res}^G_{\langle s\rangle}(K^\times)) \\
&= \text{Ker}\left(\ominus_{s\in\Gamma_\gamma}\text{Res}^G_{\langle s\rangle}(K^\times)\right) \\
&= \text{Ker}\left[\left(\oplus_{s\in\Gamma_\gamma}\text{Res}^{N_\gamma}_{\langle s\rangle}(K^\times)\right) \circ \text{Res}^G_{N_\gamma}(K^\times)\right] \\
&= \text{Ker}\left(\text{Res}^G_{N_\gamma}(K^\times)\right) \\
&\cong H^1(G/N_\gamma, K^\times)
\end{aligned}$$

where the last isomorphism follows from the inflation-restriction sequence.
■

3.3 Finite Groups with Normal Reflection Subgroups

Let G be a finite group and let A be a lattice on which G acts faithfully. Set $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ and let Γ be a G-stable set of reflections on V. Let R be the subgroup of G generated by Γ. We will use the notation of Section 2.3.

Let Φ be a suitable G-stable crystallographic root system for A and Γ. Then by Lemma 2.3.5(a), $V = \mathbf{Q}\Phi \oplus V^R$ is a $\mathbf{Q}G$ decomposition of V. Let Δ be a base for Φ and recall that $\Omega_H = \{h \in H | h\Delta = \Delta\}$. Then, by Lemma 2.3.6(b), we have that $H = \text{Im}(G \to GL(\mathbf{Q}\Phi)) = W \rtimes \Omega_H$ is a subgroup of $\text{Aut}(\Phi) = \{g \in GL(\mathbf{Q}\Phi) | g\Phi = \Phi\}$.

Lemma 3.3.1. *(a) The commutator subgroup H' of $H = W \rtimes \Omega_H$ is $P_W \rtimes \Omega'_H$ where*

$$P_W = \langle st | s \sim_H t, s, t \in \Gamma\rangle \triangleleft H$$

and $\sim_H$ denotes H-conjugacy.

(b)

$$\sigma : W \rtimes \Omega_H / P_W \rtimes \Omega'_H \to W/P_W, wtP_W\Omega'_H \mapsto wP_W$$

is a split epimorphism of groups so that

$$H/H' \cong W/P_W \times \Omega_H/\Omega'_H$$

(c)

$$W/P_W = \oplus_{s\in\Gamma/H} sP_W \cong \oplus_{s\in\Gamma/H}\mathbf{Z}/2\mathbf{Z}$$

Proof: (a) To fix notation, let $(g,h) = ghg^{-1}h^{-1}$ and $g^h = hgh^{-1}$ for all $g,h \in G$.

If $h \in H$ and $s_\alpha \sim_H s_\beta$, then $hs_\alpha s_\beta h^{-1} = s_{h\alpha}s_{h\beta}$ where $s_{h\alpha} \sim_H s_\alpha \sim_H s_\beta \sim_H s_{h\beta}$. So P_W is a normal subgroup of H.

Since H is generated by the set $\{s_\alpha | \alpha \in \Phi\} \cup \Omega_H$, H' is the normal closure of the group generated by the set

$$\{(s_\alpha, s_\beta), (s_\alpha, t), (t_1, t_2) | \alpha, \beta \in \Phi, t, t_1, t_2 \in \Omega_H\}$$

As $(s_\alpha, x) = s_\alpha s_{x\alpha}$, we see that $(s_\alpha, s_\beta), (s_\alpha, t)$ are all in P_W while $(t_1, t_2) \in \Omega'_H$. Hence the above set of generators for H' is contained in $P_W\Omega'_H$. The same relation shows that $P_W\Omega'_H$ is generated by commutators and so is contained in H'.

To show that $P_W\Omega'_H = H'$, it suffices, by the above, to show that $P_W\Omega'_H$ is a normal subgroup of H. Since $P_W \triangleleft H$, it suffices to check that $t^h \in P_W\Omega'_H$ for $t \in \Omega'_H$, $h \in W\Omega_H$. But $t^x \in \Omega'_H \leq P_W\Omega'_H$ for $x \in \Omega_H$, and $t^{s_\alpha} = (s_\alpha, t)t \in P_W\Omega'_H$ verifies this on generators of H, so (a) is proved.

(b) $\sigma : H \to W/P_W, wt \mapsto wP_W$ is a well defined map. Let $w_1, w_2 \in W$ and $t_1, t_2 \in \Omega_H$. Then since $w_1t_1w_2t_2 = w_1w_2^{t_1}t_1t_2$ and $(w_1w_2)^{-1}(w_1w_2^{t_1}) = (w_2^{-1}, t_1) \in H' \cap W \subset P_W$, we see that

$$\sigma(w_1t_1w_2t_2) = \sigma(w_1w_2^{t_1}t_1t_2) = w_1w_2^{t_1}P_W = w_1w_2P_W = \sigma(w_1t_1)\sigma(w_2t_2)$$

So σ is an epimorphism and since $P_W\Omega'_H \in \mathrm{Ker}(\sigma)$, the induced map $\overline{\sigma}$ is a group epimorphism. Define $\psi : W/P_W \to W\Omega_H/P_W\Omega'_H$ as the homomorphism induced by the inclusion map $W \hookrightarrow W\Omega_H$. Since $\overline{\sigma} \circ \psi = id_{W/P_W}$, $\overline{\sigma}$ is a split epimorphism as required.

(c) Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be our base of Φ defining Ω_H. Since H is a subgroup of $\mathrm{Aut}(\Phi)$, the H-orbit of an irreducible root system Φ_0 in Φ is a union of irreducible root systems all isomorphic to Φ_0. Decompose Φ into irreducible root systems $\Phi = \cup_{i=1}^m \Phi_i^{k_i}$ so that the H-orbit of the irreducible root system Φ_i is $\Phi_i^{k_i}$. This decomposition gives a corresponding decomposition for $\Delta = \cup_{i=1}^m \Delta_i^{k_i}$, $\Gamma = \cup_{i=1}^m \Gamma_i^{k_i}$, $W = \prod_{i=1}^m W_i^{k_i}$ and $V = \oplus_{i=1}^m V_i^{k_i}$ such that $\Delta_i^{k_i}$ is a base of the root system $\Phi_i^{k_i}$ on the $\mathbf{Q}H$ space $V_i^{k_i}$ having Weyl group $W_i^{k_i}$ generated by $\Gamma_i^{k_i}$. Note also that $P_W = \prod_{i=1}^m P_{W_i^{k_i}}$. But then

$W/P_W = \prod_{i=1}^m W_i^{k_i}/P_{W_i^{k_i}}$ and $\oplus_{s\in\Gamma/H} sP_W = \oplus_{i=1}^m \oplus_{s\in\Gamma_i^{k_i}/H} sP_{W_i^{k_i}}$ shows that we may reduce to the case when Φ is an H-orbit of an irreducible root system.

So suppose Φ is an H-orbit of an irreducible root system and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be our base of Φ defining H. Then $W = \langle s_1, \ldots, s_n\rangle$ where $s_i = s_{\alpha_i}$ [20, p.11] is the simple reflection corresponding to α_i. So $W/P_W = \langle s_1 P_W, \ldots, s_n P_W\rangle$ is an elementary abelian 2-group.. Each reflection s_α is W-conjugate to a simple reflection.

Claim: $s_i P_W = s_j P_W$ if and only if $s_i \sim_H s_j$.

Suppose $s_i \sim_H s_j$. Then $s_i s_j \in P_W$ implies that $s_i P_W = s_j P_W$.

Now suppose $s_i P_W = s_j P_W$. Then $s_i s_j \in P_W$. So $s_i s_j = \prod_{k=1}^m s_{\beta_k} s_{\tau_k}$ where $s_{\beta_k} \sim_H s_{\tau_k}$. Now each $s_{\beta_k} = w_k s_{i_k} w_k^{-1}$ and $s_{\tau_k} = v_k s_{j_k} v_k^{-1}$ for some $w_k, v_k \in W$ and some simple reflections s_{i_k}, s_{j_k}. Then $s_{i_k} \sim_W s_{\beta_k} \sim_H s_{\tau_k} \sim_W s_{j_k}$. Write each w_k and v_k as a product of simple reflections and then w_k^{-1}, v_k^{-1} as the product in reverse order of the same simple reflections. This results in an expression for $s_i s_j$ as a product of simple reflections with the special property that each simple reflection occurring has some H-conjugate appearing elsewhere. Apply the deletion condition [21, p.13] repeatedly removing a W-conjugate (and hence H-conjugate) pair at each step. Since $s_i s_j$ has length 2 (or 0) and the special property is preserved by this process, it ends with $s_i s_j = s_k s_l$ where $s_k \sim_H s_l$. We must show that this implies that $s_i \sim_H s_j$. There are four cases to consider:

Case 1: $\{i, j\} \cap \{k, l\} = \phi$

$$s_i s_j(\alpha_j) = -\alpha_j + \langle \alpha_j, \alpha_i\rangle \alpha_i$$
$$s_k s_l(\alpha_j) = \alpha_j - \langle \alpha_j, \alpha_l\rangle\alpha_l - \langle\alpha_j, \alpha_k\rangle\alpha_k + \langle\alpha_j, \alpha_l\rangle\langle\alpha_l, \alpha_k\rangle\alpha_k$$

But since $s_i s_j = s_k s_l$ and the simple roots are linearly independent this case is impossible.

Case 2: $i = k$ or $j = l$

If $i = k$ then $s_i s_j = s_k s_l$ implies $s_i = s_k \sim_H s_l = s_j$.

Case 3: $i = l$

$s_i s_j = s_k s_l$ implies $s_i = s_l \sim_H s_k = s_i s_j s_i \sim_H s_j$.

Case 4: $j = k$

$s_i s_j = s_k s_l$ implies $s_j = s_k \sim_H s_l = s_j s_i s_j \sim_H s_i$. So $s_i \sim_H s_j$.

We may conclude that $W/P_W = \langle s_\alpha P_W | s_\alpha \in \Gamma/H\rangle$. We still need to show that no non-trivial product of reflections s_α where $s_\alpha \in \Gamma/H$ is in P_W. Since we are in the case when Φ is an H-orbit of an irreducible root system

Φ_0, there can be at most 2 H-orbits of roots in Φ as there are at most 2 W-orbits of roots in Φ_0. But then there are at most 2 H-conjugacy classes of reflections in Γ. Then the above claim shows that $W/P_W = \oplus_{s_\alpha \in \Gamma/H} s_\alpha P_W$ as required. ■

Remark: A more conceptual and straightforward proof of the last Lemma was suggested by Lorenz:

Observe first that for $h \in H, s \in \Gamma$, $(h,s) = (hsh^{-1})s$ where $hsh^{-1} \in \Gamma$ and $hsh^{-1} \sim_H s$. So we have

$$P_W = \langle (h,s) | h \in H, s \in \Gamma \rangle$$

By the identity $(g,xy) = (g,y)(ygy^{-1},x)$ and the fact that Γ generates W, it follows that

$$P_W = (H,W) = \langle (h,w) | h \in H, w \in W \rangle$$

Then (a) of the Lemma follows easily from the identity $(h,w)^x = (h^x, w^x)$ for $h, x \in H, w \in W$ and the fact that W is normal in H. Part (b) of the Lemma remains the same. For (c), we note that $W' \le P_W$ induces the natural epimorphism $W/W' \twoheadrightarrow W/P_W, wW' \mapsto wP_W$ with kernel

$$\begin{aligned}
&\{wW' | w = \prod_{i=1}^{m} (h_i, w_i) \text{ for some } h_i \in H, w_i \in W\} \\
=&\{wW' | wW' = \sum_{i=1}^{m} (h_i - 1) \cdot w_i W' \text{ for some } h_i \in H, w_i \in W\} \\
=&I_H(W/W')
\end{aligned}$$

So $W/P_W \cong (W/W')/I_H(W/W') = H_0(H, W/W')$. Since W/W' is an elementary abelian 2-group generated by $\{sW' | s \in \Gamma\}$ then its homomorphic image W/P_W is too and if $s \sim_H t$ then $sP_W = tP_W$. So $W/P_W = \{sP_W | s \in \Gamma/H\}$. To show that $W/P_W = \oplus_{s \in \Gamma/H} sP_W$ we then reduce to an H-orbit of an irreducible root system and then prove our claim as before.

We now let our finite group G, acting faithfully on the lattice A, act also on a field K. Let Γ be a G-stable set of reflections on A which act trivially on the field K and let R be the normal subgroup of G generated by Γ.

Lemma 3.3.2.

$$\oplus_{s\in\Gamma/G}\mathrm{Res}^G_{\langle s\rangle} : H^1(G, K^\times) \to \oplus_{s\in\Gamma/G}H^1(\langle s\rangle, K^\times)$$

is surjective.

Proof: We first make some reductions:
Step 1: We may assume that G acts trivially on K:
Apply cohomology to the sequence

$$0 \to (K^\times)^G \to K^\times \to K^\times/(K^\times)^G \to 0$$

Then

$$\begin{array}{ccc} H^1(G,(K^\times)^G) & \longrightarrow & H^1(G,K^\times) \\ {\scriptstyle \oplus\mathrm{Res}^G_{\langle s\rangle}}\downarrow & & \downarrow{\scriptstyle \oplus\mathrm{Res}^G_{\langle s\rangle}} \\ \oplus_{s\in\Gamma/G}H^1(\langle s\rangle,(K^\times)^G) & \longrightarrow & \oplus_{s\in\Gamma/G}H^1(\langle s\rangle,K^\times) \end{array}$$

is a commutative diagram where the sums are taken over all $s \in \Gamma/G$. But the bottom horizontal arrow is surjective as $s \in \Gamma$ acts trivially on $K^\times$ and $-1 \in K^G$. So the right hand map is surjective if the left hand map is surjective.
Step 2: We may assume that Φ spans V where $\Phi = \Phi_A = \{\alpha|\mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha$ for some $s \in \Gamma\}$:

Let $H = \mathrm{Im}(G \xrightarrow{\varphi} GL(\mathbf{Q}\Phi))$ so that $W = \mathrm{Im}(R \to GL(\mathbf{Q}\Phi))$ is a normal subgroup of H. Now $G/\mathrm{Ker}(\varphi) \cong H$. Consider the commutative diagram

$$\begin{array}{ccc} H^1(H,K^\times) & \xrightarrow{\mathrm{inf}} & H^1(G,K^\times) \\ {\scriptstyle \oplus_{\bar{s}\in\Gamma/H}\mathrm{Res}^H_{\langle s\rangle}}\downarrow & & \downarrow{\scriptstyle \oplus_{s\in\Gamma/G}\mathrm{Res}^G_{\langle s\rangle}} \\ \oplus_{\bar{s}\in\Gamma/H}H^1(\langle s\rangle,K^\times) & \xrightarrow{\mathrm{inf}} & \oplus_{s\in\Gamma/G}H^1(\langle s\rangle,K^\times) \end{array}$$

Since G and H have the same orbits on Φ and hence on Γ, the bottom horizontal arrow is surjective, hence the right vertical arrow is surjective if the left one is.
Step 3: After these reductions, we may replace G, N by H, W so that $H = W \rtimes \Omega_H$ as above.

Since by Lemma 3.3.1,

$$W/P_W = \oplus_{s\in\Gamma/H}(sP_W) \cong \oplus_{s\in\Gamma/H}(sP_W\Omega'_H)$$

is a direct summand of $H/H' = W\rtimes\Omega_H/P_W\rtimes\Omega'_H$, we find that $\oplus_{s\in\Gamma/H}\mathrm{Hom}(sP_W\Omega'_H, K^\times)$ is a direct summand of $H^1(H, K^\times)$ which maps isomorphically onto $\oplus_{s\in\Gamma/H}\mathrm{Hom}(\langle s\rangle, K^\times)$ under $\oplus_{s\in\Gamma/H}\mathrm{Res}^H_{\langle s\rangle}$. ■

3.4 Main Result

Let G be a finite group acting on the field K and the lattice A. Let γ be a 1-cocycle representing $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$ and let G act on $K_\gamma[A]$ as before. Let Γ_γ, N_γ, and D_γ be given as in Section 3.2.

Remark: Although we assume in the following theorem that G acts faithfully on A, we may still apply it to the general case. Indeed, by Proposition 2.1.4, we can obtain an isomorphic twisted multiplicative invariant ring in which the group does act faithfully on the lattice.

Theorem 3.4.1. *Assume G acts faithfully on A. There is a short exact sequence*

$$H^1(G/N_\gamma, K^\times)/\partial'_G(A^G) \rightarrowtail \mathrm{Cl}((K_\gamma[A])^G) \twoheadrightarrow \mathrm{Ker}(\delta'_G) \cap H^1(G/D_\gamma, A^{D_\gamma})$$

where $\partial_G : A^G \to H^1(G, K^\times)$ and $\delta_G : H^1(G, A) \to H^2(G, K^\times)$ are the respective connecting homomorphisms associated with the short exact sequence

$$K^\times \rightarrowtail (K_\gamma[A])^\times \twoheadrightarrow A$$

and ∂'_G satisfies $\mathrm{inf}^G_{G/N_\gamma} \circ \partial'_G = \partial_G$, while $\delta'_G = \delta_G \circ \mathrm{inf}^G_{G/D_\gamma}$.

Proof: For each $s \in \Gamma_\gamma$, the exact sequence

$$K^\times \rightarrowtail (K_\gamma[A])^\times \twoheadrightarrow A$$

gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} A^G & \xrightarrow{\partial_G} & H^1(G,K^\times) & \longrightarrow & H^1(G,(K_\gamma[A])^\times) & \longrightarrow & H^1(G,A) & \xrightarrow{\delta_G} & H^2(G,K^\times) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A^{\langle s\rangle} & \xrightarrow{\partial_{\langle s\rangle}} & H^1(\langle s\rangle,K^\times) & \longrightarrow & H^1(\langle s\rangle,(K_\gamma[A])^\times) & \longrightarrow & H^1(\langle s\rangle,A) & \xrightarrow{\delta_{\langle s\rangle}} & H^2(\langle s\rangle,K^\times) \end{array}$$

where the vertical maps are given by restriction. By summing over G-conjugacy classes of reflections in Γ_γ and applying Lemma 3.2.1(c), we obtain the commutative diagram :

$$\begin{array}{ccccc} H^1(G, K^\times)/\partial_G(A^G) & \rightarrowtail & H^1(G,(K_\gamma[A])^\times) & \twoheadrightarrow & \mathrm{Ker}(\delta_G) \\ \downarrow{\scriptstyle \oplus\overline{\mathrm{Res}^G_{\langle s\rangle}(K^\times)}} & & \downarrow{\scriptstyle \oplus\mathrm{Res}^G_{\langle s\rangle}(K_\gamma[A])^\times)} & & \downarrow{\scriptstyle \oplus\mathrm{res}^G_{\langle s\rangle}(A)} \\ \oplus_{s\in\Gamma_\gamma/G} H^1(\langle s\rangle, K^\times) & \rightarrowtail & \oplus_{s\in\Gamma_\gamma/G} H^1(\langle s\rangle, ((K_\gamma[A])^\times) & \twoheadrightarrow & \oplus_{s\in\Gamma_\gamma^d/G}\mathrm{Ker}(\delta_{\langle s\rangle}) \end{array}$$

since $\partial_{\langle s\rangle} \equiv 1$ for all $s \in \Gamma_\gamma$ by Lemma 3.2.1(c). Here $\overline{\mathrm{Res}^G_{\langle s\rangle}(K^\times)}$ and $\mathrm{res}^G_{\langle s\rangle}(A)$ are induced from the respective restriction maps. From Lemma 3.3.2, we find that $\oplus_{s\in\Gamma_\gamma/G}\overline{\mathrm{Res}^G_{\langle s\rangle}(K^\times)}$ is surjective so the snake lemma applied to this diagram yields the exact sequence

$$\mathrm{Ker}(\oplus_{s\in\Gamma_\gamma/G}\overline{\mathrm{Res}^G_{\langle s\rangle}(K^\times)}) \rightarrowtail \mathrm{Ker}(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}^G_{\langle s\rangle}((K_\gamma[A])^\times)) \twoheadrightarrow \mathrm{Ker}(\oplus_{s\in\Gamma_\gamma^d/G}\mathrm{res}^G_{\langle s\rangle}(A))$$

is exact. We need only determine the terms of this sequence. From the first paragraph of Section 3.2 and Lemma 3.2.4(c), we find that

$$\begin{aligned} \mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}^G_{\langle s\rangle}(K_\gamma[A]^\times)\right) &= \cap_{s\in\Gamma_\gamma}\mathrm{Ker}(\mathrm{Res}^G_{\langle s\rangle}(K_\gamma[A]^\times)) \cong \mathrm{Cl}(K_\gamma[A]^G) \\ \mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\mathrm{res}^G_{\langle s\rangle}(A)\right) &= \mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}^G_{\langle s\rangle}(A)\right) \cap \mathrm{Ker}(\delta'_G) \\ &\cong H^1(G/D_\gamma, A^{D_\gamma}) \cap \mathrm{Ker}(\delta'_G) \\ \mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\overline{\mathrm{Res}^G_{\langle s\rangle}(K^\times)}\right) &= \mathrm{Ker}\left(\oplus_{s\in\Gamma_\gamma/G}\mathrm{Res}^G_{\langle s\rangle}(K^\times)\right)/\partial'_G(A^G) \\ &\cong H^1(G/N_\gamma, K^\times)/\partial'_G(A^G) \end{aligned}$$

We now have our exact sequence as required. ■

Chapter 4

Rationality

4.1 Twisted Farkas

In this section, we generalize the results of Farkas in [16]. Let G be a finite group and let A be a lattice on which G acts faithfully. Let Γ be a G-stable set of reflections on $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$ and let R be the normal subgroup generated by Γ. We will use the notation of Section 2.3.

Notation: Fix a suitable root system Φ for A and Γ with weight lattice $\Lambda = \Lambda(\Phi)$ and choose a base Δ for $\pi(\Phi) = \Phi$. Form the dominant weights

$$\Lambda^+ = \{\omega \in \Lambda | \langle \omega, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Delta\}$$

Also define a partial order on $\pi(V)$ by

$$x \leq y \Leftrightarrow y - x = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

where all $c_\alpha \geq 0$.

Lemma 4.1.1. *(a) Each R orbit on Λ has exactly one element in Λ^+.*
(b) If $\lambda \in \Lambda^+$ then $r\lambda \leq \lambda$ for all $r \in R$.
(c) The stabilizer $R_x = \{r \in R | rx = x\}$ of $x \in \pi(V)$ is generated by the simple reflections it contains.
(d) $\leq$ satisfies the minimum condition on Λ^+, i.e. if $\lambda \in \Lambda^+$ then $\{\mu \in \Lambda^+ | \mu \leq \lambda\}$ is finite.

Proof: Since Φ is a suitable root system for A, R induces the Weyl group W of Φ on $\mathbf{Q}\Phi$ so (a) and (b) are [20, p. 68]; (c) is [21, p. 22] and (d) is [20, p. 70]. ■

Let γ be a 1-cocycle representing a class $[\gamma] \in H^1(R, \mathrm{Hom}(A, K^\times))$. We will eventually be interested in the case when $\Gamma = \Gamma_\gamma$, i.e. the set of reflections $s \in R$ so that

(a) s acts trivially on K and

(b) $A^{(s)} \subset \mathrm{Ker}(\gamma_s)$

To start, we assume only that Γ satisfies (a).

Lemma 4.1.2. *Let* $A_0 = \{a \in A | \gamma_r(a) = 1$ *for all* $r \in R_a\}$. *Define* $X(a) = \sum_{a \in R/R_a} re(a)$ *for* $a \in A_0$. *Then*

$$\{X(a) | a \in \pi^{-1}(\Lambda^+) \cap A_0\}$$

is a K*-basis for* $K_\gamma[A]^R$.

Proof: Observe that for $a \in A_0$, $X(a)$ is well-defined (i.e. independent of the choice of representatives of R/R_a) since for $r \in R$, $t \in R_a$, $rte(a) = \gamma_{rt}(rta)e(rta) = \gamma_r(ra)\gamma_t(a) = \gamma_r(ra) = re(a)$.

The set $\{X(a) | a \in \pi^{-1}(\Lambda^+) \cap A_0\}$ is linearly independent since the R-orbits of A are disjoint and each R-orbit contains precisely one element of $\pi^{-1}(\Lambda^+) \cap A$.

It now suffices to show that our set spans $K_\gamma[A]^R$. Clearly each $X(a) \in K_\gamma[A]^R$. Let $\sum_{a \in A} c_a e(a) \in K_\gamma[A]^R$. Then

$$r(\sum_{a \in A} c_a e(a)) = \sum_{a \in A} c_a e(a)$$

for all $a \in A$ implies that $c_a \gamma_r(ra) = c_{ra}$ for all $r \in R$ and $a \in A$. Note that if $r \in R_a$ has $\gamma_r(a) \neq 1$, then $c_a \gamma_r(a) = c_a$ implies that $c_a = 0$. So

$$\begin{aligned}
&\sum_{a \in A} c_a e(a) \\
&= \sum_{a \in \pi^{-1}(\Lambda^+) \cap A_0} \sum_{r \in R/R_a} c_{ra} e(ra) \\
&= \sum_{a \in \pi^{-1}(\Lambda^+) \cap A_0} c_a (\sum_{r \in R/R_a} \gamma_r(ra) e(ra)) \\
&= \sum_{a \in \pi^{-1}(\Lambda^+) \cap A_0} c_a X(a)
\end{aligned}$$

as required. ■

Lemma 4.1.3. *$R_v = R_{\pi(v)}$ for all $v \in V$*

Proof: If $r \in R_v$, then $rv = v$ implies that $r\pi(v) = \pi(rv) = \pi(v)$ which shows that $r \in R_{\pi(v)}$

If $r \in R_{\pi(v)}$, then $r\pi(v) = \pi(v)$ so that $rv - v \in V^R$. Let $rv = v + v_0$ for some $v_0 \in V^R$. Then $r^k v = v + kv_0$. But if k is the order of r, we find that $v + kv_0 = v$ so that $v_0 = 0$ and $rv = v$ which implies that $r \in R_v$. ■

From now on, we specialize to the case $\Gamma = \Gamma_\gamma$.

Lemma 4.1.4. *If R is generated by Γ_γ then $\{X(a) | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a K-basis for $K_\gamma[A]^R$*

Proof: Note that

$$R_a \mapsto K^\times$$
$$a \to \gamma_r(a)$$

is a group homomorphism since $\gamma_{r_1 r_2}(a) = \gamma_{r_1}(a)\gamma_{r_2}(r_1^{-1}a) = \gamma_{r_1}(a)\gamma_{r_2}(a)$ where $r_1, r_2 \in R_a$.

By Lemma 4.1.2, it suffices to prove that $\gamma_r(a) = 1$ for all $a \in \pi^{-1}(\Lambda^+) \cap A$ and $r \in R_a$. So $\pi(a) \in \Lambda^+$ and $r \in R_a$. Write w as a product of simple reflections: $w = s_1 \dots s_k$. Then we find by Lemma 4.1.1(c), $s_i \in R_{\pi a}$ for all i so that by Lemma 4.1.3, $s_i \in R_a$ for all i. So it follows that $\gamma_r(a) = \gamma_{s_1}(a) \cdots \gamma_{s_k}(a) = 1$. ■

Definition: [16] A finite subset F of V is *peaked* if there exists $a \in F$ such that $\pi(b) < \pi(a)$ for all $b \in F$ with $b \neq a$. Then a is called the *peak* of F.

Definition: The *support* of an element $y = \sum_{a \in A} y_a e(a) \in K_\gamma[A]$, is the finite set $\text{supp}(y) = \{a \in A | y_a \neq 0\}$. $y \in K_\gamma[A]$ is *peaked* if its support is peaked. Denote the peak of supp(y) by $\hat{y}$.

Lemma 4.1.5. *(a) Let F, F' be peaked subsets of V with peaks a, a' respectively. Then $F + F' = \{b + b' | b \in F, b' \in F'\}$ is peaked with peak $a + a'$.*

(b) Moreover, in the notation of (a), if for $b \in F, b' \in F'$ we have $b + b' = a + a'$ then $b = a$ and $b' = a'$.

(c)[16, Theorem 5] The R-orbit of each element of A is peaked and its peak lies in $\pi^{-1}(\Lambda^+) \cap A$.

(d) If $y, z \in K_\gamma[A]$ are peaked then so is yz and $\widehat{(yz)} = \hat{y} + \hat{z}$.

Proof: (a),(b) If $b \in F$, $b' \in F'$ then $\pi(b) \leq \pi(a)$, $\pi(b') \leq \pi(a')$ implies that $\pi(b+b') = \pi(b)+\pi(b') \leq \pi(a)+\pi(a') = \pi(a+a')$. If equality holds here then $(\pi(a)-\pi(b))+(\pi(a')-\pi(b')) \geq 0$ with $\pi(a)-\pi(b) \geq 0$ and $\pi(a')-\pi(b') \geq 0$. So $\pi(b) = \pi(a)$ and $\pi(b') = \pi(a')$ by definition of $\leq$ and then $b = a, b' = a'$ by definition of peaked. Thus $b + b' = a + a'$ proving (a).

Moreover, if b, b' are as in (b) then equality holds in the above argument. So we have $b = a$ and $b' = a'$ as above, proving (b).

(c) Let $a \in A$. By Lemma 4.1.1(a), there exists $w \in R$ with $\pi(wa) = w\pi(a) \in \Lambda^+$ and by Lemma 4.1.1(b), $r\pi(a) \leq w\pi(a)$ for all $r \in R$. If $\pi(ra) = \pi(wa)$ then $w^{-1}r \in R_{\pi a} = R_a$ by Lemma 4.1.3. So $ra = wa$ as required.

(d) From $yz = \sum_{c\in A}(\sum_{a+b=c} y_a z_b)e(c)$ we see that $\mathrm{supp}(yz) \subset \mathrm{supp}(y) + \mathrm{supp}(z)$. By (a), letting a^*, b^* be the peaks of $\mathrm{supp}(y), \mathrm{supp}(z)$, we know $a^* + b^*$ is the peak of $\mathrm{supp}(y) + \mathrm{supp}(z)$. By (b), the coefficient of $e(a^* + b^*)$ is $y_{a^*} z_{b^*} \neq 0$. Thus $a^* + b^* \in \mathrm{supp}(yz)$ must be the peak of $\mathrm{supp}(yz)$. ■

Lemma 4.1.6. *Let E be a set of peaked elements in $K_\gamma[A]^R$ such that $\pi(E) = \Lambda^+ \cap \pi(A)$. Then $K_\gamma[A]^R$ is generated by E as a $K[A^R]$-module.*

Proof: Since R is generated by Γ_γ, $r \mapsto \gamma_r(a)$ is a homomorphism $R \to K^\times$ for all $a \in \cap_{s\in\Gamma_\gamma} A^{(s)} = A^R$, hence $\gamma_r|_{A^R} = 1$ for all $r \in R$. So we note that if $a \in A^R$, $e(a) \in K_\gamma[A]^R$ and hence that $K[A^R] \subset K_\gamma[A]^R$.

Let S be the $K[A^R]$-submodule of $K_\gamma[A]^R$ spanned by E. By Lemma 4.1.4, we need only show $X(a) \in S$ for all $a \in \pi^{-1}(\Lambda^+) \cap A$.

Suppose not. Then by Lemma 4.1.1(b), we can find $a \in \pi^{-1}(\Lambda^+) \cap A$ such that a is minimal subject to $X(a) \notin S$. Find $y \in E$ with $\dot{y} = a - b$ where $b \in A^R$. Then by Lemma 4.1.4, we find that $X(a) - ye(b) \in K_\gamma[A]^R$ must be a linear combination $\sum_{c\in\pi^{-1}\cap A} m_c X(c)$. the support of $X(a) - ye(b)$ consists by Lemma 4.1.5(c), (d) of elements d with $\pi(d) < \pi(a)$. In particular, $X(a) - ye(b)$ is a K linear combination of $\{X(c) | c \in \pi^{-1}(\Lambda^+) \cap A, \pi(c) < \pi(a)\} \subset S$ by minimality of $\pi(a)$. Thus $X(a) - ye(b) \in S$ and $X(a) \in S$, contradiction. ■

Proposition 4.1.7. *Let $Y_1, \ldots, Y_n$ be peaked elements of $K_\gamma[A]^R$ and let E be the multiplicative monoid they generate. If the map*

$$E \to \Lambda^+ \cap \pi(A)$$
$$Y \to \pi(\dot{Y})$$

is surjective, then $K_\gamma[A]^R$ is generated by E as a $K[A^R]$ module. Furthermore, if this map is bijective then E is a $K[A^R]$ basis of $K_\gamma[A]^R$.

Proof: The first statement follows from the last lemma.

Suppose $\sum_{Y\in E} r_Y Y = 0$ where $Y \in E$ and $r_Y \in K[A^R]$. Assuming not all the coefficients are zero, choose $Z \in E$ with $\pi(\check{Z})$ maximal in $\{\pi(\check{Y})|r_Y \neq 0\}$. If b is in the support of r_Z, then $e(b)e(\check{Z})$ must appear a second time in $\sum_{Z\neq Y\in E} r_Y Y$: i.e. there exists $Y \in E, Y \neq Z$ so that $e(b)e(\check{Z}) = e(c)e(x)$ where $x \in \text{supp}(Y), c \in \text{supp}(r_Y)$. Then $b+\check{Z} = c+x$ with $b, c \in A^R$ implies $\pi(\check{Z}) = \pi(x) \leq \pi(\check{Y})$. Since $\pi(\check{Z})$ was maximal, we find that $\pi(\check{Z}) = \pi(\check{Y})$. Using the bijection hypothesis, $Z = Y$. But then x lies in the support of Z and $\pi(x) = \pi(\check{Z})$. Then $x = \check{Z}$ by the fact that $\check{Z}$ is the peak of Z. But then $b(\check{Z})$ occurs in precisely one way. By contradiction, E is linearly independent over $K[A^R]$ as required. ∎

Definition 4.1.8. Let $\omega_1, \ldots, \omega_n$ be the fundamental dominant weights with respect to the base Δ of Φ. A submodule of Λ containing Φ is called a *stretched weight lattice* if it has a $\mathbf{Z}$-basis of the form $\{m_1\omega_1, \ldots, m_n\omega_n\}$ for some $m_1, \ldots, m_n \in \mathbf{Z}_+$.

Corollary 4.1.9. *Suppose $\pi(A) = \oplus_{i=1}^n \mathbf{Z} m_i\omega_i$ is a stretched weight lattice for a suitable root system Φ for A and choose $a_i \in A$ with $\pi(a_i) = m_i\omega_i$. Then $K_\gamma[A]^R$ is a polynomial ring over $K[A^R]$ in the variables $X(a_1), \ldots, X(a_n)$.*

Proof: Λ^+ consists of $\mathbf{Z}_{\geq 0}$ combinations of the linearly independent elements $\omega_1, \ldots, \omega_n$ hence $\pi(A) \cap \Lambda^+$ consists of $\mathbf{Z}_{\geq 0}$ combinations of $m_1\omega_1, \ldots, m_n\omega_n$. We apply Proposition 4.1.7 with $Y_i = X(a_i)$. Then $Y \in E$ means $Y = \prod_{i=1}^n X(a_i)^{k_i}$ with $k_i \in \mathbf{Z}_{\geq 0}$ hence $\check{Y} = \sum_i k_i \check{X}(a_i) = \sum_i k_i a_i$ by Lemma 4.1.5(c),(d), and $\pi(\check{Y}) = \sum_{i=1}^n k_i m_i \omega_i$ verifying the bijective hypothesis. So E is a $K[A^R]$-basis of $K_\gamma[A]^R$.

Moreover, the independence of $m_1\omega_1, \ldots, m_n\omega_n$ implies that E is freely generated by $X(a_1), \ldots, X(a_n)$ as a multiplicative monoid. Indeed, if $\prod_{i=1}^n X(a_i)^{k_i}$ for some $k_i \in \mathbf{Z}_{\geq 0}$ then $\sum_{i=1}^n k_i m_i \omega_i = \pi(\check{(}\prod_{i=1}^n X(a_i)^{k_i})) = 0$ implies that $k_i = 0$ for all $i = 1, \ldots, n$. ∎

Following Farkas, we want to embed A into a $\mathbf{Z}R$ lattice B on V to which the Corollary 4.1.9 applies. This is not as straightforward now as we also have to accommodate the cocycle γ. The next few technical lemmas will be used to determine which conditions are required to axiomatize the situation. In the following, we let B denote a $\mathbf{Z}R$ lattice on V with $A \subset B$.

Lemma 4.1.10. *(a)* $[\gamma] \in \mathrm{Im}(H^1(R, \mathrm{Hom}(B, K^\times) \to H^1(R, \mathrm{Hom}(A, K^\times))$ *is equivalent to the existence of a 1-cocycle* γ' *in* $[\gamma]$ *so that there exists a 1-cocycle* $\hat{\gamma} : R \to \mathrm{Hom}(B, K^\times)$ *such that* $\hat{\gamma}_r|_A = \gamma'_r$ *for all* $r \in R$*. This means that* $A \hookrightarrow B$ *induces* $K_{\gamma'}[A] \hookrightarrow K_{\hat{\gamma}}[B]$ *R-equivariantly.*

(b) Assume $[\gamma] \in \mathrm{Im}(H^1(R, \mathrm{Hom}(B, K^\times) \to H^1(R, \mathrm{Hom}(A, K^\times))$, $\pi(B) \subset \Lambda$ *and that* γ *is chosen as in (a) (where it's called* γ'*). Let* $b_1, \dots, b_k \in B$ *and form* $X(b_1), \dots, X(b_k) \in K_{\hat{\gamma}}[B]^R$*. Then* $X(b_1) \cdots X(b_k) \in K_\gamma[A]^R$ *if and only if* $\sum_{i=1}^n b_i \in A$*.*

Proof:

(a) The condition $[\gamma] \in \mathrm{Im}(H^1(R, \mathrm{Hom}(B, K^\times) \to H^1(R, \mathrm{Hom}(A, K^\times))$ with the map induced by the inclusion $A \hookrightarrow B$ holds iff $[\gamma]$ is the image of some $[\hat{\gamma}] \in H^1(R, \mathrm{Hom}(B, K^\times))$. Then for a 1-cocycle $\hat{\gamma}$ representing $[\hat{\gamma}]$, we see that $r \mapsto \hat{\gamma}_r|_A$ is in the same cohomology class as $r \mapsto \gamma_r$. So $\hat{\gamma}_r|_A = \gamma'_r$ with $\gamma' \sim \gamma$. By Lemma 2.1.3, replacing γ by γ' gives $K_{\gamma'}[A]^R \cong K_\gamma[A]^R$ so after this $A \hookrightarrow B$ induces $K_\gamma[A]^R \hookrightarrow K_{\hat{\gamma}}[B]^R$. We will always assume γ is so chosen.

In fact if $K_\gamma[A]^R \hookrightarrow K_{\hat{\gamma}}[B]^R$ is induced by $A \hookrightarrow B$ then

$$\sum_{r \in R} \gamma_r(ra)e(ra) = \sum_r re(a) = \sum_r \hat{\gamma}_r(ra)e(ra)$$

for all $a \in A$ so $\hat{\gamma}_r|_A = \gamma_r$ for all $r \in R$.

(b) **Claim:** R acts trivially on B/A.

Proof of Claim: If $b \in B$ then $\langle b, \alpha\rangle = \langle \pi(b), \alpha\rangle \in \mathbf{Z}$ for all $\alpha \in \Phi$ since $\pi(B) \subset \Lambda$. We need to prove $rb - b \in A$ for all $r \in R, b \in B$ by induction on the length m of r. For $m = 1$, $s_\alpha(b) - b = -\langle b, \alpha\rangle\alpha \in \Phi \subset A$. Then

$$s_{\alpha_1} \cdots s_{\alpha_m}(b) - b = s_{\alpha_1} \cdots s_{\alpha_{m-1}}(s_{\alpha_m} b - b) + (s_{\alpha_1} \cdots s_{\alpha_{m-1}} b - b) \in A$$

by the inductive hypothesis. So $rb - b \in A$ for all $r \in R$, $b \in B$, as required.

A typical element in the support of $X(b_1) \dots X(b_k)$ has the form $\sum_i r_i b_i$ for some $r_1, \dots, r_k \in R$. Since $K_{\hat{\gamma}}[B]^R \cap K_\gamma[A] = K_\gamma[A]^R$, we see that $X(b_1) \cdots X(b_k) \in K_\gamma[A]^R$ iff $X(b_1) \cdots X(b_k) \in K_\gamma[A]$ iff $\mathrm{supp}(X(b_1) \cdots X(b_k)) \subset A$ iff $\sum_{i=1}^k r_i b_i \in A$ for all $r_i \in R$. But by the claim $\sum_i r_i b_i - \sum_i b_i \in A$ so this last holds iff $\sum_i b_i \in A$. ∎

Lemma 4.1.11. *(a) The set of indecomposable elements in* $\Lambda^+ \cap \pi(A)$ *is finite.*

(b) Assume $\pi(B) = \oplus_{i=1}^{n} \mathbf{Z} m_i\omega_i$ is a stretched weight lattice for Φ and choose $b_1, \ldots, b_n \in B$ with $\pi(b_i) = m_i\omega_i$. Then every $a \in \pi^{-1}(\Lambda^+) \cap A$ can be written as $a = b_0 + \sum_{i=1}^{n} k_i b_i$ with unique $k_i \in \mathbf{Z}, k_i \geq 0$ and $b_0 \in B^R$.

Proof:

(a) Since $\mathbf{Z}\Phi = \mathbf{Z}\pi(\Phi) \subset \pi(A) \subset \Lambda$ has finite index, $d = [\Lambda : \pi(A)]$ is finite. Then $d\omega_i \in \Lambda^+ \cap \pi(A)$ for all i. So if $\sum_{i=1}^{n} k_i\omega_i \in \Lambda^+ \cap \pi(A)$ is indecomposable then $0 \leq k_i \leq d$ for all i.

(b) Since $\pi(a) = \sum_{i=1}^{n} k_i m_i \omega_i$ with unique $k_i \geq 0$. Then $a - \sum_{i=1}^{n} k_i b_i \in \mathrm{Ker}_B(\pi) = B^R$ shows that we have a unique $b_0 \in B^R$. ■

Lemma 4.1.12. *For a group H, a $\mathbf{Z}H$ lattice X, $S \subset H$ which generates a normal subgroup N of H and T a H/N-module written multiplicatively, set*

$$H^1_S(H, \mathrm{Hom}(X,T)) := \{[f] \in H^1(H, \mathrm{Hom}(X,T)) | f_s|_{X^{\langle s\rangle}} = 1 \text{ for all } s \in S\}$$

Then

(a)

$$H^1_S(H, \mathrm{Hom}(X,T)) = \cap_{s\in S} \mathrm{Ker}(i_s^* \circ \mathrm{Res}^H_{\langle s\rangle}) = \cap_{s\in S/H} \mathrm{Ker}(i_s^* \circ \mathrm{Res}^H_{\langle s\rangle})$$

is a subgroup of $H^1(H, \mathrm{Hom}(X,T))$ where $i_s : X^{\langle s\rangle} \hookrightarrow X$.
(b) An inclusion $i : X \hookrightarrow Y$ of $\mathbf{Z}H$ lattices induces a homomorphism $H^1_S(H, \mathrm{Hom}(Y,T)) \to H^1_S(H, \mathrm{Hom}(X,T))$.

Proof:

(a) Since $\langle s\rangle$ acts trivially on $\mathrm{Hom}(X^{\langle s\rangle}, T)$, $H^1(\langle s\rangle, \mathrm{Hom}(X^{\langle s\rangle}, T)) = \mathrm{Hom}(\langle s\rangle, \mathrm{Hom}(X^{\langle s\rangle}, T))$ via $[f] \leftrightarrow f_s$. This shows the first equality and proves that $H^1_S(H, \mathrm{Hom}(X,T))$ is a subgroup.

For the second equality, we need to show that $\mathrm{Ker}(i_s^* \circ \mathrm{Res}^H_{\langle s\rangle}) = \mathrm{Ker}(i_t^* \circ \mathrm{Res}^G_{\langle t\rangle})$ if $s \sim t$ in H. Let $t = hsh^{-1}$ for some $h \in H$ and let $[f] \in \mathrm{Ker}(i_s^* \circ \mathrm{Res}^H_{\langle s\rangle})$. For $x \in X^{\langle t\rangle} = hX^{\langle s\rangle}$, we have

$$\begin{aligned} f_t(x) &= f_{hsh^{-1}}(x) \\ &= f_h(x)h[f_s(h^{-1}x)]sh[f_{h^{-1}}(s^{-1}h^{-1}x)] \\ &= f_h(x)h[f_{h^{-1}}(h^{-1}x)] \\ &= f_{hh^{-1}}(x) = 1 \end{aligned}$$

shows that $[f] \in \mathrm{Ker}(i_t^* \circ \mathrm{Res}^H_{(t)})$. The reverse inclusion is symmetric. ■

Now Lemma 4.1.12 applies to the case $H = R, S = \Gamma, N = \langle\Gamma\rangle$. Now we will list the required conditions in order to obtain an embedding of $K_\gamma[A]^R$ into a UFD $K_{\hat\gamma}[B]^R$.

(EMB) Let $\Gamma = \Gamma_\gamma$ and $R = \langle\Gamma_\gamma\rangle$. Let Φ be a suitable root system for A and Γ with weight lattice $\Lambda(\Phi)$. There exists a $\mathbf{Z}R$ lattice B on V which contains A and satisfies:

(E1) $[\gamma] \in \mathrm{Im}(H^1_\Gamma(R, \mathrm{Hom}(B, K^\times)) \to H^1_\Gamma(R, \mathrm{Hom}(A, K^\times)))$. Let $\hat\gamma$ be a 1-cocycle which maps to γ.

(E2) $\pi(B) \subset \Lambda(\Phi)$

(E3) $\pi(B) = \oplus_{i=1}^n \mathbf{Z}m_i\omega_i$ is a stretched weight lattice for Φ. Choose $b_i \in B$ with $\pi(b_i) = m_i\omega_i$ for all i.

Notation: Assume (EMB). For $a \in A$, write $a = b_0 + \sum_{i=1}^n k_ib_i$ as in Lemma 4.1.11(b). Then set $X_a = X(b_0)X(b_1)^{k_1}\cdots X(b_n)^{k_n}$. Note that for $a \in \pi^{-1}(\Lambda^+) \cap A$, $X_a \in K_\gamma[A]^R$ by Lemma 4.1.10(b).

Proposition 4.1.13. *Assume (EMB). Then*
(a) $K_{\hat\gamma}[B]^R$ is a polynomial ring in $X(b_1), \ldots, X(b_n)$ over $K[B^R]$.
(b) $X_a \in K_\gamma[A]^R$ is irreducible iff $\pi(a)$ is indecomposable in $\pi(A) \cap \Lambda^+$.
(c) Let E be the multiplicative monoid $\{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$. Then E generates $K_\gamma[A]^R$ as a $K[A^R]$ module. Let $\pi(a_1), \ldots, \pi(a_N)$ be the indecomposable elements in $\Lambda^+ \cap \pi(A)$. Then $X_{a_1}, \ldots, X_{a_N}$ are irreducible in $K_\gamma[A]^R$ and generate E as a multiplicative monoid.

Proof: (a) We want to apply Corollary 4.1.9 to $K_{\hat\gamma}[B]^R$. Note that a suitable root system for A is also suitable for B since then $\Phi \subset A \subset B$ and $\pi(B) \subset \Lambda(\Phi)$ by (E2). (E1) ensures that $\Gamma = \Gamma_{\hat\gamma}$ and (E3) shows that $\pi(B)$ is a stretched weight lattice.

(b) Note that $X(b_0) = e(b_0)$ is a unit in $K_{\hat\gamma}[B]^R$. Since $X(b_1), \ldots, X(b_n)$ are irreducible elements of the UFD $K_{\hat\gamma}[B]^R$ by (a), we see that any factor of $X_a = X(b_0)X(b_1)^{k_1}\cdots X(b_n)^{k_n}$ must take the form $uX(b_1)^{l_1}\cdots X(b_n)^{l_n}$ where $0 \leq l_i \leq k_i$ for all i and $u \in (K_{\hat\gamma}[B]^R)^\times = K^\times e(B^R)$. So up to multiplication by an element of $K^\times$, $u = X(b_0')$ with $b_0' \in B^R$. This means by Lemma 4.1.11(b) that a divisor of X_a in $K_\gamma[A]^R$ must take the form $cX_{a'} =$

$cX(b_0')X(b_1)^{l_1}\cdots X(b_n)^{l_n}$ where $c \in K^\times, b_0' \in B^R$ and $b_0' + \sum_{i=1} l_i b_i = a' \in A$, $0 \le l_i \le k_i$.

Now suppose X_a is reducible in $K_\gamma[A]^R$. Then, by the above, $X_a = cX_{a'}X_{a''}$ with $c \in K^\times$. Taking peaks, we get $a = a' + a''$ and hence $\pi(a) = \pi(a') + \pi(a'')$. Since $X_{a'}, X_{a''}$ are non-units, $\pi(a'), \pi(a'')$ are both non-zero so that $\pi(a)$ is decomposable in $\Lambda^+ \cap \pi(A)$.

If $\pi(a) = \sum_{i=1}^n k_i m_i \omega_i$ is decomposable in $\Lambda^+ \cap \pi(A)$, then $\pi(a) = \pi(a') + \pi(a'')$ where $a', a'' \in \pi^{-1}(\Lambda^+) \cap A$. By Lemma 4.1.11(b), we have that $a' = b_0' + \sum_{i=1} l_i b_i$ and $a'' = b_0'' + \sum_{i=1}^n l_i' b_i$ where $b_0', b_0'' \in B^R$ and $l_i, l_i' \ge 0$. Since $\pi(a') = \sum_{i=1}^n l_i m_i \omega_i$, then $\pi(a'') = \sum_{i=1}^n (k_i - l_i) m_i \omega_i = \sum_{i=1}^n l_i' m_i \omega_i$ so that $a'' = b_0'' + \sum_{i=1}^n (k_i - l_i) b_i$. Now $a_0 = a - a' - a'' \in \mathrm{Ker}_A(\pi) = A^R$. So we find that $a_0 = b_0 - b_0' - b_0''$. But then $X_a = e(b_0)X(b_1)^{k_1}\cdots X(b_n)^{k_n} = e(a_0)X_{a'}X_{a''}$. Since $\pi(a'), \pi(a'') \ne 0$, we see that $X_{a'}, X_{a''}$ are not in $(K_\gamma[B])^R)^\times = K^\times e(B^R)$ and so cannot be units in $K_\gamma[A]^R$. Hence X_a is reducible.

(c) The map

$$E \to \Lambda^+ \cap \pi(A)$$
$$X_a \to \pi(\zeta(X_a)) = a$$

is surjective. The first statement then follows from (b) and Proposition 4.1.7. For $a \in \pi^{-1}(\Lambda^+) \cap A$, $\pi(a) = \sum_{i=1}^N k_i \pi(a_i)$ for some $k_i \ge 0$, so that $a_0 = a - \sum_{i=1}^N k_i a_i \in A^R$ and $X_a = e(a_0)\prod_{i=1}^N X(a_i)^{k_i}$ as required. ■

Proposition 4.1.14. *Assume (EMB). Then $K_\gamma(A)^R$ is rational over K.*

Proof:

By Proposition 4.1.13(a), $K_\gamma[B]^R$ is a polynomial ring in $X(b_1), \dots, X(b_n)$ over $K[B^R]$. The multiplicative subgroup M of the field of fractions $K_\gamma(B)^R$ generated by B^R and $X(b_1), \dots, X(b_n)$ is a free abelian group of finite rank whose members are linearly independent over K. Now

$$\{e(b_0)X(b_1)^{k_1}\cdots X(b_n)^{k_n} | b_0 \in B^R, k_i \in \mathbf{Z}, k_i \ge 0\}$$

is a K-basis of $K_\gamma[B]^R$. Let S be the multiplicative monoid generated by $X(b_1), \dots, X(b_n)$. After localizing at S, each $X(b_i)$ becomes a unit so that

$$\{e(b_0)X(b_1)^{k_1}\cdots X(b_n)^{k_n} | b_0 \in B^R, k_i \in \mathbf{Z}\}$$

is a K-basis for $S^{-1}K_{\bar\gamma}[B]^R$. Hence $S^{-1}K_{\bar\gamma}[B]^R$ is the group algebra $K[M]$ and so $K_{\bar\gamma}(B)^R = K(M)$.

By Proposition 4.1.13(c), $K_\gamma[A]^R$ is generated by the multiplicative monoid $E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ as a $K[A^R]$ module. Since $A^R \subset E$, we see that in fact E generates $K_\gamma[A]^R$ over K. Now localize $K_\gamma[A]^R$ at the monoid E. Consider $E^{-1}K_\gamma[A]^R \subset E^{-1}K_{\bar\gamma}[B]^R$. We first show that $E^{-1}K_{\bar\gamma}[B]^R$ is also the group algebra $K[M]$. It suffices to show that $X(b_i)$ is a unit for each i. Let $d = [B : A]$. Then $db_j \in \pi^{-1}(\Lambda^+) \cap A$. So $X(b_j)^d = X_{db_j}$ is invertible in $E^{-1}K_{\bar\gamma}[B]^R$ and hence so is $X(b_j)$. So $E^{-1}K_{\bar\gamma}[B]^R = K[M]$.

Now let L be the subgroup of M generated by E. Hence L is also free abelian of finite rank. By Proposition 4.1.13(c), $\{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a K-generating set for $K_\gamma[A]^R$. Hence L spans $E^{-1}K_\gamma[A]^R$ over K so that $E^{-1}K_\gamma[A]^R = K[L]$ is a group algebra. So $K_\gamma(A)^R \cong K(L)$ as required. ∎

Remark 4.1.15. In fact, as noted by Lorenz, the proof of 4.1.14 shows the following stronger result: Assume (EMB). Then the invariant algebra $K_\gamma[A]^R$ is an affine normal semigroup algebra over K [8]. Indeed since the multiplicative monoid E is contained in M, the elements of E are linearly independent over K. This shows that $K_\gamma[A]^R = K[E]$ is a semigroup algebra, and $K_\gamma[A]^R$ is clearly affine and normal, as an invariant subalgebra of the affine normal algebra $K[A]$ under a finite group action. Letting L denote the subgroup of M that is generated by E, we see that L is free abelian of finite rank, as M is, and $K_\gamma(A)^R = Q(K[E]) = Q(K[L]) = K(L)$ is rational over K.

Proposition 4.1.16. *Assume (EMB). If $K_\gamma[A]^R$ is a unique factorization domain then $\pi(A)$ is a stretched weight lattice.*

Proof: By Proposition 4.1.13(a), $K_{\bar\gamma}[B]^R$ is a polynomial ring over $K[B^R]$ in the variables $X(b_1), \ldots, X(b_n)$. We want to first show that the set of indecomposables in $\Lambda^+ \cap \pi(A)$ is of the form $\{k_1 m_1 \omega_1, \ldots, k_n m_n \omega_n\}$ where $k_i > 0$ for all $i = 1, \ldots, n$.

Let $a \in \pi^{-1}(\Lambda^+) \cap A$ be such that $\pi(a)$ is indecomposable. Then by Lemma 4.1.11(b), $a = b_0 + \sum_{i=1}^n k_i m_i \omega_i$ where $k_1, \ldots, k_n \geq 0, b_0 \in B^R$ and by Proposition 4.1.13(b), $X_a = X(b_0)X(b_1)^{k_1} \cdots X(b_n)^{k_n}$ is irreducible in $K_\gamma[A]^R$. Since B/A is finite, we let $d = [B : A]$. Then $db_i \in X$ for all i. So $X(b_i)^d \in K_\gamma[A]^R$. Now $X_a^d = [X(b_0)]^d[X(b_1)^d]^{k_1} \cdots [X(b_n)^d]^{k_n}$. Since X_a is an irreducible in the UFD $K_\gamma[A]^R$ then X_a divides $X(b_j)^d$ for some

$1 \leq j \leq n$. So $k_i = 0$ for all $i \neq j$ and $\pi(a) = k_j m_j \omega_j$. This shows that any indecomposable element of $\Lambda^+ \cap \pi(A)$ is of the form $k_j m_j \omega_j$. Since $db_i \in A$ for $i = 1, \ldots, n$, there exists a minimal $k_i > 0$ such that $k_i b_i \in A$. If $k_i' m_i \omega_i \in \Lambda^+ \cap \pi(A)$ then $k_i' = qk_i + r$ where $0 \leq r < k_i$. But then $r\omega_i \in \Lambda^+ \cap \pi(A)$. By minimality of k_i, $k_i' = qk_i$ so that we may conclude that $\{k_1 m_1 \omega_1, \ldots, k_n m_n \omega_n\}$ is the set of indecomposable elements in $\Lambda^+ \cap \pi(A)$.

Since $\{k_1 m_1 \omega_1, \ldots, k_n m_n \omega_n\}$ is $\mathbf{Z}$-linearly independent, we need only show that it spans $\pi(A)$. Suppose $\sum_{i=1}^n l_i m_i \omega_i \in \pi(A)$. Choose N sufficiently large so that $\frac{l_i}{k_i} \leq N$ for all i. Since $k_i m_i \omega_i \in \pi(A)$ we have $N(\sum_{i=1}^n k_i m_i \omega_i) \in \pi(A)$. Thus $\sum_{i=1}^n (Nk_i - l_i) m_i \omega_i \in \Lambda^+ \cap \pi(A)$. So $\sum_{i=1}^n (Nk_i - l_i) m_i \omega_i = \sum_{i=1}^n r_i k_i m_i \omega_i$ for some $r_i \geq 0$ since every element of $\Lambda^+ \cap \pi(A)$ is a sum of indecomposables. Hence $\sum_{i=1}^n l_i m_i \omega_i = \sum_{i=1}^n (N - r_i) k_i m_i \omega_i$ as required. ∎

4.2 Embedding

Let G be a finite group acting faithfully on a lattice A and a field K. Let $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$. Let R be the subgroup of G generated by $\Gamma \equiv \Gamma_\gamma$. We have an exact sequence of $\mathbf{Z}G$ lattices

$$0 \to A^R \to A \xrightarrow{\pi} A/A^R \to 0$$

Set $X = A/A^R$, let $\tau : X \to A$ be a $\mathbf{Z}$-splitting of $\pi : A \to X$ and let $\rho : G \to \mathrm{Hom}(X, K_\gamma[A^R]^\times)$ be the 1-cocycle of Lemma 2.1.5. Or, more accurately, let $L = K_\gamma(A^R)$ and let $\rho : G \to \mathrm{Hom}(X, L^\times)$ be obtained via $K_\gamma[A^R]^\times \hookrightarrow L$. Then by Lemma 2.1.5, $L_\rho(X)^G \cong K_\gamma(A)^G$.

Lemma 4.2.1. $\Gamma_\rho = \Gamma_\gamma = \Gamma$. *In particular,* Γ_ρ *generates* R.

Proof: Let $s \in \Gamma_\gamma$. Then s acts as a reflection on A and trivially on A^R hence as a reflection on X. For $n \in A^R$ we have $se(n) = \gamma_s(sn)e(sn) = \gamma_s(n)e(n) = e(n)$ since $A^R \subset A^{\langle s \rangle} \subset \mathrm{Ker}(\gamma_s)$. Since s also acts trivially on K, we see that s acts trivially on $K_\gamma[A^R]$ and hence on L. Finally we show that $X^{\langle s \rangle} \subset \mathrm{Ker}(\rho_s)$: If $x \in X^{\langle s \rangle}$, there exists $a \in A^{\langle s \rangle}$ such that $\pi(a) = x$ since $H^1(\langle s \rangle, A^R) = 0$. But $\pi(\tau(x)) = x$ implies that $\tau(x) - a \in A^R \subset A^{\langle s \rangle}$. So $\tau(x) \in A^{\langle s \rangle}$ implies that

$$\beta_s(x) = (s\tau)(x) - \tau(x) = s[\tau(sx)] - \tau(x) = s[\tau(x)] - \tau(x) = 0$$

and so $\rho_s(x) = \gamma_s(\tau(x))\gamma_s(\beta_s(x))e(\beta_s(x)) = \gamma_s(\tau(x)) = 1$ since $s \in \Gamma_\gamma$ and $\tau(x) \in A^{(s)}$. Thus $\Gamma_\gamma \subset \Gamma_\rho$.

Conversely let $s \in \Gamma_\rho$. Then s fixes L hence fixes K and A^R and satisfies $A^R \subset \mathrm{Ker}(\gamma_s)$. Since s acts as a reflection on X and fixes A^R, it is also a reflection on A. Finally if $x \in X^{(s)}$ then $\tau(x) \in A^{(s)}$ as above so $1 = \rho_s(x) = \gamma_s(\tau(x))\gamma_s(\beta_s(x))e(\beta_s(x))$ with $\beta_s(x) = 0$ as before. It follows that $\tau(X^{(s)}) \subset \mathrm{Ker}(\gamma_s)$ which with $A^R \subset \mathrm{Ker}(\gamma_s)$ and $\tau(X^{(s)}) \subset A^{(s)}$ implies $A^{(s)} \subset \mathrm{Ker}(\gamma_s)$. Thus $\Gamma_\rho \subset \Gamma_\gamma$. ∎

We want to embed $K_\gamma(A^R)_\rho[X]^R$ into $K_\gamma(A^R)_{\bar\rho}[Y]^R$ for a suitable stretched weight lattice Y. We first need to find an appropriate stretched weight lattice.

Notation: Let Φ be a suitable root system for $X = \pi(A)$, let $\{\alpha_1, \ldots, \alpha_n\}$ be a base for Φ and let $s_i = s_{\alpha_i}$ for all $i = 1, \ldots, n$. For all $i = 1, \ldots, n$, define m_i so that $\mathrm{Im}_X(s_i - 1) = \mathbf{Z}m_i\alpha_i$.

Lemma 4.2.2. *$I_R X = \oplus_{i=1}^n \mathbf{Z}m_i\alpha_i$ where I_R is the augmentation ideal of $\mathbf{Z}R$.*

Proof: Since $\{s_i\}$ is a set of group generators for R, then $\{s_i - 1\}$ is a set of R-module generators for I_R. So $I_R X$ is the R-module generated by $\sum_{i=1}^n \mathrm{Im}_X(s_i - 1) = \oplus_{i=1}^n \mathbf{Z}m_i\alpha_i$. It suffices to show that $\oplus_{i=1}^n \mathbf{Z}m_i\alpha_i$ is R-stable:

$$s_j(m_i\alpha_i) = m_i\alpha_i - m_i\frac{\langle\alpha_i, \alpha_j\rangle}{m_j}m_j\alpha_j \in \oplus_{i=1}^n \mathbf{Z}m_i\alpha_i$$

with $\frac{\langle\alpha_i,\alpha_j\rangle}{m_j} \in \mathbf{Z}$ since $\langle\alpha_i, \alpha_j\rangle\alpha_j \in \langle X, \alpha_j\rangle\alpha_j = \mathrm{Im}_X(s_j - 1) = \mathbf{Z}m_j\alpha_j$. ∎

Proposition 4.2.3. *For a trivial $\mathbf{Z}R$ module T written multiplicatively, there is an exact sequence*

$$\mathrm{Hom}(X, T) \xrightarrow{i^*} \mathrm{Hom}(I_R X, T) \overset{\psi_X}{\twoheadrightarrow} H^1_\Gamma(R, \mathrm{Hom}(X, T))$$

where i^* is induced by the inclusion $i : I_R X \hookrightarrow X$ and $\psi_X(\theta) = [f]$ where $[f]$ is represented by a 1-cocycle f such that $f_{s_i}(x) = \theta((s_i - 1)(x))$ for all $i = 1, \ldots, n$, $x \in X$. In particular, ψ_X induces an isomorphism

$$\overline{\psi_X} : \mathrm{Hom}(I_R X, T)/i^*(\mathrm{Hom}(X, T)) \cong H^1_\Gamma(R, \mathrm{Hom}(X, T))$$

Proof: We first show that ψ_X is well-defined. Let F be the free group on $S_1, \ldots, S_n$ and let $F \twoheadrightarrow R$ be the Coxeter presentation with kernel N, the normal closure in F of $\{(S_iS_j)^{m_{ij}} | 1 \le i, j \le n\}$ [21, p. 16]. Consider X, T as F modules via the inflation $F \twoheadrightarrow R$. Given $\theta \in \mathrm{Hom}(I_R X, T)$, set $\tilde{f}_{S_i}(x) = \theta((s_i - 1)x)$ for all $x \in X$, $i = 1, \ldots, n$. Since F is a free group on $S_1, \ldots, S_n$, there exists a unique 1-cocycle $\tilde{f} : F \to \mathrm{Hom}(X, T)$ with the above $\tilde{f}_{S_i}$ on generators. (Setting $\tilde{f}_{S_i^{-1}} = -S_i^{-1}\tilde{f}_{S_i}$, we may use the cocycle relations to define $\tilde{f}$ on reduced words.) Now

$$\tilde{f}_{S_i^2}(x) = (f_{S_i} + (S_i f_{S_i}))(x) = f_{S_i}((s_i + 1)(x)) = \theta((s_i - 1)(s_i + 1)x) = 1$$

so that $\tilde{f}_{S_i^2} = 1$. Note that $s_j s_i = (s_i s_j)^{-1}$ so that $\langle s_j s_i \rangle = \langle s_i s_j \rangle$. So we also have

$$\begin{aligned}
f_{(S_iS_j)^{m_{ij}}} &= [(1 + S_iS_j + \cdots + (S_iS_j)^{m_{ij}-1})\tilde{f}_{S_iS_j}](x) \\
&= \tilde{f}_{S_i}(N_{\langle s_i s_j \rangle}x)(S_i\tilde{f}_{S_j})(N_{\langle s_i s_j \rangle}x) \\
&= \tilde{f}_{S_i}(N_{\langle s_i s_j \rangle}x)\tilde{f}_{S_j}(s_i N_{\langle s_i s_j \rangle}x) \\
&= \theta((s_i - 1)N_{\langle s_i s_j \rangle}x)\theta((s_j - 1)s_i N_{\langle s_i s_j \rangle}x) \\
&= \theta((s_j s_i - 1)N_{\langle s_i s_j \rangle}x) = 1
\end{aligned}$$

Since N is the normal closure in F of these relators which act trivially on $\mathrm{Hom}(X, T)$, it follows that $\tilde{f}|_N = 1$.

But now the inflation-restriction sequence shows that $\tilde{f}$ is the inflation of a 1-cocycle $f : R \to \mathrm{Hom}(X, T)$ such that $f_{s_i}(x) = \tilde{f}_{S_i}(x) = \theta((s_i - 1)x)$ for all $i = 1, \ldots, n$ and all $x \in X$. Note that $f_{s_i}(y) = \theta((s_i - 1)y) = 1$ if $y \in X^{\langle s_i \rangle}$. So $\psi_X : \mathrm{Hom}(I_R X, T) \to H^1_\Gamma(R, \mathrm{Hom}(X, T))$ is a well-defined map.

It is clear from the definition that ψ_X is a homomorphism. To see that ψ_X is surjective, we first choose for each $i = 1, \ldots, n$, $x_i \in X$ such that $(s_i - 1)x_i = m_i\alpha_i$. Observe that $X = X^{\langle s_i \rangle} \oplus \mathbf{Z}x_i$ for all $i = 1, \ldots, n$. Let $[f] \in H^1_\Gamma(R, \mathrm{Hom}(X, T))$ be given. By Lemma 4.2.2, $\{(s_i - 1)x_i | i = 1, \ldots, n\}$ is a $\mathbf{Z}$ basis for $I_R X$ so we may define $\theta \in \mathrm{Hom}(I_R X, T)$ such that $\theta((s_i - 1)x_i) = f_{s_i}(x_i)$ for all $i = 1, \ldots, n$. But then $\theta((s_i - 1)x) = f_{s_i}(x)$ for all x and all i follows from the facts that $X = X^{\langle s_i \rangle} \oplus \mathbf{Z}x_i$ and that $X^{\langle s_i \rangle} \subset \mathrm{Ker}(f_{s_i})$. So $\psi_X(\theta) = [f]$ as required.

Suppose $\theta \in \mathrm{Ker}(\psi_X)$. Then the 1-cocycle $r \mapsto f_r$ satisfying $f_{s_i}(x) = \theta((s_i - 1)x)$ is a 1-coboundary and hence there exists $\hat{\theta} \in \mathrm{Hom}(X, T)$ such that $f_r = (r\hat{\theta})/\hat{\theta}$ for all $r \in R$. But then, in particular,

$$\hat{\theta}((s_i - 1)x_i) = ((s_i\hat{\theta})/\hat{\theta})(x_i) = f_{s_i}(x_i) = \theta((s_i - 1)x_i)$$

shows that $i^*(\hat{\theta}) = \theta$ since $\{(s_i - 1)x_i | i = 1, \ldots, n\}$ is a $\mathbf{Z}$-basis of $I_R X$. So $\mathrm{Ker}(\psi_X) \subset i^*(\mathrm{Hom}(X,T))$.

Conversely, suppose $\theta = i^*(\hat{\theta})$ for some $\hat{\theta} \in \mathrm{Hom}(X,T)$. Then for the 1-cocycle $r \mapsto f_r$ satisfying $f_{s_i}(x) = \theta((s_i - 1)x)$ for all i, $x \in X$ we have

$$f_{s_i}(x) = \hat{\theta}((s_i - 1)x) = ((s_i\hat{\theta})/\hat{\theta})(x)$$

But then $f_r = (r\hat{\theta})/\hat{\theta}$ for all $r \in R$ since the s_i generate R. Hence $i^*(\mathrm{Hom}(X,T)) \subset \mathrm{Ker}(\psi_X)$ as required. ∎

Lemma 4.2.4. *Let Λ be the weight lattice for a suitable root system Φ for X and let $\{\omega_i\}_{i=1}^n$ be the basis of fundamental dominant weights corresponding to the base $\{\alpha_i\}_{i=1}^n$ for Φ. Set $Y = \oplus_{i=1}^n \mathbf{Z} m_i \omega_i$. Then*
(a) $X \subset Y \subset \Lambda$ and $I_R Y = I_R X$.
(b) For a trivial $\mathbf{Z}R$ module written multiplicatively, the natural map $H^1_\Gamma(R, \mathrm{Hom}(Y,T)) \to H^1_\Gamma(R, \mathrm{Hom}(X,T))$ induced by $X \hookrightarrow Y$ is surjective.

Proof: (a) Since $\langle X, \alpha_i \rangle = m_i \mathbf{Z}$ and $x = \sum_{i=1}^n \langle x, \alpha_i \rangle \omega_i$ we have $X \subset Y$.

Since $I_R Y$ is the R-module generated by $\sum_{i=1}^n \mathrm{Im}_Y(s_i - 1)$, and $I_R X$ is the R-module generated by $\sum_{i=1}^n \mathrm{Im}_X(s_i - 1)$, it suffices to show that $\mathrm{Im}_Y(s_i - 1) = \mathrm{Im}_X(s_i - 1)$ for all i. But $s_i \omega_j = \omega_j - \delta_{ij}\alpha_i$ implies that

$$\mathrm{Im}_Y(s_i - 1) = \mathbf{Z} m_i (s_i - 1)\omega_i = \mathbf{Z} m_i \alpha_i = \mathrm{Im}_X(s_i - 1)$$

as required.

(b) Note that Lemma 4.2.2 and Proposition 4.2.3 also apply to Y since a suitable root system for X is also suitable for Y. $X \hookrightarrow Y$ induces $I_R X \hookrightarrow I_R Y$ and hence the vertical maps in the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(I_R Y, T) & \xrightarrow{\psi_Y} & H^1_\Gamma(R, \mathrm{Hom}(Y,T)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(I_R X, T) & \xrightarrow{\psi_X} & H^1_\Gamma(R, \mathrm{Hom}(X,T)) \end{array}$$

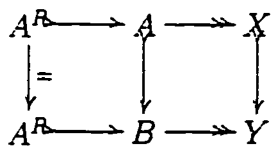
This diagram commutes: The image of $\theta \in \mathrm{Hom}(I_R Y, T)$ under ψ_Y is $[f] \in H^1_\Gamma(R, \mathrm{Hom}(Y,T))$ with representing 1-cocycle f such that $f_{s_i}(y) = \theta((s_i -$

$1)y)$ for all $y \in Y$ which is taken to $[f'] \in H^1_\Gamma(R, \mathrm{Hom}(X,T))$ such that $f'_{s_i}(x) = \theta((s_i - 1)x)$ for all $x \in X$. But $[f']$ is also the image of θ under $\mathrm{Hom}(I_R Y, T) \to \mathrm{Hom}(I_R X, T)$ followed by ψ_X. Since $I_R X = I_R Y$ by (a), (b) follows. ∎.

Proposition 4.2.5. *If R is generated by Γ_γ then $K_\gamma(A)^R$ is rational over $K(A^R)$.*

Proof: By Lemma 2.1.5 and by Lemma 4.2.1, we may replace $K_\gamma(A)^R/K(A^R)$ by $L_\rho(A)^R/L(A^R)$ where $L = K_\gamma(A^R)$. Now $[\rho] \in H^1_\Gamma(R, \mathrm{Hom}(X, L^\times))$ can be lifted to $[\tilde{\rho}] \in H^1_\Gamma(R, \mathrm{Hom}(Y, L^\times))$ by Lemma 4.2.4 (b). In order to apply Proposition 4.1.14, we need only check that the hypotheses apply to our situation. The only ones left to verify: $X \subset Y \subset \Lambda$ and Y a stretched weight lattice were arranged by Lemma 4.2.4. Then Proposition 4.1.14 shows that $L_\rho(X)^R$ is rational over L as required. ∎

Lemma 4.2.6. *Choose the suitable root system for X as Φ_X and define Y as in Lemma 4.2.4. Then*
(a) m_i = order of $H^1(\langle s_i\rangle, X)$
(b) Under the identification of a lattice M with its double dual M^{**}, $Y = (\mathbf{Z}\Phi_{X^*})^*$
(c) There exists a $\mathbf{Z}R$ lattice B such that the following diagram commutes:



Proof: (a) $H^1(\langle s_i\rangle, X) \cong \mathrm{Ker}_X(s_i+1)/\mathrm{Im}_X(s_i - 1) = \mathbf{Z}\alpha_i/\mathbf{Z}m_i\alpha_i \cong \mathbf{Z}/m_i\mathbf{Z}$
(b) Since for each i,

$$\omega_i^* = \sum_{j=1}^{n} \omega_i^*(m_j\omega_j)(m_j\omega_j)^* = m_i(m_i\omega_i)^*$$

where $\omega_1^*, \dots, \omega_n^*$ is the basis of Λ^* dual to $\omega_1, \dots, \omega_n$, we see that $Y = \oplus_{i=1}^n \mathbf{Z}m_i\omega_i$ implies that $Y^* = \oplus_{i=1}^n \mathbf{Z}\frac{1}{m_i}\omega_i^*$. So it suffices to show $\mathbf{Z}\Phi_{X^*} = \oplus_{i=1}^n \mathbf{Z}\frac{1}{m_i}\omega_i^*$.
Claim: $\mathrm{Im}_{X^*}(s_i - 1) = \mathbf{Z}\omega_i^*$

Proof: If $\eta \in X^*$ then $[(s_i - 1)\eta](\omega_j) = \eta(s_i\omega_j) - \eta(\omega_j) = \eta(\omega_j - \delta_{ij}\alpha_i) - \eta(\omega_j) = -\delta_{ij}\eta(\alpha_i)$ hence $(s_i - 1)\eta = \sum_j -\delta_{ij}\eta(\alpha_j)\omega_j^* = -\eta(\alpha_i)\omega_i^* \in \mathbf{Z}\omega_i^*$ since $\alpha_i \in X$ implies that $\eta(\alpha_i) \in \mathbf{Z}$. Conversely, $\mathbf{Z}\alpha_i = \mathrm{Ker}_X(s_i + 1)$ has $X/\mathbf{Z}\alpha_i$ torsionfree. This implies that there exists a lattice C_i such that $X = C_i \oplus \mathbf{Z}\alpha_i$ and so there exists $\eta \in X^*$ with $\eta(\alpha_i) = 1$. This η has $(s_i - 1)\eta = -\omega_i^*$ hence $\omega_i^* \in \mathrm{Im}_{X^*}(s_i - 1)$ proving the Claim.

Note that as an $\mathbf{Z}\langle s_i\rangle$ module, we have $X^* \cong X$ since this is true for each of the indecomposables $\mathbf{Z}, I_{\langle s_i\rangle} = \mathbf{Z}^-, \mathbf{Z}\langle s_i\rangle$. Hence s_i acts as a reflection on X^* and then $H^1(\langle s_i\rangle, X^*) \cong H^1(\langle s_i\rangle, X) \cong \mathbf{Z}/m_i\mathbf{Z}$. From the Claim, we have $(s_i + 1)\omega_i^* = 0$ hence $\mathrm{Ker}_{X^*}(s_i + 1) = X^* \cap \mathbf{Q}\omega_i^*$ since $\mathrm{Ker}_{X^*}(s_i + 1)$ must have rank 1. But then since $\mathrm{Ker}_{X^*}(s_i + 1)/\mathrm{Im}_{X^*}(s_i - 1) \cong \mathbf{Z}/m_i\mathbf{Z}$, we see that $\mathrm{Ker}_{X^*}(s_i + 1) = \mathbf{Z}\frac{1}{m_i}\omega_i^*$ as required.

(c) The following diagram commutes where $m = \mathrm{rank}(A^R)$:

$$\begin{array}{ccc}
H^1(R, \mathrm{Hom}(Y, A^R)) & \longrightarrow & H^1(R, \mathrm{Hom}(X, A^R)) \\
\downarrow \cong & & \downarrow \cong \\
H^1(R, Y^*)^m & \longrightarrow & H^1(R, X^*)^m
\end{array}$$

In view of Lemma 2.1.1, it suffices to show that the top arrow is surjective. Since R acts trivially on A^R, we are reduced, by the above diagram to showing that $H^1(R, Y^*) \to H^1(R, X^*)$ is surjective.

But $Y^* = \mathbf{Z}\Phi_{X^*}$ by (b), so we may appeal to Proposition 2.2.25 of the classification section to get the following commutative diagram:

$$\begin{array}{ccc}
\Lambda(\Phi_{X^*})/\mathbf{Z}\Phi_{X^*} & \xrightarrow{\cong} & H^1(R, \mathbf{Z}\Phi_{X^*}) \\
\downarrow & & \downarrow \\
\Lambda(\Phi_{X^*})/X^* & \xrightarrow{\cong} & H^1(R, X^*)
\end{array}$$

Observe that the horizontal arrows are isomorphisms since Φ_{X^*} is the root system of both $\mathbf{Z}\Phi_{X^*}$ and X^*. Since the left hand map is surjective, so is $H^1(R, \mathbf{Z}\Phi_{X^*}) \to H^1(R, X^*)$ and the proof is complete. ■

4.3 Reduction

Let G be a finite group acting faithfully on a lattice A which also acts on a field K. Let $V = \mathbf{Q} \otimes_{\mathbf{Z}} A$, $[\gamma] \in H^1(G, \mathrm{Hom}(A, K^\times))$ and let R be the normal subgroup of G generated by Γ_γ. We will use the notation of Section 2.3.

Choose a suitable crystallographic root system Φ for A and $\Gamma = \Gamma_\gamma$. Recall from Lemma 2.3.5, for $g \in G$, $\alpha \in \Phi$ and $v \in V$, $gs_\alpha g^{-1} = s_{g\alpha}$ and $\langle gv, g\alpha\rangle = \langle v, \alpha\rangle$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a base for Φ and let Λ be the weight lattice of Φ with basis of fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$ corresponding to the base Δ. Now set

$$\Omega_G = \Omega_G(\Delta) = \{g \in G | g(\Delta) = \Delta\}$$

Then from Lemma 2.3.6(a), we recall that $G \cong R \rtimes \Omega_G$.

Lemma 4.3.1. *$\mathbf{Z}\Phi$ and Λ are isomorphic $\mathbf{Z}\Omega_G$ permutation lattices.*

Proof:

$\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is a $\mathbf{Z}$ basis for the root lattice $\mathbf{Z}\Phi$ and the set of fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$ corresponding to Δ is a $\mathbf{Z}$ basis for Λ. By definition, $\langle \omega_i, \alpha_j\rangle = \delta_{ij}$ so that for $\lambda \in \Lambda$, $\lambda = \sum_{i=1}^n \langle \lambda, \alpha_i\rangle \omega_i$. Since Ω_G stabilizes the base Δ, we see that for each $t \in \Omega_G$, there exists $\sigma \in S_n$ such that $t(\alpha_i) = \alpha_{\sigma(i)}$ for $i = 1, \ldots, n$. This shows that $\mathbf{Z}\Phi$ is a permutation $\mathbf{Z}\Omega_G$ lattice. Now $\langle t\omega_i, \alpha_{\sigma(j)}\rangle = \langle t\omega_i, t\alpha_j\rangle = \langle \omega_i, \alpha_j\rangle = \delta_{ij} = \delta_{\sigma(i)\sigma(j)} = \langle \omega_{\sigma(i)}, \alpha_{\sigma(j)}\rangle$ for all j implies that $t\omega_i = \sum_{j=1}^n \langle t\omega_i, \alpha_j\rangle \omega_j = \omega_{\sigma(i)}$ for all i. This shows that $\mathbf{Z}\Phi \to \Lambda, \alpha_i \mapsto \omega_i$ is a $\mathbf{Z}\Omega_G$ isomorphism. ■

Note that Lemma 4.1.12 applies to $H = G$, $S = \Gamma$, $N = R$, $T = K^\times$.

Proposition 4.3.2. *Suppose*

$$[\gamma] \in \mathrm{Im}(H^1_\Gamma(G, \mathrm{Hom}(B, K^\times)) \to H^1_\Gamma(G, \mathrm{Hom}(A, K^\times)))$$

where $B = e_R A \oplus \Lambda$. Then the invariant fields $K_\gamma(A)^G$ and $K_\gamma(A)^{\Omega_G}$ are isomorphic under an isomorphism that is the identity on $K_\gamma(A^R)^G = K_\gamma(A^R)^{\Omega_G}$. In particular, $K_\gamma(A)^G$ is rational over $K_\gamma(A^R)^G$ if and only if $K_\gamma(A)^{\Omega_G}$ is rational over $K_\gamma(A^R)^{\Omega_G}$.

Proof: Let $A_0 = e_R A \subset V^R$ and let Λ be the weight lattice for a suitable root system Φ of A. Then $A \subset \pi(A) \oplus A_0$ and $\pi(A) \subset \Lambda$. Setting $B = A_0 \oplus \Lambda$ then gives a $\mathbf{Z}G$ lattice containing A with $B^R = A_0$ which satisfies the (EMB) hypotheses of Twisted Farkas: Indeed $\pi(B) = \Lambda$ and $[\gamma] \in \mathrm{Im}(H^1_\Gamma(G, \mathrm{Hom}(B, K^\times)) \to H^1_\Gamma(G, (A, K^\times))$ implies that $\mathrm{Res}^G_R[\gamma] \in \mathrm{Im}(H^1_\Gamma(R, \mathrm{Hom}(B, K^\times)) \to H^1_\Gamma(R, \mathrm{Hom}(A, K^\times))$.

Using the notation of the proof of Proposition 4.1.14, we let M be the multiplicative subgroup of $K_{\hat{\gamma}}(B)^R$ generated by $e(A_0)$ and $X(\omega_1), \ldots, X(\omega_n)$ (note that $\Lambda \subset B$ and $\pi(\omega_i) = \omega_i$) which is then free abelian and linearly independent over K. Recall that each $a \in \pi^{-1}(\Lambda^+) \cap A$ has a unique expression as $a = a_0 + \sum_{i=1} k_i\omega_i$ where $a_0 \in A_0$ and $k_i \geq 0$. Then $X_a \equiv e(a_0)\prod_{i=1}^n X(\omega_i)^{k_i} \in K_\gamma[A]^R$. Then $E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$ is a multiplicative monoid with $e(A^R) \subset E$. In Proposition 4.1.14, we showed that $E^{-1}K_{\hat{\gamma}}[B]^R = K[M]$ and $E^{-1}K_\gamma[A]^R = K[L]$ where L is the subgroup of M generated by E, which is hence also free abelian. Note that $K[M]$ and $K[L]$ are the K-subalgebras of $K_{\hat{\gamma}}(B)^R$, respectively $K_\gamma(A)^R$ generated by M and L. They are both group algebras of free abelian groups of finite rank written multiplicatively.

R acts trivially on $K[M] \subset K_{\hat{\gamma}}(B)^R$ and on $K[L] \subset K_\gamma(A)^R$. We will now show that $\Omega_G \cong G/R$ also acts on $K[M]$ and $K[L]$ inducing an action of G on $K[M]$ and $K[L]$ by inflation.

Let $t \in \Omega_G$. By the last lemma, we know that $t\omega_i = \omega_{\sigma(i)}$ for some $\sigma \in S_n$. Now

$$\begin{aligned}
tX(\omega_i) &= \sum_{r \in R/R_{\omega_i}} tre(\omega_i) \\
&= \sum_{r \in R/R_{\omega_i}} r^t te(\omega_i) \\
&= \sum_{r_1 \in R/R^t_{\omega_i}} r_1 te(\omega_i) \\
&= \sum_{r \in R/R_{t\omega_i}} r\hat{\gamma}_t(t\omega_i)e(t\omega_i) \\
&= \hat{\gamma}_t(t\omega_i) \sum_{r \in R/R_{t\omega_i}} re(t\omega_i) \\
&= \hat{\gamma}_t(t\omega_i)X(t\omega_i) \\
&= \hat{\gamma}_t(\omega_{\sigma(i)})X(\omega_{\sigma(i)}) \in K[M]
\end{aligned}$$

and for $b_0 \in B^R$, $te(b_0) = \hat{\gamma}_t(tb_0)e(tb_0) \in K[M]$. We may express $b \in B$ uniquely as $b = b_0 + \sum_{i=1}^n k_i\omega_i$ for $b_0 \in B^R$ and $k_i \in \mathbf{Z}$. Note that $M =$

$\{X_b | b \in B\}$ where we recall that $X_b = e(b_0)\prod_{i=1}^n X(\omega_i)^{k_i}$ Then

$$\begin{aligned} tX_b &= t(e(b_0)\prod_{i=1}^n X(\omega_i)^{k_i}) \\ &= \hat{\gamma}_t(tb_0)e(tb_0)\prod_{i=1}^n \hat{\gamma}_t(t\omega_i)^{k_i} X(t\omega_i)^{k_i} \\ &= \hat{\gamma}_t(tb_0)e(tb_0)\prod_{i=1}^n \hat{\gamma}_t(\omega_{\sigma(i)})^{k_i} X(\omega_{\sigma(i)})^{k_i} \\ &= \hat{\gamma}_t(tb)X_{tb} \end{aligned}$$

defines the action of Ω_G on $K[M]$.
Claim: $L = \langle X_a | a \in A\rangle$.

Since L is generated as a group by the multiplicative monoid

$$E = \{X_a | a \in \pi^{-1}(\Lambda^+) \cap A\}$$

then $z \in L$ can be expressed as $z = \frac{X'_a}{X_{a''}} = X_{a'-a''}$ where $a', a'' \in \pi^{-1}(\Lambda^+) \cap A$.

Conversely, if $a \in A$, then $a = a_0 + \sum_{i=1}^n k_i\omega_i$ where $a_0 \in A_0$ and $k_i \in \mathbf{Z}$. Since $\Lambda/\pi(A)$ is finite, there exists $d \in \mathbf{N}$ such that $d\omega_i \in \pi(A)$ for all $i = 1, \ldots, n$. Choose $x_i \in A$ such that $\pi(x_i) = d\omega_i$ and m_i such that $k_i + dm_i \geq 0$ and set $a' = \sum_{i=1}^n dm_i x_i$. Then $a + a', a' \in \pi^{-1}(\Lambda^+) \cap A$. So

$$X_a = \left(\frac{X_{a+a'}}{X_{a'}}\right) \in L$$

Then for $a \in A$, by the calculation above, we have $tX_a = \omega_t(ta)X_{ta} \in K[L]$ showing that Ω_G also acts on $K[L]$.

Now $B \to M, b \mapsto X_b$ is an abelian group isomorphism inducing the K-algebra isomorphism $\varphi : K_{\hat{\gamma}}[B] \to K[M], e(b) \mapsto X_b$. Note that $\varphi|_{K[B^R]} = id$. We check that φ is Ω_G-equivariant where Ω_G acts on $K_{\hat{\gamma}}[B]$ by $te(b) = \hat{\gamma}_t(tb)e(tb)$ and on $K[M]$ by $tX_b = \hat{\gamma}_t(tb)X_{tb}$ for $t \in \Omega_G$ and $b \in B$. Indeed, $\varphi(te(b)) = \varphi(\hat{\gamma}_t(tb)e(tb)) = \hat{\gamma}_t(tb)\varphi(e(tb)) = \hat{\gamma}_t(tb)X_{tb} = t\varphi(e(b))$ as required. Note that $\varphi(K_\gamma[A]) = K[L]$ since $L = \langle X_a | a \in A\rangle$, $\varphi|_{K^\times} = id$ and $\varphi(e(a)) = X_a$. So $K_{\hat{\gamma}}[B]$ and $K[M]$, respectively $K_\gamma[A]$ and $K[L]$, are isomorphic as K-algebras under an Ω_G- equivariant isomorphism which acts as the identity on $K[B^R]$, respectively $K[A^R]$.

So $K_\gamma(A)^G = (K_\gamma(A)^R)^{G/R} = (K(L))^{\Omega_G} \overset{\varphi^{-1}}{\cong} K_\gamma(A)^{\Omega_G}$ as required. Note that the above isomorphism acts as the identity on $K_\gamma(A^R)^G = K_\gamma(A^R)^{\Omega_G}$. Now if $K_\gamma(A)^{\Omega_G} = K_\gamma(A^R)^{\Omega_G}(X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ are algebraically independent over $K_\gamma(A^R)$, we find that

$$K_\gamma(A)^G = K(L)^{\Omega_G} \cong K_\gamma(A^R)(\varphi(X_1), \ldots, \varphi(X_n))$$

where $\{\varphi(X_i)\}_{i=1}^n$ are algebraically independent over $K_\gamma(A^R)$ as required. ■

Corollary 4.3.3. *The invariant fields $K(A)^G$ and $K(A)^{\Omega_G}$ are isomorphic under an isomorphism that is the identity on $K(A^R)^G = K(A^R)^{\Omega_G}$. In particular, $K(A)^G$ is rational over $K(A^R)^G$ if and only if $K(A)^{\Omega_G}$ is rational over $K(A^R)^{\Omega_G}$.*

Proof:

The condition $[\gamma] \in \mathrm{Im}(H^1_\Gamma(G, \mathrm{Hom}(B, K^\times))) \to H^1_\Gamma(G, \mathrm{Hom}(A, K^\times))$ for $B = e_R A \oplus \Lambda$ in Proposition 4.3.2 is trivial in this case since $[\gamma] = 0$. ■

4.4 The Automorphism Group of a Root System

The automorphism group G of a crystallographic root system acts on the $\mathbf{Q}$-vector space V spanned by the root system. In this section, we would like to use Corollary 4.3.3 to show that $K(A)^G$ is rational over K where A is a full $\mathbf{Z}G$ lattice on V and G acts trivially on the field K. In the next few lemmas, we will determine some information about the automorphism group of a crystallographic root system, its full reflection subgroup and the stabilizer of a base.

Lemma 4.4.1. *Let Ψ be an irreducible crystallographic root system of rank n.*

(a) There exists an irreducible crystallographic root system $\hat{\Psi}$ of rank n such that $\Psi \subset \widehat{\Psi}$, $\mathrm{Aut}(\Psi) = \mathrm{Aut}(\widehat{\Psi})$ and the full reflection subgroup of $\mathrm{Aut}(\Psi)$ is $W(\widehat{\Phi})$. In particular, for $\Psi \neq A_2, A_3, D_n$, we may take $\widehat{\Psi} = \Psi$. For the remaining cases:

$$\widehat{A_2} = G_2, \widehat{A_3} = B_3, \widehat{D_4} = F_4, \widehat{D_n} = B_n, n \geq 5,$$

we have $\mathrm{Aut}(\Psi) = W(\widehat{\Psi})$.
(b) There exists a crystallographic root system $\widehat{\Psi^l}$ *such that* $\mathrm{Aut}(\Psi^l) = \mathrm{Aut}(\widehat{\Psi^l})$ *and the reflection subgroup of* $\mathrm{Aut}(\Psi^l)$ *is* $W(\widehat{\Psi^l})$. *If* $\Psi = A_1$, *then* $\widehat{A_1^l} = B_l$; *otherwise* $\widehat{\Psi^l} = (\widehat{\Psi})^l$.

Proof:

(a) For $\Psi \neq A_n, n \geq 2, D_n, E_6$, we have $\mathrm{Aut}(\Psi) = W(\Psi)$ so the result is trivial.

For $\Psi = A_n, n \geq 2; E_6$, note that $[\mathrm{Aut}(\Psi) : W(\Psi)] = 2$. Suppose $\mathrm{Aut}(\Psi)$ were a reflection group. Then there would exist an irreducible reflection group of the same rank of size $|\mathrm{Aut}(\Psi)|$. Examining the orders of irreducible Weyl groups [20, p. 66], we see that this cannot hold. By contradiction, $\mathrm{Aut}(\Psi)$ has reflection subgroup $W(\Psi)$ in these cases.

For the remaining cases, we see that $|\mathrm{Aut}(\Psi)| = |W(\widehat{\Psi})|$, $\mathrm{Aut}(\widehat{\Psi}) = W(\widehat{\Psi})$ and $\Psi \subset \widehat{\Psi}$. [20, pp. 64–65] (Note: We may replace A_3 with its isomorphic copy D_3 with base $\{\epsilon_2 - \epsilon_3, \epsilon_1 - \epsilon_2, \epsilon_2 + \epsilon_3\}$ contained in B_3.) So we have $W(\Psi) \leq W(\widehat{\Psi})$ in each case. We need only show that the diagram automorphism group of Ψ is generated by reflections in $W(\widehat{\Psi})$ for each case.

For A_2, the diagram automorphism group is generated by $s_{\epsilon_1 - 2\epsilon_2 + \epsilon_3}$ in $W(G_2)$.

For $D_n, n = 3, n \geq 5$ (i.e. including A_3), the diagram automorphism group is generated by $s_{(\epsilon_{n-1} - \epsilon_n) - (\epsilon_{n-1} + \epsilon_n)} = s_{2\epsilon_n} \in W(B_n)$.

For D_4, the diagram automorphism group is generated by $s_{(\epsilon_1 - \epsilon_2) - (\epsilon_3 - \epsilon_4)}$ and $s_{(\epsilon_1 - \epsilon_2) - (\epsilon_3 + \epsilon_4)} \in W(F_4)$.

(b) For $\Psi \neq A_1$, suppose $s \in \mathrm{Aut}(\Psi^l) = \mathrm{Aut}(\Psi)^l \rtimes S_l$ is a reflection not in $\mathrm{Aut}(\Phi)^l$. Then $s = r\sigma$ where $r \in (\mathrm{Aut}(\Psi))^l$ and $1 \neq \sigma \in S_l$. Then $\sigma(j) \neq j$ for some j. Since $V = \mathbf{Q}\Psi$ has dimension larger than 1, there exist two linearly independent elements $x^{(j)}, y^{(j)}$ in $V^{(j)}$, the jth copy of V. But then $\mathrm{Im}_V(r\sigma - 1)$ contains $rx^{(\sigma(j))} - x^{(j)}$ and $ry^{(\sigma(j))} - y^{(j)}$. If $rx^{(\sigma(j))} - x^{(j)} = c(ry^{(\sigma(j))} - y^{(j)})$ for some $0 \neq c \in \mathbf{Q}$, then $r\sigma(x^{(j)} - cy^{(j)}) = x^{(j)} - cy^{(j)}$ and so $x^{(\sigma(j))} - cy^{(\sigma(j))} = r^{-1}(x^{(j)} - cy^{(j)}) \in V^{(\sigma(j))} \cap V^{(j)} = 0$ implies a contradiction. So the reflections in $\mathrm{Aut}(\Psi^l) = \mathrm{Aut}(\Psi)^l \rtimes S_l$ are those in $\mathrm{Aut}(\Psi)^l$ and hence generate $W(\widehat{\Psi})^l = W(\widehat{\Psi}^l)$. Thus we may take $\widehat{\Psi^l} = (\widehat{\Psi})^l$.

If $\Psi = A_1$, we note that $A_1^l \subset B_l$ so that $W(A_1)^l \leq W(B_l)$. Since the stabilizer of a base for A_1^l is generated by $s_{\epsilon_i} - s_{\epsilon_j} \in W(B_l)$, we find that $\mathrm{Aut}(A_1^l) = W(B_l)$ and hence $\widehat{A_1^l} = B_l$. ∎

Definition 4.4.2. For an arbitrary crystallographic root system Φ where $\Phi = \cup_{i=1}^{m} \Phi_i^{k_i}$ is a disjoint union of irreducible root systems with the Φ_i distinct, we define $\overline{\Phi} = \cup_{i=1}^{m} \widehat{\Phi_i^{k_i}}$ where $\widehat{\Phi_i^{k_i}}$ is as in the previous Lemma. Note that it is not necessarily true that $\text{Aut}(\Phi) = \text{Aut}(\overline{\Phi})$ as, for example, the irreducibles $\widehat{\Phi}_i$ may not be distinct.

Notation: Let Ψ be a crystallographic root system and $\Psi = \cup_{i=1}^{m} (\Psi_i)^{l_i}$ be the decomposition of Ψ into irreducible crystallographic root systems with Ψ_i distinct. Let $V_i = \mathbf{Q}\Psi_i$ so that $G = \text{Aut}(\Psi) = \prod_{i=1}^{m} \text{Aut}(\Psi_i)^{l_i} \rtimes S_{l_i}$ acts diagonally on $V = \oplus_{i=1}^{m} V_i^{l_i}$ with $\text{Aut}(\Psi_i)^{l_i}$ acting diagonally on $V_i^{l_i}$ and S_{l_i} permuting components of $V_i^{l_i}$. Let Γ_0 be the (G-stable) set of reflections in $W(\Psi) = \prod_{i=1}^{m} W(\Psi_i)^{l_i}$ and let Γ be the set of all reflections in G. Let R be the group generated by Γ. Note that R is the full reflection subgroup of G.

Lemma 4.4.3. *The full reflection subgroup R of $G = \text{Aut}(\Psi) = \prod_{i=1}^{m} (\text{Aut}(\Psi_i))^{l_i} \rtimes S_{l_i}$ on $V = \mathbf{Q}\Psi$ is $R = W(\overline{\Psi}) = \prod_{i=1}^{m} W(\widehat{\Psi^{l_i}})$. Let $\widehat{\Pi}_i$ be a base for $\widehat{\Psi}_i$ and let $\widehat{\Pi_i^{l_i}}$ be a base for $\widehat{\Psi_i^{l_i}}$ where if $\Psi_i \neq A_1$, we choose $\widehat{\Pi_i^{l_i}} = \widehat{\Pi}_i^{l_i}$. The stabilizer of a base $\overline{\Pi} = \cup_{i=1}^{m} \widehat{\Pi_i^{l_i}}$ of $\overline{\Psi}$ is $\Omega \equiv \Omega_G(\overline{\Pi}) = \prod_{i=1}^{m} \Omega_i$, where*

$$\Omega_i = \{g \in \text{Aut}(\widehat{\Psi_i^{l_i}}) | g(\widehat{\Pi_i^{l_i}}) = \widehat{\Pi_i^{l_i}}\} = \begin{cases} T_i^{l_i} \rtimes S_{l_i} & \text{if } \Psi_i \neq A_1 \\ 1 & \text{if } \Psi_i = A_1 \end{cases}$$

where $T_i = \{g \in \text{Aut}(\widehat{\Psi}_i) | g(\widehat{\Pi}_i) = \widehat{\Pi}_i\}$.

Proof: Since $\text{Aut}(\Psi) = \prod_{i=1}^{m} \text{Aut}(\Psi_i^{l_i})$ acts diagonally on $V = \oplus_{i=1}^{m} V_i^{l_i}$, an element of $s \in \text{Aut}(\Psi)$ has $\text{Im}_V(s-1) = \oplus_{i=1}^{m} \text{Im}_{V_i^{l_i}}(s-1)$ so that it can act as a reflection on V iff it is a reflection in $\text{Aut}(\Psi_i^{l_i}) = \text{Aut}(\widehat{\Psi_i^{l_i}})$ for some i. So the reflection subgroup of $\text{Aut}(\Psi)$ is $R = \prod_{i=1}^{m} R_i$ where R_i is the reflection subgroup of $\text{Aut}(\Psi_i^{l_i})$. The result follows from Lemma 4.4.1.

The stabilizer of the base $\overline{\Pi}$ in G is

$$\begin{aligned}\Omega_G(\overline{\Pi}) &= \{g \in G | g\widehat{\Pi} = \widehat{\Pi}\} \\ &= \prod_{i=1}^{m} \{g \in \mathrm{Aut}(\Psi_i^{l_i}) | g\widehat{\Pi_i^{l_i}} = \widehat{\Pi_i^{l_i}}\} \\ &= \prod_{i=1}^{m} \{g \in \mathrm{Aut}(\widehat{\Psi_i^{l_i}}) | g\widehat{\Pi_i^{l_i}} = \widehat{\Pi_i^{l_i}}\} \\ &= \prod_{i=1}^{m} \Omega_i\end{aligned}$$

where the Ω_i are as described above. ■

Notation: For a $\mathbf{Z}G$ lattice A on V and a G-stable set of reflections S, set

$$\Phi_{A,S} = \{\alpha \in V | \mathrm{Ker}_A(s+1) = \mathbf{Z}\alpha \text{ for some } s \in S\}$$

We write Φ_A for $\Phi_{A,\Gamma}$.

Lemma 4.4.4. *Let Ψ be a crystallographic root system, $G = \mathrm{Aut}(\Psi)$, $V = \mathbf{Q}\Psi$ and R be the full reflection subgroup of G acting on V. For a $\mathbf{Z}G$ lattice A on V, $\Phi_{A,\Gamma_0} \subset \Phi_A$ where $\mathrm{Aut}(\Phi_{A,\Gamma_0}) = \mathrm{Aut}(\Psi) = G$.*

Φ_A can be expressed as a disjoint union of crystallographic root systems $\Phi_A = \overline{\Phi_{A,\Gamma_0}} \equiv \cup_{i=1}^{m} \widehat{\Phi_i^{l_i}}$ where $\Phi_{A,\Gamma_0} = \cup_{i=1}^{m} \Phi_i^{l_i}$ is the decomposition of Φ_{A,Γ_0} into irreducibles with Φ_i distinct. A base Δ_A for Φ_A may be expressed as $\Delta_A = \cup_{i=1}^{m} \widehat{\Delta_i^{l_i}}$ where $\widehat{\Delta_i}$ is a base for $\widehat{\Phi}_i$ and $\widehat{\Delta_i^{l_i}}$ is a base for $\widehat{\Phi_i^{l_i}}$ chosen so that $\widehat{\Delta_i^{l_i}} = \widehat{\Delta_i}^{l_i}$ if $\Phi_i \neq A_1$. Moreover

$$\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$$

where $G = \prod_{i=1}^{m} \mathrm{Aut}(\widehat{\Phi_i^{l_i}})$, $R = \prod_{i=1}^{m} R_i$ with $R_i = W(\widehat{\Phi_i^{l_i}})$ and the stabilizer of the base Δ_A is $\Omega = \prod_{i=1}^{m} \Omega_i$ with

$$\Omega_i = \{g \in \mathrm{Aut}(\widehat{\Phi_i^{l_i}}) | g(\widehat{\Delta_i^{l_i}}) = \widehat{\Delta_i^{l_i}}\}$$

acts diagonally on $\mathbf{Z}\Phi_A = \oplus_{i=1}^{m} \mathbf{Z}\widehat{\Phi_i^{l_i}}$ and $\Lambda(\Phi_A) = \oplus_{i=1}^{m} \Lambda(\widehat{\Phi_i^{l_i}})$.

Proof:

Since Φ_A and $\overline{\Psi}$ have the same reflection set Γ, the set of reflections in $R = W(\overline{\Psi})$, we find by Lemma 2.2.10 that $\Phi_A = \{c_\alpha \alpha | \alpha \in \overline{\Psi}\}$ for some $c_\alpha \in \mathbf{Q}^+$. Since both $\overline{\Psi}$ and Φ_A are G-stable, we see that $c_{g\alpha} = c_\alpha$ for all $g \in G$ and $\alpha \in \overline{\Psi}$.

By definition, $\Phi_{A,\Gamma_0} \subset \Phi_{A,\Gamma} \equiv \Phi_A$. Since Ψ and Φ_{A,Γ_0} both have reflection set Γ_0, we have $\Phi_{A,\Gamma_0} = \{c_\alpha \alpha | \alpha \in \Psi\}$ so that $\mathrm{Aut}(\Phi_{A,\Gamma_0}) = \mathrm{Aut}(\Psi) = G$. Set $\Phi_i = \{c_\alpha \alpha | \alpha \in \Psi_i\}$, $i = 1, \ldots, m$. By Lemma 2.2.10, $\Phi_{A,\Gamma_0} = \cup_{i=1}^m \Phi_i^{l_i}$ is the decomposition of Φ_{A,Γ_0} into irreducibles with Φ_i distinct. Note that $\widehat{\Phi_i^{l_i}} = \{c_\alpha \alpha | \alpha \in \widehat{\Psi_i^{k_i}}\}$ so that $\Phi_A = \cup_{i=1}^m \widehat{\Phi_i^{l_i}}$ and $\Delta_A = \cup_{i=1}^m \widehat{\Delta_i^{l_i}}$. So we have

$$G = \mathrm{Aut}(\Psi) = \mathrm{Aut}(\Phi_{A,\Gamma_0}) = \prod_{i=1}^m \mathrm{Aut}(\widehat{\Phi_i^{l_i}}),$$

$$R = W(\Phi_A) = \prod_{i=1}^m W(\widehat{\Phi_i^{l_i}}) = \prod_{i=1}^m W(\widehat{\Psi_i^{l_i}}) = W(\overline{\Psi})$$

and

$$\Omega = \Omega_G(\Delta_A) = \prod_{i=1}^m \Omega_i$$

where

$$\begin{aligned} \Omega_i &= \{g \in \mathrm{Aut}(\widehat{\Psi_i^{l_i}}) | g(\widehat{\Pi_i^{l_i}}) = \widehat{\Pi_i^{l_i}}\} \\ &= \{g \in \mathrm{Aut}(\widehat{\Phi_i^{l_i}}) | g(\widehat{\Delta_i^{l_i}}) = \widehat{\Delta_i^{l_i}}\} \end{aligned}$$

all act diagonally on $V = \oplus_{i=1}^m V_i^{l_i} = \oplus_{i=1}^m \mathbf{Q}\widehat{\Phi_i^{l_i}}$ and hence act diagonally on $\mathbf{Z}\Phi_A = \oplus_{i=1}^m \mathbf{Z}\widehat{\Phi_i^{l_i}}$, $\Lambda(\Phi_A) = \oplus_{i=1}^m \Lambda(\widehat{\Phi_i^{l_i}})$. ∎

Remark 4.4.5. The last few lemmas were necessary to determine the structure of the crystallographic root system Φ_A. From now on, we will change the notation slightly to facilitate the proof of the remaining Lemmas and eventually the Theorem. Lemma 4.4.4 shows that Φ_A can be decomposed as $\cup_{i=1}^m \Phi_i^{k_i}$ where each Φ_i is an irreducible crystallographic root system but the Φ_i are not necessarily distinct and $k_i = l_i$ if $\Psi_i \neq A_1$, otherwise $k_i = 1$ and $\Psi_i = B_{l_i}$. (Note that $A_1 = B_1$.) Here $G = \mathrm{Aut}(\Psi) = \prod_{i=1}^m \mathrm{Aut}(\Psi_i^{l_i})$, its subgroup $R = \prod_{i=1}^m R_i$ generated by all the reflections on V in G, and the stabilizer of a base Δ_A, $\Omega = \prod_{i=1}^m \Omega_i$ each act diagonally on $V = \oplus_{i=1}^m V_i^{k_i}$ where we take $V = \mathbf{Q}\Psi = \mathbf{Q}\Phi_A$, $V_i = \mathbf{Q}\Phi_i$ so that $V_i^{k_i} = \mathbf{Q}\Phi_i^{k_i} = \mathbf{Q}\Psi_i^{l_i}$.

We will refer to $\Phi_i^{k_i}$ as the ith component of Φ_A. Taking Δ_i as a base for Φ_i, $\Delta_A = \cup_{i=1}^m \Delta_i^{k_i}$ is a base for Φ_A. Then with this new notation, $\Omega_i = T_i^{k_i} \rtimes S_{k_i}$ where T_i is the group of diagram automorphisms of the irreducible root system Φ_i with respect to the base Δ_i.

Note that by Lemmas 4.4.1, 4.4.3, and 4.4.4, we see that for a component $\Phi_i^{k_i}$ of Φ_A, Φ_i can be any irreducible crystallographic root system except $A_n, n \leq 3, D_n, n \geq 4$.

The following is a technical lemma about lattices for the wreath product $T^k \rtimes S_k$, where T is any finite group:

Lemma 4.4.6. *(a) For a* $\mathbf{Z}T$ *module* X, $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} X = \oplus_{i=1}^k X^{(i)}$ *where the inflation is with respect to the projection homomorphism* $p_k : T^k \rtimes S_{k-1} \to T$ *onto the last component of* T^k *,* $\sigma \in S_k$ *maps* $X^{(i)}$ *identically onto* $X^{(\sigma(i))}$ *and* T^k *acts componentwise on* $X = \oplus_{i=1}^k X^{(i)}$.

(b) If X *is a faithful* $\mathbf{Z}T$ *lattice, then* $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} X$ *is a faithful* $T^k \rtimes S_k$ *lattice.*

(c) If T *is the diagram automorphism group corresponding to a base* Δ *for an irreducible root system* Φ *with weight lattice* Λ *then, fixing the last copy* $\Lambda(\Phi)$ *of* $\Lambda(\Phi)^k$*, respectively* $\mathbf{Z}\Phi$ *of* $\mathbf{Z}\Phi^k$*, we have*

$$\Lambda(\Phi)^k = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \Lambda(\Phi) \qquad \mathbf{Z}\Phi^k = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z}\Phi$$

as $T^k \rtimes S_k$ *lattices.*

Proof: Identifying T^k with $\mathrm{Map}(I_k, T)$ where $I_k = \{1, \ldots, k\}$, S_k acts on T^k as $f^\sigma(i) = f(\sigma^{-1} i)$ and $T^k \rtimes S_k$ is the semi-direct product corresponding to this action. Hence elements of $T^k \rtimes S_k$ take the form $f\sigma$ with multiplication

$$f\sigma f'\sigma' = f f'^{\sigma} \sigma\sigma'$$

and inverses $(f\sigma)^{-1} = (f^{\sigma^{-1}})^{-1}\sigma$. Then $p_k : T^k \rtimes S_{k-1} \to T$ defined by $f\sigma \mapsto f(k)$ is a homomorphism since S_{k-1} centralizes the last component of T^k.

(a) Since $\{(i,k)|i = 1,\ldots,k\}$ is a transversal for $T^k \rtimes S_{k-1}$ in $T^k \rtimes S_k$ where $(k,k) = id$, take $X^{(i)} = (i,k) \otimes \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} X, i = 1,\ldots,k$. For $\sigma \in S_k, \sigma(i,k) = (\sigma(i),k)\sigma'$ where $\sigma' \in S_{k-1}$. So

$$\sigma((i,k) \otimes x) = (\sigma(i),k) \otimes \sigma' x = (\sigma(i),k) \otimes x$$

For $f \in T^k$, $f(i,k) = (i,k)(f^{(i,k)})$ so that

$$f((i,k) \otimes x) = (i,k) \otimes (f^{(i,k)}(k))x = (i,k) \otimes f(i)x$$

as required.

(b) Let $f\sigma \in T^k \rtimes S_k$ act trivially on $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} X$. Then, in particular, $f\sigma X^{(i)} = X^{(i)}, i = 1,\ldots,k$ so that $X^{(\sigma(i))} = \sigma X^{(i)} = f(i)^{-1}X^{(i)} = X^{(i)}$ for all i implies that $\sigma = 1$. Now $\sum_{i=1}^k x^{(i)} = f\sum_{i=1}^k x^{(i)} = \sum_{i=1}^k f(i)x^{(i)}$ shows that $f(i)$ acts trivially on $X^{(i)}$ which is congruent to X as a $\mathbf{Z}T$ lattice and hence that $f(i) = 1$ for all i since the action of T on X is faithful. So $f\sigma = 1$ as required.

(c) follows immediately from (a) and the description of the action of $T^k \rtimes S_k$ on $\Lambda(\Phi)^k$ and $(\mathbf{Z}\Phi)^k$. ■

By Lemma 4.3.1, $\mathbf{Z}\Phi_A$ and $\Lambda(\Phi_A)$ are both permutation lattices for Ω. The following lemma refines this description:

Lemma 4.4.7.
(a) For each i, $\mathbf{Z}\Phi_i^{k_i}$ contains $\mathbf{Z}\Omega_i$ permutation lattices P_i, Q_i' and $\Lambda(\Phi_i^{k_i})$ contains a faithful $\mathbf{Z}\Omega_i$ permutation lattice Q_i for Ω_i such that $\mathbf{Z}\Phi_i^{k_i} = P_i \oplus Q_i'$ and $\Lambda(\Phi_i^{k_i}) = P_i \oplus Q_i$ and $Q_i' \subset Q_i$.
(b)

$$Q_i \cong Q_i' \cong \begin{cases} \mathrm{Ind}_{S_{k_i-1}}^{S_{k_i}} \mathbf{Z} & \Omega_i = S_{k_i} \\ \mathrm{Ind}_{T^{k_i-1} \rtimes S_{k_i-1}}^{T^{k_i} \rtimes S_{k_i}} \mathbf{Z} & \Omega_i = T^{k_i} \rtimes S_{k_i} \\ \text{or } \mathrm{Ind}_{T^{k_i-1} \rtimes S_{k_i-1}}^{T^{k_i} \rtimes S_{k_i}} \mathbf{Z} \oplus \mathrm{Ind}_{T^{k_i} \rtimes S_{k_i-1}}^{T^{k_i} \rtimes S_{k_i}} \mathbf{Z} & \Omega_i = T^{k_i} \rtimes S_{k_i} \end{cases}$$

(c) $P_A = \oplus_{i=1}^m P_i$, $Q_A = \oplus_{i=1}^m Q_i$ and $Q_A' = \oplus_{i=1}^m Q_i'$ are $\mathbf{Z}\Omega$ permutation lattices such that $\mathbf{Z}\Phi = P_A \oplus Q_A'$ and $\Lambda(\Phi_A) = P_A \oplus Q_A$ where Ω acts faithfully on Q_A and $Q_A' \subset Q_A$.

Proof: Since $\Omega = \prod_{i=1}^{m} \Omega_i$ acts componentwise on $\Lambda(\Phi_A) = \oplus_{i=1}^{m} \Lambda(\Phi_i^{k_i})$ and $\mathbf{Z}\Phi_A = \oplus_{i=1}^{m} \mathbf{Z}\Phi_i^{k_i}$, we see that (c) follows immediately from (a).

(a), (b) To simplify notation, we suppress the subscript i. $\Omega = T^k \rtimes S_k$ where T is the group of diagram automorphisms for a base Δ of Φ and $\Phi^k = (\Phi)^k$. Then fixing the last component $\Lambda(\Phi)$, $\mathbf{Z}\Phi$ of $\Lambda(\Phi)^k, \mathbf{Z}\Phi^k$ we see that $\Lambda(\Phi)^k = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \Lambda(\Phi)$ and $\mathbf{Z}\Phi^k = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z}\Phi$ as $T^k \rtimes S_k$ lattices by Lemma 4.4.6. Since induction and inflation preserve direct sums and send permutation lattices to permutation lattices, it suffices to find a $\mathbf{Z}T$ permutation lattice K and faithful $\mathbf{Z}T$ permutation lattices $L' \subset L$ such that

$$\mathbf{Z}\Phi = K \oplus L' \qquad \Lambda(\Phi) = K \oplus L \tag{4.4.8}$$

We then set

$$P = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}}(K)$$
$$Q = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}}(L), Q' = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}}(L')$$

and note that both Q and Q' are faithful $\mathbf{Z}T^k \rtimes S_k$ lattices by Lemma 4.4.6.

We will show (4.4.8) case by case.

Case 1: $T = 1$

To show (4.4.8), we need only find a suitable $\mathbf{Z}$ decomposition. Since $\Lambda(\Phi)/\mathbf{Z}\Phi$ is cyclic, we may choose a $\mathbf{Z}$-basis $x_1, \ldots, x_n$ of $\Lambda(\Phi)$ such that $x_1, \ldots, x_{n-1}, dx_n$ is a $\mathbf{Z}$-basis of $\mathbf{Z}\Phi$. Then we may take $K = \oplus_{j=1}^{n-1} \mathbf{Z}x_j$, $L = \mathbf{Z}x_n$ and $L' = dL$.

Then $P = \mathrm{Ind}_{S_{k-1}}^{S_k}(K) \cong (\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z})^{n-1}$, $Q = \mathrm{Ind}_{S_{k-1}}^{S_k}(L) \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}, Q' = \mathrm{Ind}_{S_{k-1}}^{S_k}(dL) = dQ \cong Q$ gives the required decomposition.

Case 2: $T \neq 1$.

By [20, p. 66], the only irreducible root systems with non-trivial diagram automorphism groups are of types A_n for $n \geq 2$, E_6 or $D_n, n \geq 4$. But in the previous remark, we noted that $(D_n)^k$, $(A_2)^k$, $(A_3)^k$ cannot be components of Φ_A. So, for each component Φ^k of Φ_A such that Φ has non-trivial diagram automorphism group T, T is cyclic of order 2. We now show (4.4.8) in each case:

Case 2a: Φ has type A_n for $n \geq 4$.

Set $T = \langle t \rangle$. By [20, p. 59], the base for the root system $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ can be expressed in terms of the fundamental dominant weights $\{\omega_1, \ldots, \omega_n\}$

as

$$\alpha_1 = 2\omega_1 - \omega_2$$
$$\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1} \text{ for } 1 < i < n$$
$$\alpha_n = -\omega_{n-1} + 2\omega_n$$

The T-action is given by $t\omega_i = \omega_{n+1-i}$ and $t\alpha_i = \alpha_{n+1-i}$.

Since $\{\omega_n, \alpha_2, \ldots, \alpha_n\}$ is also a $\mathbf{Z}$ basis for $\Lambda(\Phi)$, we can take $K = \mathbf{Z}\alpha_2 \oplus \cdots \oplus \mathbf{Z}\alpha_{n-1}$ and observe that K is a $\mathbf{Z}T$ permutation lattice and $\Lambda(\Phi)/K$ is a $\mathbf{Z}T$ lattice with $\mathbf{Z}$ basis $\{\omega_n + K, \alpha_n + K\}$. To determine the T action on $\Lambda(\Phi)/K$, we recall [20, p. 69] that

$$\omega_n = \frac{\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n}{n+1} \qquad \omega_1 = \frac{n\alpha_1 + (n-1)\alpha_2 + \cdots + \alpha_n}{n+1}$$

to calculate that

$$\begin{aligned} t(\omega_n + K) = \omega_1 + K &= n\omega_n - (n-1)\alpha_n - (n-2)\alpha_{n-1} - \cdots - \alpha_2 + K \\ &= n\omega_n - (n-1)\alpha_n + K \\ t(\alpha_n + K) = \alpha_1 + K &= (n+1)\omega_n - n\alpha_n - (n-1)\alpha_{n-1} - \cdots - 2\alpha_2 + K \\ &= (n+1)\omega_n - n\alpha_n + K \end{aligned}$$

Then

$$\begin{aligned} H^1(T, \Lambda(\Phi)/K) &\cong \mathrm{Ker}_{\Lambda(\Phi)/K}(t+1)/\mathrm{Im}_{\Lambda(\Phi)/K}(t-1) = \frac{\mathbf{Z}(\overline{\omega_n} - \overline{\alpha_n})}{\gcd(n-1, n+1)\mathbf{Z}(\overline{\omega_n} - \overline{\alpha_n})} \\ &= \begin{cases} \mathbf{Z}/2\mathbf{Z} & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases} \end{aligned}$$

We see that t acts as a reflection on $\Lambda(\Phi)/K$ and

$$\Lambda(\Phi)/K \cong \begin{cases} \mathbf{Z}T & , n \text{ even} \\ \mathbf{Z} \oplus \mathbf{Z}^- & , n \text{ odd} \end{cases}$$

So if n is even,

$$K \rightarrowtail \Lambda(\Phi) \twoheadrightarrow \Lambda(\Phi)/K$$

is T- split so we get $L \cong \mathbf{Z}T$ with $\Lambda(\Phi) = K \oplus L$ and then $\mathbf{Z}\Phi = K \oplus L'$ with $L' = L \cap \mathbf{Z}\Phi \cong \mathbf{Z}\Phi/K \cong \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_n \cong \mathbf{Z}T$ as required.

Unfortunately, if n is odd, $K \cong (\mathbf{Z}C_2)^{\frac{n-1}{2}} \oplus \mathbf{Z}$ and $\Lambda(\Phi)/K \cong \mathbf{Z} \oplus \mathbf{Z}^-$. Since $\mathrm{Ext}^1_{\mathbf{Z}C_2}(\Lambda(\Phi)/K, K) = \mathrm{Ext}^1_{\mathbf{Z}C_2}(\mathbf{Z}^-, \mathbf{Z}) \neq 0$, we cannot guarantee a splitting in this case. Let's try a different approach for the odd case. Take

$$K = \oplus_{i=2, i\neq \frac{n+1}{2}}^{n-1} \mathbf{Z}\alpha_i$$

Again K is a permutation $\mathbf{Z}T$ lattice and $\Lambda(\Phi)/K$ is a $\mathbf{Z}T$ lattice with basis $\{\omega_n + K, \alpha_n + K, \alpha_{\frac{n+1}{2}} + K\}$. Once again we need to know the structure of $\Lambda(\Phi)/K$ as a $\mathbf{Z}T$ lattice. Now

$$\begin{aligned} t(\omega_n + K) &= \omega_1 + K = n\omega_n - (n-1)\alpha_n - (n-2)\alpha_{n-1} - \cdots - \alpha_2 + K \\ t(\alpha_n + K) &= \alpha_1 + K = (n+1)\omega_n - n\alpha_n - (n-1)\alpha_{n-1} - \cdots - 2\alpha_2 + K \\ t(\alpha_{\frac{n+1}{2}} + K) &= \alpha_{\frac{n+1}{2}} + K \end{aligned}$$

so that

$$\begin{aligned} t(\omega_n + K) &= n\omega_n - (n-1)\alpha_n - (\frac{n-1}{2})\alpha_{\frac{n+1}{2}} + K \\ t(\alpha_n + K) &= (n+1)\omega_n - n\alpha_n - (\frac{n+1}{2})\alpha_{\frac{n+1}{2}} + K \\ t(\alpha_{\frac{n+1}{2}} + K) &= \alpha_{\frac{n+1}{2}} + K \end{aligned}$$

and this time

$$\mathrm{Ker}_{\Lambda(\Phi)/K}(t+1) = \mathbf{Z}(2\overline{\omega}_n - 2\overline{\alpha}_n - \overline{\alpha}_{\frac{n+1}{2}}) = \mathrm{Im}_{\Lambda(\Phi)/K}(t-1)$$

Again t acts as a reflection on $\Lambda(\Phi)/K$ and $H^1(T, \Lambda(\Phi)/K) = 0$ and $\Lambda(\Phi)/K \cong \mathbf{Z} \oplus \mathbf{Z}T$. So again $K \rightarrowtail \Lambda(\Phi) \twoheadrightarrow \Lambda(\Phi)/K$ is T-split since both K and $\Lambda(\Phi)/K$ are permutation. Hence $\Lambda(\Phi) = K \oplus L$ with $L \cong \mathbf{Z} \oplus \mathbf{Z}T$ and $\mathbf{Z}\Phi = K \oplus L'$ with $L' = L \cap \mathbf{Z}\Phi$. Note that $L' \cong \mathbf{Z}\Phi/K \cong \mathbf{Z}\alpha_1 \oplus \mathbf{Z}\alpha_n \oplus \mathbf{Z}\alpha_{\frac{n+1}{2}} \cong \mathbf{Z}T \oplus \mathbf{Z}$ as required.

Case 2b: Φ has type E_6

By [20, p. 59], $\Delta = \{\alpha_1, \ldots, \alpha_6\}$ can be expressed in terms of the corresponding basis of fundamental dominant weights as follows:

$$
\begin{aligned}
\alpha_1 &= 2\omega_1 - \omega_3 \\
\alpha_2 &= 2\omega_2 - \omega_4 \\
\alpha_3 &= -\omega_1 + 2\omega_3 - \omega_4 \\
\alpha_4 &= -\omega_2 - \omega_3 + 2\omega_4 - \omega_5 \\
\alpha_5 &= -\omega_4 + 2\omega_5 - \omega_6 \\
\alpha_6 &= -\omega_5 + 2\omega_6
\end{aligned}
$$

Note that $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \omega_2, \omega_3\}$ is an alternate $\mathbf{Z}$- basis for $\Lambda(\Phi)$ since

$$
\begin{aligned}
\omega_4 &= -\alpha_2 + 2\omega_2 \\
\omega_1 &= -\alpha_3 - \omega_4 + 2\omega_3 \\
\omega_5 &= -\alpha_4 - \omega_2 - \omega_3 + 2\omega_4 \\
\omega_6 &= -\alpha_5 - \omega_4 + 2\omega_5
\end{aligned}
$$

Let $K = \mathbf{Z}\alpha_2 \oplus \mathbf{Z}\alpha_3 \oplus \mathbf{Z}\alpha_4 \oplus \mathbf{Z}\alpha_5$. Note that t acts on $\Lambda(\Phi)$ and $\mathbf{Z}\Phi$ by

$$
\begin{aligned}
t\alpha_1 &= \alpha_6 & t\omega_1 &= \omega_6 \\
t\alpha_2 &= \alpha_2 & t\omega_2 &= \omega_2 \\
t\alpha_3 &= \alpha_5 & t\omega_3 &= \omega_5 \\
t\alpha_4 &= \alpha_4 & t\omega_4 &= \omega_4 \\
t\alpha_5 &= \alpha_3 & t\omega_5 &= \omega_3 \\
t\alpha_6 &= \alpha_1 & t\omega_6 &= \omega_1
\end{aligned}
$$

So $K \cong \mathbf{Z}T \oplus \mathbf{Z}$, $\Lambda(\Phi)/K = \mathbf{Z}\overline{\alpha}_1 \oplus \mathbf{Z}\overline{\alpha}_6 \cong \mathbf{Z}T$ and $\mathbf{Z}\Phi/K \rightarrowtail \Lambda(\Phi)/K \twoheadrightarrow \Lambda(\Phi)/\mathbf{Z}\Phi$ with $\Lambda(\Phi)/\mathbf{Z}\Phi$ of order 3 implies $H^1(T, \Lambda(\Phi)/K) = 0$ hence $\Lambda(\Phi)/K \cong \mathbf{Z}T$ (as T does not act trivially on $\Lambda(\Phi)/K$). Alternatively, we compute

$$
\begin{aligned}
t(\omega_2 + K) &= \omega_2 + K \\
t(\omega_5 + K) &= \omega_5 + K \\
&= -\alpha_4 - \omega_2 - \omega_3 + 2\omega_4 + K \\
&= -\alpha_4 - \omega_2 - \omega_3 + 2(2\omega_2 - \alpha_2) + K \\
&= 3\omega_2 - \omega_3 - \alpha_4 - 2\alpha_2 + K \\
&= 3\omega_2 - \omega_3 + K
\end{aligned}
$$

so that

$$\mathrm{Ker}_{\Lambda(\Phi)/K}(t+1) = \mathbf{Z}(3\overline{\omega}_2 - 2\overline{\omega}_5) = \mathrm{Im}_{\Lambda(\Phi)/K}(t-1)$$

So t acts as a reflection on $\Lambda(\Phi)/K$ and $H^1(T, \Lambda(\Phi)/K) = 0$ imply again that $\Lambda(\Phi)/K \cong \mathbf{Z}T$. In either case, $K \rightarrowtail \Lambda(\Phi) \twoheadrightarrow \Lambda(\Phi)/K$ splits and $\Lambda(\Phi) = K \oplus L, \mathbf{Z}\Phi = K \oplus L'$ with $L' = L \cap \mathbf{Z}\Phi \cong \mathbf{Z}\Phi/K \cong \mathbf{Z}T$ as required.

Note that the first method used to show that $\Lambda(\Phi)/K \cong \mathbf{Z}T$ works also in Case 2(a) when n is even.

Since $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z} \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} \mathbf{Z}T \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, $P \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} K$, and $Q \cong Q' \cong \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L$ we see that

$$P \cong \begin{cases} (\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^{\frac{n-1}{2}}, & \Phi = A_n, n \text{ odd} \\ (\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^{\frac{n}{2}-1}, & \Phi = A_n, n \text{ even} \\ \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus (\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z})^2, & \Phi = E_6 \end{cases}$$

$$Q \cong Q' \cong \begin{cases} \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}, & \Phi = A_n, n \text{ even}, E_6 \\ \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}, & \Phi = A_n, n \text{ odd} \end{cases}$$

so that (b) is proved. ■

Theorem 4.4.9. *Let Ψ be a crystallographic root system for the $\mathbf{Q}$ space V. Then $G = \mathrm{Aut}(\Psi)$ acts faithfully on V. For any $\mathbf{Z}G$ lattice A on V, $K(A)^G$ is rational over K where G acts trivially on K.*

Proof: By Corollary 4.3.3, we need only show that $K(A)^{\Omega}$ is rational over K. By Lemma 4.4.4, we have $\mathbf{Z}\Phi_A \subset A \subset \Lambda(\Phi_A)$. By Lemma 4.4.7, we can decompose Φ_A and $\Lambda(\Phi_A)$ as $\mathbf{Z}\Phi_A = P_A \oplus Q'_A$, $\Lambda(\Phi_A) = P_A \oplus Q_A$ where P_A, Q_A, Q'_A are $\mathbf{Z}\Omega$ permutation lattices and Ω acts faithfully on $Q'_A \subset Q_A$. Take $B_A = Q_A \cap A$. Then $A = P_A \oplus B_A$ is a decomposition of $\mathbf{Z}\Omega$ lattices with Ω acting faithfully on B_A and $Q'_A \subset B_A \subset Q_A$. Then by Proposition 1.0.13, $K(A)^{\Omega}$ is rational over $K(B_A)^{\Omega}$.

It now suffices to prove the rationality of $K(B_A)^{\Omega}$ over K. In order to do this, we want to apply Corollary 4.3.3 to $K(B_A)^{\Omega}$. Recall that $\Omega = \prod_{i=1}^{k} \Omega_i$ and $Q'_A = \oplus_{i=1}^{m} Q'_i \subset B_A \subset Q_A = \oplus_{i=1}^{m} Q_i$ where $Q'_i \cong Q_i$ are $\mathbf{Z}\Omega_i$ direct summands of $\mathbf{Z}\Phi_i^{k_i}$, respectively $\Lambda(\Phi_i^{k_i})$.

Definition 4.4.10. Let Φ^k be a component in the decomposition of Φ_A. Let $Q' \cong Q$ be the $\mathbf{Z}\Omega$ direct summands of $\mathbf{Z}\Phi^k$ defined in Lemma 4.4.7.

Let T be the group of diagram automorphisms for a base Δ of Φ. Then Φ^k is said to be of type:
I if $\Omega = S_k$ and $Q' \cong Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$
II if $\Omega = T^k \rtimes S_k$ and $Q' \cong Q \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$
III if $\Omega = T^k \rtimes S_k$ and $Q' \cong Q \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$

Observe that by Lemma 4.4.7, these are all the possibilities for a component Φ occurring in the decomposition of Φ_A.

For each type of component, we need to find the reflection subgroup $R^\sharp$ of the action of Ω on Q, the stabilizer $\Omega^\sharp$ of a base of an associated root system and the structure of Q as a $\mathbf{Z}\Omega^\sharp$ lattice. We also need a decomposition of Q into permutation $\mathbf{Z}\Omega^\sharp$ lattices similar to that in Lemma 4.4.7.

Lemma 4.4.11. *(a) Let Φ^k be a component of the decomposition of Φ_A. Then with the above notation:*
Φ^k *of type I:* $R^\sharp = S_k$, $\Omega^\sharp = 1$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathbf{Z}^k$.
Φ^k *of type II:* $R^\sharp = C_2^k$, $\Omega^\sharp = S_k$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$
Φ^k *of type III:* $R^\sharp = C_2^k$, $\Omega^\sharp = S_k$, $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$

(b) For Φ^k as in (a) of any type, we have $\mathbf{Z}\Omega^\sharp$ permutation lattices C, D, D' such that there exist decompositions

$$Q = C \oplus D \qquad Q' = C \oplus D' \qquad D' \subset D$$

as $\mathbf{Z}\Omega^\sharp$ lattices where $\Omega^\sharp$ acts faithfully on D. Let d be the order of $\Lambda(\Phi)/\mathbf{Z}\Phi$. Then for Φ^k of type I, $D \cong \mathbf{Z}^k, D' = dD$ and for Φ^k of type II,III, $D \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}, D' = dD$.

(c) For each component $\Phi_i^{k_i}$ in the decomposition of Φ_A, let $R_i^\sharp$, $\Omega_i^\sharp$ be given as in (a) and C_i, D_i, D_i' be given as in (b) and set $C_A = \oplus_{i=1}^m C_i$, $D_A = \oplus_{i=1}^m D_i$, $D_A' = \oplus_{i=1}^m D_i'$. Then the reflection group acting on Q_A is $R^\sharp = \prod_{i=1}^m R_i^\sharp$ and the stabilizer of a base of the associated root system is $\Omega^\sharp = \prod_{i=1}^m \Omega_i^\sharp$. Then C_A, D_A, D_A' are $\mathbf{Z}\Omega^\sharp$ permutation lattices and there exist decompositions

$$Q_A = C_A \oplus D_A \qquad Q_A' = C_A \oplus D_A' \qquad D_A' \subset D_A$$

of $\mathbf{Z}\Omega^\sharp$ lattices where $\Omega^\sharp$ acts faithfully on D_A.

Proof:

(a) For Φ^k of type I, $\Omega = S_k$ and $Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k}\mathbf{Z} \cong \mathbf{Z}[S_k/S_{k-1}]$. Now $\{(i,k)|i = 1,\ldots,k\}$ is a transversal for S_{k-1} in S_k. For all i and $\sigma \in S_k$. $(\sigma(i),k)\sigma(i,k) \in S_{k-1}$. So $\sigma(i,k)S_{k-1} = (\sigma(i),k)S_{k-1}$. Take $u_i = (i,k)S_{k-1}$ for each i. Then $\{u_1,\ldots,u_k\}$ is a $\mathbf{Z}$-basis for Q and $\sigma(u_i) = u_{\sigma(i)}$. $\sigma \in S_k$ acts as a reflection on Q if and only if

$$\mathrm{Im}_Q(\sigma - 1) = \mathrm{Span}_{\mathbf{Z}}\{u_{\sigma(i)} - u_i|i = 1,\ldots,k\}$$

has rank 1 if and only if σ fixes $k-2$ elements of $\{1,\ldots,k\}$ if and only if σ is a transposition. So $R^\sharp = S_k$ since it is generated by the transpositions. Hence $\Omega^\sharp = 1$ and $\mathrm{Res}_{\Omega^\sharp}Q = \mathbf{Z}^k$ follow easily.

For Φ^k of type II, $\Omega = T^k \rtimes S_k$ and

$$Q \cong \mathrm{Ind}_{T^{k-1}\rtimes S_{k-1}}^{T^k\rtimes S_k}\mathbf{Z} \cong \mathbf{Z}[T^k \rtimes S_k/T^{k-1} \rtimes S_{k-1}]$$

where $T = \langle t\rangle \cong C_2$. Let $m_i \in T^k$ be such that $m_i(i) = t$ and $m_i(j) = 1$, $j \neq i$. Then

$$\{(i,k)|i = 1,\ldots,k\} \cup \{m_i(i,k)|i = 1,\ldots,k\}$$

is a transversal for $T^{k-1} \rtimes S_{k-1}$ in $T^k \rtimes S_k$. Note that $\sigma m_i\sigma^{-1} = m_{\sigma(i)}$. Then for $\sigma \in S_k, i = 1,\ldots,k$, we have

$$\sigma(i,k)T^{k-1} \rtimes S_{k-1} = (\sigma(i),k)T^{k-1} \rtimes S_{k-1}$$

since $(\sigma(i),k)\sigma(i,k) \in S_{k-1} \leq T^{k-1} \rtimes S_{k-1}$ and

$$\begin{aligned}\sigma m_i(i,k)T^{k-1} \rtimes S_{k-1} &= m_{\sigma(i)}\sigma(i,k)T^{k-1} \rtimes S_{k-1}\\ &= m_{\sigma(i)}(\sigma(i),k)T^{k-1} \rtimes S_{k-1}\end{aligned}$$

$$m_j(i,k)T^{k-1} \rtimes S_{k-1} = \begin{cases} m_i(i,k)T^{k-1} \rtimes S_{k-1} & ,j = i\\ (i,k)T^{k-1} \rtimes S_{k-1} & ,j \neq i\end{cases}$$

$$m_jm_i(i,k)T^{k-1} \rtimes S_{k-1} = \begin{cases} (i,k)T^{k-1} \rtimes S_{k-1} & ,j = i\\ m_i(i,k)T^{k-1} \rtimes S_{k-1} & ,j \neq i\end{cases}$$

Hence for each i, we may take $x_i = (i,k)T^{k-1} \rtimes S_{k-1}$, $y_i = m_i(i,k)T^{k-1} \rtimes S_{k-1}$ so that $Q = \oplus_{i=1}^{k} \oplus \mathbf{Z}x_i \oplus_{i=1}^{k} \mathbf{Z}y_i$. Here m_i interchanges x_i and y_i and fixes $x_j, y_j, j \neq i$ and $\sigma(x_i) = x_{\sigma(i)}$, $\sigma(y_i) = y_{\sigma(i)}$.

Now $f \in T^k$ is a reflection on Q iff

$$\mathrm{Im}_Q(f-1) = \mathrm{Span}_{\mathbf{Z}}\{x_i - y_i | f(i) \neq 1\}$$

has rank 1 iff $f = m_i$ for some i. Suppose $f\sigma \in T^k \rtimes S_k$ has $\sigma(i) \neq i$ for some i. Then if $f(i) = 1$, $x_{\sigma(i)} - x_i, y_{\sigma(i)} - y_i$ are in $\mathrm{Im}_Q(f\sigma - 1)$ and if $f(i) \neq 1$, $y_{\sigma(i)} - x_i, x_{\sigma(i)} - y_i$ are in $\mathrm{Im}_Q(f\sigma - 1)$. In either case, $f\sigma$ cannot be a reflection if $\sigma \neq 1$. So $R^\sharp = T^k$ since it is generated by the set of reflections $\{m_i\}$. Since Q is a permutation lattice,

$$\mathrm{Ker}_Q(m_i + 1) = \mathrm{Im}_Q(m_i - 1) = \mathbf{Z}(x_i - y_i), i = 1, \ldots, k$$

We may take a base for the associated root system to be $\{x_i - y_i | i = 1, \ldots, k\}$. Then since $\sigma(x_i - y_i) = x_{\sigma(i)} - y_{\sigma(i)}$, we see that S_k stabilizes the base. Since $\Omega^\sharp \cap T^k = 1$, we see that $\Omega^\sharp = S_k$. Finally note that $\oplus_{i=1}^{k} \mathbf{Z}x_i$ and $\oplus_{i=1}^{k} \mathbf{Z}y_i$ are $\mathbf{Z}S_k$ lattices both isomorphic to $\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$. Hence $\mathrm{Res}_{\Omega^\sharp} Q \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z} \oplus \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$.

For Φ^k of type III, $\Omega = T^k \rtimes S_k$ and $Q \cong \mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z} \oplus \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$ We have already seen that $\{(i,k) | i = 1, \ldots, k\}$ is a transversal for $T^k \rtimes S_{k-1}$ in $T^k \rtimes S_k$. Let $z_i = (i,k)T^k \rtimes S_{k-1}$. Then $\{z_i | i = 1, \ldots, k\}$ is a basis for $\mathbf{Z}[T^k \rtimes S_k / T^k \rtimes S_{k-1}]$ where T^k acts trivially on z_i and $\sigma(z_i) = z_{\sigma(i)}$. So

$$Q = \oplus_{i=1}^{k} \mathbf{Z}x_i \oplus \oplus_{i=1}^{k} \mathbf{Z}y_i \oplus \oplus_{i=1}^{k} \mathbf{Z}z_i$$

Reflections on Q must act as a reflection on one of $\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$ and trivially on the other. Now an element f of T^k acts as a reflection on $\mathrm{Ind}_{T^{k-1} \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$ iff $f = m_i$ for some i. Since all elements of T^k act trivially on $\mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathbf{Z}$, we see that the m_i are reflections on Q. Now if $\sigma \neq 1$, the rank of $\mathrm{Im}_Q(f\sigma - 1)$ was already shown to be greater than 1, hence $f\sigma$ cannot be a reflection. We see that $R^\sharp = T^k$ and as for type II, we see that $\Omega^\sharp = S_k$. Since $\oplus_{i=1}^{k} \mathbf{Z}x_i, \oplus_{i=1}^{k} \mathbf{Z}y_i, \oplus_{i=1}^{k} \mathbf{Z}z_i$ are each $\mathbf{Z}\Omega^\sharp$ lattices isomorphic to $\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$, we see that $\mathrm{Res}_{\Omega^\sharp} Q \cong (\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z})^3$ as required.

(b) For Φ^k of type I with $\Omega^\sharp = 1$, we have $Q = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L = \mathrm{Ind}_{S_{k-1}}^{S_k} L \cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z}$ and $Q' = dQ$. We may take $C = 0$, $D = Q$, $D' = Q' = dD$ to get the required decompositions.

Let Φ^k be of type II or III with $\Omega^\sharp = S_k$. Then in Lemma 4.4.7, we found $\mathbf{Z}T$ lattices L', L with $T \cong C_2$ such that

$$Q = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \qquad Q' = \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L'$$

and $L' \subset L$. Since $S_k(T^k \rtimes S_{k-1}) = T^k \rtimes S_k$, we see from the Mackey decomposition [10, p.237] that

$$\begin{aligned}
\mathrm{Res}_{S_k}^{T^k \rtimes S_k} Q &= \mathrm{Res}_{S_k}^{T^k \rtimes S_k} \mathrm{Ind}_{T^k \rtimes S_{k-1}}^{T^k \rtimes S_k} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \\
&\cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathrm{Res}_{S_{k-1}}^{T^k \rtimes S_{k-1}} \mathrm{Inf}_T^{T^k \rtimes S_{k-1}} L \\
&\cong \mathrm{Ind}_{S_{k-1}}^{S_k} \mathrm{Res}_1^{T^k \rtimes S_{k-1}} L \\
&\cong (\mathrm{Ind}_{S_{k-1}}^{S_k} \mathbf{Z})^{\mathrm{rank} L}
\end{aligned}$$

where the second last equality follows from the fact that S_{k-1} is contained in the kernel of the inflation map $T^k \rtimes S_{k-1} \to T$. The last equality gives another proof of the structure of Q as an $\Omega^\sharp$ lattice. The same calculation shows that we have a parallel statement for Q', L'. Then since restriction, induction and inflation preserve direct sums and permutation lattices, it suffices to find appropriate splittings of L', L as $\mathbf{Z}$-lattices. But $L/L' \cong \Omega/\mathbf{Z}\Phi$ is cyclic of order d. Hence there exist lattices C_0, D_0, D_0' where $D_0' = dD_0$ are of rank 1 and $L' = C_0 \oplus D_0, L = C_0 \oplus D_0$. Then we can take $C = \mathrm{Ind}_{S_{k-1}}^{S_k} C_0, D = \mathrm{Ind}_{S_{k-1}}^{S_k} D_0, D' = \mathrm{Ind}_{S_{k-1}}^{S_k} D_0'$ to get the required decompositions.

(c) Since $\Omega = \prod_{i=1}^m \Omega_i$ acts componentwise on $Q_A = \oplus_{i=1}^m Q_i$ and $Q_A' = \oplus_{i=1}^m Q_i'$, the results follow easily from (a) and (b). ■

By Corollary 4.3.3, it suffices to show that $K(B_A)^{\Omega^\sharp}$ is rational over K. Now $Q_A' \subset B_A \subset Q_A$ where there exist permutation $\mathbf{Z}\Omega^\sharp$ lattices C_A, D_A', D_A and decompositions

$$Q_A' = C_A \oplus D_A' \qquad Q_A = C_A \oplus D_A \qquad D_A' \subset D_A$$

of $\mathbf{Z}\Omega^\sharp$ lattices with $\Omega^\sharp$ acting faithfully on D_A. Setting $Y_A = B_A \cap D_A$, we see that $B_A = C_A \oplus Y_A$ is a decomposition of $\mathbf{Z}\Omega^\sharp$ lattices with C_A permutation, $D_A' \subset Y_A \subset D_A$ and $\Omega^\sharp$ acting faithfully on Y_A. So we may again use Proposition 1.0.13 to show that $K(B_A)^{\Omega^\sharp}$ is rational over $K(Y_A)^{\Omega^\sharp}$. Now we have reduced the problem to showing that $K(Y_A)^{\Omega^\sharp}$ is rational over K. But observe that $\Omega^\sharp \cong \prod_{i=1}^m \Omega_i^\sharp$ acts diagonally on $D_A = \oplus_{i=1}^m D_i$ and

$D'_A = \oplus_{i=1}^m D'_i$. From Lemma 4.4.11, we see that if i corresponds to an component of type II,III, $\Omega_i^\sharp = S_{k_i}$ acts faithfully as a group of reflections on $D_i \cong D'_i \cong \mathrm{Ind}_{S_{k_i-1}}^{S_{k_i}} \mathbf{Z}$ and if i corresponds to an irreducible of type I, $\Omega_i^\sharp = 1$ acts trivially (and also faithfully) on D_i, D'_i. If all the irreducibles are of type I, $\Omega^\sharp$ acts trivially on D'_A, D_A and hence on Y_A, so $K(Y_A)^{\Omega^\sharp} = K(Y_A)$ is clearly rational over K. Otherwise, $\Omega^\sharp$ acts faithfully as a group of reflections on D_A and D'_A and hence also on Y_A. So Farkas' result (Proposition 1.0.7) shows that $K(Y_A)^{\Omega^\sharp}$ is rational over K in this case. Hence we finally have proved the rationality of $K(A)^G$ over K. ■

Chapter 5

Centre of the Generic Division Ring

Define the $\mathbf{Z}S_n$-permutation lattices $U_n = \mathbf{Z}[S_n/S_{n-1}]$ and $V_n = \mathbf{Z}[S_n/S_{n-2}]$. Let $\{u_i | i = 1, \ldots, n\}$ be a $\mathbf{Z}S_n$ permutation basis for U_n and let $\{y_{ij} | 1 \leq i \neq j \leq n\}$ be a $\mathbf{Z}S_n$ permutation basis for V_n so that for $\sigma \in S_n$, $\sigma u_i = u_{\sigma(i)}$ and $\sigma y_{ij} = y_{\sigma(i)\sigma(j)}$. Defining $\epsilon_n : U_n \to \mathbf{Z}, u_i \mapsto 1$ and $\rho_n : V_n \to U_n$, $y_{ij} \mapsto u_i - u_j$, we get exact sequences of $\mathbf{Z}S_n$ lattices:

$$A_{n-1} \rightarrowtail U_n \stackrel{\epsilon_n}{\twoheadrightarrow} \mathbf{Z} \tag{5.0.1}$$

$$G_n \rightarrowtail V_n \stackrel{\rho_n}{\twoheadrightarrow} A_{n-1} \tag{5.0.2}$$

which define the $\mathbf{Z}S_n$ lattices A_{n-1} and G_n. By Procesi-Formanek [26, 27, 18], the centre of the division ring of generic matrices is isomorphic to the multiplicative invariant field $\mathbf{C}(U_n)(U_n \oplus G_n)^{S_n}$

Definition 5.0.3. For $\sigma = (i_1, \ldots, i_r)$ an r- cycle in S_n, define

$$c_\sigma = y_{i_1 i_2} + y_{i_2 i_3} + \cdots + y_{i_{r-1} i_r} + y_{i_r i_1} \in V_n$$

which we call a *cycle*. Note that c_σ does not depend on the way σ is presented and hence is well defined. Note also that for $\tau \in S_n$, $\tau(c_\sigma) = c_{\tau\sigma\tau^{-1}}$.

Definition 5.0.4.

$$w_{ij} = c_{(i,j)}, \qquad z_{ij} = \begin{cases} c_{(i,i+1,\ldots,j-1,j)} & \text{if } i < j \\ c_{(i,i-1,\ldots,j+1,j)} & \text{if } i > j \end{cases}$$

Lemma 5.0.5. *(a) The cycles c_σ generate G_n.*

(b)

$$B_1 := \{w_{ij} | 1 \leq i < j \leq n\} \cup \{z_{ij} | 1 \leq i, j \leq n, j - i \geq 2\}$$

and

$$B_2 := \{w_{ij} | 1 \leq i < j \leq n\} \cup \{z_{ij} | 1 \leq i, j \leq n, i - j \geq 2\}$$

are both **Z** *bases of G_n.*

Proof:

(a) Clearly every $c_\sigma \in \mathrm{Ker}(\rho_n) = G_n$. To show that they span $G_n = \mathrm{Ker}(\rho_n)$, we need only show that $\mathrm{Ker}(\rho_n) \subset \mathrm{Span}_{\mathbf{Z}}\{c_\sigma | \sigma = (i_1, \ldots, i_r)\}$. Let $a = \sum_{i \neq j} a_{ij} y_{ij} \in \mathrm{Ker}(\rho_n) = G_n$ and set $h_i(a) = \sum_{1 \leq j \leq n, j \neq i} a_{ij}$ for all i. Then $\sum_{i \neq j} a_{ij}(u_i - u_j) = 0$ implies that $h_i(a) = \sum_{1 \leq j \leq n, j \neq i} a_{ji}$. For each (i,j) with $i < j$, let $m_{ij} = \min\{a_{ij}, a_{ji}\}$. Since $\sum_{i<j} m_{ij}(y_{ij} + y_{ji}) \in \mathrm{Span}_{\mathbf{Z}}\{c_\sigma\}$, we see that

$$a \equiv a - \sum_{i<j} m_{ij}(y_{ij} + y_{ji}) \mod \mathrm{Span}_{\mathbf{Z}}(c_\sigma)$$

so that we may assume that $a_{ij} \geq 0$ and $\min\{a_{ij}, a_{ji}\} = 0$. Then every $h_i(a) \geq 0$ and we may proceed by induction on $h(a) = \sum_{i=1}^n h_i(a)$ for a satisfying

$$a = \sum_{i \neq j} a_{ij} y_{ij} \in \mathrm{Ker}(\rho_n), a_{ij} \geq 0, \min(a_{ij}, a_{ji}) = 0 \qquad (5.0.6)$$

If $h(a) = 0$, we are clearly done. If $h(a) > 0$ then consider the directed graph with vertices $\{1, \ldots, n\}$ and an edge from i to $j \Leftrightarrow a_{ij} > 0$. Note that an edge from i to j implies that there is none from j to i since $\min(a_{ij}, a_{ji}) = 0$. Note also that if i has an edge coming in, it also has one going out since $a_{ki} > 0$ implies $h_i(a) = \sum_{1 \leq j \leq n, j \neq i} a_{ji} = \sum_{1 \leq j \leq n, j \neq i} a_{ij} > 0$. Since $h(a) > 0$, there are some edges. Start at any edge and follow any next edge to form a path in the directed graph. Since the graph has finitely many vertices, the path must eventually return to some vertex: i.e. there exists $\sigma = (i_1, \ldots, i_k)$ such that $a_{i_j i_{j+1}} > 0$ for all $1 \leq j \leq k-1$ and $a_{i_k i_1} > 0$. Then $a - c_\sigma$ satisfies (5.0.6) and $h(a - c_\sigma) < h(a)$. So $a - c_\sigma$ satisfies the inductive hypothesis and hence is a **Z**-linear combination of cycles. Thus so is $a = a - c_\sigma + c_\sigma$.

(b) Note that

$$\mathrm{rank}_{\mathbf{Z}}(G_n) = \mathrm{rank}_{\mathbf{Z}}\mathbf{Z}[S_n/S_{n-2}] - \mathrm{rank}_{\mathbf{Z}} A_{n-1} = n(n-1) - (n-1) = (n-1)^2$$

Since also $|B_i| = (n-1)^2, i = 1, 2$, it suffices to show that B_1 and B_2 both span G_n. By (a), we need only show that each cycle c_σ, $\sigma = (i_1, \ldots, i_r)$ can be written as a $\mathbf{Z}$ linear combination of elements of B_1 (resp. B_2).

Setting

$$x_{ij} = \begin{cases} w_{ij} - z_{ij} & \text{if } i < j \\ z_{ji} & \text{if } i > j \end{cases}$$

we have:

$$y_{ij} - x_{ij} = \begin{cases} y_{ii+1} + \cdots + y_{j-1j} & \text{if } i < j \\ -(y_{jj+1} + \cdots + y_{i-1i}) & \text{if } i > j \end{cases}$$

Then since $i_{r+1} = i_1$, we see that $\sum_{j=1}^{r} y_{i_j i_{j+1}} - x_{i_j i_{j+1}} = 0$ so that $c_\sigma = \sum_{j=1}^{r} x_{i_j i_{j+1}}$ is a $\mathbf{Z}$ linear combination of elements of B_1 as required.

Similarly with

$$x'_{ij} = \begin{cases} w_{ij} - z_{ij} & \text{if } i > j \\ z_{ji} & \text{if } i < j \end{cases}$$

we find that $c_\sigma = \sum_{j=1}^{r} x'_{i_j i_{j+1}}$

Hence both B_1 and B_2 are $\mathbf{Z}$-bases of G_n. ■

Remark 5.0.7. For $n = 2$, $G_2 = \mathbf{Z}(y_{12} + y_{21}) \cong \mathbf{Z}$. So we see that $\mathbf{C}(U_2)(U_2 \oplus G_2)^{S_2}$ is rational over $\mathbf{C}(U_2)^{S_2}$ by Proposition 1.0.13 and hence over $\mathbf{C}$. Thus we may assume $n \geq 3$ from now on.

In order to investigate the $\mathbf{Z}S_n$ structure of G_n we first determine the irreducible components of the $\mathbf{Q}S_n$ module $\mathbf{Q}G_n$. Tensoring (5.0.2) over $\mathbf{Q}$ we obtain

$$0 \to \mathbf{Q}G_n \to \mathbf{Q}[S_n/S_{n-2}] \to \mathbf{Q}A_{n-1} \to 0$$

By Maschke's Theorem, $\mathbf{Q}[S_n/S_{n-2}] \cong \mathbf{Q}G_n \oplus \mathbf{Q}A_{n-1}$. Denote the irreducible $\mathbf{Q}S_n$ module corresponding to the partition $\lambda \mapsto n$ by S^λ. Then $\mathbf{Q}A_{n-1} = S^{(n-1,1)}$. Here we will apply Young's rule [22, p. 89], a beautiful combinatorial formula for determining the irreducible components of $\mathbf{Q}[S_n/S_\lambda]$ where S_λ is the Young subgroup of S_n corresponding to the partition $\lambda \mapsto n$. This shows that

$$\mathbf{Q}[S_n/S_{n-2}] \cong \begin{cases} S^{(n)} \oplus 2S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1^2)} & n \geq 4 \\ S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1^3)} & n = 3 \end{cases}$$

from which we deduce

$$\mathbf{Q}G_n \cong \begin{cases} S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus S^{(n-2,1^2)} & n \geq 4 \\ S^{(3)} \oplus S^{(2,1)} \oplus S^{(1^3)} & n = 3 \end{cases}$$

Lemma 5.0.8. *(a)* $\mathrm{Span}_{\mathbf{Z}}\{w_{ij} | 1 \leq i < j \leq n\}$ *is isomorphic to* $\mathbf{Z}[S_n/S_{n-2} \times S_2]$.
(b) $\psi_n : \mathbf{Z}[S_n/S_{n-2}] \to \mathbf{Z}[S_n/S_{n-2} \times S_2]$ *defined by* $\psi_n(y_{ij}) = w_{ij}$ *is a surjective* $\mathbf{Z}S_n$ *homorphism whose kernel* K_n *has basis* $\{y_{ij} - y_{ji} | 1 \leq i < j \leq n\}$.
(c) $\mathbf{Q}K_n \cong S^{(n-1,1)} \oplus S^{(n-2,1^2)}$.

Proof:

(a) The set $\{w_{ij}\}$ is linearly independent by Lemma 5.0.5. For $\sigma \in S_n$, $\sigma(w_{ij}) = w_{\sigma(i)\sigma(j)}$ so clearly $L_n = \mathrm{Span}_{\mathbf{Z}}\{w_{ij} | 1 \leq i < j \leq n\}$ is a transitive $\mathbf{Z}S_n$ permutation lattice. Since the stabilizer of $w_{ij} = y_{ij} + y_{ji}$ is isomorphic to $S_{n-2} \times S_2$, we have shown that $L_n \cong \mathbf{Z}[S_n/S_{n-2} \times S_2]$

(b) The first statement follows from (a) and $|\{y_{ij} - y_{ji} | 1 \leq i < j \leq n\}| = n(n-1)/2 = \mathrm{rank}(\mathbf{Z}[S_n/S_{n-2}]) - \mathrm{rank}(\mathbf{Z}[S_n/S_{n-2} \times S_2]) = \mathrm{rank}(K_n)$ shows that it suffices to show that these span. But $a = \sum_{i \neq j} a_{ij} y_{ij} \in \mathrm{Ker}(\psi_n)$ implies that $\sum_{i<j}(a_{ij} + a_{ji}) w_{ij} = 0$ and hence that $a = \sum_{i<j} a_{ij}(y_{ij} - y_{ji})$.

(c) By Young's rule, we have

$$\mathbf{Q}[S_n/S_{n-2} \times S_2] \cong \begin{cases} S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} & , n \geq 4 \\ S^{(3)} \oplus S^{(2,1)} & , n = 3 \end{cases}$$

Tensoring the exact sequence

$$0 \to K_n \to \mathbf{Z}[S_n/S_{n-2}] \to \mathbf{Z}[S_n/S_{n-2} \times S_2] \to 0$$

by $\mathbf{Q}$ and applying Maschke's Theorem we find that

$$\mathbf{Q}K_n \cong S^{(n-1,1)} \oplus S^{(n-2,1^2)}$$

as required. ∎

We would like to determine the unique pure $\mathbf{Z}S_n$ sublattice E_n of G_n corresponding to the irreducible for the partition $(n-2, 1^2)$.

Lemma 5.0.9. *(a)* $E_n = \mathrm{Span}_{\mathbf{Z}}\{c_\sigma - c_{\sigma^{-1}} | \sigma$ *a cycle in* $S_n\}$ *is a* $\mathbf{Z}S_n$ *sublattice of both* G_n *and* K_n *with basis* $\{z_{ij} - z_{ji} | 1 \leq i, j \leq n, j - i \geq 2\}$ *and* $\mathbf{Z}$*-rank* $\binom{n-1}{2}$.

(b) E_n is a pure sublattice of both G_n and K_n.
(c) $E_n = G_n \cap K_n$ and $\mathbf{Q}E_n \cong S^{(n-2,1^2)}$
(d) $h_n : G_n \to U_n$, $c_\sigma \mapsto \sum_{j=1}^r u_{i_j}$ for $\sigma = (i_1, \ldots, i_r)$ is a $\mathbf{Z}S_n$ epimorphism with $E_n \subset \mathrm{Ker}(h_n)$. For $n = 3$, $E_3 = \mathrm{Ker}(h_3) \cong \mathbf{Z}^-$ and $\mathrm{C}(U_3)(U_3 \oplus G_3)^{S_3}$ is rational over $\mathbf{C}$.

Proof: (a) E_n is S_n-stable since

$$\tau(c_\sigma - c_{\sigma^{-1}}) = c_{\tau\sigma\tau^{-1}} - c_{(\tau\sigma\tau^{-1})^{-1}}$$

for $\tau \in S_n$ and $\sigma = (i_1, \ldots, i_r) \in S_n$ and is contained in G_n, K_n since $\rho_n(c_\sigma) = \rho_n(c_{\sigma^{-1}})$ and $\psi_n(c_\sigma) = \psi_n(c_{\sigma^{-1}})$. The claimed set is linearly independent as for each (i,j) with $j - i \geq 2$, $z_{ij} - z_{ji}$ is the unique element with non-zero coefficient for y_{ij}. Finally, this set spans E_n because from the proof of Lemma 5.0.5, we see that

$$c_\sigma - c_{\sigma^{-1}} = \sum_{j=1}^{r} (x_{i_j i_{j+1}} - x'_{i_{j+1} i_j}) = \sum_{j=1}^{r} \mathrm{sgn}(i_j - i_{j+1})(z_{i_j i_{j+1}} - z_{i_{j+1} i_j})$$

(b) It suffices to show that the basis B_3 of E_n from (a) can be extended to bases for K_n and G_n respectively. Since for $j - i \geq 2$,

$$z_{ij} - z_{ji} = (y_{ji} - y_{ij}) + \sum_{k=i}^{j-1} (y_{kk+1} - y_{k+1k})$$

we see that $B_3 \cup \{y_{ii+1} - y_{i+1i} | i = 1, \ldots n-1\}$ is a basis for K_n. Also for $j - i \geq 2$,

$$z_{ij} = (z_{ij} - z_{ji}) - (z_{ij} - w_{ij}) + (\sum_{k=i}^{j-1} w_{kk+1})$$

shows that $B_3 \cup \{z_{ij} - w_{ij} | j - i \geq 2\} \cup \{w_{ii+1} | i = 1, \ldots, n-1\}$ is a basis for G_n.

(c) $\rho_n : \mathbf{Z}[S_n/S_{n-2}] \to A_{n-1}, y_{ij} \mapsto u_i - u_j$ maps $y_{ij} - y_{ji} \mapsto 2(u_i - u_j)$. Since $z_{ij} - z_{ji} \mapsto \sum_{k=i}^{j-1} 2(u_k - u_{k+1}) + 2(u_j - u_i) = 0$ and $\mathrm{rank}(K_n/E_n) = n - 1 = \mathrm{rank}(2A_{n-1})$ we see that $E_n = \mathrm{Ker}(\rho_n|_{K_n})$ so we have the exact sequence

$$0 \to E_n \to K_n \xrightarrow{\rho_n} 2A_{n-1} \to 0$$

with $\mathbf{Q}K_n \cong S^{(n-1,1)} \oplus S^{(n-2,1^2)}$, $\mathbf{Q} \otimes (2A_{n-1}) \cong S^{(n-1,1)}$ and so $\mathbf{Q}E_n \cong S^{(n-2,1^2)}$. From $E_n \subset G_n \cap K_n \subset K_n$ and $K_n/E_n \cong 2A_{n-1}$ an irreducible lattice, we need only rule out $G_n \cap K_n = K_n$ or equivalently $K_n \subset G_n$. But $G_n = \text{Ker}(\rho_n)$ and we just saw that $\rho_n(K_n) \neq 0$. So $E_n = G_n \cap K_n$ as required.

(d) We must first show that h_n is well defined. We define h_n on the basis

$$B_1 = \{w_{ij} | 1 \leq i < j \leq n\} \cup \{z_{ij} | 1 \leq i, j \leq n, j - i \geq 2\}$$

for G_n given in Lemma 5.0.5 and then check that this definition matches the definition on the cycles. So set

$$h_n(w_{ij}) = u_i + u_j, \qquad h_n(z_{ij}) = \sum_{k=i}^{j} u_k$$

From the proof of Lemma 5.0.5(b), we know that for $\sigma = (i_1, \ldots, i_r)$, we have $c_\sigma = \sum_{j=1}^{r} x_{i_j i_{j+1}}$ where

$$x_{ij} = \begin{cases} w_{ij} - z_{ij} & \text{if } i < j \\ z_{ji} & \text{if } j < i \end{cases}$$

Now,

$$h_n(x_{ij}) = \begin{cases} -(\sum_{k=i+1}^{j-1} u_k) & \text{if } i < j \\ \sum_{k=j}^{i} u_k & \text{if } i > j \end{cases}$$

So we have

$$h_n(c_\sigma) = \sum_{j=1}^{r} h_n(x_{i_j i_{j+1}}) = \sum_{k=1}^{n} m_k u_k$$

for some $m_k \in \mathbf{Z}$. where

$$m_k = |\{j | i_{j+1} \leq k \leq i_j\}| - |\{j | i_j < k < i_{j+1}\}|$$

But since $i_{r+1} = i_1$, we have $k \in (i_{l+1}, i_l)$ if and only if $k \in (i_m, i_{m+1})$ so that $m_k = |\{j | k = i_j\}|$ and $h_n(c_\sigma) = \sum_{j=1}^{r} u_{i_j}$ as required.

For $\tau \in S_n, \sigma = (i_1, \ldots, i_r)$, $\tau\sigma\tau^{-1} = (\tau(i_1), \ldots, \tau(i_r))$ so that

$$h_n(\tau c_\sigma) = h_n(c_{\tau\sigma\tau^{-1}}) = \sum_{j=1}^{r} u_{\tau(i_j)} = \tau(h_n(c_\sigma))$$

and hence h_n is a $\mathbf{Z}S_n$ map. $h_n(c_{(i_1,\dots,i_r)} - c_{(i_2,\dots,i_r)}) = u_{i_1}$ shows that h_n is surjective for $n \geq 3$. Since $h_n(c_\sigma) = h_n(c_{\sigma^{-1}})$ we see that $E_n \subset \text{Ker}(h_n)$. For $n = 3$, $E_3 \subset \text{Ker}(h_3)$ are both pure $\mathbf{Z}S_3$ sublattices of G_3 of rank 1 and hence $E_3 = \text{Ker}(h_3)$. Clearly $E_3 = \text{Span}(z_{13} + z_{31}) \cong \mathbf{Z}^-$. Then by Proposition 1.0.13, $\mathbf{C}(U_3)(U_3 \oplus E_3)^{S_3} \cong \mathbf{C}(E_3)(U_3 \oplus U_3)^{S_3}$ is rational over $\mathbf{C}(E_3)^{S_3}$. But since $E_3 \cong \mathbf{Z}^-$, it is a reflection lattice for S_3 and hence $\mathbf{C}(E_3)^{S_3}$ is rational over $\mathbf{C}$. So we have $\mathbf{C}(U_3)(U_3 \oplus E_3)^{S_3}$ is rational over $\mathbf{C}$ as required. ■

Remark 5.0.10. For $n = 4$, E_4 is in fact the lattice A_2 that Formanek uses in [19] to prove rationality of the centre of the division ring of 4×4 generic matrices. Formanek defines $A_2 = \text{Span}_{\mathbf{Z}}\{a_1, a_2, a_3, a_4\}$ where in our notation $a_1 = c_{(234)} - c_{(243)}$, $a_2 = c_{(143)} - c_{(134)}$, $a_3 = c_{(124)} - c_{(142)}$, $a_4 = c_{(132)} - c_{(123)}$. One easily checks by direct calculation that $a_1 + a_2 + a_3 + a_4 = 0$ and that $\{a_1, a_2, a_4\}$ is a $\mathbf{Z}$- basis for A_2. Our basis for E_4 can be expressed in terms of the basis $\{a_1, a_2, a_4\}$: $z_{14} - z_{41} = -a_2 - a_4$, $z_{13} - z_{31} = -a_4$ and $z_{24} - z_{42} = a_1$. Hence $E_4 = A_2$. Formanek describes his lattice A_2 as the dual of the kernel of the augmentation map for the signed permutation lattice: That is,

$$\sigma(a_i) = \text{sgn}(\sigma)(a_{\sigma(i)}), \sigma \in S_4$$

where $\sum_{i=1}^4 a_i = 0$ so that for the signed permutation lattice L with basis $\{e_i | i = 1, \dots, 4\}$ such that $\sigma(e_i) = \text{sgn}(\sigma)e_{\sigma(i)}$ for $\sigma \in S_4$, there is an exact sequence of $\mathbf{Z}S_4$ lattices:

$$0 \to \mathbf{Z}^- \to L \xrightarrow{\theta} E_4 \to 0$$

where $\theta : L \to E_4, e_i \mapsto a_i$ and $\text{Ker}(\theta) = \mathbf{Z}(e_1 + e_2 + e_3 + e_4) \cong \mathbf{Z}^-$.

Lemma 5.0.11. *The following is a commutative diagram with exact rows and columns:*

$$\begin{array}{ccccc}
E_n & \rightarrowtail & G_n & \twoheadrightarrow & G_n/E_n \\
\downarrow & & \downarrow & & \downarrow \\
K_n & \rightarrowtail & \mathbf{Z}[S_n/S_{n-2}] & \overset{\psi_n}{\twoheadrightarrow} & \mathbf{Z}[S_n/S_{n-2} \times S_2] \\
\downarrow & & \downarrow{\scriptstyle \rho_n} & & \downarrow \\
2A_{n-1} & \rightarrowtail & A_{n-1} & \twoheadrightarrow & A_{n-1}/2A_{n-1}
\end{array}$$

Proof: The middle row and column come from Lemma 5.0.8(b) and (5.0.2) respectively. By the proof of Lemma 5.0.9(c), restricting ρ_n to K_n gives an exact sequence $E_n \rightarrowtail K_n \overset{\rho_n}{\twoheadrightarrow} 2A_{n-1}$ So we find that

$$\begin{array}{ccc}
E_n & \rightarrowtail & G_n \\
\downarrow & & \downarrow \\
K_n & \rightarrowtail & \mathbf{Z}[S_n/S_{n-2}] \\
\downarrow{\scriptstyle \rho_n|} & & \downarrow{\scriptstyle \rho_n} \\
2A_{n-1} & \rightarrowtail & A_{n-1}
\end{array}$$

is a commutative diagram with exact rows and columns. Applying the snake lemma, we find that the cokernel sequence

$$G_n/E_n \rightarrowtail \mathbf{Z}[S_n/S_{n-2}]/K_n \twoheadrightarrow A_{n-1}/2A_{n-1}$$

is exact and fits into the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc}
G_n & \twoheadrightarrow & G_n/E_n & \xrightarrow{=} & G_n/E_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Z}[S_n/S_{n-2}] & \twoheadrightarrow & \mathbf{Z}[S_n/S_{n-2}]/K_n & \xrightarrow{\overline{\psi}_n} & \mathbf{Z}[S_n/S_{n-2} \times S_2] \\
\downarrow{\scriptstyle \rho_n} & & \downarrow & & \downarrow \\
A_{n-1} & \twoheadrightarrow & A_{n-1}/2A_{n-1} & \xrightarrow{=} & A_{n-1}/2A_{n-1}
\end{array}$$

Putting these two commutative diagrams together and observing that the composite of $\mathbf{Z}[S_n/S_{n-2}] \twoheadrightarrow \mathbf{Z}[S_n/S_{n-2}]$ and $\overline{\psi}_n$ is ψ_n, we obtain the required commutative diagram with exact rows and colums. ■

Definition 5.0.12. [9] A $\mathbf{Z}G$ lattice A is *coflasque* if $H^1(H, A) = 0$ for all subgroups H of G.

Lemma 5.0.13. *The $\mathbf{Z}S_n$ lattice G_n/E_n is coflasque iff n is odd.*

Proof: Tensoring the exact sequence of $\mathbf{Z}S_n$ lattices $0 \to A_{n-1} \to U_n \to \mathbf{Z} \to 0$ over $\mathbf{Z}/2\mathbf{Z}$ and using the natural $\mathbf{Z}G$ isomorphism $X/2X \overset{\cong}{\to} \mathbf{F}_2 X, x+2X \mapsto$

$\overline{x}$ where X is a $\mathbf{Z}G$ lattice, we obtain the following diagram of $\mathbf{Z}S_n$ modules which commutes and has exact rows and columns:

$$\begin{array}{ccccc} A_{n-1}/2A_{n-1} & \rightarrowtail & U_n/2U_n & \twoheadrightarrow & \mathbf{Z}/2\mathbf{Z} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \mathbf{F}_2A_{n-1} & \rightarrowtail & \mathbf{F}_2U_n & \twoheadrightarrow & \mathbf{F}_2 \end{array}$$

By Lemma 5.0.11, the $\mathbf{Z}S_n$ sequence

$$0 \to G_n/E_n \to \mathbf{Z}[S_n/S_{n-2} \times S_2] \to A_{n-1}/2A_{n-1} \to 0$$

is exact. Composing with the natural $\mathbf{Z}S_n$ isomorphism $A_{n-1}/2A_{n-1} \cong \mathbf{F}_2A_{n-1}$, we obtain an exact sequence

$$0 \to G_n/E_n \to \mathbf{Z}[S_n/S_{n-2} \times S_2] \to \mathbf{F}_2A_{n-1} \to 0 \tag{5.0.14}$$

where the map $\mathbf{Z}[S_n/S_{n-2} \times S_2] \to \mathbf{F}_2A_{n-1}$ is given by $w_{ij} \mapsto \overline{u}_i + \overline{u}_j$. For a subgroup H of S_n, this sequence induces the following exact sequence in cohomology:

$$\mathbf{Z}[S_n/S_{n-2} \times S_2]^H \to (\mathbf{F}_2A_{n-1})^H \twoheadrightarrow H^1(H, G_n/E_n)$$

since $H^1(H, \mathbf{Z}[S_n/S_{n-2} \times S_2]) = 0$ by Shapiro's lemma [7, p. 73]. So $H^1(H, G_n/E_n) = 0$ iff $\mathbf{Z}[S_n/S_{n-2} \times S_2]^H \to (\mathbf{F}_2A_{n-1})^H$ is surjective.

We first wish to determine $(\mathbf{F}_2A_{n-1})^H$. Denote the basis of $\mathbf{F}_2U_n$ by $\{\overline{u}_i | i = 1, \ldots, n\}$. Then $\mathbf{F}_2A_{n-1}$ has basis $\{\overline{u}_i + \overline{u}_{i+1} | i = 1, \ldots, n-1\}$. Let $\{O_1, \ldots, O_r\}$ be the set of orbits of H acting on $\{1, \ldots, n\}$. Then $(\mathbf{F}_2U_n)^H$ has basis $\{\hat{O}_i | i = 1, \ldots, r\}$ where $\hat{O}_i = \sum_{j \in O_i} \overline{u}_i$. So $\sum_{i=1}^m a_i\hat{O}_i \in (\mathbf{F}_2A_{n-1})^H = \mathbf{F}_2A_{n-1} \cap (\mathbf{F}_2U_n)^H$ iff $\sum_{i=1}^m a_i|\hat{O}_i| \equiv 0 \mod 2$. Reorder the orbits above so that the first t have even length and the remaining $r-t$ have odd length. So a basis for $(\mathbf{F}_2A_{n-1})^H$ is

$$\{\hat{O}_i | i = 1, \ldots, t\} \cup \{\hat{O}_i + \hat{O}_r | i = t+1, \ldots, r-1\}$$

Let n be odd. If H acts transitively, then our basis for $(\mathbf{F}_2A_{n-1})^H$ shows that that $(\mathbf{F}_2A_{n-1})^H = 0$ and so the fixed point map is surjective in this case.

Suppose H acts intransitively. Then since $\sum_{i=1}^{r} |O_i| = n \equiv 1 \mod 2$, we find that $|O_r|$ is odd. Then $\sum_{j\in O_i, k\in O_r} w_{jk} \in \mathbf{Z}[S_n/S_{n-2} \times S_2]^H$ maps to $\sum_{j\in O_i, k\in O_r}(\overline{u}_j + \overline{u}_k) = |O_r|\hat{O}_i + |O_i|\hat{O}_r = \hat{O}_i + |O_i|\hat{O}_r$ For $1 \le i \le t$, this shows that $\hat{O}_i$ is in the image and for $t+1 \le i \le r-1$, it says that $\hat{O}_i + \hat{O}_r$ is in the image. So our map is onto as required.

Let n be even. We will show that G_n/E_n is not coflasque by showing that the fixed point map $\mathbf{Z}[S_n/S_{n-2} \times S_2]^H \to (\mathbf{F}_2 A_{n-1})^H$ is not surjective for $H = D_{2n} = \langle \tau, \sigma \rangle$ where $\tau = (1,2)$ and $\sigma = (1,2,\ldots,n)$. Since $(\mathbf{F}_2 A_{n-1})^{D_{2n}}$ has basis $\{\sum_{i=1}^{n} \overline{u}_i\}$, we need to show that this map is zero. It suffices to show that the D_{2n} orbit of a basis element w_{ij} maps to 0. Now the D_{2n} orbit of a basis element w_{ij} must contain an element w_{1l} with $l \neq 1$ since D_{2n} acts transitively on $\{1,2,\ldots,n\}$.

Case 1: $l \neq \frac{n}{2}+1$

Since $\sigma^{\frac{n}{2}}(w_{1l}) = w_{1+\frac{n}{2}, l+\frac{n}{2}} \neq w_{1l}$, the $C_n = \langle\sigma\rangle$ orbit of w_{1l} is $\{\sigma^i w_{1l} | i = 0,\ldots,n-1\}$ and $\sum_{i=0}^{n-1} \sigma^i w_{1l} \in \mathbf{Z}[S_n/S_{n-2} \times S_2]^{C_n}$ maps to $\sum_{i=1}^{n} \sigma^i(\overline{u}_1 + \overline{u}_l) = 2\sum_{i=1}^{n} \overline{u}_i = 0$. Since C_n is normal in D_{2n} with transversal $\{1,\tau\}$, $(1+\tau)(\sum_{i=0}^{n-1} \sigma^i w_{1l}) \in \mathbf{Z}[S_n/S_{n-2} \times S_2]^{D_{2n}}$ also maps to 0 under the fixed point map.

Case 2: $l = \frac{n}{2}+1$

The D_{2n} orbit of w_{1l} also includes $\sigma^{-1}\tau w_{1,\frac{n}{2}+1} = w_{1,\frac{n}{2}}$ which reduces us to the previous case. ■

Lemma 5.0.15. *For n odd, there exists an exact sequence*

$$0 \to G_n/E_n \oplus \mathbf{Z} \to \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} \oplus U_n \to U_n \to 0$$

Proof: We have exact sequences

$$G_n/E_n \rightarrowtail \mathbf{Z}[S_n/S_{n-2} \times S_2] \twoheadrightarrow \mathbf{F}_2 A_{n-1}$$

$$\mathbf{F}_2 A_{n-1} \rightarrowtail \mathbf{F}_2 U_n \twoheadrightarrow \mathbf{F}_2$$

where $\mathbf{F}_2 U_n \to \mathbf{F}_2$ maps $\overline{u}_i \mapsto \overline{1}$. Since $(\mathbf{F}_2 U_n)^{S_n}$ has basis $\{\sum_{i=1}^{n} \overline{u}_i\}$ which goes to $\overline{n} \in \mathbf{F}_2$ under this map, we see that $\mathbf{F}_2 A_{n-1} \oplus \mathbf{Z} \to \mathbf{F}_2 U_n$, $(x,z) \mapsto x + \overline{z}\sum_{i=1}^{n} \overline{u}_i$ is surjective for n odd (and not for n even). Also for n odd, the images of $\mathbf{F}_2 A_{n-1} \to \mathbf{F}_2 U_n$ and $\mathbf{Z} \to \mathbf{F}_2 U_n$ are disjoint.

This suggests building the map $(\alpha_n, \beta_n) : \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} \twoheadrightarrow \mathbf{F}_2 U_n, \alpha_n(w_{ij}) = \overline{u}_i + \overline{u}_j, \beta_n(1) = \sum_{i=1}^{n} \overline{u}_i$. Note that $\mathrm{Im}(\alpha_n) \cap \mathrm{Im}(\beta_n) = 0$ since the images of

$\mathbf{F}_2A_{n-1} \to \mathbf{F}_2U_n$ and $\mathbf{Z} \to \mathbf{F}_2U_n$ are disjoint. But then the kernel of (α_n, β_n) is $\mathrm{Ker}(\alpha_n) \oplus \mathrm{Ker}(\beta_n) = G_n/E_n \oplus 2\mathbf{Z} \cong G_n/E_n \oplus \mathbf{Z}$. So we have an exact sequence

$$G_n/E_n \oplus \mathbf{Z} \rightarrowtail \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} \twoheadrightarrow \mathbf{F}_2U_n$$

Form the pullback square:

$$\begin{array}{ccc} X_n & \twoheadrightarrow & U_n \\ \downarrow & & \downarrow \\ \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} & \twoheadrightarrow & \mathbf{F}_2U_n \end{array}$$

We obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccc} & & U_n & \xrightarrow{=} & U_n \\ & & \downarrow & & \downarrow{\scriptstyle \times 2} \\ G_n/E_n \oplus \mathbf{Z} & \rightarrowtail & X_n & \twoheadrightarrow & U_n \\ \downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\ G_n/E_n \oplus \mathbf{Z} & \rightarrowtail & \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} & \twoheadrightarrow & \mathbf{F}_2U_n \end{array}$$

The exact sequence

$$0 \to U_n \to X_n \to \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z} \to 0$$

splits since U_n and $\mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z}$ are both permutation lattices. So $X_n \cong U_n \oplus \mathbf{Z}[S_n/S_{n-2} \times S_2] \oplus \mathbf{Z}$ and we obtain the desired exact sequence. ■

Proposition 5.0.16. *For n odd, $G_n \sim E_n$ where $\sim$ is the equivalence relation of Proposition 1.0.15. In particular, $\mathbf{C}(U_n)(U_n \oplus G_n)^{S_n}$ is stably equivalent to $\mathbf{C}(U_n)(U_n \oplus E_n)^{S_n}$. Hence, for n odd, $\mathbf{C}(U_n)(U_n \oplus G_n)^{S_n}$ is stably rational over $\mathbf{C}$ provided $\mathbf{C}(E_n)^{S_n}$ is.*

Proof: Since G_n/E_n is coflasque by Lemma 5.0.13, the exact sequence of Lemma 5.0.15 splits so that

$$G_n/E_n \oplus \mathbf{Z} \oplus U_n \cong U_n \oplus \mathbf{Z} \oplus \mathbf{Z}[S_n/S_{n-2} \times S_2] \equiv P_n$$

with P_n a permutation lattice. We can construct exact sequences

$$0 \to E_n \to G_n \oplus \mathbf{Z} \oplus U_n \to P_n \to 0$$

$$0 \to G_n \to G_n \oplus \mathbf{Z} \oplus U_n \to \mathbf{Z} \oplus U_n \to 0$$

so that we see that $G_n \sim E_n$ and hence that $U_n \oplus G_n \sim U_n \oplus E_n$. Since S_n acts faithfully on U_n and hence on $\mathbf{C}(U_n)$, we may apply Proposition 1.0.15 to prove that the above fields of tori invariants are stably equivalent. ■

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IMAGE EVALUATION TEST TARGET (QA-3)

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6"

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