MAXIMAL ABELIAN K-DIAGONALIZABLE SUBGROUPS OF REDUCTIVE GROUPS

by

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Abstract

Given an algebraic group G over a field k and a k-algebra R, the role of maximal abelian k-diagonalizable subalgebras (MAD for short) of $G(R)$ is the same as that the split maximal torus play in $G(k)$. Let G be a reductive group such that the derived subgroup is simply connected and let $\mathrm{Spec}(R)$ be a connected reduced affine scheme. This dissertation is to studying conjugacy problems related to MADs in $G(R)$. First, we provide the conjugacy theorem for regular MADs. For arbitrary MADs, the conjugacy theorem does not exist. But we give the structure of MADs in the classical groups of type A, B, C and D.

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Mathematics is both an art and a science, and pure mathematics lies at its heart. Pure math has been described as ”that part of mathematical activity that is done without explicit or immediate consideration of direct application.” Completing my PhD degree in pure math is probably the most challenging and exciting activity of my first 30 years of my life. I am deeply indebted to my supervisor, Professor Jochen Kuttler, whose patient, encouragement, and support inspired me throughout my graduate studies. I sincerely appreciate all the long discussions we had every week, where I also finally saw mathematics' marrow. He introduced me to the beautiful bridge between scheme theory and group theory, which is the heart of algebraic geometry. I also owe a great debt of gratitude to my co-supervisor, Professor Arturo Pianzola, who I not only get to learn from his wisdom but also his view of science and life.

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Contents

Notation

$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
$\mathbb{Z}_m$	the finite cyclic group $\mathbb{Z}/m\mathbb{Z}$
$\mathbb{Q}$	the field of rational numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{R}$	the field of real numbers
δ_{ij}	the Kronecker symbol, which is 1 if $i = j$ and is 0 if $i \neq j$
$\mathrm{Mat}_n(R)$	the set of $n \times n$-matrices with coefficients in the ring R
$\mathrm{Spec}(R)$	the spectrum of R

Chapter 1

Introduction

The origins of Lie theory are geometric and stem from the view of Felix Klein (1849-1925) that geometry of space is determined by the group of its symmetries. Lie algebras were introduced to study the concept of infinitesimal transformations. The term "Lie algebra" after Sophus Lie (1842-1899) was introduced by Hermann Weyl in the 1930s. Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over a field k of characteristic zero. Maximal diagonalizable subalgebras play a crucial role in understanding $\mathfrak{g}$ and its representations. From the work of Cartan and Killing we know that $\mathfrak{g}$ can be classified in terms of the weights of the adjoint representation, the so-called root system. At first it seems as though the root system may depend on the choice of split Cartan subalgebra, but it is in fact an invariant of $\mathfrak{g}$. Consequently Chevalley gave us the conjugacy theorem of split Cartan subalgebras:

Theorem 1.1 ([Che41]). All split Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action of $G(k)$, where G is the simply connected Chevalley-Demazure group corresponding to $\mathfrak{g}$.

Infinite dimensional Lie algebras emerged in the study of theoretical physics in the 1960s. They turned out to be one of the most useful mathematical tools to describe supersymmetric phenomena. V. Kac and R. Moody generalized the theory of finite dimensional simple Lie algebras to the infinite dimensional setting(see [Kac94] and [Moo68]). Let R be a k-algebra. We are interested in the infinite dimensional Lie algebra $\mathfrak{g} \otimes R$, which could be viewed as an infinite dimensional algebra over k. The element $p \in \mathfrak{g} \otimes R$ is called k-diagonalizable if $\mathrm{ad}_{\mathfrak{g}\otimes R}(p)$, when viewed as a k-linear endomorphism of $\mathfrak{g} \otimes R$), is diagonalizable. We call $\mathcal{M}$ a MAD of $\mathfrak{g} \otimes R$ if it is a maximal abelian k-diagonalizable subalgebra. In infinite dimensional Lie theory (for example, in the case of Kac-Moody Lie algebras) these type of subalgebras do play the role that the split Cartan subalgebras play in the classical theory. This is our motivation for asking when all MADs of $\mathfrak{g} \otimes R$ are conjugate under some suitable subgroup of $\mathrm{Aut}_k(\mathfrak{g} \otimes R)$. A well understood example is the case of the ring $R = k[t_1^{\pm 1}, ..., t_n^{\pm 1}]$. When $n = 1$ it is the Laurent polynomials and $\mathfrak{g} \otimes R$ is the so called loop algebra of $\mathfrak{g}$. This is interesting since Kac's loop construction reveals the key structure of affine Kac-Moody algebras as the universal central extensions (with a central element and a derivation) of loop algebras based on finite dimensional simple Lie algebras over $\mathbb{C}$(see Chapter 7,8 of [Kac94]). The appropriate version of conjugacy for this case is to be found in the work of Peterson and Kac (see [PK83]). The general case follows by an induction argument due to P. Gille. It is important in the construction of toroidal Lie algebras (see [Pia00] and [Pia02]). Pianzola gives us the conjugacy theorem over a more general algebra. Here is his result.

Theorem 1.2 (Theorem 1 of [Pia04])**.** *Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over k, and $\mathfrak{G}$ its simply connected Chevally-Demazure group scheme. Let $\mathfrak{X} = \operatorname{Spec}(R)$ be a connected affine scheme and $\mathfrak{X}_{red}$ the corresponding reduced scheme where R is an associative commutative unital k-algebra. Assume that $\mathfrak{X}(k) \neq 0$. Then*

(a) If the Picard group of $\mathfrak{X}_{red}$ is trivial then all regular maximal abelian k-diagonalizable subalgebras of $\mathfrak{g} \otimes R$ are conjugate under $\mathfrak{G}(R)$.

(b) Consider the following property on $\mathfrak{X}$.

(TLT) If $\mathfrak{L}$ is the Levi subgroup of a standard parabolic subgroup of $\mathfrak{G}$, then any locally trivial principal homogeneous space for $\mathfrak{L}$ over $\mathfrak{X}_{red}$ is trivial.

If (TLT) holds, then all maximal abelian k-diagonalizable subalgebras of $\mathfrak{g} \otimes R$ are regular (and hence all conjugate by (a)).

After proving the conjugacy theorem of MADs in Lie algebra case, it is natural to consider MADs in linear algebraic groups. Before studying MADs in linear algebraic groups, we should understand linear algebraic groups.

The interest of linear algebraic groups appeared when to establish a Galois theory for systems of differential equations by quadratures at the end of 19th century (S. Lie, E. Study, E. Picard, L. Maurer). S. Lie observed that some classical methods of integration of ordinary differential equations could be unified on the existence of a one-parameter group of transformations keeping the system invariant. In the middle of the 20th century C. Chevalley and A. Borel are central players in the next great development of linear algebraic groups in arbitrary characteristics. At the beginning, Chevalley's attempt didn't go very far since he was tied of the exponential map. But the work [Borel56] of Borel, which used global methods based on algebraic geometry, changed the picture dramatically. After Borel's work, Chevalley went forward to the classification of semisimple algebraic groups over algebraically closed field and their representations up to isomorphism [Che05]. This classification is analogous to the classification of Cartan-Killing of complex semisimple Lie algebras. The classification of Chevalley is based on the fact that in a semi-simple algebraic group one can construct analogues to the elements of the theory of Cartan-Killing, namely the Cartan subgroups, roots, etc. An important role played here are Borel subgroups and maximal tori.

In the case of an arbitrary field k, a k-split torus is a torus that is k-isomorphic to a direct product of one-dimensional groups $GL_1(k)$ which are defined over k. For a compact, connected algebraic group G, we are interested in the problem of determining its irreducible representations by considering a torus T of G that is as large as possible and relating the representation theory of G (which we don't understand) and that of T (which we do understand well). This will involve the space G/T of T-cosets of G. The geometry and topology of G/T is quite interesting and is central to the problem of finding the representations of G. The conjugacy theorem of maximal tori over an algebraically closed field was developed by Borel and Humphreys (see Section 11 of [Borel91] and Section 21 of [Hum87] for details). The needs of Lie theory, number theory, and finite group theory led to the development of a theory of reductive groups over any perfect field. Problems over local and global function fields provided motivation to remove the perfectness assumption. It is a difficult theorem that in any smooth connected affine group G over any field k, all maximal k-split tori are conjugate under $G(k)$. This is Section 20.9 of [Borel91] for reductive G.

Theorem 1.3 (Corollary 11.3 of [Borel91])**.** Let G be a smooth connected affine group over a field k. Then any two split maximal tori in G are conjugate by an element in $G(k)$.

One should note that if $k \neq k_s$ then typically there are many $G(k)$-conjugacy classes of maximal k-tori. For example, if $G = GL_n$ then the maximal k-tori in G are in a one-to-one correspondence with the maximal finite étale commutative k-subalgebras of $\mathrm{Mat}_n(k)$. In particular, two maximal k-tori are $G(k)$-conjugate if and only if the corresponding maximal finite étale commutative k-subalgebras of $\text{Mat}_n(k)$ are $GL_n(k)$-conjugate. Hence, if such k-subalgebras are not abstractly k-isomorphic then their corresponding maximal k-tori are not $G(k)$-conjugate. For example, non-isomorphic degree-n finite separable extension fields of k yield such algebras.

Let G be an affine algebraic group over k and R be a k-algebra. Put $\mathscr{G} = X \times G$ which is an affine group scheme over $X = \text{Spec}(R)$. Let $R[\mathscr{G}]$ be the coordinate ring of $\mathscr{G}$ which is considered as k-algebra. Given an element g of $\mathscr{G}(R)$, we say g is k-diagonalizable if g acts on $R[\mathscr{G}]$ k-diagonalizably (consider $R[\mathscr{G}]$ as a comdule of $\mathscr{G}$). We call $\mathcal{M} \subset \mathscr{G}(R)$ a MAD if it is a maximal abelian k-diagonalizable subgroup. The purpose of this dissertation is to study conjugacy questions of MADs. In Chapter 2, we will provide a brief review of the general theory of affine group schemes and principal bundles, which is part of the preliminaries required for our subsequent discussions. In Chapter 3, we will state some main results of the paper [Pia04] by Arturo Pianzola regarding MADs in Lie algebras, which is the inspiration and idea for this dissertation. Chapter 4 give us a toy example for conjugacy questions related to MADs in a reductive algebraic group over the polynomial ring $\mathbb{C}[X]$ with a variable X. Chapter 5 is the main body of this dissertation. In this chapter, let G be a reductive algebraic group over k such that its derived group is simply connected and let $\text{Spec}(R)$ be a connected reduced affine scheme. We will define regular MADs, connected MADs and finite MADs of $\mathscr{G}(R)$ respectively. Then we provide the conjugacy theorem for regular MADs and connected MADs and state an important property of finite maximal k-diagonalizable subgroups. Finally, we will give the structure of MADs in the classical groups of type A, B, C and D. Here are the main results:

Theorem 1.4. *Let G be a reductive algebraic group over k such that its derived group is simply connected. Let $X = \text{Spec}(R)$ be a connected reduced affine scheme, and $\mathscr{G} = G \times X$ its group scheme over X.*

(i) If Pic(X) is trivial, then all regular maximal abelian k-diagonalizable subgroups of $\mathscr{G}(R)$ are conjugate under $\mathscr{G}(R)$ (Theorem 5.23).

Consider the following property on X:

$(\mathbf{TLT})$*If $\mathscr{L}$ is the Levi subgroup of a standard parabolic subgroup of $\mathscr{G}$, then any locally trivial principal homogeneous space for $\mathscr{L}$ over X is trivial.*

(ii) Assume the above property holds, we have (Theorem 5.41)

- *if G is a group of type A, then all maximal abelian k-diagonalizable subgroups of $\mathscr{G}(R)$ are connected (and hence all conjugate by Theorem 5.28),*
- *if G is a group of type B, C, D, then any maximal abelian k-diagonalizable subgroup $\mathcal{M}$ is conjugate to the group of the form $T(k) \cdot \Gamma$ where T is a torus of G and Γ is a finite group such that $ZZ(\Gamma) = \Gamma$ in $Z_G(T)$.*

Chapter 2

Affine group scheme

This chapter is a review of the basic definitions and general theory of linear algebraic groups. Throughout this chapter, k denotes a commutative field of characteristic 0, K an algebraically closed extension of k, k_s(resp. $\bar{k}$) the separable (resp. algebraic) closure of k in K. In this chapter, all algebraic groups are affine, unless explicitly stated otherwise.

2.1 Basic Theory of Affine Group Schemes

We denote by R**-rng** the category of commutative associative unital R–algebras. We are interested in *R-functors*, that are covariant functors from R**-rng** to the category of sets. If X is an R-scheme, it defines a covariant functor

$$h_X : \ R\textbf{-rng} \to \textbf{Set}, \ \ S \mapsto X(S) := \mathrm{Hom}_R(\mathrm{Spec}(S), X).$$

Theorem 2.1 (Theorem 1.2 of [Wat79]). *Let F be an R-functor. If the elements in $F(S)$ correspond to solutions in S of some family of equations, there is a R-algebra A and natural correspondence between $F(S)$ and Hom$_R(A, S)$. The converse also holds.*

Such F is called *representable* by A.

Theorem 2.2. *(Yoneda's Lemma) Let E and F be functors representable by R-algebras A and B. The natural maps $E \to F$ correspond to R-algebra homomorphisms $B \to A$.*

Now we can define an affine group scheme over R.

Definition 2.3. Let R be a commutative ring. An *affine group scheme* $\mathbf{G}$ over R is a group object in the category of affine R-schemes:

$$\mathbf{G} : R\textbf{-rng} \to \textbf{grp}, \tag{2.1.1}$$

where **grp** is the category of groups. $\mathbf{G}$ is representable by $R[\mathbf{G}]$, which is called the coordinate ring of $\mathbf{G}$ (i.e., for any $S \in R$-**rng**, $\mathbf{G}(S) = \mathrm{Hom}_R(R[\mathbf{G}], S)$.

There are natural maps: a multiplication $m : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$, a unit $\epsilon : \mathrm{Spec}(R) \to \mathbf{G}$, and an inverse $\sigma : \mathbf{G} \to \mathbf{G}$ such that the following diagrams commute:

Associativity:

$$\begin{tikzcd} \mathbf{G} \times \mathbf{G} \times \mathbf{G} \arrow[r, "m \times id"] \arrow[d, "id \times m"'] & \mathbf{G} \times \mathbf{G} \arrow[d, "m"] \\ \mathbf{G} \times \mathbf{G} \arrow[r, "m"] & \mathbf{G} \end{tikzcd}$$

Unit:

$$\begin{tikzcd} \mathrm{Spec}(R) \times \mathbf{G} \arrow[r, "\epsilon \times id"] \arrow[d, equal] & \mathbf{G} \times \mathbf{G} \arrow[d, "m"] \\ \mathbf{G} \arrow[r, equal] & \mathbf{G} \end{tikzcd}$$

Inverse:

$$\begin{tikzcd} \mathbf{G} \arrow[r, "\sigma \times id"] \arrow[d] & \mathbf{G} \times \mathbf{G} \arrow[d, "m"] \\ \mathrm{Spec}(R) \arrow[r, "\epsilon"] & \mathbf{G} \end{tikzcd}$$

$\mathbf{G}$ is called commutative if the following diagram commutes

$$\begin{tikzcd} \mathbf{G} \times \mathbf{G} \arrow[r, "\text{switch}"] \arrow[d, "m"'] & \mathbf{G} \times \mathbf{G} \arrow[d, "m"] \\ \mathbf{G} \arrow[r, equal] & \mathbf{G} \end{tikzcd}$$

(switch: $(g, h) \mapsto (h, g)$).

By Yoneda's Lemma, the group structure on $\mathbf{G}$ is translated to a coassociative Hopf algebra structure on $R[\mathbf{G}]$. The corresponding maps are the comultiplication $\Delta = m^* : R[\mathbf{G}] \to R[\mathbf{G}] \otimes R[\mathbf{G}]$, the counit $\epsilon^* : R[\mathbf{G}] \to R$, and the coinverse $\sigma^* : R[\mathbf{G}] \to R[\mathbf{G}]$ such that the following diagrams commute:

Coassociativity:

$$\begin{tikzcd} R[\mathbf{G}] \otimes R[\mathbf{G}] \otimes R[\mathbf{G}] & R[\mathbf{G}] \otimes R[\mathbf{G}] \arrow[l, "\Delta \otimes id"'] \\ R[\mathbf{G}] \otimes R[\mathbf{G}] \arrow[u, "id \otimes \Delta"] & R[\mathbf{G}] \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

Counit:

$$\begin{tikzcd} R \otimes R[\mathbf{G}] & R[\mathbf{G}] \otimes R[\mathbf{G}] \arrow[l, "\epsilon^* \otimes id"'] \\ R[\mathbf{G}] \arrow[u, equal] \arrow[r, equal] & \mathbf{G} \arrow[u, "\Delta"'] \end{tikzcd}$$

Coinverse:

$$\begin{tikzcd} R[\mathbf{G}] & R[\mathbf{G}] \otimes R[\mathbf{G}] \arrow[l, "(\sigma^*, id)"'] \\ R \arrow[u] & R[\mathbf{G}] \arrow[l, "\epsilon^*"'] \arrow[u, "\Delta"'] \end{tikzcd}$$

For an object S in R-**rng**, we call the elements of $\mathbf{G}(S)$ the S–points of $\mathbf{G}$. A *homomorphism* $\mathbf{G} \to \mathbf{H}$ of affine group schemes is a family of homomorphisms $f(S) : \mathbf{G}(S) \to \mathbf{H}(S)$ of groups, indexed by the R-algebras S, such that, for every homomorphism $\alpha : S \to S'$ of R-algebras, the diagram

$$\begin{tikzcd} \mathbf{G}(S) \arrow[r, "f(S)"] \arrow[d, "\mathbf{G}(\alpha)"'] & \mathbf{H}(S) \arrow[d, "\mathbf{H}(\alpha)"] \\ \mathbf{G}(S') \arrow[r, "f(S')"] & \mathbf{H}(S') \end{tikzcd}$$

commutes. So it is a morphism of affine schemes preserving δ, ϵ, σ in the obvious way.

Example 2.4. Let R be a ring and S an object in R-**rng**. We present several affine group schemes by describing their S–points and their coordinate rings.

(i) The multiplicative group scheme $\mathbf{G}_m$:
$\mathbf{G}_m(S) = S^\times$ is the group of multiplicative units in S.
$R[\mathbf{G}_m] \cong R[t^{\pm 1}]$.

(ii) The additive group scheme $\mathbf{G}_a$:
$\mathbf{G}_a(S) = S$ is viewed as a group under addition.
$R[\mathbf{G}_a] \cong R[t]$.

(iii) The general linear group $\mathbf{GL}_n$ for $n \geqslant 1$:
$\mathbf{GL}_n(S)$ is the group of invertible $n \times n$–matrices with entries in S.
$R[\mathbf{GL}_n] = R[x_{ij}, \det(x_{ij})^{-1}]_{1 \leqslant i,j \leqslant n}$.

(iv) The special linear group $\mathbf{SL}_n$ for $n \geqslant 1$:
$\mathbf{SL}_n(S)$ is the group of $n \times n$–matrices with entries in S and determinant 1.
$R[\mathbf{SL}_n] = R[x_{ij}]_{1 \leqslant i,j \leqslant n} / \langle \det(x_{ij}) - 1 \rangle$.

(v) The orthogonal group $\mathbf{O}_n$ for $n \geqslant 1$:
$\mathbf{O}_n(S) = \{A \in \mathrm{Mat}_n(S) | AA^T = I_n\}$, where A^T is the transpose of A and I_n is the $n \times n$ identity matrix.
$R[\mathbf{O}_n] = R[x_{ij}]_{1 \leqslant i,j \leqslant n} / \langle \sum_{j=1}^{n} x_{il}x_{jl} - \delta_{ij} | 1 \leqslant i, j \leqslant n \rangle$.

(vi) The special orthogonal group $\mathbf{SO}_n$ for $n \geqslant 1$:
$\mathbf{SO}_n(S) = \{A \in \mathrm{Mat}_n(S) | \det A = 1, AA^T = I_n\}$,
$R[\mathbf{SO}_n] = R[\mathbf{O}_n] / \langle \det(x_{ij}) - 1 \rangle$.

(vii) The group scheme $\boldsymbol{\mu}_n$ of the n-th roots of unity for $n \geqslant 1$:
$\boldsymbol{\mu}_n(S) = \{a \in S | a^n = 1\}$.
$R[\boldsymbol{\mu}_n] = R[t] / \langle t^n - 1 \rangle$.

Example 2.5. Let Γ be a finite group. We define an R-scheme $\boldsymbol{\Gamma} := \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)$ (the disjoint union of $\mathrm{Spec}(R)$). The group structure on Γ induces a group scheme structure on $\boldsymbol{\Gamma}$ with multiplication

$$\boldsymbol{\Gamma} \times \boldsymbol{\Gamma} = \bigsqcup_{(\gamma, \gamma') \in \Gamma^2} \mathrm{Spec}(R) \to \boldsymbol{\Gamma} = \bigsqcup_{\gamma \in \Gamma} \mathrm{Spec}(R)$$

induced by mapping the component (γ, γ') to the one in the desired by $\gamma\gamma'$. The corresponding coordinate ring is $R^{\Gamma} \cong R \times R \times ... \times R$.

Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be affine group schemes over R. The affine group scheme $\mathbf{G}_1 \times \mathbf{G}_2$ over R is defined to be the functor

$$R \rightsquigarrow \mathbf{G}_1(R) \times \mathbf{G}_2(R).$$

The coordinate ring is

$$R[\mathbf{G}_1 \times \mathbf{G}_2] = R[\mathbf{G}_1] \otimes R[\mathbf{G}_2],$$

since for any R-algebra S_1, S_2, S_3, we have

$$\mathrm{Hom}_{R-\mathrm{alg}}(S_1 \otimes_R S_2, S_3) = \mathrm{Hom}_{R-\mathrm{alg}}(S_1, S_3) \times \mathrm{Hom}_{R-\mathrm{alg}}(S_2, S_3).$$

Let $\mathbf{G}_1$ $\mathbf{G}_2$ and $\mathbf{H}$ be affine group schemes over R. Let $\mathbf{G}_1 \to \mathbf{H}$ and $\mathbf{G}_2 \to \mathbf{H}$

be homomorphisms. The *fibre product* $\mathbf{G}_1 \times_{\mathbf{H}} \mathbf{G}_2$ is defined as

$$R \rightsquigarrow \mathbf{G}_1(R) \times_{\mathbf{H}(R)} \mathbf{G}_2(R)$$

and has the coordinate ring

$$R[\mathbf{G} \times_{\mathbf{H}} \mathbf{G}] = R[\mathbf{G}] \otimes_{R[\mathbf{H}]} R[\mathbf{G}].$$

We choose our base ring R somewhat arbitrarily, requiring only that the defining equations make sense in R. Suppose we take a ring homomorphism $R \to S$. An S-algebra S' can be regarded as an R-algebra through $R \to S \to S'$ and an S-algebra homomorphism becomes an R-algebra homomorphism for this structure. An affine group scheme $\mathbf{G}$ of R-algebras "restricts" to an affine group

$$\mathbf{G}_S : S' \to \mathbf{G}(S')$$

of S-algebras whose coordinate ring is $S[\mathbf{G}]$ since

$$\mathrm{Hom}_S(S \otimes R[\mathbf{G}], S') \cong \mathrm{Hom}_R(R[\mathbf{G}], S').$$

Let Y and Y' be subschemes of R-group scheme $\mathbf{G}$. The *transporter* on S-schemes from Y to Y' is defined to be

$$\mathrm{Trans}_{\mathbf{G}}(Y, Y') : S \rightsquigarrow \{g \in \mathbf{G}(S) | gY_S g^{-1} \subseteq Y'_S\}.$$

In the special case, when Y is finitely presented over R, the *normalizer* of Y is defined to be

$$N_{\mathbf{G}}(Y) : S \rightsquigarrow \{g \in \mathbf{G}(S) | gY_S g^{-1} = Y_S\}.$$

Proposition 2.6 (Proposition 2.1.2 of [Con11])**.** *Let Y and Y' be finitely presented closed subschemes of an affine R-group $\mathbf{G}$ of finite presentation. Assume Y is either a multiplicative type subgroup or is finite flat over R. Then $\mathrm{Trans}_{\mathbf{G}}(Y, Y')$ exists as a finitely presented closed subscheme of $\mathbf{G}$. In particular, the normalizer $N_{\mathbf{G}}(Y)$ exists as a finitely presented closed subscheme of $\mathbf{G}$.*
If $\mathbf{G}$ is smooth and Y and Y' are both multiplicative type subgroup of $\mathbf{G}$, then these subschemes are smooth.

In classical group theory, a morphism between two groups is an isomorphism if and only if it is bijective. But in general, this is not the case for algebraic groups.

A bijective morphism between two irreducible affine groups over k need not be an isomorphism. Let us first consider the following example.

Example 2.7. Let $H := \{(X,Y) \in k^2 | X^3 = Y^2\}$. Then $H \subseteq k^2$ is irreducible. Define $f : k \to H$ by $t \mapsto (t^2, t^3)$. This is a bijective morphism, but $f^*(k[H]) = k[T^2, T^3] \subsetneqq k[T]$ and hence this is not an isomorphism.

What then in addition to being bijective is required for a morphism to be an isomorphism? The answer to this question involves two basic definitions: birational equivalence and normal varieties.

Let X and Y be irreducible affine varieties over k and let $f : X \to Y$ be a dominant morphism. Since $k[X]$ and $k[Y]$ are integral domains, they have a corresponding field of fractions, denoted by $k(X)$ and $k(Y)$, respectively. Hence f^* induces a k-algebra homomorphism $k(X) \to k(Y)$. The morphism f is said to be a *birational equivalence* if f^* induces an isomorphism of fields $k(X) \cong k(Y)$. Clearly, if $f : X \to Y$ is an isomorphism, then f is a birational equivalence. The above example shows that the converse need not true. If X is an irreducible affine variety, then X is said to be *normal* if $k[X]$ is integrally closed in its field of fractions.

Hence we have the following proposition:

Proposition 2.8 (Theorem 5.2.8 of [Springer98]). *Let $f : X \to Y$ be a bijective and birational morphism of irreducible affine varieties and assume Y is normal, then f is an isomorphism.*

If G and G' are algebraic groups over k, a map $f : G \to G'$ is a homomorphism of algebraic groups if f is a morphism of varieties and a group homomorphism. Then f is an *isomorphism* of algebraic groups if f is an isomorphism of varieties and a group isomorphism.

2.2 Properties of algebraic groups over k

In this section, we consider only affine group schemes over a field k. Let G be an affine group scheme over k.

Proposition 2.9 (Proposition 21.9 of [KMRT98]). *Let G be an algebraic group over k and let $A = k[G]$. The following are equivalent:*

(i) A_L *is reduced for any field extension* L/k.

(ii) A_K *is reduced.*

(iii) $\mathit{dim}_k(\mathit{Lie}(G)) = \mathit{dim}(G)$.

If k *is perfect, these conditions are also equivalent to*

(iv) A *is reduced.*

An algebraic group G is said to be *smooth* if it satisfies the conditions of Proposition of 2.9.

Proposition 2.10 (Proposition 21.10 of [KMRT98])**.** *Let* G *be an algebraic group over* k.

(i) G_L *is smooth if and only if* G *is smooth, where* L/k *is a field extension.*

(ii) *If* G_1, G_2 *are smooth, then* $G_1 \times G_2$ *is smooth.*

(iii) *If* $\mathit{char}(k) = 0$, *all algebraic groups are smooth.*

(iv) *An algebraic group is smooth if and only if its connected component* G° *is smooth.*

A *character* of G is a homomorphism of algebraic groups from G to $\mathbf{G}_m$. This is equivalent to giving a homomorphism of k-algebras $k[t, t^{-1}] \to k[G]$ which respects comultiplications, which, in turn, is equivalent to specifying an element u of the coordinate ring $k[G]$ such that u is a unit in $k[G]$ and $\Delta(u) = u \otimes u$. Such elements are said to be *group-like*. Therefore, $\lambda \leftrightarrow u(\lambda)$ is a one-to-one correspondence between the characters of G and the group-like elements of $k[G]$. The set $X(G)$ of characters of G is an abelian group. Indeed, the product of two characters of G is a character of G, the inverse of a character of G is a character of G, and characters of G commute with each other.

Let M be a finitely generated commutative group. The functor

$$R \rightsquigarrow \mathit{Hom}(M, R^*)$$

is an algebraic group $\mathbf{D}(M)$ with coordinate ring:

$$k[M] = \{\Sigma_{m \in M} a_m m | a_m \in k\}.$$

An algebraic group G is said to be *diagonalizable* if G is isomorphism to $\mathbf{D}(M)$. Equivalently, G is diagonalizable if its coordinate ring is spanned (as a k-vector space) by its group-like elements. If G is diagonalizable, say $G = \mathbf{D}(M)$, then it is possible to recover M from G as the set of group-like elements in $k[G]$, i.e., $M = X(G)$.

Recall that $\mathbf{D}_n$ is the group of invertible diagonal $n \times n$-matrices; thus

$$\mathbf{D}_n \cong \mathbf{G}_m \times ... \times \mathbf{G}_m \cong \mathbf{D}(\mathbb{Z}^n).$$

A finite-dimensional representation $(V;\rho)$ of an affine group G is diagonalizable if and only if there exists a basis for V such that $\rho(G) \subset \mathbf{D}_n$. More precisely, the representation defined by an inclusion $G \subset \mathbf{GL}_n$ is diagonalizable if and only if there exists an invertible matrix p in $M_n(k)$ such that, for all k-algebras R and all $g \in G(R)$,

$$pgp^{-1} \in \left\{ \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}.$$

A character $\lambda : G \to \mathbf{G}_m$ defines a representation of G on any finite-dimensional vector space V : let $g \in G(R)$ act on V_R via multiplication by $\lambda(g) \in R^*$. For example, define a representation of G on k^n by

$$g \to \begin{bmatrix} \lambda(g) & & 0 \\ & \ddots & \\ 0 & & \lambda(g) \end{bmatrix}.$$

Let $(V;\rho)$ be a representation of G. We say that G acts on V through λ if

$$\rho(g) \cdot v = \lambda(g)v \text{ all } g \in G(R), \quad v \in V_R.$$

This means that the image of ρ is contained in the centre $\mathbf{G}_m$ of $\mathbf{GL}(V)$ and that ρ is the composite of

$$G \xrightarrow{\lambda} \mathbf{G}_m \hookrightarrow GL(V).$$

Let $\pi : V \to V \otimes k[G]$ be the action defined by ρ. Then G acts on V through the character λ if and only if

$$\pi(v) = v \otimes u(\lambda), \text{ for all } v \in V,$$

where $u(\lambda)$ is the group-like element in $k[G]$ corresponding to λ. When V is 1-dimensional, $\mathbf{GL}(V) = \mathbf{G}_m$, and so G always acts on V through some character. Let $(V;\rho)$ be a representation of G. If G acts on subspaces W and W' through the character λ, then it acts on $W \oplus W'$ through the character λ. Therefore, for each $\lambda \in X(G)$, there is a largest subspace V_λ (possibly zero) such that G acts on V_λ through λ . We have

$$V_\lambda = \{v \in V \mid \pi(v) = v \otimes u(\lambda)\}.$$

Those λ for which $V_\lambda \neq 0$ are the *weights* of G in V. In particular, let $T \subset G$ be a diagonalizable subgroup. Suppose T acts on G, then T acts on $\mathfrak{g} = \mathrm{Lie}(G)$ by $g \mapsto tgt^{-1}$, and the set $\Phi(T, G)$ of non-zero weights of T in $\mathfrak{g}$ is called the set of *roots* of G relative to T.

Proposition 2.11 (Theorem 4.7 XIV of [Mil12])**.** *The following conditions on an affine group G are equivalent:*

(i) G is diagonalizable,

(ii) Every finite-dimensional representation of G is diagonalizable,

(iii) Every representation of G is diagonalizable,

(iv) For every representation $(V;\rho)$ of G, we have

$$V = \bigoplus_{\lambda \in X(G)} V_\lambda.$$

Corollary 2.12 (Corollary 8.4 of [Borel91])**.** *Suppose G is diagonalizable. Then the same is true of each subgroup of G and of the image of G under any morphism.*

Definition 2.13 (Section 7.2 of [Wat79])**.** An algebraic group G is called a *torus* if G_{k_s} is a finite product of copies of $\mathbf{G}_m$. We call a torus *split* if it is diagonalizable, or in other words, the Galois action on the character group is trivial. At the other extreme, a torus is called *anisotropic* if there is no nontrivial map to $\mathbf{G}_m$, or equivalently the identity is the only fixed element in the character group.

A *one-parameter* subgroup of T is a homomorphism of algebraic groups from G_m to T. The set $X^*(T)$ of one-parameter subgroups is an abelian group. If $T \cong G_m$, then $X(T) = X^*(T)$ is just the set of maps $x \to x^r$, as r varies over $\mathbb{Z}$. In general, $T \cong G_m^d$ for some positive integer d, so $X(T) \cong X(G_m)^d \cong \mathbb{Z}^d \cong X^*(T)$. We have a nondegenerate pairing

$$\langle \cdot, \cdot \rangle : X(T) \times X^*(T) \to \mathbb{Z} \tag{2.2.1}$$

given by

$$\langle \chi, \lambda \rangle = r \quad \textit{where} \;\; \chi \circ \lambda(x) = x^r, x \in G_m. \tag{2.2.2}$$

Proposition 2.14 (Proposition 8.7 of [Borel91])**.** *Let G be diagonalizable and split over k. Then G is a direct product $G = G^\circ \times F$, where F is a finite group and G° is a torus defined as split over k.*

There exist unique tori T_{spl} and T_{an} of T, both defined over k, such that $T = T_{spl} \cdot T_{an}$, where T_{spl} is a k-split subgroup generated by $\{im(\lambda) | \lambda \in X(T)_k\}$ and T_{an} is k-anisotropic subgroup which is the identity component of $\cap_{\chi \in X(T)_k} ker(\chi)$.

Example 2.15. (i) Let T be the subgroup of $GL_n(\mathbb{C})$ consisting of diagonal matrices in $GL_n(\mathbb{C})$. Then T is a $\mathbb{R}$-split $\mathbb{R}$-torus.

(ii) Let T be the closed subgroup of $GL_2(\mathbb{C})$ defined by

$$T = \Big\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a, b \in \mathbb{C}, \; a^2 + b^2 \neq 0 \Big\}.$$

Then T is an $\mathbb{R}$-torus and is $\mathbb{R}$-anisotropic.

An algebraic group G is said to be *unipotent* if every nonzero representation of G has a nonzero fixed vector, or equivalently, if it is isomorphic to a closed subgroup of the group of upper triangular matrices with diagonal entries 1.

The *derived subgroup* $D(G)$ of G is the subgroup generated by all the commutators of the group. It is a normal subgroup of G and G/DG is the largest commutative quotient of G. An algebraic group G is said to be *solvable* if its derived series

$$G \supset DG \supset D^2G \supset ...$$

terminates with 1.

Remark 2.16. Let $\mathbf{G}$ be a reductive group scheme over a ring R. There is a unique semisimple closed normal R-subgroup $D(\mathbf{G}) \subset \mathbf{G}$ such that $\mathbf{G}/D(\mathbf{G})$ is a torus. Moreover, $D(\mathbf{G})$ represents the fppf-sheafification of the commutator subfunctor $S \rightsquigarrow [\mathbf{G}(S), \mathbf{G}(S)]$ on the category of R-schemes.

Quotients and extensions of unipotent groups are unipotent. Therefore, any maximal normal unipotent subgroup of G contains all other unipotent subgroups i.e., it is the (unique) largest normal unipotent subgroup of G. Similar statements hold for "solvable". Hence, among the smooth connected normal subgroups of an algebraic group G, there is a largest solvable one, called the *radical* $R(G)$ of G, and a largest unipotent one, called the *unipotent radical* $R_u(G)$ of G. The radical (resp. unipotent radical) of G_K is called the geometric radical (resp. geometric unipotent radical) of G.

A smooth connected algebraic group is *semisimple* if its geometric radical is trivial. A smooth connected algebraic group is *reductive* if its geometric unipotent radical is trivial. For example, GL_n is reductive, but is not semisimple because its center has one-dimensional torus of all scalar matrices, whereas SL_n is semisimple.

Proposition 2.17 (Theorem 5.1, Chapter XVII of [Mil12])**.** *If G is reductive, then its radical $R(G)$ is a torus. The centre $Z(G)$ of G is a group of multiplicative type whose largest subtorus is $R(G)$. The derived group $D(G)$ of G is semisimple and has the centre $Z(G) \cap D(G)$. Furthermore, $G = R(G) \cdot D(G)$. Therefore G is the almost-direct product of a torus $R(G)$ and a semisimple group $D(G)$:*

$$1 \to R(G) \cap D(G) \to R(G) \times D(G) \to G \to 1.$$

For example, the centre of SL_n is μ_n and its radical is 1. The centre of GL_n and its radical both equal D_n, its derived group is SL_n, where D_n is the diagonal matrix group. Thus the sequence is

$$1 \to \mu_n \to D_n \times SL_n \to GL_n \to 1.$$

Let G be a reductive group over k where k is an algebraically closed field of characteristic 0. We think of G as a $G \times G$-variety with the usual action given by $(g_1, g_2)h := g_1 h g_2^{-1}$. The simple $G \times G$-modules are of the form $V \otimes W$ where V, W are simple G-modules. In particular, $V \otimes V^*$ is a simple $G \times G$-module which is canonically isomorphic to End(V):

$$V \otimes V^* \cong \text{End}(V)$$

is induced by $v \otimes \lambda \mapsto f_{v,\lambda}$ where $f_{v,\lambda}(w) = \lambda(w) \cdot v$. The corresponding representation $\rho : G \to GL(V)$ gives a $G \times G$-equivariant morphism $\rho : G \to End(V)$

and thus a $G \times G$-homomorphism $\rho^* : \text{End}(V)^* \to k[G]$ which is injective because $\text{End}(V)$ is simple. It is also clear that for two equivalent representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ we have the same images $\rho_1(\text{End}(V_1)^*) = \rho_2(\text{End}(V_2)^*)$.

For every isomorphism class λ, let V_λ be a simple module of type λ.

Proposition 2.18 (Theorem 3.1.1 of [Pro05])**.** *Let G be a reductive group over k where k is an algebraically closed field of characteristic 0. Then the isotypic decomposition of $k[G]$ as a $G \times G$-module has the form*

$$k[G] = \bigoplus_{\lambda} k[G]_\lambda$$

where $k[G]_\lambda \cong End(V_\lambda)^* \cong V_\lambda \oplus V_\lambda^*$.

By a *maximal torus* of G we mean a torus of G not properly contained in any other torus. A reductive group is called *split* if it contains a split maximal torus. A reductive group over a separably closed field is automatically split, since all tori over such a field are split.

Now we have the following conjugacy theorem for maximal tori:

Theorem 2.19 (Theorem 3.22 of [Mil12])**.** *Let G be a smooth connected algebraic group over K. All maximal tori in G are conjugate by an element of $G(K)$.*

Here we mention two stronger results:

(i) *Let G be a smooth affine algebraic group over a separably closed field k. Then any two maximal tori in G are conjugate by an element of $G(k)$. [Appendix A.2.10 of [CGP10]]*

(ii) Let G be a smooth connected affine group over a field k. Then any two split maximal tori in G are conjugate by an element in $G(k)$. [Appendix C.2.3 of [CGP10]]

One should note that if $k \neq k_s$, then typically there are many $G(k)$-conjugacy classes of maximal k-tori. For example, if $G = GL_n$, then the maximal k-tori in G are in a one-to-one correspondence with the maximal finite étale commutative k-subalgebras of $\text{Mat}_n(k)$. In particular, two maximal k-tori are $G(k)$-conjugate if and only if the corresponding maximal finite étale commutative k-subalgebras of

$\text{Mat}_n(k)$ are $GL_n(k)$-conjugate. Hence, if such k-subalgebras are not abstractly k-isomorphic, then their corresponding maximal k-tori are not $G(k)$-conjugate. For example, non-isomorphic degree-n finite separable extension fields of k yield such algebras.

Let G be a semisimple algebraic group and T be a split maximal torus. Recall the definition of weights of T in a rational representation, the weights in $\mathbb{R} \otimes_{\mathbb{Z}} X(T)$ form a lattice P which contains the root lattice Q as a subgroup of finite index. A semisimple group G is *adjoint* if $X(T) = Q$ and *simply connected* if $X(T) = P$. For example, SL_2 is simply connected and PGL_2 is adjoint.

Proposition 2.20 (Theorem 9.9 of [Stein68])**.** *For a semisimple algebraic group G the following conditions are equivalent.*
(i) The group G is simply connected.
(ii) The centralizer of every semisimple element of G is connected.
(iii) Every two commuting semisimple elements of G are contained in a maximal torus.

In general, if we do not have the condition of triviality of the fundamental group, we cannot guarantee that the centralizer is connected. For example: let $t = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in SL_2(\mathbb{C})$ where $i \in \mathbb{C}^* \backslash \{\pm 1\}$. Let $\bar{t}$ be the image of t under the quotient map $\phi : SL_2(\mathbb{C}) \to PGL_2(\mathbb{C})$. Hence $\bar{t}$ is semisimple in $PGL_2(\mathbb{C})$. Suppose $Z_G(\bar{t})$ is connected, then its preimage in $SL_2(\mathbb{C})$ must be connected. Let $g \in Z_G(\bar{t})$. Considering the preimage $\tilde{g}$ of g, we have $\tilde{g}t = \pm t\tilde{g}$, it means $\tilde{g} \in N_{SL_2(\mathbb{C})}(t)$.

2.3 Principal G-bundles and principal homogeneous spaces for G

Throughout this section, we denote by G an affine algebraic group over k.

Definition 2.21. A *principal bundle under G* is a morphism of schemes $\pi : E \to X$ which satisfies the following conditions:

(i) E is equipped with an action τ of G such that π is G-invariant (G acts on X trivially).

(ii) π is faithfully flat.

(iii) The diagram

$$\begin{tikzcd} G \times E \arrow[r, "p_2"] \arrow[d, "\tau"'] & E \arrow[d, "\pi"] \\ E \arrow[r, "\pi"] & X \end{tikzcd}$$

commutes, where τ denotes the action and p_2 the projection.

In the above definition, conditions (ii) and (iii) may be replaced with:

(iv) For any point $x \in X$, there exists an open subset V of X containing x and a faithfully flat morphism $f : U \to V$ such that the pull-back morphism $E \times_X V \to V$ is isomorphic to the trivial bundle $G \times U \to U$ as a G-scheme over U.

The morphism π is also said to be a locally trivial bundle for the fppf (locally faithfully flat and of finite presentation) topology under our standing assumption of finiteness for schemes. If we assume G is smooth, any such bundle is also locally trivial for the étale topology, i.e., we may replace 'faithfully flat' with 'étale' in condition (iv). But this does not extend to an arbitrary group G.

Given a G-bundle $\pi : E \to X$ and a scheme Y equipped with a G-action, the associated bundle is a scheme W equipped with morphisms $q : E \times Y \to W$ and $\pi_1 : W \to X$ such that the diagram

$$\begin{tikzcd} E \times Y \arrow[r, "q"] \arrow[d, "p_1"'] & W \arrow[d, "\pi_1"] \\ E \arrow[r, "\pi"'] & X \end{tikzcd}$$

commutes, where p_1 is the projection. Thus q is a G-bundle relative to the diagonal action of G on $E \times Y$ by

$$g \cdot (e, y) = (e \cdot g, g^{-1} \cdot y),$$

and hence W is well defined. It is denoted as $E(Y) = E \times^G Y$. The associated bundle need not exist in general.

Any G-equivariant map $\psi : Y_1 \to Y_2$ induces a morphism $E(\psi) : E(Y_1) \to$

$E(Y_2)$. In particular, a section $s : X \to E(Y)$ is given by a morphism

$$s' : E \to Y$$

such that $s'(e \cdot g) = g^{-1} \cdot s'(e)$ and $s(x) = (e, s'(e))$, where $e \in E$ for which $\pi(e) = x$, where $\pi : E \to X$.

Proposition 2.22 (Lemma 6.1.3 of [BSU])**.** *Let $\pi : E \to X$ be a G-bundle, and Y a scheme equipped with an action of G.*

(i) The associated bundle $E(Y)$ exists if Y admits a G-equivariant embedding into the projectivization of a finite-dimensional G-module.

(ii) Given a subgroup $H \subset G$, the morphism π factors as $E \xrightarrow{\phi} Y \xrightarrow{\psi} X$ where ϕ is an H-bundle (obtained from π by the reduction of the structure group), and ψ a smooth morphism with fibres isomorphic to G/H. If H is a normal subgroup, then ψ is a G/H-bundle.

(iii) The set of sections of $E \times^G Y \to X$ is identified with Hom$_G(E, Y)$.

In particular, $E \times^G Y$ exists when Y is affine. For example, if Y is a rational finite-dimensional G-module, then the associated bundle

$$\pi_1 : E(V) := E \times^G V \to X$$

exists and moreover, is a vector bundle on X since the trivial vector bundle is the pull-back of X under the faithfully flat morphism $\pi : X \to Y$.

Remark 2.23. In the case of $G = GL_n$, we identify a principal GL_n-bundle with the associated vector bundle by taking the associated vector bundle for the standard representation. For any GL_n-bundle $E \to X$, we have a corresponding vector bundle $E \times_{GL_n} X \to X$. For any vector bundle $E \to X$, we have a GL_n-bundle $E^* \to X$, where

$$E^* = \{(v_1, v_2, ..., v_n) \in E^n |\, v_1, v_2, ..., v_n \text{ are linearly independent}\}$$

and GL_n acts on E^* by

$$g_{ij} \cdot (v_1, v_2, ..., v_n) = (v_1, v_2, ..., v_n)(g_{ij})^{-1}.$$

Let X be a k-scheme and let G be an algebraic k-group. A *G-torsor* over X (also called a *principal homogeneous space for G* over X) is an X-scheme Y on which $G_X := G \times_{\mathrm{Spec}(k)} X$ acts on the right and that is locally isomorphic to G_X for the flat topology of X (with G_X acting on itself by right multiplication), i.e., there exist flat and locally finitely presented morphisms $\psi_i : U_i \to X$ such that $X = \cup \psi_i(U_i)$ and the pullback morphism $Y \times_X U_i \cong G_X \times_X U_i$. Since G is smooth and ψ_i may be taken to be étale, we can consider the isomorphism class of a torsor Y defined as an element of $\mathrm{H}^1_{\text{ét}}(X, G_X)$. This is an isomorphism if X is affine. The set $\mathrm{H}^1_{\text{ét}}(X, G_X)$ is the same as the isomorphism class of the trivial torsor G_X acting on itself by right multiplication. If ψ_i is an open immersions, then we think of the U_i as an open cover of X in the Zariski topology, and the torsor Y over X is said to be locally trivial. Their isomorphism classes are viewed as $\mathrm{H}^1_{\text{Zar}}(X, G_X)$. (For more details see chapter 3, 4 of [DG70])

2.4 Classical groups over k

Let V be a finite dimensional vector space over k. The set of all invertible linear transformations from V to V will be denoted $GL(V)$. This set has a group structure under composition of transformations with identity element the identity transformation $Id(x) = x$ for all $x \in V$. Let $GL_n(k)$ be denote the set of $n \times n$ invertible matrices with coefficients in k. Then $GL_n(k)$ is a group under matrix multiplication with the identity matrix. We observe that if V is an n-dimensional vector space over k with basis $v_1, ..., v_n$, then the map $f : GL(V) \to GL_n(k)$, which send the transformation to the matrix of T with respect to this basis, is a group isomorphism. The group $GL_n(k)$ is called the general linear group of rank n.

Type A_n: Let V be a k-vector space of dimension $n+1$. Let $G = SL(V) = \{g \in GL(V) | \det(g) = 1\}$. If we choose a suitable basis of V, we can identify G with a subgroup of $GL_{n+1}(k)$. The diagonal matrices $T \subset G$ is a split maximal torus in G. The character $\chi_i \in T^*$ is given by

$$\chi_i(\text{dia}(t_1, t_2, ..., t_{n+1})) = t_i, \quad i = 1, 2, ..., n+1$$

and the root system of type A_n is $\{\chi_i - \chi_j | i \neq j\}$. We can show that $SL(V)$ is a simply connected simple group. The kernel of the adjoint representation of G is μ_{n+1}, thus the corresponding adjoint type is $SL(V)/\mu_{n+1} \cong PGL(V)$.

Type B_n: Let V be a k-vector space of dimension $2n+1$ with a regular quadratic form q (i.e. $q(tv) = t^2q(v)$ for all $t \in k$ and $v \in V$). Consider the group $G = SO(V) = \{g \in GL(V)|q(gv) = q(v),\ \forall\, v \in V\ \ \det(g) = 1\}$. The subgroup T of diagonal matrices of the form

$$t = \text{diag}(1, t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})$$

is a split maximal torus of G. The character $\chi_i \in T^*$ is given by $\chi_i(t) = t_i$ and the root system of type B_n is $\{\pm\chi_i, \pm\chi_j \pm \chi_j | i \neq j\}$. Thus, $SO(V, q)$ is a adjoint simple group when $\dim(V) \geq 3$. The corresponding simply connected group is $Spin(V, q)$.

Type C_n: Let V be a k-vector space of dimension $2n$ with a nondegenerate alternating form. Consider the group $G = Sp(V) = \{g \in GL_{2n}(k)|g^tJg = J\}$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. The subgroup of T of diagonal matrices of the form

$$t = \text{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})$$

is a split maximal torus in G. The character $\chi_i \in T^*$ is given by $\chi_i(t) = t_i$ and the root system of type C_n is $\{\pm 2\chi_i, \pm\chi_j \pm \chi_j | i \neq j\}$.

Type D_n: Let V be a k-vector space of dimension $2n$ with a regular quadratic form q. Consider the group $G = SO(V) = \{g \in GL(V)|q(gv) = q(v),\ \forall\, v \in V\ \ \det(g) = 1\}$. The subgroup T of diagonal matrices of the form

$$t = \text{diag}(t_1, ..., t_n, t_1^{-1}, ..., t_n^{-1})$$

is a split maximal torus of G. The character $\chi_i \in T^*$ is given by $\chi_i(t) = t_i$ and the root system of type D_n is $\{\pm\chi_j \pm \chi_j | i \neq j\}$. Thus, $SO(V, q)$ is a semisimple group (simple, if $n > 3$). The corresponding simply connected and adjoint groups are $Spin(V, q)$ and $PGSO(V, q)$.

Example 2.24. The group Pin(V) and Spin(V)

Let (V, q) be a non-singular n-dimensional quadratic space over k of characteristic 0. The *Clifford algebra* of (V, q) is the associated k-algebra

$$C(V, q) = C(q) := T(V)/\langle v \otimes v - q(v) \rangle$$

equipped with the k-linear map $V \to C(V)$ induced by the natural inclusion of V into $T(V)$, the quotient is by the 2-sided ideal in $T(V)$ generated by elements of the form $v \otimes v - q(v)$.

The decomposition

$$C(V) = C_0 \oplus C_1$$

makes $C(V)$ into a $\mathbb{Z}/2\mathbb{Z}$-graded algebra where C_0 is generated by the even products of elements of V and C_1 by the odd products. In particular, C_0 is a subalgebra of $C(V)$, sometimes called the *even Clifford algebra*.

For example: Let $V = k$ and $q(x) = cx^2$ for some $c \in k$. In this case, the tensor algebra $T(V)$ is the commutative 1-variable polynomial ring $k[t]$, with t corresponding to the element $1 \in V \subset T(V)$ and $C(V) = k[t]/(t^2 - c)$ (since $1 \in V$ is a basis and $1 \otimes 1 = q(1) = c$).

We now introduce some important subgroups of the group of units of the Clifford algebra $C(V) = C_0 \oplus C_1$ of (V, q). Let $u \in C^*$ be a unit which is homogeneous, i.e., $u \in C_0^*$ or $u \in C_1^*$. The graded inner automorphism is defined as

$$i_u : C \to C \quad \text{given by} \quad i_u(x) = (-1)^{d(u)d(x)} u x u^{-1}$$

for x homogeneous, $d(u) = 0$ if $u \in C_0$ and $d(u) = 1$ if $u \in C_1$. The *Clifford group* of (V, q) is defined to be

$$\Gamma(V) = \{u \in C^* |\ u \text{ homogeneous and } \ i_u(V) \subset V\}$$

and the subgroup

$$S\Gamma(V) = \{u \in C_0^* \mid i_u(V) \subset V\} = \Gamma \cap C_0$$

is called the *special Clifford group*.

Let σ be the standard involution of C. We define a map

$$\mu : C \to C \ \text{ by } \ \mu(x) = \sigma(x)x \ \text{ for all } x \in C,$$

where $\mu(v) = -q(v)$ for all $v \in V$.

Here the *Pin group* of (V, q) is defined as

$$Pin(V) = \{u \in \Gamma(V) \mid \mu(u) = 1\}$$

and the group

$$Spin(V) = \{u \in S\Gamma(V) \mid \mu(u) = 1\}$$

is called the *Spin group* of (V, q).

If k is a field of characteristic zero, then we have the exact sequences

$$1 \to \{\pm 1\} \to Pin(V) \to O(V) \to 1 \quad \text{and}$$

$$1 \to \{\pm 1\} \to Spin(V) \to SO(V) \to 1.$$

Chapter 3

Conjugacy theorem of MADs in $\mathfrak{g}(R)$

In this chapter we state some results of the paper [Pia04] by Arturo Pianzola regarding the MADs in Lie algebras. Throughout k will denote a field of characteristic zero.

3.1 Definition of maximal abelian k-diagonalizable subalgebra

Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over k. Let R be an associative commutative unital k-algebra and $X = \mathrm{Spec}(R)$. Consider the Lie algebra of the form $\mathfrak{g}(R) := \mathfrak{g} \otimes R$ where $\mathfrak{g}(R)$ is in general viewed as an infinite dimensional algebra over k. A well understood example is the loop algebra $\mathfrak{g}(R)$ of $\mathfrak{g}$, which is involved in the realizations of non-twisted affine Kac-Moody Lie algebras, where $R = k[t, t^{-1}]$ is the ring of Laurent polynomials.

Definition 3.1. A subalgebra $\mathfrak{a}$ of $\mathfrak{g}(R) := \mathfrak{g} \otimes R$ is called a maximal abelian k-diagonalizable subalgebra, or MAD for short, if it satisfies

(i) $\mathfrak{a}$ is abelian.

(ii) All elements of $\mathfrak{a}$ are k-diagonalizable : If p belongs to $\mathfrak{a}$ then $\mathrm{ad}_{\mathfrak{g}(R)}p$, when viewed as a k-linear endomorphism of $\mathfrak{g}(R)$, is diagonalizable.

(iii) No subalgebra of $\mathfrak{g}(R)$ satisfying (i) and (ii) above properly contains $\mathfrak{a}$.

Remark 3.2. Since these type of subalgebras play a crucial role in the understanding of $\mathfrak{g}(R)$ and its representations in both the finite dimensional and affine Kac-Moody case, it is natural and relevant to ask if all MADs of $\mathfrak{g}(R)$ are conjugate under some suitable subgroup of $\mathrm{Aut}_k(g(R))$. The natural choice for this subgroup (because of functionality on R and compatibility with the usual results in the case of a base field), is the group $\mathscr{G}(R)$ of R-points of the corresponding simply connected Chevalley-Demazure group, acting on $\mathfrak{g}(R)$ via the adjoint representation. The answer to this question is quite interesting and is related to the triviality of certain principal homogeneous spaces over $\operatorname{Spec}(R)$.

For the conjugacy theorem of MADs in $\mathfrak{g}(R)$, we should assume the following property:

$(\mathbf{TLT})$ (Triviality of locally trivial Levi torsors): If $\mathfrak{L}$ is the Levi subgroup of a standard parabolic subgroup of $\mathscr{G}$, then any locally trivial principal homogeneous space for $\mathfrak{L}$ over X_{red} is trivial.

3.2 Conjugacy theorem of MADs

Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over k and $\mathscr{G}$ its simply connected Chevalley-Demazure group scheme. Let $X = \operatorname{Spec}(R)$ be a connected affine scheme. If R is a k-algebra the residue field of an element x of $\operatorname{Spec}(R) = X$ will be denoted by $k(x)$. For convenience in what follows the group $\mathscr{G}(k(x))$ will be denoted simply by $\mathscr{G}(x)$, and the corresponding group homomorphism $\mathscr{G}(R) \to \mathscr{G}(R/x) \subset \mathscr{G}(x)$ by $p \mapsto p(x)$. The assumption regarding the existence of a rational point, namely of a maximal ideal x_0 such that $R/x_0 = k$, is central.

Proposition 3.3 (Corollary 6 of [Pia04]). *Let $\mathfrak{h}$ be a split Cartan subalgebra of $\mathfrak{g}$. Assume $X = \operatorname{Spec}(R)$ is connected. Then $\mathfrak{h}$ is the unique MAD of $\mathfrak{g}(R)$ contained in $\mathfrak{h}(R)$.*

Proposition 3.4 (Proposition 7 of [Pia04]). *Let $X = \operatorname{Spec}(R)$ be connected reduced and with a rational point. Let $p \in \mathfrak{g}(R)$ be k-diagonalizable. Fix $x_0 \in X$ such that $k(x_0) = k$ and set $p_0 := p(x_0)$. If $x \in X$, then $p(x)$ and p_0 (viewed as two elements of $\mathfrak{g}(x)$) are conjugate under $\mathscr{G}(x)$.*

Proposition 3.5 (Proposition 10 of [Pia04]). *Let X, p, and p_0 be as in Proposition 3.4. Let $J \lhd S(\mathfrak{g}^*)$ be the defining ideal of the closed subset $\mathscr{G}(k) \cdot p_0 \in \mathfrak{g}$, and let $\mathfrak{L}$ be the isotropy group of p_0 (i.e., $\mathfrak{L}(S) = \{g \in \mathscr{G}(S) : g \cdot p_0 = p_0\}$) for any k-algebra S. Then*

(i) There exists a canonical isomorphism $\mathscr{G}/\mathfrak{L} \cong \operatorname{Spec}(S(\mathfrak{g}^*)/J)$.

(ii) p vanishes on J thereby inducing a scheme morphism $\psi_p : X \to \mathscr{G}/\mathfrak{L}$.

Proposition 3.6 (Proposition 11 of [Pia04])**.** *With the notation of Proposition 3.5 the following are equivalent:*

(i) There exists $g \in \mathscr{G}(R)$ such that $p_0 = g \cdot p$.

(ii) There exists a scheme morphism $\hat{\psi}_p : X \to \mathscr{G}$ rendering the diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[d, "q"] \\ X \arrow[ur, "\hat{\psi}_p"] \arrow[r, "\psi_p"'] & \mathscr{G}/\mathfrak{L} \end{tikzcd}$$

commutative.

(iii) The pull back $pr_1 : X \times_{\mathscr{G}/\mathfrak{L}} \mathscr{G} \to X$ admits a global section.

Definition 3.7. Let $f_{\text{reg}} \in S(\mathfrak{g}^*)$ be the polynomial function defining the basic Zariski open dense set of regular elements of g (see [Bour75] Ch. VII). Since f_{reg} is defined over k, we can think of it as a polynomial function on the free R-module $\mathfrak{g}(R)$. An element p of $\mathfrak{g}(R)$ is said to be regular if $f_{\text{reg}}(p)$ is a unit of R. Finally, a MAD is said to be regular if it contains a regular element.

Theorem 3.8 (Theorem 1 of [Pia04])**.** *Let $\mathfrak{g}$ be a finite dimensional split semisimple Lie algebra over k, and $\mathscr{G}$ its simply connected Chevalley-Demazure group scheme. Let $X = \operatorname{Spec}(R)$ be a connected affine scheme and X_{red} the corresponding reduced scheme. Then*

(i) If $\mathfrak{a}$ is an abelian k-diagonalizable subalgebra of $\mathfrak{g}(R)$ then $\dim_k(\mathfrak{a}) \leq rank(\mathfrak{g})$. If this is an equality, then $\mathfrak{a}$ is maximal.

(ii) Assume that $X(k) \neq 0$.
(a) (Regular conjugacy). If the Picard group of X_{red} is trivial, then all regular maximal abelian k-diagonalizable subalgebras of $\mathfrak{g}(R)$ are conjugate under $\mathscr{G}(R)$.
(b) (Full conjugacy). Consider the following property on X:
$(\mathbf{TLT})$ *(Triviality of locally trivial Levi torsors): If $\mathfrak{L}$ is the Levi subgroup of a standard parabolic subgroup of $\mathscr{G}$, then any locally trivial principal homogeneous space for $\mathfrak{L}$ over X_{red} is trivial.*
If (TLT) holds, then all maximal abelian k-diagonalizable subalgebras of $\mathfrak{g}(R)$ are regular (and hence all conjugate by (a)).

The main idea of the proof of Theorem 3.8 is to evaluate the different prime ideals of X at a given k-diagonalizable element of $\mathfrak{g}(R)$. Each of these evaluations puts us in the finite dimensional case where the conjugacy is known to hold (Chevalley's theorem asserting the conjugacy of all split Cartan subalgebras of $\mathfrak{g}$). One then is lead to look at assumptions on X that allow all of these finite dimensional conjugacies to be glued together to create an element of $\mathscr{G}(R)$.

Chapter 4

"MADs" in $G(\mathbb{C}[X])$

In this chapter we will examine a motivating example of the conjugacy problem of MADs in a reductive algebraic group over a particular ring.

Let G be a reductive group over the field $\mathbb{C}$ of complex numbers. Let $\mathbb{C}[X]$ be the polynomial ring over $\mathbb{C}$ with the variable X. For any $s \in \mathbb{C}$, we define $\pi_s : G(\mathbb{C}[X]) \longrightarrow G(\mathbb{C})$ by $\pi_s(g) = g_s$, for all $g \in G(\mathbb{C}[X])$. In fact, this map is the evaluation of g at a point s.

Definition 4.1. Let $\mathcal{T}$ be a subset of $G(\mathbb{C}[X])$. We say $\mathcal{T}$ is a *MAD* if $\pi_s(\mathcal{T}) = \mathcal{T}_s$ is a maximal torus in $G(\mathbb{C})$ for any $s \in \mathbb{C}$. (Here "MADs" is not the same definition as Definition 5.12 in Chapter 5.)

Since all maximal tori of $G(\mathbb{C})$ are conjugate, let

$$H = \{(s, h) \in \mathbb{C} \times G(\mathbb{C}) \,|\, h\mathcal{T}_s h^{-1} = D\}$$

where D is a maximal torus in $G(\mathbb{C})$ with its reduced induced structure.

Lemma 4.2. *The set H is a subvariety of $\mathbb{C} \times G(\mathbb{C})$.*

Proof. It suffices to show H is closed. Let N be the normalizer of D in $G(\mathbb{C})$ i.e., $N = \{n \in G(\mathbb{C}) | nDn^{-1} \subset D\}$. Define the action of N on H by

$$n \cdot (s, h) = (s, nh).$$

This action is well defined, since $(nh)\mathcal{T}_s(nh)^{-1} = nh\mathcal{T}_s h^{-1}n^{-1} = nDn^{-1} = D$. Let

$$M = \{(s, xN) \in \mathbb{C} \times G(\mathbb{C})/N |\, \pi_s(t)xN = xN \ \text{ for all } t \in \mathcal{T}\}.$$

It is nonempty since H is not empty. Consider the canonical map

$$\phi : \mathbb{C} \times G(\mathbb{C}) \longrightarrow \mathbb{C} \times G(\mathbb{C})/N, \text{ by } \phi(x, g) = (x, gN).$$

If $(s, xN) \in M$, then we have $\pi_s(t)xN = xN$, for all $t \in \mathcal{T}$ and hence $\pi_s(\mathcal{T}) = \mathcal{T}_s \subseteq xNx^{-1}$. However, $\mathcal{T}_s$ is maximal and $\mathcal{T}_s = xDx^{-1}$, so $\phi^{-1}(M) \subset H$. If

$(s, h) \in H$, then $h\mathcal{T}_s h^{-1} = D$. Under the map ϕ, we have $h\mathcal{T}_s h^{-1}N = DN = N$ and hence $\mathcal{T}_s \subset hNh^{-1}$. From which it follows that $H \subset \phi^{-1}(M)$. It remains to show that M is closed. Given any $f \in \mathcal{O}(G(\mathbb{C})/N)$, we have $f(\pi_s(t)(xN)) = f(xN)$. For fixed t, the map $f_t : (s, xN) \mapsto f(\pi_x(t), xN)$ is regular. For any t, we observe $\pi(t) : \mathbb{C} \to G(\mathbb{C})$, $s \mapsto \pi_s(t)$ is a polynomial map, since every entry in the matrix t is a polynomial. Consequently, $\pi(t)$ is a morphism. Hence f_t is regular and thus $H = \phi^{-1}(M)$ is closed. $\square$

Recall Definition 2.21 of a principal G-bundle. We have the following lemma.

Lemma 4.3. *Let $\pi : X \to Y$ be a principal G-bundle where Y is a variety. If $P \subseteq X$ is a closed subvariety and G-stable, then $Q = \pi(P)$ is closed and $\pi|_P : P \to Q$ is a principal G-bundle.*

Proof. The preimage of Q under π is closed if and only if Q is closed. We will show $\pi^{-1}(\pi(P))$ is closed. Since π is G-equivariant and $Y \cong X/G$, we have that $\pi^{-1}(\pi(P)) = G \cdot P = P$, thus Q is closed.

Because $X \to Y$ is a principal G-bundle, there exists an étale morphism $Z \to Y$ such that $X \times_Y Z \to Z$ is trivial. Consider the following diagram

$$\begin{tikzcd}
Q \times_Y X = P \arrow[r] \arrow[d] & Q \arrow[d, hook] & Q \times_Y Z \arrow[l] \arrow[d] & (Q \times_Y Z) \times_Z (X \times_Y Z) \arrow[l] \arrow[d] \\
X \arrow[r, "p.b."'] & Y & Z \arrow[l, "étale"] & X \times_Y Z \arrow[l, "p.b."]
\end{tikzcd}$$

All subdiagrams commute since they represent base change. Since $Y \leftarrow Z$ is étale, we have the pull-back morphism $Q \times_Y Z$ is étale over Z and $Q \times_Y Z \to Z$ is a closed immersion. Consider the rightmost square. Since $X \times_Y Z \to Z$ is a trivial bundle, the induced bundle $Q \times_Y X \times_Y Z = (Q \times_Y Z) \times_Z (X \times_Y Z) \to Q \times_Y Z$ is trivial. This yields

$$\begin{tikzcd}
Q \times_Y Z \times_Q Q \times_Y Z = Q \times_Y X \times_Y Z \arrow[r] \arrow[d] & Q \times_Y X = P \arrow[d] \\
Q \times_Y Z \arrow[r] & Q
\end{tikzcd}$$

Let $Z' = Q \times_Y Z$. Then $P \times_Q Z' = Z' \times G$ and therefore $P \to Q$ is a principal G-bundle. $\square$

Proposition 4.4. *The morphism $H \to \mathbb{C}$ is a principal N-bundle.*

Proof. We begin with the principal N-bundle $\phi : \mathbb{C} \times G(\mathbb{C}) \to \mathbb{C} \times G(\mathbb{C})/N$, where $H \subseteq \mathbb{C} \times G(\mathbb{C})$ is closed and N-stable. By Lemma 4.3, it is enough to show $\phi(H) \cong \mathbb{C}$. Fix $s \in \mathbb{C}$, then all elements in $\phi^{-1}(s)$ are in one orbit hN. Indeed, suppose $x, y \in G(\mathbb{C})$ such that $x\mathcal{T}_s x^{-1} = D$ and $y\mathcal{T}_s y^{-1} = D$. Thus, $yx^{-1}Dxy^{-1} = D$, which gives $yx^{-1} \in N$ and hence $y \in xN$. Since the map $\phi(H) \to \mathbb{C}$ is bijective, $\phi(H) \to \mathbb{C}$ is birational. We have $\phi(H) \backsimeq \mathbb{C}$ by Proposition 2.8 and consequently the map $H \to \mathbb{C}$ is a principal N-bundle. $\square$

From the exact sequence $1 \to D \to N \to N/D \cong W \to 1$, where W is the Weyl group, we obtain the following two bundles:

(i) a principal W-bundle $\psi : H/D \to H/N \cong \mathbb{C}$ and

(ii) a principal D-bundle $\varphi : H \to H/D$.

Lemma 4.5. *The map $\psi : E = H/D \to \mathbb{C}$ is a trivial bundle.*

Proof. Since $\mathbb{C}$ is simply connected, by the lifting theorem there is a section as a topological bundle. To use an algebraic version of the implicit function theorem, we need to show ψ is a finite and étale morphism (See Section 4.66 of [KM98]). First, we have $E \times_{\mathbb{C}} E \cong E \times W$. Let $U \subseteq E$ be an open affine subset, define $\tilde{\psi}$: $E \times W \to E$, then $\tilde{\psi}^{-1}(U) = U \times W$ is affine. Now $\mathcal{O}(\tilde{\psi}^{-1}(U)) = \mathcal{O}(U) \otimes \mathcal{O}(W)$, because W is finite and $\mathcal{O}(\psi^{-1}(U))$ is a finitely generated $\mathcal{O}(U)$-module. Thus the map $E \times_{\mathbb{C}} E \to E$ is finite.

Now consider the following diagram

$$\begin{array}{ccc} \mathcal{O}(E) & \longrightarrow & \mathcal{O}(E \times E) \\ \uparrow & & \uparrow \\ \mathcal{O}(\mathbb{C}) = (\mathcal{O}(E))^W & \longrightarrow & \mathcal{O}(E) = (\mathcal{O}(E \times_{\mathbb{C}} E))^W \end{array}$$

Since $\mathcal{O}(E \times_{\mathbb{C}} E)$ is a finitely generated $\mathcal{O}(E)$-module, we can write $\mathcal{O}(E \times_{\mathbb{C}} E) = \mathcal{O}(E)[x_1, ..., x_n]$ for $x_i \in \mathcal{O}(E \times E)$. If $x \in \mathcal{O}(E \times E)$, then $x^n = \sum_{i=1}^{n-1} a_i x^i$ for some $a_i \in \mathcal{O}(E)$. Let R be the Reynolds operator (See Remark 4.6). Suppose $x \in (\mathcal{O}(E \times E))^W$, then we have

$$x^n = R(x^n) = \sum_{i=1}^{n-1} R(a_i) x^i,$$

where $R(a_i) \in \mathcal{O}(\mathbb{C})$. Therefore, $\mathcal{O}(E)$ is a finitely generated $\mathcal{O}(\mathbb{C})$-module and it follows that ψ is finite.

Since $\psi : E \to \mathbb{C}$ is a principal W-bundle, for each $s \in \mathbb{C}$, the fibre $\psi^{-1}(s) = W$. As W is a finite group, $\dim(W) = 0$. Subsequently by Theorem 10.2 of [Hart99], ψ is smooth of relative dimension 0 and hence ψ is étale. $\square$

Remark 4.6. Let V be a rational representation of a reductive group G, then there is a G-invariant linear subspace $W \subset V$ such that $V = V^G \oplus W$ (as representations). The projection

$$R_V : V \to V^G$$

is called the *Reynolds operator*.

This map $R = R_V$ has the following properties:

(i) R is G-equivairant.

(ii) $R \circ R = R$.

(iii) $\mathrm{Im}(R) = V^G$.

Conversely, if $R : V \to V^G$ is an operator which satisfies (i), (ii) and (iii), then it can be proved that $R = R_V$.

Proposition 4.7. *The principal N-bundle $H \to \mathbb{C}$ is trivial.*

Proof. Since $\psi : E = H/D \to \mathbb{C}$ is trivial, we have $H/D \cong \mathbb{C} \times W$. Let $H' = \varphi^{-1}(\mathbb{C} \times \{e\})$, where e is the identity of W, then $H' \to \mathbb{C} \times \{e\} \cong \mathbb{C}$ is a principal D-bundle. To verify this proposition, it is enough to show $H' \to \mathbb{C}$ is trivial. By Remark of 2.23, if we want to show any principal D-bundle is trivial, it suffices to show the corresponding vector bundle is trivial. We know that $D \cong (\mathbb{C}^*)^n \cong \underbrace{GL_1 \times ... \times GL_1}_{n \text{ copies}}$. Let $L_i \to \mathbb{C}$ be the line bundle corresponding the i-th GL_1-bundle, then the vector bundle $L_1 \oplus L_2 \oplus ... \oplus L_n \to \mathbb{C}$ is in one-to-one correspondence with the D-bundle $H' = L_1^* \oplus ... \oplus L_n^* \to \mathbb{C}$. There is a one-to-one correspondence between algebraic vector bundles over an affine variety and finitely generated projective modules over its coordinate ring. Since $\mathbb{C}[X]$ is an unique factorization domain, every projective $\mathbb{C}[X]$-module of rank 1 is free (See Theorem II.1.3 of [Lam00]). And consequently the map $L_i \to \mathbb{C}$ is trivial. Hence $L_1 \oplus L_2 \oplus ... \oplus L_n \to \mathbb{C}$ is trivial and we conclude that $H' \to \mathbb{C}$ is trivial. $\square$

Now we know $H \to \mathbb{C}$ is trivial which plays the central role in this chapter. Let σ be a section of H. We have the following map:

$$\mathbb{C} \xrightarrow{\sigma} H \xrightarrow{p_2} G(\mathbb{C}) \xrightarrow{q} G(\mathbb{C})/N,$$

where p_2 is the projection to the second component and q is the canonical quotient map. Let $\rho := q \circ p_2 \circ \sigma : \mathbb{C} \to G(\mathbb{C})/N$, then there exists $\tilde{\rho}$ such that the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\rho"] \arrow[d, "\sigma"] \arrow[dr, dotted, "\tilde{\rho}"] & G(\mathbb{C})/N \\
H \arrow[r, "p_2"] & G(\mathbb{C}). \arrow[u, "q"']
\end{tikzcd}$$

commutes. This enables us to redefine $H = \{(s, \tilde{\rho}) \in \mathbb{C} \times G(\mathbb{C}) | \tilde{\rho}(s)\mathcal{T}_s\tilde{\rho}(s)^{-1} = D\}$.

Remark 4.8. We can also define $\rho : \mathbb{C} \to G(\mathbb{C})/N$ by an elementary way. Recall Lemma 4.2,

$$M = \{(s, xN) \in \mathbb{C} \times G(\mathbb{C})/N | \pi_s(t)(xN) = xN \text{ for all } t \in \mathcal{T}\}.$$

Clearly the projection $\phi : M \to \mathbb{C}$ is birational as we have $\phi^* : \mathcal{K}(\mathbb{C}) = \mathbb{C}(X) \cong \mathcal{K}(M)$. Let $\psi : \mathbb{C}[X] \hookrightarrow \mathcal{O}(M) \subseteq \mathbb{C}(X)$. We will show it is surjective. Let $f \in \mathcal{O}(M)$. We can assume $f \in \mathbb{C}(X)$ and so we can write $f = \frac{g}{h}$, where $g, h \in \mathbb{C}[X]$. Also since $\mathbb{C}$ is a UFD, we can assume g, h are coprime. Now, it suffices to show h is constant. Suppose h is not, then there exists $p \in \mathbb{C}$ such that $h(p) = 0$ and $g(p) \neq 0$. Therefore $f^{-1} \in \mathcal{O}(\mathbb{C})_p$. Choose $q \in M$ such that $\phi(q) = p$, then we have

$$f^{-1} = \phi^*(f^{-1}) = \phi^*(f)^{-1} \in \mathcal{O}(M)_q$$

and hence

$$f^{-1}(q) = f^{-1} \circ \phi(q) = f^{-1}(p) = 0.$$

Since $f, f^{-1} \in \mathcal{O}(M)_q$, f, f^{-1} are units. Thus f^{-1} is not contained in the maximal ideal of $\mathcal{O}(M)_q$. However, this contradicts the fact that $f^{-1}(q) = 0$. And hence no such p exists. We have $M \cong \mathbb{C}$ and so there exists $\rho : \mathbb{C} \to G(\mathbb{C})/N$ such that M is the graph of ρ.

Theorem 4.9. *Let G be a reductive group over $\mathbb{C}$. Let $\mathbb{C}[X]$ be the polynomial* *ring with the variable X over $\mathbb{C}$. All MADs of $G(\mathbb{C}[X])$ are conjugate to D under $G(\mathbb{C}[X])$.*

Proof. Let $\Gamma(\mathbb{C} \times G(\mathbb{C}))$ be the set of sections of $\mathbb{C} \times G(\mathbb{C})$ over $\mathbb{C}$. Then $\Gamma(\mathbb{C} \times G(\mathbb{C}))$ is canonically isomorphic to $G(\mathbb{C}[X])$. Since $\tilde{\rho}$ as constructed below Lemma 4.7 is a section of $\mathbb{C} \times G(\mathbb{C})$, it is contained in $G(\mathbb{C}[X])$. Let

$$S = \{(s, \pi_s(t)) \in \mathbb{C} \times G(\mathbb{C}) \mid \text{ for all } s \in \mathbb{C} \text{ and } t \in \mathcal{T}\}$$

(Recall $\pi_s : G(\mathbb{C}[X]) \longrightarrow G(\mathbb{C})$ is defined by $\pi_s(g) = g_s$ for all $g \in G(\mathbb{C}[X])$). We have an isomorphism $\mu : \mathbb{C} \times D \cong S$ given by $\mu(s, d) = (s, \tilde{\rho}(s)^{-1} d\, \tilde{\rho}(s))$, which induces $\mu^* : \Gamma(\mathbb{C} \times D) \cong \Gamma(S)$ given by $\mu^*(\sigma) = (p_1 \circ \sigma, \tilde{\rho}^{-1} \cdot (p_2 \circ \sigma) \cdot \tilde{\rho})$, so that

$$\mu^*(\sigma)(s) = (s, \tilde{\rho}(s)^{-1}(p_2 \circ \sigma(s))\, \tilde{\rho}(s)).$$

Clearly, $\Gamma(\mathbb{C} \times D) \cong D$ and consequently $\mathcal{T} \subseteq \Gamma(S) \subseteq \tilde{\rho}^{-1} D\, \tilde{\rho}$. However, since for any $s \in \mathbb{C}$, $\mathcal{T}_s = \tilde{\rho}^{-1}(s) D\, \tilde{\rho}(s)$, we have $\mathcal{T} = \tilde{\rho}^{-1} D\, \tilde{\rho}$. $\square$

Chapter 5

Conjugacy problems of MADs in $G(R)$

In this chapter we will provide a complete description of maximal abelian k-diagonalizable subgroups of a linear algebraic group. Throughout this chapter, k denotes an algebraically closed field of characteristic zero.

5.1 AD and MAD subgroups

In this section, we will provide a definition of an abelian k-diagonalizable subgroup of an algebraic group and give some of its interesting properties.

Let G be a reductive algebraic group over k and let R be a k-algebra. We are interested in algebraic groups of the form $\mathscr{G} = G \times \mathrm{Spec}(R)$ (This is the group scheme over $\mathrm{Spec}(R)$ obtained by base change and $\mathscr{G}(R) = G(R)$). There are many ways to define an action of $\mathscr{G}(R)$. For example: G acts on itself by right multiplication or acts on its Lie algebra by the adjoint representation. Here is the way we define:

Let $R[\mathscr{G}]$ be the coordinate ring of $\mathscr{G}$. The group scheme $\mathscr{G}$ can be thought of as the scheme Spec($R[\mathscr{G}]$) or as the functor $\mathrm{Hom}_R(R[\mathscr{G}], -)$ from the category of commutative associative unital R- algebras into the category of groups. Given an element g of $\mathscr{G}(R)$, the action of g on $R[\mathscr{G}]$ is defined by

$$R[\mathscr{G}] \xrightarrow{\Delta} R[\mathscr{G}] \otimes R[\mathscr{G}] \xrightarrow{id \otimes g} R[\mathscr{G}] \otimes R = R[\mathscr{G}]$$

(Recall that $R[\mathscr{G}]$ is viewed as a comodule of $\mathscr{G}$). Since we also can consider $R[\mathscr{G}]$ as a k-algebra, then we say g is k-diagonalizable if g, when viewed as a k-linear endomorphism of $R[\mathscr{G}]$, is diagonalizable.

Definition 5.1. Let $\mathcal{H}$ be a subgroup of $\mathscr{G}(R)$. $\mathcal{H}$ is called an *abelian k-diagonalizable subgroup* of $\mathscr{G}(R)$ if $\mathcal{H}$ is abelian and acts on $R[\mathscr{G}]$ k-diagonalizably.

There is another possibility to define k-diagonalizable subgroup, namely to call $\mathcal{H}$ is abelian and every element is k-diagonalizable.

If the action of $\mathcal{H}$ on $R[\mathscr{G}]$ is k-diagonalizable, then every element of $\mathcal{H}$ is k-diagonalizable. But the converse is not so clear. We claim that if R is Noetherian, the converse holds.

Proposition 5.2. *Let $\mathcal{H}$ be an abelian subgroup of $\mathscr{G}(R)$ and let R be Noetherian. If every element of $\mathcal{H}$ is k-diagonalizable, then $\mathcal{H}$ is k-diagonalizable on $R[\mathscr{G}]$.*

Before proving this proposition, we need the following two lemmas:

Lemma 5.3. *Let M be a finitely generated R-module, where R is Noetherian. Then there is an integer n such that every decomposition of M into a direct sum of submodules has at most n nonzero summands.*

Proof. Let $M = M_1 \oplus M_2 \oplus ... \oplus M_t$ and $p \in \mathrm{Spec}(R)$. We have $p \in$ Supp(M) if and only if the localization M_p of M by p is nonzero if and only if there exists an i such that $M_i \neq 0$ in M_p if and only if there exists an i such that Ann$(M_i) \subseteq p$ if and only if Ann$(M) = \bigcap_{i=1}^{t}$ Ann$(M_i) \subseteq p$. Hence Supp$(M) = V(I)$ where I is the annihilator of M. So the support of M is closed in $\mathrm{Spec}(R)$.

By Noetherian induction, we may assume that the assertion is true for all finitely generated modules with strictly smaller support.

Let P be a minimal prime over I, corresponding to an irreducible component of $V(I)$. Now $S = R/P$ is a Noetherian integral domain, so the torsion submodule of M/PM has strictly smaller support than $V(I)$. Thus, applying the induction hypothesis to the torsion module of M/PM gives that the number of nonzero modules M_i/PM_i that has torsion for S are bounded by some integer m (only depending on M). Clearly the number of torsion free modules M_i/PM_i is bounded by dim$_K M/PM \otimes_S K =: d$, where K is the quotient field of S. Hence the number of nonzero modules M_i/PM_i is at most $m+d$. Let N be the maximum of these numbers over all irreducible components of $V(I)$ and let c be the number of components. Then the number of modules M_i such that M_i/PM_i is nonzero for some minimal prime P is bounded by cN. It remains to bound the number of modules M_i such that $M_i/PM_i = 0$ for all minimal primes P. Now if $M_i = PM_i$, then $M_i = QM_i$, for all $Q \in V(P)$. By Nakayama's Lemma, this implies that $(M_i)_Q = 0$ for all $Q \in V(P)$. As these Q range over all primes in $V(I)$, this means $M_i = 0$. $\square$

Lemma 5.4. *Let A be an abelian group. Let V be an R-module on which A acts R-linearly, where R is Noetherian. Suppose every element of A is k-diagonalizable on V , and suppose A acts locally R-finitely on V (Here we say A acts locally R-finitely, if every $v \in V$ is contained in a finitely generated A-stable R-submodule). Then A is k-diagonalizable on V.*

Proof. It suffices to show that V is generated (as a k-vector space) by eigenvectors for A. As V is R-finite, we may suppose V is a finitely generated R-module. Let n be an upper bound for the number of direct summands of V. Pick $g \in A$ such that g has multiple eigenvalues. If no such g exists, A acts diagonally and we are done. Otherwise $V = \oplus_\alpha V_\alpha$, where α ranges over the eigenvalues of g. There are at most n nonzero V_α as each V_α is an R-module. Further more, each V_α is finitely generated since it is an image of V.

Now, A acts on each V_α. For each α, let $g_\alpha \in A$ be an element with multiple eigenvalues on V_α. Then $V_\alpha = \oplus_\beta V_{\alpha\beta}$. We can continue this process at most finitely many times until one of two things happens: all elements of A act diagonally or otherwise we could increase the number of summands beyond n. $\square$

With these results in hand we are now in a position to prove Proposition 5.2.

Proof Proposition 5.2. By Theorem 3.3 of [Wat79], $R[\mathscr{G}]$ is a directed sum of finitely generated submodules. Hence, by Lemma 5.4, $\mathcal{H}$ is also k-diagonalizable on $R[\mathscr{G}]$. $\square$

By this proposition, the above two definitions of abelian k-diagonalizable subgroups are equivalent in the case where R is Noetherian.

Remark 5.5. Let V be a vector space and let g be a diagonalizable endomorphism. Then g is diagonalizable on every g-stable subspace: if $W \subset V$ is g-stable and $w \in W$, then w is contained in a finite dimensional subspace U of W that is g-stable. Now U is contained in the span of finitely many eigenvectors and hence is itself spanned by eigenvectors according to the standard theory for finite dimensional vector spaces.

From now on, R denotes a reduced commutative associative unital finitely generated k-algebra. Let $X = \mathrm{Spec}(R)$ be the corresponding affine variety.

Let $x \in X$, with corresponding prime ideal $P \in R$. The residue field of an element x of X will be denoted by $k(x)$, For convenience, in what follows the group $\mathscr{G}(k(x))$ ($\mathscr{G}(\overline{k(x)})$) will be denoted by $\mathscr{G}(x)$ (resp., $\mathscr{G}(\bar{x})$), and the corresponding group homomorphism $\mathscr{G}(R) \to \mathscr{G}(R/x) \subset \mathscr{G}(x)$ given by $p \mapsto p(x)$ (where $p(x)$ denotes for the canonical $k(x)$-point of $\mathscr{G}(x)$ defined by p). Fix a rational point $x_0 \in X$ such that $k(x_0) \cong k$ and set $p_0 = p(x_0)$ (a rational point exists since k is algebraically closed). We identify $\mathscr{G}(x_0)$ with $G(k) \subset \mathscr{G}(R)$.

Let P be a finitely generated projective module over a Noethrian ring S. The *rank* of P at a prime ideal $\mathfrak{p}$ in $\mathrm{Spec}(S)$ is the rank of the free $S_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$. This rank function is locally constant. If $\mathrm{Spec}(S)$ is connected, then P has constant rank at all primes, and we call this number the *rank* of P. In particular, when S is an integral domain, we have $\mathrm{rk}(P) = \dim_K(P \times_S K)$, where $K = \mathrm{Quot}(S)$.

Lemma 5.6. *Let L be a finitely generated projective S-module. For all $\mathfrak{p} \in Spec(S)$, we have rank$(L/\mathfrak{p}L) =$ rank(L) as an $R/\mathfrak{p}R$-module.*

Proof. Let $q \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \subset q$. We have L_q is free over R_q since a finitely generated projective module over a local ring is free. Let $\mathrm{rank}(L_q) = n$. Then

$$(L/\mathfrak{p}L)_q = L_q/\mathfrak{p}L_q = S_q^{(n)}/\mathfrak{p}S_q^{(n)} = (S_q/\mathfrak{p}S_q)^{(n)}.$$

Therefore, $(L/\mathfrak{p}L)_q = L_q/\mathfrak{p}L_q$ is free over $(R/\mathfrak{p}R)_q$, for any q. Then $\mathrm{rank}(L/\mathfrak{p}L) = \mathrm{rank}(L/\mathfrak{p}L)_q = \mathrm{rank}(L_q) = \mathrm{rank}(L) = n$. ☐

We will now introduce a crucial proposition which relates to the structure groups of the torsors involved in conjugacy.

Proposition 5.7. *Let $p \in \mathscr{G}(R)$ be k-diagonalizable. For any $x \in Spec(R)$, let $p(x)$ and p_0 be defined as above. Then they are conjugate in $\mathscr{G}(\bar{x})$.*

Proof. By Proposition 2.20, there is no loss of generality in assuming that both $p(x)$ and p_0 belong to a maximal torus $T(\bar{x})$.

By Corollary 1.2 of [Mum94], if p_0 and $p(x)$ are not conjugate under $\mathscr{G}(\bar{x})$, then they can be separated by an invariant function $f \in \overline{k(x)}[\mathscr{G}]^{\triangle}$, where $\triangle \subset \mathscr{G}(\bar{x}) \times \mathscr{G}(\bar{x})$ is the diagonal subgroup acting on $\overline{k(x)}[\mathscr{G}]$ by $(g,g) \cdot f(t) = f(gtg^{-1})$. We will now consider what the invariant function f looks like. Recall the decomposition of $k[G]$ in Proposition 2.18. $k[G]$ as a $G \times G$-module has the form

$$k[G] = \bigoplus_{\lambda} \mathrm{End}_k(V_\lambda),$$

where the V_λ are the irreducible representation of G. The map $\mathrm{End}_k(V_\lambda) \hookrightarrow k[G]$

is defined by

$$\varphi \mapsto f_\varphi(g) := \mathrm{tr}(\rho_\lambda(g) \circ \varphi).$$

Since $k \hookrightarrow R$ is flat, we can decompose $R[\mathscr{G}] = \bigoplus_\lambda \mathrm{End}_k(V_\lambda) \otimes R = \bigoplus_\lambda \mathrm{End}_R(V_\lambda \otimes R)$. By $R \to k(x) \to \overline{k(x)}$, we further get the decomposition of

$$\overline{k(x)}[\mathscr{G}] = \bigoplus_\lambda \mathrm{End}(V_\lambda \otimes \overline{k(x)}).$$

Then $\overline{k(x)}[\mathscr{G}]^\triangle = \bigoplus_\lambda \mathrm{End}_{\overline{k(x)}}(V_\lambda \otimes \overline{k(x)})^\triangle = \bigoplus_\lambda \mathrm{End}_{\mathscr{G}(\bar{x})}(V_\lambda \otimes \overline{k(x)})$. By Schur's lemma, we know $\mathrm{End}_{\mathscr{G}(\bar{x})}(V_\lambda \otimes \overline{k(x)})$ is one dimensional, i.e.,

$$\mathrm{End}_{\mathscr{G}(\bar{x})}(V_\lambda \otimes \overline{k(x)}) = \overline{k(x)} \cdot \mathrm{id} \ \text{ for all } \lambda.$$

If $f \in \mathrm{End}_{\mathscr{G}(\bar{x})}(V_\lambda \otimes \overline{k(x)})$ and $g \in \mathscr{G}(x)$ is k-diagonalizable, then

$$f(g) = \mathrm{tr}(\rho_\varphi(g(x)) \circ \mathrm{id}) = \Sigma m_i \lambda_i, \tag{5.1.1}$$

where λ_i is an eigenvalue of g and m_i is the multiplicity of the corresponding eigenvalue λ_i on $V_\lambda \otimes \overline{k(x)}$. It is enough to show p_0 and $p(x)$ have the same eigenvalues and multiplicity when acting on $V_\lambda \otimes \overline{k(x)}$. Since p is k-diagonalizable on $R[\mathscr{G}]$, we can write $\mathrm{End}_R(V_\lambda \otimes R) = \bigoplus_\mu L_\mu$, where $\mu \in k$ and $L_\mu = \{v \in \mathrm{End}_R(V_\lambda \otimes R) | p \cdot v = \mu v\}$. Since the action is R-linear, L_μ is a projective R-submodule of $\mathrm{End}_R(V_\lambda \otimes R)$. Further more, $\mathrm{End}_R(V_\lambda \otimes R)$ is a free R-module with rank $(\dim(V_\lambda))^2$ and the projection map $\mathrm{End}_R(V_\lambda \otimes R) \twoheadrightarrow L_\mu$ is surjective, so L_μ is finitely generated. Considering the map $\mathscr{G}(R) \to \mathscr{G}(R/x) \subset \mathscr{G}(x)$, for any $x \in \mathrm{Spec}(R)$, we decompose

$$\mathrm{End}(V_\lambda \otimes R)/x\mathrm{End}(V_\lambda \otimes R) = \bigoplus_\mu L_\mu / xL_\mu \text{ and}$$

$$\mathrm{End}(V_\lambda \otimes R)/x_0\mathrm{End}(V_\lambda \otimes R) = \bigoplus_\mu L_\mu / x_0 L_\mu$$

with respect to $p(x)$ (resp., p_0). From the above lemma, we see

$$\mathrm{rank}(L_\mu / xL_\mu) = \mathrm{rank}(L_\mu / x_0 L_\mu) = \mathrm{rank}(L_\mu).$$

Hence, the eigenvalues and corresponding multiplicity of $p(x)$ acting on $\mathrm{End}(V_\lambda \otimes R)/x\mathrm{End}(V_\lambda \otimes R)$ are the same as those of p_0 acting on $\mathrm{End}(V_\lambda \otimes R)/x_0\mathrm{End}(V_\lambda \otimes$

R).

By the trivially of the action of p_0 on R and $\mathrm{End}_R(V_\lambda \otimes R) = \mathrm{End}_k(V_\lambda) \otimes R$, and $k(x_0) = k = k(\bar{x})$, it is clear that the action of p_0 on $\mathrm{End}(V_\lambda \otimes R)/x\mathrm{End}(V_\lambda \otimes R)$ is the same as the action of p_0 on $\mathrm{End}(V_\lambda) \otimes_k k(\bar{x}) = \mathrm{End}(V_\lambda \otimes R)/x_0\mathrm{End}(V_\lambda \otimes R)$. Hence $f(p(x)) = f(p_0)$. Since p_0 and $p(x)$ are semisimple, their conjugacy classes are closed and therefore, $p(x)$ and p_0 are in the same orbit. $\square$

As an immediate consequence of this proposition, we obtain the following corollary.

Corollary 5.8. *If $\mathcal{H} \subset \mathscr{G}(R)$ is a k-diagonalizable subgroup, then $\bar{\mathcal{H}} \subset \mathscr{G}(R/m)$ is k-diagonalizable for any ideal m of R.*

Proof. From Proposition 5.7, $\mathcal{H}$ decomposes $R[\mathscr{G}]$ as $R[\mathscr{G}] = \bigoplus_\mu L_\mu$ where $L_\mu = \{v \in \mathrm{End}_R(V_\lambda \otimes R) | h \cdot v = \mu(h)v \text{ for all } h \in \mathcal{H}\}$. Then $\bar{\mathcal{H}}$ decomposes $(R/m)[\mathscr{G}]$ as $(R/m)[\mathscr{G}] = \bigoplus_\mu L_\mu/mL_\mu$. $\square$

As another consequence of this proposition, we have the following lemma:

Lemma 5.9. *Let $\mathcal{H}$ be a k-diagonalizable subgroup of $\mathscr{G}(R)$. Let N be the image of $\mathcal{H}$ in $\mathscr{G}(x_0)$. Then the map $\mathcal{H} \to N$ is injective.*

Proof. Suppose q is in the kernel of $\mathcal{H} \to N$, then $q(x_0) = e$ in $\mathscr{G}(x_0)$. By Proposition 5.7, we may assume $q(x) = q(x_0) = e$, for all $x \in X$. From the definition of $q(x)$ and e, we have

$$\underbrace{k[G] \xrightarrow{q} R \xrightarrow{\varepsilon_x} k(x)}_{q(x)} \hookrightarrow \overline{k(x)}$$

and

$$\underbrace{k[G] \xrightarrow{e} R \xrightarrow{\varepsilon_x} k(x)}_{e} \hookrightarrow \overline{k(x)}.$$

If $f \in k[G]$, then $q(f) = e(f)$ is in $k(x)$, and therefore $q(f) = e(f)$ is in R/x. That means $q(f) - e(f) \in x$, for all $x \in \mathrm{Spec}(R)$. But since R is reduced, we have $q(f) - e(f) = 0$, for all $f \in k[G]$. Hence $q = e$. $\square$

Remark 5.10. This lemma also holds if we remove the condition that R is reduced.

The proof is as follows. We have the following picture:

$$\begin{tikzcd} \mathscr{G}(R) \arrow[r, hook] \arrow[d] & GL_n(R) \arrow[d] \\ \mathscr{G}(R/\mathrm{Nil}(R)) \arrow[r, hook] & GL_n(R/\mathrm{Nil}(R)) \end{tikzcd} .$$

Since R is Noetherian, we have $\mathrm{Nil}(R)$ is finitely generated. For any $y \in M_n(R)$ with entries in $\mathrm{Nil}(R)$, we have $y^n = 0$ for large enough n. Thus, if $g \in \mathscr{G}(R)$ is in the kernel of $\mathscr{G}(R) \to \mathscr{G}(R/\mathrm{Nil}(R))$, g is unipotent. Thus $g = e$ as it is k-diagonalizable. Therefore, $\mathcal{H} \hookrightarrow \mathscr{G}(R/\mathrm{Nil}(R))$ and we can reason as in the proof of the reduced case.

Lemma 5.11. *Let S be the connected component of $\bar{N}$. Then S is a torus.*

Proof. By Corollary 5.8, N is k-diagonalizable in G. Let $\bar{N}$ be the closure of N in G. We have $\bar{N}$ is k-diagonalizable. Indeed, N is semisimple, $N \subset \bar{N}_s$ which is closed. Since $\bar{N}$ is abelian, we have $\bar{N} = \bar{N}_s$. So $\bar{N}$ is abelian and k-diagonalizable and subsequently the connected component S of $\bar{N}$ is a torus. ☐

After defining k-diagonalizable subgroups, it is natural to define maximal abelian k-diagonalizable subgroups.

Definition 5.12. We call $\mathcal{M} \subset \mathscr{G}(R)$ a maximal abelian k-diagonalizable subgroup, or a MAD for short, if

(i) $\mathcal{M}$ is abelian.

(ii) All elements of $\mathcal{M}$ are k-diagonalizable on $R[\mathscr{G}]$.

(iii) No subgroup of $\mathscr{G}(R)$ satisfying (i) and (ii) above properly contains $\mathcal{M}$.

Example 5.13. Let T be a split maximal torus of G. Then $T(k)$ is not in general self-centralizing, since $T(R)$ is the centralizer of $T(k)$ in $\mathscr{G}(R)$. But the k-points of any split maximal torus of G are examples of abelian k-diagonalizable subgroups of $\mathscr{G}(R)$, and in this case $T(k)$ is a MAD. Indeed, let $g \in \mathscr{G}(R)$ be k-diagonalizable and commute with $T(k)$, then $g \in T(R)$. We have a surjective map $R[\mathscr{G}] \to R[T]$ and since g is k-diagonalizable on $R[\mathscr{G}]$, it is also k-diagonalizable on $R[T]$. But then for every $\alpha \in X(T)$, we have $\alpha(g) \in k$, and hence $g \in T(k)$.

We begin by stating an important proposition of which we will make repeated use in what follows.

Proposition 5.14. *For every abelian diagonalizable subgroup $D \subset \mathscr{G}(R)$, where R is a Noetherian ring, there is a MAD containing D. In particular, every k-diagonalizable element of $\mathscr{G}(R)$ is contained in a MAD.*

Proof. Let S be the set of abelian k-diagonalizable subgroups E containing D. S is a partially ordered set. Let $B \subset S$ be a totally order subset. Define $\cup B := \cup_{E \in B} E$. Clearly, $\cup B$ is anabelian group and every element is k-diagonalizable. By Theorem 3.3 of [Wat79], $R[\mathscr{G}]$ is a directed sum of finitely generated $\mathscr{G}(R)$-stable submodules. Hence, by Lemma 5.4, $\cup B$ is also a k-diagonalizable subgroup in S, and hence it is an upper bound for B in S. By Zorn's Lemma (see Remark 5.15), the set S contains at least one maximal element. Furthermore, let $g \in \mathscr{G}(R)$ be a k-diagonalizable element which there is necessarily a MAD. Then $< g >$ is an abelian k-diagonalizable subgroup, hence contained in a MAD. □

Remark 5.15. Zorn's lemma (also known as the Kuratowski-Zorn lemma): Suppose a partially ordered set P has the property that every chain (i.e. totally ordered subset) has an upper bound in P. Then the set P contains at least one maximal element.

Let G be a reductive algebraic group over k. An element p of G is called *regular* if $\dim(Z_G(p))$ is minimal (i.e. equal to the rank of G). Similarly, let $\mathscr{G}$ be a group scheme over a scheme X. For any scheme morphism $f : Y \to X$, we have a pullback scheme $f^*\mathscr{G}$ over Y. A section p of $f^*\mathscr{G}$ is called regular if for all $y \in Y$, $p(y) \in (f^*G)(k(y)) \subseteq (f^*G)(\overline{k(y)})$ is regular.

For the regular element of G, we note the following proposition.

Proposition 5.16 (Corollary E III 1.7 of [SS1970]). *The following conditions on a semisimple element p of G are equivalent:*

(i) *p is regular.*

(ii) *$Z_G(p)^\circ$ is a torus.*

(iii) *p is contained in a unique maximal torus.*

(iv) *$\alpha(p) \neq 1$ for every root α relative to some, or every, maximal torus containing p.*

Now we have the following two definitions.

Definition 5.17. Let $\mathcal{H}$ be an AD of $\mathscr{G}(R)$. The subgroup $\mathcal{H}$ is called *regular* if it contains a regular element.

For example, the k-points of any split maximal tori ia a regular MADs.

Let $\mathcal{H}$ be an AD and N be the image of $\mathcal{H}$ in $\mathscr{G}(x_0)$. By Lemma 5.9 and 5.11, the evaluation map $\mathcal{H} \to N$ is injective and the connected component of $\bar{N}$ is a torus. We introduce the following definition:

Definition 5.18. Let $\mathcal{H}$ be an AD of $G(R)$. The subgroup $\mathcal{H}$ is called *connected* if the evaluation of $\mathcal{H}$ at rational points is a torus.

Later we will see that regular MADs are connected (refer to Theorem 5.28).

5.2 Conjugacy theorem of Regular MADs

Analogous to the Lie algebra case, we have a conjugacy theorem for regular MADs.

Throughout this section, let G be a reductive algebraic group over k such that its derived group is simply connected. Let R be a reduced commutative associative unital finitely generated k-algebra and $X = \mathrm{Spec}(R)$ be the corresponding affine variety. Let $\mathscr{G} = G \times X$ be the corresponding group scheme over X.

Remark 5.19. Let $p \in \mathscr{G}(R)$ be k-diagonalizable and set $p_0 = p(x_0)$. For any k-algebra S, define $Z(p_0)(S) = \{h \in G(S) \mid hp_0h^{-1} = p_0\}$. Since G is reductive, we have $G = Z(G) \cdot (G, G)$ and hence $G(k) = Z(G)(k) \cdot (G, G)(k)$. So, $p_0 = p_1p_2$ where $p_1 \in Z(G)(k)$ and $p_2 \in (G, G)(k)$. Therefore $Z(p_0) = Z(p_2)$.

From Proposition 2.20, we have $Z(p_0)$ is connected by the simple connectedness of (G, G), therefore $Z(p_0)$ is a connected reductive subgroup L of G. We put $\mathscr{L} = L \times X$.

Proposition 5.20. *Let X, p and p_0 be as above. Then p induces a scheme morphism $\psi_p : X \to \mathscr{G}/\mathscr{L}$.*

Proof. Let $C_G(p_0)$ be the conjugacy class of p_0 in G. By Proposition 5.7, if $\overline{k(x)}[\mathscr{G}]^{\triangle}$ is generated by the trace functions. Let J be the ideal generated by all the functions of the form $f - f(p_0)$ where f is a trace function as given above in Equation 5.1.1. Then J is defined over k and $I := I(C_G(p_0)) = \sqrt{J}$. Thus $p(x)$

vanishes on I. If $f \in I$ then

$$f(p) := p(f) \in \bigcap_{x \in \mathrm{Spec}(R)} x = (0),$$

since R is reduced. This induces a homomorphism

$$\bar{p} : k[G]/I \to R.$$

The corresponding scheme morphism is $\mathrm{Spec}(R) \to G/L$. Consequently, the identity map $\mathrm{id} : \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ induces

$$\psi_p : X \to G/L \times X = \mathscr{G}/\mathscr{L}.$$

□

Proposition 5.21. *With the notation as in Proposition 5.7, the following are equivalent.*

(i) There is a scheme morphism $\tilde{\psi_p} : X \to \mathscr{G}$ such that the diagram

$$\begin{tikzcd} & \mathscr{G} \arrow[d, "q"] \\ X \arrow[ur, "\tilde{\psi_p}"] \arrow[r, "\psi_p"'] & \mathscr{G}/\mathscr{L} \end{tikzcd}$$

is commutative.

(ii) The pull back $pr : X \times_{\mathscr{G}/\mathscr{L}} \mathscr{G} \to X$ admits a global section.

(iii) There exists $h \in \mathscr{G}(R)$ such that $p_0 = hph^{-1}$.

Proof. The equivalence between (i) and (ii) is trivial. We only need to show (i) and (iii) are equivalent. The orbit map $q(R) : \mathscr{G}(R) \to (\mathscr{G}/\mathscr{L})(R)$ is defined as $s \mapsto s^{-1}p_0 s$, where the corresponding R-algebra homomorphism $R[\mathscr{G}]/I \otimes R \to R[\mathscr{G}]$ is given by $\alpha \mapsto f_\alpha$, where

$$f_\alpha(g) = \alpha(g^{-1}p_0 g).$$

Since the orbit map

$$\mathscr{G}(R) \to \mathscr{G}(R) \times \mathscr{G}(R) \xrightarrow{R_{p_0}} \mathscr{G}(R) \times \mathscr{G}(R) \to \mathscr{G}(R)$$

is given by $s \mapsto (s^{-1}, s) \mapsto (s^{-1}p_0, s) \mapsto s^{-1}p_0 s$, the corresponding homomorphism is

$$R[\mathscr{G}] \to R[\mathscr{G}] \otimes R[\mathscr{G}] \xrightarrow{R_{p_0}} R[\mathscr{G}] \otimes R[\mathscr{G}] \to R[\mathscr{G}]$$

which is given by $f(g) \mapsto (f(g^{-1}), f(g)) \mapsto (f(g^{-1}p_0), f(g)) \mapsto f(g^{-1}p_0 g)$, where R_{P_0} is the right multiplication. The commutative diagram is equivalent to saying that there exists $h \in \mathscr{G}(R)$ such that $q(R)(h) = p$. Therefore, $q(R)(h) = h^{-1}p_0 h = p$, and so $h = \tilde{\psi}_p$. $\square$

Remark 5.22. To show that (i) and (iii) are equivalent, we can also argue as follows:

$$\begin{tikzcd}
R[\mathscr{G}] \arrow[d] \arrow[dr, "p"] & \\
R[\mathscr{G}/\mathscr{L}] \arrow[r, "\bar{p}"] \arrow[d] & R \\
R[\mathscr{G}]^{\mathscr{L}(R)} \arrow[d] & \\
R[\mathscr{G}] \arrow[uur, "h"'] &
\end{tikzcd}$$

For any $f \in R[\mathscr{G}/\mathscr{L}]$, since we have a surjection $R[\mathscr{G}] \twoheadrightarrow R[\mathscr{G}/\mathscr{L}]$, $f \in R[\mathscr{G}]$. From the commutativity of the upper triangle of the diagram, we have $p(f) = \bar{p}(f)$. Furthermore, the map $R[\mathscr{G}/\mathscr{L}] \to R[\mathscr{G}]^{\mathscr{L}(R)}$ is an isomorphism as $f(\bar{g}) \mapsto \bar{f}(g)$ where $\bar{f}(g) := f(g^{-1}p_o g)$. The whole diagram commutes, we have $f(p) = f(\bar{p}) = \bar{f}(h) = f(h^{-1}p_0 h)$. Hence p and $h^{-1}p_0 h$ are the same element.

Before stating one of our main theorems, we will recall the following definition:

The *Picard group* of a scheme X over k, denoted by Pic(X), is the group of isomorphism classes of line bundles on X, where the group operation is the tensor product.

In particular, (1) If A is a local ring then all line bundles are trivial, and so Pic(A) is trivial. (2) If A is a principal ideal domain, then Pic(A) is trivial.

Theorem 5.23. *Let G be a reductive algebraic group over k such that its derived group is simply connected. Let $X = \mathrm{Spec}(R)$ be a connected reduced affine scheme, and $\mathscr{G} = G \times X$ its group scheme over X.*
If Pic(X) is trivial, then all regular MADs of $\mathscr{G}(R)$ are conjugate under $\mathscr{G}(R)$.

Proof. Let $\mathcal{M}$ be a regular MAD and let p be a regular element of $\mathcal{M}$. By Remark 5.19, the centralizer $\mathscr{L}$ of $p_0 = p(x_0)$ over R is a maximal torus of $\mathscr{G}$, from which we obtain that $\mathscr{L}$ is a product of l copies of the multiplicative group G_m. Hence the $\mathscr{L}$-torsors over X are classified by

$$\mathrm{H}^1_{\text{ét}}(X, \mathscr{L}) \backsimeq \mathrm{H}^1(X, G_m)^l \backsimeq \mathrm{Pic}(X)^l$$

(Section 4, Chapter 4 of [Mil80]). The pull back pr in Proposition 5.21 is trivial and hence $p_0 = hph^{-1}$, for some $h \in \mathscr{G}(R)$. Hence

$$h\mathcal{M}h^{-1} \subset hZ_{\mathscr{G}(R)}(p)h^{-1} = Z_{\mathscr{G}(R)}(hph^{-1}) = Z(p_0)(R) = \mathscr{L}(R).$$

However, $\mathscr{L}(k)$ is the unique MAD of $\mathscr{G}(R)$ contained in $\mathscr{L}(R)$. Indeed, if $\mathscr{N} \subset \mathscr{L}(R)$ is an abelian k-diagonalizable subgroup of $\mathscr{G}(R)$, then $\mathscr{L}(k) \cdot \mathscr{N}$ is also abelian and k-diagonalizable. Evaluating at x_0, $\mathscr{N}(k) \subset \mathscr{L}(k)$. Since the evaluation is injective by Lemma 5.9, $\mathscr{N} \hookrightarrow \mathscr{N}(k) \subset \mathscr{L}(k)$. Therefore, by maximality, we deduced that $h\mathcal{M}h^{-1} = \mathscr{L}(k)$. $\square$

Example 5.24. (This is modelled after Example 13 of [Pia04].)

Consider $G = SL_2$. Let $S = k[SL_2] = k[x_{11}, x_{12}, x_{21}, x_{22}]/\langle x_{11}x_{22} - x_{12}x_{21} - 1\rangle$ and $R = k[SL_2]^{D(k)}$, where D is the diagonal subgroup of SL_2. Let $h = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, where $a \in k$ and $a \neq \pm 1$.

Consider the element

$$\begin{aligned} g &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix} \\ &= \begin{pmatrix} ax_{11}x_{22} - a^{-1}x_{12}x_{21} & -(a - a^{-1})x_{11}x_{21} \\ (a - a^{-1})x_{21}x_{22} & a^{-1}x_{11}x_{22} - ax_{21}x_{21} \end{pmatrix} \end{aligned}$$

where $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in SL_2(S)$.

Here h acts on $R[SL_2] = R[y_{11}, y_{12}, y_{21}, y_{22}]/\langle y_{11}y_{22} - y_{12}y_{21} - 1\rangle$ k-diagonalizably by right multiplication since

$$\begin{array}{ccccc} S[SL_2] & \xrightarrow{\Delta} & S[SL_2] \otimes S[SL_2] & \xrightarrow{id \otimes h} & S[SL_2] \\ y_{11} & \mapsto & y_{11} \otimes y_{11} + y_{12} \otimes y_{21} & \mapsto & ay_{11} \end{array}$$

$$
\begin{array}{rcccl}
y_{12} & \mapsto & y_{11} \otimes y_{12} + y_{12} \otimes y_{22} & \mapsto & a^{-1} y_{12} \\
y_{21} & \mapsto & y_{21} \otimes y_{11} + y_{22} \otimes y_{21} & \mapsto & a y_{21} \\
y_{22} & \mapsto & y_{21} \otimes y_{12} + y_{22} \otimes y_{22} & \mapsto & a^{-1} y_{22}
\end{array}
$$

and the monomials in the y_{ij} span $S[SL_2]$.

Similarly, the action of g on $R[SL_2]$ is also k-diagonalizable. We have $R[SL_2]$ is a submodule of $S[SL_2]$. g and h are conjugate automorphisms of $S[SL_2]$, so g is diagonalizable on $S[SL_2]$ and so also on any submodule. Since

$$
\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11}y_{11} + x_{21}y_{12} & x_{12}y_{11} + x_{22}y_{12} \\ x_{11}y_{21} + x_{21}y_{22} & x_{12}y_{21} + x_{22}y_{22} \end{pmatrix},
$$

$\{x_{11}y_{11} + x_{21}y_{12}, x_{12}y_{11} + x_{22}y_{12}, x_{11}y_{21} + x_{21}y_{22}, x_{12}y_{21} + x_{22}y_{22}\}$ are generators of $R[SL_2]$ which g acts on k-diagonalizably. Now we compute the image of one of the generators:

$$
\begin{aligned}
x_{11}y_{11} + x_{21}y_{12} \xrightarrow{\Delta} & x_{11}(y_{12} \otimes y_{21} + y_{11} \otimes y_{11}) + x_{21}(y_{11} \otimes y_{12} + y_{12} \otimes y_{22}) \\
\xrightarrow{id \otimes g} & x_{11}[(a - a^{-1})x_{21}x_{22}y_{12} + (a x_{11}x_{22} - a^{-1}x_{12}b_{21})y_{11}] + \\
& x_{21}[-(a - a^{-1})x_{11}x_{12}y_{11} + (-a x_{12}x_{21} + a^{-1}x_{11}b_{22})y_{12}] \\
= & a(x_{11}y_{11} + x_{21}y_{12})
\end{aligned}
$$

Observe that g and h are also contained in $SL_2(R)$ since $D(k)$ acts on g invariantly. By Proposition 5.2, we have $\langle g \rangle$ and $\langle h \rangle$ are contained in MADs $\mathcal{M}$ and $D(k)$. Both g and h are regular.

Claim: The two MADs $\mathcal{M}$ and $D(k)$ are not conjugate in $SL_2(R)$.

Since g, h act on $R[SL_2]$ k-diagonalizably, they do so on the defining representation R^2. Now S^2 decomposes as $Se_1 \oplus Se_2$ by h, and S^2 decomposes by g as

$$
S \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \oplus S \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}.
$$

We observe that x_{11}, x_{21} belong to S_α where $\alpha \in \Phi(D, SL_2)$ from which it follows that $r \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} \in R^2$ only if $r \in S_{-\alpha}$. We know that $S_{-\alpha}$ is a projective R-module of rank one but not free. Indeed, one can show that every line bundle on SL_2/D is SL_2-equivariant. If L is an SL_2-equivariant line bundle over $X \cong SL_2/D$, the fiber over a base point is a one-dimensional representation of the stabilizer D on which D acts by some character $\alpha : D \to k^*$. Conversely, given such a character α, we construct the associated line bundle $L = SL_2 \times^D k_\alpha$, which is the quotient of $SL_2 \times k_\alpha$ by the D-action

$$d \cdot (g, \lambda) = (gd^{-1}, \alpha(d) \cdot \lambda).$$

This is a line bundle over X, by the projection map

$$p(g, \lambda) = g \cdot D.$$

The group SL_2 acts on $SL_2 \times k_\lambda$ by left multiplication on the first factor. Hence L is an SL_2-equivariant line bundle and an SL_2-equivariant line bundle corresponds to a character of D. Therefore,

$$\mathbb{Z} = X(D) \cong \mathrm{Pic}(X).$$

However we observe that α and $-\alpha$ generate $X(D)$, so the line bundle corresponding to the character $-\alpha$ is not trivial, and its global sections are precisely the modules S_α and $S_{-\alpha}$.

5.3 Conjugacy theorem of connected MADs

As in the last section, let G be a reductive algebraic group over k such that its derived group is simply connected. As always R is a reduced commutative associative unital finitely generated k-algebra with the corresponding affine variety $X = \mathrm{Spec}(R)$. Let $\mathscr{G} = G \times X$ be the group scheme over X associated to G.

Let $\mathcal{M}$ be a MAD in $\mathscr{G}(R)$ and N be the image of $\mathcal{M}$ in $\mathscr{G}(x_0)$. According to Lemma 5.11, the connected component S of $\bar{N}$ is a torus. Let $\tilde{L}$ be the centralizer of S in G, then it is a Levi subgroup of G.

Proposition 5.25. *There exists $g \in \mathcal{M}$ such that the centralizer $Z_G(g_0)$ is $\tilde{L} = Z_G(S)$, where $g_0 = g(x_0)$.*

Proof. Let N be the image of $\mathcal{M}$ in $\mathscr{G}(x_0)$. We observe that $N \cap S$ is dense in S. Indeed,

$$\bar{N} = S \cup x_1 S \cup x_2 S \cup ... \cup x_k S \quad \text{where} \quad x_i \in N.$$

Hence $N \cap x_i S$ is dense in $x_i S$ for all i, otherwise, $\bar{N} \neq \cup \overline{N \cap x_i S}$.

Pick a maximal torus T containing S, with corresponding root system Φ. Let $\Phi_S := \{\alpha \in \Phi \mid \alpha|_S \neq 1\}$, then Φ_S is finite. We claim there is $g_0 \in N \cap S$ such that $\alpha(g_0) \neq 1$, for all $\alpha \in \Phi_S$. Indeed, if g_0 does not exist, then

$$N \cup S \subset \bigcap_{\alpha \in \Phi_S} \ker \alpha.$$

The right hand side is a proper closed subset of S, and hence $N \cap S$ is not dense in S. Since $\tilde{L}$ is generated by S and U_α where $\alpha \notin \Phi_S$ (U_α is the unique connected T-stable subgroup of G such that Lie$(U_\alpha) = \mathfrak{g}_\alpha$), we have

$$U_\alpha \subset \tilde{L} \Leftrightarrow \alpha \notin \Phi_S \Leftrightarrow \alpha|_S = 1 \Leftrightarrow U_\alpha \subset Z(g_0).$$

Hence the connected components of $\tilde{L}$ and $Z(g_0)$ are the same. However, as (G, G) is simply connected, we have $Z(g_0)$ is connected, and so $Z(g_0) = \tilde{L}$. Since the map $M \to N$ is injective, the preimage g of g_0 is the required element of $\mathcal{M}$. $\square$

Before continuing, we should consider the following property on X:

$(\mathbf{TLT})$(Triviality of locally trivial Levi torsors): Let L be a Levi subgroup of a standard parabolic subgroup of G. If $\mathscr{L} = X \times L$ is the corresponding Levi subgroup of a standard parabolic subgroup of $\mathscr{G}$, then any Zariski locally trivial principal homogeneous space for $\mathscr{L}$ over X is trivial (see Section 2.3 for the definition of a principal homogeneous space).

There are two important examples of rings with this property,

(i) $k[x_1, ..., x_n]$, and

(ii) $k[x_1^{\pm 1}, ..., x_n^{\pm 1}]$.

Case (i) is proven by Raghunathan and Ramanathan where Raghunathan considered the triviality of certain torsors over algebraic affine space (Theorem 2.2 of [Rgn89]). Case (ii) follows from case (i). For $n = 1$, this is immediate since every locally trivial principal homogeneous space under $\mathscr{G}$ over the punctured affine line, extends to one over the whole affine line. Note that this recovers the conjugacy theorem of Peterson-Kac in the case of untwisted affine Kac-Moody Lie algebras (see [PK83]). The general case follows by an induction argument due to Gille ([Gil01]).

Remark 5.26. Let g and g_0 be defined as above and let $C_G(g_0)$ be the conjugacy class of g_0. Then $C_G(g_0) \cong G/\tilde{L}$ and it follows that

$$C_{\mathscr{G}}(g_0) = C_G(g_0) \times X = G/\tilde{L} \times X \cong \mathscr{G}/\mathscr{L}$$

where $\mathscr{L} = \tilde{L} \times X$. By a similar argument to that given in Proposition 5.20, there is a scheme morphism $\psi_g : X \to \mathscr{G}/\mathscr{L}$ which corresponds to the morphism $\bar{g} : k[G]/I \to R$, where I is the Hopf ideal representing $C_G(g_0)$. Hence we have the following pull back diagram:

$$\begin{tikzcd} X \times_{\mathscr{G}/\mathscr{L}} \mathscr{G} \arrow[r] \arrow[d, "pr_1"'] & \mathscr{G} \arrow[d, "q"] \\ X \arrow[r, "\psi_g"'] & \mathscr{G}/\mathscr{L} \end{tikzcd}$$

Since the quotient morphism $q : \mathscr{G} \to \mathscr{G}/\mathscr{L}$ is locally trivial (See Theorem 12.1 of [Gil07]), so is the pullback $pr_1 : X \times_{\mathscr{G}/\mathscr{L}} \mathscr{G} \to X$ which is a principal homogeneous space for $\mathscr{L}$ over X. Now if we assume the property $(\mathbf{TLT})$ holds, then pr_1 is trivial by construction.

Remark 5.27. If we consider MADs in $\mathscr{G}(R)$, without loss of generality, we could assume G is semisimple. Indeed, let $g \in \mathscr{G}(R)$ be k-diagonalizable. By Proposition 5.7, there exists an $h \in \mathscr{G}(\bar{x})$ such that $hg(x)h^{-1} = g(x_0)$. Since

$$G(k) = Z(G)(k) \cdot (G,G)(k),$$

we have $g_0 = g_1 g_2$ where $g_1 \in Z(G)(k) \subseteq Z(\mathscr{G})(R)$ and $g_2 \in (G,G)(k)$. Therefore, $g(x) = h^{-1}g_1 h h^{-1} g_2 h = h^{-1} g_1 g_2 h$. Since $g_2 \in (G,G)(k)$ and (G,G) is normal, $h^{-1}g_2 h \in (\mathscr{G},\mathscr{G})(\bar{x})$. Let I be the ideal in $R[\mathscr{G}]$. Consider the following map

$$I \hookrightarrow R[G] \to R \to R/x \hookrightarrow k(\bar{x}), \text{ given by } g_1^{-1}g$$

We have $g_1^{-1}g(I) = 0$ in R/x for all x. Because R is reduced, $g_1^{-1}g(I) = 0$. Therefore $g_1^{-1}g \in (\mathscr{G},\mathscr{G})(R)$. Since g_1 and g are k-diagonalizable on $R[\mathscr{G}]$ and commute, $g_1^{-1}g$ is k-diagonalizable on $R[\mathscr{G}]$. From the fact that $R[(\mathscr{G},\mathscr{G})]$ is a quotient of $R[\mathscr{G}]$, we have $g_1^{-1}g$ is k-diagonalizable on $R[(\mathscr{G},\mathscr{G})]$. Hence $\mathcal{M} = \mathcal{M}_1\mathcal{M}_2$, where $\mathcal{M}_1 \subseteq Z(G)(k) \subseteq Z(\mathscr{G})(R)$ and $\mathcal{M}_2 \in (\mathscr{G},\mathscr{G})(R)$. Also $\mathcal{M}_2$ is a MAD in $(\mathscr{G},\mathscr{G})(R)$, since if not, then there is a bigger $\mathcal{M}_2' \in (\mathscr{G},\mathscr{G})(R)$ containing

$\mathcal{M}_2$, and $\mathcal{M} \subsetneq \mathcal{M}_1\mathcal{M}_2'$.

Recall from Definition 5.18 that a MAD $\mathcal{M}$ of $\mathscr{G}(R)$ is called connected if $\bar{N}$ is a torus.

Theorem 5.28. *Let G be a reductive algebraic group over k such that its derived group is simply connected. Let $X = \operatorname{Spec}(R)$ be a connected affine scheme, and $\mathscr{G} = G \times X$ its group scheme over X. If the condition* **TLT** *holds, then all connected MADs are conjugate to the k-points of a maximal tours, with its standard embedding.*

Proof. Let $\mathcal{M}$ be a connected MAD and let N and S be defined as in Lemma 5.9. Then by Proposition 5.25, there is $g \in \mathcal{M}$ such that

$$Z(g_0) = L = Z(S).$$

Let $\mathscr{L} = L \times X$. By Proposition 5.21 and Remark 5.26, there exists $h \in \mathscr{G}(R)$ such that $hgh^{-1} = g_0$. We have

$$h\mathcal{M}h^{-1} \subset hZ_{\mathscr{G}(R)}(g)h^{-1} = Z_{\mathscr{G}(R)}(hgh^{-1}) = Z_{\mathscr{G}(R)}(g_0) = \mathscr{L}(R).$$

Let $\mathcal{M}' = h\mathcal{M}h^{-1}$, then $\mathcal{M}'$ is k-diagonalizable on $R[\mathscr{G}]$ and the same is the case for the quotient $R[\mathscr{L}]$ of $R[\mathscr{G}]$. Hence $\mathcal{M}'$ is an abelian k-diagonalizable subgroup of $\mathscr{L}(R)$. From the map $\mathscr{L}(R) \to \mathscr{L}(x_0)$, we obtain

$$\mathcal{M}'(x_0) = h(x_0)\mathcal{M}(x_0)h^{-1}(x_0) \subset S(k),$$

since $h(x_0) \in Z(S)$. Now $S(k)$ is k-diagonalizable and commutes with $\mathcal{M}'$, $\mathcal{M}'S(k)$ is also a MAD, subsequently $\mathcal{M}'S(k) = \mathcal{M}'$ and hence $S(k) \subset \mathcal{M}'$. Therefore, $S(k) = \mathcal{M}'$, we conclude that

$$\mathcal{M} = h^{-1}(x_0)S(k)h(x_0) = S(k).$$

Indeed, since $hgh^{-1} = g_0$, $h(x_0)g(x_0)h^{-1}(x_0) = g_0$. Hence $h(x_0) \in Z(S)$. We deduced that S is a maximal torus of G, otherwise, there exists a maximal torus T such that $\mathcal{M} = S(k) \subset T(k)$, which contradicts the fact that $\mathcal{M}$ is a MAD. $\square$

Now we can see that all regular MADs and connected MADs are conjugate to the k-points of a maximal torus of G. The definition of MADs in Chapter 4 coincides with the definition of connected MADs.

5.4 Double centralizer of finite MADs

In this section, we will consider the finite part of a MAD in $\mathscr{G}(R)$. Later we will give an important property of finite k-diagonalizable subgroups in $\mathscr{G}$.

Let $\mathscr{G}$ be a smooth affine linear group scheme over X (of finite type) where $X = \mathrm{Spec}(R)$ is an affine connected k-variety. We freely switch between $X(k)$ and X whenever convenient later. By linear we mean that $\mathscr{G}$ is isomorphic to a subgroup scheme of $\underline{\mathrm{GL}}(V)$, where V is a free R-module of finite rank. Let $x \in X$, with corresponding prime ideal $P \in R$. As before, we write $\mathscr{G}_x$ for the scheme-theoretic fibre of $\mathscr{G}$ over x, that is $\mathscr{G} \times_X k(x)$, and $g(x)$ for the canonical $k(x)$-point of $\mathscr{G}(x)$ defined by g.

Proposition 5.29. *With the notation as above, let $g \in \mathscr{G}(R)$ be the element of finite order. Then g is k-diagonalizable if and only if $g(x)$ is k-diagonalizable as an element of $\mathscr{G}_x(k(x))$ for all $x \in X$. If in addition X is of finite type over k, then it is sufficient that $g(x)$ be k-diagonalizable for all closed points of X.*

The only-if part is obvious, since the image of any basis of eigenvectors for $R[\mathscr{G}]$ will span $k(x)[\mathscr{G}_x]$ over $k(x)$. Note that if X is integral, then the if-part is obvious as well: in this case, $R[\mathscr{G}]$ is a g-stable k- subspace of $K[\mathscr{G}]$, where K is the function field of X. So if $K[\mathscr{G}]$ is spanned by k-eigenvectors, so is $R[\mathscr{G}]$. However, $K[\mathscr{G}]$ is spanned by k-eigenvectors since $g(x)$ is k-diagonalizable on $K[\mathscr{G}]$ where x is the generic point. Now let $g \in \mathscr{G}(X)$ be an element of finite order, and let W be any representation of $\mathscr{G}$. By a representation we mean a homomorphism $\mathscr{G} \to \underline{\mathrm{GL}}(W)$, where $W \cong R^n$ for some n. Note that since g has finite order, $g(x)$ is semisimple of finite order and hence $g(x)$ is k-diagonalizable in $\underline{\mathrm{GL}}(W)(x) \cong \mathrm{GL}(W(x))$, where $W(x) = W(k(x)) = W \otimes_R k(x)$.

Lemma 5.30. *Let $g \in \mathscr{G}(X)$ have finite order. If $g(x)$ is k-diagonalizable for all $x \in X$, then the g-action on W is k-diagonalizable.*

Proof. We write g for the element of $\underline{\mathrm{GL}}(W)(X)$ given by g (considering $g : W \to W$) and $g(x)$ for the image of g in $\mathrm{GL}(W(x))$. Let $T_i = g - \mu_i id$ where μ_i is an n-th root of unity in k. Let $K_i = \mathrm{Im}\, T_i \subset W$. Finally, put $W_i = W/K_i$. For each x we have an exact sequence

$$K_i \otimes k(x) \to W(x) \xrightarrow{\pi_i} (W/K_i) \otimes k(x) \to 0.$$

Dualizing over $k(x)$, we find

$$0 \to ((W/K_i) \otimes k(x))^* \xrightarrow{\pi_i^*} W(x)^* \to (K_i \otimes k(x))^*$$

and the image of π_i^* is (by exactness of the first sequence) precisely the set of all $\lambda \in W(x)^*$ which are zero on the image of $K_i \otimes k(x)$ in $W(x)$. In other words, the set of all λ such that

$$\lambda(T_i(w) \otimes c) = 0$$

for all $c \in k(x)$ and all $w \in W$. Now $T_i(w) = gw - \mu_i w$ and since the image of $gw - \mu_i w$ in $W(x)$ is $(gw - \mu_i w) \otimes 1 = g(x)w(x) - \mu_i w(x)$, we end up with all λ for which

$$\lambda((g(x) - \mu_i)w) = 0,$$

for all $w \in W(x)$. In other words, it is the μ_i-eigenspace of $g(x)^*$ on $W(x)^*$. We know that $g(x)$ is diagonalizable on $W(x)$, hence $g(x)^*$ is diagonalizable on $W(x)^*$, and we find that $W(x)^*$ is isomorphic to the direct sum of the $W_i(x)^*$. But that in turn means $W(x)$ is isomorphic to $\oplus_i W_i(x)$. Consider therefore the natural map

$$W \to W_1 \oplus W_2 \oplus ... \oplus W_n.$$

Let $x \in X$, and localize at the corresponding ideal $P \subset R$. Then mod P the map

$$W_P/PW_P \to (W_{1P}/PW_{1P}) \oplus ... \oplus (W_{nP}/PW_{nP})$$

is an isomorphism by construction. By Nakayama's lemma, it is therefore surjective at P, and since P was arbitrary, it is surjective everywhere. (It is enough to check that for maximal P.) If $w \in W$ is in the kernel, then $w \in P_p W_p$ and since W is free, W/PW injects into $W \otimes k(x)$, so $w \in PW$ for all P. As R is reduced, and since W is free, this means $w = 0$. (Again, if R is finitely generated over k, then it suffices to consider maximal P.) $\square$

Since $\mathscr{G}$ is linear, we know that $R[\mathscr{G}]$ is a quotient of $R[\underline{\mathrm{GL}}(V)]$ for some V and g is k-diagonalizable on V, and hence on $R[\underline{\mathrm{GL}}(V)]$. It follows that g is k-diagonalizable on $R[\mathscr{G}]$.

Let N, S be defined as before, $L = Z_G(S)$ and $\mathscr{L} = X \times L \subset \mathscr{G}$ in the usual way. Let $\mathcal{M} \subset \mathscr{G}(R)$ be a MAD. As in the proof of Theorem 5.28, we conclude that without loss, $\mathcal{M} \subset \mathscr{L}(R)$, and since $S(k)$ and $\mathcal{M}$ commute, we must have $S(k) \subset \mathcal{M}$. Since $N = S(k) \cdot (N \cap (L, L))$, we must have $\mathcal{M} = \mathcal{M}^\circ \cdot \Gamma$, where $\mathcal{M}^\circ = S(k)$ and Γ is the preimage of $N \cap (L, L)$, and in particular finite. Note that for all $\gamma \in \Gamma$, all $x \in X$, $\gamma(x)$ and $\gamma(x_0)$ are conjugate in $\mathscr{L}(\bar{x})$, hence $\gamma(\bar{x}) \in (\mathscr{L}, \mathscr{L})(\bar{x})$ for all x, and so $\gamma \in (\mathscr{L}, \mathscr{L})(R)$, as in Remark 5.27. We claim that Γ is a MAD in $(\mathscr{L}, \mathscr{L})(R)$. Indeed, assume Γ is not a MAD in $(\mathscr{L}, \mathscr{L})(R)$ and let $\Gamma \subset \Gamma'$, where Γ' is a MAD of $(\mathscr{L}, \mathscr{L})(R)$. If Γ' is not finite, then repeating the above argument, it has a connected component, i.e., it contains the k-points of some torus S'. Consequently it contains infinitely many elements of finite order and, in particular, one that is not in Γ. Such an element is also k-diagonalizable on $\mathscr{G}(R)$ and hence $\mathcal{M} = S(k)\Gamma$ is not maximal in $\mathscr{G}(R)$. On the other hand, if Γ' is finite, then it is also k-diagonalizable on $\mathscr{G}(R)$ and $S(k)\Gamma'$ is bigger than $\mathcal{M}$. Since Γ in a MAD in $(\mathscr{L}, \mathscr{L})(R)$ and $\mathcal{M}^\circ = S(k)$ is a central torus, we have $\mathcal{M}$ is also a MAD in $\mathscr{L}(R)$.

Example 5.31. Let $G = SO_n$ and $A = \{\pm \underbrace{1\} \times \{\pm 1\} \times ... \times \{\pm}_{\text{n copies}} 1\} \in SO_n(\mathbb{C}) \mid$ even number of $-1\}$. We claim that $A \subset SO_n(\mathbb{C})$ is a MAD. To that end, since SO_n acts on $\mathbb{C}^n$, it can be written as a direct orthogonal sum as:

$$\mathbb{C}^n = L_1 \oplus L_2 \oplus ... \oplus L_n,$$

where $L_i = \mathbb{C}e_i$. Then

$$\begin{aligned} Z_{SO_n}(A) &= \{x \in O(L_1) \times O(L_2) \times ... \times O(L_n) \mid \det(x) = 1\} \\ &= \{\{\pm \underbrace{1\} \times \{\pm 1\} \times ... \times \{\pm}_{\text{n copies}} 1\} \mid \text{even numbers of } -1\} \\ &= A. \end{aligned}$$

Clearly, A is a k-diagonalizable subgroup of SO_n. Also, A is maximal since any element that commutes with $\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$ is of the form $\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, and as it is also contained in O_n, then $\lambda_i = \pm 1$.

The above example shows that there does exist a MAD which is finite, but in the algebra case, there is no such MADs.

Let Γ be the finite part of a MAD in $\mathscr{G}(R)$. We have shown that Γ is also a MAD in $\mathscr{L}(R)$ and we will now assume $\mathscr{L} = \mathscr{G}$. Now given a finite MAD of $\mathscr{G}$, we will show that the k-points of its double centralizer equals to itself (See Theorem 5.35).

Remark 5.32. Let $\Gamma = \{g_1, ..., g_n\} \subset \mathscr{G}(R)$ be a finite abelian group. We want to associate to Γ a finite commutative subgroup scheme of $\mathscr{G}$, for any $g_i \in \Gamma$, $g_i : R[\mathscr{G}] \to R$. Now we define a map

$$\psi : R[\mathscr{G}] \to R \times R \times ... \times R$$

by

$$\psi(f) := (g_1(f), g_2(f), ..., g_n(f)) \text{ where } n = |\Gamma|.$$

As we will see, this map of group schemes over R where $R \times ... \times R$ represents the constant group associated to Γ. The obvious candidate for its underlying scheme is $\underline{\Gamma} := \bigsqcup_i \Gamma_{g_i}$, where Γ_{g_i} is the image of $g_i : \operatorname{Spec}(R) \to X \times G$ which is isomorphic to $\operatorname{Spec}(R)$. We know $\underline{\Gamma}' := \bigcup_i \Gamma_{g_i} \subset \mathscr{G}$ is closed. Let I be the ideal of $R[\mathscr{G}]$ representing $\underline{\Gamma}'$. We have the exact sequence:

$$0 \to I \to R[\mathscr{G}] \to \prod R.$$

To verify that $\underline{\Gamma}$ is a group subscheme of $\mathscr{G}$ (i.e. $\underline{\Gamma} = \underline{\Gamma}'$), it is enough to show $\psi : R[\mathscr{G}] \to \prod R$ is surjective and the corresponding morphism $\underline{\Gamma} \to \mathscr{G}$ is a group homomorphism. We observe that

$$\psi \text{ is surjective} \Leftrightarrow \Gamma_{g_i} \cap (\bigsqcup_{i \neq j} \Gamma_{g_j}) = \emptyset.$$

The latter equation holds. Indeed, suppose $p \in \Gamma_{g_i} \cap \Gamma_{g_j}$, we may assume p is a closed point and since $\mathscr{G}$ is a variety, this means there exists $x \in X$ such that $g_i(x) = g_j(x)$. As X is a variety, by Proposition 5.7, $g_i^{-1}g_j(x) = g_i^{-1}g_j(x_0) = 1$ in $G(\bar{x})$. Hence $g_i^{-1}g_j = 1$ since R is reduced and so $\Gamma_{g_i} = \Gamma_{g_j}$. Hence $\underline{\Gamma}$ can be consider as k-variety: Since $\underline{\Gamma}$ is an affine scheme with coordinate ring $R[\Gamma_0]$, where $\Gamma_0 = \underline{\Gamma}(x_0)$ (geometrically that means $X \times \Gamma'$, where $\Gamma' = \operatorname{Spec}(k[\Gamma_0])$), so $\underline{\Gamma}(k) = X(k) \times \Gamma_0$. By Corollary 4.5 of [Wat79], it suffices to consider k-points, i.e., to show that for any closed point $x \in X$, the map $\underline{\Gamma}(x) \to \mathscr{G}(x)$ is a group homomorphism. However, $\underline{\Gamma}(x)$ is the image of Γ under the homomorphism

$\mathscr{G}(R) \to \mathscr{G}(x)$. So $\underline{\Gamma}(x)$ is a subgroup of $\mathscr{G}(x)$.

The multiplication $\underline{\Gamma} \times \underline{\Gamma} \to \underline{\Gamma}$ on $\underline{\Gamma}$ is given by

$$\bigsqcup_i \Gamma_{g_i} \times \bigsqcup_j \Gamma_{g_j} \to \bigsqcup_{i,j} \Gamma_{g_i g_j}.$$

The corresponding coordinate ring is the ring of functions

$$R^\Gamma := \{f : \Gamma \to R \,|\, \text{ f is a map of sets}\},$$

whose addition and multiplication are defined componentwise, and whose 0 and 1 are the constant maps with value 0 and 1 respectively. The comultiplication $\Delta : R^\Gamma \to R^\Gamma \otimes R^\Gamma \cong R^{\Gamma\times\Gamma}$ is characterized by $\Delta(f)(g_i, g_j) = f(g_i g_j)$, the counit $\epsilon^* : R^\Gamma \to R$ by $\epsilon^*(f) = f(1)$, and the coinverse $\iota : R^\Gamma \to R^\Gamma$ by $\iota(f)(g_i) = f(g_i^{-1})$.

We also observe that $\underline{\Gamma}(R) = \Gamma$. We have $\underline{\Gamma}(R) = (\bigsqcup_i \Gamma_{g_i})(R) = \bigsqcup_i \Gamma_{g_i}(R)$ since each Γ_{g_i} is a scheme defined over $\mathrm{Spec}(R)$. And $\bigsqcup_i \Gamma_{g_i}(R) = \bigsqcup_i \{g_i\} = \Gamma$.

Proposition 5.33. *Let $H \subset \mathscr{G}(R)$ be a finite k-diagonalizable subgroup. Then for every closed point $x \in X$, $H(x)$ and $H_0 = H(x_0)$ are conjugate in $\mathscr{G}(\bar{x})$.*

Proof. If H is cyclic, then the proposition follows from Proposition 5.7, since for any $a \in \Gamma$, we have $ha^n(x)h = a_0^n$. Let $S \subset H$ be a maximal subgroup for which $S(x_0)$ and $S(x)$ are conjugate in $\mathscr{G}(x)$, for all $x \in X$, then S is not trivial. Let $\underline{S}$ be defined as in Remark 5.32. Let $S_0 = S(x_0)$ and let $\underline{S}_0$ be the constant group $X \times S_0$ (where we identify the group of points $G(x_0)$ with the variety $G = G_{k(x_0)}$).

Let $T \subset \mathscr{G}$ be the transporter $T = \mathrm{Trans}(\underline{S}, S_0)$, i.e., for any R-algebra L, $T(L) = \mathrm{Tran}_{\mathscr{G}}(\underline{S}, \underline{S}_0)(L) := \{x \in \mathscr{G}(L) \,|\, x\underline{S}_L x^{-1} = \underline{S}_{0L}\}$. Its k-points are,

$$T(k) = \{(x, g) \in X(k) \times G(k) | gS(x)g^{-1} = S_0\}.$$

By Proposition 2.6, the subscheme T is a closed and smooth subscheme of $X \times G$. Since $X = \mathrm{Spec}(R)$ is a k-variety, T can be considered as a k-variety, and by assumption $T \to X$ is surjective (on k-points). Next, we have $T \times_X \mathscr{G} \cong T \times G$, since $T \times_X \mathscr{G} \cong T \times_X X \times G \cong T \times G$. Then $T \times_X \underline{S} \cong T \times_X \underline{S}_0 \cong T \times S_0$ in $T \times_X \mathscr{G} \cong T \times G$. The group H embeds canonically into $\mathscr{G}(T)$, and is still k-diagonalizable: to that end, if $T = \mathrm{Spec}(A)$, then the coordinate ring of $T \times G$ is $A \otimes_R R[G]$ and any section of $\mathscr{G}(R)$ acts only on $R[G]$.

By Definition of T, there exists $g \in (T \times G)(A) = G(A)$ such that

$$g(T \times_X \underline{S}_0)(A)g^{-1} = (T \times S_0)(A). \tag{5.4.1}$$

As H commutes with S_0, we have $gHg^{-1} \subset (T \times Z)(A)$, where Z is the centralizer of S_0 in G, then Z is reductive. Let $x \in X$ be a closed point (i.e., $x \in X(k)$). Then x lifts to a geometric point y of T such that $G(x) \cong G(y)$ and $Z(x) \cong Z(y)$ as subgroups (indeed, the fibre T_x of T over x is a variety over $k = k(x)$ and so any closed point works.). Similarly, we may choose y_0 in T, a lift of x_0. This can be done in such a way that both y and y_0 are in the same connected component T° of T: Indeed, let N be the normalizer of S_0 in G. We have T° is open and N°-stable by right action, then $T^\circ \cdot N^\circ = T$. Let $m \in H$ be an element that is not in S. Recall Equation 5.4.1, then $gmg^{-1} \in (T \times Z)(A)$. By Proposition 5.7, there exists $h \in (T \times Z)(y)$ such that

$$hg(y)m(y)g^{-1}(y)h^{-1} = g_0 m_0 g_0^{-1}.$$

This implies that

$$g_0^{-1}hg(y)m(y)g^{-1}(y)h^{-1}g_0 = m_0.$$

Let $g_0^{-1}hg(y) = p$, then

$$pS(y)p^{-1} = g_0^{-1}hg(y)S(y)g^{-1}(y)h^{-1}g_0 = g_0^{-1}hS_0h^{-1}g_0^{-1} = g_0^{-1}S_0g_0 = S_0.$$

As x is arbitrary, $\langle S, m\rangle$ is a strictly bigger subgroup containing S that satisfies the property. Contradiction. $\square$

Here is an alternate proof of Proposition 5.33: Let S, T and m be defined as above. Let $N = N_G(S_0)$ be the normalizer of S_0 in G and let $Z = C(S_0)$ be the centralizer of S_0. Finally, let $C \subset G$ be the intersection of the G-conjugacy class of m_0 with Z. Then N acts on C by conjugation. Here we need the following well-known lemma.

Lemma 5.34. *N and N° have finitely many orbits on C, all of which are closed.*

Proof. It is enough to show that every orbit of N in C is open. Indeed: If every orbit is open, the closure of each orbit in C is a union of orbits and at most one orbit can be dense in an orbit closure. So it must be the only orbit in the closure and hence is also closed.

To that end, let $y \in C$. We observe that the G-conjugacy class of m_0 and y in G are the same. Consider the map $\pi : G \to C_y$ from G onto the conjugacy class of y in G which is closed, as y is semisimple (see Proposition 18.2 of [Hum87]). We have

$$d\pi_e(X) = R_y(X) - L_y(X) \text{ for all } X \in \mathfrak{g} = \text{Lie}(G)$$

where R_y is the derivative of $x \to xy^{-1}$ and L_y is the derivative of $x \to yx$. The map: $\mathfrak{g} \to T_e(C_y)$ is Id $-$ Ad(y). Hence as a subset of $T_y(G) = R_y(\mathfrak{g})$, the tangent space of C_y at y is the set of elements of the form $R_y(X - \text{Ad}(y)(X))$. Now $T_y(C) = T_y(Z \cap C_y) \subset T_y(Z) \cap T_y(C_y)$. By Theorem 13.4 of [Hum87], we have

$$\text{Lie}(Z_G(S_0)) = z_{\mathfrak{g}}(S_0) = \{X \in \mathfrak{g} | \text{Ad}(s)(X) = X \text{ for all } s \in S_0\}.$$

Then the tangent space $T_y(Z) = R_y(z_{\mathfrak{g}}(S_0))$. So translating back to e, we get that

$$R_{y^{-1}} T_y(C) \subset R_{y^{-1}}(T_y(Z) \cap T_y(C_y)) \subset \{t \in \mathfrak{g} | t = X - \text{Ad}(y)(X) \text{ for all } X \in \mathfrak{g}\}.$$

Since the identity component $N_G(S_0)^\circ = Z_G(S_0)^\circ$, we have Lie$(N) =$ Lie$(Z_G(S_0))$. Thus the tangent space of the N-orbit N_y at y is

$$z_{\mathfrak{g}}(S_0)/z_{\mathfrak{g}}(S_0) \cap z_{\mathfrak{g}}(y)$$

(See Proposition 9.1 of [Borel91]) and so

$$R_{y^{-1}}(T_y(N_y)) = \{X - \text{Ad}(y)(X) \,|\, X \in z_{\mathfrak{g}}(S_0)\}$$

. Since y is semisimple and $y \in Z$, we may choose an Ad(y)-stable complement K of $z_{\mathfrak{g}}(S_0)$ in $\mathfrak{g}$. Let $X = k + z$ with $k \in K$, $z \in z_{\mathfrak{g}}(S_0)$, then

$$\text{Ad}(y)(X) = \text{Ad}(y)(k) + \text{Ad}(y)(z) = k' + z'$$

with $z' \in z_G(S_0)$ and $k' \in K$. If Ad$(y)(X) - X \in R_{y^{-1}}(T_y(Z) \cap T_y(C_y))$, then

$$\text{Ad}(y)(X) - X = (k' - k) + (\text{Ad}(y)(z) - z) \in z_{\mathfrak{g}}(S_0).$$

Hence $k' - k = 0$ and Ad$(y)(X) - X =$ Ad$(y)(z) - z$. We obtain that $R_{y^{-1}} T_y(C) \subset R_{y^{-1}}(T_y(N_y))$ and hence $T_y(C) \subset T_y(N_y)$. As N_y is an orbit in C, $N_y \subset C$ and $T_y(N_y) \subset T_y(C)$. Hence $T_y(N_y) = T_y(C)$. It follows that every N-orbit is open in C.

As in the first proof, we conclude that m, as an element of $\mathscr{G}(T)$, is conjugate to a section of $T\times C$, i.e., there exists $g\in\mathscr{G}(T)$ such that $h=gmg^{-1}\in(T\times C)(T)$. Moreover, we have $T\times C=(T\times C_1)\cup(T\times C_2)\cup\ldots\cup(T\times C_n)$, where the C_i are the N-orbits in C. Thus, we obtain a partition of T into closed subsets $T_1,T_2,\ldots,T_n$ where

$$T_i=h^{-1}(T\times C_i)=\{y\in T\mid h(y)\in C_i\}=\{y\in T|g(y)m(y)g(y)^{-1}\in C_i\}\text{ (on points)},$$

where we may view h as a morphism $T\to T\times C$. Here T_i is closed and open. The connected component N° of N acts on T, it preserves connected components of T and hence T_i. Thus T_i/N° is a closed and open subset of T/N° which surjectively maps to X and subsequently $T_i\to X$ is surjective for at least one i.

Now if x is a closed point of X, then x lifts to a closed point y of T_i. So $S(y)$ is $G(y)$-conjugate to S_0 by means of $g(y)$ in $G(y)$ and $m(y)$ is mapped to an element of $Z(y)$ under this conjugation, in the same $N(y)$-conjugacy class as $m(y_0)$ (y_0 is the lift of x_0). Thus, as x is arbitrary, $\langle S,m\rangle$ is a strictly bigger subgroup containing S. Contradiction. $\square$

Theorem 5.35. *Let Γ be a finite MAD of $\mathscr{G}(R)$ and let $\Gamma_0=\Gamma(x_0)$ be the image of Γ in $\mathscr{G}(x_0)\cong G$. Then $\Gamma_0=ZZ(\Gamma_0)$.*

We begin with some general observations and fixing some notation that will be used throughout the proofs of this theorem.

Lemma 5.36. *Let $X\to Y\to Z$ be morphisms of affine schemes where Z is a variety. Assume that X/Z and X/Y are smooth and surjective, and $Y\to Z$ is bijective. Then Y/Z is étale. Moreover, $Y\simeq Z$.*

Proof. Let $z\in Z$ be a closed point. We have

$$X_z:=X\times_Z k(z)=X\times_Y(Y\times_Z k(z))\overset{\text{smooth}}{\longrightarrow}Y\times_Z k(z).$$

It follows that $Y\times_Z k(z)$ is nonsingular. For the corresponding coordinate rings, we have $A\to B\to C$, where A/C and B/C are faithfully flat. Let $0\to M\to N$ be a left exact sequence of A-modules. Then

$$M\otimes_A C=M\otimes_A B\otimes_B C\hookrightarrow N\otimes_A B\otimes_B C=N\otimes_A C,$$

since A/C is flat. Hence $M \otimes_A B \hookrightarrow N \otimes_A B$, since B/C is faithfully flat and so $Y \to Z$ is flat. Also $Y \to Z$ is bijective, then the relative dimension of Y/Z is zero. By Theorem 10.2 of [Hart99], Y/Z is étale. $\square$

Let Γ be a finite subgroup of $\mathscr{G}(R)$ and let $\Gamma_0 = \Gamma(x_0)$ be the image of Γ in $\mathscr{G}(x_0) = G(k) \subset \mathscr{G}(R)$. Let $\underline{\Gamma}$ and $\underline{\Gamma}_0$ be the corresponding finite group schemes of $\mathscr{G}$ defined in Remark 5.32. For any R-algebra S, we define the normalizer $\underline{N}$ of $\underline{\Gamma}_0$ in $\mathscr{G}$ to be

$$\underline{N}(S) := \{n \in \mathscr{G}(S)\,|n\underline{\Gamma}_{0S}n^{-1} \subseteq \underline{\Gamma}_{0S}\}$$

and the transporter $\underline{T}$ from $\underline{\Gamma}$ to $\underline{\Gamma}_0$ in $\mathscr{G}$ to be

$$\underline{T}(S) = \mathrm{Tran}_{\mathscr{G}}(\underline{\Gamma}, \underline{\Gamma}_0)(S) := \{x \in \mathscr{G}(S)\,|\,x\underline{\Gamma}_S x^{-1} = \underline{\Gamma}_{0S}\}.$$

Recall the definition of Transporter of two subschemes defined in Section 2.1, the condition should be $x\underline{\Gamma}_S x^{-1} \subseteq \underline{\Gamma}_{0S}$. But γ and γ_0 are finite groups and $\underline{\Gamma}(X) \subseteq \underline{\Gamma}_0(X)$. So it is same to $x\underline{\Gamma}_S x^{-1} = \underline{\Gamma}_{0S}$. By Proposition 2.6, $\underline{N}$ and $\underline{T}$ are closed and smooth since $\underline{\Gamma}$ and $\underline{\Gamma}_0$ are both of multiplicative type.

Remark 5.37. We observe that $\underline{N}$ and $\underline{T}$ are affine subscheme of $X \times G$. Since $\underline{N}$ and $\underline{T}$ are smooth over the k-variety $X = \mathrm{Spec}(R)$, they are affine varieties over k as well. Thus $T(k) = \underline{T}(k) = \{(x, g) \in X(k) \times G(k) \mid g\Gamma(k(x))g^{-1} = \Gamma_0\}$ which is not empty by Proposition 5.33, and hence $\underline{T} \to X$ is surjective. We can think of $\underline{T}$ and $\underline{\Gamma}$ by means of their k-points. For convenience in what follows the k-variety structure of $\underline{T}$ and $\underline{\Gamma}$ will be denoted by T and $\underline{\Gamma}_k$ respectively. And let $N = \underline{N}(x_0)$. For varieties over k we freely switch between $X(k)$ and X whenever convenient.

Lemma 5.38. *The morphism $T \to X$ is a principal N-bundle.*

Proof. We have that $T \to T/N$ is a principal N-bundle. It is enough to show that $T/N \cong X$. Fix $x \in X$, let $(x, g), (x, h) \in T$ such that $g\Gamma(x)g^{-1} = \Gamma_0$ and $h\Gamma(x)h^{-1} = \Gamma_0$, then $hg^{-1}\Gamma_0 gh^{-1} = \Gamma_0$ and hence $hg^{-1} \in N$, i.e., $h \in gN$. Thus the right action of N on T only has one orbit. Hence $T/N \to X$ is bijective and so $T \to X$ is surjective. By the above lemma, we conclude that $T/N \simeq X$. Because $T \times_X T \cong T \times_k N$, we also have $T \times_X T \cong T \times_X \underline{N}$. $\square$

Lemma 5.39. *Let $\underline{\Gamma}_k$, Γ_0, T and N be defined as above. Then $T \times^N \Gamma_0 = (T \times \Gamma_0)/N \cong \underline{\Gamma}_k$.*

Proof. It is enough to consider k-points. Define the map $T \times^N \Gamma_0 \to \underline{\Gamma}_k$ by $((x,g),\gamma) \to (x, g^{-1}\gamma g)$. By the definition of T, this map is surjective. Fix $x \in X$, let $((x,g),\gamma)$ and $((x,h),\delta)$ be elements of $T \times^N \Gamma_0$ with the same image in Γ. Since T has only one N-orbit, there exists $n \in N$ such that $g = nh$. We have

$$g^{-1}\gamma g = h^{-1}n^{-1}\gamma nh = h^{-1}\delta h,$$

which yields $n^{-1}\gamma n = \delta$. So the above map is injective. The result follows because it becomes an isomorphism when pulled back under the faithfully flat map $T \to X$. $\square$

With these results in hand, we are now in a position to prove Theorem 5.35.

Proof of Theorem 5.35.

Since $\underline{T}$ and $\underline{N}$ are affine k-varieties, we can argue this by means of k-points. Let N° be the connected component of N. Since Γ_0 is discrete, N° acts on Γ_0 trivially. Let $T' = T/N^\circ$ and $N' = N/N^\circ$. It is easy to see that $T' \to T'/N' = (T/N^\circ)/(N/N^\circ) \cong X$ is a principal N'-bundle. Let T'_0 be a connected component of T'. Then we have $T'_0 \to X$ is surjective, since the image of T'_0 is open and closed and X is connected.

Let $N'_0 = \mathit{Norm}_{N'}T'_0 = \{g \in N' \mid gT'_0g^{-1} \subseteq T'_0\}$. It is easy to see that $T'_0 \times^{N'_0} N' \to T'$ by $[t,n] \mapsto nt$ is an isomorphism. It follows that

$$T'_0 \times^{N'_0} \Gamma_0 \cong T'_0 \times^{N'_0} N' \times^{N'} \Gamma_0 \cong T' \times^{N'} \Gamma_0 \cong T \times^N \Gamma_0 \cong \underline{\Gamma}_k.$$

Claim: The sections of $T'_0 \times^{N'} \Gamma_0 \to X$ are the same as the N'-equivariant sections of $T'_0 \times \Gamma_0 \to T'_0$. To that end, consider the following diagram:

$$\begin{tikzcd} T'_0 \times_k \Gamma_0 \arrow[r, "/N'"] \arrow[d] & T'_0 \times^{N'} \Gamma_0 \arrow[d] \\ T'_0 \arrow[r, "/N'"] \arrow[ur, dotted, "f"] & X \end{tikzcd}.$$

Given a section $\sigma : X \to T'_0 \times^{N'} \Gamma_0$, we can compose σ with the quotient map to obtain the map $f : T'_0 \to T'_0 \times^{N'} \Gamma_0$ defined by $t \mapsto [t, \delta_t]$. Pulling this function back we have

$$
\begin{array}{ccccccc}
f' : T_0' & \longrightarrow & T_0' \times_X (T_0' \times^{N'} \Gamma_0) & \xrightarrow{\sim} & (T_0' \times_k N') \times^{N'} \Gamma_0 & \xrightarrow{\sim} & T_0' \times_k \Gamma_0 \\
 & & & & & & \\
t & \longrightarrow & (t, [t, \delta_t]) & \longrightarrow & [(t,1), \delta_t] & \longrightarrow & (t, \delta_t)
\end{array}
$$

where $T_0' \times_X T_0' \cong T_0' \times_k N'$ is given by $(t, s) \mapsto (t, s/t)$ and $(T_0' \times_k N') \times^{N'} \Gamma_0 \cong T_0' \times_k \Gamma_0$ is given by $[(t,n), \delta] \mapsto (t, n\delta n^{-1})$. The map f' defines an N'-equivariant section. The converse is clear, because $f : T_0' \to T_0' \times \Gamma_0$ is N'-equivariant, $f' : T_0'/N' \to (T_0' \times \Gamma_0)/N'$ is the required section. Then the sections of $T_0' \times^{N'} \Gamma_0 \cong \bar{\Gamma}$ over R are precisely Γ which is an one-to-one correspondence with Γ_0. The sections of $T_0' \times \Gamma_0$ over T_0' are Γ_0. We have $\Gamma_0 = \Gamma_0^{N_0'}$. It follows that N_0' acts on Γ_0 trivially. Let N_0 be the preimage of N_0' under the map $N \to N/N^\circ$. Let T_0 be the preimage of T_0' under the map $T \to T/N^\circ$. Since the connected component N° is normal, it is easy to see that $N_0 = \mathrm{Norm}(T_0)$. Since N_0' acts on Γ_0 trivially and N^0 acts on Γ_0 trivially, we have N_0 acts on Γ_0 trivially and hence $N_0 \subset Z(\Gamma_0)$.
Let $D = ZZ(\Gamma_0) \subset G$. Again we have $T_0 \times^{N_0} D \cong T_0' \times^{N_0'} \times N_0 \times^{N_0} D \cong T_0' \times^{N_0'} D$. Suppose there is a subgroup Σ of D containing Γ_0 which is finite and k-diagonalizable on $k[G]$. Since $T \times^N G \cong \mathscr{G}$ and $T_0 \times^{N_0} N \cong T$, we have $T_0 \times^{N_0} G \cong T_0 \times^{N_0} N \times^N G \cong T \times^N G \cong \mathscr{G}$. Therefore, we have

$$T_0 \times^{N_0} \Gamma_0 \subset T_0 \times^{N_0} \Sigma \subset \mathscr{G} = X \times G.$$

Hence $(T_0 \times^{N_0} \Gamma_0)(R) \subset (T_0 \times^{N_0} \Sigma)(R) \subset \mathscr{G}(R)$. As $N_0 \subset Z(\Gamma_0)$, D commutes with N_0. This implies that

$$(T_0 \times^{N_0} \Sigma)(R) = (T_0/N_0 \times \Sigma)(R) = (X \times \Sigma)(R) \subset \mathscr{G}(R),$$

and

$$(T_0 \times^{N_0} \Gamma_0)(R) = (X \times \Gamma_0)(R) = \underline{\Gamma}(R) = \Gamma.$$

It follows that $\Gamma \subset (X \times \Sigma)(R) \cong \Sigma \subset \mathscr{G}(R)$. By Theorem 5.29, $(X \times \Sigma)(R)$ is k-diagonalizable on $R[\mathscr{G}]$ as a subgroup of $\mathscr{G}(R)$. Then $\Gamma = (X \times \Sigma)(R)$ since Γ is a maximal finite k-diagonalizable subgroup of $\mathscr{G}(R)$. Hence, $\Gamma_0 = \Sigma \subset G$. We concludes that $ZZ(\Gamma_0) = D = \Gamma_0$ as otherwise there is such a Σ $\square$

5.5 Finite subgroups in classical algebraic groups

From the last section, we know that any finite MAD Γ in $\mathscr{G}(R)$ satisfy the property $ZZ(\Gamma_0) = \Gamma_0$, where $\Gamma_0 = \Gamma(x_0)$. For each of the classical groups, we will examine whether or not there exists a finite abelian subgroup A with the property $ZZ(A) = A$. (The results of this section are elementary and well-known.)

• **Type A**: let $G = SL(V)$, where V is a k-vector space of dimension $n+1$, and let $A \subset SL(V)$ be a finite abelian subgroup. Since A is k-diagonalizable on V, we can decompose $V = V_{\alpha_1} \oplus ... \oplus V_{\alpha_t}$ with respect to A, where α_i is the corresponding character with values in k. We have

$$\begin{aligned} Z(A) &= \{g \in G \mid g(V_{\alpha_i}) = V_{\alpha_i} \text{ for all } i\} \\ &= \{g \in GL(V_{\alpha_1}) \times ... \times GL(V_{\alpha_t}) \mid \det(g) = 1\}. \end{aligned}$$

It follows that

$$A = ZZ(A) = \{(a_1, a_2, ..., a_t) \in k^* \times k^* \times ... \times k^* \mid a_1 a_2 ... a_t = 1\}$$

where we should consider $(a_1, a_2, ..., a_t)$ is a diagonal element in $SL(V)$. But as A is finite, we conclude there exists no such subgroup in type A_n.

• **Type C**: let $G = Sp(V)$, where V is a k-vector space of dimension $2n$ with non-degenerated symplectic bilinear form ω.

Let $A \subset SL(V)$ be a finite abelian subgroup. We write $V = V_0 \oplus V_{\alpha_1} \oplus ... \oplus V_{\alpha_t}$ with respect to A, where α_i is the corresponding eigenvalue. Let $v \in V_{\alpha_i}$ and $w \in V_{\alpha_j}$. For any $a \in A$, we have

$$\omega(av, aw) = \omega(\alpha_i(a)v, \alpha_j(a)w) = (\alpha_i + \alpha_j)(a)\omega(v, w) = \omega(v, w).$$

We conclude that:

(i) If $\alpha_i = -\alpha_i$, then ω is non-degenerate on V_{α_i}.

(ii) If $\alpha_i \neq -\alpha_i$, then $\omega = 0$ on V_{α_i}.

The second situation cannot occur, otherwise, in this case, ω is non-degenerated on

$V_{\alpha_i} \oplus V_{-\alpha_i}$ and

$$GL(V_{\alpha_i}) \subset Sp(V_{\alpha_i} \oplus V_{-\alpha_i}) \subset Sp(V),$$

but $GL(V_{\alpha_i})$ commutes with A and $k^* \subset Z(GL(V_{\alpha_i})) \subset ZZ(A) = A$. In addition, the trivial eigenspace cannot appear in the decomposition. If so, we would have $\{\pm 1\} = Z(Sp(V_0)) \subset ZZ(A) = A$, but the image of A in $Sp(V_0)$ is $\{1\}$.

Therefore, $V = V_{\alpha_1} \oplus V_{\alpha_2} \oplus \cdots \oplus V_{\alpha_t}$, $\alpha_i = -\alpha_i$ and dim(V_{α_i}) is even. Thus $Z(A) = Sp(V_{\alpha_1}) \times Sp(V_{\alpha_2}) \times \cdots \times Sp(V_{\alpha_t})$, and also

$$A = ZZ(A) = \{\pm I_1\} \times \{\pm I_2\} \times \cdots \times \{\pm I_t\},$$

where I_i is an identity matrix with the same dimension as V_{α_i}. Since dim(V_{α_i}) is even, the determinant of $\pm I_i$ is 1 for each i. It follows that $\{\pm I_1\} \times \{\pm I_2\} \times \cdots \times \{\pm I_t\} \subset Sp(V)$.

• $\mathbf{Type\ B}$ and $\mathbf{D}$: First we consider the case $G = SO(V)$, where V is a k-vector space of dimension n. Let $H \subset O(V)$ be a finite abelian subgroup such that $H = ZZ(H)$. By arguments similar to those given for Type C, $Z(H) = O(V_{\alpha_1}) \times O(V_{\alpha_2}) \times \cdots \times O(V_{\alpha_t})$ and hence

$$H = ZZ(H) = \{\pm I_1\} \times \{\pm I_2\} \times \cdots \times \{\pm I_t\}.$$

By the definition of the special orthogonal group, we have the short exact sequence

$$1 \to SO(V) \to O(V) \to \{\pm 1\} \to 1.$$

Let H be such that $\alpha_i \neq \det + \alpha_j$ for all $i \neq j$. We claim that $A := H \cap SO(v) = \{x \in \{\pm I_1\} \times \{\pm I_2\} \times \cdots \times \{\pm I_t\} \,|\, \det(x) = 1\}$ is a subgroup of $SO(V)$ with the property $Z_{SO(V)} Z_{SO(V)}(A) = A$. Indeed, A and H have the same eigenvalues, so $Z_{O(V)}(A) = Z_{O(V)}(H)$. Then

$$\begin{aligned} Z_{SO(V)} Z_{SO(V)}(A) &= Z_{SO(V)} Z_{O(V)}(A) = Z_{SO(V)} Z_{O(V)}(H) \\ &= SO(V) \cap Z_{O(V)} Z_{O(V)}(H) = H \cap SO(V) \\ &= A. \end{aligned}$$

How about the converse? Given a finite subgroup of $SO(V)$ whose double centralizer equal to itself, can we construct a finite group coinciding with H?

When n is odd, we have

$$O(V) = SO(V) \cup (-1)SO(V).$$

Let A be a finite diagonalizable subgroup of $SO(V)$ such that $A = ZZ(A)$. Then $H = A \cup -A$ is the corresponding subgroup in $O(V)$. Indeed, $Z_{O(V)}(A) \supset Z_{SO(V)}(A)$, so $Z_{O(V)}Z_{O(V)}(A) \subset Z_{O(V)}Z_{SO(V)}(A) \subset A \cup -A$. As -1 is also in the center of $O(V)$, $Z_{O(V)}Z_{O(V)}(-A) \subset A \cup -A$. Hence $Z_{O(V)}Z_{O(V)}(A \cup -A) = A \cup -A$.

When n is even, V is a k-vector space with even dimension. We have the decomposition $V = V_{\alpha_1} \oplus V_{\alpha_2} \cdots V_{\alpha_t}$ with respect to H. There are two situations:

(i) All the V_{α_i} have even dimension,

(ii) Some of the V_{α_i} have odd dimension.

Situation 1: We have $H = ZZ(H) = \{\pm I_1\} \times \{\pm I_2\} \times \cdots \times \{\pm I_t\}$. For each i, $\det(I_i) = 1$, so $H \subset SO(V)$.

Situation 2: Without loss of generality, we suppose $\dim(V_{\alpha_1})$ is odd. Let x be a diagonal matrix with entries $(-I_1, I_2, \cdots, I_t)$. We conclude

$$O(V) = SO(V) \cup xSO(V).$$

Clearly, x commutes with A and so $H := A \cup xA \subset Z_{O(V)}Z_{O(V)}(A \cup xA)$. By the definition of x, x preserves the decomposing of V with respect to A. Thus V has the same decomposition with respect to $A \cup xA$ and A. Consequently, $Z_{O(V)}(A \cup xA) = Z_{O(V)}(A) - Z_{SO(V)}(A) \cup xZ_{SO(V)}(A)$. It follows that

$$Z_{O(V)}Z_{O(V)}(A \cup xA) = Z_{O(V)}Z_{O(V)}(A) \subset Z_{O(V)}Z_{SO(V)}(A \cup xA) \subset A \cup xA$$

and hence $Z_O Z_O(A \cup xA) = A \cup xA$. In all cases the eigenspace of A and H coincide.

Next we will consider the simply-connected case. To that end let $G = Spin(V)$. First, we consider the Pin group (defined in Example 2.24). Let $A \subset Pin(V)$ be a finite k-diagonalizable subgroup such that $ZZ(A) = A$. Consider the following exact sequence:

$$1 \to \{\pm 1\} \to Pin(V) \xrightarrow{\psi} O(V) \to 1.$$

Let H be the image of A under ψ. Thus H is abelian and finite. Subsequently, H decomposes V as $V = V_1 \oplus V_2 \oplus ... \oplus V_n$. The centralizer of H is :

$$Z_{O(V)}(H) = O(V_1)\times O(V_2)\times ...\times O(V_t)\times GL(V_{\alpha_1}\oplus V_{-\alpha_1})\times ...\times GL(V_{\alpha_j}\oplus V_{-\alpha_j})$$

where the first t components correspond to the eigenvalues α with the property $\alpha = -\alpha$ and the last $(n-t)/2$ components correspond to the the eigenvalue α with $\alpha \neq -\alpha$. It follows that $Z_{O(V)}Z_{O(V)}(H) = \{\pm I_1\} \times ... \times \{\pm I_t\} \times k^* \times ... \times k^*$.

By Lemma 2.27 of [Mil11], we have

$$\begin{aligned} \text{Lie}(ZZ(A)) &= \text{Lie}(Spin(V))^{Z(A)} = \{X \in \mathfrak{o}(V) | zXz^{-1} = X \ \ \forall \ \ z \in Z(A)\} \\ &= \text{Lie}(A) = 0. \end{aligned}$$

Since $\psi(Z(A)) \subset Z(\psi(A)) \subset O(V)$, it follows that

$$\begin{aligned} 0 = d\psi(ZZ(A)) &= \{x \in \mathfrak{o}(V) | zXz^{-1} = X \ \ \forall \ \ z \in \psi(Z(A))\} \\ &\supseteq \{x \in \mathfrak{o}(V) | zXz^{-1} = X \ \ \forall \ \ z \in Z(H)\} = \text{Lie}(ZZ(H)) \end{aligned}$$

We deduce that $\dim(ZZ(H)) = \dim(\text{Lie}(ZZ(H)) = 0$, and therefore

$$ZZ(H) = \{\pm I_1\} \times \{\pm I_2\} \times ... \times \{\pm I_t\}.$$

For each V_i, consider the following exact sequence $1 \to \{\pm 1\} \to Pin(V_i) \xrightarrow{\psi} O(V_i) \to 1$. Let $\{\pm I\} \in O(V_i)$. Let Γ_i be the preimage of $\{\pm I\}$ in $Pin(V_i)$. We obtain

$$1 \to \{\pm 1\} \to \Gamma_i \to \{\pm 1\} \to 1.$$

We conclude that Γ_i is an abelian group of order 4 which is isomorphic to the cyclic group Z_4 when $n \equiv 1, \ 2 \bmod 4$ or the Klein four-group V_4 when $n \equiv 0, \ 3 \bmod 4$ (See [AL95] for details).

Consider the following diagram:

$$\begin{tikzcd} Pin(V_1) \times ... \times Pin(V_n) \arrow[r, "\varphi"] \arrow[d, "\phi"'] & O(V_1) \times ... \times O(V_n) \arrow[d, "i"] \\ Pin(V) \arrow[r, "\psi"] & O(V) \end{tikzcd},$$

where i is the inclusion map. Since $C(V \oplus V') \cong C(V) \otimes C(V')$, by the definition of $Pin(V)$, for each i, $Pin(V_i)$ is a subgroup of $Pin(V)$, and thus ϕ is the product map. Let $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$ be the preimage of $ZZ(H) = \{\pm I_1\} \times \{\pm I_2\} \times ... \times \{\pm I_t\}$ under φ and let $\Gamma = \mathrm{Im}(\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n)$ in $Pin(V)$. Recall A is a finite subgroup of $Pin(V)$ such that $ZZ(A) = A$. Then $\psi(A) = H \subset ZZ(H) = \{\pm I_1\} \times \{\pm I_2\} \times ... \times \{\pm I_t\}$ and hence

$$A = \psi^{-1}(H) \subset \psi^{-1}(\{\pm I_1\} \times \{\pm I_2\} \times ... \times \{\pm I_t\}) = \Gamma.$$

Therefore, $Z(A) \supseteq Z(\Gamma)$.

On the other hand, $\psi(Z(A)) \subset Z(\psi(A)) = Z(H) = O(V_1) \times O(V_2) \times ... \times O(V_n)$. Because the above diagram commutes, $Z(A) \subset \psi^{-1}(O(V_1) \times O(V_2) \times ... \times O(V_n)) = \psi(Pin(V_1) \times ... \times Pin(V_n))$. It follows that

$$A = ZZ(A) \supseteq Z(\psi(Pin(V_1) \times ... \times Pin(V_n))) \supseteq \Gamma.$$

The last inclusion holds since $\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n \subseteq Z(Pin(V_1) \times ... \times Pin(V_n))$, and $\Gamma = \psi(\Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n) \subseteq \psi(Z(Pin(V_1) \times ... \times Pin(V_n))) \subseteq Z(\psi(Pin(V_1) \times ... \times Pin(V_n)))$.

It follows that

$$A = ZZ(A) = ZZ(\Gamma) = \Gamma.$$

If $\Gamma \subset Pin(V)$ is a finite abelian subgroup such that $\Gamma = ZZ(\Gamma)$, then $\Gamma \cap Spin(V)$ is a subgroup in $Spin(V)$ with that property. Indeed, $H = \Gamma \cap Spin(V)$ satisfies $ZZ(H) = H$, if $\psi(H) \subset SO(V)$ does. We have

$$\psi(H) \subseteq \psi(ZZ(H)) \subseteq Z(\psi(Z(H))) \subseteq ZZ(\psi(H)) = \psi(H).$$

Let $A \subset Spin(V)$ be a finite abelian subgroup such that $A = ZZ(A)$. Since $Z_{Spin}(A) \subset Z_{Pin}(A)$, we have

$$A = Z_{Spin}Z_{Spin}(A) \subset Z_{Pin}Z_{Spin}(A) \subseteq Z_{Pin}Z_{Pin}(A).$$

The first inclusion has at most index 2, hence $A \subset Z_{Pin}Z_{Pin}(A)$ with at most index 2. Consequently, $Z_{Pin}Z_{Pin}(A)$ is finite.

Let $x \in Z_{Pin}Z_{Pin}(A)$. x commutes with $Z_{Pin}(A)$, but as $A \subset Z_{Pin}(A)$, we have x commutes with A, hence $x \in Z_{Pin}(A)$. It follows that x commutes with all elements in $Z_{Pin}Z_{Pin}(A)$ and so $Z_{Pin}Z_{Pin}(A)$ is abelian.

Let $\Gamma = Z_{Pin} Z_{Pin}(A)$. We claim that Γ is the finite abelian subgroup in $Pin(V)$ corresponding to A. To that end, we will show

(i) $Z_{Pin} Z_{Pin}(\Gamma) = \Gamma$,

(ii) $\Gamma \cap Spin(V) = A$.

For (i), since $A \subset \Gamma$, $Z_{Pin}(\Gamma) \subset Z_{Pin}(A)$. In addition, by the definition of Γ, Γ commutes with $Z_{Pin}(A)$, then $Z_{Pin}(A) \subset Z_{Pin}(\Gamma)$. Hence

$$Z_{Pin}(\Gamma) = Z_{Pin}(A).$$

Therefore, $\Gamma = Z_{Pin} Z_{Pin}(A) = Z_{Pin} Z_{Pin}(\Gamma)$.

For (ii), clearly, $A \subset \Gamma \cap Spin(V)$. Let $x \in \Gamma \cap Spin(V)$. Since $x \in Z_{Pin} Z_{Pin}(A)$, we have $(x, Z_{Pin}(A)) = 1$. Then $(x, Z_{Spin}(A)) = 1$. Since $x \in Spin(V)$, we conclude that $x \in Z_{Spin} Z_{Spin}(A) = A$.

5.6 Conclusion

The main thrust of this dissertation is to investigate the question of conjugacy of maximal abelian k-diagonalizable subgroups of $G(R)$ when G is a connected reductive group over k and R is a reduced associative commutative unital k-algebra. The result we aim for is in the spirit of Pianzola's work, as explained in Chapter 3. The two main theorems of this dissertation are:

Theorem 5.40 (Theorem 5.23)**.** *Let G be a reductive algebraic group over k such that its derived group is simply connected. Let $X = \mathrm{Spec}(R)$ be a connected reduced affine scheme, and $\mathscr{G} = G \times X$ its group scheme over X.*

If the Pic(X) is trivial then all regular maximal abelian k-diagonalizable subgroups of $\mathscr{G}(R)$ are conjugate under $\mathscr{G}(R)$.

Theorem 5.41. *Let $\mathscr{G}$ and X be defined as above. Consider the following property on X:*

(**TLT**)*If $\mathscr{L}$ is the Levi subgroup of a standard parabolic subgroup of $\mathscr{G}$, then any locally trivial principal homogeneous space for $\mathscr{L}$ over X is trivial.*

Assume the above property holds, we have

- *if G is a group of type A, then all maximal abelian k-diagonalizable subgroups of $\mathscr{G}(R)$ are connected (and hence all conjugate by Theorem 5.28),*

- *if G is a group of type B, C, D, then any maximal abelian k-diagonalizable subgroup $\mathcal{M}$ is conjugate to the group of the form $T(k) \cdot \Gamma$ where T is a torus of G and Γ is a finite group such that $ZZ(\Gamma) = \Gamma$ in $Z_G(T)$.*

The research presented in this thesis is part of the increasingly active investigation of understanding $G(R)$ and its representations. As with Lie algebras, the structure theory of reductive groups uses root systems arising from adjoint representation. The role of a maximal torus for reductive algebraic group over k will be played here by a MAD. The conjugacy problem for MADs are based on the viewpoint of the conjugacy theorem for maximal torus. This viewpoint may allow one to investigate the conjugacy problem for MADs over a non-perfect field k and more arbitrary groups and rings.

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