Categories of Weight Modules for Unrolled Quantum Groups and Connections to Vertex Operator Algebras

by

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Abstract

Recently, numerous connections between the categories of modules $\mathrm{Rep}_{\langle s\rangle}\mathcal{M}(r)$ for the singlet vertex operator algebra $\mathcal{M}(r)$ and $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ for the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}_2)$ at $2r$-th root of unity have been established. This has led to the conjecture that these categories are ribbon equivalent. In this thesis, we focus on extending the known connections between the singlet vertex algebra and unrolled quantum groups to the B_r vertex algebra, and developing unrolled quantum groups in higher rank. In the first portion of this thesis, we use the conjectural ribbon equivalence between $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ and $\mathrm{Rep}_{\langle s\rangle}\mathcal{M}(r)$ to identify algebra objects $\mathcal{A}_r$ associated to the B_r vertex operator algebra and show that the properties of its category of local modules, $\mathrm{Rep}^0\mathcal{A}_r$, compare nicely to that of $\mathrm{Rep}_{\langle s\rangle}B_r$.

For the second portion of this thesis, we begin by showing that the category of weight modules $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{g})$ for the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$ is generically semisimple and ribbon with trivial Müger center. We then construct families of commutative (super) algebra objects in $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{g})$ and study their categories of local modules. Their irreducible modules are determined and conditions for these categories being finite, non-degenerate, and ribbon are derived. Among these commutative algebra objects are examples whose categories of local modules are expected to compare nicely to module categories of the higher rank triplet $W_Q(r)$ and $B_Q(r)$ vertex algebras. Lastly, we restrict to the case of $\overline{U}_i^H(\mathfrak{sl}_3)$. The structure and characters of irreducible modules are determined, and Loewy diagrams for all Verma and projective modules are found.

Preface

The content of Chapters 2, 3, and 4 are based on three articles

[ACKR] - J. Auger, T. Creutzig, S. Kanade, M. Rupert, Braided Tensor Categories related to Bp Vertex Algebras, Commun. Math. Phys. (2020).

[CRu] - T. Creutzig and M. Rupert, Uprolling Unrolled Quantum Groups, [arXiv:1910.05922].

[R] - M. Rupert, Categories of Weight Modules for Unrolled Restricted Quantum Groups at Roots of Unity, [arXiv:1910.05922].

Although the first two articles are coauthored, I was responsible for the majority of manuscript and proof writing. The content of Chapter 5 will appear in an article coauthored with Thomas Creutzig and David Ridout. I am responsible for the proofs appearing in this chapter.

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Contents

Chapter 1

Introduction

The unrolled quantum groups $U_q^H(\mathfrak{g})$ associated to a finite dimensional simple complex Lie algebra $\mathfrak{g}$ were initially introduced and studied, primarily at odd roots of unity, as examples for producing link invariants through their categories of weight modules (see [GP1, GP2, GPT1]). Further connections between the unrolled quantum groups, knot invariants, and topological quantum field theories have been explored in [BCGP, D, DGP]. The study of these quantum groups has favored odd roots of unity in part due to the belief that topological applications won't differ much from those at even roots of unity. Recently, however, interest in unrolled quantum groups at even roots of unity has grown due to their potential connections to vertex operator algebras.

Relationships between module categories of vertex operator algebras and quantum groups have been studied since the early 1990's, starting with the pioneering work of Kazhdan-Lusztig [KL1, KL2, KL3, KL4]. They gave a braided equivalence between module categories for affine Lie algebras and corresponding quantum groups. These module categories for affine Lie algebras were later realized as module categories over certain affine vertex operator algebras [Fi, Zha, H5]. Motivated by logarithmic conformal field theory, the Kazhdan-Lusztig correspondence was explored in the context of the triplet vertex algebra $\mathcal{W}(r)$ ($r \in \mathbb{Z}_{\geq 2}$) and the restricted quantum group $\overline{U}_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ at $2r$-th root of unity through the work of Feigin, Gainutdinov, Semikhatov and Tipunin [FGST1, FGST2, FGST3, FGST4]. Stated therein was the conjecture that certain module categories over the triplet and restricted quantum group $\overline{U}_q(\mathfrak{sl}_2)$ are ribbon equivalent, and an abelian equivalence for these categories was later proven by Nagatomo and Tsuchiya for arbitrary r [NT]. It turns out, however, that the category of modules for the restricted quantum group $\overline{U}_q(\mathfrak{sl}_2)$ is not braidable for $r > 2$ [KS], while the triplet's module category is braided, so there cannot be a ribbon equivalence between these categories. Progress on this problem was made by Creutzig, Gainutdinov, and Runkel in [CGR] via the construction of a factorizable quasi-Hopf modification $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ of $\overline{U}_q(\mathfrak{sl}_2)$. This quasi-Hopf algebra was reconstructed as the quantum group realizing the category of local modules of some algebra object in the category $\mathcal{C}$ of weight modules for the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}_2)$. This approach was motivated by recently established connections between $\overline{U}_q^H(\mathfrak{sl}_2)$ and the singlet vertex algebra $\mathcal{M}(r)$ [CMR]. In fact, it follows from the results of [CGR] that if the module categories of $\mathcal{M}(r)$ and $\overline{U}_q^H(\mathfrak{sl}_2)$ are ribbon equivalent, then the module categories for $\mathcal{W}(r)$ and $\overline{U}_q(\mathfrak{sl}_2)$ are ribbon equivalent as well.

The singlet vertex operator algebras $\mathcal{M}(r)$ ($r \in \mathbb{Z}_{\geq 2}$) are a very prominent family of VOAs studied by many authors [A1, AM1, AM2, AM4, CM2, CMW]. The fusion rules for $\mathcal{M}(r)$ are known only for $r = 2$, but there is a conjecture for $r > 2$ [CM1]. It has been shown (see [CMR]) that if the fusion rules are as conjectured, then there exists an identification

$$\phi : \mathrm{Irr}(\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{g})) \to \mathrm{Irr}(\mathrm{Rep}_{\langle s\rangle}\mathcal{M}(r) \tag{1.0.1}$$

between sets of irreducible modules in $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ and the category $\mathrm{Rep}_{\langle s\rangle}\mathcal{M}(r)$ of modules generated by irreducible $\mathcal{M}(r)$ modules, which induces an isomorphism of Grothendieck rings. This map can be extended to indecomposables in a way which preserves Loewy diagrams [CGR] (see Definition 5.1.3). We also know [CMR, Theorem 1] that the regularized asymptotic dimensions of irreducible $\mathcal{M}(r)$-modules coincide exactly with normalized modified traces of open hopf links for the corresponding (under ϕ) $\overline{U}_q^H(\mathfrak{sl}_2)$-module. The map ϕ was the key ingredient in the construction of the quasi-Hopf algebra $\overline{U}_q^{\Phi}(\mathfrak{sl}_2)$ of [CGR]. There are a number of very interesting vertex algebras which can be constructed from the singlet, among them are the triplet $\mathcal{W}(r)$ and B_r ($r \in \mathbb{Z}_{\geq 2}$) vertex algebras. The techniques in [CGR] demonstrated that the relationship between the triplet and singlet could be exploited together with the map ϕ to construct braided tensor categories which compare nicely to the representation category $\mathrm{Rep}_{\langle s\rangle}\mathcal{W}(r)$ generated by irreducible $\mathcal{W}(r)$-modules. Naturally, we expect the same to be true for the B_r vertex algebra, which is the subject of Chapter 2. Constructing the quasi-Hopf algebra which serves as a candidate for the Kazhdan-Lusztig dual of B_r is an interesting problem for future study.

The higher rank analogues of the singlet $\mathcal{M}(r)$, triplet $\mathcal{W}(r)$, and B_r vertex algebras were introduced in [FT, C3, CM2] (see also [BM, Mi]) and are denoted by $W_Q^0(r)$, $W_Q(r)$, and $B_Q(r)$ respectively where Q is the root lattice of a simple finite dimensional complex Lie algebra $\mathfrak{g}$ of ADE type and $r \in \mathbb{Z}_{\geq 2}$. Vertex algebras in higher rank are notoriously difficult. Recently, however, some of the Feigin-Tipunin conjectures have been solved by Shoma Sugimoto for the higher rank triplet algebra $W_Q(r)$ [Su], and some connections to quantum groups have been explored by Lentner and Flandoli [Le, FL]. As in the rank one cases, their representation categories are expected to coincide with categories constructed from the category of weight modules of the corresponding unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{g})$ at $2r$-th root of unity. Understanding this category is therefore prerequisite to many interesting problems relating to the $W_Q^0(r)$, $W_Q(r)$, and $B_Q(r)$ vertex operator algebras, and is the subject of Chapters 3, 4, and 5. We now briefly describe the contents of the thesis before stating results explicity.

Argyres-Douglas theories are 4-dimensional supersymmetric field theories associated to pairs of Dynkin diagrams. The B_r vertex operator algebras appear as chiral (vertex) algebras for Argyres-Douglas theories of type (A_1, A_{2r-3}) [C3]. In Chapter 2 we construct and study a family of braided tensor categories associated to the B_r vertex algebras from the category of local modules of some corresponding algebra object. The categories constructed allow us to further probe the relationship between non-rational vertex algebras and quantum groups, and we test a conjectural Verlinde formula for the B_r algebras. Further, we show that the character of the B_r vertex operator algebra coincides with that of a Quantum-Hamiltonian reduction, as stated in [C2, Remark 5.6]. It has since been proven [ACGY, Theorem 12] that the B_r algebra is indeed a Quantum-Hamiltonian reduction, and the proof uses our result and additional results from [CHJRY].

The expected Kazhdan-Lusztig correspondence between vertex algebras and quantum groups occurs at even roots of unity, but most of the work done on unrolled quantum groups has been at odd roots of unity. It was shown in [GP1, GP2] that the category $\mathcal{C} := \mathrm{Rep}_{wt} U_q^H(\mathfrak{g})$ (see Definition 3.3.1) of finite dimensional weight modules for $U_q^H(\mathfrak{g})$ at odd roots is ribbon and generically semi-simple (see Definition 3.1.1). In Chapter 3, we study the category of weight modules $\mathcal{C}$ for the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{g})$ at arbitrary roots of unity and show that this category remains ribbon and generically semi-simple. We also establish certain self-duality properties of projective modules in $\mathcal{C}$ with respect to an appropriate duality functor (see Subsection 3.3.2). These properties are used extensively in Chapter 5 to determine the structure of projective modules in the category of weight modules for $\overline{U}_i^H(\mathfrak{sl}_3)$.

The braided tensor categories corresponding to the rank one triplet $\mathcal{M}(r)$ and B_r vertex operator algebras are realized as categories of local modules for particular commutative algebra objects. In Chapter 4, we classify commutative algebra objects and supercommutative superalgebra objects built from simple currents (see subsection 2.1.1) in $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{g})$ and study their categories of local modules. This yields a wealth of examples of non-degenerate ribbon categories of both finite and non-finite type. We take particular care with the examples which we expect to correspond to the higher rank vertex algebras $W_Q(r)$ and $B_Q(r)$. We also expect that some of the categories constructed in this way should be closely related to the representation categories of quantum groups constructed in [N, GLO].

In Chapter 5 we specialize to the case of $\overline{U}_i^H(\mathfrak{sl}_3)$ to study properties of projective covers of irreducible modules in $\mathrm{Rep}_{wt}\overline{U}_i^H(\mathfrak{sl}_3)$. We determine the structure of all irreducible modules and Loewy diagrams (see Definition 5.1.3) of Verma modules. The main result of this Chapter is a proof determining Loewy diagrams of all projective covers in $\mathrm{Rep}_{wt}\overline{U}_i^H(\mathfrak{sl}_3)$. This result is then used to determine tensor decompositions for all tensor products of irreducibles involving a projective irreducible module.

1.1 Results

We now present our results and then discuss further connections to vertex algebras in the following section. Readers unfamiliar with vertex algebras, quantum groups, or algebra objects in braided tensor categories may find it helpful to first read the background material in Subsection 2.1 of Chapter 2.

1.1.1 Chapter 2 - Braided tensor categories for B_r vertex algebras

In this subsection, we summarise the results of Chapter 2 concerning the construction of braided tensor categories related to the B_r vertex operator algebra. These results are published in [ACKR]. Let r be a positive integer at least 2 and let $q = e^{-\pi i/r}$. Define the category $\mathcal{C} := \mathcal{H}_{i\mathbb{R}}^{\oplus} \boxtimes \mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ to be the Deligne product of a category $\mathcal{H}_{i\mathbb{R}}^{\oplus}$ of $i\mathbb{R}$-graded complex vector spaces (see Subsection 2.1.1 for details) and the weight category of the quantum group $\overline{U}_q^H(\mathfrak{sl}_2)$. $\mathcal{C}$ contains simple currents $\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H$ where λ_r satsifies $\lambda_r^2 = -r/2$ and it is shown in Section 2.2 that the object in $\mathcal{C}$ corresponding to B_r under the identification

ϕ of equation (1.0.1) is

$$\mathcal{A}_r := \bigoplus_{k\in\mathbb{Z}} (\mathrm{F}_{k\lambda_r} \boxtimes \mathbb{C}^H_{kr}) \in \mathcal{C}^{\oplus} \tag{1.1.1}$$

where $\mathcal{C}^{\oplus}$ denotes an appropriate direct sum completion of $\mathcal{C}$ (see [AR]). $\mathcal{A}_r$ is a commutative algebra object with unique (up to isomorphism) algebraic structure (see Proposition 2.2.2). Hence, one can define the corresponding representation categories of modules and local modules $\mathrm{Rep}(\mathcal{A}_r)$ and $\mathrm{Rep}^0(\mathcal{A}_r)$ respectively (see Subsection 2.1.1). Since $\mathcal{A}_r$ is a direct sum of simple currents, we can determine all simple local modules using the induction functor $\mathscr{F} : \mathcal{C} \to \mathrm{Rep}\mathcal{A}_r$ of Definition 2.1.14. The categorical structure of $\mathrm{Rep}^0(\mathcal{A}_r)$ is determined in Proposition 2.2.6 and is completely inherited from $\mathcal{C}$. The following theorem lists simple and projective objects in $\mathcal{C}$, where we remark that V_α denotes the Verma modules of highest weight $\alpha + r - 1$ and $S_i \otimes \mathbb{C}^H_{\ell r}$ are the non-projective irreducible modules for $\overline{U}^H_q(\mathfrak{sl}_2)$ with highest weight $i + \ell r$ and dimension $i + 1$ (see Subsection 2.1 for details).

Theorem 1.1.1. The simple modules in $\mathrm{Rep}^0(\mathcal{A}_r)$ are (with $\lambda_r^2 := -r/2$ and $\ddot{\mathbb{C}} := (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$)

$$E^V_{\gamma,\alpha} = \mathscr{F}(\mathrm{F}_\gamma \boxtimes V_\alpha) \qquad \text{with } \alpha \in \ddot{\mathbb{C}} \text{ and } \gamma\lambda_r + \frac{\alpha + r - 1}{2} \in \mathbb{Z}\,, \tag{1.1.2}$$

$$E^S_{\gamma,i,\ell} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}^H_{\ell r})\right) \qquad \text{with } i \in \{0, \dots, r-2\}, \ell \in \mathbb{Z} \text{ and } \gamma\lambda_r + \frac{i + r\ell}{2} \in \mathbb{Z}\,. \tag{1.1.3}$$

We have families of indecomposable modules:

$$Q^V_{\gamma,\alpha} = \mathscr{F}(\mathrm{F}_\gamma \boxtimes V_\alpha) \qquad \text{with } \alpha \notin \ddot{\mathbb{C}} \text{ and } \gamma\lambda_r + \frac{\alpha + r - 1}{2} \in \mathbb{Z}\,, \tag{1.1.4}$$

$$Q^P_{\gamma,i,\ell} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (P_i \otimes \mathbb{C}^H_{\ell r})\right) \qquad \text{with } i \in \{0, \dots, r-2\}, \ell \in \mathbb{Z} \text{ and } \gamma\lambda_r + \frac{i + \ell r}{2} \in \mathbb{Z}, \tag{1.1.5}$$

with $Q^P_{\gamma,i,\ell}$ being projective, and the above modules satisfy

$$E^V_{\gamma,\alpha} \cong E^V_{\gamma+k\lambda_r,\alpha+rk}, \qquad E^S_{\gamma,i,\ell} \cong E^S_{\gamma+k\lambda_r,i,\ell+k}, \tag{1.1.6}$$

$$Q^V_{\gamma,\alpha} \cong Q^V_{\gamma+k\lambda_r,\alpha+rk}, \qquad Q^P_{\gamma,i,\ell} \cong Q^P_{\gamma+k\lambda_r,i,\ell+k}, \tag{1.1.7}$$

for all $k \in \mathbb{Z}$. Given $\alpha \notin \ddot{\mathbb{C}}$, we can write $\alpha = i + \ell r$ for some $i = 1, ..., r-1$ and $\ell \in \mathbb{Z}$. The indecomposable modules admit the following Loewy diagrams (recall Definition 2.1.15):

$$
Q^V_{\gamma,\,i+\ell r}\quad
\begin{tikzcd}
E^S_{\gamma,\,(r-1)-i,\,\ell} \arrow[d] \\
E^S_{\gamma,\,i-1,\,\ell+1}
\end{tikzcd}
\qquad\qquad
Q^P_{\gamma,\,r-1,\,\ell}\quad
\begin{tikzcd}
E^V_{\gamma,\,i+1-r+\ell r} \arrow[d] \\
E^V_{\gamma,\,r-1-i+\ell r}
\end{tikzcd}
$$

$$
\textit{For } i=1,\dots,r-2: \qquad
\begin{tikzcd}
 & E^S_{\gamma,\,i,\,\ell} \arrow[dl] \arrow[dr] & \\
E^S_{\gamma,\,(r-2)-i,\,\ell-1} \arrow[dr] & Q^P_{\gamma,\,i,\,\ell} & E^S_{\gamma,\,(r-2)-i,\,\ell+1} \arrow[dl] \\
 & E^S_{\gamma,\,i,\,\ell} &
\end{tikzcd}
.
$$

The category $\mathrm{Rep}^0(\mathcal{A}_r)$ is a rigid monoidal category with tensor product $\mathscr{F}(U)\otimes\mathscr{F}(V)\cong\mathscr{F}(U\otimes V)$. $\mathrm{Rep}^0(\mathcal{A}_r)$ is also braided with braiding $c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)}$ defined by

$$c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)} = \mathrm{Id}_{\mathcal{A}_r}\otimes c_{U,V}, \tag{1.1.8}$$

$$\tag{1.1.9}$$

where $c_{U,V}$ is the braiding on $\mathcal{C}^{\oplus}$ given by the product of the braidings on $\mathcal{H}^{\oplus}_{i\mathbb{R}}$ and $\mathrm{Rep}_{wt}\overline{U}^H_q(\mathfrak{sl}_2)$, where we have implicitly assumed the isomorphism $\mathscr{F}(U)\otimes\mathscr{F}(V)\cong\mathscr{F}(U\otimes V)$. If r is odd, then $\mathrm{Rep}^0(\mathcal{A}_r)$ has twist $\theta_{\mathcal{A}_r}$ and Hopf links $S^{\infty}_{\mathscr{F}(U),\mathscr{F}(V)}$ given by

$$\theta_{\mathscr{F}(V)} = \mathrm{Id}_{\mathcal{A}_r}\otimes\theta_V \tag{1.1.10}$$

$$\mathsf{S}^{\infty}_{\mathscr{F}(U),\mathscr{F}(V)} = \mathsf{S}^{\infty}_{U,V} \tag{1.1.11}$$

where θ_V and $S^{\infty}_{U,V}$ are the twist and Hopf links respectively on $\mathcal{C}^{\oplus}$, and we are viewing the Hopf links as the scalars by which they act.

The examples of the $\beta\gamma$ vertex operator algebra and $\mathbb{L}_{-4/3}(\mathfrak{sl}_2)$

The B_r-algebra for $r=2$ and $r=3$ are the $\beta\gamma$ vertex operator algebra and the affine vertex operator algebra $\mathbb{L}_{-4/3}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ at level $-4/3$, respectively. These vertex operator algebras have been studied in [A2, CR1, CR2, CR4, CRW, Ri, Ri2, Ri3, RW2]. We use the notation of [CRW]. These cases are of course the easiest, but also the most important for applications. We list their representation-theoretic data explicitly, including irreducibles, projectives, tensor product decompositions and braidings.

The $\beta\gamma$ vertex operator algebra

The $\beta\gamma$ vertex operator algebra has modules $\mathbb{E}_\lambda^s, \mathbb{L}_0^s$ with $s \in \mathbb{Z}$ and $\lambda \in \mathbb{R}/\mathbb{Z}$. The $\beta\gamma$ vertex operator algebra itself is $\mathbb{L}_0^0$ and this is the only highest-weight module. The $\mathbb{E}_\lambda^0$ are the relaxed-highest weight modules. That is, their conformal weight is bounded below but the top level is infinite-dimensional. The supersript s denotes the twists of these modules by an automorphism called the spectral flow. Comparing with our notation we first set $\lambda_2 = i = \sqrt{-1}$ and then under the conjectural quantum group to singlet vertex operator algebra equivalence one has

$$\mathbb{E}_\lambda^{2s-1} \leftrightarrow E^V_{\frac{2s-1-2\lambda}{2\sqrt{-1}},2\lambda} \qquad \text{and} \qquad \mathbb{L}_0^{2s} \leftrightarrow E^S_{\frac{s}{\sqrt{-1}},0,0}. \tag{1.1.12}$$

We now list the data of $\mathrm{Rep}^0(\mathcal{A}_2)$ explicitly including tensor product decompositions and braidings. We have the set $\ddot{\mathbb{C}} = (\mathbb{C} \setminus \mathbb{Z}) \cup 2\mathbb{Z}$. Hence, the simple modules in $\mathrm{Rep}^0(\mathcal{A}_2)$ are

$$\begin{aligned} E^V_{\gamma,\alpha} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) && \text{with } \alpha \in \ddot{\mathbb{C}}, \text{ and } \frac{\alpha+1}{2} + i\gamma \in \mathbb{Z}\,, \\ E^S_{\gamma,0,\ell} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes \mathbb{C}^H_{2\ell}\right) && \text{with } i\gamma, \ell \in \mathbb{Z}\,. \end{aligned}$$

We have families of indecomposable modules:

$$\begin{aligned} Q^V_{\gamma,\alpha} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) && \text{with } \alpha \in \mathbb{Z} - 2\mathbb{Z} \text{ and } \frac{\alpha+1}{2} + i\gamma \in \mathbb{Z}\,, \\ Q^P_{\gamma,0,\ell} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (P_0 \otimes \mathbb{C}^H_{2\ell})\right) && \text{with } i\gamma, \ell \in \mathbb{Z}, \end{aligned}$$

where $Q^P_{\gamma,0,\ell}$ is projective, and the above modules satisfy $E^V_{\gamma,\alpha} \cong E^V_{\gamma+ik,\alpha+2k}$, $E^S_{\gamma,0,\ell} \cong E^S_{\gamma+ik,0,\ell+k}$, $Q^V_{\gamma,\alpha} \cong Q^V_{\gamma+ik,\alpha+2k}$, and $Q^P_{\gamma,0,\ell} \cong Q^P_{\gamma+ik,0,\ell+k}$ for all $k \in \mathbb{Z}$. The indecomposable modules satisfy the short exact sequences:

$$0 \to E^S_{\gamma,0,\ell} \to Q^V_{\gamma,1+2\ell} \to E^S_{\gamma,0,\ell+1} \to 0 \qquad \text{and} \qquad 0 \to E^V_{\gamma,1+2\ell} \to Q^P_{\gamma,0,\ell} \to E^V_{\gamma,1+2(\ell-1)} \to 0.$$

The tensor decompositions for $\mathcal{C}$ can be found in [CGP1, Section 8] and it then follows that

$$E^S_{\gamma_1,0,\ell_1} \otimes E^S_{\gamma_2,0,\ell_2} \cong E^S_{\gamma_1+\gamma_2,0,\ell_1+\ell_2} \qquad \text{and} \qquad E^S_{\gamma_1,0,\ell} \otimes E^V_{\gamma_2,\alpha} \cong E^V_{\gamma_1+\gamma_2,\alpha+2\ell}$$

with $\alpha \in \ddot{\mathbb{C}}$. For $\alpha, \beta \in \ddot{\mathbb{C}}$ with $\alpha + \beta \notin \mathbb{Z}$, we have

$$E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} \cong E^V_{\gamma_1+\gamma_2,\alpha+\beta-1} \oplus E^V_{\gamma_1+\gamma_2,\alpha+\beta+1}.$$

When $\alpha+\beta = n \in \mathbb{Z}$, set $n = j+2k$ with $j = 0, 1$ and $k \in \mathbb{Z}$. The tensor decompositions for these cases do not appear in [CGP1], but are easily computed from characters for $p = 2, 3$. If $\alpha \in \ddot{\mathbb{C}}$, $n = 2\ell$, we have

$$E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} \cong Q^P_{\gamma_1+\gamma_2,0,\ell}$$

and if $n = 1 + 2\ell$, we have

$$E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} \cong E^V_{\gamma_1+\gamma_2,2\ell} \oplus E^V_{\gamma_1+\gamma_2,2(\ell+1)}$$

Let $c_{X,Y}$ denote the braiding. We have

$$c_{\mathrm{F}_{\gamma_1}\boxtimes X,\mathrm{F}_{\gamma_2}\boxtimes Y} = c_{\mathrm{F}_{\gamma_1},\mathrm{F}_{\gamma_2}} \boxtimes c_{X,Y} = e^{\pi i\gamma_1\gamma_2}\mathrm{Id} \boxtimes c_{X,Y}.$$

The braiding restricted to a simple summand is a scalar which can be computed by acting with the braiding on a highest (or lowest) weight vector in the summand and we get

$$c_{E^S_{\gamma_1,0,\ell_1},E^S_{\gamma_2,0,\ell_2}} = q^{2(\ell_1\ell_2+\gamma_1\gamma_2)}\mathrm{Id}_{E^S_{\gamma_1+\gamma_2,0,\ell_1+\ell_2}} \qquad \text{and} \qquad c_{E^S_{\gamma_1,0,\ell},E^V_{\gamma_2,\alpha}} = q^{\ell(\alpha+1)+2\gamma_1\gamma_2}\mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+2\ell}}.$$

For $\alpha, \beta \in \ddot{\mathbb{C}}$ with $\alpha + \beta \notin \mathbb{Z}$, we have

$$c_{E^V_{\gamma_1,\alpha},E^V_{\gamma_2,\beta}} = q^{\frac{1}{2}(\alpha+1)(\beta+1)+2\gamma_1\gamma_2}\mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+\beta+1}} \oplus q^{\frac{1}{2}(\alpha-1)(\beta-1)+2\gamma_1\gamma_2}\mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+\beta-1}}.$$

If $\alpha + \beta = 2\ell$, we have

$$c_{E^V_{\gamma_1,\alpha}\otimes E^V_{\gamma_2,\beta}} = q^{\frac{1}{2}(\alpha+1)(\beta+1)+2\gamma_1\gamma_2}\mathrm{Id}_{Q^P_{\gamma_1+\gamma_2,0,\ell}} \oplus n_{Q^P_{\gamma_1+\gamma_2,0,\ell}}$$

for some nilpotent endomorphism $n_{Q^P_{\gamma_1+\gamma_2,0,\ell}}$ on $Q^P_{\gamma_1+\gamma_2,0,\ell}$. If $\alpha + \beta = 1 + 2\ell$, we have

$$c_{E^V_{\gamma_1,\alpha},E^V_{\gamma_2,\beta}} = q^{\frac{1}{2}(\alpha+1)(\beta+1)+2\gamma_1\gamma_2}\mathrm{Id}_{E^V_{\gamma_1+\gamma_2,2(\ell+1)}} \oplus q^{\frac{1}{2}(\alpha-1)(\beta-1)+2\gamma_1\gamma_2}\mathrm{Id}_{E^V_{\gamma_1+\gamma_2,2\ell}}$$

The affine vertex operator algebra $\mathbb{L}_{-4/3}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ at level $-4/3$

$\mathbb{L}_{-4/3}(\mathfrak{sl}_2)$ has modules $\mathbb{E}^s_\lambda, \mathbb{L}^s_0, , \mathbb{L}^s_{-2/3}$ with $s \in \mathbb{Z}$ and $\lambda \in \mathbb{R}/2\mathbb{Z}$. As before s indicates the spectral flow twists of modules. The $\mathbb{E}^0_\lambda$ are the relaxed-highest weight modules and the affine vertex operator algebra itself is $\mathbb{L}^0_0$. There are two more highest and lowest weight modules, namely $\mathbb{L}^0_{-2/3}$ and $\mathbb{L}^1_0$ are of highest-weight $-2/3$ and $-4/3$ while $\mathbb{L}^{-1}_{-2/3}$ and $\mathbb{L}^{-1}_0$ are of lowest-weight $2/3$ and $4/3$ (here the highest and lowest weights refer to the $\mathfrak{sl}_2$-weights.). The identification with our modules is with $\lambda_3 = \sqrt{-3/2}$ given by

$$\mathbb{E}^s_\lambda \;\leftrightarrow\; E^V_{\frac{2s-\lambda}{2\sqrt{-3/2}},\lambda} \qquad \text{and} \qquad \mathbb{L}^s_0 \;\leftrightarrow\; E^S_{\frac{s}{\sqrt{-3/2}},0,0} \qquad \text{and} \qquad \mathbb{L}^s_{-2/3} \;\leftrightarrow\; E^S_{\frac{2s+1}{\sqrt{-3/2}},1,0}. \tag{1.1.13}$$

We have $\ddot{\mathbb{C}} = (\mathbb{C}\setminus\mathbb{Z})\cup 3\mathbb{Z}$. The simple modules in $\mathrm{Rep}^0(\mathcal{A}_3)$ are

$$\begin{aligned} E^V_{\gamma,\alpha} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) && \text{with } \alpha\in\ddot{\mathbb{C}} \text{ and } \alpha+2+\sqrt{6}i\gamma\in 2\mathbb{Z}\,, \\ E^S_{\gamma,j,\ell} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (S_j\otimes \mathbb{C}^H_{\ell p})\right) && \text{with } j\in\{0,1\}, \ell\in\mathbb{Z} \text{ and } j+3\ell+\sqrt{6}i\gamma\in 2\mathbb{Z}\,. \end{aligned}$$

We have families of indecomposable modules:

$$\begin{aligned} Q^V_{\gamma,\alpha} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) && \text{with } \alpha\in\mathbb{Z}\setminus 3\mathbb{Z} \text{ and } \alpha+2+\sqrt{6}i\gamma\in 2\mathbb{Z}\,, \\ Q^P_{\gamma,j,\ell} &= \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (P_j\otimes \mathbb{C}^H_{\ell p})\right) && \text{with } j\in\{0,1\}, \ell\in\mathbb{Z} \text{ and } j+3\ell+\sqrt{6}i\gamma\in 2\mathbb{Z}, \end{aligned}$$

with $Q^P_{\gamma,j,\ell}$ being projective, and the above modules satisfy $E^V_{\gamma,\alpha}\cong E^V_{\gamma+i\sqrt{3/2}k,\alpha+3k}$, $E^S_{\gamma,j,\ell}\cong E^S_{\gamma+i\sqrt{3/2}k,j,\ell+k}$, $Q^V_{\gamma,\alpha}\cong Q^V_{\gamma+i\sqrt{3/2}k,\alpha+3k}$, and $Q^P_{\gamma,j,\ell}\cong Q^P_{\gamma+i\sqrt{3/2}k,j,\ell+k}$ for all $k\in\mathbb{Z}$. The reducible indecomposables satisfy the following short exact sequences:

$$0\to E^S_{\gamma,2-j,\ell}\to Q^V_{\gamma,j+3\ell}\to E_{\gamma,j-1,\ell+1}\to 0 \qquad \text{and} \qquad 0\to E^V_{\gamma,2-j+3\ell}\to Q^P_{\gamma,j,\ell}\to E_{\gamma,1+j+3(\ell-1)}\to 0.$$

The tensor decompositions are as follows:

$$E^S_{\gamma_1,i,\ell_1}\otimes E^S_{\gamma_2,j,\ell_2}\cong \bigoplus_{\substack{k=|i-j|\\ \text{by } 2}}^{i+j} E^S_{\gamma_1+\gamma_2,k,\ell_1+\ell_2} \qquad \text{and} \qquad E^S_{\gamma_1,0,\ell}\otimes E^V_{\gamma_2,\alpha}\cong E^V_{\gamma_1+\gamma_2,\alpha+3\ell}.$$

Note that when $i=j=1$ above, we have $E^S_{\gamma_1+\gamma_2,2,\ell_1+\ell_2}\cong E^V_{\gamma_1+\gamma_2,3(\ell_1+\ell_2)}$. When $\alpha\in\mathbb{C}\setminus\mathbb{Z}$ or $\alpha=3\ell_2$, we get

$$E^S_{\gamma_1,1,\ell}\otimes E^V_{\gamma_2,\alpha}\cong E^V_{\gamma_1+\gamma_2,\alpha+1+3\ell}\oplus E^V_{\gamma_1+\gamma_2,\alpha-1+3\ell} \qquad \text{and} \qquad E^S_{\gamma_1,1,\ell_1}\otimes E^V_{\gamma_2,3\ell_2}\cong Q^P_{\gamma_1+\gamma_2,1,\ell_1+\ell_2}$$

respectively. For $\alpha, \beta \in \ddot{\mathbb{C}}$ with $\alpha + \beta \notin \mathbb{Z}$, we have

$$E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} \cong \bigoplus_{\substack{k=-2 \\ \text{by } 2}}^{2} E^V_{\gamma_1+\gamma_2,\alpha+\beta+k}.$$

For $\alpha \in \ddot{\mathbb{C}}$, $\alpha + \beta = n$ with $n = 3\ell, 1 + 3\ell$, and $2 + 3\ell$, we get

$$\begin{aligned} E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} &\cong Q^P_{\gamma_1+\gamma_2,0,\ell} \oplus E^V_{\gamma_1+\gamma_2,3\ell} \\ E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} &\cong Q^P_{\gamma_1+\gamma_2,1,\ell} \oplus E^V_{\gamma_1+\gamma_2,3(\ell+1)} \\ E^V_{\gamma_1,\alpha} \otimes E^V_{\gamma_2,\beta} &\cong Q^P_{\gamma_1+\gamma_2,1,\ell+1} \oplus E^V_{\gamma_1+\gamma_2,3\ell} \end{aligned}$$

respectively. We immediately obtain the following braidings:

$$\begin{aligned} c_{E^S_{\gamma_1,0,\ell_1}, E^S_{\gamma_2,i,\ell_2}} &= q^{\frac{1}{2}3\ell_1(3\ell_2+i)+3\gamma_1\gamma_2} \mathrm{Id}_{E^S_{\gamma_1+\gamma_2,i,\ell_1+\ell_2}} \\ c_{E^S_{\gamma_1,1,\ell_1}, E^S_{\gamma_2,1,\ell_2}} &= q^{\frac{1}{2}(3\ell_1+1)(3\ell_2+1)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,3(\ell_1+\ell_2)}} \oplus q^{\frac{1}{2}(3\ell_1-1)(3\ell_2-1)+3\gamma_1\gamma_2} \mathrm{Id}_{E^S_{\gamma_1+\gamma_2,0,\ell_1+\ell_2}} \\ c_{E^S_{\gamma_1,0,\ell}, E^V_{\gamma_2,\alpha}} &= q^{\frac{3\ell}{2}(\alpha+2)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+3\ell}} \\ c_{E^S_{\gamma_1,1,\ell} \otimes E^V_{\gamma_2,\alpha}} &= q^{\frac{1}{2}(1+3\ell)(\alpha+2)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+1+3\ell}} \oplus q^{\frac{1}{2}(-1+3\ell)(\alpha-2)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha-1+3\ell}} \\ c_{E^S_{\gamma_1,1,\ell_1} \otimes E^V_{\gamma_2,3\ell_2}} &= q^{\frac{1}{2}(1+3\ell_1)(2+3\ell_2)+3\gamma_1\gamma_2} \mathrm{Id}_{Q^P_{\gamma_1+\gamma_2,1,\ell_1+\ell_2}} \oplus n_{Q^P_{\gamma_1+\gamma_2,1,\ell_1+\ell_2}} \end{aligned}$$

with $n_{Q^P_{\gamma_1+\gamma_2,1,\ell_1+\ell_2}}$ a nilpotent endomorphism on $Q^P_{\gamma_1+\gamma_2,1,\ell_1+\ell_2}$. For $\alpha, \beta \in \ddot{\mathbb{C}}$ with $\alpha + \beta \notin \mathbb{Z}$, we have

$$\begin{aligned} c_{E^V_{\gamma_1,\alpha}, E^V_{\gamma_2,\beta}} = {} & q^{\frac{1}{2}(\alpha+2)(\beta+2)+3\gamma_2\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+\beta+2}} \oplus q^{\frac{1}{2}(\alpha-2)(\beta-2)+3\gamma_2\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+\beta-2}} \\ & \oplus \left(q^{\frac{1}{2}(\alpha+2)(\beta-2)} + q^{\frac{1}{2}\alpha\beta} + q^{\frac{1}{2}(\alpha-2)(\beta+2)} \right) q^{3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,\alpha+\beta}} \end{aligned}$$

For $\alpha \in \ddot{\mathbb{C}}$, $\alpha + \beta = n$ with $n = 3\ell, 1 + 3\ell$, and $2 + 3\ell$, we get

$$
\begin{aligned}
c_{E^V_{\gamma_1.\alpha} \otimes E^V_{\gamma_2,\beta}} &= \left(q^{\frac{1}{2}(\alpha+2)\beta} + q^{\frac{1}{2}\alpha(\beta+2)}\right) q^{3\gamma_1\gamma_2} \mathrm{Id}_{Q^P_{\gamma_1+\gamma_2,0,\ell}} \oplus n_{Q^P_{\gamma_1+\gamma_2,0,\ell}} \oplus q^{\frac{1}{2}(\alpha+2)(\beta+2)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,3\ell}} \\
c_{E^V_{\gamma_1.\alpha} \otimes E^V_{\gamma_2,\beta}} &= \left(q^{\frac{1}{2}(\alpha+2)\beta} + q^{\frac{1}{2}\alpha(\beta+2)}\right) q^{3\gamma_1\gamma_2} \mathrm{Id}_{Q^P_{\gamma_1+\gamma_2,1,\ell}} \oplus n_{Q^P_{\gamma_1+\gamma_2,1,\ell}} \oplus q^{\frac{1}{2}(\alpha+2)(\beta+2)+3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,3(\ell+1)}} \\
c_{E^V_{\gamma_1.\alpha} \otimes E^V_{\gamma_2,\beta}} &= \left(q^{\frac{1}{2}(\alpha+2)(\beta-2)} + q^{\frac{1}{2}(\alpha\beta} + q^{\frac{1}{2}(\alpha-2)(\beta+2)}\right) q^{3\gamma_1\gamma_2} \mathrm{Id}_{E^V_{\gamma_1+\gamma_2,3\ell}} \\
&\qquad \oplus q^{\frac{1}{2}(\alpha+2)(\beta+2)+3\gamma_1\gamma_2} \mathrm{Id}_{Q^P_{\gamma_1+\gamma_1,1,\ell+1}} \oplus n_{Q^P_{\gamma_1+\gamma_1,1,\ell+1}}
\end{aligned}
$$

for some nilpotent endomorphisms on the projective modules Q^P.

Modularity and Verlinde's formula

We call the modules $E^V_{\gamma,\alpha}$ typical and $E^S_{\gamma,i,\ell}$ atypical. Modules of the B_r-algebra are bigraded by conformal weight and also by the weight of the Heisenberg vertex operator algebra. The graded trace of the corresponding B_r modules turns out to only make sense as a formal power series in the case of typical module characters as formal delta distributions appear. Using the ideas of [CR2, CR3] we can compute a modular S-transformation giving a certain function on the set of typical modules that we call S-kernel S^χ.

The Verlinde algebra of characters generated by characters of atypical modules is shown to have a particular generating set which is closed under modular S-transformations. This generating set can be related to the semi-simplification of $\mathrm{Rep}^0(\mathcal{A}_r)$. For this, we recall that the semi-simplification of a category is the category obtained by quotienting negligible morphisms.

Definition 1.1.2. Let $\mathcal{C}$ be a rigid braided tensor category. Let M, N be objects, then a morphism $f : M \to N$ is negligible if for every morphism $g : N \to M$ the trace of $f \circ g$ vanishes. The semisimplification $\mathcal{C}^{\mathrm{ss}}$ of $\mathcal{C}$ is the category whose objects are those of $\mathcal{C}$ but all negligible morphism are identified with the zero morphism. An object M is called negligible if the identity on M is a negligible morphism and in a rigid tensor category the subcategory $\mathcal{N}$ whose objects are all negligible objects forms a tensor ideal.

Let $\mathcal{G}(\mathcal{C})$ be the Grothendieck ring (see Definition 4.5.2 [EGNO]) of $\mathcal{C}$ and let $\mathcal{G}(\mathcal{N})$ be the Grothendieck ring of the ideal of negligible objects then we define the ring

$$\mathcal{G}^{\mathrm{ss}}(\mathcal{C}) := \mathcal{G}(\mathcal{C})/\mathcal{G}(\mathcal{N})$$

and we note that in general $\mathcal{G}^{\mathrm{ss}}(\mathcal{C})$ is a homomorphic image of $\mathcal{G}(\mathcal{C}^{\mathrm{ss}})$.

The modular properties of characters of the B_r-algebra and Hopf links of $\mathcal{C}$ are studied and compared in Sections 2.3 and 2.4. In Section 2.3, the modular S-matrix S^χ coming from the modular action on characters for typical modules is computed and shown to agree with the S-matrix S^{∞} coming from closed Hopf-links associated to typical modules in $\mathcal{C}$ up to normalization in Proposition 2.3.2. To compare the atypical modules, the ring $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ of the category of local modules (those which are induced to $\mathrm{Rep}^0\mathcal{A}_r$ by the induction functor) is derived in Proposition 2.4.1. The corresponding matrix S^{∞} is derived and shown to agree up to normalization with the matrix S^χ determined by the modular action on the Verlinde algebra of characters generated by atypical B_r-modules when r is odd (see Proposition 2.4.2). From this, the Verlinde formula immediately follows in Corollary 2.4.3. When r is even, B_r is half-integer graded and we instead compare modular properties associated to its integer part $B_r^{\bar{0}}$, showing again that the statement analagous to the following Theorem 1.1.3 holds.

Theorem 1.1.3. (Verlinde's formula) For the parametrization of atypical simples, refer to the discussion around (2.4.16).

1. (Proposition 2.3.2) The normalized modular S-matrix S^χ and Hopf links S^{∞} of typical modules coincide:
$$\frac{\mathsf{S}^\chi_{(\nu,\ell),(\nu',\ell')}}{\mathsf{S}^\chi_{\mathbb{1},(\nu',\ell')}} = \frac{\mathsf{S}^{\infty}_{(\nu,\ell),(\nu',\ell')}}{\mathsf{S}^{\infty}_{\mathbb{1},(\nu',\ell')}}.$$

2. (Proposition 2.4.2) Let r be odd. The normalized modular S-matrix S^χ and Hopf links S^{∞} of atypical modules coincide:
$$\frac{\mathsf{S}^\chi_{(s,s'),(n,n')}}{\mathsf{S}^\chi_{\mathbb{1},(n,n')}} = \frac{\mathsf{S}^{\infty}_{(s,s'),(n,n')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}}.$$

3. (Corollary 2.4.3) Let r be odd and let Λ_r be a $\mathbb{Z}$-basis of $\mathcal{G}^{\mathrm{ss}}(\mathrm{Rep}^0\mathcal{A}_r)$ with structure constants $N^{(k,k')}_{(s,s'),(t,t')}$. That is
$$(k,k') \times (s,s') = \sum_{(n,n')\in\Lambda_p} N^{(k,k')}_{(s,s'),(t,t')}(t,t')$$

for (k,k') and (s,s') in Λ_p. Then the Verlinde formula holds

$$\sum_{(n,n')\in\Lambda_p} \frac{\mathsf{S}^{\infty}_{(s,s'),(n,n')}\mathsf{S}^{\infty}_{(t,t'),(n,n')}(\mathsf{S}^{\infty})^{-1}_{(n,n'),(k,k')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}} = N^{(k,k')}_{(s,s'),(t,t')},$$

It was conjectured in [C2, Remark 5.6] that the B_r vertex operator algebra is a quantum Hamiltonian reduction of $V_k(\mathfrak{sl}_{r-1})$ at level $k+r-1 = \frac{r-1}{r}$. Quantum Hamiltonian reductions are associated to nilpotent elements and the relevant nilpotent element f for us corresponds to the partion $(r-2,1)$ of $r-1$. We denote the corresponding simple W-algebra by $W_k(\mathfrak{sl}_{r-1}, f)$ and in Section 2.5, we show that the characters of these algebras coincide. That is, (Theorem 2.5.1):

Theorem 1.1.4. The characters of the B_r-algebra and of $W_k(\mathfrak{sl}_{r-1}, f)$ coincide for $k = -r+1+\frac{r-1}{r}$.

The problem of realizing B_r as a quantum Hamiltonian reduction has since been solved [ACGY, Theorem 12], and the above result is a necessary component of the proof, so we briefly outline the argument here. Since the Virasoro modules appearing in the decomposition of the B_r-algebra are uniquely determined by their character this Theorem actually says that the B_r-algebra and $W_k(\mathfrak{sl}_{r-1}, f)$ are isomorphic as modules for the tensor product of the Heisenberg and Virasoro vertex operator algebras. This means both are extensions of the same subalgebra and they coincide as modules for the subalgebra. We can therefore ask if such extensions are unique. Recent progress in [CHJRY] shows that the category of C_1-cofinite modules of the Virasoro algebra at any central charge has a vertex tensor category structure. Vertex tensor category structure on the Heisenberg vertex operator algebra is known [CKLR] and since vertex operator algebra extensions in a vertex tensor category are in one-to-one correspondence to commutative and associative algebra objects in the category by [HKL] the uniqueness question of vertex operator algebra extensions is equivalent to uniqueness of these algebra objects. This last point is proven in [ACGY], and as a consequence they obtain a proof that the $B_r \cong W_k(\mathfrak{sl}_{r-1}, f)$ [ACGY, Theorem 12].

1.1.2 Chapter 3 - The category of weight modules for $\overline{U}_q^H(\mathfrak{g})$

The correspondence between unrolled quantum groups and singlet vertex operator algebras occurs at even roots of unity, but the higher rank unrolled quantum groups have previously been studied only at odd roots of unity. In Chapter 3 we establish results necessary for the construction of the braided tensor categories which we expect to compare nicely to the higher rank vertex algebras $W^0_Q(r)$, $W_Q(r)$, and $B_Q(r)$. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. Let q be a primitive ℓ-th root of unity, $r = \frac{3+(-1)^\ell}{4}\ell$ and $\mathcal{C}$ the category of finite dimensional weight modules for the unrolled restricted quantum group $\overline{U}^H_q(\mathfrak{g})$ (see Definition 3.2.5 and the opening comments of Section 3.3). We describe in Section 3.2 how to construct the unrolled quantum groups as a semi-direct product $U^H_q(\mathfrak{g}) := U_q(\mathfrak{g}) \rtimes U(\mathfrak{h})$ of the Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$ and the universal enveloping algebra of the Cartan subalgebra of $\mathfrak{g}$. We also show that the action of the braid group $\mathcal{B}_\mathfrak{g}$ (see Definition 3.2.3) extends naturally from $U_q(\mathfrak{g})$ to $U^H_q(\mathfrak{g})$ (Proposition 3.2.2):

Proposition 1.1.5. The action of the braid group $\mathcal{B}_\mathfrak{g}$ on $U_q(\mathfrak{g})$ can be extended naturally to the unrolled quantum group $U^H_q(\mathfrak{g})$.

This statement is known to some, but hasn't appeared in the literature to the author's knowledge. It has been shown previously (see [CGP2]) that there is a generically semi-simple structure (see Definition 3.1.1) on the category of weight modules over the restricted unrolled quantum group at odd roots of unity where $\ell \notin 3\mathbb{Z}$ if $\mathfrak{g} = G_2$. The purpose of this restriction on the G_2 case is to guarantee that $gcd(d_i, r) = 1$ for all $i = 1, ..., n$ where $d_i = \frac{1}{2}\langle \alpha_i, \alpha_i \rangle$. This condition fails at odd roots only for G_2, but for even roots all non ADE-type Lie algebras fail this condition for some choice of ℓ. We show that when $gcd(d_i, r) \neq 1$, generic semi-simplicity can be retained if one quotients by a larger Hopf ideal (see Definition 3.2.5). In Subsection 3.3.1, we observe that $\mathcal{C}$ being ribbon follows easily from the techniques developed in [GP2] as in the case for odd roots, and we show that $\mathcal{C}$ has trivial Müger center (see Definition 3.1.2). We therefore have the following (Propositions 3.3.8, 3.3.10, and Corollary 3.3.9):

Theorem 1.1.6. $\mathcal{C}$ is a generically semi-simple ribbon category with trivial Müger center.

$\overline{U}^H_q(\mathfrak{sl}_2)$ was studied at even roots of unity in [CGP1]. Stated therein ([CGP1, Proposition 6.1]) is a generator and relations description of the projective covers of irreducible modules. One apparent property of these projective covers is that their top and socle coincide. Showing that this is a general feature of projective covers in $\mathcal{C}$ is the topic of Subsection 3.3.2. Let P^λ denote the projective cover of the irreducible module $S^\lambda \in \mathcal{C}$. We introduce a character preserving contravariant functor $M \mapsto \check{M}$, analagous to the duality functor for Lie algebras ([Hu, Subsection 3.2]). We are then able to prove the following theorem:

Theorem 1.1.7. The projective covers P^λ are self-dual under the duality functor. That is, $\check{P}^\lambda \cong P^\lambda$.

This theorem has the following corollaries (Corollary 3.3.15 and 3.3.16):

Corollary 1.1.8.
- Socle$(P^\lambda) = S^\lambda$.
- P^λ is the injective Hull of S^λ.
- $\mathcal{C}$ is unimodular.
- $\mathcal{C}$ admits a unique (up to scalar) two-sided trace on its ideal of projective modules.

These results will be used extensively in Chapter 5 to determine the Loewy diagrams of projective covers for $\overline{U}_i^H(\mathfrak{sl}_3)$.

1.1.3 Chapter 4 - Algebra objects and simple currents in $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{g})$

As described in the introduction and Chapter 2, we can construct braided tensor categories which compare nicely to categories of modules for vertex operator algebras by studying appropriately chosen commutative algebra objects. In Chapter 4, we study families of commutative algebra and supercommutative superalgebra objects which are direct sums of simple currents. These families contain the algebra objects corresponding to the higher rank vertex algebras $W_Q(r)$ and $B_Q(r)$, allowing us to make conjectures about their representation theory. Let $\mathcal{B}$ be a braided tensor category. A simple object in $\mathcal{B}$ is called a simple current if it is invertible with respect to the tensor product. The tensor product of two simple currents is a simple current, so the Grothendieck ring of the pointed subcategory of simple currents of $\mathcal{B}$ is the group algebra of an abelian group. We assume this group is free and associate to it a lattice $\mathcal{L}$ with quadratic form $\langle -,- \rangle : \mathcal{L} \times \mathcal{L} \to \mathbb{Q}/\ell\mathbb{Z}$ for some fixed integer ℓ. The simple current associated to λ in $\mathcal{L}$ is denoted by $\mathbb{C}_\lambda$ and we assume this quadratic form determines the braiding of simple currents, e.g. $c_{\mathbb{C}_\lambda,\mathbb{C}_\mu}(v_\lambda \otimes v_\mu) = q^{\langle \lambda,\mu \rangle} v_\mu \otimes v_\lambda$ for a primitive ℓ-th root of unity q. For any lattice $L \subset \mathcal{L}$, define the object

$$\mathcal{A}_L := \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \in \mathcal{B}^\oplus.$$

Then we have the following classification result:

Theorem 1.1.9. $\mathcal{A}_L$ is an associative algebra object for all $L \subset \mathcal{L}$. $\mathcal{A}_L$ is commutative if and only if $\sqrt{2/\ell}\, L$ is an even lattice, that is if and only if $\langle \lambda, \lambda \rangle \in \ell\mathbb{Z}$ and $2\langle \lambda, \mu \rangle \in \ell\mathbb{Z}$ for all $\lambda, \mu \in L$.

Let $L^\mu \subset \mathcal{L}$ be the lattice obtained by adding a generator $\mu \in \mathcal{L}$ to a lattice $L \subset \mathcal{L}$. We can then define the associated algebra object $\mathcal{A}_{L^\mu} := \bigoplus_{\lambda \in L^\mu} \mathbb{C}_\lambda$, and we have the following proposition (Proposition 4.2.3 and Corollary 4.2.5):

Proposition 1.1.10. Let $L \subset \mathcal{L}$ such that $\mathcal{A}_L$ is commutative and $\mu \in \mathcal{L}$ such that $\mu \notin L$ and $2\mu \in L$, then $\mathcal{A}_{L^\mu}$ is a superalgebra. $\mathcal{A}_{L^\mu}$ is supercommutative if and only if

$$2\langle \mu, \mu \rangle \in \ell\mathbb{Z} \setminus 2\ell\mathbb{Z} \qquad \text{and} \qquad 2\langle \mu, \lambda \rangle \in \ell\mathbb{Z}$$

for all $\lambda \in L$, and all supercommutative superalgebras $A = A^{\bar{0}} \oplus A^{\bar{1}}$ which are direct sums of simple currents such that $A^{\bar{0}} = \mathcal{A}_L$ for some lattice $L \subset \mathcal{L}$ and $A^{\bar{1}}$ is a non-trivial simple object in $\mathrm{Rep}A^{\bar{0}}$ take this form.

Given an additional assumption, which holds in the examples we consider, irreducible objects in $\mathrm{Rep}\mathcal{A}_{L^\mu}$ are given by the action of the induction functor $\mathscr{F} : \mathcal{B} \to \mathrm{Rep}\mathcal{A}_{L^\mu}$ (Definition 2.1.14) on irreducibles in $\mathcal{B}$ (Proposition 4.2.4):

Proposition 1.1.11. Suppose every indecomposable object in $\mathcal{B}$ has a simple subobject. Then $N \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is simple if and only if $N \cong \mathscr{F}(M)$ for a simple object $M \in \mathcal{B}$.

The primary example we consider is the category of weight modules $\mathcal{C}$ over the unrolled restricted quantum groups (see Definitions 3.2.1, 3.2.5 and 3.3.1) at root of unity q of order $\ell \geq 3$. The simple objects in $\mathcal{C}$ are denoted by S^λ with $\lambda \in \mathfrak{h}^*$ where $\mathfrak{h} = \mathrm{Span}\{H_1, ..., H_n\} \subset \overline{U}_q^H(\mathfrak{g})$. This category admits projective covers for each irreducible module, which we denote by P^λ, and we have the following results for induction of projective and irreducible modules (Theorem 4.3.2):

Theorem 1.1.12.

- $\mathscr{F}(P^\lambda) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $\lambda \in \frac{\ell}{2}(L^\mu)^*$.
- Let $X \in \mathcal{C}$ and let $P_X \in C$ be the projective cover of X. Then $\mathscr{F}(X) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $\mathscr{F}(P_X) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$.
- $\mathscr{F}(P^\lambda)$ is the projective cover of $\mathscr{F}(S^\lambda)$ in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$.
- The distinct irreducible objects in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ are $\{\mathscr{F}(S^\lambda) \,|\, \lambda \in \Lambda(L^\mu)\}$, where $\Lambda(L^\mu) := \frac{\ell}{2}(L^\mu)^*/\frac{\ell}{2}(L^\mu)^* \cap L^\mu$. $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is finite if and only if $\mathrm{rank}(L^\mu) = \mathrm{rank}(P)$.

Further, we can determine when the category of local modules is ribbon and has trivial Müger center (see Definition 3.1.2, Proposition 4.3.4):

Proposition 1.1.13. Let $\mathcal{A}_{L^\mu}$ be a supercommutative superalgebra, then

- $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is ribbon if $2(1-r)\langle\lambda,\rho\rangle \in \ell\mathbb{Z}$ for all $\lambda \in L$ and $2(1-r)\langle\mu,\rho\rangle \in \frac{\ell}{2}\mathbb{Z}$.
- Let $r = 2\ell/(3+(-1)^\ell)$. If $r \nmid 2d_i$, for all i, then $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ has non-trivial Müger center if and only if there exists a $\lambda \in \Lambda(L^\mu)$ such that $\langle\lambda,\gamma\rangle \in \frac{\ell}{2}\mathbb{Z}$ for all $\gamma \in \Lambda(L^\mu)$.

Note that the analogous results for commutative algebra objects are given by setting $\mu = 0$. In Subsections 4.3.1 and 4.3.2 of Chapter 3, we identify algebra objects $\mathcal{A}_{rQ} \in \mathcal{C}^\oplus$ and $\mathfrak{B}^{a_r}_{rP} \in (\mathcal{C} \boxtimes \mathcal{H})^\oplus$ where $\mathcal{H}$ is an appropriate category of modules over the Heisenberg vertex algebra (see Subsection 4.1), which correspond to the $W_Q(r)$ and $B_Q(r)$ vertex operator algebras respectively. Their categories of local modules satisfy the following properties (Proposition 4.3.5 and 4.3.7):

Proposition 1.1.14.

- $\mathrm{Rep}^0\mathcal{A}_{rQ}$ is a finite non-degenerate ribbon category (i.e. Log-Modular) with $\det(A)\cdot r^{\mathrm{rank}(\mathfrak{g})}$ distinct irreducible modules, where A is the Cartan matrix of $\mathfrak{g}$.
- $\mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ is non-degenerate, ribbon if r is odd or $\rho \in Q$, and the irreducible modules are
$$\{\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma) \mid \mu,\gamma \in \mathfrak{h}^*, \text{ and } \mu + ra_r\gamma \in Q\}$$
with relations $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma) \cong \mathscr{F}(S^{\mu+\lambda} \boxtimes \mathsf{F}_{\gamma+a_r\lambda})$ for all $\lambda \in rP$, where $a_r = \sqrt{-1/r}$.

1.1.4 Chapter 5 - Projective modules for $\overline{U}_i^H(\mathfrak{sl}_3)$

In Chapter 5 we specialize to the case $\overline{U}_i^H(\mathfrak{sl}_3)$. Our main focus is to understand the structure of projective modules and tensor decompositions for tensor products involving projective Verma modules. To determine the Loewy diagrams of projectives (see Definition 5.1.3), we need to know the structure of the irreducible and Verma modules, which are denoted S^λ and M^λ respectively, where $\lambda \in \mathfrak{h}^*$ is the highest weight. It follows from Propositions 3.3.4 and 3.3.6 that the irreducibility of M^λ is determined by the "typicality" of the scalars $\lambda_k = \langle\lambda+\rho,\alpha_k\rangle$ $(k=1,2)$ and $\lambda_3 = \lambda_1+\lambda_2$ where α_1,α_2 are the simple roots for $\mathfrak{sl}_3$, and we call λ_j typical if $\lambda_j \in \ddot{\mathbb{C}} \setminus \mathbb{Z} \cup 2\mathbb{Z}$. We call a weight $\lambda \in \mathfrak{h}^*$ typical if all λ_j are typical, and atypical otherwise. Atypical weights with at least one typical scalar will sometimes be called semi-typical. It is easy to check that there are five distinct cases:

1. λ typical (all λ_j are typical),
2. $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$,
3. $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 1 + 2\mathbb{Z}$ or $\lambda_2 \in 2\mathbb{Z}$ and $\lambda_1 \in 1 + 2\mathbb{Z}$,
4. $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$ and $\lambda_2 \in 1 + 2\mathbb{Z}$ or $\lambda_2 \in \mathbb{Z} \setminus \mathbb{Z}$ and $\lambda_1 \in 1 + 2\mathbb{Z}$,
5. $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$ and $\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$.

Each case yields a different structure for the irreducible modules S^λ. In fact, the Loewy diagrams of Verma modules and projective covers are also determined by the above cases. The structure of irreducible modules are as follows (Proposition 5.1.2):

Proposition 1.1.15.

- If $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$, then $\dim(S^\lambda) = 1$ and $\mathrm{ch}[S^\lambda] = z^\lambda$.
- If $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$ with $\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$, then $\dim(S^\lambda) = 4$ with

$$\mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-\alpha_2} + z^{\lambda-\alpha_1-\alpha_2}$$

- If λ is semi-typical with atypical index λ_i ($i \in \{1,2\}$), then if $j \neq i \in \{1,2\}$,

$$\begin{array}{lll} \dim(S^\lambda) = 3 & \mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda-\alpha_j} + z^{\lambda-(\alpha_1+\alpha_2)} & \text{if } \lambda_j \in 2\mathbb{Z} \\ \dim(S^\lambda) = 4 & \mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda-\alpha_j} + z^{\lambda-(\alpha_1+\alpha_2)} + z^{\lambda-\alpha_i-2\alpha_j} & \text{if } \lambda_j \in \mathbb{C} \setminus \mathbb{Z} \end{array}$$

- If λ is typical, then $\dim(S^\lambda) = 8$ and

$$\mathrm{ch}[S^\lambda] = \mathrm{ch}[M^\lambda] = z^\lambda \prod_{\alpha \in \Delta^-} \left(\frac{z^{2\alpha} - 1}{z - 1} \right)$$

From this, we can prove the following result (Proposition 5.1.4):

Proposition 1.1.16. The Loewy diagrams for Verma modules are as follows:

$$\begin{tikzcd}
 & {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda-\alpha_1} \arrow[dr] & M^{\lambda} & {}^{(1)}S^{\lambda-\alpha_1-2\alpha_2} \arrow[dl] \\
 & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1 + 2\mathbb{Z}$, $\lambda_2 \in 2\mathbb{Z}$.

$$\begin{tikzcd}
 & {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dr] & M^{\lambda} & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dl] \\
 & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 2\mathbb{Z}$, $\lambda_2 \in 1 + 2\mathbb{Z}$.

$$\begin{tikzcd}
M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\
 & {}^{(4)}S^{\lambda-\alpha_2}
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$,
$\lambda_2 \in 1 + 2\mathbb{Z}$.

$$\begin{tikzcd}
M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\
 & {}^{(4)}S^{\lambda-\alpha_1}
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1 + 2\mathbb{Z}$,
$\lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$.

$$\begin{tikzcd}
M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\
 & {}^{(4)}S^{\lambda-\alpha_1-\alpha_2}
\end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$
$\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$

$$\begin{tikzcd}
 & {}^{(1)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(3)}S^{\lambda-\alpha_1} \arrow[dr] & M^{\lambda} & {}^{(3)}S^{\lambda-\alpha_2} \arrow[dl] \\
 & {}^{(1)}S^{\lambda-2(\alpha_1-\alpha_2)} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$.

where left superscripts indicate the dimension of S^{λ}.

To determine the Loewy diagrams of projective modules P^{λ}, we must first determine the Verma modules which appear in its standard filtration (a standard filtration is one for which successive quotients are Verma modules). This is done using Proposition 1.1.16 to determine the multiplicity $[M^{\mu} : L^{\lambda}]$ of L^{λ} in the composition series of M^{μ} for every $\mu \in \mathfrak{h}^*$, and then applying BGG reciprocity (Proposition 3.3.12) to determine the multiplicity $(P^{\lambda} : M^{\mu})$ of each M^{μ} in the standard filtration for P^{λ}. Since we know the irreducible factors of each Verma module, we then know the irreducible factors which appear in the Loewy diagram of P^{λ}. One then exploits the self-duality of projective covers (Theorem 1.1.7) via Lemma 5.2.2 along with the Loewy diagrams of Verma modules to determine which row each irreducible factor appears in, and many of the arrows for the diagram. One then completes the diagram by computing extension groups to determine missing arrows, which yields the following:

Theorem 1.1.17. The Loewy diagrams for the projective covers are as follows:

$$\begin{tikzcd}
 & {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_1} \arrow[dl] \\
 & {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_1}$

$$\begin{tikzcd}
 & {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_2} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_2} \arrow[dl] \\
 & {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in \mathbb{C}\setminus\mathbb{Z}, \lambda_2 \in 1+2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_2}$

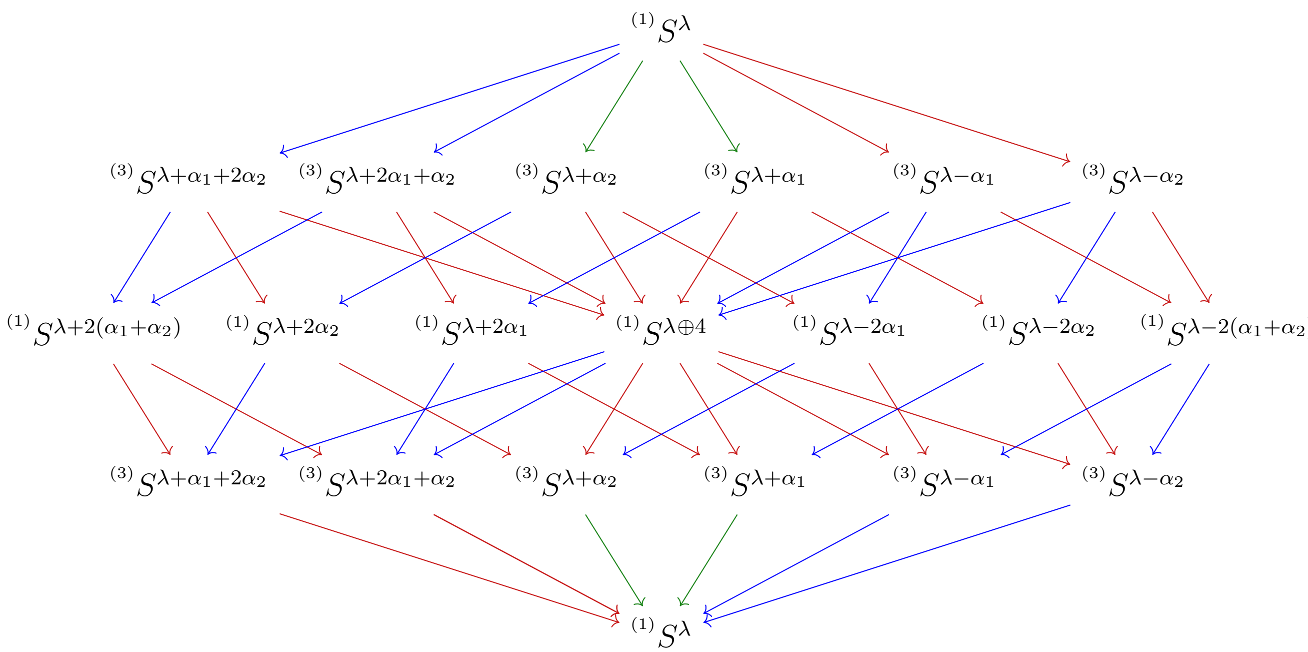
$$\begin{tikzcd}
 & {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1-\alpha_2} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dl] \\
 & {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$, $\lambda_1+\lambda_2 \in 1+2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_1+\alpha_2}$

$$\begin{tikzcd}
 & & & {}^{(3)}S^{\lambda} \arrow[dll, blue] \arrow[d, red] \arrow[drr, red] & & & \\
 & {}^{(1)}S^{\lambda+\alpha_2} \arrow[dl, blue] \arrow[dr, red] \arrow[drrr, red] & & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dlll, blue] \arrow[dr, green] \arrow[drrr, red] & & {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dlll, blue] \arrow[dl, blue] \arrow[dr, red] & \\
{}^{(3)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dr, red] \arrow[drrr, red] & & {}^{(3)}S^{\lambda+\alpha_2-\alpha_1} \arrow[dl, blue] \arrow[drrr, red] & & {}^{(3)}S^{\lambda} \arrow[dlll, blue] \arrow[dl, green] \arrow[dr, red] & & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} \arrow[dlll, blue] \arrow[dl, blue] \\
 & {}^{(1)}S^{\lambda+\alpha_2} \arrow[drr, red] & & {}^{(1)}S^{\lambda-\alpha_2} \arrow[d, red] & & {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dll, blue] & \\
 & & & {}^{(3)}S^{\lambda} & & &
\end{tikzcd}$$

Loewy diagram for $\lambda_1 \in 2\mathbb{Z}, \lambda_2 \in 1+2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_2}, M^{\lambda+\alpha_1+\alpha_2}$

The diagram for $\lambda_1 \in 1+2\mathbb{Z}$, $\lambda_2 \in 2\mathbb{Z}$ can be obtained by swapping $\alpha_1 \leftrightarrow \alpha_2$ in the simple factors.



Loewy diagram for $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_1}, M^{\lambda+\alpha_2}, M^{\lambda+2\alpha_1+\alpha_2}, M^{\lambda+\alpha_1+2\alpha_2}, M^{\lambda+2(\alpha_1+\alpha_2)}$

Applying this theorem together with the fact that projective modules with coinciding characters are isomorphic (Proposition 3.3.13), we can compute the tensor decompositions of the form $S^{\lambda} \otimes M^{\mu}$ where M^{μ} is projective.

Proposition 1.1.18. Let $i, j \in \{1, 2\}$ with $i \neq j$. If $\lambda_i \in 1 + 2\mathbb{Z}$ and $\lambda_j \in 2\mathbb{Z}$, then

$$
\begin{aligned}
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_i \in 2\mathbb{Z},\ \mu_j \in \mathbb{C}\setminus\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_j} && \text{when } \mu_i \in \mathbb{C}\setminus\mathbb{Z},\ \mu_j \in 2\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_1, \mu_2 \in 2\mathbb{Z}
\end{aligned}
$$

For $\lambda_i \in 1 + 2\mathbb{Z}$ and $\lambda_j \in \mathbb{C} \setminus \mathbb{Z}$, we get

$$
\begin{aligned}
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_j} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} && \text{when } \mu_i \in \mathbb{C}\setminus\mathbb{Z},\ \mu_j \in 2\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_1, \mu_2 \in 2\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu-2\alpha_j-\alpha_i} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \in 2\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-2\alpha_j-\alpha_i} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \in 1 + 2\mathbb{Z}\\
&S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_j} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \notin \mathbb{Z}
\end{aligned}
$$

For $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_3 = \lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$, we have

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} \qquad \text{If } \mu_1, \mu_2 \in 2\mathbb{Z}$$
$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \qquad \text{Otherwise}$$

1.1.5 Connections to vertex operator algebras

Extending the connections between $\overline{U}_q^H(\mathfrak{sl}_2)$ and the rank one singlet, triplet, and B_r vertex operator algebras to the higher rank cases $W_Q^0(r)$, $W_Q(r)$, and $B_Q(r)$ is our primary motivation. As described in Subsections 4.3.1 and 4.3.2, the vertex operator algebras $W_Q(r)$ and $B_Q(r)$ can be identified with commutative algebra objects in $(\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2))^\oplus$ and $(\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2) \boxtimes \mathcal{H})^\oplus$ respectively where $\mathcal{H}$ is the category of modules over the Heisenberg vertex operator algebra on which L_0 acts semisimply (see Subsection 4.1), and $\mathcal{C}^\oplus$ denotes the direct sum completion of $\mathcal{C}$ (see [AR]). We expect that the module categories $\mathrm{Rep}_{\langle s\rangle}W_Q(r)$ and $\mathrm{Rep}_{\langle s\rangle}B_Q(r)$ generated by irreducible $W_Q(r)$ and $B_Q(r)$ modules respectively to be ribbon equivalent to the categories of local modules over their corresponding algebra objects $\mathcal{A}_{rQ}$ and $\mathfrak{B}_{rP}^{a_r}$, and the structure of their categories of local modules are given in Propoisition 1.1.14. We therefore make the following conjectures for the corresponding VOA categories:

Conjecture 1.1.19.

1. $\mathrm{Rep}_{\langle s\rangle}\mathcal{W}_Q(r)$ is a finite non-degenerate ribbon category with $\det(A) \cdot r^{\mathrm{rank}(\mathfrak{g})}$ distinct irreducible modules, where A is the Cartan matrix of $\mathfrak{g}$.

2. $\mathrm{Rep}_{\langle s\rangle}B_Q(r)$ is non-degenerate, ribbon if r is odd or $\rho \in Q$, and the irreducible modules can be indexed as

$$\{S_\gamma^\mu \mid \mu, \gamma \in \mathfrak{h}^*, \text{ and } \langle \lambda, \mu + r a_r \gamma \rangle \in \mathbb{Z} \text{ for all } \lambda \in P\}$$

   with relations $S_\gamma^\mu \cong S_{\gamma+a_r\lambda}^{\mu+\lambda}$ for all $\lambda \in rP$, where $a_r = \sqrt{-1/r}$.

We remark that the number of irreducibles in the conjecture for $W_{\mathfrak{sl}_3}(2)$ agrees with observations of Shoma Sugimoto that he presented in a seminar talk at the University of Alberta.

The $B_{A_2}(2)$-algebra is particularly interesting, as it is isomorphic to the simple affine vertex algebra of $\mathfrak{sl}_3$ at level $-3/2$, $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ [A3], which is the easiest case beyond rank one. The associated algebra object lives in the category $(\mathrm{Rep}_{wt}\overline{U}_i^H(\mathfrak{sl}_3) \boxtimes \mathcal{H})^\oplus$, so the results of Chapter 5 on the category of weight modules for $\mathrm{Rep}_{wt}\overline{U}_i^H(\mathfrak{sl}_3)$ will be useful for studying this category. The results of Chapter 5 will appear along with the construction of the category of local modules corresponding to $B_{A_2}(2)$ and a comparison to $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$ in a future paper coauthored with Thomas Creutzig and David Ridout.

Verlinde's formula

We call a vertex operator algebra strongly rational if it is rational, C_2-cofinite, simple, and self-contragredient. Let $M_0 = V, M_1, ..., M_n$ denote the inequivalent simple modules of a strongly rational vertex operator algebra V (recall that rational vertex algebras have finitely many simples). Then the torus one-point functions are the graded trace

$$\mathrm{ch}[M_i](v,\tau) = \mathrm{tr}_{M_i}\left(o(v)q^{L_0-\frac{c}{24}}\right), \qquad q = e^{2\pi i\tau} \tag{1.1.14}$$

with $o(v) \in \mathrm{End}(M_i)$ the zero-mode corresponding to $v \in V$ and c the central charge of the vertex operator algebra. They carry an action of the modular group [Zhu] and the modular S-transformation defines the modular S-matrix, $\mathsf{S}^{\chi}_{i,j}$ by

$$\mathrm{ch}[M_i]\left(v, -\frac{1}{\tau}\right) = \tau^k \sum_{j=0}^{n} \mathsf{S}^{\chi}_{i,j}\mathrm{ch}[M_j](v,\tau) \tag{1.1.15}$$

with k the modified degree of v [Zhu]. Denote by N_{ij}^k the fusion rules, i.e.

$$M_i \boxtimes_V M_j \cong \bigoplus_{k=0}^{n} {N_{ij}}^k\, M_k. \tag{1.1.16}$$

Then, Verlinde's formula says that

$${N_{ij}}^k = \sum_{\ell=0}^{n} \frac{\mathsf{S}^{\chi}_{i,\ell}\mathsf{S}^{\chi}_{j,\ell}(\mathsf{S}^{\chi})^{-1}_{\ell,k}}{\mathsf{S}^{\chi}_{0,\ell}}. \tag{1.1.17}$$

It was long expected from physics considerations that Verlinde's formula should hold for strongly rational vertex operator algebras [V, MS]. It can be shown that Verlinde's formula follows directly from the formula

$$\frac{\mathsf{S}^{\infty}_{i,j}}{\mathsf{S}^{\infty}_{0,j}} = \frac{\mathsf{S}^{\chi}_{i,j}}{\mathsf{S}^{\chi}_{0,j}} \tag{1.1.18}$$

with the Hopf links $\mathsf{S}^{\infty}_{i,j} = \mathrm{tr}_{M_i \boxtimes_V M_j}(c_{j,i} \circ c_{i,j})$ where the $c_{i,j} : M_i \boxtimes_V M_j \xrightarrow{\cong} M_j \boxtimes_V M_i$ are the braiding isomorphisms, since in any finite rigid braided tensor category Hopf links satisfy

$$\frac{\mathsf{S}^{\infty}_{i,\ell}}{\mathsf{S}^{\infty}_{0,\ell}} \frac{\mathsf{S}^{\infty}_{j,\ell}}{\mathsf{S}^{\infty}_{0,\ell}} = \sum_{k=0}^{n} {N_{ij}}^{k} \frac{\mathsf{S}^{\infty}_{k,\ell}}{\mathsf{S}^{\infty}_{0,\ell}} \tag{1.1.19}$$

where ℓ labels a simple object. Indeed, (1.1.17) follows directly from (1.1.18), (1.1.19), and invertibility of the Hopf link S^{∞}-matrix in modular tensor categories. The famous Theorem of Yi-Zhi Huang says that (1.1.18) is true in any strongly rational vertex operator algebra [H1, H2, H3].

It is natural to ask for a variant of Verlinde's formula for vertex operator algebras with non semi-simple finite representation categories. The picture promoted in [CG1] is that there should still be a relation similar to equation (1.1.18) with some modifications: since traces on negligible objects vanish one then needs to replace Hopf links by modified Hopf links. That is, modified traces of double braidings. Similarly one also needs to take into account so-called pseudo trace functions of modules [Miy]. This picture is verified in examples based on conjectural correspondences to restricted quantum groups [CG1, CMR], see also [GR1, FGR] for further work on the Verlinde formula in this context, and [S2] for the categorical perspective.

In practice, most interesting vertex operator algebras (such as the affine vertex operator algebras at admissible level) have representation categories that are not even finite, they have uncountably many inequivalent simple objects. There exists a conjecture relating to Verlinde's formula for affine vertex operator algebras of $\mathfrak{sl}_2$ at admissible level by treating characters as formal distributions [CR2, CR3], see [RW1] for a review. This conjecture is open, except for some encouraging computations of fusion rules [Ga, Ri, AP] and a recent proof of a formula of type (1.1.18) for the finite subcategory of ordinary modules for all affine vertex operator algebras of simply-laced Lie algebras at admissible level [CHY, C1].

The conjectural Verlinde formula for the B_r-algebras is of the same type as admissible level affine $\mathfrak{sl}_2$ and actually B_3 is isomorphic to the affine vertex operator algebra of $\mathfrak{sl}_2$ at level $-4/3$. Assuming the quantum group correspondence between $\mathcal{M}(p)$ and $\overline{U}_q^H(\mathfrak{g})$ to be correct, the results of Sections 2.3 and 2.4 in Chapter 2 verify that normalized Hopf links coincide up to complex conjugation with normalized modular S^{χ}-coefficients, i.e. verify the appropriate analogue of equation (1.1.18) in this context.

Chapter 2

Braided Tensor Categories for B_r Vertex Operator Algebras

The B_r-algebras are a family of vertex operator algebras parameterized by $r \in \mathbb{Z}_{\geq 2}$. They are important examples of logarithmic conformal field theories and appear as chiral algebras in Argyres-Douglas theories of type (A_1, A_{2r-3}). The first member of this series, the B_2-algebra, is the well-known symplectic bosons also often called the $\beta\gamma$ vertex operator algebra. We study categories related to the B_r vertex operator algebras using their conjectural relation to unrolled restricted quantum groups of $\mathfrak{sl}_2$. These categories are braided, rigid and non semi-simple tensor categories. We list their simple and projective objects, their tensor products and their Hopf links. The latter are succesfully compared to modular data of characters for the B_r vertex algebras. Assuming the categories we construct and $\mathrm{Rep}_{\langle s\rangle} B_r$ are indeed ribbon equivalent, this confirms a proposed Verlinde formula of Thomas Creutzig and David Ridout.

2.1 Preliminaries

The B_r-algebra can be realized as a direct sum of simple currents in the Deligne product of representation categories of the singlet and Heisenberg vertex operator algebras (see

[C2],[CRW]). It was shown in [CMR] that there exists a bijection between a particular category of modules for the singlet and the category of weight modules for the unrolled restricted quantum group of $\mathfrak{sl}_2$, $\overline{U}_q^H(\mathfrak{sl}_2)$. Further structures as tensor products and open Hopf links were succesfully matched with conjectural fusion products on the singlet algebra and asymptotic dimensions of characters, which led to the conjecture that these categories are equivalent as ribbon categories. Further evidence for this conjecture has been given in [CGR].

In this section we recall the definition of $\overline{U}_q^H(\mathfrak{sl}_2)$ and the structure of its category of weight modules, as seen in [CGP1]. We will also recall the definition of the Heisenberg vertex operator algebra and the category of modules we are interested in, as well as some preliminaries of algebra objects and simple currents.

The unrolled restricted quantum group of $\mathfrak{sl}_2$ and its weight modules

Let $r \geq 2$ be a positive integer and

$$q = e^{-\pi i/r} \tag{2.1.1}$$

a $2r$-th root of unity. For any $x \in \mathbb{C}$ we choose the notation

$$\{x\} = q^x - q^{-x}, [x] = \frac{\{x\}}{\{1\}}, \text{ and for any } n \in \mathbb{Z},\ \{n\}! = \{n\}\{n-1\}...\{1\}. \tag{2.1.2}$$

The Drinfeld-Jimbo algebra associated to $\mathfrak{sl}_2$, $U_q(\mathfrak{sl}_2)$ is the associative algebra over $\mathbb{C}$ with generators E, F, K, K^{-1} and relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}.$$

This algebra has a Hopf algebra structure given by a counit $\epsilon : U_q(\mathfrak{sl}_2) \to \mathbb{C}$, a coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$, and an antipode $S : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ defined by

$$\begin{aligned}
&\Delta(K) = K \otimes K, && \epsilon(K) = 1, && S(K) = K^{-1},\\
&\Delta(E) = 1 \otimes E + E \otimes K, && \epsilon(E) = 0, && S(E) = -EK^{-1},\\
&\Delta(F) = K^{-1} \otimes F + F \otimes 1, && \epsilon(F) = 0, && S(F) = -KF.
\end{aligned}$$

The unrolled quantum group of $\mathfrak{sl}_2$, $U_q^H(\mathfrak{sl}_2)$, is defined by extending $U_q(\mathfrak{sl}_2)$ through the addition of a fifth generator H with relations

$$HK^{\pm 1} = K^{\pm 1}H, \qquad [H,E] = 2E, \qquad [H,F] = -2F.$$

The counit, coproduct, and antipode can be extended to $U_q^H(\mathfrak{sl}_2)$ by defining

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \epsilon(H) = 0, \qquad S(H) = -H.$$

The unrolled restricted quantum group of $\mathfrak{sl}_2$, $\overline{U}_q^H(\mathfrak{sl}_2)$, is then obtained taking the quotient of $U_q^H(\mathfrak{sl}_2)$ by the relations $E^r = F^r = 0$.
A finite dimensional $\overline{U}_q^H(\mathfrak{sl}_2)$-module V is called a weight module if it is a direct sum of its H-eigenspaces (H acts semisimply) and $K = q^H$ as an operator on V. Let $\mathrm{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ denote the category of weight modules for $\overline{U}_q^H(\mathfrak{sl}_2)$. A classification of simple and projective modules was given in [CGP1] as follows:

Given any $n \in \{0, ..., r-1\}$, let S_n be the simple highest weight module of weight n and dimension $n+1$ with basis $\{s_0, ..., s_n\}$ and action

$$Fs_i = s_{i+1}, \quad Es_i = [i][n+1-i]s_{i-1}, \quad Hs_i = (n-2i)s_i, \quad Es_0 = Fs_n = 0.$$

For any $\alpha \in \mathbb{C}$, define V_α to be the p-dimensional highest weight module of highest weight $\alpha + r - 1$, whose action is defined on a basis $\{v_0, ..., v_{r-1}\}$ as

$$Fv_i = v_{i+1}, \quad Ev_i = [i][i-\alpha]v_{i-1}, \quad Hv_i = (\alpha + r - 1 - 2i)v_i, \quad Ev_0 = Fv_{r-1} = 0.$$

V_α is called typical if $\alpha \in \ddot{\mathbb{C}} := (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$ and atypical otherwise. The typical V_α are simple since any basis vector v_i can generate a scalar multiple of every other basis vector through the action of E and F. If V_α is atypical, then we have $\alpha = rm + k$ for some $m \in \mathbb{Z}$ and $1 \leq k \leq r-1$. Hence,

$$Ev_k = -[k][k-(rm+k)]v_{k-1} = [k][rm]v_{k-1} = 0,$$

since $\{rm\} = q^{rm} - q^{-rm} = (-1)^m - (-1)^{-m} = 0$. So, when V_α is atypical, it contains a simple submodule generated by the basis elements $\{v_k, v_{k+1}, ..., v_{r-1}\}$.

For any $\ell \in \mathbb{Z}$, let $\mathbb{C}^H_{\ell r}$ denote the one dimensional module on which E and F act as zero and H acts as scalar multiplication by ℓr. Then the following holds

Proposition 2.1.1. [CGP1, Theorem 5.2 and Lemma 5.3]

1. The typical V_α ($\alpha \in \ddot{\mathbb{C}} = \mathbb{C} \setminus \mathbb{Z} \cup r\mathbb{Z}$) are projective.

2. Every simple module in $\mathrm{Rep}_{wt}\overline{U}^H_q(\mathfrak{sl}_2)$ is isomorphic to $S_n \otimes \mathbb{C}^H_{\ell r}$ for some $n \in \{0, ..., r-2\}$ and $\ell \in \mathbb{Z}$ or V_α for some $\alpha \in \ddot{\mathbb{C}}$.

A weight vector v is called dominant if $(FE)^2 v = 0$. If v is a dominant weight vector of weight i, then we denote by P_i the module generated by v with no additional relations. This module's structure is given explicitly in [CGP1, Section 6], and the following proposition was proven therein:

Proposition 2.1.2. The module P_i is projective and indecomposable with dimension $2r$. Any projective indecomposable module with integer highest weight $(\ell+1)r - i - 2$ is isomorphic to $P_i \otimes \mathbb{C}^H_{\ell r}$.

If V is an object in $\mathrm{Rep}_{wt}\overline{U}^H_q(\mathfrak{sl}_2)$ with basis $\{v_1, ..., v_n\}$, then V has the obvious dual vector space $V^* = \mathrm{Hom}_\mathbb{C}(V, \mathbb{C})$ with dual basis $\{v_1^*, ..., v_n^*\}$ and action $af(v) = f(S(a)v)$ for $f \in V^*, a \in \overline{U}^H_q(\mathfrak{sl}_2)$ and S the antipode. The left duality morphisms are given by

$$\overrightarrow{\mathrm{coev}}_V : \mathbb{C} \to V \otimes V^* \quad \text{ and } \quad \overrightarrow{\mathrm{ev}}_V : V^* \otimes V \to \mathbb{C},$$

where $\overrightarrow{\mathrm{coev}}(1) = \sum_i^n v_i \otimes v_i^*$, and $\overrightarrow{\mathrm{ev}}(f \otimes w) = f(w)$. Ohtsuki defined in [O] the R-matrix operator on $\mathrm{Rep}_{wt}\overline{U}^H_q(\mathfrak{sl}_2)$ by

$$R = q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n, \tag{2.1.3}$$

where $q^{H\otimes H/2}(v\otimes w) = q^{\lambda_v\lambda_w/2} v\otimes w$ for weight vectors v, w with weights λ_v and λ_w. The braiding on $\text{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ is then given by the family of maps $c_{V,W} : V\otimes W \to W\otimes V$ where $c_{V,W}(v\otimes w) = \tau(R(v\otimes w))$ where τ is the flip map $w\otimes v \mapsto v\otimes w$. Ohtsuki also defined an operator $\widetilde{\theta}_V : V \to V$ on each $V \in \text{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ by

$$\widetilde{\theta} = K^{r-1}\sum_{n=0}^{p-1}\frac{\{1\}^{2n}}{\{n\}!}q^{n(n-1)/2}S(F^n)q^{-H^2/2}E^n. \tag{2.1.4}$$

The twist $\theta_V : V \to V$ is then given by the operator $v \mapsto \widetilde{\theta}^{-1}v$. $\text{Rep}_{wt}\overline{U}_q^H(\mathfrak{sl}_2)$ also admits compatible right duality morphisms

$$\begin{aligned} \overleftarrow{\text{ev}}_V : V\otimes V^* :\to \mathbb{C}, &\qquad \overleftarrow{\text{ev}}_V(v\otimes f) = f(K^{1-p}v), \\ \overleftarrow{\text{coev}}_V : \mathbb{C} \to V^*\otimes V, &\qquad \overleftarrow{\text{coev}}(1) = \sum_i K^{p-1}V_i \otimes v_i^*. \end{aligned}$$

The following Lemma was proved in [Ru, Proposition 6] using [CGP1] and will be used in multiple results:

Lemma 2.1.3. For any $k \in \{1, ..., r-1\}$ there are short exact sequences of modules

$$\begin{gathered} 0 \to S_{r-1-k}\otimes \mathbb{C}^H_{\ell r} \to V_{k+\ell r} \to S_{k-1}\otimes \mathbb{C}^H_{(\ell+1)r} \to 0, \\ 0 \to V_{r-1-i+\ell r} \to P_i \otimes \mathbb{C}^H_{\ell r} \to V_{1+i-r+\ell r} \to 0. \end{gathered}$$

2.1.1 Vertex Operator Algebras

Given a ring R, we denote by $R[z]$, $R[[z]]$, and $R((z))$ the space of formal R-valued polynomials, Taylor series, and Laurent series respectively. That is,

$$R[z] = \left\{ \sum_{i=0}^{n} r_i z^i | r_i \in R, n \in \mathbb{Z}_+ \right\}$$
$$R[[z]] = \left\{ \sum_{i=0}^{\infty} r_i z^i | r_i \in R \right\}$$
$$R((z)) = \left\{ \sum_{i=-m}^{\infty} r_i z^i | r_i \in R, m \in \mathbb{Z}_+ \right\}$$

Let V be a complex vector space, and $\mathrm{End}(V)$ the collection of linear operators $f : V \to V$. The formal power series

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n}$$

with coefficients $A_n \in \mathrm{End}(V)$ is called a field if for all $v \in V$, $A(z)v$ is a Laurent series, so

$$A(z)v = \sum_{n \in \mathbb{Z}} A_n(v) z^{-n} \in V((z)).$$

This is equivalent to stating that for all $v \in V$, $A_n v = 0$ for sufficiently large $n(v)$. A $\mathbb{Z}$-graded vector space is a vector space V such that $V = \bigoplus_{n \in \mathbb{Z}} V_n$ where each V_n is itself a vector space. A linear operator $f \in \mathrm{End}(V)$ on a graded vector space V is said to be homogeneous of degree m if $f(V_n) \subset V_{n+m}$ for all $n \in \mathbb{Z}$, and we denote the degree by $\deg f$. A field $A(z)$ is then said to be homogeneous of conformal dimension m if each operator A_n is homogeneous of degree $m - n$.

Definition 2.1.4. Two fields $A(z)$ and $B(w)$ are said to be local iff there exists an $N \in \mathbb{Z}_+$ such that

$$(z - w)^N [A(z), B(w)] := (z - w)^N (A(z)B(w) - B(w)A(z)) = 0.$$

Definition 2.1.5. A vertex algebra is a vector space V equipped with the following objects:

- A distinguished vector $|0\rangle$ called the vacuum vector.
- A linear operator $T : V \to V$.
- A linear operator $Y(-, z) : V \to \mathrm{End}V[[z^{\pm 1}]]$ which sends each element $v \in V$ to a field $Y(v, z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$.

These objects are subject to the following constraints:

- For any $v \in V$, we have $Y(v, z)|0\rangle \in V[[z]]$ where $Y(v, z)|0\rangle|_{z=0} = v$ and $Y(|0\rangle, z) = Id_V$.
- For any $v \in V$, $[T, Y(v, z)] = \partial_z Y(v, z)$ and $T|0\rangle = 0$.
- All fields $Y(v, z)$ are local with respect to each other.

A vertex algebra is called $\mathbb{Z}$-graded if $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with $|0\rangle \in V_0$, T is a linear operator of degree 1, and for any $v \in V_m$, $Y(v, z)$ has conformal dimension m i.e. $\deg v_{(n)} = -n+m-1$.

Definition 2.1.6. A vertex operator algebra, or conformal vertex algebra, of central charge $c \in \mathbb{C}$ is a $\mathbb{Z}$-graded vertex algebra $V = \bigoplus_{n\in\mathbb{Z}} V_n$ with a non-zero "conformal vector" $\omega \in V_2$ such that the coefficients L_n, $n \in \mathbb{Z}$ of the associated field:

$$Y(\omega, z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy $T = L_{-1}$, $L_0|_{V_n} = n\mathrm{Id}$, and the defining relations of the Virasoro Lie algebra with central charge c. That is,

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n,-m}c.$$

Heisenberg vertex operator algebra and its Fock modules

The rank-1 Heisenberg Lie algebra, denoted by $\widehat{\mathfrak{h}}$, has vector space basis given by $\{\mathbf{c}, b_n | n \in \mathbb{Z}\}$ and bracket

$$[\mathbf{c}, b_n] = 0 \text{ and } [b_n, b_m] = n\delta_{n+m,0}\mathbf{c}.$$

Let $\widehat{\mathfrak{h}}_\pm = \mathrm{Span}_{\mathbb{C}}\{b_n \mid \pm n > 0\}$. Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of any Lie algebra $\mathfrak{g}$. Denote by F_β the usual Fock space of charge $\beta \in \mathbb{C}$ with vector space basis $\mathcal{U}(\widehat{\mathfrak{h}}_-)$ and $\widehat{\mathfrak{h}}$-action on an arbitrary element $b \in \mathsf{F}_\beta$ given by

$$\begin{aligned}
\mathbf{c} \cdot b &= b, \\
b_0 \cdot b &= \beta b, \\
b_n \cdot b &= b_n b \quad \text{for all } n < 0, \\
b_n \cdot b &= n\frac{\partial}{\partial b_{-n}} b \quad \text{for all } n > 0.
\end{aligned}$$

Definition 2.1.7. The Heisenberg vertex operator algebra $\mathsf{H} = (\mathsf{F}_0, 1, Y, T, \omega)$ is given by the following data (see [FB, Chapter 2]):

- a $\mathbb{Z}_+$-gradation $\deg(b_{j_1} \cdots b_{j_k}) = -\sum_{i=1}^k j_i$,
- a vacuum vector $|0\rangle = 1$,
- a translation operator T defined by $T(1)$=0, $[T, b_i] = -ib_{i-1}$,
- vertex operators $Y(-, z)$ defined by

$$Y(b_{j_1} \cdots b_{j_k}, z) = \frac{: \partial_z^{-j_1-1} b(z) \cdots \partial_z^{-j_k-1} b(z) :}{(-j_1-1)! \cdots (-j_k-1)!},$$

  where $: X(z)Y(z) :$ denotes the normally ordered product.
- a conformal vector $\omega = b_{-1}^2$ of central charge 1.

If $b' \in \mathsf{H}$ and $b \in \mathsf{F}_\beta$ then H acts on F_β as $b'(b) = Y(b', z)b$ as an extension of the action of $\mathfrak{h}$. We can therefore consider the representation category H-Mod generated by the Fock spaces.

This category is rigid (contains duals) and has braiding and twist. The data is

1. $\mathsf{F}_\beta^* \simeq \mathsf{F}_{-\beta}$, (Duals)

2. $\mathsf{F}_{\beta_1} \otimes \mathsf{F}_{\beta_2} \simeq \mathsf{F}_{\beta_1+\beta_2}$, (Fusion/Tensor products)

3. $c_{\mathsf{F}_{\beta_1}\otimes \mathsf{F}_{\beta_2}} = e^{\pi i \beta_1\beta_2}\mathrm{Id}_{\mathsf{F}_{\beta_1}\otimes \mathsf{F}_{\beta_2}}$, (Braidings)

4. $\theta_{\mathsf{F}_\beta} = e^{\pi i \beta^2}\mathrm{Id}_{\mathsf{F}_\beta}$, (Ribbon Twists)

Let $\mathcal{H}^\oplus$ denote the category of $\mathbb{C}$-graded complex vector spaces with finite or countable dimension and let $H_V : V \to V$ be the degree map on $V := \bigoplus\limits_{\nu\in\mathbb{C}} V_\nu$ given by $H_V|_{V_\nu} = \nu \mathrm{Id}_{V_\nu}$. $\mathcal{H}^\oplus$ can be given a (non-unique) ribbon structure by defining the braiding c and twist θ by

$$c_{U,V} := \tau_{U,V} \circ e^{\pi i H_U \otimes H_V}, \tag{2.1.5}$$

$$\theta_V := e^{\pi i H_V^2}\mathrm{Id}_V, \tag{2.1.6}$$

where $\tau_{U,V}$ is the usual flip map. Denote by $\mathcal{H}^\oplus_{i\mathbb{R}}$ the full tensor subcategory of $\mathcal{H}^\oplus$ with purely imaginary index. $\mathcal{H}^\oplus_{i\mathbb{R}}$ is braided equivalent to the full subcategory of H-Mod whose simple objects are given by Fock modules F_{iy} with $y \in \mathbb{R}$, or, $iy \in i\mathbb{R}$ (see [CGR, Subsection 2.3] for details). The equivalence is given by identifying the usual Fock space F_{iy} with the one dimensional vector space of degree iy, $\mathrm{F}_{iy} := \mathbb{C}v_{iy}$.

The Singlet vertex operator algebra $\mathcal{M}(r)$ and its category of modules

Let $r \in \mathbb{Z}_{>0}$ and $L := \sqrt{2r}\mathbb{Z}$ an even lattice. The lattice VOA $V_L := \bigoplus_{l\in L} \mathsf{F}_l$ associated to L can be constructed via the reconstruction theorem as outlined in [FB, Proposition 5.2.5]. Let $e^\gamma(z)$ denote the usual fields associated to lattice vertex operator algebras:

$$e^\gamma(z) := S_\gamma z^{\gamma b_0} \exp\left(-\gamma \sum_{n<0} \frac{b_n}{n} z^{-n}\right) \exp\left(-\gamma \sum_{n\geq 0} \frac{b_n}{n} z^{-n}\right),$$

where S_γ is the shift operator $\mathsf{F}_\beta \to \mathsf{F}_{\beta+\gamma}$. Define the screening operator $\widetilde{Q} = e_0^{-\sqrt{\frac{2}{r}}}$ where $e^\gamma(z) = \sum_{n\in\mathbb{Z}} e_n^\gamma z^{-n-1}$ is the expansion of $e^\gamma(z)$. The Singlet VOA is then defined as the kernel $\mathcal{M}(r) := \mathrm{Ker}_{\mathsf{F}_0}(\widetilde{Q})$. For $k, s \in \mathbb{Z}, 1 \leq s \leq r$, let $\alpha_{k,s} = -\frac{k-1}{2}\sqrt{2r} + \frac{s-1}{\sqrt{2r}}$. For $s = r$, the Fock space with $\mathsf{F}_{\alpha_{k,s}}$ is simple as an $\mathcal{M}(r)$-module, which we denote by $F_{\alpha_{k,s}}$. When $s \neq r$, $F_{\alpha_{k,s}}$ is reducible and we define $M_{k,s}$ to be the socle of $F_{\alpha_{k,s}}$ which is known to be a simple $\mathcal{M}(r)$-module. Note that we are always working with the category of $\mathbb{Z}_{\geq 0}$-graded modules for the singlet. These and further results on the module categories for the singlet could be found in [A1] and [AM1] (see also of [CM1, Section 3.2]).

It is expected that the module categories of $\mathcal{M}(r)$ and $\overline{U}_q^H(\mathfrak{sl}_2)$ are equivalent as monoidal (or perhaps braided) categories. This is motivated by the following statement proven in [CMR]:

Proposition 2.1.8. For $\alpha \in \ddot{\mathbb{C}} := (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$, $i \in \{0, 1, \ldots, r-2\}$ and $k \in \mathbb{Z}$, consider the map

$$\varphi : V_\alpha \mapsto F_{\frac{\alpha+r-1}{\sqrt{2r}}}, \qquad \varphi : S_i \otimes \mathbb{C}_{kr}^H \mapsto M_{1-k,i+1} \tag{2.1.7}$$

between simple modules of $\overline{U}_q^H(\mathfrak{sl}_2)$ and the ($\mathbb{Z}_{\geq 0}$-graded) simple modules for $\mathcal{M}(r)$ singlet vertex operator algebra. This map satisfies the following properties:

1. This map is a bijection of the sets of representatives of equivalence classes (under isomorphisms) of simple modules.

2. Assuming the fusion rules for $\mathcal{M}(r)$ are as conjectured in [CM1], this map induces an isomorphism from the Grothendieck ring of weight modules of $\overline{U}_q^H(\mathfrak{sl}_2)$ to the conjectured Grothendieck ring of $\mathcal{M}(r)$.

A precise conjecture on the connection between the module categories of $\mathcal{M}(r)$ and $\overline{U}_q^H(\mathfrak{sl}_2)$ is given in [CMR, Subsection 3.1] and [CGR, Conjecture 5.8].

Simple currents and algebra objects

Conformal extensions of vertex operator algebras can be studied efficiently via the notion of (super)commutative algebra objects in vertex tensor categories. The representation category for the extended vertex operator (super)algebra then corresponds to the category of local modules for the corresponding (super)commutative algebra object. This program has been developed in [KO, HKL, CKM]. It works particularly well for simple current extensions [CKL, CKLR]. For us, we are interested in algebra objects built as direct sums of simple currents which correspond to some vertex operator algebra under the identification in Proposition 2.1.8, or its higher rank analogue.

Definition 2.1.9. A simple current is a simple object which is invertible with respect to the tensor product. Objects which are their own inverse are called self-dual.

Definition 2.1.10. A commutative associative unital algebra (or just algebra, for short) in a braided monoidal category $\mathcal{C}$ is an object A in $\mathcal{C}$ with multiplication morphism $\mu : A \otimes A \to A$ and unit $\iota : \mathbb{1} \to A$ with the following assumptions:

- Associativity: $\mu \circ (\mu \otimes \mathrm{Id}_A) = \mu \circ (\mathrm{Id}_A \otimes \mu) \circ a_{A,A,A}$ where $a_{A,A,A} : (A \otimes A) \otimes A \to A \otimes (A \otimes A)$ is the associativity isomorphism.

- Unit: $\mu \circ (\iota \otimes \mathrm{Id}_A) \circ l_A^{-1} = \mathrm{Id}_A$ where $l_A : \mathbb{1} \otimes A \to A$ is the left unit isomorphism.

- Commutativity: $\mu \circ c_{A,A} = \mu$ where $c_{A,A}$ is the braiding.

- (Optional assumption) Haploid: $\dim(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)) = 1$.

Supercommutative superalgebra objects are defined as follows:

Definition 2.1.11. $A \in \mathcal{C}$ is a superalgebra if it is an algebra with a $\mathbb{Z}_2$-grading compatible with the product. That is, $A = A^{\bar{0}} \oplus A^{\bar{1}}$ such that

$$\mu(A^{\bar{i}} \otimes A^{\bar{j}}) \subset A^{\overline{i+j}}.$$

A is supercommutative if for $x \in A^{\bar{i}}, y \in A^{\bar{j}}$,

$$\mu|_{A^{\bar{i}} \otimes A^{\bar{j}}} = (-1)^{ij} \cdot \mu \circ c_{A^{\bar{i}}, A^{\bar{j}}}. \tag{2.1.8}$$

We denote by Rep$A$ the category of objects (V, μ_V) where $V \in \mathcal{C}$ is an object in $\mathcal{C}$ and $\mu_V \in \mathrm{Hom}(A \otimes V, V)$ is a $\mathcal{C}$-morphism satisfying the usual assumptions required to make V an A-module:

- $\mu_V \circ (\mathrm{Id}_A \otimes \mu_V) = \mu_V \circ (\mu \otimes \mathrm{Id}_A) \circ a^{-1}_{A,A,V}$,

- $\mu_V \circ (\iota \otimes \mathrm{Id}_V) \circ l_V^{-1} = \mathrm{Id}_V$.

Although $\mathcal{C}$ is braided, Rep$A$ need not be, but there is a full subcategory of Rep$A$ which is braided [KO]:

Definition 2.1.12. The category of local modules $\mathrm{Rep}^0 A$ is the full subcategory of Rep$A$ whose objects are given by

$$\{(V, \mu_V) \in \mathrm{Rep}A \,|\, \mu_V \circ c_{V,A} \circ c_{A,V} = \mu_V\}$$

Definition 2.1.13. [CGR, Section 2] Let $\mathcal{C}$ be a tensor category with tensor identity $\mathbb{1}$ and an algebra object A. A morphism $\omega : A \otimes A \to \mathbb{1}$ is called a non-degenerate invariant pairing if

1. The morphisms $A \otimes (A \otimes A) \xrightarrow{\mathrm{Id} \otimes \mu} A \otimes A \xrightarrow{\omega} \mathbb{1}$ and $A \otimes (A \otimes A) \xrightarrow{a^{-1}_{A,A,A}} (A \otimes A) \otimes A \xrightarrow{\mu \otimes \mathrm{Id}} A \otimes A \xrightarrow{\omega} \mathbb{1}$ coincide. (Invariance)

2. For any object V and morphism $f : V \to A$ the equalities $\omega \circ (f \otimes \mathrm{Id}_A) = 0$ or $\omega \circ (\mathrm{Id}_A \otimes f) = 0$ both imply that $f = 0$. (non-degeneracy)

The notion of simplicity of an extended vertex operator algebra corresponds precisely to the corresponding algebra object having a non-degenerate invariant pairing. This is explained in the proof of Corollary 5.9 of [CGR]. The most well behaved modules in Rep$A$ are those which can be obtained by induction.

Definition 2.1.14. Let $\mathcal{C}$ be a category with algebra object A. The induction functor $\mathscr{F} : \mathcal{C} \to \operatorname{Rep} A$ is defined by $\mathscr{F}(V) = (A \otimes V, \mu_{\mathscr{F}(V)})$ where $\mu_{\mathscr{F}(V)} = (\mu \otimes \mathrm{Id}_V) \circ a^{-1}_{A,A,V}$ (here μ is the product on A) and for any morphism f, $\mathscr{F}(f) = \mathrm{Id}_A \otimes f$.

We also have a forgetful restriction functor $\mathcal{G} : \operatorname{Rep} \mathcal{A} \to \mathcal{C}$ that sends an object (X, μ_X) to X. The induction and restriction functors satisfy Frobenius reciprocity:

$$\mathrm{Hom}_{\mathcal{C}}(X, \mathcal{G}(Y)) \cong \mathrm{Hom}_{\operatorname{Rep} A}(\mathcal{F}(X), Y) \tag{2.1.9}$$

for $X \in \mathcal{C}$ and $Y \in \operatorname{Rep} A$.

It was shown in [CKL, Theorem 3.12] that in the module category of a vertex operator algebra V satisfying certain assumptions, certain (super)-algebra objects built from simple currents have (super)-vertex operator algebra structure and give a (super)-vertex operator algebra extension V_e of V. If the module category of V is sufficiently nice, then the category $\operatorname{Rep}^0 V_e$ (with V_e viewed as a categorical (super)-algebra object) is equivalent as a braided tensor category to the category of generalized modules of V_e (with V_e now viewed as a (super)-vertex operator algebra).

Loewy Diagrams

Recall that a filtration, or series, for a module M is a family of proper submodules ordered by inclusion as

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

Loewy diagrams are defined in terms of socle filtrations.

Definition 2.1.15. The socle filtration of M is the filtration defined by $M_1 = \mathit{Socle}(M)$ (the socle of a module is its largest semi-simple submodule) and we inductively define M_k to be the largest submodule of M such that M_k/M_{k-1} is semi-simple. We define the Loewy diagram of M to be the diagram whose k-th layer from the bottom consists of composition factors of the semisimple module M_k/M_{k-1} with downward arrows indicating submodule inclusion.

2.2 The B_r-algebra as a simple current extension

Let $\mathcal{C} := \mathcal{H}_{i\mathbb{R}}^{\oplus} \boxtimes \overline{U}_q^H(\mathfrak{sl}_2)$-Mod be the Deligne product of $\mathcal{H}_{i\mathbb{R}}^{\oplus}$ (see subsection 2.1.1) and $\overline{U}_q^H(\mathfrak{sl}_2)$-Mod. The tensor product in this category is given by

$$(X \boxtimes Y) \otimes (X' \boxtimes Y') = (X \otimes X') \boxtimes (Y \otimes Y')$$

and the braiding, twist, and rigidity morphisms are given by the product of the corresponding morphisms in $\overline{U}_q^H(\mathfrak{sl}_2)$-Mod and $\mathcal{H}_{i\mathbb{R}}^{\oplus}$. Notice that for any $\lambda \in i\mathbb{R}$, we have

$$(\mathrm{F}_\lambda \boxtimes \mathbb{C}_r^H) \otimes (\mathrm{F}_{-\lambda} \boxtimes \mathbb{C}_{-r}^H) = \mathrm{F}_0 \boxtimes \mathbb{C}_0^H.$$

Hence, $\mathrm{F}_\lambda \boxtimes \mathbb{C}_r^H$ is a simple current. In what follows, we choose a λ_r that satisfies

$$\lambda_r^2 = -\frac{r}{2}. \tag{2.2.1}$$

We can define an object $\mathcal{A}_r$ of the extended category $\mathcal{C}^{\oplus}$ (which allows infinite direct sums while retaining sufficient structure, see [AR]) by

$$\mathcal{A}_r := \bigoplus_{k\in\mathbb{Z}} (\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H)^{\otimes k} \cong \bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{k\lambda_r} \boxtimes \mathbb{C}_{kr}^H. \tag{2.2.2}$$

Remark 2.2.1. The B_r-algebra is a vertex operator algebra extension of $\mathsf{H} \otimes \mathcal{M}(r)$ and it decomposes as an $\mathsf{H} \otimes \mathcal{M}(r)$-module as

$$B_r \cong \bigoplus_{k\in\mathbb{Z}} \mathsf{F}_{kr} \boxtimes M_{1-k,1} \tag{2.2.3}$$

so that under the correspondence of Proposition 2.1.8 the B_r-algebra is the image of $\mathcal{A}_r$

$$B_r = \bigoplus_{k\in\mathbb{Z}} \mathsf{F}_{kr} \boxtimes \varphi(\mathbb{C}_{kr}^H). \tag{2.2.4}$$

The following is a special case of [CGR, Proposition 2.15]. Note that the proof of that

Proposition is in Appendix A of that paper.

Proposition 2.2.2. [CGR, Proposition 2.15] $\mathcal{A}_r$ can be given a structure of a commutative algebra object in $\mathcal{C}^{\oplus}$ with non-degenerate invariant pairing. This structure is unique up to isomorphism.

We now give some criteria on analyzing certain objects in the category $\operatorname{Rep}^0 A$ associated to an algebra object A in $\mathcal{H}_{i\mathbb{R}}^{\oplus} \boxtimes \mathcal{C}$.

Lemma 2.2.3. If $X \in \mathcal{C}$ is simple, then $\mathscr{F}(X) \in \operatorname{Rep} A$ is simple. If $P \in \mathcal{C}$ is projective, then $\mathscr{F}(P)$ is projective in $\operatorname{Rep} A$.

Proof. The first statement is [CKM, Proposition 4.4]. For the second, note that $\operatorname{Hom}_{\operatorname{Rep} A}(\mathscr{F}(P), \bullet) = \operatorname{Hom}_{\mathcal{C}}(P, \bullet) \circ \mathcal{G}$ as functors, due to the Frobenius reciprocity of $\mathscr{F}$ and $\mathcal{G}$. Our forgetful restriction functor $\mathcal{G}$ is exact. Also, $\operatorname{Hom}_{\mathcal{C}}(P, \bullet)$ is exact since P is projective. Therefore the functor $\operatorname{Hom}_{\operatorname{Rep} A}(\mathscr{F}(P), \bullet)$ is exact, which proves that $\mathscr{F}(P)$ is projective.

□

Lemma 2.2.4. If $W \cong_{\mathcal{C}} \bigoplus_{\nu} F_\nu \boxtimes X_\nu$ is a simple object in $\operatorname{Rep} A$ then W is isomorphic to the induction of a simple object.

Proof. Pick and fix any ν_0 amongst the ν appearing above, and pick a non-zero morphism $f : S_{\nu_0} \to X_{\nu_0}$ where S_{ν_0} is a simple module. Consider $g : F_{\nu_0} \boxtimes S_{\nu_0} \xrightarrow{\mathrm{Id} \otimes f} F_{\nu_0} \boxtimes X_{\nu_0} \hookrightarrow \bigoplus_{\nu} F_\nu \boxtimes X_\nu$ which is a non-zero morphism as well. By Frobenius reciprocity, we obtain a non-zero morphism $h : \mathscr{F}(F_{\nu_0} \boxtimes S_{\nu_0}) \to W$.

Now, note that $\mathscr{F}(F_{\nu_0} \boxtimes S_{\nu_0})$ as an object of $\mathcal{C}$ decomposes as $\bigoplus_{k} (F_{k\lambda_r + \nu_0}) \boxtimes (\mathbb{C}_{kr}^H \otimes S_{\nu_0})$, where the summands are all (mutually inequivalent) simple objects due to the simple current property of $F_{k\lambda} \boxtimes \mathbb{C}_{kr}^H$. Now, [CKM, Proposition 4.4] applies and tells us that $\mathscr{F}(F_{\nu_0} \boxtimes S_{\nu_0})$ is infact a simple $\operatorname{Rep} A$ object. Since W is simple with a non-zero morphism $\mathscr{F}(F_{\nu_0} \boxtimes S_{\nu_0}) \to W$, W is isomorphic to the induced object $\mathscr{F}(F_{\nu_0} \boxtimes S_{\nu_0})$.

$\square$

Theorem 2.2.5. The following list of objects in $\mathcal{C}$ induce to $\mathrm{Rep}^0(\mathcal{A}_r)$ by the induction functor $\mathscr{F} : \mathcal{C} \to \mathrm{Rep}(\mathcal{A}_r)$:

1. $\mathrm{F}_\gamma \boxtimes V_\alpha$ induces to $\mathrm{Rep}^0(\mathcal{A}_r)$ if and only if $\alpha + r - 1 + 2\lambda_r\gamma \in 2\mathbb{Z}$;

2. $\mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}^H_{\ell r})$ induces to $\mathrm{Rep}^0(\mathcal{A}_r)$ if and only if $i + \ell r + 2\lambda_r\gamma \in 2\mathbb{Z}$;

3. $\mathrm{F}_\gamma \boxtimes (P_i \otimes \mathbb{C}^H_{\ell r})$ induces to $\mathrm{Rep}^0(\mathcal{A}_r)$ if and only if $i + \ell r + 2\lambda_r\gamma \in 2\mathbb{Z}$.

Proof. Recall from subsection 2.1.1 that given any $\mathrm{F}_\gamma \boxtimes X \in \mathcal{C}^\oplus$, $\mathscr{F}(\mathrm{F}_\gamma \boxtimes X) \in \mathrm{Rep}^0\,\mathcal{A}_r$ iff

$$\mu_{\mathscr{F}(\mathrm{F}_\gamma\boxtimes X)} \circ M_{\mathcal{A}_r,\mathrm{F}_\gamma\boxtimes X} = \mu_{\mathscr{F}(\mathrm{F}_\gamma\boxtimes X)}$$

where $\mu_{\mathscr{F}(\mathrm{F}_\gamma\boxtimes X)} = (\mu \otimes \mathrm{Id}_{\mathrm{F}_\gamma\boxtimes X}) \circ a^{-1}_{\mathcal{A}_r,\mathcal{A}_r,\mathrm{F}_\gamma\boxtimes X}$ and $M_{A,B} = c_{B,A} \circ c_{A,B}$ is the monodromy. By Proposition 2.2.2 we can, without loss of generality, assume that $\mu(1_u \otimes 1_v) = 1_{u+v}$ in which case the above equation holds iff $M_{\mathcal{A}_r,\mathrm{F}_\gamma\boxtimes X} = \mathrm{Id}$, but by Theorem 2.11 in [CKL] it is enough to check that $M_{\mathrm{F}_{\lambda_r}\boxtimes\mathbb{C}^H_r,\mathrm{F}_\gamma\boxtimes X} = \mathrm{Id}$. Note that we have

$$M_{\mathrm{F}_{\lambda_r}\boxtimes\mathbb{C}^H_r,\mathrm{F}_\gamma\boxtimes X} = M_{\mathrm{F}_{\lambda_r},\mathrm{F}_\gamma} \boxtimes M_{\mathbb{C}^H_r,X} \tag{2.2.5}$$

where $M_{\mathrm{F}_{\lambda_r},\mathrm{F}_\gamma} = q^{2r\lambda_r\gamma}\mathrm{Id}$ by (2.1.5). Recall that the braiding on $\overline{U}^H_q(\mathfrak{sl}_2)$-Mod is given by $\tau \circ R$ where τ is the usual flip map and

$$R = q^{H\otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n. \tag{2.2.6}$$

Since the generating vector $v_r \in \mathbb{C}^H_r$ satisfies $Ev_r = Fv_r = 0$, the braiding acts as $\tau \circ q^{H\otimes H/2}$ and hence the monodromy $M_{\mathbb{C}^H_r,X}$ acts as $q^{H\otimes H}$ on $\mathbb{C}^H_r \otimes X$. The endomorphism rings of V_α and $S_i \otimes \mathbb{C}^H_{\ell r}$ are one dimensional, so the monodromies $M_{\mathbb{C}^H_r,V_\alpha}$ and $M_{\mathbb{C}^H_r,S_i\otimes\mathbb{C}^H_{\ell r}}$ must act as scalars on $\mathbb{C}^H_r \otimes V_\alpha$ and $\mathbb{C}^H_r \otimes (S_i \otimes \mathbb{C}^H_{\ell r})$, respectively. It follows by direct computation on the usual generators of these modules that

$$M_{\mathbb{C}_r^H, V_\alpha} = q^{r(\alpha+r-1)}\mathrm{Id}, \tag{2.2.7}$$

$$M_{\mathbb{C}_r^H, (S_i \otimes \mathbb{C}_{\ell r}^H)} = q^{r(i+\ell r)}\mathrm{Id}. \tag{2.2.8}$$

It follows that

$$M_{\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H, \mathrm{F}_\gamma \boxtimes V_\alpha} = (-1)^{\alpha+r-1+2\lambda_r\gamma}\mathrm{Id}, \tag{2.2.9}$$

$$M_{\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H, \mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}_{\ell r}^H)} = (-1)^{i+\ell r+2\lambda_r\gamma}\mathrm{Id}, \tag{2.2.10}$$

and so $\mathrm{F}_\gamma \boxtimes V_\alpha$ and $\mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}_{\ell r}^H)$ lift to $\mathrm{Rep}^0 \mathcal{A}_r$ iff $\alpha + r - 1 + 2\lambda_r\gamma, i + \ell r + 2\lambda_r\gamma \in 2\mathbb{Z}$, respectively. The endomorphism ring of P_i (and hence the endomorphism ring of $\mathbb{C}_r^H \otimes (P_i \otimes \mathbb{C}_{\ell r}^H)$ is two-dimensional spanned by the identity and a nilpotent operator (see Theorem 6.2 in [CGP1]), but $M_{\mathbb{C}_r^H, P_i \otimes \mathbb{C}_{\ell r}^H}$ has no nilpotent part since it acts by $q^{H \otimes H}$, so it must act by a scalar multiple of the identity. Acting on the vector $v_r \otimes (\mathsf{w}_i \otimes v_{\ell r})$, it is easily seen that

$$M_{\mathbb{C}_r^H, P_i \otimes \mathbb{C}_{\ell r}^H} = q^{r(i+\ell r)}\mathrm{Id} \tag{2.2.11}$$

and so

$$M_{\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H, \mathrm{F}_\gamma \boxtimes (P_i \otimes \mathbb{C}_{\ell r}^H)} = (-1)^{i+\ell r+2\lambda_r\gamma}\mathrm{Id}.$$

Therefore, $\mathrm{F}_\gamma \boxtimes (P_i \otimes \mathbb{C}_{\ell r}^H)$ lifts iff $i + \ell r + 2\lambda_r\gamma \in 2\mathbb{Z}$. $\square$

The induction functor $\mathscr{F} : \mathcal{C}^\oplus \to \mathrm{Rep}(\mathcal{A}_r)$ is a tensor functor by Theorem 2.59 in [CKM] and $\mathrm{Rep}^0 \mathcal{A}_r$ is a tensor subcategory of $\mathrm{Rep}\, \mathcal{A}_r$, so for any objects $\mathscr{F}(U), \mathscr{F}(V) \in \mathrm{Rep}^0(\mathcal{A}_r)$,

$$\mathscr{F}(U) \otimes \mathscr{F}(V) \cong \mathscr{F}(U \otimes V). \tag{2.2.12}$$

$\mathrm{Rep}^0(\mathcal{A}_r)$ is rigid by Proposition 2.77 and Lemma 2.78 of [CKM], and by Proposition 2.67 of [CKM], the braiding $c^{\mathcal{A}_r}_{-,-}$ on $\mathrm{Rep}^0(\mathcal{A}_r)$ satisfies the relation

$$\mathrm{Id}_{\mathcal{A}_r} \otimes c_{U,V} = \mathscr{F}(c_{U,V}) = g_{V,U} \circ c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)} \circ f_{U,V} \tag{2.2.13}$$

where $f_{U,V} : \mathscr{F}(U \otimes V) \xrightarrow{\cong} \mathscr{F}(U) \otimes \mathscr{F}(V)$ and $g_{V,U} : \mathscr{F}(V) \otimes \mathscr{F}(U) \xrightarrow{\cong} \mathscr{F}(V \otimes U)$ are isomorphisms defined in Theorem 2.59 of [CKM]. Ultimately, we are interested in the scalars associated to the monodromy isomorphisms $c^{\mathcal{A}_r}_{\mathscr{F}(V),\mathscr{F}(U)} \circ c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)}$, therefore for calculation purposes, we ignore the f and g isomorphisms and simply take

$$c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)} = \mathrm{Id}_{\mathcal{A}_r} \otimes c_{U,V}. \tag{2.2.14}$$

We compute that $\theta_{(\mathrm{F}_\lambda \boxtimes \mathbb{C}^H_r)^{\otimes k}} = \mathrm{Id}_{(\mathrm{F}_\lambda \boxtimes \mathbb{C}^H_r)^{\otimes k}}$ when r is odd, so $\theta_{\mathcal{A}_r} = \mathrm{Id}_{\mathcal{A}_r}$ for odd r, and hence by Corollary 2.82 and Theorem 2.89 in [CKM], we have

$$\begin{aligned}\theta_{\mathscr{F}(V)} &= \mathscr{F}(\theta_V) = \mathrm{Id}_{\mathcal{A}_r} \otimes \theta_V \\ S^{\infty}_{\mathscr{F}(U),\mathscr{F}(V)} &= \varphi \circ (\mathrm{Id}_{\mathcal{A}_r} \otimes S^{\infty}_{U,V}) \circ \varphi^{-1}.\end{aligned}$$

Using Lemmas 2.2.3 and 2.2.4, and exactness of the Deligne product and induction functor together with Lemma 2.1.3, we easily obtain the following Proposition:

Proposition 2.2.6. The simple modules in $\mathrm{Rep}^0(\mathcal{A}_r)$ are

$$E^V_{\gamma,\alpha} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) \qquad \text{with } \alpha \in \ddot{\mathbb{C}} \text{ and } \gamma\lambda_r + \frac{\alpha + r - 1}{2} \in \mathbb{Z}, \tag{2.2.15}$$

$$E^S_{\gamma,i,\ell} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}^H_{\ell r})\right) \qquad \text{with } i \in \{0,\dots,r-2\} \text{ and } \gamma\lambda_r + \frac{i + r\ell}{2} \in \mathbb{Z}. \tag{2.2.16}$$

We have families of indecomposable modules:

$$Q^V_{\gamma,\alpha} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes V_\alpha\right) \qquad \text{with } \alpha \notin \ddot{\mathbb{C}} \text{ and } \gamma\lambda_r + \frac{\alpha + r - 1}{2} \in \mathbb{Z}\,, \tag{2.2.17}$$

$$Q^P_{\gamma,i,\ell} = \mathscr{F}\left(\mathrm{F}_\gamma \boxtimes (P_i \otimes \mathbb{C}^H_{\ell r})\right) \qquad \text{with } i \in \{0,\dots,r-2\} \text{ and } \gamma\lambda_r + \frac{i + \ell r}{2} \in \mathbb{Z}, \tag{2.2.18}$$

with $Q^P_{\gamma,i,\ell}$ being projective, and the above modules satisfy $E^V_{\gamma,\alpha} \cong E^V_{\gamma+k\lambda_r,\alpha+rk}$, $E^S_{\gamma,i,\ell} \cong E^S_{\gamma+k\lambda_r,i,\ell+k}$, $Q^V_{\gamma,\alpha} \cong Q^V_{\gamma+k\lambda_r,\alpha+rk}$, and $Q^P_{\gamma,i,\ell} \cong Q^P_{\gamma+k\lambda_r,i,\ell+k}$ for all $k \in \mathbb{Z}$. The indecomposable modules admit the following Loewy diagrams:

$$Q^V_{\gamma,i+\ell r} \quad \begin{tikzcd} E^S_{\gamma,(r-1)-i,\ell} \arrow[d] \\ E^S_{\gamma,i-1,\ell+1} \end{tikzcd} \qquad\qquad Q^P_{\gamma,r-1,\ell} \quad \begin{tikzcd} E^V_{\gamma,i+1-r+\ell r} \arrow[d] \\ E^V_{\gamma,r-1-i+\ell r} \end{tikzcd}$$

$$
\textit{For } i = 1, \dots, r-2: \quad
\begin{tikzcd}
 & E^S_{\gamma,\, i,\, \ell} \arrow[dl] \arrow[dr] & \\
E^S_{\gamma,\, (r-2)-i,\, \ell-1} \arrow[dr] & Q^P_{\gamma,\, i,\, \ell} & E^S_{\gamma,\, (r-2)-i,\, \ell+1} \arrow[dl] \\
 & E^S_{\gamma,\, i,\, \ell} &
\end{tikzcd} .
$$

The category $\mathrm{Rep}^0(\mathcal{A}_r)$ is a rigid monoidal category with tensor product $\mathscr{F}(U) \otimes \mathscr{F}(V) \cong \mathscr{F}(U \otimes V)$. $\mathrm{Rep}^0(\mathcal{A}_r)$ is also braided with braiding $c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)}$ defined by

$$
c^{\mathcal{A}_r}_{\mathscr{F}(U),\mathscr{F}(V)} = \mathrm{Id}_{\mathcal{A}_r} \otimes c_{U,V},
$$

where $c_{U,V}$ is the braiding on $\mathcal{C}_\oplus$ given by the product of the braidings on $\mathcal{H}$-Mod and $\overline{U}^H_q(\mathfrak{sl}_2)$-Mod. If r is odd, then $\mathrm{Rep}^0(\mathcal{A}_r)$ has twist $\theta_{\mathcal{A}_r}$ and Hopf links $S^{\varpi}_{\mathscr{F}(U),\mathscr{F}(V)}$ given by

$$
\theta_{\mathscr{F}(V)} = \mathrm{Id}_{\mathcal{A}_r} \otimes \theta_V, \tag{2.2.19}
$$

$$
\mathsf{S}^{\varpi}_{\mathscr{F}(U),\mathscr{F}(V)} = \mathsf{S}^{\varpi}_{U,V} \tag{2.2.20}
$$

where θ_V and $\mathsf{S}^{\varpi}_{U,V}$ are the twist and Hopf links respectively on $\mathcal{C}^\oplus$, and we are viewing the Hopf links as scalars.

2.3 Modular data for typical modules

In this section we compute and compare the modular data for typical and atypical modules in B_r and $\mathcal{C}$ through the correspondence $\varphi : \mathrm{Rep}_{wt}\overline{U}^H_q(\mathfrak{sl}_2) \to \mathrm{Rep}_{\langle s\rangle}\mathcal{M}(r), V_\alpha \mapsto F_{\frac{\alpha+r-1}{\sqrt{2r}}}, S_i \otimes \mathbb{C}^H_{\ell r} \mapsto M_{1-\ell,i+1}$ of Proposition 2.1.8 found in [CMR].

2.3.1 Typical modules

The typical modules in $\mathcal{C}$ take the form $\mathscr{F}(\mathrm{F}_\gamma \boxtimes V_\alpha)$ for $\alpha \in \ddot{\mathbb{C}} := (\mathbb{C} \setminus \mathbb{Z}) \cup r\mathbb{Z}$, which is associated to the B_r-Module

$$
E_{\gamma,\alpha} := \mathscr{F}\left(\mathsf{F}_\gamma \boxtimes F_{\frac{\alpha+r-1}{\sqrt{2r}}}\right) \tag{2.3.1}
$$

through the above correspondence. Recall that F_γ denotes the usual Fock space as a module of the Heisenberg VOA H, and $F_{\frac{\alpha+r-1}{\sqrt{2r}}}$ as a module of the singlet VOA $\mathcal{M}(r)$. Note that by Theorem 2.2.5, $E_{\gamma,\alpha} \in \operatorname{Rep}^0 B_r$ iff $\gamma\lambda_r + \frac{\alpha}{2} + \frac{r-1}{2} \in \mathbb{Z}$ and that since γ takes purely imaginary values, we are therefore forced to take $\alpha \in (\mathbb{R} \setminus \mathbb{Z}) \cup r\mathbb{Z}$. On the quantum group side, $\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H$ induces to the algebra object $\mathcal{A}_r$ under $\mathscr{F} : \mathcal{C} \to \mathcal{A}_r$-Mod, and on the VOA side, $\mathsf{F}_{\lambda_r} \otimes M_{0,1}$ induces to B_r. Therefore, we have

$$E_{\gamma,\alpha} \cong E_{\gamma+\lambda_r,\alpha+r}. \tag{2.3.2}$$

We re-parameterize the space of simple typicals with the following substitution:

$$\nu = \frac{2\alpha}{r}, \quad \ell = \gamma\lambda_r + \frac{\alpha}{2}. \tag{2.3.3}$$

Considering isomorphisms, our parameter space reduces to $\ell + \frac{r-1}{2} \in \mathbb{Z}, \nu \in (-1,1] \setminus \frac{2}{r}\mathbb{Z}$. This re-parametrization is done to facilitate a comparison with [CR2] and [CR3] which correspond to $r = 2$ and $r = 3$ cases. The following S^χ matrix calculations will also work with $\nu \in (-1,1]$, and therefore, for what follows we work with this slightly larger parameter space. By abuse of notation, we still let $E_{\nu,\ell} := E_{\gamma,\alpha}$. For characters, we use the following convention:

$$\operatorname{ch}[\mathsf{F}_\gamma] = \frac{z^{-4\lambda_r\gamma/r} q^{\gamma^2/2}}{\eta(q)} \tag{2.3.4}$$

where $z = e^{2\pi i\zeta}$, $q = e^{2\pi i\tau}$. We introduce another variable $y = e^{2\pi i\kappa}$ in the characters to have the S matrix come out nicely[1]. We multiply the characters by $y^{-4/r} = e^{-8\pi i\kappa/r}$. Under the S-transformations, we have:

$$\mathsf{S} : (\zeta, \tau, \kappa) \mapsto \left(\frac{\zeta}{\tau}, -\frac{1}{\tau}, \kappa - \frac{\zeta^2}{\tau}\right). \tag{2.3.5}$$

Lemma 2.3.1. Using the parametrisation (2.3.3) for the typical modules, their (super)characters

[1]Without this additional variable modular S-transformation would give a ζ and τ dependent prefactor, often called automorphy factor. The only purpose of y is to get rid of this factor.

are given by

$$\mathrm{ch}[E_{\nu,\ell}](z;q) = \frac{e^{-8\pi i\kappa/r}}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}} e^{2\pi i\tau\ell^2/r}e^{\pi i(\nu-4\ell/r)m}\delta(2\zeta+\ell\tau-m) \qquad r \text{ odd} \tag{2.3.6}$$

$$\mathrm{sch}[E_{\nu,\ell}](z;q) = \frac{e^{-8\pi i\kappa/r}}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}+(1/2)} e^{2\pi i\tau\ell^2/r}e^{\pi i(\nu-4\ell/r)m}\delta(2\zeta+\ell\tau-m) \qquad r \text{ even.} \tag{2.3.7}$$

Proof. Let r be odd. By construction, we have:

$$\begin{aligned}
\mathrm{ch}[E_{\gamma,\alpha}](z;q) &= \sum_{k\in\mathbb{Z}} e^{-8\pi i\kappa/r}\mathrm{ch}\left[\mathrm{F}_{\gamma+k\lambda_r}\boxtimes F_{\frac{\alpha+(k+1)r-1}{\sqrt{2r}}}\right] \\
&= \sum_{k\in\mathbb{Z}} \frac{e^{-8\pi i\kappa/r}z^{-4\gamma\lambda_r/r-4k\lambda_r^2/r}q^{(\gamma+k\lambda_r)^2/2}}{\eta(\tau)}\frac{q^{\frac{1}{2}\left(\frac{\alpha+(k+1)r-1}{\sqrt{2r}}-\frac{\alpha_0}{2}\right)^2}}{\eta(\tau)} \\
&= \frac{e^{-8\pi i\kappa/r}z^{-4\gamma\lambda_r/r}q^{\frac{\gamma^2}{2}+\frac{\alpha^2}{4r}}}{\eta(\tau)^2}\sum_{k\in\mathbb{Z}} z^{2k}q^{k(\gamma\lambda_r+\frac{\alpha}{2})} \\
&= \frac{e^{-8\pi i\kappa/r}e^{2\pi i\zeta(-4\gamma\lambda_r/r)}e^{2\pi i\tau\left(\frac{\gamma^2}{2}+\frac{\alpha^2}{4r}\right)}}{\eta(\tau)^2}\sum_{m\in\mathbb{Z}}\delta(2\zeta+(\gamma\lambda_r+\alpha/2)\tau-m),
\end{aligned} \tag{2.3.8}$$

where our δ-function is supported at 0. Passing to the ν,ℓ notation (2.3.3), we observe the following relations:

$$-4\lambda_r\gamma\zeta/r = \zeta\nu - 4\zeta\ell/r, \tag{2.3.9}$$

$$\gamma^2/2+\alpha^2/4r = -\ell^2/r+\nu\ell/2. \tag{2.3.10}$$

The expression for $\mathrm{ch}[E_{\nu,\ell}](z;q)$ follows straightforwardly. For r even, we introduce a factor of $e^{\pi ik}$ in the very first summation. The rest now follows similarly. □

If r is odd, we set

$$
\begin{aligned}
\mathsf{S}\left\{\mathrm{ch}[E_{\nu,\ell}]\right\} &= \frac{e^{-8\pi i(\kappa-\zeta^2/\tau)/r}}{\eta(-1/\tau)^2}\sum_{m\in\mathbb{Z}} e^{-2\pi i\ell^2/r\tau}e^{\pi i(\nu-4\ell/r)m}\delta\left(\frac{2\zeta-\ell-m\tau}{\tau}\right) \\
&= \frac{|\tau|}{-i\tau\eta(\tau)^2}e^{-8\pi i(\kappa-\zeta^2/\tau)/r}\sum_{m\in\mathbb{Z}} e^{-8\pi i\zeta^2/r\tau}e^{2\pi i m^2\tau/r}e^{\pi i\nu m}\delta\left(2\zeta-\ell-m\tau\right) \\
&= \frac{|\tau|}{-i\tau\eta(\tau)^2}e^{-8\pi i\kappa/r}\sum_{m\in\mathbb{Z}} e^{2\pi i m^2\tau/r}e^{\pi i\nu m}\delta\left(2\zeta-\ell-m\tau\right) \qquad (2.3.11)
\end{aligned}
$$

and similarly if r is even,

$$
\mathsf{S}\left\{\mathrm{sch}[E_{\nu,\ell}]\right\} = \frac{|\tau|}{-i\tau\eta(\tau)^2}e^{-8\pi i\kappa/r}\sum_{m\in\mathbb{Z}+(1/2)} e^{2\pi i m^2\tau/r}e^{\pi i\nu m}\delta\left(2\zeta-\ell-m\tau\right). \qquad (2.3.12)
$$

Regardless of the parity of r, let:

$$
\mathsf{S}^{\chi}_{(\nu,\ell),(\nu',\ell')} = \frac{|\tau|}{-i\tau}\frac{1}{2}e^{\pi i(4\ell\ell'/r-\ell\nu'-\ell'\nu)}. \qquad (2.3.13)
$$

Now we show that this S-matrix correctly gets us the S-transformations of the characters in the r odd case. In the r even case, the calculation is similar, except with characters replaced with supercharacters and summations over $\mathbb{Z}$ now changed to summations over $\mathbb{Z}+(1/2)$.

$$
\begin{aligned}
&\sum_{\ell'\in\mathbb{Z}}\int_{-1}^{1}\mathsf{S}^{\chi}_{(\nu,\ell),(\nu',\ell')}\mathrm{ch}[E_{\nu',\ell'}]d\nu' \\
&= \sum_{\ell'\in\mathbb{Z}}\int_{-1}^{1}\frac{|\tau|}{-i\tau}\frac{1}{2}e^{\pi i(4\ell\ell'/r-\ell\nu'-\ell'\nu)}\frac{e^{-8\pi i\kappa/r}}{\eta(q)^2}\sum_{m\in\mathbb{Z}} e^{2\pi i\tau\ell'^2/r}e^{\pi i(\nu'-4\ell'/r)m}\delta(2\zeta+\ell'\tau-m)d\nu' \\
&= \sum_{\ell'\in\mathbb{Z}}\frac{|\tau|}{-i\tau}e^{\pi i(4\ell\ell'/r-\ell'\nu)}\frac{e^{-8\pi i\kappa/r}}{\eta(q)^2}\sum_{m\in\mathbb{Z}} e^{2\pi i\tau\ell'^2/r}e^{\pi i(-4\ell'/r)m}\delta(2\zeta+\ell'\tau-m)\int_{-1}^{1}\frac{1}{2}e^{\pi i(\nu'm-\nu'\ell)}d\nu' \\
&= \sum_{\ell'\in\mathbb{Z}}\frac{|\tau|}{-i\tau}e^{\pi i(4\ell\ell'/r-\ell'\nu)}\frac{e^{-8\pi i\kappa/r}}{\eta(q)^2}\sum_{m\in\mathbb{Z}} e^{2\pi i\tau\ell'^2/r}e^{\pi i(-4\ell'/r)m}\delta(2\zeta+\ell'\tau-m)\delta(\ell-m) \\
&= \sum_{\ell'\in\mathbb{Z}}\frac{|\tau|}{-i\tau}e^{\pi i(-\ell'\nu)}\frac{e^{-8\pi i\kappa/r}}{\eta(q)^2}e^{2\pi i\tau\ell'^2/r}\delta(2\zeta+\ell'\tau-\ell) \\
&= \sum_{m\in\mathbb{Z}}\frac{|\tau|}{-i\tau\eta(q)^2}e^{-8\pi i\kappa/r}e^{\pi i m\nu}e^{2\pi i\tau m^2/r}\delta(2\zeta-m\tau-\ell) \\
&= \mathsf{S}^{\chi}\left\{\mathrm{ch}[E_{\nu,\ell}]\right\}. \qquad (2.3.14)
\end{aligned}
$$

All that remains now is to make an appropriate choice of normalisation. For this purpose, we obtain a resolution of the tensor identity $S_0 \otimes \mathbb{C}_0^H$, and by induction, a resolution of $\mathcal{A}_r$ in terms of other typical modules through the short exact sequences from Lemma 2.1.3. We transfer this resolution to the vertex operator algebra side and obtain a character relation for B_r from Proposition 2.1.8.

Shifting the short exact sequence in Lemma 2.1.3 by $(j, \ell) \to (j+1, \ell-1)$, we can obtain a family of linked short exact sequences

$$0 \to S_{(r-2)-j_k} \otimes \mathbb{C}^H_{(\ell_k-1)r} \to V_{(j_k+1)+(\ell_k-1)r} \xrightarrow{f_k} S_{j_k} \otimes \mathbb{C}^H_{\ell_k r} \to 0, \tag{2.3.15}$$

where $(j_{k+1}, \ell_{k+1}) = (r-2-j_k, \ell_k - 1)$. We therefore obtain a long exact sequence

$$\cdots \to V_{n_4} \xrightarrow{f_4} V_{n_3} \xrightarrow{f_3} V_{n_2} \xrightarrow{f_2} V_{n_1} \xrightarrow{f_1} V_{n_0} \xrightarrow{f_0} S_{j_0} \otimes \mathbb{C}^H_{\ell_0 r} \to 0 \tag{2.3.16}$$

where $n_k = (j_k+1) + (\ell_k - 1)r$. We can take $(j_0, \ell_0) = (0, 0)$, tensor this sequence with F_0, and apply the induction functor (see Definition 2.1.14) and lastly, the correspondence in Proposition 2.1.8. This sends $S_0 \otimes \mathbb{C}_0^H$ to B_r, so we obtain the following relation for the character of B_r:

$$\mathrm{ch}[B_r] = \sum_{m=0}^{\infty} (-1)^m \mathrm{ch}[Y_{n_m}] = \sum_{m=0}^{\infty} \big(\mathrm{ch}[Y_{n_{2m}}] - \mathrm{ch}[Y_{n_{2m+1}}]\big), \tag{2.3.17}$$

where $Y_{n_m} = \mathscr{F}(\mathsf{F}_0 \boxtimes F_{\frac{n_m+r-1}{\sqrt{2r}}})$ (recall that $F_{\frac{n_m+r-1}{\sqrt{2r}}}$ corresponds to V_{n_m} through proposition 2.1.8). Note that by Theorem 2.2.5, the module $\mathsf{F}_0 \boxtimes V_{n_m}$ lifts to $\mathrm{Rep}^0(\mathcal{A}_r)$ through the induction functor iff $n_m + r - 1 \in 2\mathbb{Z}$. Observe by a simple induction that with $(j_0, \ell_0) = (0, 0)$, the indices n_m satisfy

$$n_m = \begin{cases} 1-(m+1)r & \text{for } m \text{ even} \\ -1-mr & \text{for } m \text{ odd,} \end{cases} \tag{2.3.18}$$

so $\mathrm{F}_0 \boxtimes V_{n_m}$ lifts to $\mathrm{Rep}^0\, \mathcal{A}_r$ and $\mathsf{F}_0 \boxtimes F_{\frac{n_m+r-1}{\sqrt{2r}}}$ lifts to $\mathrm{Rep}^0\, B_r$ for all m and r. Adopting the parametrisation (2.3.3) for the modules Y_{n_m} gives the following:

$$Y_{n_m} = \begin{cases} E_{\frac{2(1-(m+1)r)}{r}, \frac{1-(m+1)r}{2}} & \text{for } m \text{ even} \\ E_{\frac{2(-1-mr)}{r}, \frac{-1-mr}{2}} & \text{for } m \text{ odd.} \end{cases} \tag{2.3.19}$$

We write $\mathbb{1}$ for the B_r-algebra and obtain from (2.3.17) the relation

$$\mathsf{S}^{\chi}_{\mathbb{1},(\nu',\ell')} = \sum_{m=0}^{\infty} \left(\mathsf{S}^{\chi}_{\left(\frac{2(1-(2m+1)r)}{r}, \frac{1-(2m+1)r}{2}\right),(\nu',\ell')} - \mathsf{S}^{\chi}_{\left(\frac{2(-1-(2m+1)r)}{r}, \frac{-1-(2m+1)r}{2}\right),(\nu',\ell')} \right). \tag{2.3.20}$$

By substituting (2.3.13) into (2.3.20) and simplifying, we deduce that

$$\begin{aligned} \mathsf{S}^{\chi}_{\mathbb{1},(\nu',\ell')} &= \frac{|\tau|}{-i\tau} \frac{1}{2} \sum_{m=0}^{\infty} \left(e^{-\pi i \frac{1-(2m+1)r}{2} \nu'} - e^{-\pi i \frac{-1-(2m+1)r}{2} \nu'} \right) \\ &= \frac{|\tau|}{-i\tau} \frac{1}{2} \sum_{m=0}^{\infty} \left(e^{\pi i (r-1) \frac{\nu'}{2}} e^{\pi i \frac{2mr}{2} \nu'} - e^{\pi i (r+1) \frac{\nu'}{2}} e^{\pi i \frac{2mr}{2} \nu'} \right). \end{aligned}$$

Setting $x = e^{\pi i \frac{\nu'}{2}}$, we see that

$$\begin{aligned} \mathsf{S}^{\chi}_{\mathbb{1},(\nu',\ell')} &= \frac{|\tau|}{-i\tau} \frac{1}{2} \left(x^{r-1} \sum_{m=0}^{\infty} x^{2mr} - x^{r+1} \sum_{m=0}^{\infty} x^{2mr} \right). \\ &= \frac{|\tau|}{-i\tau} \frac{x^r}{2} (x^{-1} - x) \sum_{m=0}^{\infty} x^{2mr} \end{aligned} \tag{2.3.21}$$

Since x lies on the unit circle this sum is not convergent. If we infinitesimally deform x to lie within the unit circle, this sum converges to

$$\frac{|\tau|}{-i\tau} \frac{1}{2} \cdot \frac{x - x^{-1}}{x^r - x^{-r}}. \tag{2.3.22}$$

This is the motivation for our choice of normalisation factor $\mathsf{S}^{\chi}_{\mathbb{1},(\nu',\ell')}$, i.e. we define it to be

$$\mathsf{S}^{\chi}_{\mathbb{1},(\nu',\ell')} := \frac{|\tau|}{-i\tau} \frac{1}{2} \cdot \frac{x - x^{-1}}{x^r - x^{-r}}, \tag{2.3.23}$$

where $x = e^{\pi i \frac{\nu'}{2}}$. As a further motivation we note that this type of regularization worked very well in the Verlinde formula story of [CR2, CR3].

We will now compare the modular S^χ matrix coming from modular transformations of characters for B_r with the S^{ϖ} matrix coming from Hopf links in $\mathcal{C}$ for typical modules calculated on the quantum group side. These quantities agree up to normalised conjugation.

Proposition 2.3.2. The normalized modular S-matrix S^χ and Hopf links S^{ϖ} agree. That is,

$$\frac{\mathsf{S}^\chi_{(\nu,\ell),(\nu',\ell')}}{\mathsf{S}^\chi_{\mathbb{1},(\nu',\ell')}} = \frac{\mathsf{S}^{\varpi}_{(\nu,\ell),(\nu',\ell')}}{\mathsf{S}^{\varpi}_{\mathbb{1},(\nu',\ell')}}, \tag{2.3.24}$$

Proof. It follows from equation 2.3.13 and Definition 2.3.23 that the normalisation of S^χ is given by

$$\frac{\mathsf{S}^\chi_{(\nu,\ell),(\nu',\ell')}}{\mathsf{S}^\chi_{\mathbb{1},(\nu',\ell')}} = \frac{x^r - x^{-r}}{x - x^{-1}} \cdot e^{\pi i(4\ell\ell'/r - \ell\nu' - \ell'\nu)} \tag{2.3.25}$$

where $x = e^{\pi i \nu'/2}$. Since the S^{ϖ} matrix is preserved under induction, we just calculate it in the category $\mathcal{C}$. The S^{ϖ}-matrix for typical modules satisfies the relation

$$\mathsf{S}^{\varpi}_{\mathrm{F}_{\gamma_1}\boxtimes V_{\alpha_1},\mathrm{F}_{\gamma_2}\boxtimes V_{\alpha_2}} = \mathsf{S}^{\varpi}_{\mathrm{F}_{\gamma_1},\mathrm{F}_{\gamma_2}} \mathsf{S}^{\varpi}_{V_{\alpha_1},V_{\alpha_2}} \tag{2.3.26}$$

where $\mathsf{S}^{\varpi}_{\mathrm{F}_{\gamma_1},\mathrm{F}_{\gamma_2}} = e^{2\pi i \gamma_1\gamma_2}$. By Lemma 6.6 in [CGP1], we have

$$\mathsf{S}^{\varpi}_{V_{\alpha_1},V_{\alpha_2}} = (-1)^{r-1} r q^{\alpha_1\alpha_2}, \qquad \mathsf{S}^{\varpi}_{S_0,V_{\alpha_2}} = (-1)^{r-1}\frac{r\{\alpha_2\}}{\{r\alpha_2\}}. \tag{2.3.27}$$

Hence,

$$\frac{\mathsf{S}^{\varpi}_{\mathrm{F}_{\gamma_1}\boxtimes V_{\alpha_1},\mathrm{F}_{\gamma_2}\boxtimes V_{\alpha_2}}}{\mathsf{S}^{\varpi}_{\mathrm{F}_0\boxtimes S_0,\mathrm{F}_{\gamma_2}\boxtimes V_{\alpha_2}}} = e^{2\pi i\gamma_1\gamma_2} q^{\alpha_1\alpha_2}\frac{\{r\alpha_2\}}{\{\alpha_2\}}. \tag{2.3.28}$$

Setting $(\nu_i,\ell) = (\frac{2\alpha_1}{r}, \lambda\gamma_1 + \frac{\alpha_1}{2})$, $(\nu',\ell') = (\frac{2\alpha_2}{r}, \lambda\gamma_2 + \frac{\alpha_2}{2})$, $x = e^{\pi i\nu'/2}$, adopting the notation $\mathsf{S}^{\varpi}_{(\nu,\ell),(\nu',\ell')} = \mathsf{S}^{\varpi}_{\mathrm{F}_{\gamma_1}\boxtimes V_{\alpha_1},\mathrm{F}_{\gamma_2}\boxtimes V_{\alpha_2}}$, $\mathbb{1} = \mathrm{F}_0 \boxtimes S_0$, and recalling that $q = e^{-\pi i/r}$, we easily see that

$$\frac{\mathsf{S}^{\varpi}_{(\nu,\ell),(\nu'\ell')}}{\mathsf{S}^{\varpi}_{\mathbb{1},(\nu',\ell')}} = \frac{x^r - x^{-r}}{x - x^{-1}} \cdot e^{\pi i(4\ell\ell'/r - \ell\nu' - \ell'\nu)}.$$

$\square$

2.4 Modular data for atypical modules

Recall that by $\mathcal{G}^{\mathrm{ss}}$ we mean the quotient of the Grothedieck ring by the ideal of the Grothendieck ring formed by the negligible objects.

In this section we derive $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ for the category of local modules $\mathcal{C}^0$ in $\mathcal{C}$ (those which are induced to $\mathrm{Rep}^0\,\mathcal{A}_r$ by the induction functor). We also compute the corresponding S^{∞}-matrix and make a comparison with the matrix S^{χ} coming from modular transformations of characters appearing in the Verlinde algebra of characters $\mathcal{V}^{\mathrm{ss}}(B_r)$ of the semisimplification of B_r-Mod. The Verlinde formula then follows from the standard categorical argument.

2.4.1 Structure of Grothendieck rings

We start by determining structure of the underlying Grothendieck rings.

Proposition 2.4.1. $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ has a $\mathbb{Z}_+$-basis with elements $\mathsf{M}_{\gamma,i,r\ell}$ of the form

$$\mathsf{M}_{\frac{2n}{2\lambda_r},2m,0}, \qquad n \in \{0,1,\ldots,\frac{r-1}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}, \tag{2.4.1}$$

$$\mathsf{M}_{\frac{2n+1}{2\lambda_r},2m,r}, \qquad n \in \{0,1,\ldots,\frac{r-3}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}, \tag{2.4.2}$$

when r is odd, and

$$\mathsf{M}_{\frac{2n}{2\lambda_r},2m,0}, \qquad n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-2}{2}\}, \tag{2.4.3}$$

$$\mathsf{M}_{\frac{2n+1}{2\lambda_r},2m+1,0}, \qquad n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-4}{2}\}, \tag{2.4.4}$$

when r is even. The product is given by

$$\mathsf{M}_{\frac{x}{2\lambda_r},i,r\ell}\cdot\mathsf{M}_{\frac{y}{2\lambda_r},j,r\ell'}\begin{cases}=\displaystyle\sum_{\substack{l=|i-j|\\ \text{by }2}}^{i+j}\mathsf{M}_{\frac{x+y \bmod \mathrm{r}}{2\lambda_r},l,r(\ell+\ell') \bmod 2\mathrm{r}} & \text{if } i+j<r,\\ \displaystyle\sum_{\substack{l=|i-j|\\ \text{by }2}}^{2r-4-i-j}\mathsf{M}_{\frac{x+y \bmod \mathrm{r}}{2\lambda_r},l,r(\ell+\ell') \bmod 2\mathrm{r}} & \text{if } i+j\geq r.\end{cases} \tag{2.4.5}$$

Proof. We first note that the quantum dimension of an object $V\in\overline{U}_q^H(\mathfrak{sl}_2)$-Mod is given by

$$\sum_{i=0}^{k}v_i^*(K^{1-r}v_i),$$

where $\{v_0,...,v_k\}$ is a basis for V. Hence, if $V=V_\alpha$, then

$$\mathrm{qdim}(V_\alpha)=\sum_{i=0}^{r-1}v_i^*(K^{1-r}v_i)=\sum_{i=0}^{r-1}q^{(r-1)(\alpha+r-1-2i)}=q^{(r-1)(\alpha+r-1)}\frac{1-q^{2r}}{1-q^2}=0. \tag{2.4.6}$$

By Lemma 2.1.3 we have the short exact sequence

$$0\to V_{r-1-i+\ell r}\to P_i\otimes\mathbb{C}^H_{\ell r}\to V_{1+i-r+\ell r}\to 0,$$

so we see that the quantum dimension of the $P_i\otimes\mathbb{C}^H_{\ell r}$ is also zero. Hence, $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ is generated by the elements corresponding to the $\mathrm{F}_\gamma\boxtimes(S_i\otimes\mathbb{C}^H_{\ell r})$ which are induced into $\mathrm{Rep}^0(\mathcal{A}_r)$ by the induction functor, which by theorem 2.2.5, are those objects such that $i+r\ell+2\lambda_r\gamma\in 2\mathbb{Z}$. We will denote the object in $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$corresponding to $\mathrm{F}_\gamma\boxtimes(S_i\otimes\mathbb{C}^H_{\ell r})$ by $\mathsf{M}_{\gamma,i,r\ell}$. Recall from Lemma 2.1.3 that for any $i\in\{1,...,r-1\}$ we have the short exact sequence

$$0\to S_{r-1-i}\otimes\mathbb{C}^H_{\ell r}\to V_{i+\ell r}\to S_{i-1}\otimes\mathbb{C}^H_{r(\ell+1)}\to 0,$$

so we have $\mathsf{M}_{\gamma,r-1-i,r\ell}=-\mathsf{M}_{\gamma,i-1,r(\ell+1)}$. So, for any $i\in\{0,...,r-2\}$, we have

$$\mathsf{M}_{\gamma,i,r\ell}=\mathsf{M}_{\gamma,r-1-(r-1-i),r\ell}=-\mathsf{M}_{\gamma,r-1-(i+1),r(\ell+1)}=\mathsf{M}_{\gamma,i,r(\ell+2)}. \tag{2.4.7}$$

Hence, $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ is generated by elements of the form $\mathsf{M}_{\gamma,i,0}$ and $\mathsf{M}_{\gamma,i,r}$ which satisfy

$$i + 2\lambda_r\gamma \in 2\mathbb{Z}, \tag{2.4.8}$$

$$i + r + 2\lambda_r\gamma \in 2\mathbb{Z}. \tag{2.4.9}$$

If i is odd in (2.4.8), then $i + 2\lambda_r\gamma$ being even implies that $\gamma = \frac{2n+1}{2\lambda_r}$ for some $n \in \mathbb{Z}$, and i even in (2.4.8) implies $\gamma = \frac{2n}{2\lambda_r}$ for some $n \in \mathbb{Z}$. If r and i are both odd or even in (2.4.9), then $i + r$ is even, so $\gamma = \frac{2n}{2\lambda_r}$ for some $n \in \mathbb{Z}$. If one of r and i is odd in (2.4.9), and one is even, then $i + r$ is odd, so $\gamma = \frac{2n+1}{2\lambda_r}$ for some $n \in \mathbb{Z}$. Notice also that $\mathrm{F}_{\lambda_r} \boxtimes \mathbb{C}_r^H$ induces to the identity in $\mathrm{Rep}^0 \mathcal{A}_r$ and hence corresponds with the identity in $\mathcal{C}^0$. So, we have the relation

$$\mathsf{M}_{\gamma,i,r\ell} = \mathsf{M}_{\gamma,i,r\ell} \cdot \mathsf{M}_{\lambda_r,0,r} = \mathsf{M}_{\gamma+\lambda_r,i,r(\ell+1)}, \tag{2.4.10}$$

and $\frac{r}{2\lambda_r} + \lambda_r = 0$, so the elements $\mathsf{M}_{\gamma,i,r\ell}$ have $\gamma \in \{0, \frac{1}{2\lambda_r}, ..., \frac{r-1}{2\lambda_r}\}$. Combining the above remarks, we see that when r is odd, $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ is generated by the objects

$$\begin{aligned}
&\mathsf{M}_{\frac{2n}{2\lambda_r},2m,0}, && n \in \{0,1,\ldots,\frac{r-1}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}\\
&\mathsf{M}_{\frac{2n}{2\lambda_r},2m+1,r}, && n \in \{0,1,\ldots,\frac{r-1}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}\\
&\mathsf{M}_{\frac{2n+1}{2\lambda_r},2m+1,0}, && n \in \{0,1,\ldots,\frac{r-3}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}\\
&\mathsf{M}_{\frac{2n+1}{2\lambda_r},2m,r}, && n \in \{0,1,\ldots,\frac{r-3}{2}\}, m \in \{0,1,\ldots,\frac{r-3}{2}\}
\end{aligned}$$

and if r is even, then $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ is generated by the objects

$$\begin{aligned}
&\mathsf{M}_{\frac{2n}{2\lambda},2m,0}, && n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-2}{2}\}\\
&\mathsf{M}_{\frac{2n}{2\lambda},2m,r}, && n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-2}{2}\}\\
&\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}, && n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-4}{2}\}\\
&\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,r}, && n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-4}{2}\}
\end{aligned}$$

The tensor products for the F_γ and S_i modules (see [CGP1, Proposition 8.4]) are given by

$$\mathrm{F}_\gamma \otimes \mathrm{F}_{\gamma'} = \mathrm{F}_{\gamma+\gamma'} \qquad \text{and} \qquad S_i \otimes S_j = \begin{cases} \displaystyle\bigoplus_{\substack{l=|i-j| \\ \text{by } 2}}^{i+j} S_l & \text{if } i+j<r, \\ \displaystyle\bigoplus_{\substack{l=|i-j| \\ \text{by } 2}}^{2r-4-i-j} S_l \oplus \bigoplus_{\substack{l=2r-2-i-j \\ \text{by } 2}}^{r-1} P_l & \text{if } i+j \geq r. \end{cases}$$

The projective indecomposable modules denoted by P_i have quantum dimension zero so their corresponding object in $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ is zero. We therefore have the product

$$\mathsf{M}_{\frac{x}{2\lambda},i,r\ell} \cdot \mathsf{M}_{\frac{y}{2\lambda},j,r\ell'} = \begin{cases} \displaystyle\sum_{\substack{l=|i-j| \\ \text{by } 2}}^{i+j} \mathsf{M}_{\frac{x+y \bmod \mathrm{r}}{2\lambda},l,r(\ell+\ell') \bmod 2\mathrm{r}} & \text{if } i+j<r, \\ \displaystyle\sum_{\substack{l=|i-j| \\ \text{by } 2}}^{2r-4-i-j} \mathsf{M}_{\frac{x+y \bmod \mathrm{r}}{2\lambda},l,r(\ell+\ell') \bmod 2\mathrm{r}} & \text{if } i+j \geq r. \end{cases} \tag{2.4.11}$$

This set can be reduced as we have not yet accounted for the relation $\mathsf{M}_{\gamma,r-1-i,r\ell} = -\mathsf{M}_{\gamma,i-1,r(\ell+1)}$. Consider the case r odd. Notice that every generator of the form $\mathsf{M}_{\frac{2n}{2\lambda},2m,0}$ corresponds to a generator of the form $\mathsf{M}_{\frac{2n}{2\lambda},2m+1,r}$, and each generator of the form $\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}$ to one of the form $\mathsf{M}_{\frac{2n+1}{2\lambda},2m,r}$ under $\mathsf{M}_{\gamma,r-1-i,r\ell} = -\mathsf{M}_{\gamma,i-1,r(\ell+1)}$. We therefore have the set

$$\begin{aligned} \mathsf{M}_{\frac{2n}{2\lambda},2m,0}, \qquad & n \in \{0,...,\frac{r-1}{2}\}, m \in \{0,...,\frac{r-3}{2}\} \\ \mathsf{M}_{\frac{2n+1}{2\lambda},2m,r}, \qquad & n \in \{0,...,\frac{r-3}{2}\}, m \in \{0,...,\frac{r-3}{2}\} \end{aligned}$$

Notice that the product of any pair of elements of this generating set will be a sum of generators of the form $\mathsf{M}_{\frac{x}{2\lambda},i,r\ell}$ where i must be even, and hence will be a sum of elements in the set with positive coefficients. Therefore, this set in fact gives a $\mathbb{Z}_+$-basis for the ring when r is odd.

When r is even, by the exact same considerations we obtain the set

$$\begin{aligned}\mathsf{M}_{\frac{2n}{2\lambda},2m,0}, \qquad & n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-2}{2}\} \\ \mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}, \qquad & n \in \{0,1,\ldots,\frac{r-2}{2}\}, m \in \{0,1,\ldots,\frac{r-4}{2}\}\end{aligned}$$

Further, the product of generators of any of these generators with another of the same form will be a sum of generators $\mathsf{M}_{\frac{2n}{2\lambda},2m,0}$ with positive coefficients. A product of a generator $\mathsf{M}_{\frac{2n}{2\lambda},2m,0}$ with another $\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}$ will then be a sum of generators $\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}$ with positive coefficients. Hence, this set is again a $\mathbb{Z}_+$-basis.

□

2.4.2 Comparison for odd r

Let $V, W \in \overline{U}_q^H(\mathfrak{sl}_2)$-Mod and $w \in W$ a highest weight vector of weight λ. For any $\gamma \in \mathbb{C}$, define $\Psi_\gamma : \mathbb{Z}[z] \to \mathbb{C}$ by $\Psi_\gamma(z^s) = q^{\gamma s}$. Then, as noted in the proof of Lemma 6.6 in [CGP1], the open Hopf links $\Phi_{V,W}$ satisfy $\Phi_{V,W}(w) = \Psi_{\lambda+1-r}(\chi(V))w$ where $\chi(V)$ is the character of V. Clearly then, when W is simple, $\Phi_{V,W} = \Psi_{\lambda+1-r}(\chi(V))\mathrm{Id}_W$. Using the fact that the Hopf link $\mathsf{S}^{\infty}_{V,W}$ is the trace of the open Hopf link $\Phi_{V,W}$ and the appropriate character formulas [CGP1, Equation (16)], it is easy to show that

$$\mathsf{S}^{\infty}_{S_i\otimes\mathbb{C}^H_{rk}, S_j\otimes\mathbb{C}^H_{r\ell}} = (-1)^{k(j+r(\ell-1)+1)+(i+1)(\ell+1)+\ell+1}\frac{\{(i+1)(j+1)\}}{\{j+1\}}\,\mathrm{tr}(\mathrm{Id}_{S_j\otimes\mathbb{C}^H_{r\ell}}) \tag{2.4.12}$$

where $\mathrm{tr}(\mathrm{Id}_{S_j\otimes\mathbb{C}^H_{r\ell}}) = (-1)^{(1-r)\ell+j}[j+1]$. The Hopf links in $\mathcal{H}_{i\mathbb{R}^\oplus}$ (see Subsection 2.1.1) are easily seen to be $\mathsf{S}^{\infty}_{\mathrm{F}_{\gamma_1},\mathrm{F}_{\gamma_2}} = e^{\pi i\gamma_1\gamma_2}e^{\pi i\gamma_2\gamma_1} = e^{2\pi i\gamma_1\gamma_2}$. Therefore, the Hopf links are given by

$$\begin{aligned}\mathsf{S}^{\infty}_{(\gamma_1,i,k),(\gamma_2,j,\ell)} &= \mathsf{S}^{\infty}_{S_i\otimes\mathbb{C}^H_{rk}, S_j\otimes\mathbb{C}^H_{r\ell}} \cdot \mathsf{S}^{\infty}_{\mathrm{F}_{\gamma_1},\mathrm{F}_{\gamma_2}} \\ &= (-1)^{k(j+r(\ell-1)+1)+(i+1)(\ell+1)+\ell+1+(1-r)\ell+j}e^{2\pi i\gamma_1\gamma_2}[(i+1)(j+1)] \\ &= (-1)^{(i+1)(\ell+1)+(j+1)(k+1)+r(k\ell+k+\ell)}e^{2\pi i\gamma_1\gamma_2}[(i+1)(j+1)].\end{aligned} \tag{2.4.13}$$

Normalizing the S^{∞} matrix and restricting to odd r (i.e., using $i, j \in 2\mathbb{Z}$ for all elements of our generating set) gives

$$\frac{\mathsf{S}^{\infty}_{(\gamma_1,i,k),(\gamma_2,j,\ell)}}{\mathsf{S}^{\infty}_{(0,0,0),(\gamma_2,j,\ell)}} = e^{\pi i(2\gamma_1\gamma_2+rkl)}\frac{\{(i+1)(j+1)\}}{\{j+1\}}.$$

Below, in order to find S^{χ}, we shall use certain modular transformation properties from [C2]. For this, it will be beneficial for us to re-parametrize by setting $(s,s') = (i+1, -2\lambda_r\gamma_1 - rk), (n,n') = (j+1, -2\lambda_r\gamma_2 - r\ell)$ and adopting the notation $\mathsf{S}^{\infty}_{(s,s'),(n,n')} := \mathsf{S}^{\infty}_{(\gamma_1,i,k),(\gamma_2,j,\ell)}$, we see that

$$\frac{\mathsf{S}^{\infty}_{(s,s'),(n,n')}}{\mathsf{S}^{\infty}_{(1,0),(n,n')}} = q^{n's'}\frac{\{ns\}}{\{n\}}. \tag{2.4.14}$$

Recall that $\mathscr{F}(\mathrm{F}_\gamma \boxtimes (S_i \otimes \mathbb{C}^H_{r\ell})) = \bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{\gamma+k\lambda_r} \boxtimes (S_i \otimes \mathbb{C}^H_{r(k+\ell)})$ which corresponds to

$$\bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{\gamma+k\lambda_r} \boxtimes M_{1-(k+\ell),i+1}$$

under the correspondence in Proposition 2.1.8. This module is the spectral flow ([C2, Subsection 3.1.2])

$$\sigma^{s'}(W_s) := \bigoplus_{\widetilde{k}\in\mathbb{Z}} \mathrm{F}_{\widetilde{\lambda_r}(\widetilde{k}-\frac{s'}{r})} \boxtimes M'_{1+\widetilde{k},s} \tag{2.4.15}$$

of the module W_s defined in [C2, Subsection 4.2], where $\widetilde{\lambda_r} = -\lambda_r$, $\widetilde{k} = -k-\ell$, and $(s,s') = (i+1, -2\lambda_r\gamma - r\ell)$. Consider the basis for odd r given in Proposition 2.4.1. By applying the relation $\mathsf{M}_{\gamma,r-1-\alpha,r\ell} = -\mathsf{M}_{\gamma,\alpha-1,r(\ell+1)}$ to the generators of the form $\mathsf{M}_{\frac{2n+1}{2\lambda_r},2m,r}$, keeping in mind that $\widetilde{\lambda_r} = -\lambda_r$ and that the correspondence in Proposition 2.1.8 preserves tensor structure up to character, it is easy to see that the Verlinde algebra of characters, $\mathcal{V}(B_r)$, generated by the atypical modules of B_r has a generating set

$$\{\mathrm{ch}[\sigma^{s'}(W_s)] \,|\, (s,s') \in \Lambda_r\}$$

where

$$\Lambda_r := \{(s,s') | 0 < s \le r-1,\ 0 \le s' \le r-1,\ s+s'+1 \in 2\mathbb{Z}\}. \tag{2.4.16}$$

This set is closed under modular transformations, and we will show that the corresponding S^{χ}-matrix agrees with the S^{∞}-matrix (2.4.14) up to normalised conjugation. From the character formula for W_s and the relation $\mathrm{ch}[\sigma^{s'}(M)](u;\tau) = q^{\frac{s'^2}{4r}-\frac{s'^2}{2}} x^{\frac{s'}{r}-2s'}\mathrm{ch}[M](u+\tau\frac{s'}{2};\tau)$ (see [C2, Subsections 4.3 and 3.1.2]), we see that the character of $\sigma^{s'}(W_s)$ is given by

$$\mathrm{ch}[\sigma^{s'}(W_s)](u;\tau) = \frac{q^{\frac{s'^2}{4r}-\frac{s'^2}{2}} x^{\frac{s'}{r}-2s'}}{\eta(\tau)^2} \sum_{n\in\mathbb{Z}} \left(\frac{q^{r(n+\frac{1}{2}-\frac{s}{2r})^2}}{1-xq^{r(n+\frac{1}{2}-\frac{s}{2r})+\frac{s'}{2}}} - \frac{q^{r(n+\frac{1}{2}+\frac{s}{2r})^2}}{1-xq^{r(n+\frac{1}{2}+\frac{s}{2r})+\frac{s'}{2}}} \right) \tag{2.4.17}$$

where $x = e^{2\pi i u}, q = e^{2\pi i\tau}$. Even though $s = 0, s = r$ are not allowed in (2.4.16), it is easy to see that with $s = 0, r$ in equation (2.4.17), for all s', we have

$$\mathrm{ch}[\sigma^{s'}(W_0)](u;\tau) = \mathrm{ch}[\sigma^{s'}(W_{-r})](u;\tau) = 0, \tag{2.4.18}$$

$$\mathrm{ch}[\sigma^{s'}(W_s)](u;\tau) + \mathrm{ch}[\sigma^{s'}(W_{-s})](u;\tau) = 0. \tag{2.4.19}$$

Recall the notation from [C2]:

$$\Pi(v;\tau) = q^{1/6}(z-z^{-1})\prod_{n=1}^{\infty}(1-z^2q^n)(1-q^n)^2(1-z^{-2}q^n) \tag{2.4.20}$$

with $z = e^{2\pi i v}$. By [C2, Theorem 3.6 and Subsection 4.3], we have the following:

$$\begin{aligned}
\mathrm{ch}[\sigma^{s'}(W_s)](\frac{u}{\tau};\frac{-1}{\tau}) &= \lim_{v\to 0}\frac{\Pi(v;\tau)}{\eta(\tau)^2}\mathrm{ch}[\sigma^{s,s'}\chi_r](\frac{u}{\tau};\frac{v}{\tau};\frac{-1}{\tau}) \\
&= \lim_{v\to 0}\frac{\Pi(v;\tau)}{\eta(\tau)^2}e^{\frac{2\pi i}{\tau}(kv^2-\frac{u^2}{r})}\sum_{(n,n')\in S_r} S_{(s,s'),(n,n')}\mathrm{ch}[\sigma^{(n,n')}(\chi_r)](u;v;\tau) \\
&= \sum_{(n,n')\in S_r} e^{\frac{-2\pi i u^2}{r\tau}} S_{(s,s'),(n,n')}\lim_{v\to 0}\frac{\Pi(v;\tau)}{\eta(\tau)^2}\mathrm{ch}[\sigma^{(n,n')}(\chi_r)](u;v;\tau) \\
&= \sum_{(n,n')\in S_r} e^{\frac{-2\pi i u^2}{r\tau}} S_{(s,s'),(n,n')}\mathrm{ch}[\sigma^{n'}(W_n)](u,\tau),
\end{aligned} \tag{2.4.21}$$

with $S_{(s,s'),(n,n')} = \frac{-1}{r}e^{-\frac{2\pi i}{2r}(sn-s'n')}$ and $S_r := \{(n,n')| -r \le n \le r-1, 0 \le n' \le r-1, n+n'+1 \in 2\mathbb{Z}\}$. It follows that the character of $\sigma^{s'}(W_s)$ satisfies

$$\mathrm{ch}[\sigma^{s'}(W_s)](\frac{u}{\tau};\frac{-1}{\tau}) = \sum_{(n,n')\in\Lambda_r} e^{\frac{-2\pi i u^2}{r\tau}}\left(S_{(s,s'),(n,n')} - S_{(s,s'),(-n,n')}\right)\mathrm{ch}[\sigma^{n'}(W_n)](u,\tau), \tag{2.4.22}$$

and therefore,

$$\begin{aligned}
S^{\chi}_{(s,s'),(n,n')} &= e^{\frac{-2\pi i u^2}{r\tau}}\left(S_{(s,s'),(n,n')} - S_{(s,s'),(-n,n')}\right) \\
&= -\frac{1}{r} e^{\frac{-2\pi i u^2}{r\tau}} e^{\frac{2\pi i}{2r} n's'}\left(e^{-\frac{2\pi i}{r} ns} - e^{\frac{2\pi i}{r} ns}\right) \\
&= -\frac{1}{r} e^{\frac{-2\pi i u^2}{r\tau}} q^{n's'}\{ns\}. \qquad (2.4.23)
\end{aligned}$$

The unit object in $\mathcal{C}$ is $\mathrm{F}_0 \boxtimes (S_0 \otimes \mathbb{C}^H_0)$ which induces to $\bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{k\lambda} \boxtimes (S_0 \otimes \mathbb{C}^H_{rk})$, so the identity object corresponds to $\sigma^0(W_1)$, which we will simply denote by $\mathbb{1}$ in the index. Hence, the normalized S^{χ}-matrix is given by

$$\frac{\mathsf{S}^{\chi}_{(s,s'),(n,n')}}{\mathsf{S}^{\chi}_{\mathbb{1},(n,n')}} = q^{n's'}\frac{[ns]}{[n]}. \qquad (2.4.24)$$

The following proposition follows by comparing (2.4.24) and (2.4.14):

Proposition 2.4.2. For atypical modules, the matrices S^{∞} and S^{χ} are in agreement up to normalization. That is,

$$\frac{\mathsf{S}^{\chi}_{(s,s'),(n,n')}}{\mathsf{S}^{\chi}_{\mathbb{1},(n,n')}} = \frac{\mathsf{S}^{\infty}_{(s,s'),(n,n')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}} \qquad (2.4.25)$$

for $(s,s'),(n,n') \in \Lambda_r$.

It then follows that the matrix S^{∞} is invertible since S^{χ} is, and by the standard argument the categorical Verlinde formula holds. We include this argument here for completeness:

Recall that the Hopf links give a one dimensional representation of the fusion ring and therefore satisfy

$$\frac{\mathsf{S}^{\infty}_{(s,s'),(n,n')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}}\frac{\mathsf{S}^{\infty}_{(t,t'),(n,n')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}} = \sum_{(m,m')} N^{(m,m')}_{(s,s'),(t,t')}\frac{\mathsf{S}^{\infty}_{(m,m'),(n,n')}}{\mathsf{S}^{\infty}_{\mathbb{1},(n,n')}}$$

where the sum runs over the pairs corresponding to the basis of $\mathcal{G}^{\mathrm{ss}}(\mathcal{C}^0)$ in Proposition 2.4.1.

The matrix S^χ is invertible, so by the above corollary, S^∞ is also invertible. Cancelling out the $\mathsf{S}^\infty_{\mathbb{1},(n,n')}$ terms and multiplying by $(\mathsf{S}^\infty)^{-1}_{(k,k'),(n,n')}$ for any fixed (k,k') yields

$$\frac{\mathsf{S}^\infty_{(s,s'),(n,n')}\mathsf{S}^\infty_{(t,t'),(n,n')}}{\mathsf{S}^\infty_{\mathbb{1},(n,n')}}(\mathsf{S}^\infty)^{-1}_{(n,n'),(k,k')} = \sum_{(m,m')} N^{(m,m')}_{(s,s'),(t,t')}\mathsf{S}^\infty_{(m,m'),(n,n')}(\mathsf{S}^\infty)^{-1}_{(n,n'),(k,k')}.$$

Summing over the index (n,n') then gives

$$\begin{aligned}\sum_{(n,n')}\frac{\mathsf{S}^\infty_{(s,s'),(n,n')}\mathsf{S}^\infty_{(t,t'),(n,n')}(\mathsf{S}^\infty)^{-1}_{(n,n'),(k,k')}}{\mathsf{S}^\infty_{\mathbb{1},(n,n')}} &= \sum_{(m,m')} N^{(m,m')}_{(s,s'),(t,t')}\left(\mathsf{S}^\infty(\mathsf{S}^\infty)^{-1}\right)_{(m,m'),(k,k')} \\ &= \sum_{(m,m')} N^{(m,m')}_{(s,s'),(t,t')}\delta_{(m,m'),(k,k')} \\ &= N^{(k,k')}_{(s,s'),(t,t')} \qquad (2.4.26)\end{aligned}$$

which is the Verlinde formula and by Proposition 2.4.2, the Verlinde formula also holds for atypical B_r modules:

Corollary 2.4.3. The Verlinde formula holds for the Verlinde algebra of characters generated by atypical modules of B_r when r is odd. That is,

$$\sum_{(n,n')\in\Lambda_r}\frac{\mathsf{S}^\infty_{(s,s'),(n,n')}\mathsf{S}^\infty_{(t,t'),(n,n')}(\mathsf{S}^\infty)^{-1}_{(n,n'),(k,k')}}{\mathsf{S}^\infty_{\mathbb{1},(n,n')}} = N^{(k,k')}_{(s,s'),(t,t')}$$

where Λ_r is given in (2.4.16).

2.4.3 Comparison for even r

When r is even, B_r has half integer $L(0)$-grading and $\mathbb{Z}_2$-grading given by

$$B_r = B_r^{\bar{0}} \oplus B_r^{\bar{1}},$$

where $B_r^{\bar{0}}$ is the integer part of the $\frac{1}{2}\mathbb{Z}$-grading and $B_r^{\bar{1}}$ is the non-integer part. In this case, we instead compare the ring $\mathcal{G}^{ss}(\mathcal{C}^0_{\bar{0}})$ for the modules which lift to $B_r^{\bar{0}}$ modules with the Verlinde algebra of characters of $B_r^{\bar{0}}$. Note that when we computed $\mathcal{G}^{ss}(\mathcal{C}^0)$ we used the relation $\mathsf{M}_{\gamma,i,\ell r} = \mathsf{M}_{\gamma+\lambda,i,(\ell+1)r}$ which no longer holds. To remedy this, we need only notice that the objects $\mathsf{M}_{\gamma,i,\ell r}$ and $\mathsf{M}_{\gamma+2\lambda,i,(\ell+2)r}$ induce to the same $B_r^{\bar{0}}$ module and are therefore equal in $\mathcal{G}^{ss}(\mathcal{C}^0_{\bar{0}})$. Hence, it is enough to add to the basis for $\mathcal{G}^{ss}(\mathcal{C}^0)$ with r even obtained in Proposition 2.4.1 the shifted elements $\mathsf{M}_{\gamma+\lambda,i,(\ell+1)r}$ for each basis element $\mathsf{M}_{\gamma,i,\ell r}$. Hence, a basis for $\mathcal{G}^{ss}(\mathcal{C}^0_{\bar{0}})$ is given by

$$\begin{aligned}
\mathsf{M}_{\frac{2n}{2\lambda},2m,0}, \mathsf{M}_{\frac{2n-r}{2\lambda},2m,r} \qquad & n\in\{0,1,\ldots,\frac{r-2}{2}\}, m\in\{0,1,\ldots,\frac{r-2}{2}\},\\
\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}, \mathsf{M}_{\frac{2n+1-r}{2\lambda},2m+1,r} \qquad & n\in\{0,1,\ldots,\frac{r-2}{2}\}, m\in\{0,1,\ldots,\frac{r-4}{2}\}.
\end{aligned}$$

This is not a $\mathbb{Z}_+$ basis with respect to the product

$$\mathsf{M}_{\frac{x}{2\lambda},i,r\ell}\cdot\mathsf{M}_{\frac{y}{2\lambda},j,r\ell'} = \begin{cases} \sum_{\substack{l=|i-j|\\ \text{by }2}}^{i+j} \mathsf{M}_{\frac{x+y \bmod 2r}{2\lambda},l,r(\ell+\ell') \bmod 2r} & \text{if } i+j<r,\\ \\ \sum_{\substack{l=|i-j|\\ \text{by }2}}^{2r-4-i-j} \mathsf{M}_{\frac{x+y \bmod 2r}{2\lambda},l,r(\ell+\ell') \bmod 2r} & \text{if } i+j\geq r.\end{cases} \tag{2.4.27}$$

Applying the relation $\mathsf{M}_{\gamma,j-1,r(\ell+1)} = -\mathsf{M}_{\gamma,r-1-j,r\ell}$, we obtain a basis

$$\begin{aligned}
\mathsf{M}_{\frac{2n}{2\lambda},2m,0}, \mathsf{M}_{\frac{2n-r}{2\lambda},r-2-2m,0} \qquad & n\in\{0,1,\ldots,\frac{r-2}{2}\}, m\in\{0,1,\ldots,\frac{r-2}{2}\},\\
\mathsf{M}_{\frac{2n+1}{2\lambda},2m+1,0}, \mathsf{M}_{\frac{2n+1-r}{2\lambda},r-2-(2m+1),0} \qquad & n\in\{0,1,\ldots,\frac{r-2}{2}\}, m\in\{0,1,\ldots,\frac{r-4}{2}\},
\end{aligned}$$

which is easily seen to be a $\mathbb{Z}_+$-basis. We have already seen that the Hopf links satisfy equation (2.4.13). Restricting this equation to r even ($k=\ell=0$) gives

$$(\mathsf{S}^{\infty}_{B^{\bar{0}}_r})_{(s,s'),(n,n')} = (-1)^{s+n} q^{n's'}\{ns\}, \tag{2.4.28}$$

once again making the substitution $(s,s') = (i+1,-2\lambda_r\gamma_1 - rk)$, $(n,n') = (j+1,-2\lambda_r\gamma_2 - r\ell)$ and the indices run over the set $\Lambda_r\cup\tilde{\Lambda}_r$ where $\tilde{\Lambda}_r := \{(s,s'-r)|(s,s')\in\Lambda_r\}$. We therefore have four cases:

$((s,s'),(n,n'))$	$(\mathsf{S}^{\infty}_{B_r^{\bar{0}}})_{(s,s'),(n,n')}$
$\Lambda_r \times \Lambda_r$	$(-1)^{s+n} q^{n's'} \{ns\}$
$\Lambda_r \times \tilde{\Lambda}_r$	$(-1)^{n+1} q^{n's'} \{ns\}$
$\tilde{\Lambda}_r \times \Lambda_r$	$(-1)^{s+1} q^{n's'} \{ns\}$
$\tilde{\Lambda}_r \times \tilde{\Lambda}_r$	$q^{n's'} \{ns\}$

Following the analysis in Subsection 2.4.2 and replacing characters with supercharacters where necessary, we see that the S^{χ} matrix for the atypical B_r modules $\{\sigma^{s'}(W_s) | (s,s') \in \Lambda_r\}$ are still given by equation (2.4.23). However, each B_r module $\sigma^{s'}(W_s)$ splits into two $B_r^{\bar{0}}$ modules $\sigma^{s'}(W_s)^{\bar{0}}$ and $\sigma^{s'}(W_s)^{\bar{1}}$, where we recall that $\sigma^{s'}(W_s)$ is given by equation (2.4.15) and $\sigma^{s'}(W_s)^{\bar{0}}, \sigma^{s'}(W_s)^{\bar{1}}$ are given by the even and odd summands respectively. It then follows from equation (2.4.23) and [C2, Equation 4.9] that these matrices agree up to normalised conjugation.

2.5 The characters of B_r and of a QH-reduction

In this section, we set $n = r-1$ and compare the character of B_r with a certain Quantum-Hamiltonian reduction of $\mathfrak{sl}_n$ as announced in [C2, Remark 5.6]. For more details about Quantum-Hamiltonian reduction, see [Ara] for instance.

Recall that $\mathfrak{sl}_{n-1}$ can be embedded in $\mathfrak{sl}_n$ such that

$$\mathfrak{sl}_n \cong \mathfrak{sl}_{n-1} \oplus \rho_{n-1} \oplus \overline{\rho_{n-1}} \oplus \mathbb{C}$$

where ρ_{n-1} is the standard representation of $\mathfrak{sl}_{n-1}$ and $\overline{\rho_{n-1}}$ its conjugate. To carry out computations, we see $\mathfrak{sl}_{n-1}$ as embedded in the upper-left square of the matrix realisation of $\mathfrak{sl}_n$. Fix the $\mathfrak{sl}_2$-triplet $\{F, H, E\}$ in $\mathfrak{sl}_{n-1}$ as follows

$$F := \sum_{i=1}^{n-2} e_{i+1,i}, \qquad H := \frac{1}{2}\sum_{i=1}^{n-1}(n-2i)e_{i,i}, \qquad E := \sum_{i=1}^{n-2} e_{i,i+1} \tag{2.5.1}$$

Notice that for any matrix $h = \sum_{j=1}^{n} \lambda_j e_{j,j} \in \widehat{\mathfrak{h}}$, we have

$$[F, h] = \sum_{i=1}^{n-2} (\lambda_i - \lambda_{i+1}) e_{i+1,i},$$

and the matrix must be of trace zero, so $0 = (n-1)\lambda_1 + \lambda_n$. It follows that the subalgebra of $\widehat{\mathfrak{h}}$ annihilated by F under the adjoint action is $\mathrm{Span}_{\mathbb{C}}\{A\}$ where A is the diagonal matrix $A := \mathrm{diag}\{\frac{1}{n}, ..., \frac{1}{n}, \frac{1-n}{n}\}$. Let $K = Av$ for complex v and set $x = e^v$. Let Δ_+ denote the set of positive roots of the full $\mathfrak{sl}_n$, $\Delta_+^0 \subset \Delta_+$ the set of positive roots such that $\alpha(H) = 0$, and $\Delta_+^{\frac{1}{2}} \subset \Delta_+$ the positive roots for which $\beta(H) = 1/2$. By [KW, Equation (11)], the character of the Quantum-Hamiltonian reduction associated to (2.5.1) is given by the following formula:

$$(-i)^{\frac{r(r+1)}{2}} q^{\frac{(r^2-1)(r-2)}{24}} \frac{\eta(r\tau)^{-\frac{1}{2}r^2+\frac{5}{2}r-3}}{\eta(\tau)^{r-3}} \frac{\prod_{\alpha\in\Delta_+} \vartheta_{11}(r\tau, \alpha(K - \tau H))}{\left(\prod_{\alpha\in\Delta_+^0} \vartheta_{11}(\tau, \alpha(K))\right)\left(\prod_{\beta\in\Delta^{\frac{1}{2}}} \vartheta_{01}(\tau, \beta(K))\right)^{1/2}} \tag{2.5.2}$$

where ϑ_{11} and ϑ_{01} are the standard Jacobi theta functions:

$$\vartheta_{11}(\tau, z) = -iq^{1/12} u^{-1/2} \eta(\tau) \prod_{k=1}^{\infty} (1 - u^{-1}q^k)(1 - uq^{k-1}) \;, \tag{2.5.3}$$

$$\vartheta_{01}(\tau, z) = \prod_{k=1}^{\infty} (1 - u^{-1}q^{k-1/2})(1 - q^k)(1 - uq^{k-1/2}) \tag{2.5.4}$$

where $u = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$.

Theorem 2.5.1. Let $r \in \mathbb{Z}$. Then the character of B_r is given by equation (2.5.2).

Proof. As for proving [C2, Theorem 5.5], a character is viewed as a formal power series where we admit $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$. Two formal power series X, Y will be seen as equivalent if $X = \gamma q^a x^b Y$ for $\gamma \in \mathbb{C}$ and $a, b \in \mathbb{Q}$. Also here, it is enough to show that $\mathrm{ch}[B_r](x; q)$ is equivalent to (2.5.2) since both are formal power series of the form $q^{-\frac{c_r}{24}}(1 + \cdots)$.

Let's then make the formula (2.5.2) explicit, keeping in mind that $n = r-1$ here. First let's fix a standard choice of positive roots for $\mathfrak{sl}_n = \mathfrak{sl}_{r-1}$ and compute Δ_+^0 and $\Delta^{\frac{1}{2}}$:

- $\Delta_+ = \Delta_+(\mathfrak{sl}_n) = \left\{ \alpha_{i,j} = \sum_{\ell=i}^{j} \alpha_\ell \mid 1 \le i \le j \le n-1 \right\}$ where $\alpha_\ell(e_{ii}) = \delta_{\ell,i} - \delta_{\ell+1,i}$;

- $\alpha_{i,j}(K) = 0$ if $j < n-1$;

- $\alpha_{i,n-1}(K) = v$;

- $\alpha_{i,j}(H) = j - i + 1$ if $j < n-1$;

- $\alpha_{i,n-1}(H) = \frac{n}{2} - i$;

- $\Delta_+^0 = \{\alpha_{\frac{n}{2},n-1}\}$ and $\Delta^{\frac{1}{2}} = \emptyset$ if n is even (r odd);

- $\Delta_+^0 = \emptyset$ and $\Delta^{\frac{1}{2}} = \{\alpha_{\frac{n-1}{2},n-1}, -\alpha_{\frac{n+1}{2},n-1}\}$ if n is odd (r even).

We shall now examine each factor in (2.5.2). For any choice of n, since $\alpha_{i,j}(K - \tau H) = \tau(i-j-1)$ if $i < j$ and $\alpha_{i,n-1}(K - \tau H) = v - \tau(\frac{n}{2} - i)$ one obtains:

$$\frac{\vartheta_{11}(r\tau, \alpha_{i,j}(K - \tau H))}{\eta(r\tau)} \sim \prod_{k=1}^{\infty} (1 - q^{r(k-1)+(j-i+1)})(1 - q^{rk-(j-i+1)}) \qquad (j < n-1)$$

$$\frac{\vartheta_{11}(r\tau, \alpha_{i,n-1}(K - \tau H))}{\eta(r\tau)} \sim \prod_{k=1}^{\infty} (1 - x^{-1} q^{r(k-1)+(\frac{n}{2}-i)})(1 - x q^{rk-(\frac{n}{2}-i)}),$$

where we used the observation $\prod_{k=1}^{\infty}(1-u^{-1}q^k)(1-uq^{k-1}) \sim \prod_{k=1}^{\infty}(1-uq^k)(1-u^{-1}q^{k-1})$ on each line. Then, aiming to find the product over Δ_+ as in the numerator of (2.5.2), we write:

$$\begin{aligned}
&\prod_{k=1}^{\infty}\prod_{1\le i\le j<n-1}\frac{\vartheta_{11}(r\tau,\alpha_{i,j}(K-\tau H))}{\eta(r\tau)} \sim \prod_{k=1}^{\infty}\prod_{1\le i\le j<n-1}(1-q^{rk+(j-i+1-r)})(1-q^{rk-(j-i+1)})\\
&\sim \prod_{k=1}^{\infty}\prod_{1\le a\le n-2}(1-q^{rk-(r-a)})^{r-2-a}(1-q^{rk-a})^{r-2-a}\\
&\sim \prod_{k=1}^{\infty}(1-q^{rk-1})(1-q^{r(k-1)+1})\prod_{1\le b\le n}(1-q^{rk-b})^{r-4}\\
&\sim \prod_{k=1}^{\infty}(1-q^{rk-1})(1-q^{r(k-1)+1})\prod_{1\le b\le n}(1-q^{rk-b})^{r-4}\\
&\sim \prod_{k=1}^{\infty}(1-q^{rk-1})(1-q^{r(k-1)+1})\left(\frac{1-q^k}{1-q^{rk}}\right)^{r-4}\\
&\sim \left(\frac{\eta(\tau)}{\eta(r\tau)}\right)^{r-4}\cdot\prod_{k=1}^{\infty}(1-q^{rk-1})(1-q^{r(k-1)+1}).
\end{aligned}\tag{2.5.5}$$

We also will need the following for the numerator:

$$\begin{aligned}
\prod_{1\le i\le n-1}\frac{\vartheta_{11}(r\tau,\alpha_{i,n-1}(K-\tau H))}{\eta(r\tau)} &\sim \prod_{k=1}^{\infty}\prod_{1\le i\le n-1}(1-x^{-1}q^{r(k-1)+(\frac{n}{2}-i)})(1-xq^{rk-(\frac{n}{2}-i)})\\
&=\prod_{k=0}^{\infty}\frac{(1-xq^{k+\frac{r}{2}+\frac{3}{2}})(1-x^{-1}q^{k-\frac{r}{2}+\frac{3}{2}})}{(1-xq^{rk+\frac{3}{2}r-\frac{1}{2}})(1-xq^{rk+\frac{3}{2}r+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})}.
\end{aligned}\tag{2.5.6}$$

Finally completing the product of ϑ_{11}s over Δ_+ in the numerator of (2.5.2) yields:

$$\begin{aligned}
\prod_{\alpha\in\Delta_+}\vartheta_{11}(r\tau,\alpha(K-\tau H)) &\sim \eta(r\tau)^{\frac{(n-2)(n-1)}{2}+(n-1)-(n-3)}\eta(\tau)^{n-3}\cdot\prod_{k=1}^{\infty}(1-q^{rk-1})(1-q^{r(k-1)+1})\\
&\times\prod_{k=0}^{\infty}\frac{(1-xq^{k+\frac{r}{2}+\frac{3}{2}})(1-x^{-1}q^{k-\frac{r}{2}+\frac{3}{2}})}{(1-xq^{rk+\frac{3}{2}r-\frac{1}{2}})(1-xq^{rk+\frac{3}{2}r+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})}.
\end{aligned}\tag{2.5.7}$$

The η factors written in terms of r instead of n read:

$$\eta(r\tau)^{\frac{(r-3)(r-2)}{2}-(r-4)+r-2}\eta(\tau)^{r-4}=\eta(r\tau)^{\frac{r^2}{2}-\frac{5r}{2}+5}\eta(\tau)^{r-4}.\tag{2.5.8}$$

The denominator of (2.5.2) is given up to equivalence by

$$\eta(\tau)\prod_{k=0}^{\infty}(1-x^{-1}q^{k+1})(1-x^{n}q^{k}) \qquad r \text{ odd}, \tag{2.5.9}$$

$$\eta(\tau)\prod_{k=0}^{\infty}(1-x^{-1}q^{k+\frac{1}{2}})(1-xq^{k+\frac{1}{2}}) \qquad r \text{ even}. \tag{2.5.10}$$

In effect, collecting various η factors arising from ϑs now gets us: $\eta(r\tau)^{\frac{r^2}{2}-\frac{5r}{2}+5}\eta(\tau)^{r-5}$. Canceling these with the η factors already present in the (2.5.2), we are left with simply $\dfrac{\eta(r\tau)^2}{\eta(\tau)^2}$.

We now show the rest of the calculation for r be odd, the other case is similar. In this case, the character, up to $\sim$ equivalence has now simplified to:

$$\begin{aligned}
&\frac{\eta(r\tau)^2}{\eta(\tau)^2}\prod_{k=0}^{\infty}\frac{(1-q^{r(k+1)-1})(1-q^{rk+1})(1-xq^{k+\frac{r}{2}+\frac{3}{2}})(1-x^{-1}q^{k-\frac{r}{2}+\frac{3}{2}})}{(1-xq^{rk+\frac{3}{2}r-\frac{1}{2}})(1-xq^{rk+\frac{3}{2}r+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})(1-x^{-1}q^{k+1})(1-xq^{k})}\\
&=\frac{\eta(r\tau)^2}{\eta(\tau)^2}\prod_{k=0}^{\infty}\frac{(1-q^{r(k+1)-1})(1-q^{rk+1})}{(1-xq^{rk+\frac{3}{2}r-\frac{1}{2}})(1-xq^{rk+\frac{3}{2}r+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})}\\
&\qquad\times\frac{\prod_{k=-\frac{r}{2}+\frac{3}{2}}^{0}(1-x^{-1}q^{k})}{\prod_{k=0}^{\frac{r}{2}+\frac{1}{2}}(1-xq^{k})}\\
&\sim\frac{\eta(r\tau)^2}{\eta(\tau)^2}\left(\prod_{k=0}^{\infty}\frac{(1-q^{r(k+1)-1})(1-q^{rk+1})}{(1-xq^{rk+\frac{3}{2}r-\frac{1}{2}})(1-xq^{rk+\frac{3}{2}r+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})}\right)\\
&\qquad\times\frac{1}{(1-xq^{\frac{r}{2}-\frac{1}{2}})(1-xq^{\frac{r}{2}+\frac{1}{2}})}\\
&=\frac{\eta(r\tau)^2}{\eta(\tau)^2}\prod_{k=0}^{\infty}\frac{(1-q^{r(k+1)-1})(1-q^{rk+1})}{(1-xq^{rk+\frac{r}{2}-\frac{1}{2}})(1-xq^{rk+\frac{r}{2}+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})}.
\end{aligned} \tag{2.5.11}$$

The character of B_r can be written in a product form using $B_r = W_1$ and the [C2, Subsection

4.1, Proposition 5.2] as follows:

$$
\begin{aligned}
&\mathrm{ch}[B_r](x;\tau) = \mathrm{ch}[W_1](x;\tau) = \lim_{z\to 1} \frac{\Pi(zq^{\frac{1}{2}};\tau)}{\eta(\tau)^2} q^{\frac{1}{4r}} \mathrm{ch}[\chi_r](x;zq^{\frac{1}{2}};\tau) \\
&= \frac{1}{\eta(\tau)^2} q^{\frac{1}{4r}} q^{r/4-1/6} \prod_{k=0}^{\infty} \frac{(1-q^{rk+1})(1-q^{r(k+1)})^2(1-q^{r(k+1)-1})}{(1-xq^{r(k+\frac{1}{2})+\frac{1}{2}})(1-xq^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})} \\
&\sim \frac{\eta(r\tau)^2}{\eta(\tau)^2} \prod_{k=0}^{\infty} \frac{(1-q^{rk+1})(1-q^{r(k+1)-1})}{(1-xq^{r(k+\frac{1}{2})+\frac{1}{2}})(1-xq^{r(k+\frac{1}{2})-\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})+\frac{1}{2}})(1-x^{-1}q^{r(k+\frac{1}{2})-\frac{1}{2}})}.
\end{aligned}
\tag{2.5.12}
$$

This matches with (2.5.11), completing the proof.

$\square$

Theorem 2.5.1 shows that the character of B_r coincides with that of a quantum Hamiltonian reduction, as suggested in [C2, Remark 5.6].

Chapter 3

Categories of Weight Modules for Unrolled Restricted Quantum Groups at Roots of Unity

Motivated by connections to the singlet vertex operator algebra in the $\mathfrak{g} = \mathfrak{sl}_2$ case, we study the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{g})$ at arbitrary roots of unity with a focus on its category of weight modules. We show that the braid group action on the Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$ naturally extends to the unrolled quantum groups and that the category of weight modules is a generically semi-simple ribbon category (previously known only for odd roots) with trivial Müger center and self-dual projective modules.

3.1 Preliminaries

Generically semisimple categories

Let $\Bbbk$ be a field. A $\Bbbk$-category is a category $\mathcal{C}$ such that its Hom-sets are left $\Bbbk$-modules, and morphism composition is $\Bbbk$-bilinear. A pivotal $\Bbbk$-linear category $\mathcal{C}$ is said to be $\mathcal{G}$-graded for some group $\mathcal{G}$ if for each $g \in \mathcal{G}$ we have a non-empty full subcategory $\mathcal{C}_g$ stable under retract such that such that

- $\mathcal{C} = \bigoplus_{g \in \mathcal{G}} \mathcal{C}_g$,

- $V \in \mathcal{C}_g \implies V^* \in \mathcal{C}_{g^{-1}}$,

- $V \in \mathcal{C}_g$ and $W \in \mathcal{C}_{g'} \implies V \otimes W \in \mathcal{C}_{gg'}$,

- $V \in \mathcal{C}_g, W \in \mathcal{C}_{g'}$ and $\mathrm{Hom}(V, W) \neq 0 \implies g = g'$.

A subset $\mathcal{X} \subset \mathcal{G}$ is called symmetric if $\mathcal{X}^{-1} = \mathcal{X}$ and small if it cannot cover $\mathcal{G}$ by finite translations, i.e. for any $n \in \mathbb{N}$ and $\forall g_1, ..., g_n \in \mathcal{G}$, $\bigcup_{i=1}^{n} g_i\mathcal{X} \neq \mathcal{G}$.

Definition 3.1.1. A $\mathcal{G}$-graded category $\mathcal{C}$ is called generically $\mathcal{G}$-semisimple if there exists a small symmetric subset $\mathcal{X} \subset \mathcal{G}$ such that for all $g \in \mathcal{G} \setminus \mathcal{X}$, $\mathcal{C}_g$ is semisimple. $\mathcal{X}$ is referred to as the singular locus of $\mathcal{C}$ and simple objects in $\mathcal{C}_g$ with $g \in \mathcal{G} \setminus \mathcal{X}$ are called generic.

Generically semisimple categories appeared in [GP1, CGP1] and were used in [GP2] to prove that representation categories of unrolled quantum groups at odd roots of unity are ribbon. If $\mathcal{C}$ has braiding $c_{-,-}$, then an object $Y \in \mathcal{C}$ is said to be transparent if $c_{Y,X} \circ c_{X,Y} = \mathrm{Id}_{X \otimes Y}$ for all $X \in \mathcal{C}$.

Definition 3.1.2. The Müger center of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ consisting of all transparent objects.

Triviality of the Müger center should be viewed as a non-degeneracy condition. Indeed, for finite braided tensor categories triviality of the Müger center is equivalent to the usual notions of non-degeneracy (see [S, Theorem 1.1]).

3.2 The Unrolled Restricted Quantum Group $\overline{U}_q^H(\mathfrak{g})$

We first fix our notations. Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra of rank n and dimension $n + 2N$ with Cartan martix $A = (a_{ij})_{1 \leq i,j \leq n}$ and Cartan subalgebra $\mathfrak{h}$. Let $\Delta := \{\alpha_1, ..., \alpha_n\} \subset \mathfrak{h}^*$ be the set of simple roots of $\mathfrak{g}$, Δ^+ (Δ^-) the set of positive (negative) roots, and $Q := \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$ the integer root lattice. Let $\{H_1, ..., H_n\}$ be the basis of

$\mathfrak{h}$ such that $\alpha_j(H_i) = a_{ij}$ and $\langle , \rangle$ the form defined by $\langle \alpha_i, \alpha_j \rangle = d_i a_{ij}$ where $d_i = \langle \alpha_i, \alpha_i \rangle / 2$ and normalized such that short roots have length 2. Let $P := \bigoplus_{i=1}^{n} \mathbb{Z}\omega_i$ be the weight lattice generated by the dual basis $\{\omega_1, ..., \omega_n\} \subset \mathfrak{h}^*$ of $\{d_1 H_1, ..., d_n H_n\} \subset \mathfrak{h}$, and $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \in P$ the Weyl vector.

Now, let $\ell \geq 3$ and $r = \ell$ when ℓ is odd, $r = \frac{1}{2}\ell$ when ℓ is even (i.e. set $r = 2\ell/(3 + (-1)^\ell)$). Let $q \in \mathbb{C}$ be an primitive ℓ-th root of unity, $q_i = q^{d_i}$, and fix the notation

$$\{x\} = q^x - q^{-x}, \quad [x] = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad [n]! = [n][n-1]...[1], \quad \binom{n}{m} = \frac{\{n\}!}{\{m\}!\{n-m\}!}, \tag{3.2.1}$$

$$[j;q] = \frac{1-q^j}{1-q}, \qquad [j;q]! = [j;q][j-1;q]\cdots[1;q]. \tag{3.2.2}$$

We will often use a subscript i, e.g. $[x]_i$, to denote the substitution $q \mapsto q_i$ in the above formulas.

Definition 3.2.1. Let L be a lattice such that $Q \subset L \subset P$. The unrolled quantum group $U_q^H(\mathfrak{g})$ associated to L is the $\mathbb{C}$-algebra with generators $K_\gamma, X_i, X_{-i}, H_i$, (we will often let $K_{\alpha_i} := K_i$) with $i = 1, ..., n$, $\gamma \in L$, and relations

$$K_0 = 1, \quad K_{\gamma_1} K_{\gamma_2} = K_{\gamma_1 + \gamma_2}, \quad K_\gamma X_{\pm j} K_{-\gamma} = q^{\pm \langle \gamma, \alpha_j \rangle} X_{\sigma j}, \tag{3.2.3}$$

$$[H_i, H_j] = 0, \quad [H_i, K_\gamma] = 0, \qquad [H_i, X_{\pm j}] = \pm a_{ij} X_{\pm j}, \tag{3.2.4}$$

$$[X_i, X_{-j}] = \delta_{i,j} \frac{K_{\alpha_j} - K_{\alpha_j}^{-1}}{q_j - q_j^{-1}}, \tag{3.2.5}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} X_{\pm i}^k X_{\pm j} X_{\pm i}^{1-a_{ij}-k} = 0 \qquad \text{if } i \neq j, \tag{3.2.6}$$

There is a Hopf-algebra structure on $U_q^H(\mathfrak{g})$ with coproduct Δ, counit ϵ, and antipode S defined by

$$\Delta(K_\gamma) = K_\gamma \otimes K_\gamma, \qquad \epsilon(K_\gamma) = 1, \qquad S(K_\gamma) = K_{-\gamma} \tag{3.2.7}$$

$$\Delta(X_i) = 1 \otimes X_i + X_i \otimes K_{\alpha_i} \qquad \epsilon(X_i) = 0, \qquad S(X_i) = -X_i K_{-\alpha_i}, \tag{3.2.8}$$

$$\Delta(X_{-i}) = K_{-\alpha_i} \otimes X_{-i} + X_{-i} \otimes 1, \qquad \epsilon(X_{-i}) = 0, \qquad S(X_{-i}) = -K_{\alpha_i} X_{-i}., \tag{3.2.9}$$

$$\Delta(H_i) = 1 \otimes H_i + H_i \otimes 1, \qquad \epsilon(H_i) = 0, \qquad S(H_i) = -H_i. \tag{3.2.10}$$

It is easy to see that the subalgebra generated by K_γ and $X_{\pm i}$ is the usual Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$. The unrolled quantum group is actually a smash product of the Drinfeld-Jimbo algebra with the universal enveloping algebra of $\mathfrak{h}$, which we will briefly recall. Let the generators $H_1, ..., H_n \in \mathfrak{h}$ act on $U_q(\mathfrak{g})$ by the derivation $\partial_{H_i} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ defined by

$$\partial_{H_i} X_{\pm j} = \pm a_{ij} X_{\pm j}, \qquad \partial_{H_i} K_\gamma = 0. \tag{3.2.11}$$

It is easy to see that these operators commute, so they do indeed define an action of $\mathfrak{h}$ on $U_q(\mathfrak{g})$. It is easy to check the relations

$$(\Delta \circ \partial_{H_i})(X_j) = (\mathrm{Id} \otimes \partial_{H_i} + \partial_{H_i} \otimes \mathrm{Id}) \circ \Delta(X_j)$$

$$(\Delta \circ \partial_{H_i})(X_{-j}) = (\mathrm{Id} \otimes \partial_{H_i} + \partial_{H_i} \otimes \mathrm{Id}) \circ \Delta(X_{-j})$$

and $\Delta \circ \partial_{H_i}(K_\gamma) = 0 = (\partial_{H_i} \otimes 1 + 1 \otimes \partial_{H_i}) \circ \Delta(K_\gamma)$, so $\mathfrak{h}$ acts on $U_q(\mathfrak{g})$ by $\mathbb{C}$-biderivations. It follows from [AS, Lemma 2.6] that $U_q(\mathfrak{g}) \rtimes U(\mathfrak{h}) := U_q(\mathfrak{g}) \otimes U(\mathfrak{h})$ is a Hopf algebra with algebra structure coming from the smash product (the unit here is $1 \otimes 1$) and coalgebra structure coming from the tensor product:

$$\begin{aligned}(X \otimes H) \cdot (Y \otimes H') &= X(\partial_{H_{(1)}} Y) \otimes H_{(2)} H' \\ \Delta(X \otimes H) &= (\mathrm{Id} \otimes \tau \otimes \mathrm{Id}) \circ (\Delta_{U_q(\mathfrak{g})} \otimes \Delta_{U(\mathfrak{h})})(X \otimes H) \\ \epsilon &= \epsilon_{U_q(\mathfrak{g})} \otimes \epsilon_{U(\mathfrak{h})}\end{aligned}$$

where $\tau : U_q(\mathfrak{g}) \otimes U(\mathfrak{h}) \to U(\mathfrak{h}) \otimes U_q(\mathfrak{g})$ is the usual flip map and $\Delta(H) = \sum_{(H)} H_{(1)} \otimes H_{(2)}$ is the Sweedler notation for the coproduct. We then see that for any $X \in U_q(\mathfrak{g})$ and $H_i \in U(\mathfrak{h})$, we have

$$\begin{aligned}(X \otimes 1) \cdot (1 \otimes H_i) &= X \otimes H_i & \qquad (X \otimes 1) \cdot (Y \otimes 1) &= XY \otimes 1 \\ (1 \otimes H_i) \cdot (X \otimes 1) &= \partial_{H_i}(X) \otimes H_i & \qquad (1 \otimes H_i) \cdot (1 \otimes H_j) &= 1 \otimes H_i H_j,\end{aligned}$$

so $U_q(\mathfrak{g}) \rtimes U(\mathfrak{h})$ is generated by the elements $\{X_{\pm i} \otimes 1, K_\gamma \otimes 1, 1 \otimes H_i \,|\, i = 1, ..., n, \gamma \in L\}$. By abuse of notation, we set $X_{\pm i} := (X_{\pm_i} \otimes 1), K_\gamma := (K_\gamma \otimes 1)$, and $H_i := (1 \otimes H_i)$, and we see that $U_q(\mathfrak{g}) \rtimes U(\mathfrak{h})$ is generated by $X_{\pm i}, K_\gamma$ $(\gamma \in L)$, and H_i, $i = 1, ..., n$ with defining relations (3.2.3)-(3.2.6) and Hopf algebra structure given by equations (3.2.7)-(3.2.10). This is precisely the unrolled quantum group of Definition 3.2.1.

There exists an automorphism of $U_q(\mathfrak{g})$ which swaps X_i with X_{-i} and inverts K_i (see [J, Lemma 4.6]). It is easily checked that this automorphism can be extended to $U_q^H(\mathfrak{g})$ by defining $\omega(H_i) = -H_i$, so there exists an automorphism $\omega : U_q^H(\mathfrak{g}) \to U_q^H(\mathfrak{g})$ defined by

$$\omega(X_{\pm i}) = X_{\mp i}, \qquad \omega(K_\gamma) = K_{-\gamma}, \qquad \omega(H_i) = -H_i. \tag{3.2.12}$$

This automorphism will appear in Section 3.3 in the definition of a Hermitian form on Verma modules introduced in [DCK], and the definition of a contravariant functor analogous to the duality functor for Lie algebras.

3.2.1 Braid Group Action on $U_q^H(\mathfrak{g})$

Recall that for a finite dimensional Lie algebra the scalars $a_{ij}a_{ji}$ are equal to $0, 1, 2,$ or 3 for $i \neq j$ and for each case let m_{ij} be $2, 3, 4, 6$ respectively. Then,

Definition 3.2.2. The braid group $\mathcal{B}_\mathfrak{g}$ associated to $\mathfrak{g}$ has generators T_i with $1 \leq i \leq n$ and defining relations

$$T_iT_jT_i \cdots = T_jT_iT_j \cdots$$

for $i \neq j$ where each side of the equation is a product of m_{ij} generators.

It is well known (see [KlS, CP]) that the braid group of $\mathfrak{g}$ acts on the quantum group $U_q(\mathfrak{g})$ by algebra automorphisms defined as follows:

$$T_i(K_j) = K_jK_i^{-a_{ij}}, \qquad T_i(X_i) = -X_{-i}K_i, \qquad T_i(X_{-i}) = -K_i^{-1}X_i \tag{3.2.13}$$

$$T_i(X_j) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{-t} X_i^{(-a_{ij}-t)} X_j X_i^{(t)} \quad i \neq j, \tag{3.2.14}$$

$$T_i(X_{-j}) = \sum_{t=0}^{-a_{ij}} (-1)^{t-a_{ij}} q_i^{t} X_{-i}^{(t)} X_{-j} X_i^{(-a_{ij}-t)}, \quad i \neq j, \tag{3.2.15}$$

where $X_{\pm i}^{(n)} = X_{\pm i}^n/[n]_{q_i}!$. Therefore, if we extend the action of $\mathcal{B}_{\mathfrak{g}}$ to $U_q^H(\mathfrak{g})$ by

$$T_i(H_j) = H_j - a_{ji}H_i,$$

it is enough to check that the automorphisms T_i, $i = 1, ..., n$ respect equations (3.2.4) and the braid group relations when acting on the H_i. We first note that the relations

$$[T_k(H_i), T_k(H_j)] = [T_k(H_i), T_k(K_\gamma)] = 0$$

follow trivially from $[H_i, H_j] = [H_i, K_\gamma] = 0$. We must therefore show that

$$[T_k(H_i), T_k(X_{\pm j})] = \pm a_{ij}T_k(X_{\pm j}).$$

We will prove the statement for positive index, as the negative index case is identical. Suppose $k = j$, then $T_j(X_j) = -X_{-j}K_j$ and we have

$$H_n(-X_{-j}K_j) = -(X_{-j}H_n - a_{nj}X_{-j})K_j = -X_{-j}K_jH_n + a_{nj}X_{-j}K_j,$$

so we see that $[H_n, -X_{-j}K_j] = a_{nj}X_{-j}K_j$. Hence,

$$\begin{aligned}
[T_j(H_i), T_j(X_j)] &= [H_i - a_{ij}H_j, -X_{-j}K_j] \\
&= [H_i, -X_{-j}K_j] - a_{ij}[H_j, X_{-j}K_j] \\
&= a_{ij}X_{-j}K_j - 2a_{ij}X_{-j}K_j \\
&= -a_{ij}X_{-j}K_j = a_{ij}T_j(X_j).
\end{aligned}$$

Suppose now that $k \neq j$. We then see that we must show

$$\sum_{t=0}^{-a_{kj}} (-1)^{t-a_{kj}} q_k^{-t} [T_k(H_i), X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] = \sum_{t=0}^{-a_{kj}} (-1)^{t-a_{kj}} q_k^{-t} a_{ij} X_k^{(-a_{kj}-t)} X_j X_k^{(t)}.$$

Clearly then, it is enough to show that

$$[T_k(H_i), X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] = a_{ij} X_k^{(-a_{kj}-t)} X_j X_k^{(t)}$$

for each $t = 0, ..., -a_{kj}$. It follows easily from equation (3.2.4) that

$$[H_n, X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] = (-a_{kj}a_{nk} + a_{nj}) X_k^{(-a_{kj}-t)} X_j X_k^{(t)}.$$

Therefore,

$$\begin{aligned}
[T_k(H_i), X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] &= [H_i, X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] - a_{ik}[H_k, X_k^{(-a_{kj}-t)} X_j X_k^{(t)}] \\
&= (-a_{kj}a_{ik} + a_{ij} - a_{ik}(-2a_{kj} + a_{kj})) X_k^{(-a_{kj}-t)} X_j X_k^{(t)} \\
&= a_{ij} X_k^{(-a_{kj}-t)} X_j X_k^{(t)}.
\end{aligned}$$

We see then that equations (3.2.13)-(3.2.16) define an automorphism of $U_q^H(\mathfrak{g})$. To show that these automorphisms give an action of the braid group, we need only show that they satisfy the braid relations as operators on $U_q^H(\mathfrak{g})$. We know these relations are satisfied for the elements of $U_q(\mathfrak{g})$, so we need only check the H_i. This amounts to showing that

$$T_i T_j \cdots (H_k) = T_j T_i \cdots (H_k).$$

One therefore computes $T_i T_j \cdots (H_k)$ and checks that the result is symmetric in i and j, giving the following proposition:

Proposition 3.2.3. The elements T_i of the braid group $\mathcal{B}_{\mathfrak{g}}$ act on $U_q^H(\mathfrak{g})$ by automorphisms given by relations (3.2.13)-(3.2.15) and

$$T_i(H_j) = H_j - a_{ji} H_i. \tag{3.2.16}$$

Let W denote the Weyl group of $\mathfrak{g}$ and $\{s_i | i = 1, ..., n\}$ the simple reflections generating W. Let $s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of the longest element ω_0 of W. Then, $\beta_k := s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$, $k = 1, ..., N$ gives a total ordering on the set of positive roots Δ^+ of $\mathfrak{g}$ and for each β_i, $i = 1, ..., N$, we can associate the root vectors $X_{\pm\beta_i} \in U_q(\mathfrak{g})$ as seen in [CP, Subsections 8.1 and 9.1]. We have the following PBW theorem [CP]:

Theorem 3.2.4. The multiplication operation in $U_q(\mathfrak{g})$ defines a vector space isomorphism

$$U_q(\eta^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\eta^-) \cong U_q(\mathfrak{g})$$

where $U_q(\eta^{\pm})$ is the subalgebra generated by the $X_{\pm\alpha_i}$ and $U_q(\mathfrak{h})$ the subalgebra generated by the K_γ. The set $\{X_{\pm\beta_1}^{k_1} X_{\pm\beta_2}^{k_2} \cdots X_{\pm\beta_N}^{k_N} | k_i \in \mathbb{Z}_{\geq 0}\}$ is a basis of $U_q(\eta^{\pm})$.

It can be shown by induction on s [J] that

$$[X_i, X_{-i}^s] = [s]_i X_{-i}^{s-1}[K_i; d_i(1-s)], \tag{3.2.17}$$

where $[K_i; n] = (K_i q^n - K_i^{-1} q^{-n})/(q_i - q_i^{-1})$. Let $d_\alpha = \frac{1}{2}\langle \alpha, \alpha \rangle$, and define

$$r_\alpha := r/gcd(d_\alpha, r). \tag{3.2.18}$$

Then $[r_{\alpha_i}]_i = [r] = 0$, so it follows from equations (3.2.5) and (3.2.17) that $[X_j, X_{-i}^{r_{\alpha_i}}] = 0$ for all i, j. Applying the braid group action then gives $[X_\beta, X_{-\alpha}^{r_\alpha}] = 0$ for all $\alpha, \beta \in \Delta^+$, where we have used the fact that $d_\alpha = d_i$ when α lies in the Weyl orbit of α_i. It follows that given any maximal vector v (i.e. $X_i v = 0$ for all i) in some $\overline{U}_q^H(\mathfrak{g})$-module V, $X_{-\alpha}^{r_\alpha} v$ is also maximal. In particular, all Verma modules of $\overline{U}_q^H(\mathfrak{g})$ will be reducible and therefore there will be no projective irreducible modules (as all irreducibles are quotients of Vermas). Obtaining a category of representations which is generically semisimple is the motivation for our choice of definition of the unrolled restricted quantum group at arbitrary roots and to do this, we quotient out $\{X_{\pm\alpha}^{r_\alpha}\}_{\alpha \in \Delta^+}$. It follows from equation (3.2.3) that $X_i \otimes K_i X_i = q^{2d_i} X_i \otimes X_i K_i$, so equation (3.2.8) and the q-binomial formula tell us that

$$\Delta(X_i^{r_{\alpha_i}}) = \sum_{k=0}^{r_{\alpha_i}} \binom{r_{\alpha_i}}{k}_{q^{2d_i}} X_i^n \otimes K_i^n X_i^{r_{\alpha_i}-n} = 1 \otimes X_i^{r_{\alpha_i}} + X_i^{r_{\alpha_i}} \otimes K_i^{r_{\alpha_i}}$$

since $\binom{r_{\alpha_i}}{k} = 1$ if $k = 0, r_{\alpha_i}$ and zero otherwise. We can perform the same computation for $X_{-i}^{r_{\alpha_i}}$, so we see that the two-sided ideal generated by $\{X_{\pm\alpha_i}^{r_{\alpha_i}}\}_{\alpha_i \in \Delta}$ is a Hopf ideal (it follows immediately from equations (3.2.3) and (3.2.8) that this ideal is invariant under the antipode S).

Definition 3.2.5. The unrolled restricted quantum group of $\mathfrak{g}$, $\overline{U}_q^H(\mathfrak{g})$, is defined to be the unrolled quantum group $U_q^H(\mathfrak{g})$ of Definition 3.2.1 quotiented by the Hopf ideal generated by $\{X_{\pm\alpha_i}^{r_{\alpha_i}}\}_{\alpha_i \in \Delta}$.

This definition is very closely related to that of the small quantum group in [L]. It follows trivially from the braid relations that $X_{\pm\alpha}^{r_\alpha} = 0$ in $\overline{U}_q^H(\mathfrak{g})$ for every root vector $\alpha \in \Delta^+$, and it follows from the PBW theorem that $\{X_{\pm\beta_1}^{k_1} X_{\pm\beta_2}^{k_2} \cdots X_{\pm\beta_N}^{k_N} \mid 0 \leq k_i < r_{\beta_i}\}$ is a basis of $\overline{U}_q^H(\eta^\pm)$.

3.3 Representation Theory of $\overline{U}_q^H(\mathfrak{g})$

For each module V of $\overline{U}_q^H(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$, define the set $V(\lambda) := \{v \in V \mid H_i v = \lambda(H_i)v\}$. If $V(\lambda) \neq 0$, then we call λ a weight of V, $V(\lambda)$ its weight space, and any $v \in V(\lambda)$ a weight vector of weight λ.

Definition 3.3.1. A $\overline{U}_q^H(\mathfrak{g})$-module V is called a weight module if V splits as a direct sum of weight spaces and for each $\gamma = \sum_{i=1}^{n} k_i\alpha_i \in L$, $K_\gamma = \prod_{i=1}^{n} q_i^{k_i H_i}$ as operators on V. We define $\mathcal{C}$ to be the category of finite dimensional weight modules for $\overline{U}_q^H(\mathfrak{g})$.

Given any $V \in \mathcal{C}$, we denote by $\Gamma(V)$ the set of weights of V. That is,

$$\Gamma(V) = \{\lambda \in \mathfrak{h}^* \mid V(\lambda) \neq 0\}. \tag{3.3.1}$$

We define the character of a module $V \in \mathcal{C}$ using the dimensions of the H-eigenspaces as

$$\mathrm{ch}[V] = \sum_{\lambda \in \mathfrak{h}^*} \dim V(\lambda) z^\lambda. \tag{3.3.2}$$

It is easy to show that for any module V and $\lambda \in \mathfrak{h}^*$,

$$X_{\pm j} V(\lambda) \subset V(\lambda \pm \alpha_j). \tag{3.3.3}$$

We define the usual partial order "$\geq$" on $\mathfrak{h}^*$ by $\lambda_1 \geq \lambda_2$ iff $\lambda_1 = \lambda_2 + \sum_{i=1}^{n} k_i\alpha_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$. A weight λ of V is said to be highest weight if it is maximal with respect to the partial order among the weights of M. A vector $v \in V$ is called maximal if $X_i v = 0$ for each i, and a module generated by a maximal vector will be called highest weight.

Given a weight $\lambda \in \mathfrak{h}^*$, denote by I^λ the ideal of $\overline{U}_q^H(\mathfrak{g})$ generated by the relations $H_i \cdot 1 = \lambda(H_i)$, $K_\gamma \cdot 1 = \prod_{i=1}^{n} q_i^{k_i\lambda(H_i)}$ for $\gamma = \sum_{i=1}^{n} k_i\alpha_i \in L$ and $X_i \cdot 1 = 0$ for each i.

Definition 3.3.2. Define $M^\lambda := \overline{U}_q^H(\mathfrak{g})/I^\lambda$. M^λ is generated as a module by the coset

$v_\lambda := 1 + I^\lambda$ with relations

$$X_i v_\lambda = 0, \qquad H_i v_\lambda = \lambda(H_i) v_\lambda, \qquad K_\gamma v_\lambda = \prod_{i=1}^{n} q_i^{k_i \lambda(H_i)} v_\lambda,$$

where $\gamma = \sum k_i \alpha_i \in L$. It follows from Theorem 3.2.4 that M^λ has basis $\{X_{\beta_1}^{k_1} X_{\beta_2}^{k_2} \cdots X_{\beta_N}^{k_N} v_\lambda \mid 0 \leq k_i < r_{\beta_i}\}$.

Clearly, $M^\lambda \in \mathcal{C}$ and is universal with respect to highest weight modules in $\mathcal{C}$, that is, for any module $M \in \mathcal{C}$ generated by a highest weight vector of weight λ, there exists a surjection $M^\lambda \twoheadrightarrow M$. Each proper submodule N of M^λ is a direct sum of its weight spaces and has $N(\lambda) = \emptyset$, so the union of all proper submodules is a maximal proper submodule. Hence, each reducible M^λ has a unique maximal proper submodule N^λ and unique irreducible quotient S^λ of highest weight λ. We therefore refer to M^λ as the Verma (or universal highest weight) module of highest weight λ and we have the following proposition by standard arguments:

Proposition 3.3.3. *$V \in \mathcal{C}$ is irreducible iff $V \cong S^\lambda$ for some $\lambda \in \mathfrak{h}^*$.*

It is clear that every module in $\mathcal{C}$ is a module over $U_q(\mathfrak{g})$ and since the H_i act semi-simply, M^λ is irreducible iff it is irreducible as a $U_q(\mathfrak{g})$-module. Kac and De Concini defined a Hermitian form H on M^λ [DCK, Equation 1.9.2] by

$$H(v_\lambda, v_\lambda) = 1 \quad \text{and} \quad H(Xu, v) = H(u, \omega(X)v)$$

for all $X \in \overline{U}_q^H(\mathfrak{g})$ and $u, v \in M^\lambda$ where ω is the automorphism defined in equation (3.2.12). Let $\eta \in \Delta^+$ and denote by $\det_\eta(\lambda)$ the determinant of the Gram matrix of H restricted to $M^\lambda(\lambda - \eta)$ in the basis consisting of elements $F_{\beta_1}^{k_1} \cdots F_{\beta_N}^{k_N} v_\lambda$ with $\vec{k} = (k_1, ..., k_N) \in \mathrm{Par}(\eta) := \{\vec{k} \in \mathbb{Z}^N \mid \sum k_i \beta_i = \eta,\ 0 \leq k_i < r_{\beta_i}\}$. The determinant of H vanishes precisely on the maximal submodule of M^λ and is given on $M^\lambda(\lambda - \eta)$ by [DCK, Equation 1.9.3]

$$\det_\eta(\lambda) = \prod_{\alpha \in \Delta^+} \prod_{m=0}^{r_\alpha - 1} \left(\frac{\{m d_\alpha\}}{\{d_\alpha\}^2} \right)^{|\mathrm{Par}(\eta - m\alpha)|} \left(\lambda(K_\alpha) q^{\langle \rho, \alpha \rangle - \frac{m}{2}\langle \alpha, \alpha \rangle} - \lambda(K_\alpha^{-1}) q^{-\langle \rho, \alpha \rangle + \frac{m}{2}\langle \alpha, \alpha \rangle}\right)^{|\mathrm{Par}(\eta - m\alpha)|}$$

where ρ is the Weyl vector. It then follows as in [DCK, Theorem 3.2] that we have the following:

Proposition 3.3.4. M^λ is irreducible iff $q^{2\langle\lambda+\rho,\alpha\rangle-k\langle\alpha,\alpha\rangle}\neq 1$ for all $\alpha\in\Delta^+$ and $k=1,...,r_\alpha-1$.

For each $\alpha\in\Delta^+$ we associate to λ the scalars $\lambda_\alpha\in\mathbb{C}$ defined by

$$\lambda_\alpha := \langle\lambda+\rho,\alpha\rangle.$$

Notice that M^λ is reducible iff for some $\alpha\in\Delta^+$ we have

$$2(\lambda_\alpha - k_\alpha^\lambda d_\alpha) = n_\alpha^\lambda \ell \tag{3.3.4}$$

for some $k_\alpha^\lambda\in\{1,...,r_\alpha-1\}$, $n_\alpha^\lambda\in\mathbb{Z}$, where $d_\alpha := \frac{1}{2}\langle\alpha,\alpha\rangle$. This motivates the following definition:

Definition 3.3.5. We call the scalar λ_α typical if $2(\lambda_\alpha - kd_\alpha)\neq 0 \mod \ell$ for all $k=1,...,r_\alpha-1$ and atypical otherwise. We call $\lambda\in\mathfrak{h}^*$ typical if λ_α is typical for all $\alpha\in\Delta^+$ and atypical otherwise.

Clearly, M^λ is irreducible iff λ is typical. We can rewrite the atypicality condition into a more convenient form, which will be useful in the next subsection:

Proposition 3.3.6. λ_α is typical iff $\lambda_\alpha\in\ddot{\mathbb{C}}_\alpha$ where

$$\ddot{\mathbb{C}}_\alpha := \begin{cases} (\mathbb{C}\setminus g_\alpha\mathbb{Z})\cup r\mathbb{Z} & \text{if } \ell \text{ is even} \\ (\mathbb{C}\setminus \frac{g_\alpha}{2}\mathbb{Z})\cup \frac{r}{2}\mathbb{Z} & \text{if } \ell \text{ is odd} \end{cases}$$

where $g_\alpha = gcd(d_\alpha, r)$.

Proof. By Proposition 3.3.4 and the following comments, λ_α is atypical iff $2(\lambda_\alpha - kd_\alpha) = 0 \mod \ell$ for some $k=1,...,r_\alpha-1$. That is, iff

$$\lambda_\alpha \in \bigcup_{n\in\mathbb{Z}}\bigcup_{k=1}^{r_\alpha-1}\frac{n\ell+2kd_\alpha}{2}$$

Assume now that $gcd(d_i, r) = 1$ for all i. Note that each non-simple root α lies in the Weyl orbit of some simple root α_i and that $d_\alpha = d_{\alpha_i}$, so $gcd(d_\alpha, r) = 1$ for all $\alpha \in \Delta^+$ (so $r_\alpha = r$ for all α). Let $r = \ell$ be odd, then we have $\lambda_\alpha \in \bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^{r-1} \frac{nr+2kd_\alpha}{2}$ which is clearly a subset of $\frac{1}{2}\mathbb{Z}$. Suppose $nr + 2kd_\alpha = rm$ for some $m \in \mathbb{Z}$. Then $r(m-n) = 2kd_\alpha$ and we have $gcd(r, 2d_\alpha) = 1$ so we must have $2d_\alpha | m-n$. However, we have $k \in \{1, ..., r-1\}$ so $|r(m-n)| > |2kd_\alpha|$, a contradiction. Hence, $\lambda_\alpha \in \frac{1}{2}\mathbb{Z} \setminus \frac{r}{2}\mathbb{Z}$. Let $x \in \mathbb{Z} \setminus r\mathbb{Z}$. Since $gcd(r, 2d_\alpha) = 1$, there exist $a, b \in \mathbb{Z}$ such that $2d_\alpha a + br = 1$. Since $x \notin r\mathbb{Z}$, we have $ax \notin r\mathbb{Z}$, otherwise $x = 2d_\alpha ax + brx \in r\mathbb{Z}$. Therefore, there exist $m, k \in \mathbb{Z}$ with $k = 1, ..., r-1$ such that $ax = mr + k$. Then, $2d_\alpha ax = 2d_\alpha mr + 2d_\alpha k$, so $x = 2d_\alpha k \bmod \mathrm{r}$ (since $2d_\alpha a = 1 \bmod \mathrm{r}$) and so $x \in \bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^{r-1} rn + 2d_\alpha k$. Hence, we have shown

$$\bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^{r-1} \frac{nr + 2d_\alpha k}{2} = \frac{1}{2}\mathbb{Z} \setminus \frac{r}{2}\mathbb{Z}$$

A similar argument shows that $\bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^{r-1} \frac{n\ell+2kd_\alpha}{2} = \mathbb{Z} \setminus r\mathbb{Z}$ when ℓ is even. Suppose now that $gcd(d_\alpha, r) \neq 0$. Then we have

$$\bigcup_{n\in\mathbb{Z}} \bigcup_{k=1}^{r_\alpha-1} \frac{(nr_\alpha + 2k)d_\alpha}{2} = \begin{cases} d_\alpha\mathbb{Z} \setminus r\mathbb{Z} & \ell \text{ even} \\ \frac{d_\alpha}{2}\mathbb{Z} \setminus \frac{r}{2}\mathbb{Z} & \ell \text{ odd} \end{cases}$$

□

Remark 3.3.7. The invertible objects in $\mathcal{C}$ are clearly the 1-dimensional S^λ. Note that we have

$$X_j X_{-i} v_\lambda = \delta_{i,j}[\lambda(H_i)]_i v_\lambda$$

so S^λ is 1-dimensional iff $2\lambda(H_i)d_i = 0 \bmod \ell$ i.e. $\lambda(H_i) \in \frac{\ell}{2d_i}\mathbb{Z}$ for all i. These objects played a crucial role in [CGR] for the construction of certain quasi-Hopf algebras $\overline{U}_q^\Phi(\mathfrak{sl}_2)$ whose representation theory related to the triplet VOA. We expect this to remain true in the higher rank case, and will be investigated in future work. We also expect the higher rank analogues of $\overline{U}_q^\Phi(\mathfrak{sl}_2)$ to be closely related to those quantum groups which appear in [N, GLO].

3.3.1 Categorical Structure

We first remark that it is easily seen (as in [GP1, Subsection 5.6]), that the square of the antipode acts as conjugation by $K_{2\rho}^{1-r}$, i.e.

$$S^2(x) = K_{2\rho}^{1-r} x K_{2\rho}^{r-1}$$

for $x \in \overline{U}_q^H(\mathfrak{g})$ where $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$ is the Weyl vector. A Hopf algebra in which the square of the antipode acts as conjugation by a group-like element is pivotal (see [B, Proposition 2.9]), so $\mathcal{C}$ is pivotal. It is clear from equation (3.3.3) that given any $v \in V(\lambda)$, every weight for the submodule $\langle v \rangle \subset V$ generated by v has the form $\lambda + \sum_{i=1}^{n} k_i \alpha_i$ for some $k_i \in \mathbb{Z}$. That is, every weight vector in $\langle v \rangle$ has weight differing from λ by an element of the root lattice Q of $\mathfrak{g}$. We can quotient $\mathfrak{h}^*$ by Q to obtain the group $\mathfrak{t} := \mathfrak{h}^*/Q$. We then define $\mathcal{C}_{\overline{\lambda}}$ for $\overline{\lambda} \in \mathfrak{t}$ to be the full subcategory of $\mathcal{C}$ consisting of modules whose weights differ from λ by an element of Q. Clearly,

$$\mathcal{C} = \bigoplus_{\overline{\lambda} \in \mathfrak{t}} \mathcal{C}_{\overline{\lambda}} \tag{3.3.5}$$

and it is easy to see that this gives $\mathcal{C}$ a $\mathfrak{t}$-grading as in Section 3.1. In fact, we have the following:

Proposition 3.3.8. $\mathcal{C}$ *is generically* $\mathfrak{t}$*-semisimple.*

Proof. Let $\mathcal{X}$ be the subset of $\mathfrak{t}$ consisting of equivalence classes $\overline{\lambda}$ corresponding to weights λ such that $\lambda_\alpha \in \frac{(3+(-1)^\ell)g_\alpha}{4}\mathbb{Z}$ for some $\alpha \in \Delta^+$. Notice that this implies that $\mu_\alpha \in \frac{(3+(-1)^\ell)g_\alpha}{4}\mathbb{Z}$ for all $\mu \in \overline{\lambda}$ since μ being comparable to λ implies that

$$\mu_\alpha = \lambda_\alpha \text{ mod } d_\alpha \mathbb{Z}. \tag{3.3.6}$$

Equation (3.3.6) is easy to see for simple roots and for non-simple roots, one uses invariance of $\langle -, - \rangle$ under the action of the Weyl group. $\mathcal{X}$ is clearly symmetric. To see χ is small, consider the subset

$$A := \{\overline{\mu^a} \in \mathfrak{t} \mid (\mu^a)_\alpha = ai \text{ for some } a \neq 0 \in \mathbb{R} \text{ and all } \alpha \in \Delta^+\} \subset \mathfrak{t}.$$

Each $\overline{\mu^a}$ is distinct since the corresponding weights do not differ by elements of the root lattice. Suppose that $\overline{\mu^a} + \mathcal{X} = \overline{\mu^b} + \mathcal{X}$ for some $a \neq b$, both non-zero. Then $\overline{\mu^a} - \overline{\mu^b} \in \mathcal{X}$, a contradiction since any element $\mu \in \overline{\mu^a} - \overline{\mu^b}$ will have purely imaginary μ_α for each α, so μ cannot belong to $\mathcal{X}$. Therefore, A cannot be covered by finitely many translations of $\mathcal{X}$ and $\mathcal{X}$ is small. Notice that by construction, $\mathfrak{t} \setminus \mathcal{X}$ consists of equivalence classes $\overline{\lambda}$ such that every weight of every module $V \in \mathcal{C}_{\overline{\lambda}}$ is typical.

The argument in [CGP2, Lemma 7.1] can be applied to our setting to show that the irreducibles of typical weight are projective. Recall that $\{X_{\pm\beta_1}^{k_1} X_{\pm\beta_2}^{k_2} \cdots X_{\pm\beta_N}^{k_N} \mid 0 \leq k_i < r_{\beta_i}\}$ is a basis of $\overline{U}_q^H(\eta^\pm)$. Let $X_+ := \prod_{k=1}^{N} X_{\beta_k}^{r_{\beta_k}-1}$ and $X_- := \prod_{k=1}^{N} X_{\beta_k}^{r_{\beta_k}-1}$ denote the highest and lowest weight vectors of $\overline{U}_q^H(\mathfrak{g})$. Suppose $\lambda \in \mathfrak{h}^*$ is of typical weight and that there is a surjection $f : M \to M^\lambda$ for some $M \in \mathcal{C}$. Since λ is typical, M^λ is irreducible so X_+X_- acts on the generator v_λ of M^λ by a scalar, which we denote ν. Then there is a vector $w \in f^{-1}(\frac{1}{\nu}v_\lambda)$. The vector $w' := X_+X_-w$ is maximal since X_+ is maximal in $\overline{U}_q^H(\eta^+)$ (i.e. $XX_+ = 0$ for all $X \in \overline{U}_q^H(\eta^+)$) and non-zero since $f(w') = v_\lambda$. Therefore, by the universal property of Verma modules there is a map $g : M^\lambda \to M$ such that $g(v_\lambda) = w'$ and $f : M \to M^\lambda$ splits. Hence, every $\mathcal{C}_{\overline{\lambda}}$ with $\overline{\lambda} \in \mathfrak{t} \setminus \mathcal{X}$ contains only projective irreducible modules and is therefore semisimple. $\square$

It is well known (see [GP1, Section 5.6], for example) that the duality morphisms are given by

$$\begin{aligned}
\overrightarrow{\mathrm{coev}}_V &: \mathbb{1} \to V \otimes V^*, && 1 \mapsto \sum_{i \in I} v_i \otimes v_i^*,\\
\overrightarrow{\mathrm{ev}}_V &: V^* \otimes V \to \mathbb{1}, && f \otimes v \mapsto f(v),\\
\overleftarrow{\mathrm{coev}}_V &: \mathbb{1} \to V^* \otimes V, && 1 \mapsto \sum_{i \in I} v_i^* \otimes K_{2\rho}^{r-1} v_i,\\
\overleftarrow{\mathrm{ev}}_V &: V \otimes V^* \to \mathbb{1}, && v \otimes f \mapsto f(K_{2\rho}^{1-r} v),
\end{aligned}$$

where $\mathbb{1}$ is the 1-dimensional module of weight zero and $\{v_i\}_{i \in I}$, $\{v_i^*\}_{i \in I}$ are dual bases of V and V^*. The pivotal structure on $\mathcal{C}$ is the monoidal natural transformation $\delta : \mathrm{Id}_{\mathcal{C}} \to (-)^{**}$ defined by components

$$\delta_V = \psi_V \circ \ell_{K_{2\rho}^{1-r}} : V \to V^{**}$$

where $\psi_V : V \to V^{**}$ is the canonical embedding $\psi_V(v)(f) = f(v)$ and $\ell_x(v) = xv$ denotes left multiplication. It was shown in [GP1, Subsection 5.8] (see also [GP2, Subsection 4.2]) that the unrolled quantum group $U_q^H(\mathfrak{g})$ is braided. The proof of this statement is given for odd roots, but holds for even roots as well with very minor adjustments. The proof uses a projection map $p : U_h(\mathfrak{g}) \to U^<$ from the h-adic quantum group $U_h(\mathfrak{g})$ to the $\mathbb{C}[[h]]$-module generated by the monomials

$$\prod_{i=1}^{n} H_i^{m_i} \prod_{j_1=1}^{n} X_{\beta_{j_1}}^{k_{j_1}} \prod_{j_2=1}^{n} X_{-\beta_{j_2}}^{k_{j_2}} \tag{3.3.7}$$

with $m_i \in \mathbb{Z}_{\geq 0}$, $0 \leq k_{j_1}, k_{j_2} < r$. If we generalize this by defining $U^<$ to be the $\mathbb{C}[[h]]$-module generated by the monomials in equation (3.3.7) with $m_i \in \mathbb{Z}_{\geq 0}$ and $0 \leq k_{j_s} < r_{\beta_{j_s}}$, then the proof follows verbatim as in [GP1, Subsection 5.8]. This yields an R-matrix $R := \mathscr{H}\mathscr{R}$ where

$$\mathscr{H} := q^{\sum\limits_{i,j} d_i (A^{-1})_{ij} H_i \otimes H_j}, \tag{3.3.8}$$

$$\mathscr{R} := \prod_{i=1}^{N} \left(\sum_{j=0}^{r_{\beta_i}-1} \frac{\left((q_{\beta_i} - q_{\beta_i}^{-1}) X_{\beta_i} \otimes X_{-\beta_i}\right)^j}{[j; q_{\beta_i}^{-2}]!} \right), \tag{3.3.9}$$

where $\{\beta_i\}_{i=1}^N$ is the ordered bases for Δ^+ as described in Subsection 3.2.1, $[j;q]!$ is defined in equation (3.2.2), and $q_\beta = q^{\langle \beta, \beta \rangle /2}$. Recall that $\overline{U}_q^H(\mathfrak{g})$ is the quotient of $U_q^H(\mathfrak{g})$ by the Hopf ideal I generated by the set $\{X_{\pm i}^{r_{\alpha_i}}\}$ so the braided structure on $U_q^H(\mathfrak{g})$ induces a braided structure on the quotient $\overline{U}_q^H(\mathfrak{g})$. We then obtain a braiding on $\mathcal{C}$ given by $c = \tau \circ R$ where $\tau : V \otimes W \to W \otimes V$ for any $V, W \in \mathcal{C}$ is the usual flip map.

Let $M \in \mathcal{C}$ be simple with maximal vector $m \in M(\lambda)$, and define the family of morphisms $\theta_V : V \to V$ by

$$\theta_V := (Id_V \otimes \overleftarrow{\mathrm{ev}}_V) \circ (c_{V,V} \otimes Id_{V^*}) \circ (Id_V \otimes \overrightarrow{\mathrm{coev}}_V)$$

where $c_{V,V}$ is the braiding. An easy computation shows that

$$\theta_M(m) = q^{\langle \lambda, \lambda + 2(1-r)\rho \rangle} m.$$

Hence, on any simple module $M \in \mathcal{C}$ with maximal vector $m \in M(\lambda)$,

$$\theta_M = q^{\langle \lambda, \lambda + 2(1-r)\rho \rangle} Id_M \tag{3.3.10}$$

We then observe, as in [GP2][Subsection 4.4], that $\theta_{(S^\lambda)^*} = (\theta_{S^\lambda})^*$ for all generic simple modules (S^λ such that $\overline{\lambda} \in \mathfrak{t} \setminus \chi$), where f^* denotes the right dual. So, by [GP2, Theorem 9], $\mathcal{C}$ is ribbon.

Corollary 3.3.9. $\mathcal{C}$ is a ribbon category.

We also observe here that $\mathcal{C}$ has trivial Müger center (recall Definition 3.1.2):

Proposition 3.3.10. $\mathcal{C}$ has no non-trivial transparent objects (i.e. $\mathcal{C}$ has trivial Müger center).

Proof. Given a pair of irreducible modules $S^\lambda, S^\mu \in \mathcal{C}$, it is easy to see that the braiding $c = \tau \circ \mathscr{H}\mathscr{R}$ acts as $\tau \circ \mathscr{H}$ on the product of highest weights $v_\lambda \otimes v_\mu \in S^\lambda \otimes S^\mu$. We therefore see that

$$c_{S^\mu, S^\lambda} \circ c_{S^\lambda, S^\mu}(v_\lambda \otimes v_\mu) = q^{2\langle \lambda, \mu \rangle} Id.$$

Hence, there is no irreducible object transparent to all other irreducible objects since there is no weight $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \mu \rangle \in \frac{\ell}{2}\mathbb{Z}$ for all $\mu \in \mathfrak{h}^*$. In particular, there are no non-trivial irreducible transparent objects. If some $M \in \mathcal{C}$ were transparent, then all of its subquotients and, in particular, the factors appearing in its composition series must also be transparent, so any transparent object has composition factors isomorphic to $\mathbb{1}$. Any such module has character $\mathrm{ch}[M] = 0$ (recall Equation (3.3.2)), but any module in $\mathcal{C}$ with vanishing character is a direct sum of $\dim(M)$ copies of $\mathbb{1}$. Indeed, the elements $H_1, ..., H_n$ act semisimply on M (as they do on all modules in $\mathcal{C}$) so they must act by zero since $\mathrm{ch}[M] = 0$, and it follows from Equation (3.3.3) that $X_{\pm i} m \in M(\pm\alpha_i) = \emptyset$, so $X_{\pm i} m = 0$ for all $m \in M$. Recalling that $K_\gamma = \prod_{i=1}^{n} q_i^{k_i H_i}$ as operators on $\mathcal{C}$, we see that

$$X_{\pm i} m = H_i m = 0 \quad \text{and} \quad K_\gamma m = 1$$

for all $m \in M$. Hence, M a direct sum of $\dim(M)$ copies of $\mathbb{1}$, and $\mathcal{C}$ has trivial Müger center. $\square$

3.3.2 Duality

Given any $M \in \mathcal{C}$, the antipode $S : \overline{U}_q^H(\mathfrak{g}) \to \overline{U}_q^H(\mathfrak{g})$ defines a module structure on the dual $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ by

$$(x \cdot f)(m) = f(S(x)m)$$

for each $f \in M^*, m \in M$, and $x \in \overline{U}_q^H(\mathfrak{g})$. Let M^*_ω be the module obtained by twisting M^* by the automorphism ω defined in equation (3.2.12), allowing $x \in \overline{U}_q^H(\mathfrak{g})$ to act on M^* as $\omega(x)$ and for convenience set $\check{M} := M^*_\omega$, which is easily seen to lie in $\mathcal{C}$. Note that $\check{M}$ is therefore the dual defined with respect to the anti-homomorphism $S \circ \omega : \overline{U}_q^H(\mathfrak{g}) \to \overline{U}_q^H(\mathfrak{g})$. It is easy to check that this map is an involution, i.e. $S \circ \omega \circ S \circ \omega = Id$. We therefore have that the canonical map $\phi : M \to \check{\check{M}}$ is an isomorphism, since

$$X \cdot \phi(v)(f) = \phi(v)(S(w(X)) \cdot f) = \phi(v)(f \circ \Pi(X)) = f(X \cdot v) = \phi(X \cdot v)(f),$$

where $\Pi : \overline{U}_q^H(\mathfrak{g}) \to \mathrm{End}(M)$ is the representation defining the action on M. Hence, $\check{\check{M}} \cong M$. The (contravariant) functor $M \mapsto \check{M}$ is exact as the composition of exact functors (taking duals in a tensor category and twisting by automorphisms), and one sees immediately (since $S(H_i) = -H_i$) that $\dim \check{M}(\lambda) = \dim M(\lambda)$, so $\mathrm{ch}[\check{M}] = \mathrm{ch}[M]$. Exactness implies that M is simple iff $\check{M}$ is, and then $\mathrm{ch}[\check{S}^\lambda] = \mathrm{ch}[S^\lambda]$ implies $\check{S}^\lambda \cong S^\lambda$. We therefore obtain the following proposition:

Proposition 3.3.11.

- The contravariant functor $M \mapsto \check{M}$ is exact and $\check{\check{M}} \cong M$.
- $\mathrm{ch}[\check{M}] = \mathrm{ch}[M]$ for all $M \in \mathcal{C}$.
- $\check{M}$ is simple iff M is simple.
- $\check{S}^\lambda \cong S^\lambda$ for all $\lambda \in \mathfrak{h}^*$.

Recall that a filtration, or series, for a module M is a family of proper submodules ordered by inclusion

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

A series for a module M is called a composition series if successive quotients are irreducible modules: $M_k/M_{k-1} \cong S^{\lambda_k}$ for some $\lambda_k \in \mathfrak{h}^*$. Similarly, a series is called a Verma (or standard) series if successive quotients are Verma modules: $M_k/M_{k-1} \cong M^{\lambda_k}$ for some $\lambda_k \in \mathfrak{h}^*$.

We have already observed in Proposition 3.3.8 that irreducible modules of typical weight are projective. Given any $V \in \mathcal{C}$ and $\lambda \in \mathfrak{h}^*$ typical, we have a surjection

$$\overleftarrow{\mathrm{ev}}_{S^\lambda} \otimes Id_V : S^\lambda \otimes (S^\lambda)^* \otimes V \to \mathbb{1} \otimes V \cong V \tag{3.3.11}$$

where $S^\lambda \otimes (S^\lambda)^* \otimes V$ is projective since projective modules form an ideal in pivotal categories (see [GPV, Lemma 17]), so $\mathcal{C}$ has enough projectives and since every module in $\mathcal{C}$ is finite, every module in $\mathcal{C}$ has a projective cover. We denote by P^λ the projective cover of S^λ, and it follows easily from the defining property of projective modules that P^λ is also the projective cover of M^λ. Replacing V in Equation (3.3.11) by an arbitrary Verma module M^μ and noting $M^\lambda \cong S^\lambda$ for typical λ (recall Proposition 3.3.4 and 3.3.6), we obtain a surjection from the projective module $M^\lambda \otimes (M^\lambda)^* \otimes M^\mu$ onto M^μ. It then follows that P^μ appears in the decomposition of $M^\lambda \otimes (M^\lambda)^* \otimes M^\mu$ into a direct sum of projective covers. It can be shown that $M^\lambda \otimes (M^\lambda)^* \otimes M^\mu$ has a standard filtration by the argument in [Hu, Theorem 3.6], and P^μ then has a standard filtration by the argument in [Hu, Proposition 3.7 (b)] since it is a summand of a module admitting a standard filtration. We denote by $(P^\lambda : M^\mu)$ the multiplicity of M^μ in the standard filtration of P^λ, and $[M^\mu : S^\lambda]$ the multiplicity of S^λ in the composition series of M^μ. With the existence of the duality functor $M \to \check{M}$ satisfying the properties in Proposition 3.3.11, BGG reciprocity follows as in [Hu]:

Proposition 3.3.12. BGG reciprocity holds in $\mathcal{C}$. That is, we have $(P^\lambda : M^\mu) = [M^\mu : S^\lambda]$.

Let $P \in \mathcal{C}$ be projective, then P is isomorphic to a direct sum of projective covers of irreducible modules: $P \cong \bigoplus_{\lambda_k \in \mathfrak{h}^*} c_{\lambda_k} P^{\lambda_k}$ for some $c_{\lambda_k} \in \mathbb{Z}_+$. Since unrolling the quantum group gives us additive weights, rather than multiplicative weights, the argument of [Hu, Corollary 3.10] can be used to show that projectives in $\mathcal{C}$ are determined up to isomorphism by their characters, which we include here for convenience. It is clearly enough to show that the characters determine the coefficients c_{λ_k} of P, since then two projective modules with coinciding characters will both be isomorphic to the same sum of projective covers. We proceed by induction on length of standard filtrations. If P has length 1, it is a Verma module and the statement is trivial. If P has length > 1, let

$$0 \subset M_1 \subset \cdots \subset M_n = P$$

with $M_k/M_{k-1} \cong M^{\mu_k}$ denote a standard filtration of P, so $\mathrm{ch}[P] = \sum_{i=1}^n d_{\mu_i}\mathrm{ch}[M^{\mu_i}]$. Let λ be minimal s.t. $d_\lambda \neq 0$. By BGG reciprocity, $(P^\mu : M^\lambda) \neq 0$ iff $[M^\lambda : S^\mu] \neq 0$, so $\mu \leq \lambda$ and therefore by minimality of λ, P^λ appears in the decomposition of P with multiplicity d_λ, that is, $P = d_\lambda P^\lambda \oplus \tilde{P}$ for some projective module $\tilde{P}$. By the induction assumption, the coefficients of $\tilde{P}$ are determined by its character, so P is determined by up to isomorphism by its character.

Proposition 3.3.13. Projective modules in $\mathcal{C}$ are isomorphic if their characters coincide.

It is apparent from the construction of projective covers in [CGP1] for $\overline{U}_q^H(\mathfrak{sl}_2)$ that the socle and top of projective covers coincide. This is actually a general feature of $\mathcal{C}$ for any $\overline{U}_q^H(\mathfrak{g})$ and is a consquence of the following theorem:

Theorem 3.3.14. P^λ is self-dual ($\check{P}^\lambda \cong P^\lambda$).

Proof. We first recall that any indecomposable module is a quotient of a direct sum of projective covers. Indeed, any cyclic indecomposable module M has a unique maximal submodule and irreducible quotient S^λ. This gives a surjection $\phi : M \twoheadrightarrow S^\lambda$ and projectivity of P^λ guarantees the existence of a map $\varphi : P^\lambda \to M$ such that $\phi \circ \varphi = q^\lambda$ where $q^\lambda : P^\lambda \to S^\lambda$ is the essential surjection. It follows that there exists a $v \in P^\lambda$ such that $\phi \circ \varphi(v) = v_\lambda \in S^\lambda$ and since $\phi : M \twoheadrightarrow S^\lambda$ is the quotient map (by the maximal ideal of M), we see that $\varphi(v)$ lies in the top of M and therefore generates M (otherwise it lies in the maximal submodule). Hence, $\varphi : P^\lambda \to M$ is surjective. Any indecomposable $M \in \mathcal{C}$ is finitely generated by some $\{v_1, ..., v_n\}$, so there exists a canonical surjection $\Phi : \bigoplus_{k=1}^{n} P^{\lambda_k} \to M$ given by mapping each P^{λ_k} onto the cyclic submodules $\langle v_k \rangle$ of M. We therefore have for each $\check{P}^\lambda$ a short exact sequence

$$0 \to N^\lambda \to \bigoplus_{k=1}^{m} P^{\lambda_k} \to \check{P}^\lambda \to 0$$

for some $\lambda_1, ..., \lambda_m \in \mathfrak{h}^*$ and some submodule N^λ of $\bigoplus_{k=1}^{m} P^{\lambda_k}$. Applying the duality functor, we obtain an exact sequence

$$0 \to P^\lambda \to \bigoplus_{k=1}^{m} \check{P}^{\lambda_k} \to \check{N}^\lambda \to 0.$$

$\mathcal{C}$ is pivotal so by [GPV, Lemma 17], projective and injective objects coincide in $\mathcal{C}$. Therefore, the sequence splits and P^λ is a summand of $\bigoplus_{k=1}^{m} \check{P}^{\lambda_k}$. Further, it is easy to see that the functor $X \to \check{X}$ preserves indecomposability since taking duals and twisting by ω preserve indecomposability in $\mathcal{C}$. Since the $\check{P}^{\lambda_k}$ are indecomposable, we have $P^\lambda \cong \check{P}^{\lambda_k}$ for some $\lambda_k \in \mathfrak{h}^*$ and $\mathrm{ch}[\check{P}^\lambda] = \mathrm{ch}[P^\lambda]$, so we must have $\lambda_k = \lambda$. That is, $\check{P}^\lambda \cong P^\lambda$. □

Theorem 3.3.14 has the following immediate corollary

Corollary 3.3.15.
- $\mathrm{Socle}(P^\lambda) = S^\lambda$.
- P^λ is the injective hull of S^λ.
- $\mathcal{C}$ is Unimodular.

Unimodularity follows from P^0 being the injective hull of $\mathbb{1}$ (see [ENO, EGNO]), so by [GKP, Corollary 3.2.1], we see that there exists a right trace on the ideal of projective modules in $\mathcal{C}$ (for details on categorical traces see [GP2, Subsection 1.3].). It then follows exactly as in [GP2, Theorem 22] that this right trace is in fact a two-sided trace:

Corollary 3.3.16. $\mathcal{C}$ admits a unique non-zero two-sided trace on the ideal *Proj* of projective modules.

Chapter 4

Algebra Objects and Simple Currents in $\mathrm{Rep}\overline{U}_q^H(\mathfrak{g})^{wt}$

In this Chapter we construct families of commutative (super) algebra objects in the category of weight modules for the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ at roots of unity, and study their categories of local modules. We determine their simple modules and derive conditions for these categories being finite, non-degenerate, and ribbon. Motivated by numerous examples in the $\mathfrak{g} = \mathfrak{sl}_2$ case, we expect some of these categories to compare nicely to categories of modules for vertex operator algebras. We focus in particular on examples expected to correspond to the higher rank triplet vertex algebra $W_Q(r)$ of Feigin and Tipunin [FT] and the $B_Q(r)$ algebras of [C1].

4.1 Preliminaries

The identification in Proposition 2.1.8 between irreducible $\overline{U}_q^H(\mathfrak{sl}_2)$ and $W^0_{A_1}(r)$-modules preserved twists. The higher rank singlet $W^0_Q(r)$ (see [CM2] for details) is a subalgebra of the Heisenberg vertex algebra whose rank is that of Q and for $\lambda \in \sqrt{r}P$ it is expected that F_λ has a simple $W^0_Q(r)$-submodule M_λ that is a simple current. This Conjecture would follow from [M, Cor. 4.8] together with [Su] if one could prove that $W_Q(r)$ is a simple vertex algebra and braided tensor category exists on a category of $W^0_Q(r)$ that contains all the M_λ. The twist is determined from the action of $e^{2\pi i L_0}$ with L_0 the Virasoro zero-mode of the vertex algebra. The twist acts on Fock spaces F_λ by the scalar $\theta_{F_\lambda} = e^{\pi i \langle \lambda, \lambda + \frac{2(1-r)}{\sqrt{r}}\rho\rangle}$. It then follows from Equation 3.3.10, that the twists on $\mathrm{Rep}_{\langle s\rangle} W_Q^0(r)$ and $\mathrm{Rep}\, \overline{U}_q^H(\mathfrak{g})^{wt}$ act on F_λ and $S^{\sqrt{r}\lambda}$ by the same scalar. We therefore make the identification

$$\phi : \mathrm{Rep}_{\langle s\rangle} W_Q^0(r) \mapsto \mathrm{Rep}_{wt} \overline{U}_q^H(\mathfrak{g}), \qquad M_\lambda \subset F_\lambda \to S^{\sqrt{r}\lambda} \tag{4.1.1}$$

for $\lambda \in \sqrt{r}P$. Note that the simple currents in $\mathrm{Rep}_{\langle s\rangle} W_Q^0(r)$ are conjecturally precisely the Fock spaces $M_\lambda \subset F_\lambda$ with $\lambda \in \sqrt{r}P$ while the simple currents in $\mathrm{Rep}_{wt} \overline{U}_q^H(\mathfrak{g})$ (for $\mathfrak{g}$ of ADE type) are the irreducibles S^λ with $\lambda \in rP$ (see [R, Remark 4.7]), so ϕ identifies simple currents.

Higher rank Heisenberg vertex operator algebra

One of the examples we are interested in is the Deligne product $\mathcal{C} \boxtimes \mathcal{H}$ of the category $\mathcal{C}$ of $\overline{U}_q^H(\mathfrak{g})$ weight modules with a semi-simple category $\mathcal{H}$ of modules over the Heisenberg vertex operator algebra. We define and review the category $\mathcal{H}$ in this section following [DN]. Tensor category structure is due to [CKLR].

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ the corresponding affine Lie algebra. Let $\lambda \in \mathfrak{h}^*$ and denote by F_λ the the usual Fock space

$$\mathsf{F}_\lambda := U(\hat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}K)} \mathbb{C}$$

where $\mathfrak{h} \otimes t\mathbb{C}[t]$ acts trivially on $\mathbb{C}$, $\mathfrak{h}$ acts as $\lambda(h)$ for all $h \in \mathfrak{h}$, and K acts as 1. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ we adopt the notation $h(n) := h \otimes t^n \in \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]$ and define

$$h(z) := \sum_{n\in\mathbb{Z}} h(n) z^{-n-1}.$$

The Fock space $\mathsf{H} := \mathsf{F}_0$ carries the structure of a vertex operator algebra. Set $\mathbb{1} := 1 \otimes 1$ and for any $v := h_1(-n_1) \cdots h_m(-n_m)\mathbb{1} \in \mathsf{H}$ where $h_1, ..., h_m \in \mathfrak{h}$, we define the vertex operator $Y(-, z)$ acting on F_λ by

$$Y(v, z) :=: [\partial^{n_1-1} h_1(z)] \cdots [\partial^{n_m-1} h_m(z)] : ,$$

where $\partial^k = \frac{1}{n!}(\frac{d}{dz})^k$ and $: XY :$ denotes the normal ordering of two fields X, Y. Set $\omega :=$

$\frac{1}{2}\sum_{k=1}^{n} H_k(-1)^2\mathbf{1}$ where $H_1, ..., H_n$ is an orthonormal basis of $\mathfrak{h}$ with respect to the Killing form, and we denote $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-1}$. Then, $\mathsf{H} := \mathsf{F}_0$ is a vertex operator algebra and it has vertex tensor categories of Fock modules, but for that one has to ensure that conformal weight is real (see [CKLR, Theorem 2.3], which requires the action of the H_k to be either real or purely imaginary. For us the latter case is relevant, e.g.

Theorem 4.1.1. $(\mathsf{H}, Y, \mathbb{1}, \omega)$ is a simple vertex operator algebra called the Heisenberg vertex operator algebra with vacuum $\mathbf{1}$, state-field correspondence $Y(-,z)$, and Virasoro element ω. We define $\mathcal{H}$ to be the category of H-modules on which $\mathfrak{h}$ acts semisimply and $\lambda(H_k) \in i\mathbb{R}$ for all $k = 1, \dots, n$. This category is semisimple and generated by the Fock modules F_λ, $\lambda \in \mathfrak{h}^*$. Tensor products are additive in the index ($\mathsf{F}_{\lambda_1} \otimes \mathsf{F}_{\lambda_2} \cong \mathsf{F}_{\lambda_1+\lambda_2}$), so all Fock spaces are simple currents with $\mathsf{F}_{-\lambda}$ the inverse of F_λ. The braiding and twist on $\mathcal{H}$ are given by

$$\text{Braiding} \qquad c_{\mathsf{F}_{\lambda_1},\mathsf{F}_{\lambda_2}} = \tau \circ e^{\pi i\langle\lambda_1,\lambda_2\rangle} \tag{4.1.2}$$

$$\text{Twist} \qquad \theta_{\mathsf{F}_\lambda} = e^{\pi i\langle\lambda,\lambda\rangle}\mathrm{Id}_{\mathsf{F}_\lambda} \tag{4.1.3}$$

where τ is the usual flip map.

4.2 Simple Current Extensions

We want to construct and study braided tensor categories related to the module categories of the higher rank triplet $W_Q(r)$ (see [BM, Mi, CM2]) and "Bp" $B_Q(r)$ (see [C3]) vertex operator algebras associated to the root lattice Q of a simple finite dimensional complex Lie algebra $\mathfrak{g}$ of ADE type. The triplet $W_Q(r)$ is an infinite order simple current extension of the singlet algebra inside $\mathrm{Rep}_{\langle s\rangle} W_Q^0(r)^{\oplus}$, while $B_Q(r)$ is a simple current extension in the Deligne product $(\mathrm{Rep}W_Q^0(r)\boxtimes\mathcal{H})^{\oplus}$ where $\mathcal{H}$ is the category in Definition 4.1.1. In the $\mathfrak{g} = \mathfrak{sl}_2$ case (see [CGR, ACKR]), such categories can be constructed from the category of weight modules over the unrolled restricted quantum group $\overline{U}_q^H(\mathfrak{sl}_2)$ through the identification of simple $\overline{U}_q^H(\mathfrak{sl}_2)$ and $W_{A_1}^0(p)$-modules found in [CMR, Theorem 1]. We therefore study algebra objects in the category $\mathcal{C}$ of weight modules over $\overline{U}_q^H(\mathfrak{g})$ and $\mathcal{C}\boxtimes\mathcal{H}$. Both cases can be handled simultaneously by considering categories of a particular form.

Throughout this section, let $\mathcal{B}$ be a braided tensor category whose simple currents $\{\mathbb{C}_\lambda \mid \lambda \in$

$\mathcal{L}\}$ are indexed by a normed lattice $(\mathcal{L}, \langle -, - \rangle)$ such that the braiding on simple currents is given by $c_{\mathbb{C}_{\lambda_1},\mathbb{C}_{\lambda_2}}(v_{\lambda_1} \otimes v_{\lambda_2}) = q^{\langle \lambda_1, \lambda_2 \rangle} v_{\lambda_2} \otimes v_{\lambda_1}$ for a primitive ℓ-th root of unity q and $\mathbb{C}_{\lambda_1} \otimes \mathbb{C}_{\lambda_2} \cong \mathbb{C}_{\lambda_1+\lambda_2}$. We assume also that objects in $\mathcal{B}$ carry vector space structure and $\dim\mathrm{End}(\mathbb{C}_\lambda) = 1$ for all $\lambda \in \mathcal{L}$. For any lattice $L \subset \mathcal{L}$, define the object

$$\mathcal{A}_L := \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \in \mathcal{B}^{\oplus}.$$

where $\mathcal{B}^{\oplus}$ denotes an appropriate direct sum completion of $\mathcal{B}$ (see [AR]).

Theorem 4.2.1. $\mathcal{A}_L$ is an associative algebra object for all $L \subset \mathcal{L}$. $\mathcal{A}_L$ is commutative if and only if $\sqrt{2/\ell}L$ is an even lattice. That is, if and only if $\langle \lambda, \lambda \rangle \in \ell\mathbb{Z}$ and $2\langle \lambda, \mu \rangle \in \ell\mathbb{Z}$ for all $\lambda, \mu \in L$.

Proof. To realize $\mathcal{A}_L$ as a commutative algebra object in $\mathcal{B}^{\oplus}$, we must define a product $\mu : \mathcal{A}_L \otimes \mathcal{A}_L \to \mathcal{A}_L$ and unit $\iota : \mathbb{C}_0 \to \mathcal{A}_L$ satisfying the associativity, unit, and commutativity constraints:

$$\mu \circ (\mu \otimes \mathrm{Id}_{\mathcal{A}_L}) = \mu \circ (\mathrm{Id}_{\mathcal{A}_L} \otimes \mu) \circ a_{\mathcal{A}_L,\mathcal{A}_L,\mathcal{A}_L}, \tag{4.2.1}$$

$$\mu \circ (\iota \otimes \mathrm{Id}_{\mathcal{A}_L}) \circ l_{\mathcal{A}_L}^{-1} = \mathrm{Id}_{\mathcal{A}_L}, \tag{4.2.2}$$

$$\mu \circ c_{\mathcal{A}_L,\mathcal{A}_L} = \mu, \tag{4.2.3}$$

where $a_{\mathcal{A}_L,\mathcal{A}_L,\mathcal{A}_L}$ is the associativity constraint, $l_{\mathcal{A}_L}$ the left unit constraint, and $c_{\mathcal{A}_L,\mathcal{A}_L}$ the braiding. Denote by 1_λ a generator of $\mathbb{C}_\lambda$. We may assume that $\iota(1) = 1_0$. Since $\dim\mathrm{End}(\mathbb{C}_\lambda) = 1$, we have

$$\mu(1_{\lambda_1} \otimes 1_{\lambda_2}) = t_{\lambda_1,\lambda_2} 1_{\lambda_1+\lambda_2} \tag{4.2.4}$$

for some $t_{\lambda_1,\lambda_2} \in \mathbb{C}$ and it is easy to see that $\mathcal{A}_L$ is a commutative algebra object if and only if these scalars satisfy

$$\begin{aligned} t_{\lambda_1+\lambda_2,\lambda_3} t_{\lambda_1,\lambda_2} &= t_{\lambda_1,\lambda_2+\lambda_3} t_{\lambda_2,\lambda_3}, \\ t_{\lambda,0} = t_{0,\lambda} &= 1, \\ t_{\lambda_1,\lambda_2} &= t_{\lambda_2,\lambda_1} q^{\langle \lambda_1, \lambda_2 \rangle}. \end{aligned} \tag{4.2.5}$$

Let $\mathcal{A}'_L$ denote the same object with algebra structure given by structure constants t'_{λ_1,λ_2}. It is easy to see that $\mathcal{A}_L$ and $\mathcal{A}'_L$ are isomorphic if and only if there exist scalars $\phi_\lambda, \lambda \in L$ such that

$$t'_{\lambda_1,\lambda_2} = \frac{\phi_{\lambda_1+\lambda_2}}{\phi_{\lambda_1}\phi_{\lambda_2}} t_{\lambda_1,\lambda_2}. \tag{4.2.6}$$

Denote by $\{\gamma_i\}_{i=1}^m$ the generators of L. Then we can write any $\lambda \in L$ as $\lambda = \sum_{i=1}^m n_i\gamma_i$ for some $n_i \in \mathbb{Z}$, and we fix the notation

$$\lambda^{>k} = \sum_{i>k} n_i\gamma_i, \qquad \lambda^{\geq k} = \sum_{i\geq k} n_i\gamma_i, \qquad \lambda^k = n_k\gamma_k. \tag{4.2.7}$$

with $\lambda^{<k}, \lambda^{\leq k}$ definied similarly. Fix non-zero scalars ϕ_{γ_i} and define the scalar ϕ_λ for any $\lambda \in L$ recursively by the relations

$$\phi_0 = 1 \qquad\qquad \phi_{n\gamma_i} = \frac{\phi_{(n-1)\gamma_i}\phi_{\gamma_i}}{t_{(n-1)\gamma_i,\gamma_i}} \qquad\qquad \phi_\lambda = \frac{\phi_{\lambda^{<m}}\phi_{\lambda^m}}{t_{\lambda^{<m},\lambda^m}}$$

and let t'_{λ_1,λ_2} be the scalars determined by equation (4.2.6). We have the following:

$$t'_{0,\lambda} = t'_{\lambda,0} = 1, \quad t'_{n\gamma_i,\gamma_i} = \frac{\phi_{(n+1)\gamma_i}}{\phi_{n\gamma_i}\phi_{\gamma_i}} t_{n\gamma_i,\gamma_i} = 1, \quad t'_{\lambda^{<k},n\gamma_{k'}} = \frac{\phi_{\lambda^{<k}+n\gamma_{k'}}}{\phi_{\lambda^{<k}}\phi_{n\gamma_{k'}}} t_{\lambda^{<k},n\gamma_{k'}} = 1, \tag{4.2.8}$$

for all $n \in \mathbb{Z}, i = 1, ...m, k' \geq k$. By associativity, we then have

$$t'_{n\gamma_i,m\gamma_i} = \frac{t'_{(n+m-1)\gamma_i,\gamma_i} t'_{n\gamma_i,(m-1)\gamma_i}}{t'_{(m-1)\gamma_i,\gamma_i}} = t'_{n\gamma_i,(m-1)\gamma_i}.$$

We therefore see that $t'_{n\gamma_i,m\gamma_i} = 1$ for all $n, m \in \mathbb{Z}$ and $i = 1, ..., m$. We can translate this and equation (4.2.8) into the notation of (4.2.7) as

$$t_{\lambda^k,\mu^k} = 1 \qquad\qquad t_{\lambda^{<k},\mu^k} = 1$$

for all $\lambda, \mu \in L$ and $k = 1, ..., m$. We then see that

$$t'_{\lambda^{\leq k},\mu^k} = \frac{t'_{\lambda^k,\mu^k} t'_{\lambda^{<k},\lambda^k+\mu^k}}{t'_{\lambda^{<k},\lambda^k}} = 1 \tag{4.2.9}$$

for any $\lambda, \mu \in L$. It follows again by associativity and equation (4.2.9) that for any k,

$t'_{\lambda,\mu^{<k}} = t'_{\lambda+\mu^{<k-1},\mu^{k-1}} t'_{\lambda,\mu^{<k-1}}$. Then,

$$\begin{aligned} t'_{\lambda,\mu} &= t'_{\lambda+\mu^{<m},\mu^m} t'_{\lambda,\mu^{<m}} \\ &= t'_{\lambda+\mu^{<m},\mu^m} t'_{\lambda+\mu^{<m-1},\mu^{m-1}} t'_{\lambda,\mu^{<m-2}} \\ &= \cdots = \prod_{k=1}^{m} t'_{\lambda+\mu^{<k},\mu^k} \end{aligned} \tag{4.2.10}$$

Then, for any $\lambda,\mu \in L$ and any $k = 1,...,m$ we have

$$\begin{aligned} t'_{\lambda^{\leq k},\mu^{>k}} &= \prod_{s=1}^{m} t'_{\lambda^{\leq k}+(\mu^{>k})^{<s},(\mu^{>k})^s} \\ &= \prod_{s=k+1}^{m} t'_{\lambda^{\leq k}+(\mu^{>k})^{<s}),\mu^s} \\ &= \prod_{s=k+1}^{m} t'_{(\lambda^{\leq k}+\mu^{>k})^{<s},\mu^s} = 1 \end{aligned}$$

where we have used equation (4.2.8) and (4.2.9), and the fact that $s > k$ in the last line. So, we have $t'_{\lambda,\mu^k} = t'_{\lambda^{\leq k}+\lambda^{>k},\mu^k} t'_{\lambda^{\leq k},\lambda^{>k}} = t'_{\lambda^{\leq k},\lambda^{>k}+\mu^k} t'_{\lambda^{>k},\mu^k} = t'_{\lambda^{>k},\mu^k}$ for any $\lambda,\mu \in L$. In particular, $t'_{\lambda+\mu^{<k},\mu^k} = t'_{\lambda^{>k},\mu^k}$, so equation (4.2.10) can be rewritten as

$$t'_{\lambda,\mu} = \prod_{k=1}^{m} t'_{\lambda^{>k},\mu^k} = \prod_{k=1}^{m} q^{\langle \lambda^{>k},\mu^k \rangle} t'_{\mu^k,\lambda^{>k}} = \prod_{k=1}^{m} q^{\langle \lambda^{>k},\mu^k \rangle} \tag{4.2.11}$$

We are now ready to evaluate equations 4.2.5. Let $\lambda_i = \sum_{j=1}^{m} n_j^i \gamma_j \in L$ $(i = 1,2,3)$. Then, we have

$$\begin{aligned} & t'_{\lambda_1+\lambda_2,\lambda_3} t'_{\lambda_1,\lambda_2} = t'_{\lambda_1,\lambda_2+\lambda_3} t'_{\lambda_2,\lambda_3} \\ \Leftrightarrow\quad & \left(\prod_{k=1}^{m} t'_{\lambda_1+\lambda_2,\lambda_3^k}\right)\left(\prod_{k=1}^{m} t'_{\lambda_1,\lambda_2^k}\right) = \left(\prod_{k=1}^{m} t'_{\lambda_1,\lambda_2^k+\lambda_3^k}\right)\left(\prod_{k=1}^{m} t'_{\lambda_2,\lambda_3^k}\right) \\ \Leftrightarrow\quad & \left(\prod_{k=1}^{m} q^{\langle \lambda_1^{>k}+\lambda_2^{>k},\lambda_3^k \rangle}\right)\left(\prod_{k=1}^{m} q^{\langle \lambda_1^{>k},\lambda_2^k \rangle}\right) = \left(\prod_{k=1}^{m} q^{\langle \lambda_1^{>k},\lambda_2^k+\lambda_3^k \rangle}\right)\left(\prod_{k=1}^{m} q^{\langle \lambda_2^{>k},\lambda_3^k \rangle}\right) \\ \Leftrightarrow\quad & \prod_{k=1}^{m} q^{\langle \lambda_1^{>k},\lambda_3^k \rangle} q^{\langle \lambda_2^{>k},\lambda_3^k \rangle} q^{\langle \lambda_1^{>k},\lambda_2^k \rangle} = \prod_{k=1}^{m} q^{\langle \lambda_1^{>k},\lambda_2^k \rangle} q^{\langle \lambda_1^{>k},\lambda_3^k \rangle} q^{\langle \lambda_2^{>k},\lambda_3^k \rangle} \end{aligned}$$

which is trivially true for all $\lambda_i \in L$. Therefore, $\mathcal{A}_L$ is an algebra object for all $L \subset \mathcal{L}$. For commutativity, we obtain the following:

$$
\begin{aligned}
& & t'_{\lambda_1,\lambda_2} &= t'_{\lambda_2,\lambda_1} q^{\langle\lambda_1,\lambda_2\rangle} \\
&\Leftrightarrow & \prod_{k=1}^{m} t'_{\lambda_1,\lambda_2^k} &= q^{\langle\lambda_1,\lambda_2\rangle} \prod_{k=1}^{m} t'_{\lambda_2,\lambda_1^k} \\
&\Leftrightarrow & \prod_{k=1}^{m} q^{\langle\lambda_1^{>k},\lambda_2^k\rangle} &= q^{\langle\lambda_1,\lambda_2\rangle} \prod_{k=1}^{m} q^{\langle\lambda_2^{>k},\lambda_1^k\rangle} \\
&\Leftrightarrow & q^{\sum\limits_{k=1}^{m}\sum\limits_{j>k} n_j^1 n_k^2 \langle\gamma_j,\gamma_k\rangle} &= q^{\sum\limits_{k=1}^{m}\sum\limits_{j>k}(n_k^1 n_j^2 + n_j^1 n_k^2)\langle\gamma_j,\gamma_k\rangle + \sum\limits_{k=1}^{m} n_k^1 n_k^2 \langle\gamma_k,\gamma_k\rangle} q^{\sum\limits_{k=1}^{m}\sum\limits_{j>k} n_j^2 n_k^1 \langle\gamma_j,\gamma_k\rangle} \\
&\Leftrightarrow & 1 &= q^{\sum\limits_{k=1}^{m} n_k^1 n_k^2 \langle\gamma_k,\gamma_k\rangle} q^{\sum\limits_{k=1}^{m}\sum\limits_{j>k} 2 n_k^1 n_j^2 \langle\gamma_j,\gamma_k\rangle}.
\end{aligned}
$$

Clearly, this holds for all $n_k^1, n_j^2 \in \mathbb{Z}$ if $\langle\gamma_k,\gamma_k\rangle \in \ell\mathbb{Z}$ for all k, and $2\langle\gamma_j,\gamma_k\rangle \in \ell\mathbb{Z}$ whenever $j \neq k$. It is also easy to see that we can choose appropriate coefficients to obtain the equations $q^{\langle\gamma_k,\gamma_k\rangle} = 1$ and $q^{2\langle\gamma_j,\gamma_k\rangle} = 1$. Hence, commutativity holds if and only if

$$
\begin{aligned}
\langle\gamma_i,\gamma_j\rangle &\in \ell\mathbb{Z} \quad \text{if } i = j \\
2\langle\gamma_i,\gamma_j\rangle &\in \ell\mathbb{Z} \quad \text{if } i \neq j
\end{aligned}
$$

It is easy to check that this conditions implies

$$
\langle\lambda,\lambda\rangle \in \ell\mathbb{Z}
$$

$$
2\langle\lambda,\mu\rangle \in \ell\mathbb{Z}
$$

for all $\lambda, \mu \in L$. □

We have the following useful corollary of the proof of Theorem 4.2.1:

Corollary 4.2.2. The scalars t_{λ_1,λ_2} of equation (4.2.4) defining the product $\mu : \mathcal{A}_L \otimes \mathcal{A}_L \to \mathcal{A}_L$ are non-zero.

Let L^μ be the lattice defined by adding a generator $\mu \in \mathcal{L}$ to a lattice $L \subset \mathcal{L}$, and let

$$\mathcal{A}_{L^\mu} = \bigoplus_{\lambda \in L^\mu} \mathbb{C}_\lambda$$

be the corresponding algebra object. Note that if $\mu = 0$, we recover the usual algebra objects defined above. If $\mu \notin L$ and $2\mu \in L$, then every element in L^μ is in L or $\mu + L$, so we can decompose $\mathcal{A}_{L^\mu}$ as $\mathcal{A}^{\bar{0}}_{L^\mu} \oplus \mathcal{A}^{\bar{1}}_{L^\mu}$ where

$$\mathcal{A}^{\bar{0}}_{L^\mu} = \bigoplus_{\lambda \in L} \mathbb{C}_\lambda = \mathcal{A}_L, \qquad \mathcal{A}^{\bar{1}}_{L^\mu} = \bigoplus_{\lambda \in L} \mathbb{C}_{\mu+\lambda},$$

and it is easy to show that $\mu(\mathcal{A}^{\bar{i}}_{L^\mu} \otimes \mathcal{A}^{\bar{j}}_{L^\mu}) \subset \mathcal{A}^{\overline{i+j}}_{L^\mu}$ since $\dim\mathrm{Hom}(\mathbb{C}_{\lambda_1}, \mathbb{C}_{\lambda_2}) = \delta_{\lambda_1,\lambda_2}$. Hence, $\mathcal{A}_{L^\mu}$ is a superalgebra object, and we have shown the first part of the following proposition:

Proposition 4.2.3. Let $L \subset \mathcal{L}$ such that $\mathcal{A}_L$ is commutative and $\mu \in \mathcal{L}$ such that $\mu \notin L$ and $2\mu \in L$, then $\mathcal{A}_{L^\mu}$ is a superalgebra. $\mathcal{A}_{L^\mu}$ is supercommutative if and only if

$$2\langle \mu, \mu \rangle \in \ell\mathbb{Z} \setminus 2\ell\mathbb{Z} \qquad \text{and} \qquad 2\langle \mu, \lambda \rangle \in \ell\mathbb{Z}$$

for all $\lambda \in L$.

Proof. It follows from the proof of Theorem 4.2.1 that $\mathcal{A}_{L^\mu}$ is isomorphic to an algebra object with structure constants $t_{\lambda,\mu}$ as in equation (4.2.4) satisfying equation (4.2.11). Recall that

$$\mathcal{A}_{L^\mu} = \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \oplus \bigoplus_{\lambda \in L} \mathbb{C}_{\mu+\lambda}$$

where $\bigoplus_{\lambda \in L} \mathbb{C}_\lambda$ and $\bigoplus_{\lambda \in L} \mathbb{C}_{\mu+\lambda}$ are the $\bar{0}$ and $\bar{1}$ components of the $\mathbb{Z}_2$-grading respectively. For $\mathcal{A}_{L^\mu}$ to be supercommutative, $\mathcal{A}_L$ must be commutative which holds if and only if $\langle \lambda, \lambda \rangle \in \ell\mathbb{Z}$ and $2\langle \lambda, \lambda' \rangle \in \ell\mathbb{Z}$ by Theorem 4.2.1. It then follows from equation (2.1.8) that to show $\mathcal{A}_{L^\mu}$ is supercommutative, we need to show that

$$t_{\mu+\lambda_1,\mu+\lambda_2} = -q^{\langle \mu+\lambda_1, \mu+\lambda_2 \rangle} t_{\mu+\lambda_2,\mu+\lambda_1} \tag{4.2.12}$$

$$t_{\mu+\lambda_1,\lambda_2} = q^{\langle \mu+\lambda_1, \lambda_2 \rangle} t_{\lambda_2,\mu+\lambda_1} \tag{4.2.13}$$

$$t_{\lambda_1,\mu+\lambda_2} = q^{\langle \lambda_1, \mu+\lambda_2 \rangle} t_{\mu+\lambda_2,\lambda_1} \tag{4.2.14}$$

for all $\lambda_1, \lambda_2 \in L$. Let $\gamma_1, ..., \gamma_m$ be a generating set for L, and $\gamma_{m+1} = \mu$. We can apply Equation (4.2.11) to obtain the following relations

$$\begin{aligned} t_{\mu+\lambda_1,\mu+\lambda_2} &= q^{\langle \mu,\lambda_2\rangle} t_{\lambda_1,\lambda_2}, \\ t_{\mu+\lambda_1,\lambda_2} &= q^{\langle \mu,\lambda_2\rangle} t_{\lambda_1,\lambda_2}, \\ t_{\lambda_1,\mu+\lambda_2} &= t_{\lambda_1,\lambda_2}, \end{aligned}$$

for all $\lambda_1, \lambda_2 \in L$. It then follows from an easy computation using the fact that $t_{\lambda_1,\lambda_2} = q^{\langle \lambda_1,\lambda_2\rangle} t_{\lambda_2,\lambda_1}$ (from the proof of Theorem 4.2.1) that $\mathcal{A}_{L^\mu}$ is supercommutative if and only if

$$2\langle \mu, \mu\rangle \in \ell\mathbb{Z} \setminus 2\ell\mathbb{Z} \qquad \text{and} \qquad 2\langle \mu, \lambda\rangle \in \ell\mathbb{Z}$$

for all $\lambda \in L$. □

The categories $\mathcal{C}$ and $\mathcal{C} \boxtimes \mathcal{H}$ consist of finite length modules and therefore all indecomposables have simple submodules. If we assume this property for $\mathcal{B}$, then we have the following proposition which was proven in [CKM, Proposition 4.4] for module categories of vertex operator algebras, but holds more generally:

Proposition 4.2.4. Suppose every indecomposable object in $\mathcal{B}$ has a simple subobject. Then $N \in \mathrm{Rep}^0 \mathcal{A}_{L^\mu}$ is simple if and only if $N \cong \mathscr{F}(M)$ for a simple object $M \in \mathcal{B}$.

Proof. Let $M \in \mathcal{B}$ be simple and suppose $\mathscr{F}(M) \in \mathrm{Rep}\mathcal{A}_{L^\mu}$ is reducible with proper subobject X. We then have a $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$-morphism $X \hookrightarrow \mathscr{F}(M)$ which gives a $\mathcal{B}$-morphism

$$\mathcal{G}(X) \hookrightarrow \mathcal{G}(\mathscr{F}(M)) = \bigoplus_{\lambda \in L^\mu} M^\lambda \in \mathcal{B}^\oplus$$

by Frobenius reciprocity, where we have adopted the notation $M^\lambda := \mathbb{C}_\lambda \otimes M$. Since every $M^\lambda \in \mathcal{B}$ is simple, we have that

$$\mathcal{G}(X) = \bigoplus_{\lambda \in T} M^\lambda$$

for some non-empty subset $T \subset L^\mu$. By Corollary 4.2.2, the scalars t_{λ_1,λ_2} defining the product on $\mathcal{A}_{L^\mu}$ are non-zero, so we see that the action of $\mathcal{A}_{L^\mu}$ on $\mathscr{F}(M)$ satisfies

$$\mathbb{C}_\lambda \cdot M^\gamma \cong M^{\lambda+\gamma}.$$

In particular, any summand M^γ generates $\mathscr{F}(M)$, so $X = \mathscr{F}(M)$ and we see that $\mathscr{F}(M)$ has no non-trivial proper subobjects. It follows that the induction $\mathscr{F}(M)$ of a simple object $M \in \mathcal{B}$ is always simple.

Let $N \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ be simple and N' an indecomposable summand in $\mathcal{G}(N) \in \mathcal{B}^\oplus$ (so $N' \in \mathcal{B}$). Then N' contains a simple submodule M and we therefore have a non-zero map $f : M \to N' \hookrightarrow \mathcal{G}(N)$. By Frobenius reciprocity, we obtain a non-zero $\mathrm{Rep}\mathcal{A}_{L^\mu}$-morphism $g : \mathscr{F}(M) \to N$, which is an isomorphism because N and $\mathscr{F}(M)$ are both simple. $\square$

Given any superalgebra object $A = A^{\bar{0}} \oplus A^{\bar{1}}$, it is easy to check that $A^{\bar{1}} \in \mathrm{Rep}A^{\bar{0}}$ with action given by the restriction $\mu|_{A^{\bar{0}} \otimes A^{\bar{1}}} : A^{\bar{0}} \otimes A^{\bar{1}} \to A^{\bar{1}}$ of the product $\mu : A \otimes A \to A$. Suppose A is supercommutative and a direct sum of simple currents such that $A^{\bar{0}} = \mathcal{A}_L$ for some lattice $L \subset \mathcal{L}$ and $A^{\bar{1}}$ is a non-trivial simple object in $\mathrm{Rep}A^{\bar{0}}$. Since A is supercommutative, $A^{\bar{0}} = \mathcal{A}_L$ is commutative and because $A^{\bar{1}}$ is simple in $\mathrm{Rep}A^{\bar{0}}$, we know $A^{\bar{1}} \cong \mathscr{F}(M)$ for some simple module $M \in \mathcal{B}$ by Proposition 4.2.4. Since A is a direct sum of simple currents, $A^{\bar{1}} \cong \mathscr{F}(M) = \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \otimes M$ is a direct sum of simple currents, so we must have $M \cong \mathbb{C}_\mu$ for some $\mu \in \mathcal{L}$. Hence, we have

$$A = A^{\bar{0}} \oplus A^{\bar{1}} \cong \mathcal{A}_L \oplus \mathcal{G}(\mathscr{F}(\mathbb{C}_\mu)) = \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \oplus \bigoplus_{\lambda \in L} \mathbb{C}_{\mu+\lambda} = \mathcal{A}_{L^\mu}.$$

Since $m(\mathbb{C}_\mu \otimes \mathbb{C}_\mu) \subset \mathcal{A}_L$ we have $2\mu \in L$, and $\mu \notin L$ otherwise $A^{\bar{1}} = \mathcal{G}(\mathscr{F}(\mathbb{C}_\mu)) = \mathcal{A}_L$ is trivial in $\mathrm{Rep}\mathcal{A}_L$. We therefore have the following:

Corollary 4.2.5. Let $A = A^{\bar{0}} \oplus A^{\bar{1}}$ be a supercommutative superalgebra which is a direct sum of simple currents such that $A^{\bar{0}} = \mathcal{A}_L$ for some lattice $L \subset \mathcal{L}$ and $A^{\bar{1}}$ is a non-trivial simple object in $\mathrm{Rep}A^{\bar{0}}$. Then $A = \mathcal{A}_{L^\mu}$ for some lattice $L^\mu \subset \mathcal{L}$ is of the type described in Proposition 4.2.3.

4.3 Examples

Recall from Definition 3.3.1 that a $\overline{U}_q^H(\mathfrak{g})$-module V is called a weight module if the H_i act semi-simply on V and we have $K_\gamma = q^{\sum_{i=1}^{n} k_i d_i H_i}$ as operators on V for $\gamma = \sum_{i=1}^{n} k_i \alpha_i \in \mathsf{L}$, and $\mathcal{C}$ denotes the category of weight modules for $\overline{U}_q^H(\mathfrak{g})$ at ℓ-th root of unity. We denote by M^λ and S^λ the Verma and irreducible modules of highest weight $\lambda \in \mathfrak{h}^*$ in $\mathcal{C}$, and by P^λ the projective cover of S^λ. As noted in [R, Remark 4.6], the simple-currents in $\mathcal{C}$ are given by the set

$$\{S^\lambda \,|\, \lambda \in \mathcal{L}\} \tag{4.3.1}$$

where $\mathcal{L} := \{\lambda \in \mathfrak{h}^* \,|\, \lambda(H_i) \in \frac{\ell}{2d_i}\mathbb{Z}\}$. Throughout this section, we will denote by $\Gamma(X)$ the set of weights of a module $X \in \mathcal{C}$, and we adopt the notation $\mathbb{C}_\lambda := S^\lambda$ when $\lambda \in \mathcal{L}$.

Lemma 4.3.1. For any $\lambda \in \mathcal{L}$, the braiding $c_{-,-}$ acts as $\tau \circ \mathscr{H}$ (recall Equation 3.3.8) on $X \otimes \mathbb{C}_\lambda, \mathbb{C}_\lambda \otimes X \in \mathcal{C}$ for all $X \in \mathcal{C}$.

Proof. Recall that the braiding in $\mathcal{C}$ is given by $\tau \circ \mathscr{H}\mathscr{R}$ where τ is the flip map and $\mathscr{H}$ and $\mathscr{R}$ are defined by Equation 3.3.8 and 3.3.9. It follows immediately from equation (3.3.9) that action of the braiding coincides with that of $\tau \circ \mathscr{H}$ on $X \otimes \mathbb{C}_\lambda$ and $\mathbb{C}_\lambda \otimes X \in \mathcal{C}$ since for all i, $X_{\pm i}$ acts as zero on $\mathbb{C}_\lambda$. $\square$

Theorem 4.3.2.

- $\mathscr{F}(P^\lambda) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $\lambda \in \frac{\ell}{2}(L^\mu)^*$.
- Let $X \in \mathcal{C}$ and let $P_X \in C$ be the projective cover of X. Then $\mathscr{F}(X) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $\mathscr{F}(P_X) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$.
- $\mathscr{F}(P^\lambda)$ is the projective cover of $\mathscr{F}(S^\lambda)$ in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$.
- The distinct irreducible objects in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ are given by the set $\{\mathscr{F}(S^\lambda) \,|\, \lambda \in \Lambda(L^\mu)\}$ where $\Lambda(L^\mu) := \frac{\ell}{2}(L^\mu)^*/\frac{\ell}{2}(L^\mu)^* \cap L^\mu$. $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is finite if and only if $\mathrm{rank}(L^\mu) = \mathrm{rank}(P)$.

Proof. Recall from Definition 2.1.12 that $\mathscr{F}(X) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $M_{\mathbb{C}_\lambda, X} = c_{X,\mathbb{C}_\lambda} \circ$

$c_{\mathbb{C}_\lambda,X} = \mathrm{Id}$ for all $\lambda \in L^\mu$. It follows from Lemma 4.3.1 and equation (3.3.8) that for any vector $w_\gamma \in X$ of weight $\gamma \in \mathfrak{h}^*$ we have

$$M_{\mathbb{C}_\lambda,X}(v_\lambda \otimes w_\gamma) = q^{2\langle\lambda,\gamma\rangle}\mathrm{Id} \tag{4.3.2}$$

We therefore have X local if and only if $2\langle\lambda,\gamma\rangle \in \ell\mathbb{Z}$ for all $\gamma \in \Gamma(X)$ and $\lambda \in L^\mu$, which holds if and only if $\gamma \in \frac{\ell}{2}(L^\mu)^*$ for all $\gamma \in \Gamma(X)$. Recall, however, that $L^\mu \subset \mathcal{L}$, so it is easy to see that $\alpha \in \frac{\ell}{2}(L^\mu)^*$ for all $\alpha \in Q$. For any cyclic indecomposable $X \in \mathcal{C}$, all of the weights of X differ by an element of Q, so $\mathscr{F}(X) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ if and only if any one of its weights $\gamma \in \Gamma(X)$ satisfies equation (4.3.2). In particular, $\mathscr{F}(P^\lambda) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ if and only if $\lambda \in \frac{\ell}{2}(L^\mu)^*$ and $\mathscr{F}(X) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ if and only if $\mathscr{F}(P_X) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ as their weights dif and only ifer by elements in Q.

Define $G := \{\gamma \in \mathfrak{h}^* |\ 2\langle\lambda,\gamma\rangle \in \ell\mathbb{Z} \quad \forall\lambda \in L^\mu\} = \frac{\ell}{2}(L^\mu)^*$, so $\mathscr{F}(P^\gamma) \in \mathrm{Rep}^0(\mathcal{A}_{L^\mu})$ if and only if $\gamma \in G$. It is easy to see that two irreducible modules $S^{\gamma_1}, S^{\gamma_2}$ induce to the same module if and only if $\gamma_1 - \gamma_2 \in L^\mu$. In particular, the set of distinct $\mathscr{F}(L^\gamma)$ is in bijective correspondence with the quotient of free abelian groups $G/(G \cap L^\mu)$, which is finite if and only if $\mathrm{rank}(G) = \mathrm{rank}(\mathrm{G} \cap \mathrm{L}^\mu)$, which holds if and only if L^μ has full rank ($\mathrm{rank}(L^\mu) = \mathrm{rank}(P)$).

It follows immediately from Frobenius reciprocity that $\mathscr{F}(P^\lambda) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is projective, and the surjection $f : P^\lambda \twoheadrightarrow S^\lambda$ induces to a surjection $\mathscr{F}(f) = Id_{\mathcal{A}_L} \otimes f : \mathscr{F}(P^\lambda) \to \mathscr{F}(S^\lambda)$. If $\mathscr{F}(P^\lambda)$ is not the projective cover of $\mathscr{F}(S^\lambda)$, then $\mathscr{F}(f)$ must not be essential. That is, there exists a proper submodule $A \subset \mathscr{F}(P^\lambda)$ such that $\mathscr{F}(f)(A) = \mathscr{F}(S^\lambda)$. Note that

$$\mathscr{F}(P^\lambda) \cong \bigoplus_{\gamma\in L} P^{\lambda+\gamma} \qquad\qquad \mathscr{F}(S^\lambda) \cong \bigoplus_{\gamma\in L} S^{\lambda+\gamma}$$

and we must have $\mathscr{F}(f)|_{P^{\lambda+\gamma}} : P^{\lambda+\gamma} \to S^{\lambda+\gamma}$. Therefore, if f is not essential, there exists a P^γ with a submodule A' which surjects onto S^γ, contradicting the fact that P^γ is the projective cover of S^γ. $\square$

Recall that $r = \ell$ if ℓ is odd and $r = \ell/2$ if ℓ is even. We have the following:

Lemma 4.3.3. If $r \nmid 2d_i$ for all i, then $\mathrm{Ext}^1(\mathbb{C}_{\lambda_1}, \mathbb{C}_{\lambda_2}) = 0$ for all $\lambda_1, \lambda_2 \in \mathcal{L}$.

Proof. Let $N \in \mathrm{Ext}^1(\mathbb{C}_{\lambda_1}, \mathbb{C}_{\lambda_2})$, that is,

$$0 \to \mathbb{C}_{\lambda_2} \to N \to \mathbb{C}_{\lambda_1} \to 0. \tag{4.3.3}$$

If $\lambda_1 = \lambda_2$, then the sequence splits since $X_{\pm i}$ acts as zero for all i as $\mathrm{ch}[N] = 2z^\lambda$ and the H_i act semisimply. Suppose now that $\lambda_1 > \lambda_2$. If the sequence does not split, we must have $\lambda_1 = \lambda_2 - \alpha_i$ for some i (otherwise, $X_{\pm i} v_{\lambda_1} = 0$ for all i and the sequence splits as above). Any vector $n_{\lambda_1} \in N(\lambda_1)$ is highest weight and generates N. We therefore have a quotient map $\phi : M^{\lambda_1} \twoheadrightarrow N \; v_{\lambda_1} \mapsto n_{\lambda_1}$, from the Verma module M^{λ_1} of highest weight λ_1 onto N. However, $\lambda_1 \in \mathcal{L}$, so it follows from equation (3.2.5) that we have

$$X_i X_{-i}^2 v_{\lambda_1} = [\lambda_1(H_i) - 2]_i v_{\lambda_1} = -[2]_i X_{-i} v_{\lambda_1}.$$

$[2]_i = 0$ if and only if $r | 2d_i$. We then see that $X_{-i}^2 v_{\lambda_1} \neq 0$, a contradiction, so the module N cannot exist if $r \nmid 2d_i$ for all i. If $\lambda_1 < \lambda_2$, then the above argument tells us that the dualized sequence

$$0 \to \mathbb{C}_{\lambda_1} \to \check{N} \to \mathbb{C}_{\lambda_2} \to 0$$

splits, where $(\check{-})$ is the exact contravariant functor defined in [R, Subsection 4.2], hence the original sequence splits. $\square$

Let $L^\mu \subset \mathcal{L}$ such that $\mathcal{A}_{L^\mu}$ is a supercommutative superalgebra object. Recall from Equation (3.3.10), that the twist acts on $\mathbb{C}_\gamma$ as

$$\theta_{\mathbb{C}_\gamma} = q^{\langle \gamma, \gamma + 2(1-r)\rho \rangle} \mathrm{Id} = q^{\langle \gamma, \gamma \rangle + 2(1-r)\langle \gamma, \rho \rangle} \mathrm{Id}.$$

It then follows from [CKM, Proposition 2.86] and Proposition 4.2.3 that $\mathrm{Rep}^0 \mathcal{A}_{L^\mu}$ is ribbon if $2(1-r)\langle \lambda, \rho \rangle \in \ell\mathbb{Z}$ for all $\lambda \in L$ and $2(1-r)\langle \mu, \rho \rangle \in \frac{\ell}{2}\mathbb{Z}$. This gives us the first statement of the following proposition, where we recall that $\Lambda(L^\mu) := \frac{\ell}{2}(L^\mu)^* / \frac{\ell}{2}(L^\mu)^* \cap L^\mu$.

Proposition 4.3.4. Let $\mathcal{A}_{L^\mu}$ be a supercommutative superalgebra. Then

- $\mathrm{Rep}^0 \mathcal{A}_{L^\mu}$ is ribbon if $2(1-r)\langle \lambda, \rho \rangle \in \ell\mathbb{Z}$ for all $\lambda \in L$ and $2(1-r)\langle \mu, \rho \rangle \in \frac{\ell}{2}\mathbb{Z}$.

- If $r \nmid 2d_i$ for all i, then $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ has non-trivial Müger center if and only if there exists a $\lambda \in \Lambda(L^\mu)$ such that $\langle \lambda, \gamma \rangle \in \frac{\ell}{2}\mathbb{Z}$ for all $\gamma \in \Lambda(L^\mu)$.

Proof. As the braiding on induced modules (hence all simple modules) in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ descends from that of $\mathcal{C}^\oplus$ [CKM, Theorem 2.67] (i.e. acts by the same scalar), an irreducible module $\mathscr{F}(S^\lambda) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is transparent to all other irreducible modules $\mathscr{F}(S^{\lambda'}) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ if and only if it satisfies

$$c_{\mathscr{F}(S^\lambda),\mathscr{F}(S^{\lambda'})} \circ c_{\mathscr{F}(S^{\lambda'}),\mathscr{F}(S^\lambda)} = q^{2\langle \lambda,\lambda' \rangle}\mathrm{Id}_{\mathscr{F}(S^\lambda)\otimes\mathscr{F}(S^{\lambda'})} = \mathrm{Id}_{\mathscr{F}(S^\lambda)\otimes\mathscr{F}(S^{\lambda'})}$$

for all $\mathscr{F}(S^{\lambda'}) \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$. Recall from Theorem 4.3.2 that irreducibles in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ are given by the set $\{\mathscr{F}(S^\lambda) \,|\, \lambda \in \Lambda(L^\mu)\}$ where $\Lambda(L^\mu) = \frac{\ell}{2}(L^\mu)^*/\frac{\ell}{2}(L^\mu)^* \cap L^\mu$. Hence, $\mathscr{F}(L^\lambda)$ is transparent if and only if $\langle \lambda, \lambda' \rangle \in \frac{\ell}{2}\mathbb{Z}$ for all $\lambda' \in \Lambda(L^\mu)$. It follows that if there is no such $\lambda \in \Lambda(L^\mu)$, then the only transparent irreducible object is the unit object $\mathcal{A}_{L^\mu}$. If an indecomposable module $X \in \mathrm{Rep}^0\mathcal{A}_{L^\mu}$ is transparent, then all of the irreducible factors in its composition series are. Therefore, all transparent objects in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ are extensions of $\mathcal{A}_{L^\mu}$. Suppose now that we have an object $N \in \mathrm{Ext}^1_{\mathrm{Rep}^0\mathcal{A}^\mu_L}(\mathcal{A}_{L^\mu}, \mathcal{A}_{L^\mu})$, that is,

$$0 \to \mathcal{A}_{L^\mu} \to N \to \mathcal{A}_{L^\mu} \to 0.$$

This sequence splits if and only if $\dim \mathrm{Hom}(\mathcal{A}_{L^\mu}, N) = 2$. The restriction functor $\mathcal{G}$: $\mathrm{Rep}\mathcal{A}_{L^\mu} \to \mathcal{C}$ then yields a corresponding sequence

$$0 \to \bigoplus_{\lambda \in L^\mu} \mathbb{C}_\lambda \xrightarrow{\iota} \mathcal{G}(N) \xrightarrow{\pi} \bigoplus_{\lambda \in L^\mu} \mathbb{C}_\lambda \to 0$$

in $\mathcal{C}^\oplus$. Let $\iota_\gamma := \iota|_{\mathbb{C}_\gamma}$ and N^γ the indecomposable factor in $\mathcal{G}(N)$ containing $\mathrm{Im}(\iota_\gamma)$. Then we have an exact sequence

$$0 \to \mathbb{C}_\gamma \xrightarrow{\iota_\gamma} N^\gamma \xrightarrow{\pi|_{N^\gamma}} \bigoplus_{\lambda \in T} \mathbb{C}_\lambda \to 0$$

for some finite $T \subset \mathcal{L}$. It follows from Lemma 4.3.3 that this sequence splits for all γ, so $\dim \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}_\gamma, \mathcal{G}(N)) \geq 2$. By Frobenius reciprocity, we have $\dim \mathrm{Hom}(\mathcal{A}_{L^\mu}, X) \geq 2$, so the original sequence splits. It follows that all transparent objects in $\mathrm{Rep}^0\mathcal{A}_{L^\mu}$ are trivial (direct sums of $\mathcal{A}_{L^\mu}$). $\square$

4.3.1 Braided Tensor Categories for $W_Q(r)$

The triplet vertex operator algebra $W_Q(r)$ associated to the root lattice Q of a finite dimensional complex simple Lie algebra $\mathfrak{g}$ of ADE type can be written as a simple current extension of the singlet as

$$W_Q(r) = \bigoplus_{\lambda \in \sqrt{r}Q} F_\lambda \in (\mathrm{Rep}_{\langle s \rangle} W_Q^0(r))^{\oplus}.$$

The algebra object in $\mathcal{C}^{\oplus}$ corresponding to the triplet vertex operator algebra under the map $\phi : \mathrm{Rep}_{\langle s \rangle} W_Q^0(r) \to \mathrm{Rep}\overline{U}_q^H(\mathfrak{g})^{wt}$ of Equation (4.1.1) is

$$\mathcal{A}_{rQ} = \bigoplus_{\lambda \in rQ} \mathbb{C}_\lambda.$$

It follows from Theorem 4.3.2 that the irreducible modules in $\mathrm{Rep}^0 \mathcal{A}_L$ are

$$\{\mathscr{F}(S^\lambda) \,|\, \lambda \in P/rQ\}$$

Recall that the order $|P/rQ|$ is equal to the determinant of the change of basis matrix $A_{rQ \leftarrow P}$, which is easily seen to be rA where A is the Cartan matrix of $\mathfrak{g}$, so $\mathrm{Rep}^0\mathcal{A}_{rQ}$ has $\mathrm{Det}(A) \cdot r^{\mathrm{rank}(\mathfrak{g})}$ distinct irreducible objects. Further, we have $\langle \rho, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in Q$, so it follows from Proposition 4.3.4 that $\mathrm{Rep}^0\mathcal{A}_{rQ}$ is ribbon. Suppose $\lambda \in \mathrm{Rep}^0\mathcal{A}_L$ is transparent, i.e. $\langle \lambda, \gamma \rangle \in r\mathbb{Z}$ for all $\gamma \in P/rQ$. It follows that $\langle \lambda, \omega_k \rangle \in r\mathbb{Z}$ for all fundamental weights $\omega_k \in P$, so $\lambda \in Q/rQ$ and we can write $\lambda = \sum_{k=1}^{n} m_i \alpha_i$ where $\sum_{k=1}^{n} m_i \in r\mathbb{Z}$ and $0 \le m_i < r$. Then, if $\langle \lambda, \rho \rangle \in r\mathbb{Z}$, choose j such that $n_j \neq 0$, then $\langle \lambda, \rho + \omega_j \rangle = \langle \lambda, \rho \rangle + m_j \notin r\mathbb{Z}$. Hence, $\mathrm{Rep}^0\mathcal{A}_{rQ}$ has trivial Müger center by Proposition 4.3.4.

Proposition 4.3.5. $\mathrm{Rep}^0\mathcal{A}_{rQ}$ is a finite non-degenerate ribbon category (i.e. Log-Modular) with $\det(A) \cdot r^{\mathrm{rank}(\mathfrak{g})}$ distinct irreducible modules.

We expect that $\mathrm{Rep}^0\mathcal{A}_{rQ}$ and the category $\mathrm{Rep}_{\langle s \rangle} W_Q(r)$ generated by irreducible $W_Q(r)$-modules are ribbon equivalent.

4.3.2 Braided Tensor Categories for $B_Q(r)$

Let H denote the Heisenberg vertex operator algebra of Subsection 4.1 whose rank coincides with that of $\mathfrak{g}$, and $\mathcal{H}$ the category of modules on which the zero-mode L_0 of the Virasoro element acts semi-simply. This category is semisimple with irreducible modules F_λ, $\lambda \in \mathfrak{h}^*$ and fusion rules

$$\mathsf{F}_\lambda \otimes \mathsf{F}_\mu \cong \mathsf{F}_{\lambda+\mu}.$$

We see then every object $\mathsf{F}_\lambda \in \mathcal{H}$ is a simple current with tensor inverse $\mathsf{F}_{-\lambda}$. Hence, any object $\mathbb{C}_\lambda \boxtimes \mathsf{F}_\gamma \in \mathcal{C} \boxtimes \mathcal{H}$ in the Deligne product of $\mathcal{C}$ $\mathcal{H}$ with $\gamma \in \mathfrak{h}^*$ and $\lambda \in \mathcal{L} = \{\lambda \in \mathfrak{h}^* \,|\, \lambda(H_i) \in \frac{\ell}{2d_i}\mathbb{Z}\}$ is a simple current. We can therefore define a family $\mathfrak{B}^a_L$ of objects in $(\mathcal{C} \boxtimes \mathcal{H})^\oplus$ where $a \in \mathbb{C}$ as:

$$\mathfrak{B}^a_L := \bigoplus_{\lambda \in L} \mathbb{C}_\lambda \boxtimes \mathsf{F}_{a\lambda}$$

It follows from Lemma 4.3.1 and Equation (4.1.2) that the braiding in $\mathcal{C} \boxtimes \mathcal{H}$ acts on simple currents as $c_{\mathbb{C}_\lambda \boxtimes \mathsf{F}_{a\lambda}, \mathbb{C}_\mu \boxtimes \mathsf{F}_{a\mu}} = q^{(1+pa^2)\langle \lambda, \mu\rangle} Id$. We then have the following corollary of Theorem 4.2.1:

Corollary 4.3.6. $\mathfrak{B}^a_L \in (\mathcal{C} \boxtimes \mathcal{H})^\oplus$ is a commutative algebra object if and only if

$$(1+pa^2)\langle \lambda, \lambda' \rangle \in p\mathbb{Z},$$
$$(1+pa^2)\langle \lambda, \lambda \rangle \in 2p\mathbb{Z},$$

for all $\lambda, \lambda' \in L$.

The higher rank $B_Q(r)$ vertex operator algebras of ADE type can be realized as (see [C3, Section 3])

$$B_Q(r) = \bigoplus_{\lambda \in P} M_{\sqrt{r}\lambda} \otimes \mathsf{F}_{\sqrt{-r}\lambda} \in \mathrm{Rep}_{\langle s \rangle} \mathcal{W}^0(r)_\mathfrak{g} \otimes \mathcal{H} \tag{4.3.4}$$

where $M_{\sqrt{r}\lambda}$ is a simple current for $W^0_Q(r)$, which corresponds to $\mathbb{C}_{r\lambda}$ under the correspondence in equation (4.1.1). $B_Q(r)$ therefore corresponds to the object

$$\mathfrak{B}^{a_r}_{rP} := \bigoplus_{\lambda \in P} \mathbb{C}_{r\lambda} \boxtimes \mathsf{F}_{\sqrt{-r}\lambda} = \bigoplus_{\lambda \in rP} \mathbb{C}_\lambda \boxtimes \mathsf{F}_{a_r\lambda}$$

where $a_r = \sqrt{-1/r}$, which clearly satisfies the conditions of Corollary 4.3.6 since $1 + ra_r^2 = 0$.

It follows from Equations (3.3.10) and (4.1.3) that the twist acts on $\mathbb{C}_\lambda \boxtimes \mathsf{F}_{a_r\lambda}$ as

$$\theta_{\mathbb{C}_\lambda \boxtimes \mathsf{F}_{a_r\lambda}} = \theta_{\mathbb{C}_\lambda} \boxtimes \theta_{\mathsf{F}_{a_r\lambda}} = q^{\langle \lambda, \lambda+2(1-r)\rho\rangle - \langle \lambda,\lambda\rangle} = q^{2(1-r)\langle \lambda,\rho\rangle}.$$

Hence, by [CKM, Proposition 2.86] $\mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ is ribbon if and only if $(1-r)\langle\lambda,\rho\rangle \in \mathbb{Z}$ for all $\lambda \in P$. $\rho \in \frac{1}{2}Q$, so this holds if r is odd, or $\rho \in Q$. The irreducible modules in $\mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ are $\{\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma) \,|\, \langle \lambda, \mu + r a_r \gamma\rangle \in \mathbb{Z} \text{ for all } \lambda \in P\}$ with relations $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma) \cong \mathscr{F}(S^{\mu+\lambda} \boxtimes \mathsf{F}_{\gamma + a_r\lambda})$ for all $\lambda \in rP$. The monodromy (double braiding) acts on pairs of irreducible modules $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma)$, $\mathscr{F}(S^{\mu'} \boxtimes \mathsf{F}_{\gamma'})$ as $M_{\mathscr{F}(S^\mu\boxtimes\mathsf{F}_\gamma),\mathscr{F}(S^{\mu'}\boxtimes\mathsf{F}_{\gamma'})} = q^{2\langle\mu,\mu'\rangle + 2r\langle\gamma,\gamma'\rangle}$, so $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma)$ is transparent if and only if

$$2\langle\mu,\mu'\rangle + 2r\langle\gamma,\gamma'\rangle \in 2r\mathbb{Z} \tag{4.3.5}$$

for all $\mu',\gamma' \in \mathfrak{h}^*$ such that $\langle\lambda, \mu' + r a_r\gamma'\rangle \in \mathbb{Z}$ for all $\lambda \in P$. It is easy to see that $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma) \in \mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ implies $\mathscr{F}(S^{\mu+\alpha} \boxtimes \mathsf{F}_{\gamma + \frac{1}{ra_r}\beta}) \in \mathrm{Rep}^0\mathcal{A}_{rP}$ for all $\alpha,\beta \in Q$. It then follows from Equation (4.3.5) that $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma)$ is transparent if and only if $\mu \in rP$ and $\gamma \in ra_rP$. Let $\mu = r\tilde{\mu}$ and $\gamma = ra_r\tilde{\gamma}$ where $\tilde{\mu},\tilde{\gamma} \in P$. Then,

$$\begin{aligned} & & 2\langle\mu,\mu'\rangle + 2r\langle\gamma,\gamma'\rangle \in 2r\mathbb{Z} \\ \Leftrightarrow & & 2r\langle\tilde{\mu},\mu'\rangle + 2r^2a_r\langle\tilde{\gamma},\gamma'\rangle \in 2r\mathbb{Z} \\ \Leftrightarrow & & \langle\tilde{\mu},\mu'\rangle + \langle\tilde{\gamma},\mu'\rangle - \langle\tilde{\gamma},\mu'\rangle + ra_r\langle\tilde{\gamma},\gamma'\rangle \in \mathbb{Z} \\ \Leftrightarrow & & \langle\tilde{\mu}-\tilde{\gamma},\mu'\rangle + \langle\tilde{\gamma},\mu' + ra_r\gamma'\rangle \in \mathbb{Z} \\ \Leftrightarrow & & \langle\tilde{\mu}-\tilde{\gamma},\mu'\rangle \in \mathbb{Z} \end{aligned}$$

where we have used the fact that $\langle\lambda,\mu' + ra_r\gamma'\rangle \in \mathbb{Z}$ for all $\lambda \in P$ and $\gamma' \in P$. Hence, $\mathscr{F}(S^\mu \boxtimes \mathsf{F}_\gamma)$ is transparent if and only if $\langle\tilde{\mu}-\tilde{\gamma},\mu'\rangle \in \mathbb{Z}$ for all μ' such that $\mathscr{F}(S^{\mu'} \boxtimes \mathsf{F}_{\gamma'}) \in \mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ for some $\gamma' \in \mathfrak{h}^*$. This clearly never holds as we can always choose a μ' not in the root lattice Q. Hence, there are no transparent objects and the argument in Proposition 4.3.4 can be applied to show that the Müger center is trivial. We therefore make the following conjecture for the full subcategory $\mathrm{Rep}_{\langle s\rangle}B_Q(r)$ of $B_Q(r)$-modules generated by irreducibles:

Proposition 4.3.7. $\mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ is non-degenerate, ribbon if r is odd or $\rho \in Q$, and the irreducible modules can be indexed as

$$\{S^{\mu}_{\gamma} \mid \mu, \gamma \in \mathfrak{h}^*, \text{ and } \mu + r a_r \gamma \in Q\ \}$$

with relations $S^{\mu}_{\gamma} \cong S^{\mu+\lambda}_{\gamma+a_r\lambda}$ for all $\lambda \in rP$, where $a_r = \sqrt{-1/r}$.

We expect that $\mathrm{Rep}^0\mathfrak{B}^{a_r}_{rP}$ and the category $\mathrm{Rep}_{\langle s\rangle}B_Q(r)$ generated by irreducible $B_Q(r)$-modules are ribbon equivalent.

Chapter 5

Projective modules for $\overline{U}_i^H(\mathfrak{sl}_3)$

In this Chapter we consider the special case of $\overline{U}_i^H(\mathfrak{sl}_3)$, which is the easiest case beyond rank 1. The category of weight modules for $\overline{U}_i^H(\mathfrak{sl}_3)$ is expected to be ribbon equivalent to the category of modules generated by irreducibles for the singlet vertex algebra $W^0_{A_2}(2)$, $\mathrm{Rep}_{\langle s\rangle} W^0_{A_2}(2)$. The corresponding triplet $W_{A_2}(2)$ and $B_{A_2}(2)$ algebras can be constructed from this category. $B_{A_2}(2)$ is particularly interesting because it is isomorphic to the simple affine vertex algebra $L_{-\frac{3}{2}}(\mathfrak{sl}_3)$. Our primary focus is to study the structure of projective modules in the category of weight modules for $\overline{U}_i^H(\mathfrak{sl}_3)$. To reach this goal, we determine the structure and characters of all irreducible modules, and the Loewy diagrams for Verma modules. Combining these results with the self-duality of projective covers (Theorem 3.3.14) allows us to determine the Loewy diagrams of all projective covers. Knowing the Loewy diagrams also allows us to compute all tensor product $S^\lambda \otimes M^\mu$ between irreducible modules and projective Verma modules. The results of this chapter will appear in a future paper coauthored with Thomas Creutzig and David Ridout, along with the construction of braided tensor categories for $W_{A_2}(2)$ and $B_{A_2}(2)$, and a comparison with the known properties of $\mathrm{Rep}_{\langle s\rangle} W_{A_2}(2)$ and $\mathrm{Rep}_{\langle s\rangle} L_{-\frac{3}{2}}(\mathfrak{sl}_3)$.

5.1 Irreducible and Verma modules

Throughout this chapter we consider the category of weight modules $\mathcal{C}$ for the unrolled restricted quantum group $\overline{U}_i^H(\mathfrak{sl}_3)$. Recall from Propositions 3.3.4 and 3.3.6 that the irreducibility of a Verma module M^λ is determined by the typicality $\lambda \in \mathfrak{h}^*$. This can be translated into the $\mathfrak{g} = \mathfrak{sl}_3$ case as follows:

Definition 5.1.1. For $\lambda \in \mathfrak{h}^*$, let $\lambda_j := \langle \lambda + \rho, \alpha_j \rangle$ for $j = 1, 2$ and $\lambda_3 = \lambda_1 + \lambda_2$. We call λ_k ($k \in \{1, 2, 3\}$) typical if $\lambda_k \in \ddot{\mathbb{C}} := \mathbb{C} \setminus \mathbb{Z} \cup 2\mathbb{Z}$. We call a weight $\lambda \in \mathfrak{h}^*$ typical if λ_k is typical for $k = 1, 2, 3$, and semi-typical if some, but not all λ_k are typical. Note that

$$\lambda_k = \langle \lambda + \rho, \alpha_k \rangle = \langle \lambda, \alpha_k \rangle + 1 = \lambda(H_k) + 1. \tag{5.1.1}$$

It is easy to check that there are five distinct cases:

1. λ typical,
2. $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$,
3. $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 1 + 2\mathbb{Z}$ or $\lambda_2 \in 2\mathbb{Z}$ and $\lambda_1 \in 1 + 2\mathbb{Z}$,
4. $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$ and $\lambda_2 \in 1 + 2\mathbb{Z}$ or $\lambda_2 \in \mathbb{Z} \setminus \mathbb{Z}$ and $\lambda_1 \in 1 + 2\mathbb{Z}$,
5. $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$ and $\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$.

Each case yields a different structure for S^λ, as described in the following proposition:

Proposition 5.1.2.

- If $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$, then $\dim(S^\lambda) = 1$ and $\mathrm{ch}[S^\lambda] = z^\lambda$.
- If $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$ with $\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$, then $\dim(S^\lambda) = 4$ with

$$\mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda - \alpha_1} + z^{\lambda - \alpha_2} + z^{\lambda - \alpha_1 - \alpha_2}$$

- If λ is semi-typical with atypical index λ_i ($i \in \{1, 2\}$), then if $j \neq i \in \{1, 2\}$,

$$\dim(S^\lambda) = 3 \qquad \mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda - \alpha_j} + z^{\lambda - (\alpha_1 + \alpha_2)} \qquad \text{if } \lambda_j \in 2\mathbb{Z}$$

$$\dim(S^\lambda) = 4 \qquad \mathrm{ch}[S^\lambda] = z^\lambda + z^{\lambda-\alpha_j} + z^{\lambda-(\alpha_1+\alpha_2)} + z^{\lambda-\alpha_i-2\alpha_j} \qquad \text{if } \lambda_j \in \mathbb{C}\setminus\mathbb{Z}$$

- If λ is typical, then $\dim(S^\lambda) = 8$ and

$$\mathrm{ch}[S^\lambda] = \mathrm{ch}[M^\lambda] = z^\lambda \prod_{\alpha\in\Delta^-}\left(\frac{z^{2\alpha}-1}{z-1}\right)$$

Proof. Let $v_\lambda \in S^\lambda$ be a vector of weight λ. If $\lambda_1, \lambda_2 \in 1+2\mathbb{Z}$, then $X_{-1}v_\lambda$ and $X_{-2}v_\lambda$ are maximal since $X_k X_{-k'} v_\lambda = \delta_{k,k'}[\lambda(H_k)]v_\lambda = 0$ by Equation (5.1.1), and are therefore contained in the maximal submodule of M^λ. Hence, S^λ is one-dimensional with character z^λ.

Suppose that $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$ and $\lambda_3 \in 1+2\mathbb{Z}$. Then we have $\lambda(H_1)+\lambda(H_2) = \lambda_1+\lambda_2-2 \in 1+2\mathbb{Z}$ by Equation (5.1.1). We have the following relation:

$$\begin{aligned}
X_1([\lambda(H_2)-1]X_{-1}X_{-2}v_\lambda + [\lambda(H_2)]X_{-2}X_{-1}v_\lambda) &= ([\lambda(H_2)-1][\lambda(H_1)+1] + [\lambda(H_2)][\lambda(H_1)])X_{-2}v_\lambda \\
&= 2(i^{\lambda(H_1+H_2)} + i^{-\lambda(H_1+H_2)})X_{-2}v_\lambda \\
&= 2(i^{\lambda(H_1+H_2)} + (-1)^{\lambda(H_1+H_2)} i^{\lambda(H_1+H_2)})X_{-2}v_\lambda \\
&= 2(i^{\lambda(H_1+H_2)} - i^{\lambda(H_1+H_2)})X_{-2}v_\lambda = 0
\end{aligned}$$

where the second line follows by expanding terms and the final line follows from $\lambda(H_1) + \lambda(H_2) \in 1+2\mathbb{Z}$. We also have the relation

$$\begin{aligned}
X_2([\lambda(H_2)-1]X_{-1}X_{-2}v_\lambda + [\lambda(H_2)]X_{-2}X_{-1}v_\lambda) &= [\lambda(H_2)]([\lambda(H_2)-1] + [\lambda(H_2)+1])X_{-1}v_\lambda \\
&= \frac{[\lambda(H_2)]}{\{1\}}(i^{\lambda(H_2)}(i+i^{-1}) - i^{-\lambda(H_2)}(i+i^{-1})) = 0
\end{aligned}$$

since $i^{-1} = -i$. We therefore see that $v = [\lambda(H_2)-1]X_{-1}X_{-2}v_\lambda + [\lambda(H_2)]X_{-2}X_{-1}v_\lambda \in M^\lambda(\lambda-\alpha_1-\alpha_2)$ is maximal and generates a proper submodule which must also contain $X_{-1}v, X_{-2}v$, and $X_{-1}X_{-2}v$. Further, since $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$, we easily check that $X_{-1}v_\lambda, X_{-2}v_\lambda$, and $X_{-1}X_{-2}v_\lambda$ are not in the maximal submodule of M^λ. Hence, S^λ has dimension 4 and character $z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-\alpha_2} + z^{\lambda-\alpha_1-\alpha_2}$.

Let $\lambda_1 \in \ddot{\mathbb{C}}$, $\lambda_2 \in 1 + 2\mathbb{Z}$. Then, $X_{-2}v_\lambda$ is maximal since $X_k X_{-2} v_\lambda = \delta_{k,2}[\lambda(H_2)]v_\lambda = 0$. Therefore, $X_{-2}v_\lambda$, $X_{-1}X_{-2}v_\lambda$, $X_{-2}X_{-1}X_{-2}v_\lambda$, and $X_{-1}X_{-2}X_{-1}X_{-2}v_\lambda$ are all contained in the maximal submodule of M^λ. We have the following:

$$\begin{aligned}
X_1 X_{-1} v_\lambda &= [\lambda(H_1)]v_\lambda, \\
X_2 X_{-2} X_{-1} v_\lambda &= [\lambda(H_2) + 1]X_{-1}v_\lambda, \\
X_1 X_{-1} X_{-2} X_{-1} v_\lambda &= [\lambda(H_1) - 1]X_{-2}X_{-1}v_\lambda + [\lambda(H_1)]X_{-1}X_{-2}v_\lambda \\
X_2 X_{-1} X_{-2} X_{-1} v_\lambda &= 0.
\end{aligned}$$

We therefore see that $X_{-1}v_\lambda$ and $X_{-2}X_{-1}v_\lambda$ are not contained in the maximal submodule of M^λ since $[\lambda(H_1)]$ and $[\lambda(H_2)+1]$ are non-zero, and $X_{-1}X_{-2}X_{-1}v_\lambda$ is contained in the maximal submodule iff $\lambda(H_1) - 1 = \lambda_1 - 2 \in 2\mathbb{Z}$, so the result follows. The case $\lambda_2 \in \ddot{\mathbb{C}}, \lambda_1 \notin 1 + 2\mathbb{Z}$ is similar.

If λ is typical, then $S^\lambda = M^\lambda$. The Verma module M^λ is a free $\overline{U}_q^H(\eta^-)$-module, so by Proposition 3.2.4 has a basis given by $\{X_{-1}^{n_1}X_{-3}^{n_3}X_{-2}^{n_2}v_\lambda \,|\, n_k = 0,1\}$ where $X_{-3} := -X_{-2}X_{-1} + qX_{-1}X_{-2}$ is the vector associated to the root $-\alpha_1 - \alpha_2$. Therefore, $\dim(S^\lambda) = 2^3 = 8$ and the character is given by

$$\mathrm{ch}[S^\lambda] = \mathrm{ch}[M^\lambda] = \sum_{\substack{n_\alpha = 0 \\ \alpha \in \Delta^+}}^{1} z^{\lambda - \sum_{\alpha\in\Delta^+} n_\alpha \alpha} = z^\lambda \prod_{\alpha \in \Delta^+} \left(\sum_{n_\alpha = 0}^{1} z^{n_\alpha \alpha} \right) = z^\lambda \prod_{\alpha\in\Delta^+} \left(\frac{z^{2\alpha} - 1}{z - 1} \right).$$

where we have adopted the notation $n_k = n_{\alpha_k}$ for $k = 1, 2$ and $n_3 = n_{\alpha_1 + \alpha_2}$ □

We are now ready to begin constructing Loewy diagrams, which we now define. Recall that a filtration, or series, for a module M is a family of proper submodules ordered by inclusion as

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

Such a series is a composition (standard) series if successive quotients are irreducible (Verma) modules: $M_k/M_{k-1} \cong S^{\lambda_k}$ ($M_k/M_{k-1} \cong M^{\lambda_k}$) for some $\lambda_k \in \mathfrak{h}^*$. Loewy diagrams are defined in terms of socle filtrations.

Definition 5.1.3. The socle filtration of M is the filtration defined by $M_1 = \mathit{Socle}(M)$

(the socle of a module is its largest semi-simple submodule) and we inductively define M_k to be the largest submodule of M such that M_k/M_{k-1} is semi-simple. We define the Loewy diagram of M to be the diagram whose k-th layer from the bottom consists of composition factors of the semisimple module M_k/M_{k-1} with downward arrows indicating submodule inclusion.

Knowing the structure of irreducibles in each case is sufficient for determining the Loewy diagrams of Verma modules.

Proposition 5.1.4. The Loewy diagrams for the reducible M^λ are as follows:

$$\begin{tikzcd} & {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\ {}^{(1)}S^{\lambda-\alpha_1} \arrow[dr] & M^{\lambda} & {}^{(1)}S^{\lambda-\alpha_1-2\alpha_2} \arrow[dl] \\ & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} & \end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}$, $\lambda_2 \in 2\mathbb{Z}$.

$$\begin{tikzcd} & {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\ {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dr] & M^{\lambda} & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dl] \\ & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} & \end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 2\mathbb{Z}$, $\lambda_2 \in 1+2\mathbb{Z}$.

$$\begin{tikzcd} M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\ & {}^{(4)}S^{\lambda-\alpha_2} \end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_2 \in 1+2\mathbb{Z}$.
$\lambda_2 \in 1+2\mathbb{Z}$.

$$\begin{tikzcd} M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\ & {}^{(4)}S^{\lambda-\alpha_1} \end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}$,
$\lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$.

$$\begin{tikzcd} M^{\lambda} & {}^{(4)}S^{\lambda} \arrow[d] \\ & {}^{(4)}S^{\lambda-\alpha_1-\alpha_2} \end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$
$\lambda_1 + \lambda_2 \in 1+2\mathbb{Z}$

$$\begin{tikzcd} & {}^{(1)}S^{\lambda} \arrow[dl] \arrow[dr] & \\ {}^{(3)}S^{\lambda-\alpha_1} \arrow[dr] & M^{\lambda} & {}^{(3)}S^{\lambda-\alpha_2} \arrow[dl] \\ & {}^{(1)}S^{\lambda-2(\alpha_1-\alpha_2)} & \end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in 1+2\mathbb{Z}$.

where left superscripts indicate the dimension of S^λ.

Proof. Let $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 1+2\mathbb{Z}$. It follows from Proposition 5.1.2 that the top of M^λ is a 3 dimensional irreducible module spanned by v_λ, $X_{-1}v_\lambda$, and $X_{-2}X_{-1}v_\lambda$. One easily checks that $X_{-2}v_\lambda$, and $X_{-1}X_{-2}v_\lambda$ are maximal. Each of these vectors generate submodules of M^λ and we have

$$\begin{aligned}
(\lambda-\alpha_2)_1 &= \lambda_1+1 \in 1+2\mathbb{Z} & \qquad (\lambda-\alpha_2)_2 &= \lambda_2-2 \in 1+2\mathbb{Z}\\
(\lambda-\alpha_1-\alpha_2)_1 &= \lambda_1-1 \in 1+2\mathbb{Z} & \qquad (\lambda-\alpha_1-\alpha_2)_2 &= \lambda_2-1 \in 2\mathbb{Z}\\
(\lambda-2\alpha_1-\alpha_2)_1 &= \lambda_1-3 \in 1+2\mathbb{Z} & \qquad (\lambda-2\alpha_1-\alpha_2)_2 &= \lambda_2 \in 1+2\mathbb{Z}
\end{aligned}$$

Therefore by Proposition 5.1.2 $X_{-2}v_\lambda$ generates submodule whose irreducible quotient is one dimensional, and $X_{-1}X_{-2}v_\lambda$ generates an irreducible 3-dimensional submodule spanned by $X_{-1}X_{-2}v_\lambda, X_{-2}X_{-1}X_{-2}v_\lambda$, and $X_{-1}X_{-2}X_{-1}X_{-2}v_\lambda$. We therefore see that $X_{-1}X_{-2}v_\lambda$ generates the socle of M^λ, and it is easy to check that the image of $X_{-1}X_{-2}X_{-1}v_\lambda$ is maximal in $M^\lambda/Soc(M^\lambda)$ and generates a 1-dimensional irreducible submodule of $M^\lambda/Soc(M^\lambda)$ of weight $\lambda-2\alpha_1-\alpha_2$. The Loewy diagram then follows:

$$\begin{tikzcd}
& {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda-\alpha_1} \arrow[dr] \arrow[rr, phantom, "M^\lambda"] & & {}^{(1)}S^{\lambda-\alpha_1-2\alpha_2} \arrow[dl] \\
& {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}$, $\lambda_2 \in 2\mathbb{Z}$.

$$\begin{tikzcd}
& {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dr] \arrow[rr, phantom, "M^\lambda"] & & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dl] \\
& {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 2\mathbb{Z}$, $\lambda_2 \in 1+2\mathbb{Z}$.

where the left superscript indicates dimension. Let $\lambda_1 \in \mathbb{C}\setminus\mathbb{Z}$ and $\lambda_2 \in 1+2\mathbb{Z}$. Then by Proposition 5.1.2, the irreducible quotient of M^λ is 4-dimensional with basis given by the images of v_λ, $X_{-1}v_\lambda, X_{-2}X_{-1}v_\lambda$, and $X_{-1}X_{-2}X_{-1}v_\lambda$. We also have that $X_{-2}v_\lambda$ is maximal and

$$(\lambda-\alpha_2)_1 = \lambda_1+1 \in \mathbb{C}\setminus\mathbb{Z}, \qquad (\lambda-\alpha_2)_2 = \lambda_2-2 \in 1+2\mathbb{Z}.$$

Therefore, it follows again from Proposition 5.1.2 that $X_{-2}v_\lambda$ generates an irreducible submodule with highest weight $\lambda-\alpha_2$ and dimension 4. The $\lambda_1 \in 1+2\mathbb{Z}, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$ case is similar. We therefore have the following Loewy diagrams:

$$\begin{tikzcd} & {}^{(4)}S^{\lambda} \arrow[d] \\ M^{\lambda} & \\ & {}^{(4)}S^{\lambda-\alpha_2} \end{tikzcd} \qquad \begin{tikzcd} & {}^{(4)}S^{\lambda} \arrow[d] \\ M^{\lambda} & \\ & {}^{(4)}S^{\lambda-\alpha_1} \end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_2 \in 1 + 2\mathbb{Z}$.

Loewy Diagram for $\lambda_1 \in 1 + 2\mathbb{Z}$, $\lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$.

Let $\lambda_1, \lambda_2 \in 1+2\mathbb{Z}$, then it is easy to check as above that $X_{-1}v_\lambda$, $X_{-2}v_\lambda$, and $X_{-1}X_{-2}X_{-1}X_{-2}v_\lambda$ are maximal and generate submodules whose irreducible quotients have dimension 3, 3, and 1 respectively by Proposition 5.1.2, while the irreducible quotient of M^λ is also dimension 1. It is clear then that $X_{-1}X_{-2}X_{-1}X_{-2}v_\lambda$ generates the socle of M^λ and it is easy to check that the submodule generated by $X_{-1}v_\lambda$ does not contain $X_{-2}v_\lambda$ and vice versa. We therefore have the following Loewy diagram:

$$\begin{tikzcd} & {}^{(1)}S^{\lambda} \arrow[dl] \arrow[dr] & \\ {}^{(3)}S^{\lambda-\alpha_1} \arrow[dr] & M^{\lambda} & {}^{(3)}S^{\lambda-\alpha_2} \arrow[dl] \\ & {}^{(1)}S^{\lambda-2(\alpha_1+\alpha_2)} & \end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$.

If $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$, $\lambda_1+\lambda_2 \in 1+2\mathbb{Z}$, then it follows from Proposition 5.1.2 that the irreducible quotient S^λ of M^λ has character $z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-\alpha_2} + z^{\lambda-\alpha_1-\alpha_2}$. It was observed in the proof of Proposition 5.1.2 that there exists a maximal vector v of weight $\lambda - \alpha_1 - \alpha_2$. It is easily seen that $\lambda - \alpha_1 - \alpha_2$ is of the same weight type as λ and therefore v generates a 4-dimensional irreducible submodule. The result follows. $\square$

5.2 Projective Modules

Recall from Theorem 3.3.14 that projective covers in the category of weight modules of $\overline{U}_q^H(\mathfrak{g})$ are self-dual under the exact contravariant duality functor $(\check{-})$ of Subsection 3.3.2. It turns out that for $\overline{U}_i^H(\mathfrak{sl}_3)$, with a few additional observations, this constrains the projective covers enough to determine their Loewy diagrams. We begin by establishing two necessary lemmas.

Lemma 5.2.1. For $\lambda \in \mathfrak{h}^*$ such that $\lambda_1 \in 2\mathbb{Z}$, $\lambda_2 \in 1 + 2\mathbb{Z}$, there does not exist a module with a Loewy diagram of the form

$$\begin{tikzcd}
 & {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda+\alpha_2} \arrow[dr] & & {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dl] \\
 & {}^{(3)}S^{\lambda} &
\end{tikzcd}$$

Proof. Suppose there exists such a module M. Then by Proposition 5.1.2, we have $\mathrm{ch}[M] = 2(z^{\lambda}+z^{\lambda-\alpha_1}+z^{\lambda-\alpha_1-\alpha_2})+z^{\lambda+\alpha_2}+z^{\lambda-2\alpha_1-\alpha_2}$. Let $\{v_\lambda^1, v_\lambda^2, v_{\lambda-\alpha_1}^1, v_{\lambda-\alpha_1}^2, v_{\lambda-\alpha_1-\alpha_2}^1, v_{\lambda-\alpha_1-\alpha_2}^2, v_{\lambda+\alpha_2}, v_{\lambda-2\alpha_1-\alpha_2}\}$ be a basis for M where the v_μ^1 generate the top S^λ and v_μ^2 generate the socle. We immediately see that $X_{-2}v_\lambda^i = 0$ since $\lambda - \alpha_2$ is not a weight of M, and from the form of the Loewy diagram we have $X_2 v_\lambda^1 = a v_{\lambda+\alpha_2}$, $X_{-2}v_{\lambda+\alpha_2} = b v_\lambda^2$ for some non-zero scalars a, b. It follows from the identity $X_2 X_{-2} = X_{-2}X_2 + \frac{K_2 - K_2^{-1}}{q-q^{-1}}$ that

$$0 = X_2 X_{-2} v_\lambda^1 = X_{-2} X_2 v_\lambda^1 + [\lambda(H_2)] v_\lambda^1 = X_{-2} X_2 v_\lambda^1 = ab v_\lambda^2,$$

a contradiction. $\square$

Lemma 5.2.2. Suppose the socle filtration of P^λ has length n. Then we have the following:

- An irreducible module S^μ appears in the k-th row of the Loewy diagram of P^λ iff it also appears in the $(n+1-k)$-th row.

- There is an arrow from S^{μ_1} in the $(k-1)$-st row to S^{μ_2} in the k-th row of the Loewy diagram of P^λ iff there is an arrow from S^{μ_2} in the $(n+1-k)$-th row to S^{μ_1} in the $(n+2-k)$-th row.

Proof. We first establish the isomorphism $\mathrm{Socle}(M) \cong \mathrm{Top}(\check{M})$ for any module $M \in \mathcal{C}$, where $\mathrm{Top}(N)$ is the largest semisimple quotient of N. Suppose S^μ is an irreducible factor appearing in $\mathrm{Socle}(M)$. Then we have a short exact sequence

$$0 \to S^\mu \to M \to M/S^\mu \to 0.$$

The duality functor is involutive and preserves irreducibles by [R, Proposition 4.11], so this short exact sequence holds if and only if the short exact sequence

$$0 \to (M\check{/}S^\mu) \to \check{M} \to S^\mu \to 0.$$

holds as well. Hence, S^μ appears in $\mathrm{Socle}(M)$ if and only if it appears in $\mathrm{Top}(\check{M})$ with the same multiplicity. Now, let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = P^\lambda$$

be a socle filtration for P^λ as defined in Subsection 5.1.3. If S^μ appears in the $(n+1-k)$-th row of the Loewy diagram of P^λ, then by definition it appears in $\mathrm{Top}(M_k)$ and therefore appears in $\mathrm{Socle}(\check{M}_k)$. We have a short exact sequence

$$0 \to M_k \to P^\lambda \to P^\lambda/M_k \to 0$$

which yields another exact sequence

$$0 \to (P^\lambda\check{/}M_k) \to P^\lambda \to \check{M}_k \to 0,$$

where we have used Theorem 3.3.14, so $\check{M}_k$ is a quotient of P^λ whose socle filtration has length k (since M_k has length k). In particular, this means that the irreducible factors appearing in the socle of $\check{M}_k$ must appear in the k-th row of the Loewy diagram of P^λ, hence S^μ appears there. The other direction is given by replacing k with $n+1-k$ above.

Suppose now that there is an arrow from S^{μ_2} in the $(n+1-k)$-th row to S^{μ_1} in the $(n+2-k)$-

th row. In particular, this means that there exist cyclic submodules $N_1 \subset N_2 \subset P^\lambda$ whose socle filtrations have length have length $k-1$ and k respectively and $\mathrm{Top}(N_1) = S^{\mu_1}$ and $\mathrm{Top}(N_2) = S^{\mu_2}$. In particular, $N_1 \subset N_2$ implies that $\check{N}_1$ is a quotient of $\check{N}_2$. Since $\check{N}_1$ has socle filtration length $k-1$, $\mathrm{Socle}(\check{N}_1) = S^{\mu_1}$ appears in the $(k-1)$-st row of the Loewy diagram of $\check{N}_2$ and has an arrow to $\mathrm{Socle}(\check{N}_2) = S^{\mu_1}$ in the k-th row of the diagram since by definition all factors in the second last row of a diagram must have an arrow to the socle in the final row (otherwise they would be irreducible, and hence in the socle i.e. the final row). The statement then follows since $\check{N}_2$ is a quotient of P^λ. The other direction follows by replacing $n+1-k$ and $n+2-k$ with k and $k-1$ above. $\square$

With these two Lemma's, we are now ready to prove the main theorem for this section.

Theorem 5.2.3. The Loewy diagrams and Verma factors (in the standard filtration) for the projective covers are:

$$\begin{tikzcd}
& {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_1} \arrow[dl] \\
& {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$
Verma factors: $M^\lambda, M^{\lambda+\alpha_1}$

$$\begin{tikzcd}
& {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_2} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_2} \arrow[dl] \\
& {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}, \lambda_2 \in 1+2\mathbb{Z}$
Verma factors: $M^\lambda, M^{\lambda+\alpha_2}$

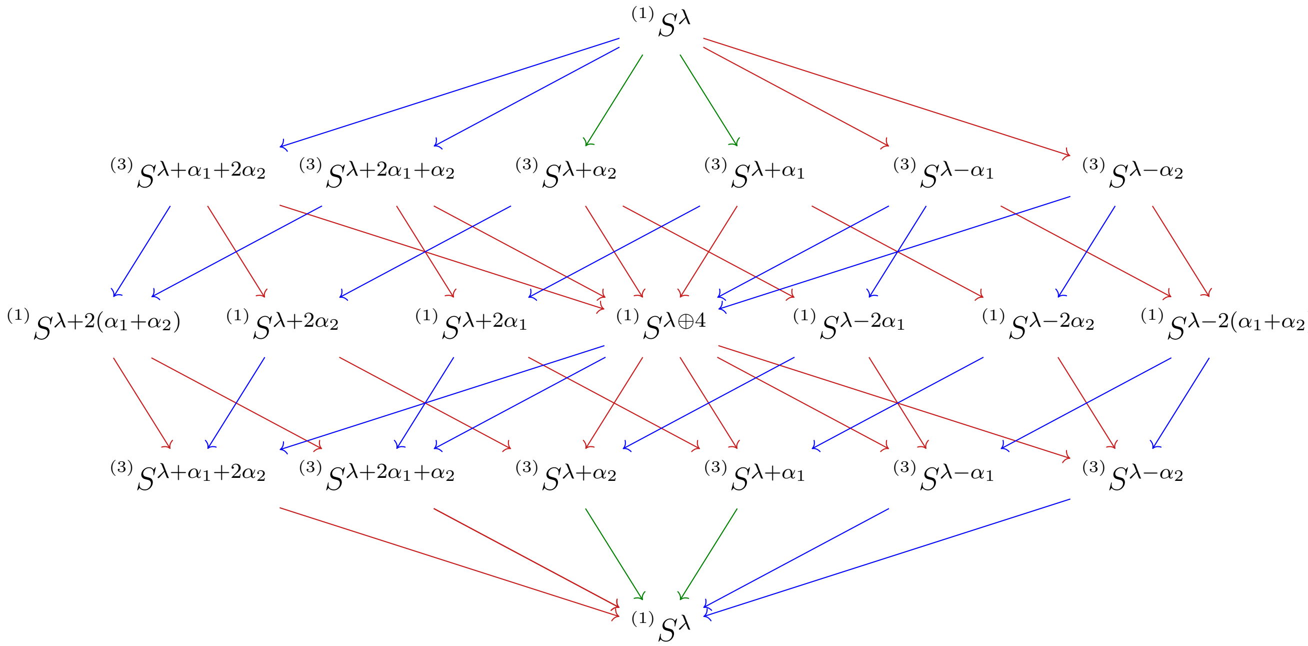
$$\begin{tikzcd}
& {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1-\alpha_2} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dl] \\
& {}^{(4)}S^{\lambda} &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_1 + \lambda_2 \in 1+2\mathbb{Z}$
Verma factors: $M^\lambda, M^{\lambda+\alpha_1+\alpha_2}$

$$
\begin{tikzcd}
 & & & {}^{(3)}S^{\lambda} \arrow[dll, blue] \arrow[d, red] \arrow[drr, red] & & & \\
 & {}^{(1)}S^{\lambda+\alpha_2} \arrow[dl, blue] \arrow[dr, red] \arrow[drrr, red] & & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dlll, blue] \arrow[dr, green] \arrow[drrr, red] & & {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dlll, blue] \arrow[dl, blue] \arrow[dr, red] & \\
{}^{(3)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dr, red] \arrow[drrr, red] & & {}^{(3)}S^{\lambda+\alpha_2-\alpha_1} \arrow[dl, blue] \arrow[drrr, red] & & {}^{(3)}S^{\lambda} \arrow[dlll, blue] \arrow[dl, green] \arrow[dr, red] & & {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} \arrow[dlll, blue] \arrow[dl, blue] \\
 & {}^{(1)}S^{\lambda+\alpha_2} \arrow[drr, red] & & {}^{(1)}S^{\lambda-\alpha_2} \arrow[d, red] & & {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dll, blue] & \\
 & & & {}^{(3)}S^{\lambda} & & &
\end{tikzcd}
$$

Loewy diagram for $\lambda_1 \in 2\mathbb{Z}, \lambda_2 \in 1 + 2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_2}, M^{\lambda+\alpha_1+\alpha_2}$

The diagram for $\lambda_1 \in 1 + 2\mathbb{Z}$, $\lambda_2 \in 2\mathbb{Z}$ can be obtained by swapping $\alpha_1 \leftrightarrow \alpha_2$ in the simple factors.



Loewy diagram for $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$
Verma factors: $M^{\lambda}, M^{\lambda+\alpha_1}, M^{\lambda+\alpha_2}, M^{\lambda+2\alpha_1+\alpha_2}, M^{\lambda+\alpha_1+2\alpha_2}, M^{\lambda+2(\alpha_1+\alpha_2)}$

Proof. We first need to determine the Verma factors of P^{λ} for any $\lambda \in \mathfrak{h}^*$. By BGG reciprocity (Proposition 3.3.12), it is sufficient to determine which Verma modules M^{μ} contain S^{λ} in their standard filtrations. Recall that M^{μ} has a basis $\{X_{-1}^{n_1} X_{-3}^{n_3} X_{-2}^{n_2} v_{\mu} \mid n_k \in \{0,1\}\}$, so the weights of M^{μ} are

$$\{\mu, \mu - \alpha_1, \mu - \alpha_2, \mu - \alpha_1 - \alpha_2, \mu - 2\alpha_1 - \alpha_2, \mu - \alpha_1 - 2\alpha_2, \mu - 2(\alpha_1 + \alpha_2)\}.$$

We therefore see that the only possibilities for μ such that $[M^{\mu} : S^{\lambda}] \neq 0$ are those in the set

$$\Gamma(\lambda) := \{\lambda, \lambda + \alpha_1, \lambda + \alpha_2, \lambda + \alpha_1 + \alpha_2, \lambda + 2\alpha_1 + \alpha_2, \lambda + \alpha_1 + 2\alpha_2, \lambda + 2(\alpha_1 + \alpha_2)\}.$$

The corresponding scalars for the weights in $\Gamma(\lambda)$ are

$$(\lambda + \alpha_1)_1 = \lambda_1 + 2 \qquad (\lambda + \alpha_1)_2 = \lambda_2 - 1 \tag{5.2.1}$$

$$(\lambda + \alpha_2)_1 = \lambda_1 - 1 \qquad (\lambda + \alpha_2)_2 = \lambda_2 + 2 \tag{5.2.2}$$

$$(\lambda + \alpha_1 + \alpha_2)_1 = \lambda_1 + 1 \qquad (\lambda + \alpha_1 + \alpha_2)_2 = \lambda_2 + 1 \tag{5.2.3}$$

$$(\lambda + 2\alpha_1 + \alpha_2)_1 = \lambda_1 + 3 \qquad (\lambda + 2\alpha_1 + \alpha_2)_2 = \lambda_2 \tag{5.2.4}$$

$$(\lambda+\alpha_1+2\alpha_2)_1 = \lambda_1 \qquad\qquad (\lambda+\alpha_1+2\alpha_2)_2 = \lambda_2+3 \tag{5.2.5}$$

$$(\lambda+2\alpha_1+2\alpha_2)_1 = \lambda_1+2 \qquad\qquad (\lambda+2\alpha_1+2\alpha_2)_2 = \lambda_2+2 \tag{5.2.6}$$

Suppose that $\lambda_1 \in 1+2\mathbb{Z}, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$. Then each weight $\mu \in \Gamma(\lambda)$ satisfies $\mu_1 \in \mathbb{Z}, \mu_2 \in \mathbb{C}\setminus\mathbb{Z}$. Since we are looking for those M^μ such that $[M^\mu, S^\lambda] \neq 0$, μ must be atypical, hence $\mu_1 \in 1+2\mathbb{Z}$. It then follows from Equations (5.2.1)-(5.2.6) that the only weights in $\Gamma(\lambda)$ of this type are $\lambda, \lambda+\alpha_1, \lambda+\alpha_1+2\alpha_2$, and $\lambda+2(\alpha_1+\alpha_2)$. It then follows from Proposition 5.1.4 that the only weights in this list whose corresponding Verma module contains S^λ as an irreducible factor is λ and $\lambda+\alpha_1$. Hence, P^λ has Verma length two with factors M^λ and $M^{\lambda+\alpha_1}$, and that the irreducible factors of P^λ are ${}^{(4)}S^\lambda$ with multiplicity two and ${}^{(4)}S^{\lambda+\alpha_1}, {}^{(4)}S^{\lambda-\alpha_2}$ with multiplicity one. It follows immediately from Lemma 5.2.2 that the Loewy diagram of P^λ is:

$$\begin{tikzcd}
& {}^{(4)}S^\lambda \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1} \arrow[dr] & P^\lambda & {}^{(4)}S^{\lambda+\alpha_1} \arrow[dl] \\
& {}^{(4)}S^\lambda &
\end{tikzcd}$$

Loewy Diagram for $\lambda_1 \in 1+2\mathbb{Z}, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$

The argument for $\lambda_1 \in \mathbb{C}\setminus\mathbb{Z}$, $\lambda_2 \in 1+2\mathbb{Z}$ is identical. Suppose that $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{Z}$ and $\lambda_1+\lambda_2 \in 1+2\mathbb{Z}$. Then every $\mu \in \Gamma(\lambda)$ has $\mu_1, \mu_2 \in \mathbb{C}\setminus\mathbb{Z}$, and we easily check using Equation (5.2.1)-(5.2.6) that only $\mu = \lambda, \lambda+\alpha_1+\alpha_2$, and $\lambda+2(\alpha_1+\alpha_2)$ satisfy $\mu_1+\mu_2 \in 1+2\mathbb{Z}$. By Proposition 5.1.4, S^λ does not appear in the Loewy diagram of $M^{\lambda+2(\alpha_1+\alpha_2)}$, but does appear in the diagrams for M^λ and $M^{\lambda+\alpha_1+\alpha_2}$. So, the Verma factors of P^λ are M^λ and $M^{\lambda+\alpha_1+\alpha_2}$ and the irreducible factors are ${}^{(4)}S^\lambda$ with multiplicity two and ${}^{(4)}S^{\lambda-\alpha_1-\alpha_2}, {}^{(4)}S^{\lambda+\alpha_1+\alpha_2}$ with multiplicty 1. It again follows immediately from Lemma 5.2.2 that the Loewy diagram of P^λ is given by

$$
\begin{tikzcd}
& {}^{(4)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(4)}S^{\lambda-\alpha_1-\alpha_2} \arrow[dr] & P^{\lambda} & {}^{(4)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dl] \\
& {}^{(4)}S^{\lambda} &
\end{tikzcd}
$$

Loewy Diagram for $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$
$\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$

Suppose now that $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 1+2\mathbb{Z}$. Then, one checks again using Equations (5.2.1)-(5.2.6) that the atypical weights in $\Gamma(\lambda)$ are $\lambda, \lambda+\alpha_2, \lambda+\alpha_1+\alpha_2, \lambda+2\alpha_1+\alpha_2, \lambda+2\alpha_1+2\alpha_2$. It then follows from Proposition 5.1.4 and BGG reciprocity in the usual way that the Verma factors are $M^{\lambda}, M^{\lambda+\alpha_1}$, and $M^{\lambda+\alpha_1+\alpha_2}$ where

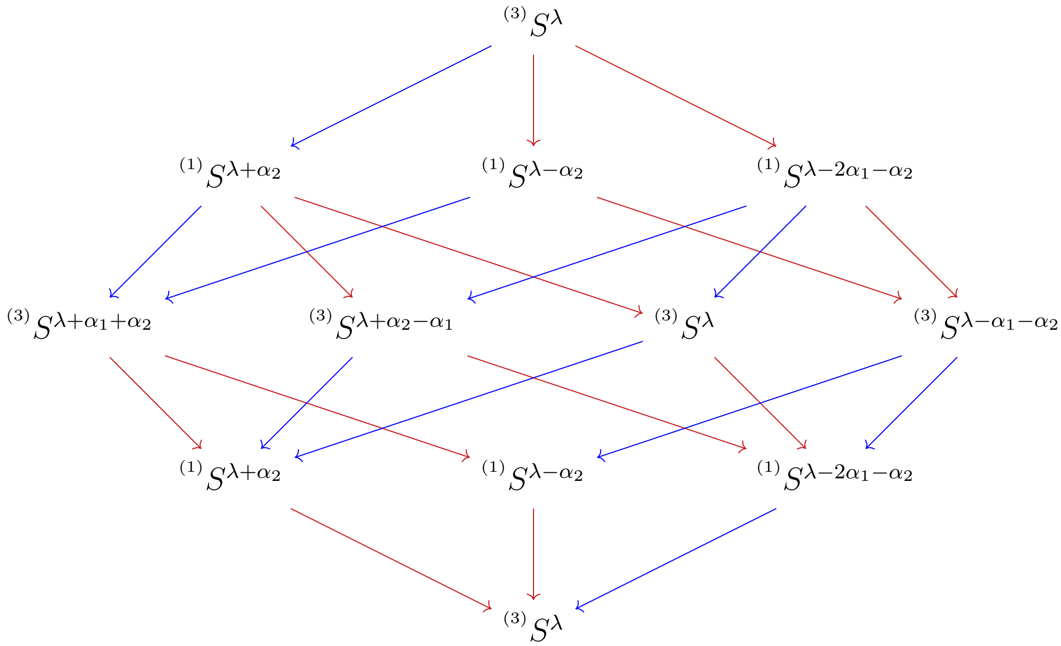
$$
\begin{aligned}
(\lambda+\alpha_2)_1 &\in 1+2\mathbb{Z} & \qquad (\lambda+\alpha_2)_2 &\in 1+2\mathbb{Z} \\
(\lambda+\alpha_1+\alpha_2)_1 &\in 1+2\mathbb{Z} & \qquad (\lambda+\alpha_1+\alpha_2)_2 &\in 2\mathbb{Z}
\end{aligned}
$$

and the Loewy diagrams for the Verma factors are therefore given by

$$
\begin{tikzcd}
& {}^{(3)}S^{\lambda} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} \arrow[dr] & M^{\lambda} & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dl] \\
& {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& {}^{(1)}S^{\lambda+\alpha_2} \arrow[dl] \arrow[dr] & \\
{}^{(3)}S^{\lambda+\alpha_2-\alpha_1} \arrow[dr] & M^{\lambda+\alpha_1} & {}^{(3)}S^{\lambda} \arrow[dl] \\
& {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& {}^{(3)}S^{\lambda+\alpha_1+\alpha_2} \arrow[dl] \arrow[dr] & \\
{}^{(1)}S^{\lambda+\alpha_2} \arrow[dr] & M^{\lambda+\alpha_1+\alpha_2} & {}^{(1)}S^{\lambda-\alpha_2} \arrow[dl] \\
& {}^{(3)}S^{\lambda} &
\end{tikzcd}
$$

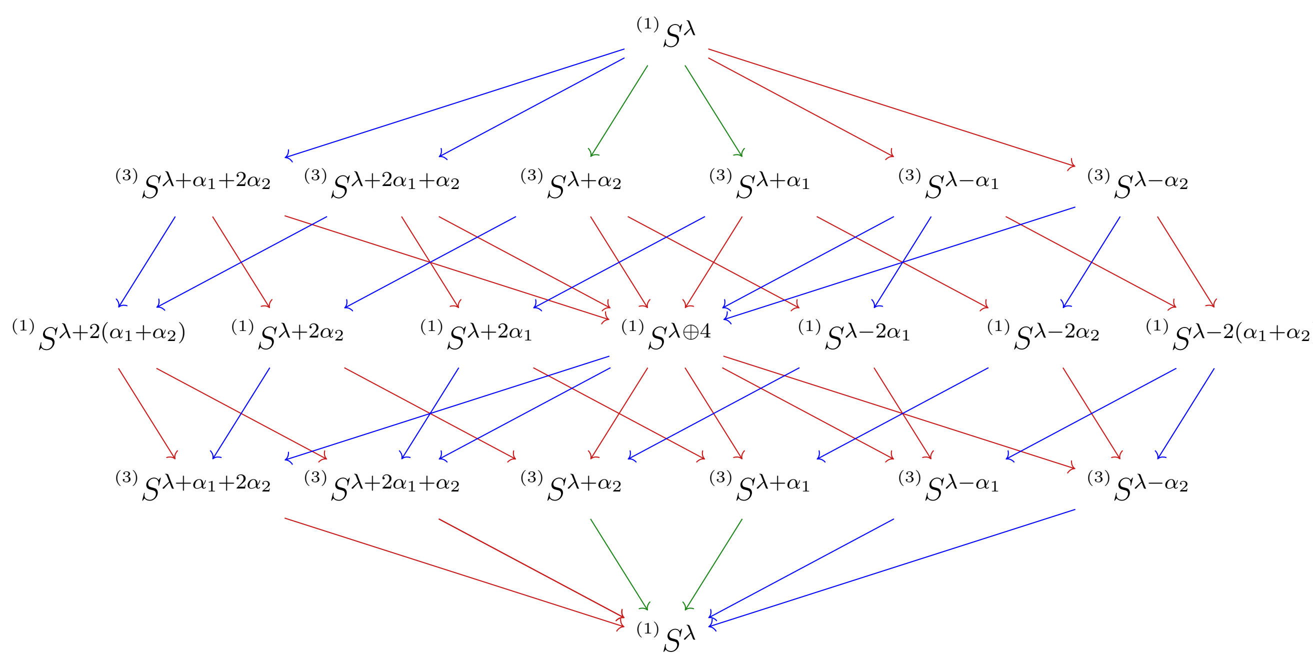
We therefore see that the factors in the Loewy diagram of P^{λ} are ${}^{(3)}S^{\lambda}$ with multiplicity 3, ${}^{(1)}S^{\lambda-\alpha_2}$, ${}^{(1)}S^{\lambda+\alpha_2}$, and ${}^{(1)}S^{\lambda-2\alpha_1-\alpha_2}$ with multiplicity 2, and ${}^{(3)}S^{\lambda+\alpha_2-\alpha_1}$, ${}^{(3)}S^{\lambda+\alpha_1+\alpha_2}$, and ${}^{(3)}S^{\lambda-\alpha_1-\alpha_2}$ with multiplicity 1. It then follows immediately from Lemma 5.2.2 that the Loewy diagram for P^{λ} (possibly with missing arrows) is given by



Partial Loewy diagram for P^{λ}

where we have filled in the arrows we get for free from the Loewy diagrams of Verma modules and the dual arrows from Lemma 5.2.2. It remains only to determine if there exist arrows from the second to third row in the diagram which are currently missing (arrows from the third to fourth row are then determined by Lemma 5.2.2). Suppose there exists a module $M \in \mathcal{C}$ such that $0 \to {}^{(3)}S^{\lambda-\alpha_1-\alpha_2} \to M \to {}^{(1)}S^{\lambda+\alpha_2} \to 0$ holds. It follows from Proposition 5.1.2 that $\mathrm{ch}[M] = z^{\lambda-\alpha_1-\alpha_2} + z^{\lambda-\alpha_1-2\alpha_2} + z^{\lambda-2(\alpha_1+\alpha_2} + z^{\lambda+\alpha_2}$. Hence, $X_{\pm i} \cdot M(\lambda+\alpha_2) \subset M(\lambda+\alpha_2 \pm \alpha_i) = \varnothing$ for $i = 1, 2$ and therefore ${}^{(1)}S^{\lambda+\alpha_2}$ must be a summand. That is, $\mathrm{Ext}^1({}^{(3)}S^{\lambda-\alpha_1-\alpha_2}, {}^{(1)}S^{\lambda+\alpha_2}) = \varnothing$ and an identical argument shows that $\mathrm{Ext}^1({}^{(3)}S^{\lambda+\alpha_1+\alpha_2}, {}^{(1)}S^{\lambda-2\alpha_1-\alpha_2}) = \varnothing$. Therefore, there are no arrows from ${}^{(1)}S^{\lambda+\alpha_2}$ to ${}^{(3)}S^{\lambda-\alpha_1-\alpha_2}$ or from ${}^{(1)}S^{\lambda-2\alpha_1-\alpha_2}$ to ${}^{(3)}S^{\lambda+\alpha_1+\alpha_2}$. Hence, the Loewy diagram of P^{λ} is given by the partial diagram above with an additional arrow from ${}^{(1)}S^{\lambda-\alpha_2}$ in the second row to ${}^{(3)}S^{\lambda}$ in the third row, and an arrow from ${}^{(3)}S^{\lambda}$ in the third row to ${}^{(1)}S^{\lambda-\alpha_2}$ in the fourth. We must now determine the arrows originating from ${}^{(1)}S^{\lambda-\alpha_2}$ in the second row. However, by the same argument as above we easily find that $\mathrm{Ext}^1({}^{(3)}S^{\lambda+\alpha_2-\alpha_1}, {}^{(1)}S^{\lambda-\alpha_2}) = \varnothing$. To complete the proof, we observe that Lemma 5.2.1 implies the existence of an arrow from ${}^{(1)}S^{\lambda-\alpha_2}$ in the second row to ${}^{(3)}S^{\lambda}$ in the third, since if such an arrow does not exist, there would be a quotient of P^{λ} whose Loewy diagram contradicts the Lemma.

To determine the Verma length of totally atypical P^{λ} ($\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$), one checks by the usual method that the atypical weights in $\Gamma(\lambda)$ are $\lambda, \lambda+\alpha_1, \lambda+\alpha_2, \lambda+2\alpha_1+\alpha_2, \lambda+\alpha_1+2\alpha_2, \lambda+2(\alpha_1+\alpha_2)$. All corresponding Verma modules contain S^λ with multiplicity one in their Loewy diagrams by Proposition 5.1.4. Therefore P^λ has Verma length 6. It follows again from Proposition 5.1.4 that the irreducible factors for P^λ are ${}^{(1)}S^{\lambda+2(\alpha_1+\alpha_2)}$, ${}^{(1)}S^{\lambda-2(\alpha_1+\alpha_2)}$, ${}^{(1)}S^{\lambda\pm 2\alpha_1}$, ${}^{(1)}S^{\lambda\pm 2\alpha_2}$ with multiplicity 1, ${}^{(3)}S^{\lambda+\alpha_1+2\alpha_2}$, ${}^{(3)}S^{\lambda+2\alpha_1+\alpha_2}$, ${}^{(3)}S^{\lambda\pm\alpha_1}$, ${}^{(3)}S^{\lambda\pm\alpha_2}$ with multiplicity 2, and ${}^{(1)}S^\lambda$ with multiplicity 6. By Lemma 5.2.2, the Loewy diagram of P^λ must have an odd number of rows (since otherwise all factors would have even multiplicity), and therefore the multiplicity one factors must all lie in the center row. Further, we know that $M^{\lambda+2(\alpha_1+\alpha_2)}$ is a submodule of P^λ by the argument in [Hu, Proposition 3.7 (a)] since $\lambda+2(\alpha_2+\alpha_2)$ is maximal among the weights of P^λ, so the irreducible submodule ${}^{(1)}S^\lambda$ of $M^{\lambda+2(\alpha_1+\alpha_2)}$ is the socle of P^λ. Note that $S^{\lambda+2(\alpha_1+\alpha_2)}$ has multiplicity 1 in P^λ and therefore lies in the center row, so the socle filtration of P^λ has length 5. It therefore follows from Lemma 5.2.2 that the Loewy diagram of P^λ must take the following form:



Here, we have filled in the arrows we get for free from the Loewy diagrams in Proposition 5.1.4 of the Verma factors of P^λ (red arrows), Lemma 5.2.2 (blue arrows), and the definition of Loewy diagram (green arrows). One can easily check using the same character argument as the previous case that there are no missing arrows from the second row to the third row, and hence no missing arrows from the third row to the fourth row by Lemma 5.2.2.

□

5.3 Tensor Structure

Here we compute all tensor products $S^\lambda \otimes M^\mu$ when M^μ is projective. To do this, we use Proposition 3.3.13 and the fact that projectives form a tensor ideal in $\mathcal{C}$ since $\mathcal{C}$ is pivotal (see [GPV, Lemma 17]). The argument in [CGP2, Lemma 7.1] can be easily adapted to our setting to show that M^μ is projective iff μ is typical. Recall from Proposition 5.1.2 that the character of M^μ is given by

$$\mathrm{ch}[M^\lambda] = \sum_{\substack{n_\alpha=0\\ \alpha\in\Delta^+}}^{1} z^{\lambda - \sum\limits_{\alpha\in\Delta^+} n_\alpha \alpha} = z^\lambda \prod_{\alpha\in\Delta^-}\left(\frac{z^{2\alpha}-1}{z-1}\right).$$

where $\Delta^\pm$ are the sets of positive and negative roots. It follows from the above equation that the character of tensor product of Verma modules is given by

$$\mathrm{ch}[M^\lambda \otimes M^\mu] = \sum_{n_1=0}^{1}\sum_{n_2=0}^{1}\sum_{n_3=0}^{1} \mathrm{ch}[M^{\lambda+\mu-n_1\alpha_1-n_2\alpha_2-n_3(\alpha_1+\alpha_2)}] \tag{5.3.1}$$

To determine tensor products, we must determine the typicality of the weights appearing in the character formula. We first note the following identities:

$$(\lambda+\mu)_1 = \lambda_1+\mu_1-1 \qquad (\lambda+\mu)_2 = \lambda_2+\mu_2-1 \tag{5.3.2}$$
$$(\lambda+\mu-\alpha_1)_1 = \lambda_1+\mu_1-3 \qquad (\lambda+\mu-\alpha_1)_2 = \lambda_2+\mu_2 \tag{5.3.3}$$
$$(\lambda+\mu-\alpha_2)_1 = \lambda_1+\mu_1 \qquad (\lambda+\mu-\alpha_2)_2 = \lambda_2+\mu_2-3 \tag{5.3.4}$$
$$(\lambda+\mu-\alpha_1-\alpha_2)_1 = \lambda_1+\mu_1-2 \qquad (\lambda+\mu-\alpha_1-\alpha_2)_2 = \lambda_2+\mu_2-2 \tag{5.3.5}$$
$$(\lambda+\mu-2\alpha_1-\alpha_2)_1 = \lambda_1+\mu_1-4 \qquad (\lambda+\mu-2\alpha_1-\alpha_2)_2 = \lambda_2+\mu_2-1 \tag{5.3.6}$$
$$(\lambda+\mu-\alpha_1-2\alpha_2)_1 = \lambda_1+\mu_1-1 \qquad (\lambda+\mu-\alpha_1-2\alpha_2)_2 = \lambda_2+\mu_2-4 \tag{5.3.7}$$
$$(\lambda+\mu-2\alpha_1-2\alpha_2)_1 = \lambda_1+\mu_1-3 \qquad (\lambda+\mu-2\alpha_1-2\alpha_2)_2 = \lambda_2+\mu_2-3 \tag{5.3.8}$$

If all of the above weights are typical, then we have

$$M^\lambda \otimes M^\mu \cong \bigoplus_{\substack{n_i=0 \\ i=1,2,3}}^{1} M^{\lambda+\mu-n_1\alpha_1-n_2\alpha_2-n_3(\alpha_1+\alpha_2)} \tag{5.3.9}$$

by Proposition 3.3.13. Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda_1 \in 2\mathbb{Z}$ and $\lambda_2 \in 1+2\mathbb{Z}$, so S^λ has dimension 3 and character $z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-(\alpha_1+\alpha_2)}$ by Proposition 5.1.2. Then, we have

$$\begin{aligned} \mathrm{ch}[S^\lambda \otimes M^\mu] &= \left(z^{\lambda+\mu} + z^{\lambda+\mu-\alpha_1} + z^{\lambda+\mu-(\alpha_1+\alpha_2)}\right) \prod_{\alpha\in\Delta^-} \left(\frac{z^{2\alpha}-1}{z-1}\right) \\ &= \mathrm{ch}[M^{\lambda+\mu}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1-\alpha_2}] \end{aligned}$$

Therefore, if $\mu_1 \in \mathbb{C}\setminus\mathbb{Z}$ and $\mu_2 \in 2\mathbb{Z}$, then by equations (5.3.2), (5.3.3), and (5.3.5), $\lambda+\mu$ is typical and $\lambda+\mu-\alpha_1, \lambda+\mu-\alpha_1-\alpha_2$ are atypical with $(\lambda+\mu-\alpha_1-\alpha_2)_1 \in \mathbb{C}\setminus\mathbb{Z}$ and $(\lambda+\mu-\alpha_1-\alpha_2)_2 \in 1+2\mathbb{Z}$. It follows from Theorem 5.2.3 that the Verma factors of $P^{\lambda+\mu-\alpha_1-\alpha_2}$ are $M^{\lambda+\mu-\alpha_1-\alpha_2}$ and $M^{\lambda+\mu-\alpha_1}$. Hence, we have $\mathrm{ch}[S^\lambda \otimes M^\mu] = \mathrm{ch}[M^\lambda \oplus P^{\lambda+\mu-\alpha_1-\alpha_2}]$. It then follows from Proposition 3.3.13 that

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2}.$$

If $\mu_1 \in 2\mathbb{Z}$ and $\mu_2 \in \mathbb{C}\setminus\mathbb{Z}$, then by equations (5.3.2), (5.3.3), and (5.3.5), $\lambda+\mu-\alpha_1-\alpha_2$ is typical and $\lambda+\mu, \lambda+\mu-\alpha_1$ are atypical. By the same argument above, we see that

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1}$$

If $\mu_1, \mu_2 \in 2\mathbb{Z}$, then in the same way as before, we see that $\lambda+\mu, \lambda+\mu-\alpha_1$, and $\lambda+\mu-\alpha_1-\alpha_2$ are all atypical and $(\lambda+\mu-\alpha_1-\alpha_2)_1 \in 2\mathbb{Z}$, $(\lambda+\mu-\alpha_1-\alpha_2)_2 \in 1+2\mathbb{Z}$. It then follows from Theorem 5.2.3 that the Verma factors of $P^{\lambda+\mu-\alpha_1-\alpha_2}$ are $M^{\lambda+\mu}$, $M^{\lambda+\mu-\alpha_1}$, and $M^{\lambda+\mu-\alpha_1-\alpha_2}$. We therefore have $\mathrm{ch}[S^\lambda \otimes M^\mu] = \mathrm{ch}[P^{\lambda+\mu-\alpha_1-\alpha_2}]$, so we see that

$$S^\lambda \otimes M^\mu \cong P^{\lambda+\mu-\alpha_1-\alpha_2}.$$

Letting $\lambda_1 \in 1+2\mathbb{Z}$ and $\lambda_2 \in 2\mathbb{Z}$, and performing the same analysis, we obtain the following

$$\begin{aligned} &S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_1 \in 2\mathbb{Z},\ \mu_2 \in \mathbb{C}\setminus\mathbb{Z} \\ &S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_2} && \text{when } \mu_1 \in \mathbb{C}\setminus\mathbb{Z},\ \mu_2 \in 2\mathbb{Z} \end{aligned}$$

$$S^\lambda \otimes M^\mu \cong P^{\lambda+\mu-\alpha_1-\alpha_2} \qquad\qquad \text{when } \mu_1, \mu_2 \in 2\mathbb{Z}$$

Now, let $\lambda \in \mathfrak{h}^*$ with $\lambda_1 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_2 \in 1 + 2\mathbb{Z}$. Then by Proposition 5.1.2, $\dim(S^\lambda) = 4$ and has character $z^\lambda + z^{\lambda-\alpha_1} + z^{\lambda-\alpha_1-\alpha_2} + z^{\lambda-2\alpha_1-\alpha_2}$. We then have

$$\begin{aligned}\mathrm{ch}[S^\lambda \otimes M^\mu] &= \left(z^{\lambda+\mu} + z^{\lambda+\mu-\alpha_1} + z^{\lambda+\mu-\alpha_1-\alpha_2} + z^{\lambda+\mu-2\alpha_1-\alpha_2}\right) \prod_{\alpha \in \Delta^-} \left(\frac{z^{2\alpha}-1}{z-1}\right) \\ &= \mathrm{ch}[M^{\lambda+\mu}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1-\alpha_2}] + \mathrm{ch}[M^{\lambda+\mu-2\alpha_1-\alpha_2}].\end{aligned}$$

If $\mu_1 \in 2\mathbb{Z}$ and $\mu_2 \in \mathbb{C} \setminus \mathbb{Z}$, then the highest weights of the Verma modules appearing in the character are all typical, so their direct sum is projective and we have

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-2\alpha_1-\alpha_2}. \tag{5.3.10}$$

If $\mu_1, \mu_2 \in 2\mathbb{Z}$, then it follows from equations (5.3.2), (5.3.3), (5.3.5), and (5.3.6) that $\lambda + \mu$ and $\lambda + \mu - 2\alpha_1 - \alpha_2$ are typical, while $\lambda + \mu - \alpha_1$ and $\lambda + \mu - \alpha_1 - \alpha_2$ are atypical with $(\lambda + \mu - \alpha_1 - \alpha_2)_1 \in \mathbb{C} \setminus \mathbb{Z}$, $(\lambda + \mu - \alpha_1 - \alpha_2)_2 \in 1 + 2\mathbb{Z}$. It follows from Theorem 5.1.4 that $P^{\lambda+\mu-\alpha_1-\alpha_2}$ has Verma factors $M^{\lambda+\mu-\alpha_1-\alpha_2}$ and $M^{\lambda+\mu-\alpha_1}$. Hence, $\mathrm{ch}[S^\lambda \otimes M^\mu] = \mathrm{ch}[M^\lambda] + \mathrm{ch}[M^{\lambda+\mu-2\alpha_1-\alpha_2}] + \mathrm{ch}[P^{\lambda_\mu-\alpha_1-\alpha_2}]$ so we have

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-2\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2}.$$

If $\mu_1 \in \mathbb{C} \setminus \mathbb{Z}$ and $\mu_2 \in 2\mathbb{Z}$, then we have three distinct cases: $\lambda_1 + \mu_1 \notin \mathbb{Z}$, $\lambda_1 + \mu_1 \in 2\mathbb{Z}$, and $\lambda_1 + \mu_1 \in 1 + 2\mathbb{Z}$. In the first case, the highest weights of the Verma modules appearing in the character of $S^\lambda \otimes M^\mu$ are again typical, so the tensor product is given by equation (5.3.10). In the second case, use equations (5.3.2)-(5.3.8) again to determine that $\lambda+\mu-2\alpha_1-\alpha_2$ is typical while the other weights are atypical and $(\lambda+\mu-\alpha_1-\alpha_2)_1 \in 2\mathbb{Z}$, $(\lambda+\mu-\alpha_1-\alpha_2)_2 \in 1+2\mathbb{Z}$. It follows from Proposition 5.2.3 that $\mathrm{ch}[S^\lambda \otimes M^\mu] = \mathrm{ch}[M^{\lambda+\mu-2\alpha_1-\alpha_2}] + \mathrm{ch}[P^{\lambda+\mu-\alpha_1-\alpha_2}]$, so we have

$$S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-2\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2}.$$

In the third case of $\lambda_1 + \mu_1 \in 1 + 2\mathbb{Z}$, by the usual argument we see that $\lambda + \mu$ is the only typical weight while $(\lambda + \mu - 2\alpha_1 - \alpha_2)_1 \in 2\mathbb{Z}$, $(\lambda + \mu - 2\alpha_1 - \alpha_2)_2 \in 1 + 2\mathbb{Z}$, so we get an equivalence of characters from Theorem 5.2.3 and

$$S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-2\alpha_1-\alpha_2}$$

Suppose now that $\mu_1, \mu_2 \in \mathbb{C} \setminus \mathbb{Z}$. If $\lambda_1 + \mu_1 \notin \mathbb{Z}$, all highest weights of the Verma modules appearing in the character of $S^{\lambda} \otimes M^{\mu}$ are typical and the tensor product is again given by equation (5.3.10). If $\lambda_1 + \mu_1 \in \mathbb{Z}$, then an analysis identical to the analagous cases above gives

$$\begin{aligned} &S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-2\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1} && \text{if } \lambda_1 + \mu_1 \in 2\mathbb{Z} \\ &S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_1} \oplus P^{\lambda+\mu-2\alpha_1-\alpha_2} && \text{if } \lambda_1 + \mu_1 \in 1 + 2\mathbb{Z} \end{aligned}$$

The cases for $\lambda_1 \in 2\mathbb{Z}$, $\lambda_2 \in 1 + 2\mathbb{Z}$ are symmetric. If $\lambda_1, \lambda_2 \in 1 + 2\mathbb{Z}$, then S^{λ} is one dimensional and we have $S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu}$. We are therefore left only with the case $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$. By Proposition 5.1.2, we have

$$\begin{aligned} \mathrm{ch}[S^{\lambda} \otimes M^{\mu}] &= (z^{\lambda+\mu} + z^{\lambda+\mu-\alpha_1} + z^{\lambda+\mu-\alpha_2} + z^{\lambda+\mu-\alpha_1-\alpha_2}) \prod_{\alpha \in \Delta^-} \left(\frac{z^{2\alpha} - 1}{z - 1} \right) \\ &= \mathrm{ch}[M^{\lambda+\mu}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_2}] + \mathrm{ch}[M^{\lambda+\mu-\alpha_1-\alpha_2}]. \end{aligned}$$

If $\mu_1, \mu_2 \in 2\mathbb{Z}$, then we see immediately from equations (5.3.2)-(5.3.8) that $\gamma_1, \gamma_2 \in \mathbb{C} \setminus \mathbb{Z}$ for $\gamma = \lambda + \mu, \lambda + \mu - \alpha_1, \lambda + \mu - \alpha_2, \lambda + \mu - \alpha_1 - \alpha_2$. Further, we have

$$\begin{aligned} (\lambda + \mu)_3 &\in 1 + 2\mathbb{Z} \\ (\lambda + \mu - \alpha_1)_3 &\in 2\mathbb{Z} \\ (\lambda + \mu - \alpha_2)_3 &\in 2\mathbb{Z} \\ (\lambda + \mu - \alpha_1 - \alpha_2)_3 &\in 1 + 2\mathbb{Z} \end{aligned}$$

so we know $\lambda + \mu - \alpha_k$, $k = 1, 2$ are typical. It follows from Theorem 5.2.3 that $P^{\lambda+\mu-\alpha_1-\alpha_2}$ has Verma factors $M^{\lambda+\mu-\alpha_1-\alpha_2}$ and $M^{\lambda+\mu}$, so the characters of $S^{\lambda} \otimes M^{\mu}$ and $M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2}$ coincide and the modules are isomorphic. In the same way, one can show that in the other cases for μ typical, all weights are typical and

$$S^{\lambda} \otimes M^{\mu} \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2}$$

Proposition 5.3.1. Let $i, j \in \{1, 2\}$ with $i \neq j$. If $\lambda_i \in 1 + 2\mathbb{Z}$ and $\lambda_j \in 2\mathbb{Z}$, then

$$
\begin{aligned}
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_i \in 2\mathbb{Z},\ \mu_j \in \mathbb{C} \setminus \mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus P^{\lambda+\mu-\alpha_j} && \text{when } \mu_i \in \mathbb{C} \setminus \mathbb{Z},\ \mu_j \in 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_1, \mu_2 \in 2\mathbb{Z}
\end{aligned}
$$

For $\lambda_i \in 1 + 2\mathbb{Z}$ and $\lambda_j \in \mathbb{C} \setminus \mathbb{Z}$, we get

$$
\begin{aligned}
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_j} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} && \text{when } \mu_i \in \mathbb{C} \setminus \mathbb{Z},\ \mu_j \in 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_1, \mu_2 \in 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-2\alpha_j-\alpha_i} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \in 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus P^{\lambda+\mu-2\alpha_j-\alpha_i} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \in 1 + 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_j} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} \oplus M^{\lambda+\mu-\alpha_i-2\alpha_j} && \text{when } \mu_i \in 2\mathbb{Z},\ \lambda_j + \mu_j \notin \mathbb{Z}
\end{aligned}
$$

For $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}$, $\lambda_3 = \lambda_1 + \lambda_2 \in 1 + 2\mathbb{Z}$, we have

$$
\begin{aligned}
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus P^{\lambda+\mu-\alpha_1-\alpha_2} && \text{If } \mu_1, \mu_2 \in 2\mathbb{Z} \\
&S^\lambda \otimes M^\mu \cong M^{\lambda+\mu} \oplus M^{\lambda+\mu-\alpha_1} \oplus M^{\lambda+\mu-\alpha_2} \oplus M^{\lambda+\mu-\alpha_1-\alpha_2} && \text{Otherwise}
\end{aligned}
$$

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