University of Alberta

IRREDUCIBLE OPERATORS AND SEMIGROUPS ON BANACH LATTICES

by

Niushan Gao

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

Department of Mathematical and Statistical Sciences

Fall 2013
Edmonton, Alberta

In memory of my father, and
for the love of my mother

Abstract

We study ideal irreducible and band irreducible positive operators and semigroups of positive operators on Banach lattices.

In Chapter 1, we consider irreducible positive operators. In particular, we prove the following *comparison theorem*. Let T, S be two operators on a Banach lattice such that $0 \leq S \leq T$ and $r(T) = r(S)$. Suppose that $r(S)$ is a pole of the resolvent $R(\cdot, S)$. Then $T = S$ if either S is ideal irreducible, or S is band irreducible with a σ-order continuous functional $x_0^* > 0$ such that $S^* x_0^* = r(S) x_0^*$ and T is σ-order continuous. We also provide some applications of comparison theorems. We prove that if two positive operators are semi-commuting with one of them compact and the other one irreducible then they are commuting. We also prove that if two positive operators are semi-commuting with one of them compact then their commutator is quasinilpotent. These results answer questions in [1, 9].

In Chapter 2, we consider irreducible semigroups of positive operators. We prove that for an ideal irreducible $\overline{\mathbb{R}^+}$-closed semigroup $\mathscr{S}$ containing a non-zero compact operator, if all the minimal-rank projections of $\mathscr{S}$ have the same range, then there exist positive disjoint vectors $x_1, \dots, x_r$ such that every operator in $\mathscr{S}$ acts as a positive scalar multiple of a permutation on $x_1, \dots, x_r$; in particular, the operators in $\mathscr{S}$ have a common eigenvector. We also establish the dual version of this result for ideal irreducible $\overline{\mathbb{R}^+}$-closed semigroups which have a unique minimal-rank projection. Both of these results apply to commutative semigroups. This study extends the approach invented in [36, 37] and improves some results of [1] about peripheral spectra of irreducible operators.

In Chapter 3, we consider positive operators which have irreducible super-commutants. We prove that if a compact operator $K > 0$ has ideal irreducible super left-commutant or super right-commutant then it is non-quasinilpotent and it acts as a positive scalar multiple of a permutation on some positive disjoint vectors $x_1, \dots, x_r$ which span the peripheral spectral subspace of K. We also show that $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ for any $x > 0$ and that every operator semi-commuting with K commutes with it; in particular, $[K\rangle = \langle K] = L_+(X) \cap \{K\}'$. Similar properties are shown valid for K^*. We also prove that the positive operators $S > 0$ in the following three chains have positive eigenvectors: $T \leftrightarrow K \leftrightarrow S$, $T \leftrightarrow S \leftrightarrow K$ and $S \leftrightarrow T \leftrightarrow K$, where $T > 0$ is ideal irreducible, $K > 0$ is compact and $\leftrightarrow$ stands for commutation.

This thesis is based on [19, 20, 21].

Preface

A positive matrix A is said to be irreducible if it does not have a block form $\left[\begin{smallmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{smallmatrix}\right]$ under any permutation of the standard basis. The classical Perron-Frobenius theory explores the peripheral spectrum of irreducible matrices. We present it following [1, Chapter 8].

Theorem. *Let $A > 0$ be an irreducible matrix on $\mathbb{R}^n$ $(n \geq 2)$.*

(i) The spectral radius $r(A)$ is non-zero and is a simple root of its characteristic polynomial f_A, and $\ker(r(A) - A) = \mathrm{Span}\{x_0\}$ for some strictly positive vector x_0.

(ii) The peripheral spectrum $\sigma_{\mathrm{per}}(A) = r(A)G$ where G is the set of all k-th roots for some $k \geq 1$, and each point in $\sigma_{\mathrm{per}}(A)$ is a simple root of f_A with one-dimensional eigenspace.

(iii) If $0 \leq B \leq A$ and $r(A) = r(B)$, then $A = B$.

This theory has been extensively studied and generalized to irreducible operators on arbitrary Banach lattices. Following [23], we refer to results of style of (i) as *Jentzsch-Perron Theorem* and to results of style of (ii) as *Frobenius Theorem.* We also follow [30] to refer to results of style of (iii) as *comparison theorems.*

For ideal irreducible positive operators whose spectral radius is a pole of the resolvent, the Jentzsch-Perron part was established in [40, 38], the Frobenius part was established via the various efforts of [33, 34, 39, 35], and the comparison theorem was established in [10]. Similar properties have also been established for band irreducible σ-order continuous operators whose spectral radius is a pole of the resolvent, under some additional conditions, through the efforts of [28, 4, 5]. We refer the reader to [23, 26], etc for more results and history in this direction.

However, it had not been known whether these results applied to compact operators until de Pagter proved the celebrated result that compact ideal irreducible operators are non-quasinilpotent ([12]). The band irreducible version was proved in [43, 22]. de Pagter's Theorem has been extensively studied. It was extended in [2, 3] to commuting and semi-commuting positive operators that possess irreducibility and compactness in one sense or another (cf. also [1, Chapters 9 and 10]). It was also extended in [14] to collections and semi-groups of positive operators. In particular, Drnovšek proved that if a compact operator $K > 0$ is locally quasinilpotent at a non-zero positive vector then its super right-commutant $[K\rangle = \{S \geq 0 : SK \geq KS\}$ is ideal reducible.

In Chapter 1 of this thesis, which is based on [19, 20], we consider the Jentzsch-Perron theorem for irreducible operators under some less restrictive conditions and we develop a uniform approach to several versions of comparison theorems including the versions in [10, 4, 5]. We also show that the conditions imposed on the band irreducible operators in [4] and [5] are equivalent (cf. Theorem 1.14). Finally, we provide some applications of comparison theorems. In particular, we use a comparison theorem to show that the semi-commuting condition is equivalent to commuting in some results of [1]. Specifically, we prove that if a compact operator $K > 0$ semi-commutes with an ideal irreducible operator $T > 0$ then it commutes with T.

This result is improved to a much general form in Chapter 3, where we prove that if a compact operator $K > 0$ has ideal irreducible super right-commutant or super left-commutant then every operator semi-commuting with K commutes with K. In particular, this implies that in the aforementioned theorem of Drnovšek, we can replace the super right-commutant with super left-commutant. A more thorough study of such operators K is conducted in Chapter 3. We actually establish a Jentzsch-Perron-Frobenius theorem for such operators K. Chapter 3 is based on [20].

The Perron-Frobenius theory has also been extended to semigroups of positive operators on Banach lattices. An approach was invented in [36] for irreducible semigroups of positive matrices and then was extended to irreducible semigroups of positive compact operators on $L_p(\mu)\,(1 \leq p < \infty)$ in [37, Section 8.7]. This approach was later also used in [27] to extend some of the results to irreducible semigroups of positive compact operators on order continuous Banach lattices.

In Chapter 2 of this thesis, we fully extend the approach to irreducible semigroups of positive operators on arbitrary Banach lattices. The compactness condition is weakened to that the semigroup contains a non-zero compact positive operator. As an application of this study, we give an alternative proof of the Frobenius theorems for irreducible operators whose spectral radius is a pole of the resolvent, which are originally due to [35, 4, 5]. Chapter 2 is based on [21].

Acknowledgements

I am deeply grateful to my supervisor Dr. Vladimir Troitsky for his constant help, encouragement and support. Without his guidance, this work would have never existed. I am also grateful to Dr. Alexander Litvak for many friendly and beneficial conversations. I am greatly indebted to Dr. Kungyang Wang, Dr. Yingbo Zhang and Dr. Anthony To-Ming Lau for encouraging and inspiring me to study mathematics.

I want to express my sincere gratitude to my Sensei Sean Bowen from whom I have learned to be a better human being. I also want to acknowledge the help of Senseis Dan Wilm, Douglas Aoki, Beau Sapach and Sam Maloney. My special thanks are due to Senseis Betty Mah, Lauren Sergy and Glen Sergy for their friendliness.

I would like to thank my friends Valerie Budd, Zhihua Chang, Ying Chen, Dan Cheng, Lu Cheng, Wei Cheng, Sherri Kessels, Xihui Lin, Junfeng Ma, Jie Ren, Tara Schuetz, Xiaobai Tan, Huijuan Wang, Sheng Wang, Foivos Xanthos and Yakun Zhao for their precious friendships which have accompanied me through many difficult times.

Contents

Chapter 0

Preliminaries

0.1 Basic notions on spectral theory

We briefly review some fundamental concepts on Banach spaces and spectral theory. Standard references are [11, 1].

Throughout this chapter, $\mathbb{F}$ stands for either the set $\mathbb{R}$ of real scalars or the set $\mathbb{C}$ of complex scalars.

Definition 0.1. A real-valued function $\|\cdot\|$ on a vector space X over $\mathbb{F}$ is called a ***norm*** if it satisfies the following:

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{F}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In case $\|\cdot\|$ is a norm, the pair $(X, \|\cdot\|)$ is called a ***normed space***. If, in addition, $(X, \|\cdot\|)$ is complete, then it is called a ***Banach space***.

When no ambiguity arises, we write X for $(X, \|\cdot\|)$ for simplicity. We refer to [11, Chapters III, VI and V] for basic properties of normed and Banach spaces, especially, for their weak and weak-star topologies.

Definition 0.2. Let X and Y be two normed spaces over $\mathbb{F}$.

(i) A linear operator $T : X \to Y$ is ***bounded*** (or ***continuous***) if the following is satisfied:

$$\|T\| := \sup_{\|x\| \leq 1} \left\|Tx\right\| < \infty. \tag{1}$$

We denote by $L(X, Y)$ the space of all bounded operators from X to Y endowed with the operator norm defined by (1). In case $Y = \mathbb{F}$, we write X^* for $L(X, \mathbb{F})$ and call it the ***dual space*** of X.

(ii) An operator $T \in L(X,Y)$ is ***invertible*** if there exists $S \in L(Y,X)$ such that $TS = \mathrm{Id_Y}$ and $ST = \mathrm{Id_X}$. In this case, we write $S := T^{-1}$.

(iii) For $T \in L(X,Y)$, its ***adjoint operator*** $T^* \in L(Y^*, X^*)$ is defined by $T^*y^* := y^* \circ T$ for any $y^* \in Y^*$.

If $X = Y$, we write $L(X,X)$ as $L(X)$ for simplicity.

Definition 0.3. Let X be a complex Banach space. For an operator $T \in L(X)$, we define its

(i) ***spectrum*** by $\sigma(T) := \{\lambda \in \mathbb{C} : \lambda - T \text{ is not invertible}\}$,

(ii) ***spectral radius*** by $r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$,

(iii) ***peripheral spectrum*** by $\sigma_{per}(T) := \{\lambda \in \sigma(T) : |\lambda| = r(T)\}$,

(iv) ***resolvent set*** by $\rho(T) := \mathbb{C}\backslash\sigma(T)$,

(v) ***resolvent*** by $R(\cdot, T) : \rho(T) \to L(X); \lambda \mapsto (\lambda - T)^{-1}$.

We refer to [11, Chapter VII] and [1, Chapter 6] for general spectral theory of operators on complex Banach spaces. For operators on real Banach spaces, we pass to their complexifications when considering spectral properties; cf. [1, Section 1].

The following is a useful result on continuity of spectral radius.

Proposition 0.4 ([19])**.** *Let X be a Banach space and (T_n) a sequence in $L(X)$ converging to $T \in L(X)$. If $\sigma_{per}(T)$ is a spectral set then $r(T_n) \to r(T)$.*

Proof. It is well known that $\limsup_n r(T_n) \le r(T)$; cf. [32, Chapter 1, Theorem 31]. It remains to prove $r(T) \le \liminf_n r(T_n)$. We imitate the proof of continuity of spectral radius on compact operators. Assume $r(T) > \liminf_n r(T_n)$.

Take $\varepsilon > 0$ such that $r(T) > \liminf_n r(T_n) + 2\varepsilon$. By passing to a subsequence, we may assume $r(T) > r(T_n) + \varepsilon$ for all $n \geq 1$.

Since $\sigma(T)\backslash\sigma_{per}(T)$ is closed, we can take $\delta > 0$ small enough such that $2\delta < \varepsilon$ and $\sigma(T)\backslash\sigma_{per}(T) \subset \{z : |z| < r(T) - 2\delta\}$. Now for $j = \pm 1$, define curves $\gamma_j(t) = [r(T) + j\delta]e^{it}$, $0 \leq t \leq 2\pi$. By Cauchy integral theorem, we have

$$\int_{\gamma_1 - \gamma_{-1}} R(\lambda, T_n)\mathrm{d}\lambda = 0, \ \forall\, n \geq 1.$$

On the other hand, we know that in a unital Banach algebra, for a invertible and x with $||x|| < \|a^{-1}\|^{-1}$, one has $||(a-x)^{-1} - a^{-1}|| \leq \frac{||x||\,||a^{-1}||^2}{1-||x||\,||a^{-1}||}$; cf. [32, p. 5]. Using this, one can easily verify that $R(\cdot, T_n) \to R(\cdot, T)$ uniformly on γ_j's. Therefore,

$$P = \frac{1}{2\pi i}\int_{\gamma_1 - \gamma_{-1}} R(\lambda, T)\mathrm{d}\lambda = \lim_n \frac{1}{2\pi i}\int_{\gamma_1 - \gamma_{-1}} R(\lambda, T_n)\mathrm{d}\lambda = 0,$$

where P is the spectral projection of T for $\sigma_{per}(T)$. This is absurd. $\square$

We now introduce several important classes of operators.

Definition 0.5. Let X and Y be two normed spaces. An operator $T \in L(X, Y)$ is said to be

(i) ***finite-rank*** if its range Range(T) is finite dimensional,

(ii) ***compact*** if $T(B_X)$ is relatively compact in Y where B_X is the closed unit ball of X,

(iii) ***weakly compact*** if $T(B_X)$ is relatively weakly compact in Y,

(iv) ***strictly singular*** if $T|_Z : Z \to T(Z)$ is not invertible for any infinite-dimensional closed subspace Z of X.

In case $X = Y$, an operator $T \in L(X)$ is said to be ***power compact*** (or weakly compact, strictly singular, etc) if there exists $k \geq 1$ such that T^k satisfies the property.

Definition 0.6. Let X be a Banach space. An operator $T \in L(X)$ is said to be

(i) ***Riesz*** if for any $\varepsilon > 0$, $\{\lambda \in \sigma(T) : |\lambda| > \varepsilon\}$ [1] is a spectral set with finite-dimensional spectral subspace,

(ii) ***peripherally Riesz*** if $r(T) > 0$ and $\sigma_{per}(T)$ is a spectral set with finite-dimensional spectral subspace.

The class of peripherally Riesz operators was introduced in [21]. We refer to [11, Chapter VI] and [1, Chapters 2, 4 and 7] for relations and (spectral) properties of these classes of operators. In particular, we refer to [1, Section 7.5] for equivalent characterizations of Riesz and peripherally Riesz operators in terms of their ***essential spectra***.

0.2 Asymptotic behaviors of peripherally Riesz operators

We consider asymptotic behaviors of peripherally Riesz operators in this section. We are motivated by [36] and the results are based on [21].

Throughout this section, X stands for a Banach space. Recall that we pass to the complexifications whenever considering spectral properties of operators on real Banach spaces.

[1]This set could possibly be empty.

A set of operators on X is said to be $\overline{\mathbb{R}^+}$***-closed*** if it is norm closed and it is closed under multiplication by positive scalars. Given a set $\mathcal{A}$ of operators on a Banach space X, we write $\overline{\mathbb{R}^+}\mathcal{A}$ for the smallest $\overline{\mathbb{R}^+}$-closed (multiplicative) semigroup containing $\mathcal{A}$. In particular, if T is an operator on X, we write $\overline{\mathbb{R}^+}T$ for the $\overline{\mathbb{R}^+}$-closed semigroup generated by T. Clearly, $\overline{\mathbb{R}^+}T$ consists of all positive scalar multiples of powers of T and of all the operators of form $\lim_j b_j T^{n_j}$ for some sequence (b_j) in $\mathbb{R}_+$ and some strictly increasing sequence (n_j) in $\mathbb{N}$; these limit operators form the ***asymptotic part*** of $\overline{\mathbb{R}^+}T$.

Given a semigroup $\mathscr{S}$ in $L(X)$, we denote by $\min\operatorname{rank}\mathscr{S}$ the minimal rank of non-zero elements of $\mathscr{S}$; we write $\min\operatorname{rank}\mathscr{S} = +\infty$ if $\mathscr{S}$ contains no non-zero operators of finite rank.

The following observations on matrices are based on [36, Lemma 1] and are critical to our study. We need the following standard lemma. Recall that a vector $u \in \mathbb{C}^n$ is said to be ***unimodular*** if $|u_i| = 1$ for all $1 \le i \le n$. Let $\mathbb{U}_n$ denote the set of all unimodular vectors in $\mathbb{C}^n$. Clearly, $\mathbb{U}_n$ is a group with unit $(1, \dots, 1)$, with respect to the coordinate-wise product.

Lemma 0.7. *If $u \in \mathbb{U}_n$ then there exists a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $u^{m_j} \to (1, \dots, 1)$.*

Proof. Since $\mathbb{U}_n$ is compact, we can find a subsequence $u^{k_j} \to v$ for some $v \in \mathbb{U}_n$. Passing to a subsequence, we may assume that $m_j := k_{j+1} - k_j$ is strictly increasing. Then $u^{m_j} = u^{k_{j+1}} u^{-k_j} \to vv^{-1} = (1, \dots, 1)$. $\square$

Let A be a square matrix with $r(A) = 1$ and $\sigma(A) = \sigma_{\mathrm{per}}(A)$. Using Jordan decomposition of A, we can write $A = U + N$ where U is ***unimodular*** (i.e. there is a basis under which U is diagonal and the diagonal is a unimodular vector), N is nilpotent, and $UN = NU$. By Lemma 0.7, we can find a strictly increasing sequence (m_j) such that $U^{m_j} \to I$.

Case $N = 0$. In this case, $A = U$, so that $A^{m_j} \to I$.

Case $N \neq 0$. Let k be such that $N^k \neq 0$ but $N^{k+1} = 0$. Then

$$A^n = (U+N)^n = U^n + \binom{n}{1} U^{n-1} N + \cdots + \binom{n}{k} U^{n-k} N^k. \tag{2}$$

Note that $\lim_n \binom{n}{i} / \binom{n}{k} = 0$ whenever $i < k$. Therefore, if we divide (2) by $\binom{n}{k}$, then every term in the sum except the last one converges to zero as $n \to \infty$. Denote $r_j = m_j + k$ and $c_j = 1/\binom{r_j}{k}$, then $\lim_j c_j A^{r_j} = \lim_j U^{r_j - k} N^k = N^k$. Moreover, we claim that if $B = \lim_j b_j A^{n_j}$ with (n_j) strictly increasing, then B is nilpotent (even square-zero) and $b_j \to 0$. Indeed, let $e_1, \dots, e_n$ be a basis under which the matrix U is diagonal and $\operatorname{diag}(U)$ is a unimodular vector. For $x = \sum_{i=1}^n x_i e_i$, put $\|x\| = \sum_{i=1}^n |x_i|$. Clearly, this is a norm on $\mathbb{C}^n$ and U is an isometry with respect to this norm. It follows from (2) that $\binom{n}{k}^{-1} A^n - U^{n-k} N^k \to 0$ as $n \to \infty$. Since U is an isometry and $N^k \neq 0$, the sequence $\left(\|U^{n-k} N^k\|\right)_n$, and therefore $\left(\binom{n}{k}^{-1} \|A^n\|\right)_n$, is bounded and bounded away from zero. It follows from $b_j A^{n_j} \to B$ that the sequence $\left(b_j \binom{n_j}{k}\right)_j$ is bounded, hence $b_j \to 0$. It also follows that $b_j \binom{n_j}{k} U^{n_j - k} N^k \to B$ so that $B^2 = \lim_j \left(b_j \binom{n_j}{k} U^{n_j-k} N^k\right)^2 = 0$ because $UN = NU$ and $N^{2k} = 0$. This proves the claim.

Now we can obtain the following two possible structures of the asymptotic part of $\overline{\mathbb{R}^+ T}$ when T is peripherally Riesz.

Proposition 0.8 ([21]). *Suppose that T is peripherally Riesz with $r(T) = 1$. Let $X = X_1 \oplus X_2$, where X_1 and X_2 are the spectral subspaces for $\sigma_{\mathrm{per}}(T)$ and* *its complement in $\sigma(T)$, respectively. Let P be the spectral projection onto X_1. Then exactly one of the following holds:*

(i) ("Unimodular" case) $T_{|X_1}$ *is unimodular, and each operator in the asymptotic part of $\overline{\mathbb{R}^+T}$ is of form $cU \oplus 0$, where $c \geq 0$ and U is unimodular. Some sequence (T^{m_j}) of powers of T converges to P, P is the only non-zero projection in $\overline{\mathbb{R}^+T}$, and $\overline{\mathbb{R}^+T}$ contains no non-zero quasinilpotent operators,*

(ii) ("Nilpotent" case) *The asymptotic part of $\overline{\mathbb{R}^+T}$ is non-trivial. For each operator $S = \lim_j b_j T^{n_j}$ with (n_j) strictly increasing, we have $S = B \oplus 0$ where $B \in L(X_1)$ is nilpotent (even square-zero) and $b_j \to 0$. Moreover, $\overline{\mathbb{R}^+T}$ contains no non-zero projections.*

Proof. Let $T_1 = T_{|X_1}$ and $T_2 = T_{|X_2}$. Then $r(T_1) = 1$ and $r(T_2) < 1$. Since $\sigma(T_1) = \sigma_{\text{per}}(T_1) = \sigma_{\text{per}}(T)$, the preceding observation applies to T_1.

(i) Suppose that T_1 has a zero nilpotent part. Then T_1 is unimodular. Take any $S \in \overline{\mathbb{R}^+T}$. As X_1 and X_2 are invariant under S, we can write $S = S_1 \oplus S_2$. Suppose that $S = \lim_j b_j T^{n_j}$ for some sequence (b_j) in $\mathbb{R}_+$ and some strictly increasing sequence (n_j) in $\mathbb{N}$. Then $b_j T_1^{n_j} \to S_1$. Since T_1 is unimodular, we have that $\{\|T_1^n\|\}_n$ is bounded above and bounded away from 0. It follows that (b_j) is bounded and hence has a convergent subsequence. By passing to a subsequence, we assume $b_j \to b$. If $b = 0$, then $S_1 = 0$; if $b \neq 0$, then $S_1/b = \lim_j T^{n_j}$ is also unimodular. Thus, S_1 is always a scalar multiple of a unimodular matrix. It also follows from $r(T_2) < 1$ that $S_2 = \lim_j b_j T_2^{n_j} = 0$. So S has the form $cU \oplus 0$. Furthermore, for every non-zero $S \in \overline{\mathbb{R}^+T}$, the restriction $S_{|X_1}$ is a non-zero scalar multiple of a unimodular matrix, so that S is not quasinilpotent.

By the observation preceding this proposition, $(T_1^{m_j})$ converges to the identity of X_1 for some strictly increasing sequence (m_j). Since $r(T_2) < 1$, we have $T_2^{m_j} \to 0$. Therefore, $T^{m_j} \to P$. Finally, we show that P is the only non-zero projection in $\overline{\mathbb{R}^+ T}$. Let $Q \in \overline{\mathbb{R}^+ T}$ be a projection. Suppose first that $Q = cT^n$ for some $c > 0$ and $n \in \mathbb{N}$. Then $\frac{1}{c^{m_j}} Q = \left(\frac{1}{c} Q\right)^{m_j} = T^{n m_j} \to P^n = P$; it follows that $c = 1$ and $Q = P$. Suppose now that Q is in the asymptotic part of $\overline{\mathbb{R}^+ T}$. By the preceding paragraph, it is easily seen that $Q = Q_1 \oplus 0$ where Q_1 is unimodular and is a projection in $L(X_1)$. Hence, Q_1 is the identity on X_1 and, therefore, $Q = P$.

(ii) Suppose now that T_1 has a non-trivial nilpotent part. By the observation preceding this proposition, there exist sequences (c_j) in $\mathbb{R}_+$ and (r_j) in $\mathbb{N}$ such that $c_j \to 0$, (r_j) is strictly increasing, and $(c_j T_1^{r_j})$ converges to a non-zero square-zero operator C on X_1. It follows from $c_j \to 0$ and $r(T_2) < 1$ that $c_j T_2^{r_j} \to 0$. Therefore, $c_j T^{r_j} \to C \oplus 0$. It follows that $C \oplus 0 \neq 0$ is in the asymptotic part of $\overline{\mathbb{R}^+ T}$.

Suppose that $S = \lim_j b_j T^{n_j}$ for some (b_j) in $\mathbb{R}_+$ and some strictly increasing (n_j). The observation preceding this proposition applied with $A = T_1$ guarantees that $b_j \to 0$ and $S_{|X_1}$ is a square-zero operator. Furthermore, $r(T_2) < 1$ implies $S_{|X_2} = \lim_j b_j T_2^{n_j} = 0$. Thus, S has the required form. In particular, S cannot be a projection.

It is left to show that if $Q = cT^n$ for some $c > 0$ and $n \in \mathbb{N}$ then Q is not a projection. Suppose it is. It follows from $r(Q) = 1 = r(T^n)$ that $c = 1$, so $Q = T^n$. Hence, the set of all distinct powers of T is finite. It follows from $c_j \to 0$ that $c_j T^{r_j} \to 0$, contradicting that $c_j T^{r_j} \to C \oplus 0 \neq 0$. $\square$

Remark 0.9. Suppose that, in addition, $\operatorname{rank} T = \min \operatorname{rank} \overline{\mathbb{R}^+ T} < \infty$. Then the nilpotent case in Proposition 0.8 is impossible. Indeed, otherwise, $\overline{\mathbb{R}^+}T$ would contain an operator of the form $C \oplus 0$ where C is a non-zero nilpotent operator in $L(X_1)$, hence

$$0 < \operatorname{rank} C \oplus 0 = \operatorname{rank} C < \dim X_1 \leq \operatorname{rank} T$$

since T is an isomorphism on X_1; a contradiction. Thus, we have $P \in \overline{\mathbb{R}^+}T$, where P is the spectral projection onto X_1. It follows that $\operatorname{rank} T = \operatorname{rank} P = \dim X_1$, so that $T_{|X_2} = 0$. Hence, $\operatorname{Range} T = X_1$, $\ker T = X_2$, and $\sigma(T)$ consists of $\sigma_{\text{per}}(T)$ and, possibly, zero.

We now apply the results to $\overline{\mathbb{R}^+}$-closed semigroups of operators on Banach spaces. The following is immediate by Proposition 0.8.

Proposition 0.10 ([21])**.** *If an $\overline{\mathbb{R}^+}$-closed semigroup $\mathscr{S}$ contains a peripherally Riesz operator then $\mathscr{S}$ contains a finite-rank operator.*

In particular, this proposition applies when $\mathscr{S}$ contains a non-quasinilpotent compact or even strictly singular operator.

Can we find not just a finite-rank operator in $\mathscr{S}$ but a finite-rank projection? As in Remark 0.9, if there is a $T \in \mathscr{S}$ such that $\operatorname{rank} T = \min \operatorname{rank} \mathscr{S} < +\infty$ and T is not nilpotent then the spectral projection P for $\sigma_{\text{per}}(T)$ is in $\mathscr{S}$ and $\operatorname{rank} P = \operatorname{rank} T$. The next result shows that in this case $\mathscr{S}$ contains "many" projections.

Proposition 0.11 ([21])**.** *Let $\mathscr{S}$ be an $\overline{\mathbb{R}^+}$-closed semigroup. Suppose that $S \in \mathscr{S}$ satisfies $r := \operatorname{rank} S = \min \operatorname{rank} \mathscr{S} < \infty$ and S is not nilpotent. Then there exist projections P and Q in $\mathscr{S}$ with $\operatorname{rank} P = \operatorname{rank} Q = r$ and*

$PS = SQ = S$. Moreover, the condition "S is not nilpotent" may be replaced with "AS is not nilpotent for some $A \in \mathscr{S}$".

Proof. Suppose AS is not nilpotent for some $A \in \mathscr{S}$ or $A = I$. Then $r(SA) = r(AS) \neq 0$. Clearly, $\operatorname{rank} AS = \operatorname{rank} SA = r$. It follows from $\operatorname{Range} SA \subseteq \operatorname{Range} S$ and $\operatorname{rank} SA = \operatorname{rank} S$ that $\operatorname{Range} SA = \operatorname{Range} S$. By Remark 0.9 applied with $T = SA$, the peripheral spectral projection P of SA is in $\mathscr{S}$, $\operatorname{rank} P = r$, and $\operatorname{Range} P = \operatorname{Range} SA = \operatorname{Range} S$, hence $PS = S$.

In order to find Q, we pass to the dual semigroup $\mathscr{S}^* = \{T^* \mid T \in \mathscr{S}\}$. Note that $\mathscr{S}^*$, S^*, and A^* still satisfy all the assumptions of the lemma, so we can find a projection $R \in \mathscr{S}^*$ such that $\operatorname{rank} R = r$ and $RS^* = S^*$. Then $R = Q^*$ for some projection $Q \in \mathscr{S}$ with $\operatorname{rank} Q = r$ and $SQ = S$. $\square$

0.3 Banach lattices and positive operators

In this section, we review some basic notions on Banach lattices and positive operators. Standard references are [42, 31, 1, 6].

Definition 0.12. A ***vector lattice*** (or ***Riesz space***) is a real vector space X with a partial order "$\leq$" that satisfies the following:

(i) for any $x, y, z \in X$, $x \leq y$ implies $x + z \leq y + z$,

(ii) for any $x, y \in X$ and $0 \leq \alpha \in \mathbb{R}$, $x \leq y$ implies $\alpha x \leq \alpha y$,

(iii) for any $x, y \in X$, their supremum, denoted by $x \vee y$, exists in X.

Some conventional notations will be used without specific explanations. For example, "$x \geq y$" means $y \leq x$, $x \wedge y$ means the infimum of x and y,

$|x| := x \vee (-x)$ means the modulus of x, and $x_+ := x \vee 0$ means the positive part of x.

Definition 0.13. A net $(x_\alpha)_{\alpha\in\Gamma}$ in a vector lattice X is said to be ***order convergent*** to x, written as $x_\alpha \xrightarrow{o} x$, if there exists another net $(z_\beta)_{\beta\in\Lambda}$ in X satisfying the following:

(i) (z_β) is decreasing (i.e. $\beta \le \beta'$ implies $z_\beta \ge z_{\beta'}$) and $\inf_\beta z_\beta = 0$,

(ii) for any $\beta \in \Lambda$, there exists $\alpha_0 \in \Gamma$ such that $|x_\alpha - x| \le z_\beta$ for all $\alpha \ge \alpha_0$.

Definition 0.14. A vector subspace Y of a vector lattice X is called

(i) a ***lattice subspace*** if Y with the order inherited from X is a vector lattice in its own right,

(ii) a ***sublattice*** (or ***Riesz subspace***) if it is a lattice subspace and for any pair of vectors in Y, their suprema in X and Y are equal,

(iii) an ***ideal*** if it is a sublattice and for any $x \in X$ and $y \in Y$ with $|x| \le |y|$, one has $x \in Y$,

(iv) a ***band*** if it is an ideal and is order closed, i.e. for any net $(x_\alpha) \subset Y$, $x_\alpha \xrightarrow{o} x$ in X implies $x \in Y$.

Definition 0.15. (i) A ***normed lattice*** X is a vector lattice with a norm $\|\cdot\|$ such that $|x| \le |y|$ implies $\|x\| \le \|y\|$. If, in addition, $(X, \|\cdot\|)$ is (norm) complete, then X is called a ***Banach lattice***.

(ii) A Banach lattice is said to be ***order continuous*** or ***have order continuous norm*** if $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \xrightarrow{\|\cdot\|} x$.

For $A \subset X$, we denote by I_A (respectively, B_A) the ideal (respectively, band) generated by A in the vector lattice X.

Definition 0.16. Let X be a vector lattice. A vector $x > 0$ is called a

(i) ***weak unit*** if the ideal I_x generated by x is order dense in X, or equivalently, the generated band $B_x = X$,

(ii) ***quasi-interior point*** if X is a normed lattice and the ideal I_x generated by x is norm dense in X.

Note that in normed lattices, bands are always (norm) closed. Hence, quasi-interior points are always weak units. Note also that in order continuous Banach lattices, the notions of bands and closed ideals coincide. Thus, the notions of weak units and quasi-interior points also coincide.

We refer to [6, Chapters 3 and 4] for fundamental properties of Banach lattices, especially, for equivalent characterizations of order continuous Banach lattices.

We now introduce some basic notions on positive operators.

Definition 0.17. A linear mapping T on a vector lattice X is said to be

(i) ***positive*** if T maps positive vectors to positive vectors; in this case, we write $T \geq 0$,

(ii) ***strictly positive*** if T is positive and vanishes at no non-zero positive vectors,

(iii) ***σ-order continuous*** if T maps order null sequences to order null sequences,

(iv) ***order continuous*** if T maps order null nets to order null nets.

We denote by $L_+(X)$ the set of all positive operators on X. If T is positive and non-zero, we write $T > 0$. We refer to [6] for general theory of positive operators on Banach lattices. In particular, note that positive operators on Banach lattices are (automatically) bounded. Note also that a positive operator is zero if it vanishes at a quasi-interior point or if it is σ-order continuous and vanishes at a weak unit. It can also be easily verified that if $T > 0$ is σ-order continuous then so is $\sum_1^\infty \frac{T^n}{\lambda^n}$ for all $\lambda > r(T)$.

For the rest of this section, X stands for a Banach lattice. Recall that the dual space of a Banach lattice is a Banach lattice under the natural order.

Proposition 0.18. *Let X be a Banach lattice and $\mathscr{C} \neq \{0\}$ a collection of positive functionals on X.*

(i) If $B_{\mathscr{C}} = X^*$, then for any $x > 0$, there exists $x^* \in \mathscr{C}$ such that $x^*(x) > 0$.

(ii) Suppose X is order continuous. Then $B_{\mathscr{C}} = X^*$ if and only if for any $x > 0$, there exists $x^* \in \mathscr{C}$ such that $x^*(x) > 0$.

Proof. (i) Suppose there exists $x > 0$ such that $x^*(x) = 0$ for all $x^* \in \mathscr{C}$. Then it is easily seen that $B_{\mathscr{C}}$ vanishes at x. Thus, $B_{\mathscr{C}} \neq X^*$.

(ii) It remains to prove the "if" part. Suppose that X is order continuous and $B_{\mathscr{C}} \neq X^*$. Let P be the band projection from X^* onto $B_{\mathscr{C}}$. Then $0 \neq P \neq I$. By [6, Theorem 3.59], there exists a band projection Q on X such that $P = Q^*$. Clearly, $0 \neq Q \neq I$. Take any $x > 0$ in the range of $I - Q$. We have $x^*(x) = (Px^*)(x) = x^*(Qx) = 0$ for all $x^* \in \mathscr{C}$. $\square$

Corollary 0.19. *Let x^* be a positive functional on a Banach lattice X.*

(i) If x^* is a weak unit of X^* then x^* is strictly positive.

(ii) Suppose X is order continuous. Then x^* is a weak unit of X^* iff it is strictly positive.

The following are two simple technical lemmas. Recall that two operators T and S are said to ***semi-commute*** if their commutator $TS - ST$ is either positive or negative.

Lemma 0.20 ([19])**.** *Let $T, S \in L(X)$ be two semi-commuting operators. Suppose $Tx_0 = \lambda x_0$ and $T^* x_0^* = \lambda x_0^*$ for some vector $x_0 > 0$ and some strictly positive functional $x_0^* > 0$ and some $\lambda \in \mathbb{R}$. Then $TS = ST$ if any of the following are satisfied:*

(i) x_0 is a quasi-interior point,

(ii) x_0 is a weak unit, and T and S are both σ-order continuous.

Proof. We only prove (ii); the proof of (i) is similar. Note that $x_0^*((TS - ST)x_0) = (T^* x_0^*)(Sx_0) - x_0^*(STx_0) = \lambda x_0^*(Sx_0) - x_0^*(S\lambda x_0) = 0$. Since x_0^* is strictly positive, we have $(TS - ST)x_0 = 0$. Note also that $TS - ST$ is σ-order continuous. Thus, x_0 being a weak unit yields $TS - ST = 0$. □

Lemma 0.21 ([20])**.** *For $T > 0$ on X, the following hold.*

(i) If $Tx_0 = \lambda x_0$ for some quasi-interior point $x_0 > 0$, then $\liminf_n \|T^{n*} x^*\|^{\frac{1}{n}} \geq \lambda$ for any $x^* > 0$. If, in addition, $r(T)$ is an eigenvalue of T^*, then $\lambda = r(T)$.

(ii) If $Tx_0 = \lambda x_0$ for some weak unit $x_0 > 0$, then $\liminf_n \|T^{n*} x^*\|^{\frac{1}{n}} \geq \lambda$ for any σ-order continuous $x^* > 0$. If, in addition, $r(T)$ is an eigenvalue of T^* with a σ-order continuous eigenfunctional, then $\lambda = r(T)$.

(iii) If $T^*x_0^* = \lambda x_0^*$ for some strictly positive $x_0^* > 0$, then $\liminf_n \|T^n x\|^{\frac{1}{n}} \geq \lambda$ for any $x > 0$. If, in addition, $r(T)$ is an eigenvalue of T, then $\lambda = r(T)$.

Proof. We only prove (i); the proofs of (ii) and (iii) are similar. Suppose that $x^* > 0$. Then x_0 being quasi-interior implies $x^*(x_0) > 0$. Note that $\lambda^n x^*(x_0) = x^*(T^n x_0) = T^{n*}x^*(x_0) \leq \|T^{n*}x^*\| \, \|x_0\|$. Thus, $\|T^{n*}x^*\|^{\frac{1}{n}} \geq \lambda \sqrt[n]{x^*(x_0)/\|x_0\|}$. Letting $n \to \infty$, we have $\liminf_n \|T^{n*}x^*\|^{\frac{1}{n}} \geq \lambda$.

Suppose now $T^*x^* = r(T)x^*$ for some $x^* \neq 0$. Then $\lambda|x^*| \leq r(T)|x^*| \leq T^*|x^*|$. Note that $(T^*|x^*| - \lambda|x^*|)(x_0) = |x^*|(Tx_0) - \lambda|x^*|(x_0) = 0$. Hence, x_0 being quasi-interior yields $T^*|x^*| = \lambda|x^*|$. It follows that $\lambda|x^*| = r(T)|x^*| = T^*|x^*|$. In particular, $\lambda = r(T)$. $\square$

We now include some spectral properties of positive operators on Banach lattices. Recall that we pass to their complexifications whenever spectral theory is considered. Observe first that if $T > 0$ then $r(T) \in \sigma(T)$.

Lemma 0.22 ([25])**.** *For an operator $T \in L_+(X)$, if $r(T) = 1$ is a simple pole of $R(\cdot, T)$, then $\frac{1}{n}\sum_{i=1}^n T^i$ converges to the spectral projection of T for $r(T) = 1$.*

Recall that *Krein-Rutman Theorem* asserts that any non-quasinilpotent positive compact operator has a positive eigenvector for the spectral radius. The following is a generalization.

Lemma 0.23 ([40])**.** *For an operator $T \in L_+(X)$, if $r(T)$ is a pole of $R(\cdot, T)$, then the leading coefficient of the Laurent expansion of $R(\cdot, T)$ at $r(T)$ is positive. Moreover, T as well as T^* has a positive eigenvector for $r(T)$.*

We will need the following deep result essentially due to Lotz and Schaefer. As has been observed in [7], the proof is a revision of that of [42, Theorem 5.5,

p. 331].

Lemma 0.24 (Lotz-Schaefer)**.** *Let $T \in L_+(X)$ be a non-quasinilpotent operator. Suppose $r(T)$ is a* ***Riesz point*** *(i.e. $r(T)$ is a pole of $R(\cdot, T)$ with finite-dimensional spectral subspace). Then T is peripherally Riesz.*

We end this section with a structural result on finite-rank positive projections on Banach lattices.

0.25. The following observation is based on [42, Proposition 11.5, p. 214]. Let P be a positive projection on a Banach lattice X; let $Y = \operatorname{Range} P$. It is easy to see that Y is a lattice subspace of X with lattice operations $x \overset{*}{\wedge} y = P(x \wedge y)$ and $x \overset{*}{\vee} y = P(x \vee y)$ for any $x, y \in Y$. We denote this vector lattice by X_P. Note that this lattice structure is determined by Y, so that if Q is another positive projection on X with $\operatorname{Range} Q = Y$ then it generates the same lattice structure on Y.

Suppose, in addition, that $n := \operatorname{rank} P < \infty$. Being a finite-dimensional Archimedean vector lattice, X_P is lattice isomorphic to $\mathbb{R}^n$ with the standard order; cf. [42, Corollary 1, p. 70]. Thus, we can find positive *-disjoint $x_1, \dots, x_n \in X_P$ that form a basis of X_P. Furthermore, we can find positive $y_1^* \dots, y_n^* \in X_P^*$ such that $y_i^*(x_j) = \delta_{ij}$. Put $x_i^* = y_i^* \circ P$, then $x_1^* \dots, x_n^* \in X_+^*$ and $x_i^*(x_j) = \delta_{ij}$. It is easy to see that $P = \sum_{i=1}^n x_i^* \otimes x_i$.

0.4 Irreducible operators and semigroups

We now discuss some elementary properties of irreducible operators and semigroups which are the main objects in this thesis. Throughout this section,

X stands for a Banach lattice with $\dim X > 1$ and T stands for a non-zero positive operator on X.

Definition 0.26. (i) A collection $\mathscr{C}$ of positive operators on X is said to be ***ideal irreducible*** if there is no non-trivial (that is, different from $\{0\}$ and X) closed ideal which is invariant under each member of $\mathscr{C}$.

(ii) $\mathscr{C}$ is said to be ***band irreducible*** if there is no non-trivial band which is invariant under each member of $\mathscr{C}$.

(iii) In particular, an operator $T > 0$ is said to be ideal irreducible (respectively, band irreducible) if the singleton $\{T\}$ is ideal irreducible (respectively, band irreducible).

(iv) A positive operator T is said to be ***strongly expanding*** (respectively, ***expanding***) if it sends non-zero positive vectors to quasi-interior points (respectively, weak units).

Recall that the notions of bands and closed ideals coincide in order continuous Banach lattices. Thus, the notions of ideal irreducibility and band irreducibility also coincide on such spaces. In particular, they coincide on $L_p(\mu)$-spaces for $1 \leq p < \infty$ and on $\mathbb{R}^n$. Note also that strongly expanding (respectively, expanding) operators are ideal irreducible (respectively, band irreducible).

We collect some elementary facts about irreducible operators.

Lemma 0.27. *Fix $\lambda > r(T)$. The following statements are equivalent:*

(i) T is ideal irreducible,

(ii) $\sum_1^\infty \frac{T^n}{\lambda^n}$ is strongly expanding,

(iii) $\sum_1^\infty \frac{T^{n*}}{\lambda^n} x^*$ *is strictly positive for any* $x^* > 0$.

Proof. The equivalence of (i) and (ii) is shown in [41, p. 317]. For (ii) $\Leftrightarrow$ (iii), simply note that $x^*\big(\sum_1^\infty \frac{T^n}{\lambda^n} x\big) = \big(\sum_1^\infty \frac{T^{n*}}{\lambda^n} x^*\big)(x)$ and that $y > 0$ is a quasi-interior point if and only if $x^*(y) > 0$ for any $x^* > 0$; cf. [6, Theorem 4.85]. $\square$

Lemma 0.28. *Let* $T > 0$ *be ideal irreducible.*

(i) If $Tx = \lambda x$ *for some* $x > 0$ *and* $\lambda \in \mathbb{R}$, *then* x *is a quasi-interior point and* $\lambda > 0$.

(ii) If $T^* x^* = \lambda x^*$ *for some* $x^* > 0$ *and* $\lambda \in \mathbb{R}$, *then* x^* *is strictly positive and* $\lambda > 0$.

(iii) T *is strictly positive.*

Proof. (i) It is clear that $\lambda \geq 0$. Pick any $\delta > r(T)$. By Lemma 0.27 (ii), we have that $\sum_1^\infty \frac{T^n}{\delta^n} x = \big(\sum_1^\infty \frac{\lambda^n}{\delta^n}\big) x$ is a quasi-interior point. Thus, $\lambda > 0$ and x is a quasi-interior point. (ii) can be proved similarly using Lemma 0.27 (iii). (iii) follows immediately from Lemma 0.27(ii). $\square$

Lemma 0.29. *Let* $T > 0$ *be* σ*-order continuous. Fix* $\lambda > r(T)$. *The following two statements are equivalent:*

(i) T *is band irreducible,*

(ii) $\sum_1^\infty \frac{T^n}{\lambda^n}$ *is expanding.*

Any of these two statements implies the following:

(iii) $\sum_1^\infty \frac{T^{n*}}{\lambda^n} x^*$ *is strictly positive for any* σ*-order continuous* $x^* > 0$.

Proof. (i) $\Leftrightarrow$ (ii) can be proved similarly as for ideal irreducible operators; cf. [41, p. 317]. For the last assertion, simply note that if $y > 0$ is a weak unit then $x^*(y) > 0$ for any σ-order continuous $x^* > 0$. $\square$

Lemma 0.30. *Let $T > 0$ be band irreducible and σ-order continuous.*

(i) If $Tx = \lambda x$ for some $x > 0$ and $\lambda \in \mathbb{R}$, then x is a weak unit and $\lambda > 0$.

(ii) If $T^*x^* = \lambda x^*$ for some σ-order continuous $x^* > 0$ and $\lambda \in \mathbb{R}$, then x^* is strictly positive and $\lambda > 0$.

(iii) T is strictly positive.

The following lemma handles relations between irreducibility of T and T^*; cf. [1, p. 356, Exercise 16].

Lemma 0.31. *(i) If T^* is band irreducible, then T is ideal irreducible.*

(ii) Suppose X is order continuous. Then T is ideal irreducible if and only if T^* is band irreducible.

Proof. (i) Suppose T^* is band irreducible. Being the adjoint of a positive operator, it is order continuous. Pick any $\lambda > r(T)$ and $x^* > 0$. We have that $\sum_1^\infty \frac{T^{n*}}{\lambda^n} x^*$ is a weak unit of X^* by Lemma 0.29 and thus is strictly positive on X by Corollary 0.19. It follows that T is ideal irreducible by Lemma 0.27. (ii) can be proved using similar machineries. $\square$

We end this section with the semigroup analogues of these properties. **For the rest of this section, $\mathscr{S}$ stands for a non-zero (multiplicative) semigroup of positive operators on X.** For $x \in X$, the ***orbit*** of x under $\mathscr{S}$ is defined as $\mathscr{S}x = \{Sx \mid S \in \mathscr{S}\}$. The following proposition is well known; cf. [37, Lemma 8.7.6] and [15, Proposition 2.1].

Proposition 0.32. *The following are equivalent:*

(i) $\mathscr{S}$ is ideal irreducible,

(ii) every non-zero algebraic ideal in $\mathscr{S}$ is ideal irreducible,

(iii) for any non-zero $x \in X_+$ and $x^* \in X^*_+$ there exists $S \in \mathscr{S}$ such that $\langle x^*, Sx \rangle \neq 0$,

(iv) $A\mathscr{S}B \neq \{0\}$ for any non-zero $A, B \in L(X)_+$,

(v) for any $x > 0$, the ideal generated in X by the orbit $\mathscr{S}x$ is dense in X.

Proof. The equivalence of (i) through (iv) is [15, Proposition 2.1]. It is easy to see that (i)⇒(v)⇒(iii). □

The band irreducible versions are as follows.

Proposition 0.33. *Suppose $\mathscr{S}$ consists of order continuous operators. The following are equivalent.*

(i) $\mathscr{S}$ is band irreducible,

(ii) $B_{\mathscr{S}x} = X$ for all $x > 0$,

(iii) every non-zero algebraic ideal of $\mathscr{S}$ is band irreducible.

Proof. (i)⇒(ii) Suppose that $\mathscr{S}$ is band irreducible. It is easy to see that $B_{\mathscr{S}x}$ is $\mathscr{S}$-invariant for every $x > 0$, so it suffices to prove that $\mathscr{S}x \neq \{0\}$. For each $S \in \mathscr{S}$, since S is order continuous, its null ideal $\operatorname{Null}(S) = \big\{x \in X : S|x| = 0\big\}$ is a band. Therefore, $\bigcap_{S\in\mathscr{S}} \operatorname{Null}(S)$ is a band. It is easy to see that the intersection is $\mathscr{S}$-invariant, hence it is zero. It follows that for every $x > 0$ there exists $S \in \mathscr{S}$ such that $Sx > 0$, so that $\mathscr{S}x$, and therefore $B_{\mathscr{S}x}$, is non-zero.

For (ii)⇒(i), suppose that B is a non-zero proper $\mathscr{S}$-invariant band. For each $0 < x \in B$ we have $B_{\mathscr{S}x} \subseteq B$, hence $B_{\mathscr{S}x} \neq X$.

(i)$\Rightarrow$(iii) Let $\mathcal{J}$ be a non-zero algebraic ideal in $\mathscr{S}$. Take any $x > 0$. Then $y \in I_{\mathcal{J}x}$ if and only if there exist $S_1, \dots, S_n \in \mathcal{J}$ and $\lambda \in \mathbb{R}_+$ such that $|y| \leq \lambda(S_1 + \cdots + S_n)x$. In this case, for any $S \in \mathscr{S}$ we have $|Sy| \leq \lambda(SS_1x + \cdots + SS_n)x$, so that Sy is in $I_{\mathcal{J}x}$. It follows that $I_{\mathcal{J}x}$ and, therefore, $B_{\mathcal{J}x}$ is $\mathscr{S}$-invariant.

Observe that $\mathcal{J}x$ and, therefore, $B_{\mathcal{J}x}$, is non-zero. Indeed, suppose that $\mathcal{J}x = \{0\}$ and fix any non-zero $T \in \mathcal{J}$. Then for every $S \in \mathscr{S}$ we have $TS \in \mathcal{J}$ so that $TSx = 0$. It follows that T vanishes on $\mathscr{S}x$ and, therefore, on $B_{\mathscr{S}x}$. But $B_{\mathscr{S}x} = X$ by the equivalence of (i) and (ii), so that $T = 0$; a contradiction.

Thus, the band $B_{\mathcal{J}x}$ is $\mathscr{S}$-invariant and non-zero, hence $B_{\mathcal{J}x} = X$. Now the required result follows from the equivalence of (i) and (ii) again.

The implication (iii)$\Rightarrow$(i) is obvious. $\square$

Corollary 0.34. *Suppose that $\mathscr{S}$ is band irreducible and consists of order continuous operators. The following statements hold.*

- *(i) For any $x > 0$ in X and any order continuous $x^* > 0$ in X^*, there exists $S \in \mathscr{S}$ such that $x^*(Sx) \neq 0$.*
- *(ii) $U\mathscr{S}V \neq \{0\}$ for any non-zero $U, V \in L(X)_+$ provided that U is order continuous.*

Proof. (i) Suppose not. Then x^* vanishes on $\mathscr{S}x$, hence on $B_{\mathscr{S}x} = X$, so that $x^* = 0$; a contradiction.

(ii) Suppose not, suppose $U\mathscr{S}V = \{0\}$. Since $V \neq 0$, there exists $x > 0$ with $Vx > 0$. Then U vanishes on $\mathscr{S}Vx$ and, therefore, on $B_{\mathscr{S}Vx}$, so that, by Proposition 0.33, $U = 0$; a contradiction. $\square$

We define the dual semigroup of $\mathscr{S}$ by $\mathscr{S}^* = \{S^* : S \in \mathscr{S}\}$.

Proposition 0.35. *(i) If $\mathscr{S}^*$ is band irreducible then $\mathscr{S}$ is ideal irreducible.*

(ii) Suppose X is order continuous. Then $\mathscr{S}$ is ideal irreducible if and only if $\mathscr{S}^*$ is band irreducible

Proof. (i) Pick any $x > 0$ and $x^* > 0$. Since $\mathscr{S}^*$ is band irreducible and consists of order continuous operators, we have $B_{\mathscr{S}^* x^*} = X^*$ by Proposition 0.33. Hence, there exists $S \in \mathscr{S}$ such that $x^*(Sx) = S^*x^*(x) > 0$ by Proposition 0.18. It follows from Proposition 0.32 that $\mathscr{S}$ is ideal irreducible. (ii) can be proved using similar machineries. $\square$

Chapter 1

Irreducible Positive Operators

1.1 Introduction

The results in this chapter are mainly taken from [19] and [20]. Throughout this chapter, X stands for a real Banach lattice with $\dim X > 1$ and T stands for a non-zero positive operator on X.

Many efforts have been made to extend the classical Perron-Frobenius theory (c.f. Theorem in Preface) to irreducible positive operators on arbitrary Banach lattices. For example, combining the results in [40, 38, 35, 10], we have the following.

Theorem 1.1. *Let $T > 0$ be an ideal irreducible operator on X such that $r(T)$ is a pole of the resolvent $R(\cdot, T)$.*

(i) $r(T)$ is non-zero and is a simple pole of $R(\cdot, T)$, $\ker(r(T) - T) = \mathrm{Span}\{x_0\}$ for some quasi-interior point $x_0 > 0$ and $\ker(r(T) - T^*) = \mathrm{Span}\{x_0^*\}$ for some strictly positive functional $x_0^* > 0$.

(ii) $\sigma_{per}(T) = r(T)G$ where G is the set of all k-th roots of unity for some $k \geq 1$, and each point in $\sigma_{\mathrm{per}}(T)$ is a simple pole of $R(\cdot, T)$ with one-dimensional eigenspace.

(iii) If $0 \leq S \leq T$ and $r(T) = r(S)$ then $T = S$.

Analogous results have also been established for band irreducible operators. We would like to mention the following.

Theorem 1.2 ([4, 5])**.** *Let $T > 0$ be a σ-order continuous band irreducible operator on X such that $r(T)$ is a pole of $R(\cdot, T)$. If any of the following are satisfied:*

(i) there exists a non-zero σ-order continuous functional on X,

(ii) T as well as the leading coefficient of the Laurent expansion of $R(\cdot, T)$ at $r(T)$ is order continuous,

then the results in Theorem 1.1 hold (in the Jentzsch-Perron part, x_0 is now a weak unit and x_0^* is σ-order continuous and strictly positive).

It deserves mentioning that Theorems 1.1 and 1.2 are applicable to compact irreducible operators due to the following theorem. The ideal irreducible case is conventionally referred to as *de Pagter's Theorem* and the band irreducible case is conventionally referred to as *Schaefer-Grobler Theorem*.

Theorem 1.3 ([12, 43, 22])**.** *Let $K > 0$ be a compact operator on X. If K is either ideal irreducible or σ-order continuous and band irreducible, then $r(K) > 0$.*

In this chapter, we establish some more versions of the Jentzsch-Perron theorems and comparison theorems for irreducible operators.

Section 1.2 is devoted to the Jentzsch-Perron theorems. Observe first that they hold under much weaker conditions than those in Theorems 1.1 and 1.2. For example, the following can be found in [42, Theorem 5.2, p. 329] for ideal irreducible operators and in [23, Theorem 4.12] for band irreducible operators.

Theorem 1.4 ([42, 23])**.** *Suppose $Tx_0 = r(T)x_0$ and $T^*x_0^* = r(T)x_0^*$ for some $x_0 > 0$ and some $x_0^* > 0$. Then $\ker(r(T) - T) = \operatorname{Span}\{x_0\}$ if any of the following are satisfied:*

(i) T is ideal irreducible,

(ii) T is band irreducible and σ-order continuous and x_0^* is strictly positive.

We are particularly interested in the dual version of Theorem 1.4. Namely, we want to know when the eigenspaces of the adjoint operators are one-dimensional. We provide several sufficient conditions for this property in Proposition 1.10 and Theorem 1.12. We are also interested in the spectral behaviors of irreducible operators at their spectral radii (Proposition 1.13). As a consequence, we prove that the conditions (i) and (ii) in Theorem 1.2 are equivalent (Theorem 1.14).

Section 1.3 is devoted to comparison theorems. When the dominating operator is ideal irreducible with spectral radius being a pole of the resolvent, the comparison theorem (Theorem 1.1(iii)) was established in [10] and was applied to prove the following.

Theorem 1.5 ([10]). *Suppose* $0 \leq S \leq T$ *and* $r(T) = r(S)$. *If* $r(T)$ *is a Riesz point of* $R(\cdot, T)$*, then it is also a Riesz point of* $R(\cdot, S)$.

Using this theorem, Alekhno established the comparison theorems in Theorem 1.2. We also mention the following result (cf. [8, Theorem 2.8 and Corollary 2.12]).

Theorem 1.6 ([8]). *Suppose* $0 \leq S \leq T$ *on* $L^p(\mu)$ *where* $1 < p < \infty$ *and* μ *is* σ*-finite. Suppose also* $r(T) = r(S)$. *Then* $T = S$ *if either* T *or* S *is irreducible and power compact.*

We show in Corollary 1.19 that Theorem 1.6 remains valid for general Banach lattices. We also show in Theorem 1.21 that the power compactness condition in Theorem 1.6 may be replaced with the (weaker) condition that the spectral radius is a pole of the resolvent. Thus, Theorem 1.21 not only includes the comparison theorems in Theorems 1.1 and 1.2 but also allows to move the assumptions for T to the dominated operator S.

Section 1.4 is devoted to some application of comparison theorems. The following is an extension of de Pagter Theorem; cf. [1, Theorems 10.25 and 10.26].

Theorem 1.7 ([1])**.** *Let $T, K > 0$ be such that T is ideal irreducible and K is compact.*

(i) If $TK \geq KT$ then $\liminf_n \|K^n x\|^{\frac{1}{n}} > 0$ for any $x > 0$.

(ii) If $TK \leq KT$ then $r(K) > 0$.

The authors asked the following question ([1, p. 402]): can we obtain local non-quasinilpotency of K in (ii) as we do in (i)? We prove that the answer is affirmative. We actually prove a stronger result that in both cases of Theorem 1.7 the operators T and K are commuting (Theorem 1.24). Finally, we apply this result to prove that the commutator of two semi-commuting positive operators is quasinilpotent if one of the operators is compact. This answers a question in [9].

1.2 Jentzsch-Perron theorems

In this section, we establish several more versions of the Jentzsch-Perron Theorem. In particular, we are interested in the eigenspaces of the adjoint operators of irreducible operators. We also prove some auxiliary lemmas that will be of frequent use in the sequel.

Recall that we always assume $T > 0$. For convenience, we denote by $x^*_\pm$ the positive/negative parts of a functional x^*. We need the following technical lemma.

Lemma 1.8. *(i) Suppose $T^*x_0^* = \lambda x_0^*$ for some strictly positive functional x_0^*. Then for any $x \in X$ such that $Tx \geq \lambda x$ or $Tx \leq \lambda x$, we have $Tx_\pm = \lambda x_\pm$.*

(ii) Suppose $Tx_0 = \lambda x_0$ for some quasi-interior point $x_0 > 0$. Then for any $x^* \in X^*$ such that $T^*x^* \geq \lambda x^*$ or $T^*x^* \leq \lambda x^*$, we have $T^*x_\pm^* = \lambda x_\pm^*$.

(iii) Let T be σ-order continuous. Suppose $Tx_0 = \lambda x_0$ for some weak unit $x_0 > 0$. Then for any σ-order continuous $x^* \in X^*$ such that $T^*x^* \geq \lambda x^*$ or $T^*x^* \leq \lambda x^*$, we have $T^*x_\pm^* = \lambda x_\pm^*$.

Proof. (i) Note that $0 = (T^*x_0^* - \lambda x_0^*)(x) = x_0^*(Tx - \lambda x)$. Since x_0^* is strictly positive, we have $Tx = \lambda x$. This in turn implies $\lambda|x| \leq T|x|$. Using what we have just proved, we have $T|x| = \lambda|x|$. Hence, $Tx_\pm = \lambda x_\pm$.

(iii) Note that $(T^*x^* - \lambda x^*)(x_0) = x^*(Tx_0) - \lambda x^*(x_0) = 0$. Since $T^*x^* - \lambda x^*$ is σ-order continuous and x_0 is a weak unit, we have $T^*x^* - \lambda x^* = 0$. This in turn implies $\lambda|x^*| \leq T|x^*|$. Since $|x^*|$ is also σ-order continuous, applying what we have just proved, we have $T^*|x^*| = \lambda|x^*|$. Hence, $Tx_\pm^* = \lambda x_\pm^*$.

(ii) can be proved either similarly as (iii), or via (i) since x_0 acts as a strictly positive functional on X^* such that $T^{**}x_0 = \lambda x_0$. $\square$

One can replace the spectral radius in Theorem 1.4 with a positive eigenvalue as follows. The proof is analogous to that of [42, Theorem 5.2, p. 329].

Lemma 1.9. *Suppose $Tx_0 = \lambda x_0$ and $T^*x_0^* = \lambda x_0^*$ for some $x_0 > 0$ and $x_0^* > 0$ and $\lambda \in \mathbb{R}$. Then $\ker(\lambda - T) = \operatorname{Span}\{x_0\}$ if any of the following are satisfied:*

(i) T is ideal irreducible,

(ii) T is band irreducible and σ-order continuous and x_0^* is strictly positive.

Proof. We only prove (ii); the proof of (i) is similar. Suppose, otherwise, $\dim\ker(\lambda - T) > 1$. Then there exists $x \in \ker(\lambda - T)$ such that $x_\pm > 0$. By Lemma 1.8 (i), $x_\pm \in \ker(\lambda - T)$. Therefore, they are both weak units by Lemma 0.30 (ii). This is absurd since $x_+ \perp x_-$. $\square$

We deduce the following "double" version of Jentzsch-Perron theorem.

Proposition 1.10. *Suppose $Tx_0 = \lambda x_0$ and $T^*x_0^* = \lambda x_0^*$ for some $x_0 > 0$ and $x_0^* > 0$ and $\lambda \in \mathbb{R}$. If T^* is band irreducible (in particular, X has order continuous norm and T is ideal irreducible), then* $\ker(\lambda - T) = \mathrm{Span}\{x_0\}$ *and* $\ker(\lambda - T^*) = \mathrm{Span}\{x_0^*\}$.

Proof. Suppose that T^* is band irreducible. Then T is ideal irreducible by Lemma 0.31. Applying Lemma 1.9(i) to T, we have $\ker(\lambda - T) = \mathrm{Span}\{x_0\}$. By Lemma 0.28(i), x_0 is a quasi-interior point. Thus, it acts as a strictly positive functional on X^* (cf. [6, Theorem 4.85]) such that $T^{**}x_0 = \lambda x_0$. Being the adjoint of a positive operator, T^* is order continuous. Hence, applying Lemma 1.9(ii) to T^*, we have $\ker(\lambda - T^*) = \mathrm{Span}\{x_0^*\}$. Finally, observe that if X is order continuous and T is ideal irreducible, then T^* is band irreducible by Lemma 0.31 again. $\square$

We now establish another double version of Jentzsch-Perron theorem. For this purpose, we need the following technical lemma on σ-order continuity of eigenfunctionals. The idea of the proof has appeared in [43, 22, 23, 2]. Recall that an operator $S \in L(X)$ is **order weakly compact** if $S[0, x]$ is relatively weakly compact for all $x > 0$. This is a large class of operators containing all compact operators, AM-compact operators and weakly compact operators. It also includes strictly singular operators; cf. [31, Corollary 3.4.5].

Lemma 1.11. *(i) If T is σ-order continuous and order weakly compact, then T is σ-order-to-norm continuous (i.e. $x_n \downarrow 0$ implies $\|Tx_n\| \to 0$).*

(ii) If T is power σ-order-to-norm continuous and $T^*x^* = \lambda x^*$ for some $\lambda \neq 0$, then x^* is σ-order continuous.

(iii) If T is σ-order continuous and power order weakly compact and $T^*x^* = \lambda x^*$ for some $\lambda \neq 0$, then x^* is σ-order continuous.

Proof. (i) Take $x_n \downarrow 0$. Then $Tx_n \downarrow 0$. Since $T[0, x_1]$ is relatively weakly compact, we have, by Eberlein-Smulian theorem, $Tx_{n_j} \to y$ weakly for some (n_j) and $y \in X$. Since Tx_{n_j} is decreasing, it is straightforward verifications that $y = \inf_j Tx_{n_j} = \inf_n Tx_n = 0$. Hence, $||Tx_{n_j}|| \to 0$ by Dini theorem (cf. [6, Theorem 3.52]). This in turn implies that $||Tx_n|| \to 0$.

(ii) Suppose T^k is σ-order-to-norm continuous for some $k \geq 1$. For any $x_n \downarrow 0$, we have $\|T^k x_n\| \to 0$. Thus, $\lambda^k x^*(x_n) = T^{k*}x^*(x_n) = x^*(T^k x_n) \to 0$. It follows that $x^*(x_n) \to 0$. By [1, Theorem 1.26], x^* is σ-order continuous.
(iii) follows from (i) and (ii). □

Theorem 1.12. *Suppose T is power order weakly compact, and $Tx_0 = \lambda x_0$ and $T^*x_0^* = \lambda x_0^*$ for some $x_0 > 0$ and $x_0^* > 0$ and $\lambda \in \mathbb{R}$. If T is either ideal irreducible or band irreducible and σ-order continuous, then* $\ker(\lambda - T) = \operatorname{Span}\{x_0\}$ *and* $\ker(\lambda - T^*) = \operatorname{Span}\{x_0^*\}$.

Proof. We only prove the band irreducible case; the other case can be proved similarly. By Lemma 0.30 (i), x_0 is a weak unit and $\lambda > 0$. Thus, x_0^* is σ-order continuous by Lemma 1.11 (iii), and is strictly positive by Lemma 0.30 (ii). It follows from Lemma 1.9 (ii) that $\ker(\lambda - T) = \operatorname{Span}\{x_0\}$.

It remains to prove $\ker(\lambda - T^*) = \mathrm{Span}\{x_0^*\}$. Without loss of generality, assume $\lambda = 1$. Suppose $\dim\ker(1-T^*) > 1$. Then there exists $x^* \in \ker(1-T^*)$ with $x_+^* > 0$ and $x_-^* > 0$. By Lemma 1.11 (iii) again, x^* is σ-order continuous. Since x_0 is a weak unit, Lemma 1.8 (iii) implies that $x_\pm^* \in \ker(1 - T^*)$. Let $k \geq 1$ be such that T^k is order weakly compact. Then $(T^k)^* x_\pm^* = x_\pm^*$. It follows that

$$\begin{aligned} 0 = x_+^* \wedge x_-^*(x_0) &= \inf_{0 \leq x \leq x_0} (x_+^*(x_0 - x) + x_-^*(x)) \\ &= \inf_{0 \leq x \leq x_0} ((T^k)^* x_+^*(x_0 - x) + (T^k)^* x_-^*(x)) \\ &= \inf_{0 \leq x \leq x_0} (x_+^*(x_0 - T^k x) + x_-^*(T^k x)). \end{aligned}$$

So we can take $(x_n) \subset [0, x_0]$ such that $x_+^*(x_0 - T^k x_n) + x_-^*(T^k x_n) \to 0$. Using order weak compactness of T^k and Eberlein-Smulian theorem, we can assume, by passing to a subsequence, that $T^k x_n \to y \in [0, T^k x_0] = [0, x_0]$ weakly. In particular, we have $x_+^*(x_0 - y) + x_-^*(y) = 0$. It follows that $x_+^*(x_0 - y) = x_-^*(y) = 0$. Since $x_\pm^*$ are both σ-order continuous and lie in $\ker(1 - T^*)$, they are both strictly positive by Lemma 0.30 (ii). This forces $x_0 - y = y = 0$. Thus, $x_0 = 0$, which is absurd. Therefore, $\dim\ker(\lambda - T^*) = 1$, and consequently, $\ker(\lambda - T^*) = \mathrm{Span}\{x_0^*\}$. □

We now study the spectral behaviors of irreducible operators at their spectral radii. Suppose that $r(T)$ is a pole of $R(\cdot, T)$ of order m. Let A_{-n} be the coefficient of $(\lambda - r(T))^{-n}$ in the Laurent expansion of $R(\cdot, T)$ at $r(T)$ and P be the corresponding spectral projection. Then $P = A_{-1}$ and $A_{-n} = (T - r(T))^{n-1} P$; cf. [1, Lemmas 6.37 and 6.38]. It follows, in particular, that if $m > 1$ then $A_{-m}^2 = 0$. Recall also that if $m = 1$ then $PX = \ker(\lambda - T)$;

cf. [1, Corollary 6.40].

Proposition 1.13. *Suppose $r(T)$ is a pole of $R(\cdot,T)$. Let P be the corresponding spectral projection. If T satisfies any of the following:*

(i) T is ideal irreducible,

(ii) T is band irreducible and σ-order continuous and $T^*x_0^* = r(T)x_0^*$ for some strictly positive functional x_0^*[1]*,*

then $r(T) > 0$ and it is a simple pole of $R(\cdot,T)$, $PX = \ker(r(T) - T) = \mathrm{Span}\{x_0\}$ for some $x_0 > 0$, and $P^*X^* = \ker(r(T) - T^*) = \mathrm{Span}\{x_0^*\}$. In particular, $P = \frac{1}{x_0^*(x_0)} x_0^* \otimes x_0$.

(i) is in [38, Theorems 1 and 2] (and in [40, Theorem 2] except the last two assertions); see also Theorem 1.1. Variants of (ii) can be found in [22, 26, 4, 5]. We include here a simple proof combining the techniques of [38] and [40].

Proof. We prove (ii) first. By Lemma 0.23, there exists $x_0 > 0$ such that $Tx_0 = r(T)x_0$. By Lemma 0.30 (i), $r(T) > 0$.

Let $r(T)$ be a pole of order m. Denote by A_{-n} the coefficient of $(\lambda - r(T))^{-n}$ in the Laurent expansion of $R(\lambda,T)$ at $r(T)$. Observe that P^* is the spectral projection of T^* for $r(T)$. Since $T^*x_0^* = r(T)x_0^*$, we have $x_0^* \in P^*X^*$ and $P^*x_0^* = x_0^*$. By Lemma 0.23, we can take $x > 0$ such that $A_{-m}x > 0$. Then $0 < x_0^*(A_{-m}x) = (T^* - r(T))^{m-1}(P^*x_0^*)(x) = [(T^* - r(T))^{m-1}x_0^*](x)$. If $m \geq 2$, then $(T^* - r(T))^{m-1}x_0^* = 0$, yielding a contradiction! Hence, $m = 1$, that is, $r(T)$ is a simple pole.

By the remark preceding this proposition and Theorem 1.4 (ii), $PX = \ker(r(T) - T) = \mathrm{Span}\{x_0\}$. Thus, $\mathrm{rank}(P^*) = \mathrm{rank}(P) = 1$. It follows from

[1] By Lemma 0.23, what we really require here is strict positivity of x_0^*, not its existence.

$0 \neq \ker(r(T) - T^*) \subset P^*X^*$ that $P^*X^* = \ker(r(T) - T^*) = \mathrm{Span}\{x_0^*\}$. Now it is straightforward verifications that $P = \frac{1}{x_0^*(x_0)} x_0^* \otimes x_0$.

The proof of (i) is similar, since in view of Lemmas 0.23 and 0.28, there exist $x_0 > 0$ such that $Tx_0 = r(T)x_0$ and a strictly positive functional x_0^* such that $T^*x_0^* = r(T)x_0^*$. $\square$

As a consequence of this proposition, we prove that the conditions (i) and (ii) in Theorem 1.2 are equivalent.

Theorem 1.14. *Suppose T is band irreducible and σ-order continuous and $r(T)$ is a pole of the resolvent $R(\cdot, T)$. The following are equivalent:*

(i) there exists a non-zero σ-order continuous functional on X,

(ii) the set of σ-order continuous functionals separates the points of X,

(iii) there exists a σ-order continuous functional $x_0^* > 0$ such that $T^*x_0^* = r(T)x_0^*$,

(iv) the leading coefficient of the Laurent expansion of $R(\cdot, T)$ at $r(T)$ is σ-order continuous,

(v) any positive operator that commutes with T is order continuous.

Proof. (i)$\Rightarrow$(ii) follows from [43, Lemma 1]; (ii)$\Rightarrow$(i) is obvious. (i)$\Rightarrow$(iii) follows from [4, Theorem 1(c)]; (iii)$\Rightarrow$(i) is obvious. This proves (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

Now assume (iii) holds. By Lemma 0.30, x_0^* is strictly positive. Hence, Proposition 1.13 implies that the spectral projection P of T for $r(T)$ is σ-order continuous, strictly positive and compact. Observe that each positive operator that commutes with T also commutes with P. Hence, (v) follows from

[1, Lemma 9.30]. (v)$\Rightarrow$(iv) follows because the leading coefficient commutes with T and is positive by Lemma 0.23.

It remains to prove (iv)$\Rightarrow$(iii). The proof is analogous to that of [5, Theorem 4.11]. We claim that the spectral projection P at $r(T)$ is strictly positive and σ-order continuous. Let m be the order of $r(T)$ and A_{-n} be the coefficient of $(\lambda - r(T))^{-n}$ in the Laurent expansion. If $A_{-m}x = 0$ for some $x > 0$, then $A_{-m}\sum_1^\infty \frac{T^n}{\delta^n}x = \sum_1^\infty \frac{T^n}{\delta^n}A_{-m}x = 0$ for any fixed $\delta > r(T)$. By Lemma 0.29, $\sum_1^\infty \frac{T^n}{\delta^n}x$ is a weak unit. Thus, since A_{-m} is σ-order continuous, we have $A_{-m} = 0$, which is absurd. It follows that A_{-m} is strictly positive. It also follows that $m = 1$, since, otherwise, $A_{-m}^2 = 0$, contradicting its strict positivity. This finishes the proof of the claim, since $P = A_{-1} = A_{-m}$ now.

We now claim that P has rank 1. Since $m = 1$, we have $A_{-2} = (r(T) - T)P = 0$. Therefore, $PT = r(T)P$, yielding that $\sum_1^\infty \frac{T^n}{\delta^n}P = \left(\sum_1^\infty \frac{r(T)^n}{\delta^n}\right)P$. Since P is strictly positive and $\sum_1^\infty \frac{T^n}{\delta^n}$ is expanding, it follows that $\sum_1^\infty \frac{T^n}{\delta^n}P$, and therefore, P, is also expanding. If $\operatorname{rank} P > 1$ then there exists $x \in \operatorname{Range} P$ such that $x^\pm > 0$. Since P is strictly positive, $\operatorname{Range} P$ is a sublattice of X; cf. [1, Theorem 5.59]. Therefore, $x^\pm \in \operatorname{Range} P$. Since P is expanding, they are both weak units, which is absurd, since $x^+ \perp x^-$.

It follows that $P = x_0^* \otimes x_0$ for some $x_0^* > 0$ and $x_0 > 0$. Clearly, $T^*x_0^* = r(T)x_0^*$. Since P is σ-order continuous, so is x_0^*. This proves (iv)$\Rightarrow$(iii). $\square$

1.3 Comparison theorems

In this section, we establish several versions of comparison theorems. The following lemma is straightforward to verify.

Lemma 1.15. *Suppose $0 \le S \le T$. Then $T = S$ if any of the following are satisfied:*

(i) $T^*x_0^* = S^*x_0^*$ for some strictly positive functional x_0^*,

(ii) $Tx_0 = Sx_0$ for some quasi-interior point $x_0 > 0$,

(iii) T is σ-order continuous and $Tx_0 = Sx_0$ for some weak unit $x_0 > 0$.

The following is a generalization of [29, Theorem 4.3].

Lemma 1.16. *Suppose $0 \le S \le T$ and T is ideal irreducible. Then $T = S$ if any of the following are satisfied:*

(i) $T^*x_0^* = \lambda x_0^*$ for some $x_0^* > 0$ and $Sx_0 \ge \lambda x_0$ for some $x_0 > 0$,

(ii) $Tx_0 = \lambda x_0$ for some $x_0 > 0$ and $S^*x_0^* \ge \lambda x_0^*$ for some $x_0^* > 0$.

Suppose $0 \le S \le T$ and T is band irreducible and σ-order continuous. Then $T = S$ if any of the following are satisfied:

(i') $T^*x_0^* = \lambda x_0^*$ for some strictly positive $x_0^* > 0$ and $Sx_0 \ge \lambda x_0$ for some $x_0 > 0$,

(ii') $Tx_0 = \lambda x_0$ for some $x_0 > 0$ and $S^*x_0^* \ge \lambda x_0^*$ for some σ-order continuous $x_0^* > 0$.

Proof. We only prove (ii'); the other cases can be proved in a similar fashion. By Lemma 0.30 (i), we know that x_0 is a weak unit. Therefore, it follows from $\lambda x_0^* \le S^*x_0^* \le T^*x_0^*$ and Lemma 1.8 (iii) that $\lambda x_0^* = S^*x_0^* = T^*x_0^*$. By Lemma 0.30 (ii), x_0^* is strictly positive. Hence, it follows from $T^*x_0^* = S^*x_0^*$ and Lemma 1.15 (i) that $T = S$. □

Lemma 1.17. *Suppose T and S are compact, $0 \leq S \leq T$ and $r(T) = r(S)$. Then $T = S$ if T is either ideal irreducible, or band irreducible and σ-order continuous.*

Proof. Suppose that T is band irreducible and σ-order continuous. By Schaefer-Grobler Theorem 1.3, we have $r(T) > 0$. Since T and S are both compact, we have, by Krein-Rutman Theorem, that there exist $x_0 > 0$ and $x_0^* > 0$ such that $Tx_0 = r(T)x_0$ and $S^*x_0^* = r(S)x_0^*$. Since S is also σ-order continuous, we know that x_0^* is σ-order continuous by Lemma 1.11 (iii). It follows from Lemma 1.16 (ii') that $T = S$. The case when T is ideal irreducible can be proved similarly using de Pagter's Theorem. $\square$

Theorem 1.18. *Suppose $0 \leq S \leq T$, $r(T) = r(S)$, and S^k is non-zero and compact for some $k \geq 1$. Then $T = S$ if T is either ideal irreducible or band irreducible and σ-order continuous.*

Proof. We only prove the band irreducible case; the other case can be proved similarly. Without loss of generality, assume $||T|| < 1$. Put $\widetilde{T} = \sum_1^\infty T^m$ and $\widetilde{S} = \sum_1^\infty S^m$. Recall that $r(T) \in \sigma(T)$ and $r(S) \in \sigma(S)$. Thus, by the spectral mapping theorem, it is easily seen that

$$r(\widetilde{T}T^k\widetilde{T}) = r(T)^k \left(\sum_1^\infty r(T)^m \right)^2 = r(S)^k \left(\sum_1^\infty r(S)^m \right)^2 = r(\widetilde{S}S^k\widetilde{S}).$$

It follows from $\widetilde{T}T^k\widetilde{T} \geq \widetilde{T}S^k\widetilde{T} \geq \widetilde{S}S^k\widetilde{S}$ that $r(\widetilde{T}S^k\widetilde{T}) = r(\widetilde{S}S^k\widetilde{S})$.

Recall that $\widetilde{T}$ is σ-order continuous. Hence, so is $\widetilde{T}S^k\widetilde{T}$. By Lemma 0.29 (ii), $\widetilde{T}$ is expanding, hence so is $\widetilde{T}S^k\widetilde{T}$; in particular, $\widetilde{T}S^k\widetilde{T}$ is band irreducible. Finally, note that since S^k is compact, so are $\widetilde{T}S^k\widetilde{T}$ and $\widetilde{S}S^k\widetilde{S}$. Applying Lemma 1.17 to $0 \leq \widetilde{S}S^k\widetilde{S} \leq \widetilde{T}S^k\widetilde{T}$, we have $\widetilde{T}S^k\widetilde{T} = \widetilde{S}S^k\widetilde{S}$.

It follows from $\widetilde{T}S^k\widetilde{T} \geq \widetilde{T}S^k\widetilde{S} \geq \widetilde{S}S^k\widetilde{S}$ that $\widetilde{T}S^k\widetilde{T} = \widetilde{T}S^k\widetilde{S}$. Since $\widetilde{T}$ is strictly positive and $S^k\widetilde{T} \geq S^k\widetilde{S}$, we have $S^k\widetilde{T} = S^k\widetilde{S}$. If $S^kx = 0$ for some $x > 0$, then $S^k\widetilde{T}x = \widetilde{S}S^kx = 0$. But $\widetilde{T}x$ is a weak unit, forcing $S^k = 0$, which is absurd. Hence, S^k is strictly positive. Now it follows from $S^k\widetilde{T} = S^k\widetilde{S}$ and $\widetilde{T} \geq \widetilde{S}$ that $\widetilde{T} = \widetilde{S}$. This in turn implies $T = S$. □

We are now ready to present a generalization of Theorem 1.6 to operators on arbitrary Banach lattices.

Corollary 1.19. *Suppose* $0 \leq S \leq T$ *and* $r(T) = r(S)$. *Then* $T = S$ *if any of the following are satisfied:*

(i) T *is power compact, and is either ideal irreducible or band irreducible and* σ*-order continuous,*

(ii) S *is power compact, and either* S *is ideal irreducible, or* S *is band irreducible and* T *is* σ*-order continuous.*

Proof. (i) By Lemma 0.28 (iii) or Lemma 0.30 (iii), each power of T is non-zero, hence $r(S) = r(T) > 0$ by [31, Corollary 4.2.6]. In particular, this implies that each power of S is non-zero. By Aliprantis-Burkinshaw's Cube theorem ([6, Theorem 5.14]), we know that S is also power compact. Thus the desired result follows from Theorem 1.18.

(ii) Assume S^k is compact. By Lemma 0.28 (iii) or Lemma 0.30 (iii), $S^k > 0$. Since S is irreducible, so is T. Now apply Theorem 1.18 again. □

Remark 1.20. Lemma 1.17, Theorem 1.18 and Corollary 1.19 still hold if we replace compactness involved by strict singularity and assume $r(T) > 0$. The same lines of arguments with minor modifications will work. For example, let's look at the band irreducible case of Lemma 1.17. Suppose S and T are now strictly singular. Since $0 < r(T) \in \sigma(T)$ and $0 < r(S) \in \sigma(S)$, $r(T)$ and $r(S)$ are poles of $R(\cdot, T)$ and $R(\cdot, S)$, respectively; cf. [1, Exercise 8, p. 314 and Corollary 7.49]. Replacing Krein-Rutman Theorem with Lemma 0.23, we get x_0 and x_0^* as before. Since S is order weakly compact, Lemma 1.11 again implies that x_0^* is σ-order continuous. Thus, Lemma 1.17 holds. Theorem 1.18 holds because the set of strictly singular operators also forms an ideal of $L(X)$; cf. [1, Corollary 4.62]. Corollary 1.19 holds because the power property also holds for strictly singular operators (that is, if $0 \leq S \leq T$ and T is strictly singular, then S^4 is strictly singular; see Corollary 4.2, [18]).

Motivated by an idea from [8], we can also prove a variant of Corollary 1.19 replacing the power compactness condition with that the spectral radius is a pole of the resolvent.

Theorem 1.21. *Let $0 \leq S \leq T$ be such that $r(T) = r(S)$.*

(i) Suppose that $r(T)$ is a pole of $R(\cdot, T)$. Then $T = S$ if T is either ideal irreducible, or band irreducible and σ-order continuous with σ-order continuous[2] *$x_0^* > 0$ such that $T^* x_0^* = r(T) x_0^*$.*

(ii) Suppose that $r(S)$ is a pole of $R(\cdot, S)$. Then $T = S$ if either S is ideal irreducible, or S is band irreducible with σ-order continuous $x_0^* > 0$ such that $S^* x_0^* = r(S) x_0^*$ and T is σ-order continuous.

Proof. (i) Suppose first that T is band irreducible and σ-order continuous and $T^* x_0^* = r(T) x_0^*$ for some σ-order continuous $x_0^* > 0$. Let P be the spectral projection of T for $r(T)$. By Lemma 0.30 (ii), x_0^* is strictly positive. Hence, by

[2] By Lemma 0.23, what we require here is the σ-order continuity of x_0^*, not its existence.

Proposition 1.13, $r(T) > 0$ is a simple pole of $R(\cdot, T)$ and $P = \frac{1}{x_0^*(x_0)} x_0^* \otimes x_0$ for some $x_0 > 0$. Without loss of generality, assume $r(T) = 1$. Then $PT = P$ is compact, σ-order continuous and expanding (in particular, band irreducible).

Since $r(T)$ is a simple pole of $R(\cdot, T)$, Karlin's Theorem (Lemma 0.22) implies $\frac{1}{n}\sum_1^n T^i \to P$. Therefore, since PS and PT are compact, we have, by Proposition 0.4,

$$\begin{aligned} 1 =& \lim_n r\left(\frac{1}{n}\sum_1^n S^i S\right) \leq \lim_n r\left(\frac{1}{n}\sum_1^n T^i S\right) = r(PS) \\ \leq& r(PT) = \lim_n r\left(\frac{1}{n}\sum_1^n T^i T\right) = 1. \end{aligned}$$

It follows that $r(PT) = r(PS) = 1$. Now applying Lemma 1.17 to $0 \leq PS \leq PT$, we have $PT = PS$. Thus, $T = S$ due to strict positivity of P. The case when T is ideal irreducible can be proved similarly.

For (ii), let P be the spectral projection of S for $r(S)$. As before, one can show that PS, and therefore PT, is (strongly) expanding. A similar argument also gives $r(PT) = r(PS)$. Now applying Lemma 1.17 to $0 \leq PS \leq PT$ again, one obtains $PT = PS$. Thus, $T = S$ due to strict positivity of P again. □

Remark 1.22. Recall that power compact operators are essentially quasinilpotent; cf. [1, Definition 7.47]. Recall also that power compact ideal irreducible (or band irreducible and σ-order continuous) operators are non-quasinilpotent; cf. [31, Corollary 4.2.6]. Hence, the spectral radius of such an operator is a pole of the resolvent; cf. [1, Corollary 7.49]. Therefore, it is easily seen that Corollary 1.19 and Theorem 1.6 can also be deduced from Theorem 1.21.

We remark that the ideal irreducible case in Theorem 1.21(i) is just Theorem 1.1(iii). Note also that the band irreducible case in Theorem 1.21(i) is equivalent to the two comparison theorems in Theorem 1.2. Indeed, it is immediate by Theorem 1.14.

We refer to a recent preprint [24] for some other versions of comparison theorems.

1.4 Quasinilpotency of positive commutators

Recall that we say two operators $U, V \in L(X)$ ***semi-commute*** if their commutator $UV - VU$ is either positive or negative.

Lemma 1.23. *Suppose $T, S > 0$ semi-commute and $S^2 = 0$. Then T has a non-trivial invariant closed ideal. If, in addition, both T and S are σ-order continuous, then T has a non-trivial invariant band.*

Proof. Without loss of generality, assume $\|T\| < 1$. Since $S > 0$ and $S^2 = 0$, it is easily seen that the null ideal $\mathrm{Null}(S) := \{x \in X : S|x| = 0\}$ and the closed ideal I generated by $\mathrm{Range}(S)$ are both nontrivial closed ideals of X.

Suppose first $TS \geq ST$. One can easily verify that $\mathrm{Null}(S)$ is invariant under T. Suppose now T and S are σ-order continuous. Observe that although $\mathrm{Null}(S)$ may fail to be a band, it contains all the bands generated by its elements. Pick any $0 < x \in \mathrm{Null}(S)$. If $Tx = 0$, then the band generated by x is non-trivial and invariant under T. If $Tx \neq 0$, consider the band generated by $z := \sum_1^\infty T^n x > 0$. It is clearly invariant under T. Since $\mathrm{Null}(S)$ is invariant under T, we have $z \in \mathrm{Null}(S)$. Thus, the generated band is also non-trivial.

Suppose now $TS \leq ST$. Then one can easily verify that I is invariant under T. Suppose now T and S are σ-order continuous. Pick any $0 < x \in I$. If $Tx = 0$, consider the band generated by x; if $Tx \neq 0$, consider the band generated by $z := \sum_1^\infty T^n x \in I$. They are invariant under T. Since $S > 0$ and $S^2 = 0$, we have $I \subset \mathrm{Null}(S)$. Thus the generated bands are contained in $\mathrm{Null}(S)$ and, therefore, are non-trivial. $\square$

Theorem 1.24. *Suppose $T, K > 0$ semi-commute and K is compact. Then $TK = KT$ if any of the following are satisfied:*

(i) T is ideal irreducible,

(ii) T is band irreducible and both T and K are σ-order continuous.

Proof. We only prove (ii); the proof of (i) is similar. Without loss of generality, assume $\|T\| < 1$. Then $\widetilde{T} := \sum_1^\infty T^n$ is σ-order continuous and expanding. We claim that $K\widetilde{T}$ and $\widetilde{T}K$ do not have a common non-trivial invariant band. Otherwise, let B be such a band. If there exists $0 < x \in B$ such that $Kx > 0$, then $\widetilde{T}(Kx)$ is a weak unit. But $\widetilde{T}Kx \in \widetilde{T}K(B) \subset B$ implies $B = X$, a contradiction. Hence, $B \subset \ker K$. Now pick any $0 < x \in B$. Then $\widetilde{T}x > 0$ is a weak unit. From $K(K\widetilde{T}x) \in K(K\widetilde{T}(B)) \subset K(B) = \{0\}$, it follows that $K^2 = 0$. This contradicts band irreducibility of T by Lemma 1.23, and thus completes the proof of the claim.

Now assume $TK \geq KT$. Then $T^nK \geq KT^n$ for all $n \geq 1$. Thus $\widetilde{T}K \geq K\widetilde{T} \geq 0$. If $\widetilde{T}K$ has a non-trivial invariant band then it is also invariant under $K\widetilde{T}$, contradicting the preceding claim. Thus, $\widetilde{T}K$ is band irreducible. Note that $\widetilde{T}K$ and $K\widetilde{T}$ are both compact and σ-order continuous and that $r(\widetilde{T}K) = r(K\widetilde{T})$. By Lemma 1.17, we have $\widetilde{T}K = K\widetilde{T}$, and therefore, $TK = KT$. For $TK \leq KT$, we have $K\widetilde{T} \geq \widetilde{T}K$. The same argument yields $K\widetilde{T} = \widetilde{T}K$ and $TK = KT$. $\square$

We now see that the question following Theorem 1.7 has an affirmative answer; namely, in Theorem 1.7(ii) we can obtain local non-quasinilpotency of K at non-zero positive vectors. See Chapter 3 for more properties of such operators T and K.

We now turn to an application of Theorem 1.24. In [9], it is proved that if two operators $T > 0$ and $S > 0$ semi-commute and are both compact, then their commutator $TS - ST$ is quasinilpotent. It is also shown there that there exist two semi-commuting operators $T > 0$ and $S > 0$, neither of which is compact, such that $TS - ST$ is not quasinilpotent. The authors asked the following question.

Question. *Suppose $T, S > 0$ semi-commute and one of them is compact. Is $TS - ST$ necessarily quasinilpotent?*

A partial solution of this question was given in [16, Theorem 3.6] and asserts that the commutator is indeed quasinilpotent provided that, in addition, it semi-commutes with the compact operator. We now prove that the question has an affirmative answer in general.

To this end, we first recall some necessary notions. A collection $\mathcal{C}$ of closed subspaces of X is called a chain if it is totally ordered under inclusion. For any $M \in \mathcal{C}$, the predecessor M_- of M in $\mathcal{C}$ is defined to be the closed linear span of all proper closed subspaces of M that belong to $\mathcal{C}$. The following lemma is straightforward to verify.

Lemma 1.25. *Let $\mathcal{C}$ be a chain of closed ideals of X, $M \in \mathcal{C}$. Then M_- is a closed ideal of X, $M_- \subset M$ and $\mathcal{C} \cup \{M_-\}$ is chain.*

Lemma 1.26 ([13])**.** *Let $\mathcal{C}$ be a chain of closed ideals of X. Then it is maximal as a chain of closed subspaces of X if and only if it is maximal as a chain of closed ideals of X.*

Recall that a collection $\mathcal{S}$ of positive operators is called **ideal triangulations** if there exists a chain of closed ideals of X such that each member in the chain is invariant under $\mathcal{S}$ and the chain itself is maximal as a chain of closed subspaces of X (cf. Lemma 1.26). Such a chain is called an ideal triangularizing chain for $\mathcal{S}$.

Theorem 1.27. *Suppose T and K are two non-zero positive semi-commuting operators such that K is compact. Then $S := TK - KT$ is quasinilpotent.*

Proof. Since replacing T with $T+K$ does not change the commutator, we can assume $T \geq K > 0$. Let $\mathcal{C}$ be a maximal chain of invariant closed ideals of T (existence of such a chain follows from Zorn's lemma). Take any $M \in \mathcal{C}$. It is easily seen that M_- is invariant under T. Hence, by Lemma 1.25, $M_- \in \mathcal{C}$. We claim that the induced quotient operator $\widetilde{T}$ on M/M_- is ideal irreducible. Suppose that, otherwise, J is a non-trivial closed ideal of M/M_- invariant under $\widetilde{T}$. We consider $\pi^{-1}(J) = \{x \in M : \pi(x) \in J\}$, where π is the canonical quotient mapping from M onto M/M_-. By [42, Proposition 1.3, p. 156], $\pi^{-1}(J)$ is a closed ideal of M, and thus is a closed ideal of X. It is clearly invariant under T, properly contains M_- and is properly contained in M. Thus, it is easily seen that $\pi^{-1}(J)$ is comparable with members of $\mathcal{C}$. But $\pi^{-1}(J) \notin \mathcal{C}$, contradicting maximality of $\mathcal{C}$.

It follows that $\widetilde{T}$ is ideal irreducible on M/M_-. Since $T \geq K \geq 0$, both M and M_- are invariant under K; hence the quotient operator $\widetilde{K}$ is well defined on M/M_-. Theorem 1.24 implies $\widetilde{S} = \widetilde{T}\widetilde{K} - \widetilde{K}\widetilde{T} = 0$.

For each $M \in \mathcal{C}$, let $\widetilde{\mathcal{C}_M}$ be a maximal chain of closed ideals of M/M_- (existence of such a chain follows from Zorn's lemma). Put $\mathcal{C}_M = \{\pi^{-1}(J) : J \in \widetilde{\mathcal{C}_M}\}$. Then $\mathcal{C}_M$ consists of closed ideals of X each of which contains M_-

and is contained in M. Since $\widetilde{S}=0$ on M/M_-, each member of $\mathcal{C}_M$ is invariant under S. Thus, it is easily seen that $\mathcal{C}_1=\mathcal{C}\cup_{M\in\mathcal{C}}\mathcal{C}_M$ is a **chain** of closed ideals of X each of which is invariant under S.

We claim that $\mathcal{C}_1$ is an ideal triangularizing chain for S. It remains to prove $\mathcal{C}_1$ is maximal as a chain of closed subspaces of X. Suppose, otherwise, there exists a closed subspace $Y\notin\mathcal{C}_1$ such that $\mathcal{C}_1\cup\{Y\}$ is still a chain. Consider $M:=\cap_{J\in\mathcal{C},J\supset Y}J$ and $N:=\overline{\cup_{J\in\mathcal{C},J\subset Y}J}$. They are well defined since $\{0\},X\in\mathcal{C}$. Note also that $N\subset Y\subset M$. It is easily seen that they are closed ideals of X invariant under T. Since each member of $\mathcal{C}$ is comparable with Y, it is easy to see that each $J\in\mathcal{C}$ either is contained in N or contains M. Hence, by maximality of $\mathcal{C}$, we have $M,N\in\mathcal{C}$. It also follows that $N=M$ or M_-, the predecessor of M in $\mathcal{C}$. The first case is impossible, since it forces $Y=N=M\in\mathcal{C}\subset\mathcal{C}_1$. Hence, $M_-=N\subsetneqq Y\subsetneqq M$. Note that Y/M_- is a closed subspace of M/M_-. Clearly, $Y/M_-\notin\widetilde{\mathcal{C}_M}$. Since Y is comparable with each member of $\mathcal{C}_1$, Y/M_- is comparable with each member of $\widetilde{\mathcal{C}_M}$, contradicting maximality of $\widetilde{\mathcal{C}_M}$, by Lemma 1.26. This proves the claim.

Now note that for any $N\in\mathcal{C}_1$, we can find $M\in\mathcal{C}$ such that $M_-\subset N_-\subset N\subset M$, where M_- is the predecessor of M in $\mathcal{C}$ and N_- is the predecessor of N in $\mathcal{C}_1$. Since $\widetilde{S}=0$ on M/M_-, $\widetilde{S}=0$ on N/N_-. Hence, by Ringrose's theorem ([37, Theorem 7.2.3]), $\sigma(S)=0$, i.e. S is quasinilpotent. $\square$

We remark that an independent proof of this theorem can be found in [17].

Chapter 2

Irreducible Semigroups of Positive Operators

2.1 Introduction

The results of this chapter are based on [21]. Throughout this chapter, X stands for a real Banach lattice with $\dim X > 1$ and $\mathscr{S}$ stands for a non-zero semigroup of positive operators on X.

The classical Perron-Frobenius theory was extended to irreducible semigroups of positive matrices in [36] and [37, Section 5.2] and to irreducible semigroups of compact positive operators on L_p-spaces in [37, Section 8.7]. In particular, they have the following results for L_p-spaces.

Theorem 2.1. *Let $\mathscr{S}$ be an irreducible semigroup of compact positive operators on $L_p(\Omega;\mu)$ ($1 \le p < \infty$ and μ is σ-finite). Let r be the minimal rank of non-zero operators in $\overline{\mathbb{R}^+\mathscr{S}}$. Suppose that all the rank r projections in $\overline{\mathbb{R}^+\mathscr{S}}$ have the same range.*

(i) There exists $f \in L_p$, almost everywhere positive, such that $Sf = r(S)f$ for all $S \in \mathscr{S}$. Moreover, f is unique up to scalar multiples.

(ii) Every non-zero $S \in \mathscr{S}$ has at least r eigenvalues of modulus $r(S)$, counting geometric multiplicities, all of which are of form $r(S)\theta$ with $\theta^{r!} = 1$.

(iii) There are r pairwise disjoint measurable subsets Ω_i of Ω such that $\Omega = \cup_1^r \Omega_i$ and the $r \times r$ matrix representation of every non-zero $S \in \mathscr{S}$ relative to the partition $L_p(\Omega_1) \oplus \cdots \oplus L_p(\Omega_r)$ has exactly one non-zero block in each block row and in each block column.

The same approach was employed in [27] to extend some of the results in [36, 37] to irreducible semigroups of compact positive operators on order continuous Banach lattices.

In this chapter, we extend the Perron-Frobenius theory to irreducible semigroups of positive operators on arbitrary Banach lattices. The approach and some of the ideas we use are parallel to those used in [36] and [37], but in many cases we had to develop completely new techniques. Moreover, we weaken the condition that the semigroup consists entirely of compact operators; we only require that the semigroup contains a non-zero compact operator or even a peripherally Riesz operator.

The structure of this chapter is as follows. In Section 2.2, we study $\overline{\mathbb{R}^+}$-closed ideal irreducible semigroups containing a peripherally Riesz operator or a non-zero compact operator. We show that such a semigroup contains finite-rank operators and contains "sufficiently many" projections of rank r, where r is the minimal non-zero rank of operators in the semigroup. In Section 2.3, we discuss the special case when all such projections have the same range (this is the case when $\mathscr{S}$ is commutative, in particular, when $\mathscr{S}$ is generated by a single operator). We show that, in this case, there are disjoint vectors $x_1, \dots, x_r$ in X_+ such that each operator in the semigroup acts on these vectors as a scalar multiple of a permutation. In particular, $x_0 := x_1 + \dots + x_r$ is a common eigenvector for $\mathscr{S}$. In Section 2.4, we show that the dual semigroup $\{S^* \mid S \in \mathscr{S}\}$ has the same properties under the somewhat stronger condition that $\mathscr{S}$ has a unique projection of rank r (which is still satisfied when $\mathscr{S}$ is commutative). In Section 2.5, we apply our results to finitely generated semigroups. We completely characterize $\mathscr{S}$ in the case when it is generated by a single peripherally Riesz ideal irreducible operator T; we show that T acts as a scalar multiple of a cyclic permutation on $x_1, \dots, x_r$ which span the peripheral spectral subspace of T. In particular, we deduce the Frobenius part in Theorem 1.1. We improve [1, Corollary 9.21] that if S and K are two positive commuting operators such that K is compact and S is ideal irreducible then $r(K) > 0$ and $r(S) > 0$; we show that in this case $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ and $\liminf_n \|S^n x\|^{\frac{1}{n}} > 0$ whenever $x > 0$. In Section 2.6, we extend the results of the preceding sections to band irreducible semigroups of order continuous operators.

2.2 Semigroups containing finite-rank operators

The following generalization of de Pagter's Theorem is of fundamental importance to our study and is often used together with Proposition 0.32.

Theorem 2.2 ([14]; cf. also [1, Corollary 10.47])**.** *If $\mathscr{S}$ consists of compact quasinilpotent operators then $\mathscr{S}$ is ideal reducible.*

Suppose that $r := \min \operatorname{rank} \mathscr{S} < +\infty$; let $\mathscr{S}_r$ be the set of all operators of rank r in $\mathscr{S}$ and zero. Then $\mathscr{S}_r$ is a non-zero ideal of $\mathscr{S}$, so that $\mathscr{S}$ is ideal irreducible if and only if $\mathscr{S}_r$ is ideal irreducible by Proposition 0.32. Also, since the set of all operators of rank less than or equal to r is closed in $L(X)$, it is easily seen that $\mathscr{S}_r$ is $\overline{\mathbb{R}^+}$-closed whenever so is $\mathscr{S}$.

Proposition 2.3. *If $\mathscr{S}$ is ideal irreducible $\overline{\mathbb{R}^+}$-closed and contains a peripherally Riesz operator, then* $\min \operatorname{rank} \mathscr{S} < +\infty$ *and $\mathscr{S}$ contains a projection P with* $\operatorname{rank} P = \min \operatorname{rank} \mathscr{S}$.

Proof. By Proposition 0.10, $r := \min \operatorname{rank} \mathscr{S}$ is finite. By the preceding remark, $\mathscr{S}_r$ is ideal irreducible, and therefore, Theorem 2.2 guarantees that $\mathscr{S}_r$ contains a non-(quasi)nilpotent operator. Now apply Proposition 0.11. $\square$

The following example shows that, in general, for a peripherally Riesz operator $T \in \mathscr{S}$, the peripheral spectral projection of T need not be in $\mathscr{S}$.

Example 2.4. Let $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, and let $\mathscr{S} = \overline{\mathbb{R}^+}\{A, B\}$. Clearly, $\mathscr{S}$ is irreducible and the peripheral spectral projection of A is the identity. We claim that $I \notin \mathscr{S}$. Indeed, $\mathscr{S}$ consists of all positive scalar multiples of products of A and B and their limits. Any product that involves B has rank one or zero; since the set of matrices of rank one or zero is closed, any limit of products involving B is also of rank one or zero. On the other hand, it follows from $A^n = \left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$ that if $S = \lim b_j A^{n_j}$ then S is a scalar multiple of $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. Therefore, the only elements of $\mathscr{S}$ of rank two are the scalar multiples of powers of A. Hence, $I \notin \mathscr{S}$.

Corollary 2.5. *If $\mathscr{S}$ is ideal irreducible $\overline{\mathbb{R}^+}$-closed and contains a non-zero compact operator, then* $\min \operatorname{rank} \mathscr{S} < +\infty$ *and $\mathscr{S}$ contains a projection P with* $\operatorname{rank} P = \min \operatorname{rank} \mathscr{S}$.

Proof. By Proposition 2.3, it suffices to show that $\mathscr{S}$ contains a non-quasinilpotent compact operator. The set of all compact operators in $\mathscr{S}$ is a non-zero ideal, hence is ideal irreducible by Proposition 0.32(ii). Then it contains a non-quasinilpotent operator by Theorem 2.2. $\square$

Throughout the rest of this section, we assume that $\mathscr{S}$ is an ideal irreducible $\overline{\mathbb{R}^+}$-closed semigroup with $r := \min \operatorname{rank} \mathscr{S} < +\infty$. We denote by $\mathscr{S}_r$ for the ideal of all operators of rank r in $\mathscr{S}$ or zero; we will write $\mathscr{P}_r$ for the (non-empty) set of all projections of rank r in $\mathscr{S}$.

Lemma 2.6. *For every non-zero $S \in \mathscr{S}_r$ there exists $A \in \mathscr{S}$ such that AS is not nilpotent.*

Proof. Let $\mathcal{J} = \mathscr{S}S\mathscr{S}$. Then $\mathcal{J}$ consists of operators of finite rank, hence compact. Note that $\mathcal{J}$ is non-zero by Proposition 0.32(iv) and ideal irreducible by Proposition 0.32(ii). Hence, by Theorem 2.2, $\mathcal{J}$ contains a non-quasi-nilpotent operator. That is, there exist $A_1, A_2 \in \mathscr{S}$ such that $0 \neq r(A_1SA_2) = r(A_2A_1S) = r(AS)$ where $A = A_2A_1$. $\square$

Combining this lemma with Proposition 0.11, we show that $\mathscr{S}$ contains "sufficiently many" rank r projections (cf. [37, Lemmas 5.2.2 and 8.7.17]).

Proposition 2.7. *For every $S \in \mathscr{S}_r$, there exist $P, Q \in \mathscr{P}_r$ such that $PS = SQ = S$.*

Corollary 2.8. *For every non-zero $x \in X_+$ and $x^* \in X_+^*$, there exist $P, Q \in \mathscr{P}_r$ such that $Qx \neq 0$ and $P^*x^* \neq 0$.*

Proof. Since $\mathscr{S}_r$ is ideal irreducible, by Proposition 0.32(iii) there exists $S \in \mathscr{S}_r$ such that $x^*(Sx) \neq 0$. Now take P and Q as in Proposition 2.7. $\square$

Let P be a non-zero positive projection on X. Let X_P be the lattice subspace of X as in 0.25. Consider $\mathscr{S}_P = \{PSP_{|X_P} \mid S \in \mathscr{S}\}$, so that $\mathscr{S}_P \subseteq L_+(X_P)$ (note that P need not be in $\mathscr{S}$). The following proposition extends [37, Lemmas 5.2.1 and 8.7.16].

Proposition 2.9. *If P is a positive finite-rank projection and $P\mathscr{S}P \subseteq \mathscr{S}$ then $\mathscr{S}_P$ is an ideal irreducible $\overline{\mathbb{R}^+}$-closed semigroup in $L_+(X_P)$.*

Proof. It follows from $P\mathscr{S}P \subseteq \mathscr{S}$ that $\mathscr{S}_P$ is a semigroup. Let $P = \sum_{i=1}^n x_i^* \otimes x_i$ as in 0.25; relative to the basis $x_1, \dots, x_n$, we can view $\mathscr{S}_P$ as a semigroup of positive $n \times n$ matrices. Since $\mathscr{S}$ is ideal irreducible, by Proposition 0.32(iii), for each i,j there exists $S \in \mathscr{S}$ such that $x_i^*(Sx_j) \neq 0$, i.e., the (ij)-th entry of the matrix of $PSP_{|X_P}$ is non-zero. Hence, $\mathscr{S}_P$ is ideal irreducible by Proposition 0.32 again.

To show that $\mathscr{S}_P$ is closed, suppose that $PS_nP_{|X_P} \to A$ for some sequence (S_n) in $\mathscr{S}$ and some $A \in L(X_P)$. Put $S = PAP \in L(X)$. Then $PS_nP \to S$, so that $S \in \mathscr{S}$ because $\mathscr{S}$ is closed. Now $A = PSP_{|X_P}$ yields $A \in \mathscr{S}_P$. $\square$

Of course, the assumption that $P\mathscr{S}P \subseteq \mathscr{S}$ is satisfied when $P \in \mathscr{S}$. If, in addition, $\operatorname{rank} P = r$, we get the following much stronger result. We begin with a technical lemma.

Lemma 2.10. *Suppose that $\mathscr{S}$ is an $\overline{\mathbb{R}^+}$-closed semigroup of matrices such that every non-zero matrix in $\mathscr{S}$ is invertible. Then $\big\{A \in \mathscr{S} \mid r(A) = 1\big\}$ is a closed group.*

Proof. Let $\mathscr{S}^\times := \big\{A \in \mathscr{S} \mid r(A) = 1\big\}$. Take any $A \in \mathscr{S}^\times$. Since $\mathscr{S}$ contains no non-zero nilpotent matrices, the nilpotent case in Proposition 0.8 is impossible, hence some sequence of powers A^{m_j} converges to the peripheral spectral projection P of A. In particular, $P \in \mathscr{S}$, hence invertible, so that $P = I$ and $\sigma(A)$ is contained in the unit circle. This yields that A is unimodular. It follows from $A^{m_j-1} = A^{-1}A^{m_j} \to A^{-1}$ that $A^{-1} \in \mathscr{S}$. Clearly, $\sigma(A^{-1})$ is also contained in the unit circle, so that $A^{-1} \in \mathscr{S}^\times$.

Suppose that $0 \neq A \in \mathscr{S}$. Then $\frac{1}{r(A)}A \in \mathscr{S}^\times$, and the later matrix is unimodular, so that $\lvert\det A\rvert = r(A)^n$. It follows that for $A \in \mathscr{S}$ we have $A \in \mathscr{S}^\times$ if and only if $\lvert\det A\rvert = 1$. Therefore, $\mathscr{S}^\times$ is closed under multiplication. It also follows that $\mathscr{S}^\times$ is closed. $\square$

We write $\mathcal{G}_P := \big\{PSP_{|X_P} \mid S \in \mathscr{S} \text{ and } r(PSP) = 1\big\}$.

Proposition 2.11. *Suppose that $P \in \mathscr{P}_r$. Then every non-zero element of $\mathscr{S}_P$ is invertible and, after appropriately scaling the basis vectors of X_P, $\mathcal{G}_P$ is a transitive*[1] *group of permutation matrices.*

Proof. By Proposition 2.9, $\mathscr{S}_P$ is irreducible and $\overline{\mathbb{R}^+}$-closed. Since $r = \min \operatorname{rank} \mathscr{S}$, every non-zero element of $\mathscr{S}_P$ is invertible. It follows from Lemma 2.10 that $\mathcal{G}_P$ is a group. In particular, each matrix in $\mathcal{G}_P$ has a positive inverse. It is known that a positive matrix A in $M_r(\mathbb{R})$ has a positive inverse if and only if it is a weighted permutation matrix with positive weights, i.e., there exist positive weights $w_1, \dots, w_r$ and a permutation σ of $\{1, \dots, r\}$ such that $Ax_i = w_i x_{\sigma(i)}$ for each $i = 1, \dots, r$.

It is left to show that, after scaling x_i's, we may assume that all the weights are equal to one (for all $S \in \mathcal{G}_P$). We essentially follow the proof of [37, Lemma 5.1.11]. Since $\mathscr{S}_P$ is an irreducible semigroup of matrices, for each $i, j \leq r$ there exists $A \in \mathscr{S}_P$ such that Ax_i is a scalar multiple of x_j. Put $A_1 = I$. For each $i = 1, \dots, r$ fix $A_i \in \mathcal{G}_P$ such that $A_i x_1 = \mu_i x_i$ for some $\mu_i > 0$. Replacing x_i with $\mu_i x_i$ for $i = 2, \dots, r$, we have $A_i x_1 = x_i$. It suffices to show that with respect to these modified x_i's, all the matrices in $\mathcal{G}_P$ are permutation matrices. Let $B \in \mathcal{G}_P$. We know that B is a weighted permutation matrix. Take any i and j such that $\lambda := b_{ij}$ is non-zero. Put $C = A_i^{-1} B A_j$. Then $C \in \mathcal{G}_P$ and $Cx_1 = \lambda x_1$, so that $\lambda = c_{11} \leq r(C) = 1$. Similarly, λ^{-1} is the $(1,1)$'s entry of C^{-1}, hence $\lambda^{-1} \leq 1$ as well, so that $\lambda = 1$.

Finally, transitivity of $\mathcal{G}_P$ follows from the irreducibility of $\mathscr{S}_P$. $\square$

Remark 2.12. It follows that the vector $x_0 = x_1 + \cdots + x_r$ is invariant under $\mathcal{G}_P$. Furthermore, for each $S \in \mathscr{S}$, if $PSP \neq 0$ then the minimality of

[1] Transitive in the sense that for each i and j there exists $A \in \mathcal{G}_P$ such that $Ax_i = x_j$.

rank implies that PSP is an isomorphism on X_P, so that $r(PSP) \neq 0$ and, therefore, a scalar multiple of PSP is in $\mathcal{G}_P$. It follows that x_0 is a common eigenvector for $\mathscr{S}_P$ with $PSPx_0 = r(PSP)x_0$.

2.3 Semigroups with all the rank r projections having the same range

As in the previous section, $\mathscr{S}$ will stand for an $\overline{\mathbb{R}^+}$-closed ideal irreducible semigroup of positive operators on a Banach lattice, with $r := \min \operatorname{rank} \mathscr{S} < \infty$. We will write $\mathscr{S}_r$ for the (ideal irreducible) ideal of all operators of rank r in $\mathscr{S}$ and zero, and $\mathscr{P}_r$ for the set of all projections of rank r in $\mathscr{S}$ (which is non-empty by, e.g., Corollary 2.5).

Let $P \in \mathscr{P}_r$ and x_0 be as in Remark 2.12. For x_0 to be a common eigenvector of the entire semigroup $\mathscr{S}$, it would suffice that (a) $\operatorname{Range} P$ is invariant under $\mathscr{S}$ and (b) $PSP \neq 0$ for every non-zero $S \in \mathscr{S}$. We will see that, surprisingly, (a) implies (b). The following proposition extends [37, Lemmas 5.2.4 and 8.7.18].

Proposition 2.13. *The following are equivalent:*

(i) *all projections in $\mathscr{P}_r$ have the same range,*

(ii) *all non-zero operators in $\mathscr{S}_r$ have the same range,*

(iii) $S(\operatorname{Range} P) = \operatorname{Range} P$ *for all non-zero $S \in \mathscr{S}$ and $P \in \mathscr{P}_r$,*

(iv) *The range of some $P \in \mathscr{P}_r$ is $\mathscr{S}$-invariant.*

Proof. (i)⇒(ii) follows from Proposition 2.7.

(ii)$\Rightarrow$(iii) Let $S \in \mathscr{S}$ and $P \in \mathscr{P}_r$. Since $\mathscr{S}_r$ is ideal irreducible, $S\mathscr{S}_r \neq \{0\}$, so that $ST \neq 0$ for some $T \in \mathscr{S}_r$. It follows from $\operatorname{Range} T = \operatorname{Range} P$ that $SP \neq 0$. Since $SP \in \mathscr{S}_r$, we have $\operatorname{Range} SP = \operatorname{Range} P$.

(iii)$\Rightarrow$(iv) is trivial.

(iv)$\Rightarrow$(i) Suppose that $\operatorname{Range} P$ is $\mathscr{S}$-invariant for some $P \in \mathscr{P}_r$. Take any $Q \in \mathscr{P}_r$. We have $Q\mathscr{S}P \neq \{0\}$ by Proposition 0.32(iv), so that $QSP \neq 0$ for some $S \in \mathscr{S}$. By assumption, $SP = PSP$, so that $QPSP \neq 0$, hence $QP \neq 0$. This yields $\operatorname{rank} QP = r$. By assumption, $\operatorname{Range} QP = Q(\operatorname{Range} P) \subseteq \operatorname{Range} P$, but, trivially, $\operatorname{Range} QP \subseteq \operatorname{Range} Q$. Since all the three ranges are r-dimensional, the inclusions are, in fact, equalities, so that $\operatorname{Range} P = \operatorname{Range} QP = \operatorname{Range} Q$. $\square$

Next, we would like to provide a few examples.

Example 2.14. Suppose that $x, y \in X_+$ and $x^*, y^* \in X_+^*$ such that $x^*(x) = y^*(x) = x^*(y) = y^*(y) = 1$. Let $\mathscr{P} = \{x^* \otimes x, y^* \otimes x, x^* \otimes y, y^* \otimes y\}$. Then $\mathscr{P}$ is a semigroup of projections. Let $\mathscr{S} = \overline{\mathbb{R}^+}\mathscr{P}$, the semigroup of all non-negative scalar multiples of the elements of $\mathscr{P}$. Clearly, $\mathscr{P}$ is exactly the set $\mathscr{P}_r$ of the minimal rank projections in $\mathscr{S}$ with $r = 1$, and the ranges of the elements of $\mathscr{S}$ are $\operatorname{Span} x$ and $\operatorname{Span} y$. In particular, all the ranges are the same if and only if $x = y$.

Example 2.15. More specifically, take in Example 2.14 $X = \mathbb{R}^2$, $x = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $y = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$, and $x^* = y^* = [1, 1]$. Then $\mathscr{P} = \{P, Q\}$ where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ are ideal irreducible and have different ranges.

Example 2.16. Again in Example 2.14, take $X = \mathbb{R}^2$, $x = y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x^* = \left[\frac{1}{2}, \frac{1}{2}\right]$, and $y^* = \left[\frac{1}{3}, \frac{2}{3}\right]$. Then $\mathscr{P} = \{P, Q\}$ where $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$ are both ideal irreducible and have the same range.

Example 2.17. Again in Example 2.14, take $X = \mathbb{R}^2$, $x = y = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $x^* = [1, 0]$, and $y^* = [0, 1]$. Then $\mathscr{P} = \{P, Q\}$ where $P = \left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]$ and $Q = \left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right]$. Even though neither P nor Q is ideal irreducible, they generate an ideal irreducible semigroup. Note that P and Q have the same range.

For the rest of this section, we assume that all the projections in $\mathscr{P}_r$ have the same range. This condition looks rather strong at the first glance. However, it will follow immediately from Proposition 2.28 that it is satisfied for commutative semigroups, and, in particular, for semigroups generated by a single operator.

We now prove a Banach lattice version of [37, Lemmas 5.2.5 and 8.7.19,Theorems 5.2.6 and 8.7.20]. Denote by Y the common range of the projections in $\mathscr{P}_r$. For a non-zero $S \in \mathscr{S}$ we denote by S_Y the restriction of S to Y; we write $\mathscr{S}_Y = \{S_Y \mid S \in \mathscr{S}\}$ and $\mathcal{G} := \big\{S_Y \mid S \in \mathscr{S},\ r(S_Y) = 1\big\}$. Note that $\mathscr{S}_Y = \mathscr{S}_P$ and $\mathcal{G} = \mathcal{G}_P$ for every $P \in \mathscr{P}_r$, cf. 0.25 and Proposition 2.11. In particular, $\mathcal{G}$ is a transitive group of permutation matrices in the appropriate positive basis $x_1, \dots, x_r$ of Y. The following lemma follows immediately from Proposition 2.13(iii).

Lemma 2.18. *For each non-zero $S \in \mathscr{S}$, the restriction S_Y is an isomorphism of Y. In particular, $r(S_Y) > 0$ and $\frac{1}{r(S_Y)} S_Y \in \mathcal{G}$.*

It follows, in particular, that $\mathscr{S}$ contains no zero divisors and no non-zero quasinilpotent operators.

Theorem 2.19. *There exist disjoint positive vectors $x_1, \dots, x_r$ such that every $S \in \mathscr{S}$ acts as a scalar multiple of a permutation on x_i's.*

Proof. The statement follows immediately from Lemma 2.18 and Proposition 2.11 except for the disjointness of x_i's. By 0.25, we know that Y is a lattice subspace of X, and the positive vectors $x_1, \dots, x_r$ form a basis of Y and are disjoint in Y. The latter means that for each $i, j \leq r$ with $i \neq j$, we have $P(x_i \wedge x_j) = 0$ for every $P \in \mathscr{P}_r$. It now follows from Corollary 2.8 that $x_i \perp x_j$ in X. $\square$

Corollary 2.20. *All the operators in $\mathscr{S}$ have a unique common eigenvector x_0. Namely, $Sx_0 = r(S_Y)x_0$ for each $S \in \mathscr{S}$. Furthermore, x_0 is positive and quasi-interior.*

Proof. Let $x_1, \dots, x_r$ be as in the theorem. Put $x_0 = x_1 + \cdots + x_r$. Since each $S \in \mathscr{S}$ is just a scalar multiple of a permutation on x_i's, it follows that x_0 is a common eigenvector for $\mathscr{S}$. Clearly, the ideal I_{x_0} generated by x_0 is invariant under $\mathscr{S}$, hence is dense in X; it follows that x_0 is quasi-interior. It is left to verify uniqueness (of course, up to scaling). Indeed, suppose that y is also a common eigenvector for $\mathscr{S}$. Then for each $P \in \mathscr{P}_r$ we have $y \in \operatorname{Range} P = Y$. It follows that y is a linear combination of x_i's. In particular, viewed as an element of $\mathbb{R}^r$, it is a common eigenvector of the transitive group of permutations $\mathcal{G}$, so that it has to be of the form $(\lambda, \dots, \lambda)$; it follows that $y = \lambda x_0$. $\square$

Note that the semigroup in Example 2.15 has no common eigenvectors.

2.21. *Other eigenvalues of $\mathscr{S}$.* Since every element of $\mathcal{G}$ is a permutation matrix with respect to the basis $x_1, \dots, x_r$ of Y, its Jordan form is diagonal and unimodular. It follows that every non-zero $S \in \mathscr{S}$ has at least r eigenvalues of modulus $r(S_Y)$ (counting geometric multiplicities). If we scale S so that $r(S_Y) = 1$ then $(S_Y)^{r!}$ is the identity of Y; it follows that these eigenvalues satisfy $\lambda^{r!} = 1$.

2.22. *Block-matrix structure of $\mathscr{S}$.* Let $X_i = \overline{I_{x_i}}$ for each $i = 1, \dots, r$. Then $X = X_1 \oplus \cdots \oplus X_r$ is a decomposition of X into pair-wise disjoint closed ideals, and for every non-zero $S \in \mathscr{S}$ the block-matrix of S with respect to this decomposition has exactly one non-zero block in each row and in each column.

Proposition 2.23. *If $T \in \mathscr{S}$ is peripherally Riesz then $r(T_Y) = r(T)$. Furthermore, if $r(T) = 1$ then the component of T corresponding to $\sigma_{\mathrm{per}}(T)$ is unimodular.*

Proof. Without loss of generality, $r(T) = 1$. By Lemma 2.18, $\mathscr{S}$ has no non-zero nilpotent elements. It follows that the nilpotent case in Proposition 0.8 is impossible, hence the peripheral spectral projection P of T is in $\mathscr{S}$ and there is an increasing sequence (m_j) in $\mathbb{N}$ with $T^{m_j} \to P$. In particular, $(T_Y)^{m_j} \to P_Y$. It follows from $r(T) = 1$ that $r(T_Y) \leq 1$. Suppose that $r(T_Y) < 1$. Then $(T_Y)^{m_j} \to 0$, hence $P_Y = 0$. But this contradicts P_Y being an isomorphism by Lemma 2.18. $\square$

Corollary 2.24. *If every non-zero operator in $\mathscr{S}$ is peripherally Riesz then spectral radius is multiplicative on $\mathscr{S}$.*

Proof. Let $S, T \in \mathscr{S}$. By Proposition 2.23, $r(S) = r(S_Y)$, $r(T) = r(T_Y)$, and $r(ST) = r(S_Y T_Y)$. Since S_Y and T_Y are scalar multiples of permutation matrices by Theorem 2.19, it follows that $r(S_Y T_Y) = r(S_Y) r(T_Y)$. $\square$

For each non-zero $S \in \mathscr{S}$ we have $r(S^*) = r(S) \geq r(S_Y) > 0$ by Lemma 2.18. The following refinement is immediate by Corollary 2.20 and Lemma 0.21.

Corollary 2.25. *For every non-zero $S \in \mathscr{S}$ and non-zero $x^* \in X_+^*$, we have $\liminf_n \|S^{*n} x^*\|^{\frac{1}{n}} \geq r(S_Y)$. In particular, S^* is strictly positive.*

Example 2.26. In Example 2.14, take $X = \mathbb{R}^2$, $x = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $y = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and $x^* = y^* = [1,1]$. Then $\mathscr{P} = \{P, Q\}$ where $P = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$ and $Q = \left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$. Now the semigroup generated by P and Q is irreducible, however P and Q have different ranges and neither P^* nor Q^* is strictly positive.

Remark 2.27. Let $x_1, \dots, x_r$ be a disjoint positive basis of Y as before. Suppose that $P \in \mathscr{P}_r$, then, as in 0.25, we have $P = \sum_{i=1}^r x_i^* \otimes x_i$ for some positive functionals $x_1^* \dots, x_r^*$. Observe that these functionals are disjoint. Indeed, by Riesz-Kantorovich formula, if $i \neq j$ then

$$\begin{aligned}(x_i^* \wedge x_j^*)(x_0) =& \inf\bigl\{x_i^*(u) + x_j^*(v) \mid u, v \in [0, x_0], u + v = x_0\bigr\}\\ \leq& x_i^*(x_j) + x_j^*(x_0 - x_j) = 0,\end{aligned}$$

hence $(x_i^* \wedge x_j^*)(x_0) = 0$. Since x_0 is quasi-interior, it follows that $x_i^* \wedge x_j^* = 0$.

2.4 Semigroups with a unique rank r projection

As before, we assume that $\mathscr{S}$ is an ideal irreducible $\overline{\mathbb{R}^+}$-closed semigroup of positive operators on a Banach lattice X with $r = \min \operatorname{rank} \mathscr{S} < +\infty$.

In the previous section we showed that if all the rank r projections have the same range then $\mathscr{S}$ has some nice properties. In this section, we will show that many of these properties are also enjoyed by the dual semigroup $\mathscr{S}^* = \{S^* \mid S \in \mathscr{S}\}$ provided that $\mathscr{S}$ has a *unique* projection of rank r. Even though this is, obviously, a stronger assumption, the following proposition implies that it is still satisfied for commutative semigroups. It is analogous to

[37, Lemmas 5.2.7 and 8.7.21].

Proposition 2.28. *The following are equivalent:*

(i) $\mathscr{P}_r$ *consists of a single projection,*

(ii) every $P \in \mathscr{P}_r$ *commutes with* $\mathscr{S}$*,*

(iii) some $P \in \mathscr{P}_r$ *commutes with* $\mathscr{S}$*.*

Proof. (i)⇒(ii) Suppose that $\mathscr{P}_r = \{P\}$ and let $0 \neq S \in \mathscr{S}$. It follows from Proposition 2.13(iii) that $PSP \neq 0$. Hence, PS and SP are non-zero elements of $\mathscr{S}_r$. Applying Proposition 2.7 to PS and SP we get $PS = PSP = SP$.

(ii)⇒(iii) is trivial.

(iii)⇒(i) Suppose $P \in \mathscr{P}_r$ commutes with $\mathscr{S}$. It follows that $PSP = SP$ for all $S \in \mathscr{S}$, hence by Proposition 2.13(iv), all the projections in $\mathscr{P}_r$ have the same range. Therefore, $P = QP = PQ = Q$ for every $Q \in \mathscr{P}_r$. $\square$

Throughout the rest of the section, we assume that $\mathscr{S}$ has a unique projection P of rank r. This condition allows us to "dualize" the results of Section 2.3 for $\mathscr{S}^*$, even though $\mathscr{S}^*$ may not be ideal irreducible.

Suppose $\mathscr{P}_r = \{P\}$. As in Section 2.3, we denote $Y = \operatorname{Range} P = X_P$. We can write it as $P = \sum_{i=1}^{r} x_i^* \otimes x_i$ as in Remark 2.27. It is easy to see that P^* is a projection onto $X_{P^*} := \operatorname{Range} P^* = \operatorname{Span}\{x_1^*, \dots, x_r^*\}$ in X^*. For every non-zero $S \in \mathscr{S}$, it follows from Proposition 2.28 that $PSP = SP = PS$, so that $P^*S^*P^* = S^*P^*$, and, therefore, X_{P^*} in invariant under S^*. Note that $r(P^*S^*P^*) = r(PSP) = r(S_Y) \neq 0$ by Lemma 2.18. As in Section 2.3, if $r(S_Y) = 1$ then $S \in \mathcal{G}$ (since P is unique, we write $\mathcal{G}_P = \mathcal{G}$) and S acts as a permutation matrix on $x_1, \dots, x_r$. It follows from $x_i^*(x_j) = \delta_{ij}$ that S^* acts as a permutation matrix on $x_1^*, \dots, x_r^*$ (namely, as the transpose of the matrix of S on $x_1, \ldots, x_r$). Moreover, since $\mathcal{G}$ is transitive on $x_1, \ldots, x_r$, the group $\mathcal{G}^* := \{S^* \mid S \in \mathcal{G}\}$ is transitive on $x_1^*, \ldots, x_r^*$. In particular, we have $S^* x_0^* = x_0^*$, where $x_0^* = x_1^* + \cdots + x_r^*$.

Corollary 2.29. *For every non-zero $S \in \mathscr{S}$, the operator $\frac{1}{r(S_Y)} S^*$ acts as a permutation of $x_1^*, \ldots, x_r^*$. In particular, $S^* x_0^* = r(S_Y) x_0^*$ for each non-zero $S \in \mathscr{S}$. The functional x_0^* is strictly positive and is a unique common eigenfunctional for $\mathscr{S}^*$.*

Proof. Uniqueness is proved exactly as in Corollary 2.20. It is left to prove that x_0^* is strictly positive. Fix $x > 0$. By Proposition 0.32(iii), there exists $S \in \mathscr{S}$ with $x_0^*(Sx) \neq 0$. Since $r(S_Y) \neq 0$ by Lemma 2.18 and $x_0^*(Sx) = (S^* x_0^*)x = r(S_Y) x_0^*(x)$, we have $x_0^*(x) \neq 0$. □

The following proposition should be compared with Proposition 2.23.

Proposition 2.30. *Let $0 \neq S \in \mathscr{S}$. If $r(S)$ is an eigenvalue of S or S^* then $r(S_Y) = r(S)$.*

Proof. We know that $Sx_0 = r(S_Y) x_0$ and $S^* x_0^* = r(S_Y) x_0^*$. Since x_0^* is strictly positive and x_0 is quasi-interior, the proposition follows from Lemma 0.21. □

In view of Corollary 2.29, the following fact is the dual version of Corollary 2.25 and also follows from Lemma 0.21. Clearly, Corollaries 2.29 and 2.31 extend [37, Lemma 5.2.8 and Corollary 8.7.22].

Corollary 2.31. *For every $x > 0$ and every non-zero $S \in \mathscr{S}$ we have $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq r(S_Y)$. In particular, S is strictly positive.*

This means that not only every non-zero $S \in \mathscr{S}$ is not quasinilpotent, but it is not even *locally* quasinilpotent.

We would like to point out that Corollaries 2.29 and 2.31 generally fail if instead of assuming that $\mathscr{S}$ has a unique minimal projection we only assume, as in Section 2.3, that all the rank r projections in $\mathscr{S}$ have the same range. Indeed, the semigroups in Examples 2.16 and 2.17 are irreducible, $\overline{\mathbb{R}^+}$-closed, have exactly two distinct projections P and Q of rank r each, and they have the same range. Nevertheless it is easy to see that the dual semigroup $\mathscr{S}^*$ in Example 2.16 has no common positive eigenfunctionals (as P^* and Q^* have no common eigenfunctionals), while the operators P and Q in Example 2.17 are not strictly positive.

Recall that a positive operator T is strongly expanding if Tx is quasi-interior whenever $x > 0$.

Corollary 2.32. *The projection P is strongly expanding iff $r = 1$.*

Proof. Note that P is strictly positive by Corollary 2.31. If $r = 1$ then Range P is the span of x_0, hence Px is a positive scalar multiple of x_0 whenever $x > 0$. On the other hand, if $r > 1$ then $Px_1 = x_1 \perp x_2$, hence Px_1 is not quasi-interior. $\square$

Example 2.33. Fix $n > 2$ and let $\mathscr{S}$ be the semigroup of all positive scalar multiples of all permutation matrices in $M_n(\mathbb{R})$. Then $\mathscr{S}$ is not commutative; nevertheless, the identity matrix is the unique element of $\mathscr{P}_r$.

Commutative semigroups

In view of Proposition 2.28, all the results of Sections 2.3 and 2.4 apply to commutative semigroups. In particular, the group $\mathcal{G}$ is a commutative transitive semigroup of permutation matrices. Every matrix in such a group is a direct sum of cycles of equal lengths; it follows, in particular, that $S^r_{|Y}$ is a multiple of the identity on Y for each $S \in \mathscr{S}$. See [37, Lemma 5.2.11] for a proof and further properties of such groups of matrices.

2.5 Applications to finitely generated semigroups

Singly generated semigroups

Suppose that T is a positive ideal irreducible peripherally Riesz operator on a Banach lattice X. We now present a version of Perron-Frobenius Theorem for T, extending [37, Corollaries 5.2.3 and 8.7.24]. In addition, we completely describe $\overline{\mathbb{R}^{+}T}$ (cf. Proposition 0.8). For simplicity, scaling T if necessary, we assume that $r(T) = 1$. Let $X = X_1 \oplus X_2$ be the spectral decomposition for T where X_1 is the subspace for $\sigma_{\mathrm{per}}(T)$, and $T = T_1 \oplus T_2$ the corresponding decomposition of T. Clearly, $\overline{\mathbb{R}^{+}T}$ is ideal irreducible. Since it is commutative, all the results of Sections 2.3 and 2.4 apply to it. We will see that, surprisingly, the asymptotic part of $\overline{\mathbb{R}^{+}T}$ is very small: it consists of finitely many operators and their positive scalar multiples.

Proposition 2.34. *Under the preceding assumptions,* $\dim X_1 = \min \operatorname{rank} \overline{\mathbb{R}^{+}T}$*,* X_1 *has a basis of disjoint positive vectors* $x_1, \dots, x_r$ *such that* T_1 *is a cyclic permutation of* $x_1, \dots, x_r$*, and* $\overline{\mathbb{R}^{+}T}$ *consists precisely of all the powers of* T*, of the operators* $T_1^k \oplus 0$ *for* $k = 0, \dots, r-1$*, and of their positive scalar multiples (and zero).*

Proof. By Proposition 2.23, T_1 is unimodular. Hence, we are in the unimodular case of Proposition 0.8. In particular, the peripheral spectral projection P is the only non-zero projection in the semigroup. It follows that $r := \min \operatorname{rank} \overline{\mathbb{R}^{+}T} = \dim X_1$, $\mathscr{P}_r = \{P\}$, and X_1 coincides with Y in the notation of Section 2.3. This implies by Theorem 2.19 that X_1 is a sublattice generated by some disjoint sequence $x_1, \dots, x_r$ and T_1 is a permutation of x_i's. We claim that this permutation is a cycle of full length r. Indeed, otherwise, T_1 has a cycle of length $m < r$, i.e., after re-numbering the basis vectors, T_1 acts as a cycle on $x_1, \dots, x_m$. But then the closed ideal generated by $x_1, \dots, x_m$ is invariant under T and is proper as it is disjoint with $x_{m+1}, \dots, x_r$.

It follows that T_1^r is the identity of X_1, so that the set of the distinct powers of T_1 is, in fact, finite. Suppose that $0 \neq S = \lim_j b_j T^{n_j}$ for some (b_j) in $\mathbb{R}_+$ and some strictly increasing (n_j) in $\mathbb{N}$. By Proposition 0.8, $S_{|X_2} = 0$ and $S_{|X_1} = \lim_j b_j T_1^{n_j}$. Since the set of the distinct powers of T_1 is finite, it follows that $S_{|X_1}$ is a scalar multiple of a power of T_1. $\square$

Remark 2.35. Suppose T is an ideal irreducible operator such that $r(T)$ is a pole of the resolvent $R(\cdot, T)$. By Proposition 1.13 and Lemma 0.24, we know that T is peripherally Riesz. Thus, the Frobenius part in Theorem 1.1 follows immediately from Proposition 2.34.

Semigroups generated by two commuting operators

In [1, Corollary 9.21], the following extension of de Pagter's theorem was established: suppose that S and K are two non-zero positive commuting operators such that S is ideal irreducible and K is compact, then $r(S) > 0$ and $r(K) > 0$. Moreover, K is not even locally quasinilpotent at any positive non-zero vector x, i.e., $\liminf_n \|K^n x\|^{\frac{1}{n}} > 0$; cf. [1, Theorem 9.19]. Using the results of the preceding sections, we can now strengthen this conclusion even further.

Proposition 2.36. *Under the preceding assumptions on S and K, there exists a quasi-interior vector $x_0 \in X_+$, a strictly positive functional x_0^*, and a positive* *real λ such that $Sx_0 = \lambda x_0$, $S^*x_0^* = \lambda x_0^*$, $Kx_0 = r(K)x_0$, and $K^*x_0^* = r(K)x_0^*$. Furthermore,* $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ *and* $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K) > 0$ *whenever* $x > 0$.

Proof. Let $\mathscr{S} = \overline{\mathbb{R}^+}\{S, K\}$. Then $\mathscr{S}$ is ideal irreducible and commutative, so that all the results of Sections 2.3 and 2.4 apply. In particular, by Corollaries 2.20 and 2.29 there exist a quasi-interior vector $x_0 \in X_+$ and a strictly positive functional x_0^* such that $Sx_0 = r(S_Y)x_0$, $S^*x_0^* = r(S_Y)x_0^*$, $Kx_0 = r(K_Y)x_0$, and $K^*x_0^* = r(K_Y)x_0^*$. Now put $\lambda := r(S_Y)$ and note that $r(K_Y) = r(K)$ by Proposition 2.23. Also, observe that $r(S) \geq \lambda > 0$ and $r(K) > 0$ by Lemma 2.18.

It is left to show the "furthermore" clause. Fix $x > 0$. It follows from Corollary 2.31 that $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ and $\liminf_n \|K^n x\|^{\frac{1}{n}} \geq r(K)$. However, we clearly have $\limsup_n \|K^n x\|^{\frac{1}{n}} \leq r(K)$, so that $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$. $\square$

Remark 2.37. (i) It is easy to see that $\limsup_n \|T^n x\|^{\frac{1}{n}} \leq r(T)$ for every operator T and every vector x. Therefore, the conclusion $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ in the theorem is sharp.

(ii) Corollary 2.25 yields $\liminf_n \|S^{*n} x^*\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^{*n} x^*\|^{\frac{1}{n}} = r(K)$ whenever $x^* > 0$.

(iii) Clearly, the result (and the proof) remains valid if we require that K is ideal irreducible instead of S. Moreover, the result can be extended to any ideal irreducible commutative collection of operators containing a compact or a peripherally Riesz operator. In this case, the result will still be valid for every operator S in the collection (with λ depending on S).

2.6 Band irreducible semigroups

In this section, we will show that most of the results of the preceding sections remain valid if we replace ideal irreducibility with band irreducibility under the additional assumption that all the operators in $\mathscr{S}$ are order continuous. This additional assumption is justified by the following two facts. For $A \subseteq X$ we write I_A and B_A for the ideal and the band generated by A, respectively. Suppose that S is a positive order continuous operator. If S vanishes on a set $A \subseteq X_+$ then S also vanishes on B_A. Furthermore, if J is an S-invariant ideal then the band B_J is still S-invariant.

For the rest of this section, we will assume that $\mathscr{S}$ is a semigroup of positive order continuous operators on a Banach lattice X. We start with a variant of Theorem 2.2 for band irreducible case using the idea of the proof of Lemma 1.11.

Proposition 2.38. *If all the operators in $\mathscr{S}$ are compact and quasinilpotent then $\mathscr{S}$ is band reducible.*

Proof. Let F be the closed ideal generated by the union of the ranges of all the operators in $\mathscr{S}$. We may assume, without loss of generality, that $\dim F > 1$ as, otherwise, F is a band and we are done. Applying Theorem 2.2 to the restriction of $\mathscr{S}$ to F, we find a non-zero closed ideal $J \subsetneq F$ such that J is $\mathscr{S}$-invariant. It follows that B_J is $\mathscr{S}$-invariant. It is left to show that B_J is proper. Suppose that $B_J = X$. Then for any $x \in X_+$ we have $x_\alpha \uparrow x$ for some net (x_α) in J_+. Let $S \in \mathscr{S}$. Since S is order continuous, we have $Sx_\alpha \uparrow Sx$. Since S is compact, after passing to a subnet we know that (Sx_α) converges in norm; hence $Sx_\alpha \to Sx$ in norm. It follows that $Sx \in J$. Since $x > 0$ was arbitrary, it follows that $F \subseteq J$; a contradiction. $\square$

2.39. One can now easily verify that the results of the previous sections remain true for band irreducible semigroups of order continuous operators with the following straightforward modifications.

- In Corollary 2.8, one has to assume that x^* is *order continuous*.
- In 0.25, we now only consider order continuous projections. It is easy to see that the functionals $x_1^*, \dots, x_n^*$ defined there are also order continuous.
- Proposition 2.9 extends as long as P is order continuous.
- In Corollary 2.20, we replace "quasi-interior" with "a weak unit".
- In 2.22, we replace $\overline{I_{x_i}}$ with B_{x_i}.
- In Corollary 2.25 we need to assume that x^* is σ-order continuous, because in this case we still have $x^*(x_0) > 0$ (recall that x_0 is now a weak unit). In particular, S^* is strictly positive on σ-order continuous functionals.
- In Corollary 2.29, the functional x_0^* is now order continuous because $x_0^* = x_1^* + \cdots + x_r^*$ and $x_1^*, \dots, x_r^*$ are order continuous.
- Proposition 2.30 remains valid for S. For S^* we can only say that if there is a σ-order continuous eigenfunctional x^* for $r(S)$ then $r(S_Y) = r(S)$. Apply Lemma 0.21(ii).

Next, we consider finitely generated semigroups. Let T be a band irreducible σ-order continuous operator such that $r(T)$ is a pole of $R(\cdot, T)$. To apply our results, we need to require that $\overline{\mathbb{R}^+ T}$ consists of order continuous operators. For this purpose, it suffices to assume that there exists a σ-order continuous functional $x_0^* > 0$ such that $T^*x_0^* = r(T)x_0^*$ (cf. Theorem 1.14). By Lemma 0.30, $r(T) > 0$ and x_0^* is strictly positive. Hence, T is peripherally Riesz by Proposition 1.13 and Lemma 0.24. Assume $r(T) = 1$. Then all the conclusions of Proposition 2.34 remain valid. In particular, the Frobenius part of Theorem 1.2 follows.

Finally, we can also extend Proposition 2.36 as follows (it can also be viewed as an extension of [1, Corollary 9.34]).

Lemma 2.40. *Let S and T be two commuting non-zero positive σ-order continuous operators. If T is band irreducible then S is strictly positive.*

Proof. Suppose not, suppose $Sx = 0$ for some $x > 0$. Without loss of generality, $\|T\| < 1$, so that $z := \sum_{n=0}^{\infty} T^n x$ exists. Clearly, $Tz \leq z$. It follows that B_z in invariant under T and, therefore, $B_z = X$. On the other hand, we have $Sz = 0$, so that S vanishes on B_z, so $S = 0$; a contradiction. $\square$

Proposition 2.41. *Suppose that S and K are two non-zero commuting positive operators such that K is compact, σ-order continuous and band irreducible. Then there exists a weak unit $x_0 \in X_+$, a strictly positive functional x_0^*, and a positive real λ such that $Sx_0 = \lambda x_0$, $S^*x_0^* = \lambda x_0^*$, $Kx_0 = r(K)x_0$, and $K^*x_0^* = r(K)x_0^*$. Furthermore,* $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda$ *and* $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K)$ *whenever $x > 0$.*

Proof. Let $\mathscr{S} = \overline{\mathbb{R}^+}\{S, K\}$. Then $\mathscr{S}$ is commutative and band irreducible. By Lemma 2.40, K is strictly positive. Thus, by [1, Lemma 9.30] that all the operators in $\mathscr{S}$ are order continuous. Hence, all the results of Sections 2.3 and 2.4 apply with the modifications described in 2.39. The rest of the proof is exactly as in Proposition 2.36 with the only exception that, instead of being quasi-interior, x_0 is now a weak unit. $\square$

Remark 2.42. Using Corollary 2.25, which remains valid for band-irreducible semigroups if x^* is σ-order continuous, we can show, as in Remark 2.37(ii), that $\liminf_n \|S^{*n}x^*\|^{\frac{1}{n}} \geq \lambda$ and $\lim_n \|K^{*n}x^*\|^{\frac{1}{n}} = r(K)$ whenever $x^* > 0$ is σ-order continuous.

Remark 2.43. As in Proposition 2.36, the result can be extended to any commutative semigroup of σ-order continuous operators containing a band irreducible operator and a non-zero compact operator. Indeed, by Lemma 2.40, the compact operator is strictly positive, so that all the operators in the semigroup are order continuous by [1, Lemma 9.30]. Now we can apply results of Sections 2.3 and 2.4.

Chapter 3

Positive Operators with Irreducible Super-commutants

3.1 Introduction

The results in this chapter are mainly taken from [20]. Throughout this chapter, X stands for a real Banach lattice with $\dim X > 1$.

For a positive operator T, recall from [1] that its **super right-commutant** is defined by $[T\rangle := \{S \geq 0 : ST \geq TS\}$ and its **super left-commutant** is defined by $\langle T] := \{S \geq 0 : ST \leq TS\}$. The main goal of this chapter is to study spectral properties of positive operators which have irreducible super-commutants.

Let $\{K\}' := \{S \in L(X) : SK = KS\}$ be the commutant of K. We would like to mention the following variant of de Pagter's theorem.

Theorem 3.1 ([1, Theorem 9.19])**.** *Let $K > 0$ be a compact operator. If $\lim_n \|K^n x\|^{\frac{1}{n}} = 0$ for some $x > 0$, then $L_+(X) \cap \{K\}'$ is ideal reducible.*

This can be viewed as a special case of [14, Theorem 4.3].

Theorem 3.2 ([14])**.** *Let $K > 0$ be AM-compact (i.e. K maps order intervals to relatively compact subsets of X). If $\lim_n \|K^n x\|^{\frac{1}{n}} = 0$ for some $x > 0$, then $[K\rangle$ is ideal reducible.*

A natural question following Theorem 3.2 is that whether we can replace the super right-commutant of K with the super left-commutant. Generally, the answer is no. For example, take $X = \ell_2$ and let L and R be the left and right shifts on X, respectively. Then $Le_1 = 0$ where $e_1 = (1, 0, 0, \cdots)$; in particular, $\lim_n \|L^n e_1\|^{\frac{1}{n}} = 0$. Since the order intervals in ℓ_2 are compact, L is AM-compact. Put $T = L + R$. Then it is straightforward verifications that $T \in \langle L]$ and T is ideal irreducible; in particular, $\langle L]$ is ideal irreducible.

However, it is surprising that the question has an affirmative answer when K is compact. In this case, we can actually prove that if either the super right-commutant or super left-commutant of K is ideal irreducible, then $\lim_n \|K^n x\|^{\frac{1}{n}} = r(K) > 0$ for all $x > 0$ and every operator semi-commuting with K commutes with it; in particular, $[K\rangle = \langle K] = L_+(X) \cap \{K\}'$.

These results follow from a Jentzsch-Perron-Frobenius Theorem for such operators K. Recall from Proposition 2.34 that an ideal irreducible peripherally Riesz operator T acts as a positive scalar multiple of a cyclic permutation on some positive disjoint vectors $x_1, \ldots, x_r$ which span the peripheral spectral subspace of T. We prove in Section 3.2 that similar results remain valid for peripherally Riesz operators which have ideal irreducible super commutants (cf. Remark 3.6 and Theorem 3.5). In particular, they hold for compact operators with ideal irreducible super commutants.

It deserves mentioning that although the super commutants are always $\overline{\mathbb{R}^+}$-closed semigroups, the results of Chapter 2 generally do not apply to them. This is simply because the minimal-rank projections in the super commutants may fail to have the same range. For example, one can consider the identity on $\mathbb{R}^n$ ($n \geq 2$). The super left/right commutant is the set of all positive matrices and is clearly irreducible. But it is easily seen that all the band projections P_i onto $\mathrm{Span}\{e_i\}$ ($1 \leq i \leq n$) are minimal-rank projections in the super commutants with different ranges.

In Section 3.3, we establish some Jentzsch-Perron theorems for operators related to irreducible operators and compact operators. In particular, we prove that the operators $S > 0$ in the following three chains have positive eigenvectors: $T \leftrightarrow K \leftrightarrow S$, $S \leftrightarrow T \leftrightarrow K$ and $T \leftrightarrow S \leftrightarrow K$, where $T > 0$ is ideal irreducible, $K > 0$ is compact, and $\leftrightarrow$ stands for commutation. In the second and third cases, we can actually choose the eigenvectors to be quasi-interior points. This implies, in particular, that $r(S) > 0$. Therefore, it extends [1, Corollary 9.28]:

Theorem 3.3 ([1])**.** *Let $T, S, K > 0$ be such that $T \leftrightarrow S \leftrightarrow K$, T is ideal irreducible and K is compact. Then $r(S) > 0$.*

In Section 3.4, we include the band irreducible analogues.

3.2 A Jentzsch-Perron-Frobenius theorem

Note that, for a positive operator T, the super commutants $[T\rangle$ and $\langle T]$ are both (multiplicative) semigroups containing the identity.

We also need the following fact. Suppose $T, S > 0$ are semi-commuting. If $TS \leq ST$ then $(TS)^n \leq S^n T^n$ for all n; if $ST \leq TS$ then $(ST)^n \leq T^n S^n$ for all n. Thus, in either case, we have $r(TS) = r(ST) \leq r(T)r(S)$.

Lemma 3.4. *Let $K > 0$ be a compact operator such that either $[K\rangle$ or $\langle K]$ is ideal irreducible. Then $r(K) > 0$.*

Proof. Suppose first that $[K\rangle$ is ideal irreducible and $r(K) = 0$. Let $\mathcal{K}$ be the algebraic ideal generated by K in the semigroup $[K\rangle$. Then each member in $\mathcal{K}$ is of the form $S_1 K S_2$ for some $S_i \in [K\rangle$. It is clear that $S_1 K S_2 \leq S_1 S_2 K$. Moreover, it follows from the remarks preceding this lemma that $S_1 S_2 \in [K\rangle$ and $r(S_1 S_2 K) \leq r(K) r(S_1 S_2) = 0$. Therefore, $r(S_1 K S_2) = 0$. It follows that $\mathcal{K}$ consists of quasi-nilpotent compact operators. Hence, $\mathcal{K}$ is ideal reducible by Theorem 2.2 and, therefore, so is $[K\rangle$ by Proposition 0.32. This contradicts our assumption. Similar arguments work for the other case. $\square$

For the rest of this section, we assume that K is a non-zero compact positive operator such that either $[K\rangle$ or $\langle K]$ is ideal irreducible. Then $r(K) > 0$ by the preceding lemma. **We scale K so that $r(K) = 1$.**

Theorem 3.5. *Let P be the spectral projection of K for $\sigma_{per}(K)$.*

(i) There exist a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that

$$Kx_0 = x_0 \ \text{ and } \ K^*x_0^* = x_0^*.$$

(ii) There exist disjoint positive vectors $\{x_i\}_1^r$ and disjoint positive functionals $\{x_i^*\}_1^r$, where $r = \operatorname{rank}(P)$, such that

$$P = \sum_1^r x_i^* \otimes x_i; \quad x_i^*(x_j) = \delta_{ij}, \ \forall\, i, j.$$

(iii) $K|_{PX}$ is a permutation on $\{x_i\}_1^r$ and $K^*|_{P^*X^*}$ is a permutation on $\{x_i^*\}_1^r$. In particular, there exists $m \geq 1$ such that $P = \lim_n K^{nm}$.

Proof. Suppose first that $[K\rangle$ is ideal irreducible. We apply Proposition 0.8 to K. We claim that the nilpotent case in Proposition 0.8 is impossible. Indeed, otherwise, $c_jK^{n_j}$ converges to a non-zero finite-rank nilpotent operator N for some $n_j \uparrow \infty$ and positive reals $c_j \downarrow 0$. Clearly, N is positive and compact. Thus, Lemma 3.4 implies that $[N\rangle$ is ideal reducible. It is easy to verify that $[K\rangle \subset [N\rangle$. Hence, $[K\rangle$ is also ideal reducible, a contradiction. This proves the claim. Using Proposition 0.8 again, we have $P = \lim_j K^{n_j}$ for some $n_j \uparrow \infty$. In particular, $P > 0$. It follows from 0.25 that the range PX is a lattice subspace of X with lattice operations $x \overset{*}{\wedge} y = P(x \wedge y)$ and there exist positive *-disjoint vectors $x_i \in PX$ $(i = 1, \ldots, r)$ and $x_i^* \in X_+^*$ such that $P = \sum_1^r x_i^* \otimes x_i$ and $x_i^*(x_j) = \delta_{ij}$.

Being a spectral subspace, PX is invariant under K. Note that $K|_{PX}$ is positive on the lattice subspace PX. Moreover, it follows from $P = \lim_j K^{n_j}$ that $I|_{PX} = \lim_j (K|_{PX})^{n_j}$. Thus, $(K|_{PX})^{-1} = \lim_j (K|_{PX})^{n_j - 1}$ is also positive on PX. It is well known that a positive operator on $\mathbb{R}^r$ has a positive inverse if and only if it is a weighted permutation on the standard basis with positive weights, if and only if it is a direct sum of weighted cyclic permutations with positive weights. Since $\sigma(K|_{PX}) = \sigma_{per}(K) \subset \{z \in \mathbb{C} : |z| = 1\}$, it is easily seen that after appropriately scaling the basis vectors x_i's, $K|_{PX}$ is a permutation on x_i's. We accordingly scale x_i^*'s so that we still have $x_i^*(x_j) = \delta_{ij}$ and $P = \sum_1^r x_i^* \otimes x_j$. Note that $Kx_j = KPx_j = PKx_j = \sum_{i=1}^r x_i^*(Kx_j)x_i$ and that $K^*x_j^* = K^*P^*x_j^* = P^*K^*x_j^* = \sum_{i=1}^r x_j^*(Kx_i)x_i^*$. Hence, the matrix of $K^*|_{P^*X^*}$ relative to $\{x_i^*\}$ is the transpose of the matrix of $K|_{PX}$ relative to $\{x_i\}$. It follows that $K^*|_{P^*X^*}$ is a permutation on x_i^*'s. Put $x_0 = \sum_1^r x_i$ and $x_0^* = \sum_1^r x_i^*$. It is clear that $Kx_0 = x_0$ and $K^*x_0^* = x_0^*$.

Since $K|_{PX}$ is a permutation on x_i's, we can take $m \geq 1$ such that $(K|_{PX})^m = I|_{PX}$. Denote by Q the spectral projection of K for $\sigma(K) \backslash \sigma_{per}(K)$. Then $r(K|_{QX}) < 1$. Thus $(K|_{QX})^n \to 0$ as $n \to \infty$. It follows that $K^{mn} = (K|_{PX})^{mn} \oplus (K|_{QX})^{mn} \to I|_{PX} \oplus 0 = P$ as $n \to \infty$.

Since $P = \sum_1^r x_i^* \otimes x_i$, it is easy to see that P is strictly positive if and only if x_0^* is strictly positive and that P^* is strictly positive if and only if x_0 is quasi-interior. We now prove that both P and P^* are strictly positive. It is easy to verify that the null ideal $\text{Null}(P) := \{x \in X : P|x| = 0\}$ is a closed ideal invariant under $[P\rangle$. Since $[K\rangle \subset [P\rangle$, we know $\text{Null}(P)$ is also invariant under $[K\rangle$. From this it follows easily that $\text{Null}(P) = \{0\}$. Thus, P is strictly positive, and so is x_0^*. Now for any $T \in [K\rangle$, we have $x_0^*((TK - KT)x_0) = x_0^*(TKx_0) - (K^*x_0^*)(Tx_0) = 0$. By strict positivity of x_0^*, we have $KTx_0 = TKx_0 = Tx_0$. Thus, $Tx_0 \in \ker(1-K) \subset PX \subset \overline{I_{x_0}}$. This implies that $\overline{I_{x_0}}$ is invariant under $[K\rangle$, hence $\overline{I_{x_0}} = X$. It follows that x_0 is quasi-interior and thus P^* is strictly positive.

Since P is strictly positive, it follows from $0 = x_i \overset{*}{\wedge} x_j = P(x_i \wedge x_j)$ that $x_i \perp x_j$ whenever $i \neq j$. We now prove that x_i^*'s are disjoint. Indeed, by Riesz-Kantorovich formulas, for $i \neq j$,

$$0 \leq (x_i^* \wedge x_j^*)(x_0) = \inf_{0 \leq u \leq x_0} \{x_i^*(u) + x_j^*(x_0 - u)\} \leq x_i^*(x_j) + x_j^*(x_0 - x_j) = 0.$$

Thus, $(x_i^* \wedge x_j^*)(x_0) = 0$, yielding that $x_i^* \wedge x_j^* = 0$ since x_0 is quasi-interior.

Now assume that $\langle K]$ is ideal irreducible. We shall apply similar arguments. In fact, we only need to modify the proof of strict positivity of P and P^*. It is easy to verify that the ideal $I_{PX} \neq \{0\}$ is invariant under $\langle P]$ and thus is invariant under $\langle K]$. Therefore, $\overline{I_{PX}} = X$. On the other hand, we clearly have $I_{PX} = I_{x_0}$. Hence, $\overline{I_{x_0}} = X$. It follows that x_0 is quasi-interior and thus P^* is strictly positive. We claim that x_0^* is strictly positive. Suppose, otherwise, $x_0^*(x) = 0$ for some $x > 0$. Then $x_i^*(x) = 0$ for $1 \leq i \leq r$. Note that, for any $T \in \langle K]$, $\langle (K^*T^* - T^*K^*)x_0^*, x_0 \rangle = x_0^*(TKx_0) - x_0^*(KTx_0) = 0$. Hence, $K^*T^*x_0^* = T^*K^*x_0^* = T^*x_0^*$. In particular, $T^*x_0^* \in \ker(1-K^*) \subset P^*X^* = \mathrm{Span}\{x_i^*\}_1^r$. It follows that $x_0^*(Tx) = (T^*x_0^*)(x) = 0$ for any $T \in \langle K]$. By Proposition 0.32, $\langle K]$ is ideal reducible, a contradiction. It follows that x_0^* and P are both strictly positive. $\square$

Remark 3.6. Note that the operator K in Theorem 3.5 can be replaced with a peripherally Riesz operator $R > 0$. The same proof goes along. One can also deduce Proposition 2.34 from Theorem 3.5. Indeed, let $T > 0$ be an ideal irreducible peripherally Riesz operator with $r(T) = 1$. It suffices to prove that

$T|_{PX}$ is a cyclic permutation on the disjoint vectors x_i's. Suppose, otherwise, $T|_{PX}$ has a cycle of length $m < r$. Without loss of generality, assume that $T|_{PX}$ has a cycle on $x_1, \ldots, x_m$. Then the closed ideal $\overline{I_{\{x_1,\ldots,x_m\}}}$ is non-zero and invariant under T. Since $\overline{I_{\{x_1,\ldots,x_m\}}}$ is disjoint from x_r, it is proper. This contradicts ideal irreducibility of T.

Modifying the following example, one can see that for the operator K in Theorem 3.5, $K|_{PX}$ can be an arbitrary permutation.

Example 3.7. Consider $T = \begin{bmatrix} 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \\ 1&1&1&1 \end{bmatrix}$ and $K = \begin{bmatrix} 0&1&& \\ 1&0&& \\ &&1&0 \\ &&0&1 \end{bmatrix}$. Then $K \leftrightarrow T$ and T is irreducible (in particular, $[K\rangle$ and $\langle K]$ are both irreducible). Note also that the peripheral spectral projection of K is the identity and thus $K|_{PX} = K$.

The following corollary shows that K has nice local behaviors.

Corollary 3.8. *For any $x > 0$ and $x^* > 0$, we have*

$$0 < \inf_n \|K^n x\| \leq \sup_n \|K^n x\| < \infty,$$

$$0 < \inf_n \|K^{n*} x^*\| \leq \sup_n \|K^{n*} x^*\| < \infty.$$

In particular, we have $\lim_n \|K^n x\|^{\frac{1}{n}} = 1$ for all $x > 0$ and $\lim_n \|K^{n*} x^*\|^{\frac{1}{n}} = 1$ for all $x^* > 0$.

Proof. Pick any $x > 0$. Since P is strictly positive, we have $Px > 0$. Therefore, $K|_{PX}$ being a permutation implies that

$$0 < \liminf_n \big\|(K|_{PX})^n Px\big\| \leq \limsup_n \big\|(K|_{PX})^n Px\big\| < \infty. \tag{3.1}$$

Let Q be the spectral projection of K for $\sigma(K) \backslash \sigma_{per}(K)$. Since $(K|_{QX})^n \to 0$, it follows from (3.1) and $K^n x = (K|_{PX})^n Px + (K|_{QX})^n Qx$ that

$$0 < \liminf_n \|K^n x\| \leq \limsup_n \|K^n x\| < \infty.$$

This implies, in particular, that K is strictly positive. Therefore, we have

$$0 < \inf_n \|K^n x\| \leq \sup_n \|K^n x\| < \infty.$$

Taking the n-th root, we have $\lim_n \|K^n x\|^{\frac{1}{n}} = 1$ for all $x > 0$. The dual case follows from a similar argument. $\square$

The following is immediate by Theorem 3.5 and Lemma 0.20.

Corollary 3.9. *Every operator semi-commuting with K commutes with K. In particular,*

$$[K\rangle = \langle K] = L_+(X) \cap \{K\}'.$$

We remark that the ideal irreducible case in Theorem 1.24 is an immediate consequence of Corollary 3.9.

Corollary 3.10. *For any $0 < S \leftrightarrow K$, there exist $\lambda_S \geq 0$, $x > 0$ and $x^* > 0$ such that*

$$Sx = \lambda_S x \ \ \textit{and} \ \ S^* x^* = \lambda_S x^*.$$

Proof. Since $SP = PS$, the lattice subspace PX is invariant under S. It is straightforward verifications that the matrix of $S|_{PX}$ relative to $\{x_i\}$ is $(x_i^*(Sx_j))_{i,j}$ and the matrix of $S^*|_{P^*X^*}$ relative to $\{x_i^*\}$ is $(x_i^*(Sx_j))_{j,i}$, the transpose of $(x_i^*(Sx_j))_{i,j}$. Since both matrices are positive, they have positive eigenvectors for the spectral radius. It follows that there exist $0 < x \in PX$ and $0 < x^* \in P^*X^*$ such that $Sx = r(S|_{PX})x$ and $S^*x^* = r(S|_{PX})x^*$. $\square$

The following example shows that we can not expect $\lambda_S > 0$.

Example 3.11. Let K be as in Example 3.7. Put $S = \begin{bmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} & \\ & \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}$. Then $S \leftrightarrow K$, but S is nilpotent.

3.3 More Jentzsch-Perron type theorems

Let $T > 0$ be ideal irreducible and $K > 0$ be compact. Corollary 3.10 implies that if $T \leftrightarrow K \leftrightarrow S$ and $S > 0$ then S has a positive eigenvector. We now prove that if $S \leftrightarrow T \leftrightarrow K$ then S also has a positive eigenvector.

The following proposition has been proved in Section 2.5 using semigroup techniques. We now give a direct elementary proof.

Proposition 3.12. *Let $T, K > 0$ be such that T is ideal irreducible, K is compact and $T \leftrightarrow K$. Then there exist $0 < \lambda \leq r(T)$, a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that*

$$Tx_0 = \lambda x_0,\ T^*x_0^* = \lambda x_0^*; \quad Kx_0 = r(K)x_0,\ K^*x_0^* = r(K)x_0^*.$$

Proof. Without loss of generality, assume $\|T\| < 1$. Then $\sum_1^\infty T^n$ is strongly expanding by Lemma 0.27. Hence, so is $\widetilde{K} := (\sum_1^\infty T^n)K(\sum_1^\infty T^n)$. It follows that $\widetilde{K}$ is an ideal irreducible compact operator. By de Pagter's Theorem, $r(\widetilde{K}) > 0$. Applying Theorem 1.1 to $\widetilde{K}$, we obtain a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that

$$\ker\big(r(\widetilde{K}) - \widetilde{K}\big) = \mathrm{Span}\{x_0\} \text{ and } \ker\big(r(\widetilde{K}) - \widetilde{K}^*\big) = \mathrm{Span}\{x_0^*\}.$$

Since $T \leftrightarrow \widetilde{K}$, these one-dimensional spaces $\ker\big(r(\widetilde{K}) - \widetilde{K}\big)$ and $\ker\big(r(\widetilde{K}) -$

$\widetilde{K}^*\big)$ are invariant under T and T^*, respectively. Thus, there exist $\lambda, \delta \in \mathbb{R}$ such that

$$Tx_0 = \lambda x_0 \text{ and } T^* x_0^* = \delta x_0^*.$$

Note that $\delta x_0^*(x_0) = T^* x_0^*(x_0) = x_0^*(Tx_0) = \lambda x_0^*(x_0)$. Hence, $\delta = \lambda$. Since $T > 0$ can not vanish on quasi-interior points, we have $r(T) \geq \lambda > 0$. Since $K \leftrightarrow \widetilde{K}$, a similar argument yields $0 < \mu \leq r(K)$ such that

$$Kx_0 = \mu x_0 \text{ and } K^* x_0^* = \mu x_0^*.$$

By Krein-Rutman Theorem, $r(K)$ is an eigenvalue of K. Thus, $\mu = r(K)$ by Lemma 0.21. $\square$

The following is immediate by Proposition 3.12 and Lemma 0.20.

Corollary 3.13. *Under the conditions of Proposition 3.12, every operator semi-commuting with T commutes with T. In particular,*

$$[T\rangle = \langle T] = L_+(X) \cap \{T\}'.$$

Corollary 3.14. *Under the conditions of Proposition 3.12, for any $S \in \{T\}'$, $Sx_0 = \lambda_S x_0$ for some $\lambda_S \in \mathbb{R}$. If, in addition, $S > 0$, then $r(S) > 0$.*

Proof. By Proposition 3.12 and Lemma 1.9, we have $\ker(\lambda - T) = \operatorname{Span}\{x_0\}$. Since $S \leftrightarrow T$, we know $Sx_0 \in \ker(\lambda - T)$. Thus, $Sx_0 = \lambda_S x_0$ for some $\lambda_S \in \mathbb{R}$. If $S > 0$ then $Sx_0 > 0$ since x_0 is quasi-interior. It follows that $r(S) \geq \lambda_S > 0$. $\square$

Remark 3.15. Suppose X is order continuous. Then for the operator T in Proposition 3.12, we have $\ker(\lambda - T^*) = \operatorname{Span}\{x_0^*\}$ by Proposition 1.10. Thus if $R \in \{T^*\}'$, then $Rx_0^* = \lambda_R x_0^*$ for some $\lambda_R \in \mathbb{R}$. Note also that x_0^* is a weak unit of X^* by Proposition 0.18. Hence, if, in addition, R is non-zero, positive and σ-order continuous, then $\lambda_R > 0$. In particular, $r(R) > 0$.

Remark 3.16. Sirotkin in [44] proved a Lomonosov-type theorem for positive operators on real Banach lattices, which implies that if $T > 0$ is non-scalar, $K > 0$ is compact and $S \leftrightarrow T \leftrightarrow K$, then S has a non-trivial invariant closed subspace; cf. [44, Corollary 2.4]. Proposition 3.12(3.14) implies that, in such a chain, if S is also non-scalar, then either T has a non-trivial invariant closed ideal, or S has a non-trivial *hyperinvariant* closed subspace (namely, the eigenspace of S for λ_S).

We now prove that if $T \leftrightarrow S \leftrightarrow K$ then S also has a positive eigenvector.

Proposition 3.17. *Suppose $T > 0$ is ideal irreducible and $K > 0$ is compact. Then there exist a quasi-interior point $x_0 > 0$ and a strictly positive functional x_0^* such that for any $S \in L(X)$ with $T \leftrightarrow S \leftrightarrow K$, one has*

$$Sx_0 = \lambda_S x_0, \quad S^* x_0^* = \lambda_S x_0^*,$$

for some $\lambda_S \in \mathbb{R}$.

Proof. Without loss of generality, assume $\|T\| < 1$. As before, it is easily seen that $\widetilde{K} := \left(\sum_1^\infty T^n\right) K \left(\sum_1^\infty T^n\right)$ is a compact ideal irreducible operator. Thus by Theorem 1.1, there exist a quasi-interior point $x_0 > 0$ and a strictly positive functional $x_0^* > 0$ such that

$$\ker\left(r(\widetilde{K}) - \widetilde{K}\right) = \operatorname{Span}\{x_0\} \text{ and } \ker\left(r(\widetilde{K}) - \widetilde{K}^*\right) = \operatorname{Span}\{x_0^*\}.$$

Since $S \leftrightarrow \widetilde{K}$, these one-dimensional spaces are invariant under S and S^*, respectively. Now the proposition follows easily. $\square$

The following is immediate by Proposition 3.17 and Lemma 0.21.

Corollary 3.18. *Under the conditions of Proposition 3.17, if $S > 0$, then $r(S) \geq \lambda_S > 0$, $\liminf_n \|S^n x\|^{\frac{1}{n}} \geq \lambda_S$ for any $x > 0$, and $\liminf_n \|S^{n*} x^*\|^{\frac{1}{n}} \geq \lambda_S$ for any $x^* > 0$. Moreover, $\lambda_S = r(S)$ if $r(S)$ is an eigenvalue of either S or S^*.*

Corollary 3.18 clearly improves Theorem 3.3.

Remark 3.19. In Propositions 3.12 and 3.17, the operator K can also be replaced with a peripherally Riesz operator $R > 0$. Simply note that Proposition 0.8 yields a compact operator to take the position of R.

We end this section with a few interesting criteria on ideal reducibility improving Theorems 1.7 and 3.2 when $K > 0$ is compact. The following is immediate by Lemma 3.4, Theorem 3.5 and its corollaries.

Proposition 3.20. *Let $K > 0$ be compact. Then $[K\rangle$ and $\langle K]$ are both ideal reducible if any of the following are satisfied:*

(i) $r(K) = 0$, or $\liminf_n \|K^n x\|^{\frac{1}{n}} < r(K)$ for some $x > 0$, or $\liminf_n \|K^{n*} x^*\|^{\frac{1}{n}} < r(K)$ for some $x^* > 0$;

(ii) there exists $S \in L(X)$ such that either $SK > KS$ or $SK < KS$.

Recall that if $T, S > 0$ are semi-commuting then $r(TS) \leq r(T)r(S)$.

Proposition 3.21. *Suppose $T, K > 0$ are semi-commuting and K is compact. Then T is ideal reducible if any of the following are satisfied:*

(i) $r(K) = 0$, *or* $\liminf_n \|K^n x\|^{\frac{1}{n}} < r(K)$ *for some* $x > 0$, *or* $\liminf_n \|K^{n*} x^*\|^{\frac{1}{n}} < r(K)$ *for some* $x^* > 0$*;*

(ii) $r(TK) = 0$, *or* $\liminf_n \|T^n x\|^{\frac{1}{n}} < \frac{r(TK)}{r(K)}$ *for some* $x > 0$, *or* $\liminf_n \|T^{n*} x^*\|^{\frac{1}{n}} < \frac{r(TK)}{r(K)}$ *for some* $x^* > 0$*;*

(iii) there exists a quasinilpotent operator $S > 0$ *semi-commuting with* T*;*

(iv) there exists $S \in L(X)$ *such that* $ST < TS$ *or* $ST > TS$*.*

Proof. (i) is clear by Proposition 3.20. For the other assertions, we prove by way of contradiction. Assume that T is ideal irreducible. Then $TK = KT$ by Theorem 1.24. It is easy to verify that $TK > 0$. Since $TK \leftrightarrow T$, we have $r(TK) > 0$ by Corollary 3.14. Replacing K with TK in Proposition 3.12, we have

$$Tx_0 = \lambda x_0,\ T^* x_0^* = \lambda x_0^*;\ TKx_0 = r(TK)x_0,\ (TK)^* x_0^* = r(TK)x_0^*.$$

Applying Lemma 0.21 to TK, we have, for any $x > 0$,

$$r(TK) = \lim_n \|(TK)^n x\|^{\frac{1}{n}} \leq r(K) \liminf_n \|T^n x\|^{\frac{1}{n}}.$$

The dual case can be proved similarly. This proves (ii). (iii) follows from Corollaries 3.13 and 3.14. (iv) follows from Corollary 3.13. □

3.4 Band irreducible analogues

Many results in Sections 3.2 and 3.3 remain valid if we replace ideal irreducibility with band irreducibility and assume additionally that all the operators and functionals involved are (σ-)order continuous.

We start with the modifications for Section 3.2. For this purpose, we need a few technical lemmas. The first one is a band irreducible version of Lemma 3.4. The proof is analogous except that instead of Theorem 2.2, we use Proposition 2.38.

Lemma 3.22. *Let $K > 0$ be a compact operator and $\mathscr{S}$ a band irreducible semigroup of order continuous operators containing K. Suppose $\mathscr{S}$ is contained in either $[K\rangle$ or $\langle K]$. Then $r(K) > 0$.*

The second one is a semigroup version of Lemma 1.23.

Lemma 3.23. *Let $N > 0$ be an order continuous nilpotent operator and $\mathscr{S}$ a semigroup of order continuous operators. Suppose $\mathscr{S}$ is contained in either $[N\rangle$ or $\langle N]$. Then $\mathscr{S}$ is band reducible.*

Proof. Suppose $N^k > 0$ but $N^{k+1} = 0$ for some $k \geq 1$. Replacing N with N^k, we may assume $N^2 = 0$. As in the proof of Lemma 1.23, it is straightforward to verify that if $\mathscr{S} \subset [N\rangle$ then $\mathrm{Null}(N)$ is a nontrivial band invariant under $\mathscr{S}$ and if $\mathscr{S} \subset \langle N]$ then the band generated by the range of N is nontrivial and invariant under $\mathscr{S}$. $\square$

The third one handles order continuity of limit operators.

Lemma 3.24. *Suppose that $T \in L_+(X)$ is order-to-norm continuous (i.e. T maps order null nets to norm null nets). Then every operator in $\overline{\mathbb{R}^+ T}$ is order-to-norm continuous (in particular, order continuous).*

Proof. It suffices to consider operators in the asymptotic part of $\overline{\mathbb{R}^+ T}$. Suppose $S = \lim_j b_j T^{n_j}$ for some positive reals (b_j) and strictly increasing (n_j). Pick any net (x_α) such that $x_\alpha \downarrow 0$. By passing to a tail, we may assume that $(x_\alpha) \subset [0,x]$ for some $x > 0$. Fix any $\varepsilon > 0$. Pick j such that $\|S - b_j T^{n_j}\| < \varepsilon$. Then $\|Sx_\alpha\| \le \big\|S - b_j T^{n_j}\big\|\|x_\alpha\| + \big\|b_j T^{n_j} x_\alpha\big\| \le \varepsilon\|x\| + \big\|b_j T^{n_j} x_\alpha\big\|$. Therefore, $\limsup_\alpha \|Sx_\alpha\| \le \varepsilon\|x\|$. It follows that $Sx_\alpha \to 0$. □

The last one is an analogous version of Lemma 1.11. The proof is similar and thus is omitted.

Lemma 3.25. *Let $K > 0$ be order continuous and AM-compact.*

(i) K is order-to-norm continuous.

(ii) If $K^* x^* = \lambda x^*$ for some $\lambda \neq 0$ then x^* is order continuous.

Now we replace the common assumption in Section 3.2 with that **either $[K\rangle$ or $\langle K]$ contains a band irreducible semigroup $\mathscr{S}$ of order continuous operators containing** K. The reason of this assumption is that the super commutants need not consist of order continuous operators even when K is order continuous.

Observe that now we still have $r(K) > 0$ by Lemma 3.22. As before, we scale K so that $r(K) = 1$.

3.26. The other modifications for Section 3.2 are listed below.

- Theorem 3.5 remains valid except that x_0 is now a weak unit and x_0^* is order continuous and strictly positive.

  The proof goes along similar lines. We claim as before that the nilpotent case in Proposition 0.8 is impossible. Indeed, otherwise, N would be a positive nilpotent operator such that $\mathscr{S} \subset [N\rangle$ or $\mathscr{S} \subset \langle N]$ according to $\mathscr{S} \subset [K\rangle$ or $\mathscr{S} \subset \langle K]$, respectively. By Lemma 3.25, K is order-to-norm continuous. Thus, N is order continuous by Lemma 3.24. It follows from Lemma 3.23 that $\mathscr{S}$ is band reducible, a contradiction.

That x_0^* is order continuous follows from $K^*x_0^* = x_0^*$ and Lemma 3.25. All the other arguments work as before except that we can not expect P^* to be strictly positive now. It is only strictly positive on σ-order continuous positive functionals.

- In Corollary 3.8, we need to assume that x^* is σ-order continuous.
- In Corollary 3.9, we only have that every σ-order continuous operator semi-commuting with K commutes with K.

It deserves mentioning that for the super right-commutant we can simply take $\mathscr{S} = [K\rangle$. It follows from the following lemma which is an extension of [1, Lemma 9.30].

Lemma 3.27. *Let $K > 0$ be compact and order continuous. If K is strictly positive (in particular, $[K\rangle$ is band irreducible), then $[K\rangle$ consists of order continuous operators.*

Proof. Let K be strictly positive. Suppose $T \in [K\rangle$. Take $x_\alpha \downarrow 0$ and let y be such that $Tx_\alpha \geq y \geq 0$ for all α. By Lemma 3.25, K is order-to-norm continuous. Thus, $\|Kx_\alpha\| \to 0$ and therefore, $\|TKx_\alpha\| \to 0$. It follows that $\|KTx_\alpha\| \to 0$. Therefore, $Ky = 0$. Since K is strictly positive, we have $y = 0$. This proves T is order continuous. It remains to show that if $[K\rangle$ is band irreducible then K is strictly positive. Indeed, otherwise, $\mathrm{Null}(K)$ would be a nontrivial invariant band of $[K\rangle$. $\square$

The modifications for Section 3.3 are very minor and are listed below.

3.28. • In Proposition 3.12, we need to assume that $T > 0$ is σ-order continuous and band irreducible and $K > 0$ is σ-order continuous and compact. Now x_0 is a weak unit and x_0^* is strictly positive and order continuous.

- In Corollary 3.13, we only have that every σ-order continuous operator semi-commuting with T commutes with T.

- In Corollary 3.14, the "in addition" case extends as long as S is σ-order continuous.

- In Proposition 3.17, we need to assume that $T > 0$ is σ-order continuous and band irreducible and $K > 0$ is σ-order continuous and compact. Now x_0 is a weak unit and x_0^* is strictly positive and order continuous.

- In Corollary 3.18, we need to assume that x^* is σ-order continuous. For the dual case in the "moreover" part, we only have that if $r(S)$ is an eigenvalue of S^* with a σ-order continuous eigenfunctional then $\lambda_S = r(S)$.

  Note that S is automatically order continuous in this case. Indeed, it is immediate by [1, Lemma 9.30] since S commutes with the σ-order continuous, strictly positive and compact operator $\widetilde{K}$.

- In Proposition 3.20, we replace the super-commutants by a semigroup of order continuous operators which contains K and is contained in one of the super-commutants. In (i), we assume $x^* > 0$ is σ-order continuous. In (ii), we assume S is σ-order continuous.

- In Proposition 3.21, T is band reducible if all the operators and functionals involved are assumed to be σ-order continuous.

Bibliography

[1] Y. A. Abramovich and C. D. Aliprantis. *An invitation to operator theory*, volume 50 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw. On the spectral radius of positive operators. *Math. Z.*, 211(4):593–607, 1992.

[3] Y. A. Abramovich, C. D. Aliprantis, and O. Burkinshaw. Invariant subspaces of operators on l_p-spaces. *J. Funct. Anal.*, 115(2):418–424, 1993.

[4] E. A. Alekhno. Spectral properties of band irreducible operators. In *Positivity IV—theory and applications*, pages 5–14. Tech. Univ. Dresden, Dresden, 2006.

[5] E. A. Alekhno. The irreducibility in ordered Banach algebras. *Positivity*, 16(1):143–176, 2012.

[6] C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006. Reprint of the 1985 original.

[7] F. Aràndiga and V. Caselles. Approximations of positive operators and continuity of the spectral radius. *J. Operator Theory*, 26(1):53–71, 1991.

[8] J. Bernik, L. W. Marcoux, and H. Radjavi. Spectral conditions and band reducibility of operators. *J. Lond. Math. Soc. (2)*, 86(1):214–234, 2012.

[9] J. Bračič, R. Drnovšek, Y. B. Farforovskaya, E. L. Rabkin, and J. Zemánek. On positive commutators. *Positivity*, 14(3):431–439, 2010.

[10] V. Caselles. On the peripheral spectrum of positive operators. *Israel J. Math.*, 58(2):144–160, 1987.

[11] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1990.

[12] B. de Pagter. Irreducible compact operators. *Math. Z.*, 192(1):149–153, 1986.

[13] R. Drnovšek. Triangularizing semigroups of positive operators on an atomic normed Riesz space. *Proc. Edinburgh Math. Soc. (2)*, 43(1):43–55, 2000.

[14] R. Drnovšek. Common invariant subspaces for collections of operators. *Integral Equations Operator Theory*, 39(3):253–266, 2001.

[15] R. Drnovšek and M. Kandić. Ideal-triangularizability of semigroups of positive operators. *Integral Equations Operator Theory*, 64(4):539–552, 2009.

[16] R. Drnovšek and M. Kandić. More on positive commutators. *J. Math. Anal. Appl.*, 373(2):580–584, 2011.

[17] Roman Drnovšek. Once more on positive commutators. *Studia Math.*, 211(3):241–245, 2012.

[18] J. Flores, F. L. Hernández, and P. Tradacete. Powers of operators dominated by strictly singular operators. *Q. J. Math.*, 59(3):321–334, 2008.

[19] N. Gao. Extensions of Perron-Frobenius theory. *Positivity*, to appear.

[20] N. Gao. On commuting and semi-commuting positive operators. *Proceeding of the American Mathematical Society*, to appear.

[21] N. Gao and V. G. Troitsky. Irreducible semigroups of positive operators on Banach lattices. *Linear and Multilinear Algebras*, to appear.

[22] J. J. Grobler. Band irreducible operators. *Nederl. Akad. Wetensch. Indag. Math.*, 48(4):405–409, 1986.

[23] J. J. Grobler. Spectral theory in Banach lattices. In *Operator theory in function spaces and Banach lattices*, volume 75 of *Oper. Theory Adv. Appl.*, pages 133–172. Birkhäuser, Basel, 1995.

[24] D. W. Hadwin, A. K. Kitover, and M. Orhon. Strong monotonicity of spectral radius of positive operators. *preprint.*

[25] S. Karlin. Positive operators. *J. Math. Mech.*, 8:907–937, 1959.

[26] A. K. Kitover. A generalized Jentzsch theorem. *Positivity*, 9(3):501–509, 2005.

[27] A. Levin. *Extension of the Perron-Frobenius theorem to semigroups of positive operators on order-continuous Banach lattices. Master's thesis.* University of Alberta, Deptartment of Mathematical and Statistical Sciences, 2009.

[28] H. P. Lotz. Über das Spektrum positiver Operatoren. *Math. Z.*, 108:15–32, 1968.

[29] I. Marek. A contribution to the Frobenius theory of positive operators. Comparison theorems. *Math. Systems Theory*, 4:46–59, 1970.

[30] I. Marek. Frobenius theory of positive operators: Comparison theorems and applications. *SIAM J. Appl. Math.*, 19:607–628, 1970.

[31] P. Meyer-Nieberg. *Banach lattices*. Universitext. Springer-Verlag, Berlin, 1991.

[32] V. Müller. *Spectral theory of linear operators and spectral systems in Banach algebras*, volume 139 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, second edition, 2007.

[33] F. Niiro. On indecomposable operators in $l_p\,(1 < p < \infty)$ and a problem of H. Schaefer. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 14:165–179, 1964.

[34] F. Niiro. On indecomposable operators in $L_p\,(1 < p < \infty)$ and a problem of H. H. Schaefer. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 16:1–24, 1966.

[35] F. Niiro and I. Sawashima. On the spectral properties of positive irreducible operators in an arbitrary Banach lattice and problems of H. H. Schaefer. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 16:145–183, 1966.

[36] H. Radjavi. The Perron-Frobenius theorem revisited. *Positivity*, 3(4):317–331, 1999.

[37] H. Radjavi and P. Rosenthal. *Simultaneous triangularization*. Universitext. Springer-Verlag, New York, 2000.

[38] I. Sawashima. On spectral properties of some positive operators. *Natur. Sci. Rep. Ochanomizu Univ.*, 15:53–64, 1964.

[39] I. Sawashima. On spectral properties of positive irreducible operators in $C(S)$ and a problem of H. H. Schaefer. *Natur. Sci. Rep. Ochanomizu Univ.*, 17:1–15, 1966.

[40] H. H. Schaefer. Some spectral properties of positive linear operators. *Pacific J. Math.*, 10:1009–1019, 1960.

[41] H. H. Schaefer. *Topological vector spaces*. Springer-Verlag, New York, 1971. Third printing corrected, Graduate Texts in Mathematics, Vol. 3.

[42] H. H. Schaefer. *Banach lattices and positive operators.* Springer-Verlag, New York, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 215.

[43] H. H. Schaefer. On theorems of de Pagter and Ando-Krieger. *Math. Z.*, 192(1):155–157, 1986.

[44] G. Sirotkin. A version of the Lomonosov invariant subspace theorem for real Banach spaces. *Indiana Univ. Math. J.*, 54(1):257–262, 2005.