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Interacting With Implicit Knowing in the Mathematics Classroom

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Abstract

This study explores Grade Seven students' experiences of doubt and certainty in mathematics. During nine months of (bi-monthly) sessions, students responded to several mathematical prompts; their interactions with each other and with the teacher-researcher were video-taped, transcribed, and coded for learners' evolving perceptions of what was (a) sufficient to define certainty (including what was experienced as intuitive or counter-intuitive and ways such certainty was *interrupted*), (b) relevant to the tasks (including understandings that initially dwelled on the periphery of awareness), and (c) mathematically connected. The study is conceptualized within an enactivist view of cognition that emphasizes autonomous, co-emergent, and embodied knowing (Thompson, 2007; Varela, Thompson, & Rosch, 1991), and classes were designed with these principles in mind. It became clear that doubt and certainty emerge from a broader, holistic, understanding that is largely beneath ordinary awareness and is deeply implicated in what we experience as "repeatable context" (Bateson (1964/1972). An important aspect of the study was to bring more of this understanding to awareness. In doing so, Varela's (Varela & Scharmer, 2000) notion of researcher as empathic coach and Gendlin's notions of "felt sense" (1962, 1978) and "implicit intricacy" (1991; 2009a) assumed importance. By attending to the holistic sense that points to implicit understanding, it was possible to broaden the scope of what was deemed relevant in selected contexts. It was found that previously subconscious understandings nonetheless influenced learning. Once named (even broadly), implicit understanding co-evolved with language in developing mathematical understanding. By attending to external indicators of felt meaning, learners interacted with each others' implicit understanding, thereby bringing it closer to consciousness and into conversation. Prematurely insisting on clarity and logic precluded awareness of the implicit.

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1. Introduction: Finding Doubt Spaces

I sat with a small group of graduate students, all mathematics teachers, around a seminar table working on a problem the professor had posed. The problem was one you might find in any middle school textbook. It was concisely stated and relatively straightforward: "In a warehouse you obtain a 20% discount but you must pay a 15% sales tax. Which would you prefer to have calculated first, discount or tax?" (Mason, Burton, & Stacey, 1982, p. 1).

The members of the group soon began sharing their responses: "It does not matter which you calculate first." "It is the same." "No preference, it will cost the same." I noted that order matters if you're the seller or the tax-collector, but not if you're the customer. We shared arguments and examples to explain our responses, and there appeared to be little disagreement.

But at least one person was bothered. He commented that although he agreed with both the deductive arguments and the large number of examples developed by the others, "It just feels wrong that the solutions should come out the same either way."

I, too, was unsatisfied with the seemingly obvious resolution to this problem. As I pondered this experience, many loosely connected strands in my quest to more deeply engage with teaching, learning, and mathematics began to converge. Over time, they became the focus of this study: *How do learners experience doubt and certainty in the mathematics classroom? More specifically, how might learners deepen their awareness of partially conscious feelings associated with doubt and certainty and use them as gateways to deeper understanding?*

Gregory Bateson (1979/2002) once said, “A story is a little knot or complex of that species of connectedness which we call relevance…. [A]ny A is relevant to any B if both A and B are parts or components of the same ‘story’” (p. 12). Stories have been important to the framing and development of this study. Before providing an overview, then, I would like to offer a chronology detailing a piece of the story that brought me to this place. I then share two more stories that helped focus and define it. As I develop my research frame in Chapter 2, I relate Dave Hewitt’s (1996) story about learning to sail to Thompson’s (2000) description of a bacterium in a sucrose gradient and contrast these with my own experience of “teaching” with a pre-packaged environmental mystery. In Chapter 3, I share a number of vignettes relevant to my efforts to deepen the awareness needed to become an effective empathic second-person researcher. Stories from the classroom resurface in Chapter 4. As Bateson and Bateson (1987) so aptly put it, “[T]he real trick [I hope] is what happens when the stories are set side by side” (p. 35).

Early Stories

Holding together investments, beliefs, and ways of being that seem tenuously related or even contradictory can be challenging. But, *my* investments, even the most contradictory, are *somehow* connected in my own present moment, united in a unique history. This project, then, represents part of my ongoing commitment to bring into deeper conversation a number of enduring and recurring personal investments and to explore what still sometimes feels like a surprising affinity to mathematics education. Throughout this study, then, the interaction of I-the-researcher with I-the-teacher and I-the-learner-of-mathematics are necessarily deeply intertwined and co-implicating. The methodology described in Chapter 3 supports the interdependence of these roles.

In the Book of Life, the Answers Are Not in the Back

My first teaching job was in a small rural community in Alberta, where I was responsible for Grades 7-9 Math, Science, and Social Studies (the Junior High formed a single class) and for Grades 5-6 Science, Social Studies, and later Math (along with numerous options and extra-curricular responsibilities). And no prep time. Full of new-teacher ideals and despite an overwhelming course load, I was determined that all subjects should be built around modes of inquiry that allowed students to discover the things they needed to know. I was discouraged when this did not happen. The problem was particularly noticeable in science and math, likely in part because I was least familiar with their modes of inquiry. I quickly grew frustrated with the so-called scientific method (narrowly defined for use in school), which not only failed to provide deep understanding of science concepts, but also limited the very nature of the questions that we might ask. I wanted inquiry to be doing what it takes to find and justify answers to meaningful questions. And if it could not do that, should the methods not be adjusted?

Elementary science, with fewer curriculum expectations (than junior high) and eager students, soon became a rich arena for exploring these questions. Together the students and I learned to tap more deeply into our own creativity to generate hypotheses and find ways to (rationally) evaluate and / or (empirically) test them. As we did so, we started to develop our own list of criteria for good science. I was very excited by our dawning recognition of both the power and the fallibility of science. The students generated many tentative explanations and often found sophisticated ways to test (and revise and test again) their ideas. Eventually, one sixth-grader announced in frustration:

> [T]here's zillions of ways that you could, that one person could think of, that explain why or how these two things [vinegar and baking soda] mix or what they do when they mix. And this is just one in a sea of zillions, so it could be right, it could be wrong, it could be a little of both. (as cited in Schmidt, 1999, p. 216)

This student's reflections followed extensive interactions with classmates in the context of a particular type of experience. His frustration echoes those philosophers of science and mathematics who claim that knowledge is theory-dependent and can, at best, be falsified (*cf.* Kuhn, 1970; Lakatos, 1976; Popper, 1959).

Some students seemed to come up with creative hypotheses and interesting ways to test them almost effortlessly. But if I wanted forming and testing hypotheses to be a central part of my science program–and if I wanted to evaluate the students on their ability to do so–I felt I should be able to *teach* those things rather than merely identify who was already good at them[1]. To my surprise, analogical reasoning emerged as a powerful factor in students' generation and evaluation of their ideas (Metz, 2009; Schmidt, 1999). I began to recognize metaphor as a powerful catalyst for new insights and questions. It also became evident that metaphor, when used without awareness, could lead to overgeneralization and assumption and that it lies at the heart of much prejudice and pseudoscience. I became convinced that greater awareness of our use (and misuse) of analogical reasoning could ultimately have tremendous impact on the way we interact with our worlds. In fact, as students gradually became more aware of their use of analogical reasoning, their use of it did evolve. At first, there were instances

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[1] Gendlin's (1978) description of his efforts to identify the elusive qualities of successful psychotherapy patients seems very similar to this endeavor. Like Gendlin's patients, some students seemed to approach learning in qualitatively different ways. Like Gendlin, one of my longstanding goals as a teacher has been to identify and teach those ways.

where they used their dawning awareness to explain away non-working parts of an analogy with, "It's okay if not everything matches up–it's just an analogy." *This* awareness allowed us to consider more deeply which connections did, in fact, need to hold up in order for the analogy to be used to support a conclusion.

I originally conceived of analogical reasoning as central to idea-generation, so it came as a surprise to me that even *evaluating* proposed analogs (particularly by identifying implications of a proposed idea) was deeply dependent on analogical reasoning (nonetheless, my inclination to oppose creativity and skepticism re-emerged in this study; I further problematize this tendency in Chapter 6).

Moving beyond individual knowing, I became fascinated with how group dynamics are implicated in the evolution of belief structures:

> [W]hen we ask questions, everybody gets them, and then they think of questions, and then people get answers from them, and then more people think of questions from the answers, and then the argument goes on again. (Grade 6 student; as cited in Schmidt, 1999, p. 107)

How might children's awareness of this dynamic influence their understanding of knowledge, how it is created, and what is knowable?

Sometimes it strikes me a bit odd that my Masters thesis was built around understanding electric circuits and the reaction between vinegar and baking soda. My own interests have tended to cluster more densely around ecology and social justice. Despite our intense focus on understanding the objects of our investigations, however, I attended most deeply to understanding the nature of knowledge claims, of limits on our ability to know, and to the role of the collective in generating that knowledge.

How Thinking Goes Wrong

> I have taught various branches of behavioral biology and cultural anthropology to American students, ranging from college freshmen to psychiatric residents, in various schools and teaching hospitals, and I have encountered a very strange gap in their thinking that springs from a lack of certain tools of thought. This lack is rather equally distributed at all levels of education, among students of both sexes and among humanists as well as scientists. Specifically, it is lack of knowledge of the presuppositions not only of science but also of everyday life….
> Those who lack all idea that it is possible to be wrong can learn nothing except know-how. (Bateson, 1979/2002, pp. 23-24)

Although knowledge construction in my classrooms became a very social endeavor, I am not sure that I invited adequate consideration of the value of tradition or the potential dangers of unbridled personal agency (even if *the person* sometimes felt more like *the class*). After all, sometimes tradition protects much of great value, and I cannot say that any of us had deep awareness of the complex web with which we were tampering or of the ways the allure of independence and agency can be abused; Oppenheimer candidly spoke of his *need* to build an atomic bomb so that he might test the ideas he had developed (Margulis, 1997). Unaware of these dangers, by this time I had also become concerned with what I perceived as the dangers of blind adherence to ideology, particularly in the forms of pseudoscience and religion. This seemed to add an urgent moral imperative to use the classroom as a site for developing the sort of thinking that might help separate right from wrong–both true from false and good from bad.

Inspired by Richard Feynman's (1985) definition of science (which I had not yet seriously considered as a potential ideology in its own right) as "a long history of learning how not to fool ourselves" (p. 338) and Carl Sagan's (1996) "Baloney-Detection Kit," I had my fourth-graders start compiling a list of "how thinking goes wrong" (which we later expanded to include "what counts as good evidence"). I stayed with many of the same students over the next four years, and we continued to expand these lists, often straying far beyond science.

Whenever we had a little time, I liked to share stories that depicted the implications of blindly following allegedly scientific claims or of uncritically accepting propaganda of various sorts. Many of the students found stories about the Holocaust and the Arab-Israeli conflict particularly thought-provoking, and these became powerful motivators for considering the potential consequences of unexamined political, economic, and religious assumptions and of failing to appreciate other points of view. One of my favorite books was *The War Within*, by Carol Matas (2003), which portrays a Jewish family living in the American South during the Civil War. The characters are sensitively developed as both oppressed and oppressor–oppressed by the southern authorities, even as they slowly come to terms with what it means to be slave owners.

I was often inspired by how the students learned to make connections between their own behavior and the events in the novels we shared. In particular, I remember the reflections several students offered after revolting against a student teacher when I was out of the room. After working through her anger, one girl spoke candidly and with remarkable insight about how her rage had made her feel justified, about how strong she felt as she stood and shouted, about how the support of the class gave her a sense of power, and about how easy it was to get caught up in the emotion of the moment. Her honesty prompted others to speak, and many acknowledged that they had let themselves get out of control–that just because the student teacher had behaved in ways they felt were disrespectful did not mean they had to handle the situation the way they did–*even if he was wrong*. The lead girl spontaneously connected the event to a Hitler rally. She was not being overdramatic; she had gained a small glimpse into just how easy it might be, under the right circumstances, for a charismatic leader to stir up the passions of people convinced that they are *right*. I, too, learned a very important lesson that day, for it was clear to me that the students fully expected me to take their side–perhaps even to be proud of their courage in standing up for themselves.

Ironically, over-reliance on rational-empirical thought has become part of tradition in our culture. We have seen dangers in its excesses, but it is important to consider what we risk losing as we interrupt its supremacy. Can we do this in such a way that our need for security and certainty do not respond to the loss of one form of certainty with a turn to another? Can we teach students to appreciate what science and math *do* allow us to know, thereby helping them better understand distortions of their fallibility by those who do not like the conclusions they support? Sagan (1996) illustrates potential impacts of these dangers in *The Demon-Haunted World: Science as a Candle in the Dark*. Although I do not agree that rational-empirical modes of thought should be *the* antidote to fear generated by loss of certainty, if efforts to make students more aware of the fallibility of this knowledge lead them to reject those forms of knowledge outright, we must be alert to ways to better support their journeys.

Rebel Without a Clue?

> Our German teacher, an enthusiastic democrat, often read aloud to us from the liberal Frankfurter Zeitung. But for this teacher I would have remained altogether nonpolitical in school. For we were being educated in terms of a conservative bourgeois view of the world. In spite of the revolution which had brought in the Weimar Republic, it was still impressed upon us that the distribution of power in society and the traditional authorities were part of the God-given order of things. We remained largely untouched by the currents stirring everywhere during the early twenties. In school, there could be no criticism of courses of subject matter, let alone of the ruling powers in the state. Unconditional faith in the authority of the school was required. It never even occurred to us to doubt the order of things, for as students we were subjected to the dictates of a virtually absolutist system. Moreover, there were no subjects such as sociology which might have sharpened our political judgments. Even in our senior year, German class assignments called solely for essays on literary subjects, which actually prevented us from giving any thought to the problems of society. Nor did all these restrictions in school impel us to take positions on political events during extracurricular activities or outside of school. (Speer,[2] 1970, pp. 8-9)

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[2] Albert Speer worked for Hitler as an architect and eventually became Minister of Armaments and War Production for the Third Reich. He was convicted at Nuremberg and remained in Spandau Prison until

It seems my early investments in teaching mathematics as a form of inquiry were deeply rooted in my resistance to the authority of textbook and tradition, as I attempted to dig beneath the rote methods that had dominated my own education in math. It soon became apparent that much of math seemed based on *necessary* implications rather than *arbitrary* rules (see Hewitt, 1999; 2001a; 2001b for a helpful exploration of the arbitrary and the necessary in mathematics). I began to make frequent reminders to my students that in math, seemingly complicated ideas were natural consequences of simple things they all understood. This appealed to me, and I wanted them to learn how to make these connections. Math was a place of independence where I could allow (and expect) students to think for themselves (although typically within the context of a strong classroom community), to figure out how to do things their own way, and not to have to wait to be told by someone in-the-know.

Nonetheless, my early years of teaching math involved a lot of me *explaining* the necessity of mathematical ideas–not so much because I thought this was a good approach, but because I had not yet developed an alternative. I had to learn a lot of math in order to offer explanations for what I had until then learned only procedurally–a venture that consumed vast amounts of precious planning time in a whirlwind first year of teaching. Despite having achieved a perfect score on my Grade 12 diploma exam in Math 30, I had always felt that somehow I had faked my way through–I knew that I did not really understand what I was doing, and I assumed that those who were *genuinely* good at math did. It was in developing what I considered satisfying explanations for familiar procedures that I first experienced the epiphany of mathematics as connected and exciting. Nonetheless, I was alerted to the limitations of teaching-by-explaining through an unpleasant experience with summative evaluation: At the end of my first

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1966, after which he wrote two books of memoirs. In them, he accepted responsibility for his role in the Nazi regime and explored the psychological complexities of his involvement.

year, I decided that the division-wide math tests did not really test for understanding, and I designed my own. Although my tests were based on what I thought I had taught that year, even my top math students performed poorly. Most failed miserably. I knew it was not fair to blame the students, and I decided to give them the division's tests for comparison. Suddenly, the distribution of marks looked reasonable. It seemed that my teaching was likely quite normal, but I knew something was terribly wrong. This was confirmed the following year when I taught most of the same students again (I was teaching a combined Grade 7-9 class): I could not blame the previous year's teacher for the things my students did not know. I had learned a lot about math by figuring out why various procedures worked, but how could I engage my students in similar experiences? Framing my goal in this way did not radically transform my teaching practice over night, but it set me on a new course that affected how I designed and presented tasks, what I chose to adjust within and between lessons, and how I wove insights from professional development activities and readings into my approach (at that time, constructivism and cooperative learning were big influences). Over the next twenty years, commitment to and refinement of this goal has indeed prompted a radical (and ongoing) transformation of my understanding of mathematics, of teaching, and of what it means to learn.

In my most recent teaching assignment, I had the luxury of focusing almost exclusively on math and science, even teaching the same topics to two groups of students–quite a contrast to my first years. When I started this position, I had just finished my M.Ed. in science education and was particularly excited about teaching science. But over the next seven years, I found my practice *drifting* into a more dominant focus on mathematics. I was increasingly seeing the importance of mathematizing what we were doing in science, which in turn prompted deeper understanding of the required mathematics. Although there were times when I could have taught the students how to use mathematical tools required by a particular context, more often than not, the procedures themselves did not offer the necessary insight. The contextual need for the mathematics informed its development and use.

Over time, and with the support of people in the mathematical community who shared with me a different sort of mathematics than I had been exposed to in my own school experience, it became increasingly apparent that math can be a wonderfully creative endeavor with ample room for new ideas. I began to get excited when the students (and I) found new ways to understand, represent, and connect the mathematics we discovered in a wide variety of (scientific, ecological, social, and purely mathematical) contexts and to use mathematics to more deeply understand these situations. The students developed insights I have not seen presented in any textbook: *Not just* “anyone could be here in [our] place[s], and things would proceed identically” (Jardine, 1994/2006, p. 187).

Often, the value of new ideas was not immediately clear, and we had to practice patient, careful listening and deep questioning to recognize the significance of each other’s offerings. Sometimes ideas that seemed different could be seen as identical. Sometimes new methods were not correct in an ideal or universal sense but were adequate–even ingenious–within a particular context.[3] Conversely, sometimes mathematical evidence that would have been considered sound by conventional standards failed to convince.[4] As we learned to negotiate understanding of each other’s ideas within these mathematical spaces, we learned much about mathematics and much about creating and working within a supportive community.

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[3] Ethnomathematics has much to contribute to this discussion (*cf*. d’Ambrosio, 1990, 2006).

[4] Hanna & Jahnke (1996) and Reid (2002) offered further examples and interesting insights into “proof” that fails to convince

Again, Where Do Ideas *Come* From?

As I learned to involve the students more meaningfully in generating their own mathematical understandings, my earlier struggles with the origins of scientific hypotheses again assumed significance. Creativity is required to think of appropriate (or inappropriate) inductive generalizations or deductive conclusions. What allows someone to perceive a pattern in a sea of numbers? Where do potential explanations for those patterns come from? Even if the answer is “from established truth claims,” what brings the appropriate premises to mind? Again, I found it difficult to teach students to generate their own conjectures or to make connections between various topics; some seemed naturally good at it, but how could I support its development in those who struggled? I was not satisfied with the notion that some students are naturally more logical, more creative, more intuitive, or better at some other vague construct that I was unable to name, never mind teach.

Once again, the role of analogy assumed significance, and I began to dig deeper into how what I then called “subconscious analogy” might be implicated in learning mathematics; I was increasingly interested in associations that influenced action without our being aware of them. If analogy was as pervasive as I believed it to be, it should not be hard to find, but I needed to pay attention. In fact, it seems to be deeply implicated in the elusive notion of mathematical intuition, as broadly characterized by Burton (1999) in her interviews with practicing mathematicians. Like Burton, I wonder, “Why is intuition so important to mathematics but [for the most part] missing from mathematics education?”

According to Polya, analogical thinking is central to human thought. He described its role in both his discussion of inductive (plausible) reasoning (1954) and in his problem solving heuristics (1945/2004), which strongly emphasized connections to prior knowledge and the identification of similar/related problems. Central to effective analogical reasoning is what Polya (1954) called clarifying the analogy, which results in "clearly definable relations of…respective parts" (p. 13). While this is helpful, it has little to say about helping learners find ways to better *identify* potential analogs.

Clement (1981) identified three methods of analogy generation:

1. generation from a principle: the analogy occurs as a result of perceived similarities between the abstractions of two or more different objects or situations; i.e. two objects or situations are perceived as belonging to a common and more general class of phenomena
2. generative transformation: some aspect of the original situation is modified to create a more familiar situation that can then be analyzed
3. associative leap: aspects of the target and base are perceived as similar.

Always, however, the question of where original ideas come from remained; i.e. Why do we perceive objects or situations as belonging to the same class? How do we identify *aspects* of a situation that can be modified? How do we perceive similarities between aspects of objects or situations? I now add: What do we perceive as possible or interesting? In Chapter 3, I argue that analogy is experienced at the level of perception: The connections we perceive must emerge from the pre-logical and the intuitive before they can be subjected to analogical *reasoning*. This has important implications for how we engage with it in the classroom.

Where might we find reference to analogical reasoning in mathematical curricula? Kilpatrick, Swafford, and Findell (2001) identified five strands of mathematical proficiency:

- Conceptual Understanding: comprehension of mathematical concepts, operations, and relations
- Procedural Fluency: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- Strategic Competence: ability to formulate, represent, and solve mathematical problems
- Adaptive Reasoning: capacity for logical thought, reflection, explanation, and justification
- Productive Disposition: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 116)

A brief mention of analogy was included in their discussion of adaptive reasoning, but there was little consideration of how its development might be supported in the classroom:

> Many conceptions of mathematical reasoning have been confined to formal proof and other forms of deductive reasoning. Our notion of adaptive reasoning is much broader, including not only informal explanation and justification but also intuitive and inductive reasoning based on pattern, analogy, and metaphor. (p. 129)

Their discussion of productive disposition emphasized the need for students to experience mathematics as sense-making rather than as memorizing (p. 131) and for them to see mathematical ability as malleable in response to experience. *But willingness to listen to the tentative, vague, and intuitive that may lie at the very heart of doing mathematics was not mentioned*.

Both the NCTM (National Council of Teachers of Mathematics) Standards (2000) and the WNCP (Western and Northern Canadian Protocol) for Mathematics (2006) emphasized the importance of connections; i.e. relating mathematical ideas to each other, to other school subjects, and to personal experience. Both also emphasized the importance of multiple representations and diverse methods of solving problems. But how might we help students *distinguish (notice)* and *name* relevant ideas and *make* connections between them?

Two Defining Stories

At this point, I would like to share two stories that helped me conceptualize and implement this study. The first is a personal reflection on my own work with the warehouse problem, prompted by the events in the opening anecdote; here, I had to dig deep to understand potential influences on my own thinking. The second is a reflection on a classroom experience based on students' attempts to compare the Fahrenheit and Celsius temperature scales; here, I had to dig deep in an attempt to understand potential influences on students' thinking (a precursor to my chosen methodology; i.e. one that defines researcher as empathic second-person coach). In both cases, I asked:

- Where did I recognize and act upon learners' doubt (in one case, my own) in ways that facilitated deepening understanding?
- Conversely, where did my lack of awareness-in-the-moment shut down opportunities for deepening understanding?

Attending to aspects of understanding that dwelled on the periphery of consciousness–even after the stated problems were solved–was central to developing deeper understanding. This is a key theme that emerges time and again throughout this study.

The Warehouse Problem

Again, consider the warehouse problem (Mason, Burton, & Stacey, 1982) described in the opening anecdote: With a 20% discount and 15% tax, which would you prefer to have calculated first? The teacher who noted that the solution *felt* wrong did not dismiss the feeling as *irrelevant* when he realized that it could not be *right*. He located the source of his discomfort in a perceived connection to another familiar situation: "It seems like those problems where you add 20%, then subtract 20%, and you do not come back to the original price." This resonated with my own initial doubts, even though I, too, was very confident that my solution to the problem was correct.

My first reaction to discovering that the order of discount and tax did not seem to matter was that my choice of a starting number ($100) might somehow have produced an anomalous solution. Surprisingly (at least to me), despite many subsequent experiences with special cases that produced unusual results, this suspicion seems most strongly rooted in an analogy to the similarity between 2 + 2 and 2 X 2 that sometimes leads students to believe 3 + 3 = 3 x 3! As I attended more closely to the operations I had performed on my chosen starting price of $100, it became clear that I had subtracted a larger percentage of a smaller amount and added a smaller percentage of a larger amount; i.e.:

- 20% of $100 (discount) = $20, so the discounted price = $100 - $20 = $80
- 15% of $80 (tax) = $12, so the final price = $80 + $12 = $92

Alternatively:

- 15% of $100 (tax) = $15, so the original price + tax = $100 + $15 = $115
- 20% of $115 (discount) = $23, so the discounted price = $115 - $23 = $92

But it was still not obvious to me why what was gained and what was lost should perfectly balance. In fact, I was immediately *suspicious* of this potential symmetry, and I was able to locate my suspicion in an analogy to work my students had done in developing ways to compare fractions: In comparing 2/3 and 3/4, it does not work to say that since three quarters (of something) can be represented as more but smaller pieces than two thirds (of the same thing) that the fractions are equivalent.

I decided to condense what I had done in each case: First, I calculated 80% of the total amount, then 115% of what was left: .8 * $100 * 1.15 = $92. Then I calculated 115% of the original amount and took 80% of what was left: 1.15 * $100 * .8 = $92. It now seemed obvious that the only difference was order of multiplication–a difference I knew did not matter–and that this would be true regardless of what number I chose for a starting number, for tax, or for the discount.

I was convinced, but even now something remained unresolved in my broader understanding. I decided to explore the connection between this problem and the +20%, -20% problem that my colleague had identified. As I expected, when I added 20% then subtracted 20%, I did not end up where I had started; e.g.:

Assume a starting price of $100.

Add 20% of $100 ($20): $100 + $20 = $120

Subtract 20% of $120 ($24): $120 - $24 = $96 (*not* the $100 I started with)

But why did such a seemingly trivial result seem problematic? It was only after working out the details of what I considered a routine problem then considering them *alongside* the warehouse problem that I realized that a fair comparison of the two problems would not ask, “Did I end up where I started?”, but rather “Does it matter in which order I add and subtract?” In the warehouse problem, I did not end up where I started (i.e. $100), either. Exposing the source of my faulty intuition did little to bolster my confidence in my original solution (in which I was already very confident), but it gave me a sense of *peace* (for the time being) in the consistency of my broader understanding. In other words, *intentional* use of analogy prompted me to consider more deeply the connections between these problems and resulted in a clarified understanding of both. Had I been content to merely *solve* the warehouse problem, I would not have addressed the niggling doubt that made the solution counter-intuitive. Exploring that doubt more deeply helped me connect my mathematical understanding in new ways.[5]

Converting Temperature: The Thermometer Problem

This story took place during my work with a group of eighth-graders who were exploring the relationship between Celsius and Fahrenheit temperature scales. To do so, most students began by marking intervals of 10°C and 18°F between boiling and freezing on twin thermometer diagrams (the distance between boiling and freezing can be described as either 100°C or 180°F). Someone expressed *surprise* that -40°F lands in the same place as -40°C. In the moment, I did not think to question the source of this surprise: Were students troubled that different starting points followed by different intervals arrived in the same place? Or, more generally, by the way difference produced similarity?

I pursued the idea by asking the class whether they thought there might be another match. I was surprised by the diversity of responses: A few students argued that the two scales *must* continue to diverge and could never come back to the same point. One explained that the only reason the scales met once was that the Celsius scale essentially had a "head start": It starts at 0° and counts down by 10s to -40°, whereas the Fahrenheit scale starts at 32° and counts down by 18s. Others noted that the Celsius and Fahrenheit scales were even further apart by the time they reached absolute zero, but they did not seem to recognize the steadily increasing divergence of the two scales, much less the *necessity* of such divergence. Of particular interest to this

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[5] Interestingly, the notion that adding 10% to a particular value and then subtracting 10% from the result does *not* effect a return to the starting point is itself counter-intuitive to many learners. Stavy and Tirosh (2000) made a strong case for the notion that this is due to an intuitive rule that they deem "Same A–Same B," which they claim leads to the assumption that both 10%s are equivalent (and further affects thinking in a wide variety of problem contexts). Learners who have not worked through this idea would probably not be surprised to find that it makes no difference in which order tax and discount are calculated in the warehouse problem.

discussion, many seemed to have a vague notion that "if it happened once, it could happen again." What lay at the root of such a belief, and what made it so compelling? Was it rooted in everyday experience? Past mathematical experience?

One girl who thought the two scales would meet again was able to locate her idea in a comparison to counting by 2s and 5s: They keep *meeting up* (i.e. Both 2 and 5 and 10 and 18 have common multiples). This student abandoned her argument after grasping the necessity of divergence, and I doubt she ever explored the root of her original intuition or its connection to the thermometer. I now wonder whether doing so might have deepened her understanding of common multiples, diverging sequences and thermometers. To help motivate further exploration, I could have asked questions like:

- How are the 10s and the 18s on the thermometer like 2s and 5s? Are those similarities sufficient to determine that the two scales will keep meeting? Why did your comment that "It's like 2s and 5s" seem so convincing?
- Just because something happens once, does that guarantee that it will happen again? In this case, why does that seem to you to be the case? Where else might this notion rightly or wrongly influence your thinking?
- Why does the fact that Celsius and Fahrenheit are further apart by absolute zero mean that such divergence will continue?

As is central to my own interest here, offering prompts like these might also have alerted the students to the manner in which doubt and certainty are more broadly implicated in the way they interpret–or even perceive–their engagement with mathematics.

After reflecting on ways in which I attended to or *could have* attended to emerging understanding in the warehouse problem and the thermometer problem, I decided it might be fruitful to attend more closely to how doubt and certainty are experienced in response to engagement with mathematics. In doing so, I was particularly interested in influences that dwelled beneath ordinary awareness. Doing so has become the central purpose of this work.

Framing a Research Question

My study, then, emerges from a long-standing interest in what it means to know– a personal journey out of Cartesian dualism in which I have tried to balance a high regard for reason with a growing awareness of its limitations. It is within this space that I ask: *How might learners develop deeper awareness of partially conscious feelings associated with doubt and certainty and use them as gateways to deeper understanding?* A significant theme that has emerged in this work is the importance of *finding* significant doubt spaces. I will return to this notion many times throughout the presentation and analysis of my data. Analogy, broadly conceptualized in terms of perceived similarity (at varying levels of consciousness), also emerges time and again as deeply implicated in learners' experiences of doubt and certainty. If learners already, necessarily, and often sub-consciously think (in the full sense of knowing / doing) with their own analogs, with or without attention to their appropriateness, what happens when they develop a deeper awareness of when and how such analogs are operating?

Overview of Dissertation

To explore experiences of doubt and certainty in learning mathematics, I engaged students in tasks that allowed sufficient time and space for their ideas to emerge and interact and for students to attend to subtleties in their work that they might otherwise overlook. In designing a classroom environment, I tried to remain mindful of the implications of an enactivist view of cognition (Maturana & Varela, 1987; Thompson, 2007; Varela, Thompson, & Rosch, 1991), which further interrupts the Cartesian separation of mind and body by emphasizing embodied, non-representational knowing and the autonomy of the learner in creating his or her own world of relevance. This is the subject of Chapter 2. Accordingly, I selected problems with the potential to prompt various experiences of doubt and certainty; some had provable solutions, while one relied on justifying assumptions about what aspects of a situation were relevant and on the use of statistical evidence. The tasks did not yield quick, easy, and/or complete solutions. They announced important mathematical questions without prescribing narrow solutions. In this way, mathematics served as a context in which students could deepen their awareness of doubt and certainty.

Both my interactions with the students and my analysis of the data that emerged from those interactions were further guided by Varela's notion of researcher as "empathic second person"; i.e. one who is neither a third-person objective observer nor has first-person direct access to learners' consciousness (Depraz, Varela, & Vermersch, 2003, Varela & Scharmer, 2000; Varela & Shear, 1999). Rather, because of my own experience, I could empathize with learners' perspectives, and doing so directly affected our interactions. Because the study is rooted in learners' experiences of knowing, it became important to ask what (and how) aspects of such knowing might be brought to awareness. In Chapter 3, I first consider the role of analogy in enabling perception of environmental regularities. I then turn to phenomenological literature that speaks to awareness of consciousness, including work done in the emerging field of neurophenomenology. In drawing deeper attention to potential indicators of doubt and certainty, I directed student awareness in ways that students likely would not have done on their own. At the same time, I continued to deepen my own awareness of how I experience mathematical doubt and certainty. The recursive interactions between our experiences were essential to my role as second-person teacher-researcher; here, the inseparability of I-the-teacher, I-the-researcher, and I-the-learner-of-mathematics becomes significant.

American philosopher and psychotherapist Eugene Gendlin does not use the term enactivism to describe his work, but his description of explicit knowing as emerging from a more “implicit intricacy” (1991) that is undifferentiated by symbols and fully-embodied knowing has significantly shaped both my understanding of enactivism (Chapter 2) and the methods I developed for this study (Chapter 3).

In Chapter 4, I share stories from the classroom. Here, I hope to draw the reader into relevant aspects of both the students’ and my own experiences with the chosen problems. In each case, finding a doubt space was essential to further work and typically involved working with aspects of the problem that felt counter-intuitive. Each problem offered unique insights, which I elaborate in the more thematic analysis offered in Chapter 5. In each case, I attempt to show how vague meaning (verbal or non-verbal) emerged from the implicit and continued to evolve.

In Chapter 6, I develop a model to describe the interaction between implicit and explicit understanding, consider how such interaction is implicated in our experiences of doubt and certainty, and set these experiences alongside relevant findings in neuroscience. While I do not claim expertise in neuroscience, the pieces I invoke here provide a generative space in which the findings of neuroscience can speak back to my own findings and prompt deeper insight into the experiential and observable world of the classroom. Philosopher and neuroscientist Walter Freeman’s (2000) work to understand how brains generate meaning and neuropsychiatrist Iain McGilchrist’s (2009) more speculative exploration of how hemispheric modes of attention may be implicated in our reciprocal experience of and creation of our worlds were particularly helpful here; both resonate strongly with the enactivist frame I adopted for the study.

Even though I began with the intent to honor the implicit, quiet, intuitive, and vague, there were many times I might have more effectively coaxed these to the surface. More strongly, there are times when my lack of sensitivity to emerging meaning likely blocked potential insights. Through recursive analysis of classroom / interview transcripts and my own mathematical experiences, I have tried to broaden my awareness of indicators of further sensitize myself to the depth of just-below-conscious processing so that I might become a better coach. I take these notions up in Chapter 7, where I reflect on both my use of enactivism to design the learning environment and on my adoption of a second-person stance; both of these also have implications for teaching.

Finally, throughout this project, I have puzzled about the significance of mathematics to my endeavor. While I have no clear answers to this question, I have come to more deeply appreciate mathematics as a fascinating window to self and consciousness. In my preliminary observations of the students and particularly in my own experience, it seems that mathematics has a special (though probably not unique) power to amplify experiences that make the implications of these ideas more visible and more broadly applicable.

A Brief Note on Doubt and Certainty

As in my opening examples (regarding the warehouse and thermometer problems), I have conceived of doubt and certainty as more than confidence in a particular solution. While learners' experience of proof or justification is an important theme in mathematics education (*cf*. Hanna & Jahnke, 1996; Reid, 2002), here I am interested in the role doubt and certainty play in how learners identify and select among possibilities for further action. I intentionally defined both doubt and certainty broadly: The word *doubt* comes from the Latin *dubitare*, meaning *hesitate*; here, such hesitation may be either a gut feeling of wrongness or counter-intuitiveness or a vague sense of possibility. *Certainty*, on the other hand, points to a confidence that inspires movement (of thought and / or physical movement, including speech, gesture, and facial expression) in a particular direction (which includes stopping); it may emerge from prejudice or habit, from a logically developed argument, or, like doubt, from a (literal) gut feeling. Both doubt and certainty are also implicated in what we deem *significant*. In this sense, doubt might also be cast as a dim awareness of potential relevance; in this sense, the hesitation of doubt has to do with the vagueness of its referent rather than a negative response to an articulated idea.[6] Both doubt and certainty may have the effect of shutting down or opening up new spaces of possibility: Doubt may result in directing attention toward or away from a potentially significant idea, and certainty may result in either the uncritical acceptance or rejection of the same idea (at the expense of other possibilities). Mark Johnson's (2007) discussion of furtherance and hindrance offers a helpful description of doubt and certainty:

> Feelings of "furtherance" and "hindrance" in our thinking play a key role in how we know *what follows from what* in our thinking. Thinking moves in a direction, *from* one thought *to* another, and we have corresponding feelings of how this movement is going: we feel the halt in our thinking, we feel the tension as we entertain possible ways to *go on* thinking, and we feel the consummation when a line of thought runs its course to satisfactory conclusion. (p. 97; emphasis in original)

I am interested in how learners experience such movement in their thought and particularly in how attending more closely to such experiences can open spaces where new understanding can develop.

As I worked with my data, it became increasingly clear that much of what prompted experiences of doubt and certainty–or furtherance and hindrance–occurred

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[6] Walter Freeman (2000) described how a new stimulant (in his case, an odorant) that does not match expectations can lead to a synaptic "burst suppression" that may be experienced as "I don't what it is, but it may be important" (p. 79).

beneath ordinary awareness. When I speak of ideas, then, I am not referring only to fully formed or named concepts. In fact, I have largely recast my initial framing of doubt and certainty in terms of what Gendlin (1962, 1978) called “felt meaning,” which emerges from a more holistic implicit intricacy. Understanding how students attended to (or not), interpreted, described, and interacted with such felt meaning has become central to this project. The recursive relationship between felt meaning and its explication are key to Gendlin’s (1978) exploration of “focusing,” which I describe more fully in Chapter 3. Importantly, I wish to consider the role of felt meaning not just as it pertains to proposed solutions, but also as it is implicated throughout the process of meaning-making.

2. Enactivism: Embodied, Co-Emergent, & Autonomous Knowing

In this chapter, I introduce key principles of the theory of cognition known as enactivism and explain their relevance to this study, particularly in terms of designing the learning environment in which the study took place. In doing so, I discuss the importance of embodiment, the co-emergence of meaning and language, and the nature of learner as autonomous agent who specifies his or her own world of relevance. Throughout the chapter, I weave together my discussion of enactivism with insights from Gendlin's philosophy of the implicit.

According to Gendlin (1967):

> [Heidegger] calls modern science mathematical, not because it so widely employs mathematics but because this basic plan of uniform units makes it possible to quantify everything one studies. It makes everything amenable to mathematics. (p. 265)

But mathematics is not just about manipulating bits. It also involves creating the bits; i.e. it requires an observer making distinctions:

> An observer is a human being, a person, a living system who can make distinctions and specify that which he or she distinguishes as a unity, as an entity different from himself or herself that can be used for manipulations or descriptions in interactions with other observers. An observer can make distinctions in actions and thoughts, recursively, and is able to operate as if he or she were external to (distinct from) the circumstances in which the observer finds himself or herself. Everything said is said by an observer to another observer, who can be himself or herself. (Maturana, 1987, p. 31)

This is a central tenet of enactivism.

Enactivism challenges representationalist views of cognition, emphasizing instead the embodiment of mind. It was developed by Chilean biologists Humberto Maturana and Francisco Varela (drawing from their work in autopoiesis) along with philosopher Evan Thompson and psychologist Eleanor Rosch. It also has significant roots in Gregory Bateson's work in cybernetics to bridge ecology, evolution, and epistemology (1964/1972; 1979/2002) and to Merleau-Ponty's phenomenology (*cf.* 1945/2002; 1963/1942). Thompson (2007) summarized key features of an enactivist perspective as follows:

1. [L]iving beings are autonomous agents that actively generate and maintain themselves, and thereby also enact or bring forth their own cognitive domains.
2. [T]he nervous system is an autonomous dynamic system: It actively generates and maintains its own coherent and meaningful patterns of activity, according to its operation as a circular and reentrant network of interacting neurons.
3. [C]ognition is the exercise of skillful know-how in situated and embodied action. Cognitive structures and processes emerge from recurrent sensorimotor patterns of perception and action.
4. [A] cognitive being's world is not a prespecified, external realm, represented internally by its brain, but a relational domain enacted or brought forth by that being's autonomous agency and mode of coupling with the environment.
5. [E]xperience is not an epiphenomenal side issue, but central to any understanding of the mind, and needs to be investigated in a careful phenomenological manner.

(p. 13; numeration added)

The first three points in this list are consistent with the radical constructivist views of Piaget (*cf.* 1954; 1970/1968) and von Glasersfeld (1990) in that they support (a) the notion of knowledge "actively built up by the cognizing subject," (b) the adaptive tendency of cognition toward viability, and (c) emphasis on "the subject's organization of the experiential world" (von Glasersfeld, 1990, pp. 22-23). In fact, enactivism may be seen to form a neurocognitive basis for constructivism; in the words of Maturana and Varela's (1987) subtitle, they speak to "the biological roots of human understanding." In terms similar to Bateson's (1979/2002) criteria for mental process (with which he describes epistemology, evolution, and epigenesis), Maturana and Varela attended to cognition as a form of organization common to all living beings.

The third and fourth points in the list describe an important shift in neuroscience away from representationalist views of knowledge (Varela, Thompson, & Rosch, 1991), a shift that moves beyond constructivism. Perception and action are not *encoded* in neural cognates: They are forms of cognition in their own right. In Maturana and Varela's (1987) key phrase, "*All doing is knowing and all knowing is doing* (p. 27). While Piaget (1970/1968) emphasized the centrality of perception and action in the sensorimotor activity of children and did not deny its role beyond childhood, he stressed the primacy of what he called "the semiotic function" (p. 46) in children's "passage from intelligence that is acted out to intelligence that is thought" (p. 45). The fourth point addresses an important gap in constructivist discourses, which have little to say about where new ideas (i.e. novel behavior) come from (Schmidt, 1999; Freeman & Mrazek, 2001; Metz, 2009), either in terms of inherited cognitive structures (phylogeny) or those that grow from inherited structures as an organism and its environment couple and co-evolve (ontogeny).[7] Finally, from an enactivist perspective, it is not just that we do not have *access* to knowledge of an external world (as posited by constructivism): Knower and known are dynamic and co-determining. By our knowing, we change the world, which in turn changes our knowing, and so on.

The centrality of first-hand experience of mind described in the fifth point, while perhaps a more radical departure from tradition in the field of neuroscience (*cf*. Depraz, Varela, & Vermersch, 2003; Varela & Shear, 1999) than in the fields of psychology and education, prompts important insight into teaching and learning. This is key to my own work; in Chapter 3, I explore how as researcher I adopt an empathic second-person point of view to gain valuable insight into first-person experience.

Enactivism, then, does more than provide a neurological basis for the evolution of knowledge: It also seeks to describe the *experience* of knowing. In this study, I attend to experiences of knowing in the particular realm of mathematical doubt and certainty. While my own reason for adopting a neurophenomenological approach is primarily pedagogical, careful descriptions of student experience may also further inform an enactivist theory of cognition.

Embodied Knowing: Holistic, Physical, and More-Than-Symbolic

French phenomenologist Merleau-Ponty's (1942/1963) emphasis on the nature of receptors, thresholds of nerve centers, and movement of organs highlighted the importance of embodiment central to human experience. His work has become influential in naturalizing phenomenology, whereby phenomenology draws insight from

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[7] This is not to say that adopting a radical constructivist view of knowing *precludes* attending to the manner in which new ideas are generated. Duckworth's (1996) essay about "*The Having of Wonderful Ideas"* is clear evidence of this. She explicitly asked the question that educators and educational researchers might ask of whatever theories they are using: "Even if I did believe that Piaget was right, how could he be helpful?" (p. 3).

biology and cognitive science. Varela, Thompson, and Rosch (1991) elaborated on embodied cognition:

> [C]ognition depends upon the kinds of experience that come from having a body with various sensorimotor capacities, and…these individual sensorimotor capacities are themselves embedded in a more encompassing biological, psychological, and cultural context. By using the term *action* we mean to emphasize…that sensory and motor processes, perception and action, are fundamentally inseparable in lived cognition. (p. 173).

In Gendlin's (1962) terms, perception / action serves to identify or set apart aspects of experience–i.e. to symbolize:

> There are visual and kinaesthetic "symbols" and in this sense even action, objects, and situations can be "symbols." For direct reference, a "symbol" is **anything that performs the function of marking off or specifying "a" feeling**, and thus making our attention (or reference) to it possible. (p. 97; emphasis added)

In this sense, symbolization is not dependent on language. Meaning is defined as symbolized experience: "Feelings are felt meanings only insofar as they function in such a relationship with symbols" (p. 110). Philosopher Mark Johnson (2007) and linguist George Lakoff (*cf.* Lakoff & Johnson, 1980) have done much to bridge cognitive science and cognitive linguistics by emphasizing the embodied nature of meaning that goes far deeper than words and concepts.

Although it may seem that much of mathematics has been abstracted and divorced from concrete experience in a physical world, Lakoff and Núñez (2000) showed that mathematics is deeply grounded in embodied experience. More recently, Núñez (2007a, 2007b) argued for the "psychological reality" of conceptual metaphors (i.e. that the metaphorical bases continue to influence thinking), but so far his focus has been primarily on mapping a stable (living metaphorical) conceptual structure for mathematical ideas (as evidenced in mathematicians) rather than on how people learn (or, frequently, do not learn) them. A significant aspect of this work is that the very metaphors *necessary* for constructing particular mathematical understandings can break down at the level of mathematical formalism (Núñez, 2009); in such cases, lack of awareness of the metaphorical relationship to the scaffold may provide a predictable source of difficulty for learners. I am interested in (a) how embodied experience and the many metaphors built upon it (which may or may not be part of a symbolic hierarchy) interact directly with developing understanding of mathematics and (b) how the experience of knowing is itself a bodily experience.

The notion of embodied cognition clearly emphasizes the non-representational nature of cognition:

> We propose as a name the term *enactive* to emphasize the growing conviction that cognition is not the representation of a pregiven world by a pregiven mind but is rather the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs. (Varela, Thompson, & Rosch, 1991, p. 9)

This view is in stark contrast to the notion that minds behave like computers. Such a view is reinforced by a dangerous positive feedback loop that directs attention to aspects of cognition that fit this model while obscuring awareness of mental processes that are not at all computer-like.[8]

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[8] Dehaene (1997) clearly outlined a number of inconsistencies apparent in considering the brain purely as a logical machine; in particular, humans cannot begin to compete with computers in the realm of computation, computers cannot recognize shapes or attribute meaning (both areas in which the human brain excels), and computers cannot simulate emotion, which play a powerful role in human reasoning (Damasio, 1994). Conceptualizing mind-as-computer (or symbol processor) is a double-edged sword. In the field of artificial intelligence, it has proven exceedingly difficult to program common sense, because its symbols can only

In fact, much of our thinking seems to involve large chunks of vague meaning, which are not comprised of distinct symbols at all (Polanyi, 1967). How these chunks function in mathematical thought and how they come (or do not come) to awareness is important to this study. Importantly, chunks are not stored as discrete bits, and they change with new experience; larger chunks of meaning may be called to mind as our histories complexify, but the wholes are in constant flux. They are often vague and contradictory. When we attempt to name them, both they and the words we use to name them change: As Gendlin (1995) put it, words are always “crossing” with broader, implicit knowing.

Here, what we perceive–what we experience and act upon–is dependent upon what we experience (consciously or not) as repeatable. In other words, what we mark off as significant is based on our history of engagement in a world. Because this notion is so significant to my analysis, I have set aside a section at the beginning of Chapter 3 where I explain how my early understanding of analogy has broadened into the notion of “repeatable context” (Bateson, 1964/1972). For now, I continue my discussion of enactivism by focusing on co-emergent knowing.

Co-Emergent Knowing: Relating Action and Description

Throughout this study, I attend to the recursive interactions of action and description[9] in mathematics students’ (and my own) encounters with the selected tasks and with each other. As Davis and Sumara (2006) put it, “The point here is not that

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contain the specific meanings programmed into the system. On the flip side, treating learners as symbol processors ignores their need for meaning, which cannot be pre-programmed and somehow attached to the symbols. In short, attempting to treat computers-as-minds cannot work, because we cannot attribute mind-like meaning to static, pre-defined symbols. Treating minds-as-computers cannot work, because we cannot program static, pre-defined meaning into human minds.

[9] This may also be phrased in terms of Gendlin’s (1962) discussion of functional relationships between felt sense (the unspecified *all that*) and symbolization (broadly defined as anything that specifies, which includes both action and description). New symbolizations prompt new or call forth previously existing felt meanings, which may be further specified / symbolized by other actions or descriptions. Here, meaning is the ongoing emergence of ways of being (see p. 110).

things change by virtue of how they are described, but that the actions of the describer are affected by the descriptions" (p. 315). Here, I find Gendlin's (2009a) notion of "carrying forward" helpful:

> When people explicate something implicit they usually say that their words "match" their experience, as if they were comparing two forms. But an implicit sense does not have the kind of form that could match words or concepts. What people call "matching" is indeed an important relation between implicit and explicit but the relation is not representation. It is rather the characteristic continuity we experience when new sentences and then new concepts articulate and explain what we had understood only implicitly. We call this relation "carrying forward." (p. 150)

Furthermore: "The words mean the change that saying them makes in a situation. *Words do not represent; they do something.* They mean what they do" (pp. 149-150; emphasis in original). In this sense, words are inputs that interact with a much deeper system of implicit understanding:

> Inputs are described as perturbations to the system's intrinsic dynamics, rather than as instructions to be followed, and internal states are described as self-organized compensations triggered by perturbations, rather than as representations of external states of affairs. (Thompson, 2007, p. 11)

My own descriptions similarly emphasize the co-emergence of knower and knowledge in the experience of knowing, as such experience unfolds within webs of "personal thought, social relationships, and cultural forms, all at once" (Simmt, 2000, p. 155).

By attending to the manner in which the students (and I) engage in mathematical activity, I hope to use mathematics as a site for developing mindfulness / awareness:

> [O]ur scientific culture has only just begun to consider the possibility of pragmatic and progressive approaches to experience that would enable us to learn to transform our deep-seated and emotional grasping after a ground. (Varela, Thompson, & Rosch, 1991, p. 244)

It is important to acknowledge at the outset, then, that I, as learner-teacher-researcher, do not claim to have reached a state of awareness that is free from such emotional grasping. Necessarily, the students and I embarked on this journey together. The validation of work within this tradition must ultimately be "derived from its ability to transform progressively our lived experience and self-understanding" (Varela, Thompson, & Rosch, 1991, p. xix).

Autonomous Knowing: Learner-Specified Relevance

According to Tom Kieren ("Enactivism," n.d.), one of the first in the mathematics education community to explore the implications of an enactivist view of cognition, such a view:

> …must [1] consider and trace the patterns of mathematical activity and understanding as it occurs, [2] must look at the mechanisms and beliefs by which persons act mathematically, [3] must attempt to account for the ways in which the environment occasions or creates space for personal mathematical activities, and [4] must account for the interactions and conversations through which mathematical activity occurs and by which it is bounded.

In my discussion of autonomous[10] knowing, I am particularly interested in Kieren's third point; i.e. I consider ways in which I was able to influence the classroom environment to allow space for students to specify their own mathematical activity.

Varela, Thompson, and Rosch (1991) referred to the ability "to pose, *within broad constraints*, the relevant issues that need to be addressed at each moment" as "the greatest ability of living cognition" (p. 145; emphasis added). Effective teaching and learning, then, should broaden the contexts within which learners are able to choose appropriate action. So that students might experience mathematics as a space in which to encounter their own perceptions, beliefs, and motivations, I attempted to define the mathematical spaces I offered for their consideration broadly enough that students could select spaces of relevance within which to work. In contrast, behaviorism might be seen as operating within contexts defined narrowly enough that perturbations produce predictable responses rather than *sets of alternatives*:

> [W]e may regard "context" as a collective term for all those events which tell the organism among what *set* of alternatives he must make his next choice. (Bateson, 1964/1972, p. 289)

If effective learning involves broadening the contexts within which we are able choose appropriate action, good teaching should nurture this sort of level-jumping. Do I teach my young son to stay out of the kitchen when I am cooking? To stay away from the

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[10] "Autonomous" may seem a strange word to describe knowing that is deeply embedded in social, cultural, and historical context. Here, however, autonomy means "having a point of view" that influences the nature of interaction within these contexts. In enactivist terms, this point of view attends more to the structural constraints imposed by its multi-layered embodiment than it does to a sense of personal will or responsibility. In discussing what he has termed the "virtual self," Varela (Varela & Scharmer, 2000) helped explain this: "[The virtual self] is real in the sense that it can effectively tackle the world with which it's coping. But that coping is constantly updating itself or renewing itself, submitted to all kinds of changes, both endogenous and exogenous" (p. 10). The virtual self is an emergent self: "There is a way in which the whole brain, if you focus on it, will show you how this emergence happens. But it would be a mistake to think that that's it, because there is a network of causality" (p. 11). This network includes social networks as well as a "whole network of other causalities in the longer time of evolution, in genetics and the molecular constraints…. [I]t is like the multiple levels which articulate onto one another an constrain one another" (p. 11).

stove? To stay away from the stove when it is hot (and therefore to learn to distinguish between hot and cold)? Not to touch hot items *on* the stove? Each item in this list requires greater discrimination than the one preceding it. If I choose the first option, he will not have an opportunity to learn how to work independently in the kitchen. If I choose the latter *too soon*, he may end up with a bad burn. There are always trade-offs when shifting to broader contexts, and helping students negotiate this boundary is central to the art of teaching.

For every role that I assume as a teacher, I might also consider how to encourage students to do the same. At any given moment, do I summarize, ask for a summary or ask students to think about the manner in which summarizing contributes to their thinking? Do I point out how their current activity is implicated in a broader context (or identified problem space), ask them to consider how what they are doing is so implicated, or teach them to ask themselves this question? Do I present a variation (such as a simpler case) for students to consider, ask *them* to generate variations, and/or ask them to think about how variations contribute to their thinking? Do I point out their errors, teach them specific strategies for checking (or proving) their work, or help them become more broadly aware of their own fallibility?

As students take on more of the latter items in each set (items of higher logical type), it is important to be aware that teachers (or other students) unaccustomed to their doing so may interpret their behavior as arrogant, presumptuous, or disrespectful. It is also important to maintain a teacher presence: As the person (usually) with the broadest overview of both the mathematical and pedagogical terrains, the teacher is the person who is aware of and directs attention toward the highest levels in a particular interaction. Different rules apply at different levels. Even when students are allowed to bend the rules, the teacher is ultimately the one who allows such bending and specifies how far the rules can bend.[11] Students must learn to make choices within ever-widening contexts. Teachers must make choices about how and how much choice to make available. For students to experience doubt and certainty, they must have opportunities to work within spaces where next steps are not tightly prescribed. From an enactivist point of view, opportunities for decision-making must allow consideration not only of what-to-do in response to a narrowly defined problem, but also of what sorts of problems a particular mathematical problem space might offer (or, more broadly, what sorts of mathematics a particular space might offer). It is within the space of personal relevance that students' own knowledge–their own ways of being–can be brought to bear, and it is within this space that doubt and certainty necessarily reside.

Bounding a Space of Relevance

Although Lesh and Doerr (2006) do not explicitly locate their work within an enactivist framework, the distinctions they draw between traditional views of mathematical problem solving and their own modeling approach to mathematics education are helpful in elaborating the importance of learners specifying what is relevant within a given learning context:

> Some of the most important aspects of real life problem solving involve developing useful ways to *interpret* the nature of givens, goals, possible solution paths, and patterns and regularities beneath the surface of things. Solutions typically involve several "modeling cycles" in which descriptions, explanations, and predictions are gradually refined and elaborated. (p. 31; emphasis in original)

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[11] I often tell my own students that they have to "know the rules well enough to break them." So long as they're in my classroom, however, I'm the one who gets to decide whether their knowledge is sufficient to make this choice.

Lesh and Doerr's use of the word *interpret* seems consistent with Varela, Thompson, and Rosch's (1991) description of interpretation as "understood widely to mean the enactment of a domain of distinctions out of a background" (p. 156). From an enactivist point of view, "adequate conduct" (Maturana, 1987) within the selected problem spaces must be defined according to sufficiency rather than optimality. Here, Zack and Reid's (2003) description of "good-enough understanding" as that which is "good enough to support [students'] continuing engagement in the mathematical activity of the class" (p. 46) is also helpful. Thus, adequate conduct is partially defined by the classroom community, including the teacher. In such a community, *learner* may not be easily described in terms of individual students. What is deemed relevant and how what is deemed relevant is acted upon are often decided in conversation, and ideas are seldom traceable to a single student (Davis & Simmt, 2003).

The sorts of tasks used in this study might in fact be seen as problems-to-be-solved. However, they do not pre-define a narrow set of relevant factors or methods that severely constrict the range of permissible activity. Although examples of narrowly defined mathematical problems abound, one instance in which the difficulty with pre-specified relevance became starkly apparent to me involved my attempt to use a pre-packaged environmental mystery with my students. The students' intrigue with the opportunity to participate in solving a mystery quickly subsided when they realized that the clues they might use to investigate what had gone wrong in a fictitious river were all pre-defined and very limited. They asked many potentially relevant questions that the available clues could not answer. In other words, the constraints built into the task did not allow the students to specify what was relevant or to contextualize what they knew in relation to a larger picture. Biologically, this type of restriction would likely mean death:

> Doubtless, in order to be able to subsist, [the organism] must encounter a certain number of physical and chemical agents in its surroundings. But it is the organism itself–according to the proper nature of its receptors, the thresholds of its nerve centers and the movements of the organs–which chooses the stimuli in the physical world to which it will be sensitive. (Merleau-Ponty, 1942/1963, p. 13)

In the case of the environmental mystery, the students encountered various agents (clues), but few of them matched their questions (receptors). They were not sensitive to the pre-packaged stimuli, and little meaningful interaction took place.

In addition to being defined broadly enough to allow learners to specify what is relevant, contexts for learning must be broad enough to inform local effects within them–i.e. to provide meaningful feedback. Consider the following description of a bacterium's activity in a sucrose gradient:

> In order to give a full and complete account of the bacterium's activity in swimming up the sucrose gradient, it is not sufficient to refer simply to the local molecular effects of sucrose as it traverses the membrane and gets taken up internally. Although these local effects are indeed crucial, they are at every step *subordinated* to and regulated by the global maintenance of autopoiesis. In other words, the local molecular effects happen as they do because of the global and organizational context in which they are embedded. And it is this global level that defines the bacterium as a biological individual and sucrose as food. (Thompson, 2007, p. 75; emphasis added)

I find it helpful to set it alongside Hewitt's (1996) description of how skill development needs to be subordinated to a broader task. The notion of feedback is central to this discussion.

Subordinating Skill Development

In developing the notion of subordination, Hewitt (1996) offered a reflection on his experience of learning to sail: To control the angle of the sail, he had to coordinate body position, the rope attached to the sail, and the rudder (which had the added complication of requiring arm motion opposite of boat motion):

> If I concentrated on any one of the rudder, rope, or body position, I forgot about the other two. My attention was darting from one to another. On the arrival of my attention to any one of the three, I found that I needed to make large corrective adjustments before turning my attention to the next, which, in turn, needed large adjustments also. I felt frantic and unable to cope. (p. 30)

By heeding his instructors' advice to attend to the sail rather than the three factors affecting it, he found that he:

> …was able to make increasingly finer adjustments with the rudder, rope, and my body position such that he sail just did not flap. I became more relaxed and felt in control of not only the boat but also the possibilities of moving the rudder, rope, and body position. (p. 30)

Hewitt then described the angle of the sail as being at a higher subordinate level than rudder, rope, or body position. He offered three criteria for effective subordination of a skill to a task:

1. The skill to be practiced is required by another task.
2. The task provides feedback regarding how effectively the skill is used.
3. The task can be understood before mastering the skill.

Here, as with Thompson's (2007) bacterium in the sucrose gradient, it is the global level of organization–i.e. the structural coupling of sailor-boat-water-wind in the cognitive act of sailing–that defines appropriate action for a cognizing subject. The local effects of rudder, rope, and body position are surely crucial, but they behave as they do because of the larger context in which they are embedded. In this sense, rudder, rope, and body position need not be treated as *separate* locations of action any more than further fragmentations of any of these might be. For example, in sailing a boat, I probably would not consciously evaluate the role of, say, my pinky finger. Feedback *pertaining to the learner's intention* (in this case, steering a boat) occurs at the global level of boat stability and direction. Gendlin's (1978) description of "felt sense" (which he also refers to as "felt meaning") further captures the significance of this global level of awareness:

> It would be impossible for them to *think* all the details of location, surrounding environment and body movement that are woven into aiming. But the body knows the complex set of coordinated movements it must make to swing. The single felt sense of the situation incorporates the problem and the bodily-known situation…. When a golfer swings, several hundred different muscles must all work together in a precise way, each coming into action at a certain microsecond, each exerting just the right amount of pull on the right bone, for the right length of time. The body feels all this as a whole. (p. 82)

Hewitt (1996) argued that when children direct their attention to a task, they progress to new understandings while practicing the old, gaining fluency in subordinated skills without having to attend to them directly. I think it would be fair to say that when a skill is subordinated, it becomes a means to an end rather than an end in itself. Importantly, it is a means that is informed by progress toward a broader end. Hewitt went so far as to say that it is “precisely because they [children] do not spend their time attending to the act of walking along flat floors *and nothing else*” (p. 29; emphasis added) that they learn to walk. I struggled at first with this; if ignoring something helps you learn it, there are many things we should know a great deal about. But I think the *nothing else* is key to this phrase. Concepts learned in a context that makes the need for them apparent strengthens the associated learning, but this may have more to do with where attention *is* than it does with where attention *is not*. When a task makes the need for a particular concept evident, children’s attention (and intention) is directed to the global level of that need. When concepts are developed in response to that need, their purpose becomes an integral part of the child’s understanding *of that concept*. It is no longer simply a *how-to-do* for its own sake: It is a *how-to-do-something*. It is this *something* that gives power to the developing understanding, and this is why it is so important that the task itself be understood. This may also help explain why concepts learned or skills practiced while conscious attention is directed to a task to which they are subordinated tend to be retained longer: Understanding a task deepens understanding of various aspects needed to engage it by placing them in a particular relation to one another. These aspects may or may not be separated, marked off, or named as identifiable concepts or skills.

In this sense, it could be argued that attention remains on the subordinated concepts but is directed to their deeper structure as announced by the meaning-framework of the task: A deep understanding of why a particular concept is useful is an essential part of understanding that concept. For example, when a context creates a *need* for central tendency (as in the ice melt problem that I elaborate in Chapter 4), understanding that need informs evolving understanding of the *nature* of central tendency. To borrow structure from Thompson's (2007, p. 75) discussion of the bacterium in the sugar gradient: *It is the global level of the problem space that defines the child as a mathematizing agent and central tendency as a discrete mathematical concept*. If students develop understanding of, say, mean as "a technique that allows us to reduce a large collection of fluctuating data to a single representative value that balances out the highs and lows" as opposed to "what we get when we add up a list of numbers and divide by the number of values," they have developed fundamentally different kinds of understanding.

Perhaps more than in any other subject, teachers feel that concepts addressed in a mathematics classroom must be mastered to be of value. If we take seriously the notion of subordination, we must move past this idea, for the main focus of attention includes but is always at least one logical level removed from subordinated concepts and skills. As Zack and Reid (2003) so nicely articulated:

> [L]earners work with 'good enough for the moment' ideas as placeholders, that is, when confronted by many complex ideas for the first time through learners seem to be wading in heavy water, making many tentative temporary decisions, keeping diverse and at times contradictory possibilities 'in the air' and waiting at times to the end to make sense of what has transpired. (p. 43)

It is in this messy, exploratory space that children can become deeply engaged in their attempts to resolve uncertainty. And it is when they are engaged in this manner that they experience doubt and certainty rooted in their own ways of being. It is also in this messy, exploratory space that they can become frustrated and give up. Different students have different comfort levels with ambiguity and may also differ in terms of the amount of contingent information that they are able to hold in or close to consciousness at any given time.

Offering Real-World Contexts For Mathematics

Although there has been much consideration of the importance of problem-solving contexts (real-life, relevant, or otherwise; *cf.* Boaler, 1993; Kilpatrick, 1985; Nicol & Crespo, 2005; Nyabanyaba, 1999), it seems to me that little has been done to clarify the distinction between context that subordinates mathematical ideas and context that merely uses them. One does not have to look far to find supposedly mathematical contexts that in fact do not subordinate the mathematics that they make use of. A recent focus issue of *Mathematics Teaching in the Middle School* dedicated to “Connecting Mathematics and Science” (NCTM, 2006) described potentially rich problem spaces that could easily *become* sites for mathematical subordination, but most of the articles described ways that teachers might use the contexts as rationales for telling children how to carry out mathematical procedures that properly belong in the realm of what Hewitt (2001a), following Gattegno (1970), called awareness. In one instance designed to help develop understanding of area and ratio, students were asked to calculate the “sinking value” of various animals. However, rather than letting the context inform the mathematics required by the situation, teachers were advised to *tell* students how to calculate area and to *provide* a formula to calculate sinking value. The activity demonstrates a *use* for area and ratio, but as presented does not prompt deep understanding of either. In Hewitt’s and Gattegno’s terms, area and ratio are not subordinated to the task of determining sinking value.

Again, a notable exception to this view of context may be found in the work of Lesh and Doerr (2003). Although they do not draw explicitly on the notion of subordination, their description of model-eliciting activities seems consistent with its principles:

> A point emphasized throughout this book is that model-eliciting activities can be designed to lead to significant forms of learning. This is true because model-eliciting activities usually involve mathematizing–by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities. Consequently, in cases where the conceptual systems that students develop are mathematically significant sense-making systems, the constructs that are extended, revised, or refined may involve situated versions of some of the most powerful elementary–but deep–constructs that provide the foundations for elementary mathematical reasoning. (p. 5)

Their focus includes the structure of the mathematics, not the utility of mathematical tools to achieve a pragmatic goal; i.e., the mathematics is genuinely subordinated to the context.

Implications For Task Selection & Design

In summary, I would like to emphasize four points relevant to staging potential encounters with doubt and certainty in mathematics that I see following from an enactivist view of learning.

First, if knowing is inseparable from doing, it makes little sense to insist that students have a clear plan of action before they are allowed to begin work within a problem space or that they follow all leads to completion. This is not to say that planning should not be encouraged, but actions must be allowed to emerge and change in the context of doing and reflecting on what has been done (i.e. in response to students' experiences of doubt and certainty).

Second, if students are to access the richness of their own embodied knowing in the context of mathematical problem spaces, they need the opportunity to *mathematize* and to test their mathematized models against the real world; i.e. that which they model. Note that *real-world* need not imply a physical world, but it does need to be one that has not been artificially constrained to the point that students have no role in defining relevance. During the ice melt problem, students were asked to use accumulating melt water to figure out when a funnel full of ice had started melting. Although the task was initially designed as a partial model for radiometric decay, I doubt that anyone would engage in this activity outside of a mathematics (or science) classroom[12]. What is real about it is that the ice *really melts*: Students were able to select aspects of the situation that they deemed relevant, mathematize them in ways appropriate to their chosen goals, and return to the original situation to select different aspects as needed. This allowed a much different experience than, say, providing students with a data table detailing the volume of accumulated melt water at specified time intervals and then asking them to project back. In the latter case, the data pre-defines relevance. In fact, the aspect of students' engagement with this problem that I primarily focus on in this dissertation is students' concern with whether the amount of starting ice matters–something that would be completely overlooked if students were not involved in mathematizing selected aspects of the manner in which the ice melts. In the consecutive integer problem (also elaborated in Chapter 4), students created, defined, re-defined, and connected their own mathematical objects from the wholeness of a broadly defined problem–itself a created

[12] Note that the students in this study did not consider the ice as a model radiometric decay.

object, but one for which the boundaries were (at least loosely) marked off by the teacher.

Third, classroom interactions must center on interaction of *ideas* (Davis and Simmt, 2003; Fosnot & Dolk, 2001a, 200b, 2002). A good discussion is not simply one where each child gets a chance to speak and where children speak politely to one another (which is not to say these are not important). What makes a discussion rich is when ideas are connected in a rich web of associations; i.e. when the feedback children offer one another is allowed to influence the ongoing evolution of their ideas and their choices about what to do next. Students need to be able to respond to one another in an ongoing fashion, not simply comment and then move on to a new topic.

Finally, what students perceive as doubt or certainty–and as possibilities for action–may be rooted in physical, emotional, kinesthetic, sensory, and / or conceptual metaphor (by no means discrete categories) that often influences cognition beneath ordinary consciousness. It was important to encourage and allow space for such vague or implicit understanding to gradually come into focus as explicit (though perhaps temporary) objects that could be more consciously manipulated and used. Developing a deeper understanding of how this happens was central to this study. As I will attempt to show, there were times when I helped facilitate this and times when my actions as a teacher obstructed the elaboration of implicit understanding.

3. Deepening Awareness of Doubt & Certainty

> [Enactivism and phenomenology] share a view of the mind as having to constitute its objects.… Things show up, as it were, having the features they do, because of how they are disclosed and brought to awareness by the intentional activities of our minds. Such constitution is not apparent to us in everyday life but requires systematic analysis to disclose. (Thompson, 2007, p.15).

Because I am interested in aspects of doubt and certainty that are typically not spontaneously available to conscious awareness, it became important to consider how I might bring these experiences to awareness. *I am particularly interested in how students' perceptions of similarity are implicated in their experiences of doubt and certainty and in how their deepening awareness of their own pre-reflective experiences of similarity might expand the domain of actions they perceive as possible and justified.* What students experienced as similar was partially constrained by the tasks-as-given, but their perceptions of ways they might engage with the tasks evolved throughout their engagement with them. Varela, Thompson, and Rosch (1991) noted:

> [T]he species brings forth and specifies its own domain of problems to be solved by satisficing; this domain does not exist "out there" in an environment that acts as a landing pad for organisms that somehow drop or parachute into the world. Instead, living beings and their environments stand in relation to each other through *mutual specification* or *codetermination*…. Environmental regularities are the result of a conjoint history, a congruence that unfolds from a long history of codetermination. (pp. 198-199; emphasis in original)

In this chapter, I explore a broader consideration of analogy as a primary vehicle for experiencing similarity then consider how this is connected to current understanding in mathematics education research. I then turn to a significant methodological approach that addresses how both the participants and I might deepen our awareness of when and how what I have broadly called analogy is operating. Here, I draw particularly from Clement's (1994) study of creativity in physicists, Gendlin's (1978) development of the psychotherapeutic practice of "focusing" and Varela's notion of researcher as empathic second-person coach (Varela & Scharmer, 2000; Varela & Shear, 1999). Here, my own deepening awareness of doubt and certainty becomes key to my being able to direct student attention to this potential in their experience. Following this, I lay out the details of how I took up this work in the classroom.

Repeatable Context: The Role of Analogy in Perceiving *As*

While each moment in the coupling of knower-known embodies a unique history, for a learner to bring forth his or her own world of significance, each moment must contain what Bateson (1964/1972) called "repeatable context":

> This notion [of repeatable context]…contains the implicit hypothesis that for the organisms which we study, the sequence of life experience, action, etc., is *somehow* segmented or punctuated into subsequences or "contexts" which may be equated or differentiated by the organisms. (p. 292, emphasis added)

Gendlin (1962) provides further insight into Bateson's *somehow* in his discussion of "functional equality":

> When different schemes can be equated, the reason is that, at the point in the discourse at which they are equated, they refer to (select, differentiate, symbolize) the same aspect of experience–which can be referred to also by "direct reference." (p. 214)

Johnson (2007) noted that language marks distinctions that allow us to recognize similiarities. Gendlin (1962) argued further that language is merely one way of symbolizing similarities first recognized as what he refers to as a “felt sense”:

> Ordinary sense perception is usually a case of recognition feeling. We see a tree, or we see brown, or we see a shape. “Oh,” we say, “that’s brown.” Or we don’t even say that. The recognition feeling is simply called forth and is the meaning to us of what we perceive. (p. 103)

At its most basic level, the recognition of similar contexts is necessarily *analogical*; i.e. it requires (implicit or explicit) “judgment” (often not in the usual sense of judgment resulting from conscious deliberation) regarding whether a particular situation is *sufficiently* like another to be recognized as such or to be acted upon in the same manner. In this sense, analogy is a primary enabler of perception and likely lies at the heart of what Fischbein (1982) called intuition (as well as our experiences of doubt and certainty):

> *[I]ntuition is to be the homologue of perception at the symbolic level, having the same task as perception: to prepare and to guide action (mental or external).* (p. 11; emphasis in original)

However, he struggled to distinguish perception and intuition:

> Intuition and perception have essential common features and for this reason the term *intuitive knowledge* is sometimes used to designate both categories. Both are global, direct, effective forms of cognition. The difference between intuition–as a specific form of knowledge–and perception is that intuition does not directly reflect objects or events with all their concrete qualities. Intuition is mostly a form of *interpretation,* a *solution* to a problem, i.e. a *derived form* of knowledge, like symbolic knowledge. (p. 11; emphasis in original)

Again, Gendlin's (1962) broad definition of symbolic as anything that has been singled out as *an experience* seems pertinent here. In this sense, perception, too, is a form of interpretation. An enactivist view of cognition rejects the notion that perception "directly reflects objects or events with all their concrete qualities," further clouding Fischbein's (1982) distinction between perception and intuition.

Pimm's (1988) broad description of metaphor as involving "the seeing (and therefore the understanding) of one thing in terms of another" and as a "conceptual rather than solely a linguistic phenomenon" (p. 30) is well-suited to my work here. For my purposes, however, I do not limit this definition to "deliberate extension…across previously established category boundaries" (Winner, as cited in Pimm, 1988, p. 30). I intentionally collapse distinctions sometimes drawn between metaphors, analogies, examples (including extreme cases), models, and representations as well as those between within and between-domain analogies. In Merleau-Ponty's (1942/1963) terms, these might be considered "virtual" distinctions; i.e. their status depends on the structure of the perceiver (at a given moment). I wish to attend more generally to the nature of meaning transfer that might occur in any of these cases–particularly to the notion of similarity sufficient to allow transfer (often at a subconscious level). With Sfard (1999), I also observe that analogical transfer is a recursive process, impacting understanding of both source and target analogs.

In the manner described here, language itself is inherently analogical (see also Lakoff & Johnson, 1980), and logic is dependent on analogic (Bateson, 1979/2002; McGilchrist, 2009). Therefore, close attention to language is important to this work. Every noun (except proper nouns), verb, adjective, or adverb we use refers to a class of similar objects, actions, or descriptors. The word *cat* is the name of the analog for all cats–to determine (consciously or otherwise) whether to call an animal that I have never seen before a cat means that I have to decide (again, though, often not in the conscious sense of "decide") either (a) whether the characteristics of the new animal are sufficiently like other cats I have seen (an act of analogic that requires consideration of what constitutes *sufficiently*) or (b) is consistent with generalized features that comprise my existing definition of cat (an act of logic dependent on a prior act of analogic)[13].

Alternatively, a new animal perceived as having both much in common with as well significant differences from my generalized working definition of cat might prompt me to revise my criteria for cat. Thus:

> What in *Principia* appears as a ladder made of steps that are all alike (names of names of names and so on) will become an alternation of two species of steps. To get from the *name* to the *name of the name*, we must go through the *process* of naming the name. There must always be a generative process whereby the classes are created before they can be named. (Bateson, 1979/2002, p. 174)

Categories draw attention to aspects of experience that can never be fully conceptualized; they are necessarily incomplete. American author, philosopher, and semioticist Walker Percy (1954/1975) considered the value of namings that seem to have no connection to that which they name in that they are better able to point to the ineffability of certain experiences in ways that precise categorical labels cannot. This provides a stark contrast to the heady feeling of rightness that can be part of *recognition* or *perception of something as* a member of a particular class.

Percy (1954/1975) also noted: "The modern semioticist is scandalized by the metaphor *Flesh is grass*;[14] but he is also scandalized by the naming sentence "*This is*

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[13] When this act of recognition takes place subconsciously, it may be argued that the neural processes that determine sufficient similarity (e.g. recognition of a common felt sense) stretch the definition of "logic" to the breaking point. It is therefore important to explicitly recognize my broad use of this term.

[14] This is likely a reference to Bateson's (1980/1987) syllogism for metaphor: "Grass dies. Men die. Men are grass" (p. 44).

flesh" (p. 72). Again, the difficulty posed by such naming is twofold: "Is *this* sufficiently like flesh?" and "In light of new information, does my definition of *flesh* need to be more narrowly or tightly constrained?" Again, much questioning of this nature occurs beneath ordinary consciousness. In fact, every time we use a word, it may be argued that its meaning for us changes (if only subtly) by incorporating another experience into its referential realm and into the felt sense to which it refers (as becomes apparent in Chapter 4). Yet only if we agreed to hold our analogically developed categories rigid (if that were even possible) could we proceed in a purely logical fashion:

> We are employing the term "logical" to apply to uniquely symbolized concepts. A "logical relationship" is one that is entirely in terms of uniquely specified concepts. Whatever occurs in the creation, specification, or symbolization of concepts is obviously prior to their properties as finished products. (Gendlin, 1962, p. 141)

Again, even our pre-reflective perceptions of the world may be understood as analogical.

The analogical nature of perception may also be observed in animals and young children:

> The thrust of my argument is that the very process of perception is an act of logical typing. Every image is a complex of many-leveled coding and mapping. And surely the dogs and cats have their visual images. When they look at you, surely they see "you." When a flea bites, surely the dog has an image of an "itch" located "there." (Bateson, 1979/2002, p. 178)

My own everyday observations of animals also suggest that they categorize, although of course I must be careful not to ascribe human intentions to animal behaviors: A friend's dog refused to go on the linoleum when they moved to a new house; in the old house, this was the beginning of forbidden territory. My brother's dog would not eat his new dog food, seemingly because anything different than the old dog food was cat food and would lead to trouble if he ate it. Ben Gadd, an expert on the Canadian Rockies, described the following encounter with a moose:

> Valerius Geist, a well-known mammalogist from the University of Calgary, once attracted the wrath of a bull while filming it in the fall. The bull paid little attention to Val and his co-worker at first, so the two men decided to imitate another bull in order to get some action. Val rolled up his sleeves, exposing his white arms–white like antlers. He spread his arms out, tilting them as a moose does. This ploy was a little too successful; it provoked a charge. (Gadd, 1986, p. 735)

According to neuroscientist Lise Eliot, young children's language development is similarly categorical:

> Just imagine all the possibilities that could be going on in Nathan's mind when his mother responds to his pointing with the word *bottle*. It could refer to the vessel's contents, *milk*; to some piece of it, like the *nipple*; to one of its properties, like its *purple* color or its long *cylindrical* shape; to that specific *long*, *cylindrical purple bottle with the slicone nipple containing milk*; or finally to the generalized object that we all know as a *baby bottle*…. They do so, scientists believe, because their brains are innately biased to assume three things about words: (1) that they refer to whole objects [often designated by common movement and by stripes, which show edges], as opposed to their parts or properties; (2) that they designate classes of items, rather than individual members of the class (the purple bottle, the clear bottle, the small bottle, and so on); and (3) that objects have only one name. (Eliot, 1999, p. 372)

These descriptions nicely complement Gendlin's description of symbolization as anything that specifies experience. It is important to note, however, that it is unlikely that animals or small children are aware of the ways they are generalizing. Humans eventually (and to varying degrees) learn when metaphor is operating; animals likely do not. According to Gendlin (1991), "[A]nimals cannot respond to pictures as pictures. A bird cannot react to both at once: The bird reacts either by pecking at the piece of cardboard, or–with fear–by fleeing from the cat" (p. 41). So it was with the sparrow that flew at his own reflection in my living room window all day every day for seven days this summer and with the peacock on my parents' farm that fought his own reflection until he shattered the glass in which it appeared. A real bear does not think like Winnie-the-Pooh:

> "That buzzing-noise means something. If there's a buzzing noise, somebody's making a buzzing-noise, and the only reason for making a buzzing-noise that I know of is because you're a bee. And the only reason for being a bee that I know of is making honey. And the only reason for making honey is so as I can eat it." So he began to climb the tree. (Milne, 1926, p. 6)

Humans, on the other hand, can get very excited about signs! I remember puzzling over a strange track in the snow and the rush I felt at the moment of insight when I recognized it as wing print from a magpie taking off. While I enjoy watching birds, I get no such *rush* from simply watching a magpie.

Although I have now offered a rather lengthy rationale for my broad definition of metaphor, neuropsychiatrist Iain McGilchrist (2009) explored evidence that metaphor that crosses accepted boundaries and metaphor that has become clichéd (i.e. much of language) is processed differently in the brain. It appears that there is, in fact, a neurological difference between what might be loosely categorized as within- and between-domain analogies. Importantly, however, the individual cognizing agent defines such domains.[15] This also seems consistent with McGilchrist's description of hemispheric differences in *modes of attention* rather than neurological differences in the way different metaphors are processed; i.e. the right hemisphere's broad attention allows more far-reaching connections than the focused and language-dependent mode of attention typical of the left hemisphere. This also helps resolve the puzzle of how metaphor can be at once both the basis of logical typing (and therefore everything familiar) and the source of all that is new. In other words, if it is appropriate to adopt the broad view of metaphor that I have done here, then how can the left hemisphere–so crucial to language and logic–be blind to it? McGilchrist (2009) provided an intriguing possibility:

> An archetype may be familiar to us without our ever having come across it in experience…. It is not that one or other hemisphere 'specialises in', or perhaps even 'prefers', whatever it may be, but that each hemisphere has its own disposition towards it, which makes one or another aspect of it come forward–and it is that *aspect* which is brought out in the world of that hemisphere. The particular table at which I work, in all its individual givenness, is familiar to me as part of 'my' world and everything that matters to me (right hemisphere); tables generically are familiar precisely because they are generic (left hemisphere)–in the sense that there is nothing new or strange to come to terms with. (pp. 172-173)

While speaking of the right hemisphere's openness to the new–its "reconnection to the world which familiarity had veiled" (p. 173), McGilchrist emphasized that deliberately

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[15] I think it is important to note that the individual cognizing agent is also embedded in a collective involving many interacting cognizing agents–humans with shared goals and shared capacities for perceiving, reasoning, and moving. Within such collectives, shared metaphors allow a seemingly objective designation regarding what is within- and between-domain. As Lakoff and Núñez (2000) developed in their exploration of "how the embodied mind brings mathematics into being," this forms the basis for what has often been seen as a mysterious correlation between mathematics and the structure of the physical world.

"recombining already known elements in bizarre ways, thus breaking the conventions of our shared reality" (p. 173) deals with breaking the whole of experience into parts, and therefore lies within the domain of the left hemisphere. In other words, left-hemisphere newness is about recombining parts in new ways, while right-hemisphere newness involves seeing the familiar without the blinders of habitual categories. To me, it seems a sort of paradox emerges here: Often generalities are used to broaden attention; here the holistic perspective of the right hemisphere also attends to the particular. The left hemisphere attends to clearly defined contexts, but it does so through the abstractions and generalities inherent in language.

Mathematics and Analogical Reasoning

In recent years, the fundamental role that analogical reasoning plays in mathematical thinking (*cf.* English, 1999, 2004; Lakoff & Núñez, 2000) and cognition in general (*cf.* Lakoff & Johnson, 1980) has been increasingly recognized. As English (2004) pointed out, however, studies of children's use of analogical reasoning in mathematics have typically focused on classical analogies (A:B : : C:D), problem analogies (transferring solution strategies between problems recognized as similar; *cf.* Schoenfeld & Hermann, 1982), and pedagogical analogies (where the teacher selects analogies deemed useful and emphasizes where they work and where they break down). In each of these scenarios, students are expected to develop understandings that conform to predetermined expectations and transfer is based on conscious reasoning.

Even within the realm of conscious reasoning, adopting the sense of analogy as repeatable context greatly expands the terrain in which it might be seen to be operating. In this sense, it may be seen to play an important role in Watson and Mason's (2005) discussion of learner-generated examples and Lesh and Doerr's (2003) discussion of mathematical modeling. Both attend to (1) learners' responses to a particular problem or prompt, (2) the importance of considering how initial responses are appropriate / inappropriate to the situation at hand, and (3) whether conclusions based on those responses are justifiable. In so doing, explicit attention is directed to the structure of mathematical understanding. Does my understanding of *this case* justify conclusion(s) about *a larger class*? What else belongs in such a class? Should I reject an example, alter a definition, or both? How do extreme and special cases inform this understanding? Does my understanding of a particular model justify conclusion(s) about the situation being modeled? If not, should I alter my mathematical model, alter my understanding of the situation that the model is designed to represent, or both?

As students develop and refine their own models, they gain access to the power of mathematics as a tool for recognizing similarity among seemingly disparate ideas: "The best mathematics has a curious kind of universality, so that ideas derived from some simple problem turn out to illuminate a lot of others" (Stewart, 2006a, p. x; also see Wigner, 1960). Like Watson and Mason's (2005) work with learner-generated examples, Lesh and Doerr's (2003) description of mathematical modeling emphasizes the importance of attending to those aspects of a relationship that must be preserved to allow justifiable conclusions. Once a situation has been mathematized, the mathematized situation must be treated as an *analog* to the original, and conclusions based on the model cannot be *assumed* to apply to the situation from which the model was abstracted: "The map is not the territory" (Bateson, 1979/2002). Furthermore, symbolic and / or geometric methods developed in response to modeling may be extended and modified in ways that are not immediately relevant to the situation that prompted them, and this may be facilitated by attending to the dimensions of possible variation and range of permissible change emphasized in Watson and Mason's work with learner-generated example spaces.

Analogy as I have conceived of it here also demands attention to understanding that dwells on the periphery of conscious awareness. Perceptions of similarity at varying levels of consciousness are deeply implicated in learners' experiences of doubt and certainty.

Awareness of Analogy

If children already, necessarily, and often sub-consciously think (in the full sense of knowing / doing) with their own analogs, with or without attention to their appropriateness, what happens if they develop a deeper awareness of when this is happening and if they develop the habit of considering the strengths, weaknesses, and applicability of these analogs?

I have become increasingly sensitive to a *feeling* that often alerts me to my own subconscious use of analogy. Sometimes it is a niggling feeling that something is not quite right…like a floor tile that sits slightly askew, a piece of music without the closure of a final chord, a wrinkle in a bed cover, a picture hanging slightly off balance…. Like an itch that begs to be scratched, none of these experiences is exactly painful, but each may prompt a response as strong as if it were.[16] Sometimes it begins to emerge with a vague sense of, "I think it has something to do with…" or "That's sort of like…" or "It seems like…."

For me, these feelings also have much in common with having somebody's name "on the tip of my tongue." The actual analog, however, has a feeling of its own that is distinct from feelings of rightness, wrongness, or sameness: These *accompany* the feeling of the influencing analog. Thus I can be very sure that the name I am seeking is "Chris-ish" or that it starts with "C"… Crystal? That seems *almost* right–so

[16] In fact, the feeling is not unlike Haase's (2002) description of the obsessive-compulsive experience of things being "not just right" (p. 81).

right that I begin to doubt my doubt…. Later, it occurs to me: It is Cristin! And I am *sure*. It *feels* exactly right in the same way that locating the analog behind a particular conviction, doubt, or sense of similarity feels right. For example, when I first read the following passage, it somehow seemed very *familiar* (and therefore plausible) to me, despite my knowing very little about embryology:

> Imagine that a particular locality–a small region–of the developing embryo must receive some version of the instructions governing these multiple limbs. The version that specifies ‘five’ or ‘ten’ [pairs of appendages on a crab] can be of no use to the specific restricted locality. How shall this spot–this bunch of cells in this particular region of the developing embryo–know about a number that is in fact embodied in the larger aggregate of tissues developing those appendages?” (Bateson & Bateson, 1987, p. 115)

It also left me with the feeling that there was a *deeper meaning* embedded in the passage. I went *searching* for the source of that feeling and *located* it in a problem that I had explored with one of my students a few years ago (a problem that had both of us stumped): “Since uranium does not decompose at a steady rate, how does it ‘know’ when half of it has decayed?” Such “locating” (rightly or wrongly) was announced by yet another feeling–one of contentment–I had found a match. Having thus located what I believed to be one source of confidence in my understanding of Bateson’s passage (which I have learned to distinguish from confidence that the analog is *applicable*), I was then able to consider whether uranium really was a good source of confidence for my understanding (or some part of it): Was I justified in feeling that I understood it? (No.)

So far, I have distinguished the feeling that a *subconscious analogy is at work* from the feeling that accompanies *locating the precise analog*. For me, the actual *search for such an analog* describes a third type of feeling that is separate from both of these and is much like the feeling that sometimes accompanies an actual physical search: “What am I forgetting?” or “I know I saw that screwdriver here somewhere…. It feels like I will find it on a shelf…. No, not this sort of shelf…. I know! It is on the bookshelf in the bedroom. *Aha*! There it is.” And again, the *feeling* of contentment that comes with locating the object (or realizing that I suddenly know with certainty where I will find it) is very much like the one that accompanies *recognition* of the right analog. It seems that the three feelings I have here distinguished are similar to those identified by Gendlin (1962):

> Here we note a felt meaning that functions (a) to indicate something potentially known, (b) to indicate incorrect and correct formulations of it, (c) in the process of recall, because to recall something, we concentrate our attention on the “feel” of it. (p. 76)

Gendlin (1978) described a sort of infallibility in relation to such bodily felt sense:

> Your body, with its sense of rightness, knows what would feel right. The feelings of “bad” or “wrong” inside you are, in effect, your body’s measurement of the distance between ‘perfect’ and the way it actually feels. It knows the direction. It knows this just as surely as you know which way to move a crooked picture. (p. 75)

I resisted this notion of rightness at first–probably because it triggered a deep-rooted suspicion of infallibility. But a right felt sense does not necessarily mean right in an absolute or objective sense. It refers to a rightly described subjective experience, as identified by the experiencing subject. Such a description is far from static; in fact, in being so brought to awareness, it often changes.

Johnson (2007) provided a fascinating description of the bodily experience of doubt (based on the work of William James and Charles Sanders Peirce):

> [O]ne’s experience of *doubt* is a fully-embodied experience of hesitation, withholding of assent, felt bodily tension, and general bodily restriction. Such felt bodily experiences are not merely accompaniments of doubt; rather they *are* your doubt. (p. 53)

As I have learned to attend to the feeling that (for me) often announces the presence of a subconscious analog, it has gradually transformed from an indicator or doubt or certainty to one of *possibility* (as in, “Hey! There’s something worth exploring here”).

Approaching the First-Person Through Second-Person Mediation: Becoming an Effective Second-Person Coach

Exploration of subjective experiences of mathematical doubt and certainty requires disciplined inquiry into first-person experience. Recent work in this area has emphasized commonalities among three first-person approaches to inquiry: introspection (from psychology), phenomenology, and contemplative practice (Depraz, Varela, & Vermersch, 2003; Thompson, 2007; Varela & Scharmer, 2000; Varela & Shear, 1999). Each of these involves refining the individual’s access to his or her own experience, and each requires careful cultivation (to varying degrees) of three basic elements: suspension (of habitual thought patterns), redirection (attending to what emerges in the space thus created), and letting go (not fixating on what emerges):

> Now in phenomenology, for example, there's a tremendous emphasis on suspension. The letting go is less present, because it's much more directed to philosophical result. In Buddhism,[17] instead, the letting go and suspension are paramount. But the redirection is not emphasized. In contrast, in introspective psychology or experimental psychology, redirection is really the most important thing. In fact, there is very little suspension or letting go. The subject is not at all encouraged to do that. (Varela & Scharmer, 2000, p. 5)

Varela emphasized the need for a more “universal tool” that acknowledges the importance of each these elements and which requires a deepening appreciation of what he called the “virtual self”–i.e. a self in constant flux. Such a tool must rely on second-person (inter-subjective) mediation:

> In contrast [to first and third-person approaches]**, a more interesting second person is really empathetic.** He admits that you have in your mind an access to your experience, but this person himself knows the kind of experience you're talking about and therefore acts as a coach. (p. 7; emphasis in original)

It must also employ second-person validation; i.e. “accounts amenable to intersubjective feedback” (p. 11).

Gendlin’s (1978) description of focusing and Petitmengin-Peugeot’s (1999) exploration of intuitive experience provide valuable insight into how we might bring pre-reflective experiences of doubt and certainty to consciousness. In his study of sources of creativity in physicists, Clement (1994) further identified a number of helpful indicators

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[17] In addition to being a biologist and philosopher, Francisco Varela trained with meditation masters Chögyam Trungpa Rinpoche and Tulku Urgyen Rinpoche. He spent many years as a Tibetan Buddhist.

for identifying when intuitive processing is happening so that it may be flagged for in-the-moment reflection. I now consider aspects of each that are of particular relevance to this study.

Gendlin's Focusing

Gendlin's (1978) approach to psychotherapy grew from his observations that clients who referred directly to the implicit made progress in their therapy, while those who did not could spend years in therapy without progress. Having recognized this, he began to teach patients how to refer to the implicit–this is what he called focusing. Gendlin's experience with this impasse resonates with my experience of teaching mathematics: Those who get it often keep getting better–but what exactly is the *it* that they get, and can it be taught? Gray and Tall (1994) identified the *it* as a "proceptual divide." According to their research, successful students recognize symbols as indicative of both process and concept, while less successful students treat symbols as prompts for inflexible procedures. But *why* do some students remain trapped in procedure alone?

Focusing contains important insights for making math class a place where students may engage deeply with their experiences of doubt and certainty. It involves the following steps (from Gendlin, 1978):

- Get comfortable.
- Push the jumble of problems aside.
- Attend to the worst one–what does "all that" feel like?
- What's the worst of it? (the "crux")
- Allow words to come forth.
- Check words against feeling.
- Go deeper.

Like Varela, Gendlin (1996) emphasized being “friendly” to whatever comes to awareness in the process of focusing (including gaps when it may seem that nothing significant is forthcoming). Gendlin described sensing the “unclear edge” in self and others (p. 15), and attending to “positive stirrings” and “green shoots” (p. 22) that emerge when we allow words to come forth. Mathematically, we do not attend to what Gendlin calls “the worst of it” (a reference to personal problems in psychotherapy), but we *can* attend more closely to a particular sort of doubt that points to significant implicit information.

Students must be given time and support to describe their experiencing in their own terms:

> The client’s own discovery and grappling with the feelings and experiences within him takes a long time and consists of feelings, not of concepts–consists of his experienced particulars, not of conceptual generalizations. Hence, all forms of therapy consist of a person’s efforts to experience more deeply and to come to grips with and symbolize his own felt experience for himself. (Gendlin, 1962, p. 78)

Such support may include time and space, questions that direct attention to the felt sense of experience, and encouragement to find ways to refer to such experience. For example, Gendlin emphasized “direct reference,” which he claimed allows “a whole mass of data [to become] available” (p. 78).

“Direct reference” may also allow students to proceed with vague understanding in the math classroom. Rowland (1992) noted an unusual prevalence of the pronoun *it* in his mathematical discussions with nine-year old Susie. He wondered what exactly *it* was referring to, then noted that “Susie makes effective use of the pronoun to point to ideas of a general nature *which neither she nor I have named*” (p. 46; emphasis in original). In my experience, students often have trouble articulating the referent of their pronouns. At times, they seem to refer to felt senses that are not yet clearly pinned down by a particular word. As such, they both (a) allow students to proceed with partial understanding and (b) provide potential points of entry for students to explore their understanding more deeply.

Importantly, all of this takes time:

> You reach a point where you say, "Well, I haven't beaten this problem yet, but I'm at a stopping place that feels pretty good. I need a day to let my body live with this much changed, and perhaps also to go out into the world and see what happens." Steps of focusing and steps of outward action often alternate. Each aids the other. (Gendlin, 1962, p. 63)

I further develop Gendlin's ideas in the analysis of my data in Chapter 5.

Clement's Studies of Intuition

Clement's (1994) work with physicists offers further insight into how I might attend to experiences of doubt and certainty. He described an "elemental physical intuition" as "…'direct' knowledge of the behavior of a physical object or system–knowledge that does not depend on a formal symbol system" and as "knowledge structures that reside in long-term memory and can be activated to provide an interpretation or an expectation about a physical phenomenon" (p. 212). While my work does not distinguish between knowledge that does or does not depend on a formal symbol system or between phenomena that are or are not strictly physical, Clement's attention to sensorimotor knowing (particularly in experts) is significant, and he provided a number of indicators that this sort of intuition is taking place:

- "intuition reports" (p. 211): The subject states that he or she is making a prediction based on an intuition.
- unjustified statements: A physical intuition is unquestioned; it is something that he or she thinks with rather than thinks about
- statements that refer to situations that are "more general than the memory of a specific incident" (p. 211), "do not refer to a specific episode in the past" (p. 221), and prompt "an expectation for what will happen over a wide range of circumstances" (p. 221) [In my own experience, this does not preclude the possibility that specific episodes may lurk beneath the general statements that come more easily to mind.]
- self-evaluation of plausibility: confidence is based on internal rather than external authority.
- orientation to concrete objects: "Subjects usually speak of an intuition as referring directly to objects and physical phenomena, not to abstract equations." (p. 211) [Again, in this work, I do not draw a clear line between reference to physical and abstract phenomena.]

Later, he also discussed the significance of "spontaneous imagery reports" (p. 215):

> To anticipate, the view that I propose here is that although imagery plays a role in physical intuition, elemental physical intuitions do not just consist of specific images. Rather, they involve a general schema, often an action-oriented, perceptual motor schema accompanied by kinesthetic as well as visual imagery. (p. 213)

It seems that movement often accompanies kinesthetic imagery, as when people attempt to act out ideas for which they have no words. Here, I often think of a student who moved his hands to simulate the *feeling* of trying to move like poles of magnets together as he attempted to form an explanation for what might be happening in an electric circuit; while doing so, he commented: “Then they...like…shake. Almost. Yeah. Pretty sure” (Schmidt, 1999, p. 238). Here, the objects the student was attempting to describe *were* physical. During this study, I also observed students use gesture to indicate *balance* in reference to an imagined number line.

Children may or may not use the word *intuition*, but there are other indicators that students are grappling with understanding that is bigger than available words. For example, they may say, “I think it has something to do with….” or, “It seems like….” They may use a pronoun with an unclear referent or a gesture to fill-in-the-blank of an intuitive sense for which they have no specific verbal referent. Surprise or puzzlement may also indicate the influence of something counter-intuitive.

Petitmengin-Peugeot’s Studies of Intuition

Claire Petitmengin-Peugot (1999) pushed the traditional boundaries of psychological introspection by adopting Varela’s empathic second-person to describe the subjective experience of intuition; both her methods and her resulting description of intuition are significant to my work here. Her descriptions of what she called the “listening phase” of intuition resonate strongly with what I have called subconscious analogy. In particular, she cited descriptors that Theodor Reik used for the “third ear” of his psychoanalytic practice:

- “a seismograph reacting to a faint subterranean variation”
- “almost imperceptible undertones”
- “slight unnevenesses not apparent to the eye but perceptible to testing hands that glide slowly and carefully over the fabric” (p. 66)

She summarized her description of the “threshold of awareness” of intuition as follows:

> The intuition does not always emerge in a precise, complete, immediately understandable form. Most often it first caresses the consciousness as a hazy image, a vague sensation, diffuse, a line of interior force. (p. 71)

This description has much in common with Gendlin's (1978) description of felt sense as a "large, vague feeling" (p. 19).

Petitmengin-Peugeot (1999) noted that "the gestures which prepare and follow the emergence of an intuition belong to that dimension of experience which is not a part of thought-out consciousness" and, significantly to my own work, that such non-conscious knowledge "seems to be present even at the centre of our most abstract activities, those most conceptualized, those most lacking in affectivity[18]" (p. 45). To bring such experience to awareness, she used what she calls a "three-stage explicitation process": (1) "bringing the subject to the point of living, or reliving, the action or experience to be explored"; (2) "helping him [her] to operate a 'thinking-through' of his experience, that is, to pass his [her] know-how from the level of action to the level of representation"; and (3) "enabling him [her] to put into words, to clarify, this represented experience" (p. 46).

Like van Manen (1990) and Mason (1994; 2002), she emphasized the importance of attending to a *particular* experience and the difficulty of keeping participants from drifting into abstract language indicative of *generalized* experience. She used a number of strategies to encourage this:

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[18] Here, as in my work, "lacking in affectivity" seems to refer to the abstract content rather than a particular *experience*, which she noted are often infused with deep meaning and importance, "even when the intuitions have an innocuous content" (p. 71).

- encouraging attention to associated images, sensations, sounds
- encouraging participants to slow the rhythm of their speech and to welcome moments of silence
- asking questions that can only be answered by recalling the particular experience; e.g. how-questions rather than why-questions, which encourage abstract speech
- referring to experience without referring to content (to avoid influencing choice of words)

She identified perseverance as key to the success of her interviews:

> We were surprised to see how difficult it was for the interviewer to maintain the interviewee within the limits of his own experience, how much one needed both firmness and gentleness to guide the other person on the fine line of here and now. When the subject stopped fleeing to abstract levels and let himself live, or relive in the present, a singular intuition, he frequently began by stating: "I'm not doing anything" or "I don't know what I'm doing"…. When the subject "lets go", gives up his representations, beliefs, and judgements about intuition, and begins speaking slowly, from this place inside himself where he is in contact with his lived experience, the words he says seemed to us each time extremely precious, in their smallest details. (Petitmengin-Peugeot, 1999, p. 73)

Like Clement (1994), she emphasized gestures as indicators of unconscious, corporeal knowledge. In addition, she identified direction of gaze and the use of generalizations and nominalizations as significant pointers to deeper meaning.

Intuition is often portrayed as a near-mystical component of mathematical or scientific genius; in mathematics, this may be seen in frequent references to Poincaré's (1913) and Hadamard's (1945) reflections on mathematical creativity. A key assumption underlying my own work is that we all rely on intuition on an ongoing basis. A significant difference between everyday intuition and intuitive genius may be that those working beyond the experience of most others have greater difficulty finding symbols that can call forth what Gendlin (1962) called recognition of the felt meaning associated with their work. To put it another way, what Gendlin called "explication" is impossible, because adequate symbols (at least for communication with others) do not exist. While existing symbols may be used to create and symbolize new felt meanings (what Gendlin called metaphor[19]), this can be very difficult, and there is no way to ensure that such newly created symbols call forth the intended meaning in others.

Hadamard (1945) further emphasized that even logical inference is a creative process, and he listed several examples of missed insights that in hindsight seemed obvious implications of his work:

> Two theorems were such obvious and immediate consequences of the ideas contained therein that, years later, other authors imputed them to me, and I was obliged to confess that, evident as they were, I had not perceived them. (p. 51)

As Hadamard considered "attempts to govern our unconscious," he noted that it "may be detrimental to scatter our attention too much, while overstraining it too strongly in one particular direction may also be harmful to discovery" (p. 54).

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[19] Gendlin defines metaphor as follows:

> Metaphors differ from ordinary meaningful symbols in that they do not simply refer–as ordinary symbols do–to their habitual felt meaning. Rather, the metaphor applies the symbols and their ordinary felt meaning to a new area of experience, and thereby creates a new meaning [i.e. a new relation between felt sense and symbols], and a new vehicle of expression. (1962, p. 113)

Having elaborated the conceptual underpinnings of my work and method, I now turn to a more pragmatic consideration of how these were enacted in the classroom.

Pragmatic Considerations

Objectives & Scope

To carry out this work, I spent nine months of (approximately bi-monthly) sessions in a seventh-grade mathematics classroom, transcribed the video-taped classes and follow-up interviews, and maintained a reflective journal pertaining to my own mathematical experiences with these and other problems. I spent approximately half a day per week at the research site. This included teaching a 110-minute Grade 7 mathematics class approximately once a week from October 2009 to June 2010. Following most classes, I interviewed a small group for another 40 minutes to better understand work they had done during that day's class and to observe and interact with students as they continued their efforts. By attending closely to students' perceptions of what they considered (a) relevant to the task, (b) intuitive or counter-intuitive, and (c) mathematically connected, I also hoped that this work would help students to identify ways to continue engaging with challenging problem spaces, to engage more deeply with understandings they experienced as counter-intuitive, and to develop more connected understandings of mathematics. The study elaborates and extends work done during my 15 years as a classroom teacher (Middle School Mathematics, Science, and Social Studies) and for my M.Ed. (exploring the development of children's reasoning in science).

Methods & Procedures

Over the course of the nine months in the classroom, I posed six tasks for the learners, as outlined in Table 1.

Time Frame	Prompt	# of Classes	Data Collection	Data
Nov.	Chocolate Fix®	2	Class Discussion Small-Group Work Small-Group Interviews Journal Response	Video Student Journals
Dec., Feb.- Mar.	Lamp	5	Class Discussion Small-Group Work Small-Group Interviews Journal Response	Video Student Journals
Jan. - Feb.	Ice Melt	5	Class Discussion Small-Group Interviews	Video
Mar. - May	Consecutive Sums	6	Class Discussion Small-Group Interview Journal Response	Video Student Journals
May - June	Pythagorean Proof	3	Class Discussion Small-Group Interviews	Video
June	Wason Test	2	Class Discussion	Video

Table 1: Data collection.

I offer a deep analysis of three of these tasks:

Lamp Problem:

If I buy a lamp for $7, sell it for $8, buy it back for $9, and sell it for $10, how much profit do I make? (adapted from Schultz, 1977/1982, p. 12)

Ice Melt:

Students arrived in class to find a funnel of ice melting into a graduated cylinder. They were asked to figure out what time the ice started melting and to be prepared to justify their conclusion to the class. (adapted from Wise, 1990).

Consecutive Sums:

Which numbers can be written as the sum of consecutive, positive, integers? In how many ways? (from Mason, Burton, & Stacey, 1982)

I introduced each prompt and allowed the students time to work on them. They were seated in groups of four and had the option to work independently or to talk quietly with other group members. There was also considerable freedom for students to check in with other groups as they chose. I asked them to reach agreement within their groups then post responses (labeled with their group number) on the whiteboard so that others could seek them out to ask questions, extend their thinking, or dispute their claims. Throughout, I emphasized that I would not be the arbiter of right and wrong–they would have to convince themselves and each other of their ideas based on the strength of their arguments. Initially, I asked the students to note times when they were aware of their own doubt or certainty, but I found they were typically too absorbed in their work to attend closely to these.

During most classes, I set aside time for a whole-class discussion to allow greater interaction among ideas students had been developing that day and to allow time for focused reflection on doubt and certainty. A number of strategies helped promote meaningful interaction during this time:

- “Pass-it-on”: When one student was finished speaking, he or she could call the next person in.
- Posting results, and encouraging discussion of discrepancies
- Insisting students address opposing ideas rather than talking around them with alternatives
- Emphasizing sharing as a means of bringing ideas together rather than as a tool for ensuring accountability; i.e. I did not insist that every student share their ideas, particularly if they had nothing to add that hadn’t already been said.
- Defining and accepting partial conclusions; i.e. not all students solved all parts of all the problems. However, I often tried to draw attention to how their work might be contextualized within the larger problem space.

Each week, I also interviewed individual and / or small groups of students regarding their experiences of doubt and certainty. I used students’ contributions during regular class time as well as their written work and reflections as a starting place for these discussions, then proceeded in a fashion similar to the whole-class discussion described above.

I videotaped both the regular classroom (mostly during whole-group discussion) and individual / small-group interviews. I transcribed these videos during the week between visits, taking note of (a) named experiences of doubt and certainty and (b) potential sites for further exploration of doubt and certainty. These observations informed subsequent interactions with the class and allowed me to compare students' actions and descriptions with an evolving list of indicators.

Throughout, my role as researcher was tightly bound with my role as teacher. As researcher, I wanted learners to attend explicitly to their experiences of doubt and certainty. Were I acting as a classroom teacher, this is something that I would now incorporate into that role; i.e. it is a move that reflects my own developing awareness of teaching and learning (and a move that nurtures the sort of level-jumping that I described in Chapter 2).

Teacher as Empathic Second Person

As I described earlier, I adopted the role of Varela's empathic second-person (Varela & Scharmer, 2000); i.e. a coach with some degree of understanding of what students were experiencing, yet ever-mindful of how I might (a) direct attention and (b) help them develop adequate descriptions of their own experiences. As much as my own awareness permitted, I allowed myself to be "…willingly corrected, since [I was] not concerned with what is true about the client [here, student] in general but only with what is part of experiencing right here and now" (Gendlin, 1962, p. 81).

To observe experiences of doubt and certainty in mathematical problem spaces, it was important to select problem spaces that were likely to evoke and contrast such experiences, thereby providing contexts that allowed deeper awareness of their presence:

- The lamp problem invoked a counter-intuitive response in most learners, thereby interrupting habitual thought patterns and providing a window for deeper awareness.
- The ice melt problem required decisions regarding what and how to measure and regarding how measurements might justifiably be extended back in time. Many students believed that the amount[20] of starting ice was significant to solving the problem.
- In the consecutive integer problem, students formed, tested, generalized (to various degrees), and connected a wide variety of numerical conjectures.

In each of these cases, I also found important spaces for personal exploration and reflection, which greatly influenced my role as second-person coach.

[20] Here, amount was vaguely conceptualized; students did not distinguish between mass or volume.

4. Classroom Stories: Finding Doubt Spaces

I now invite the reader to join me in the classroom. Here, I share experiences I selected to form the heart of this study. I illustrate the evolution of learners' ideas by offering a narrative chronology of student and teacher interactions with three prompts. Here the chronology is largely defined by the classroom collective (Davis & Simmt, 2003): Much of my data is pulled from classroom discussions that were interspersed among the individual and group work students engaged in, and I have woven interview data into the broader classroom narrative that both inspired and was influenced by those conversations. My hope is that this will help the reader experience aspects of the progression of ideas as we did.

There is a great deal of detail here; my analysis required careful consideration of emerging understanding as visible through language (broadly defined to include body language such as facial expression, gesture, posture, etc.) that was often fragmented, tentative, and meandering yet occasionally emerged as extended outbursts that were equally difficult to follow. I flag certain events and themes that I wish to draw attention to (both by what I have chosen to include and by the labels I have attached to these experiences). In each story, I emphasize the importance of students *finding* doubt spaces (which could then become spaces of personal relevance); in each of the three stories I tell here, this involved attending to aspects of the problems that could easily have been bypassed (and indeed *had* been bypassed, by both the students and by me). In all three stories, the significance of deeply *empathizing* with student doubt was central to prompting attention to significant doubt spaces. The notion of *bridging*, whereby some aspect of the original problems were modified to help broaden students' conceptions of the given contexts (and thereby provoked doubt in their initial responses to them) was important to both the lamp problem and the ice melt problem. The manner in which meaning and language co-evolved, often in subtle ways, is also important to all three stories. Finally, I draw attention to teacher-researcher moves that *blocked* students' awareness of doubt and certainty by directing attention too narrowly along paths whose relevance was largely defined by *my* experience of the problems.

In Chapter 4, however, my *primary* goal is to maintain a sense of story. At times, it may be difficult to follow the stories, and the reader may find engaging deeply with the problems and appreciating learners' responses challenging. The cases I include here remain long, so I have divided Chapter 4 into three sections.[21] But I hope that this chapter will provide sufficient background to allow the reader to better relate to the more thematic analysis that I offer in Chapter 5. The lamp problem provided a rich starting point in the way it surfaced the limits of language and logic for resolving what to many seemed irreconcilable differences. The ice melt problem prompted the predicted experiences of statistical uncertainty, but it also led to deeper exploration of counter-intuitive implications of melt rate, largely through what I here call bridging. The consecutive integer problem involved creating and connecting a variety of mathematical objects as the students and I gradually worked from many disparate observations toward a more encompassing solution. The overarching notion that I attempt to illustrate (and explain in Chapter 5) is how vague meaning (verbal or non-verbal) emerges from the implicit and changes as it interacts with other meaning in ever-evolving contexts. Within this space, I pay particular attention to interactions between students and between the students and myself as teacher-researcher.

To avoid false connotations that a particular name might evoke, I refer to some students by number, used as a proper noun to minimize the depersonalizing effect it

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[21] In addition to the stories I discuss here, I offer for the interested reader a fourth in Appendix A; it likely requires more effort to engage with the mathematical ideas it contains and is not needed to explain key themes, but it provides rich detail to support them. Interested readers might refer to it after reading Chapter 5.

might otherwise have. The numbers are assigned to students in order of their appearance in the stories and contain no further significance. A few voices recur throughout and were powerful in shaping the emerging classroom narratives: For these, I have selected key words that connect them to a particular event in which they played a salient role; the names I chose are meant to be metonymous rather than encompassing: I hope they will serve as anchors that make it easier to recognize particular characters as they appear in various contexts.

The student to whom I refer as "Prime" was a strong voice throughout the study. I chose his name in reference to his solution to the consecutive integers problem: "All multiples of all primes can be written as the sum of consecutive, positive integers." Most students found various groups of numbers that can be so written (i.e. numbers that "work"); a few combined these as they recognized that all multiple odds work. Only Prime was troubled by the redundancy that "multiple odds" entails, and he sought a more parsimonious description.[22] He was consistently enthusiastic about sharing his own emerging understanding and was incredibly persistent in deepening and extending his initial ideas. Although he claimed to love debate (especially winning) and once claimed that he never backs down from an argument (which was clearly in evidence throughout the study!), he worked very hard to understand and engage deeply with other students' ideas. Perhaps more than any other, he was fascinated by the mathematical relationships he developed and was happy to take part in interviews that allowed him to continue exploring. He emanated a sense of calm: I don't recall him ever becoming visibly frustrated with either the mathematics or his peers. Despite his deep interest in

[22] While doing so eliminated the need to consider multiples of, say 6 and 9 since they are already accounted for as multiples of 3, he did not seem to realize that multiple primes could still identify the same number more than once; e.g. 35 is a multiple of both 5 and 7.

and sustained attention to the mathematics we explored, his excitement never emerged in loud speech or quick movement.

Although much less vocal, “Double-Check” similarly engaged deeply and patiently with both her own and her peers’ developing understandings. She noted her own tendency to think things through before speaking and expressed a persistent need to “double and triple-check.” Both her movements and her speech tended to be slow and steady. Like Prime, she was very calm and focused and maintained deep attention on the tasks long past when others began to lose interest.

“Felix” is named for Harry Potter’s good-luck spell, “Felix felicis,” in reference to the confidence and optimism he typically exuded (I further explain this connection in Chapter 5). In contrast to Prime and Double-Check, Felix’s speech and body language was highly animated as he interacted with others. He was seldom seen without a smile. He willingly shared many, many ideas that helped feed a rich classroom discussion, but he was less likely than either Prime or Double-Check to critically examine his own or others’ ideas. Nonetheless, he often asked questions and worked hard to understand others’ offerings; he showed great respect for other students and their ideas and offered deeply insightful reflections on the nature of interactions between students in his group and within the larger class.

“Jolt” is named for his reflections on the ice melt problem during which he described an experience of insight as a bodily jolt. In fact, the insight was clearly visible in his face and body. He did not speak often; when he did, he was generally confident and passionate about what he shared (though he sometimes lost contact with whether others were following him). During group discussion, he sometimes appeared lost in thought and disengaged from what others were saying, but when he spoke, he seemed to be very aware of the ideas being discussed. He would “snap” to attention and then speak an entire paragraph with few hesitations–a manner of speaking that was very rare in the study transcripts. Like Prime, Double-Check, and Felix, he worked hard to understand others, both in his group and during whole-class discussions.

"Quick-Start" spoke and moved slowly and often appeared tired. He engaged fully in the early stages of the problems; once he tired of them, however, he could sometimes be seen with his head resting on his hands on his desk, apparently disengaged (particularly if he was also tired that day or if he was at an impasse with his work). But he thought deeply about the problems and developed many insights. When a new insight emerged, he, like Jolt, would snap to attention and re-engage. Also like Jolt, this was very evident in his reflections on the ice melt problem.

By introducing this select group, I am not suggesting that other class members did not participate or make important contributions to the class dynamic. In selecting data that I hope will best illustrate my points in the context of a narrative that I hope will allow the reader to enter most vividly into our experiences, I have necessarily omitted many rich examples.

The Lamp Problem: Opposing Certainties

During the initial class, students (working in groups of four) responded to the following prompt:

> *Suppose you buy an antique lamp for $7, then sell it for $8. You buy the same lamp back for $9, then sell it for $10. How much profit do you make?* (adapted from Schultz, 1977/1982, p.12)

Each group posted their answer on the board, along with a number representing their degree of certainty in their conclusion.[23] Initially, every group posted one-dollar profit with four-level certainty as their solution, and there was little conversation or debate

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[23] 4: completely sure that it's right; 3: pretty sure that it's right; 2: seems right, but….; 1: probably wrong

(though later two students indicated that they were not comfortable with their group's first response).

Finding a Doubt Space

I had to stir things up by asking, "Would it matter if there were two different lamps?" This worked as a *bridge;* i.e. as a subtle shift of context that worked to cast doubt upon the one-dollar argument and allow consideration of alternatives. When two lamps were involved, there seemed to be general agreement that the profit would be two dollars. This prompted some students to change their minds about the solution to the problem as originally phrased. Nonetheless, one student loudly insisted, "There's only one lamp in the problem! There's only one lamp in the problem!" To him, adding a second lamp significantly altered the problem. He went on to articulate what became known as "One's Argument"–the only argument for one dollar that stood against many for two dollars, but one that many supported, nobody could effectively refute, and that was repeated many times over the following weeks:

> One: Cause, like, you're buying it for $7 and then you sell it back for $8, right? You make $1 profit there. But then you buy it back for $9, so you lose your $1 profit. Then you sell it for $10 so you end up with $1 profit. (Class 1)

This helped most students restore confidence in their original (wrong) conclusions, but a few were now adamant that the answer should be two dollars; it just did not make sense to them that two lamps should have a different result than one.

A little later, another student introduced time as a potentially significant variable: "Well, it just says you buy the same lamp back. It doesn't say WHEN you buy the same lamp back" (Class 1). For those who found time of sale and/or number of lamps relevant, their potential significance emerged in the sense of "*something-to-do-with*" (time or number of lamps). However, I do not think the students *recognized* the vagueness of these variables; they only tried to defend their signficance when they were directly challenged. Here, implicit understanding likely influenced their arguments in ways that operated beneath the level of ordinary consciousness. *Something* in the nature of buying the *same* lamp back for more than it had *just* been sold *somehow* seemed to diminish profit. The manner in which it did so eventually rose to the surface– but not in a way that justified one-dollar profit.

Sidestepping: Conflicting Certainties With No Point of Contact

We continued this work during a second session about a week later and during two more sessions about two months later. About half of the students also participated in at least one small-group interview during which they further grappled with the problem. The main arguments that emerged for two dollars were as follows:

Two:
Total spent = $7 + $9 = $16
Total income = $8 + $10 = $18
$18 is $2 more than $16, so there is a profit of $2.
(Two reverted to One's argument soon after presenting this one)

Prime:
Choose *any* starting amount of money.
Subtract $7, add $8, subtract $9, and add $10.
You come out $2 ahead.
e.g.:
$100 - $7 = $93
$93 + 8 = $101
$101 - $9 = $92
$92 + $10 = $102
$102 is $2 more than $100, so there is a profit of $2.
This works for any number: e.g. 87, 64, 1000.

Two (apparently not seeing his argument as a special case of Prime's, or perhaps seeing it as a more convincing example):
Start with $7.
Buy a lamp for $7, then sell it for $8; now you have $8.
Buy lamp for $9, leaving you with -$1.
Sell for $10, leaving you with $9, which is $2 more than you started with.

> One (also apparently not seeing his argument as a special case of Prime's; he initially argued with Two that he should start with $0, not $7 until after talking through his own argument):
> Start with $0, spend $7, then you have -$7.
> Add $8 to get $1.
> Subtract $9 to get -$8.
> Add $10 to get $2.
> (At this point, he switched to $2, but his original argument retained his name).
>
> Three: It doesn't matter in what order you buy and sell; you could sell for $10 first and $8 later.

Between these arguments, supporters of One's Argument continued to emphasize their point, mainly by restating it with surprisingly little variation. Though usually confident in their own arguments, supporters of two dollars had trouble specifying exactly what the problem with One's Argument was; attempting to do so became the main focus of the remainder of the time we spent with the problem. A few students were caught in the middle: They understood the many arguments for two dollars but found One's original formulation compelling enough to remain undecided. Among these were students who decided that time of purchase, number of lamps, or source of money *must be*[24] significant; it did not seem right that such things should matter, but the qualifiers were more plausible than rejecting One's Argument.

Co-evolving Meaning and Language

I challenged supporters of the one-dollar argument to find other ways to explain their answer and supporters of both sides to directly refute the aspects of the arguments they disagreed with rather than merely proposing alternatives. Prime worked very hard to resolve the disagreements that emerged in this problem space. In the following example, Two insisted that he had already proven that two dollars was the correct answer and did not see the point of further effort until Prime engaged the argument more

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[24] In Chapter 5, I further distinguish between a strong and weak sense of *must be*; the former refers to a necessary conclusion, while the latter indicates doubt accompanied by an inability to think of anything better.

seriously. Then both struggled to pinpoint the problem they were so sure existed. Note the half-finished sentences and their struggle to find words that worked to express their conviction:

> Prime: But from what he's saying, he just loses the dollar. It's no–he's just saying that, like, say he went down–he's saying that you lost your profit, and it's got nothing to do with, like–that profit is just completely gone, nev–never was there, won't affect anything more. Because–
>
> Ms. M.: You're saying it will more.
>
> Prime: Yes.
>
> Ms. M.: Why?
>
> Prime: Because…. I lost my train of thought.
>
> Two: Oh! So where he's wrong is–he doesn't take into account as…. He just uses 8 of this number, which is minus. Then he uses the 10 to go–take it back. And it should be $10…. No. Wait. So he loses his profit here. This is 8 of the 9. Even if…he were able to minus this from–this the last number of the one number left for the 10, it would still be 9…9 [tapping hands on table]
>
> Ms. M.: It's hard to argue, isn't it?
>
> Two: I know, but…. It's simple, but…. It doesn't take any bit of math! I mean…. It doesn't use–it just uses, like, Grade 2 or 3 math!
>
> (Class 1 Interview)

Two and Prime also talked about how One's Argument seemed to assume that when you lose your profit, you go back to zero dollars. Prime later clarified this with the help of the phrase "original money," a notion that likely encompassed a more complex argument that he still could not fully articulate; I wonder if things would have unfolded differently had he been able to say that the discrepancy had *something to do with* original money, thus marking a need for further explication. Interestingly, however, Prime often attempted to refer to meaning he thought *others* might have held implicitly:

> Ms. M.: Okay. Is that how you interpret One's? That he has–that when the profit's gone, you're at–you have zero dollars?
>
> Four: Yeah. [Eight nods]

Prime: What I'm more…. What I'm more thinking? Is you're back to your–**he's saying that you're back to your original money**? Cause–

Ms. M.: He didn't talk about original money, though.

Prime: No, but what I think. He didn't really say? He just said you lost your profit. He didn't say you went back to zero.

Cause technically–

Four: But when he says he lost his profit, it just means zero.

Two: No! No!

Prime: It means–he means zero profit, it doesn't mean the number zero. Cause then you'd actually have just lost seven dollars.

(Class 1 Interview)

I find Prime's insight about original money quite remarkable. He had to find new words to articulate something that was quite likely still un-named even for One. I read and re-read this dialogue as I attempted to understand each student's perspective and to dispel the intuitive tug that I, too, felt in response to One's Argument.

Empathizing With Students' Doubt: Why Does it Feel Like There Is a Missing Dollar?

Though fully convinced that two dollars was correct, I wanted to understand why the lost-profit idea was so compelling; as it turned out, attending more deeply to students' doubt in this manner had a very significant impact on my role as empathic second-person researcher and coach. I found it very difficult to find words to explain (to myself) the problem with one-dollar profit:

> He didn't LOSE his profit; he spent it (i.e. invested it)–plus another $8–to buy back the lamp. He's not out the money, because he has the lamp. The value of the lamp is up for grabs at this point–it all depends what he sells it for. If he tosses the $9 lamp in the garbage, he's out $8 *from his original position*–he only "loses" the profit (i.e. breaks even) if he sells the lamp for $8. If he sells for $8, he's out $2 from the *potential $10-sale position*, and *any possible amount from the best-buyer scenario*. (Research Diary)

This was a moment of insight, even though it felt as though I were articulating something I had known at some level all along. In fact, it is evident a little further into *my own* notes that even after this reflection, I *still* felt the tug of lost profit and the need to dig a little deeper:

> Another thing…. You DO lose your profit from the perspective of lost opportunity–if you knew you could eventually sell the lamp for $10, you should buy @ $7, sell @ $10 for a profit of $3. In the given scenario, you only make $2 and DO lose $1 profit. But not in the manner described by most of the kids. (Research Diary)

According to my research diary, I had also worked through the same ideas months earlier. At that time, I recognized a connection to feelings I had regarding a recent land purchase my husband and I had made. The land was more expensive when we bought it than it was when we first considered doing so several years prior. Although we both felt (or at least rationalized) that the land was a *good investment* even at the higher price, I could never shake a *feeling of lost profit* that closely matched my feeling about buying the lamp for nine dollars. From this beginning, I developed a deeper understanding of the *I-could-have-made-three-dollars* argument, which seems to be related to what Kahneman, Slovic and Tversky (1982) referred to as loss aversion: People often reduce their odds of making money because they tend to worry more about potential loss than they anticipate potential gain.[25]

Bridging Money Spent and Money Lost

At the beginning of the next session, ten students indicated they were still unsure of the correct answer; nobody claimed total certainty in one dollar, and the rest were now

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[25] Note that my efforts here are indicative of the inseparability of my roles as learner, teacher, and researcher. My efforts to more deeply understand the students' confusion were essential to (a) my role as learner deepening my own mathematical understanding, (b) my role as mathematics teacher attempting to provoke deeper understanding in the students, and (c) my role as second-person researcher attempting to empathize with the students' first-person experience.

sure of two dollars. Even after all the work Prime did during his interview, his attempt to explain why One's Argument was flawed seemed to lose some of its emerging clarity:

> Prime: You're…. What that argument is saying is pretty much you buy–you buy it for 7, um, sorry 8, and then once you sell it for 9, those two numbers never existed. They don't affect it all. That's what you're saying. (Class 2)

In any case, his efforts failed to convince: He was again met with the repeated refrain of the one-dollar supporters: "Where do you get the other dollar?" Here I wonder why Prime did not explain exactly *how* the seven and eight would affect the outcome; i.e. Why did he not explicitly state that the nine dollars spent on the second lamp–including the alleged dollar in lost profit–were all recouped in the ten-dollar sale, along with an extra dollar? Was this not clear to him, or did he not see how this could be a point of confusion? It may seem that this understanding is *implied*, but I am not convinced. "What happens with the seven and the eight" seems at this point to be centered around a vague notion that "when you buy the lamp back, you're not just spending your profit, but also a bunch of other money that nobody seems concerned about":

> Prime: But they…. Yeah, but you don't go back down to–you don't go back to the starting number.
>
> Five: All I'm asking -
>
> Prime: **Losing the profit could mean that you lose $500**. [Five: All I'm asking-] You're still losing the profit.
>
> Five: All I'm asking is how…did you get the $2 profit?
>
> (Class 2; emphasis added)

It is possible to appreciate that "you could lose $500" without explicitly recognizing how this hypothetical scenario might relate to the alleged missing dollar (note this is another *bridge* that subtly shifts the context of the original prompt). It seems to me that Prime is trying to point out that some students are confusing "spending" with "losing profit"; i.e. their way of thinking that buy-eight involves "losing profit" could be understood to imply that buy-500 would result in losing $500 in profit, even though the second purchase would still only include one dollar in profit from the buy-seven-sell-eight part of the scenario. In this sense, he might have prefaced his statement with, "*If you look at it that way*, losing the profit could mean that you lose $500." I think he was trying to problematize the one-dollar supporters' separation of the money used to buy back the lamp into "profit from the first sale" and "other money"; by their reasoning, the money can be seen as "lost" either way. My own appreciation (or interpretation) of this argument first emerged as a *tip-of-tongue* understanding, another indicator of the influence of implicit understanding that I take up in greater detail in Chapter 5.

Two then repeated an earlier argument, with what may be a significant (to Two, though unnoticed by Five) modification that acknowledged that eight of the nine dollars used to buy back the lamp were received from the first sale of the lamp. Like Prime, though, he does not explicitly recognize that all nine dollars spent on the second lamp–including the *one* dollar of profit from the sale of the first lamp–gets *recovered*. In fact, he seems to refer to all eight dollars as profit-dollars, conflating income with profit in much the same way that one-dollar supporters conflated spending with lost profit:

> Two: Okay, so you have **$8 you use out of the first profit**–the first part of the profit. You minus 9. You get minus 1. Then you–and then you add 10, so you get 9. You started with 7. That's $2 more. (Class 2; emphasis added)

His efforts were also met with confusion from the one-dollar supporters; he attempted to clarify, but he became frustrated and decided to call someone else in to help out. Three seemed to have at least a vague idea that the seemingly lost profit was now included in the nine dollars used to buy back the lamp, but, like Prime and Two, he was unable to connect with the one-dollar supporters:

> Three: Okay. So. I've got 7 bucks. And I–I buy this lamp for $7. And I sell that same lamp for $8, so I have a buck. And I buy the same lamp from a different guy–I think it's the same lamp for $9, and I lost my dollar profit. Then I sell for 10 **and then I get my $1 profit back**.

Student: Where do you get that other dollar?

Several students: YEAH!

(Class 2; emphasis added)

I do not know if he meant, “I get my $1 back, along with one more.” I also find it interesting that he added in the piece about “from a different guy” and expressed uncertainty as to whether it was the same lamp–somehow, these variables still seemed to change his intuitive sense of what was happening.

The notion that the seemingly missing dollar was somehow connected to the fact that you are not just spending profit when you buy back the lamp came up yet again as Six struggled to articulate what also seemed to be an emerging understanding of how money spent is distinct from lost profit:

> Six: For Eight? For the people who think it’s $1? They’re not thinking about **how you lose when you sell it for the first time**? Cause if you didn’t lose any money, then you would get $1 profit. [pause] Cause it’s like–it’s like, um… [sits back, hand on face, smiles; someone loudly whispers “What!?”] **I don’t know how to (?). It’s just–you guys aren’t thinking about the money you lose**. (Class 2; emphasis added)

It is tempting to think that this may have been a good place for me to help students find words to fit emerging ideas, but it is impossible to say whether my words would have matched the feelings that they were struggling to articulate. In any case, it was only in hindsight as I reflected on the transcript that I was able to further clarify my *own* understanding:

> Here may have been a good time to point out that both of the dollars–*i.e. the one gained in Buy 7–Sell 8 and another from Buy 9–Sell 10*– are included in the $10…. *The $10 replenishes the $8 I “lost” when I re-bought the lamp for $9 and adds another.* But I don’t think I was clear enough on this to help them out!! Even now, I feel confident in the general notion of “included in the 10,” but when I went back and tried to articulate that more clearly, I really had to think about it!! This may also have been a good time to further discuss the argument re: “I could have made $3.” (Research Diary)

Throughout our work on the lamp problem, I often reminded students that they needed to do more than restate old arguments. I did not always recognize the potential significance in some of the subtle shifts in students' wording as they attempted to convince the others that two dollars was the correct answer. The notion of what happens to the lost dollar seems to have been understood by many at a deeply implicit level. As I explore more deeply in Chapter 5, it is times like this when Gendlin's (1962/1968) focusing may have been helpful in that it may have allowed space for emerging words to come forth for comparison with students' felt sense of the situation, (and to be revised as necessary until a match was found).

Attending to Subtle Shifts

As I reflect on the transcripts, I find it surprising how very small shifts gradually emerged as students wrestled with contradictions and inadequate language:

> Prime: …They don't count what the 7, 8 did to the amount of numbers and where… where it puts you at. It just–instead of… you… lose 7, lose 9. There's no 7 or 8 anymore because you sort of compound, because you lost your profit. So suddenly there is no 7 and 8, it's 9 and 10 and $1 difference there. But what they…. But the problem with that is…is…. When you do 7 or 8, it doesn't put you into square–it **doesn't put you into square one**, it puts you into an **altogether new square**. So there–
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> Ms. M.: So when you say "square one," how does that fit (?). Where is square one that you're talking about?
>
> Prime: Whatever number you start with. 100…7…0….
>
> Ms. M.: Okay, but they're not…. Using this argument, what are they starting with? So if you can…just so that I'm just…not arguing with your argument, but how can you meet to this?
>
> Prime: What they're just doing.… What they're just saying is they're…. They're saying lose 7? For you to lose 7, you had to have something before that?
>
> Ms. M.: And so you're saying they're not saying what you're starting with?
>
> Prime: No. It doesn't really matter what they start with?
>
> Ms. M.: Okay.

> Prime: But what I'm saying is there was some number at the beginning [Ms. M.: Mm-hmm] for you to lose and then gain? And what they're saying? Is you lose…is you lose gain and then go straight back to square one and losing gain again. **It's not lose-gain…go to a different square…and then…then, um…losing gain again.**
>
> Ms. M.: So your…your argument is with this point, then. You're saying you don't lose your profit.
>
> Prime: You lose your profit there, but not to…not to the same number. **You lose your profit to a different number**, so when you get more money back, you get 2 more dollars than whatever number you started with.
>
> (Class 2; emphasis added)

Three was very motivated to act out the scenario. It seems he believed that acting it out would demonstrate a simple set of transactions that would clarify all the doubt surrounding the problem. However, others had already described the same transactions numerically, and nobody disputed the arithmetic: Neither the numbers nor the enactment directly confronted the inadequacies of One's Argument. As Three and his group attempted to act out the situation, what had previously seemed straightforward transactions became muddled in the students' attempts to move the money around. The notion of debt was especially problematic. As Prime had noted earlier ("Losing the profit could mean that you lose $500"), if you see purchasing as losing money, you lose all nine dollars when you buy back the lamp:

> Three: So here's a pretty lamp and I bought it for $7, so, I have minus $7. And then I sold it for 8, and now I have 1 dollar. **So I have a dollar profit. I buy for 9, and so I lost the one. And then that....** [several voices in the background] What? [Prime says "you're at –8"; lots of advice from class] **I'm in debt? What?** And then I –then I–then I sell it for ten? What? I make no sense. I make no sense. [sits down] (Class 2; emphasis added)

The group continued their work with the act-it-out scenario in the post-class interview. When they finally sorted out how much money various people should start with, who was in debt, and who needed to be paid pack after borrowing, they concluded that they ended up with nine dollars, which was two dollars more than the seven they had started with. They were excited to show their method to the class, and Three insisted that it would "clear everything up." Ten noted (as a sort of aside) that "I would have made three dollars profit if only I didn't owe Fifteen money." She was able to share this with the class with some prompting, but later seemed to completely forget this insight. Nobody used the simulation to directly confront One's Argument.

Referring to the Implicit in Others' Work

After breaking for Christmas, we started work on a different problem. During an interview, however, One noted that he was still waiting for resolution of the lamp problem. I decided to bring it up again, two months after we had last discussed it. I summarized the key arguments that had been presented, then invited discussion. At first, little new emerged. I asked the class to consider whether it would make a difference if the scenario were "Buy 7, sell 8, buy 98, sell 100." Nobody engaged. One group began testing the impact of various starting amounts of money and concluded (mainly, it seems, on the strength of many examples) that it did not.

One group presented what seemed to be a variation of what Prime had been trying to say about the illusion of losing profit. Again, as the students attempted to explain what others were trying to say, they seemed to get more deeply in touch with their own implicit understanding.[26] It seems that the tangible gap between the supporters of one dollar and two dollars was easier to point to than their own difficulties with refuting one dollar; attending seriously to differing viewpoints provided a powerful motivation to refer to the implicit (although again, it is not clear that anybody was aware of doing so):

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[26] In a sense, they were becoming empathic, second-person coaches for each other.

> Double-Check: And so this [pointing to 1] **doesn't disappear**. This [pointing to 8] stays there. It just adds on to the other numbers, so when you go to subtract…. When you go to subtract 9 [from 8]…9, this number? Isn't disappearing. **It's making this number bigger, so you subtract, actually, makes it seem like you're subtracting less**.
>
> [pause, then somebody leads applause]
>
> Jolt: Okay, so **what Double-Check's trying to say** is you have $0 first of all. We buy a lamp for $7. Now we have $-7. We buy it for $8, to get a $1 profit. Okay, now we want to subtract it by 8 [somebody corrects to 9]. Oh yeah, sorry. So one dollar subtracted by 9 equals 8. Now plus 10 equals $2 profit. We're not using the $1 profit. We're subtracting from it so that we can get a bigger profit when we add the $10.
>
> Ms. M.: So both…Double-Check and Jolt, you're arguing that there's a flaw in that first statement where it says you lose your profit.
>
> Jolt: Yeah.
>
> [noisy confusion]
>
> (Class 4; emphasis added)

Prime, Double-Check, and Jolt all worked very hard to describe what happens to the seemingly missing dollar, but many students did not understand their arguments.

After hearing these arguments, Prime used a very detailed explanation in an attempt to connect his previous explanations to theirs:

> Prime: So we agree with these two groups that it is the–that it is the…fact that in the first argument…. **They think** that it's just you lose your profit? But you don't actually lose your profit? You, um, go to a…to a different number. Cause say you didn't start with…what other number…. [he decides to start with 10 and writes a column showing the sequence of calculations as he talks]. And then, take away 7, you have 3, plus 8, you have 11. **What the first argument says** when you minus 9? You just…. **What they're saying** is that when you minus 9? The $1 profit that you got here? Just goes, "Bye, bye." And then when you add 10 again? When you add ten again? You go up to one dollar more than 9. **I can see where they think that.** But the problem is? If you do the math? So then (?). You add 10, you have twelve, so…. You don't lose your profit? [see below]. Cause if you just lost your profit, that'd be like saying, 9…. If you lose your profit, that's pretty much taking this one going from 11…to 10 [crossing out 11]. So 10 minus 9 would be 1. Plus 10 is 11. Which would be one-dollar profit. **So they're saying** when you're–when they're saying minus–when they're saying, "Do minus 9," which is lose your

profit? To lo[*se*][27]–to lose your profit, **they're saying** is you're going back–you're saying minus 10. You just lost your profit. You just lost your profit? So then you go back to 1.

10
-7
3
+8
~~*11*~~ ***10*** **[The $1 that you got here? Just goes, "Bye, bye."]**
-9
~~*2*~~ *1*
10
12 **[You add 10, you have twelve, so…. You don't lose your profit?** … **So 10 minus 9 would be 1. Plus 10 is 11. Which would be one-dollar profit.]**

(Class 4)

On contemplating this argument, Prime painstakingly attempted to address the exact point in the argument that seemed to be causing all the controversy. But it still was not enough to capture or represent his meaning, nor did it adequately capture the reasoning of the one-dollar supporters. Though the language is very precise and seems to me to mirror the "lose your one-dollar profit when you buy the nine-dollar lamp" argument, I do not think the one-dollar supporters saw that the dollar actually disappeared from the eleven dollars *before* they bought the nine-dollar lamp. To them, the dollar disappeared *within* the nine dollars–and in fact, after this transaction, there *is* one less dollar than after buying the seven-dollar lamp. But it gets recovered in the ten-dollar sale.

Thwarting Awareness of the Implicit

I personally find the lost-dollar point of view much easier to understand when I consider that students do not recognize that the dollar they supposedly lost can be considered part of the value of the lamp, provided they are able to re-sell it for the nine dollars they paid for it; i.e. re-selling for nine dollars would still result in a profit of one

---

[27] I used this transcript notation to indicate when a speaker started to say a word but stopped part-way into it; the brackets indicate what I consider the most plausible ending for the unfinished word.

dollar, returning the seller to the position after the first sale. In fact, one student followed Prime's explanation with the following comment:

> Five: You buy the lamp for $7. Then you sell it for 8. So then… you make a $1 profit? But when–like–and that–when you just think about it, **you add the $1 profit to the 9**? (Class 4; emphasis added)

Unfortunately, his comment was lost as I unsuccessfully tried to connect it to re-investing: For him, the words I chose did not seem to resonate with the understanding emerging from *his* implicit. Worse, his efforts to understand what I was saying interrupted his efforts to articulate and extend his own understanding.

The notion of re-investing might be considered just another way of expressing Prime's argument, but it feels much different to me. The fact that the value of the lamp is fluid seems very hard for some to grasp. Here, I notice that even my own choice of the phrase "fluid value" points to a subtle shift in understanding from my earlier consideration of re-investing.

At the time, I wondered if explicitly shifting frames from losing to re-investing might help some students make a shift in their understanding, but it seems only the students already in agreement with Prime's arguments seemed to understand the distinction. Nonetheless, thinking in terms of investing may have impacted the manner in which *those* students continued to move their explanations forward.

Six suggested that you could also consider spending seven dollars on the lamp losing your profit. When sharing with the class, group members Four and Seven prefaced their explanation in a way that seems to indicate an awareness of the significance of subtle shifts in wording. Also note indicators of shifts in Seven's implicit understanding as she made hand gestures and struggled to finish sentences; at times, she could also be seen talking quietly to herself:

> Seven: Basically, um, our explanation, our procedure, is **like the previous group, from like, Nine's and One's, but…um….**

Four: **Basically, we're just gonna explain it, not a different way? But another way our group looked at it.**

[Six writing on board; pause while group waits for him to finish]

20
- 7
13
+ 8 [**You lose your–you lose your profit right here and right here.]**
21
- 9
12
+10
22

Six: If you start off with 20 bucks and then you sell it for...7…7 bucks, and then you have 13 bucks left. But then you sell it for 8 bucks, and you get– and you then have 21 bucks? And then…you sell it for 9 bucks–and you sell it for 10 bucks and you have 22.

Four: Basically, that's like 2 numbers higher than the original number.

Six: And we did that for, like, other numbers, and it was all 2 higher–2 numbers higher…than the original.

Ms. M.: So using that one…. If you were using the…"you-lose-your-profit-argument," where would that–at what point in that ch[*ain*]–that…list would you have lost your profit?

Group member: Umm….

Ms. M.: Basically, it's…when you buy for…. According to the original argument, you lose your profit when you buy for–buy it back for....

Seven: I think it would be… [**hand moving in circles**] I'm not really sure. But I'm just….

Six: You lose your–you lose your profit right here and right here [indicating the part of the equation where they subtract 7 and subtract 9].

Ms. M.: Okay, but that's new. Originally, we didn't say we lost our profit when we bought the lamp for 7 dollars. Right?

(Class 4; emphasis added)

At this point in the conversation, I felt a connection to what they were saying and attempted to paraphrase rather than listen; I wonder now what might have emerged had allowed their own meaning to unfold. In particular, consider how my presentation of a dichotomy likely interrupted this process:

> Ms. M.: That wasn't part of the argument that when you buy a lamp for 7 dollars, you lose your profit. And on this one, it kind of does look that way, doesn't it? So when you buy a lamp for 7 dollars, what are you–**are you losing your profit? Or are you losing 7 dollars out of your wallet**?
>
> Student: Seven dollars out of your wallet.
>
> Ms. M.: Is there a difference?
>
> Student: No.
>
> Ms. M.: Okay. So….
>
> Seven: **Yes, actually [?–talking quietly at board]**
>
> Student: It depends.
>
> [several quiet comments; doubt in the room is almost tangible]
>
> (Class 4; emphasis added)

Noting that something new was emerging may have been significant, and perhaps even *offering* "money out of your wallet" could have been useful. But phrasing the question about losing profit vs. dollars out of your wallet as a dichotomy was likely limiting. From there, my questioning became even more directive (although I tried to keep connecting what I was saying to their explanation):

> Ms. M.: My question was **are you losing profit, are you losing money out of your wallet, or is that the same diff**?
>
> Six: (?)
>
> Three: No, but it's money out of your wallet.
>
> Ms. M.: So if the first one is money out of your wallet, what's minus 9? **Is that profit, or is that money out of your wallet**?
>
> Student: What happens if you don't have any money in your wallet to start off with?
>
> Prime: You go into debt.

One: You get a loan!

Ms. M.: So when you buy a lamp for 9 dollars, are you spending your profit, I guess is part of the question.

[Some yes's, some no's]

Six: Yeah, you are.

Ms. M.: Why?

Six: Because… because the profit…. Wait. You earn a profit when you sell it for 8 bucks. You earn $1 profit, then you sell it for 9 bucks and you use that money (?).

[short break in dialogue for a non-participant]

Ms. M.: When you buy a lamp for $9, **did you lose your dollar, or did you use it to buy a lamp**?

[various comments; "You used it to buy a lamp."]

Ms. M.: So it's not like you burned your dollar. You got a lamp out of it.

Student: You just re-invested it.

Ms. M.: So if you still have that lamp….

Student: Yeah. You lost your dollar in buying (?).

Ms. M.: You lost your dollar in buying the lamp, but **do you get that dollar back plus another dollar? Or do you only get $1 back when you sell it for $10?**

[some disagreement; quiet comments around the room; break in tape; some students were visibly frustrated with those who were unconvinced by the arguments against $1]

At this point, class was almost over, so I had the students write down their conclusions. In their written responses, fifteen students indicated 100% certainty in the two-dollar solution, and five were "pretty sure" that two dollars was correct. Two students did not respond. Of the fifteen who indicated 100% certainty, at least four did so because of the strength (and in at least one case, popularity) of arguments for two dollars rather than because they understood the flaw in the arguments for one dollar. Their continued inability to effectively refute the one-dollar argument was an ongoing source of doubt in their understanding of the situation *even when they were certain of the answer.*

After class, I still felt the need to further clarify my own thoughts about investing and finally was able to do so to the point where my doubt-feeling was fully resolved. At that time, comments my dad had made about "sitting on all the wealth" of his farmland (as opposed to having disposable income) came to mind; I wondered what brought this to awareness and how it might have been influencing my understanding. My parents do not have large amounts of money to spare. Their land is valuable, but they need it to keep farming, and they are still paying for some of it. So how exactly did the lamp problem connect to my dad's comment?

> I guess it's that the land (or the lamp) doesn't present itself to me as money because it's not available to spend. It is also true that the cash value of either depends on resale, but that may be an after-the-fact rationalization of the connection rather than the intuitive connection that brought Dad's comment to mind in the first place. (Research Diary)

When I was doubting whether my and my husband's own land purchase had been a good idea, I tried to comfort myself with the notion that at least we had the land–we could always sell it. But that came with its own set of doubts about whether the land would hold its value; i.e. it is only worth what we can resell it for. Here, I can locate another doubt rooted in the idea that "It already made a big jump in value prior to our purchasing of it and is therefore unlikely to do so again"–a personal bias not rooted in a rational appraisal of real estate trends that might be summed up as "we missed out." The notion of investing is likely more obvious to others than it is to me; investing-for-resale is not something I've ever spent a lot of time thinking about. This may also be the case for many seventh-graders.

Although contrary to the enactivist framing of the work, persistent students convinced me to share with them the "official answer." First, however, I decided to offer an argument of my own. I wanted students to attend to what for me was a strong discomfort with buying a lamp for more than I had just sold it and for them to consider whether this feeling might be influencing *their* thinking; I thought this might provide a window through which they might develop a deeper appreciation of how implicit understanding might be coloring their ways of engaging with the problem.

My first attempt was similar to one that Ten had suggested during an interview but that had not received much attention in the larger group: We *could have* made three dollars if we had bought for seven dollars and sold for ten dollars without engaging in the buy-seven-sell-eight transaction in between. While most seemed to understand where the three dollars came from, this did not seem to prompt deeper insight into the supposedly missing dollar. I later tried two more *bridging* tactics:

- I asked the students to consider what happens if you buy for $7, sell for $8, then buy for $1000, and sell for $1001. I also asked them to consider what would happen if they sold it back for $1000.
- I talked a bit about how to me it felt like a mistake to buy for $9 right after I had sold for $8, then asked what would happen if I went directly from buy for $7 to sell for $10. I told them that for me, the feeling of the missing dollar was connected to the feeling of losing a potential extra dollar.

I asked if my arguments helped anybody; some said yes, others said no. One student was very concerned to know whether I used my profit when I bought the lamp back for $1000, and a short discussion regarding why that would matter ensued. But we did not reach any firm resolutions. I acknowledged that other people's explanations–including mine–were not sufficient. I told the students that the correct answer was two dollars, and nobody indicated surprise. But in the after-class interview, one student commented:

> Eight: And yeah, that $3 thing? Kind of caught me off guard. Cause sometimes, like, the answer that you didn't think about is the right answer. (Class 4 Interview)

Of course, three dollars was not the right answer. But that was how she interpreted my explanation.

The Ice Melt Problem: What Would Happen If…?

Finding a Doubt Space

The first day that the students worked on the ice-melt problem, they came to class to find funnels full of ice melting and dripping into a graduated cylinder (problem adapted from Wise, 1990). Each group set to work figuring out what time the ice started melting. I asked them to keep track of their efforts so they could justify their conclusions to the rest of the class. Some students immediately wanted to know how much ice there was to start with. When I would not tell them, most soon came up with a way to measure the rate at which water was dripping into the graduated cylinder and used that to figure out how long all of the accumulated water might have taken to collect. Two weeks later, the students presented their conclusions to the class, reviewed written summaries of their strategies, addressed questions aimed at their group, revised their conclusions, and formed improved plans for a second trial.

Questions mostly focused on the impact of rounding (especially before multiplying) and on the assumption of a relatively steady melt rate, which also made choice of interval size a relevant consideration. Some questions focused on practical matters (is it easier to count milliliters or drops?) and calculation errors (especially involving time). Many students found it difficult to figure out whether they should adjust their times forward or backward to account for a rounding error or for a proposed speeding or slowing of melting. There was much to account for here, and some students expressed frustration with generating answers that they felt confident in.

Once all groups had developed and refined methods for finding a start time, it would have been easy for me to assume that they had recognized the insignificance of the amount of starting ice and to focus attention on other sources of doubt. As became particularly clear in one group interview, however, many students had merely sidestepped their initial concern with the significance of the initial amount of ice. It is this story that I elaborate here, as it provides fascinating insight into the role of the implicit and into the way subtle variations can help bring the counter-intuitive to awareness where it can co-evolve with the language used to describe it.

Empathizing With Students' Doubt: Does Starting Ice Matter?

While the amount of starting ice used in the initial setup of this problem may have affected the melt rate of the ice, it should not have affected calculated start times. It may also have affected the initial delay that occurred before water started to drip into the cylinders, but this did not seem to be the primary source of the students' belief that the amount of ice was significant. So why *did* they feel it was important? Did they have a felt need to control all variables? Was the feeling rooted in an unevaluated feeling that more ice equals more time? That more ice would produce more water? More ice probably *would* take longer to melt and probably *would* produce more water, but both of these variables are accounted for when time values are assigned to particular intervals of melt water.

Although I was fully confident that the amount of starting ice should not impact start-time calculations, when I made a point to engage more deeply with students' doubts, I recognized a niggling doubt that alerted me to the possibility that I probably had not formed an explicit rationale for this. In fact, there is a point in one of the transcripts where I puzzled over the fact that two groups measured very different melt rates yet calculated nearly identical start times! Why was this surprising? I remember experiencing a momentary doubt that this situation might not fit the *category* of *starting-ice-doesn't-matter* as I had conceived it, but why? In hindsight, it seems like a perfect example: It demonstrates how different melt rates can count back to the same start time when different time values are attached to each interval of water. Conversely, when Prime commented that his group had spilled their ice part way through the experiment, my immediate internal response was, "Oh, that shouldn't matter–starting ice is irrelevant." But of course in this case, it *does* matter–my immediate questioning of whether *this* situation belongs in the *starting-ice-doesn't-matter* category was entirely functional. Here, the accumulated water was dependent on the melt rate associated with *that* ice.

While these insights seem obvious in the relative quiet of my writing room, I was able to empathize with the counter-intuitiveness of the notion that starting ice was not relevant. I have used this problem many times in the past, and somehow both the students and I always sidestepped the issue of starting ice (which was almost always mentioned by at least one student or group of students). Taking seriously the quiet doubts that could easily have been overlooked as seemingly irrelevant was again (as in the lamp problem) essential to my role as empathic second-person coach.

Bridging Amount of Melting and Rate of Melting

Prime was the only student who (on his own) articulated a more nuanced understanding of these ideas:

> Ms. M.: Does the size of the handful of ice make a difference?
>
> Eight, Four: Yeah.
>
> Prime: Yes. **[A]** Cause the… cause the more ice that there is? The colder that it will be at the start? So it will take longer for it to start melting? **[B]** And then the other ice will also slow down when it does start melting? It'll also slow it down ever so slightly? Which could affect (?).
>
> (Class 3 Interview)

While Prime may have been right about the impact of initial delay (A), a slower melt rate (B) should be accounted for by attributing less time to a particular volume of melt water. When I suggested this, Prime explained that he was concerned that the rate might *change*; at first, I thought he had not recognized that a changing rate could also be accounted for. But he had easily solved a set of hypothetical scenarios–including two with non-linear melt rates–that we had worked through during that day's class (see Appendix B). As he had done in the lamp problem, he created his own *bridge* or hypothetical scenario to help make his point:

> Prime: It'll start going faster. And, say, once a whole ice cube melts? Then it might go at a different level? Even if it's consistent then? So say at… 8? Say when you put it in at 8:12? You put in 5? But then by the time we were checking it at 9? One whole ice cube had melted? So therefore it might be going–even if it was going consistently at 9? **It would be–it might be consistent pace… different than it was at 8.**
>
> (Class 3 Interview; emphasis added)

He is right in noting that an *unobserved* change in melt rate (i.e. one that occurred before the students arrived in class) would impact start-time calculations if the *change* in melt rate was not consistent; i.e. if, as he suggested, it jumped to a different consistent rate.

For the second trial, I gave one group a very large funnel of ice dripping into a one-liter (as opposed to 100 mL) cylinder. Many students commented on this setup as they arrived in the room. During the post-class interview, I asked the students whether the large amount of ice likely affected the final time determined by their calculations. The interviewees were not from the group with the giant funnel, but they had met with that group and knew that it had not made much of a difference even to melt *rate*: Both groups measured 8 seconds/drop.[28] I actually thought this a bit strange: I expected more water from the larger setup. As some of the students pointed out, however, the larger volume of ice likely kept everything cool longer. Over a longer time frame, a different result may have been observed. In any case, the students proceeded to talk about how smaller chunks of ice with more air circulation would likely affect not only melt rate, but also start-time predictions, if only slightly:

> Quick-Start: Mmmm…. I don't think that would affect it. The only thing that might happen is if there was more ice cubes, it would be cooler in the… jar, and then it would melt… **[quietly] slower….**
>
> …
>
> Quick-Start: Maybe **a little bit**. **Kind of**. But. **If they were reasonably the same**, then I don't think it would affect it [start time] **much**.
>
> (Class 4 Interview; emphasis added)

I wanted to explore Quick-Start's reasons for hesitating, for suggesting the starting ice needed to be *reasonably* the same, and for claiming that it would not affect start time by *much*. Why should different starting ice matter *at all*? Was Quick-Start's prediction of a *small* difference more an indicator of doubt than an indicator of impact? At this point, It seemed to me that the students had *totally missed the point that a higher melt rate would result in larger amounts of water being assigned to the same time interval than would be the case in a situation with a slower melt rate.* The students correctly stated that the large funnel would not make much difference to start time, but their reasons for doing so seemed to betray a fundamental misunderstanding of ratio. Even if melt rate changed dramatically, start time predictions should not be affected.

To further probe their thinking on this matter, I invented a series of *what-would-happen-ifs* to direct attention to implications of melt rate. I had to keep removing

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[28] I was surprised how many students chose to count and time drops rather than mL. This led to some challenging calculations as they tried to reconcile drops / time interval with accumulated milliliters.

distractors in my hypothetical examples (italicized in the sequence of dialogues that follow), as the students kept explaining away differences using what were for my purposes non-essential features; i.e. they either (a) minimized the potential effect by saying the difference in amount of ice would not be enough to make a difference or (b) continued to focus on how the changes would affect *melt rate*, thereby avoiding contact with the key idea I wanted them to grapple with; i.e. would the changing conditions affect *start-time predictions*? Although it took a lot of back-and-forthing, eventually I framed the question in terms of a large container of ice melting quickly compared with a small container of ice melting slowly. In other words, “Don’t worry about all the reasons why it might speed up or slow down and assume it *does* speed up” (and assume the ice was placed in the funnels at the same time). *Then* will the start-times be the same? It seemed that as soon as Quick-Start realized what I was asking, he had an *aha* moment, and that *aha* continued to evolve into another a few lines later:

> *Ms. M.: Okay. Okay, here’s another suppose. Suppose it does go much–let’s–let’s say it goes really fast because there’s so much ice. And there’s… and let’s say… a lot more water comes out of that funnel. Should you still… get the same answer–would you get the same answer or a different answer than the rest of the class?*
>
> Quick-Start: [**sitting up quickly**] You’d get… No, you’d–you’d get the same answer. It doesn’t matter how much… water there is in there. You just–if you get it out–if you count the drop times the same as us, you should be able to you should be able to **backtrack it just in bigger numbers** to… what time it started.
>
> (Class 4 Interview; emphasis added)

Jolt seemed to misinterpret Quick-Start’s point, so I kept pushing with my line of questioning:

> Jolt: Yeah, as long as the ice is the same size, and the room temperature’s the same, it doesn’t matter how big the beaker is and how…much ice you put in.
>
> *Ms. M.: Why does the size of the ice matter?*

> Jolt: Because if the ice is really bulk, then it might take longer a longer time for it to melt.
>
> *Ms. M.: Okay, so let's–let's say that happens. Let's say I get one solid chunk of ice to fill the funnel. So there's no ice cubes, it's just one solid chunk of ice. It's probably gonna melt really slow, right?*
>
> Jolt: Yeah.
>
> Ms. M.: Would you still get the same answer as the rest of your class?
>
> Quick-Start: [**suddenly sitting up**] **Yes, it would!** It would just take a longer time.
>
> Ms. M.: Kay, so you say yes. Double-Check, you're nodding.
>
> Double-Check: Yeah, I think it would, too.
>
> Ms. M.: And–and–what's your–why do you think you'd still get the same amount of time?
>
> Double-Check: Because if you're taking–it's…. **No matter wh[at]–how fast or slow it's going? You're gonna measure it out, so it's still gonna give you your answer**. It's still gonna be the same.
>
> Quick-Start: Around the same.
>
> Double-Check: No more than like a 5 minute difference. Or 2 minutes.

As the students struggled to describe their changing understanding, their speech was filled with stops and starts, half-finished words and sentences, and hedges like "or whatever":

> Ms. M.: Okay. What do you think of that, Jolt. We have two opinions going now. And I'm not sure which one you have.
>
> Twelve: I'm kind of like…with Double-Check and Quick-Start, I think the same.
>
> Jolt: I don't know. I think maybe yes, and maybe no [**staring, concentrating, hand on chin; sudden change of attention, sound of voice**]. Well, actually si[*nce*]–I'd say yes, because it's one big chunk of ice? So…the room temp…. So the heat isn't worrying about–so the heat isn't, like, working on two pieces of ice? It's only working on one solid piece? So the melting rate should be the same? But since it's bigger, the time would be a little bit different. About a few minutes or something.

Ms. M.: No, I'm saying suppose it DID melt really, really slow. Like it IS melting very slow. Because it's in a solid chunk. You're right, it–it could balance out, but let's suppose that it melted very, very slowly. Would that change your final answer?

Jolt: Maybe.

Ms. M.: You say no [to Quick-Start]. Why?

Quick-Start: Well, because. **No m[*atter*]–if–l[*ike*]–if–kay–if. No matter wh[*at*] how much…how big it is…or whatever…or what condition it is…in. You'd be able to get the drop per minute and you'd be able to… backtrack to when it started**. Kay, let's say our class got… what you gave us–like, uh… and then another class got a class full of big beakers or whatever? And you started them at the same time. They would all, like…people…. They would just be…. They would just backtrack different. Like you'd get different numbers.

It seems to me that this is a major breakthrough in understanding. I find the hesitations in Quick-Start's speech very interesting here. He seemed quite confident in changing his stance, yet he struggled to explain what lay behind his conviction.

While Quick-Start was talking, Jolt was glancing around the room, stretching, and seemed not be paying attention. But just as Quick-Start finished, Jolt sat up suddenly and pointed at me, talking fast:

Jolt: I'm with Quick-Start on that one! I'd say it would be the same, because the melting rate would be slower. For example, it take–it might take, um…. Maybe it's still 8 drops per mL? But the time to get 1 drop–maybe it's like 5 minutes just to make a mL instead of 3. So the same time you have to backtrack from the evidence that you have to get the same–to get the time. So I'm with Quick-Start.

…

Ms. M.: Okay. [to Jolt] Now… just a minute ago, you just suddenly –you leaned forward and you went [imitating gesture]. Like, did it–j[*ust*]–what just happened there?

Jolt: **I just remembered about the melting rate, the drops per mL, and the drop time. I just remembered that that's one of the big parts of the answer**.

(Class 4 Interview)

Both Quick-Start and Jolt visibly jumped when they suddenly realized why the times should come out the same. Three of the four students excitedly developed more detailed explanations for their insight and in doing so developed some interesting articulations of the significance of proportion.

Co-evolving Meaning and Language

It was interesting to observe evolving definitions of *ratio* and *rate*: In particular, Quick-Start and Jolt (respectively) used these words in ways that seemed to quite suddenly take on an altered but initially vague meaning. Double-Check also seemed to make the shift, although she spoke less; Twelve did not seem to understand the change the other three were experiencing. All four hung on to the idea that there might be a small difference in time, but they no longer attributed that difference to the notion that different melt rates would produce different start times.

As the students tried to more clearly articulate their insight, old language evidenced its limitations. Quick-Start was very vocal as he tried to articulate his shifting understanding. He seemed to recognize something in his deeper understanding that he struggled to capture in words; he kept coming back to the notion of *backtracking*, and he eventually connected *his* use of backtracking with “drop ratios or whatever”:

> Quick-Start: **And let’s say the other class did… I don’t know, really big… mounds.** It would take longer…. They’d get different **drop ratios or whatever**, but they would still be able to **backtrack** and get the same answer as we got. [emphasis added]
>
> (Class 4 Interview)

Here, I get the sense that Quick-Start was developing a different understanding of ratio. Until then, ratio was more of a procedure, but here it became a significant feature of the melting ice. He seems unsure whether the word *ratio* pointed to the distinction he was trying to make, hence his use of “or whatever.”

After exploring a hypothetical case, Jolt announced, “What I think is it’s the melting rate that makes all the difference” (Class 4 Interview). Later, he seems to use *melt rate* to describe both his old and his new understanding:

> Jolt: Can I say something? I think that we didn’t really consider the **melting rate** of the dr[*op*]–of the ice–the–of the two pieces of ice that we were considering. We were only thinking of the **melting rate**. We didn’t consider how fast or how slow it was melting. And the amount of time it takes to create a mL. [emphasis added]

Earlier, Jolt claimed he remembered that melting rate is a big part of the answer, yet he had been using some version of melting rate throughout. Here, he used two versions of in the same explanation! In noting that the group *only* considered the rate, not the time nor the volume of water, it seems that one version of rate had somehow lost its connection to *both* volume and time.

As with the lamp problem, I find it intriguing how much of the students’ understanding is implicit and how difficult it can be to understand their understanding.

Experiencing “I” As Separate From Brain and Body

Immediately following the initial breakthroughs, three of the students offered insightful reflections to describe the shift in their understanding. Interestingly, Double-Check separated conflicting understandings by referring to what her brain thinks as distinct from what *she* thinks:

> Ms. M.: Now what’s–what’s interesting to me is how… when–when you first came in here, it seemed very–
>
> Quick-Start: We had a totally different perspective.
>
> Ms. M.: Yeah! All of you had a different answer to that question. What do you think makes that original answer so convincing? Like, why did it feel like, um…a bigger–like a slower ice should give you a different answer?

> Double-Check: Probably because in the beginning when the ice is melting slower, you automatically–**your brain just thinks**, **"**Oh, it's going slower, so it's going to be a longer amount of time[29]," and you're gonna get a totally different answer. **So that's automatically what your brain thinks**. But then when we got into it, you start to think, oh, so if the drops/mL and the drop time is changing–is changing and it's all going back, you're gonna be able to trace it back the same way.
>
> (Class 4 Interview; emphasis added)

In fact, the common phrase, "I changed my mind" also describes this experience of separation–an *I* who controls the mind.

As they reflected on their *ahas* about melt rate and ratio, Quick-Start and Jolt described bodily feelings of sudden change; Double-Check described a more subtle and cautious shift that included a need to "check and double-check," which she said prevented a "jolt" like that described by the other two:

> Ms. M.: Can anybody… remember that moment where it suddenly… where your idea flipped? **[Jolt snaps his fingers]** I saw it in Quick-Start and Jolt. I saw the moment, but… how would you describe that moment where suddenly you changed your mind?
>
> Jolt: **I just felt a jolt in my body**, and, uh…. I just sort of felt **an impulse to say**, "I changed my mind!" [**shaking finger**; Twelve giggling]
>
> Ms. M.: You felt a jolt in your body? Oh, that's very interesting.
>
> Jolt: **Yeah, I felt a jolt in my mind that I need to–that my–that my answer is wrong**. And I felt **an impulse in my body to…um…say, "I need to say what my answer is wr[*ong*] changed"** and stuff.
>
> Ms. M.: That's interesting.
>
> Quick-Start: Well, I…I was kind of thinking about it…. I was like, well…. I did that
> and then…. **OH!!!** [**jumps up from head resting on arms on table to sitting bolt upright in chair**]
>
> Ms. M.: There you just did right there. Was that [laughs]…. Would you…. Do you agree with the jolt-in-your-body description?

---

[29] This seems like an interesting variation of Stavy and Tirosh's (2000) intuitive rule "More A-More B," a variation of the "Same A-Same B" rule I mentioned in Chapter 1.

Quick-Start: I was just thinking [Twelve quietly in background: It just pops up in your mind] and then I was just like, **"Oh, that makes sense! That makes sense!"** [**pitch rising as he repeats his statement**]

Jolt: I just had **a funny feeling and then an impulse to say that I changed my mind**.

Ms. M.: Now interesting–you said two things–you said that you felt a jolt that it was wrong and then you felt an impulse to change it.

Jolt: Yeah.

Ms. M.: Now, Quick-Start, was your–what–when you did the jolt, was it–was it a jolt that the new answer was the jolt or the old wrong answer was the jolt, or both?

Quick-Start: Both. Kind of. Yeah, both.

Jolt: For me, **I felt a jolt that it was wrong and I felt the impulse of the right answer**.

Ms. M.: Okay. And what about Double-Check? Did you…. I didn't see the…. I didn't…I did…. With those two, I saw the moment. I didn't see it with you.

Double-Check: Um…. Usually, I just think about it, and then think about it and then I realize that I'm wrong, and then I just…. I don't immediately think, "Oh! I'm– this is how it's going." **I think about how it's right and double-check it and triple-check it and make sure it's right**.

Ms. M.: So you were still double, triple-checking, and it wasn't quite such a sudden–

Double-Check: **I realized it? And then I wanted to make sure** that it's–cause sometimes you can realize something? And then **you can go on and on about it thinking you're so right, and then realize, "Oh. I forgot that**." And then you're like, "Oh no."

Ms. M.: So right away there was something…. The double-check [Double-Check nodding] came in immediately.

Double-Check: **My mind likes to double-check things**.

Ms. M.: Okay. Twelve?

Twelve: Um…well…. Like, hearing all their different ideas and, like, them explaining it? It kind of makes, like, sense once you, like, know…like, what they're talking about? And… so it all just makes sense. And then it just kind of comes to you, and you're like, oh right, they're right [rolling eyes]. And. So.

Ms. M.: So for you, there wasn't that sudden moment of–

Twelve: No [smiling]. I guess not!

Quick-Start: Oh my God! It's recess already.

(Class 4 Interview; emphasis added)

There is some evidence that Double-Check may have had a clearer sense of ratio early on, so it is also possible that the change was not really so much of an *aha* for her, but her stated need to double and triple-check things is consistent with the manner she seemed to approach her work on all of the problems. I am not sure Twelve ever appreciated how different melt rates could backtrack to the same start time. She was reluctant to talk about her understanding, and I did not want to press her with too many questions when she already appeared uncomfortable.

The Consecutive Integer Problem: Connecting Objects

I chose to include the consecutive integer problem in my description and analysis because here the implicit was central to creating and connecting mathematical ideas and to considering the extent to which those ideas might be generalized. Here, attending to implicit understanding gets right to the heart of mathematical thinking.

On Day 1 of the problem, I presented the following prompt:

Part 1: Some numbers can be expressed as the sum of positive, consecutive integers. Which numbers have this property? For example:
9 = 2 + 3 + 4
11 = 5 + 6
18 = 3 + 4 + 5 + 6

Part 2: In how many ways can any given number be expressed as the sum of positive, consecutive integers? For example:
21 = 1 + 2 + 3 + 4 + 5
21 = 6 + 7 + 8
21 = 10 + 11

Are there other ways? Can you find a way to predict how many ways any given integer could be so expressed?

(from Mason, Burton, & Stacey, 1982, pp. 68, 182)

Most students focused primarily on Part 1 for the entire duration of our work (six forty-minute classes spread over about a month and a half). All students started by working their way up from one, looking for a single method for each number, and thereby showing which numbers "worked" (i.e. could be written as the sums of consecutive positive integers):

1: (no solution)

2: (no solution)

3: 1 + 2

4: (no solution)

5: 2 + 3

6: 1 + 2 + 3

7: 3 + 4

8: (no solution)

9: 4 + 5

10: 1 + 2 + 3 + 4

11: 5 + 6

12: 3 + 4 + 5

13: 6 + 7

14: 2 + 3 + 4 + 5

15: 1 + 2 + 3 + 4 + 5 OR 4 + 5 + 6

16: (no solution)

(etc.)

By the end of the first day, everybody seemed to recognize that all odd numbers (except 1) could always be expressed as the sum of two consecutive, positive integers (or CPIs).

Finding a Doubt Space

There were important differences in how students described this result: Three groups noted that you could keep extending the series 1 + 2, 2 + 3, 3 + 4, 4 + 5, etc. to describe sums for all odd numbers, while two groups described a method that allowed them to find the addends for *any* odd number: "When any odd number is split into half, it gives you the pair ½ above and below" (i.e. the integers that are 0.5 above and below the middle are consecutive and positive). During class discussion, I asked all students to test a random large number to see which method would be more helpful in finding the addend pair, to explain *why* this method *must* work for all odd numbers, and to consider whether their method might also work for even numbers. Moving beyond addend-pairs that sum to odd numbers, one group noted that *most* multiples of three, five, and seven could be written as the sum of consecutive positive integers; at first, this was based on extension of the following patterns:

1 + 2 + 3 = 6	1 + 2 + 3 + 4 + 5 = 15	1 + 2 + 3 + 4 + 5 + 6 + 7 = 28
2 + 3 + 4 = 9	2 + 3 + 4 + 5 + 6 = 20	2 + 3 + 4 + 5 + 6 + 7 + 8 = 35
3 + 4 + 5 = 12	3 + 4 + 5 + 6 + 7 = 25	3 + 4 + 5 + 6 + 7 + 8 + 9 = 42

Here, they easily (and without prompting) generalized their methods from three to five to seven addends. But why *most* multiples of three, five, and seven (unless perhaps they were noting that the smallest multiples of each seem to be missing from the lists), and why were they satisfied to stop at seven? Also, until I questioned whether 330 could be written as the sum of three integers, they did not look for a method that would work without counting up from known examples.

Once thus challenged, someone determined that you could divide a multiple of three by three to find three equal addends, then subtract one from the high number and add it to the low number (i.e. 330 = 110 + 110 + 110 = 109 + 110 + 111). Unprompted, few (if any) considered whether a similar divide-and-redistribute method would work for multiples of other odd numbers. As they worked with these ideas, it seems that a general notion of *balancing* or *making up for* contributed to a *sense of* a more general case that applied to both odd numbers and multiples of three. Note the hesitations, the statements of inadequacy, the qualifiers ("pretty", "or something"), and the use of a single example the speaker hopes the listener will allow to stand for a whole class (i.e. 4 + 4 + 4 = 12):

> Six: It's because, um, when you divide a number by 2, you get a half? And that half time[*sed*]–by, um–added by itself? Would equal the number? So–but, so if you go one below that number and one higher, it's gonna be the same thing, because…like…the…. Like, for this one, it's 31.5? If you go up .5 ahead? And .5 below, they both…**make up for each other?** So they make up for the .5. **I don't know how to explain it.**
>
> Ms. M.: Do you understand what he's saying?
>
> Seven: So it's like **one point is going higher and one point is going lower**. So it's like when you said [**gesturing quote signs with fingers**] the **losing profit or something**. Losing profit, so…. One number is losing a profit, one is gaining, so when you add it together, it **balances out**.
>
> [a little later, while discussing multiples of 3]
>
> Six: I don't know. [pause] Ah, **it's pretty**…. It's the **same thing as when you divide an odd number by 2? 4 + 4 + 4 equals, um, 12?**
>
> Ms. M.: Okay. So if we–just let me follow you along here. So if you did 4, 4, 4, we–you know you'd get 12, right?
>
> Six: So if you go one high–if you go one below? Again you have to **make up for that**, so you have to go one higher? And that–in another place?
>
> (CPI Class 1 Interview; emphasis added)

Throughout the six weeks we spent on the problem, I encouraged students broaden the scope of applicability of their conjectures; sometimes, they quickly saw ways that a particular idea might apply more broadly, while at other times what seemed to me obvious connections were not experienced as such by the students. At the beginning of each class, I gave students an updated list of conjectures / examples that showed the ongoing evolution of various groups' ideas (see Appendix C for the final summary). I encouraged them to note which they agreed / disagreed with, whether those ideas were always true, and to extend the list as necessary.

There were occasions when students inappropriately generalized with high certainty. One student formed a long series of seemingly ad hoc conjectures that seemed to have little point *that I could see*; I take this up in the section on forced movement. I now wonder if this student might have collected all of his notions under a vague conjecture that emphasized the significance of odd and even numbers of odd and even addends.

The students used or invented a variety of categories to describe what worked and did not work; these included odds, "four-numbers" (i.e. sums of four consecutive addends, as in 1 + 2 + 3+ 4 = 10; 2 + 3 + 4 + 5 = 14; 3 + 4 + 5 + 6 = 18; keep going up by four to get 22, 26, etc.), "six-numbers" (i.e. 21, 27, 33, …), and eventually multiple odds. Again, had I helped direct attention more broadly toward the emerging significance of odd and even numbers of odd and even addends, we might have more systematically explored various combinations and might have had less trouble connecting the many diverse explorations and mathematical objects that emerged (such connections become an important topic of discussion in Chapter 5). Nonetheless, muddling through the profusion of categories was central to our experience. In the story that follows, I focus primarily on the evolution of *powers of two* as a mathematical object; others are included in Appendix D.

Empathizing With Students' Doubt

Most groups recognized a pattern when they found that two, four, eight, and sixteen were the first four numbers that they could not find a way to write as the sum of consecutive positive integers. I introduced *powers of two* as a name for these numbers, but this seemed to be new to most of the students. To them, doubling the previous number in the list, not multiplying together a string of twos, generated the sequence. Nobody seemed particularly interested in figuring out *why* powers of two were excluded: The group that first posted this pattern noted that it was “unexplainable totally.” Even much later after *all multiples of odd numbers* was proposed as a concise way of stating what works, many students did not see a connection to the 2-4-8-16 sequence as what was left. I pushed them to consider what multiple-odds omitted and how these might be connected to the missing powers of two that so far we had no explanation for. One group of students noted that as the powers of two continued, the gaps got very large; they were sure there *must be* (in the weak sense) others that did not work between 128 and 256 and dedicated much of one class to finding a counter-example. Even if students had recognized the 2-4-8-16 sequence as powers of two, however, I doubt most students would have realized that this was the *only* way those numbers can be written as the product of prime factors or that having only twos as prime factors *necessitates* the absence of odd factors.[30]

As with my reasons for refuting one-dollar profit in the lamp problem and my rationale for why starting ice should not matter in the ice melt problem, my understanding of the connection between odd factors and powers of two remained vague and unarticulated until I took the time to consider it more deeply. My exploration was prompted and informed by my conviction that odd factors were important and that powers of two *must not* (again, in the weak sense) have any odd factors. After all, I had already explained (to myself) the importance of odd factors in terms of (a) their allowing a middle addend (which indicates an odd number of addends that can be rearranged

[30] A teacher who read a draft of this document noted that this resonated with her experience as well–that it was “huge” for her. She further noted that it prompted her to test whether there were other ways to write factors for 16, and suggested that I use this as an example; e.g. “16 = 2 x 2 x 2 x 2, but can it be written another way? Why not?”

around the middle addend to form equal addends) or (b) the way they emerge from odd pair-sums, starting with a middle pair of (consecutive) addends (one even and one odd, meaning the sum is always odd, as is each pair surrounding it).[31]

In fact, despite having done plenty of factoring, making prime factor trees, and finding lowest-common-multiples and greatest-common-factors (which had become narrowly-defined objects in their own right) in my own elementary school experience[32], unique prime factorization is something I explored for the first time as I attempted to solve the consecutive integers problem. I did not find it difficult to convince myself of its truth, but it was something I had never had occasion to think about before. Even with the long hours I had spent on this problem, I had set aside my issue with the powers of two to focus on other aspects that I was more uncertain of; after all, I fully believed that powers of two *were* excluded; I just had a niggling doubt that there were holes in my justification for doing so. While it can be useful (and necessary) to set aside contingent understanding (Zack & Reid, 2003; 2004) to allow focused attention in other areas, revisiting this niggling doubt was very fruitful.

Later, as I struggled to understand Euclid's proof for the infinitude of primes (Devlin, 2000), I discovered that unique prime factorization is important enough to have been dubbed "The Fundamental Theorem of Arithmetic"! I wondered why this was the

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[31] i.e. either:
(a) The number can be written with an **odd** number of consecutive addends (which can be rearranged so that each addend matches the middle addend).

3 + 4 + *5* + **6** + 7 + 8 + 9 = 42
6 + 6 + 6 + **6** + 6 + 6 + 6 = 42

(b) The number can be written with an **even** number of consecutive addends (which can be rearranged as equal pairs of odd numbers).

3 + 4 + *5* + **6 + 7** + *8* + 9 + 10 = 52
4 x 13 = 52

Both methods require an odd factor:

- For (a), you need a middle number, which means you need an odd number of addends.
- For (b), each pair of addends always adds to an odd number (the middle 2 are consecutive, meaning one is odd and one is even; the surrounding pairs also always include one odd and one even).

[32] Notably, they were sometimes objectified *as* LCMs and GCFs, as opposed to lowest-common-multiples and greatest-common-factors; i.e. the meaning of the acronyms became lost in the procedures I used to find them.

first time I had found a need for something so fundamental. Curious, I emailed the following questions to a fellow math teacher:

> *Does anything come immediately to mind in response to either of the following questions?*
>
> *Which numbers have no odd factors?*
>
> *Which numbers are never multiples of odd numbers?*

To my relief, she replied that nothing jumped to mind. I suspect that this just is not how we typically think of powers of two in school. In fact, this definition of powers of two seemed remarkable to me; I had spent many hours with the consecutive integers problem on a number of occasions over a span of six years. From the beginning, I had noticed the absence of powers of two, determined that every odd factor produced a solution, and identified multiple odds as significant.[33] But significant links between these ideas were missing.

Again, it seems my doubt re: powers of two had more to do with my lack of intuitive and immediate feel for *why* powers of two can never have an odd factor. After all, other even numbers can be written as products of odd and even numbers (e.g. 10 = 2 x 5). What makes powers of two so special? I did not experience *never having an odd factor* and *powers of two* as the same for a long time: The necessary *complementarity* of multiple odds and powers of two had never come to my attention. Now, this seems so

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[33] However, it took a long time before I could clearly articulate my already-strong conviction that each odd factor produced a *unique* solution. I knew that an odd factor could be used to produce an odd number of addends that could be rearranged into consecutive integers–even if this involved negative integers cancelling with their additive inverses, and even if the remaining number of addends was even. But I also knew that I could use an odd factor to make a *pair* of consecutive addends in the middle of the sequence, then build out from there; e.g.: 35 = 7 x 5, so dividing by 5 to get 7, I could find:

35 = 7 + 7 + 7 + 7 + 7 = 5 + 6 + 7 + 8 + 9

35 = -1 + 0 + 1 + 2 + **3 + 4** + 5 + 6 + 7 + 8 = 2 + 3 + 4 + 5 + 6 + 7 + 8

I recognized that the latter method ended up with 7 addends (the same result I would have got by dividing by 7 (i.e. 35 = 5 + 5 + 5 + 5 + 5 + 5 + 5), but I didn't at first see why these were *necessarily* complementary in that manner. And who was to say there wasn't yet another method?

obvious that I am embarrassed to admit it was ever a stumbling block![34] But being able to state the following *was* somehow a big enough leap that I felt it worthy of note:

> Numbers with no odd factors must be powers of 2; every other number has at least one odd factor. Prime factorization makes this clear: If you put anything other than a 2 or a power of 2 in a series of factors, the factors must include at least one odd.[35] (Research Diary)

Nonetheless, I doubt *hearing* the same words early in my work on the problem would have had much impact on my understanding.[36]

Experiencing Mathematical Objects

Recognizing this connection was also problematic for the students. One eventually figured out that all even groups of evens are excluded; she was momentarily stumped as to why 24 (as 6 x 4) and 12 (as 2 x 6) worked, but then recognized that these numbers can *also* be expressed in terms of an even number multiplied by an odd number (i.e. 24 = 3 x 8; 12 = 3 x 4). She recognized that numbers that do not work can *only* be expressed as the product of evens. At this point, however, she did not connect this recognition to *powers of two*.

The following dialogue demonstrates one student's efforts to more fully justify his conviction that powers of two can never be written as consecutive positive integers. It would have been easy to simply accept *no odd factors* as an appropriate definition of powers of two, but notice his hesitation and how, when questioned further, he allowed new language to come forth to express his deepening understanding:

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[34] Later as I explored which numbers can be written as the difference of two squares, I also discovered that I had somehow partially conflated square numbers and powers of two in my mind. On a conscious level, certainly I knew that square numbers are the sequence $\{\ldots(-2)^2, (-1)^2, 0^2, 1^2, 2^2\ldots\}$ and that powers of two are the sequence $\{\ldots 2^{-1}, 2^{-1}, 2^0, 2^1, 2^2\ldots\}$. But in my experience, it's not the conscious definitions that tend to interfere. There is more to each sequence than the list of numbers they contain.

[35] Using "complementary" to describe the relationship between multiple odds and powers of two occurred even later, and it was very satisfying in that doing so took something that I felt clearly but had struggled to articulate and made it concise and complete.

[36] Here, a teacher reader commented, "YES YES YES! I'm reading and understanding but not convinced!!!"

Ms. M.: Do you have any reason why, then, the powers of 2 would NOT work?

Prime: Cause they've got no odd, um…factors.

Ms. M.: And are you sure of that?

Prime: Yes.

Ms. M.: Like, how do we know that there's never a power of 2 that has an odd factor?

Prime: Because, um, they're powers of 2, so you times it by 2, so there's no odds... in it [**voice trailing off**].

Ms. M.: So…. Two…. What do you mean you times it by 2? How do you–

Prime: **I'm not actually sure… just, the way it is, is all the numbers that you have–any odd number when you times it by 2? You would, um…. It equals…an…even**.

Ms. M.: But those ones still work. Odd times even still work, right?

Prime: Yes.

Ms. M.: Like 30…works.

Prime: Yes. But in this…. There's no odd. **I'm not actually sure where I'm going with this**. I know there is no odd…there is no odd factors? Cause it goes…. [writing, talking to himself]. These are…. Say with 64. Factors of 64 are 1, 2, 4, 8, 16, 32. **The numbers before are its factors, so.** So 128…. 1, 2, 4, 8, 16, 32, 64, 128.[37]

(Class 5 Interview; emphasis added)

In the last line, I think Prime recognized something new about the structure of powers of two. Given more time, his understanding likely would have deepened further. I think it is significant that he ended the sentence with a conclusive "so" (rather than a tentative or

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[37] Interestingly, even after all my work with summing consecutive positive integers and even after the surprising *aha* of realizing that "all multiples of odd numbers" is an exact complement to "powers of two," I was recently *momentarily* surprised to realize that the factors of 64 are all powers of two! I was working on a different problem (trying to find a number with exactly 11 divisors). In a flash, it was obvious, but I find that even such momentary hesitation points to the sort of mental stumbling block that can greatly interfere with my ability to hold everything I need in consciousness when solving a difficult problem. Resolving even minor doubts seems to greatly enhance what I am able to work with.

trailing “so….”). To me (likely because I use it this way), this seems to indicate that the words are inadequate–that further explication is needed but not immediately forthcoming; it implies a hope that the listener will recognize the non-verbal meaning called forth by the words, so that I do not have to find better ones. I might follow it with, “Do you know what I mean?” In my own reflections, I referred to this as “pointing and hoping.”

Few students pushed their thinking even to this level; in their final summaries, many indicated that they were pretty sure that powers of two would never work, because they had been unable to find any that did. Some noted that powers of two have no odd factors, but nobody offered an explanation for how they knew this. A few still confused powers of two with the larger set of even numbers, perhaps conflating two sorts of doubling.

Thwarting Awareness of the Implicit: Extreme Skepticism

As with the lamp problem, reflecting on my role as teacher-researcher helped me identify ways that my own actions impeded attention to implicit understanding. During Class 4, Felix spent a good deal of time considering which even numbers can be written as the sum of consecutive positive integers. Initially, he claimed that 402, 408, 502, 508, 602, 608, “etc.” would not work. I am not sure what he considered the defining characteristics of this set, but he did say that 402 breaks into two equal numbers (201 + 201) that cannot be rearranged. But this structure applies to all even numbers, and had he been thinking only of even numbers, I do not think he would have chosen the examples that he did. In any case, I countered that 402 is also a multiple of three and should therefore work. When he asked how I knew it was a multiple of three, I told him that since 4 + 2 = 6, and six is a multiple of three, 402 must also be a multiple of three. Since this rule seemed unfamiliar to him, I also noted that 133 + 134 + 135 = 402. This was the start of a long series of ad-hoc revisions and counterexamples in which overgeneralizations of the divisibility rule for three seemed to play a (likely subconscious) role. A summary of our arguments follows:

> Felix: All multiples of four must have four addends, and all multiples of five have five addends. (He said he was totally sure of this.)
>
> I presented a counter-example.
>
> Felix: The rule only works if the sum of the digits is even; i.e. 508 cannot be written as the sum of consecutive positive integers, because 5 + 8 = 13.
>
> Ms. M.: 60 + 61 + 62 + **63 + 64** + 65 + 66 + 67 = 508 (In presenting this, I marked off pairs to indicate 4 groups of 127.)
>
> Felix: You could take any number, add the digits, find the factors of the sum, and use the factors to find solutions. (Again, he said he was 100% sure this would work and presented 408 as an example: 4 + 8 = 12; since the factors of 12 are 1, 2, 3, 4, and 6, 408 can be written as 2, 3, 4, or 6 addends.)
>
> Ms. M.: 408 is even and therefore cannot be written as the sum of 2.
>
> Felix: 808: 8 + 8 = 16; 16's factors are 2, 4, and 8, but the 2 does not count.
>
> Ms. M.: 43 + 44 + 45 + … + 58 = 808 (Here, I marked off pairs to indicate 8 groups of 101.)
>
> Felix: 404 = 4 + 4 = 8; 8 ÷ 2 = 4; the answer equals the last digit [i.e. 404 can be written as the sum of 4 consecutive positive integers]; this [?] is also true of 808: 8 + 8 = 16; 16 ÷ 2 = 8.

Felix's conjectures in this sequence of exchanges greatly surprised me; in many ways, he seemed to have a good understanding of work that had been presented so far. He often made insightful comments during class discussion. But at this point, Felix's original (partly implicit) idea seems to have been lost in the revision process. He took part in an interview immediately following class, and I asked him to sum up his conclusions from his work that day. He replied:

> [Y]ou add up the digits, and then you divide. You divide it by whatever factors it has. And then that's what–that's how many addends you would have. But then we found out that 2 wouldn't work. And…so…. (Class 4 Interview)

Again, he tried to prove that if a number is divisible by four, it should be possible to write it as the sum of four addends but got stumped when he tried to do this for 12. At this point, he recalled more from that day's class: "Remember what we said about in…if you have an even number, you can't have a middle number? So it doesn't really work?" Rather than asking him to elaborate, I again narrowly focused his attention by asking whether it would be possible to use a number of addends not evident in the factor list:

> Felix: Because that's the numbers that are…can–can be multiplied to get 12. Because if 5 doesn't get 12 [i.e. if 5 isn't a factor of 12], then it's impossible that [?[38]] will try to get–it's impossible that 12 can be expressed in 5 numbers. (Class 4 Interview)

Here, it seems he was overgeneralizing his observation that multiples of three can be written as the sum of three addends, multiples of five can be written as the sum of five addends; every odd factor allows a solution. But even factors do not produce solutions with a matching number of addends; e.g. four-numbers (or double-odds) have four addends but are not multiples of four, and multiples of four cannot be written as the sum of four addends.

Following this, Felix returned again to his example of 408 and reiterated his persistent belief in the significance of the sum of its digits for generating factors, which in turn could generate addends that could be rearranged to form consecutive positive integers. Since 12 (and therefore its factors) do in fact all divide evenly into 408, I suggested we work with 808, since 808 is not divisible by 16 (i.e. the sum of digits). He agreed, but easily dismissed it as a counter-example by saying that the last factor does not count: "So no wonder, that's why."[39] I continued with a series of other numbers that I hoped would serve as counter-examples, but each time he rejected them on the basis of some particularity of the example. Finally, he again declared that numbers that are

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[38] (inaudible)
[39] Apparently I hadn't noticed that in all of his previous examples, he excluded the number itself from the list of factors. Given this, 808 was not a particularly helpful counter-example.

divisible by four all work and can be written as the sum of four addends. He still did not offer any examples to support this, and he did not seem to see a connection to work he had done earlier in the day to figure out why the non-working powers of two (all of which are multiples of four) skipped so many numbers as the numbers got bigger.

Even so, when I asked him to re-focus the investigation (in an attempt escape our back-and-forth conjectures and refutations), this is the very work he turned to. He was emphatic that no powers of two work, but he suspected that there were additional non-working numbers between them–particularly between 128 and 256, which seemed to him like too big a jump to have no non-working numbers. Then he talked about multiples of three, claiming that all can be written as the sum of three addends. At first he used a counting-up argument (i.e. if 1 + 2 + 3, 2 + 3 + 4, 3 + 4 + 5, etc.), then explained that multiples of three also allow you to divide by three and rearrange. Then he returned to his argument that multiples of four can be written as the sum of four addends. This time, I asked him to test it, and he quickly refuted his own idea:

> Felix: Okay. So, 8 (writing) 3, 4, 5…. No. 2 + 3 + 4 = 9. 1 + 2 + 3 = 6…. Yeah, I don't think that works, then. Oh, wait! But these are 4 numbers! Sorry. 1 + 2 + 3 + 4… [adding out loud] 7, 9, 10…. No. Doesn't work. (Class 4 Interview)

Just as quickly, he created another overgeneralization:

> Felix: No. Doesn't work…. Cause this is an even number, right? So… I guess… another rule would be, if it's an even number, it cannot be written in 4, because it doesn't't' have a middle number.

I reminded him that 10 = 1 + 2 + 3 + 4 (which he had just tested), and he simply said "Weird." It is true that this series of consecutive integers cannot be generated by dividing 10 into an even number of addends and then redistributing around a middle number, but he did not consider other possible methods when making his statement. The dissonance indicated by his surprise would likely have been worth pursuing; something was clearly conflicted in his implicit understanding of the situation.

But the interview was over, and rather that direct attention to this dissonance, I asked him to summarize his work so far and continue testing his ideas. In doing so, he reiterated that the only way you can get an even number is by the sum of an odd number of addends, then re-affirmed that all odd multiples work because they have a middle number. I commented that today he had made a variety of conjectures that had been falsified with counter-examples, and I suggested that testing his own idea might be a good place to begin during our next class.

At the beginning of the next class, he restated his idea: “The only way you can get an even number is by the sum of an odd number of addends.” Again, rather than having him explore this idea more deeply,[40] this time I countered with 2 + 3 + 4 + 5 = 14. He said that this is not really what he meant and revised his wording: “An odd number of addends will always equal an even number.” I countered with 4 + 5 + 6 + 7 + 8 = 29. Jolt (his partner) then suggested, “An equation with an odd number of odd numbers has an answer that is a consecutive number.” I pointed out that an equation with an odd number of odd numbers (i.e. of equal addends) would always be an odd number and that we had already agreed that all odd numbers work. Here Jolt offered yet another opportunity to consider their thinking about the problem more broadly; in naming both the oddness or evenness of the addends and the oddness or evenness of the number of addends as significant, these could have been varied and explored more systematically. Had I been more alert to the possibility in his statement–or more generally to the possibility of allowing *something to do with…* to open a deeper investigative space–this

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[40] His idea makes sense if you consider that each odd factor can produce a string of equal addends that can be redistributed. Sometimes, the string extends to include zero and possibly negative integers. The zero and the inverse pairs sum to zero and leave behind an even number of addends. For example:
14 ÷ 7 = 2, so
2 + 2 + 2 + 2 + 2 +2 + 2 = 14, which rearranges to
-1 + 0 + 1 + 2 + 3 + 4 + 5 = 14, which can be seen as
2 + 3 + 4 + 5 = 14
He hadn’t explored this yet, but attending to the conjecture as he stated it rather than simply refuting it may have allowed him to look at what was *right* in his thinking.

could have opened a very rich investigation. In this moment, I again chose to focus narrowly rather than broadly; having recognized this, I might now choose otherwise, even if I were not fully aware of the significance of oddness and evenness (which requires some level of familiarity with the problem). I take up this idea in greater detail in Chapter 5 in my discussion of defining and connecting mathematical ideas.

In reflecting on my dialogues with Felix, it seemed that he generated many, many creative ideas, while I primarily played the roles of skeptic and focuser; i.e. I searched for counter-examples and counter-arguments, and I asked how the results contributed to the solution of the broader problem. While seeking counter-examples is important, I now wonder whether my role as extreme skeptic prevented Felix from more clearly articulating beginning ideas that could have been better developed before being subjected to such scrutiny. In particular, I wonder now whether a vaguer, less certain notion that encompassed *all that*, might have dwelled beneath Felix's shifting assertions. I wonder how he might have proceeded had he claimed (and had I given him the space to do so) that the solution had *something to do with* whether the number of addends is even or odd and / or *something to do with* whether the addends are even or odd (as Jolt started to do above). Despite my stated intention to honor the implicit and vague, this is not something that I adequately encouraged in this exchange. On the contrary, my own confrontational approach (though he seemed to enjoy the banter) may have prevented Felix from interacting with his own emerging understanding.

Craving Elegance

Considering experiences of doubt and certainty alongside the notion of mathematical elegance provides interesting insight into both, particularly when elegance is considered in terms of an absence of conflicting understandings (often implicit) that seem to be so much a part of experiences of doubt. I say more about this in Chapter 5;

for now, I maintain the narrative of how understanding evolved in the context of the consecutive integer task.

Why Only Evens?

During the final interview, one group further explored the relationship between powers of two and odd factors. One confidently asserted that powers of two would never work because they have no odd factors, and both Prime and One indicated strong agreement with a conjecture that Twelve and Double-Check had shared during that day's class: *Odd factors always produce a solution for a particular number*. But there still seemed to be some confusion regarding why some numbers *could* be written as the sum of an even number of addends:

> Prime: You have to divide it by an odd number to get…um…the amount of num[*bers*]…to…get the amount of numbers that you need for…to add it up? Um…. Cause sometimes you *can* add it up with 4 numbers? **But that's a lot harder to find than 5 numbers. Cause there is no middle… number.**[41]
>
> Ms. M.: And in class today, there was an example where…. You said 6 +7 + 8 + 9 equals…
>
> Prime: 30.
>
> Ms. M.: But in fact…in that case–
>
> Prime: That's 15. It's…it's 2 + 3 + 4 + 5 +…. Sorry. It's…-5 + -4 + -3 + -2 + -1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 +7 + 8 + 9. I think that's 15….
>
> Ms. M.: So…. Even though you ended up with 4 addends [i.e. 6 + 7 + 8 + 9], you got there by dividing by an odd factor.
>
> Prime: Yeah.
>
> (Class 5 Interview; emphasis added)

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[41] I wonder now if persistent use of the term "middle number" (as opposed to middle addend) may have created some confusion. I have caught myself subconsciously partially conflating the *middle* of an odd number with the middle *addend* of a multiple odd. Even more strangely, having articulated this, I'm not even sure what I meant by the middle of an odd number; technically, it would be something-point-five, but the vague idea seems more like the number in the middle when you list 1, 2, 3, 4, 5… up to the odd number in question! And that makes no sense at all. A key point here is that the confounding "middle-of-an-odd" was never a clear concept at all.

I reminded them that Double-Check had specified that her method should only work for even numbers, and Prime suggested we try it with 45. He suggested that her use of even was likely just because we already had a rule for odds; i.e. we could write any odd number as the sum of two addends.

Prime also commented that although every odd factor would produce a solution, it was also possible–but harder without a middle number–to find solutions with an even number of addends. Apparently, despite his own example and my attempt to draw attention to it, he had not considered that dividing by an odd number *can* yield an even number of addends (i.e. when the zero and the additive inverse pairs are removed from the sequence).

Every Odd Factor… Except Itself?!

While odd factors allow an odd number of addends–and therefore a middle addend–that can be redistributed, sometimes this requires negative integers, which are not allowed. As this group tested each factor of 45, Prime had another insight. Since 45 is an odd factor of 45, he reasoned he should be able to write it as the sum of 45 consecutive positive integers and proceeded to do so: He put a 1 in the middle (since 45 divided by 45 is 1) and wrote out 22 addends on either side. At one point, I suggested that he simply add/subtract 22 from 1 rather than write out the whole list, but he insisted on writing each number:

~~-21 -20 -19 -18 -17 -16 -15 -14 -13 -12 -11 -10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0~~ ~~**1**~~
~~2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21~~ 22 23

As he cancelled the additive inverses, both Prime and One indicated surprise:

Prime: (long, low whistle; pause) Negative 21, so, 22 and 23.

Ms. M.: So you're left with only 22 and 23?

Prime: Yeah.

One: Wow.

(Class 5 Interview)

That all odd numbers could be written as the sum of two consecutives located just above and below the half-way point was one of the very first insights that had emerged in response to this problem, yet here it was again, arising in a completely different fashion!

As the others examined and discussed his work, Prime suddenly sat up and interrupted the conversation with an emphatic: "Wait!!" He continued (apparently in response to a question of his own):

Prime: Actually, yes! It works for all these num[*bers*]–well, cause I said add 2 numbers? When you take the exact numbers, it gets to 2 numbers, 22 and 23.

Ms. M.: And is that just in this case? Or will that always happen that div[*ide*]–doing this big, long thing will end up with the same 2 numbers that you would get–

Prime: **I think it would, cause when you divide a number by… itself, it gets to 1…. And then 1, 0 is the two–is two numbers that [gestures to indicate placement of numbers on an imagined number line] (?) two negatives that get (?).** So let's just try this…for 15. So 1…. There'll be 7 on each side, so…. [counting, writing while others watch]

…

Ms. M.: …. So would it HAVE to work every time?

Prime: Yeah, I'd say so, cause what's happening? Is here's the number… here? So…. **You've got the middle number? And this is immediately going up into positives? So you've got 2? But here it's gotta go zero, THEN negative one. So it's a jump of 2? (?) just a jump of 1 to start going up. Jump of 2 to get rid of 1, jump of 1. To…circle. So it's like…jump from 2…. 2 to get to the first negative, so 2 numbers left.**

(Class 5 Interview; emphasis added)

As he struggled to articulate his insight, I suggested they each test a number. They worked independently for a few minutes, and I helped those who were not clear about Prime's intent. Prime continued to talk out loud as he worked, then announced "Yep," and sat up:

> Ms. M.: (to Prime) Are you convinced that–that–you know–last time you said it–it's always–the ze[*ro*]–because of the zero and the…negative 1, was it?
>
> Prime: Yeah.
>
> Ms. M.: It's always going to leave 2?
>
> Prime: **Uh, it's not really cause the zero AND the negative one? It's because there's 2 jumps? Till…the negative 1.** [Ms. M.: Okay.] So two numbers left. Well that's the reason why I'm thinking of it. [to himself] Let's try…. [appears to be testing another case]

As they continued testing, Prime, Twelve, and One showed growing confidence that the pattern would always hold. But they found it difficult to articulate their understanding and explain their conviction:

> Ms. M. [to One]: Would that HAVE to be true that you would ALWAYS be left with 2 numbers?
>
> One: Mooost of the time (smiling).
>
> [pause]
>
> Prime: Yep. So I think this method works for odds AND evens. **The–all the odd numbers and (?) are how many times you can do it.**
>
> Ms. M.: So every odd factor will give you one method.
>
> Prime: Yeah.
>
> …
>
> Ms. M.: And do you–what do you [Twelve] think, like, would that–is there a reason why there's always 2 left when you do that?
>
> Twelve: **Uh, well…. It starts at 1, so…. Positive…. It's always going to be…2 steps higher than the…negative.**
>
> Ms. M.: Which would mean that there's 2 left that don't cancel?

Twelve: Yeah.

...

19: **There's a zero (pointing to his work). And that just… (?) one number up here. (?) an extra number.**

Ms. M.: Which means there's always 2 left?

One: Yeah.

…

Prime: **The reason why this…. I–the reason why, um… evens don't work? Is the fact that… there is no middle number for them. Cause… for 10 it'd have to have 5 on either side, so. It wouldn't really work. Say you divide by…it by 10? Then it'd be 1? But it wouldn't be equal on the other side. It'd be like 3 (?) and 4 (?)**.

Ms. M.: And yet 10 does work.

Prime: **Yeah. But not this…. But not by itself. Cause this would mean…. Cause if you did use this method? Divide by itself? Then you would get 2 numbers. And 2 numbers can never equal, uh, an even number, so**.

Here, the phrase "evens don't work" meant "evens don't work with this method," and Prime seems clear about this. Notably, Prime was the only student who had been bothered that the number itself did not (seem to) count as an odd factor that would allow a solution. When this method produced the same result as dividing by two to find the two consecutive integers surrounding the middle of an odd, both he and One were visibly excited, and Prime was motivated to more clearly articulate the emerging insight and to consider whether it would apply to *any* odd number. By this time, most students had lost interest in the problem.

Summary

Having recounted a series of stories from the classroom, I turn now to a more analytical consideration of how experiences of doubt and certainty can provide important indicators of implicit understanding. I further consider how learners (including me as the teacher / researcher) might interact with their own and each other's tentative understanding as it emerges from the implicit and how doing so allows the creation and elaboration of mathematical objects.

5. Attending to Implicit Meaning

> In this context there is not a human word, not a gesture, even one which is the outcome of habit or absentmindedness, which has not some meaning. For example, I may have been under the impression that I lapsed into silence through weariness, or some minister may have thought he had uttered merely an appropriate platitude, yet my silence or his words immediately take on a significance, because my fatigue or his falling back upon a ready-made formula are not accidental, for they express a certain lack of interest, and hence some degree of adoption of a definite position in relation to the situation. (Merleau-Ponty, 1954/2002, p. xx-xxi)

Gallop (2000) suggested that as ethical readers, we need to pay attention to details that are actually on the page, and she gives five examples of particular sorts of details to which we might attend: (a) unusual vocabulary, (b) words that seem unnecessarily repeated, (c) images or metaphors, (d) text that is set off by italics or parentheses, and (e) long footnotes. She argued that “detail is the best possible safeguard against projection. It is the main idea or the general shape which is most likely to correspond to our preconceptions about the book” (p. 11). An underlying assumption here is that the words an author uses can only point to a deeper underlying meaning–and it is to that meaning an ethical reader hopes to connect using various cues as guides to deeper insight. In this study, the interactions took place in real-time, but the notion that the *ways* words are used can become cues to appreciating what lies beneath them is helpful; here, non-verbal cues also assume relevance. In this chapter, I consider more deeply how vague meaning emerges from the implicit and co-evolves with language in mathematical contexts.

In reflecting on my own mathematical experience over the course of this study, I have become increasingly aware of the role of non-verbal processing. It is often claimed that mathematicians, especially gifted ones, do much of their thinking outside of the realm of words (Dehaene, 1997; Devlin, 2000; Hadamard, 1954). I do not think this distinction should be reserved for experts, however, never mind for only the brilliant. I am not a mathematician, and I have to wrestle with problems that even a novice mathematician would find easy. In doing so, I have been surprised to notice how much happens on a non-verbal level, how difficult it can be to describe what is happening, and the incredible volume of words needed to describe even relatively simple mathematical explorations. So what happens if we insist that students *always* show their work and explain their thinking? According to McGilchrist (2009):

> The biggest problem of explicitness … is that it returns us to what we already know. It reduces a unique experience, person or thing to a bunch of abstracted, therefore central, concepts that we could have found already anywhere else–and indeed *had already*. Knowing, the sense of seeing clearly, is always seeing 'as' a something *already known*, and therefore not present but re-presented. Fruitful ambiguity is forced into being one thing or another. (p. 180)

If we stopped at the implicit, however, language could never point to new insight. So how might we allow (and encourage) implicit understanding to interact fruitfully with the verbal, the specified, and the symbolic such that both continue to evolve? Here, Gendlin's work is of particular significance. Learning to attend to *all that*, to *allow words to come forth*, and to attend to how articulated meaning co-evolves with implicit meaning allows powerful shifts in understanding.

In this chapter, I characterize doubt in two distinct but overlapping roles: doubt-as-vagueness and doubt-as-conflict. In the former, doubt emerges from the absence of a clear sense of a next step, whereas in the latter, doubt has more to do with a felt need to resolve opposing next steps. As I shall attempt to show, vague understanding may also contribute to a sense of doubt-as-conflict. When attended to and articulated, both forms of doubt can prompt deeper understanding.

I begin this chapter by sharing students' descriptions of doubt and certainty. I then consider doubts that surfaced as students began to recognize the limitations of language in developing mathematical understanding. Here, the dual nature of doubt-as-vagueness and doubt-as-conflict becomes particularly apparent: Solutions in direct opposition to one another both seemed compelling to many students. Even for those who were strongly convinced one way or another, this opposition was very difficult to reconcile. Making explicit just what it was that was opposed was the dominant activity as students worked on the lamp problem.

Following this, I identify a number of ways that felt meaning can alert us to its presence prior to naming: "Niggling doubts", "tip-of-tongue understandings", "all that's," and moments that simply "seem significant" all begin as doubtful in the sense of vague and unarticulated; niggling doubts also involve doubt-as-conflict. The ways each of these come to awareness are significant to this work. Vague language that indicated "something-to-do-with" and thereby cast a net over a chunk of felt meaning was evident throughout the transcripts. Even less conscious moves such as facial expressions, body movement, and tone and pace of speech sometimes provided clues about the felt meaning from which they emerged and to which they contributed.

Finally, I discuss the significance of naming mathematical objects, how naming made it possible to negate or assign attributes, and how named attributes could be systematically and / or strategically (as in bridging) varied. Importantly, these objects emerged from a deeper and more holistic felt meaning that is harder to talk about; in fact, it is through naming that felt meaning becomes an object in the first place.

While some of these themes have been addressed in other literature, I try to weave new connections between them, to voice key ideas through the reflections of seventh-graders, and especially to maintain an ear to that which has not been fully formed or verbalized–to the felt meanings that point to a holistic and non-symbolic implicit intricacy. My hope is that each of the themes will be meaningful to the reader in its own right, as well as assume deeper meaning in Chapter 6 as I incorporate them into a broader explanatory structure that considers their role in the interplay between the implicit and explicit, the named and more-than-named, and the broadly and narrowly focused. Here, Varela's (Varela & Scharmer, 2000) paradoxically fragmented "virtual" self" that nonetheless ordinarily experiences itself as unified is a key theme.

Experiencing Doubt and Certainty

In *Harry Potter and the Half-Blood Prince*, J. K. Rowling (2005) introduced a spell called *Felix Felicis* (from the Latin for happy, lucky):

> Felix Felicis makes the drinker lucky for a period of time, depending on how much was taken, during which time everything they attempted would be successful. It was to be used sparingly, however, because if taken in excess it caused giddiness, recklessness, and dangerous overconfidence. (Harry Potter Wiki, 2011)

Furthermore:

> Felix Felicis possibly works by providing the drinker with the best possible scenario. This usually registers in the drinker's mind in the form of an unusual urge to take a certain action, or as a voice telling him to do so.

Luck and happiness, then, are not just about the positive impact of good choices, but about the sense of (unambiguously) knowing what to do next. This seems much like Johnson's (2007) description of furtherance and hindrance (see Chapter 1).

In reflecting on their small group interview about the lamp problem, one student commented that their interview video "would make an interesting talk show" (Lamp Class 2 Interview). To me, this emphasized the narrative structure of their progress through the problems, with seeming *movement* through a complex plot. The shifting context of the problems provided backdrops against which such movement could be perceived. Although there seemed to be a class narrative, different students experienced their own movement through the stories through varying combinations of solving, winning, and connecting (i.e. with others). Impeded progress within these plot lines was often experienced as confusion, anger, sadness, or even shame. Passions ran particularly high as students debated their solutions to the lamp problem, prompting one student to comment (with hands up in mock defeat): “It’s okay! It’s just math” (Lamp Class 1). A little later, her own competitive spirit emerged: “Okay, so we either argue it now and stay strong or WE will DIE (shaking fist).” She was joking, but it does seem that thwarted certainty, even in the context of a math problem, was able to conjure up feelings that diminished the sense of surviving or belonging.

In their reflections on a simple number problem (one I do not elaborate here), students described certainty in terms of completion, resolution, accomplishment, and sometimes one-up-man-ship. After their experience with the lamp problem, several noted feeling competitive (“GO ME!”) and angry, especially when certainty was combined with an inability to convince others (“Are you blind!”). One student noted that certainty is associated with both happiness and being smart, while two carefully distinguished *happy* from *satisfied* or *accomplished*. Several students described certainty in terms of knowing a next step and moving forward rather than in terms of final resolution; one described certainty as an “almost sort of mood.”

Associating certainty with confidence also carries a connotation of movement in that confidence implies a confidence to do something. The energy associated with certainty (and contrasting doubt-lethargy) was evident throughout the transcripts. In reviewing the videos of class discussions and small-group interviews, it is fascinating to observe the flow of emerging understanding as it manifested itself in non-conscious speech patterns and bodily movement–places where (for example) the words rushed out faster than the speaker could monitor, where a tapping pencil accelerated and indicated an oncoming *aha*, or where a student suddenly sat up straight and unintentionally spoke loudly over top of somebody else. In coding my own research diary, I noted a variety of codes that contained the word *rush* (some of which overlap): rush of confirmation, rush of category, rush of metaphor, rush of pattern, rush of new explication. I will not elaborate the subtle distinctions between each but rather note that all were experienced as fast-forwards of sorts. One student actually used the phrase *tap-my-feet* to describe certainty. Gendlin (1978), too, refers to shifts in understanding as movement. He further noted surprise (1996, p. 87), sighs, (p. 99), laughter (p. 94), sudden movement, and nodding (p. 84) as indicators of felt meaning.

Doubt was often characterized as lack of motion, being stalled, quiet, feeling frustrated and/or confused into inaction, feeling dumb ("I am so stupid!"), feeling "fickle," and even feeling that others were mad or sad. Sometimes, the students described the co-existence of doubt and certainty:

> Nine: When (I forgot who) tried to convince everybody that the answer was one-dollar profit I had to rethink things. Like: Hmm… well, does their idea make sense? I noticed that I was getting quieter because I was thinking more about the question and possible answer. My reactions when people told me I was wrong I was pretty confused / annoyed. (Journal Reflection)

It is interesting to consider why some students sometimes experienced doubt as possibility, while others at other times experienced it as stagnation and shut down. Initially, I found myself referring to students with labels like over- or under-confident, but this was very limiting. As I showed in the ice melt story, Double-Check was cautious but confident, and very aware of her tendency to think and re-think, while both Jolt and Quick-Start exhibited strong jolts of insight. As Felix developed the long string of conjectures for the consecutive integer problem, he displayed an exuberant confidence; in this case, he was also quick to accept anything that sounded plausible with very little (and sometimes despite conflicting) evidence. Prime confidently developed and rigorously examined many ideas; he was very persistent in the face of difficulty. One student who associated certainty with smartness tended to be very hesitant to approach the problems and to share her ideas; another claimed, “Whenever I’m in doubt, my brain shuts down” (Class 2 Interview). A more useful consideration than confidence, then, might be how learners are swayed by successes and setbacks.

Finding (Fruitful) Doubt Spaces

> The role of proof in the classroom is different from its role in research. In research its role is to *convince.* In the classroom, convincing is no problem. Students are all too easily convinced. Two special cases will do it. (Hersh, 1993, p. 396; emphasis in original)

A recurring theme in my analysis is how what is doubtful in the sense of vague and unarticulated can, when recognized, serve a powerful role in deepening mathematical understanding. This helps to address Clement and Konold’s (1989) observation that, in contrast to experts:

> Novices often ignore feelings of confusion with the rationale that since they always feel confusion when working on problems, such feelings ought to be ignored. When the confusion becomes too strong to ignore, they take it as a sign that no progress can be made and abandon work on the problem…. The necessary process of engaging in a cycle of conjecture, evaluation, and correction requires both the attribution of confusion to problem-specific causes and breaking away from the common belief that problem solving always consists of recalling a well-defined procedure and executing it. (pp. 28-29)

Clement and Konold identified “using confusion as a signal to rethink part of the solution” (p. 27) as an important problem-solving skill. Mason, Burton, and Stacey (1982) also emphasized being stuck as a potentially fruitful state, and they provided a number of strategies for moving forward. My aim is to elaborate such strategies by considering ways teachers and students in the classroom might directly attend to aspects of understanding that are often vague, unarticulated, and unnoticed–particularly for non-mathematicians, whose need to prove (at least initially) often seems to emerge from a different space than it does for mathematicians.[42]

As I attempted to show in each of the problem narratives, *much of the work that challenged the boundaries of students' understanding only began in earnest once I found ways to interrupt the feelings of adequacy they attached to their initial solutions.*[43] The entire class quickly reached consensus that a lamp bought for seven dollars, sold for eight dollars, purchased back for nine dollars, and sold for ten dollars would result in a one-dollar profit. When mounting evidence for two-dollar profit swayed their conviction, they did not feel the need to *refute* the one-dollar argument unless doing so was necessary to convince someone else. At first, I was the only skeptic they needed to convince. When attempting to determine what time a funnel full of ice dripping into a graduated cylinder started melting, most students assumed a linear melt rate; on their own, they did not feel a need to gather data to support this claim. Furthermore (and of greatest significance to this study), they were willing to suspend their doubt regarding the significance of the amount of starting ice as soon as they found alternative approaches to solve the problem. As they attempted to determine which numbers could

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[42] This is one of the issues that can emerge when teachers want all mathematics to be presented in the context of practical applications to daily life. I worked with someone once who calculated supplies required for a construction job by measuring blueprints. I was appalled to discover that he treated 3'4" as 3.4" (so he could punch it into his calculator). He wasn't particularly interested in my explanation for why this was wrong, and I supposed the difference between four tenths and four twelfths might not be too serious. When I asked what he entered for 3 feet 11 inches (fully expecting him to reply with the more serious error of 3.11 inches), he said, "I just round it to 4!" For his purposes, this level of accuracy was sufficient, and he saw no need to change what he was doing.

[43] This is also true of the two stories I shared in my introduction to the study in Chapter 1.

be written as the sum of consecutive, positive integers, many students noted patterns and simply assumed they would continue. Most were satisfied to find *some* ways and did not feel the need to identify them all. Many noted that “2, 4, 8, 16, etc.” always seemed to be missing from the list of numbers that work, but few were concerned with *why* this might be the case and were happy to trust the pattern.

Gendlin (1962) talked about the impact of bypassing rather than resolving conflicting impulses:

> We know this from how difficult it is to devise courses of action and interpretations that take account of all in the situation and leave us feeling whole and unconflicted…. There are always plenty of easy alternatives for saying and doing something that fails to resolve anything.
>
> To really resolve the "hang-ups" is a very different and far more difficult matter than just picking one or another of the many available schemes and actions that will not resolve anything. (p. 293)

While there are likely significant differences between unresolved hang-ups in therapy and those in understanding math, exploring unresolved mathematical hang-ups did open meaningful spaces for deepening understanding. These doubt spaces may also have helped address Boaler’s (2008) concern that many students with whom she worked were unwilling to alter problems before working on them, as doing so “went against their *ladder of rules* view of mathematics” (p. 192). The doubt spaces that emerged in this study were prompted by awareness of limitations within students’ own ways of thinking about the problems. This was particularly true of the lamp problem, which students could easily see did not involve what they considered hard math.

In Chapter 3, I used Thompson’s (2007) description of a bacterium swimming in a sucrose gradient to emphasize the importance of the global and organizational context in defining the bacterium as a biological individual and sucrose as food. In each of the three problems considered here, the contexts had to be shifted before the students recognized mathematical possibility within them (as I elaborate in the section on bridging). With characteristic eloquence, Hewitt (1986) compared the way he engages with mathematics with the way he engages with playing the guitar:

> I can play, develop and practice techniques, explore and test ideas, continually set challenges, involve myself physically, intellectually and emotionally. The new builds on the old, the old is mastered as it is required by the new. What makes me a guitarist is not my technical ability on the guitar but the fact that I am aware of the potential, and actual, adventure it holds for me and that I engage in *it.* (Hewitt, 1986, p. 34)

Appreciating the "potential adventure" in mathematics is deeply tied to attending to the implicit understanding that so often announces the spaces of personal relevance central to an enactivist view of cognition.

Appreciating the Limits of Language and Logic

I recently attempted to Google an unfamiliar bird I had spotted and wanted to identify. As I watched the bird, I could only pick out a few features in the deepening twilight; what really caught my attention was a haunting call that carried through the forest. I found it difficult to find words that were uniquely identifying to describe it. Even a clear view would not have made for an easy task–the bird had a dark V-shaped stripe across his breast–but how would someone else describe it? Is that marking uniquely identifying? Some plant and animal identification guides offer broad visual categories (such as silhouettes) to help narrow searches like this, but the task often remains difficult. I also find the dichotomous keys sometimes used in field guides quite difficult to use; they, too tend to rely on what is namable (or at least sketch-able).

In math, describing visuals–or “saying what you see” (Hewitt, 1989)–can also be difficult. At times, a visual may serve as a generic example that represents a whole class of examples; as Reid and Knipping (2010) pointed out, this may be “unavoidable if one does not have a method of representing unspecified numbers symbolically” (p. 20). To further complicate matters, a felt meaning need not be particularly visual or auditory; describing it becomes a matter of “saying what you feel” in a very holistic sense that is not connected to a particular sensory modality. As many have explored in far greater depth than I will here, words (and even images) must somehow call forth intended meaning, which always transcends the words.

As the students struggled to convince one another that their answer to the lamp problem was correct, many came to new realizations about the limits of language and, by extension, the limits of logical argument, which is dependent on language. Recognizing this in a seemingly simple context seemed to be a startling (or at least disconcerting) insight for some of the seventh-graders who took part in the study. Others were very reluctant to accept the insufficiency of their words; those who did not understand were either being intentionally difficult or not trying hard enough. Consider the following dialogue:

Two: But my theory PROVED it’s $2–it’s not $1.

[Prime attempts a more detailed rebuttal of One’s Argument]

Two: And my theory PROVES that he’s wrong, because as I said before–

Eight & Four: (laughing; Four puts her fist in opposite hand)

Four: Well BAM! So confident.

> Two: Well bam! It proves that he's wrong, because 7, 0, 8, negative 1, 9 (indicating totals on paper) proves that he's wrong. And as Prime said, he didn't take any of the math into account. He just like–when–he–minused –used–well–okay. He started with a 7, wen[*t*]–went with the 8, so–sold it for 8 bucks–he didn't take any of the math into account–he didn't–they didn't take into account negative 1. They just went to 0. That was their, like, stopping point.
>
> (Lamp Class 1 Interview)

Eight and Four were unconvinced. But Two did not see this as an issue with what he considered his proof. Prime offered similar sentiments in a journal response (though his annoyance seldom if ever showed in actual conversation):

> [What is it like to convince others?] It is extremely annoying and tiresome when someone ignores facts and goes against logic. It is even more annoying to try and correct them when they are extremely stubborn.

Contrast this statement with the following:

> [What is it like to convince others?] I think it is pretty hard because you know what you're thinking but when you try to explain it others don't because you already know what yourself means but others don't. (Thirteen, Journal Response)
>
> Okay, so, um…. Um, when you want to explain something to somebody? And they don't understand it? It's, um…. Cause then you have to reword your entire thing and…. And usually, most people only have one way to say something. Like, usually people only have one way, like. Oh, yeah, it's because blah-blah-blah-blah-blah. But could–did they ever think of, like, um… how come it's like that? And… like, what happens after that? All that stuff? But a lot of people don't, like, don't want to reword what they say, so that it make only sense them. And not to everyone else. (Nine, Lamp Class 4 Interview)

These students were clear that the meaning they were trying to convey was bigger than available words, and they did not blame others for not understanding. As students learned to more deeply appreciate the nature of doubt and certainty–in particular the limited power of words and logic to convey it–many worked much harder to understand others and to communicate clearly.

Some students recognized a tendency to resist new ideas that was quite distinct from deliberate antagonism; in the examples below, the first student noted that *other* people tended to resist change, while the second and third were prepared to accept increasing responsibility for blind spots in *their own* thinking:

> [What is it like to convince others?] It's pretty hard because some people are so sure, they are not willing to change their minds. (Fourteen, Lamp Class 4, Journal Response)
>
> Felix: ….Cause–we all–we should respect what they think? And how…how…their way could actually be right? Because, um… sometimes you just have one little calculation problem that can mess your whole problem up? And…um…but I think when it comes to convincing people, you just have to be open-minded? And know that they–there are other people too that can make different opinions, and you should respect them, and…maybe even…use them sometimes if you feel that your answer's wrong. (Class 4 Lamp Interview)
>
> [What is it like to convince others?] To understand different points of views is actually sometimes really difficult, because it's hard opening your mind to something new when you think your answer is right. (Eight, Lamp Class 4, Journal Response)

In its seeming simplicity, the lamp problem offered a space within which students' interactions starkly illustrated the limitations of language. Initially, many students seemed to think their arguments were self-evident, and they struggled to comprehend how repeating them over and over did not somehow awaken similar understanding in the minds of their classmates. Nobody could blame too many steps, difficult vocabulary, difficult calculations, or complex lines of reasoning for the disconnects they experienced.[44] From an enactivist point of view, the complex histories embedded in differents students' understandings of the arguments were not compatible. Notably, however, all students aligned with only two major positions.

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[44] Another example: When asked to compare 8 sub sandwiches for 7 students and 5 sub sandwiches for 4 students, some students quickly divide and announce that 5 for 4 means more food. Others are adamant that the ratios are equal, as both involve one more sub than person. However, even those who use division and are confident in their answer don't generally have a quick refutation for the alternative. Deep understanding of the situation needs both.

Developing Awareness of Necessity: Two Faces of *Must-Be*

> "*You learn what he* ***is by knowing what he isn't***, her dad used to say. This was not a gray fox, and not a red fox. Coyote. A big one, probably male. Alpha's mate. (Kingsolver, 2000, p. 60; bold emphasis added)

> *This tree* ***must be–must, must, must be****–a honey tree.* (Berenstain & Berenstain, 1962/2002, p. 49; emphasis added)

> She *had* left the lid ajar, no "**must have**" about it. Living alone leaves you no one to curse but yourself when the toilet paper grins its empty cardboard jeer at you in the outhouse, or when the cornbread is peppered with [mouse] poop. (Kingsolver, 2000, p. 65; bold emphasis added)

Gendlin (1996) explained how the use of *must be* by patients in therapy often indicates a lack of connection with felt meaning:

> To say, "It must be" is an inference. People do not say "It must be" when they are directly in touch with the connection. They say, "It is," or " I feel." Of course this verbal difference only indicates the real difference. (p. 8)

I recently misplaced my cell phone. I noted that it was not in my pocket, and it was not on the counter. The only other plausible place *I could think to look* was in a purse buried deep in my backpack. It was not in the two most obvious places, so I reasoned that it *must be* there. Here, the *must be* indicated a lack of confidence in my conclusion, even though it *seemed to be* the only remaining available option. Papa Bear's repetition of the word *must* in the opening quote also indicates doubt, for if the first *must* did not produce the expected result, why would the second or third? Is "I promise times 100" more or less convincing than a simple, "I promise" or "I will"?

When *must be* is used in terms of *must-be-because-I-can't-think-of-anything-better* (the weak sense), it points to a doubtful stance. It provides a notable contrast with the *must-be* (strong sense) of mathematical necessity and certainty, which may be either *must-be-because-of-deduction-from-trusted-premises* or *must-be-because-there-are-no-*

other-possible-options. Yet neither of these is satisfying without the *feeling* that the meaning they point to is somehow sensible (Hanna & Jahnke, 1996; Reid, 2002). In this way, there is some overlap between *no-other-possible-options* and *I-can't-think-of-anything-better*: Even if only a single option from an exhaustive list remains or if a chain of logic seems unassailable, the implied conclusion may not *feel* sensible. A mathematical *must be*, however, is not satisfied with the *because-I-can't-think-of-anything-better* aspect of the colloquial use of *must be*:

> Ms. M.: Are you 100% sure that it would always work?
>
> Six: I can't see it not working.
>
> Ms. M.: Okay. Can anyone think of a reason why it would *have* to work?
>
> (CPI Class 1)

This is not to say that tentative conclusions based on a weak sense of *must be* do not play an important role in mathematical thinking. Zack and Reid (2003; 2004) showed how tentative conclusions may be "good-enough" to allow continued forward movement on a problem, but good enough for continued engagement is not the same as good enough to allow confidence in a final conclusion. Lakatos (1976) showed that even mathematicians must acknowledge that *because-I-can't-think-of-anything-better* plays a role in their conclusions. How can we be sure there is not an alternative that we have not thought of, that all premises are accurately defined or that all links in a chain of logic are sound? It may seem that there are some situations where laying out all possibilities is feasible. But when I watched my son deducing the location of "Clifford's Buried Treasure" (PBS Kids, n.d.) on a website that presented a variety of choices that he could eliminate one by one, it was obvious that mistakes are possible even in that simple context. He was only two, but what arbitrary level designates worthiness to decide? If, as Davis and Hersh's (1981) "Ideal Mathematician" claimed, a proof is "an argument that convinces someone who knows the subject" (p. 40), then the student debating him/her is correct: “Then the definition of proof is subjective. Before I can decide if something is a proof, I have to decide who the experts are” (p. 40). Lakatos said this more strongly:

> Nothing is more characteristic of a dogmatist epistemology that its theory of error. For if some truths are manifest, one must explain how anyone can be mistaken about them, in other words, why the truths are not manifest to everybody. According to its particular theory of error, each dogmatist epistemology offers its particular therapeutics to purge minds from error. (p. 31)

In some ways, perhaps it is easier for students, who have not reached the level of certainty or confidence held by mathematicians, to appreciate fallibility. But mathematicians work hard to eliminate the sort of ambiguity that plagues the conclusions in *Proofs and Refutations*. Many students, on the other hand, are quite willing to adopt the more colloquial sense of *must be*, even when working in mathematical contexts. They often seem not to recognize (or concern themselves with) the loopholes this can leave in their conclusions.

Rejecting the impossible and moving forward with whatever is left is essentially the basis of the game Chocolate Fix®, which the students played early in this study. Players must place nine plastic chocolates (three shapes in three colors) in a three by three grid according to a set of prescriptive clues. Here, no solution has a deeper sense about it; i.e. there is nothing inherently more or less plausible about placing a particular shape or color of (fake) chocolate in any of the nine available spaces. While students often berate algebraic symbolism as being without meaning, most enjoyed Chocolate Fix® and did not perceive the pure logic it requires as mathematical:

> Six: With *Chocolate Fix*®, you use, um… more common sense. No… no, like adding numbers, no division, no addition, no subtraction. It’s just common sense.

> Seven: It's something different, cause, like, for the Chocolate Fix® one, it's like, I guess, I just used my common sense, I guess. Like…. And it's fun to do, because it's trying to put the shape together and make…fix the puzzle. But this one [Consecutive Integers] you have to, like do math–like, intense math. [Six: Yeah.] Well, not intense, but, it's like….
>
> (CPI Class 1 Interview)

Similarly, I have seen Sudoku puzzles printed with the caption, "No math required!" In the lower levels of Chocolate Fix®, the certainty that comes with *must be* is based on a clear presentation of available choices, and conclusions about which moves are possible (or not) are relatively easily confirmed without need for background knowledge beyond three simple colors and shapes and the ability to recognize position on a 3x3 grid. At higher levels, the possibilities expand due to a loosening of constraints that results in more levels of contingency. While the puzzles allow only one possible location for a particular piece, it becomes less *immediately* obvious which choices lead to contradiction. At the highest levels, potential moves may be seen in terms of learned types, but the problem space remains clearly defined. I think this is what Twelve was pointing to in the following:

> Twelve: Well, Chocolate Fix® had, like, more rules and less freedom? You didn't have to go and search for new, like… ideas or theories? You just had to… try and make all the things work with all the hints you were given? But with this problem [summing consecutive integers] you had to come up with all these theories? And… test em out? And…. Yeah. (CPI Class 5 Interview)

In distinct contrast to the apathy many students claim to feel in response to what they perceive as meaningless math, *Chocolate Fix*® seemed to evoke strong emotional responses in many students; unlike the other problems, these reactions seemed to have less to do with interpersonal relations. As Six worked on the problem, he repeatedly chanted, "Why are you obsessed with this?" Another student kept exclaiming "Oh my God!" while deducing possible moves from given clues. When I played, I noted a strange comfort that sometimes came from having only one option open–a feeling that I also experience in day-to-day life, provided the available option is not a terrible one.

Similar to the manner in which students were often willing to accept something that *must be* true (in the weak sense), many were generally happy with conclusions that were merely plausible. They were happy to accept further conclusions unless direct conflict (intra or inter-personal) prompted re-evaluation:

> Felix: Maybe cause you're more into it and you're more trying to ex–(?). And something like what they said, um, how–you think you're right, you want to know you're right, and you're kind of desperate to know you're right. So you just go all the way with it…. And then you find out you're wrong? And you're like, "What happened?" (Chocolate Fix Class 2)

Learning to Refer to the Implicit

> With training, people become able to go from no direct reference at all, to reference just in passing, to palpable direct reference. This range is measurable by characteristic modes of language. What people can tell us with training is not what was there before. Direct reference carries the IU [implicit understanding] forward. But now we can study this 'carrying forward' itself. At first most people don't report much of what happens. We don't need to trust what the reports are about. *Talking is not only about something: talking is behavior and can be studied*. (Gendlin, 2009b, p. 349; emphasis added)

As I contemplated Gendlin's (1978) description of focusing, I wondered: Do some students refer to the implicit in mathematics? If so, how, and to what extent? Students' response to the lamp problem offers unique insight into this question. Those who could not get past the words *buy-for-seven-sell-for-eight-buy-for-nine-lose-your-profit-sell-for-ten-get-it-back* had a very hard time gaining deeper understanding. Students willing to struggle to reconcile conflicting views were forced into a space where words were not sufficient.

Throughout my time with the class, Prime was particularly willing to enter the murky waters of the implicit–and to share his journey. He often seemed to have complex ideas that were difficult to articulate, yet he worked hard to express them and did not seem intimidated by stalls and hesitations. In fact, his speech was slow and halting much of the time. He was not afraid to say, "I'm not sure where I'm going with this." Double-Check also seemed to enter deeply into the implicit, although she was less likely to vocalize partially formed ideas. Not surprisingly, Prime and Double-Check likely developed the most complete responses to and deepest understanding of the problems we explored together. It is unclear to me whether their willingness to attend to the implicit was somehow connected to a particularly high need to resolve doubts that others tended to ignore, whether those doubts became apparent in response to their attention to the implicit or (as I consider most likely) whether both factors were co-implicated in the way Prime and Double-Check engaged with mathematics.

As teacher-researcher, one of my goals was to support those who struggled to articulate vague understanding; here the interaction of Varela's empathic second-person coach with Gendlin's insights into implicit understanding and with his process of focusing proved very fruitful both in terms of classroom actions and retrospective analysis. As I illustrated in Chapter 4, sometimes my actions as coach seemed to help students attend to aspects of knowing that helped broaden and deepen their understanding; at other times, my actions likely hindered emerging understanding by directing attention too narrowly. The manner in which the students interacted with one another was also very significant to the manner in which implicit understanding came to the surface. I now turn to particular ways in which referring to the implicit assumed relevance during this study.

Attending to Non-Conscious Speech Patterns and Body Movement

In describing how words emerge in conversation, Gendlin (2009b) noted that we typically do not plan out our words–we simply have a sense of what [we are] about to say then allow the words to come forth as we speak:

We can see the bodily nature of language if we ask: How do the words

> come to us to say? I open my mouth and the words come. They come already organized in phrases and sentences that say what I want to say in this situation without my having to consider all possible words and combinations…. Situations don't first exist and are then "signified" by language. We don't "symbolize" by attaching "signifiers" to external things…. *The symbols and the situation are internally connected because a situation is inherently the implying of a cluster of possible sayings and doings*. (Gendlin, 2009b, pp. 355-356; emphasis in original)

One of the students also articulated the manner in which words can come forth from a situation:

> Before I went up [to the whiteboard], I was thinking I'm lukewarm, I'm just going to sort of act it out and see what it was and now I'm $2, because I– it seems I proved to MYSELF that it IS… a $2 profit? (Three; Lamp Class 2 Interview)

It seems something similar may be going on when students back up and restart when the words that emerge are perceived as not *working*; there are many instances in the transcripts where students trailed off mid-sentence, then restarted a sentence or phrase, almost as if they needed to take a run at the idea they were trying to express, or perhaps step back from a path that was *not* working to find a new starting point that allowed other alternatives. For me, this feels very much like being unable to re-start a piece of music (when singing or playing guitar or piano) from a random point. Word-blends that seem to occur faster than a conscious change-of-mind regarding word choice also became evident as I was transcribing. Interestingly, I find that these are easy to miss when simply *listening* to the dialogue. I automatically perceive meaning in a more holistic sense:

> Fourteen: Because there's **two-already** two squares there, so this can't be a square. Or–
>
> Nine: Yeah, yeah, yeah! Okay, so–

> Quick-Start: No, no! Why don't **we-you** put this on the bottom–brown, pink, kay…. Where's the pink– [adjusting pieces]
>
> (Chocolate Fix Class 2 Interview; emphasis added)

There was no pause between two-already or we-you–they flowed together seamlessly.

In addition, there were many instances in the transcripts where a pause indicated the insufficiency of words. For some, the repeated use of *like* seemed to act as filler when words did not flow as easily as they might. While this is partially a learned habit of speech,[45] I do not think anybody carefully plans just where they will insert the *likes*:

> Nine: And another thing is that they showed us–**like**, in other ones too? They told us, **like**, exactly where things would go? **Like** they said…. **Like**, remember on the first one we did with **like** the T? And **like** the T could only fit in one spot? Right? And then the other one that we did with the little, **like**, squiggly thing or whatever? [Fourteen: Yeah, yeah.] Cause they could only fit it one spot. And then also on the other ones **like**, cause there were two, **like** two pinks and they could only fit in one spot because there was, **like** other–[Fourteen: Other… yeah.] Cause there was only–either that or, **like**, two circles or **like** two squares or two triangles. (Chocolate Fix Class 2 Interview; emphasis added.)

Hedging with gestured quote marks sometimes indicated that the chosen word was not quite right for the situation–and sometimes also seemed to suggest a question regarding whether the listener was picking up what the speaker was saying. Other hand motions (broad and sweeping, hand shaking) were used to indicate missing words or inadequate language.

All of these tentative stops and starts require an interactive classroom environment that supports the ongoing evolution of developing meaning; I do not think they could emerge in an environment where clarity and precision were constantly expected. As I attempted to show in my description of the conversation with Felix regarding his long sequence of ad hoc conjectures for the consecutive integers problem, finding a helpful balance can be difficult.

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[45] In fact, it is a very particular learned habit of speech. As one reader commented, other words seem to serve the same purpose in different individuals or communities.

Recent studies of interoception, or awareness of bodily signals, suggest that such awareness may be correlated with our tendency to follow intuitive hunches. According to Robson (2011), participants with the strongest interoception did both the best and worst on the experimental test–a card game where participants had to guess which of four decks would next present a desired color, not (consciously) knowing that two of the decks had better odds than the others. This leads me to wonder whether some of the students in this study were more aware of their own bodily movements and speech patterns, or if helping them become more aware might have impacted the ways they were able to tap intuitive understanding in the math classroom. The only incident where such awareness became the explicit focus of discussion was when two students had visible jolts of insight regarding significance of the volume of starting ice in the ice melt problem. Here the students reflected on the bodily sensation of *aha*, but it is unclear whether awareness developed through their reflections impacted later work. In my own life (within and beyond mathematics), I have become increasingly aware of the need to treat any sort of rush feeling with great caution. This is also true of a sensation I call *brake burn*–a conflicted feeling that I experience as being pulled in too many directions. The felt tension is literal in a fully embodied sense: I feel driven to attend to conflicting ideas simultaneously. This can result in a restlessness that drives me to pacing, making quick movements, and starting and stopping many things (and likely shows up in my face as stress). By attending more closely to the physical condition of restlessness, sometimes I can recognize a need to step away and attend to the conflicting pieces.

Co-evolving Meaning and Language

Gendlin (1991) talked about words that come into “slots,” which are gaps or pauses in speech where words are not readily available. In the transcripts, these often show up as the "…" where a sentence trails off. They are one of the easier indicators of implicit functioning to spot (at least in the slow-time of the transcripts); for me, they also help make apparent how easy it is to insert *myself* into these spaces rather than allowing the time and encouragement for students to *allow* words to come that might provide a starting place for continued evolution of their own felt meaning:

> To come into a slot, the words must work. It must make sense, but that requires a function of the implicit intricacy of the slot, together with the implicit intricacy that the word brings. These cross. You can grasp the function performed by their crossing. (p. 55)

The crossing of words with the implicit intricacy was most noticeable to me as Quick-Start connected his emerging understanding of melt rate to *backtracking* and eventually to *drop ratios or whatever* during the ice melt problem. He must have understood enough about both situations that the word *ratio* came to mind, but it seems unlikely that either was clear prior to their coming together. Something implicit in his understanding of the melting ice and in his understanding of the word *ratio* crossed, allowing deeper insight to emerge. This was also evident in the evolution of students' (and my own) understanding of *original money* in the lamp problem and *powers of two* in the consecutive integer problem: Very subtle changes in wording sometimes pointed to deeper understandings.

Rowland (1995) explained how uncertainty in language may be considered in terms of hedges, which may be either shields or approximators. Shields indicate degree of certainty, as in *maybe x* (a plausibility shield) or *she says x* (an attribution shield). Approximators indicate degree of clarity, as in *approximately x* (a rounder) or *x-ish* (an adaptor, which blurs the boundary of a category). Whereas Rowland focused primarily on students' use of shields to protect themselves from being wrong, their use of approximators–particularly adaptors–is more pertinent to this study. In fact, I have begun to wonder whether shields that are employed defensively might be better *treated as* approximators that reduce commitment to a *specific* answer but *also* carry new meaning forward. At times, what appear to be linguistic shields may in fact indicate vague thinking; i.e. students *state* a degree of (un)certainty, but doing so need not be interpreted as distancing, as it may be rooted in a vague sense that could be opened for greater exploration. In this sense, the frustration and confusion that often accompany uncertainty may be considered in terms of doubt-as-vagueness, which may then be acted upon in particular ways. This was the nature of the clue that helped prompt the opening of the ice melt problem: Students claimed that they did not think the amount of ice would make *much* difference. Here, *much* was likely used (at least partly) as a plausibility shield; for some students, treating it as an adaptor allowed it to be opened for further elaboration that prompted an important leap in their understanding of proportional reasoning. The underlying message here is that uncertainty is nothing to be ashamed of–on the contrary, it is a powerful catalyst to deeper insight.

Ticklish Ideas: Niggling Doubts and Tip-of-Tongue Understandings

> The vast majority of languages, from Afrikaans to Hindi to Arabic, … rely on tongue metaphors to describe the tip-of-the-tongue moment.) **But here's the mystery: If we've forgotten a person's name, then why are we so convinced that we remember it? What does it mean to know something without being able to access it?** (Lehrer, 2008a; emphasis added)

I titled this section "ticklish ideas" because of the manner in which they may often come to awareness; a niggling doubt may do nothing more than cast a doubtful mood or a sense of non-confidence; it can be brushed away without really noticing it, much like a fly buzzing near your ear.[46] Nonetheless, it can continue to interfere on a non-conscious

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[46] I recently worked on a problem that required me to label coordinates identified with East-West-North-South on a grid. In converting the directions to integers, I found it strange that in some cases I was

level–something like the constant irritation of a noisy fan that I only recognize as an irritant when I finally become aware enough of it to switch it off. Similarly but on a more positive (or forward-moving) note, a *tip-of-tongue* understanding may invoke a vague sense of possibility before (or without) coming to awareness, particularly if conscious attention is occupied with more articulated aspects of a problem.[47]

For me, learning to appreciate the potential hidden within seemingly quiet or unobtrusive niggling doubts and tip-of-tongue experiences has opened many powerful learning experiences. As I tried to show in both my early examples of the warehouse problem and the thermometer problem in Chapter 1 and throughout the narratives in Chapter 4, attending to such experiences can expose deep misunderstandings, broaden the contexts in which problems are understood to be embedded, and uncover creative new ways of working within those contexts. Whenever perception of relevant context shifts, the entire space of possibility bounded by the problem (including what counts as relevant feedback) also shifts (recall once again the bacterium in the sucrose gradient). The thought processes sometimes experienced as ticklish likely influence experience,

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struggling to do what I thought should be a straightforward task. Something was *interfering*. I was able to locate this in a difficulty associating West with negative, seemingly because I live "in the West" (Alberta)! This was shocking to me. On a conscious level, I certainly was not conflating connotations of negative integers with negative-as-unpleasant (nor do I have a strong sense of West-is-best geographically). Yet my progress was noticeably slowed by the subconscious association with West-as-bad.

[47] Lehrer (2008b) described Jung-Beeman et al.'s (2004) observations of participants completing "Compound Remote Association Puzzles," by trying to think of a word that could be combined with each of three prompts; e.g. pine / crab / sauce (apple→ pineapple, crabapple, and applesauce). The answer was associated with a rush of gamma waves in the right hemisphere. Lehrer commented: "On the one hand, an epiphany is a surprising event; we are startled by what we've just discovered. Some part of our brain, however, clearly isn't surprised at all, which is why we are able to instantly recognize the insight" (p. 44). Lehrer also described Sheth, Sandkühler, and Bhattacharya's (2009) observations of the *timing* of insight: They used steady alpha waves in participants' right hemispheres to predict insight eight seconds prior to the subjects' own stated awareness of it. Although in one sense this seems almost incredible, certainly a tip-of-tongue experience or a niggling doubt is preceded by the *feeling* of an idea coming.

consciously or otherwise.[48] When they become the subject of attention, they create spaces for significant learning.

In this study, I found it easier to identify niggling doubts than tip-of-tongue experiences in the students' speech behavior, in part because I (in my role as empathic second-person teacher-researcher) played a strong role in helping them find doubt spaces significant enough to prompt deeper exploration and in part because I had not yet adequately tuned my own awareness to notice the latter (I suspect this will take some practice). I do not know if the students who participated in this study will now attend more closely to their own niggling doubts and/or whether they have experienced enough of the potential pitfalls that would give such doubts stronger voices. My own doubts played a significant role in the classroom stories, however, in that they strongly influenced what aspects of students' understanding I was able to perceive and therefore the areas to which I directed attention for deeper consideration.

Referring to *All That*

When I attend to felt meaning, it often emerges as a vague notion–an *all that*–for which first attempts to describe tend to be in the form of *something to do with*. A *something to do with* is more like a *conceptual* net cast wide–a fuzzy concept much like Rowland's (1995) description of adaptors. An *all that* may be accompanied by a sweeping hand gesture, and it may be described using a particular example to indicate

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[48] In *Still Alice,* Lisa Genova (2007) described a fictional character's experience of the progression of Alzheimer's disease. At first, she would lose a word, which would later return. Later, it seemed "more totally gone," with no first letter or clues. In another instance, she knew she was looking for something "life-and-death important" but could not name it or identify any features–yet still she was sure she'd recognize it when she saw it. As the disease progressed, she had trouble recalling even basic information important to daily functioning: "She had a loose sense of it, like the feeling of awakening from a night's sleep and knowing she'd had a dream, maybe even knowing it was about a particular thing" (p. 243). I find this description a very apt description of my own experience of "niggling doubts" and having something on the "tip of my tongue." A key difference, however, is that such experiences are not becoming progressively worse. If I give them enough time and space, I can usually "find" what I'm looking for.

something to do with (what Mason & Pimm, 1984 called "seeing the general in the particular").

An *all that* is more holistic and less differentiated yet more definitely bounded than a *something to do with*. It can bring with it the sense of knowing something quite particular without being able to describe the details. For example, I can easily distinguish a lynx from a cougar, but in the absence of one or both, I would have a hard time calling up particular descriptors for either. An *all that* may but need not be visual, and describing it involves more than picking out identifying parts. Sometimes it vanishes before I can identify anything particular–an experience I tagged *fleeting*. I also noted an experience I called *stepping stones* in which it felt like my mind was running away with a whole series of *all that's,* producing a sense of *too much contingency*; it felt careless and sloppy, like a chain of logic with intact operators but fuzzy premises. But it also allowed me to hold more in consciousness than I would otherwise have been able and to go deeper into particular ideas than I could have with well-articulated pieces. On other occasions, what started off as *stepping-stones*, seemed to simply vanish when I could no longer "hold" their unity in mind all at once (a special case of *fleeting*). It was as if somehow the *feeling* of the wholeness suddenly wasn't coherent enough to sustain itself, and the whole thing disintegrated. I had to restart with bits that I *had* articulated. I suspect *this* may have *something to do with* a pattern of neuronal activity that *almost* emerged from the chaotic background of global brain activity, and then dissolved as new activity interfered with its fragile signal.

By using a *something to do with*, I can avoid pigeonholing emerging ideas with existing labels, allow new labels to remain dynamic, and allow partial or vague understanding to function in my thinking. Zack and Reid (2003; 2004) spoke of the importance of allowing tentative or contingent ideas to function in the development of mathematical understanding; here, I also emphasize the importance of allowing ideas that are not-yet-formed to function in thought and to gradually emerge as more clearly defined entities. Whereas layers of contingency can lead to an exponential increase in the number of possibilities that must be kept in mind, *all that* and *something to do with* allow a more global perspective. Again, a shift in perspective of this nature alters perception of what is relevant and what might therefore come to mind.

While it is possible to find words or phrases to describe *all that*, the meaning evoked by the words and phrases typically changes as we do so. Again, even if a word or phrase feels exactly right, it points to a match between feeling and name, not to a match between feeling and objective truth.[49] Meaning and language continue to co-evolve:

> The felt meanings that function in experienced creation of meanings are always just *these* (directly referred-to) felt meanings, having whatever meaning they have. They are not indeterminate, they are merely capable of further symbolization. Their functioning in symbolization is our subject of inquiry. (Gendlin, 1962, p. 148)

[49] An *all that* may also present itself in a mood-like way that seems able to color whatever else enters thought. A familiar example is the "feeling of having forgotten something" important–a nagging feeling that has dissociated from the "what" that needs doing. But I've found that this dissociation of feeling and object is more common than I thought. One day as I drove away from the house, I noticed that my cell phone battery was almost dead. I took note, but didn't (consciously) dwell on this observation. As I was merging onto a major thoroughfare, I suddenly felt uneasy–a general *feeling* of "not enough" came to my attention. It prompted me to check my gas gauge, which was fine. It continued to nag until I *located* the feeling in the cell phone battery. The feeling (of "not enough" or, more specifically, "I'm going to run out") had returned and attached itself momentarily to my gas tank! I wonder what would have happened had I not been privy to the instant feedback my gas gauge provided–would I have stopped at the nearest gas station? I don't think so. More likely, I would have remained unsatisfied and therefore motivated to keep searching for a better match. This experience reminds me of Gazzaniga's (1978) split-brain research, where the left-hemisphere "interpreter" generated a plausible explanation for a message shown only to the right hemisphere. While my experience of *not enough* was large and vague, *some* part of me was clearly dissatisfied with the merely plausible suggestion that perhaps it was the gas tank that needed filling.

Referring to Others' Implicit Understanding

As the students and I attempted to negotiate common understanding, there were many times when someone would attempt to paraphrase what someone else had just said. Often, their efforts (like mine) fell into the trap of what I call *aggressive paraphrasing*, which in effect tended to put (wrong) words into someone else's mouth without providing a patient space for the other to first consider whether they were applicable. But I was amazed how hard some of the students worked to enter each other's experience and how willingly they were often corrected.

Sometimes it was helpful to *gently paraphrase* with vague language or to cast a wide swath with *something to do with….* Referring to others' implicit was easier when questions were broad enough to invite more than narrow response. There were a number of times during student interviews when I felt that a line of questioning was becoming too directed and stepped back with broader invitations. This was evident in my presentation of false dichotomies in the lamp problem and in my aggressive banter with Felix during the consecutive integer problem. In the latter case, broadening the level of questioning (though perhaps too little too late) seemed to help students return to their *own* problem narratives and escape from narrow tracks along which I attempted to direct them. There were also times when broad questions encouraged students to pan out and consider how their own current focus fit into a broader context.

Pointing and Hoping

Both the students and I sometimes put forth vague ideas with what seemed to me like unspoken *invitations* for help in articulating; in my research diary, I referred to this as "pointing and hoping." I experienced this as hope that somebody else would recognize my felt sense and be better able to articulate it than I was. In this way, *pointing and hoping* is closely related to *referring to others' implicit*. In both cases, the co-emergence of meaning and language sometimes became a very inter-subjective experience.

I turn now to discussion of how referring to the implicit is involved in mathematical generalizing and abstracting and how these are implicated in the creation and extension of mathematical of mathematical objects.

Extending and Connecting Mathematical Ideas

Perceptions of Generality: Feeling the General Case

> The teacher's invitation to skepticism about patterns seems like rather an empty gesture. Most of the time, a few special cases *do* point the way to eternity. The point of the teacher's proof-provoking question is not to achieve certainty–which is already assured–but to encourage the quest for insight. (Rowland, 1999, p. 24)

As I discussed in relation to the need to interrupt certainty, students in this study often did not attempt to express their ideas in general terms. In some cases, this may have been because they could already "feel the general case" or because they could "see the general in the particular" (Mason & Pimm, 1984). The latter was particularly evident as they looked for ways to express numbers as sums of consecutive, positive, integers. For example, is it not *obvious* that the following pattern will continue?

$1 + 2 = 3$

$2 + 3 = 5$

$3 + 4 = 7$

When pressed to explain why, students made important insights–but if *certainty* was the goal, doing so seemed quite unnecessary.

Here, I am interested in how implicit understanding might be implicated in students' perceptions of generality. In telling the story of our experience with the consecutive integers problem (Chapter 4), I included an instance where students first noted a general sense of *balancing* or *making up for* as they struggled to describe a more general *sense* of how both odd numbers and multiples of three could be divided into equal addends and then redistributed; e.g.:

5 → 2.5 + 2.5 → 2 + 3[50]

6 → 2 + 2 + 2 → 1 + 2 + 3

One student explicitly noted a lack of words to explain his sense of *make up for each other*, one related it to how gaining and losing profit (in the lamp problem) caused different amounts to *balance*, and one used a particular example that he noted would apply in other cases.

As others have developed (see especially Gattegno, 1973), understanding words-as-generalizations is a natural part of human development. Seeing the general in the particular is, in fact, essential to learning language at all. Eliot (1999) named three key principles that she claimed underlie all language development: (1) We see whole objects (she uses milk + bottle + nipple to illustrate the object *baby bottle*), (2) we perceive names as classes (all baby bottles, not just *this* one), and (3) we associate one name with one such class. But what distinguishes the object as object in the first place? If a baby bottle is described at a practical level, it is the whole that allows the experience of eating.[51] Here, the *need* to eat defines both the food-seeking child and the food–the bacterium and the sucrose.

My son (then three years old) once asked me if blowing would make his cereal softer. It seems this idea was closely related to the fact that if his food was too hot, we often blew on it so that he could eat sooner, but I do not think his suggestion was the

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[50]As I write, I'm wondering why nobody (to my knowledge) tried breaking, say, four-numbers (i.e. numbers that are the sum of four consecutive addends) into four equal but non-whole numbers; e.g. 14 → 3.5 + 3.5 + 3.5 + 3.5 → 2 + 3 + 4 + 5. I suppose, though, that this might not seem particularly helpful, since four-numbers, six-numbers, eight-numbers, etc. were not easily recognizable to the students in the way that odd numbers or (perhaps to a lesser extent) multiples of whole numbers were.

[51] Originally, I phrased this as "the whole that provides milk," but this presupposes *milk* as a separate object.

result of a conscious attempt to solve the problem of crunchy cereal. He likely experienced a familiar impatience with having to wait to eat and applied the strategy he knew best for dealing with that *feeling*. At the level of impatient-to-eat, the situations were identical and he did not need to carry out an *act* of generalizing at all–the generalizing occurred in the way he perceived the situation. Consider the following:

> Arnheim argues that "it became evident that over-all structural features are the primary data of perception, so that triangularity is not a late product of intellectual abstraction but a direct and more elementary experience than the recording of individual detail. The young child sees 'doggishness' before he is able to distinguish one dog from another" (ibid., 35). Moreover, since every act of perception already involves a capacity for abstraction–that is, an ability to select significant structure–it follows that visual perception is, at its heart, a form of thinking: "In the perception of shape lies the beginnings of concept formation" (Arnheim 1969, p. 27). (Johnson, 2007, p. 228)

But why do so many children seem to have difficulty selecting significant *mathematical* structure–in experiencing *mathematical* generality? As Mason and Pimm (1984) asked, "If examples are always examples *of* something, and counterexamples counter *to* something, how can students become aware of the 'something' which is being exemplified?" (p. 286). Is there a difference between, say, Eliot's description of how children learn the word *bottle* and how they develop understanding of *powers of two*? In other words, how do learners distinguish mathematical wholes?

I suspect this has much to do with what is perceived as movable within experienced situations. Children typically learn color-words later than words designating objects that move together or are designated by edges (and number-words in reference to cardinality still later[52]). We can only perceive a mathematical object "moving" as a unity if we use it in some sort of relevant context; this is key to the mathematical notions of invariance and change. And even when something like color is experienced as relevant in such a way that we mark it for attention, it is hard to point to; first, we must refer to our implicit sense of the situation. When the *something* has shared relevance, like, say, *blue* or *ten*, it becomes part of everyday language. When we experience it in a wide variety of contexts, our understanding deepens and becomes more nuanced. We all know that a blue sky and blue water are both instances of blueness, but we can't provide an experiential description of *blue*. It is not so easy to find ways to make the identity of *numbers with no odd factors* and *powers of two* apparent; they are both instances of… *what*ness? What makes *blue* and *ten* universally (at least in western culture) relevant such that we all find it meaningful to point to and name them. Why do so few find similar relevance in the *something*-ness that encompasses powers of two and numbers-with-no-odd-factors?

Experiencing What *Seems Significant*

> I get the eerie feeling that I must be calculating something quite general about the nature of systems, and not just compiling the individualized numbers of a particular idiosyncratic institution. (Gould, 1996, p. 122)
>
> I have often been impatient with colleagues who seemed unable to discern the difference between the trivial and the profound. But when students have asked me to define that difference, I have been struck dumb. I have said vaguely that any study which throws light upon the nature of "order" or "pattern" in the universe is surely nontrivial. (Bateson, 1964/1972, p. xvi)

For me, a sense of uncovering great truths has been a big part of my deepening interest in mathematics. I did not hear the students who participated in this study

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[52] Many children can recite numbers and even point to objects while reciting those numbers long before they appreciate cardinality; i.e. the notion that a number refers to a quantity that describes a set of objects.

articulate a similar motivation, but there are a number of instances in the classroom stories where a mathematical idea animated them and led them to make excited proclamations that had little to do with the content of the problem they were working on (e.g. in response to where little pieces of plastic chocolate should sit in a tray). But what *are* the “big ideas” that give Gould his eerie *feeling* and Bateson his *vague sense* of order and pattern?

Over the past summer, I became briefly obsessed with building my family tree. I have long been interested in my family background, but there was something that *felt* even more profound in the structure of lineage–something that had little to do with my actual family story and something that made me ask questions like what a second-cousin-once-removed might mean in other contexts. I also like *things* like feedback loops and filters and have identified them in a wide variety of diverse instances; I find new instances of these exciting simply because they belong rather than because of a particular interest in the situation from which they emerge. According to Devlin (2000):

> You can think of a mathematician’s abstract patterns as “skeletons” of things in the world. The mathematician takes some aspect of the world, say a flower or a game of poker, picks some particular feature of it, and then discards all the particulars, leaving just an abstract skeleton. In the case of a flower, that abstract skeleton might be its symmetry. For a poker game, it might be the distribution of the cards of the pattern of betting. (p. 77)

In my own experience, the abstract structure is part of the original experience, even if the name is not (though they continue to co-evolve). Often, the abstraction emerges from a *tip-of-tongue* sensation that alerts me to the presence of a broader class of whatever it is I am pondering; notably, this class may not yet have a name. Giving it one is a form of referring to the implicit. Once identified (with or without naming), that abstract structure might call forth other instances, but I cannot point to an intentional act of abstracting that created the class in the first place. So what alerts me to the potential significance of my original experience? Do I have a collection of implicit examples that exist in my mind and that somehow merge to announce a significant common structure–before I am aware of their connectedness? Or could this be like Arnheim's "doggishness," where generalities are directly perceived? For a young child, I suspect *blue* and *ten* qualify as big ideas. Again, however, why do so many directly perceive doggishness, blueness, and ten-ness before reaching school age, yet struggle long past then to perceive even elementary mathematical objects?

Here, further insight can be gleaned from Watson and Mason's (2006) distinction between generalizing and abstracting:

> We see *generalization* as sensing the possible variation in a relationship, and *abstraction* as shifting from seeing relationships as specific to the situation, to seeing them as potential properties of similar situations. (p. 94)

Initially, this distinction felt artificial to me, in much the same way that distinguishing within and between-domain analogies seemed artificial (as discussed in Chapter 3); i.e. the boundaries of "a situation" seemed arbitrary. Then I realized that this very distinction is mirrored in two experiential categories I was struggling to relate: *feeling the general case* and *seems significant*.

At the same time, I was struggling to understand research that separated these experiences at a neurological level: McGilchrist (2009) cited a number of studies that emphasized the right hemisphere's role in interpreting unfamiliar phrases and metaphor–but only when the definition of metaphor retains the distinction between novel and clichéd connections (which of course vary by individual). In this sense, the students' developing notion of *balancing* (discussed in the section on *feeling the general case*) might be experienced as *significant structure*–it is more general than their original extension of the list of consecutive pairs, and it allowed a transfer of meaning to cases with three addends. As in my experience, however, it does not seem to be the result of a deliberate act of abstracting. Although *number of addends* could also be understood as variation *within* a relationship (i.e. generalizing), it seems the students directly perceived the situation as being somehow about balancing, and this is likely what prompted them to recognize number of addends as a modifiable variable.

The distinction between generalizing and abstracting may also shed some light on the distinction sometimes drawn between math-in-context and pure math. Certain contexts seem powerful for their ability to bring certain mathematical relationships to awareness (in a manner consistent with what I am here calling abstracting). For me, the ice melt problem was one of these. I distinguish this from context that merely requires application of previously learned procedures. Contexts like the ice melt problem can *also* provide opportunities for varying attributes *within* newly discovered structure in a manner more consistent with what we are here calling "generalizing," as when the students explored various hypothetical melt rates with no connection to the way ice actually melts. Abstracting and generalizing, then, are essential to the identification and variation of mathematical objects.

Defining Mathematical Objects

> No object has a name per se, and the name of an object means only something in the code (the language) that one has accepted. But an object, name aside, has a meaning of its own, and all of us have had the good sense from our crib and later on, even without speech, to recognize meaning, to gain access to meaning. And once we have a general access to meaning, then we can put different labels on it, and the labels will stick to the meaning. (Gattegno, 1970, pp. 17-18)

> The phenomenon is understood as the manifestation of the thing itself, and phenomenology is therefore a philosophical reflection on the way in which objects show themselves–how objects appear or manifest themselves–and on the conditions of possibility for this appearance. (Zahavi, 2003, p. 55)

For purposes of this discussion, I use the term *mathematical object* to refer to a named or nameable (i.e. already separated by use) notion *as used and experienced in a particular context.* This is consistent with Watson and Mason's (2006) description of a mathematical object:

> [W]e use it to mean a thing on which a learner focuses and acts intelligently and mathematically by observing, analyzing, exploring, questioning, transforming, and so on. Thus an object could be a symbol, some text, a diagram, a theorem, a line of a theorem, a material object, an equation and so on. It is the "this" for which a teacher might say "look at this" or for which a learner might say "I am looking at this" or even "I am thinking about this." (p. 101)

Here, the inclusion of "look at this" seems to allow room for consideration of as-yet-ill-defined objects–it is broad enough to include anything that becomes a focus of attention. Attending to such objects can be a form of referring to the implicit. This seems similar to Mason's (1989) identification of a delicate shift of attention that takes place between abstracting and abstraction-as-mathematical-object. We all do this, but it seems we do so with different levels of awareness and intentionality in different circumstances.

In the narrative describing my students' and my experiences with the consecutive integers problem, I described an *aha* that took place as my own understanding of *powers of two* shifted (or expanded) from *two-to-the-nth* to *numbers with no odd factors*. Although the sets thus described are identical and their connection seems obvious in hindsight, until this shift happened, *powers of two* would not have manifested themselves to me as *numbers with no odd factors*. It was neither the problem nor I, but our interaction that provided the conditions of possibility for the appearance of a new conception that connected two previously separate (for me) mathematical objects. It may even have allowed for the emergence of the latter as object, as I am not sure I had ever had occasion to think about *numbers with no odd factors* prior to my work on this problem. Both definitions were transformed by their interaction, and though I still tend to refer to this set as *powers of two*, the term now evokes a richer meaning space than it did before.

Precision is important to logic, but it can lead to the false impression that what is clearly stated is *all that matters*. A definition can be 100% true and leave out a lot. Most people would say that they know what an even number is. But if you know that an even number ends in 0, 2, 4, 6, or 8, does that mean you know what an even number is? Is it important to recognize that all even numbers can be put into pairs? That they can be divided by two? That they cannot be written as the sum of two consecutive integers? That doubling always results in an even number? It would be easy to argue that these are different ways of saying the same thing, but they do not always manifest themselves in the same way, and it is entirely possible to know one without realizing one or more of the others, especially not in a way that makes a particular instance come to mind at an opportune time. For some, *even numbers* might encompass all of the points above in a single object. For others, *doubles* and *even numbers* might remain separate. *Even* comes to mean new things as old definitions interact with new contexts–another example of what Gendlin (1995) called “crossing and dipping.” In this study, the dynamic nature of mathematical objects as they shifted between implicit and explicit was particularly salient in the students’ work with summing consecutive integers, as they (unknowingly) used a variety of different descriptors for the same sets of numbers (see Appendix D for a more detailed discussion of how these objects emerged in the classroom):

- Even-numbers, numbers-that-can-be-divided-by-two
- Multiples-of-three, numbers-with-three-equal-addends
- four-numbers (i.e. 1+2+3+4, 2+3+4+5, …), multiples-of-four-plus-minus-two, every second multiple of two, double-odds, numbers-that-can’t-be-the-difference-of-two-squares
- six-numbers (i.e. 1+2+3+4+5+6, 2+3+4+5+6+7, …), multiples-of-six-plus-minus-3, triple-odds
- doubles-from-two, powers-of-two, numbers-that-can’t-be-expressed-as-multiple-odds, numbers-with-no-odd-factors

Conversely, in some cases the mathematical objects students used blurred important distinctions as they conflated sets of numbers that were not the same:

- four-numbers, multiples of four
- powers of two, square numbers, even numbers
- Odds, multiple-odds

During one interview, the students identified *multiples of 6 by adding 12* as significant:

$$6$$
$$6 + 6 + 6$$
$$6 + 6 + 6 + 6 + 6$$

I used the term *odd multiples of six* to refer to the same set; by this, I meant *odd numbers of sixes*. I did not recognize the confusion this description caused until one of the students commented that these were not odd; i.e. 1 x 6 = 6, 3 x 6 = 18, 5 x 6 = 30, etc. If the products are even, why, then, was I calling them odd multiples of six? These examples emphasize the complexity of understanding embedded in even seemingly simple concepts.

As I described in Chapter 4, articulating *odd number of odds* turned it into a mathematical object that allowed contrasting possibilities to come to mind while at the same time offering a perceptual framework with the potential to connect many diverse explorations. Directing attention to this as a significant object during class discussion might have helped prompt more systematic consideration of odd numbers of evens, even numbers of odds and even numbers of evens as connected within a larger space of possibility. In other words, the use of *something to do with* in this manner might have allowed both a broadening and constraining of the space of possibility this problem offered, and it may have helped connect the many conjectures put forward during our collective work on this problem.

Of course, a name can also shut down possibilities; knowing that a *power of two* is *two multiplied by itself any number of times*, likely prevented me from recognizing it as *a number with no odd factors*, especially when I had never explicitly thought about whether every number has a unique set of prime factors. Anthony de Mello's (1990) reflection on labels strongly resonates with my own experience:

> The concept always misses or omits something extremely important, something precious that is only found in reality, which is concrete uniqueness. The great Krishnamurti put it so well when he said, "The day you teach the child the name of the bird, the child will never see that bird again." How true! The first time the child sees that fluffy, alive, moving object, and you say to him, "Sparrow," then tomorrow when the child sees

> another fluffy, moving object similar to it he says, "Oh, sparrows. I've seen sparrows. I'm *bored* by sparrows. (p. 121)

The feeling of recognition when we meet a familiar bird or a familiar concept can in fact be very heady rather than boring–so much so that it is easy to leap to conclusions based on a perceived match. But the headiness is based on a feeling of rightness that includes a sense of strong forward movement. While coding the transcripts and my own research diary, I dubbed this a *rush-of-category*, highlighting an experiential connection with the *rush-of-right* that often is part of the experience of certainty that something has resolved itself.

This reminds me of driving with a friend past a very smelly feedlot when he became visually agitated and repeatedly declared, "That's hormone-fed beef!!" I was sure that the smell was silage being used to feed the animals, but he was so excited at having a familiar *name* for a new experience that he did not even hear me. Perhaps the sense of connection between a label and an experience prompted the rush. I wonder if this rush is what Dewey was talking about when he said:

> All thought in every subject begins with just such an unanalyzed whole. When the subject-matter is reasonably familiar, relevant distinctions speedily offer themselves, and sheer qualitativeness may not remain long enough to be readily recalled (Dewey 1934/1987, p. 249; as cited in Johnson, 2007, p. 75)

Whether through boredom or a heady sense of connection, then, the feeling of knowing bestowed by a name may reduce its complexity to a label. *Name* here must be taken in the broad sense of symbolized as somehow set aside in experience, whether or not a label is attached to that which was set-apart. My son has an "I Spy" game where he is supposed to find cleverly hidden objects amidst a complex visual scene. In one such scene, we were looking for a king's sword. It took us a very long time to find it, although it was in plain sight in the King of Spades' hand: I had perceived the playing card as a whole and ignored its attributes. Until then, *King of Spades* had been defined (for me) for its role in various card games and/or tricks that I had experienced–these defined the card in its entirety as the "highest appropriate pragmatic level of description" (Gallagher & Zahavi, 2008, p. 160); the king's sword had no pragmatic relevance in my experience of playing cards (the bacterium in the sugar gradient likely wouldn't have noticed carbon, hydrogen, or oxygen in its version of *I Spy*). In much the same way, *numbers with no odd factors* had no pragmatic relevance until they emerged as significant in the consecutive integers problem. This makes me wonder: Most of us learn numbers by counting, so perhaps we have to relearn the value of decomposing and recomposing them in contexts where factors and prime factors play a significant role (or learn to count in contexts that make these significant).

A conscious familiarity with names may disguise aspects of the things (and people) we think we know (and even love) and may thereby help us to appreciate them more deeply. When we are able to assign a new experience to an existing category, it can provide a sense of rightness and confidence. It can also limit new experience to existing categories, thereby resulting in a loss of the implicit intricacy that preceded the name.

Before leaving this topic, it is also important to acknowledge what a name *can* bestow–provided that its limitations are recognized. Had the students (or I) not recognized *powers of two* at all, they would not have even come to our attention. In an essay entitled, "The Scripture of Maps, The Names of Trees," Stephen Trimble recollected the summer he spent conducting a census of trees in a Colorado forest. As he learned to identify and name the different species, he reflected: "Never before had I noticed the specificity of trees in their environment." The names bestowed a sense of familiarity and closeness that allowed each to stand out from the forest:

> Ever since, I have seen these trees as my friends. When they grow along my path, I reach out to them, draw their needles through my hands, and smile. I say their names–an acknowledgement of kinship–like a formal genealogy, another chapter of Scripture. *Pseudotsuga. Picea. Juniperus. Abies concolor. Pinus flexilus. "Pseudotsuga.* Douglas fir. I am here, too." (Nabhan & Trimble, 1994, p. 30).

Mathematicians sometimes speak of particular numbers, patterns or defined relationships with a similar sense of kinship and reverence.

Reaching For the Intuitive Through Bridging: What Would Happen If…?

Once the felt meaning of a situation has been articulated and becomes an object, it can be negated and/or assigned significant attributes that can be varied systematically. Watson and Mason (2006) emphasized the importance of exploring the variance permitted within a particular mathematical situation; this is the essence of what they called generalizing. Here, I am interested in a very particular sort of variation: There were a number of occasions during this study where someone (usually me) had a particular agenda in manipulating the bounds of permissible change, usually to generate enough doubt that students began to recognize–or feel–the need for further evidence. Here, variation allowed a subtle shift of context that *altered the intuitive feel of what was happening.*[53]

Particularly during the lamp and ice melt problems, I often found myself asking, "What would happen if…?" In doing so, I created variants of the original problem that shifted attention in significant (though often subtle) ways. During the lamp problem,

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[53] Clement (1981) described bridging as a type of analogy generated as an intermediary used to evaluate the validity of the relation between a potential source and target; the bridge contains features of both. Here, the "target" is vaguely formed; the bridges are intended to clarify the target. Extreme or special cases also serve this purpose; in fact, the bridges used here might be seen as special cases.

students perceived a number of variations as somehow distinct from the original formulation of the problem: *What would happen if* I bought one lamp for $7 and sold it for $8 and another lamp for $9 and sold it for $10? *What would happen if* I bought the two lamps on different days? *What would happen if* I reversed the order of the two purchases? *What would happen if* I bought the lamp for $7 then immediately sold it for $10? *What would happen if* I bought for $7, sold for $8, then bought for $98 and sold for $100? And so on.

The bridges used in the ice melt context were (a) idealized data sets that made potential melt trends (i.e. steady, oscillating, accelerating) easier to recognize and (b) alternate scenarios that invited students to consider whether their conclusions would hold up under varying conditions that they deemed significant and I did not–in particular an extremely large amount of fast-melting ice. In both cases, the bridges were based on a destination that I as the teacher had in mind, and I used them to push student thinking in a particular direction.[54] Once engaged, the students also created bridges to deepen their understanding of the situation. While my pedagogical reason for using these bridges was to persuade, I also used them to explore and deepen my own understanding.

In this study, then, bridging of this nature was *primarily* developed as a *persuasive* tool. I suspect that it is possible to help students become more aware of the possibilities bridging might open in mathematical contexts. Although I did not consider starting ice a significant variable in the ice melt problem, I somehow recognized that it *could* feel significant. Perhaps it was a very, very quiet niggling doubt, easily silenced by stronger arguments. By varying the nature of the starting ice (in my mind), I was able to

[54] Sometimes, though, subtle shifts of context can catch me unaware: I recently had an unsettling experience while driving. As I was turning left at a major intersection, I suddenly had the uncomfortable sensation that I was running a red light - i.e. the one I was turning into. Perhaps it was the larger-than-normal size of the intersection that interrupted my sense of confidence in a normally automatic maneuver. In any case, I was momentarily baffled. Thankfully, I recovered in time and didn't slam on my brakes.

understand more deeply the role it played in the problem and, importantly for my role as teacher-researcher, why some students might find it difficult. In this sense, bridging might also be considered an *empathic* tool.

Experiencing Elegance as Non-Conflict, Beauty as Harmony

Objects and their variations emerge from a broader context. To begin to appreciate the complex network of relationships among them involves attending to the implicit. From this holistic space, gaps and contradictions can become apparent–often in the form of a niggling doubt. For example, it bothers me when an explanation seems unwieldy. But what makes it *feel* that way? Sometimes it seems that there's an unspoken, "It should be simpler than that" which seems to imply an implicit knowledge of that simpler case. Or perhaps it points to an implicit awareness of aspects of understanding that are not compatible, or even just to discomfort with having to keep too many pieces in mind to make an accurate assessment. The more complex an explanation is, the more room there is for potential contradiction; i.e. when something is clear and concise, requires few assumptions, brings together formerly disparate areas, or generalizes widely, then areas of potential conflict are minimized or eliminated. Perhaps this is why an explanation that is considered mathematically elegant may be experienced as beauty.

Many of the students seemed untroubled by the many disparate rules and strategies that emerged during their collective work on summing consecutive integers. Perhaps this was because so much of their attention was dedicated to particular focus areas that they had little time to wonder how four-numbers (or multiples-of-four-plus-two), six-numbers (or multiples-of-six-plus-three), odds, multiple odds, and powers of two fit together. I was troubled by overlap and gaps in a broader context; I wanted all the pieces to fit together like a puzzle.

One student, however, even expressed dissatisfaction with *multiple odd* as an expression of what worked–he preferred *multiple-prime*, which he recognized as eliminating an overlap. I am not sure why this had not occurred to me, though I suspect it has something to do with the clarity of space *multiple odds* describes.[55] As I showed in Chapter 4, the same student was very excited (as I had been) to figure out that *every* odd factor–including the number itself–produces one way of writing consecutive, positive, integers. I further felt the need to prove to myself that there was one *and only one* way.

The only instance in the transcripts where a student used the word *beautiful* was when that student and her partner were trying to explain a proof for the Pythagorean Theorem (elaborated in Appendix A). They were arranging pieces of squares on the sides of a given triangle and found that the pieces of b^2 fit *perfectly* on c^2 (with a square hole matching a^2 somewhere inside). They had not expected this result, partly because the pieces they were working with included both very small and very large pieces (i.e. they were highly asymmetrical). They were also very aware of a contrast with a previous set of pieces that *almost* fit and likely left a niggling doubt as to whether that example counted. The perfect fit allowed the doubt to be resolved, and it may be that lack of conflict that the student perceived as beautiful.

Such beauty (harmony?) may be contrasted with what I referred to in my research diary as *brake burn*, or the experience of competing impulses, which I find not unlike the (stressful) feeling of trying to attend to two conversations (or to the competing demands of two children) at once.[56] It may also be contrasted with what I called

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[55] In fact, I still prefer that phrasing, as odds are easier to identify than primes. Besides, some numbers are the product of more than one prime; for example 35 is a multiple of both five and seven. Also, *every* odd factor allows another way of writing a particular number as the sum of consecutive positive integers.

[56] It also feels similar to doing a "Stroop Test," where you have to identify the color a word is printed in when the word itself names a conflicting color; e.g. the word green is printed in the color blue (a Google search will

overload–in such cases, there is not necessarily anything that feels *wrong*, but there is so much information that I cannot hold it all together at once such that I might perceive it as a larger unity. At a certain point, the whole collapses. It is not just that the last piece of information I try to include in my consideration does not fit: By trying to add one more piece, the whole structure collapses.

In the following chapter, I take a more speculative look at how conflicting information might contribute to experiences of doubt and certainty and how such conflict might look at the neurological level. I am particularly interested in what might be happening at the explicit-implicit interface.

produce many examples). “Project Implicit” (I.A.T. Corp., 1998-2001) relies on a similar phenomenon to identify implicit biases by requiring participants to sort words and pictures into categories; we sort faster when the categories match our biases–even when we don’t consciously recognize those biases. Experientially, they emerge in the way they make sorting slightly more difficult. A key difference between these tests and the experience of *brake burn*, however, is that in the latter case, “I” don’t have a clear sense of which competing impulse “I” *want* or *should* be attending to for the purpose of the selected task.

6. Dynamic Meaning at the Implicit-Explicit Interface

> Sometimes I like to imagine a Two-Headed Monster of mathematical criticism. The first head demands a logically airtight explanation, one with absolutely no gaps in the reasoning or any fuzzy "hand-waving." This head is a stickler, and is utterly merciless. We all hate its constant nagging, but in our hearts we know it is right. The second head wants to see simple beauty and elegance, to be charmed and delighted, to attain not just verification but a deeper level of understanding. Usually this is the more difficult head to satisfy. (Lockhart, 2009, pp. 111-112)

> Nagarjuna… deconstructs all pairs of opposites, such as action and inaction, rest and motion, and finds that both elements of each pair are empty–*sunya*–that is to say, each exists only in relation to the other. (Varela, 1999, p. 34)

Because so much of what we communicate is verbal, it is easy to lose contact with the pre-symbolic meaning from which language emerges. In this work, I have looked for ways to help students describe and work with the dynamic meaning that arises from this more holistic space. Broadly, the explicit mode of being involves more or less *defined symbols* within a *focused space of inquiry* and chains of reasoning involving *clear rules of induction/deduction*. Here, symbols have largely lost their metaphorical character in the sense that the space they bound is relatively static. In the implicit mode of being, *attention is broad* and *meaning is holistic*; i.e. it has not yet been explicated (as in the *niggling doubts*, *tip-of-tongue-understandings*, *all-thats*, *something-to-do-withs*, and *seems significants* described in Chapter 5).[57] These are in constant interaction and may be contradictory, but they are paradoxically merged in a unified experience of the world. While this may seem obvious, it goes against much in popular culture, which seems to emphasize either the importance of listening to your gut (e.g.

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[57] As Gendlin (1962; 1978) repeatedly emphasized, however, symbols can be used to name broad, holistic experience:

> In the present instance, we have a symbol: "that," which doesn't symbolize by rendering the meaning in symbols, but which only serves as a grip on the felt meanings performing essential functions in the process of solving a problem. (Gendlin, 1962, p. 74)

Gigerenzer, 2007; Gladwell, 2005) or maintaining a constant, vigilant skepticism (e.g. Shermer, 2011).

McGilchrist (2009) noted that “the hemispheres of the brain can be seen as, at the very least, a metaphor” for “consistent ways of being that persist across the history of the Western world, that are fundamentally opposed, though complementary, in what they reveal to us” (p. 461). In his view, the connection between left and right is far more complex than a one-way portal for transfer of information from the so-called creative right to the so-called analytical left for scrutiny and potential acceptance. I use McGilchrist’s metaphor to support a generative dialogue between neuroscience and my own observations and analysis. Following my consideration of how experiences of doubt and certainty might arise from neurological disagreement, I similarly consider how neurological considerations might be implicated in experienced relevance.

Following Gendlin (1962; 1978), I begin this chapter with a consideration of the dynamic interface between implicit and explicit understanding. The model I develop to do so (see Table 2) rejects both the separation of creativity and skepticism and the primacy of either of what are often loosely referred to as (explicit) logic or (implicit) intuition. In a nutshell, the symbols used in logical formulations emerge from (careful or cautious) explication of implicit understanding. Both logic and intuition may be approached carefully or cautiously, and the level of agreement within and between these can impact the level of caution with which each are treated. Conflicting voices compete for primacy and may or may not reach the level of full awareness; these comprise Varela’s “virtual self” (Varela & Scharmer, 2000).

Explicit (Symbolic / Focused)	Interface (Awareness, Explication)	Implicit (Pre-Symbolic / Holistic)
Infallible Logic	← Careless Explication →	Infallible Intuition
↕	*Logical Contradiction or Agreement* *Conflicting or Harmonizing Intuitions* *Intrapersonal Conflict or Agreement*	↕
Cautious Logic	← Cautious Explication →	Cautious Intuition

Table 2. Implicit/explicit interactions.

Creativity vs. Skepticism?

In a preliminary set of codes for my data, I played with the possibility of categories that opposed creativity and skepticism. This is a fairly common distinction, and some findings from neuroscience seem to back it up:

> The creative ability to construct plausible-sounding responses and some ability to verify those responses seem to be separate in the human brain. Confabulatory patients retain the first ability, but brain damage has compromised the second. One of the characters involved in an inner dialogue has fallen silent as the other rambles on unchecked, it appears. (Hirstein, 2005, pp. 4-5)

In reflecting on the experiences documented for this study, however, such a dichotomy often seemed an awkward opposition. Skeptical arguments were often vague or intuitive: "That just doesn't feel right." On the other hand, the feeling of *aha!* and the alarm of *wrong!* feel (at least to me) more similar to each other than do *aha!* and *something-to-do-with,* both of which I previously saw as aspects of creativity. Similarly, *something-to-do-with* and *niggling doubts* might be seen as opposite in the sense that one bolsters and one negates, but both are similarly doubtful in their vagueness and in the (perhaps learned) way they can alert me to understandings that (for the time being) dwell beneath awareness. In Table 3, I present a more nuanced description of doubt and certainty that names strong and weak experiences of furtherance and hindrance.

	Hindrance	**Furtherance**
Strong	Wrong!	Aha!
Weak	Niggling Doubt	Something-To-Do-With

Table 3: Experiencing doubt and certainty.

A more useful distinction than creativity and skepticism (or imagination and rigor, as it is sometimes posed), then, may be that of distinct modes of attention–one broad, vague, and non-symbolic (the implicit) and the other narrow, more clearly defined, and dependent on language (the explicit). This seems consistent with McGilchrist's (2009) characterization of the experiential worlds of the left and right hemispheres:

> The world of the left hemisphere, dependent on denotative language and abstraction, yields clarity and power to manipulate things that are known, fixed, static, isolated, decontextualised, explicit, disembodied, general in nature, but ultimately lifeless. The right hemisphere, by contrast, yields a world of individual, changing, evolving, interconnected, implicit, incarnate, living beings within the context of the lived world, but in the nature of things never fully graspable, always imperfectly known–and to this world it exists in a relationship of care. (McGilchrist, 2009, pp. 174-75)

Notably, these modes of being are in *constant* interaction; nobody is a left-brainer or a right-brainer (unless perhaps they have a damaged or missing hemisphere, and even then, the brain's plasticity allows the healthy hemisphere to compensate).

To reiterate, then, I do not wish to dichotomize abilities to hemispheres: Descriptors such as implicit / explicit or logical / intuitive are useful, but they cannot be separated in the brain, never mind in the mind-body of lived experience:

> Even seemingly simple functions call for the coordination of a large number of cerebral areas, each making a modest and mechanical contribution to cognitive processing. Ten or twenty cerebral areas are activated when a subject reads words, ponders over their meaning, imagines a scene, or performs a calculation…. Neither an isolated neuron, nor a cortical column, nor even a cerebral area can "think." Only by combining the capacities of several million neurons, spread out in distributed cortical and subcortical networks, does the brain attain its impressive computational power. (Dehaene, 1997, p. 217)

Dichotomizing particular abilities to either hemisphere is problematic even in cases like language that have sometimes been treated as highly localized:

> [T]he fact that a left-hemisphere stroke in the "language areas" disrupts language function in 98% of right handers does not mean that the function of the left hemisphere is language. Rather, it means that the left hemisphere executes instructions that are required for normal language functions. (Kolb & Whishaw, 2009, p. 290)

One could also say that severe damage to the eyes disrupts the ability to read in 100% of people. This does not mean that the function of the eyes is to read or that eyes are sufficient for reading to take place.

At times, it has been tempting to return to the creativity-rigor duality. The non-verbal right hemisphere allows attention to that which exists outside the shackles of the symbols we have developed for talking or thinking about them. Clear arguments are put forth in logical, symbolic terms, and they form the basis of *definitive refutations*. But niggling feelings of wrong[58] can also play a powerful role in our thinking, and faulty

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[58] These may correspond to McGilchrist's (2009) right-hemisphere "bullshit detector" (p. 193).

conclusions can seem to be logically deduced.

According to McGilchrist, the left hemisphere will provide the *logically* correct answer to a logical syllogism, regardless of well-understood factual errors in a premise. The right hemisphere retains common sense and will not accept a counter-intuitive conclusion that follows logically from a faulty premise. It might be said that both modes of being can be skeptical but inhibit on the basis of different sorts of evidence. Perhaps feedback loops that allow logic to negate itself are easier to articulate, whereas feedback regarding right hemisphere ideas may not even reach awareness. When this happens, good ideas may get ignored and negative influences may persist. This might help us understand why ideas are that are clearly articulated can seem very convincing.

Ramachandran (2006) recently argued, “More harm has been done in science by those who make a fetish out of skepticism, aborting ideas before they are born, than by those who gullibly accept untested theories” (p. 48). His title retains the dichotomy between creativity and skepticism (“Creativity versus Skepticism within Science”), but perhaps the way we view *both* is problematic. In fact, he qualifies his definition of harmful skepticism:

> By *skeptic*, I mean one who adopts an overall skeptical attitude, being unreceptive to anything new–not one who practices legitimate skepticism toward claims that are empirically unproven. (p. 49)

This is a nice description of what healthy skepticism *is not*; skepticism that attends to the implicit in a disciplined manner might contribute to a deeper understanding of what it *is*.

How We Disagree With Ourselves

Walter Freeman is a theoretical neuroscientist and philosopher who (like Varela) rejects the representationalist view of cognition in favor of a neurodynamic model[59] that attends to meaning rather than symbols alone. He described the emergence of hemispheric-wide goal-states and neural patterns of activation which in turn are implicated in global patterns (Freeman, 2000). Within these complex interactions, experiences of doubt and certainty likely arise result from relative levels of strength and agreement among neural circuits, only some of whose activity typically reaches the level of awareness. What gets carried forward in behavior does not emerge from fully harmonized impulses.

In fact, even a repeated stimulus "does not induce precisely the same pattern in the same brain, let alone in any other brain" (Freeman, 2000, p. 22). Furthermore:

> "[B]rain activity patterns are constantly dissolving, reforming, and changing, particularly in relation to one another. When an animal learns to respond to a new odor, there is a shift in all other patterns, even if they are not directly involved in the learning" (p. 22).

Here, Freeman was not discussing language, but his comments seem consistent with the observed evolution of meaning and language that has been so evident in my own work and that is so central to Gendlin's work. Even if there were a particular neuron associated with a particular word, the word itself is just a label attached to a constantly shifting, complex network of meaning. Our minds deal with a situation in its entirety, not just with the words we choose to describe it. In this sense, it is not surprising that what may seem from the outside to be the same stimulus produces different patterns of neural

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[59] That is, a model that uses the language of dynamics to describe the activities of neurons.

activation (at what Freeman calls the “macroscopic” level of neuron populations rather than at the microscopic level of individual neurons) each time a person encounters it.

By contrast, interactions between implicit and explicit modes of being are often presented in linear terms:

> Our brains are not like computers, working systematically and logically. They are metaphor machines that leap to creative conclusions and belatedly shore them up with logical narratives. (Stewart, 2006b, pp. 22-23)

McGilchrist (2009) presented an interesting way of considering hemispheric differences as bringing different aspects of experience to awareness: At the very least, conclusions based on the narrow attention of the left must make sense within the more global perspective of the right:

> Some very subtle research by David McNeill, amongst others, confirms that thought originates in the right hemisphere, is processed for expression in speech by the left hemisphere, and the meaning integrated again by the right (which alone understands the overall meaning of a complex utterance, taking everything into account). (McGilchrist, 2010)

While many of the oppositions salient to this discussion do seem to fit within descriptions of hemispheric differences, there are likely many ways we might disagree with ourselves:

> [The brain] contains mutually opposed elements whose contrary influence make possible finely calibrated responses to complex situations. Kinsbourne points to three such oppositional pairings within the brain that are likely to be of significance. These could be loosely described as ‘up/down’ (the inhibiting effects of the cortex on the more basic automatic responses of the subcortical regions), ‘front/back’ (the inhibiting effects of

> the frontal lobes on the posterior cortex) and ‘right/left’ (the influence of the two hemispheres on one another). (McGilchrist, 2009, p. 9)

Conflict also occurs *within* these large-scale opponent processors: Even at what Freeman (2000) calls the microscopic level, individual neurons may be excitatory or inhibitory. The *experience* of uncertainty may arise when broader patterns of activation (i.e. at the macro level) contradict one another at some level of awareness. Perhaps the more areas thus deemed relevant,[60] the more difficult it becomes to achieve the level of agreement (experienced as confidence) to motivate a particular behavior.

To me, the conflict between one and two dollar solutions to the lamp problem *felt* much like the optical illusion where you can see either faces or vases–not in the sense that both were adequate interpretations of the situation, but in the *feeling* that shifting between the two arguments produced. I wonder now if the feeling of shifting might be literal–in the neural competition for primacy, either may win. But if both produce strong signals, perhaps neither maintains its status, and the answer seems to oscillate. To many students, conflicting responses to the lamp problem seemed perfectly reasonable; either response–on its own–would have been sufficient to prompt certainty. Unlike the face and vase, though, one dollar and two dollars cannot both describe the profit made from the sale and re-sale of the lamp.

It is likely not necessary for explicit logic and implicit intuition to communicate–perhaps they just lend confidence or doubt to conclusions such that each can re-evaluate on its own terms. For example, intuitive confidence in two dollars does not help the logical argument for two dollars, but it does support it in the sense that the conclusions do not conflict with one another. And it does not provide an argument against one dollar, but it provides a reason to look for one. Neither are logic nor intuition

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[60] Freeman (2000) identifies the limbic system as key to intentionality.

self-contained chunks with no internal arguments. Doubt is not just a two-way conversation.

Referring back to Table 2, logic may be challenged by competing logic (as when students experienced competing logical arguments for two dollars as logically sound) or by the counter-intuitive (as when they started to get a vague sense of what might be wrong with the argument for one dollar). Cautious logic involves checking terms, premises, and rules of deduction. Logical consequences only imply truth within the assumptions made by the metaphorical act of categorizing. Categories rely on subjective perceptions and on decisions regarding what is significant in a particular context. The intuitive may be challenged by competing logic or conflicting intuitions. Cautious intuition involves attending to the implicit, allowing it to be named and subjected to logic, and allowing what has been named to remain dynamic (in contrast to uncritically accepting the rush that can come with seeming recognition).

The middle space in Table 2 can be experienced as either a healthy balance or, when differences seem irreconcilable, as a sort of paralysis. At times, it can feel as though the required certainty threshold is somehow set very high and doubt becomes a pervasive mood–a deep rut that allows no movement. But doubt need not feel the same as *no* way forward–it is often a lack of conviction regarding the possibilities that do present themselves. At extreme levels, both extreme doubt and extreme certainty can be pathological. Pathological certainty is evident in confabulation:

> Confabulation involves absence of doubt about something one should doubt: one's memory, one's ability to move an arm, one's ability to see, and so on. It is a sort of pathological certainty about ill-grounded thoughts and utterances. (Hirstein, 2005, p. 4)

Pathological doubt is a key feature of obsessive-compulsive disorder:

> Although the phenomenology of obsessive-compulsive disorder appears to be quite diverse, with many distinct kinds of obsessions and compulsions, there are three important core features: abnormal risk assessment, pathologic doubt, and incompleteness. (Psychiatry.HealthSE.com, 2004-2005)

Some students seemed to reach a state of moving-forward-certainty very easily, while others tended to be much more careful. Still others went back and forth between powerful *ahas* and checking their ideas, yet were able to find confidence in their checks. In this study, I observed students in each of these categories who were willing (even eager) to engage deeply with complex problems and make strong efforts to understand ideas presented by their peers. Others, however, seemed to be in a near-constant state of doubt. I am not sure if this is because the tasks were particularly difficult for them or because they entailed more doubt than they were comfortable dealing with (although this may be one of the ways that something seems hard to begin with). While this may seem to support the notion that creative and skeptical faculties are somehow separate, perhaps it is the *need* to check that is compromised in confabulatory patients. Or perhaps confabulation occurs when the *relative* strength of rewards for new ideas exceeds the need to check: “After all, it is a matter of temperament whether one approaches truth through ever false over-statements or through ever true under-statements” (Lakatos, 1976, p. 58). Type I errors are Lakatos’ “ever-false overstatements; i.e. being too gullible, too lenient, having a filter with too-big (or otherwise permissive) holes, speaking too broadly, wrongly believing something that is false. Type II errors are Lakatos’ “ever-false understatements”; i.e. being too skeptical, too rigid, having filters with too-small (or otherwise restrictive) holes (or barriers), speaking too narrowly, rejecting something that is true).

Observable Disagreement

A person with alien hand syndrome (often someone with a severed corpus callosum) whose hands work independently of, or even at cross-purposes with, one another (with one hand experienced as outside of conscious control) provides a powerful image of a mind that disagrees with itself. It also emphasizes the fully-embodied nature of a mind that becomes aware of itself through observable behavior.

Dehaene (1997) provided a fascinating account of a study by Gazzaniga and Hillyard in which split-brain patients were unable to name digits presented only to the right hemisphere via the left visual field. One patient was able to defy this rule, likely via bodily cues:

> [He] appeared to recite the number sequence slowly and covertly until he had reached a numeral that "stuck out"–those were his own words–and which he then uttered aloud. Nobody knows exactly how the right hemisphere managed to signal that the number it had seen had been reached. It might have been some kind of hand movement, a contraction of the face, or some other cueing artifice that split-brain patients often devise for themselves. (Dehaene, 1997, p. 183)

The right hemisphere recognized a name that felt like a match, but it could not do the naming. Here, the importance of our bodily interactions with an environment to our experience of both the logical and the intuitive becomes apparent, and the bodily feeling of forward movement likely does not distinguish between them. In response to examples of cross-cuing used by split-brain patients (whereby behaviors performed by one side of the body provide visible cues as to the action of the opposite side of the brain), Gazzaniga (1978) concluded:

> [I]t is the verbal system that is the final arbiter of our multiple mental systems, many of which we come to know only by actually behaving.

> Emitted and elicited behaviors are important ways of discovering the multiples selves dwelling inside. Behavior is a key way that these separate information systems can communicate with each other. (p. 157).

As Gendlin (1978) made clear, attending to much more common experiences of bodily tension can alert us to sources of conflict that might otherwise remain beneath ordinary consciousness.[61] Focusing, with its emphasis on checking and modifying symbolization against felt meaning (both of which influence each other and contribute to a dynamic interpretation of meaning) provides an intentional way of attending to more complex interactions between various aspects of mind (i.e. brain-body-environment) that emerge–if we pay attention–as a unified felt sense.

Experiential Narrative: What Gets Carried Forward?

Competing voices all arise within the context of an experiential narrative that defines relevance. Freeman (2000) described how each hemisphere constructs goal states through emergent neural activity patterns. These then prime the sensory cortex to select the predicted consequences. This is what we experience as attention and expectation (p. 33). Both, it seems, are also key aspects of relevance.

In terms of this study, relevance may emerge from a learner's interaction with the stated intentions of a particular problem, from personal tendencies toward some combination of solving, winning, connecting with others, and likely from a complex array of other motivations as well. While the sense of agency that accompanies these goals

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[61] Perhaps it is the absence of such conflict that helps characterize the visible peace de Mello (2003) attributes to "The Master":

> It was a joy to behold the Master perform the simplest acts – sit or walk or drink a cup of tea or drive away a fly. There was a grace in all he did that made him seem in harmony with nature, as though his actions were produced not by him but by the universe. (de Mello, 2003, p. 30)

Or as Varela (1999) put it, "[H]ere the paradox of non-action in action vanishes when the actor becomes the action, that is to say, when the action becomes nondual" (p. 34).

may feel like a top-down process, that sense likely also emerges from a competitive process that occurs largely beneath consciousness. How awareness interacts with such goals is a fascinating question, but one that I do not address here in great depth. I do know that having attended closely to the implicit, I now have a greater tendency to direct (or so it seems) my attention more broadly at certain times,[62] and I now attend more closely to the niggling doubts and tip-of-tongue experiences that I described in Chapter 5. But I do not know why they come to my attention in the first place, and I do not know what prompts me to switch modes of attention at a particular time. Thinking back to my experience of turning left into a red light in a particularly large intersection and suddenly feeling that I should be stopping (Chapter 5), the interruption of flow in my driving precipitated by my perception of running a red light certainly made it feel as though *I* were suddenly involved in deciding what to do next. But perhaps it was just that the dissonance was loud enough that *I* became aware of it. It does seem that having reached awareness, it is now likely to influence subsequent behavior.

Closing Thoughts

It is unclear to me how intentional awareness is implicated in what we are able to bring to consciousness and how being an object of awareness is implicated in the ongoing evolution of behavioral possibility. From a pragmatic point of view, however, it does seem that we can choose to direct our attention and that how we do so alters our attentional patterns. Our attentional patterns are also altered through our interactions

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[62] While working on an earlier project, I started wondering whether I could “will” metaphor to mind. While working on a design project in science, I tried this out in terms of a more general search string; i.e. “I need something cylindrical.” I tried imagining a generic cylindrical-ness to see if it would call specific cases to mind; it seemed to work. I wonder if this is a sort of “right-brain” search–I’m thinking here of comments in Lehrer’s (2008b) article that describe right-hemisphere cells as more broadly tuned, with longer branches and more dendritic spines; i.e. neurons that collect from a larger area of cortical space. Or perhaps my search fits into the logical hierarchy evident in “20-Question” type games. At age three, my son did not differentiate between the power of asking, say, “Is it alive?” and “Is it a golden retriever?”

with others. I now return to the classroom to discuss the significance this work may have within the research community that informed it.

7. Reflections & Implications

> I hold to the presupposition that our loss of the sense of aesthetic unity was, quite simply, an epistemological mistake. I believe that that mistake may be more serious than all the minor insanities that characterize those older epistemologies which agreed upon the fundamental unity. (Bateson, 1979/2002, p. 17)

I return now to the questions that motivated and oriented this study:

- *How do learners experience doubt and certainty in the mathematics classroom?*
- *More specifically, how might learners develop deeper awareness of partially conscious feelings associated with doubt and certainty and use them as gateways to deeper understanding?*

In my efforts to explore these questions, I used the principles of enactivism to inform the design of the learning environment as well as my role as empathic teacher-researcher. In doing so, I attempted to bring Gendlin's philosophy of the implicit and practice of focusing into conversation with selected aspects of mathematics education. I now offer closing reflections–as learner, teacher, and researcher–on these aspects of my work. In doing so, I note situations where (in hindsight) I can see that I came on too strong in my interactions with the students, and I consider how I might have engaged more productively. However, to do so *in the moment* is the tricky part, and it is here that Varela's attention the *development* of the empathic second person seems to have a very promising role in mathematics education:

> The enactive approach is theory. You can theorize about know-how or [another] intrinsic action. Nevertheless, again, we're back to theories and practices. The practices can only be done by doing them. You can theorize and have a nice understanding, which is also good, but it ain't the same thing. So it's not that the enactive approach is closer to that

> transformation of learning. It can be useful, it can be a pointer to the right direction, but it ain't it. But for practice, the only thing is to do it. (Varela & Scharmer, 2000, p. 13)

Reflections on the Learning Environment

The manner in which problem-spaces were defined was highly significant to students' experience of doubt and certainty in mathematics. This was particularly evident in how students negotiated the space between problem-as-stated, personal entry points, and the entry points chosen by other students in the class.

Defining and Connecting Sub-Problems

While working on the ice melt problem, some students had to learn how to calculate with time *so that* they could average their measurements *so that* they could identify an appropriate time for a particular interval of meltwater *so that* they could predict the time that the ice started melting. They also had to come up with a strategy for *taking* the measurements. In the consecutive integer problem, students explored a wide variety of conjectures regarding particular groups of numbers that did or did not work. As a teacher, then, when (if ever) should my role include mediating confusion by helping students chart a path? Had I asked particular questions and pointed out significant objects, I likely could have guided many students toward a conclusion for consecutive integers in a few class periods. But this is a problem space within which I've been engaging on and off for five years. Too much guidance would surely have provided a misleading sense of the difficulty of the problem, created the misperception that the path along which I directed students was the only one available, and have stolen any excitement the students might have experienced. Furthermore, if working in multi-leveled problem spaces is an important goal (as students learn to broaden the contexts within which they can choose appropriate action), then I would be doing for the students the very thing that I (as teacher) want them to practice and that I (as researcher) want to understand. Yet I doubt few (if any) of the students would spend four years on a problem such as this.

Panning and Zooming

Although most students were initially enthusiastic about the consecutive integers problem, by the end of the time we dedicated to working on it, most had lost interest. In Chapter 5, I shared Twelve's comments contrasting Chocolate Fix® with the consecutive integers problem: He noted that the former is easier because it has more rules and less freedom, does not require a search for new ideas or theories, and supplies all the necessary hints to solve the problem. His comment points to the difficulty of shifting between multiple levels. After all, nobody had to learn new mathematical procedures to engage in the consecutive integers problem; in a sense, all the clues were supplied there as well. Another student elaborated that Chocolate Fix® was "more straightforward" and "easier"–even when it was difficult. It seems that the students enjoyed piecing together clues, but wanted to have confidence in the clues themselves. In the consecutive integers problem, the clues required considerable effort to develop; one student noted that he was not always confident in their accuracy.

Perhaps when sub-tasks do not demand too much attention, it is possible to attend to a problem structure with more "degrees of separation" from the problem-as-stated. In discussing the video game *Zelda*, Johnson (2005) noted:

> On the simplest level, the *Zelda* player learns how to grow bombs out of flowers. But the collateral learning of the experience offers a far more profound reward: the ability to probe and telescope in difficult and ever-

> changing situations. It's not *what* the player is thinking about, but the *way* she's thinking. (Johnson, 2005, p. 60)

Part of the appeal of *Zelda* may be that the sub-tasks themselves are easy enough that such telescoping becomes more do-able within the limited space available for conscious attention. Chocolate Fix®, on the other hand, can be difficult, but it is harder to lose sight of the main goal.

I suspect greater exposure to complex problems might help students develop the skill of zooming / panning their attention as they alternately attend to a defined problem space (a) holistically (including consideration of potential relationships between selected pieces) and (b) more narrowly on *selected* aspects of that problem space. In this study, I often assumed the role of directing attention back toward the broadest context, as defined by the given problem. Doing so required constant monitoring whether, at a given moment, the struggle to understand the pieces was likely to be more motivating or meaningful than the struggle to consider how the pieces fit together.

In Chapter 2, I claimed that for every role that I assume as a teacher, I need to consider how I might encourage students to do the same. So to what extent do I constrain a problem, and to what extent do I encourage students to tighten and/or loosen their own constraints? Toward the end of the consecutive integers problem, I asked very pointed questions: If multiple odds all work, what's left? How many ways can 27 be written as the sum of consecutive, positive, integers? The class was much more focused, but it seemed that *these questions* then defined their highest level of attention. It is not at all clear that earlier experiences within the broader spaces of the problems-as-originally-posed allowed them to place these questions in context or even to recognize the importance of defining constraints. In another instance, might they define their own? Whether or to what degree students might learn this skill remains an open question. I find students' responses to the consecutive integer problem fascinating in that their work involved mostly elementary-level calculations. But I do not think an older or more experienced group of students would find it much easier than this group did.

Personal Relevance in a Meaningful Collective

During this study, I as the teacher-researcher set the boundaries of activity *for the class as a whole* in the ways that I defined and monitored the boundaries of permissible activity within this space. Students defined their own meaning-spaces by engaging in sub-problems that contributed to the resolution of the stated problems. I provided opportunities and direction for individual meaning-spaces to interact with one another. What became part of the broader classroom narrative was greatly influenced by how I directed attention: I frequently asked students or groups of students to consider how their work contributed to the broader goal. I selected which ideas would become the focus of discussion, although students were generally free to converse (usually by calling each other in to the conversation) within those bounds. I often drew attention to points of connection between various student ideas and insisted that particular ideas be clarified or elaborated. Sometimes I summarized ideas so that (I hoped) students could hold more of the big picture in mind. Sometimes I glossed over misunderstandings so that what seemed (to me) a minor point did not escalate into a major distraction; i.e. I decided (rightly or wrongly) whether understanding was "good-enough" (Zack & Reid; 2003; 2004) to allow continued engagement. At other times, I insisted that potential distractors be dealt with; it is hard to say whether the events that I identified as potentially distracting were in fact experienced that way by a significant number of students. Decisions such as these require great sensitivity to the state of understanding in the classroom. I do not have a clear sense of how they should be made, but it is important to recognize them *as decisions* that could be made otherwise (Mason, 2002).

A Second-Person View: How Empathic Research Makes Sense in the Mathematics Classroom

During this work, the inseparability of my roles as learner, teacher, and researcher became very apparent. While all teaching requires empathy with the learners one is hoping to influence, Varela's attention to bringing aspects of cognition that typically dwell beneath consciousness to awareness is well-suited–even essential– both to understanding and improving learning. This may be particularly true in learning mathematics, which *by its very nature* requires attending to and naming implicit understanding: *Many of the objects that comprise the domain of shared understanding in mathematics dwell only in that space*. Gendlin's attention to describing felt meaning, particularly through the process of focusing, was of great value here.

A significant theme throughout this study has been the importance of finding meaningful doubt spaces. With increasing experience, I find that I tend to experience both mathematical doubt and certainty as *possibility*. Niggling doubts signal the need for further testing, more refined explication, or attention to some sort of conflicting understanding that somehow interferes. They often open broad spaces of new understanding. *Aha*s call forth previous *ahas* that felt right but did not work out, and they signal a need for caution. What *does* feel certain is more clearly embedded in a complex web of new questions and associations that more comes more easily to awareness. Awareness of doubt and certainty, then, has amplified their significance and thereby made it much easier to attend to them and to the broader implicit understandings of which they are part. Becoming more alert to my own doubt spaces has also heightened my awareness to ways I might effectively provoke doubt in students and to ways I might interact with cues that signal their own doubt; in other words, it has made me a more empathic second-person coach.

It is less clear the extent to which the students who participated in this study might now experience similar shifts in awareness of their own doubt spaces. This is a promising avenue for further investigation–one that would be best undertaken in a research environment that allows more regular and sustained involvement with learners.

Interacting With Implicit Knowing in Mathematics

In this section, I draw together particulars that emerged as significant in my efforts to bring learners' implicit understanding (including my own) to awareness and into conversation. What I offer is surely a very partial list, constrained by my own evolving style of interaction with students and by what my attentional bias allowed me to notice. Throughout, I note potential directions for further research. A common theme in doing so is to question the potential implications of students attending more closely to their own interactions with implicit knowing.

Attending to My Own Niggling Doubts

As I worked through each of the problems, I (as teacher, learner, and researcher) became more attuned to my own niggling doubts. When I brought these forward to the students, they opened doubt spaces that seemed relevant to the students as well. While this may not always be the case, it seems a promising source of insight. In this study, attending to my own doubts greatly contributed to my being able to respond as an empathic second-person coach. It is important to emphasize that I was highly confident in my responses to the problems-as-stated without attending to the doubt spaces that emerged as most significant. I was sure of the answer to the lamp problem long before I explored its counter-intuitiveness, as were many of the students. I was confident that starting ice was insignificant long before I opened it for deeper exploration; most of the students had at least moved on *as though* starting ice were not significant. I was fully confident that every odd factor produced a solution to the consecutive integer problem before I was able to explain the connection between *powers of two* and *no odd factors*.

Directly Addressing Conflicting Ideas

Especially during the lamp problem, I repeatedly encouraged students to figure out exactly how opposing arguments broke down rather than merely collecting arguments to support their own view. Doing so pushed all of us to confront difficulties that dwelled beneath ordinary consciousness. This was very evident in the great difficulty we all had in refuting one-dollar profit in the lamp problem.

Avoiding Aggressive Logic

I find that I easily play the role of the skeptic; it feels productive in the sense that it does seem to push students' thinking deeper. But it often directs inquiry along a narrow path. Students often took on this role as well–perhaps in response to my modeling and perhaps also because it is easier to make a confident, public statement when you have a clear argument in mind. The voice of seemingly clear logic often dominates. As several students noted, this was clear in the persistence of the one-dollar explanation for the lamp problem. As I illustrated in the narrative for summing consecutive integers (Chapter 4), an aggressive line of questioning, even if it addresses legitimate issues and even if participants are enjoying the banter, can prematurely shift attention away from the broad, holistic, implicit space from which deeper, more connected understanding might emerge.

Referring to Others' Implicit

As students learned to appreciate and attempted to understand the deeper meaning that dwelled beneath what their classmates were saying, they would often say things like, "I think what she means is…." Sometimes, they (like me) fell into the trap of aggressive paraphrasing, but there were many times when our attention was directed toward creating shared meaning rather than simply getting our points across. Students, too, developed skill in empathic coaching, although the manner in which that coaching was implicated in their interactions seldom became a direct object of reflection. This would be a promising avenue for future study.

Pointing and Hoping

Closely related to referring to others' implicit, there were times when students seemed to invite others' help in articulating their emerging felt sense of a situation. Here, the experience of referring to the implicit comes full circle and becomes a very intersubjective experience. I referred to this as pointing (with a vague verbal or non-verbal referent) and hoping (that somebody else might have words to name the space left by the gap in my words).

Encouraging Long Enough Spaces

Closely related to the need to directly address conflicting ideas is the broader requirement that students directly interact with each other's ideas rather than simply add their own to the collection of offerings. As students got excited about contributing their ideas and arguments, they became very enthusiastic about speaking and sometimes forget to listen well. Even when not interrupting one another, they often sat poised and waiting for an opportunity to jump in with their own ideas rather than responding to those that had already been presented. A Grade 6 student I worked with in an earlier study explained what it is like to be cut off in the midst of a still-forming idea: "If you are pausing for about two seconds or something, and somebody jumps in, and you're just thinking of another way to come at so the other person understands it, then you feel kind of like 'Hey, I was talking. I'm not finished yet'" (Schmidt, 1999, p. 102). He proposed a five or ten-second rule for jumping in, and the class strongly supported the need to make sure that the last person speaking had not just stopped to think. This is particularly important when students are undertaking the hard work of articulating implicit understanding–work that is necessarily filled with the sorts of spaces that are easy to jump into.

Seeing Past Words

Particularly in the lamp problem, students deepened their understanding of the notion that not getting what someone is saying does not mean that they are "stupid" or "bad at math." Furthermore, many came to recognize that nobody could do the work for them: There was no magic explanation that somebody *should* be able to provide. Even as a teacher who would have made similar claims prior to this work, I found that the strengths of the impasses between competing solutions to the (deceptively simple) lamp problem deepened my appreciation for the limitations of language to describe mathematical understanding. It also heightened my awareness of how very subtle shifts in students' word choices were sometimes indicative of important shifts in underlying understanding. Attending more closely to such shifts may have allowed me to more effectively coax new understanding to the surface.

Responding to Plausibility Shields

Rowland (1995) described students' use of "plausibility shields" both to distance themselves from potential error and to create cognitive space. In light of the findings of this study, I think it makes sense to say that students also use such words to indicate and begin to articulate an emerging felt sense; i.e. the vagueness is not always a deliberate tactic to protect themselves from wrongness. Sometimes, it is all that is available at the time. Left alone, students may not elaborate (or recognize the value in elaborating) these tentative beginnings. Again, felt meaning and language co-evolve, and teachers need to encourage and provide space for ideas to emerge rather than always insisting on clarity and challenging inconsistencies. Even when shields *are* used defensively, they may be attended to in ways that allow uncertainty to unfold into richer understanding.

Allowing Broader Meanings For Common Words

As students attempted to articulate emerging understandings, they often used old words in new ways. This was evident in the discussion of the evolving meanings of words and phrases like *original money* (in the lamp problem) and *backtracking*, *melting rate*, and *ratio* (in the ice melt problem). At times, it was productive to allow the words to retain their vague referents, while at other times, questioning their meaning helped that meaning to evolve. Most significant from a teaching perspective, I think, is to be aware of the great variation in ways words might be used.

Attending To Non-Verbal Indicators of Doubt

There were many occasions when students' speech was full of stops and starts and where a hand motion (often shaking or sweeping) was used to indicate a gap where words were not readily available. Sometimes quote motions were used to hedge; i.e. to indicate that the word actually used was not quite the right one. Attending to such gestures can provide cues to slow down the pace of a discussion and allow space for the right words to come forth to articulate those gestures.

Ignoring Trivial Errors

Sometimes it is important to ignore trivial errors and attend to the broader intent of an emerging idea. I find that there's an interesting dilemma here: I often feel a need to resolve small details of discrepancy so that my mind is not troubled by irritating conflict and is more free to embrace the bigger ideas. But sometimes this makes it hard for people I am talking to maintain *their* line of thought. It is important for both students and teacher to attend to this dilemma so that they can make intentional choices regarding which doubts are better set aside (at least temporarily).

Generalizing

Mason (1996) described generalizing as “the heartbeat of mathematics,” and claimed, “If teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place” (p. 65). In this study, it became clear that generalizing requires reference to the implicit. By extension, if *that* is not happening, all that’s left is a set of empty procedures.

Naming and Varying

There were a number of instances where *naming*, even at a broad level, allowed variation of the attribute named (variation of starting ice, variation of hypothetical melt rates, moving from rules for multiples of three and five to rules for multiples of other odd numbers). As I attempted to illustrate in my discussion of bridging in Chapter 5, I often played a significant role in pointing out the possibility that named attributes might be otherwise, and this had the effect of interrupting habitual thought patterns. In other instances, I missed opportunities to direct attention to potential variation in mathematical objects that the students named; for example, I did not recognize the significance of one student’s comment that writing a number as consecutive, positive integers has *something to do with* an *odd number of odd addends*. Having now reflected upon this (in the general sense), hopefully it will come to mind more easily in other situations. It might be useful to make such bridging a direct object of reflection with students in the classroom.

Attending to Bodily Indicators of Understanding

In the Chapter 4 narrative for the ice melt problem, I described students who were able to recognize more bodily senses of cognitive experience (e.g. jolts of right or wrong). As Robson (2011) found in his studies of interoception, greater attention to bodily experience may be helpful here. I am fascinated with the notion that even for a novice, attending to a golf club rather than my arms, a spot in front of me rather than my body on skis, or the water moving past me as I swim rather than my arms can help me to improve my performance. What might this mean in the context of a mathematical problem? What might constitute such external spots?

Encouraging Eavesdropping

There were a number of instances where students referred to their own thinking as though it belonged to someone else (e.g. "so that's automatically what your brain thinks"). In my own experience, I referred to this as *eavesdropping*. During my time in the classroom, I neither noticed nor drew attention to times when students' comments indicated that this might be happening. As instances like these become objects of reflection, I suspect they could prompt deeper awareness of the implicit:

> [T]he student first begins to see in a precise fashion what the mind is doing, its restless, perpetual grasping, moment to moment. This enables the student to cut some of the automaticity of his habitual patterns, which leads to further mindfulness, and he begins to realize that there is no self in any of his actual experience. (Varela, Thompson, & Rosch, 1991, p. 247)

Continuing the Work

As Varela (1999) noted, "[N]ow a new problem arises: we seem to be losing our grip on something that is undeniably close and familiar–our sense of self" (p. 39). *I* can learn to listen to the quiet voices and bring them closer to awareness. In doing so, I can set my search strings to fuzzy or focused. But who am *I*, and why do *I* choose to do this? Does conscious attention (through no intent of *my* own) merely seem to help *me* focus, when in fact it brings to awareness a positive feedback loop whereby that which has already captured *my* attention continues to fire above the consciousness threshold? And why does it matter?

In developing awareness of selflessness, Varela (1999) started to develop a connection between cognitive science and meditative traditions like Buddhism and Taoism. I am intrigued by this connection and by how it might be implicated in mathematics. I see mathematics as one context in which it is possible to observe the grasping mind and have seen that expanding awareness (even in simple ways) can broaden the scope of what I am able to understand. But I am not in search of some elusive mathematical nirvana; mathematics and wisdom have significant differences. Varela (1999) unpacked the analogy as follows:

> In *some* ways skillful means in Buddhism are like our more familiar notion of a sensorimotor skill: the student practices ("plants good seeds"), that is, avoids harmful actions, performs beneficial ones, meditates, and extends his behavior to a wider and wider range. However, unlike mastery of an ordinary skill, mastery of the skillful means of ethical expertise results in the elimination of *all* habits so that the practitioner can realize that wisdom and compassion can arise directly and spontaneously out of wisdom. (p, 72)

Wisdom transcends fair laws (ethical or mathematical), which, by definition (!), are an attempt to define general principles:

> [O]ne of the main characteristics of spontaneous compassion, which is not a characteristic of volitional action based on habitual patterns, is that it follows no rules. It is not derived from an axiomatic ethical system or even from pragmatic moral injunctions. Its highest aspiration is to be responsive to the needs of the particular situation. (Varela, 1999, p. 71)

Becoming aware of the habitual patterns invoked as we engage in mathematics is central to this work.

Closing Remarks: Mathematics as "Building a Symbolic House"

The mathematical insights that I developed through my work on the problems I shared in this study were the result of *many* hours of work and concentration–I spent far more time with the problems than the students did. Devlin (2000) described the intense concentration required to do mathematics:

> I am sure that this is a factor that prevents many people from becoming proficient at mathematics. It isn't that they are incapable of intense concentration. Rather, they don't appreciate in advance the degree of concentration required. Hence, instead of giving it that concentration, they assume they just don't have the math gene. I am not claiming that other pursuits do not require intense concentration. But by and large, they require a high level of concentration *to do them well*. Mathematics requires intense concentration *in order to do it at all.* Without the concentration, the brain does not construct the symbolic house. And without the symbolic framework, the best anyone can hope for is to learn to perform various manipulations of *linguistic* symbols–the marks on the

> paper. The result is the all-too-familiar impression that mathematics is a collection of seemingly arbitrary rules to be applied in an uncreative, essentially mindless fashion. (pp. 131-132)

In terms of this study, what Devlin refers to as the symbolic house emerges from reference to implicit understanding. If we expect students to take on that level of commitment to mathematics, they need extensive amounts of class of time to do so. Even then, they sometimes lose interest, as many did while working on the consecutive integers problem. It is entirely possible that this would have been helped by not having such long gaps between sessions, but it is also important to remember that partial conclusions are acceptable–so long as the students know that they are partial and have some awareness of how the partial results might be embedded within ever-broadening understanding. Perhaps, though, if we can encourage them to keep going a little deeper into their own doubts and hunches (while helping them remain aware of where their conclusions remain partial), they will get a taste for the joy of deep mathematical understanding. In doing so, they may move deeper into the space where they are constructing their own symbolic houses, with symbols deeply tied to their own implicit understanding.

References

Bateson, G. (1972). *Steps to an ecology of mind: A revolutionary approach to man's understanding of himself.* New York: Ballantine. (Original work published 1964)

Bateson, G. (1979/2002). *Mind and nature*. Cresskill, NJ: Hampton Press. (Original work published 1979)

Bateson, G. (1987). Men are grass: Metaphor and the world of mental process. In W.I. Thompson (Ed.), *Gaia: A way of knowing: Political implications of the new biology* (pp. 37-47). Hudson, NY: Lindisfarne Press. (Original work published 1980)

Bateson, G., & Bateson, M. C. (1987). *Angels fear.* Toronto: Bantam.

Berenstain, S. & Berenstain, J. (1962). *The big honey hunt.* New York: Beginner Books.

Boaler, J. (1993). The role of contexts in the mathematics classroom: Do they make mathematics more "real"? In *For the Learning of Mathematics 13*(2), 12-17.

Boaler, J. (2008). *What's math go to do with it? Helping children learn to love their least favorite subject–and why it's important for America.* New York: Viking.

Burns, M. (1997). *Spaghetti and meatballs for all*. New York: Scholastic.

Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? In *For the Learning of Mathematics 19*(3), 27-32.

Clement, J. (1981). Analogy generation in scientific problem solving. *Proceedings of the Third Annual Conference of the Cognitive Science Society*, 137-140.

Clement, J. (1994). Use of physical intuition and imagistic simulation in expert problem solving. In D. Tirosh (Ed.), *Implicit and explicit knowledge: An educational approach* (pp. 205-244). Norwood, NJ: Ablex.

Clement & Konold (1989). Fostering basic problem-solving skills in mathematics. In *For the Learning of Mathematics 9*(3), pp. 26-30.

Damasio, A. (1994). *Descartes' error: Emotion, reason, and the human brain.* New York: Avon.

d'Ambrosio, U. (1990). The role of mathematics education in building a democratic and just society. *For the Learning of Mathematics 10* (3), pp. 20-23.

d'Ambrosio, U. (2006). *Ethnomathematics: Link between traditions and modernity*. Rotterdam: Sense Publishers.

Davis, B. & Simmt, E. (2003). Understanding learning systems: Mathematics education and complexity science. In *Journal for Research in Mathematics Education 34*, 137-167.

Davis, B., & Sumara, D. (2006). *Complexity and education: inquiries into learning, teaching, and research.* Mahweh, NJ: Erlbaum.

Davis, P. J. & Hersh, R. (1981). *The mathematical experience*. Boston: Houghton Mifflin.

de Mello, A. (1990). *Awareness: The perils and opportunities of reality.* Toronto: Doubleday.

de Mello, A. (2003). *Awakening: Conversations with the masters.* Toronto: Image Books)

Dehaene, S. (1997). *The number sense: How the mind creates mathematics*. Toronto: Oxford.

Depraz, N., Varela, F., & Vermersch, P. (Eds.). (2003). *On becoming aware.* Amsterdam: John Benjamin's Publishing Co.

Devlin, K. (2000). *The math gene: How mathematical thinking evolved and why numbers are like gossip.* New York: Basic Books.

Dewey, J. (1987). Art as experience. In J. Boydston, Ed., *The later works, 1925-1953, Vol. 10*. Carbondale: Southern Illinois University Press. (Original work published 1934)

Duckworth, E. (1996). "*The having of wonderful ideas" and other essays on teaching and learning.* New York: Teachers College Press.

Eliot, L. (1999). *What's going on in there? How the brain and mind develop in the first five years of life.* Toronto: Bantam.

English, L. (1999). Reasoning by analogy: A fundamental process in children's mathematical learning. In L. V. Stiff & F. F. Curcio (Eds.), *Developing mathematical reasoning in grades K- 12* [1999 yearbook] (pp. 22-36). Reston, VA: National Council of Teachers of Mathematics.

English, L. (2004). *Mathematical and analogical reasoning of young learners.* Mahwah, NJ: Lawrence Erlbaum.

Felix felicis. (n.d.) Retrieved from The Harry Potter Wiki: http://harrypotter.wikia.com/wiki/Felix_Felicis

Feynman, R. (1985). *Surely you're joking, Mr. Feynman: Adventures of a curious character.* New York: W. W. Norton and Company.

Fischbein, E. (1982). Intuition and proof. In *For the Learning of Mathematics 3*(2), 9-18, 24.

Fosnot, C. & Dolk, M. (2001a). *Young mathematicians at work: Constructing number sense, addition and subtraction.* Portsmouth, NH: Heinemann.

Fosnot, C. & Dolk, M. (2001b). *Young mathematicians at work: Constructing multiplication and division.* Portsmouth, NH: Heinemann.

Fosnot, C. & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals and percents.* Portsmouth, NH: Heinemann.

Freeman, M. and Mrazek, R. (2001). Methods used by elementary students to generate and evaluate theories. *Alberta Science Education Journal, 34*(1), 3-10.

Freeman, W. (2000). *How brains make up their minds*. New York: Columbia University Press.

Gadd, B. (1986). *Handbook of the Canadian Rockies*. Jasper, AB: Corax Press.

Gallagher, S. & Zahavi, D. (2008). *The phenomenological mind: An introduction to philosophy of mind and cognitive science.* New York: Routledge.

Gallop, J. (2000). The ethics of reading: Close encounters. In *Journal of Curriculum Theorizing 16*(3), 7-17.

Gattegno, C. (1970). *What we owe children: The subordination of teaching to learning.* New York: Outerbridge & Dienstfrey.

Gattegno, C. (1973). *In the beginning there were no words: The universe of babies*. New York: Educational Solutions, Inc.

Gazzaniga, M.S. & LeDoux, J.E. (1978). *The integrated mind*. New York: Plenum Press.

Gendlin, E.T. (1962). *Experiencing and the creation of meaning: A philosophical and psychological approach to the subjective.* Evanston, IL: Northwestern University Press.

Gendlin, E.T. (1967). An analysis of *What is a thing?* In M. Heidegger, *What is a thing?* (W.B. Barton & V. Deutsch, Trans., pp. 247-296). Chicago: Henry Regnery. Retrieved from http://www.focusing.org/gendlin/docs/gol_2041.html

Gendlin, E.T. (1978). *Focusing: A step-by-step technique that takes you past getting in touch with your feelings–to change them and solve your personal problems.* New York: Everest House.

Gendlin, E.T. (1991). Thinking beyond patterns: Body, language and situations. In B. den Ouden & M. Moen (Eds.), *The presence of feeling in thought* (pp. 25-151). New York: Peter Lang.

Gendlin, E.T. (1995). Crossing and dipping: Some terms for approaching the interface between natural understanding and logical formulation. In *Mind and Machines 5*, 547-560.

Gendlin, E. T. (1996). *Focusing-oriented psychotherapy: A manual of the experiential method*. New York: The Guilford Press.

Gendlin, E.T. (2009a). We can think with the implicit as well as with fully-formed concepts. In K. Leidlmair (Ed.), *After cognitivism* (pp. 147-161). New York: Springer.

Gendlin, E.T. (2009b). What first and third person processes really are. In *Journal of Consciousness Studies 16*(10-12), 332–62.

Genova, L. (2007). *Still Alice.* Toronto: Simon & Schuster.

Gigerenzer, G. (2007). Gut feelings: The intelligence of the unconscious. New York: Viking.

Gladwell, M. (2005*). Blink: The power of thinking without thinking.* Boston: Little, Brown & Company.

Gould, S. (1996). Full house: The spread of excellence from Plato to Darwin. New York: Three Rivers Press.

Gray, E. & Tall, D. (1994). Duality, ambiguity, and flexibility: A “proceptual” view of simple arithmetic. In *Journal for Research in Mathematics Education 25*(2), 116-140.

Haase, M. (2002). Living with “obsessive compulsive disorder.” In M. van Manen (Ed.), *Writing in the dark: Phenomenological studies in interpretive inquiry.* (pp. 62-83). London, ON: Althouse Press.

Hadamard, J. (1954). *The psychology of invention in the mathematical field*. Mineola, NY: Dover.

Hanna, G. & Jahnke, N. (1996). Proof and proving. In A. J. Bishop (Ed.), *International handbook of mathematics education* (pp. 877-908). Dordrecht: Kluwer.

Hersh, R. (1993). Proving is convincing and explaining. In *Educational Studies in Mathematics 24*, 389-399.

Hewitt, D. (1986). HSM Coxeter is a geometer. In *Mathematics Teaching 117*(12), 34-35.

Hewitt, D. (1989). Asking pupils to work geometrically. In *Mathematics Teaching 129*(12), 50-55.

Hewitt, D. (1996). Mathematical fluency: The nature of practice and the role of subordination. In *For the learning of mathematics 16*(2), 28-35.

Hewitt, D. (1999). Arbitrary and necessary part 1: A way of viewing the mathematics curriculum. In *For the Learning of Mathematics 19*(3), 2-51.

Hewitt, D. (2001a). Arbitrary and necessary: Part 2 assisting memory. In *For the Learning of Mathematics 21*(1), 44-51.

Hewitt, D. (2001b). Arbitrary and necessary: Part 3 educating awareness. In *For the Learning of Mathematics 21*(2), 37-49.

Hirstein, W. (2005). *Brain fiction: Self-deception and the riddle of confabulation.* Cambridge: The MIT Press.

Jardine, D.W. (2006). On the ecologies of mathematical language and the rhythms of the earth. In D.W. Jardine, S. Friesen, & P. Clifford, *Curriculum in abundance (*pp. 187-199). New Jersey: Lawrence Erlbaum. (Original work purlibshed 1994)

Johnson, M. (2007). *The meaning of the body: Aesthetics of human understanding.* Chicago: The University of Chicago Press.

Johnson, S. (2005). *Everything bad is good for you: How today's popular culture is actually making us smarter.* New York: Riverhead.

Jung-Beeman, M., Bowden, E.M., Haberman, J., Frymiare, J.L., Arambel-Liu, S., Greenblatt, R., … Kounios, J. (2004). Neural activity when people solve verbal problems with insight. In *PLoS Biology* 2(4): e97. doi:10.1371/journal.pbio.0020097

Kahneman, D., Slovic, P., & Tversky, A. (Eds.). (1982). *Judgment under uncertainty: Heuristics and biases.* New York: Cambridge University Press.

Kieren, T. (n.d.). Enactivism and mathematics education. Retrieved from http://plato.acadiau.ca/COURSES/Educ/Reid/enactivism/EnactivismMathEd.html.

Kilpatrick, J. (1985). A retrospective account of the past 25 years of research on teaching mathematical problem solving. In E. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 1-15). Hillsdale, NJ: Erlbaum.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Kingsolver, B. (2000). *Prodigal summer*. New York: Harper Collins.

Kolb, B. & Whishaw, I.Q. (2009). *Fundamentals of human neuropsychology (6th Ed.).* New York: Worth.

Kuhn, T. (1970). *The structure of scientific revolutions*. Chicago: The University of Chicago Press.

Lakatos, I. (1976). *Proofs and refutations*. New York: Cambridge University Press.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by*. Chicago: The University of Chicago Press.

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics to life.* New York: Basic Books.

Lehrer, J. (2008a, June 1). What's that name? In *The Boston Globe*. Retrieved from http://www.boston.com/bostonglobe/ideas/articles/2008/06/01/whats_that_name/?page=full

Lehrer, J. (2008b, July 28). Annals of science: The Eureka hunt. In *The New Yorker,* 40-45.

Lesh, R. & Doerr, H. (Eds.). (2003). *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, teaching, and learning.* Mahwah, NJ: Erlbaum.

Lockhart, P. (2009). *A mathematician's lament: How school cheats us out of our most fascinating and imaginative art form.* New York: Belleview Literary Press.

Margulis, L. (1997). *Slanted truths: Essays on Gaia, symbiosis, and evolution.* New York: Springer-Verlag.

Mason, J. (1989). Mathematical abstraction seen as a delicate shift of attention. In *For the Learning of Mathematics 9*(2), 2-8.

Mason, J. (1994). Researching from the inside in mathematics education: Locating an I-you relationship. In J.P. da Ponte & J.F. Matos (Eds.), *Proceedings of the 18th annual conference of the international group for the psychology of mathematics education* (pp. 176-194). Lisbon, Portugal: University of Lisbon.

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bednarz, C. Kieran, and L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Boston: Kluwer.

Mason, J. (2002). *Researching your own practice: The discipline of noticing.* New York: Routledge.

Mason, J., Burton, L, & Stacey, K. (1982). *Thinking mathematically.* Toronto: Addison-Wesley.

Mason, J. & Pimm, D. (1984). Generic examples: Seeing the general in the particular. In *Education Studies in Mathematics 15*(3), 277-289.

Matas, C. (2003). *The war within.* Markham, ON: Scholastic.

Maturana, H. (1987). Everything is said by an observer. In W.I. Thompson (Ed.), *Gaia: A way of knowing: Political implications of the new biology* (pp. 65-82). New York: Lindisfarne Press.

Maturana, H. & Varela, F. (1987). *The tree of knowledge: The biological roots of human understanding*. Boston: Shambhala.

McGilchrist, I. (2009). *The master and his emissary: The divided brain and the making of the western world.* New Haven, CT: Yale University Press.

McGilchrist, I. (2010, Feb. 4). Interview with Iain McGilchrist. Retrieved from http://frontierpsychiatrist.co.uk/interview-with-iain-mcgilchrist/

Merleau-Ponty, M. (1963). *The structure of behaviour.* (A. L. Fisher, Trans.). Boston: Beacon Press. (Original work published 1942)

Merleau-Ponty, M. (2002). *The phenomenology of perception.* (C. Smith, Trans.). New York: Routledge & Kegan Paul. (Original work published 1954)

Metz, M. (2009). “Jordan, do you know what an analogy is??” Why analogical reasoning deserves a central place in the science classroom and beyond. *Alberta Science Education Journal 39*(2), 10-15.

Milne, A. A. (1926). *Winnie-the-Pooh.* Toronto: McClelland & Stewart.

Nabhan, G. P. & Trimble, S. (1994). *The geography of childhood: Why children need wild spaces.* Boston: Beacon Press.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (2006, April). *Mathematics Teaching in the Middle School.* Reston, VA: Author.

Nicol, C. & Crespo, S. (2005). Exploring mathematics in imaginative places: Rethinking what counts as meaningful contexts for learning mathematics. In *School Science and Mathematics 105*, 240-251.

Núñez R. (2007a). The cognitive science of mathematics: Why is it relevant for mathematics education? In R. Lesh, E. Hamilton, & J. Kaput (Eds.), *Foundations for the Future in Mathematics Education* (pp. 127-154). Mahwah, NJ: Lawrence Erlbaum Associates.

Núñez R. (2007b). A fresh look at the foundations of mathematics: Gesture and the psychological reality of conceptual metaphor. In A. Cienki & C. Müller (Eds.), *Gesture and metaphor* (pp. 93-114). Amsterdam: John Benjamins.

Núñez R. (2009). Gesture, abstraction, and the embodied nature of mathematics. In W.M. Roth (Ed.), *Mathematical representation at the interface of body and culture* (pp. 309-328). Charlotte NC: Information Age Publishing.

Nyabanyaba, T. (1999). Whither relevance? Mathematics teachers’ discussion of the use of ‘real-life’ contexts in school mathematics. In *For the Learning of Mathematics 19*(3), 10-14.

PBS Kids (n.d.). Clifford’s buried treasure. Retrieved from http://pbskids.org/clifford/games/buried-treasure.html

Percy, W. (1975). *The message in the bottle: How queer man is, how queer language is, and what one has to do with the other.* New York: Picador. (Original work published 1954)

Petitmengin-Peugeot, C. (1999). The intuitive experience. In F. Varela & J. Shear (Eds.), *The view from within: First-person approaches to the study of consciousness* (pp. 43-77).

Piaget, J. (1954). *The construction of reality in the child.* (M. Cook, Trans.). New York: Basic Books.

Piaget, J. (1970). *Genetic epistemology*. (E. Duckworth, Trans.). Toronto: George J. McLeod, Ltd. (Original lectures delivered 1968.)

Pimm, D. (1988). Mathematical metaphor. *For the Learning of Mathematics 8*(1), 30-34*.* New York: Routledge & Kegan Paul.

Poincaré, H. (1913). *The foundations of science*. (G. B. Halsted, Trans.). New York: The Science Press.

Polanyi, M. (1967). *The tacit dimension.* New York: Anchor Books.

Polya, G. (1954). *Mathematics and plausible reasoning: Volume I: Induction and analogy in mathematics.* Princeton, NJ: Princeton University Press.

Polya, G. (2004). *How to solve it.* Princeton, NJ: Princeton University Press. (Original work published 1945)

Popper, K. (1959). *The logic of scientific discovery.* New York: Routledge.

Project Implicit® (1998-2011). I.A.T. Corp. Retrieved from https://implicit.harvard.edu/implicit/

Psychiatry.HealthSE.com. (2004-2005). Current diagnosis and treatment in psychiatry: Obsessive compulsive disorder. Retrieved from http://psychiatry.healthse.com/psy/more/obsessive_compulsive_disorder_ocd/

Ramachandran, V.S. (2006). Creativity versus skepticism within science. In *The Skeptical Inquirer 30*(6), 48-49.

Reid, D. (2002). Elements in accepting an explanation. In *Journal of Mathematical Behavior 20*, 527-547.

Reid, D. & Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching.* Boston: Sense Publishers.

Robson, D. (2011, Oct. 12). Your clever body: Thinking from head to toe. In *New Scientist 2834*. Retrieved from http://www.newscientist.com/article/ mg21228341.500-your-clever-body-thinking-from-head-to-toe.html?full=true#bx283415B3.

Rowland, T. (1992). Pointing with pronouns. In *For the Learning of Mathematics 12*(2), 44-48.

Rowland, T. (1995). Hedges in mathematics talk: Linguistic pointers to uncertainty. *Educational Studies in Mathematics 29*, 327-353.

Rowland, T. (1999). ‘i’ is for induction. In *Mathematics Teaching 167*, 23-27.

Rowling, J.K. (2005). *Harry Potter and the half-blood prince*. Vancouver: Raincoast Books.

Sagan, C. (1996). *The demon-haunted world: Science as a candle in the dark.* Random House: New York.

Schmidt, M. (1999). *Putting rational constraint on divergent thought: The development of scientific reasoning.* Unpublished master’s thesis, University of Lethbridge, Lethbridge, Alberta, Canada.

Schoenfeld, A. H. & Herrmann, D. J. (1982). Problem perception and knowledge structure in expert and novice mathematical problem solvers. In *Journal of Experimental Psychology: Learning, Memory, and Cognition 8*, 484-494.

Schultz, J. (1982). *Mathematics for elementary school teachers (2nd Ed.).* Columbus, OH: C. E. Merrill.

Sesame Workshop (2012). Sesame Street. Shape Play. Retrieved from http://www.sesamestreet.org/game_player/-/pgpv/gameplayer/0/f1a0bbfd-27c6-450a-98db-1b246becd43d/shape_play

Sfard, A. (1999). Commentary: On metaphorical roots of conceptual growth. In L. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 339-371). Mahwah, NJ: Lawrence Erlbaum.

Shermer, M. (2011). *The believing brain: From ghosts and gods to politics and conspiracies–How we construct beliefs and reinforce them as truths.* New York: Times Books.

Sheth, B.R., Sandkühler, S. & Bhattacharya, J. (2009). Posterior beta and anterior gamma oscillations predict cognitive insight. In *Journal of Cognitive Neuroscience 21*(7)*,* 1269-1279. http://dx.doi.org/10.1162/jocn.2009.21069

Simmt, E. (2000). *Mathematics knowing in action: A fully embodied phenomenon.* Unpublished doctoral dissertation, University of Alberta, Edmonton, AB.

Speer, A. (1970). *Inside the Third Reich: Memoirs.* New York: Simon & Schuster.

Stavy, R., & Tirosh, D. (2000). *How students (mis)understand science & mathematics: Intuitive rules.* New York: Teachers College Press.

Stewart, I. (2006a). *How to cut a cake and other mathematical conundrums.* New York: Oxford.

Stewart, I. (2006b). *Letters to a young mathematician*. New York: Basic Books.

Thompson, E. (2007). *Mind in life.* Cambridge, MA: Harvard University Press.

van Manen, M. (1990). *Researching lived experience: Human science for an action sensitive pedagogy*. London, ON: The University of Western Ontario.

Varela, F. (1999). *Ethical know-how: Action, wisdom, and cognition*. Stanford, CA: Stanford University Press.

Varela, F. & Scharmer, C. O. (2000, Jan. 12). Three gestures of becoming aware [Interview transcript]. Retrieved from http://www.presencing.com/presencing/dol/Varela.shtml.

Varela, F. & Shear, J. (1999). *The view from within: First-person approaches to the study of consciousness.* Bowling Green, OH: Imprint Academic.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience.* Cambridge: MIT Press.

von Glasersfeld, E. (1990). An exposition on constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (Journal for Research in Mathematics Education Monograph Number 4; pp. 19-29). Reston, VA: The National Council of Teachers of Mathematics.

Watson, A. & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples.* Mahwah, NJ: Erlbaum.

Watson, A. & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. In *Mathematical Thinking and Learning 8*(2), 91-111.

Western and Northern Canadian Protocol. (2006, Sept.). *The common curriculum framework for K-9 Mathematics.* Edmonton, Alberta: Author.

Wigner, E. (1960). The unreasonable effectiveness of mathematics in the natural sciences. In *Communications in Pure and Applied Mathematics 13*(1), 1-12.

Wise, D.U. (1990). Using melting ice to teach radiometric dating. In *Journal of Geological Education (38)*, 38-40, 69.

Zack, V. & Reid, D. (2003). Good-enough understanding: Theorising about the learning of complex ideas (part 1). In *For the Learning of Mathematics, 23*(3). 43-50.

Zack, V. & Reid, D. (2004). Good-enough understanding: Theorising about the learning of complex ideas (part 2). In *For the Learning of Mathematics, 24*(1). 25-28.

Zahavi, D. (2003). *Husserl's phenomenology*. Stanford, CA: Stanford University Press.

Appendix A: Pythagorean Proof

The Pythagoras Problem: The Feeling of What Works

To introduce this problem, I showed students a diagram of a right triangle with squares drawn on each side and cut lines as indicated on b^2 (see Figure 1). I explained that this diagram has been presented as a proof for the Pythagorean Theorem; i.e. if a square drawn on Side b is cut *in the manner shown*, the pieces can be rearranged to form a square that matches the one on Side c, leaving a hole in the middle that exactly matches the square on Side a.

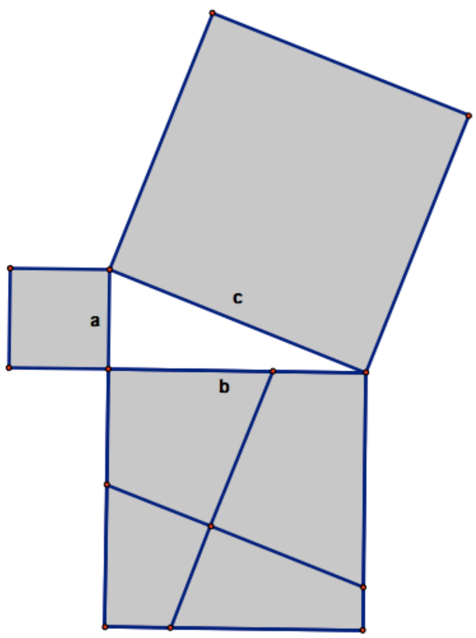


Figure 1. Does this diagram prove the Pythagorean Theorem?

But what is meant by “*the manner shown*”? As Mason (1989) pointed out, “The first task is to get a sense of the this in the statement that might ‘always happen’” (p. 4). I wanted the students to consider what the significant aspects of the lines might be, so I asked (a) if it would matter how they drew the cut-lines and (b) if the method would work for any right triangle. As an initial constraint, however, I prepared a triangle on grid paper with $a = 3$ and $b = 7$ and gave copies to each group to use as a common test model that would make comparisons easier (see Figure 2). Throughout work on this task, I found myself directing attention much more than I was comfortable with. I had a nagging sense that I was controlling too much of the problem narrative, but at the time I felt the need to keep pushing the students forward to keep them engaged.

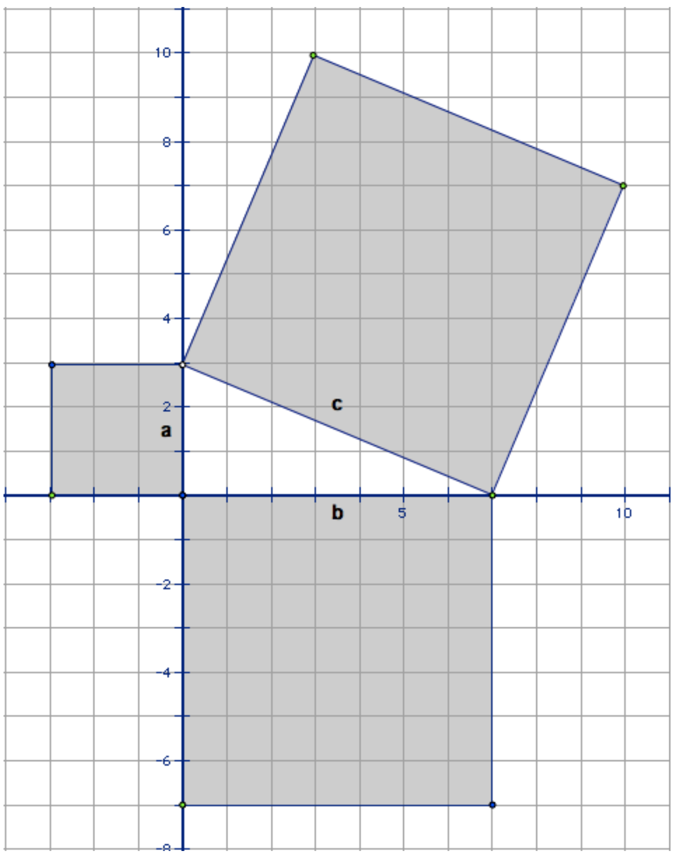


Figure 2. The test model.

Finding a Doubt Space

One group of students quickly became convinced that so long as the cut lines were perpendicular they would work; they recognized that the square corners formed by perpendicular cut lines could be rearranged to form the square corners of the larger square (i.e. c^2). All the students in the group drew cut lines on their models,[63] but so sure were they of their idea that they were reluctant to test them. In an effort to shake their confidence, I went around the table from student to student quickly indicating, "yes, yes, no, no, yes" in response to the lines they had drawn. They were quite surprised that I vetoed any of the examples, and this did in fact prompt them to get out their scissors. They did not keep track of their various efforts, however. During the following session, I provided each group with a summary sheet to keep track of their work. I also demonstrated a model in Geometer's Sketchpad that allowed students to slide the two cut lines and observe the resulting changes in how the pieces fit onto c^2 (see Figure 3). I designed one cut line such that it always matched the slope of the hypotenuse, which could only be varied by changing the relative lengths of *a* and *b*. I constructed the other cut line perpendicular to the first. In other words, it was impossible to create non-working cut lines with this dynamic drawing (I talk more about my challenges in making this drawing a little later).

[63] I'm not sure how this group managed to draw their perpendiculars; during an interview with a different group after today's class, it became evident to me that this is in itself a more complex task that I had anticipated.

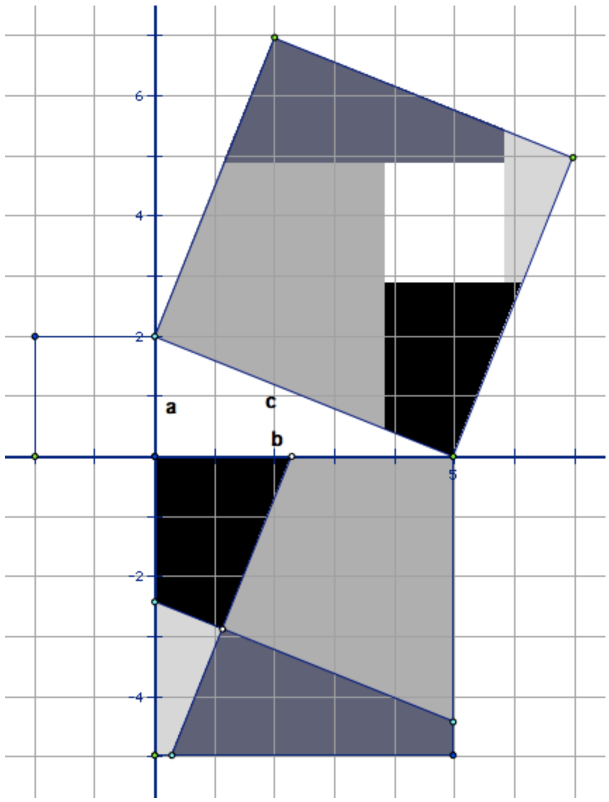


Figure 3. How do the pieces of b^2 fit onto c^2? *Must* the hole in c^2 match a^2?

Uneven Pieces Do Not Work: *Something To Do With* Symmetry?

As I attempted to show two students how to draw perpendicular lines (a difficulty I had not anticipated), I (not intentionally) chose cut lines that resulted in very uneven-sized pieces. Interestingly, both students taking part in the interview were quite sure that these pieces could not rearrange to form a square (even though the image they had manipulated in Geometer's Sketchpad allowed the easy creation of very large and very small pieces, as in Figures 3 and 4). They seemed to have an implicit belief that symmetry was a significant variable, and I wish I had asked them to elaborate on this rather than simply asking them to cut out the pieces. In any case, they were surprised when they were able to rearrange the four pieces to form a large square surrounding a smaller square hole. As became apparent a little later, however, whatever it was in their broader understanding that made the piece-size-discrepancy seem so important had not been adequately dealt with.

"Inside Out": Something To Do With Rotation?

After the students acknowledged that not all perpendicular lines produce side lengths that match c^2, I marked the cut lines so that we could better see where the sides along either side of the cut lines ended up after the pieces were rearranged. Double-Check commented, "It's kinda like you're flipping it inside out" (Class 1 Interview). I wonder now whether exploring just *how* the pieces might be considered inside out might have been valuable (Here, I was tempted to say "valuable diversion," but this again betrays a commitment to my own plot line). I, too, had a sense of inside out; for me, this was a strong sense that the pieces were *somehow* rotating out from the middle. It was a big *aha* for me when I realized I could construct the Geometer's Sketchpad image simply by *sliding* the pieces from b^2 onto c^2 (more on this a little later); I had unknowingly used a 360° rotation in combination with the slide, and the rotation diverted attention from the slide.

Are Parallel Cut Lines Are Always the Same Length?

At one point, I asked the students to consider what sort of cut lines would result in a square that was too big (a case they had experienced). I reminded them of the over-seven-up-four cut line I had used when demonstrating how to make perpendicular lines and asked if there were others with the same length. I also noted that the hypotenuse was a seven-three line and asked if other seven-three lines would have the same length. Double-Check readily agreed, but Nine was initially unconvinced. At first, she seemed to think the length of the cut lines on b^2 would match Side b. She easily recognized the flaw in this when I pointed it out, but recognizing the necessity of parallel cuts being equal took a little longer. The moment she saw the connection, it seemed as though it was suddenly very obvious to her. Nonetheless, I am not sure this recognition stayed with either of the students. They were also dealing with a number of other uncertainties, and they did not spend a lot of time considering whether lines with equal slope must have equal length. Moving on, they were still unsure how to *draw* a 7-3 line perpendicular to the first, so I helped with this. Although the following sequence of steps was highly directed, the students now appreciated the point of the question, and their surprise and interest in the working solution was apparent. Interestingly, their discomfort with uneven-sized pieces persisted:

> Ms. M.: Now again we have the problem…. You didn't like that some of the pieces were too big.
>
> Double-Check: Yeah.
>
> Nine: They seem bigger now.
>
> Ms. M.: Now it's even bigger? Mm-hmm. Well, this is a different one, so maybe it is bigger. Do you think this will work or not?
>
> Nine: No (laughs).
>
> Double-Check: No.
>
> Ms. M.: No?
>
> Nine: (?) too big [Double-Check: Yeah] Can we try it now?
>
> Ms. M.: Sure if you want to try it. Either now, or…. First of all, will it give us square corners?
>
> Nine: No, it won't–yeah–no actually it will–(?) [Double-Check: It looks like it will] 90-degree angle, 90-degree angle. Yeah, it will.
>
> Ms. M.: Are the side lengths the right length… to match this? To match this?
>
> Double-Check: Yeah.
>
> Nine: Yeah.
>
> Ms. M.: How do we know?
>
> Nine: Cause they're 7 by…

Double-Check: 7 by 3.

Nine: The diagonal lengths.

Ms. M.: But you don't think it will work, cause some of the pieces are too big and some are too small? [Double-Check nodding]

Ms. M.: Okay.

[Double-Check cuts out the pieces, then they start to arrange them]

Nine: These two probably go together.

Double-Check: That's huge!

Nine: Yeah.

Ms. M.: If you want to turn it inside out, then the pen lines should be in the corners, right?

[They continue their work.]

Double-Check: Found a right angle hopefully. Yay!

Ms. M.: Okay, so you got your corners, anyways.

[They continue their work.]

Nine: Well, it'll be, like, kind of off in the center.

[They continue their work.]

Double-Check: We might have a square!

[They continue their work.]

Double-Check: Yay!

Ms. M.: Did it work? And this time, it's exact, or is it still overlapping?

Double-Check: It's exact.

Nine: It's exact.

Double-Check: It's just kind of hard to get them perfect.

Ms. M.: So is that okay that the square is off to the side?

Nine: Yeah.

Double-Check: Yeah.

Ms. M.: Does it still show that $a^2 + b^2 = c^2$?

Double-Check: [nodding emphatically] Yeah.

Nine: So that little square [pointing]….

Ms. M.: So does this look like it would exactly fit in there?

Nine: Yeah, it looks 3 by 3.

Double-Check: It’s beautiful!

(Class 1 Interview)

To me, it seemed intuitively obvious that lines of the same slope cutting b^2 would have the same length whether parallel or perpendicular. At the time, I felt that my questioning was merely drawing attention to the obvious. But as I reflected on this work, imagining the vertical movement of a slope line called forth an earlier image from work on an unrelated problem; looking back through old files, I found that I had constructed a dynamic image in Geometer’s Sketchpad that allowed me to separately move a slope line and its perpendicular within a square in order to *test* whether they were the same length. It seems I should easily have justified this relationship with a simple application of similar triangles, but apparently, the intuitive similarity of the lines was not as obvious (even to me) as it now feels.

Generalizing the Solution

When I asked the students if they thought they had a solution to the larger problem, they worked to articulate a more explicit and generalized conclusion. Again, note the evolution of their language:

Ms. M.: Now you’ve got one that works here. But…. Have you answered the question?

Double-Check: No.

Nine: No.

Double-Check: Because you have to figure out how you can make it any one.

Nine: Yeah. How you can make every single one. Because… well, you kind of answered that question–we kind of answered that question? With you (?) the, um, triangle? **This and this [pointing]. So. As long as, like… As long as the tri–as long as they, um, match the middle length–like this length [pointing to hypotenuse]. I don't know how to explain it.**

Ms. M.: So as long as the cut length matches–

Double-Check: **Matches that… area… thing**. **If it's… the points are proper and the angle is right, then it works. But it has to also be a right angle, and has to be an x**.

(Class 1 Interview)

Despite the vagueness of their wording, both Nine and Double-Check indicated high confidence in this conclusion, and both claimed that the work of forming the general statement was more challenging than frustrating. Both agreed, however, that trying to explain their conclusion to others would likely be frustrating. In the brief time remaining for their interview, I asked them to try to clarify their earlier conclusion to make it easier to communicate it to the rest of the class:

Double-Check: **It has to be…. It has be an x, it has to have right angles, and it also has to match the diagonal that… that *A*… the triangle *ABC* makes [i.e. the hypotenuse]?**

Ms. M.: This diagonal? Okay, that's even clearer now.

Nine: **Cause already…. We've already labeled that this–that this length [one of the cut lines]? Here? Is the length of *c* that–that obviously means that this is also the length of *c*.**

At this point, both seemed to clearly recognize that the cut lines had to be the same *length* as the hypotenuse, but it is not clear that they associated this with a common *slope*; nowhere did they use the word *parallel* nor indicate that this is part of what they meant by *match*.

The Feeling of Symmetry

When the entire class reconvened just over a week later, I gave each student a copy of six identical right triangles (a = 3 and b = 7) with squares drawn on each side (see Figure 4). In each diagram, b^2 was cut in a different way; I asked the students to predict and test which cuts would allow b^2 to be rearranged such that it matched c^2 with an a^2-hole.

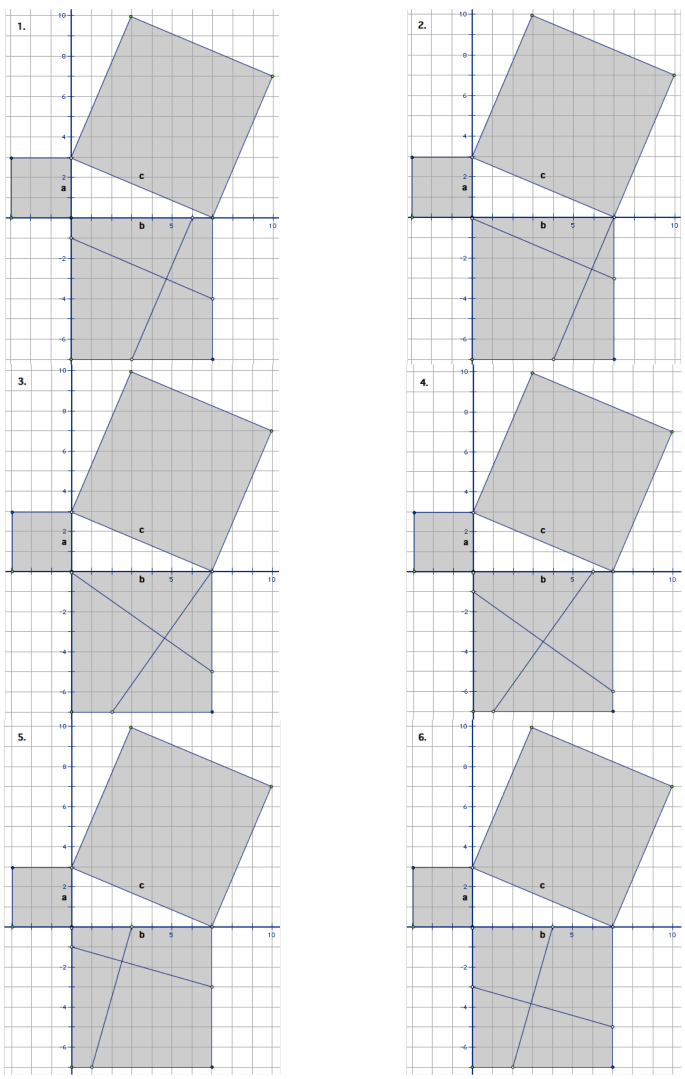


Figure 4. Which cut lines work?

Interestingly, *all* predicted that Case 6 would work and Case 2 would not, which might indicate a continued and more widespread belief in the significance of relative piece size. In any case, all students reversed their predictions after testing. I then asked them to consider how Cases 1 and 2 (the only working examples) were different from the others. One student proposed that if the two cut lines partitioned the sides of b^2 in the same way (e.g. if both broke a side into 1 and 6, 2 and 5, or 3 and 4), it would work. I refuted this with the Geometers' Sketchpad demo, and Felix pointed out that Examples 3, 4, and 5 could also serve as counter-examples.

Eventually (knowing that we had limited time left to spend on this problem), I pointed out the relationship between the length of the cut lines and the length of the hypotenuse and explained that the cut lines would become the sides of c^2. Although there were lots of "Oh's!" at this, I am sure there were students who did not recognize what I was talking about. Nonetheless, I asked the students to check the lengths of the lines on the sample sheet to see if they matched the cut-line-matches-hypotenuse theory. Many started measuring. *A potential relationship between slope and distance did not occur to anybody*–even the two students who participated in the interview a week prior did not use slope as an indicator of distance. One student did note that in working cases, one of the cut lines must be parallel to the hypotenuse, but it was unclear whether he saw this as a necessary condition for equal length.

Although it did not seem to matter a great deal to the students, I felt the need to bring some sense of closure to our investigations. I summarized the conclusions we had discussed and noted that we still had not accounted for *why* the white square in the middle always matches a^2. As I expected, most students seemed content to leave this unsolved.

The Feeling of Rotating vs. Sliding

In my own efforts to solve this problem, I fairly quickly recognized the importance of the slope (which to me also constrained length) of the cut lines. For a brief moment, I could not see how two cut lines could yield all four sides of c^2 and had a small *aha* when I realized one cut would yield two sides of the square. In connection with this, I remember a tip-of-tongue metaphor that I did not bring to fully to consciousness until later. As I re-read my journal notes a year-and-a-half later, the notion of moving tables apart to make a larger perimeter seemed to match; in particular, the story *Spaghetti and Meatballs For All* (Burns, 1997) came to mind. In the story, Mrs. Comfort carefully arranges eight square tables to seat 32 guests. As more and more people arrive in small groups, they keep pushing tables together so that larger groups can sit together. Soon, they run out of place settings and start separating the tables again to expose more table sides: *For each separation, two places settings are created*.

Although the puzzle of where the four sides came from resolved itself very easily, the niggling doubt that I had not explained why the hole in c^2 matched a^2 was more difficult. I had a sense that this also had *something to do with* slope, though I am not sure why–perhaps because it was the only variable that appeared to be changing? But this vague notion was enough to prompt me to construct two sets of 10 by 10 squares–one having perpendicular cut-lines with varying slope (see Figure 5) and one having cut lines with constant slope but varying position (see Figure 6).

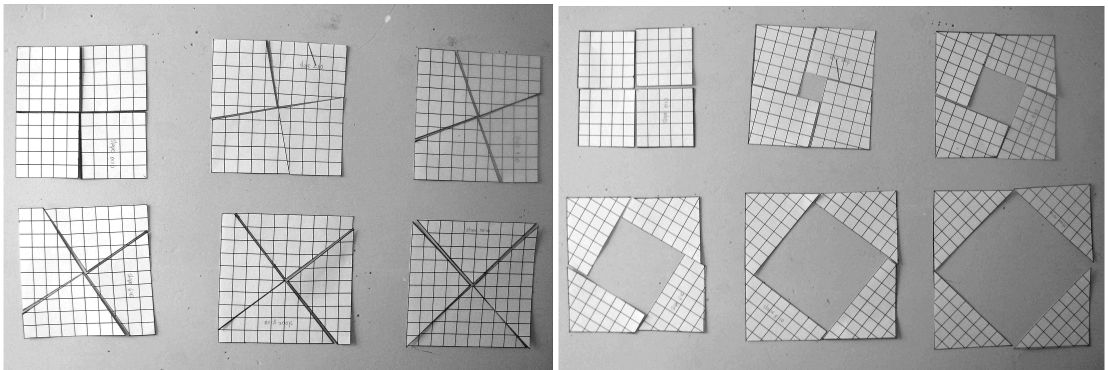


Figure 5: Perpendicular cut-lines with varying slope.[64]

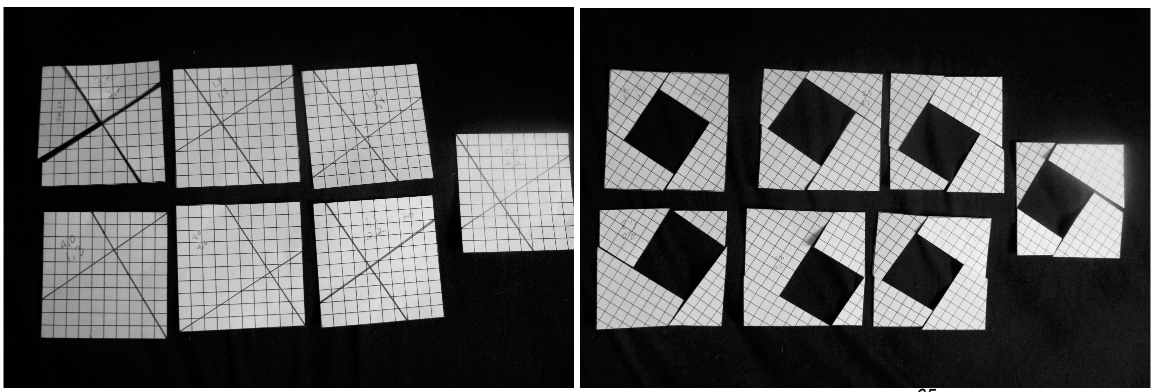


Figure 6: Perpendicular cut-lines with constant slope, varying position.[65]

At this point, I still thought that the *middle corners* (i.e. the four right angles formed at the point where the perpendicular lines intersect) had to be rotated (rather than slid) to make the corners of c^2. Because I was also trying to create a Geometer's Sketchpad demonstration to show how the pieces moved when they were rearranged, I struggled for quite a long time to find the exact angle of rotation. I could SEE it happening as I performed the rotation (I even color-coded the edges red and the cuts blue so that I could distinguish them throughout the transformation), but it was almost

---

[64] 5,5 and 5,5; 4,6 and 6,4; 3,7 and 7,3; 2,8 and 8,2; 1,9 and 9,1; 0,10 and 10,0; the first pair in each set of coordinates describes the line connecting the top and bottom of the square. The numbers denote how many units the line is from the left. The second pair in each set of coordinates describes a line from left to right in terms of units from the top.

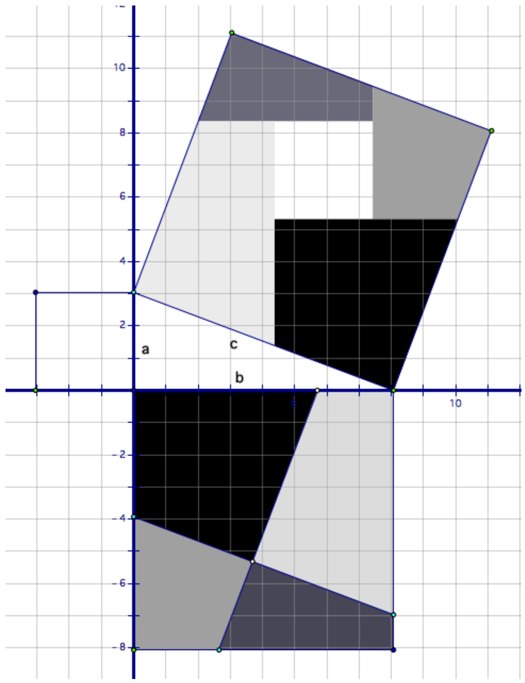
[65] 2,8 and 8,2; 1,7 and 7,1; 1,7 and 9,3; 0,6 and 8,2; 4,10 and 7,0 (a mistake–these aren't perpendicular!); 4,10 and 10,4; 1,7 and 8,2; 0,6 and 8,2; I can't remember why I chose these particular instances; there are many other whole-number combinations that would allow common slope on a 10x10 grid.

like there was a magic flip-point where (metaphorically) the vase turned into a face, and I could not quite *catch* what happened at the interface. I did, however, recognize that when the pieces were translated, opposite sides lined up, creating a square with sides equal to the difference in line position; e.g. a 3,7 line (or 2,6 or 1,5 or 0,4 line) would create a square with side lengths of 4 (7-3; 6-2; 5-1; 4-0). This confirmed my hunch that slope would determine the size of the internal square. But I was not satisfied:

> So I wonder if now I can find a more intuitive way to "see" the necessity of the a^2-hole.... STDW [*something to do with*] the slope change.... This is somewhere that I don't yet know how to take most of the kids with me. Once they've "solved" the problem, they aren't too interested in finding the why, especially if it's difficult and there are no immediate leads. For most, being willing to sit on it and obsess over it requires some further anticipated reward than this does. Not sure if more experience with the intellectual reward of solving difficult puzzles would up the ante–in their busy lives, I'm skeptical…. For me, I've LEARNED that the quiet doubts can open up big spaces of new understanding, and I'm motivated to develop that understanding. Most students are not similarly motivated when it doesn't involve required work. Even if they enjoy the work, it won't (thankfully) take precedence over music lessons, hockey practice, or limited family time after school. (Research Diary)

As I struggled to find a way to *rotate* the pieces so I could fit them onto c^2, it finally occurred to me that I was *already* rotating c^2 by the same angle as the cut lines in b^2 and therefore should not need to bother; slides were all that was needed! I had yet to figure out the exact vector for the slides, but at this point, I was deeply surprised that I did not need to rotate the pieces at all. I noted:

> Wait.... After all my mucking about [designing the image in Geometer's Sketchpad], I see that my original logical explanation isn't so complicated after all–maybe because now I have an easier way to keep the before and after shapes in my mind simultaneously (I just have to slide them up–no rotation required). CE - AD must have a length of (say 3, keeping with a=3, b=7 triangle we used in class), because that's how the slope line was drawn. Same for the other sides. This is more intuitive than my first similar explanation, because now I see the slides more clearly–how the outside edges of b^2 end up forming the edges of the a^2 hole (as Double-Check said, the square is turned "inside out," although that could mean a lot of things, I think). Now I want to actually cut them out, switch the Left/Right pieces, then switch the Top/Bottom–and to see how order influences the way the slides appear in their stages.



Pieces numbered CW 1,2,3,4 from top left:
4 stays anchored
2, 3 go L7,U3
2, 1 go L3, D7 (from new positions)
Combined slide for 2: L(7+3), D(7-3) = L10, D4

Generally:
4 stays anchored
3 goes L(b), U(a)
2 goes L(a+b), D(b-a)
1 goes L(a), D(b)

I don't think this will necessarily align with c^2; that should be a simple slide to adjust, though.

After some mucking with program features, it works! Kinda cool how the slides seemed very complex until sorted into 3 sets of moves (switch right with left; switch top with bottom; move whole shape onto c^2).

(Research Diary)

The *all-of-that* sense of *transformation-by-overlapping-slide* that developed through this work created a feeling of its own, which I only recognized when it jumped to mind as I watched my son manipulate the pieces for the puzzle in Figure 7 (he had to click on the colored shapes and drag them onto the gray pinwheel):

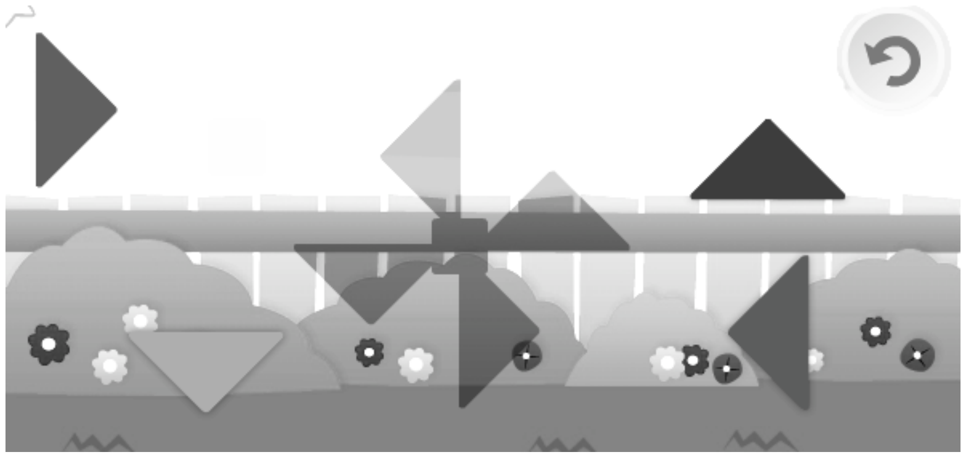


Figure 7: Slide puzzle.
(Sesame Workshop, 2012)

Both situations involve pieces sliding over one another to fit together in a new orientation. Both (for me) involve an *illusion* of rotation.

Similarly, the *feeling* of *sliding along a slope line* that emerged during this problem came back as I pondered the puzzle in Figure 8:

> I had an immediate but fuzzy and unarticulated sense that the pieces wouldn't fit together along the diagonal slope-line in the second diagram; when I listened to the TOT [*tip-of-tongue*] feeling, the shapes also called to mind Pythagorean Proof, but I wasn't entirely sure why. Had I ignored these feelings for being too fuzzy and not a well-defined procedure, I likely would not have solved the problem. (Research Diary, July 11/11)

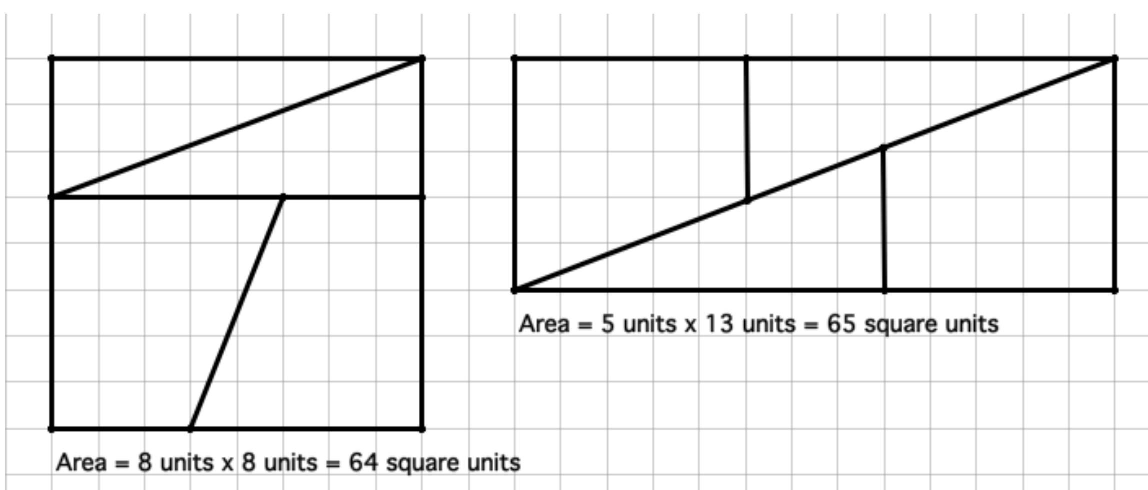
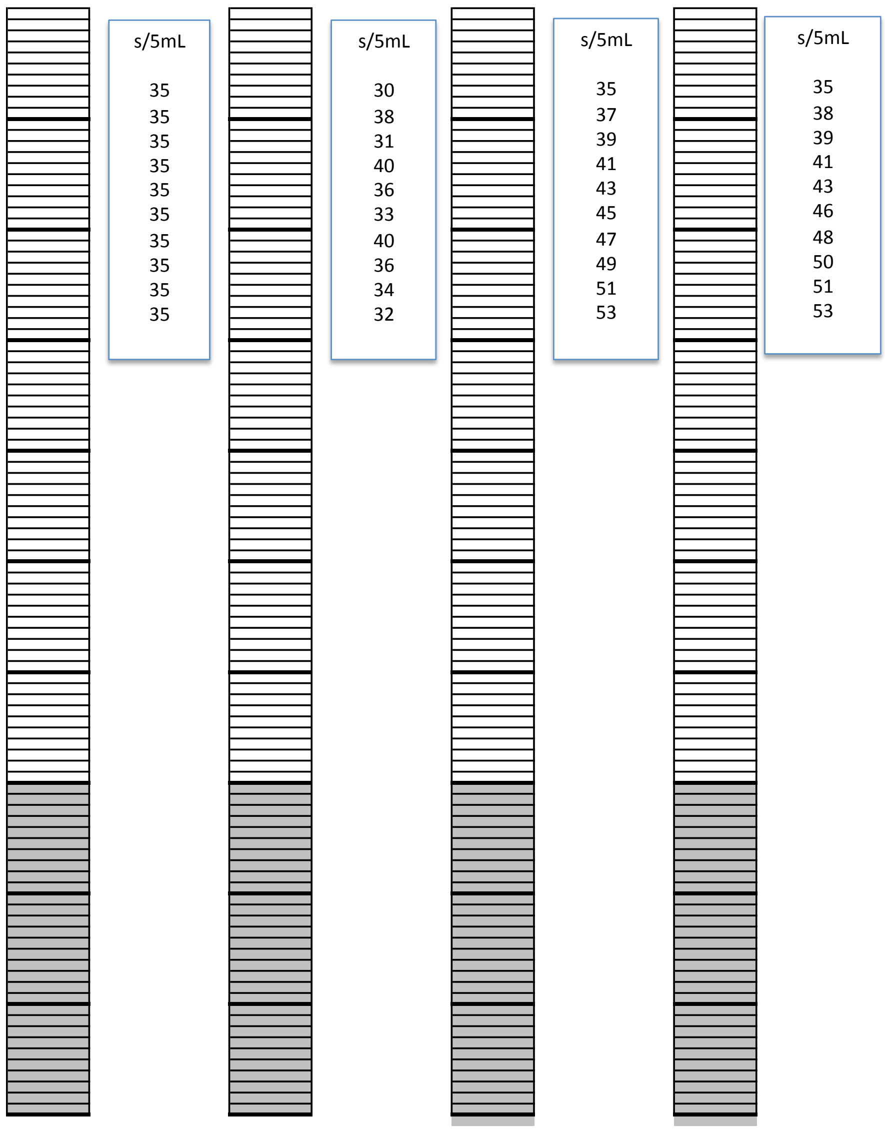


Figure 8: Where did the extra square come from?

Having made this connection, I compared the slopes of the diagonal in different parts of the second diagram, thus confirming my vague suspicion. One thing I find interesting in this example is that neither the feeling of rotation nor “the feeling of sliding along a slope line are the sorts of understanding that you’d find in a list of curriculum outcomes (nor should they; there would need to be an infinite number of such descriptors, and cataloguing and teaching them would destroy their power). The explicit knowledge I used to compare slopes *does* appear in such lists, but without the feeling, I would not have had need for that knowledge.

Appendix B: Hypothetical Melt Rates

If the time @ 30 mL was 9:10, what time did the ice start melting?



Appendix C: Summary of Student Arguments for CPIs

1. Which numbers can be written as the sum of consecutive, positive, integers?

2. In how many ways can a given number be written as the sum of consecutive, positive, integers? (Test: 24? 30? 45? Others?)

*1. **ALL** odd #s (and **ONLY** odd numbers) can be written as sum of 2.*
(Groups 2, 3, 4, 5, 6)
a) 1+2=3
2+3=5
3+4=7
***etc.?** (Yes: In **EACH** line, each addend goes up by 1, so the sum goes up by 2. **ANY** odd # plus 2 makes another odd number. **ALL** even #s break into 2 whole numbers, which can **NEVER** be rearranged to form 2 consecutive numbers.)*

*b) Divide # by 2, then round up and down to get 2 consecutive #s. **ALWAYS?** (Yes: **EVERY** odd number divided by 2 makes two equal ".5" numbers. When one is rounded up and the other down, this **ALWAYS** creates 2 consecutive #s.)*

*c) This is **ONLY** true of odd numbers: **ALL** even numbers divide into two equal addends, which **CANNOT** be rearranged into consecutive numbers.*

(included: all odd numbers to any power work; all odd numbers of odd addends work)

*2. **ALL** 4#s after 10 work.*
1+2+3+4 = 10
2+3+4+5 = 14
3+4+5+6 = 18
etc.?

*3. **ALL** 6#s after 21 work.*
1+2+3+4+5+6 = 21
2+3+4+5+6+7 = 27
etc.?

*4. a) **ALL** multiples of 3 work.*
1+2 = 3
*1+**2**+3 = 6*
4+5 = 9
*3+**4**+5 = 12*
etc.?
7+8 = 15
*5+**6**+7 = 18*
10+11 = 21
*7+**8**+9 = 24*
13+14 = 27
*9+**10**+11 = 30*

etc.?

*<u>b) **ALL** multiples of 3 can be written as sum of 3.</u>*
Divide # by 3, then go one number below and one number above; e.g.:
6 ÷3 = 2; put 2 in the middle, then go one up and one down:
6 = 1 + 2 + 3
Others:
9 = 2 + 3 + 4
12 = 3 + 4 + 5
15 = 4 + 5 + 6
18 = 5 + 6 + 7
***etc.?** (Yes: **ALL** multiples of 3 can be written as 3 equal addends; these can **ALWAYS** be rearranged into 3 consecutive addends by taking one from the highest and giving it to the lowest.)*

*<u>5. a) **MOST** multiples of 3, 5, 7, **(etc.?)** work.</u>*
Take a multiple of 5.
*Divide by 5. (**ALL** multiples of 5 divide evenly by 5.)*
*Write as 5 equal addends. (**ALL** multiples of 5 can be written this way.)*
*Rearrange to make consecutive addends. (Can you **ALWAYS** do this?)*
(same for 5, 7)

*b) <u>**ALL** multiples of **ALL** odd numbers work</u>.*
*Can **ANY** number of odd addends be rearranged so that they are consecutive?*
*What numbers **DO NOT** fall into this category (i.e. what numbers cannot be written as multiples of odd numbers)?*

*c) <u>**ALL** and **ONLY** multiples of odd primes work.</u>*

d) Does this strategy work for multiples of even numbers?

*e) What numbers are **NOT** included in this list?*

*<u>6.a) **NO** powers of 2 work</u>; e.g. 2, 4, 8, 16, 32, 64, **etc.?***

b) These numbers quickly get very far apart: 64, 128, 256–do **ALL** of the numbers in between work?

c) Do any powers of 2 have odd factors?

d) Does this pattern (factors of powers of 2) continue?
2: 1, 2
4: 1, 2, 4
8: 1, 2, 4, 8
16: 1, 2, 4, 8, 16
etc.?

*<u>7. For **ALL** even (**ONLY** even?) numbers: **EVERY** odd factor provides one way of writing a number as the sum of consecutive integers.</u>*

Appendix D: Mathematical Objects Pertaining to CPIs

Four-Numbers and Six-Numbers

Early in the investigation, one group systematically tested various combinations of four addends and six addends. These followed the same pattern as the odd numbers above, but the group recognized that the resulting sums were not multiples of four or six. They named these *4-numbers after 10* and *6-numbers after 21*:

1 + 2 + 3 + 4 = 10

2 + 3 + 4 + 5 = 14

3 + 4 + 5 + 6 = 18

(and so on to yield 22, 26, 30, etc.)

***

1 + 2 + 3 + 4 + 5 + 6 = 21

2 + 3 + 4 + 5 + 6 + 7 = 27

3 + 4 + 5 + 6 + 7 + 8 = 33

(and so on to yield 39, 45, 51, etc.).

At this point, nobody recognized *four-numbers* as *multiples of four plus/minus two* or as *double-odds*."[66] Similarly, nobody recognized *six-numbers* as *multiples of six plus/minus three* or as *triple-odds*. It was much later that anyone recognized *all multiples of odd numbers* as a concise statement of what works, and I do not think anybody connected this idea back to *four-numbers* or *six-numbers*. Again, it seemed this group (and others commenting on their work) would have been content to stop here; nobody commented

---

[66] Or, perhaps more obscurely, as "numbers that are neither odd nor multiples of four," a definition that assumed relevance as I explored another problem: "Which numbers can be written as the difference of two squares?" It was only after I wrote out this sequence that I recognized the "four-numbers" or the "double-odds" familiar to me from my work on CPIs. This opened new insight as I explored why *those* definitions might be relevant to the new problem.

on *eight-numbers* or *ten-numbers* or tried to articulate a more general rule, nor did anybody try to find a way to tell whether a number presented out of sequence was a *four-number* or a *six-number*. Certainly, the importance of the oddness or evenness of the number of addends was now explicitly apparent. As I will attempt to show, however, this understanding became muddled as our language began to conflate *odd (or even) numbers of addends* with *odd (or even) numbers*. Despite having what I thought was a clear distinction between these in my own mind, even my word choice sometimes reveals a subconscious bias that does not fully recognize the distinction.

Four-Numbers* and *Multiples of Four

During the small-group interview following the second class, one group considered whether the divide-and-redistribute method (developed for multiples of two and three) might work for multiples of four:

> Ms. M.: We now have one that works if there's 2 numbers. We know that all odd numbers can be written as the sum of 2. All multiples of 3 can be written as the sum of 3. Where else… might we take that?
>
> Fifteen: Maybe all multiples of 4 can be written as the sum of 4.
>
> (Class 2 Interview)

Before continuing, I emphasized the difference between *four-numbers* and *multiples of four*, which was causing a bit of confusion. I asked the group if there were any ways of describing the set of four-numbers that would help them determine whether a particular number was a four-number without having to count up by fours from previous known examples. Fifteen noted that all of the examples seemed to be multiples of four plus two, which the group tested and confirmed. I asked them whether 115 would fit this description. At first, they still wanted to divide by four to test it: I suspect their plan was to add two after dividing (rather than before), but unfortunately, I simply corrected them and we moved on. Here, I directed the conversation more than I am happy with. In fact, there were a number of occasions when I attached words to what were likely vague, still-forming ideas rather than allowing the students the space to do so:

> Fifteen: No… because…. Well…. For every number, it would be… 2 ahhhhead of a (?) number [talking slowly] of every number. Cause like… 16 [talking to himself, hard to hear]. Well, and then the next one would be 3, 3, this one…. So it would be one ahead of this one, and then…. I don't know if it would be the same.
>
> Ms. M.: How did you test this one? You subtr[*acted*]….
>
> Fifteen: Subtracted 2, and then divided by 4. And if it comes out… ah… a whole number… it's… ah….
>
> Ms. M.: So there's your test. Subtract 2, divide by 4. Could you apply that test to these numbers?
>
> (Class 2 Interview)

Following this, I presented *powers of two* as follows:

2 = 2

4 = 2 * 2

8 = 2 * 2 * 2

16 = 2 * 2 * 2 * 2

Having considered them in this manner, group members seemed confident that all powers of two (except two) are multiples of four, that *no* multiples of four could ever appear in the list of four-numbers, and that therefore the four-numbers do not include powers of two.

However, these were largely answers to my questions, not theirs; the answers were likely fragments that found no home in their own narratives related to this problem.

We still had not considered whether *some* multiples of four *can* be written as the sum of consecutive positive integers or whether the divide-and-redistribute method might be useful for multiples of four. Recognizing this, I redirected the conversation back to this idea. In working through this question, the students further developed the notion of *middle number* as significant. However, it seems they may also have begun to conflate *odd number of addends* with *odd number*, which may have contributed to confusion between *odds* and *multiple-odds*, which can be even (e.g. all double odds are even). In fact, *my* choice of words likely amplified this confusion, even as at least one student recognized something amiss and attempted to resolve it:

> Ms. M.: Can you just rearrange them and get 4 there? [i.e. can you rearrange 4 equal addends to form consecutive integers?]
>
> Fifteen: Ummm…..
>
> Fourteen: You… cause you–**No! Because it has to be an odd number. So you've got a middle.**
>
> Fifteen: It has to be an odd number, because there's no middle on this one. It would be like… it would be both of these two.
>
> Ms. M.: So the problem here is–
>
> Fourteen: It would work for 5, but it wouldn't work for 4.
>
> Fifteen: It works for all numbers.
>
> Fourteen: It works for all odd numbers.
>
> Ms. M.: **Kay, so can we try some more odd numbers, then?** [Here, my language supports the confusion between odd and multiple-odd.]
>
> Fourteen: 5.
> …
>
> Ms. M.: Um…. So will this work for any multiple of 5?
>
> Fifteen: Yeah.
>
> Fourteen: Yeah. It should. **It should work for any odd number. Not odd. Just [waves hand] yeah**. Multiple of 5. [last part spoken very quietly–is she unconvinced that "multiple of 5" matches the meaning her hand gesture indicated?]
>
> Fifteen: **It has to be a multiple of an odd number?**
>
> Fourteen: Yeah.
>
> Ms. M.: **And why does it have to be an odd number?**
>
> Fifteen, Fourteen: **You can't find the middle of an even number**.

Fourteen: Like if it's 4 [addends], there's no middle.

(Class 2 Interview)

Fourteen seems to have recognized the inadequacy of odd, at least at the level of an ND, and Fifteen helped her find better wording. *My* words could easily have further confused this, and their subsequent use of *middle of an even number* rather than *middle addend of an even number of addends* followed my pattern.[67]

Rather than continue exploring ways in which the divide-and-redistribute rule for three might be generalized or help the students articulate an emerging and important distinction between odds and multiple-odds, the difficulty and significance of which I had not yet fully recognized, I then asked them whether six-numbers (i.e. 21, 27, 33,…) and eight-numbers (i.e. 36, 44, 52,…) would ever include powers of two; nobody was sure, so they started writing out a list of six-numbers. They rejected six-numbers as all odd, but weren't sure about eight-numbers. I suggested that *this* might be something to consider when their group met again. Had I been more in tune with the indicators that Fourteen was struggling to surface a vague understanding (e.g. her sudden drop in volume and hand-waving), perhaps I would have responded more appropriately.

---

[67] Similarly, (and despite my explicit attempt to help students make the connection between multiples-of-three and three-equal-addends), I was appalled to find this comment in the transcripts:

> Ms. M.: Okay. So…. If we…. Can I write over…. So, using 6 as your example, then, you said you're gonna take 6 divided by 3, then you put 2 in the middle. So first of all, will *that* work for any, um, multiple of 3? Can you always divide it by 3 and get an even number?
> (Class 1 Interview)

What I *meant* was, "Can you always divide by 3 and get a WHOLE number," not an EVEN number; but here I conflated language for "divides evenly" with even numbers!! I suspect this sort of language does point to a deeper confusion that can occur beneath ordinary consciousness, and this may have played a role in the difficulty many had in working fluently with even numbers that can be expressed as multiples of odd numbers. In this case (thankfully), Six correctly responded, "Only for… even numbers" and was able to continue building a connection between multiples-of-three and three-equal-addends that encompassed both odd and even multiples of three.

Odds* and *Multiple Odds

When we reconvened in April (a month later, with a two-week vacation in between), it was somewhat difficult to re-engage with the problem. However, after working in their groups for the first part of the class, two groups became convinced of a more general rule. One concluded that "all odd numbers and all multiples of odd numbers work," and one student stated this even more succinctly: "All multiples of odd primes work." He explained:

> Prime: Uh, the non-primes–the non-primes, um…. It doesn't really…. It's just repeating, cause say 9? Is just a multiple of 3? And you'd get… say 18? You could get that with the 3. (Class 3)

Still, one group tested multiples of 13, so presumably these students were not fully convinced of the generality of their method.

In the larger group, however, an unstated confusion about the distinction between odds and multiple-odds remained. Rather than recognizing the significance of this confusion, my suggestion to "do 40" likely bypassed it:

> Prime: It [the divide-and-redistribute method for 3] also works for 5 and 7. So 5…. Say you've got…. Someone give me a multiple of 5. Felix?
>
> Felix: 33. No 35!! 35.
>
> Prime: 35. Divide by 5…. 7…. So then you've got 7. You take 2 on either side. 6 and 5….
> *35 ÷ 5 = 7*
> *5 6 7 8 9*
> …
>
> Felix: Didn't we decide that before–divide by 2, then go up by 1, half go up by half 1 (?)? [presumably divide by 2, then go up .5 and down .5 to get 2 consecutive integers]
>
> Prime: Yeah. That works for, uh….
>
> Felix: All odd numbers.
>
> Prime: Yeah, all odd numbers.
>
> Felix: 3, 5, and 7 are all odd numbers.

Ms. M.: **Let's pick an even multiple of 5, then. Do 40.**

Prime: 40?

Ms. M.: Sure.

Prime: 40…. You divide it by…. You divide it by 5, you get 8. So you got 8… 7… 6… 9, and 10. Add these all up… 6, 13, plus 8 is 21… plus 9 is 30… plus 10 is 40. Adds to 40. So it works for that. As well as all others.
40 ÷ 5 = 8
6 7 8 9 10

Ms. M.: So you're saying all multiples of 5 would fit into that pattern.

Prime: Yes.

Ms. M.: Any comments?

Prime: Jolt.

Jolt: **Even numbers also work.**

(Class 3; emphasis added)

Here, Jolt did not yet seem to distinguish between even-multiples-of-odds and even-numbers. In the following comment, though, he began to articulate a clearer distinction between odds, odd-multiples-of-odds and even-multiples-of-odds:

> Jolt: Can I write something on the board? Kay, I agree with Prime? And, um…. But I'd like to add something to his? I'd say that even even numbers? When they have an o*dd*–when they're multiplied by an odd number, so for example 2 times 3? 2 times 3 is 6, so there's 3… 2 sets of 3? This is one 3… 1 plus 2 plus 3 is 6. And, um, another thing that I noticed? And I'm not exactly sure if it works for all numbers? Is that **even even sets of even numbers could also work**. So for example, 2 plus 2 plus 2 plus 2 would equal 8? But if you change this one to 3, it changes this one to 1? This is 4… plus 4 is 8 still. (Class 3; emphasis added)

It might have been worthwhile at this point to emphasize the objects Jolt identified as relevant: He was the first to articulate that the solution has STDW even or odd numbers of even or odd addends. This global-yet-vague look at the problem was bypassed as Prime zoomed in on refuting the statement that "even sets of even numbers could also work" (four twos cannot be redistributed as consecutive positive integers, because there's no middle number).

Prime did go on to show how his method applied to *any* multiple of seven (odd or even), then extended his conclusion to include all multiples of primes. This included a brief foray into how to handle negative integers that sometimes result when redistributing equal addends around a middle addend. I then directed the conversation back to what gets missed by odds and multiple-odds. It was in contemplating this that *I* came to the in-hindsight-surprising-*aha* that powers of two might in fact be *defined as* the set of numbers with no odd factors–this long after having recognized that multiple odds and powers of two must *somehow* be complementary.

Although it still seems surprising that this could have been such a revelation to me, it was even less obvious to the students. In response to my questions regarding what was missed by multiples of primes, one student responded, "Even." I presented 10 and 18 as even numbers that *are* multiple-odds and asked *which* even numbers do not fit. This topic became a dominant theme in the remaining three classes and two interviews. The lack of a clear distinction between (or clear names for) even-groups-of-odds, odd-groups-of-odds, odd-groups-of-evens, and even-groups-of-evens continued to create confusion. Looking back, I wonder how things might have unfolded differently if we had earlier and more clearly recognized the potential significance of Jolt's emerging insight: The solution has STDW odd or even numbers of odd or even addends. As it was, this confusion persisted through three more classes and two associated interviews. Of course, it is easy to recognize significance in hindsight, but I wonder how a more general awareness of *the importance of* the vague and the global (and their indicators) might amplify their significance *in the moment*. In the following section, I further consider mathematical objects that might be considered identical yet remain separated in the students' and sometimes my own perception. I wonder whether more explicit attention to *something to do with odd or even numbers of odd or even addends* might have helped us connect these sooner.

***Multiples of 5*, *Multiples of 6*, or *Multiples of 12*?**

Nowhere was the confusion between *odd* and *multiple-odd* more apparent than during the Class 3 interview with Double-Check and Twelve. They started by sharing work done in class showing that all multiples of five can be expressed as the sum of five equal addends that can be redistributed as consecutive positive integers. Double-Check explained that different group members had tested 9, 11, 5, and 7. I wondered: Why only odd multiples of five? When Twelve added, "I did 6," Double-Check asked her if she meant multiples of six. Even Twelve was not sure, and the interview shifted to focus on multiples of six and twelve instead of multiples of five:

> Twelve: Yeah… yeah….. Like, um. [opening book] So I did, like, um–well, actually, I did, like, 5 divided by 30 to get 6?
>
> Ms. M.: Mm-hm.
>
> Twelve: And, then, like, you were there, when, like, we lined it up [i.e. 6 6 6 6 6 → 4 5 6 7 8]. And then we can take the 4–so, like, basically, it would all equal 12 [i.e. 4 + 8 = 12; 5 + 7 = 12; ignored the middle 6]? **So… like, multiples of 12?** So…. You'd have to, like… do 4 + 5 + 6 +7 +8.
> Ms. M.: Okay. And so **does this show that multiples of 5 work or multiples of 6 work**? Like this is a multiple of 5 and a multiple of 6, right? But…. Would it con[tinue]–if you…. Are you saying that all multiples of 6 work, or that all multiples of 5 work? Or neither? [Here I attempt to direct attention back to the original question, but in so doing ignore Twelve's perception that multiples of 12 are relevant.]
>
> Twelve: Um…. Well, like, I divided. **I used 5 to divide it**?
>
> Ms. M.: Mm-hm.
>
> Twelve: And then, so…. **Um, probably… all multiples of 6 work**. **Like, well at least the ones like [pointing to book] 4**.
>
> Ms. M.: Okay.
>
> Twelve: Kind of hard to explain.

Ms. M. [looking at book]: So let's see. Oh, so you're saying if you add another pair of, um… that makes 12?

Twelve: And then if you had… a few more 6s here, like 3 and, um, 9 [i.e. added to each end of the sequence], it equals 12?

Ms. M.: Okay.

Twelve: So…. Yeah.

Ms. M.: And so to do that, you would need…. That would be like another 6 here and another 6 here.

Twelve: Yeah. **Yeah, so**.

Ms. M.: Okay. So you had to add 2 more 6s to make this new one, then, right?

Twelve: Yeah.

Ms. M.: And that would be…. 30 plus… 2 more 6s. So that would be 42. And then if you put 2 more 6s…. That would be another 12, right? That what would that be…. 54? So…. From that, can you conclude that all multiples of 6 work?

Twelve: [nodding slightly] Yeah. **Just…. Yeah**.

(Class 3 Interview)

Early in this exchange, I totally missed Twelve's suggestion that she was dealing with multiples of 12, and offered her a choice between multiples of five and six. She did not seem to recognize the significance of 6 + 6 + 6 + 6 + 6 as a multiple of five, nor did she realize that by adding two sixes, she was skipping every second multiple of six. I wonder what would happen if I had more seriously considered her comment about multiples of 12; more generally, I wonder what would happen if in my mind I learned to add an *ish* to hesitant comments. They were not *quite* multiples of 12, but they had *something to do with* multiples of 12.

Twelve extended her list of sixes, even so far as to include negative integers; she had no trouble working with the additive inverses this created:

~~-1 0 1~~ 2 3 4 5 **6** 7 8 9 10 11 12 13
6 6 6 6 6 6 6 **6** 6 6 6 6 6 6 6

I drew attention back to the fact that she was in fact adding *two* sixes with each extension, after which Double-Check also asked, "Isn't it multiples of 12, then?" I pointed out the lone six left in the middle of the pairs of 12. I agreed that all were multiples of six, but noted that in jumping from 30 to 42 (five sixes to seven sixes), we added 12. Double-Check recognized that 36 was missing, and re-named the sequence *all multiples of six by adding 12s.* This sequence might also have been named all *odd numbers multiplied by six,* though this took a little longer:

> Twelve: So it's kind of like… **you go by like, every like, odd number times 6, kind of like, works**.
> Ms. M.: Okay.
>
> Twelve: Like, 3, 5, 7, 9.
>
> Ms. M.: So if you have three 6s it works, if you have five 6s it works, if you have seven 6s it works.
>
> Twelve: (nods)
>
> (Class 3 Interview)

Interestingly, I suggested using the term *odd multiples of six*, to describe this set, and Double-Check noted that 18, 30, and 42 are even multiples of six that belong to the set. In my mind, I was referring to any odd number multiplied by six, and somehow I did not even recognize the problem with my word choice. Although certainly I knew that an odd number times an even number can result in an even, it is almost as though at a deeper level, there is an intuitive tug that suggests "any odd factor results in an odd product." The difficulty does not emerge when *odd times even* is the direct object of consciousness, but somehow it plays a different role in the subconscious!

Even *after* explicitly recognizing the confusion this distinction caused for some students, I caught myself developing the following faulty line of reasoning: "An even number of addends will always form even + odd pairs, because two consecutive integers always include one even and one odd. Any even plus any odd always yields an odd."

So far so good…. But for a short time, it seemed that this implied that the sum of the addends must be odd! Somehow, an even number of odd pairs yielding an even sum did not come easily to mind. It is almost as though the odd pair-sums created an *interference* with what I knew at some other level to be true.

It was only in hindsight that it occurred to me that the set created when the students kept adding sixes was analogous to the four-numbers and six-numbers that Double-Check introduced during the very first class and might also have been called *12-numbers*. Double-Check noted that to get multiples of six, then, you would probably need to use threes; i.e. start with one three in the middle and two more each time. But we did not explore the implications of this notion, which would have provided a nice bridge back to Double-Check's six-numbers (i.e. multiples of six plus/minus three).

Here, we have drifted far from the original discussion about multiples of five, which was never fully resolved: Does Six's rule for threes (divide into groups of three, rearrange around a middle number) apply to fives? I asked them to generate a series of cases that varied the number of sixes instead of the number of fives. Double-Check was immediately confident that all multiples of five can be written as consecutive positive integers, while Twelve still wanted to consider 10 + 10 + 10 + 10 + 10 *as a multiple of ten;* i.e. by adding pairs of tens.

Double-Check tested several cases on her own. Still unsure that both understood the pattern, I asked:

> Ms. M.: Okay. So we're getting lots of examples where if you take 5 numbers in a row, it does convert to 5 consecutives, right? But will it work no matter what 5 numbers you put in that row? [pause] If you put 100, 100, 100, 100, 100, could you still circle the middle number and rearrange them?
>
> Twelve: …. Um….. Yeah, I think so.
>
> Double-Check: [talking while testing 100; confirms it works] And again, it's every… skip multiple, like it's every odd group.

(Class 3 Interview)

It seems that Double-Check moved from multiples of five to multiples of 100, but it is likely that this was deliberate; i.e. she understood the difference and explored both dimensions. By now, I seemed more aware of the potential confusion this might engender and commented:

> Ms. M.: Okay. And I think we're–we've got 2… 2 things happening here. We can look at this as multiples of 100, right? Or we can look at it as–if we've got 5 of them, we can look at it as a multiple of 5.
>
> Double-Check: Either way.
>
> Ms. M.: Yeah. So you're saying–every…. If we keep going out by 2s, every second one will make another multiple of… what?
>
> Double-Check: 100 or–
>
> Ms. M.: Okay.
>
> Double-Check: 3. No. [shakes head]
>
> Ms. M.: Now, does any…. Can you take any–can I just borrow that pen for a sec? If I go 1, 1, 1, 1, 1 or 2, 2, 2, 2, 2, or 3, 3, 3, 3, 3, all of these are multiples of what? Like, these are [Double-Check: 5] multiples of 1 [i.e. 1+1+1+1+1], multiples of 2, multiples of 3, multiples of 4 [Double-Check: 5], but all of them are multiples of 5, right? Because there's 5 in each group? This is 5 times 1, 5 times 2, 5 times 3, 5 times 4, 5 times 5? Can all of these be rearranged to make 5 consecutive numbers?
>
> (Class 3 Interview)

Pushing further, I had them consider multiples of seven as follows:

> 1 1 1 1 1 1 1
> 2 2 2 2 2 2 2
> 3 3 3 3 3 3 3
> 4 4 4 4 4 4 4
> 5 5 5 5 5 5 5

Double-Check immediately recognized that these would work, "because seven's got a middle number." At this point, she asked, "Doesn't that mean it's all odd numbers and every odd group of even?" I asked why 24 (six times four) and 12 (two times six) work, and she quickly noted that in both of those cases, the number could also be given in terms of three addends. However, she did not identify *even groups of evens* as what is left, nor did she connect this set with *powers of two*.