UNIVERSITY OF ALBERTA

MATHEMATICS KNOWING IN ACTION: A FULLY EMBODIED INTERPRETATION

by

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A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled, MATHEMATICS KNOWING IN ACTION: A FULLY EMBODIED INTERPRETATION submitted by ELAINE SIMMT in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

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Abstract

This study is an interpretation of mathematics knowing in action, from an enactivist perspective. It is a perspective that views cognition as perceptually-guided action in which a person brings forth a world of significance with other people. The dissertation consists of accounts and enactivist interpretations of the mathematics knowing of parents and children who participated in an extra-curricular mathematics program.

Enactivism is introduced as a possible means of reconceptualizing mathematics as knowing in action. Cases of parents and children engaging in mathematical activity together are used to illustrate how shifting one's view of mathematics knowing, from knowing as simply problem solving to knowing as bringing forth a world of significance, has useful and significant implications for mathematics educators. In particular, such a shift in focus suggests that students are more than problem solvers; they are fully embodied knowers.

In the discussion, I begin by making a distinction between understanding behaviour as caused by features in the environment to thinking about it as occasioned by the person's interactions with the environment. I then consider various sites and sources of perturbations for mathematics knowing: the interaction among people, interaction with the physical environment, interaction with one's own thoughts, and interaction with the interactions of others. From this, I develop a model which illustrates the roles of interaction in bringing forth a world of significance. Finally, I explore how the knower is observed to be brought forth in his or her mathematics knowing.

My research suggests that when people engage in mathematical activity, our activity intersects with the personal, social and cultural domains of our lives. In action, we bring forth worlds of significance with others; and in doing so, we bring forth ourselves. In each act of bringing forth a world of significance and our "selves", we anticipated the future as our spheres of behavioural possibilities expand making possible our next utterance,

movement, action, and thought. And, because we bring forth worlds of significance with others, what we do, what we say, and what we know makes a difference, not only for ourselves, but for the other.

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Table of Contents

List of Figures

Introduction

One night as I was going through the papers my son had brought home from school, I came across a worksheet. I barely glanced at it because it was just another one of many sheets of paper that he brings home. On it was a typewritten paragraph with a number of the words circled in pencil. As I went to add it to the week's paper stack, I was struck by a notation made on the right hand corner of the sheet. Written in ink, in what must have been the teacher's hand, was my son's name with a question mark. Obviously, the student who completed the worksheet had forgotten to put his or her name on it. I looked at the circles that the child had put around the selected words and wondered, "How did the teacher know this was my son's work?"—any child could have drawn those circles. There were no hints on the page that it was Kevin, who with pencil in hand and great contemplation, identified the words he understood to be nouns.

I was reminded of the many times I had to determine which student forgot his or her name on a piece of work he or she handed in to me. The first clue was always the handwriting. It never took long before I could identify many of the teens as easily by their handwriting as by the sounds of their voices or the shapes of their faces. Sometimes, however, I would have to read the student's response and ask myself if what was written was something that this person might say. But with many of the tests I gave my students, I would not be able to tell one student from another based on the pencil marks on the sheet of paper. Those were the forced response tests where the only indication of a student was the circle drawn around the "best response." Now I look back and wonder what I was doing when I used such means to assess what my students knew about mathematics.

> Knowledge which can be turned into words or pictures can be 'objectified' in various ways for communication purposes. Some part of knowledge, however, must be left behind in the body that gathered it, felt it, stored it and massaged it. Only abstractions (print, ideographs) and models can be objectified and made into machine-readable representations. Indeed, 'representations' is just the word to use (Neill, 1990, pp. 11 -12).

When writing, it is hard to know what to say first. In the case of my dissertation, it would be so much easier to discuss my understanding of mathematics knowing in action if I could offer the reader some examples first. However, when the examples come first, the reader might ask me why I am offering those particular examples since I have said nothing about the nature of my inquiry or why I am offering those particular examples. If only I could say everything I need to say all at once. Alternatively, it would be helpful if the reader could tell me where a good place to begin might be. Since that cannot happen, I must make a choice— actually, the choice has already been made. The first thing has already been said. The reader has already been oriented before turning a page. The reader has already been pointed in the direction of my study.

Umberto Eco (1983/1984), in the postscript to his novel, *The Name of the Rose*, reminds us that for the reader a title is a key to interpretation; the reader cannot escape the notions prompted by the title. So what about my title, *Mathematics Knowing in Action: A Fully Embodied Interpretation*—where does it point?

One of my graduate students, for whom English is a second language, asked me to clarify how the words mathematics and mathematical are used in English. She said she understood that mathematics was a noun and mathematical was used to modify a noun or a verb. Therefore, shouldn't one say *mathematical* knowing rather than mathematics knowing and doesn't one more commonly speak of mathematical *knowledge* rather than knowing? My choice of a noun for an adjective and a verb for a noun is deliberate. If I had used the phrase *mathematical knowing,* then the reader might be oriented to interpret my work as a study about a particular mode of thought (Davis, 1996). On the other hand, if I had used the word *knowledge*, the reader might have thought my work was about some commodity that is dispensed to children in schools (von Foerster, 1981) or some socially inert information that has a material existence independent of the knower (Davis, 1996).

The word action is also an interesting choice—a noun used to define verbs. "Verbs are *action* words," I was told by my one of my language arts teachers. I use the phrase "in action" because the view that mathematics is something done with minimal action and/or minimal interaction is all too common. When I ask prospective teachers to describe a mathematics classroom, an image of students working independently at desks with textbooks opened and scribbling things in notebooks is frequently offered. Mathematics teachers whose classes are noisy and active are likely to explain the situation for me; I assume they do not want me to leave with the impression that their students were "off-task."

Finally, what might the phrase "fully embodied interpretation" mean? It seems to me that it is all too common for researchers, educators and policy makers to separate themselves from the mathematics knowers they serve by scratching at disembodied surfaces (like standardized forced-response tests) when making decisions that have a profound impact on those knowers, our children and youth. There is a belief that our analyses and interpretations can be "objective," independent of who we are and who the mathematics knowers are. They could be about "anybody." It is as though who the "bodies" are does not matter.

The purpose of this dissertation is to offer accounts of mathematics knowing in action and interpretations of those accounts. In the context of my research, my observing and listening has served three purposes: to create a curriculum for a mathematics program for parents and children; to interact with parents and children and their mathematics knowing; and to create models for listening and observing mathematics knowing in action. I explored the questions: *How might I characterise the mathematics knowing that is brought forth in the actions and interactions of the parents and children in this mathematics* *program? How might I understand the knowers' activity that brings forth mathematics? How might I understand the knowers that are brought forth when doing mathematics?* I believe in responding to these questions, I provide mathematics educators possibilities for observing and listening for mathematics knowing in their interactions with others.

The very act of writing this dissertation is itself an act of knowing. It is at once my knowing in action and an expression of my knowing in action. It is important to me to acknowledge that, by posing statements of my intentions and research questions in some polished form, I am making explicit in this writing that which was not at all explicit for me until I came to the last stages of this dissertation. Merleau-Ponty (1962) helps me understand this.

> The process of expression, when it is successful, does not merely leave for the reader and the writer himself a kind of reminder, it brings the meaning into existence as a thing at the very heart of the text, it brings it to life in an organism of words, establishing it in the writer or the reader as a new sense organ, opening a new field or a new dimension to our experience (p. 182).

Finally, I offer this dissertation with the understanding that there are traditional ways to write, traditional features to seek out, traditional responses to expect; but because the path I wandered along swayed from the traditional from time to time, the reader may have to read my work differently than other dissertations. I begin with a story about how the research developed from my experiences facilitating an extra-curricular mathematics program for parents and children. In the second chapter, I discuss the research and research methods as they developed in conjunction with the parent-child mathematics program. In the third chapter, I offer an illustration of the mathematical actions and interactions of two parent-child pairs and then interpret their activity from a number of theoretical perspectives found in mathematics education research. Enactivism, the theoretical perspective from which I interpret mathematical activity, is introduced in chapter four. In chapter five, I distinguish between understanding behaviour as caused by features or constraints in the environment to thinking about it as occasioned by the person's interactions with the environment. In the sixth chapter, I discuss sites of interaction where I distinguish mathematical knowing and sources of perturbations for mathematical activity. Chapter seven is an exploration of the ways in which the knower is brought forth in mathematics knowing. In the last chapter, I explore the spaces of my research knowing and, in doing so, address the question of mathematics knowing in action yet one more time.

Chapter One

A STORY OF A RESEARCH STUDY

Teachers are people with a desire to transform.
Scientists are people with a desire to explain.

Vignette 1-1. The Math Connection: a mathematics program for parents and children[1]

"Welcome to The Math Connection. I am Elaine Simmt, and I will be facilitating this program for you. Before we get into doing some mathematics, let's introduce ourselves to each other." I approached a boy that I knew from a previous program and put out my hand. "Hi, Joss. How are you today?" I said as we shook hands.

"Good."

"What grade are you in now?" I asked.

"Three."

"This is your mom?" I offered her my hand.

"Desie. Desie Merten." She introduced herself.

"Since we are going to be working together, maybe everyone could get up, introduce yourself to the others and shake hands." I suggested. "Don't forget to shake hands," I reminded them....

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"Great, now that you all have met, I would like each of you to think of the last mathematical thing you did today. Sean, will you start?"

"My math homework," Sean replied.

"What about your dad? What was the last mathematical thing you did today?" I asked Sean's father.

His dad thought for a brief moment and then responded, "I guess it was figuring out when I would have to leave in order to get here on time."

In the same way, the other people in the room told the rest of us what the last mathematical thing they did was. For the most part, the adults said something about money, and the children said either math class or math homework. In the voices of a few, I could hear either some enthusiasm for mathematics or some anxiety about it. My initial goal would be to encourage the enthusiasm towards mathematics and to try to help settle the anxiety. To do that, we would begin by considering the number of handshakes that just happened.

"I have a problem for you. We just shook hands with each other. There are 11 of us here tonight. Well, how many handshakes were there? It looks like

[1] Photograph is used with permission.

everyone has a pencil or something to write with. Here is some paper. So let's get started. Try to figure out how many handshakes there were when we all shook hands with each other. Everyone has to think about this—adults too."

I walked towards Joss and his mother. "Any ideas Joss?"

"One each?"

"Yeah. One handshake among us—each," I said.

"With each person once," Desie repeated. "So there is how many?" She asked Joss.

"Um, I think 22."

"You think so?" Desie did not sound convinced.

"Yeah."

"If I shook hands with everybody in this room?"

"'Cause one plus one is two and 10 plus 10 is 20. So 22," Joss explained.

"Okay, so you shook hands with everybody in this room?"

"Uh huh." Joss agreed.

"You would have shaken 11 times—you would have shaken 10 times right?

"Oh, um."

"So then each person would shake hands how many times?" Desie asked.

"Ten."

"Ten. Right. So there's 11 people, and each one shakes hands how many times?"

"Ten," Joss followed his mom's line of reasoning.

"So how many would there be?" she asked again.

"Um. 101."

This vignette was created from transcripts and my recollections of the first and the subsequent times an extracurricular mathematics program for parents and children, called *The Math Connection*, was offered. It is a program I developed in conjunction with a local school board in response to a need expressed by some parents in my neighborhood who were looking for ways to supplement their children's mathematics education. The intent of the program was to encourage and challenge students, between 8 and 14 years of age, and their parents to engage in mathematical thinking and problem solving and, at the same time, to educate parents to help their children with school mathematics.

The parent-child mathematics program was designed to consist of 10 sessions, each 90 minutes in length. As noted in Appendix A, the program ran as such 4 times since 1994. These sessions were held in a predominantly middle-class suburb of Edmonton, Alberta. On two other occasions, low enrollments resulted in the program being held at the University of Alberta campus. Both times the program consisted of only 5 sessions. In a third instance, two parent-child pairs were invited on campus for a single session. The purpose of this session was a to conduct a clinical interview in which the two parent-child pairs worked on a prompt. (See Appendices B and C for more information about the participants.)

Each session began with a short opener such as a number game, a game of strategy, or a puzzle which took the first 10-15 minutes. Then, for the rest of the evening parents and children worked together in response to a mathematical prompt (see Appendix D). In this part of the session, the participants might be thought of as doing problem-solving although I believe some of them (adults and children alike) saw themselves as playing with mathematics rather than solving problems. I have come to think of their activity as "bringing forth a world of mathematics." I will discuss what I mean by this later in the dissertation.

I have begun this dissertation by introducing the parent-child mathematics program because it was my primary source of data for my exploration and study of mathematics knowing in action. At the time when the program was first offered, it was simply an extracurricular mathematics program, not a site for my doctoral research. But, as the weeks went by, I began to understand how this program had the potential to be a valuable research site. Reflecting on the program and my research, it becomes evident that my doctoral research co-emerged[2] with the parent-child mathematics program. I brought forth a research study wrapped around and, hence, fully complicit with the mathematics program for parents and children.

In this chapter, I look back over the course of events from the conception of the program and identify some of the key moments in the development of my research program. I explain how, in the first year, I was making sense of what might constitute appropriate activities for the program and how I became sensitized to the variety of

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[2] The notion of co-emergence is central to my research. I will discuss it later in this dissertation.

emotions and issues that, for some of the participants, were tied up with doing mathematics. In the second year, I began to use the site for researching mathematics knowing in action. A fellow researcher (Gordon Calvert, 1999) and I documented our experiences in this setting through the use of a research response journal. By the third term, I was collecting more extensive data from the site as I focused on the interactions between parents and children and the mathematical activity in which they engaged. For four years, the parent-child mathematics program was both a site for parent-child mathematical activity and a site for research into mathematics knowing

A Mathematics Program and a Research Site

> *The math club is different from the clinical interview we[3] were doing in our other research projects....[Further], I do not view the math club as an experiment. It provides us with a place to do research but its true intention was, and is, to have parents do mathematics with their children.* (Simmt, Journal Entry, 1995)

Imagining the Parent-child Mathematics Program

I want to tell my story of research by first elaborating on the origins of the parent-child mathematics program. The program was first conceived in a conversation I had with a neighbor. Knowing I was at the university and studying mathematics education, she asked me if I could recommend some interesting books that she could purchase for her son—a nine year old with a real interest in things mathematical. She told me that he and his dad would often discuss things like infinity and googols. I knew of a few books that had interesting mathematical problems, usually historical problems—Theoni Pappas's *Joy of Mathematics* and Martin Gardener's books, for example. I checked out a high quality children's bookstore to see what was available. Unfortunately, I did not find anything else appropriate. My neighbor and I talked a little more, and the possibility of a mathematics program for children who liked doing mathematics came up. In our community, there are

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[3] We refers to the members of the enactivist research group: Tom Kieren, Lynn Gordon Calvert, David Reid and myself.

plenty of opportunities for children to participate in sport, drama and music; but there was nothing for children who liked doing mathematics. While investigating the possibilities of a parent-child mathematics program, I came across *Family Math*, a program developed in California (Stenmark, Thompson, Cossey, Hill, 1986). In the *Family Math* program, teams made up of parents, children and teachers facilitate sessions in which families are encouraged to do mathematics together. *Family Math* has published a set of videos and workbooks that offer suggestions based on the everyday activities of the home that can be used to foster mathematical thinking and teach mathematics: cooking, sewing, building, painting, and the like.

Based on the conversations with my neighbor and my research into mathematics for parents and children, I proposed a program in which parents and children would do mathematics together. At first, I anticipated that the program would involve taking opportunities that present themselves in the children's and the parents' experiences and thinking about them mathematically. For example, in what way do children's toys and games present opportunities for mathematical thinking? Instead of providing workbooks that the parents could give their children to do at home, it made sense to provide a program in which I might help parents and children start seeing the world mathematically and asking questions that are mathematical. As I came to learn, offering recreational mathematics problems is a great place to begin. The use of games, common household activities and so on were offered as suggestions, but they never did become the primary prompts for mathematical thinking in the sessions; rather they become extensions I offered as suggestions to parents and children for when they returned home.

As I contemplated the program I asked myself: Who could participate? and What might the sessions involve? I did not want this program to replicate the mathematics activity that students likely had already experienced or would experience in school. Nor did I want my program to become tutoring sessions (these were already available). It struck me that one of the things I might do that was different from the other activities already available in the community was to insist that parents participate with their children. (I can see how even this feature of my program grew out of the conversation with my neighbor. She was looking for something that her husband and son could do together.)

Recognizing a Potential Site for Research

Part way through the first term of the parent-child mathematics program, I began to recognize how rich the parent-child site was for studying persons' mathematical understanding. Frequently, I would return to the university from one of my parent-child sessions with examples to share with my colleagues of how a parent and child interacted with each other and how their mathematical understanding of some situation developed. I was excited by my observations of the ways in which the interactions between the parents and children sustained their mathematical activity. At the same time I wondered: Why are the participants able to engage in mathematics in this program when they say that this is not the case at school? What is it about the kinds of questions that I ask that is fostering rich mathematical activity?

The second time the program ran, I kept a research journal; it was important to my growing understanding of the parent-child program as a research site. The journal provided me with the opportunity to pay attention to my observations and how I was interpreting those observations.

> *Is this confidence a key factor in doing mathematics? I really think so. It seems to me that "knowing" you are capable of doing things is an important aspect of one's structure. If we are structure-determined creatures, then it is this kind of structure that acts on the "problem triggers" (prompts) in a meaningful way (with respect to mathematics).*
>
> *Language usage could be another whole story. By removing the textbooks, by including adults (not just any adult but a parent in particular), by using problems, etc., there may be more room for talk and non-mathematical language. This is an issue for me because I'm not sure how much formal [technical] language I should use. I don't want to 1) turn people off; 2) generate anxiety; 3) nor do I want to trivialize the mathematical language. I have to think more about this.* (Simmt, Journal Entry, 1995)

It became evident to me that the parent-child mathematics program could provide the experiential basis for my doctoral study. It was a site from which I could collect data that would help me explore the nature of mathematics knowing in action.

Facilitating the Mathematics Program

For the most part, my responsibility in the program was to bring to the group different prompts with which the participants could work during the session. I tried to vary these prompts such that the parents and children worked in different areas of mathematics over the course of the program. I also tried to come up with activities that tied into their school work or related to the school curriculum but did not look like typical school mathematics. In order to facilitate the diversity among participants (in terms of their ages, background knowledge, and experiences in mathematics), I used what I have come to call *variable-entry prompts*. These are prompts which open a space for a variety of actions and mathematical activities at varying levels of mathematical sophistication. Variable-entry prompts are such that the participants do not require specific background knowledge or specific mathematical skills; however, the prompts must be intriguing, and they should lead to important mathematical ideas, concepts and processes (Lampert, 1991; Schoenfeld, 1994). The handshake prompt referred to in the opening vignette has turned out to be a particularly good variable-entry prompt. (I will discuss variable-entry prompts more fully later in this chapter.)

My role in the math program was different from my role in previous teaching experiences. I did less explaining, and I usually did not provide answers to the problems participants worked on (although I offered some of my solutions after we had exhausted the participants' solutions). After I provided an initial prompt for activity, I would circulate among the participants, listening to them, asking them to explain what they were doing and sometimes requesting that they share their thinking with the rest of the group by putting their work up on the board. Early in the term, the participants (sometimes children, sometimes parents) were likely to ask me what they might do next or if what they were doing was right. I responded to their requests either with direct answers to their questions or with questions that pointed them in a suitable direction for exploring or answering their own questions. I found that the more sessions in which they participated, the fewer questions they had for me. I often found myself wandering around looking over shoulders, listening to the interaction between the parents and the children or to explanations they offered and talking to parents and children about school mathematics. I wrote in my journal:

> *I try to leave people to work—not because I am trying to be a non-intrusive observer but because I want them to work in directions that seem appropriate to themselves.* (Simmt, Journal Entry, February 7, 1995)

For the most part, the children and the parents made sense of the activity together (and sometimes apart) and, in doing so, learned new mathematics. It is the case, though, that I had to provide more guidance to some participants than others. For some pairs, I had to generate many small tasks within the general prompt. This was the case for Greg and his mom.

Vignette 1-2. Facilitating mathematical activity

When Greg participated in the program, he was in grade four. He said he found math hard and boring. It was not surprising to me that he did not like mathematics. He indicated his goal in the program was to "speed up". Greg's mother said she hoped that the program would help Greg understand and enjoy mathematics.

After working with Greg for just a few weeks, I could imagine what he might be like in math class at school—fidgeting in his desk and imagining the greatest adventures with the simplest of things that cross his desk. When I brought in any kind of manipulative for the evening, he would spend much of his time playing with it. Whether it was Lego or straws, bingo chips or dice, Greg could find some creative way of using the items that was different from the way I had intended they be used. In the beginning, this annoyed his mother so I spent a little time each night talking to her about school mathematics and what I observed Greg's strengths to be as he played with the toys. His playing around bothered her less once she was able to recognize the mathematical things he did.

Greg was very good at seeing patterns but not so good at defining a task for himself. The prompts that I offered in the sessions had a specific question to be answered but, in most cases, the participants had to structure and define the task for themselves. Greg was not very good at that. His mother tried to encourage him to think of more to do with the initial prompts I offered but he resisted her suggestions and would play *until I posed specific tasks or problems for him.*

One night I brought in Lego. The participants were instructed to take five pieces, each of a different colour, and determine the number of different towers could be built from just five pieces. I demonstrated how a tower, from the ground up, might be ordered red, blue, yellow, green, and white and suggested that it was to be considered different from a tower that was ordered red, white, blue, green, and yellow. After giving the prompt, Greg eagerly started building the different towers. It did not take long for him to ask for more blocks, but there were no more. I encouraged him to think of another way he could figure out the answer without having to build all of the towers of height five. I suggested that he consider towers that weren't so tall. Simplifying the problem was a general strategy I tried to encourage and had suggested before.

He did what I suggested and took three blocks. That was simple enough for him and a few minutes later, he showed me the 6 different towers he made. Again he sat back as though this is all he could do. I asked his mother to keep track of his towers on a piece of paper. "When there are 3 tiles," I said, "you can make 6 towers. Now why don't you try 2 blocks. Don't forget to write it down. It will help you look for a number pattern."

Only then, once I specifically asked him to look for a pattern, did he do so. By the time he worked through the cases for one, two, three and four tiles he was able to correctly predict the number of towers he could build with 5 tiles.

The last night of the program, Greg's mom commented that she was glad they had come. She had gotten to know her son in a way that she had never known him before. As cryptic as her comment was, I interpreted it to mean that doing mathematics with Greg was a new experience for her and, like all new experiences, it offered her an opportunity to reflect and to come to understand her son's high activity level as something that could be appreciated rather than dreaded.

As much as I tried to leave the participants alone to make sense of the activities, if they paused for too long or got distracted, like Greg did, I found myself interrupting them and/or pushing them in one direction or another—my old classroom behaviour "reared its ugly head" (Simmt, Journal Entry, 1995). I used this description of my behaviour in my journal after a session in which I directed participants to "move on" from a warm-up activity in which they were still actively engaging so that they could get on to the main activity planned for the evening. In reading the journal entry, I realize how I bring my history as a classroom teacher and my desires as a teacher with me to the parent-child mathematics program. Just like when I taught, in this new setting I also found myself wanting to keep the "students" on task.

I find myself making sure they are doing something—I think Ralph [a fellow grad student] would have something to say about the students' autonomy. [On one hand] I want them to be taken up and engaged by the mathematics; but[on the other hand] I do step in quickly to point them in certain directions. [I need to have more] wait time. They [the participants] need to learn how to persist. (Simmt, Journal Entry, 1995)

I also considered it my responsibility to teach parents how they might interact with their children in the context of mathematical activity. For the most part, this meant both modeling the kinds of questions that parents might ask their children when doing mathematics and making explicit those questions. I used questions like: "How did you do that?"; "Why did you do that?"; "Are you sure about that?"; and "Explain that to me, please." It usually takes a few classes to convince some of the parents they do not need to tell their children "the answer" but, rather, they might simply pay attention to their children's thinking. When it came to new and unfamiliar mathematics or prompts on which the parents had never worked before (which was frequently the case), I suggested that one of the best strategies for parents to help children is simply to do mathematics together with them. Finally, I tried to recommend books (see Appendix D) and activities that the children and parents might find interesting to do at home.

Although my role in the program in some ways remained the same over the years, the changes it underwent were a result of my growing understanding of the participants and how I could interact with them.

Creating a Curriculum

All organized learning programs or events have a curriculum, and my parent-child mathematics program also needed to be "about something." It needed a "topic of conversation" (Gordon Calvert, 1999). However, the parent-child mathematics program had no prescribed curriculum or mandated program of studies, just a promise of providing mathematical spaces for parents and children and identifying ways for parents to help their children with school mathematics. Such a promise is already a curricular act. That is, it anticipates what might occur, and it is an historical act which pre-figures later actions.

Rather than challenging various notions or conceptions of curriculum, I work from the assumption that the curriculum is not "distinct (but coherent) knowledge bits" but has to do with the existential qualities of participating in the program. Curriculum "is conceived as the interpretation of lived experience" and from a reconceptualist perspective, it "can only be discussed in retrospect, as the path that was taken, in all of its experiential richness" (Davis, 1996, p. 90-91). I will elaborate my view of curriculum by discussing the curriculum of this program through recounting it as a lived experience—my lived experience.

There is privilege in working in a context where the curriculum (by almost anyone's definition) is not *pre*scribed—a rule of action written by others—but *a*scribed—authored by the participants in action. In my work, I am suggesting that curriculum is a space—a space that is defined in action and one that co-emerges with the mathematical knowing of the participants who are acting and interacting within it and to form it. In the parent-child mathematics program, the curriculum involves adults and children paired together and myself (the facilitator), and it includes the prompts and questions that I bring to the sessions as well as those generated by the parents and children who participate.

In the following section, I tell about three important lessons in which I learned about the curriculum of this program and show how my learning contributed to the ongoing creation of the curriculum. The first lesson suggests to me that when needs are not satisfied there is no curriculum. The second lesson I learned was that avoiding prompts that look, sound, or feel like school mathematics encourages new behaviours rather than old patterns of behaviour to emerge. My third lesson was that variable-entry prompts are needed so that everyone can participate and contribute in the emergent curriculum.

Participant needs

Each time the program ran, I asked the participants why they were participating in the program and what they hoped to get out of it. (See Appendix E for comments made by the participants to which I explicitly refer in this dissertation.) The participants had different histories and motivations for participating in the program; hence, it was not surprising that they had different expectations. A few children indicated they wanted to improve their math grade in school. Cathy, for example, expressed her desire. "I hope I can bring my grades up and understanding of what the question is asking." Others wanted to make math easier. "I want to find shortcuts and hints to make it [school math] easier," Carly, a grade 8 girl, wrote. Remy's comment was similar. "I hope it can give me a better view of math and explain problems clearly." The children's writing sometimes tied emotive reasons to cognitive ones for coming to the math program. "Let me understand so that I won't be frustrated," Sharon wrote. "I hope to be faster so my mom won't annoy me about being slow," wrote Brad. "Let me speed up," said at least one primary school student each time the program ran. Obviously, some primary teachers were having children do timed tests of basic computational skills—Mad Minutes, as the children called them. There were a few students who came to the program because they wanted to learn more advanced mathematics or because they liked doing mathematics and wanted more time to do it. Kerri explained to me that she wanted to learn new things that she won't learn in school until a higher grade.

The parents' comments generally fell into one of four themes: they wanted to learn more about what their children were learning at school; they wanted some "tools" to help their children with school math; they wanted to foster positive experiences with mathematics so that their children would grow up to find mathematics enjoyable; and there were a couple of parents that simply wanted to spend time with their children. One mom said it this way: "I want to spend quality time with my daughter, and this seems like a unique opportunity to do this."

As I envisioned a curriculum for the parent-child mathematics program, I wanted one that would address the participants' expressed needs, but it was also to be a mathematical space—as I defined mathematics. Reflecting on the selection of prompts and the design of the structure of each session, I can observe how what I valued in mathematics strongly influenced the "curriculum." Although participants came to the program with specific needs in mind, and many of the participants left the program satisfied that their needs had been met (at least in part), I suppressed some of their expressed needs by convincing the participants to do what I thought was important. It was only in reflecting on the "curriculum" that I realized the discrepancy between their felt needs and their needs as I understood them.

Vignette 1-3. A lesson about satisfying needs

When invited to express their ideas of what we should do in the program, one pair, a woman (Roberta) and a 13 year old girl (Kristina), indicated they would like to do problem solving. "Great," I told them, "this is exactly what we do in the program." Four weeks passed, and each week they came and participated very actively in the session; but on a few occasions one of them would ask, "When are we going to do problems?". For the fifth session Roberta and Kristina arrived a few minutes early. Kristina had her textbook with her. She asked if I could help her with a problem. Since there were a few minutes before the session was to begin, I agreed. The question came from the review exercises at the end of a chapter about fractions:

> The journey from Calgary to Toronto takes 3 1/2 d by train and 4 1/5 h by airplane. How many times faster is the airplane than the train? (*Journeys*, 1987, p.288)

It is not surprising that Kristina found this word-problem difficult. It involves mixed fractions, mixing time in days with hours, an unusual fraction for time (1/5 of an hour), and proportional thinking.

I first suggested that they might begin by simplifying the problem. This is a strategy that I had encouraged many other times over the course of the program and one that seemed reasonable to me in this context. By the time they began to work on the problem, other participants had arrived and we were ready to begin the session. I made a decision to break my "rule" about doing school mathematics and asked everyone to think about Kristina's problem. Each pair worked at the problem for the next twenty minutes. All of the parent-child pairs came up with strategies for thinking about the problem even though it was clearly "content" that the younger children may or may not have "covered" in math class.

When Roberta and Kristina finished the problem, I talked to them about it and problem solving. I had already learned that Kristina was not Roberta's daughter but the daughter of a friend. What I had not yet learned was that it was Roberta who asked Kristina to come to the program with her. Roberta explained that it is her dream to become an police officer, but she had failed one of the requirements of the preliminary screening —the math test. She told me she "just can't do problems." When I pointed out that we do problems every week, she indicated that it was problems like the one we worked on this night with which she had difficulties. She wondered if we could do more of these. I explained that she could always simplify a given problem; this is a generic skill that would likely help her do the kinds of word problems that appeared on the screening test. Instead of contemplating her request any further, I told her that I did not anticipate doing any more of the textbook *problems. Roberta responded that, although she was learning to do problems like I offered in class, she correctly pointed out that on a timed test there is not time to figure out how to do the problem; there is only time to calculate an answer. She said she needed help to learn to do problems (like the train problem) more quickly. Although Kristina and Roberta had fully participated in all of the activities up until that week, they did not come back for the last five sessions.*

My view of the mathematical activity that was important for the participants of the program did not always coincide with what they viewed as important. In the case of Roberta and Kristina, I had failed to demonstrate how mathematics requires a certain kind of thinking in which, in fact, they were engaging. The experiences I set up for the program had the potential to make a difference to the participants; but Roberta was right—for her, problem solving took time, and the police screening test did not give her the time she needed to solve problems. She would not be able to finish the math test in the time allocated if she had to reason about and approach the problems in the way she was doing in our classes. Roberta knew what she needed; hence, it is reasonable that she never came back. The program I offered would not satisfy her needs. Although their withdrawal was disappointing to me, in retrospect, it is also gratifying in that Roberta understood her situation and acted in a way that, I assume, kept her on her path towards who she wants to be. As Davis (1996) reminds us, the curriculum is part of who we are.

> The notion of curriculum, then, involves more than a study of particular ideas; it becomes an integral part of the constantly emerging text of our existence as enacted in the relatedness of the classroom. Issues of knowledge and understanding are thus woven into and cannot be considered apart from the notion of identity (p. 99).

Choosing prompts

Roberta and Kristina (the pair discussed in the vignette above) participated in the program the second time it was offered. By that time I had already generalised from some of my experiences in the first term and formulated a "rule" about the prompts that I would use in the program—they were not to resemble school mathematics. Each time through the program I worked at trying to understand the needs of the participants and selecting appropriate activities for the parents and children. It is a task I know well, being a secondary school teacher, but one upon which I never spent quite as much time reflecting as I had to do in the context of this extracurricular program. For this program, I needed prompts that were appropriate for children between the ages of 8 and 14 and for which there are few specific prerequisite mathematical skills required. Further, I believed that the prompts should come from a variety of areas in mathematics. However, just as importantly, I learned that for some participants the extracurricular program should not look, feel, or sound too much like school mathematics. I did not come to the program with this belief; it emerged out of my experiences with the parents and children. One event in particular stands out for me.

Vignette 1-4. Learning that worksheets can provoke school-like behaviour

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Dan and his daughter Kerri worked on the same permutation prompt that Greg and his mother worked on. They, too, were given Lego and asked to determined how many towers could be built from 5 different colours. They responded to the prompt by simplifying the problem and considering the cases of towers of height 1, 2, 3. They built a tower for each case and made a record of each tower they built. By the end of the session they had determined that they could generate 120 towers with the five colours.

Because I wanted to help parents help their children on an ongoing basis, I decided I would offer suggestions of things they could do at home together. At the end of this particular session, I passed out a worksheet that I had found in a teaching magazine and one that was specifically designed to be taken home. It involved a permutation problem similar to the one with the blocks. It asked how many different cones could one make from three different flavors of ice cream. I indicated to the participants that they might want to work together on this problem sometime over the next week at home.

I thought nothing more of the "homework" until the next week when I was preparing for the session. I had planned for the participants to construct some polyhedra from straws. As I passed out the straws, Dan took the worksheet, from the previous week out of his bag, and he and his daughter turned their attention to it. When I went to put some straws on their desk, Dan picked them up and passed them back to me indicating they hadn't finished the homework from the previous week. After Dan and Kerri worked on the sheet for about 20 minutes, I returned to their desks with the straws in hand and offered them to them once more. This time Dan turned to Kerri and asked her if she thought they should move on to the new activity. With a smile, Kerri nodded her head and her dad put away the worksheet.

In the setting of the parent-child mathematics program, it was not unusual for the parent to take on the role of 'teacher'. I neither explicitly encouraged this nor discouraged it (although I always encouraged parents to do mathematics with their children rather than watch their child "do the work"). I respected this father's decision to complete the worksheet, but I was bothered by it. [4] It immediately struck me that this pair were not engaging in mathematics in this setting as much as they were engaging in school (Bauersfeld, 1995b; Voigt, 1995; Berieter, 1990). After all, worksheets, pages of exercises, word problems and homework are common experiences in school and constitute one of the most persistent manifestations of mathematics that many, if not all, of these people had experienced in their lives.[5] It bothered me that a view of mathematics so deeply saturated with 'schooled' expectations might restrain the possibilities for parents and children to learn mathematics together. From that point forward, I deliberately attempted to use activities that did not look, sound, and feel much like school mathematics. It is clear that my actions were based on an interpretation of the situation and one that I, as the facilitator, might interpret differently than the children, parents or another teacher. However, it seemed to me that worksheets and textbook exercises were too school-like. In the end, it was these forms that I avoided using in the program, not necessarily topics or concepts which are usually developed in school mathematics. I did continue to make suggestions for things they might do at home, but I never handed them out on paper.

Promoting mathematical behaviour

The decision to offer this program to children between the ages of 8 and 14 had significant implications for the kinds of activities and prompts I was able to choose for use in the program. Teaching in the parent-child math program is unlike teaching school mathematics. In school settings, it is usual to teach students at a particular grade level and

[4] This posed an interesting dilemma for me. In school math classes, teachers direct student activity. In this situation, I felt as though I did not have any authority over the parent to justify asking him to put away the worksheet and do the new activity. As teachers in the usual classroom setting, we have the institutional authority to do just that. In my situation, I was faced with sharing my authority or having the parent grant me the authority to direct his and the child's activity.
[5] I am reminded of the day, when asked what he had done in math class, my son replied, "copy and completes," referring to the structure of his exercises rather than the mathematics he had done.

in cases where there is more than one grade level in the classroom, it is common to teach with differentiated curricula. In the mathematics program I had designed, I intentionally chose to have children of different grade levels participate and to use the same basic prompts. Of course, with the children came adults who had just as varied mathematical experiences. Thus, the activities I chose for the program had to be appropriate for this diverse group of participants. The fact that parents and children were to work together was one of the reasons I felt this strategy could work. I anticipated that, when a child needed some form of assistance, either the parent or I would be able to provide it. On the other hand, when a parent needed some assistance, either the child or I would be able to provide it. Activities were selected from a variety of resources and on the basis of whether the children in the program could enter into activity with very little direction. I was successful in selecting appropriate prompts most of the time, but not always. (A complete list of prompts used in the program can be found in Appendix D.)

Vignette 1-5. Struggling to identify qualities of good prompts

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For one session, I had the participants play a game that involved directed magnitude. The game, called "Walk the Plank," (Lovitt and Lowe, 1993) uses a number line taped to the floor to represent the plank on a pirate ship. Players roll two dice to determine the moves along the plank. The first die denotes the direction the player is to move, and the second die indicates how far the player is to move. The game is either won or lost based on the players getting back to the ship or falling into the sea. The object of playing the game in the session was to come up with an ideal size of plank; that is, a plank for which the game lasted an appropriate length of time. If the plank was too short, the game ended too quickly; if it was too long, the game took too long.

One of the younger children in the group, Cathy had difficulty working out that she needed both a direction and a magnitude to make a move, and her father had trouble clarifying for her how the moves worked. Another child, Brian, found it difficult to determine how long the plank should be. He could make this judgment on a case by case basis—this one is too long, this one is too short—but he did not have a basis from which to make this decision in general. This was not as difficult for some of the adults and older children who used the notion of average to determine a more general solution.

I selected the prompt for its potential to foster mathematical thinking. In particular, it involved some significant mathematics concepts including directed magnitude, integers, and average; further, it required participants to reason about what they were doing. Finally, there was the possibility that the parents and children could play the game at home. In spite of the positive attributes of the game, from my point of view, it did not turn out to be a very good prompt for the younger (8 to 10 year olds) participants and a few of the adults in the group. I commented in my journal:

> *The class didn't go as well as I expected it to. Although everyone played the game, I'm not sure they had any idea as to how they could determine which was the best length for the game... The activity was too complex, complicated, sophisticated, difficult (what is the word that best fits) for many of the participants.* (Simmt, Journal Entry, 1995)

My research partner responded:

> *I didn't see the activity of finding the best length too difficult—except when there wasn't a negotiation going on and exclusionary methods were used. I wasn't with the other groups but I think this was something that made them stop and think—and it wasn't immediately obvious to them. Is this your thing again about feeling a need to jump in quickly? I really think that even Calvin would have come up with something eventually.* (Gordon Calvert, Journal Entry, 1995)

As I reflect back on the prompt and the reflections in my journal of what the participants did given the prompt, I am struck by how many of the participants could not make a judgment about the "optimal" plank. In as much as all the participants could play the game (it took a few minutes for most participants to understand the directed magnitude part of the game), in my view, little mathematical thinking was prompted by it. On the other hand, when this same group of people were offered the "handshake problem" (described at the beginning of this chapter), they worked for an extended period of time, and even the youngest of children were able, with their parents, to either act the situation out, make tables or draw pictures and note number patterns. Another example of an activity into which even the youngest participants were able to enter and sustain mathematical activity (from my point of view) involved the construction of a fractal-like object and the search for number patterns that could be noted from that object. Unlike the game, these two activities seemed much more accessible to all the participants, not just a few. I noted in my journal:

Fractal cards[6]*—A much better activity for the math club. More of a* variable-entry *one I would say. (Simmt, Journal Entry, February 21, 1995)*

The prompts that I chose for the sessions needed to be accessible to all of the participants at some level—and I wanted that level to be mathematical. Thus, I began to select and develop prompts that I believed could lead to significant mathematical activity even though the participants might enter into that activity by means of different actions and strategies. At the same time, my research colleague's comments continued to resonate. Mathematics problems are truly problems when the means to the end is not obvious (Polya, 1962; 1980). I needed prompts (a curriculum) that challenged participants and encouraged them to develop new strategies for doing mathematics at the same time as building their confidence so that they believed they could solve the mathematics problems they might encounter, whether in school or in other aspects of their daily lives. I found myself asking, what are some of the features of prompts that encourage all the participants to engage in mathematical thinking?

Variable-entry Prompts

Schoenfeld (1994) and Lampert (1991), who both use problem-tasks in their research, have articulated the features of what they believe are good problems. Lampert suggests that good problems are ones which provoke discussion, reveal how students are thinking, are a safe domain in which to solicit student's participation, and have the potential to lead students into "unfamiliar and important mathematical territory" (p. 126). Schoenfeld has what he calls a problem aesthetic which consists of five criteria. Problems should be accessible; should be solvable; should illustrate important ideas; should not have trick solutions (something you haven't seen before and are not likely to use again); and should be extendible and generalisible (p. 44).

[6] A description of this activity can be found in Simmt and Davis (1998).

Polya's (1980) articulation of what constitutes a problem: "If the end by its simple presence does not instantaneously suggest the means, if, therefore, we have to search for the means, reflecting consciously how to attain the end we have to solve a problem" (pg. 1); what it means to solve a problem— "Solving a problem is finding the unknown means to a distinctly conceived end" (pg. 1); and the heuristics involved in problem solving—understand the problem, devise a plan to solve the problem, carry out the plan and look back to examine the solution—are regarded as standard in educators' discussions of problem solving.[7]

Because each of the participants who came to the sessions has a unique lived history, it was important that the prompts for the parent-child mathematics program were perturbations on which each person could act. At the same time, it was important that the prompts were perturbations on which the participants with significantly more experience in mathematics were motivated to act. The label, *variable-entry*, points to two essential features of the prompts. The first is that they are triggers to prompt mathematical activity. That is, some features of the prompts could be selected and considered by each of the participants in a way that an observer identifies as mathematical. In this way, variable-entry prompts are prompts, not problems *per se*. Secondly, variable-entry prompts have multiple and varied entry points into mathematical activity. Given a prompt, there are a variety of actions a learner might take to enter into activity that the observer might identify as mathematical.

The distinction between variable-entry prompts and open-ended problems is significant. With variable-entry prompts, the intent is that the students will have a beginning place; that is, entry into suitable activity—mathematical activity. However, the prompt may be such that the students converge on a solution. The term open-ended, on the other hand, suggests something about the solution is relevant, in particular that the ending

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[7]One only needs to note the frequency with which these heuristics are cited and treated as skills to be learned in many mathematics texts to recognize the impact Polya's (1957, 1962, 1980) conception of problem

is open. This usually means there is more than a single solution path, or there is more than one solution. For example, the question, "Show me how you can make 10", might be thought of as open-ended. In contrast, the question, "How many ways can you arrange 3 children in a line?" is not open-ended. There are only 6 distinct ways. However, it is variable-entry. Young children could act the question out or draw a picture to come to the solution, older children might manipulate symbols representing the 3 children and older children yet might recognize this as a combination problem and use factorials to solve the problem. The basis on which I select prompts is not on the possibility of many solutions but with the understanding that there are many possible ways in which to approach the question or task and to formulate some solution. The problems defined by the students in acting on the prompt may have many solution strategies (which is consistent with open-ended problems), but this is not a necessary feature. Variable-entry prompts are similar to Bauersfeld's (1995b) open tasks which he describes as tasks that open the chance for students to employ and develop their own interpretations (p. 281).

Kieren and Pirie (1992) found that it is the student's answer which determines the question to which the student responded. That is, while it is easy to classify questions and suggest that they carry some inherent meaning, Pirie and Kieren's research suggests that the student's responses point to the question's meaning for that student. As teachers, we pose a question to a student anticipating a particular kind of response, only to listen and hear that the question was something quite different from what we intended. Bloom's (Bloom et al., 1956) taxonomy may tell us that we can categorize educational objectives at different cognitive levels (knowledge, comprehension, application, analysis, synthesis, and evaluation), but it is more appropriate to categorize the students' responses if you want to understand the nature of their knowing. Thinking about mathematical knowing in this way demands we observe the student's mathematical actions. Thus, mathematical knowing and understanding are thought of not in acquisition terms but in behavioural ones. We are

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solving has had on mathematics education.

unable to specify in advance of the student acting that there even exists a perturbation that might occasion a problem for the student. It is only in observing the individual's behaviour that we can say she or he has a problem or is working on a problem. Furthermore, it is only once we observe the student acting in some way that *we recognize as problem solving or mathematics* that we can make such a claim. I return to the ways in which prompts are implicated in a person's mathematical knowing throughout the thesis.

I distinguish my notion of variable-entry prompts from either Schoenfeld's or Lampert's based on the assumption that I cannot pose problems for students; I can only provide a task or a prompt which has the potential to occasion their mathematical understanding. Hence, I operate under the assumption that I do not (because I cannot) pose problems for my students: I can only trigger their actions. Thus, in the context of studying mathematical understanding, even if we admit that people solve problems given perturbations in the environment, it is not appropriate to call the perturbations problems in advance of the person's actions.

Learning from Experience

My understanding of my role of facilitator and what constituted worthwhile tasks and curricula changed as I experienced the parent-child mathematics program. New awarenesses arose for me throughout the experience and as I reflected on that experience. Roberta and Kristina taught me that diverse motivations are part of people's acts of knowing mathematics and, thus, are part of the curriculum that is lived in classrooms. Dan and Kerri taught me that one's patterns of behaviour are triggered by familiarity and that the activities in the program could not be so similar to activities they did in school that they treated the program like school. Finally, I learned to distinguish between open-ended problems and variable-entry prompts; that is, there is a difference between tasks that one can enter into meaningful activity and tasks that have many possible solutions. As I have described, the curriculum was not prescribe but co-emerged with the mathematics knowing of the participants and myself. Thus, I might say that the parent-child mathematics program itself emerged in action.

So, too, did my research program. Making sense of the curriculum of the parent-child mathematics program was just one aspect of my research activity. As I indicated earlier, I found myself returning to discussions about the nature of mathematics knowing with illustrations and questions from my observations of the parent and child actions and interactions. As a researcher, I found within myself and my experiences a phenomenon to be explained—a question that I desired to answer (Maturana, 1991, p. 36).

How might I characterise the mathematics knowing that is brought forth in the actions and interactions of the parents and children in this mathematics program?

Chapter Two

RESEARCHING MATHEMATICS KNOWING

For decades, researchers have been attempting to find models for various problems in education, from human thinking and development to the study of classrooms as social systems. Mathematics is an interpretive framework that helps us understand various phenomena in our world, including educational phenomena. Although linear, exponential models and more recently probabilistic models have dominated educational research, recently educational researchers have begun to consider complexity and chaos theories (both of which involve the study of dynamical systems) to help understand teaching, learning, curriculum and educational research itself (Davis and Sumara, 1997; Robinson and Yaden, 1993; Cziko, 1989; Doll, 1988). As is the case with any interpretative framework, our understanding is both restrained and liberated by the tools and metaphors offered by the framework. Thus, new theories and interpretive frameworks offer theorists and researchers new "tools" for understanding "old problems" and for generating new interests.

In my research, I have meandered along a landscape that is well defined by traditions of what counts as research. At the same time, the landscape is continually changing. It is changing under the influence of those who walk through it, those who are curious about unexplored areas and those who have notions of how they might travel differently than those who have gone before them. The dissertation is a feature of this journey. Like a piece of art, the dissertation is an expression of who the researcher is and what she feels—it is a representation of her knowing in action. But, unlike the novel that tells a story of love or betrayal, that attempts to provoke an emotional response such as joy, fear and hope, a dissertation assumes that those reading it bring to the reading a shared emotion—the desire to explain.

Maturana (1991) claims that the desire to explain is the very emotion that specifies the domain of actions in which science takes place as a human activity. Historically, research in mathematics education has used scientific models, involving experimental or quasi-experimental designs which explore factors and relationships implicated in the mathematical education of our children and youth. Statistical treatment of data lends itself well to the traditional dissertation format. It is presumed that the analysis will demonstrate the causal factors related to the particular problem one was studying. From such studies, it is hoped that various aspects and problems of education can be predicted and controlled. Further, statistical analyses are useful for describing various behavioral patterns involving large numbers of people and global phenomena. However, individual behaviour and local phenomena resist statistical treatment; hence, they are not well understood through such studies (Capra, 1996; Cohen and Stewart, 1994; Glieck, 1987). If educational research is to be used to understand local phenomena or the behaviour of individuals, then we need to take seriously alternative forms of research—forms that have the potential to shape experience and enlarge understanding of the particular (see van Manen, 1990; Pinar, 1988; Eisner and Peshkin, 1990).

As modes of inquiry change so do the ways in which the research is disseminated. The dissertation is just one example of a research report that is undergoing a metamorphosis. The research community is trying to understand in what ways new forms of inquiry and/or reporting research satisfy academic standards. My work is not a theory of causes and effects, nor is it theory to explain statistical findings from a particular population. Rather, it is an interpretation of people's mathematical knowing in action. Through my interpretations, I offer models for observing mathematics knowing in action. This dissertation is an expression of my knowing for others within an academic tradition. It is, in the first place, my expressed knowing and, in the second, a representation of my knowing.

One might ask, "Is my work replicable? Is it generalisable to other populations? Does it have predictive power?" These are questions for which scientific research has traditionally been required to respond. Can research for which we cannot answer these questions affirmatively still be called educational research? Kessels and Korthagen (1996) suggest it can. In fact, they argue that accounts of the particular have an important place in educational research. They point out that quantitative research which has claimed to answer these questions "lacks flesh and blood," the very substance that helps us understand the human condition. My work, in contrast, is highly contingent. Who the people are and what they do in a particular moment matters.

Following the lead of the constructivists in mathematics education research (e.g. Pirie and Kieren, 1992; Steffe, 1990, Confrey, 1990), my work focuses on particular people's mathematics knowing in action or, as Davis (1996) distinguished, the mathematical. The mathematical includes ways of thinking which involve actions such as counting, comparing, pattern noticing, and reasoning. In contrast to the mathematical is mathematics—that which is commonly thought of as the object(s) of those processes or the objects that arise from mathematical thinking. Mathematics is commonly understood to include things like number, variable, function, and geometrical forms, for example. Rather than attempt to fully define mathematics knowing in action (which, I would argue, is not possible), I will use the concept throughout the dissertation and explore its meaning in use by investigating mathematics knowing that is brought forth in action.

In the illustrations I offer as part of my explanation of mathematics knowing in action, I describe and interpret the actions and interactions of parents and children who participated in my mathematics program by engaging in mathematical activity. I take into account the objects of their activity and focus on their actions and interactions which bring forth those objects and the implications of that bringing-forth (Maturana, 1988; Blumer, 1969). Thus, one component of this dissertation is the development of an explanation of the ways in which people bring forth mathematics through actions and interactions that I view as mathematical.

Experience, Explanations and Conceptions

Maturana (1991) suggests that any explanation must satisfy two general conditions to be accepted as an explanation. First of all, an explanation must present an experience (phenomenon) to be explained in terms of what an observer must do in order to experience the phenomenon. Secondly, the explanation must offer a reformulation of the phenomenon in terms of a generative mechanism such that, if the mechanism were to operate the way described, the observer would experience the phenomenon that is being explained (pg. 32). Maturana suggests that all explanations must satisfy these two criteria in order to be accepted as explanations (in contrast to a description or algorithm, for example); however, explanations may also have to satisfy other criteria within a particular discourse or community. For example, scientific explanations must satisfy criteria specified by the scientific community and specific to the domain of science. In contrast, if the explanation is in another domain, Christianity, for example, then the explanation must satisfy the constraints placed by the community in which it is offered, in this case the community of Christians. In other words, explanations are not independent of the humans that are posing the explanations and the humans that are listening for the explanations. Explanations do not describe a reality independent of the human observer, but, rather are a phenomenon of the human domain. Therefore, explanations are relational phenomena (Maturana, 1991). If one accepts the explanation of another then one accepts the other as a member of the community in which the explanation is offered. On the other hand, if the explanation offered is not accepted by the listener then, in the same act, the explainer is not accepted as a member of the community (presuming the explainer does not accept the criteria of acceptance). Understood this way, explanations exist in communities in which the members of the community remain in relation to each other by means of their criteria for acceptance of an explanation.

Understanding scientific explanations in this way means that science is not validated by its correspondence to an objective world. Maturana (1991) writes:

> [O]ntologically, in its manner of constitution as a cognitive domain, science is no different from other cognitive domains because it is defined and constituted as all cognitive domains are, namely as a domain of actions defined by a criterion of validation or acceptability used by an observer or by the members of a community of observers to accept those actions as valid in a domain of actions defined by that very same criterion of acceptability (p. 39).

This statement implies that scientific explanations are not valid because they explain an objective (observer independent reality), but they are valid because they meet criteria of the scientific community which were developed by the scientific community

Blumer's (1969) explanation of the functions a concept serves in science is similar to Maturana's articulation of the criteria of validation of scientific explanations. Blumer proposes that a scientific concept serves three functions: "(1) it introduces a new orientation or point of view; (2) it serves as a tool, or as a means of transacting business; (3) it makes possible deductive reasoning and so the anticipation of new experience" (p. 163).

In both Maturana's (1991) and Blumer's (1969) proposals, our attention is turned to the role of the explanation in the former case and concept in the latter in our understanding or meaning making. Certainly, in educational research, this is key. Ongoing debate as to what counts as research in education (Donmoyer, 1996) has led to a multifaceted view of how our understanding is elaborated in education. Eisner (1997) argues that whatever form educational research takes, its goal is to be generative. Descriptions, explanations and conceptions all are forms that have the potential to shape experience and broaden understanding of the particular. The explanations and concepts I propose are offered with the intention to further our understanding of mathematics knowing in action. Developed out of my interpretations of mathematics knowing in action, their value lies in their potential to occasion the knowing of members of the mathematics education community.

The Observer

Von Foerster (1992) believes that scientists have adhered to that principle of the independent observer for fear that "paradoxes would arise when the observers were allowed to enter the universe of their explanations" (p. 10). However, in some areas of scientific study (cybernetics, computer science, and biology, for example), the study of systems in which the observer is a part of the system is afforded by the use of powerful conceptual tools such as circularity, reflexivity and self-reference.

A conception of observer-dependent research results in explanations that are no longer of reality independent of the observer but, rather, a reality that includes the observer. Work in cognitive science and neuroscience suggest that there are no qualities of things that exist independently of a perceiver. Lakoff and Johnson (1999) note, "the quality of things as we can experience and comprehend them depend crucially on our neural makeup, our bodily interactions with them, and our purposes and interests" (p. 26). Von Foerster (1992) also notes that the demand in scientific discourse to separate the observer from the observed wherein the idea that "the properties of the observer shall not enter into the descriptions of his observations" is problematic (p. 10). If I, as observer, am independent of the universe, then I observe it from a distance watching it unfold. In contrast, if I am part of the universe, then whenever I act I am changing both it and myself. As a teacher and a researcher in the domain of human science, I find myself asking: "Am I an observer of a world independent of myself? Or, am I, as von Foerster asks, part of that which I am observing?" This is not a trivial distinction. If I am independent of the world I am observing, then my actions and my explanations of it make no difference to it. However, if I am part of that which I am observing then my actions and my explanations change the world in which I participate.

The study of human cognition, Maturana (1988) believes, is the study of the observer. When we explain human cognition, we explain the observer.

> Observing is both the ultimate starting point and the most fundamental question in any attempt to understand reality and reason as phenomenon of the human domain. Indeed, everything said is said by an observer to another observer that could be him- or herself, and the observer is a human being (Maturana, 1988, p. 27).

I wish to make it clear that I understand myself (as observer) to be fully complicit in the research. I am part of the interactions and the mathematics knowing that co-emerged among the participants in the parent-child mathematics program. Through my various actions such as selecting prompts, talking to participants, making (or not making) suggestions when participants asked for help (or saying nothing), I am implicated in their mathematics knowing, and they are implicated in my mathematics knowing, and none of this is separate from my research and my knowing about mathematical cognition. There lies complicity.

Maturana's (1988) asserts that "scientific explanations do not explain an independent world, they explain the *experience of the observer*, and that is the world that he or she lives" (emphasis added, p. 38). As von Glasersfeld (1999) describes, "What is observed are not things, properties or relations of a world that exists as such; but rather the results of distinctions made by the observer him or herself" (http://www.oikos.org. vonobserv.htm, p. 3). I do not believe this is the same as the claim made by ethnographers who recognize their participation in a culture or situation may have an impact (intended or not) on that culture. Maturana's point is that I am implicated not simply because I am another variable in the situation or culture that I am studying but because my knowing and my explanations constitute the worlds I bring forth with others (Maturana, 1980, 1988, 1991; Maturana and Varela, 1991; von Forester, 1981; Merleau-Ponty, 1962). Thus, I believe the ethnographer's concern of changing the culture or situation is a different one than that to which I am pointing in my work when I suggest that my observations are not independent of me, the observer.

The domain of languaging (acting in language) and explaining is also the domain in which mathematics knowing exists and, because this is the domain I am studying, I am fully complicit in the phenomena my observations and explanations participate in shaping. Merleau-Ponty (1962) explains that although a phenomenon may be expressed in terms of some internal law:

> [the] law must not be considered as a model on which the phenomena of a structure are built up. Their experience is not the external unfolding of a preexisting reason. It is not *because* the 'form' produces a certain state of equilibrium solving a problem of maximum coherence and, in the Kantian sense, making a world possible, that it enjoys a privileged place in our perception; it is the very appearance of the world and not the condition of its possibility; it is the birth of a norm and is realized according to a norm; it is the identity of the external and the internal and not the projection of the internal in the external (emphasis original, p. 60).

In the next section, I describe my role as the observer whose distinctions and explanations are being offered in this work. I describe how I conducted my research (made observations) so that the reader can make sense of the domain in which I am making explanations.

The Methods

In this section, I provide an account of the research as it pertains to issues of methodology. I discuss the methodology in retrospect as though it were all thought through in advance and carefully planned to "mine" the rich data that existed in the site. Research belongs to the human domain and, therefore, exists as an act of languaging among observers after the fact, not prior to its occurrence. Had I written this section without the story of the co-emergence of the parent-child mathematics program and the research study, I might create the impression that the research methods were clearly conceived prior to the research rather than in conjunction with the research. This is not the case. (However, having conducted qualitative studies prior to my doctoral study, I was familiar with qualitative data collection and analysis methods and used a variety of these in my research.)

This is also true of the research questions. They were not specified in advance but arose in action as my understanding transformed with my experiences.

In human science research, a tradition of statistical studies and a recent history which includes diverse qualitative methods leave the researcher with many choices in terms of research methodology; but the complexity of the phenomena we study and our complicity (that is, our co-implications) as researchers often mean any single method will not do. The researcher must be able to develop research methods specific to her search and the questions she is posing. Since I am posing the question of how I might explain my observations of mathematics knowing in action, and since I am implicated both in the action and the explanation, I need methods that allow for the observation of non-linear, complex, and interactive events and processes all at once; as well, I need methods that take advantage of my participation.

Fractal Research Cycles

As is often the case when teachers research their teaching practices in the context of their own classes, I found myself in the midst of research without a series of pre-specified research questions to be explored. Rather, I had various experiences within a particular situation (parent-child mathematics program) that I was trying to develop and understand and out of which questions emerged. In chapter one, I discussed how questions about the nature of the curriculum for the parent-child program arose in action and were addressed in action. Although I began with a general question of what the curriculum of this program should entail, the specific questions that guided my inquiry co-emerged with the inquiry. In this chapter, I will discuss how questions of how to study mathematics knowing in action co-emerged in the context of living the parent-child mathematics program and living the research.

Each chapter of this dissertation explores questions that arose for me throughout various stages of the research. In the beginning, these were questions about the nature of the activities I might use in the program and, by the end of the research, these were questions about the concepts and models that I used to observe mathematics knowing in action. My research questions co-emerged with developing the program and with my understanding of it. In other words, my research involved asking the question: *what questions arise when I observe mathematics knowing in action?*

As I reflect on the first chapter, it suggests to me a research cycle, particularly as it pertained to the ongoing development of the curriculum of the parent-child mathematics program. While immersed in the activity of the program, I was constantly reflecting on the situation I was in: making observations, analysing them, and formulating questions which guided my subsequent observations. I want to describe this reflective process as an action research cycle because it happened in action, yet I am well aware that this phrase is more commonly understood to involve research that is intended to have an impact on participants' social conditions. Hence, I will refer to it as a research-in-action cycle (Figure 2-1).

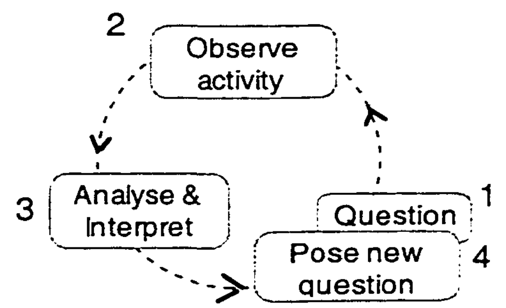

Figure 2-1. Research-in-action cycle demonstrating reflective process used in both the program development and research

An example of my research in action is the selection and development of the prompts I used in the sessions. My initial problem (question) was to identify a set of prompts suitable for the parent-child mathematics program (Figure 2-2, 1). Recall from the opening vignette that, on the first day of the program, I used a classical prompt from mathematics, the "handshake problem." I observed that all of the participants responded with activity which I interpret as mathematical (Figure 2-2, 2). My reflection on their activity validated my assumption that there are such prompts suitable for the parent-child mathematics program (Figure 2-2, 3). Over the next few weeks, the prompts I selected for the program were quite appropriate for the participants in that the parents and children engaged in mathematical activity together; hence, my observations and analyses led me to continue to seek out more prompts. However, in the fourth week of the program, when I used the activity, "Walk the Plank," the cycle was interrupted because not all of the parents and children were able to engage in what I viewed as appropriate activity. Upon observing and reflecting on the participants' actions and utterances, I made the judgment that this particular prompt was not suitable for all the participants, and my original question, "Does there exist a set of suitable prompts for the parent-child mathematics program?" changed to, "What are the features of such prompts?" (Figure 2-2, 4). At this point, the original question loses some significance, and a cycle builds around the new question (Figure 2-3). I propose, then, that a fractal is a much better image for the research cycle than a circle or a spiral. (See Appendix G for a discussion about fractals.)

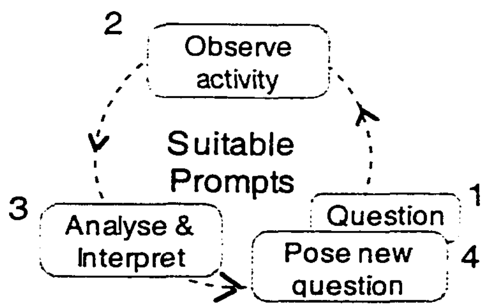

Figure 2-2. Cyclical model for observing the development of research questions about suitable prompts

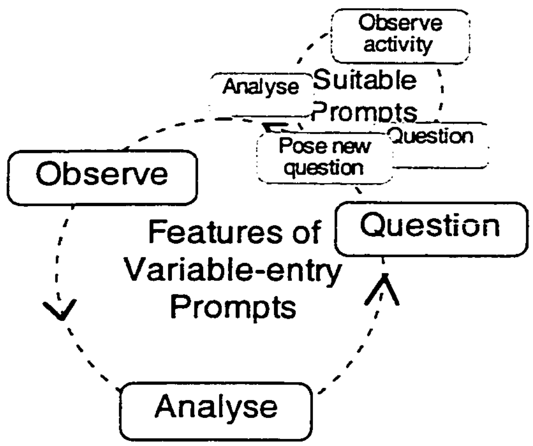

Figure 2-3. Fractal model for observing research process where new questions call previous ones forward into consideration

The development of the concept of a variable-entry prompt is an example of a product of the research cycle in which I engaged. As I selected prompts, I reflected on the ways in which they were played out in the program. From that information, I was able to generate a set of features of variable-entry prompts. Once this set of features was part of my understanding of variable-entry prompts, I understood the specific prompts differently.

Figure 2-2 and Figure 2-3 are models for observing the research cycles I experienced in developing my notion of variable-entry prompts and explicating their features. This example demonstrates one of the aspects in my research which I can explain using a fractal model; the development of my research questions, for example, can also be described with such a model. This points to one of the recurrent themes in my dissertation: the representations or artifacts of my interpretation of the recursive, dynamic and co-emergent nature of human knowing in action, be it mathematical, curricular, pedagogical, or relational, are fractal-like images. If I zoom out on my interpretive activity, I notice how the particular research cycles noted above are only two of many in the project but, more than that, they are implicated in other domains and cycles of the research. No cycle is independent of another; in this way, the various cycles can be thought of as complicit. I cannot separate my knowing in terms of variable-entry prompts from the way in which I understand parent-child interaction, or their understanding of a particular concept, or, in fact, how I am understanding the nature of mathematics cognition. Figure 2-4 is offered as an illustration of my observations of my research knowing. I understand the fractal cycle to be present throughout my research even though I delineate it by writing about it in language. Understanding the research process as dynamic, which includes ongoing recursive interpretation and the fractal form, is a methodological feature of the inquiry used in this study.

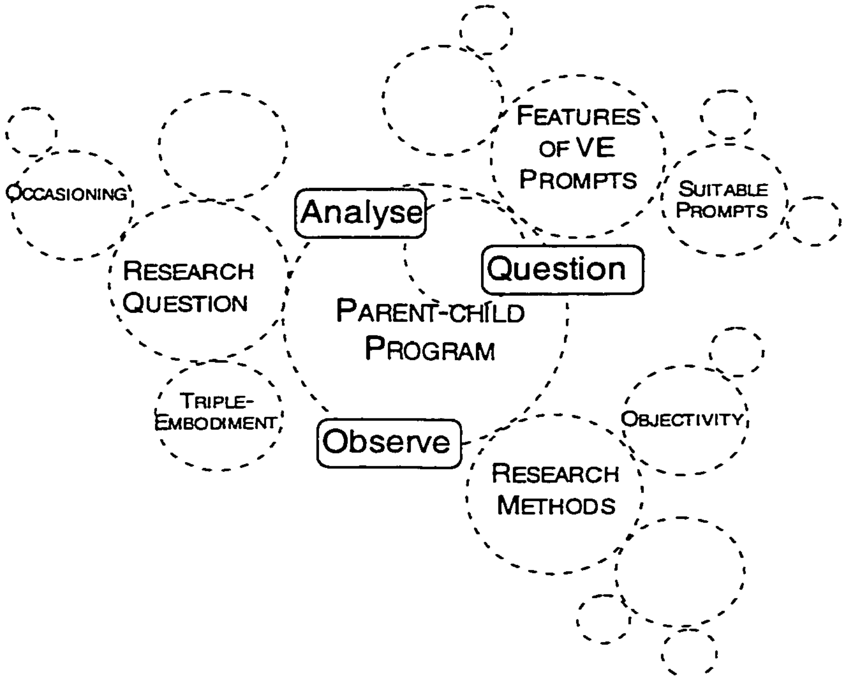

Figure 2-4. Illustration of Fractal Research

Data Collection and Analysis

My task is to explore the general question: *How might I characterise and explain the mathematics knowing that is brought forth in the actions and interactions of people?* However, I do so with the understanding that I am offering an explanation of mathematical knowing based on my experiences observing and interacting with the parents and children who participated in the program. Further, I understand my offering as a social act (interacting with others) in a community of mathematics education researchers who might be thought of as "like-minded" observers.[1] My approach is necessarily interpretive because it assumes that any explanation is a human act of understanding the coherences of one's experiences. I have described the parent-child mathematics program but I will repeat a few details here in order to move on to describe the ways in which it was treated particularly as a research site.

Over the course of the research, artifacts were created out of two kinds of activity: 1) through developing and observing particular events in which parents and children engaged in mathematical activity together; and 2) by analysing and interpreting data (this includes sharing data and interpretations at conferences and through interactions about publications) (see Figure 2-5). In terms of the first type of activity, data was collected in some (but not all) of these forms: video and audio tapes of parent-child pairs engaging in mathematical activity together, field notes, researcher response journals, participants' working papers (see Appendix A). I will call these first-order data. From the first-order data, other research artifacts were created (second-order data). In particular, all audio tapes were transcribed (verbal utterances), and still photographs were taken from the video tapes at a rate of 1 shot/min to produce non-verbal transcripts (I call these body language traces). Mathematical activity traces (Reid, 1995) were produced by viewing the video tapes and making note, in chronological order, of the various mathematical activities that were observed. These second-order data were integrated with the field notes, questionnaires, journal entries and participants' working papers to create a data file for each parent-child pair in each session.

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[1] Like-minded does not mean the observers agree with the explanation *per se*; rather, they share a common set of criteria by which they judge such explanations.

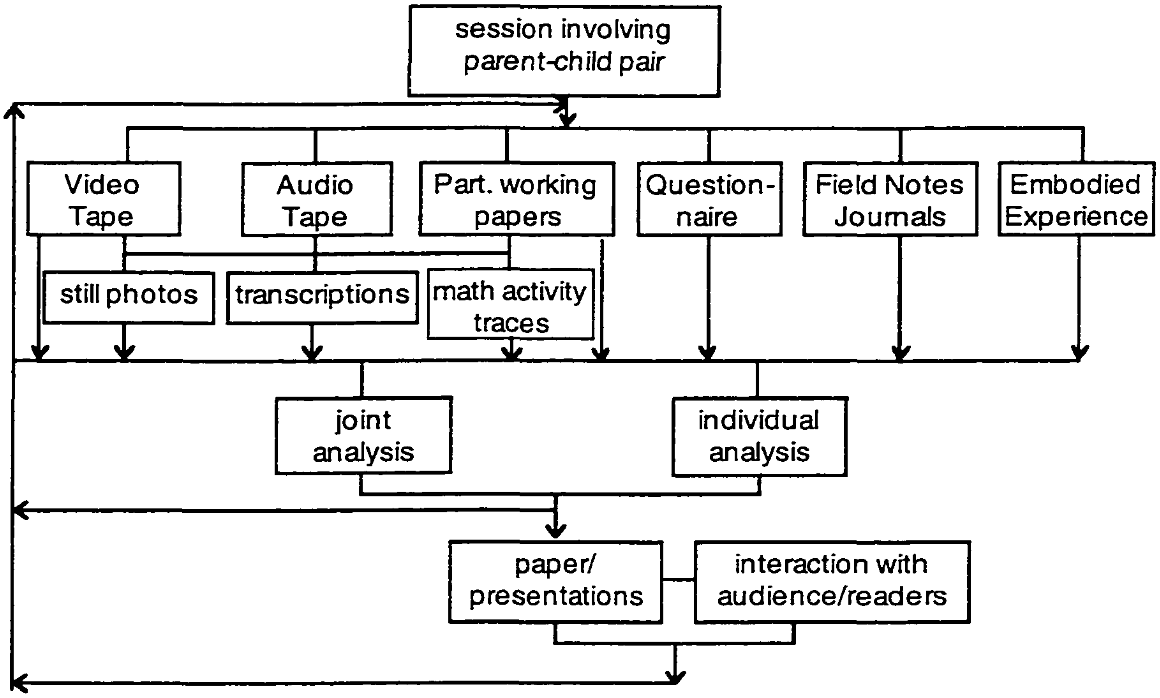

Figure 2-5. Data creation, analysis and interpretation

The data files are used in conjunction with my research journal and the questionnaires that participants complete at the beginning of the term. As part of the ongoing data analysis, the artifacts in each file are studied from a number of perspectives and at different times throughout the research. For example, I made two distinct uses of the files when I developed the concept of variable-entry prompt. (1) I studied all of the parent-child files which involved a particular prompt, and (2) I studied the files of particular parent-child pairs for their responses to various prompts used in the program. Of course, I also studied the ways in which other researchers and mathematics educators conceived prompts (Schoenfeld, 1994; Lampert, 1991; Polya, 1957).

The development of research papers, professional papers, and presentations is yet another transformation of data which results in new artifacts that further inform the study. Often these are created by two or more researchers who together analyse and interpret the first-order and second-order data from their own perspectives (understandings of mathematics knowing). Because these papers are part of the researchers' ongoing research programs, they are rarely thought of as end products but, rather, as research journeys. These research papers (and notes from presentations at conferences or conversations) are also research artifacts and, as such, they have the potential to become new data (third-order data) for further research.

I wish to be explicit here. All research artifacts form the *corpus* of data for my ongoing analysis and interpretations of mathematics knowing in action. This is particularly important since much of the "data" that I present here I have written about elsewhere (see Appendix E). The writing of papers about various aspects of the parent-child program are as much part of the research as setting up the video tapes and the taking of field notes. When any researcher returns to the first- and second-order data, the research papers and presentations are then part of the researcher's history and, thus, are necessarily implicated in subsequent data analysis and interpretation. This is why I understand them as data (information). Hence, research done the way I have described is a recursive[2] process just as all knowing acts are (Maturana and Varela, 1992).

Reflecting back on the development of the research methodology, I note two things: the first is that the methodology was not laid out in advance of the study, and the second is that I can make distinctions of phases of the research and identify various forms of artifacts and layers of interpretation within each phase. The phases are not distinct or separated as they are lived but, upon reflection, I am able to make distinctions and think about the work as having phases. Layers of interpretation are developed as phase upon phase of research is distinguished.[3] Each phase of the research and each layer of interpretation provide a certain kind of artifact which, in turn, becomes data for further research.

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[2] Recursion is used here in a way similar to the way Kieren and Pirie (1991) use it to explain children's mathematics knowing. It is not just that previous data, results, papers, etc. are iteratively the inputs for new papers or activities. Rather, these new activities at once call and alter the nature of the past activities which exist in both the internal and external artifacts and memory of the researcher.
[3] This layered research knowing is conceptually similar to the pedagogical knowing I have already discussed and the mathematics knowing that I discuss in the forthcoming chapters.

Research Accounts and Interpretations

The "results" of my inquiry and research of mathematics knowing (this dissertation) include both accounts of my experience of people's knowing in action and semantic descriptions (Maturana and Varela, 1992, p. 211) of the cognitive mechanisms observed when parents and children engage in mathematical thinking. Through the use of illustrative cases, I pose and reflect on a series of questions about mathematics knowing in action:

- How might we understand the actions and inter-actions of children and parents engaged in mathematical activity if we interpret them using theories of cognition commonly articulated in the mathematics education research?
- What does the enactive perspective of cognition offer the analysis and interpretation of these same phenomena?
- What questions do we ponder when we study cognition from an enactivist perspective?
- How is mathematics knowing triggered? How might we explain the relationship between the person and his or her environment?
- Where is mathematics knowing observed, and what are the sources of perturbations for mathematics knowing?
- In what ways is the knower brought forth by doing mathematics?
- In what ways do we observe mathematics knowing as a fully embodied phenomenon?

In each chapter, illustrative cases are offered not only to shape a context from which meaning can be created but also to suggest that knowing is brought forth by someone in relation with another (the other may be oneself) in a particular situation. Each of the illustrations involves a different kind of distinction whether it is one between a parent and a child or between parent-child pairs or between prompts. In this work, I am constantly reinterpreting various levels of activity, trying to make sense of the whole by focusing in on details which are in themselves whole pieces, only to discover that there is integrity at many levels of interpretation.[4] As these examples are analysed and interpreted, they are transformed into what I call illustrative cases of mathematics knowing in action. Beginning with the activity itself and followed by multiple readings of the transcript and multiple interpretations the illustrative cases are constituted. In chapter three, the cases are used as the site of multiple interpretations of a mathematics knowing from a variety of perspectives on knowing. In chapter five, the illustrative cases are used to develop and elaborate on the concept of structural determinism, structural coupling and occasioning and, in chapter six, the case is used to explore the sites and sources of perturbations for mathematics knowing. In chapter seven, an illustrative case is used as a place around which I try to understand how the knower is brought forth in his or her mathematical actions and interactions. Finally in chapter eight, I bring together the conceptions, models and interpretations I developed in my study of mathematics knowing in action. However, even the drawing together of my utterances cannot form a summary in the traditional sense but occasion my knowing yet again. As I discuss throughout this dissertation, every act of knowing makes possible further acts of knowing. Research always involves re-searching.

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[4] I come back to this notion in the last chapter.

Chapter Three

EXPLAINING MATHEMATICS KNOWING FROM A VARIETY OF PERSPECTIVES

Traditionally, people have viewed mathematics as an individual activity centered in one's head. Consider the stereotypic image of a mathematician—he (and I mean he, not she) is one who works alone in an office, solving problems using only paper and pencil, writing proof after theorem in abstract symbols meaningful to only a few others who live and think like he does.[1] There exists a similar view of children's mathematical activity. A common image is a child sitting at a desk working alone with pencil in hand solving problems; the only evidence of the child's mathematical understanding is the marks on a sheet of paper.

In spite of fairly widespread agreement among mathematics educational researchers when it comes to identifying knowledge or an act of cognition as mathematical[2] there are quite diverse explanations for that knowledge or act. Much of the research in mathematics education over the past 25 years (see Steffe and Kieren, 1994) has focused on children's mathematical cognition from a constructivist theory of knowing. This theory can be traced back to the work of Piaget (1970/1971) and is one that tries to understand what is going on "in the child's head" when he or she constructs mathematical knowledge. More recently, a number of researchers have been focusing on the importance of the social interaction in the development of a child's mathematical understanding (Yackel and Cobb, 1996; Lerman, 1996, Bauersfeld, 1995a). Vygotsky's (1934/1986) social theory of development provides a basis for much of this work (Lerman, 1996; Smagorinsky, 1995; Bauersfeld, 1995a). At

[1] For a discussion of mathematicians and their work, see Davis and Hersh's book *The Mathematical Experience* (1981).

the same time, there is some discussion that what the field needs is a reformulated theory of cognition (Cobb and Bauersfeld, 1995b, Confrey, 1995b; Bruner, 1996). In mathematics education, there are a number of researchers attempting to do just that (Confrey, 1995b: Cobb, 1994; Cobb and Bauersfeld, 1995a; Davis, 1996; Kieren, Gordon Calvert, Reid and Simmt, 1995).

In this dissertation, I build on the research that uses enactivism as a theoretical framework for studying mathematics knowing by using this theory to interpret the actions and interactions of parents and children who engage in mathematical activity together. Enactivism,[3] as articulated by Varela, Thompson and Rosch (1991) and in the work of Maturana and Varela (1980, 1992), provides a biological account for the social behavior of humans at the same time as explicating the inter-personal and the intra-personal nature of human knowing. Unlike theories that view cognition as information processing (input/output model) and suggest its goal is problem solving (Fodor, 1975, 1983; Guilford, 1967), an enactivist theory of cognition is based on the premise that cognition is embodied action that brings forth a world of significance.

However, prior to elaborating on enactivism, I review the literature by considering a multiplicity of perspectives from which mathematics knowing is studied. To do this, two cases which illustrate mathematics knowing in action are viewed through the lenses of theories of knowing that are prevalent in the current mathematics education literature. Then, some of the key concepts articulated in enactivism are introduced in the context of the same two cases.

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[2] This is one of the critiques Confrey (1999a) makes about studies of mathematics knowing; that is, the "mathematics" of mathematics knowing often goes unquestioned. It is assumed to be of a certain kind—a kind that is consistent with Eurocentric views.

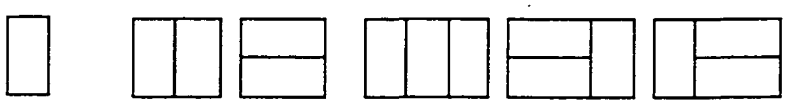
[3] I briefly outline the theory here in order to offer an illustration from my research. Following the illustration, I further develop the discussion of functionalist, constructivist and socioculturalist theories of cognition and end with a more detailed introduction to the "language" of enactivism that I use throughout the dissertation.

Two Cases of Mathematics Knowing

The two cases around which this chapter is developed are both taken from a particular session of the parent-child mathematics program. They involve a father and his 11 year old daughter and a mother and her 9 year old daughter. In this section, pieces of the transcripts from that session are used to illustrate the nature of the mathematical activity of each pair and of each person in each pair. The transcripts are not complete; the participants did much more than is stated here. However, the transcripts and samples of participants' working papers should be sufficient to begin a conversation about the participants' mathematical thinking. In order to facilitate this conversation, I will begin by situating their activity that night.

On the particular night from which these transcripts were taken, the participants were given some graph paper, pencils, a box of dominoes (which have the property of being rectangular with sides in the ratio of 2:1) and the following prompt (Figure 3-1). The participants worked in pairs, each consisting of a child and her parent. Dan and Kerri had participated in the parent-child mathematics program for two terms (20 sessions) prior to this session. The second parent-child pair, Rebecca and Cathy, had participated for one term. Although Dan and Kerri regularly participated in the mathematics program together, Cathy did not always come to the sessions with her mother as she did on this occasion; more than half of the time she attended with her father.

Given a set of dominoes, how many different ways can you tile a path that can be at most two tiles wide?



1 tile 1 way 2 tiles 2 ways 3 tiles 3 ways

How many ways for n tiles?

Figure 3-1. The tiling prompt

Dan and Kerri

When working together, it was common for Dan and Kerri to take specific and distinct roles and work on the task together. This night was no different. Immediately after the prompt was given, Dan turned to his daughter and asked, "What do you think **we** should do?" Her reply, "As **we** do it, **we** should look for a pattern" [emphasis added], foreshadows the nature of their interaction throughout the session and the kind of mathematical activity and mathematics they developed as they worked together.

Transcript 3-1. Dan (D) and Kerri's (K) inter-action occasioned by the tiling prompt

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Dan and Kerri began by redoing the examples given in the prompt. Kerri manipulated the dominoes and arranged them on the table in front of her. As she did that, Dan kept track of the tile patterns she made in a table where he also kept track of the number of patterns for each set of tiles (Figure 3-2).

D: *Okay. So 3 [tiles].*

K: *Next 4 [tiles].*

D: *Four seems like a good number. Good as any to do next.*

K: *If I am right about this, then, if 4 follows my theory -- Okay, let's see. Do this. [Kerri pointed to the set of tiles in front of her that she had just arranged.]*

D: *Do that? Okay. 1, 2, 3, 4. Okay, got it. [Dan drew | | | |]*

K: *Okay, let's see. You can do that. [She made a new arrangement.]*

D: *Okay. 1, 2, 3, 4. [Dan recorded the pattern | | = in his table.]*

K: *Do that.*

D: *Okay, 1, 2, 3, 4. [He drew = | |.]*

It's good that we are being consistent. Like if we are treating those as being different ones then. Okay, now what do we do— Oh, I think that I can see another way.

K: *[Kerri moved the tiles around and stopped when she had another tiling.]*

D: *Oh! Yeah. I didn't see that one. That's not the one I was thinking of. That's 1, 2, 3, 4. Yep, that's good. [He drew = = .]*

K: *What's the one you were thinking of?*

D: *Oh, see if you can get it.*

K: *Hum. [She moved the tiles around and then looked at her dad's sheet.]*

D: *Yeah, we have that one.*

K: *Yeah, I know. [She started moving the tiles about again.]*

D: *Oh, you're good. That's the one I was thinking of. Okay, so we have 1, 2, 3, 4. [| = |]*

K: *Okay, I don't think there's any others.*

T: *I think that's a neat one. It's kind of a frame—a picture frame. [T. Kieren (another researcher present) pointed to one of the patterns.]*

D: *[|=|] It's kind of symmetrical isn't it?*

T: *Oh, it's really nice.*

D: *So how many do we have?*

DK *[Together, they counted.] 1, 2, 3, 4, 5.*

K: *Shoot! That doesn't follow my theory. See I thought, 1 - 1, 2 - 2, 3 - 3, 4 - [her voice fades].*

D: *No. So far it just blew it out of the water. It looked pretty close. I guess we can't stop yet, but we might find a pattern yet. Okay -- one with five. Unless you want to try something else?*

K: *1, 2, 3, 4, 5. [She counted out five dominoes.]*

D: *Okay.*

K: *Okay, you could do this one.[| = | |]*

D: *Okay. What is that one? Okay, 1—.*

K: *1, going, going.*

D: *1, going, going, going, going. Okay. [| | = |]*

D: *Okay, you got your going, going. Yeah, that's the same as your other one, just turned around.*

D: *Okay, so what do you have?*

K: *1, 1, 1 --*

D: *1, 1, 1 --*

K: *2 sideways.*

D: *Blip blip. [| | | =]*

D: *Okay, what do you call the one sideways or something else?*

K: *Blip blip.*

D: *Okay, the blip, blips.*

K: *Blip blip, 1, 1, 1.*

D: *Blip, blip, 1, 1, 1. [= | | |]*

K: *Comma.*

D: *Comma. Thanks. That's a good recording technique. What are we doing now?*

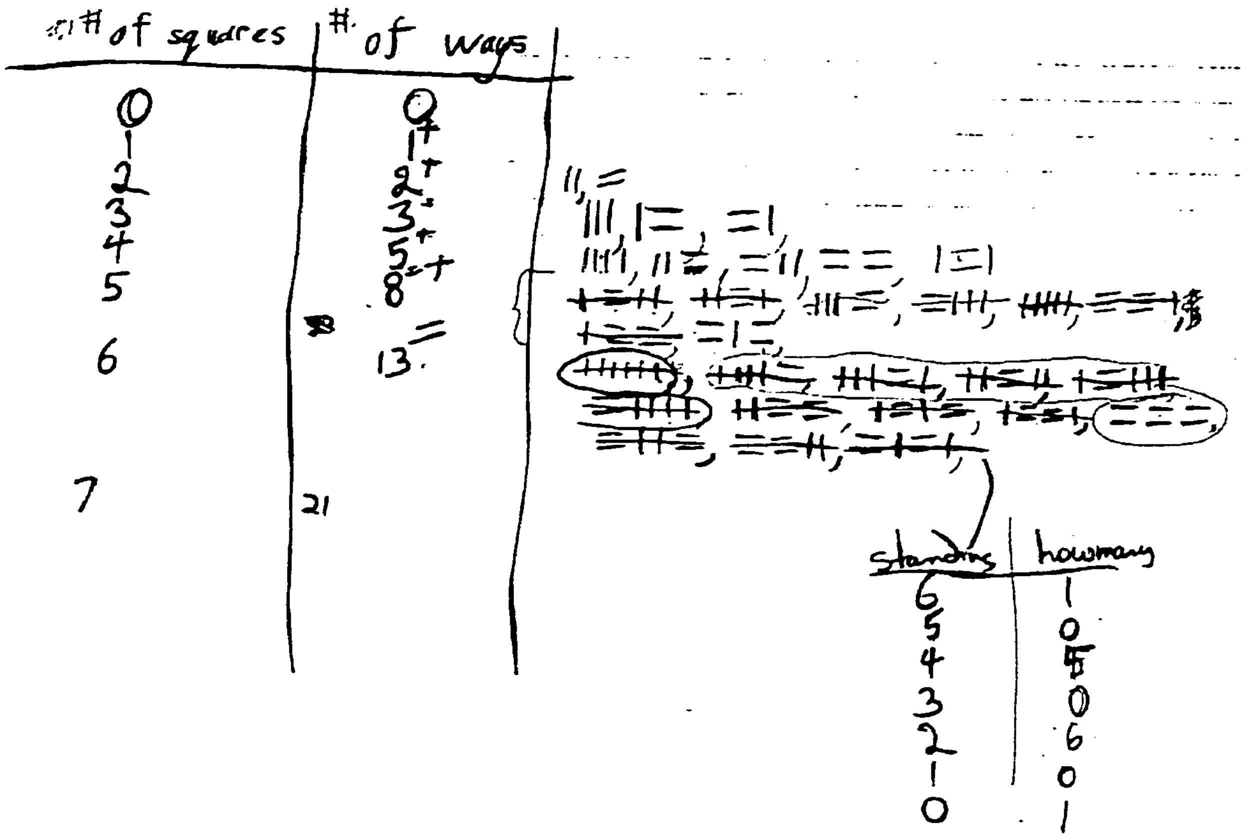

Figure 3-2. Dan and Kerri's Table

I begin an analysis of their interaction and their mathematics knowing by considering the implications of keeping a record in the form of a table (Figure 3-2). By constructing a table and placing it within eyesight, Dan and Kerri had the opportunity to see the numbers and notice patterns—which, in fact, Kerri did almost immediately. She said out loud, but apparently to herself, "If I am right about this, then— If four follows my theory" (Transcript 3-1, line 4). Although Kerri did not articulate her "theory" out loud, she had already noticed a pattern from the records they were keeping and predicted that the number of tiles would equal the number of arrangements.

From the transcript it can be demonstrated that their mathematical cognition emerged together with their verbal interaction. Notice the rhythm in their conversation (Kieren, Simmt, Gordon Calvert and Reid, 1996) (Transcript 3-1, lines 4-12) and how it was disturbed once the number of tiles on which to report grew to five (Transcript 3-1, lines 41-51). No longer could Dan understand what Kerri was reporting with her utterances of "one" and "going-going" (to indicate the vertically and horizontally laid tiles, respectively). He had to look over to the arrangements Kerri had made in order to record what she called out. This break in rhythm became an occasion for the creation of a new word and language for reporting the arrangements. After Dan struggled to record Kerri's tilings, he asked her, "What do you call the one sideways?" He was asking that she name the horizontal tiles differently than upright tiles. Once they named the tile placement, then he was able to more easily record the arrangements for five tiles as she called the patterns out to him. The horizontal pair of tiles from then on were referred to as "blip-blip," and the rhythm of their conversation was reestablished (Transcript 3-1, lines 53-57). It is important to note here that not only did Dan and Kerri have a new word added to their vocabulary, but the creation and the use of that word changed the world for them in that it changed their sphere of behavioral possibilities. The new word not only made it possible for them to communicate more fluently, but it changed the world with which they were interacting and, in doing so, new possibilities for acting mathematically were opened up.

Their need for new label was an immediate one; it provided Dan and Kerri with a means of communicating the arrangements. However, the new label also served to steer their mathematical thinking in a particular direction—which was evident a little later in the session. Dan and Kerri's reasoning took a 'deductive turn' as they tried to demonstrate why the number of arrangements was growing the way it was (note the table in the bottom right hand corner of their working paper reproduced in (Figure 3-2). Having a move defined as a blip-blip occasioned them to consider two kinds of legal moves in this task: the single vertical tile and the double horizontal tile (blip-blip). They noted that only these two moves are possible when constructing new arrangements from previously found ones. Although they did not come up with a proof for their conjecture of how to predict the number of tilings of a particular set, their activities could be seen as satisfying a need to prove (Reid, 1995).

Here I would like to explore how Dan's and Kerri's actions might be seen as related to the body of mathematics in terms of knowledge and practices. From their table making acts, we can see how Dan and Kerri take what can be viewed as a geometrical situation (covering a space in a regular way) and develop numerical patterns to describe the number of patterns that can be formed. This leads me to observe that they were looking for a function to portray the relationship between the numbers they were generating. That is, they wanted a "rule" that could be used to predict how many patterns are possible for n tiles without making all of those patterns. In this way, their actions were connected and contribute to a well defined area of mathematics—that of functions and relations. In itself, the connection is not much more than a distinction of their behavior in a very local context, made by the researcher (an observer of their behavior). However, if their activity is viewed in a more global context, then we note that Dan and Kerri are using a functional way of thinking—one that places one thing in relation to another. This, in turn, perpetuates a particular form of thinking (functional) within the community; thus, the actions of parents and children (and more commonly mathematics students in general) can be seen to be implicated in the perpetuation of the mathematical culture of the community. Another practice of the mathematics culture is noticeable in Kerri's actions. At the very beginning of their activity, Kerri made a conjecture given a very limited number of cases and then refuted her own conjecture a little later with a counter example. As is the case in the mathematics community, the generation of negative instances or counter examples are fundamental to the act of conjecturing and proving (acts in the construction of mathematical objects). Dan's and Kerri's acts are mathematical not simply because they use shapes and symbols related to mathematics, but, because both conceptually and as a matter of practice they make distinctions which would be accepted as part of the historical concern and practices of mathematicians.

Rebecca and Cathy

The second illustration involves a mother (Rebecca) and her 9 year old daughter (Cathy) who worked with the same prompt as Dan and Kerri but whose mathematical activity was quite distinct from that described above. Immediately after the prompt was given, once again, we see the parent ask the child how they should begin. In this case, the daughter, Cathy, responded that **she** wanted to look for the patterns for 4 tiles (Transcript 3-2, lines 1-4). Rebecca misunderstood what Cathy meant and also began to arrange 4 dominoes. When Cathy noticed her mother was working on arrangements for 4 tiles, she blurted out, "Four, too!" (Transcript 3-2, line 18) and then convinced her mom to work with 5 tiles instead of 4 (Transcript 3-2, lines 21-23).

Transcript 3-2. Rebecca (R) and Cathy (C) respond to tiling prompt

R: *Okay, so we are going to work on this starting from 4?*
T: *Check 3. [T. Kieren]*
C: *I want to do 4.*
R: *You want to do 4?*
C: *Yeah, I want to do 4.*
R: *Okay, you can actually take these [dominoes] out and see what you are doing. So there's your 4. 1,2,3,4.*
C: *Cool. This is obviously one way. It is obviously — Are you taping us here?*
T: *Uh hum.*
C: *Let's see.*
R: *It doesn't matter if I use a pencil or a pen, I guess. Okay, so we will draw it on our graph paper? Follow Cathy.*
C: *What are you doing mom?*
R: *Okay, well, I'm doing this one.*
C: *Which?*
R: *This one.*
C: *Four too!*
R: *Yeah. 1,2,3,4— see. Look two up and two this way.*
C: *Oh, so you are doing 4 too?*
R: *Oh! You want me to do 5, and you do 4?*

C: *Yeah, that's what I was thinking. So you are doing 5.*

R: *Okay. 5 tiles....*

[Later in the session] Rebecca just finished finding 7 tilings for 5 tiles, and Cathy has drawn the arrangements for 4 tiles.

C: *I'm doing 7 you can do 6.*

C: *I couldn't think of any other possible ways. [She looked up to the researcher who was looking over her shoulder.]*

R: *Oh, wait a minute. Two on this side. Two up and—you know what I mean, Cath? You know what I mean.*

C: *I know what she means.*

R: *Yeah. Mirrored. Reversed. Do I have the mirror of this one?*

C: *There is no mirror of that one.*

R: *[After checking] No, there isn't.*

C: *You can't mirror it. I can't remember what—*

R: *Yeah, sure. One up and then the four the other way.*

C: *You can't mirror that one I mean [pointing to another picture].*

R: *No. That's the one you can't mirror. That's right. 'Cause it's a mirror of itself. 'Kay. So, that's a mirror of that and that can't mirror and that can't mirror and then that's the mirror of that one. Ah ha! So that's what you look for too. It's mirror patterns. If you can mirror them.*

[A couple of minutes passed as they drew more images.]

C: *Hey! I know. [Writes-Look for miror patterns (sic)].*

****

Later in the session I asked them if they could predict a pattern between the number of tiles in a set and the number of tilings that set would produce.

E: *Cathy can you guess how many there are going to be for 7 [Elaine]*

R: *[mumbling] 5 plus 8*

C: *There could be 20 of them.*

R: *Well, that's what we are trying to do. We're trying to figure out some kind of—*

C: *7, 8, 9, 10, 11, 12, 13, 14, 15. So far I count 15.*

E: *'Kay. You know how many there are for—Yeah. Yeah. There is that section up there.*

R: *For 6 you mean?*

E: *Yeah. So 6 has 13 and I am wondering if you guys can figure out how many ahead of time, before you do them all. How many do you think 7 will be if you look at what has happened already for the number of tiles and the number of ways. Have you written that anywhere in one place?*

C: *Nope.*

R: *5 tiles, 8 patterns. [She writes the number 8 beside the drawings for 5* *tiles.] 4 tiles—What did we put? 5 patterns.*

L: *[Another researcher Gordon Calvert, who had been looking on prompts Rebecca to make a list] Do you want to write it on a list sort of?*

R: *Okay. [She takes another sheet of paper and writes as she calls out.] 1-1, 2— What was it? 2. 3-3, 4-5, 5-8, 6-13. Okay. And Elaine says there is a pattern here?*

C: *I am the one that predicted 5 for 5 and 3— I mean 6 for 6 and 7 for 7—*

E: *Right. But as it turns out this is not the case. So, what if you look at what happens —*

R: *has 5 and 5 has 3—*

C: *But it's 8.*

R: *5 has 8— [mumbling] Shoot!*

E: *Show your mom how this works Cathy.*

R: *Yeah. Show your mom.*

C: *[She has been working on her own table.] 1 for 1, 2 for 2, 3 for 3, 4 for 4—*

R: *4 for 5. 5 for 8 6 for 13—*

C: *6. Hold on. [mumbling]*

R: *Oh! Wait a minute. Would this be because this adds up to 8? And 5 is 8. Add up to— So this would be 21. [laughing]*

C: *[unclear] 8 would be— 8 would be what?*

E: *What would 8 be Cathy?*

C: *It would be [unclear]*

E: *Cathy, you have a few to look at for 7. Don't you!*

R: *[laughs]*

C: *I think I am running out of ideas. 'Kay people. There is supposed to be 21 for 7. This is going to be ugly.*

By separating the task in the way that they did, Cathy and Rebecca's pattern of interaction did not lead to a back and forth weaving of ideas and building on each other's work. In contrast, I note that they work in parallel—each on her own task. Not observable from the transcripts here are the many silent spaces in which mom and daughter each quietly worked on their own tilings and each keep their own records (Figure 3-3). This is not to say that they did not interact but most of their interaction, instead of being a way of working together to build on each other's efforts, was a way of checking up on the other person's activity and progress (Transcript 3-2, lines 13-15 for example), or confirming something for the other person (Transcript 3-2, lines 32-34 for example).

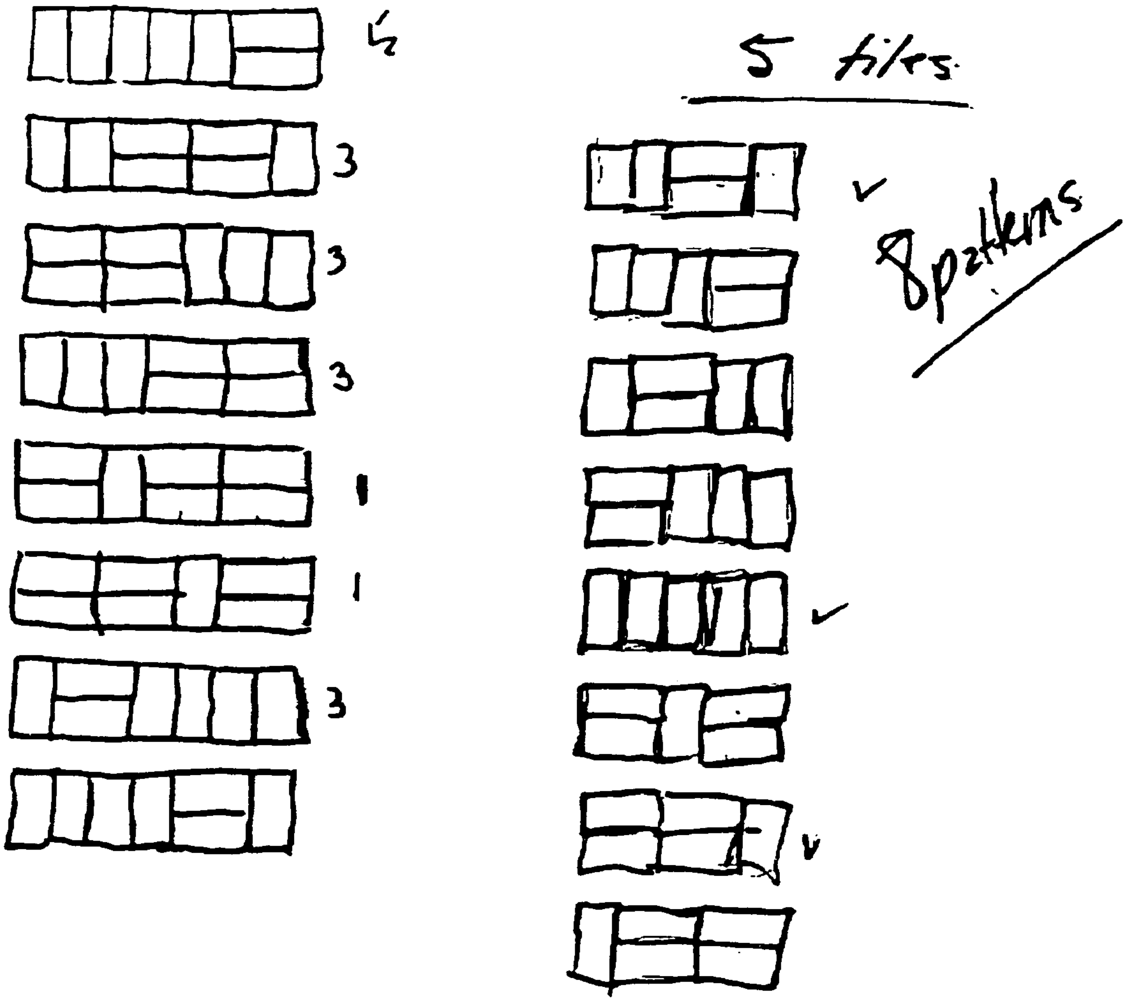

Figure 3-3. Cathy's (right) and Rebecca's (left) sketches

There were times in the session when Rebecca and Cathy did work off each other's thinking, and I note in these instances how their mathematics knowing co-emerged with their interaction. The possibility for creating different mathematics is fostered by the interaction and, not surprisingly, this made a difference to both Rebecca and Cathy. For example, Rebecca, in trying to explain how she found one of the patterns she was missing, described the pattern as a mirror image (Transcript 3-2, line 32). Both she and Cathy began to use this way of talking about the arrangements (Transcript 3-2, lines 32 - 43). There is, at least for a short period, a consensual coordination[4] of their actions through the use of language. They both begin to seek out these mirror images and start to work together—if only for a few minutes. Although Cathy was very active in looking for mirror images, a couple of minutes elapsed between the moment when her mother mentioned mirror images and the moment when she is observed to mark her own noticing by saying out loud, "Hey, I know" (Transcript 3-2, line 43) and then writing on the top of her sheet, "Look for miror patterns" [sic].

Throughout the session, Rebecca and Cathy asked about each other's progress and shared ideas with each other. Because they were not sharing the task in a way that meant they had to communicate continuously and efficiently with one another, they often had to rephrase comments and show what they meant with the dominoes or a picture. They did not have an efficient verbal language (other than the word "mirroring") to discuss their patterns. Although they both made some arrangements with the dominoes when they first began the task, Cathy stopped using them and did not use them again until trying to show her mom something. Then she used the tiles simply to demonstrate to Rebecca what she was trying to say with words.

The mathematical world of significance that Rebecca and Cathy brought forth was geometrical rather than functional. Their use of symmetry to explore a situation, for example, is a common practice in mathematics.[5] Regardless of the area in which a mathematician may work, symmetry is often a key concept in building an understanding of a mathematical object (Stewart and Golubitsky, 1992). Rather than focusing on the number

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[4] The consensual coordination of their actions is not in the sense that they agree to act in a particular way but in the sense that their actions fit together, overlap and interact to form a pattern discernible to this observer.

[5] Note how one of the researchers (Tom Kieren) commented to Dan and Kerri that he liked a tiling that was symmetric (Transcript 3-1, line 28). Neither Dan nor Kerri were prompted to consider the symmetry even with Kieren's comment.

of tilings, Rebecca and Cathy focused instead on the tilings themselves and the geometry of those tilings. After the first couple of arrangements, both of them stopped using the tiles and drew from their imagination; frequently, Cathy would look up and sketch a pattern in the air before she committed it to paper. It is interesting to note that Rebecca's sketches of the tilings look almost art-like (Figure 3-3). Notably, neither Rebecca nor Cathy kept track of how many arrangements they made for a set of tiles until I prompted them to do so; and, when they did, they each kept their own table noting the same thing (Figure 3-4).

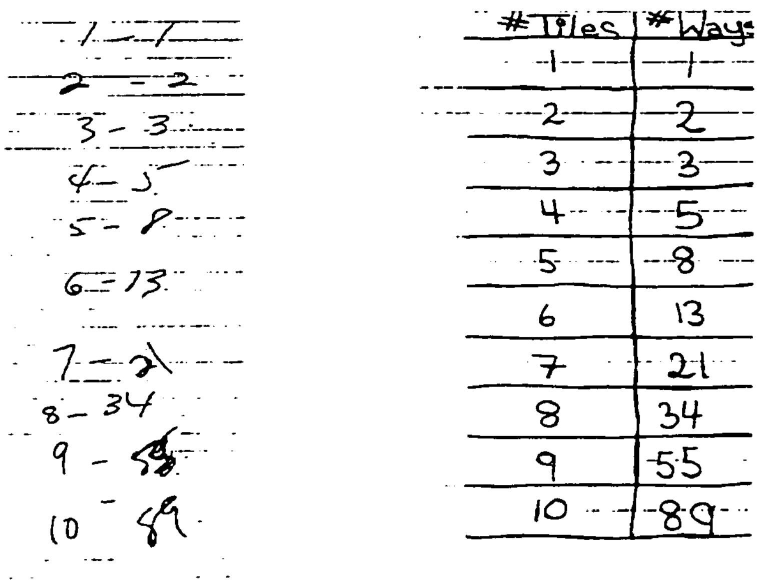

Figure 3-4. Rebecca's (left) and Cathy's (right) tables

They were far less interested than Dan and Kerri were in the potential relationship between the number of tiles in a set and the number of arrangements that could be made. Interestingly, Rebecca does not view herself as very apt in mathematics, yet this self-employed woman was not only excited by the way in which symmetry could be used to determine if she had a complete set of tilings but also understood that it was a means of solving this problem.[6]

[6] I am reminded here of a comment that David Henderson (1996), a geometer, made about his own history with mathematics. "I have always loved geometry... but I did not realize that the geometry I loved was mathematics" (p. 27). A couple of times each year I too am reminded of some people's narrow conception

Observing for Mathematical Thinking

To the observer watching these parents and children, it is obvious that they are unique individuals with distinct talents and skills; after all, each person has a different lived history—biologically, phenomenologically, and socially. Hence, it is not surprising that they came up with quite distinct solutions to the problem. On the other hand, some observers may note that they were given the same prompt and had very similar interactions with the researchers therefore, it is not surprising that both pairs solved the problem and had the same solution. In this section, I explore multiple interpretations of their activities and the mathematics knowing that was made evident in their actions.

When we consider Dan and Kerri's and Rebecca and Cathy's responses to the tiling prompt, we note that they engaged in quite different mathematics (Rebecca and Cathy—geometry and Dan and Kerri—function). How could this be the case if they shared a common environment and were given the same prompt and instruction? How do we explain the nature of their activity, their mathematics knowing? What is the role of the prompt and their interaction in their mathematics knowing? If their actions are different, then why is it that at least part of the "product" of these distinct actions is so similar?

The intent of presenting these two cases side by side and discussing them is not to evaluate and rate Dan and Kerri's mathematical knowing with Rebecca and Cathy's; rather, it is done to provide an opportunity to observe some of the features of mathematics knowing in action and to facilitate a discussion of various perspectives from which one can study mathematics knowing: information processing (Anderson, Reder and Simon, 1996a; 1996b), representational constructivist (Spiro, et al., 1995), radical constructivist (Steffe, 1988), sociocultural (Wetstch and Toma, 1995); and interactionist (Bauersfeld, 1988, 1995b).

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of mathematics when students come to my office to discuss their programs. After failing a required geometry course for the second time, a student came to me and tried to make a case for a program exemption by explaining, "I am good at mathematics, but I am not good at geometry."

Differing Perspectives of Mathematics Cognition

The question of how we come to know and how we create knowledge is studied in a variety of disciplines: philosophy, cognitive psychology, linguistics, neuroscience and artificial intelligence. Today, these contribute to an interdisciplinary area of study called cognitive science (Varela et al., 1991). From my perspective, educational research about cognitive processes and the nature of cognition constitutes an applied cognitive science. It is taking the theories and results of studies from the interdisciplinary field of cognitive science and using them to inform the study of mathematics knowing in an educational context. Focusing my attention on this applied cognitive science, I distinguish between a number of theories of cognition that are common in the mathematics education research literature.

In this section, I discuss theories of cognition that inform mathematics education by interpreting the two cases presented above from these various perspectives. It is important to note that I am interested in theories as conceptual tools for observing people engaging in what I expect to be mathematical thinking. (I am assuming that each of the perspectives I introduce is suitable for such a task.) With this kind of analysis, it is easy to fall into the trap of pointing out what a theory cannot do rather than recognizing a theorist's deliberate choice to do one thing rather than another, given his or her values and beliefs about the nature of knowledge and the purpose of mathematics education. Hence, things which I point out a theory cannot do may be better stated as that which it does not do. At the same time, when I point to these things, I am pointing to that which is important to me; thus, my pointing says as much about my beliefs and values as it does about (what may be interpreted as) deficiencies in the theories.

Information Processing Models of Mind

One of the representationalist theories from cognitive science that has had a great deal of influence in cognitive science is Putnam's "functionalism." Putnam (1991) was one of the first philosophers in the 1960s to advance the "thesis that the computer is the right model for the mind" (pg. xi). In this view of mind, thinking is understood as "symbol manipulation according to rules." From a strictly functionalist point of view, symbolic computation can be carried out by any device that can support and manipulate discrete functional elements. It follows that it doesn't make any difference who or what is doing the thinking; providing the machine (human or otherwise) has the 'right' representation, it will be able to solve the problem (Putnam, 1995). In this view, thinking is analogous to running a program. If you want to understand cognition, you need to understand the program (Churchland, 1995, p. 22 - 23). This has been called the cognitive science of the disembodied mind (Lakoff and Johnson, 1999). Although Putnam himself has changed his mind and no longer understands thinking in functionalist terms, the computational metaphor he proposed some time ago continues to permeate cognitive science (see Born, 1989) and is implicated in some educational research (Anderson et al., 1996a).

Functionalism is one of a number of theories (see also Fodor, 1995; Newell and Simon, 1971) that form a broad category known as "information processing" theories of cognition (Varela et al., 1991). From the information processing perspective, it is assumed that we can learn to teach humans better by studying computer models. This work involves trying to develop computer programs which can solve human problems of perception, pattern recognition and problem-solving (Dreyfus, 1989), and then taking what is learned from building the computer models to create learning tasks and instructional sequences that can be used with humans. These are theories "of the ways in which knowledge is represented internally, and the ways in which such internal representations are acquired" (Anderson, Reder and Simon, 1996b, http://act.psy.cmu.edu/personal/ja/misapplied.html, p. 12). Interestingly, the history of information-process models of mind indicate that, at first, human features of human intellect were ascribed to computers. They were said to "have memories", "retrieve information", and "solve problems" (von Foerster, 1981). Today, these human properties are understood in terms of what they mean for computers, and information-processing theories of mind take that understanding and apply it back to human knowing.

Computational models of mind understand cognition as the successful solution of a problem given to the system by the manipulation of symbols which represent features of an objective (or pre-given) world (Varela et al., 1991, p. 42-43). Fodor (1995) explains that the mind (because it is rational) must be a mechanism that has "representational capacities—mental states that represent states of the world—and that can operate on these mental states by virtue of its syntactical properties" (p. 88). Said differently, there exists information in the environment (mathematical knowledge in our case) which the individual (or device) is able to take in via sensory organs and process in the mind to generate a representation (symbolic structure), which is stored in memory to be accessed, retrieved, and, if necessary, modified, when a problem is encountered (Guilford, 1967) (Figure 3-5). Sawada (1991) suggests that the information-processing model is based on an input/output metaphor—a metaphor that is "deeply and tacitly embedded in the very texture of our language and culture" (p. 351).

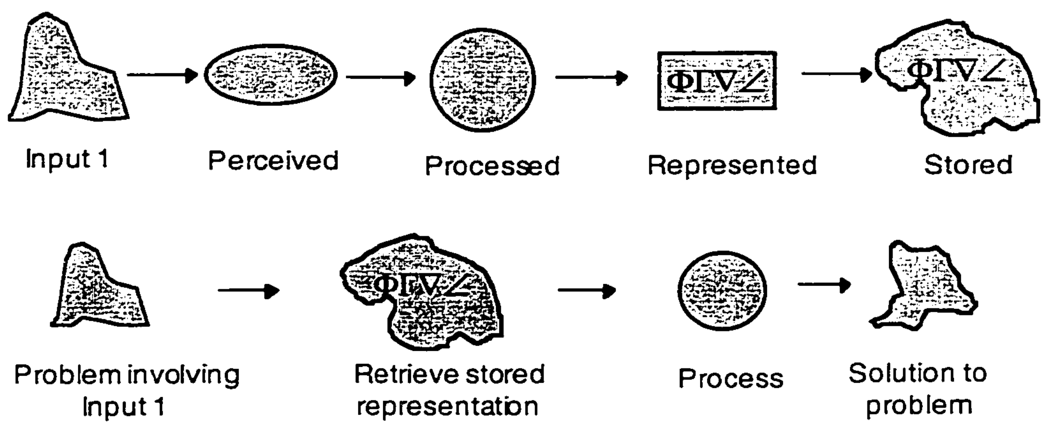

Figure 3-5. Input/Output Model of Problem Solving

Three assumptions of the computational view of mind are can be explored given our illustrative cases:

- knowledge can be decomposed into units and their relations (Anderson, Reder and Simon, 1997)

- it doesn't make any difference who (or what) is doing the thinking; providing it has the "right" representation, it will be able to solve the problem (Putnam, 1995)

- a one-to-one mapping between one individual's mental states and another individual's mental states is possible (Putnam, 1995).

I will discuss these points by going back to the two illustrations offered earlier in the chapter and thinking about the mathematics knowing that is observed in the actions of the two parent-child pairs from the perspective of the information-processing model.

Anderson et al. (1997) suggest that task analysis—reducing a problem down into its units and relations—is one of the valuable contributions of an information-processing view of mind to education. From this perspective, I as the teacher must understand the demands of the task environment and observe the student's behaviour for departures from the perfectly rational approach which an expert would use to solve the problem (Newell and Simon, 1971). Prior to observing Dan and Kerri and Rebecca and Cathy, I understood the tiling prompt as one that involves the relation between the number of tiles in a particular set and the number of tilings that were possible from that set. In fact, when I offered the prompt, I did so by showing that there was only one possibility with one tile, two possibilities with two tiles, and three possibilities with three tiles. I drew each of the possible arrangements on the board, and then I asked the participants to figure out how many different arrangements were possible for any given number of tiles. The number pattern that should be found if the task is done correctly (as per my task analysis) is $1,2,3,5,8,\ldots a_n$ where the $a_n = a_{n-2} + a_{n-1}$ (a Fibonacci sequence).

This task requires that the student is able to arrange the tiles in the appropriate way, find various tilings and deduce if all the tiling patterns have been found. I anticipated that the student would do this for a number of sets of tiles and then, using the cases generated, generalize the pattern from the particular cases. Finally, the student would formalize the generalization by stating a mathematical relationship that predicts the a_n^{th} term of the sequence.

An alternative task analysis is based on a deductive approach to the problem. Given the first three terms of the sequence, the student deduces there are only two possible "moves"—a single vertical tile and two horizontal tiles. Then the student can show that there are only two possibilities for generating the arrangements for 4 tiles. That is, take the set of arrangements for 3 tiles and add a vertical tile to the right of each tiling; this generates 3 arrangements. Now to make the arrangements with the 2 horizontal tiles, you must take the arrangements for 2 tiles and add the 2 horizontal tiles to the right of each. The 3 tilings from the first set of moves and the 2 tilings from the second set of moves produces all possible arrangements. This can now be done for 4 + 1 tiles and then 5 + 1 tiles and so on. Of the dozens of people that I have seen do this problem, not one person initially solved it this way or through this form of reasoning (Figure 3-6).

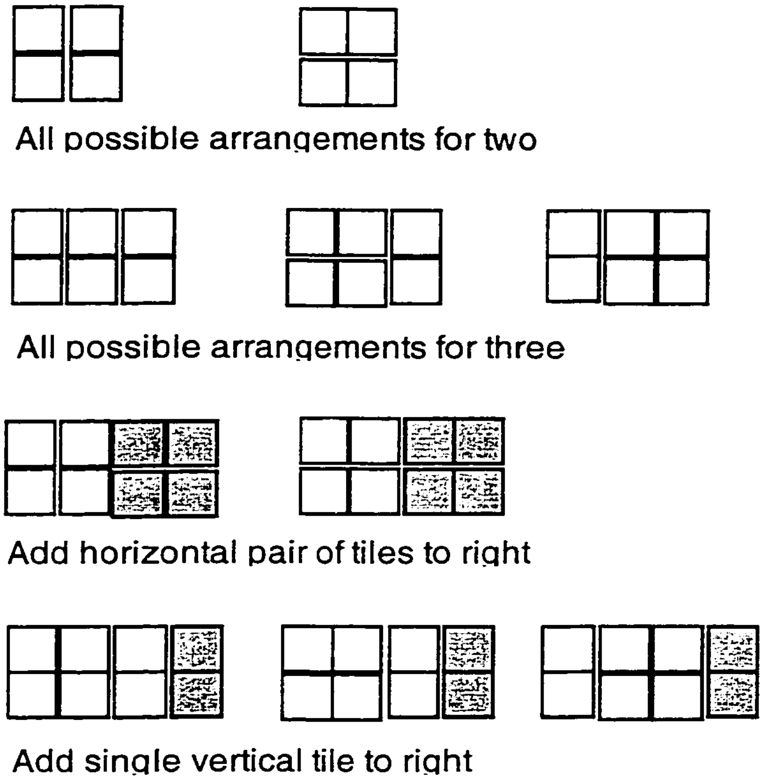

Figure 3-6. Deductive Explanation of the Growth in the Number of Tiling Patterns

Focusing first on Dan and Kerri's work (Transcript 3-1, Figure 3-2), it can be seen that their thinking resembles what I expected from my task analysis. As I suggested by my example, they used the tiles to create patterns, took note of those patterns in a table, and then looked for a pattern among the patterns. From their table of values, they noted that each term was the sum of the two previous terms—they "found" the Fibonacci sequence. From their activity I make the following assessment: (1) my instructions were appropriate; (2) they recovered the needed data from the environment; (3) they had and used appropriate mental processes to solve this problem; and (4) they now had a mental representation for the Fibonacci sequence.

In contrast, an analysis of Rebecca and Cathy's actions indicates a poor fit with my task analysis. Although both pairs listened to the same instructions and were given the same examples, these instructions were not as instructive for Rebecca and Cathy as they were for Dan and Kerri. Note that Rebecca and Cathy did not even focus on the functional features of the problem. They did not use the tiles to create multiple patterns, nor did they construct a table until they were explicitly asked to do so late in the session. For most of the session, their thinking about this tiling problem was geometrical; it focused on the symmetry and possible transformations of the arrangements rather than on the relationship between the number of tiles and the number of arrangements formed.

Proponents of an information-processing view would agree that there is more than one way to solve this problem (which I already knew). Hence, another task analysis based on Rebecca and Cathy's geometrical strategy is needed. I concur, but I add that any "knowledge" of the task is only knowable once someone knows it. The knowledge is not inherent in the task. I was unable to come up with the task analysis for the geometrical strategy until I knew it. Hence, I did not consider it prior to observing Rebecca's and Cathy's actions. Therefore, I am caught in a bind. Of what value is a task analysis to direct the student's learning when there exists more than one adequate way of solving a problem, and the person doing the task analysis neither knows all the ways in which the problem can be solved nor which students (given their existing representations) will favor a particular strategy?

Further, from an information processing perspective, I cannot account for the fact that Rebecca and Cathy were not focused on the problem that I intended to pose with my prompt. That is, I intended this to be a problem about the Fibonacci Sequence rather than a problem of the geometry of the tilings. This perspective assumes that recovery of features (data) in the environment is non-problematic (Dreyfus, 1989) and that the problem is in the environment (Searle, 1992). That is, given my instruction, examples, the tiles, and so on, one would expect the student to interact with the tiles, keep a record and then recover the patterns inherent in the tiles and the record by using the strategy I modeled to solve the problem. For Rececca and Cathy, the features of the data that were relevant for them involved the geometrical symmetry of the tilings rather than the sequential growth of the number of tilings possible. They attended to the spatial rather than the numerical.

Newell and Simon (1971) suggest that one way to deal with this problem in the task analysis when it is caused by the problem solver's capabilities or inabilities is to redraw the boundaries of the task. The solution involves viewing the problem solver's abilities as part of the task environment rather than as a property of the problem solver (p. 81). Their solution is not without its own problems. Newell and Simon suggest caution when using such a strategy.

> We must exercise caution, however, in shifting the boundary between problem solver and environment. If we remove particular operators and classify them with the task environment, there is a danger that the problem solver will disappear entirely, and that there will be no room at all for a theory of him. For example, how shall we treat the problem solver's capabilities and (and inabilities) for doing arithmetic? Is it a description of the problem solver that he can do mental multiplications at a certain speed in solving the problem? Or is this a specification of the environment (as we might want to regard it if there were a question of the availability of paper and pencil or desk calculators)? And how shall we treat the problem solver's capacity for attempting goals? If we follow the path of assigning all means to the environment, there will be nothing left of the problem solver: he will do what he does because all that he is—being means—is specified by the environment.
>
> These examples suggest that a suitable way to fix the boundary is to regard possibilities of actual physical actions as part of the description of the environment, but to regard the information processing activities of the problem solver—the processes for searching through his internal problem space—as describing him. We must now consider more carefully whether such a strategy provides at least a usable and pragmatically tenable boundary between IPS [information processing system] and environment. (p. 81)

A disregard for the "body" has been one of the fundamental problems with the information-processing models of mind. The issue of embodiment is one of the persistent problems in artificial intelligence (Varela et al., 1991).

One of the strongest criticisms of informational-processing theories involves the issue of extracting meaning from the environment and is pronounced in the failures of early artificial intelligence to build machines that could recognize patterns and solve problems (Dreyfus, 1989; Putnam, 1989). Human beings (in context) specify (moment by moment) the problems that we observe them to solve. Dreyfus (1989) comments that "our sense of the situation we are in determines how we interpret things, what significance we place on the facts, and even what counts as facts for us at any given time" (p. 44). How does one determine in advance the features of the environment that will be selected by the cognizing agent? How does a teacher predict to what the student will attend or what the student will select? How does the teacher know what problem the student will define? Our body both places us in the world and faces the world (Merleau-Ponty, 1962).

I do not want to lose sight of the fact that eventually Rebecca and Cathy did come up with the Fibonnaci sequence. When they did not seek out the relationship between the number of tiles in a set and the number of tilings that could be produced, I explicitly asked them to do so. They quickly did what I asked. They compiled a table, noted the sequence and could predict the next term. A final question with respect to the information-processing view comes to mind. At this point, have Rebecca and Cathy acquired the same mental representation of the Fibonacci sequence as Dan and Kerri? That is, are there essential features of the situation that all four people must have extracted now that they are all able to predict a next term in the sequence or to explain how the sequence works? Given that they have come to this point from a very different route, I don't think their understanding of the sequence could be the same although it may share some features. It seems to me that it is unlikely that these four people share the same representation. Not only did the pairs come to the solution in very different ways but, individually, their histories are so distinct that even acting in a similar fashion like Rebecca and Cathy did is very unlikely to lead to the same representation. This observation is a serious problem for those with a computational view of mind (Dreyfus, 1995; Bruner, 1990). When human beings are provided with non-trivial problems such as this one (and it is not nearly as complex as much of our daily experience, nor does its representation here capture its complexity), it is very difficult to predict the features to which they will attend—and if we could, such success is probably because we have simplified or trivialized the problem to such an extent that there is only one possible path (von Foerster, 1981). Such a problem is not very interesting for computer programmers let alone for educational and cognitive researchers.

Finally, I want to point out that from the information-processing point of view there are things in the parents' and children's activity, for example, the creation of the word "blip-blip," for which I am unable to account (Transcript 3-1, lines 52-58). Why did they create this term? How did it come about? What function did it serve? As far as I know, there are no computer models of cognition that account for the creation of ways of interacting with other computers. Computational theories do not address questions about the role of social interaction in the processing of information (Cicourel, 1995; Anderson et al., 1996a), nor do they ask questions about how culture is involved in shaping mind (Cicourel, 1995, Bruner, 1990). Further, information-processing models of mind tend to focus on solving a particular problem; hence, from these perspectives there is little to say about any diversions that the parents and children took up.

In summary, the information-processing or computational view of mind offers the educator notions of mental representations and task analysis. Both of these are useful when trying to understand a particular task and to plan for instruction. On the other hand, this perspective does not address issues of embodiment and lived history, intentionality, social interaction and cultural influence (Maturana and Varela, 1992, Dreyfus, 1995; Fodor, 1995; Searle, 1992).

Representational Constructivism

One might argue that human knowing cannot be understood without taking into account who is doing the knowing, the way the person perceives the problem, and his or her intentionality. Such thinking is indicative of constructivism—albeit there are many forms of constructivism (see Steffe and Gale, 1995). Constructivism is based on the premise that knowledge is not passively received from the environment but actively built up. Knowledge is a matter of interpretive construction on the part of the knower. It is a process by which the subject's experiential world is organized rather than a process by which an objective world is discovered (von Glasersfeld, 1995).

From a constructivist perspective, I would argue that it is not surprising Rebecca and Cathy and Dan and Kerri came up with different responses; after all, they have different histories and know different things. Hence, their construction of the Fibonacci sequence is a process of integrating the experiences with the tiles into the cognitive schemata they already have by either accommodation or assimilation (Piaget, 1970/1971). Ernest (1995) observes that there are some theorists who operate on the principle that there exists objective knowledge which is actively built by the cognizing agent (for example, Spiro, Feltovich, Jacobson and Coulson, 1995). Such knowledge can be taught and evaluated in terms of the correspondence between the subject's knowledge and the objective knowledge. In the case of our example, the Fibonacci sequence is the objective knowledge.

I distinguish a form of constructivism that views knowing as subjective but knowledge as objective. The educational task involves teaching that facilitates the students' construction of pre-existing knowledge. This form of constructivism has been called trivial constructivism[7] to signal the distinction between it and radical constructivism (Ernest, 1995; Kilpatrick, 1987). Kieren (personal correspondence) believes that the use of the adjective trivial is unfortunate since the distinction between the two forms of constructivism is based on a distinction between representationalist and non-representationalist views of reality. Both forms of constructivism assert that "knowledge is not passively received either through the senses or by way of communication"; rather, "knowledge is actively built up by the cognizant subject" (von Glasersfeld, 1995, p. 51). However, only radical constructivists assert that "cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (p. 51). The labeling of a representationalist form of constructivism seems apt.

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[7] Ernest (1995) points out that there are multiple forms of trivial constructivism. Not all would correspond to the distinction I have made.

Representational constructivism offers a more satisfying explanation of Rebecca and Cathy's geometrical activity.[8] From this point of view, both Dan and Kerri's and Rebecca and Cathy's actions and interactions are appropriate and fit with the constraints of the prompt. From a representational constructivist view, I am not surprised by alternative conceptions; however, as a teacher, I would be expected to provide experiences that would lead to the correct conception as I understood it.[9]

In the case of the tiling problem,[10] I note my own behaviour as facilitator could be interpreted as fitting with representational constructivism. It is evident by my actions that I brought my understanding of the mathematics of the problem to the session. Seeing Dan and Kerri working on the problem as I expected, I left them free to pursue that path of thinking. In contrast, after observing Rebecca and Cathy work on the geometry of the problem for most of the session, I specifically requested that they look for a pattern that would help them predict how many tilings would be generated for a given set of tiles (Transcript 3-2, lines 45 - 83). I expected them to "construct" the Fibonacci Sequence, and I was confident that they would find this sequence once I offered an appropriate instruction or scaffold. As I intended, my prompt turned out to be a perturbation that resulted in Rebecca and Cathy constructing a table and finding the Fibonacci sequence—the "solution" to the "problem." If one makes the assumption that knowledge is out there and must be perceived and internalized *vis à vis* some constructive process, then the observations developed thus far are adequate. However, if one denies that there are facts independent of a knower, then we are left with trying to account for the mathematics knowing that was observed in the two cases given.

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[8] The use of an emotion here is intentional. It points to the notion that one's perspective on cognition and its nature is an emotional choice, not a rational one (Maturana, 1988). This will be discussed in more detail in chapter seven.

[9] For the moment, I am not arguing that this is an inappropriate stance for teachers.

[10] The use of the word problem fits better than prompt with a representational perspective.

Radical Constructivism

Students as constructors of knowledge has been a prevalent theme in mathematics education research since the 1980s (Steffe and Kieren, 1994); however, its radical form (von Glasersfeld, 1995), which challenges the view that cognition is representation and the underlying belief that reality is independent of personal interpretations, is not as widely embraced. Radical constructivists (Steffe, 1988; Confrey, 1990, 1993; Cobb, Wood and Yackel, 1990; Pirie and Kieren, 1992) give up an observer-independent account of reality. "[C]ognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (von Glasersfeld, 1995, p. 51). Radical constructivism rejects a "picture theory of knowledge (that we are processing toward an increasingly accurate view of the 'way things really are')" and asserts that "to know something is to act on it, so that all knowledge consists of actions and reflection on those actions" (Confrey, 1995c, p. 195). From a radical constructivist perspective, knowledge is not thought to be passively received either through perception or by way of communication; rather, knowledge is actively built up by the cognizing subject (von Glasersfeld, 1995). Further, the function of cognition is adaptive, in the biological sense of the term, tending towards fit or viability (Piaget, 1974/1980). Finally, cognition is understood to serve the subject's organization of the experiential world, not the discovery of an objective ontological reality (Piaget, 1970/1971; von Glasersfeld, 1995). Piaget (1970/1971) concludes from his studies:

> On the one hand, knowledge is never derived exclusively from sensation or perception but also from schemes of action or from operatory schemes of various levels, both irreducible to perception alone. On the other hand, perception itself does not consist in a mere recording of sensorial data but includes an active organization in which decisions and preinferences intervene and which is due to the influence on perception as such of this schematism of action or of operations (p. 87).

Observing mathematics knowing from a radical constructivist perspective, one would not make a claim about knowledge (the mathematics of the situation *per se*) independent of a knowing agent because the knowledge does not exist independent of the knower. It is constructed in action and, as a result, is a phenomenon of the mind. Piaget (1970/1971) explains,

> The logico-mathematical experience, on the other hand, consists in acting on objects, but with abstraction of knowledge based on action and no longer on objects themselves. In this case, action begins by conferring on objects characteristics they did not have (and which moreover retain their previous characteristics). In this sense, knowledge is abstracted from action as such and not from the physical characteristics of the object (p. 71).

Let us return to the two cases once again and consider them from a radical constructivist perspective. Kerri arranged the tiles with her fingers attentive to the fact that she was arranging them into patterns. At some point, once the set of tiles was all used up, she distinguished a pattern. Once she had done this, she and her father compared it to other arrangements she had already made which had been recorded with iconic representations on a sheet of paper placed in front of them (see Figure 3-2). The images committed to paper were then compared. This involved counting the arrangements for a set of tiles and keeping track of how many they generated for each set. Then Dan and Kerri compared the number of tilings for one set with the number of tilings for the next set.

From an objectivist perspective, one would argue that the pattern is given—it exists in the materials and the rules of the prompt independently of the Dan and Kerri whereas the radical constructivist would explain that the pattern is a construction of Dan and Kerri's mental experiences that is viable given their experiences with the tiles and their previous knowing. The radical constructivist claim is that the pattern is not in the materials but is a mental experience. From this perspective, the subject's experience and reflection of the experience is primary—not the thing in itself.

From a Piagetian perspective, the pattern is a construct not drawn from the external objects but one that arises from "internal logico-mathematical activity engendered by the coordination of the individual's action. By serving as an assimilatory framework, then, these structures are added to the properties of the external object, but without being extracted from it" (Piaget, 1974/1980, p. 80). Von Glasersfeld (1995) elaborates, "[T]he sensory material in the agent's perceptual field can supply clues as to the action required at a given point .... in the re-presentational mode, however, attention cannot focus on actual perceptual material and pick from it cues about what to do next, because the sensory material itself has to be generated. A re-presentation—at least when it is a spontaneous one—is wholly self-generated" (von Glasersfeld, 1995, p. 97).[11]

From a radical constructivist view it is neither surprising nor bothersome that, as an observer, I distinguished very different mathematical activity between Dan and Kerri and Rebecca and Cathy. This is expected. Von Glasersfeld (1995) writes:

> knowledge does not mean knowledge of an experiencer-independent world. From this perspective [Piaget's], cognitive structures—action schemes, concepts, rules, theories, and laws—are evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism's efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations (p. 73 - 74).

On the other hand, the similarities in Dan and Kerri's and Rebecca and Cathy's actions might be explained as the result of coming to the session with similar schema and encountering common perturbations.

From a radical constructivist perspective, we note that the Fibonacci Sequence is not in the tiles and discovered rather each knower constructed the sequence for him or herself.[12] Put simply, there is no privileged information (the Fibonacci sequence) in the tiles themselves to be assimilated into the parents' and children's cognitive structures, only energy-rich matter (tiles in 2:1 ratio and arranged tiles) with which the participants interacted. Through those interactions, the energy-rich matter was transformed and integrated into the participants' structures, and pattern was imposed on the tiles (see Figure 3-7).

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[11] This example also demonstrates a key cognitive mechanism described by Piaget called abstraction (see Von Glasersfeld, 1995). Piaget proposed that there are a variety of forms of abstraction. One is called empirical because it abstracts sensorimotor properties from experiential situations. The first of the three forms of reflective abstraction projects and reorganizes, on another conceptual level, a coordination or pattern of the subject's own activities or operations. The next is similar in that it also involves patterns of activities or operations, but it includes the subject's awareness of what has been abstracted and is thus called reflected abstraction.

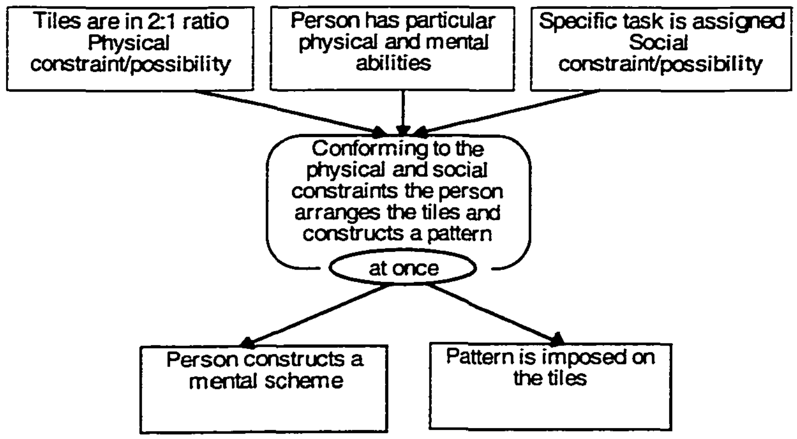

Figure 3-7. Summary of actions observed from a radical constructivist perspective

Historically, in mathematics education, the task of a number of radical constructivist researchers has been to identify mechanisms which explain cognitive processes related to mathematical thinking and mathematics (Steffe, 1988; Confrey, 1993; Kieren and Pirie, 1991; see also discussion in Steffe and Kieren, 1994). The activity of our two parent-child pairs might be observed and explained in these terms. For example, Dan and Kerri's behaviour demonstrates function mechanism at work—one which enables them to construct the Fibonacci sequence from the table of values they created. With respect to Rebecca and Cathy's actions, we might distinguish a symmetry mechanism; that is, a mechanism by which Rebecca at first and Cathy later used to check for and generate unique tilings based on the geometrical property of symmetry. Observing the parents and children working together in pairs does not change the interest of the radical constructivist. He or she continues to be focused on the individual's cognitive mechanisms rather than the interaction between the two or the negotiated meaning, for example.

It is claimed that one of the problems with radical constructivism is that it has allowed the role of the environment to slip into the background and go unexamined (Kieren, 1995). However, another problem with constructivist research is that the unit of analysis has not been the student at all but, instead, the student's cognitive schema. Some

[12] Understanding it as the Fibonacci sequence is another act of knowing, one that is related to having some

researchers have suggested that the unit of analysis could (others say should) be the class (teacher and students) rather than the student or the student's cognitive structures (Bauersfeld, 1995a; Driver, Asoko, Leach, Mortimer, Scott, 1994; Lerman, 1996). A change of perspective from the individual's mental structures to a concern with the context in which learning takes place and the discursive practices that bring it about distinguishes sociocultural theories from constructivist theories (Davis, 1996).

Sociocultural Perspectives

Just as there are a variety of theories that might be grouped under the banner of information-processing and constructivism, so too are there a variety of sociocultural perspectives of the way knowledge is constructed/acquired (see discussion in Cobb and Bauersfeld, 1995a; Ernest, 1995). Unlike those theorists who view knowing as an individual phenomenon, sociocultural theorists view knowing as a social phenomenon, one that emerges from the interaction of individuals in a social group (Lerman, 1996; Werscht and Toma, 1995). In sociocultural research, "[m]ental functioning is assumed to be inherently situated with regard to cultural, historical and institutional contexts" (Wertsch and Toma, 1995, p. 159).

A number of sociocultural theorists trace their intellectual origins back to the work of Vygotsky[13] (1934/1986, 1978). Their perspectives include the following: situated cognition (Lave and Wenger, 1991; Nunes, Schiemann and Carraher, 1993; Seely Brown, Collins and Duguid, 1989), interactionism (Bauersfeld, 1995a; 1995b) and social constructivism (Lerman, 1994; Ernest, 1991; Driver, Asoko, Leach, Mortimer, and Scott, 1994; Driver, 1995). Vygotsky's (1978) oft quoted passage suggests why proponents of sociocultural theories focus on the social plane to explain human knowing.

> Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first between people (interpyschological),

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prior experience to the cultural artifacts of mathematics. This is further discussed in chapters six and seven.
[13] Note that Vygotsky's thinking might be viewed as a cultural artifact.

> and then inside the child (intrapyschological). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals [emphasis original] (p. 57).

Wertsch and Toma (1995) suggest that Vygotsky's claim is more radical than many realize.

> It does not simply mean that mental functioning in the individual somehow emerges out of participation in social life. Instead it entails a redefinition of the very notion of a mental function.... A fundamental assumption [of most psychological theories]... is that, unless otherwise marked, terms such as cognition and memory automatically apply to the individual and individual alone. In contrast... Vygotsky's formulation ... presupposes that mental function such as memory and thinking occur on both the intermental and intramental planes. (p. 162)

A second feature of Vygotsky's theory that has been used to shape recent sociocultural theories of cognition is his notion that human mental functioning is mediated by tools and signs (with language playing a significant part) (Wertsch and Toma, 1995). In fact, he claims "that the kind of mental functioning that makes us human inherently involves the use of mediational means as well as the person (intramental functioning) or persons (intermental functioning) using them" (p. 163). This is an instrumental function of language (Bauersfeld, 1992). For our purposes, Vygotsky's work on higher concept development and the acquisition of adult speech (language) is of particular relevance in mathematics education research (Vygotsky, 1978).

Social interactionism, another of the theoretical bases being used by mathematics educational researchers (Cobb and Bauersfeld, 1995a) uses Vygotsky's work and Blumer's (1969) symbolic interactionism (which is built on the work of George Mead). Social interactionism rests on three premises: (1) human beings act toward things on the basis of the meanings that the things have for them; (2) the meanings of such things is derived from or arises out of the social interaction that one has with one's fellows; (3) these meanings are handled in and modified through an interpretative process used by the person in dealing with the things he encounters. (Blumer, 1969, p. 2).

Rather than discussing the many forms of sociocultural research into human knowing, I briefly describe a selection of research perspectives that are found in the mathematics education literature and comment on some of the features of those perspectives by doing yet another analysis and interpretation of the illustrative cases I have been developing.

Situated cognition involves the study of people ('just plain folks' as Lave and Wenger (1991) called them) in their day to day situations. In the case of adults this has included studying the way people learn and use mathematics in their employment and their business interactions (Lave and Wenger, 1991; Chalkin and Lave, 1993). In the case of children, this has included the study of children both inside and outside of the school situation (Nunes, Schiemann and Carraher, 1993; Saxe, 1995). Such research has observed that members of a particular culture participate in the activities of that culture first as "legitimate peripheral participants" (Lave, 1988) restricted to at first observing and later assisting. In doing so they acquire the cultural tools and skills of the group until they are skilled enough to participate fully (Lave and Wenger, 1991; Cole, Gay, Glick, Sharp, 1971). Many of these studies have focused on the individual's mathematical knowledge in the context of his or her activities outside of institutionalized schooling. For example, Saxe has explored the development of mathematical understandings linked to economic exchange in Brazilian child candy sellers. Nunes et al. studied what they call street mathematics of both children and adults. The common thread in these studies is that mathematics knowing is situated in the context in which it is developed and used and cannot be understood except in the context in which it emerges (Seely Bown, Collins and Duguid, 1989).

Other sociocultural theories study children's mathematical understanding in school where school is defined as the situation or context (Cobb and Bauersfeld, 1995b; Saxe, 1995). The work of Bauersfeld (1992; 1995a; 1995b) and his colleagues (Voigt, 1995; Krummheuer, 1995) is notable. Using an interactionist perspective (c.f. Blumer, 1969), they focus their attention not on the mathematical practices institutionalized by wider society but on the classroom microculture and mathematical practices constituted by the teacher and students in the course of their interactions (Cobb and Bauersfeld, 1995b, p. 9). From this view, "[s]tudents arrive at what they know about mathematics mainly through participating in the social practice in the classroom, rather than through discovering external structures existing independent of the students" (Bauersfeld, 1995a, p. 151). This work has resulted in better understandings of interaction patterns and the development of sociomathematical norms (Voigt, 1995) such as taken-as-shared understandings (Yackel and Cobb, 1996).

Yet other theorists and researchers study the relationship between what the individual knows and what is known in the culture in more general terms. These theorists offer explanations of the ways in which subjective knowledge becomes both objectified and a goal of school. The work of Ernest (1991; 1995) and Driver et al. (1994) provide two examples of theories that attempt to understand knowledge as both personal and social. Driver et al. (1994) comment:

> [E]ven in relatively simple domains of science, the concepts used to describe and model the domain are not revealed in an obvious way by reading a 'book of nature'. Rather, they are constructs that have been invented and imposed on phenomena in attempts to interpret and explain them, often as results of considerable intellectual struggles... Scientific knowledge as public knowledge is constructed and communicated through the culture and social institutions of science.... the view of scientific knowledge as socially constructed and validated has important implications for science education. It means that learning science involves being initiated into scientific ways of knowing. (p. 6)

Drawing on both this social view of the construction of knowledge and Piaget's theory of individual knowing, Driver et al. conclude that learning science involves both "personal and social processes" (p. 8) and that "an important way in which novices are introduced to a community of knowledge is through discourse in the context of relevant tasks" (p. 9). With respect to mathematics, Ernest (1991) elaborates, with arguments similar to Driver's, a social constructivist philosophy of mathematics. "A central thesis of social constructivism," he claims, "is that the unique subjective meanings and theories constructed by individuals are developed to 'fit' the social and physical worlds" (p. 105) and then that knowledge is objectified in the mathematics community. He explains how subjective individual mathematics knowledge becomes objectified social knowledge.

> The social constructivist view is that objective knowledge of mathematics is social, and is not contained in tests or other recorded materials, nor in some ideal realm. Objective knowledge of mathematics resides in the shared rules, conventions, understanding and meaning of the individual members of society, and in their interaction (and consequently, their social institutions). Thus objective knowledge of mathematics is continually recreated and renewed by the growth of subjective knowledge of mathematics in the minds of countless individuals. (p. 82)

For the social constructivists, the construction of knowledge requires social engagement in conversation and activity. "Making meaning is thus a dialogic process involving person-in-conversation, and learning is seen as a process by which individuals are introduced to a culture by more skilled members" (Driver et al., 1994, p. 7).

The work of Walkerdine (1990) and Gergen (1995), for example, suggests yet another sociocultural perspective. Both emphasize the role of language; hence, they are said to fit within a sociolinguistic tradition. "Language acquires both its social value and its meaning largely from the way it is used by people in specific contexts" (Gergen , 1995, p. 35). From this perspective, the knower does not have (knowledge) but participates in it. 'Knowledgeable and rational statements are not external expressions of the internal mind, but are integers in the ongoing flow of communal interchange" (p. 33). Walkerdine explicitly identifies school mathematics as a discursive practice into which children are initiated. She argues that "the purpose of doing mathematics in school is to produce formal statements that do not signify anything beyond themselves" (c.f. Cobb and Bauersfeld, 1995b, p. 6).

I do not claim to have exhausted the many distinct sociocultural theories of knowledge; however, I think from the ones offered, the features of these theories that are most prevalent are noted—in particular, the significance of cultural knowledge, social interaction and context. Also important for understanding mathematics knowing is the assumption that it is the adult who knows and who teaches the child. In other words, childhood involves initiation into the adult social practices of the community; thus, there is an explicit need for the child to be in interaction with adult members of the culture for this to be possible. The delimitations of my research (the study of parent-child pairs in an extracurricular mathematics program) and the particular cases I have used to illustrate the ideas in this chapter preclude me from interpreting the cases from all of the distinct sociocultural perspectives that I have introduced. I will, however, briefly consider a couple of key events in the cases to demonstrate the social nature of the knowing that emerged in the parent and child interactions and how the concepts developed from social constructivist theories might feature in interpretations and in explanations of these mathematical knowing events.

It is clear that I served as the adult (the expert) member of the community. That is, I offered the cultural artifacts (the prompts) and my understanding of the mathematical practices of the community to the parents and children (the novices). In almost all of the sessions, the prompts were novel to both parents and children; hence, the parent was not in an especially privileged position of "knowing" the mathematics of the particular situation. On the other hand, the parents have background knowledge that may have been relevant with respect to doing mathematics in general.

Vygotsky (1978) suggests that we analyse adult-child interaction to study the child's concept development and acquisition of cultural tools. An exchange between Rebecca and Cathy provides a nice example. Recall, yet again, when Rebecca checked for geometric reflections to see if she had all the possible images for the relevant tilings. Recall that after interacting with her mom, Cathy also began looking for the reflections. This action and her comment, "Hey I know. Look for mirror patterns," is evidence of the impact that the interactions between her mother and herself were having on Cathy's thinking. The mirror images strategy could be understood as a tool that mediates Cathy's thinking, her mother's thinking, and the interaction between the two. Because the notion of mirror images (reflections) could be understood as part of the mother's cultural tool kit, one might, after observing their behaviour, make a claim about Cathy acquiring or possibly constructing this cultural tool. From a constructivist perspective, we are interested in symmetry as a mechanism constructed by the child; from a sociocultural perspective, our interested is focused on symmetry as an important cultural tool or possibly a cultural artifact of mathematics. Note that features of both points of view can be used to comment on Rebecca and Cathy's mathematics knowing. It appears Cathy does 'appropriate' the cultural artifact of symmetry offered by her mother in their interaction. Yet Cathy's comment suggests that she has constructed a way of looking for patterns which she labels with her mother's language.

In the case of Dan and Kerri, we note that the father leads and encourages his daughter. For example, although it is Kerri who says, "We should keep a table", it is Dan who does so. It is he who not only keeps track of the actual tilings but also counts and checks to see that they have all the patterns for a set of tiles. He is not only modeling a very disciplined approach to working but, through these actions, the records are available to Kerri for her consideration. Not included in the transcript is a diversion where Dan and Kerri try to explain why the pattern is working in the way it is. This is Dan's idea, and although Kerri could follow his logic, she did not initiate it. It seems to me that it is unlikely she would have come to this point on her own. [14]

From yet another sociocultural perspective, I might note that Dan and Kerri and Rebecca and Cathy had established ways of working together. Both parents encouraged their children to suggest what first should be done—in this case, each child offered a strategy that involved specializing. I might also note that both parents frequently deferred decision-making to their daughters. [15] What might this imply about the distribution or influence of power in their relationships? Are the decisions the children make serious or minor ones? Do these interaction patterns extended into other situations? These are questions that might interest socioculturalists, like Walkerdine, who focus on discursive

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[14] Vygotsky (1978) defines a zone of proximal development; that is, the difference between what a child could do unassisted and what the child can do in the presence of an adult or capable peer.

[15] In a subsequent session, I observed Dan trying to direct Kerri away from a strategy that he appears to have viewed as inappropriate (this was my assessment of her strategy). She refused even though he gently tried to persuade her a number of times. Ultimately, he let her continue in a 'wrong-headed' direction.

social practices. In my study, I am not addressing these questions directly, but I do acknowledge their significance.

I think it is evident that there are a number of sociocultural theories from which we might analyse and interpret the interactions and mathematics knowing that emerged in the parent-child sessions. The sociocultural perspectives described give us lenses different from information-processing lenses, representational constructivist ones and radical constructivist ones.

Many Theories, Many Interpretations

Looking back over the various perspectives from which we can analyse the mathematics knowing of the parent-child pairs in this study, each perspective provides a particular view, emphasizes certain things and, at the same time, allows yet other features to slip into the background. The radical constructivists (von Glasersfeld, 1995; Steffe, 1990; Kieren and Pirie, 1992; Pirie and Kieren, 1994) have a history of investigating and explaining cognition as a function of an individual's mental structures and mechanisms. The sociocultural cognitivists (Lerman, 1996; Chaiklin and Lave, 1993) point to the primacy of social interaction and cultural context for cognition. Cobb (1994) describes the polarization between the constructivist and the sociocultural trends:

> Two major trends can be identified in mathematics education research during the past decade. The first is the generally accepted view that students actively construct their mathematical ways of knowing as they strive to be effective by restoring coherence to the world of their personal experience.... [This] can be contrasted with a second trend that emphasizes the socially and culturally situated nature of mathematical activity. (p. 13)

Among the theories I reviewed, the distinctions are not simply between focusing on the social or focusing on the individual but, in a number of cases, a more fundamental division exists; that is, a distinction between how the theorists view reality and, consequently, the nature of the explanations that they make. There is, as I observe it, a distinction between representationalist and non-representationalist explanations of reality, knowledge and knowing.

Viewing Reality

Maturana (1988) suggests that there are two possible views of reality that can be taken when one develops an explanation. He calls these stances 'objectivity-without-parentheses' and 'objectivity-with-parentheses'. Those theories of cognition that I have classified as representationalist involve explanations that assume objectivity-without-parentheses or a transcendental reality. Such theories suggest that objects exist independently of the cognizing person regardless of whether or not a person can know about those objects through perception or reason (Maturana, 1988, p. 28-29). These theories are based on the assumption that there exists an objective and independent reality which can be constructed and represented in the mind in such a way there is a correspondence between that pre-given reality and the individual's mental state. Although this assumption is rarely stated explicitly (hence, it is an assumption), Von Glasersfeld (1995) comments it is a persistent view of knowledge in Western thought.

> For some 2500 years the western world has manifested an overwhelming tendency to think of knowledge as the representation of a world outside and independent of the knower. The representation was supposed to reflect at least part of the world's structure and the principles according to which it works. Although the picture might not yet be quite perfect, it was thought to be perfectible in principle. (p. 113)

We note from an objectivity-without-parentheses perspective that the Fibonacci Sequence exists in the tilings independently of the patterning and generalizing in which the parents, children, and the researcher engaged. In contrast, in the path of objectivity-with-parentheses, existence of the object is constituted with what the person does. The person "brings forth the object that he or she distinguishes in a recursive act of distinguishing distinctions of distinctions in language" (Maturana, 1988, p. 30). Hence, from this path, I note that Dan and Kerri's and Rebecca and Cathy's actions with the tile arrangements progressed from manipulating the tiles, to distinguishing a set of patterns for a given number of tiles, to a generalization about the number of possible tilings for any given set of tiles, to an explanation of how the pattern worked. In this view, the Fibonacci sequence was brought forth (from this observer's observations) by perceptually guided action and in social intercourse in language (Figure 3-8).[16] I believe this to be a subtle point but one that is fundamental to understanding the enactivist perspective. I will develop an argument to clarify my point.

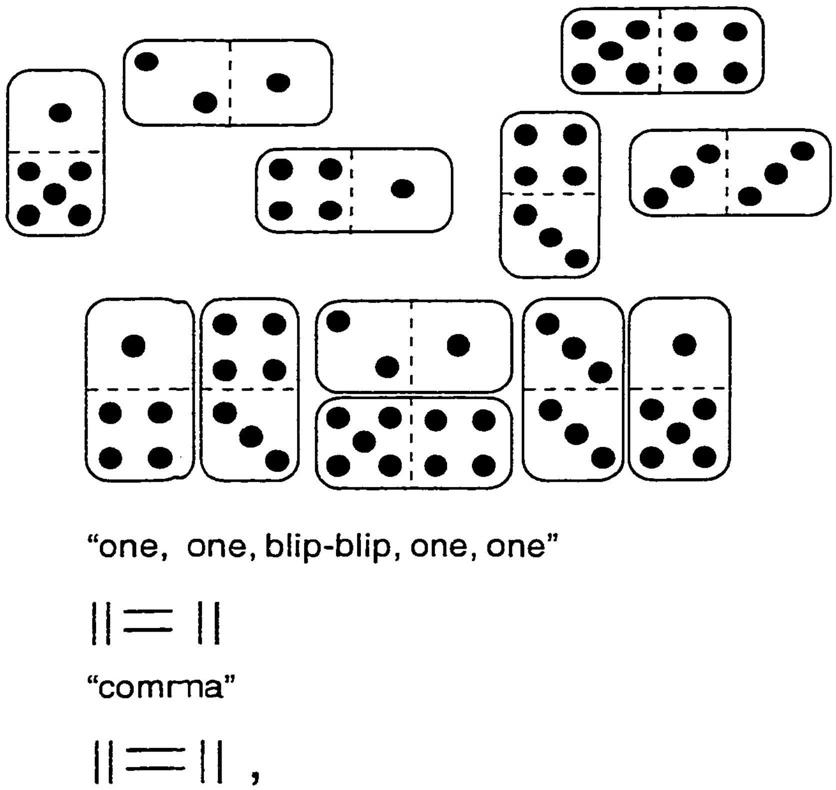

Figure 3-8. Pattern with tiles, uttered in language, and noted pictorially

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[16] Maturana (1988, 1998) calls this languaging and explains it as such:

> Language is a manner of living together in a flow of consensual coordination of coordinations of consensual behaviors, and it is as such a domain of coordinations of coordinations of doings. So, all that we human beings do we do it in language; the different worlds that we live arise as manners of coordination of our doings in language; the different domains of doings that we live as different kinds of human activities, be those in concrete or abstract, manipulative or imagined, practical or theoretical, occur as the domains of consensual coordinations of coordinations of doings in the different domains of doing that arise in our living in language. So languaging is our manner of existence as human beings (Maturana, 1998 http://www.inteco.articulos/metadesign-parte1.html).

If we take a transcendental path of objectivity, then, we should be able to identify where precisely the Fibonacci Sequence is located or what precisely it is, and both of these will be independent of a human agent (be it observer or actor). The word 'Fibonacci' is clearly a human construction; thus, I will back up and call it simply a sequence. However, a sequence is a label in human language used to refer to a particular kind of pattern, of numbers in this case. Labels are human constructs. To avoid this problem with the human label, let us refer to the number pattern that was identified as 'it' and agree that 'it' refers to the number pattern prior to its labeling. Now I might ask, "To what extent is 'it' independent of the human observer?" To find a number pattern requires a human agent so let us move one step prior to the number pattern to the numbers and suggest that the pattern is in the numbers. However, the numbers are nothing more than labels (a languaging act) to denote how many tilings were generated for particular sets of tiles, and labeling has already been marked as a human action. We are caught in a loop or infinite regress. From one perspective, this loop is at the level of making a distinction in language, so the way out is to move away from languaging.

Let us move away from the set of observations that are about language and languaging to the level of sensorimotor actions. The task posed to the participants included the question, "How many patterns can one make with a set of *n* tiles?" *n* tiles are just that—some number of tiles in a pile with no order. Kerri put order to those tiles through manipulating them with her fingers until such a point that she realized (made real) something that satisfied the constraints of the prompt. She made all of the arrangements, her dad recorded them on the paper, and together they counted them. Then they compared the results from one set of tiles to the others. Even beginning from sensorimotor activities, clearly 'it' is not independent of human action, and if we admit that the task itself is a social one, hence, not independent of the human beings, then there is no level at which I can observe 'it' to exist independently of human action. The Fibonacci Sequence does not exist independently of human action; thus, a path of objectivity-without-parentheses cannot account for 'it' even if one says that the pattern at which one eventually arrived matches a pre-given physical reality that there is to match. Suppose, instead, that one says the prompt and the patterns are pre-given by the teacher and form an external reality in that sense. It seems that the differences in actions, thoughts, and explanations between pairs counter any such claim.

In the previous example, it is tempting to make the objectivity-without-parentheses claim that the sequence is there to be discovered rather than brought forth in the actions of humans; after all, the Fibonacci Sequence is a well-known result. However, this is not so obvious if we think about the '*blip-blip*' that Dan and Kerri brought forth in their actions. This word first arose when Dan drew two marks to represent a pair of horizontal tiles and uttered, "*Blip, blip.*" *Blip-blip* was then assigned meaning (a pointing function) when Dan and Kerri explicitly discussed and decided that this word would be used to denote a pair of horizontal tiles. How might we think about the emergence of the *blip-blips*? Was it a social construction that was first intermental (that is between people on the social plane) and then transformed to become intramental (internal knowing for each person) for both Dan and Kerri as a Vygotskian perspective might suggest? Does it exist independently of Dan and Kerri?

A tiled path is observed to consist of horizontal and vertical tiles where the horizontal tiles need always be placed in groups of two to satisfy the restraints of the prompt. Dan and Kerri's actions must satisfy the conditions of the prompt, but even those are part of the interactive domain among participants in the session. *Blip-blip* is Dan and Kerri's invention; it co-emerged with their activity and satisfied a need that they had—a more efficient way of communicating with each other because of their highly interactive approach to the task. *Blip-blip* is a label which was created in the social interaction between this parent and child and thus, is social; but it is not a cultural convention that the more knowing participant introduced to the less knowing participant—a common focus of social constructivism.[17] Clearly, *blip-blip* is an object that was brought forth in Dan and Kerri's interaction; thus, it is difficult to understand it as part of a transcendental objective reality. On the other hand, *blip-blip* did not arise on the social plane from nothing. Dan uttered it as he made two strokes of his pencil. In the subsequent instant, Dan expressed a need for a label; Kerri, having heard this utterance, uttered it back; and a label (object) was created—an illustration of objectivity-with-parentheses. Maturana calls this inter-objectivity to suggest that we observe objects arise in the interaction among humans.

Vygotsky (1978), in contrast, suggests that knowledge exists first on the social plane (intermentally) and then that which is external is internalized and becomes intramental. This is not an adequate explanation of what I observed to be knowing on both the inter- and intramental planes at once. An enactivist interpretation involves the notion of co-emergence (Kieren, 1994) or codependent arising (Varela et al., 1991). The utterance Dan made was at once intermental and intramental. Intermental in that Dan uttered it and in doing so it became energy-rich matter in the sphere of possibilities from which Kerri could select and integrate into her own structure. Its dual nature is evident by noting that Dan and Kerri worked for a few minutes using the word communicatively, thus demonstrating a taken-as-shared meaning (Yackel, 1995) for *blip-blip*. This is its social nature. However, the taken-as-shared meaning broke down when Dan, reading the pattern [==] back to Kerri, said, "Four *blips*". (This is not in the transcript.) Kerri immediately respondedto his utterance with, "No. *Blip-blip*, *blip-blip*." I interpret their actions as indicating she saw a *blip-blip* as an unit whereas Dan was using the term to mean two horizontal tiles. If *blip-blip* existed only on the social plane, we would not have seen Kerri make the distinction she did. An enactivist explanation observes the social and personal knowing (inter- and intramental) as co-emergent. *Blip-blip* is not a bit of knowledge in the social plane to be acquired or even reconstructed on a mental plane; rather, it is part of an act of knowing that

[17] Vygotsky's work was focused on adult-child interactions in which the adult has the social or cultural knowledge that is to be passed on or constructed by the child.

was brought forth in the actions and interactions of Dan and Kerri within a set of environmental constraints.

From a path of objectivity-without-parentheses or representationist perspectives, there appears to be confusion about knowledge and information. Von Forester (1981) claims, "It has become matter of fact to confuse process with substance, relations with predicates, and quality with quantity" (p. 193). In particular, "information" and "knowledge" are "persistently taken as commodities, that is as substance" rather than process (p. 193). He explains:

> Information is, of course, the process by which knowledge is acquired, and knowledge is the processes that integrate past and present experiences to form new activities, either as nervous activity internally perceived as thought and will, or externally perceivable as speech and movements. Neither of these processes can be "passed on" as we are told in phrases like "...Universities are depositories of Knowledge which is passed on from generation to generation"... for your nervous activity is just your nervous activity and, alas, not mine. (p. 193 - 194)

Instead of asking how can we get x (some knowledge of some object in the environment) into y (some learner), we might instead ask, "What is the relationship between x and y (the person and the environment) and, how is that person able to select from the energy rich environment and transform it into his or her own structure for his or her own use?"

A distinction between observer independent and observer dependent views of knowledge is an important one to be made when researching human cognition. If there is objective knowledge, then it follows that there is something for the students to acquire. This easily leads to the notion of transmission of knowledge rather than the construction of knowledge. Recall my insistence that Rebecca and Cathy make a table and look for a pattern. My actions are part of a discursive practice which children experience early in their schooling. Within such discursive practices it is clear to everyone that there is something to be obtained—the Fibonacci Sequence in this case. Again, we need only look at the activities of the two parent-child pairs in our illustrative case to understand why these discursive practices are detrimental in that they alter and constrain mathematics knowing. When I act as though what is known (in this case the Fibonacci Sequence) exists independently of the actions of the knowers (Dan and Kerri and Rebecca and Cathy), then I am treating the knowers as "trivial machines" (von Foerster, 1981)—ones in which the input (tilings, instructions or some other 'pre-given') will produce similar output in these people independent of their context and their histories.[18] Humans are not trivial machines precisely because they operate in light of their histories. [19] Their actions are not invariant (from event to event), nor are they caused by the environment; rather, they are co-determined by the person's (ever changing) structure and the energy-rich environment from which they select and integrate into their structures (von Foerster, 1981). This is the principle of *structure determinism*[20] (Maturana and Varela, 1992) which is a key explanation in an enactivist theory of cognition.

In this section, I discussed the development of Dan and Kerri's *blip-blip* and commented on how Vygotskian or social constructivist perspective do not address the co-emergent features of human knowing. Now, I would like to comment on some problems with a purely radical constructivist account. From my readings, I suggest the radical constructivist account allows the social features of the knowing to slip into the background. This leads to a serious omission in terms of understanding the reflexive relationship between knowing on the social plane and the implications of such knowing. That is, experience not only contributes to the person's knowing, but it changes one's personal sphere of behavioural possibilities at the same time as it potentially changes the spheres of possibilities of others. If we assume that mathematics knowing arises in interaction between a person and his or her environment, then we must acknowledge the implications this has for others (because they are a significant part of the environment). With the illustrative cases, it simply is not possible to account for the creation of *blip-blip* except by

[18] None of the perspectives I have mentioned in this chapter would be satisfied with such a treatment, yet it is easy to find educational practices which promote this view of the human learner.
[19] The machine metaphor for the human body has been common in medicine for centuries. As I have already discussed, the human mind as a information-processing machine is a metaphor that has arisen with the emergence of computers (see discussion in von Foerster, 1981).
[20] Structure determinism is elaborated in the next chapter.

studying both the individual knowing and the knowing in the interaction between parent and child. Varela (1992) comes to a similar conclusion that "cognition cannot be properly understood without common sense, and this is none other that our bodily and social history, the inevitable conclusion is that knower and known, subject and object, stand in relation to each other as mutual specification: they arise together" (p. 253)

Enactivism is a theory of cognition which studies the social and individual features of knowing in action within a non-objectivist perspective. Thus, enactivism precludes neither neo-Piagetian theories nor neo-Vygotskian theories; however, by taking seriously the claim that knowing is at once both social and personal, different explanations and mechanisms may be generated that could not have been from simply adding the two theories together. This is why Cobb's (1994) suggestion to bridge these two perspectives by offering joint interpretation which includes both radical and social constructivist views of the mathematics understanding will not result in an adequate theory of knowing. Such an additive approach may enlarge our understanding of mathematics understanding, but it cannot transform our understanding because it does not offer conceptual tools to help us understand the phenomenon differently. Although this additive approach might be thought of as bridging the radical and social constructivist theories of mathematics knowing, I do not think this would satisfy calls for more inclusive and comprehensive theories of mathematics knowing.

Lerman (1996) makes an important observation when he argues that the two perspectives cannot be merged because they are different in kind; that is, "Vygotsky's and Piaget's programs have fundamentally different orientations, the former placing the social life as primary and the latter placing the individual as primary" (p. 133). These different orientations lead to distinct views of the source of meaning. "The difference is encapsulated in their identification of the source of meaning, the one identifying the cognizing individual (Piaget's view) and the other cultural and discursive practices" (p. 147).

Maturana (1988) makes a point that Lerman fails to recognize (however, one that Vygotsky makes and then lets slip into the background); that is, human beings are first and foremost biological beings. It is because of our biology that we are social beings.[21] Confrey (1995b) has suggested that we should draw on the complementary features of Piaget's and Vygotsky's theories of intellectual development to forge a revised theory (p. 44). As I asserted, simply adding the two theories together by joint interpretation is inadequate. Such an approach will not address points I have raised through my illustrative examples. Neither will it address some of the key issues that Confrey has pointed out which have been largely overlooked by current studies of intellectual development in the context of mathematics education: 1) Human development depends on the environment; 2) The self is both autonomous and communal; 3) Emotional intelligence is acknowledged; 4) Diversity and dissent are anticipated; 5) Abstraction is reconceptualized and placed in a dialectic; 6) Learning is viewed as reciprocal activity; 7) Classrooms are studied as interactions among interactions (Confrey 1995b, p. 36).

Rather than merging or bridging these two perspectives, the premise of my work is that new theoretical underpinnings are needed. Those theoretical underpinnings can be found in complexity theory[22] (Gliek, 1987), biology (Lewontin, 1991; Maturana, 1980, 1988, 1991), second-order cybernetics (von Foerster, 1981), cognitive linguistics (Lakoff, 1987) and enactivism (Varela et al., 1991).

Cognition is a complex phenomenon. It is dynamic; anything of interest may be noted in the instant but is unrecoverable after it has occurred. This is not to say that it is haphazard, but it cannot be predicted in the sense of a prior event causing a proceeding

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[21] In Maturana and Varela's (1991) theory of cognition, we find a complete and biological explanation for social behavior (language included).

[22] Recent work in chaos theory (Kellert, 1993; Prigogine and Stengers, 1984) and complexity theory (Cohen and Stewart, 1994; Stewart and Cohen, 1997; Kauffman, 1995; Waldrop, 1992) makes a distinction between complicated systems, which can be understood in terms of their parts and the casual relationship among the parts and complex systems which are best understood holistically (Cohen cited in Waldrop, 1992) or contextually (Stewart and Cohen, 1997) and in action (Kellert, 1993; Varela et al., 1991). As

event. Of particular significance is that cognition cannot be reduced to the sum of its parts. Thus, our understanding of mathematics knowing and how to prompt or trigger mathematics knowing cannot be understood by considering only those studies that attempt to control and factor out variables in order to deal with the complexity of the teaching and/or learning processes. Also problematic is that much of the research in mathematics education implicitly assumes that cognition is problem solving, and the criteria for mathematical understanding is the successful representation of an external world, which is pre-given, usually as a problem-solving situation. Varela (1992) argues that such an assumption is incomplete. He suggests that "precisely the greatest ability of all living cognition is to pose, within broad limits, relevant issues to be addressed at each moment of our life. They are not pre-given but enacted or brought forth from a background" (p. 250). Because cognition is a process for which "the relevant issues that need to be addressed at each moment are posed and enacted from a background of action" (Varela et al., 1991), we need to consider the individual, the environment, and the interaction between the two in our attempts to understand human understanding.

Cicourel (1995), a cognitive sociologist strongly influenced by psychology, comments, "When you take for granted culture and the way it is reflected in a local social ecology, you eliminate the contexts within which the development of human reasoning occurs" (p. 50-51). In mathematics education, researchers such as Kieren (1999), Confrey, (1995b; 1999a) and Cobb (1994) challenge us to pay attention to both that which we allow to fade into the background and that which has for so long has gone, if not unnoticed, then certainly unmentioned.

Enactivism, the alternative proposed and elaborated on throughout this dissertation, has been called a way in the middle (Varela et al., 1991; Kieren, 1995; Davis, 1996) because it demands a move away from understanding various phenomena as either one

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Poincare (see Klien, 1985) noted 100 years ago, dynamical systems can be understood if we consider their phase space.

thing or another (individual/social, mental/physical) to understanding them as mutually specified or codependent. Enactivism understands human lived experience of the world as a phenomenon of world-and-mind where mind and world co-emerge out of the interactions of humans (Varela et al., 1991; Kieren, Gordon Calvert, Reid, Simmt, 1995; Davis, 1996). In the next chapter, I develop enactivism as an explanatory proposition for human cognition in the particular case of doing mathematics and engaging in the mathematical.

Chapter Four

ENACTIVISM: A WAY IN THE MIDDLE

All knowing is incomplete, Tom Kieren (personal correspondence, 1998) claims—incomplete in at least two ways. (I call this Kieren's incompleteness theorem.) In the first place, knowing is incomplete in that as humans who live in language we exist in a multiverse (not a universe) whereby each act of knowing brings forth a world of significance—one world of the multiverse. Therefore, each act of knowing is incomplete in the many other worlds in which it is not yet (and likely never will be) realized (made real in an act of knowing). For Kieren this has two immediate consequences. A notion of multiversal incompleteness prompts humility and openness to other world views, and it promotes a research method of re-turning to and re-searching the artifacts of our research seeking different aspects or layers of mathematics knowing in action. The second way in which act of knowing is incomplete is in the moment of knowing further acts of cognition are made possible. Kieren refers to this as "occasionally" incomplete; that is, every cognitive act occasions further knowing. To elaborate, each act of knowing brings forth a world of significance and in doing so expands the *sphere of behavioural possibilities* in which the knower exists. This makes possible new acts of cognition. Methodologically, this is a second reason why returning to the research artifacts is useful; because the researcher is changed, new possibilities for understanding arise. Paradoxically, in as much as each act of knowing is incomplete, each act of knowing is fully complete in the first order domain of experience, i.e., in the temporal moment of the experience. "Moment by moment new experiences happen and are gone" (Varela, Thompson and Rosch, 1991). An act of knowing is necessarily complete, if from moment to moment we continue living.

I would like to invoke 'Kieren's incompleteness theorem' in thinking about my own acts of cognition with respect to trying to understand mathematical knowing. In chapter three, I offered two vignettes of people engaged in mathematical activity. In those vignettes, we saw a father and his daughter and a girl and her mother engage in actions and interactions which many researchers would agree are mathematical, even though the explanations of the mathematics knowing offered by various theorists might differ. Theorists, whose explanations suggest computational views of mind, propose that knowing involves taking information from the environment and processing it to create mental representations which match a pre-existing reality. In representational forms of constructivism, knowing is thought to involve assimilating and accommodating environmental stimuli into existing and newly constructed schemata and structures. Such a view suggests that the knowing of the child becomes more like that of those people who know or more closely match features of or objects in the environment. Radical constructivism suggests that, through experience in an environment, schemata and structures are constructed in such a way that they fit the personally constructed constraints of the environment rather than match features of it or objects in it. That is, the person acts to maintain the coherence of his or her thinking in the face of experience. The socio-cultural theorists explain that knowledge is constructed first on the social plane (or in social interaction), and then knowledge is internalized by the individual. Yet others, like the critical theorists, view coming to know as enculturation into the discursive practices of society. As I suggested in chapter three, each of these theories can be used to interpret the vignettes of the two parent-child pairs in order to come to an understanding of mathematics knowing and mathematical actions. The multiple interpretations in the previous chapter are an example of the first part of Kieren's incompleteness theorem. That is, the particular community (mathematics education research), in which the those interpretations are offered, has criteria which must be met by the interpretations offered within it; but, these same interpretations may not address all the criteria required by other communities, since the interpretations were not offered with any other community in mind.

Studying the actions and interactions of the two parent-child pairs from these varied perspectives and studying enactivism (a theory of cognition), expands my cognitive domain and occasions further knowing acts (Kieren's incompleteness theorem part 2). Therefore, I return to the vignettes from yet another perspective and ask how might we think about Dan's and Kerri's and Rebecca's and Cathy's mathematics knowing if we understand knowing as a co-emergent phenomenon; that is, something which co-dependently arises with the embodied actions of these people in their environments.

In this chapter, I elaborate on notions central to enactivism. In doing so, I propose that enactivism offers a perspective from which we might observe mathematics knowing in action and interaction. The discussion involves a distinction of the person in interaction with his or her environment, and the ways in which a world of significance including mathematics is brought forth in that interaction. I use the two illustrative cases developed in the previous chapter to explicate the explanatory proposition for cognition, articulated in the work of Maturana and Varela (1992) and Varela et al. (1991), which has become known as enactivism.

A Theory of Human Knowing

Enactivism, as interpreted in education, is related to radical constructivism (see von Glasersfeld, 1995; Steffe and Kieren, 1994), in that it examines the embodied structural dynamics of individuals, and is related to social constructivism (Ernest, 1995), in that it explores the ways in which knowing is a social act. It differs from both forms of constructivism in its emphasis on co-determination; that is, knowing is, at once, structurally determined and environmentally constrained (Varela, 1992, p. 254). Further, enactivism explicitly rejects representational views of knowing that are assumed in cognitivist and other information-processing models of the mind. Observations made from an enactivist perspective are explained as belonging in the domain of the observer, and hence, reflect a view of constructed realities (or objectivity-in-parentheses). Enactivism is similar to interactionism, the perspective promoted by the research of Bauersfeld and his colleagues in the mathematics education community (see Bauersfeld and Cobb, 1995a), in that both perspectives explain knowing as a phenomenon which arises when people interact with each other and emergent cultural artifacts. Enactivism differs from the constructivist perspectives in that rather than focusing on "constructions" and "constructive" mechanisms, we turn our attention to knowing as it is enacted and the knower/known relationship that is brought forth in acts of knowing.

From an enactivist perspective, mathematical cognition is not merely a property of the individual nor is it simply a product of environmental influence; instead, it is a phenomenon which *co-emerges* in the interaction between the two (Varela et al., 1991; Davis, Sumara and Kieren, 1996). According to Varela et al., cognition is neither the recovery of a pre-given world nor is it the projection of a pre-given inner world. Both of these positions are based on representationalist views of mind. Rather, the world and perceiver specify one another, i.e., co-emerge (p. 182). When an observer points to the person in an environment, the person is noted in a temporal relation which is distinguished by the observer and not a pre-existing reality.

In enactivist explanations of cognition, it is assumed that cognition is not simply a process by which representations of facts or objects from the environment are stored in our head. Rather, enactivism explains human knowing as enaction or embodied action, whereby a world of significance is brought forth in our doings. "The experience of anything out there is validated in a special way by the human structure, which makes possible 'the thing' that arises in the description" (Maturana and Varela, 1992, p. 26). *Cognition is embodied action which brings forth a world of significance.*

Varela et al. (1991) use the notion of embodiment as Merleau-Ponty (1962) does, where the body is understood as both a lived, experiential structure and the context of or milieu of cognitive mechanisms (pg. 238). "[W]e are in the world through our body, and in so far as we perceive the world with our body" (Merleau-Ponty, 1962, p. 206). Embodiment, Lakoff and Johnson (1999) assert, is a feature of human knowing that has been repeatedly demonstrated by cognitive science yet continuously denied in "mainstream Western philosophy" where "human reason and human concepts are mind-, brain-, and body-free and characterize objective, external reality" (pg. 22). Lakoff and Johnson argue that:

> Reason is not disembodied, as the tradition has largely held, but arises from the nature of our brains, bodies, and bodily experience. This is not just the innocuous and obvious claim that we need a body to reason; rather, it is the striking claim that the very structure of reason itself comes from the details of our embodiment. The same neural and cognitive mechanisms that allow us to perceive and move around also create our conceptual systems and modes of reason. (pg. 4)

The enactivist perspective explicitly embraces the notion that human knowing is a fully embodied phenomenon, and describes cognition as perceptually guided action in which a world of significance is brought forth. In other words, perception is not the grasping of an external reality, but rather, the specification of one (Maturana and Varela, 1981, p. xv) in the actions and interactions of our bodies (Merleau-Ponty, 1962). Zaner (1983) reminds us that Merleau-Ponty understood our bodies as our general means of having a world.

> [T]o be embodied, and thus to be sensuously perceptive of objects, and to be able to act on them, is to belong the world in the sense of being engaged in a body which places me at things themselves, with no intermediation, no 'representatives' or 'representations' of them (p. 183).

Varela, Thompson and Rosch (1991) elaborate on what they mean by embodied action.

> By using the term embodied we mean to highlight two points: first, that cognition depends on the kinds of experience that comes from having a body with various sensorimotor capacities, and second, that these individual sensorimotor capacities are themselves embedded in a more encompassing *biological, psychological* and *cultural* context. By using the term action we mean to emphasize once again that sensory and motor process, perception and action are fundamentally inseparable in lived cognition. Indeed, the two are not merely contingently linked in individuals; they have also evolved together. (p.173) [emphasis added]

If, as enactivism suggests, "cognition is not the representation of a pre-given world by a pre-given mind but is, rather, the enactment of a world and a mind on the basis of a history of the variety of actions that a being in the world performs" (Varela, Thompson and

Rosch, 1991, p. 9), then I come back to a key question, for me. *How does mathematics knowing arise*? Further, if cognition is the enactment or bringing forth of a world by a viable history of interaction with the environment (Varela, Thompson and Rosch, 1991, p. 27), then the study of mathematics knowing must involve the study of the individual-in-its-environment or, in other words, the individual and his or her environment as they co-dependently arise in interaction (Figure 4-1).

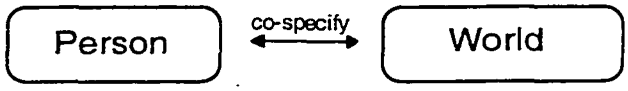

Figure 4-1. Person and world are co-specified in action

For my study of mathematics knowing in action, I distinguish a knower, a human being who lives in language with other human beings and who engages in mathematical activity, and I observe the knower in that activity. The study of mathematics knowing in action, then, becomes a meaning making activity for me, the observer. The claims I make about mathematics knowing in action are those which make coherent my experiences as an observer of the individual-in-its-environment. My claims exist in the domain of the observed rather than the first order domain of experience.

Structure Determinism

Human beings (like all living things) are characterized by their *autopoietic organization* (Maturana and Varela, 1992). As autopoietic entities they are fully autonomous and closed systems which interact with a medium but are self-referencing, self-organizing, and self-producing. Human beings differ from each other in their structure (the components and relations that actually constitute a particular human being), but are alike in their organization (those relations that must exist among the parts of a system for it to be a human being). Autopoietic organization is a key notion because it means that the being and doing of the human are inseparable— "their only product is themselves, with no separation between producer and product" (Maturana and Varela, 1992, pg. 49).

Following from the assertion that human beings are autopoietic is the notion that we are *structurally determined* systems: that is, systems in which all structural changes are made possible by the system itself as its components interact and it interacts with its medium (or environment). Like all living systems, humans are changeable (some changes are observed as learning), and change (learn) as a result of their interactions (Figure 4-2). In other words, a person participates in interactions if his or her structure is such that he or she can participate. At the same time, the person's structure is modified in those interactions. "The notion of structure is composed on three key ideas; the idea of wholeness, the idea of transformation and the idea of self-regulation" (Piaget cited in Casti, 1994, pg. 212).

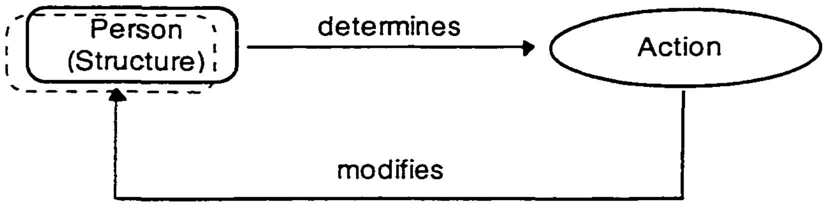

Figure 4-2. Structure determinism suggests that a person can only participate in actions which are permitted by his or her structure

There are two triggers for ongoing structural change: interactions with the environment in which it [the learner] exists and the structure's own internal dynamics (Maturana and Varela, 1992, pg. 74). As long as a human being continues to live, he or she undergoes structural change as a result of his or her interactions.

All knowing is determined, *but not fixed*, by a person's structure. "This means that nothing external to a structure determined system can specify the structural changes that it undergoes as a consequence of an interaction. An external agent that interacts with a structure determined system can only trigger in it structural changes determined in it" (Maturana, 1988, p. 36). However, any changes the system undergoes must be viable given the nature of the environment. If a change occurs that is not viable for the system, it will cease to interact in that environment.[1]

Knowing is inferred by the observer when he or she notes a behaviour, which may be an utterance (written or spoken), gesture, movement or other bodily response (like the blushing of one's face) which arises in the interaction. Because I am interested in the human mathematics learner, I find it useful to think of structure as that which is formed through ongoing personal[2], social and cultural interactions. Those are the domains of human interaction that I observe (Varela et al., 1991).

Memory

Dewey said every experience lives on in further experiences (1933). In enactivist terms, our bodies both make possible our experiences and are marked by them. With each experience (which involves spatial, temporal and relational dimensions), our whole being is modified because our structure is modified. We are changed physically, mentally, and in our relations to things (and people) in our environment. Our bodies shape our experiences and are marked by our experiences. In this way, our knowing is fully embodied. I might represent the historic nature of human knowing, in such a way, to show that the human learner brings his or her history of experiences into the moment of the present experience (Figure 4-3).

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[1] Because I am dealing with mathematics knowing, I am not talking about the death of the person but his or her metaphoric death in the realm of mathematics.

[2]The personal includes both physiological and the phenomenological. Because I am interested in mathematics knowing, I will not focus on the neurological level. However, I will discuss the personal at the levels of the sensorimotor and the phenomenological.

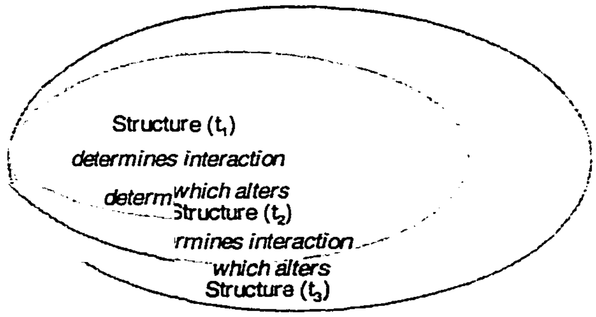

Figure 4-3. One's history of experiences is marked by changes in his or her structure

A human being's history of interaction is incorporated into his or her structure which constrains, moment by moment, how he or she acts in a given setting and under various perturbations. This might be thought of as a structure-determined understanding of memory (von Foerster, 1981). As von Glasersfeld (1995) suggests, memories are re-presentations, in the sense that they are re-constructions of our past experiences in the moment. Thus, our memories, like all of our experiences, are experienced in the moment.

When humans do mathematics we note that their structure, which embodies their mathematics "memory" (history of structural changes as a result of experiences), both constrains and makes possible the mathematics knowing that is observed. At the same time, a person's actions, and the consequences of those actions, are recursively implicated in his or her structure and result in a change to it (and hence memory). Varela (1992) tells us that our "cognitive capacities are inextricably linked to a history that is lived, much like a path that does not exist but is laid down in walking" (p. 255). Merleau-Ponty (1962) suggests that "to remember is not to bring into the focus of consciousness a self-subsistent picture of the past; it is to thrust deeply into the horizon of the past and take apart step by step the interlocked perspective until the experiences which it epitomizes are as if relived in their temporal setting. To perceive is not to remember" (p.22). Hence, the enactivist perspective asserts that present experiences are determined by one's living and lived body (which is made up of the traces left by a history of interactions); at the same time, one's interactions in the present change one's body in the moment of knowing.

This view of structural change offers an alternative to the view which suggests knowing is constructing representations of a pre-existing world which are "stored" in memory, to be retrieved when needed to solve a problem posed by the environment. In the non-representationalist, structure-determined view of knowing I am offering here, *knowing is understood to be acting and interacting in the present moment within a sphere of possibilities which is jointly defined by the person and his or her environment.* Individuals may be observed to be present together in a shared physical setting and be part of the social interaction of that setting, however, a person's range of possible behavior in the moment is co-determined by his or her history of interactions, and the context in which he or she is embedded.

One's embodied history specifies the realms of interaction possible for the person (Maturana and Varela, 1992, pg. 171), although it does not pre-determine a particular action.

> [Memory is] a field which is always at the disposal of consciousness and one which, for that very reason, surrounds and envelops perceptions, an atmosphere, a horizon, or if, you will, given 'sets' which provide it [consciousness] with a temporal situation, such is the way in which the past is present, making distinct acts of perception and recollection possible. To perceive is not to experience a host of impressions accompanied by memories capable of clinching them; it is to see, standing forth from a cluster of data, an immanent significance without which no appeal to memory is possible (Merleau-Ponty, 1962, pg. 22).

Returning to the two cases discussed in the previous chapter, I note that, although the parents and children were offered the same prompt (to tile a $2 \times n$ path with dominoes) the two pairs were observed to respond differently. Kerri's and Dan's response was to make a table. This is very consistent with what I observed when Kerri was given a variety of other prompts throughout the course of the parent-child mathematics program. Other observers might propose that the reason she responded this way could be explained by suggesting it was the most appropriate way to respond, given the prompt she was offered. This is a plausible explanation, given it was a reasonable way to proceed, in this particular situation. However, in another session, Kerri tried desperately to make sense of a situation by using a table to help her find a function to describe the situation, when by my assessment it was a very poor strategy. Given this evidence, I am more likely to believe that the strategy she used had as much to do with Kerri and what she knew (her history of interactions) as it had to do with the prompt which she now encountered.

Finally, when I suggest that a person's knowing is structurally determined, I am not saying that the person's actions are predetermined. Rather, I am saying that a person can only act/think in ways that fit with his or her history of biological, social and cultural experiences. For example, if one does not have a history that includes multiplying, then a teacher's instructions (perturbation) to a student to find the square a number by multiplying a number by itself cannot cause the student to know how to square a number (in a way that has meaning for the student). In the same way, if a student has poor fine motor coordination and cannot draw a straight line, a pencil in his or her hand and a sheet of paper will not cause the student to draw a straight line. In both of these examples, the person's structure limits what is and is not possible in that moment. However, this does not explain how certain affordances or conditions in the environment can be observed to make a difference in the person's actions. For example, the student who is unable to draw a straight line free hand might be able to do so given a straight edge. (The word might is intentional since I have seen a number of children who even with a ruler do not draw a straight line.) Or, the student who cannot multiply could find the square of a number if an adult (for example) provided adequate scaffolding for the student by suggesting, let's say, the student draw a square on graph paper and count the unit squares from which it is composed. As Vygotsky (1978) suggested, there is a zone of proximal development in which a child, in interaction with a more knowledgeable person, is able to answer questions the child could not answer on his or her own. Hence, these examples suggest structure determinism is only part of an explanation of human knowing; the role of the environment also needs to be considered.

Environment

When a human observer distinguishes a living system (in this case a human learner), at the same time he or she distinguishes the environment with which the system interacts. An alternative way of approaching the study of knowing has been to focus on that with which the learner interacts—the information in the environment, for example. Nørretranders (1991/1998) explores the role of information (or lack of it) in meaning making. His thesis is that in any communication between people there is informing without full information in the environment. The missing "bits" in the environment are what he calls *exformation.* He illustrates his point with the following story.

> Victor Hugo—famous for writing *The Hunchback of Notre Dame*—had gone on holiday following the publication of his great novel *Les Misérables*. But Hugo could not restrain himself from asking how the book was doing. So he wrote the following letter to his publisher: "?"
>
> His publisher was not to be outdone and replied fully in keeping with the truth: "!" (Nørretranders 1991/1998, p. 91)

Nørretranders goes on to suggest that there are very few "bits" of information in this correspondence; however, the communication was successful.

> It was not the number of bits transmitted that was decisive, but the context of that transmission.... Both messages represent many considerations,—thoughts feelings, and facts—which are not present yet nevertheless are. Information that is not there yet nevertheless is. The correspondence refers to a plethora of information—otherwise it would not be full of meaning....
>
> Hugo's question mark is the result of an explicit discarding of information. Not merely a discarding of information: He has not simply forgotten it all. He refers explicitly to what he has discarded, but from the point of view of the correspondence it is still discarded. For the purposes of this book, we will call such explicitly discarded information *exformation* (pg. 92).

As it appears for observers of Hugo's interaction with his publisher, it may seem as though the environment contains information (the message in the correspondence) prior to a person's realization of it. However, as Merleau-Ponty (1962) explained and Nørretrander's example suggests, the world appears to be there before reflection begins but that world is not separate from the person (Varela et al., 1991, p. 3).

While the word environment commonly is used to mean everything outside of the organism (person), this "everythingness" is precisely what some theorists critique (Maturana and Varela, 1992; Lewontin, 1991). The environment with which the organism can interact is not everything outside of itself. Rather, the organism's environment is precisely that with which the organism can interact. The environment is a relational phenomenon which consists of "an organized set of relationships among bits and pieces of the world which is selected by the individual him- or herself" (Lewontin, 1991, pg. 86). Varela et al. (1991) explain that the "local situations constantly change as a result of the perceiver's activity, the reference point for understanding perception is no longer a pre-given perceiver independent world but rather the sensorimotor structure of the perceiver" (p. 173). For any living organism, its environment is not pre-specified but is specified by the organism itself as it acts; this is, for Varela et al., enaction.[3]

That which "exists" in the environment only exists for a person in so far as the person enacts it. In other words, perturbations that an observer distinguishes in the environment are part of the environment precisely because the observed person brought them forth in action. Varela et al., (1991) elaborate :

> This insistence on the co-determination or mutual specification of organism and environment should not be confused with the more commonplace view that different perceiving organism simply have different perspectives on the world. This view continues to treat the world as pre-given; it simply allows that this pre-given world can be viewed from a variety of vantage points. The point we are making, however, is fundamentally different. We are claiming that organism and

[3] Lewontin (1991) tells the story of a Mars exploration in which the scientists were trying to determine if there was life on the planet. They designed an experiment in which some dust from the surface was placed on a radioactive medium. "Everyone was convinced that there was life on Mars when radioactive carbon dioxide was produced. But were confused when abruptly the production of the compound ceased. Lewontin says, the problem of this experiment arises precisely from the fact that organisms define their own environment. How can we know whether there is life on Mars? We present Martian life with an environment and see if it can live. But how can we know what the environment of Martian life is unless we have seen Martian organisms?... We may know the temperature, the gas content of the atmosphere, the humidity and something about the soil on Mars, but we do not know what a Martian environment is like because the environment it does not consist of temperature gas, moisture and soil. It consists of an organized set of relationships among bits and pieces of the world, which organization has been created by living Martian organisms themselves. ... It is not that organisms find environments and either adapt themselves to the environments or die. They actually construct their environment out of bits and pieces" (p.85 - 86).

> environment are mutually enfolded in multiple ways, and so what constitutes the world of a given organism is enacted by that organism's history of structural coupling (p. 202).

As a number of researchers (Lakoff and Johnson, 1999; Varela et al., 1991; Maturana and Varela, 1992, Capra, 1996) note, it is the *relationships between the human being and his or her environment* that are of significance to our understanding of human cognition.

When observing the environment and the unity I have distinguished from the environment, I note that the environment also is involved in interactions. In the same way that a structure changes by virtue of its interactions, so does its environment (Figure 4-4). Just as it is for the human learner, each time an environment is involved in an interaction there is potential for change in that environment. In this way, the environment is observed to have a "structural dynamics independent of the systems that it contains although it is modulated through its encounters with them" (Maturana, 1998, http://www.inteco.cl/ articulos/ metadesign_parte1.html).

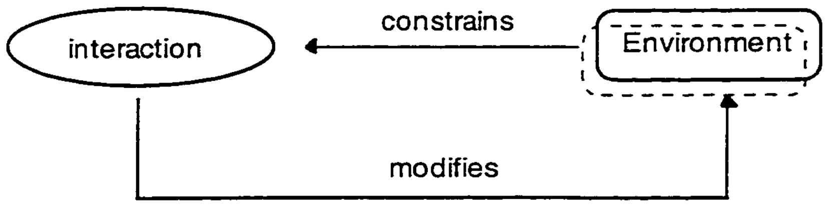

Figure 4-4. Environment constrains interaction and is modified by it

Returning to the illustrations in chapter three, the prompt, the dominoes, the sheet of paper on which Dan was keeping track of the tilings, the drawings of the tilings and the numbers recorded in the table were all part of Dan and Kerri's environment. Take for example, the tiles. Their arrangement underwent a number of transformations over the course of the evening as a result of Kerri moving them about on the table. Thus, the positions of the tiles changed constantly, and Dan and Kerri noted various patterns. Those patterns were co-specified by Dan and Kerri and the tiles themselves. The piece of paper on which Dan was writing was also changed but in a more permanent way. Each time Dan put his pen to the paper it left a mark which became part of Dan and Kerri's environment because it had significance for them. Not only did physical features of the environment undergo constant change because of the interactions of which it was a part, but so too did the relational features of the environment. The environment was transformed in terms how Dan and Kerri related to it and the perturbations it generated for them. Once they distinguished a pattern, both their own knowings and the environment were altered such that there were more patterns to distinguish (Piaget, 1974/1980). Kerri and Dan's actions changed as the environment was modified in their interactions with it. Further, in as much as Dan is part of the environment for Kerri, and Kerri for Dan, any changes in each of them resulted in a changed environment for the other. Knodt (1995) describes how further possibilities for meaning are constructed in interaction.

> No longer grounded in an external reality—as a representations or mirroring of that reality—meaning resides in the self-referential structure of a consciousness that consists solely in and through its autopoietic operations and that, in selecting from a self-generated horizon of surplus references, reproduces that horizon without ever exhausting its possibilities or transgressing its boundaries (p. xxvii).

When the individual (human learner) is in ongoing interaction with his or her environment, both the learner and the environment are observed to change together. Thus, the study of the learner must involve the study of the learner in interaction with his or her environment. Maturana and Varela (1992) posit the notion of *structural coupling* to explain the relationship between a living organism and its environment.

Structural Coupling

In the previous sections, I suggested that all interactions involving a human being may be viewed as structure-determined and as constrained by the environment, both of which are brought forth in that interaction. The enactivist view asserts that the environment does not instruct nor specify particular changes in an individual; rather, the person's interactions with the environment act as perturbations to trigger potential changes that are co-determined by the living being's structure. When a person and his or her environment are seen by the observer to co-dependently arise through mutual specification and ongoing interaction, the observer notes structural coupling (Figure 4-5).

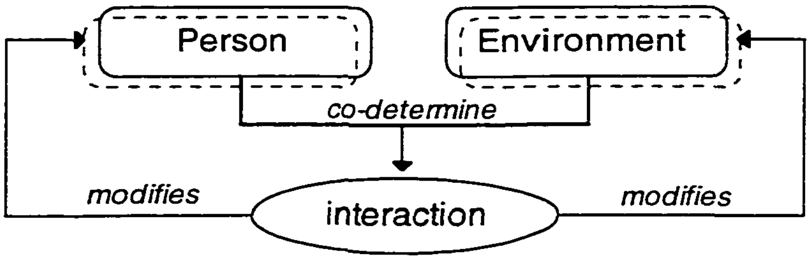

Figure 4-5. Structural coupling arises as recurrent interaction between person and environment

Maturana and Varela (1992) explain.

> [E]ach living being begins with an initial structure. This structure conditions the course of its interactions and restricts the structural changes that the interactions may trigger in it. At the same time, it is born in a particular place, in a medium that constitutes the ambiance in which it emerges and in which it interacts. This ambiance appears to have structural dynamics of its own, operationally distinct from the living being... We have thus distinguished two structures that are going to be considered operationally independent of each other: living being and environment.... In the interactions between the living being and the environment within this structural congruence, the perturbations of the environment do not determine what happens to the living being; rather, it is the structure of the living being that determines what change occurs in it.... The same holds true for the environment: the living being is a source of perturbations and not of instructions. (Maturana and Varela, 1992, p. 95 - 96)

Structural coupling occurs when, in the history of interaction, an individual and its environment interact as independent systems but systems that provide triggers for structural change in the other. Structural coupling is a phenomenon that takes place whenever an individual undergoes "recurrent interactions with structural change but without loss of organization [class identity], and which also includes changes in the environment or domain of interactions" (Maturana, 1981, p. xxi). This is useful in the case of observing the interaction among people.

> [W]hen two or more structure determined systems interact recurrently with each other in a particular medium, they enter in a history of congruent structural changes that follows a course that arises moment after moment contingent on their recurrent interactions, to their own internal structural dynamics, and to their interactions with the medium, and which lasts until one or both of them disintegrate, or they separate (Maturana, 1988, p. 46).

Our interactions are made possible by our structures, and will continue in so far as those interactions are viable. Varela et al. (1991) invoke a proscriptive logic to explain how cognition does not involve an optimal match between the system's action and the environment; rather, cognition involves satisfactory or viable action within broad environmental or contextual constraints such that anything that is not forbidden is allowed. Hence, there are many adequate actions that maintain structural coupling. Structural coupling will be destroyed however, by forbidden actions.

In as much as, human knowing is not the recovery of a pre-given world but viable action which realizes a world, knowing is adequate conduct or viable action in a co-emergent context. Therefore, knowing is better understood in terms of a proscriptive logic rather than a prescriptive one (Varela et al., 1991; Kieren, 1995). There is no particular thing that must happen in order for an observer to say that the learner knows; rather, there must a viable action within the sphere of behavioral possibilities that the observer understands as adequate conduct (Maturana, 1987). There are some actions that are forbidden (actions outside of the sphere of behavioural possibilities) and other actions that are judged, by the observer, as inadequate conduct.

Another way to think about this is to imagine that at each moment there is a sphere of behavioural possibilities for each of Dan and Kerri. Given his or her state of being (structure) in an environment at any particular moment, there are many things he or she could do next (within the sphere) and some things that he or she could not do next (outside of the sphere). As Dan and Kerri interact, not only do they personally undergo change (their structures change), but, their spheres of behavioural possibilities undergo transformations because of their interactions and the changes to their structures. In this way, their mathematics knowing co-emerges, not within, but with the context of their activity. Hence, we speak of the changing sphere of behavioural possibilities. Further, as context and structure change for one person, there is potential change for the other because the two people are interacting. Kieren (1995) cautions that, "if one is thinking about the sphere of possibilities in solely environmental terms then one looks at properties per se in isolation of the individual's action in it. [However] if one looks at the "sphere of possibilities" in ecological terms, then such isolated viewing is inappropriate. That is not to say that the sphere of possibilities does not have properties, but simply that the properties of interest manifest themselves in the interactions with the individuals and further that these properties are seen by the observer as properties of the relationship".

Structural coupling can be used to explain the distinction we observers (teachers in particular) call learning (structural change)[4]. The interaction between a person and the person's environment (including other people) creates perturbations which have the potential to trigger knowing—embodied action. When an observer notes a relation between the individual's mathematical behavior and a perturbation (or feature) in that individual's environment, we say that the perturbation in the environment *occasioned* the student's mathematical cognition (Simmt, 1996b). (I discuss this further in chapter five.)

The Social Domain

There is a fundamental circularity (Varela, 1984) in the mutual specification between the organism and its environment. This has deep implications for social organisms such as humans. For humans, it is through recurrent interactions (structural coupling) with other human beings, in language, that the social domain emerges (Figure 4-6).

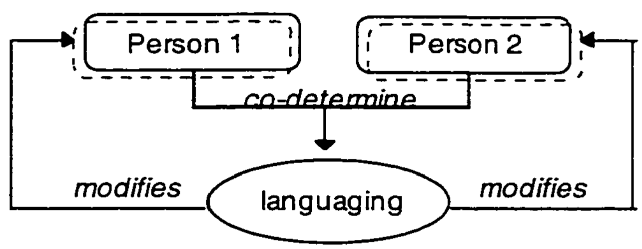

Figure 4-6. A special case of structural coupling: social interaction

[4] Learning as structural change does not involve an evaluative stance. That is, any structural change, whether an observer would call it learning or misunderstanding is learning from the enactivist perspective. Of course, in teaching situations, teachers teach towards particular outcomes, and observe for behaviours that indicate those outcomes. On the other hand, the students' structures are changed in many ways that may or may not be related to the teacher's intended outcomes, and may not be observed by the teacher at all.

In the case of humans, structural coupling is the mechanism which makes possible social behavior and the social domain. That is, our recurrent interactions with other human beings are part of our living. This form of living is what we, as observers, distinguish as social.

> What is peculiar to us as human beings though, is that we exist as such in language as the operational space in which we realize our living as such. That is, we exist in the flow of living together in the recursive coordinations of behaviour that language is (Maturana, http://www.inteco.c1/articulos/metadesign.html).

As Maturana and Varela (1992) explain, the social domain expands our cognitive domain.

> Biology also shows us that we can expand our cognitive domain. This arises through a novel experience brought forth through reasoning, through the encounter with a stranger [not ourselves], or more directly, through the expression of a biological interpersonal congruence that lets us *see* the other person and open up for him room for existence beside us. This act is called *love*, or, if we prefer a milder expression, the acceptance of the other person beside us in our daily living. This is the biological foundation of social phenomena: without love, without the acceptance of others living beside us, there is no social process and, therefore, no humanness. [emphasis original] (pg. 246)

When, humans interact with others, the worlds brought forth in that interaction have the potential to overlap with each other and with others. Hence, each time a person's world is modified by an interaction there is the possibility of change in, not only her world, but, the world of another as well. This was observed when Rebecca and Cathy interacted and Dan and Kerri interacted. When Dan uttered "blip-blip," this utterance became part of Kerri's environment, thus changing her sphere of behavioral possibilities. Or, when Rebecca described to Cathy how she was looking for the mirror images of the tilings her description changed the sphere of behavioral possibilities for Cathy. It is not as though Rebecca's actions changed Cathy; instead Rebecca's utterances become energy-rich matter from which Cathy selected. Note that, a few minutes passed before Cathy blurted out (as though it were her idea), "Hey I know" (and wrote) *look for mirror patterns*. Further, consider how Dan's utterances were energy-rich matter from which Kerri selected. At first the words, blip blip, were used as a recording and conversational tool. However, we later see Kerri use blip-blip (one word for her new concept) as a tool for mathematical thought. The blip-blips emerged as two distinct objects or two different actions, within the constraints of the prompt. It is in the recursion (Kerri's selecting of her selection) that we see an illustration of the social domain as it is complicit with the person's knowing. In the structure determined act of selecting blip-blip, and integrating it into her knowing, Kerri's embodied knowing was changed but further changes (from the point of mathematics knowing) were indicated when she used blip-blip differently from her father. Her utterances and actions suggest a transformation of blip blip (two horizontal tiles) into blip-blip (a unit consisting of two horizontal tiles).

An enactivist perspective, which asserts knowing is embodied action, encourages the observer to consider both the individual's structure and the environment (and the special case of other human beings) with which the individual interacts when interpreting mathematical activity. Once I do this, I begin to understand how the complexity of cognition arises. Our interactions are in language and include personal, social and cultural dimensions of knowing, all at once. Our bodies are both the source and the intersection of those multiple dimensions.

Bringing Forth a World of Significance

Although it has been useful in the study of children's mathematics to focus on either the individual (as radical constructivists do) or the social features of human knowing (as the socio-cultural theorists do), such approaches tend to lead to a polarization, whereby we observe for one thing or for another. This was, in part, Confrey's (1994, 1995a, 1995b) concern when she called for a reformulated theory of mathematics knowing which includes both the theories of Piaget and Vygotsky. Enactivism offers some possibilities in this regard. Its view of cognition as embodied action which brings forth a world of significance is a way in the middle that suggests knowing is at once individual and social.

In summary, enactivism explains mathematics knowing as perceptually guided action in language in which people bring forth intertwined worlds of significance which includes mathematics (see Figure 4-7).

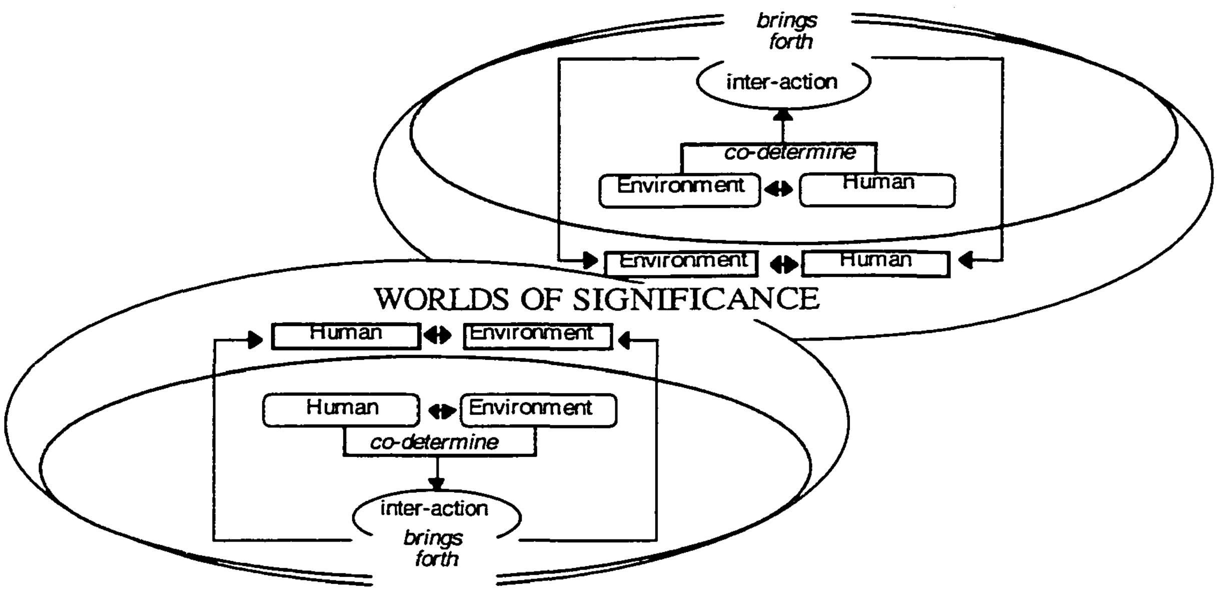

Figure 4-7. A model for observing cognition as bringing forth intertwined worlds of significance with others

The organism and the environment are observed to co-emerge in the world that is brought forth. Hence, in enactivist inquiry, not only is cognition studied as a feature of the individual, but, it is also studied in the inter-actional dynamics of the person in an environment. While the person's structure determines any world building actions which he or she takes, it is the coupling with the environment (inter-action) which constitutes the space for such actions and provides the possibilities for them—it creates a sphere of behavioural possibilities. Because we are social beings, the worlds we bring forth are intertwined with the worlds of others. When we act, our actions have the potential to alter the sphere of behavioural possibilities of others. Hence, an enactivist perspective of cognition has ethical implications. (These are further discussed in chapter eight.)

In the enactivist view of cognition, the environment is not thought to instruct or specify particular changes in an individual; rather, the person's and environment's inter-actions act as perturbations to trigger potential changes in both the organism and its environment. With respect to humans, cognition is an act of specifying (moment by moment) the relevant features of the environment with which the person interacts; and in doing so, the person creates or brings forth a world. At the same time, the person is brought forth in that world—the person is changed by his or her own cognition. As is the case with complex phenomena, complexity breeds transformation and further complexity. But even in complex phenomena, the human observer can distinguish patterns. Hence, in studying mathematics knowing in action, there is no need to reduce the phenomenon of cognition to simpler parts; an enactivist perspective provides us conceptual tools for observing human knowing in all of its complexity.

The task of the observer is multifaceted: on one hand, as an observer I need to explore the ways in which the environment constrains the knower, and the ways these constraints are specified by the sensorimotor structure of the knower him or herself. On the other hand, I also need to explore the relationship between the knower and the known, both of which are brought forth in interaction. Further, there is a need to consider the implications of a perspective, such as enactivism, which is necessarily ecological. The purpose of the remaining chapters is to use the enactivist principles developed in this chapter and the model posed in this chapter to observe and interpret mathematics knowing in action so that I might characterise it.

Chapter Five

OCCASIONING MATHEMATICS KNOWING

When an observer notices that a student acts in a particular way when a specific event or feature of the environment is present, it is not uncommon for the observer to suggest that the event or feature of the environment caused the person's behaviour. After all, it is common sense that knowing does not arise from nothing; it must be provoked. "Thinking is not a case of spontaneous combustion," Dewey said. "There is something that occasions and evokes it" (Dewey 1933, pg. 15).

When I think about some "thing," that thing arises as an object for me to think about. An enactivist perspective resists the temptation of looking for that which comes first: the thing or the thought about the thing. Instead, the thing and the thought about it are understood as co-emergent phenomena—they co-dependently arise in action. It follows, from an enactivist perspective, that one avoids seeking 'causes' for a person's thinking in favor of exploring the relationship between the person and his or her environment.

When we watch a child listening to a teacher who is offering a strategy for solving a problem and the student solves a subsequent problem, it appears to us as though the teacher caused the student's learning. In fact, we distinguish teaching and learning as complementary activities that take place between teachers and learners. However, if this is the case, then why is it that teaching does not necessarily result in a student's learning? Why is it that given 30 children in the same classroom, apparently offered the same instruction and educational opportunities, that some students "learn" what is "taught," and some of their peers do not "learn" what is "taught"?

Maturana's and Varela's (1992) explanations of structure determinism and structural coupling are useful for understanding this question. They have demonstrated that all knowing is both made possible and constrained by the person's structure and provoked through interactions with their environments (which includes other human beings). Hence, the person learns (changes) through his or her lived experiences—biological, social and cultural. Given this explanation, *in what ways can we characterise the observed relationship between a person and his or her environment? How do we interpret the interactions between a person and his or her environment?* In this chapter, I elaborate on those interactions among people and with physical and conceptual "objects," that are observed to co-dependently arise with mathematics knowing.

Occasioning

The old saying, "you can lead a horse to water but you can't make him drink," is worth pondering over. Von Glasersfeld (1995) asks, "Is it the horse's thirst or is it the water that "causes" the horse to drink?". We might respond that the presence of water and the nature of water are responsible. Or we might respond, that it is the horse's thirst that is responsible for the horse drinking water. A response from an enactive perspective is that the water and the horse's thirst are co-responsible for the horse drinking water. Rather than discuss the mathematical activity in terms of its causes and effects, I wish to re-frame the way in which we might observe actions and interactions by inquiring about the significance of co-responsibility. *How does it happen that a person comes to act mathematically? In particular, what does it mean "to occasion" mathematical cognition?*

The word occasion is commonly used as a noun to refer to a special event. It is less common to use the word as a verb; but when it is used in this way, it means to cause in a subsidiary manner. Heidegger (1977), in contrast, takes up the verb, "to occasion", and uses it in a more primary way. He invokes the four senses of causality (of being responsible for something coming about) the early Greeks spoke about: the *causa materialis* (the matter from which something is made); the *causa formalis* (the form or shape which the material eventually takes); the *causa finalis* (the end or purpose for which the thing is created); and the *causa efficiens* (that which brings about the effect).[1] Occasioning, he claims, has to do with causing in the sense of starting something on its way or in the sense of co-responsibility.

I first heard the term, occasion, used as a verb by Kieren. "A person's mathematical knowing [is] determined by that person's structure, yet co-emergent with and occasioned by the environment in which this active knowing is done" (Kieren, 1994, p. 134). Also intrigued by Kieren's use of the verb 'to occasion,' Sumara (1996) explored the notion of occasioning. He began by tracing the etiology of the word. "According to the *Oxford English Dictionary*... the original Latin meaning of occasion (*occasion-em*) has more to do with an opportunity arising from "a falling of things towards each other"—something that presents itself in the middle of a set of circumstances (Sumara, 1996, p. 200-201). Sumara offers this interpretation. "Occasion, understood in this way, is more like a hap—more like the kind of situation that is not predictable but which if taken up, can lead to a new and previously unknown path of understanding" (p. 201).

I offer occasioning as an explanation of the coherences of experiences of the person who observes another person and his or her environment as co-responsible for the mathematical knowing that emerges in the interaction between them. This notion of occasioning returns us to action. Occasioning exists in the realm of the observer. It is an explanation of the observer's experience of seeing a person and his or her environment in relation, not of the learner's immediate experience with the environment. (However, the learner can observe him- or herself, and ask about what causes his or her thinking.) The observer distinguishes an occasion, when the observer sees a behavior and a corresponding feature of the environment. From the perspective of the learner, she or he is simply acting in a way which preserves her or his relation to the environment. In other words, an observer says something occasions a person's action just when the observer observes that

[1] In modern science when we speak of cause we generally are speaking about the *causa efficiens*.

the thing exists in the person's presence, the person selects elements of or features of it, and the person acts on those features. Because this selection from and action on the thing potentially changes the person, the thing and the person's action itself, we say that the person, the thing and the person's action are co-emergent phenomena.

Occasioning is a conceptual tool which points to the relationship between the knower and the "observable" triggers for his or her knowing. It helps the observer understand a person acting on and with what the observer infers as energy-rich matter.

> [Heinz von Foerster] coined the phrase "order from noise" to indicate that a self-organizing system does not just "import" order from its environment, but takes in new energy-rich matter, integrates it into its own structure and thereby increases its order (Capra, 1996, p. 84).

From an enactivist perspective, it is important to emphasize that the individual does not simply receive objective information from outside itself as a stimulus to which it responds (Brier, 1992). Rather, a person is "informed" by something in the environment, if his or her structure is such that it can "perceive" the perturbation and change in order to maintain itself given the perturbation. One might suggest, as Nørretranders (1991/1998) did, that all knowing is informed by the exformation which is part of the interaction, or suggest as Von Foerster (1981) did, "the environment contains no information; the environment is as it is" (p. 263). Information is not something outside of the person existing in the environment; it is a phenomenon created by individuals in interaction with their environment at a particular moment under particular circumstances[2] (p. 193-194). Occasioning is like information, in von Foerster's sense. However, occasioning is as an explanation for observing a person's knowing in action rather than an explanation of how knowledge is acquired. How, then, does occasioning come about?

From an enactivist perspective, one could think about the question of occasioning in terms of structural coupling. That is, if two systems are structurally coupled, then the

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[2] The statement about information assumes objectivity-in-parenthesis (see chapter three). It is a radically contingent statement. (Kieren, personal correspondence, October, 1999).

observer would note a pattern of recurrent inter-action in which there are modifications to the interacting systems; hence, there is *co-occasioning*. In the previous chapter, I illustrated the structural coupling between a person (structurally determined system) and his or her environment (which, for the purpose of discussion may be restricted to another person) with the diagram which I have reproduced here (Figure 5-1).

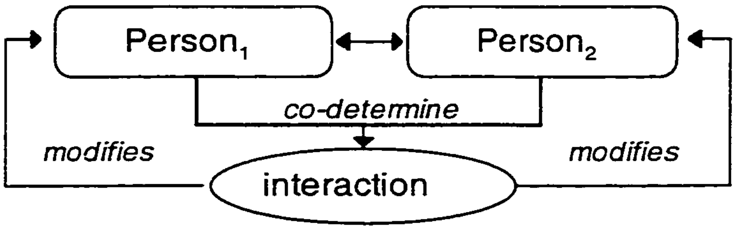

Figure 5-1. Structural coupling

In this chapter, I use this model for observing mathematics knowing in action: both to interpret and to represent my observations of people engaged in what I distinguish to be mathematical activity. I add to the diagram bubbles which represent the bringing forth of a world of significance (Figure 5-2). In each bubble, I point to that which is brought forth and that which I subsequently observed to occasion further knowing. Of course, as human beings our interactions are complex; one may observe many things brought forth, in any interaction, simply by looking for knowing in a variety of domains or from various perspectives. At the same time, there are many features of knowing in action that cannot be observed. Only knowing that is made evident in some behaviour can be observed.

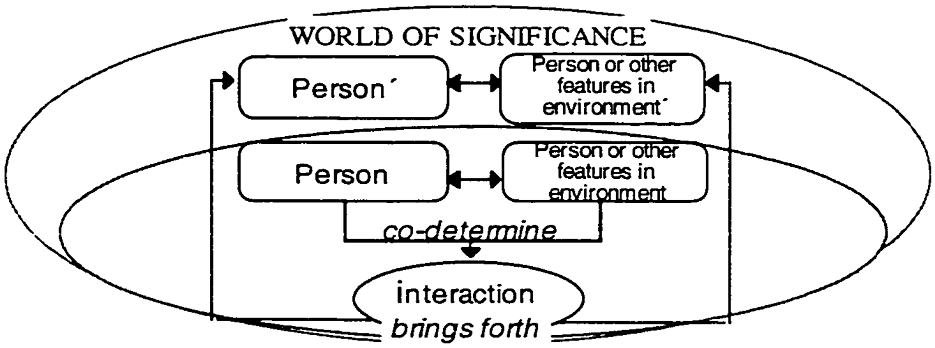

Figure 5-2. Diagram to illustrate the process of bringing forth a world of significance through action with and in one's world of significance

Observing Mathematics Knowing in Action

In this section, I offer a set of examples to illustrate how the participants in the parent-child mathematics program were occasioned to engage in mathematical thinking. I show how, for a small group of parents and children, a variable-entry prompt occasioned a wide range of responses. For me, these examples nicely illustrate how the differences in people's structures result in very different actions given what appears to be the same prompt. I also use the examples to show how one's mathematical thinking develops through recurrent interactions and ongoing occasioning. In other words, I will show how the variable-entry prompt is just one feature of the environment that propels mathematical actions.

One evening, I offered the following prompt (Figure 5-3), which I will refer to as the diagonal intruder (Stevenson, 1986), to four parent-child pairs participating in my extracurricular mathematics program.

> *Mark off a rectangle on your graph paper. Now draw in a diagonal. The object of this activity is to determine how many unit squares the diagonal passes through. For example, in this 3 by 5 (3x5) rectangle the diagonal passes through 7 unit squares.*

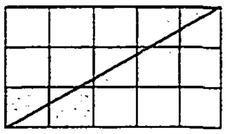

Figure 5-3. The diagonal intruder prompt

As usual, after I introduced the prompt to the group, the parent-child pairs were invited to work on it together. Each pair worked independently of the other pairs until the end of the evening when they were asked to explain what they had learned.[3]

[3] On this particular night, I audio taped only one pair, Jake and Cathy. I will discuss their activity in the next chapter. For the other pairs, I am reconstructing the events based on field notes in my research journal and the participants' working papers.

Dan and Kerri

Dan and Kerri, the father daughter pair introduced earlier, had more experience doing mathematics together than did the other pairs who responded to this prompt. These two often began by specializing (Mason, Burton and Stacey, 1982), and systematically building a set of cases. This method worked quite well for them, most of the time. In this session, Dan and Kerri generated a list of cases by beginning with a rectangle of area one square unit, then, considered all the rectangles of area two square units, and so on. They investigated the rectangles for the areas from 1 - 16 units^2 (Figure 5-4; Figure 5-5, a).

AREA	number of rooms intruder passes through		
1	1		
2	2		
3	3		
4	4	2	
5	5		
6	6	4	
7	7		
8	8	4	
9	9	3	
10	10	6	
11	11		
12	12	6	6
13	13		
14	14	8	
15	15	7	
16	16	8	4
17	17		
16	18	10	6

Figure 5-4. Dan and Kerri's record

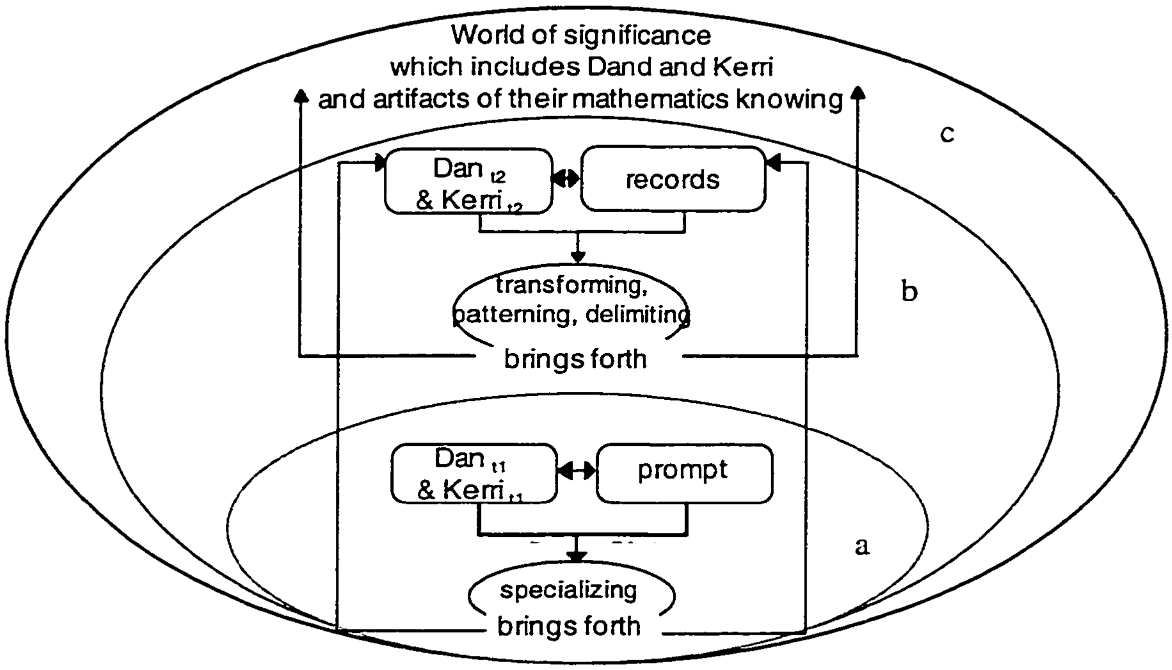

Figure 5-5. Mathematical actions were occasioned and from them a world of significance was brought forth

Using their records, Dan and Kerri looked for patterns (Figure 5-5, b) and conditions for the patterns, and were able to make some generalizations (Figure 5-5, c). For example, they noted that the diagonal of all of the $1 \times n$ rectangles would pass through n unit squares. They also noted that, if the area of the rectangle was not a prime number, then, there would be at least one rectangle for which there would be at least one other distinct quantity of unit squares intersected by the diagonal.

From the many cases they generated, they were able to state some generalizations, explain to others, and sometimes support their generalizations with deductive reasoning. Because they had participated in the parent-child mathematics program for some time prior to this session and because I know they had acted this way many times before, I am suggesting that specializing could be viewed as a structurally determined action. But at the same time, it was triggered by the diagonal intruder prompt. In other words, their specializing activity was occasioned by the prompt. Their actions brought forth a world of significance which included their own actions (thinking) and the artifacts of that thinking.

My observations suggest that their actions were viable in the context and therefore made possible further mathematical activity (Figure 5-5).

Roberta and Kristina

A second pair, Kristina (a 13 year old) and Roberta worked more slowly than did Dan and Kerri, and investigated only rectangles of size *3 x n*.[4] (The example worked was a *3 x 5* rectangle.) Roberta and Kristina each kept their own records. They would mark off a rectangle, draw in the diagonal and record the number of unit squares through which the diagonal passed (Figure 5-6).

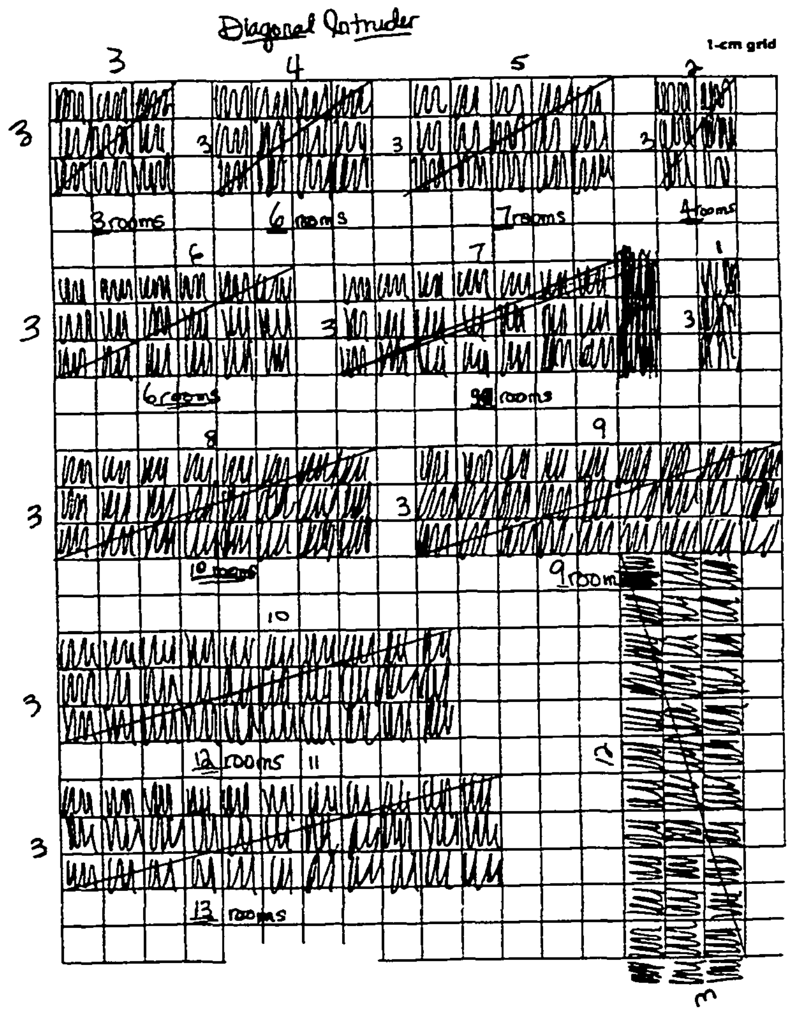

Figure 5-6. Roberta's rectangles

[4] This pair had less experience in the program (and with this type of inquiry) than did Dan and Kerri.

After they each filled up a sheet of graph paper with a number of cases for 3 x n rectangles, Roberta asked me what they should do next. I suggested she consider alternative ways to represent her data. We talked about this, and I offered her some suggestions. Roberta decided to construct a graph that compared the number of squares the diagonal passed through with the length of the variable side (Figure 5-7). She set up her graph (with my assistance), and then realized that she didn't have the *3 x 1* case and the *3 x 2* case. So she added them to her graph paper (see right hand corner of Figure 5-6). She, then, was able to plot the points for all of the *3 x 1* through *3 x 12* rectangles.

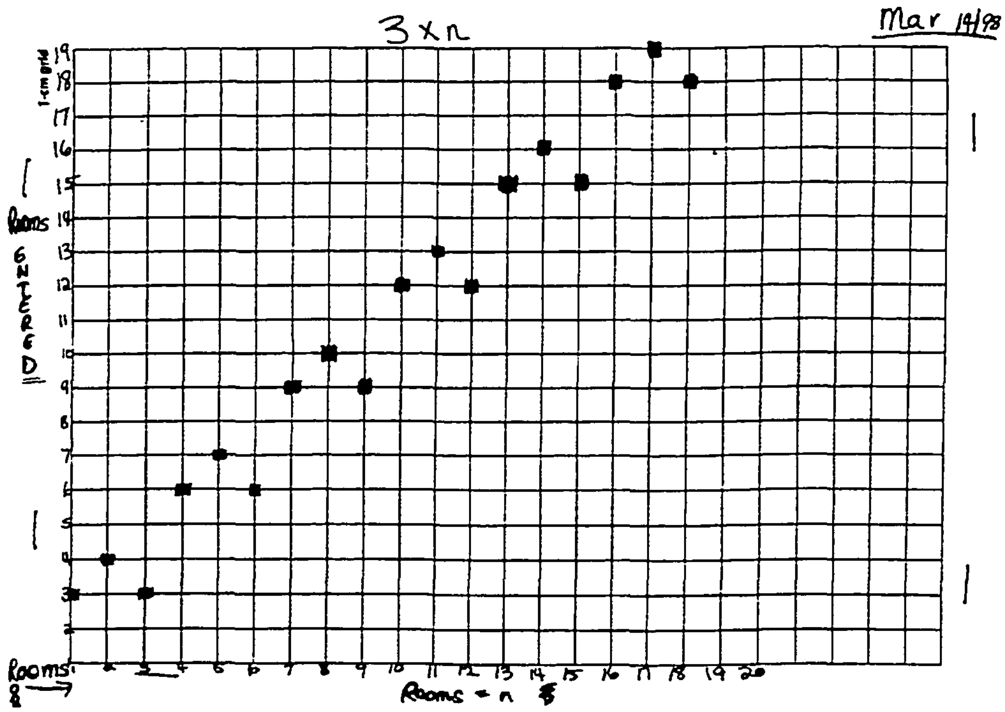

Figure 5-7. Roberta's graph of the relationship between length of variable side and number of rooms entered

While Roberta constructed the graph, Kristina continued to generate more cases of *3 x n* rectangles. Roberta called me back a while later to show me her graph. She was excited by it, and so was I. The pattern was striking. I asked her if the graph could help her predict the number of squares the diagonal would pass though for other rectangles. She confidently responded that she could and proceeded to mark points on the graph for rectangles she had not drawn by repeating the pattern she had noticed. It is interesting to note that the rectangle she drew for the *3 x 12* has the diagonal marked across only through 11 columns of unit squares. Yet, her graph correctly shows the diagonal intersecting 12 unit squares.

This series of actions and inter-actions might be illustrated as follows (Figure 5-8).

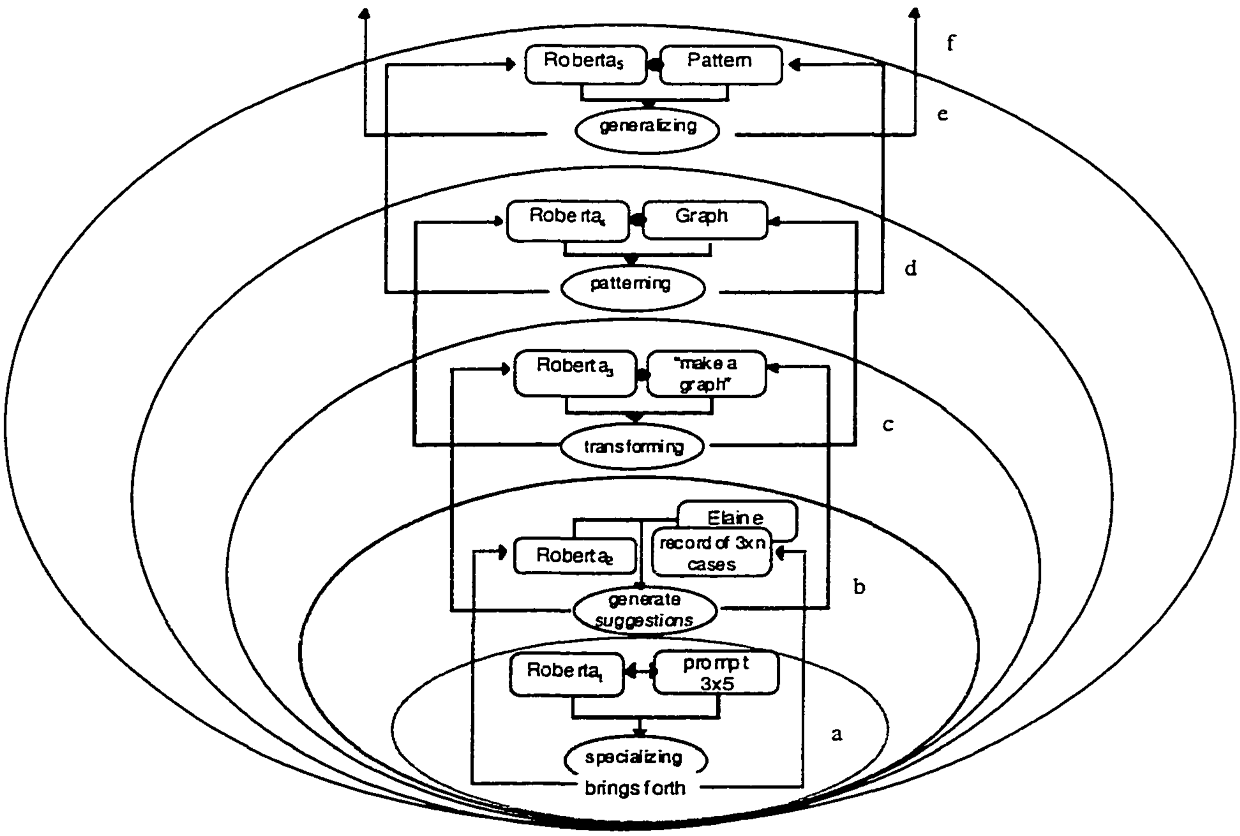

Figure 5-8. Observing Roberta's actions and interactions that brought forth her world of significance

Notice how Roberta (and Kristina) began by specializing (Figure 5-8, a); they considered a set of cases involving rectangles whose dimensions were *3 x n*. Like Dan and Kerri, Roberta and Kristina's actions were occasioned by the prompt; but, unlike Dan and Kerri, Robert and Kristina placed significance on the dimensions of the particular example offered. Their actions suggest that they defined the task to be the construction of a set of *3 x n* rectangles. Once they had completed that task, no further mathematical actions were occasioned for them (even though they each had a set of records in front of themselves) until I further interacted with them (Figure 5-8, b). Roberta asked me what should they do next. In the inter-action between us, a number of possibilities for further mathematical activity were proposed. Roberta choose, from that set of possibilities, to make a graph. In that process, she transformed her record of the number of rooms (unit squares), for each rectangle, into an ordered pair and plotted it on a coordinate system (Figure 5-8, c). (This is my explanation of what she did. I doubt she would have explained her actions this way at all.) The graph occasioned her patterning (Figure 5-8, d), from which she subsequently generalized (Figure 5-8, e) and predicted the number of squares on the diagonals of rectangles she had not yet (and would not) draw (Figure 5-8, f). It needs to be said that although Roberta could predict how many unit squares the diagonal intersected, given her actions, I do not think she considered why the pattern worked. Finally, it is worth knowing that the interactions could have been analysed by focusing on my interactions with Roberta (Figure 5-9). That is, I was occasioned to suggest that Roberta could proceed by making a table or drawing a graph.

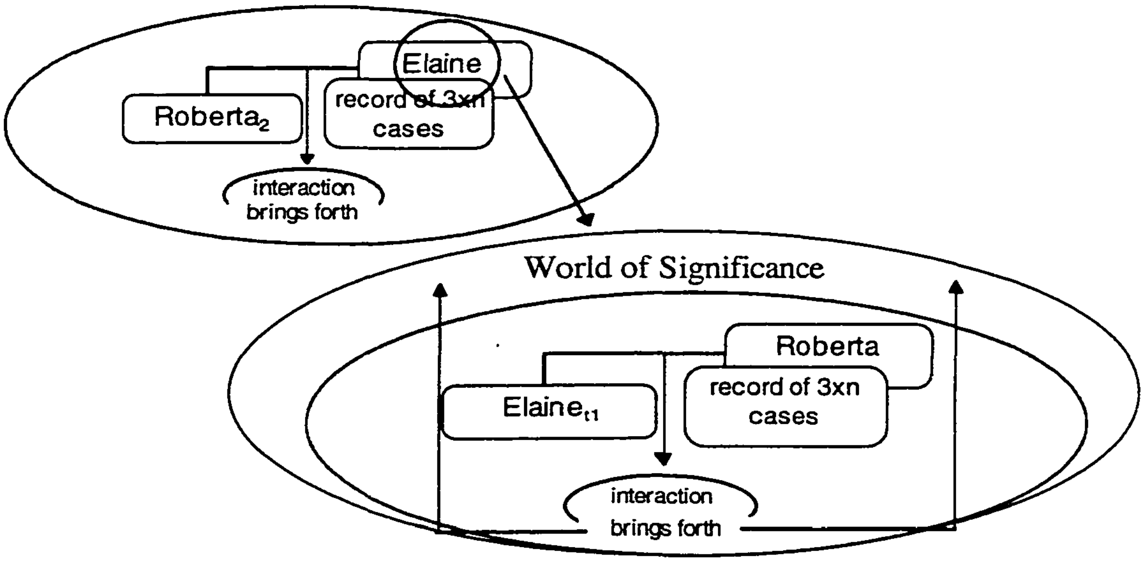

Figure 5-9. Researcher's world of significance included Roberta and the artifacts of her work

Jolene and Calvin

Whereas the prompt was the first in a series of triggers which occasioned the two pairs, we have seen so far, to look for either number patterns or a relationship between the dimensions of the rectangle and the number of unit squares the diagonal passed through, Calvin and Jolene, a mother-son pair, spent most of the evening trying to determine if, in fact, the diagonal passed through any particular unit square or through the point where four unit squares met (therefore not passing through any one of them). Their actions were constrained by Calvin's difficulty with drawing a straight line, and their need for a straight line to carry out the task. Inhibited by Calvin's fine motor skills, he and his mother, with the help of Lynn Gordon Calvert (a fellow researcher), spent most of their time that night developing a strategy to determine how they could be sure of where exactly the diagonal was to be placed. The technique they developed involved the rectangle's symmetry. They began by identifying the midpoint of the diagonal by counting unit squares both vertically and horizontally to find the centre of the rectangle. Calvin would mark that point and then they would look for the mid-point between the center and the corner of the rectangle (Figure 5-10).

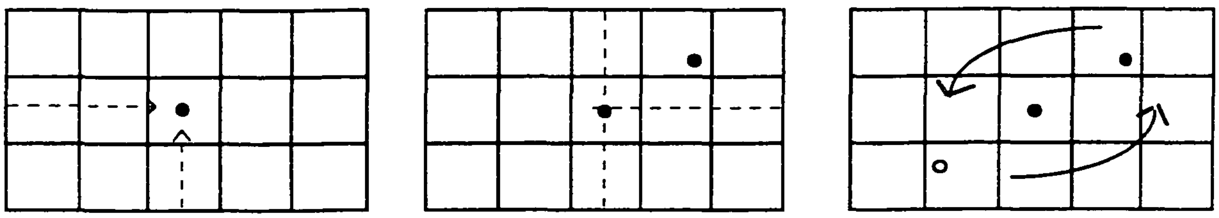

Figure 5-10. Calvin and Jolene's technique for finding the position of the diagonal

Once they had identified that point, Calvin marked it and then turned his sheet upside down to find the corresponding point on the other side of the centre point. Lynn encouraged Calvin and Jolene to use properties of the symmetry to find the corresponding point rather than turning the paper upside down and this became a key feature of the way in which they acted that night. Using symmetry Calvin and Jolene were able to determine which of the unit squares the diagonal passed through and which of those it missed. This mother and son pair never did look for the relationship between the dimensions of the rectangle and the number of unit squares the diagonal passed through. However, they did try to understand both how and why the symmetry worked, not just that it worked.

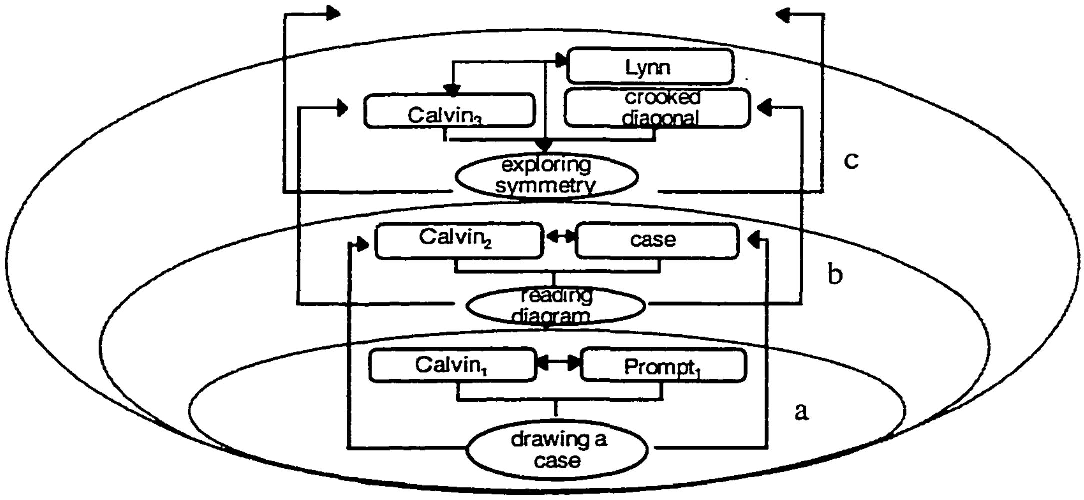

Figure 5-11. Calvin's interaction with prompt, materials and researcher

Tracing their activity we observe that, like the others, they began by drawing a specific case (Figure 5-11, a). Immediately, however, their activity was constrained by Calvin's difficulty drawing a straight line. Hence, when they tried to read the diagram (that is, count the number of unit squares that the diagonal passed through) they could not decide whether the diagonal was passing through the point between four squares or one of the squares itself (Figure 5-11, b). Lynn, occasioned by Calvin's actions, interjected with a suggestion which in turned occasioned Calvin (and Jolene) to explore the symmetry each time they considered a new example (Figure 5-11, c).

Explaining Occasioning

In each of the examples above, what could be said to occasion and sustain the mathematics activity that was observed? To what is their mathematical understanding indebted? Von Forester (1981) says that the environment is a source of energy-rich matter from which the individual can select, transform, and integrate into his or her structure. Given that all the participants worked on this prompt for more than an hour, we can assume that the environment was indeed energy-rich. However, the environment did not appear to contain the same matter for all the individuals; or if it did, that matter was taken up and transformed in very different ways. That is, each pair and in some cases each person in each pair was occasioned, by what was for them energy-rich matter, to bring forth different mathematical knowing. Each person integrated some matter into their unique structures and transformed it through their actions and inter-actions. This is knowing in action, or perceptually guided action that brings forth a world of significance which includes mathematics.

The way in which structure determinism played out, in the occasioning of mathematics knowing, was made evident to the observer by the participants' variety of actions and inter-actions given the prompt. Dan and Kerri's structures (we don't know which[5]) were such that this prompt was a problem about rectangles with common areas and involved number patterns. In contrast, for Calvin and Jolene, the mathematics was in the creation of patterns of symmetry from the geometry of the rectangles. For Dan and Kerri, patterns were created from relationships between numbers. Roberta[6] distinguished visual patterns in a graph. Although there were a range of activities in response to the prompt offered that evening, from my perspective, all of the participants' actions were mathematical and brought forth "objects" that constitute mathematics. Their mathematical knowing was occasioned, initially by the prompt and then by a variety of inter-actions throughout the session: some involving the artifacts of their thinking, others, the conversations with their partner or the facilitator (myself and fellow researcher). As an observer, I noted that each person solved problems that were created, moment by moment, in inter-action with features of the environment and their own thinking.

[5]In some of the cases, it is very difficult to distinguish who did what, Therefore, I am attributing the mathematical understanding to the pair.
[6] I am unable to say how Kristine's thinking was involved because I don't have adequate records of the interaction between Roberta and Kristine.

When I further consider the role the environment played in people's mathematical understanding, I note that the environment, or more specifically, features of it were necessary for all of the mathematical activity that occurred. Clearly, without the prompt the participants would not have thought about such a question. But, it is also clear that there were a number of interactions which people had with each other that also propelled their activity and thinking in one direction or another. One might ask, "If the prompt played such a significant role, why do I insist on denying that it caused their thinking?" My answer is that the prompt (like other events and things in the environment) triggered their thinking but, they, themselves brought forth significance out of the prompt as they interacted in this particular context. (In fact, I observed them pose and "solve" quite distinct problems and in doing so brought forth several different but potentially intersecting worlds.) Hence, I think it makes better sense to suggest that the prompt did not cause their thinking but occasioned it. The explanation of my observations is more coherent when I understand the person and the environment as co-responsible for the mathematics knowing that emerged in interaction rather than understanding the environment or the individual as being solely responsible.

As was observed in the illustrations offered in this chapter, when a person engages in mathematical activity, the world (multiverse) changes. It changes as the person brings forth his or her world of significance, and the worlds of others change because the person's world is interlaced with their worlds.

An Ecological Perspective

Throughout this chapter, I have claimed that the primary mechanism for human knowing is co-dependent interaction (structural coupling) with one's environment. The environment is not simply the place where cognition takes place nor is cognition action on the environment. Rather, my claim is that the person and the environment are, at once, brought forth in their recurrent inter-actions and bring forth those same inter-actions. This is knowing in enactivist terms.

As already alluded to, enactivism invokes an ecological logic where knower and known are seen to exist in relationship (see Davis, 1996). There are two immediate implications of this logic for my work. The first is that the enactivist perspective requires we reconsider using the person as the unit of analysis for studying mathematics knowing. Although cognition has traditionally been studied by psychologists as an individual process, enactivism suggests that cognition is better understood if we study the person-in-an-environment (see also the work of Bauersfeld and Cobb, 1995a; Lave, 1988; and Bruner, 1996; Confrey, 1995b). I propose we should consider the following sites of interaction when we study mathematics knowing in action: a person interacting with his or her own thoughts, two (or more) people interacting with each other, and a person interacting with the interactions of others. When we attempt to understand cognition, not in terms of the individual learner, but, in terms of the person-in-an-environment, we observe that any change to the person or the environment has the potential to change to the system. Hence, it is important to study mathematics knowing in ecological terms. In the next chapter, I develop an illustrative case of one parent-child pair who were also offered the diagonal intruder prompt by interpreting their knowing in these various sites of interaction.

Chapter Six

SITES OF INTERACTION AND SOURCES OF PERTURBATIONS

> Perception does not present itself in the first place as an event in the world to which the category of causality, for example, can be applied, but as a recreation or re-constitution of the world at every moment. In so far as we believe in the world's past, in the physical world, in 'stimuli', in the organism as our books depict it, it is first of all because we have present at this moment to us a perceptual field, a surface in contact with the world, a permanent rootedness in it, and because the world ceaselessly assails and beleaguers subjectivity as waves wash round a wreck on shore. All knowledge takes its place within the horizons opened up by perception. Merleau-Ponty (1962, p. 207).

A person's mathematics knowing has the potential to be occasioned, when the person is interacting with aspects/features of his or her world. As Merleau-Ponty (1962) taught us, knowing happens in action as our percepts interact with those features of the environment that they can experience. Recognizing that cognition is perceptually guided action which brings forth a world of significance, I find myself asking: *Where does one locate such action, and how does it come about? Where might I focus my observing, if I want to understand the relationship between knower and known?* The purpose of this chapter is to identify and describe the sites (situations) which co-emerge with mathematics knowing using an illustration from the parent-child mathematics program, and to investigate the relationship between the person and his or her environment to better understand the ways in which mathematics knowing comes about.

I am specifically interested in the 'micro-sites' of mathematics knowing, and how, in the local situation, mathematics knowing is triggered, propelled and sustained. This is in contrast to researchers who study situated cognition, which involves the either the micro-culture of the classroom (Bauersfeld, 1995a), or what might be thought of as the macro-culture; that is, in the context of well established communal or societal routines in which people participate such as work or school (Lave, 1988; Nunes, Schliemann and Caraher, 1993). Through multiple analyses of parent-child interaction, I have identified four categories of sites of interaction which appear to afford opportunities for mathematics knowing. They include interaction with other people, interaction with physical material, interaction with cultural artifacts and interaction with one's own thoughts. As suggested by the enactivist conception of environment, sites of interaction arise for me, the observer, as distinctions of my coherences of experience. Like knowing itself which is observed in action, sites of interaction arise and retreat in spaces created through people's knowing in action. This is a circular phenomenon, in that, knowing points to interaction which points to knowing. I, personally, see no reason to simplify the circularity or to reduce it to a linear phenomenon; instead, I see potential in acknowledging the circularity and embracing it. However, in order to discuss mathematics knowing, sites of interaction, or even the knower (as I do in the next chapter), I must jump in and begin talking about something. Hence, I do so with the proviso that these are mutually dependent or co-dependent phenomena that co-emerge in action and exist in relationship.

Observing the Sites of Mathematics Knowing in Action

Where might I fix my gaze in order to observe mathematics knowing in action? Where will I find it? Varela et al. (1991) offer some direction. They suggest that knowledge is "found in the interface between mind, society and culture rather than in one place or all of them. "*Knowledge,*" they say, "*does not exist in any one place or form but is enacted in particular situations*" (emphasis added, pg. 179). Hence, I look for mathematics knowing or mathematical behaviours in particular situations. Once I locate what I understand to be mathematics, then, I can step back and ask, "What is the nature of the interaction that brought the mathematics forth?".

Mathematics knowing is enacted in particular situations; it arises when one interacts with others, when one thinks about one's own thoughts, when one interacts with material features of the physical environment, and when one interacts with cultural artifacts which may be physical or conceptual or patterns of behaviour enacted with others. These sites are potential sources of energy-rich matter for mathematics knowing. When I speak of the learner's sphere of behavioural possibilities, it includes all of these dimensions because it is in interaction, in all of these modes, that a person enacts mathematics knowing. I must be cautious however. By distinguishing, hence separating, the sites of interaction and sources of energy-rich matter in the way that I am in the process of writing about it, I may occasion the belief that these sites are distinct from one another, are hierarchical or occur in a linear way. I do not wish to suggest this. In fact, my observations lead me to understand the sites of interaction as fractal spaces in which mathematics knowing at any particular moment involves interaction in all of these spaces at once.[1] The distinctions I make are made for explanatory purposes; it is simply too difficult to speak of all of the dimensions of mathematics knowing at once. When I distinguish a particular form of interaction, I must keep in mind that the interactions in the other dimensions are co-implicated (complicit) in the person's knowing at that moment, even though I am not articulating that complicity.

Sites are identified, when I observe some expression or behaviour that I understand as mathematics knowing. In other words, when I distinguish a mathematical action, I can identify the interactive space with which it co-emerged. These interactive spaces both orient the knower and are orientations of the knower. Hence, when I observe a particular action it is often just as sensible to distinguish it as a site of interaction as it is to distinguish the interaction that brings forth the site.

Social interaction

The mathematics knowing and mathematical activity of parents and children in my research always involved the parents and children interacting with each other; that is, they shared tasks, talked to each other, asked questions, posed conjectures and expressed their

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[1] I will elaborate on this further in chapter eight.

thinking for the other. I refer to this form of interaction among people, in language and other purposeful communicative action, as social interaction, and I claim it is a site of and for mathematics knowing (Figure 6-1).

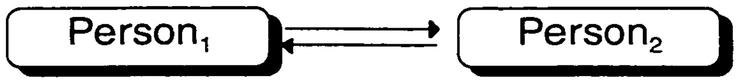

Figure 6-1. Interactions with another person a site of mathematics knowing

Social interaction includes body language and verbal utterances; anything that coordinates people's actions. Therefore, in my study of mathematics knowing in action, social interaction is not restricted to the verbal domain; however, it is primary. Of course, social interaction happens in a physical domain but I allow physical features of the verbal interaction to slip into the background (but I do not want to dismiss them). Any interaction distinguished as social necessarily involves a physical component (sound, image, movement, etc.); but by distinguishing it as social, I mean to highlight the significance of the interaction that keeps the people in relationship with each other, and how this plays a role in the people's mathematics knowing in action.

Clearly, the parent-child mathematics program was designed to take advantage of social interaction. As was intended, children and parents did mathematics together; there was also some interaction among the pairs (for example, when I asked them to put their results up on the board). There was also the interaction between the participants and myself which can be classified as social.

The Interactions of Others

A second source of energy-rich matter, for a person, is the interaction among other people. There are times when, although I am not interacting with another to maintain our relationship, his or her actions and interactions are realized in my environment, and subsequently occasion my knowing. Consider overhearing a conversation or reading a book. I do not want to call these interactions social, since I am keeping that term to refer to interaction in which two, or more, people intend to offer and respond to each other in a particular situation and maintain a relationship. Yet, the interactions of others, whether they are in the moment, as they would be in a classroom discussion, or they are artifacts of past interactions of others, such as a book, a worksheet or a piece of someone's work, are all forms of interaction which have the potential to occasion the mathematics knowing of a third person.

When a person overhears a conversation or a snippet of a conversation among others, this person is not in social interaction with the people he or she overhears in the same way that he or she is interaction with others when conversing with them. Further, a person reading a book is not in interaction with the author, as he or she would be if talking to the author directly. Neither is that person in interaction with the physical environment in the way that he or she might be when manipulating physical objects, like dominoes or algebra tiles. Yet over hearing a conversation, reading a book or studying the artifacts of someone else's work all have the potential to occasion a person's knowing. I diagram these forms of interaction in the following ways (Figure 6-2).

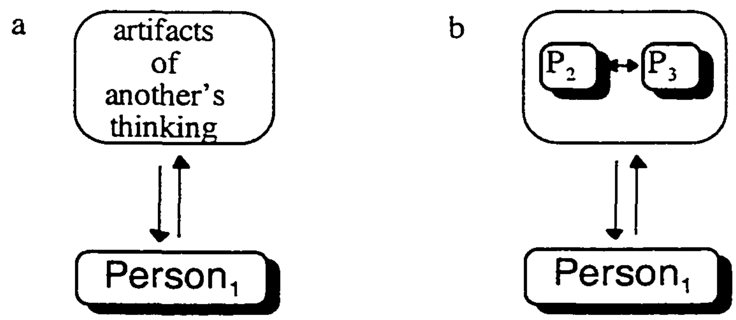

Figure 6-2. The interactions of others is a source of energy-rich matter whether that interaction is with a) the artifacts of someone's thinking or b) the interactions of others themselves

The interactions of others is an important site for mathematics knowing. We know, from experience, the possibility of learning from reading a book or studying a diagram or someone else's proof without that person present. Although the mathematics program for parents and children did not build on the interactions of others as much as it could have, this site of interaction was evident in the parents' and children's knowing in action.

Physical materials

The fact that people can act with and on physical materials in their environments, as well as produce physical materials with their actions, is obvious. Although it is easy to observe the objects students manipulate with their bodies, it is not so easy to observe how the manipulation is a co-ordination of perception, movement, and thought and is itself a mathematical act and mathematics knowing. Further, it is easy to forget that physical features of the environment are just like all other features of the environment in that they do not exist for all people. Rather, the physical environment is understood in relationship to a particular person and develops as it is brought forth in action.

The physical environment, then, becomes a third site of interaction with which mathematics knowing co-emerges. A person's mathematics knowing can be observed to be occasioned by features and/or objects in his or her physical environment. Thus, I reason that the physical environment is a source of energy-rich matter from which the person selects, and this energy-rich matter is transformed (through the creation of signs and tools for example) and integrated into the person's structure. My understanding of the physical environment as site of interaction and source of energy-rich matter is represented with the diagram below (Figure 6-3).

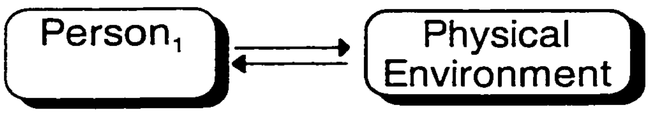

Figure 6-3. Physical materials are a source of energy-rich matter

One's own thoughts

It is obvious, just by reflecting on our own thinking, that things can "come to mind" or we can "call things to mind" without their immediate presence. Think about the internal conversations we have with ourselves, remembering something or the flashes of insight that come to us seemingly from nowhere. Because we are able to think about our own thinking, one site of interaction is our own thoughts. [2] This is not to say that one's own thoughts are separate from one's interactions with others, physical materials, and cultural artifacts. Rather, it is to say that in the moment (because indeed one has had other interactions) one can think about "something" quite independently of an interaction with something or someone outside of one's self. I illustrate a person acting on his or her thoughts in this way (Figure 6-4).

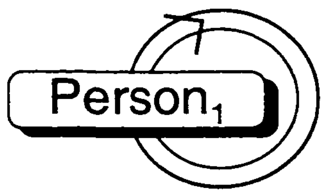

Figure 6-4. Person's knowing can be occasioned by his or her own thoughts

I have found this to be the most difficult of the sites of interaction to observe in others. Yet it seems obvious when I observe myself. There are multiple reasons for this. The first reason is that when one's thinking is occasioned and propelled by one's own thoughts there may not be any external behaviours that can be observed. Secondly, one's thoughts may be triggered by the unconscious (Hadamard, 1945/1954) and the interaction is not observable, even to the knower him or herself. The third reason is that all thinking constitutes what we observe as one's own thoughts, regardless of what occasions the thought. Hence, it is difficult to isolate fragments of a person's knowing and say this is precisely when the person is interacting with his or her own thoughts.

Human beings exist in social and cultural relations; thus, even when we observe a person thinking about his or her own thoughts, in what appears to be isolation from others, the social and cultural dimensions of his or her knowing is called into the moment. One's

[2] Von Foerster (1981) reminds us that there are more neurons in contact with each other than there are

history of social and cultural relations are co-responsible for the person's structure, at the moment he or she is observed. This is the nature of structure determined living beings.

When I point to an observation and suggest that this is where I see a person's knowing occasioned by his or her own thoughts, I am suggesting that there are no other physical, social or cultural interactions observed to have occasioned the thought. On this basis, I infer that the expressed thought was triggered by the person thinking about his or her own thinking.

Context

As human beings, we observe ourselves and others, and in doing so we make distinctions that our thinking is: personal—"It's just the way I think about it;" physical—"I found the pattern in the numbers;" social—"I got that idea from Tom;" or cultural—"That is a problem about relatively prime numbers." The distinctions that I have made about these sites of interaction, or that which triggers an idea, occur not in first order experience (the interaction itself) but in the ordering of my experiences through languaging. I propose that the distinctions I have made can be used for observing mathematics knowing in action because they point to where we can look to find mathematics knowing in action. In this way, the "sites of interaction" can be viewed as a model for observing mathematics knowing in action (Figure 6-5).

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sensory neurons connecting us with our environment.

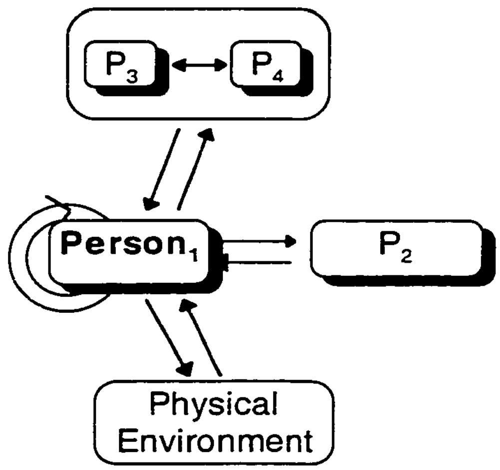


Figure 6-5. A model for observing interaction in context

In the following section, I offer illustrations of these four sites of interaction. It is important to note that usually when observing mathematics knowing in action we do not observe a person acting in the physical domain at one moment, then, in the cultural at the next. Human knowing is a complex phenomenon which happens all at once. We might observe these sites as braided together, in that complexity which is commonly called *context.* Context is very similar to the notions of environment and world. However, in the third chapter where I discussed environment, I did not suggest that one's own thoughts constitute a feature of one's environment. Here, I explicitly include one's own thoughts and suggest that, in as much as they are something with which one can interact, they can occasion mathematics knowing. Hence, they are part of the context.

Context, then, is co-emergent with human knowing. Both are manifested in action, in the temporal now and are historical phenomena. Context co-dependently arises with the human knower through a series of recurrent interactions in an environment. Thus, when a person interacts in an environment and with his or her own thoughts, the person himself or herself has a history which makes possible those thoughts. This is true of all interactions. Whenever a person interacts with the objects and features of his or her environment, the person brings into the moment of interaction a lived history of interactions, and context arises as a complex of the person-in-an-environment.

When two or more people interact, they bring their lived histories of interacting into the moment and site of interaction and context arises. Context both makes possible cultural knowledge and includes culture. Without context, new members of the social group would not have an interactive space in which the cultural knowledge is present in the form of energy-rich matter from which he or she could select. In other words, culture is made possible by interacting with others who interact and have interacted with others. The interactive space is precisely where mathematics knowledge exists. Mathematics knowledge is as Davis (1996) describes, "our established and mutable patterns of acting.... Our mathematical knowledge... is neither 'out there' nor 'in here,' but exists and consists of our acting" (pg. 79).

Interpreting and Illustrating Sites of Interaction

In the next section, I offer the case of Jake and Cathy as a means of illustrating the various sites of interaction. I discuss how this father and his daughter's mathematical actions are observed to co-emerge with the personal, physical, cultural and social interactions in which they participate.

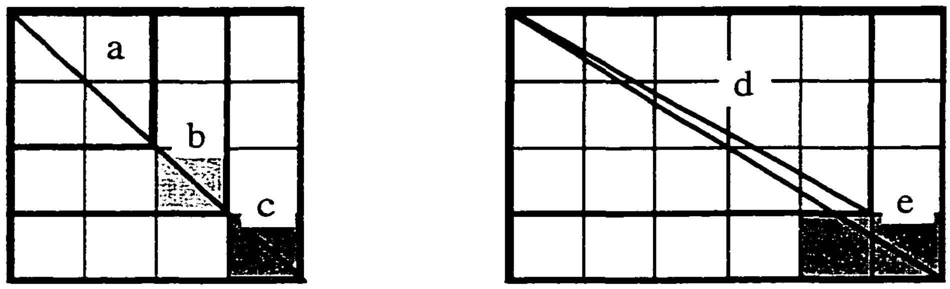
Using artifacts from the activity of a father-daughter pair who considered the diagonal intruder prompt introduced in the last chapter, I elaborate on the sites of their interaction. In order to portray the complexity of their activity, I offer a narrative constructed with a lengthy transcript and a commentary based on my observations and interpretations of this pair's activity. For organizational purposes, I have broken the relevant pieces of the transcript into chunks and have interspersed my comments.

Jake and Cathy

Jake and his daughter, Cathy (whom was first introduced in chapter two), were part of the group described in the last chapter who responded to the diagonal intruder prompt (see Chapter Five). Jake and Cathy began by working together using a single sheet of paper. Cathy acted as the scribe. However, differences in the way they desired to proceed, and the offer of a second sheet of paper had the effect of their joint activity falling apart only ten minutes into the session. For the rest of the session, they worked separately but interacted to check on what the other was doing or to share something that one of them had noticed.

Transcript 6-1. Jake and Cathy respond to diagonal intruder prompt

For the first ten or so minutes that they worked together, Cathy drew the rectangles on a sheet of paper and Jake instructed her and kept track of what she was doing.	These actions are social. Working together, Jake offered directions and Cathy responded with actions that resulted in physical artifacts of their acting and interacting.
You've got to draw your diagonal yet. Corner to corner. 'Kay," Jake directed Cathy. *"Everyone—" Cathy did not complete her sentence.*	
"There's two. It should go right through the middle. 'Kay. If this was drawn properly, it would go right through the middle. Like this," Jake indicated. "Okay?"[see Figure 6-6, a]	Jake's comment to Cathy (social) appears to have been occasioned by what were discrepancies, for him, in the material artifacts Cathy had created. Her diagonal line was not straight. The notion of 'straight' can be observed as a cultural distinction.



2x2, 3x3 and 4x4 squares

3x5 and 4x 6 recatangles

Figure 6-6. Reconstruction Cathy and Jake's first records

"Okay," Cathy responded.	
"So, it would be two. Okay, this was two by two. Let's go to three by three." Jake directed.	Social interaction is maintained as Jake continued to direct Cathy.
"Here you can use another space," Cathy said.	Cathy acted within the affordances of the physical materials.
"Three by three," they said in unison.	
"It will go [through] three," Jake noted. [Figure 6-6, b]	Jake commented on his observation of the physical properties of the diagram, which was drawn on the page in front of him. His utterance seems be more for himself than for Cathy.
"Okay."	
"—equals four by four. You are through the middle, so you can only go 1,2,3,4." [Figure 6-6, c]	Jake continued to talk out loud, seemingly for his own thinking; but the utterances are part of the physical space in which Cathy is also interacting.
"How many—" Cathy started to say.	Cathy began to interject but she was interrupted by her dad.
"Just a minute. Just a minute. You go two by two, you go through two. Three by three is three. Four by four is four, 'cause you just have	Jake articulated his personal reasoning out loud. Once he understood what was

to go the middle one, cross diagonally each one. Okay?"	happening he turned his actions back toward Cathy.

Jake and Cathy began by specializing, as is evident by their working papers (Figure 6-6). They first considered 3 examples of perfect squares. Although Cathy had trouble drawing a straight line, this did not seem to be problematic for them since Jake was able to deduce where the diagonal should go if it were straight. As seen in the illustrative case involving Cathy and her mother in chapter three, Cathy had a tendency to work with geometric images. Her acute visualization skills meant that she did not need to be convinced by a very accurate drawing to know where the diagonal passed; like her dad, she could imagine it. Cathy and Jake generated only three examples of squares before their activities were diverted away from the special case of squares; however, they did appear to have generalized from those cases (Transcript 6-1, lines 18, 25-28).

Transcript 6-2. Tension arises between father and daughter

"Can I just show you something?" Cathy asked as she began to draw another rectangle on the graph paper. [Figure 6-6, d]	Cathy acted in the social space.
"Yeah. Let's not do one too big," Jake said. "Let's do this step-by-step. Okay?" Again the line Cathy drew for the diagonal wasn't very straight.	Jake and Cathy were in social interaction and with the physical space. But this context also includes the cultural. I can interpret Jake's 'step-by-step' comment as pointing to a systematicity that is highly valued in mathematics.
"That doesn't work. Let's use a straight line." Jake commented. He redrew the diagonal on the three by five rectangle Cathy had constructed.	
When he finished Cathy counted under her breath, "—Seven."	Cathy acted on the physical artifacts their thinking.
"That's ... a rectangle," Jake commented. "That's two, four, five by three."	
Cathy mumbled.	
"Okay, this is [five] by three. You want one,	Jake addressed Cathy.

two, three, four, five, six? Okay."	
Cathy began to draw a bigger rectangle on another part of the paper.	Cathy acted in the physical space.
"Cathy. Just wait a minute. Cathy. Cathy." Jake said trying to stop her from continuing. "Don't go all over the place. What we want is to go one over the top of the other one and just keep expanding and it'll still be the same thing, because otherwise we will not have enough room. Okay. What do you have here?" [Figure 6-6, e]	Jake's actions were occasioned by his observation of Cathy's actions.
Cathy was not pleased to be drawing the four by six rectangle on top of the three by five. On the audio tape she can be heard to whisper, "This can be our ugly sheet, an ugly sheet, our rough copy or something." She took a fresh sheet of paper and started to draw a rectangle. "This isn't as complicating."	Cathy's actions here are complex. In response to her dad's request, Cathy erased the rectangle; but under her breath she expressed her dissatisfaction with the quality of the records that she was producing. On one hand, we might observe her utterances about the records to be personal, or for herself. On the other, hand she uttered this loud enough to be heard; hence, her comments had the potential to heard by her father.
"Let's count this back. One, two, three, four, five, six, seven, eight. Three by six." Jake noticed Cathy drawing on a new sheet of paper. "Don't start on that. You can still get lots on here," he said pointing to the sheet they had been using. "Okay?"	Cathy turned her actions inward (personal domain) and removed herself from the social space; however Jake kept her in the social domain by asking her to work on the common record they were keeping.

It was about ten minutes into the class when I stopped to look at the records they were keeping. I immediately noticed that the rectangles they had drawn were superimposed on one another. Unaware of the conversation Cathy had been having with her dad about saving space on the graph paper, I gave her another sheet of paper on which to write. She happily accepted it. Once Jake and Cathy each had their own sheet of paper, they no longer worked together in the same way;[3] instead, they each carried out their own inquiry, and referred to each other's work to see what the other was doing or to compare results.

As the transcript of their conversation indicates, Cathy likes to keep neat and orderly records (Transcript 6-2, lines 55 - 58). Once offered her own sheet of paper, she immediately wanted to recopy all of the work she had done and throw out the "ugly" copies. I suggested she not do that but, instead, continue on from where she was.[4] She took the new sheet and carefully drew the *4 x n* rectangles, each on their own part of the paper. Then, she noted the size of each rectangle and the number of squares the diagonal passed through just under the corresponding diagram (Figure 6-7). In her words, this was not nearly as "complicating" (Transcript 6-2, line 61).

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[3] This is something that we have seen many times in our research. Often when people have only one sheet of paper between them, they work together but the same people when given two sheets of paper work in alone or in partial isolation of the other.
[4] Interestingly, she did throw the messy working papers out that night in spite of my suggestion. Thus, I do not have those records.

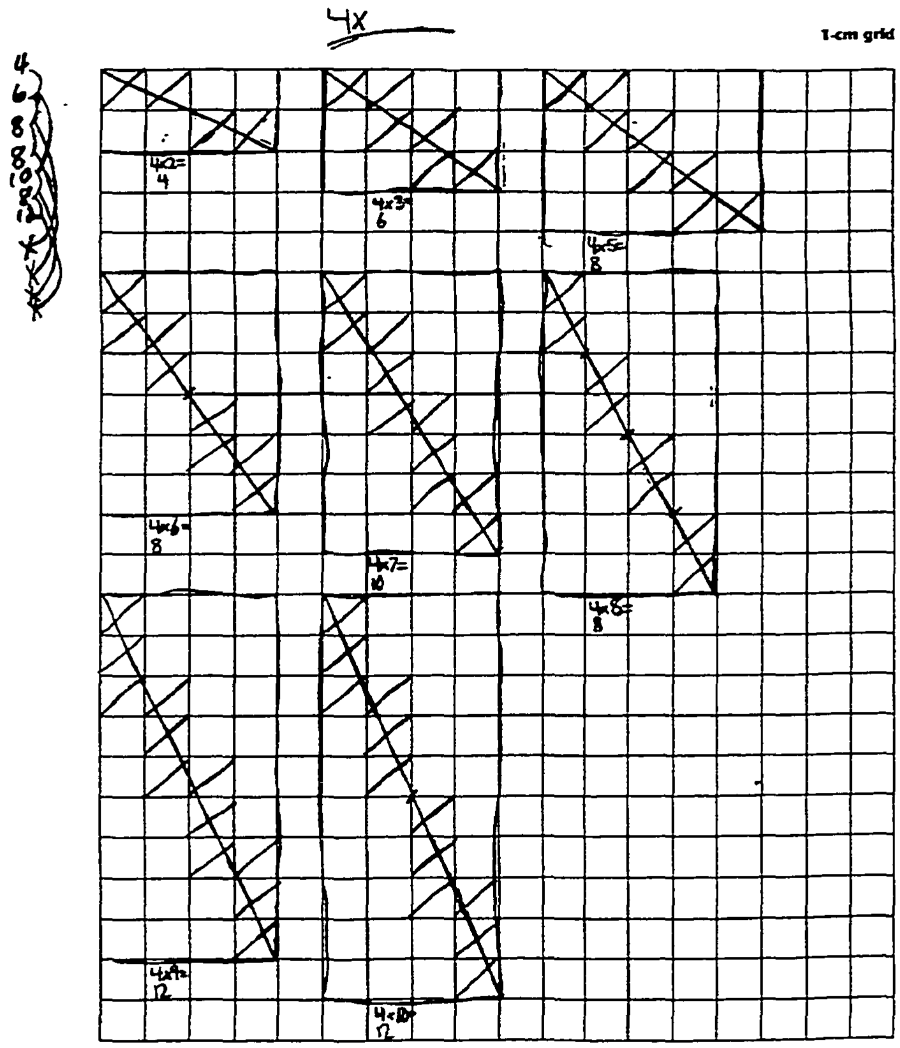

Figure 6-7. Cathy's working paper

Cathy's emotions (likes, dislikes and frustrations) played an important role in her actions (and hence her knowing) that night. Not only were they part of the reason she and her father worked separately, but her emotions were fully implicated in her mathematics knowing. Her desire for neatness, for example, was manifested in her mathematics knowing. When her working papers were messy, she found them complicating. Hence, she simply did not draw rectangles and count squares but she drew them neatly and erased any, of what she viewed, as extraneous marks. She kept a track of the number of squares the diagonal passed through, under each rectangle, and refused to make a table of them on a sheet of scrap of paper (Transcript 6-2, lines 50 - 55). Working on her own meant, she was able to keep her records the way she liked them, and in a way that made sense to her.

Transcript 6-3. Prompting Cathy to look for a pattern in the numbers she was generating

"If you just wrote those numbers down [squares on diagonal]. Just write them down here," I pointed to the side of Cathy's sheet. She hesitated. "You don't like doing that," I acknowledged. "Just write them on this piece of paper," I said as I offered her a scrap.	I entered into social interaction with Cathy as I tried to convince her to keep a table.
"Hm," she uttered as she wrote the numbers down in a list on the side of her sheet (Figure 6-7).	
"Four, six, eight, eight, ten, eight. Is that what they are?" I asked. "You and your dad are getting different answers," I said as I looked over his records. "Oh. He didn't do a four by two."	I continued to interact with Cathy but drew her attention to her father by observing what he had been doing. In this way I interacted with the artifacts of his actions.
"He didn't do a four by two? But I bet he did a two by four," Cathy said looking over at his sheet. "Not that either!"	Cathy too interacted with the artifacts of his actions.
"Oh, I did it. But I just looked at it, so I could see it. Four by two? Cathy, I got one, two, three, four," he said as he counted.	Jake interacted with Cathy and me in the social domain which then included the three of us.
"The answer is four, six, eight. It is like you are counting by twos. But it is four, six, eight. Two, four, six, eight," she skip counted. "Four, six, eight."	Cathy was trying to reason about the numbers she had just written down.
"But then look. Four, six, eight, eight," I responded.	I interacted with her and her artifacts and noted my observation.
"Yeah," Cathy knew her pattern did not continue. "But in the beginning."	Cathy expressed her personal thought in the social domain.
"Yeah, in the beginning." I agreed.	
"Four, six, eight, eight, ten, eight," Cathy read the numbers out loud.	The utterance put forth her thinking into the physical domain; hence, it had the potential to occasion an other's actions.
"Okay, how about a four by nine?" I prompted her to generate some more cases. "Do you have enough room?"	In spite of where her thoughts were directed or to whom they were intended, I responded to them with a suggestion she do

		another example.
100	*Cathy went to erase the table of numbers she had just made.*	Cathy put order to the physical space.
	"Keep your table there. That's a good thing to have," I told her.	Social interaction was sustained as I directed her to keep her table.

When I saw the way in which Cathy was keeping track of the numbers, I worried that she would find it difficult to find a relationship between the dimensions of the rectangle and the number of unit squares the diagonal passed through. (I did not notice that her diagrams were pointing to other possible relationships.) So, I encouraged her to work with the number of unit squares the diagonals passed through, rather than the ratio of the sides, for example. That record is on the side of her sheet (Figure 6-7). As was the case when she and her mother were working on the tiling prompt (Chapter Three), Cathy only made a table after I requested she do so. It was just as unlikely for Cathy to construct a table as it was likely for Dan and Kerri to begin with one.

Jake, on the other hand, had been looking for a number pattern all along. He worked at the task quickly and intensely. His working papers (Figure 6-8) demonstrate how systematically he worked. Notice, however, that as orderly as he was, he was not bothered by turning his paper upside down to make more space. Furthermore, his records indicate to me that he imagined the sides of the rectangles since he did not draw them. Obviously, this method was not complicated for him like it was for Cathy.

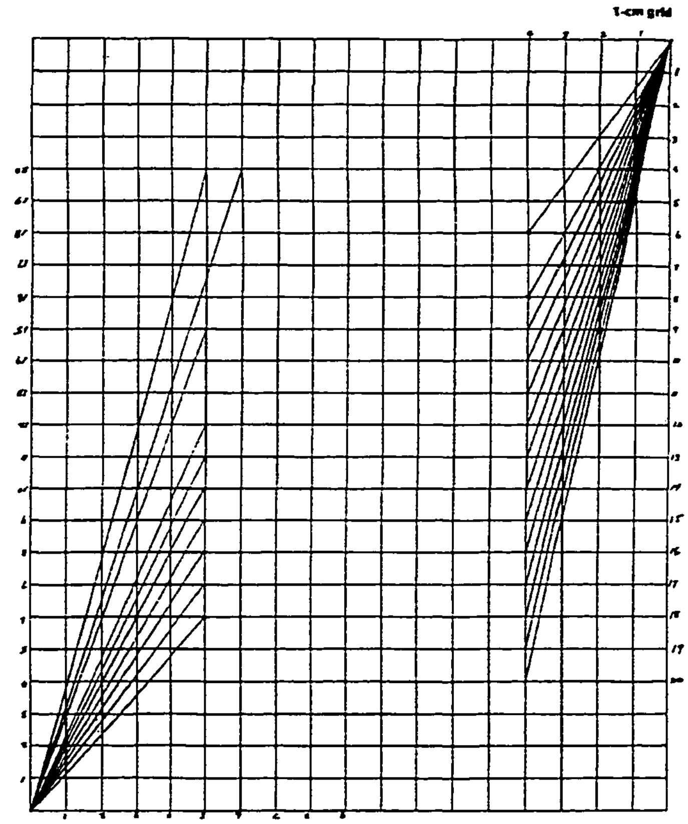

Figure 6-8. Jake's record of the rectangles he considered

I might interpret Jake's actions as simply trying to conserve paper but this was likely only part of the reason he worked the way he did. The rectangles he drew were (primarily) a preliminary step towards resolving the problem that he was investigating. Although he drew the diagonals for many rectangles (Figure 6-8), his records suggest, to me, that his pattern noticing was based on the tables he was generating on a separate sheet of paper (Figure 6-9) not on his drawings. (This is quite a contrast to the attention his wife and daughter paid to the geometry of the tiling situation they investigated—see chapter three). At least in the beginning, Jake's drawings were not the focus of his search for pattern, his number tables were. Later in the session, however, Jake did refer back to his diagrams as he attempted to relate the generalizations he made about the numbers in the table to the geometry of the situation. It was then that his diagrams became significant again.

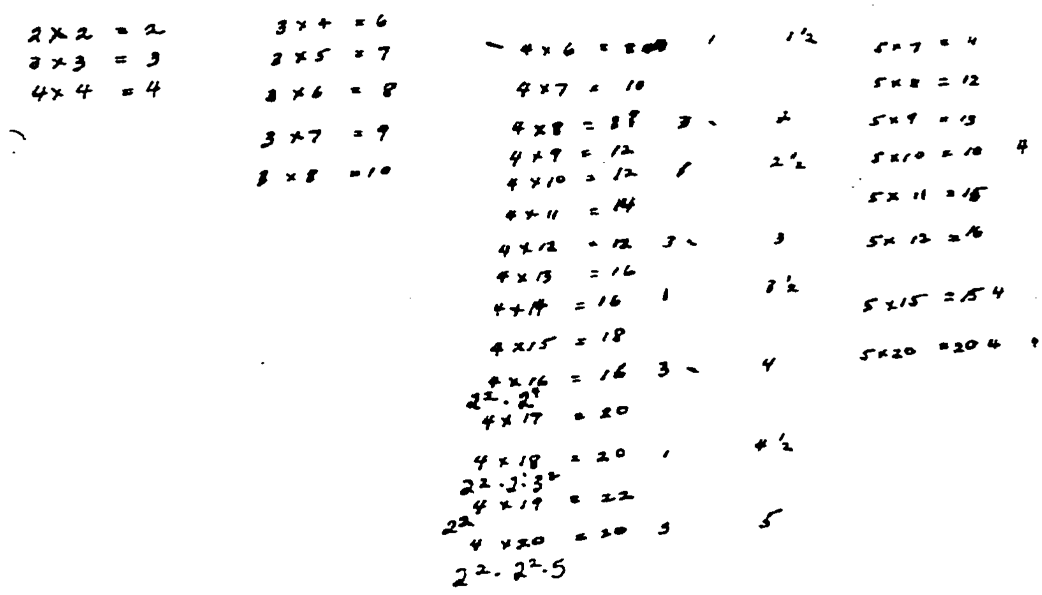

Figure 6-9. The records of the tables Jake used to understand the number of unit squares a diagonal passed through for each distinct rectangle

Over the course of the evening, Jake generated many instances as he studied the results for the *1 x n* cases, then the *2 x n* cases, then *3 x n* and so on. He would generate examples just until he could formulate a generalization based on some pattern he noticed. For example, after just three cases he noted that for the *n* x *n* rectangles the diagonal crossed through *n* squares and went on to another type of rectangle. Working much more quickly than his daughter, he found many patterns, and realized that rectangles of different *k* x *n* (where *k* is constant) generated different patterns, and that these patterns were related if *k* and *n* had a common factor. It also is interesting to note that, although he used conventional notation for his records, he did not use the notation conventionally. His table looks like a list of multiplication equations; yet, these are not products. The way he used the notation would make it very difficult, if not impossible, for a person who did not understand the context of Jake's activity to know how *2 x 2 = 2* or *3 x 4 = 6*. On the other hand, his notation was clear to me, as I had been interacting with him and the artifacts of his thinking. His artifacts were part of my context; hence, they had meaning for me.

Transcript 6-4. Jake formulates a rule

"This works the same as the other one: four plus eleven minus one. I go back to this pattern." *"So, the fifteen you are going to throw away because it just follows the one pattern?" I asked.* *"Well—"* *"Multiply and subtract one?" I mixed up his pattern. It was to add the dimensions and then subtract one.* *"Well, I am going to do it. But you have to add." Jake corrected me.*	Jake offered his explanation to me. This was within the social space.

Cathy was unable to keep up her father's pace but she was influenced, at least in part, by his work. Figure 6-10 illustrates the way in which Cathy tried to made sense of the numbers she was generating from the rectangle pictures. Although I had instructed her to make a table, neither I nor Jake told her to compare the numbers by adding or subtracting—not directly in any case. However, Cathy was present when Jake explained, to me, the pattern he found for rectangles whose dimensions were relatively prime (Transcript 6-4). His rule was to add the dimensions of the width and the length and then subtract one. It was immediately after Jake explained to me what he had found that Cathy further reflected on her numbers by constructing a second chart (Figure 6-10) in which she added the consecutive pairs of the numbers from her first chart (Figure 6-9).

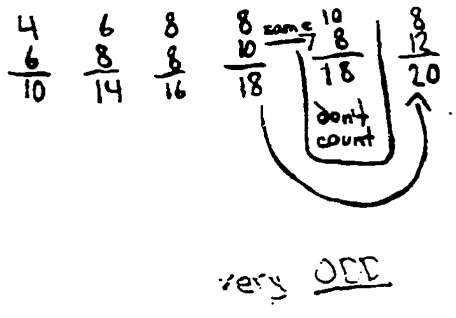

Figure 6-10. Cathy's record of her search for a number pattern

Her chart suggests to me that her focus shifted from the number of unit squares on the diagonals of the rectangles to a pattern in the numbers themselves. Her manipulation of the numbers and her comment "don't count" are nice examples of how a person orders the world for oneself. Her comment "very ODD" both uttered and written suggests to me that her emotions are implicated in her act of ordering. An exchange between Cathy and her dad nicely demonstrates the way in which her patterning developed, and how her mathematics knowing was laden with emotion.

Transcript 6-5. Cathy showing her work to her father

"I think I just figured something out," Cathy said partly to me but obviously intended for her father who had been working on his own.	Cathy offered her observation in the social space.
"Dad. Do you see a pattern in here?" She asked as she showed him her list (Figure 6-10).	When her dad did not respond, she specifically asked him to interact with her.
"It goes up by two every time," Jake offered.	Jake read from the artifacts of Cathy's thinking and expressed his knowing to her. In doing so he maintained their social interaction.
"Well, ten and eight is eighteen anyway. Next one just goes up by ten, eight— eighteen," she said. "And then I will do eight and twelve. Eight and twelve. Twelve—" Cathy counted from twelve up to twenty to herself. "Okay dad don't count that one," she said as she pointed to the second set of ten and eight. "See if there is a pattern in these."	
"Yeah," Jake replied after glancing at her sheet.	
"Cool! I just figured something out," Cathy said excitedly.	
"Well, you keep going like that because that is very good," Jake commented.	
Cathy continued to connect the numbers on her list with lines (Figure 6-10). "And then eight and eight is sixteen. Okay, and six and eight is something that I haven't got to yet. And then eight and eight is something I haven't got to yet and ten and eight is something that I haven't got. Eight and twelve is something I haven't got to yet." She paused as if in thought. "Very odd," she said as began to work quietly on something. "—two, four, six, eight, ten. It works! It actually works! Six, seven, eight, nine, ten. I'm right. Yes it works! There is a pattern in here. That solves my problem to all the fours. And then it would be equal to twenty-four. Cool. And what	Cathy worked with her own (personal) thoughts and the (physical) records of her activity.

ever this one is, it would equal to twenty-four, which is how many blocks?" She paused and then commented, "eight, six, ten. I can't figure out how this works."	

Sites of interaction and sources of perturbations

With this case, I have illustrated how Cathy and Jake were occasioned by the prompt, my comments and each other to act mathematically. Also illustrated was how, at the same time, their mathematical actions were very much influenced by their personal dispositions and histories of experiences both within the domain of mathematics and outside of it—in other words, by their structures. From an enactivist perspective, it is assumed that Cathy and Jake's distinct histories of interaction with mathematics (and in general) meant that they would act differently in the circumstances. However, because they worked closely together and were able to see and hear what the other was doing their work shared many similarities. Notably, their working papers and actions looked more like each others' than those of the other parents and children in the room. Like the others, they specialized by taking particular cases. Because they did this systematically, and because they reflected on their own thinking, they were able to make some generalizations about the relationship between the dimensions of a rectangle and the number of unit squares its diagonal would pass through. Jake sought a generalization from the beginning. However, Cathy was prompted to do so only after certain interactions with me and her father; first when I suggested she make a table and then when she "noticed" her father's records and overheard him explain his thinking to me. She did not "copy" her father's thinking but was occasioned by it and the artifacts of it.

It is evident, from the examples provided throughout the transcript, that social interaction was a significant site of interaction and, hence, source of perturbations for Jake and Cathy as mathematics knowers, in spite of the fact that they worked somewhat independently of each other. In the transcript, there are many instances of how the social domain was a site for interaction that occasioned their mathematical knowing.

A social act, as simple as my suggesting to Cathy that she record her numbers in a table (Transcript 6-3, lines 68 - 74), occasioned a predictable (student) response—she made a table. The table she constructed made possible her skip counting. (At once, she and the world were transformed by her actions.) Making, and then having, a table changed Cathy's sphere of behavioural possibilities. As simple as this seems, constructing a table was a significant action in her context because it made possible some acts of knowing and prevented others. It needs to be acknowledged that my directions prevented her from acting on the records she, on her own, had begun to generate. That is, Cathy's interest in the features of the geometry of the rectangles and their diagonals suggest that other mathematics was possible for Cathy. This situation did not need to be about the number pattern that was generated. Cathy's working papers suggest that she had begun to make distinctions about the different ways in which the diagonal crossed the unit squares creating "subrectangles". It appears that there were mathematics that went unexplored because I did not observe the distinctions Cathy was making.

In the parent-child mathematics program, social interaction sometimes involved "telling" another person what to do, as just discussed, but it also involved checking what the other was doing or explaining one's thinking for the other. Such interaction expands the person's sphere of behavioural possibilities, by transforming the context of interaction through the addition of energy-rich material. As noted in the transcript, both Cathy and Jake expressed their knowing for themselves, each other, and me in verbal utterances (as well as with markings on their working papers). For example, Jake explained to me how he could predict the number of squares on the diagonal for any rectangle whose sides are relatively prime (Transcript 6-4, lines 105 - 111). My mathematics knowing was occasioned in this interaction with Jake. I might describe our interaction as coming to a taken-as-shared understanding (Yackel, 9995) or, using the model I have developed, I could describe his and my worlds of significance intersecting (Figure 6-11). This intersection does not imply some objective "reality" but instead suggests the generation of energy-rich matter (utterances, working papers, relationships, etc.) from which we selected, based on our own structures.

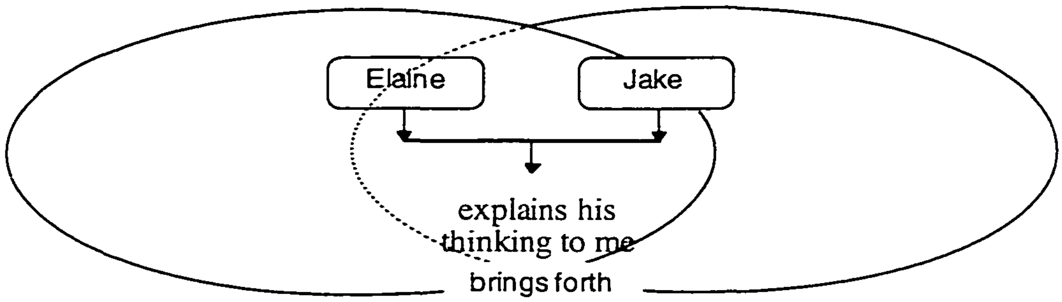

Figure 6-11. When Jake explains his thinking for me our worlds of significance have the potential to be expanded; and at the same time our worlds of significance overlap

Recall that Cathy and her father worked quite independently of each other yet Cathy's expression of her knowing about the cases she generated (Figure 6-7, Figure 6-10) bears a resemblance to Jake's. It appears that, up until the point when Cathy overheard her dad explaining his rule to me, she had not looked for a way of predicting, in advance of drawing the rectangles, how many squares the diagonal would pass through. Occasioned by the social interaction between her dad and me and the presence of her table on the side of her working paper (Figure 6-7), Cathy selected from all that was happening and transformed the energy-rich material and integrated it into her own structure, as we see in her meaning making activity (Transcript 6-5, Figure 6-12). She used the table she had already made and studied it to make some "thing" out of the numbers in it. I conjecture that overhearing her father was a significant perturbation for Cathy. However, without the table she had constructed at my request, it is unlikely that she would have done what she did. (As I have already noted, it appears she was on to something that I did not notice but I inadvertently blocked that activity.) She took her list of unit squares that the diagonals passed through for the *4 x n* rectangles and subtracted the results of one rectangle from the next in an attempt to see if she could predict the next unknown case. This is not exactly what Jake had done but his comment to me included the utterance, "Four plus eleven minus one. I go back to this pattern." Unlike the situation where social interaction with others triggers the mathematics knowing of those interacting together, this example seems to suggest that the work of others (their actions and interactions) is potentially energy-rich matter for a third person even when that third person is not part of the "social" interaction *per se*.

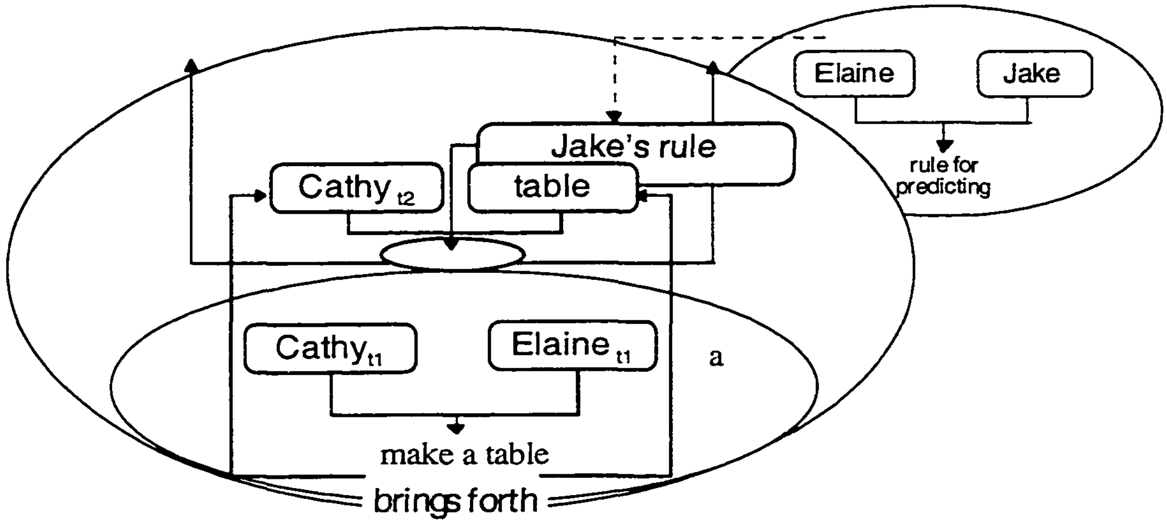

Figure 6-12. Cathy made a table at my suggestion which was present for her to act on and with when she overheard her father's rule

Cathy and Jake created and transformed the unit squares on the diagonals through a process of action, perception, representation and re-presentation. Specifically, I know Jake noticed (perceived) a pattern in the number tables he was keeping because he reported it to me. Once a pattern is re-presented in an utterance or markings on paper, the representations which become something as a result of action, at the same time, become a potential source of energy-rich matter which can be acted with and on. The physical environment is changed in the interaction and now present is a new bit of matter which can be perceived, transformed and integrated into a person's knowing.

I observed Jake acting with his table in meaning making activities. He counted the unit squares on the diagonals he constructed for each rectangle, he systematically kept records, he looked for patterns in his lists of numbers, and he articulated his generalization in the form of a rule. In this way, the record Jake produced could be understood to expand his sphere of behavioural possibilities (and had the potential to do the same for Cathy's sphere of behavioural possibilities). Without "holding in memory" all of the patterns he found, Jake was able to use the records as a memory of the patterns, and articulate the more general pattern "add the sides and subtract one". The energy-rich matter originally from the physical environment was transformed and became part of their mathematics knowing.[5]

Extending this discussion further, I note that the interaction among people provides the basis for cultural phenomena. (That is, the behaviours that persist from one generation to the next. Or, in other words, those behaviours that continue to be noted as particular members of the group are replaced.) Much of the energy-rich matter for propagating knowledge from one generation to the next depends on artifacts of the interactions of others (e.g. books, video and television programs, pictures etc.) and are brought into the sphere of possibilities for students by others (generally teachers).

The prompt, that I offered the parents and children in the large group setting, turned out to be a significant perturbation and one that implicated the cultural dimensions of knowing in a number of ways. In the first place, both the selection and the offering of the prompt were constrained by my history of experiences which included mathematics knowing and patterns of acting with students (Figure 6-13). It was an act in which the body of mathematics was "pulled through" (Gordon Calvert, 1999) me into the moment as part of my interaction with Cathy and Jake. In the second place, it was a deliberate act which, since it was taken up by the participants, set us in relation to one another

[5] Vygotsky suggests, "Even such comparatively simple operations as tying a knot or marking a stick as a reminder change the psychological structure of the memory process. They extend the operation of memory beyond the biological dimensions of the human nervous system and permit it to incorporate artificial or self-generated stimuli, which we call *signs*" (P. 39).

(teacher/student, expert/novice, etc.). I claim that, in my role as the "mathematics teacher," my act of bringing the prompt into their context and distinguishing their actions as mathematical, facilitated our collective patterns of acting—those that are understood as mathematics.

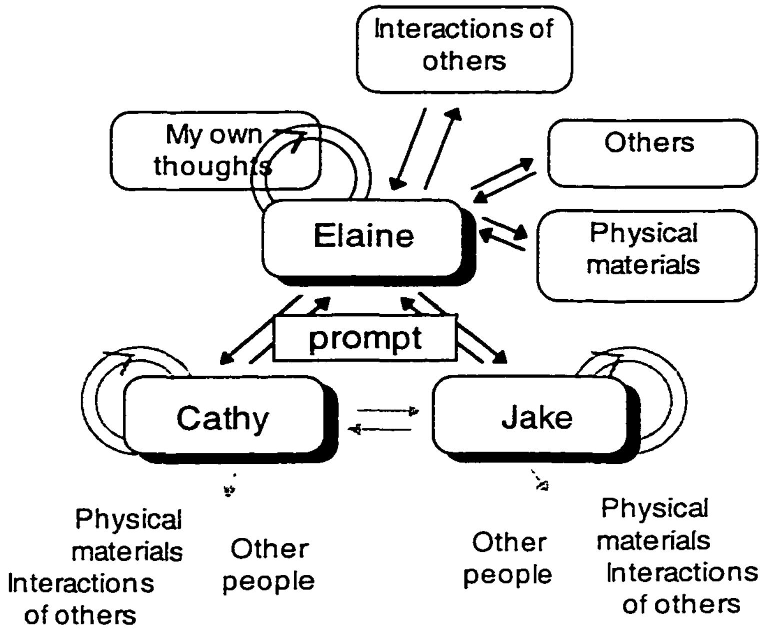

Figure 6-13. Researcher offers a prompt based on her history of experiences teaching mathematics, interacting with students, selecting prompts, etc.

However, the prompt was not a static thing; it too only existed only to the extent that it was realized and transformed in action. Returning to the transcript of Jake and Cathy's activity, one can observed how the prompt itself evolved throughout the night. Early in the evening, Cathy and Jake drew rectangles and diagonals. A little later Jake began to organize his rectangles in a very specific way (as did Cathy), and then began to keep records which compared the dimensions of the rectangle to the number of unit squares on the diagonal. Still later, he noted a pattern within the lists of the unit squares. As Jake and Cathy acted, their knowing made possible further actions and ongoing reformulation of the prompt. Although we might say that the prompt changed over the course of the evening, we could just as well say that the prompt addressed them differently throughout the evening as their knowing changed.

In the case of the parent-child mathematics program, I selected prompts for the participants, and I offered suggestions to them in terms of how they might proceed at various times in our sessions. Through my actions, the culture of mathematics (our patterns of acting) was brought into the moment as site for interaction. The diagonal intruder prompt and my suggestion to make a table are two examples of how the body of mathematics was present in the moment of interaction. If I were to ask Jake and Cathy, they might agree that in fact they were doing mathematics, since they were drawing rectangles, diagonals and counting unit squares. They would understand this as mathematics from their personal histories of interaction within a particular culture which identifies mathematics with certain practices and/or a body of knowledge. For most of us, mathematics includes rectangles, diagonals, and unit squares. But, Jake's instance on proceeding in a step-by-step manner and both his and Cathy's systematic generation of examples can also be seen as part of the culture of mathematics. Where did they learn to act in such a way except by interacting with others and with the artifacts of other's activities in spaces that we define as mathematics?

In the first chapter, I discussed the Maturanian and interactionist views that objects arise in our interactions with them. Now I elaborate. To the extent that the object which has arisen from the student's interaction is changed, it potentially changes for all of us in the community. Because human beings are historical, our actions in the present always call our interactions of the past in and to the moment. Further, because our mathematics knowing is part of the social domain, the history of interactions of the persons with whom we interact are implicated our interactions with them. Two people in conversation may not even know they have been overheard but the world is changed because their conversation was overheard by someone. An author may not know when her work is read but the world is changed by its having been read. To the extent that people live in community with others, human culture is changed as objects arise in the interaction among people that brings forth their worlds of significance.

As an enactivist perspective anticipates, the worlds of significance that are brought forth in our actions and interactions have the potential to expand and overlap with the worlds of others hence complexifying the space of the possible (Stewart and Cohen, 1994). Jake and Cathy's coordinations of actions in the social domain, which involved verbal utterances and gestures, occasioned and sustained their mathematical thinking. In this chapter and the previous one, I elaborated on enactivism, an ecological theory of cognition, by illustrating how people interact to bring forth worlds of significance which includes mathematics and how they themselves are brought forth in those worlds. Enactivism suggests that people learn (their structures' change) through perceptually guided action in language with others. I have demonstrated that there are a variety of sites for interaction: people can interact with physical materials in the environment; they can interact with their own thoughts; two or more people can interact with each other; and people can interact with the interactions of others. My conjecture, from my observations and interpretations of Jake's and Cathy's actions and interactions, is that mathematics knowing in action is at once personal, social and cultural.

Over the last two chapters, I explored the relationship between the knower and the known, and I identified the various micro-sites of interaction where I observed mathematics knowing emerged. I noted that a person's actions in the moment made possible subsequent actions by changing the person and the environment hence transforming the person's sphere of behavioural possibilities. But these observations still are not sufficient for an enactivist account of mathematics knowing because as enactivism suggests *knowing and knower* are co-emergent phenomena. In the next chapter, I turn my attention to the knower that is brought forth in mathematics, and explore the ways in which mathematics knowing is at once personal, social and cultural.

Chapter Seven

BRINGING FORTH THE KNOWER

> Our bodies are shaped by the world that they participate in shaping; they render mind-and-body, subject-and-object, individual-and-collective, mental-and-physical inseparable (Davis, 1996, p. 78).

An enactivist perspective views knower and known as phenomena which co-emerge from embodied action. In my research of mathematics knowing in action, I have distinguished a knower who co-emerges with his or her knowing, and I have explored the ways in which this is observed to happen when people interact with each other and other features of their environment. However, I have said little about the knower that co-emerges with the mathematics knowing. Davis (1996) claims that "the issue of who we are is not separate from where we are, what we are doing, who we are with and what we know" (p. 236). In this section, I turn my focus to the knower, and the implications of mathematics knowing on the body as it is understood as the node of existence not only in the physiological domain but in the human domain as well.

Merleau-Ponty's (1962) profound recognition that not only do we shape the world but the world shapes us is a key premise of an enactivist perspective. Knowledge does not simply emerge out of a person's actions or in the person's interactions but *co-emerges* with the knower him or herself who gives significance to his or her acts of knowing. An enactivist account of mathematics knowing in action prompts me to take a reflexive turn, and consider the ways in which the knower is brought forth from a world of action and interaction. In other words, I ask the question, *"In what ways is the person transformed by knowing mathematics?"*

Maturana (1998) suggests the "bodyhood" is the condition of possibility of the living system. "The manner of its constitution and continuous realization is itself continuously modulated by the flow of the living of the living system in the domain in which it operates as a totality." (Maturana, 1998, http://www.inteco.cl/articulos/ metadesign.htm). For human beings, embodiment encompasses emotions and language—or, in Maturana's terms, emotioning and languaging (c. f. Maturana, 1988). For Maturana (1988), languaging is the consensual coordination of consensual coordinations of behaviours among humans (refer to chapter three); emotions are "dynamic body dispositions for actions" which specify, moment by moment, the domains of the person's actions (p. 49). When observing for the knower that is brought forth in mathematics knowing, I attend to his or her languaging and emotioning as not only acts of knowing but manifestations of the knower him or herself.

Merleau-Ponty (1962) describes the body (the knower) as that which arises in our living—our knowing.

> [The body] is not a collection of particles, each one remaining in itself, nor yet a network of processes defined once and for all—it is not where it is, nor what it is—since we see it secreting in itself a 'significance" which comes to it from nowhere, projecting that significance upon its material surrounding, and communicating it to other embodied subjects. It has always been observed that speech or gesture transfigure the body... The fact was overlooked that, in order to express it [speech or thought], the body must in the last analysis become the thought or intention that it signifies for us. It is the body which points out, and which speaks (p. 197).

Maturana suggests that we are part of many conversations (where conversation is understood as the flow of coordinations of actions and emotions that we observers distinguish as taking place between human beings that interact recurrently in language). When human beings are in conversation, their bodyhoods change in a congruent manner. Maturana's use of the word bodyhood rather than body is deliberate. With it he points out that, as humans, our social and cultural interactions are as much a part of our living as our interactions in the physiological domain, and that interactions in all of these domains are fully complicit.

Maturana (1988) describes the bodyhood as the node of our conversations. This is useful to me for understanding how knowers are brought forth in their mathematics knowing.

> Changes in the bodyhoods of the participants follow a path contingent on the co-ordinations of actions and emotions that take place along the conversation, and the co-ordinations and actions and emotions that constitute the conversation follow a path contingent on the bodyhood changes that occur in the participants along it while generating it (p. 51).

This suggests (once again) a reflexive relationship whereby, at once, the bodyhood determines the conversation and the conversation determines the bodyhood. Since in a conversation the bodyhood is changed, it follows that there are implications for subsequent conversations—whether or not they are with the same people and in the same domain. The bodyhood is the node in which all conversations intersect, and each conversation leaves its trace in the bodyhood (Figure 7-1).

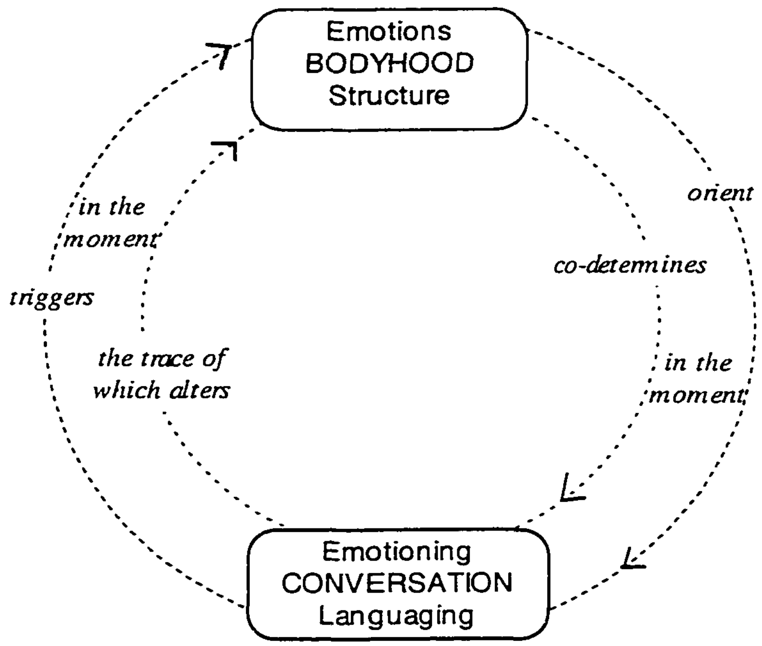

Figure 7-1. The reflexive relationship between conversation and emotions and structure[1]

In order to investigate the ways in which the bodyhood, or the knower, arises and is transformed by knowing in our multiple domains of existence (or in Maturana's terms, transformed in our multiple conversations), I do as I have done throughout the thesis, and offer an illustration of mathematics knowing in action. I discuss how the bodyhood is

[1] In the diagram, I note both emotions and structure. I do not mean to suggest that emotion is distinct from structure; however, because I wish to emphasize the role of emotion I am noting it separately. One possibility would be to contrast emotion with cognition however, I think this also is problematic, since

observed to be the node of the many conversations of which the person is a part. As I interpret the mathematics knowing of yet another parent-child pair, I do not simply use distinctions which I have made previously but enact them in this new interpretation as I attempt to understand how the body is brought forth in knowing; the conceptions and distinctions that I previously made are transformed as I begin to understand that which emerges from interaction as not simply knowing but as knower and knowing in relation to the other.

In the illustration that follows, I discuss how Desie and Joss (a mother-son pair), after listening to a story, engaged in mathematical activity, which included an exploration of big numbers, an introduction to exponents, a discussion of the conventions of mathematical notation and much more that might be viewed as mathematics knowing. I use narrative passages created from the transcripts of that evening. They demonstrate how this parent-child pair worked for an extended period of time trying to solve the problem posed in the story that was read to them. However, it is also evident that much of their activity was more than that. If we only observe their actions from the point of view that they were solving a particular problem (in this case, the problem as it was developed in the story), then we risk missing out on the complexity, complicity and contextuality of their activity and the fully embodied nature of their knowing. Hence, I follow my initial interpretation of their activity as problem solving with a detailed account of their activity, broken into fragments, and interspersed with interpretations of the ways in which they, as human knowers, were brought forth in their mathematical activity. I conclude the chapter with a discussion of how they were transformed in multiple dimensions of their existence. By doing this, I observe how this mother and son's mathematics knowing can be understood as implicated in and implicating their relationship with each other, school mathematics, the

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behaviour is often portrayed as either emotional or cognitive. I wish to emphasis that all knowing is both cognitive and emotional at once.

boy's school teacher and friends, and the mother and son's views of themselves as mathematics knowers.

Doing Mathematics Together

The night this event took place, we read from a children's story, *The Token Gift,* written by Hugh William McKibbon and illustrated by Scott Cameron (1996). It is a story about how the game of chess (Chaturanga as it was called in the story) was invented, and the pleasure it brought the king and his people. In the story, the king was so grateful for the game that he offered the man who created it whatever in the kingdom he wished. The man humbly declined a gift but the king wouldn't hear of it, and insisted the man name his reward. The children, in the class that night, suggested they thought money, jewels or a house might make a suitable reward—one child thought a golden chess game would make a great reward. But, the man had a different request.

> In the centre of the court, built into the floor, was a Chaturanga game board made up of sixty-four ceramic tiles. As his eye fell upon the game board he had an idea. He said, "Your majesty, I ask only for a token. Give me one grain of rice to represent the first square of the Chaturanga board, two for the second, four for the third, eight for the fourth and so on, doubling each time until all of the sixty-four squares have been accounted for."
>
> Then the king made a mistake. "Is that all?" he asked. "That does not seem like much of a reward to me."
>
> "It is all I wish for Your Majesty."
>
> So the king sent for a bag of rice and ordered a servant to distribute it on the ceramic tiles of the game board according to the Rajrishi's request.
>
> *The Token Gift*, Mc Kibbon and Cameron (1996)

At this point in the story, the children and parents were asked to figure out how much rice the king would need to fulfill the request. Each participant had a sheet of *2 cm x 2 cm* graph paper. I suggested they mark off an *8 x 8* grid to represent the chess board, select a square as the first one and begin.

> *"So the first one is — How many grains of rice go here?" the mother asked her son as she pointed to the square at the top left corner of their 'chess board'.*

"One," replied the boy.

"Okay. How many go here, if we are doubling it?" She asked pointing to the next square.

"Two."

"So what is doubling two?"

"Four."

"What's doubling four?"

"Eight."

"Double of eight?"

"Sixteen."

"Double of sixteen?"

"Um, thirty-two."

"Good. And double of thirty-two?"

"I have this idea." Their rhythm was interrupted.

"Oh. Are you trying to figure something out," his mom responded.

"Sixty-four," said the boy after just a brief pause.

"Good! And the double of that?"

"We are in the hundreds."

"We are. We are," his mom nodded.

The boy hesitated, "Do you have a calculator?"

As just described, Desie and her son Joss worked together, each making contributions as they considered the prompt offered by the story. They shared the task quite simply; to begin with, Desie asked Joss for the number that should be written on each of the squares. he quickly computed each double (mentally) and she recorded it on the graph paper (Figure 7-2). It only took a few doublings and the numbers grew big enough that Joss asked for a calculator to do the computations. It didn't take long and the doubling was even too much for the calculator. Desie and Joss spent the next 40 minutes trying to make sense of big numbers and the nature of the big number that would be needed for the 64th square.

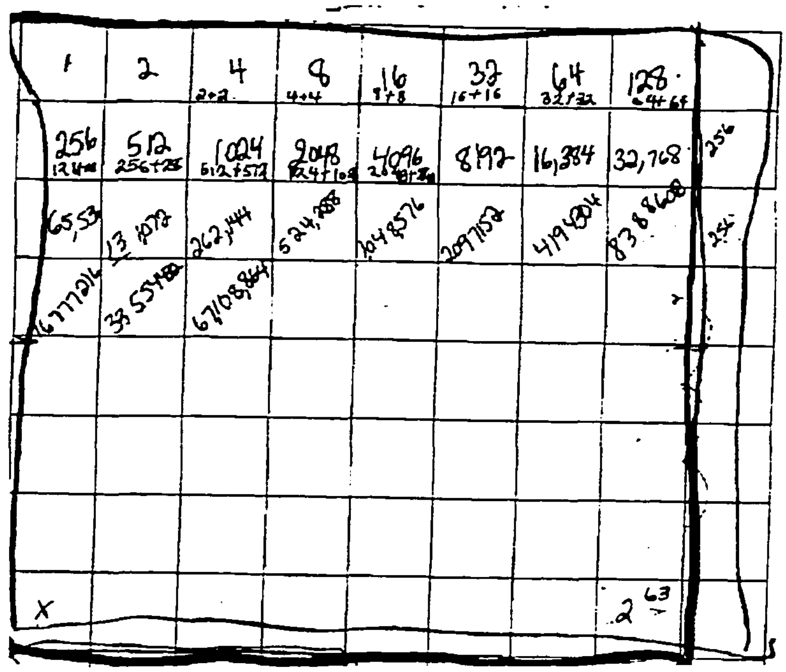

Figure 7-2. Desie and Joss's working paper of the chess board

Mathematics Cognition and Solving Problems

Polya's (1980) conception of what makes a problem a problem is commonly used by mathematics educators. He says, "Solving a problem is finding the unknown means to a distinctly conceived end. If the end by its simple presence does not instantaneously suggest the means, if, therefore, we have to search for the means, reflecting consciously how to attain the end, we have to solve a problem" (p.1). When I think about Desie and Joss's actions and interactions in terms of solving the king's problem, it appears that neither the king in the story, Joss, nor his mother had any difficulty at all in knowing what to do (to find a way where no way is known) in the circumstances with which they were faced. Begin with one grain of rice (the king's solution) or the number one (Joss and Desie's solution) and double. Thus, at first glance I might say that there wasn't a problem at all but rather a task or some computations to carry out. Of course, the king in the story has a very real problem since he cannot fulfill the request; but what was the problem that arose for Desie and Joss in their interactions? Tracing their activity from the point when the king's problem was read to them, I find a set of embedded problems and solutions that suggests this problem is not what it first appeared to be. On the surface, I might note that Desie and Joss did find an answer to the king's problem but, by observing their activity closely, I see that there was much more to this problem than this answer. Figure 7-3 is a trace of the questions on which Desie and Joss's activity over the course of the evening was oriented.

I. How much rice is needed to fill the chess board? (lines 1 - 384)
   1. What are the doubles for numbers < 100? (lines 1 - 22)
   2. What are the double for numbers > 100? (lines 1 - 379)
      a) How does the calculator work? (lines 23 - 36)
      b) How do you double a number by multiplying? (lines 29 - 36)
   3. What are the doubles for numbers too big for the calculator? (lines 65 - 379)

B. Is there a relationship between the rows of the chess board? (lines 121 - 312)
   1. What are the so called magic numbers? (line 169 - 275)
   2. Is there a relationship between rows for different sizes of squares? (lines 183 - 275)
      a) Why is there a relationship between rows for the different sizes of squares? (line 185)

C. How can we express big numbers? (lines 285 - 312, lines 355 - 385)
   1. What are powers? (lines 280 - 312)
   2. How big is 2^{63}? (lines 377 - 379)

Figure 7-3. An illustration of the embedded problems that arose for Desie and Joss

Many of the questions noted above were embedded in other questions. Inner level questions were triggered by outer levels ones. In some instances, there was a need to solve inner level questions before the outer level questions could be answered, and in other cases inner questions remained unanswered. An important feature of this embeddedness is that when Desie and Joss returned to outer-layer questions after considering inner-layer ones they did so with new understandings of not only the questions still to be answered but also to questions they had already answered. New understanding (of possibly quite different things) had the potential to alter previous understandings thereby changing the "problem" Desie and Joss considered. For example, at one point in the evening, Desie and Joss began to explore what we called magic numbers. This took them away from their problem of finding the value of the number that would go in the last square on the chess board but it also changed the possibility of how that number could be found and the form that number might take. This move to magic numbers was a change in the mathematics they were doing. For example, at first Desie and Joss were simply doubling numbers either by adding a number to itself or by multiplying by 2. However, when they turned to the magic numbers they began to consider relationships between the rows for various squares.

Tracing the questions Desie and Joss explored, begins to suggest how an explanation constructed by noticing fractal features (self-similarity, recursion, scale) of the knower's activity is useful to our understanding of mathematics cognition. Each problem was not simply a new problem distinct from those already posed by Desie and Joss nor was it a sub-problem, in the sense that once solved they were one step closer to solving the original problem. A fractal metaphor helps us see that, at once, the problems which arose in the interaction were part of Desie's and Joss's knowing but they also represent whole pieces of their knowing: creating various sized squares and finding the ratio between the rows; the doubling rice problem; and the need for a way of finding and expressing large numbers. This brief analysis, of the problems that were brought forth in Desie and Joss's activity, suggests that studying any one of a number of small parts of their activity has great potential to inform us about their knowing that co-emerged over the course of the evening. I would like to suggest that this is not a reductionist argument. That is, I am not suggesting that Desie and Joss's mathematical activity can be reduced to some basic level. Rather, each layer of mathematical activity has its own unique features at the same time as demonstrating some forms of similarity.

Again an enactivist interpretation suggests we go further than just studying the problems that Desie and Joss investigated or solved because knowing and knower co-emerge as a world of significance is brought forth. Hence, enactivism requires that we think about Desie and Joss's actions and interactions as they occurred in relation to the ways in which their knowing is embodied—how it forms and transforms their bodies. In order to think about the knower that is brought forth in knowing, I return to the transcript of their activity and study it in much more detail.

Layering Interpretations of Desie's and Joss's Activity

Returning to lines 1 - 22 of the transcript of Joss and Desie's activity, it appears as though they did not have a problem with respect to doubling numbers, at least in the beginning. However, when the numbers got into the hundreds Joss hesitated (line 20). What was his hesitation? One interpretation might be that Joss couldn't compute the doubles of the larger numbers. It is unlikely that this is a strong constraint for him since it appears as though he was more than capable of doing the computations (if not in his head then with a paper and pencil). Another interpretation might be that Joss did not want to compute the numbers in is head anymore; he wanted a more efficient means of computing. This raises a question. In what ways are one's desires implicated in his or her mathematics knowing? As will become evident, throughout the session Desie too was constantly looking for a better way of doing what she was already doing. It could be that one's desire to find an simpler (often described as more elegant) way of doing something is an emotion prompted by mathematical knowing.[2] In any case, we note that Joss's mathematics knowing provokes a bodily response which orients his and his mother's subsequent actions.

Doubling as multiplying by two or adding a number to itself

Using a calculator triggered Joss's first difficulty with the task. His mom's response was to explain the difference between how to multiply with a calculator and how to add with one.

"Times?" Joss frowned as he examined the calculator.

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[2] The mathematician, Leibniz , is quoted as saying, "I am not satisfied with algebra because it does not give the shortest method or the most beautiful construction in geometry." (Kliener and Movshovitz, 1997, p. 18).

"Times 2" she replied.

"I don't know what a times looks like."

"This one right here," she said as she reach over and touched the calculator. "You haven't started times in school yet have you?"

"No," said Joss as he pressed the keys and continued. "1 2 8 times 1 2 8—"

"No, no, sweety. If you are doing— It can't be that. It has to be 128 times 2 or 128 ***plus*** *128." She turned the calculator toward herself and pressed the keys.*

"That's what I had."

"You didn't times though." Without laboring the point she continued to compute the sums, "256 plus 256 is 512. So 512 plus 512..."

Joss leaned over his mom to watch her operate the calculator. "This is a neat calculator. If I had something like this I would carry it everywhere."

Once Desie began doubling with the calculator, they moved along quite quickly. Joss watched intently as his mom used the calculator. It was a small four function model but Joss admired it just the same. "This is a neat calculator," he said. "If I had something like this I would carry it everywhere." I am struck by his desire to carry a calculator everywhere. Why would he want to do that? What is it about this tool that Joss finds so desirable? One interpretation is that Joss thought this device would allow him to do many computations very quickly. After all, this is what he was witnessing as he watched his mother. From this perspective, I note that the calculator is a tool with which Joss could expand his computational power. However, if I interpret his utterances this way, I am led to question, "Why an eight year old boy would want or need this kind of computational power?" Consider his world: his home, the front street and backyard, hockey arenas (possibly), and school. Although Joss said he would carry the calculator everywhere, I do not think that he would find much need for it in his backyard, on the front street, or in a hockey arena—at home, maybe—at school, definitely. I believe it is only at school where he would require computational power and speed exceeding his own ability.

This interpretation makes more sense to me when I integrate it with what I know about Joss. He and his mom came to the program because Joss was having difficulties with school mathematics. Being a third grader in the province of Alberta means that, he will write a provincial achievement test at the end of the year; part of that test includes a timed set of number facts. One of the first things he told me when he began the program was that he was not good at the timed tests. Knowing this about Joss, it is easier to understand why he might have been so taken with the calculator. Using a calculator would enable him to compute sums (and differences) quickly and efficiently—a school task.

Base ten and rapid growth

"We're in the thousands," Joss wiggled in his chair and laughed. "And soon we will be in the millions!"

"16 384 plus 16 384 is 32 768."

"I can wait until we get right there," he giggled as he pointed to the last position on the calculator display.

When they did get to a million it was Desie who commented, "We just hit a million."

When Joss went to double 64 he hesitated and commented, "We are in the hundreds" (line 20). In this passage, once again, we hear him mark the magnitude of the numbers they were computing. Although Joss was no longer calculating the doubles, it is clear that by following along as his mother did the computations his understanding of big numbers and rapid growth was engaged. He did not sit back and wait for his mom to print the numbers on her sheet instead he followed along watching the calculator closely to catch sight of the numbers as they appeared on screen. As the number grew, he called out, "we are in the thousands." There was excitement in his voice. His understanding was unformulated and anticipatory; it was expressed in his body and his utterances. His knowing was enacted in a giggle, a gesture and his utterance, "I can't wait until we get right there." Desie too got caught up in the excitement of the rapid growth, "we are in the millions," she commented before Joss had a chance to shout it out.

Also of note is the fact that neither Desie nor Joss commented on the magnitude of just any number. There were no responses marking the number 512, for example. Rather, their voices revealed excitement when the numbers were ones which we distinguish with and in our language and our mathematics. Their utterances reflect the based ten number system and those powers of ten for which English has specific names. Desie and Joss's actions suggest to me that cultural forms are implicated in the personal knowing Joss and Desie brought forth in their activity. In as much as Desie and Joss responded the way they did to these numbers, I suggest that this can be understood as the cultural forms bringing forth the knowers. In other words, this suggests that the culture (body of mathematics) occasions mathematics knowing in action.

Mathematical Conventions

"Do you think there is some other way we can do this?" Desie asked Joss as she looked over the numbers she had been writing.

They both could see that the numbers were getting bigger and bigger—in fact too big for the calculator—but Joss did not want to stop. He suggested to his mom that she continue in the same way. So Desie worked on the next one. As she was writing [13, 1072] she realized she had put the comma in the wrong position.

"No. One hundred and thirty-one thousand," she said out loud as she marked the comma in the correct place.

Joss crawled up on his chair to get a closer look over his mom's shoulder. "Mom, we don't use those," he said as he pointed to her number written with a comma. "The teacher told us not to use marks like that."

"But they make it easier to tell what you are doing," she replied.

"I know, but that is what the teacher said. She said, 'if you see your mom and dad doing it. They are just old fashioned'." Joss giggled again.

"Yea, but if you didn't divide the numbers then it is hard to tell what you are looking at."

"No. I think they put—like— Instead they use spaces," he said as he sat back.

Joss's ongoing engagement in the activity and his joint action with his mother is evident in his attention to her comments, actions and written notes. When she corrected herself and uttered, "No. One hundred thirty-one thousand," Joss was well aware of what happened. Even though he would not have written the number the way she did, he understood what she wrote, and why she wrote it in the way that she had. When he saw her use a comma, he was reminded that his teacher had told him that in the old days, when his parents went to school, numbers were written differently. Joss learned to write large numbers with spaces separating the hundreds from thousands rather than commas. His comment reveals that his mathematics is different than his mother's and he recognizes this. "You're old fashioned mom."

On being smart

With only a few more computations Desie and Joss came up against the calculator's capacity for displaying numbers and they were left trying to figure out how they might proceed.

"If we had [the numbers computed] up to there," Desie pointed to the 32nd square, "we would have half of it right?" She conjectured that if she knew the values for the first half of the chess board she could multiply by some number and she would be able to compute with just one computation the 64th square. However, she realized she was still stuck—the 64th number would still be too large to compute.

Desie asked Joss again if he had any ideas how to proceed. But Joss did not respond with a strategy, instead he said to his mom. "Know what? Chris [by Joss's assessment the smartest boy in his class at school] taught us this math question that is a regular math question. Chris said, 'You think this is hard. I think this is easy.' I'm like, 'Chris. We might think it is easy too.' Then he writes down, 2 times 24. I'm like, 'Chris that is easy.'"

"So he thought you didn't know it, hey?" Desie replied.

"It's 48 Chris! He's like, 'Oh.'" Joss said with a grin.

The experiences of this particular night and now the interruption of their progress because of the need for a new way to proceed occasioned Joss to comment on an event that took place at school. He told his mom a story about how he demonstrated his mathematical prowess to a smart friend at school. His mathematics knowing is not separate from who he is at school, what he does, and with whom he interacts. As Joss develops as a mathematical knower, he changes as a classmate and a student. Maybe this is why Joss expressed the desire to carry a calculator everywhere. There is some indication that for Joss being good (fast) at mathematics changes who he is in relation to his peers. What might it mean to a boy who has trouble beating the clock to have a tool which would enable him to finish ahead of the clock and maybe even ahead of Chris, "the smartest kid in the class"? Joss might imagine himself interacting differently with the smart children in his class (on their terms) thus, altering his social position in the class—all by virtue of having a calculator. Or, it simply may be that he imagines himself being better at tests. In either case, Joss's comment about Chris and the calculator helps me understand how his mathematical knowing is woven into other aspects of his life.

Being mathematical

Desie smiled but only for an instant. The concentration on her face was evident as she continued to think about another way of working on this problem. Joss leaned over the papers and took a closer look. Desie reviewed out loud what they had found out thus far. "So each one of these is the number before it."

"Uh um."

"So two of these makes this. So would it be four of these that make this?" She asked herself out loud as she looked at the relationship between every second number.

Joss yawned.

"Because four of these make this," she looked at Joss.

"Uh huh," he agreed.

"So then it would be sixty-four." Desie continued to utter her unformulated thinking. "What's this? So this number here Joss, this number here is two of this number right? And it is four of this number and eight of this number, And 16 of this number and 32 of this number and 64 of this number and 128 of this number—"

Joss was watching his mom attentively and giggled at her finding.

"Boy, I just don't know," she said as she sat back in her chair.

"A calculator can't fix this one. Unless it is this long," he gestured with his hands.

"Yeah, it's a lot, eh. But if we had up to here then we would have half of it. Right?"

"Uh huh."

"So if we had half of it we still couldn't do it. Desie turned to Joss who seemed to be concentrating on something. "Boy I don't know what you are thinking. Are you thinking of something mathematically?" she asked him.

"I'm not mathematical."

"You are too! You showed me that last weekend when you showed me the circle and how to get the handshakes. So you are mathematical too."

It is not obvious why Joss denied being mathematical at this point in the evening. Even though Joss had just bragged about his cleverness at school, here he acknowledged that he does not see himself as mathematical. Clearly, he was able to make plenty out of the activity in which he and his mother were engaged. He even giggled when he recognized that his mom had found the same doubling pattern they had begun with: 2, 4, 8, 16, 32... On the other hand, it may be that he was responding with, what in the past had been, an acceptable reply to a request to do something difficult in mathematics. What ever the case, it took his mother's reminder of his clever solution to the problem they had considered the previous week before he generated a new idea as to how they might proceed.

Knowing you can know

Joss suggested they calculate half the table and then add all the numbers from half and put the sum in the 64th square. When his mom noted that would be a lot of big numbers, Joss responded with yet another strategy to take half of the table and then another half of it then add it together, "Thirty-two half of it, half of it and then add all those numbers together and then that big number and put it in the bottom," Joss giggled at his own suggestion.

"Let's say we did just half of it which is up to here." Desie contemplated Joss' suggestion.

"Yep. So then we would know—

"Then we would only know half of it," his mom said. "[We] still couldn't say half this and this together. It wouldn't work. Every single number is twice as much as the one before it. So this would be lined up just hugely."

"That's what I mean. If we take a big number and add it together then da, da, da."

"What's that," she said laughing at Joss's sound effects. "Yeah, but this one here. Say we have 67 million. So the next one is double of 67 million. One trillion. I don't even know what it is. One billion definitely— So I don't even know."

"Six something," Joss offered.

"It would be a billion two hundred million or— You know. I don't even know. I don't even know those numbers."

"It would be something," Joss reasoned.

"Well, yeah. It would be something," Desie agreed.

"If we get high enough we'll end up going billion, something after that, something after that, something after that— And I don't even know."

"We could write the number though," Desie said. "Because it doesn't matter if we know the name or not. We could still write the number. Right?" She thought for a moment, "Maybe, if we break it down to a four by four square" (Figure 7-4).

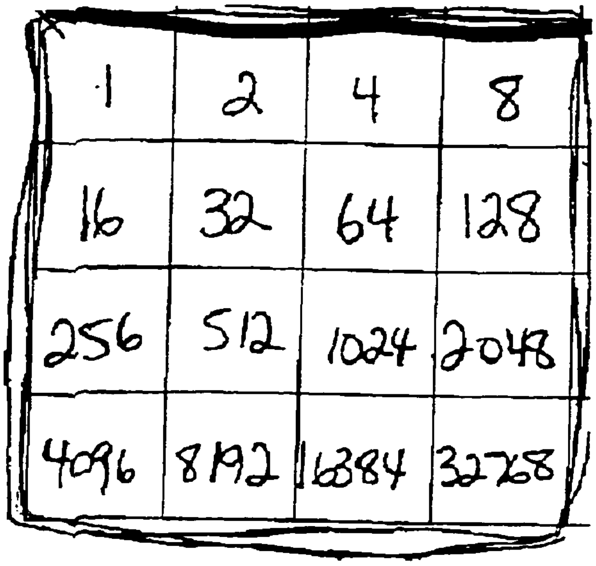

Figure 7-4. The 4 by 4 case is an example of how Desie and Joss specialized

The back and forth exchange of ideas helped keep both Desie and Joss motivated to stay in the activity. In this passage, Desie and Joss come to realize that they will end up with a number so large that they won't know "what it is" although they admit they would be able to write it. This is a significant problem for them; not only are they struggling to find a method to compute the number that belongs in the 64th square, but if they find it they will not even know what to call it. They do not have names for numbers so large. It is interesting that this does not stop them from seeking the large number. They are aware that the number exists and they know they could write it but as will be demonstrated later, even once they have a way to "write" the big number, this does not satisfy their desire to know what the big number is.

A new strategy

"I've got an idea!" Joss interjected. "If we finish off this row [the third row of the 8x8 chart] and erase that [numbers in the fourth row], we can add these numbers [the numbers from the end of the previous rows] and look at the next number— da, da." Joss said as he taped the squares at the end of two rows. "Then we go like that—da, da, dum." He suggested they add 128 to 32 768 to obtain the value for the last square in the third row.

"You and your da, da, dums. So what you are thinking is, if we had this number and figured how many times it went into this number and see how many

times in went into this number—" Desie, used Joss's idea but realized that she would have to multiply rather than add.

"Yea. There. There. There." He said as he touched the last square in each of three consecutive rows.

*"I don't know if it's going to work; but we can try it out down there," she pointed to the 4 x 4 grid she just made. "1,2,4,**8**,16,32,128,**256**. Divide by 8. You've got something buddy. See how it works on this?" Desie laughed and reached over to give Joss a quick squeeze. "You are smart. Let's see if it works up here [on the 8x8 board]. Eight hundred, 8 388 608 divided by 32 768 equals 256. But it's much smaller. 32 768 divided by the number above it, right?"*

"Uh-huh."

"128."

"No. It is supposed to be down, down, down," Joss tried to explain what he meant but Desie was busy dividing to find the common ratio so she could compute the next number.

"No, it's working out. 256. So then we can take this and multiply by 256 and get this." She said as she pointed to the 8 388 608 and the end of the next row. "But my calculator still can't do it." She sat back. "But, 256, what kind of number is that? 256?"

"I thought it was 16," Joss tried to figure out what his mom just did.

"16 is this little bitty one." She pointed to the 4x4 square and then to the 8x8 square. "See this is 64 squares. But wait—" Desie picked up her pencil and pointed to the 4 x 4 grid. "This is only 16 squares." She moved her pencil to the 16 x 16 grid. "But this is not 256, or is it 256 squares. So how does it— I wonder why it is 16 and this is —"

Joss continued to make suggestions as to how they might proceed. I could take this opportunity to observe this interaction between his mother and him with the models I developed in the last two chapters. There is ample evidence of how Desie's requests for Joss's participation occasioned his suggestions as to how they might proceed, for example. Or, I could demonstrate how Joss's suggestions, although somewhat unformulated, occasioned his mother actions. However, as I indicated earlier in this chapter, I believe there is a need to purposely make different distinctions, ones that suggest the transformation of the knower rather than just the known. In the previous chapter, I might have used a diagram like Figure 7-5.

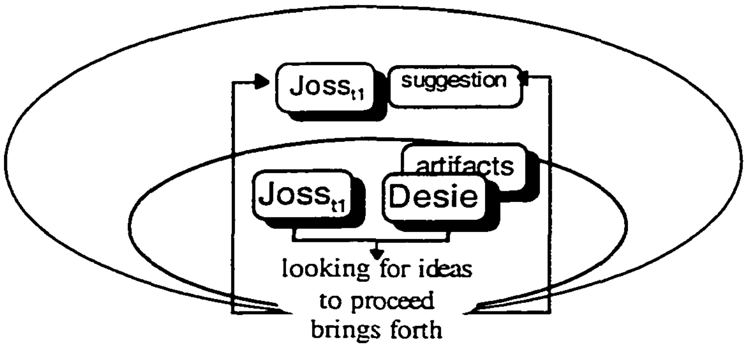

Figure 7-5. Joss, in interaction with his mother, utters the suggestion that they consider what happens between rows of the chess board.

The records Joss's mother had been keeping and her requests for his participation occasioned his thinking. He used his mother's records to explain how his new idea worked. As he explained it, he touched the end of each row and chanted, "da-dum, da-dum, da-dum." There was rhythm in his pattern making. When his mother realized she could do something with Joss's suggestion, she laughed and hugged him. Her knowing, like his, was laced with emotion. In order to highlight the rhythm in Joss's utterance and the emotional qualities of it, I model it differently than I have been (Figure 7-6). Rather than simply note that Joss makes a suggestion in the interaction between him and his mother and with the records she was keeping, the new addition to the model I have been using attempts to illustrate how Joss is brought forth in the conversation. Recalling Merleau-Ponty's (1962) words, "In order to express it, the body must in the last analysis become the thought or intention that it signifies for us" (p. 197), I am pushed to think about how Joss's bodily response, his utterances, gestures and emotions, for example, are his thoughts—they are his knowing in action.

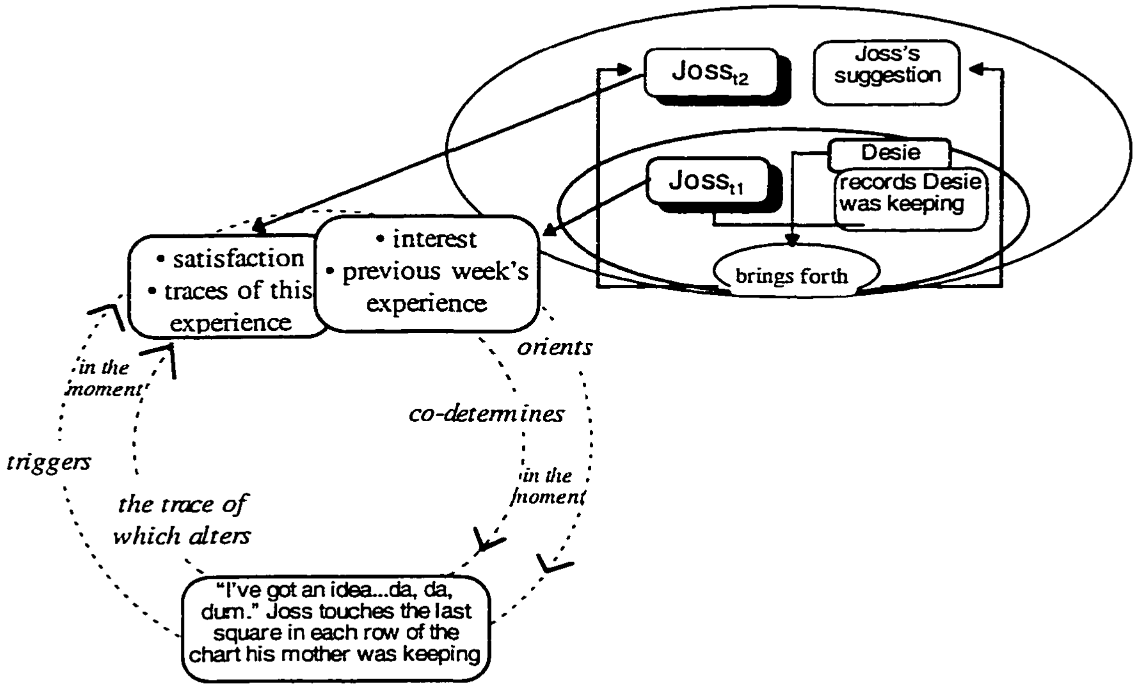

Figure 7-6. Joss is brought forth as he chants, gestures and utters his mathematics knowing

Explaining their new strategy

I approached their table to listen in on their conversation but I interrupted them instead.

"What did you discover?" I asked Joss.

"We added these together," he tried to explain.

"We divided," Desie corrected him. Then she explained what they had noted.

"That is interesting." My curiosity had been aroused. What were these 'magic numbers' they were playing with. I had used prompts like this one many times before but I had never seen anyone consider the ratio between the lines for the different size boards. I wasn't sure what to make of it but my general strategy for working with these kinds of problems is to try another case. So, I suggested they generate some more cases. Desie made up a 2x2 grid and found the magic number of 4.

"This had got 16 squares and 16. Four is four. So how come?" Desie looked at me.

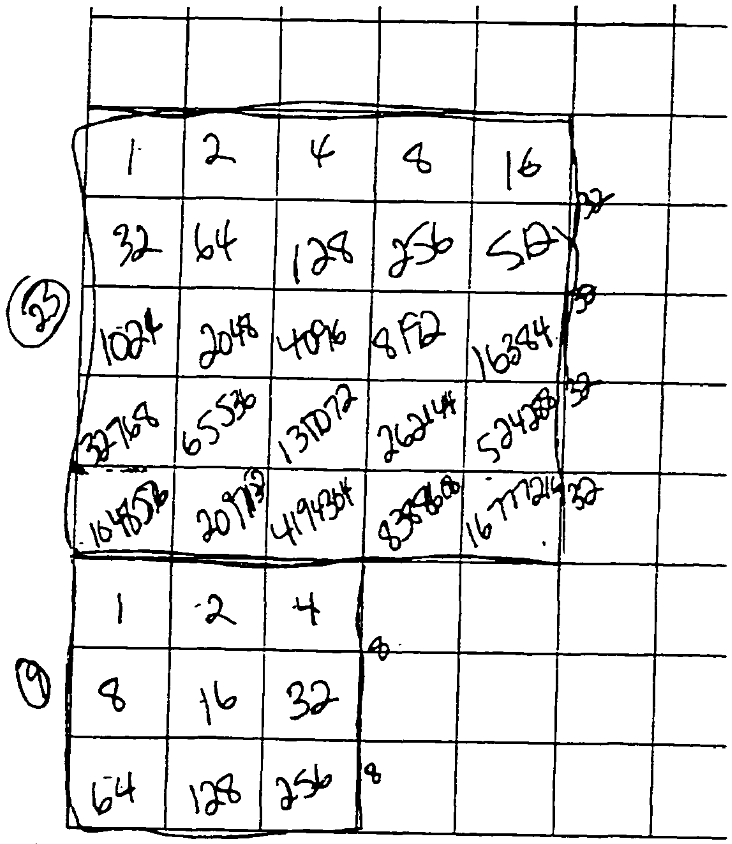

Figure 7-7. Two more cases to explore magic numbers

"Well, I don't know," I said. "I've never thought of this problem in my whole life."

"But that's what we are wondering," Desie said.

"This is interesting. No— I've never thought of this problem," I replied.

"He's got a strange mind, this one. A very different mind." Desie smiled at her son who had been listening to us.

I commented how he was acting like a mathematician and then talked about the ratio, π. Then suggested they find the magic number for a few more squares.

Desie and Joss worked out the magic number for a three by three grid and then reflected on their records.

"Look at that. Something is different here." For third time Desie was repeating the patterns of the four by four and the eight by eight grids.

"This is getting boring."

"Well this is what she asked us to figure out." Desie persisted with the task. "Here the magic number is 8, but how many squares are there? 1,2,3,4,5,6,7,8,9. So it doesn't fit. Just like this one with 64 squares, but the magic number is 256. So it's different, hey?"

"That one is not right," Joss said referring to the three by three grid.

"They are right. We just don't understand why they work that way. So we did a 2, we did a 4, we did a 3."

"I think I know."

"Yea, what's that sweety?"

"That has to be a six or under—maybe," Josh hesitated.

"Should we try a fiver?"[5x 5 square]

"—'cause like sixty, the answer is 64. So it probably has to have a 64 in it?"

"We should try a fiver. 1,2,4,6,16,32—"

"What time is it."

"—64, 128, 256."

"What time is it mom?"

"I'm not sure." Desie replied without looking up from the sheet of paper.

When Joss doubled, he took the number given and added it to itself. I first saw this when, Joss tried to double on the calculator by taking the number pressing the multiplication key and then entering the number again rather than multiplying by two (lines 25-34). Throughout the session, his understanding of doubling strongly constrained the ways in which he was able to think about the king's problem. Joss's attempt at an explanation of what he and his mother were doing (line 176) reflects that he continued to use an additive strategy late into the evening when thinking about this problem. Although he was following his mom's actions very closely, it seems that Joss's knowing was not growing (his structure was not changing) in such a way to include the multiplicative and proportional reasoning that geometric growth involves. Yet, his actions and utterances suggest that he was able to anticipate the rapid growth (as I already indicated) and he was caught up in the magic numbers they were generating. Hence, in some way, his knowing, albeit unformulated, was occasioned. Further, his mother's actions must have had enough significance for him that he thought he understood and was willing to explain. Therefore, I might expect that the potential to expand his cognitive domain was increasing. However, as will become evident, as rich as these interactions were, Joss's structure was such that over the course of this evening he did not come to engage in explicit multiplicative or proportional thinking. I anticipate there were other changes to Joss's structure but as an observer I did not note them.

More problems

"We have another problem," said Joss.

Desie didn't respond

"What time is it?" Joss tried again.

"We don't care. They will tell when our time is up. Let's find out what our magic number is on this one [5x5]. 167 772 316 divided by 542 288. The magic number is 32."

"What's this?" Joss asked.

"So, 5224 288 divided by 16 384 is 32. We've got a magic number of 32 out of this one. 16 384 divided by 512 is 32."

"25?" Joss did not follow what his mom was up to.

"5 times 5 is 25. So that's wacko too, eh? 'cause here's the 9 and the magic number is 8 and here there are 25 and the magic number is 32." Desie was comparing the number of unit squares with the magic number. It worked for the 2 x 2 square (magic number of 4) and the 4 x 4 square (magic number of 16) but it did not work for the 3 x 3 and the 5 x 5 squares.

"Four, sixteen."

"We could figure like if we had a big huge calculator we could figure this out. Like we could just multiply this by 256 to get this and so on." Desie explained how the technique Joss suggested would work. "But we don't and that's the problem."

"I got to go the bathroom."

"We've got a half an hour. Is it an emergency?" Without waiting for his response she added, "You better hurry back though."

In contrast to my growing excitement, by this point in the evening Joss appeared to be losing interest (Figure 7-8, 1). Because Joss has a poorly developed concept of multiplication he was not developing a taken-as-shared understanding with his mother (or me) and his engagement in their joint activity is wavering. For the first time Joss said to his mom, "This is getting boring" (Figure 7-8, 2). He was no longer able to make sense of what his mother was doing. Although he tried to follow along, and he did make some distinctions, he did not put them together in a meaningful way for himself. Desie and Joss began to uncouple as the mathematical interactions between them became less meaningful for Joss. Notice how Joss repeatedly asked his mother for the time, and when she finally responded with, "We don't care," he tried to get back into the mathematical activity but with little success. Eventually, he asked to go to the bathroom, and he physically removed himself from the room and the mathematics. Joss and Desie's world's of significance overlapped less (with respect to mathematics) as their taken-as-shared meanings collapsed and the mathematical interactions between them ceased.

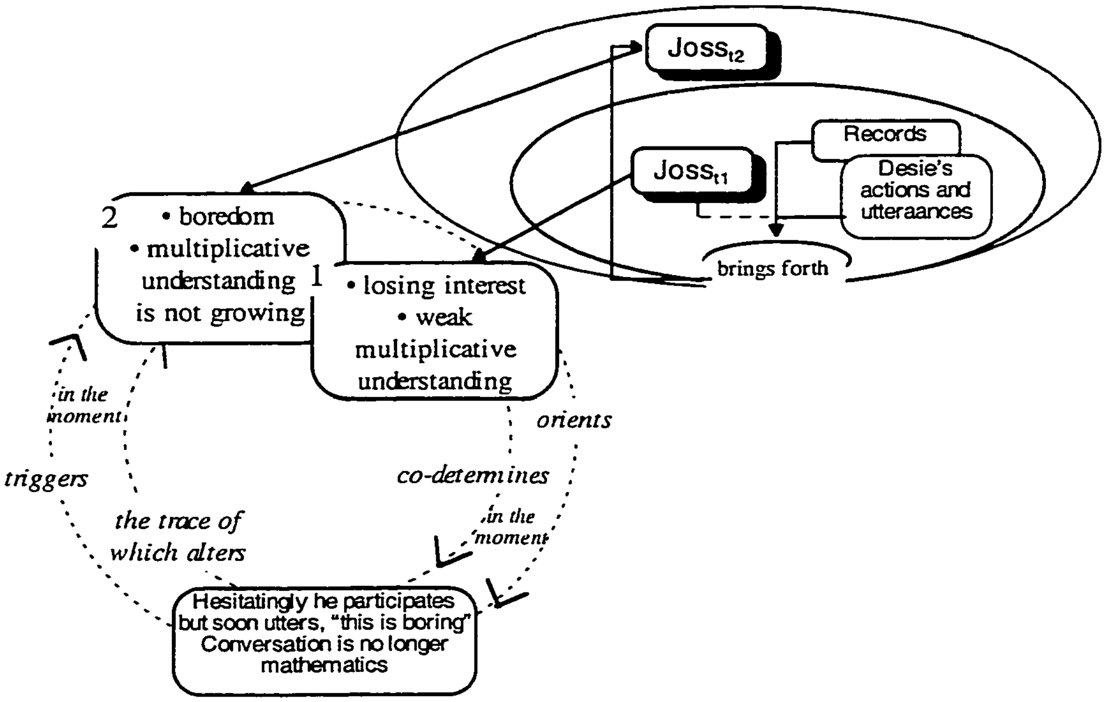

Figure 7-8. Joss's structure is such that he did not bring forth a world of significance which included his own multiplicative thinking.

Another Pattern

While Joss was out of the room Desie and I looked at the squares and magic numbers she found.

"Those numbers you are coming up with are very interesting numbers," I said to Joss when he came back. "Do you see how they are interesting?"

Desie answered for him. "No. He says he is bored."

"But wait until you see what these numbers do." I said excited to show them what I had noticed. "Because when you have a 1 by 1, well that is not very interesting —it is just one. But you did a 2 by 2, right, and when you did the 2 by 2 your number was 4. And then— Where's your 3 by 3? When you did a 3 by 3 your magic number was 8 and when you did a—" As I spoke I constructed a table. (Figure 7-9)

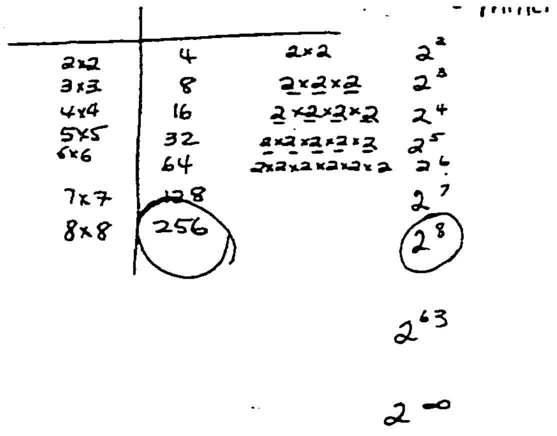

Figure 7-9. My rendering of the patterns I observed in Desie's and Joss's magic squares

"4 by 4," said Desie.

"And what was your magic number?"

Joss was following along, "16."

"Oh, this is interesting isn't it," Desie said.

"4 by 4 is 16," I continued. "Then you did— Did you do a 5 by 5?"

"Yep," said Joss.

"What was your magic number?"

"24. No. 32." Joss corrected himself.

"Now let me show you something, I've never seen this before. Not in my whole life. This is new. When you did a 2 by 2 it was 4, then it was 8, then it was 16, then it was 32. Do you know what the next one is going to be?'" I asked Joss.

"Do you have a guess for a 6 by 6?" his mom asked.

Joss looked at the sheet on which I was keeping a column of these numbers.

"Where are we moving here?" his mom prompted him.

"64?" he said tentatively.

"You see, these numbers are doubling for some reason." I said to him.

"Isn't that cool." Desie said.

Joss was still thinking. "128"

"Yes, 128 for a 7 by 7." Desie had it figured out.

"Right and guess what?" I prompted them.

"We know what our 8 by 8 is already," said Desie.

"Well, I guess the magic number would help us." Desie was still concerned with finding an easy way of computing the number for the 64th square. "We

would only have to multiply it that many times," she said as she pointed down the rows.

Joss picked up the calculator.

"Except our calculator won't do that." Desie said as she put her pencil down.

Although Joss seemed to have lost interest in doing mathematics prior to leaving the classroom, when he returned to the classroom, I was anxious to show him the pattern I had noticed in their magic numbers. As the teacher, my responsibility to the participants was to find ways to help them stay engaged with mathematics over the course of the evening. So I expressed my knowing (that was occasioned by interacting with Joss and Desie and the artifacts of their interaction) for them, anticipating that they might be occasioned to continue on with their investigation after seeing this new pattern. I showed them how when the magic numbers were ordered from smallest to largest they also formed a doubling pattern. Even though there is plenty of evidence to suggest that Joss did not reason multiplicatively (in general), he was able to work with the special case of doubling. Once again, Joss followed along with description, and Desie admitted that maybe their magic number could help them. Joss picked up the calculator again, as if he thought it might now be useful, but Desie realized that even with the magic number the calculator would be of little use. For her the doubling patterns formed a general case, and it did not matter that this was a new situation—their problem was with the large numbers that arise very quickly when doubling.

I might also demonstrate how the knower is brought forth in the interactions by talking about my actions in this situation. Based on the last three pieces of transcript, it is quite clear that Desie's and Joss's difficulties late in the evening brought forth me, the teacher (Figure 7-10). Earlier in the evening, I was brought forth as a mathematics knower excited by the potential for exploring the rice problem by specializing with these things we called magic numbers (Figure 7-10, 1). However, later in the evening, it became clear to me that Joss was no longer acting as "mathematically" as he had been early. My response was to Joss's disinterest more so than to the mathematics he had been exploring with his mother (Figure 7-10, 2). I had recognized the doubling pattern with the magic numbers and saw it as a way to bring Joss back into the activity (here my desire to teach oriented my behaviour). Later, when I notice that exponents could be used to express the magic numbers I thought that this might be a way to engage Joss once again.

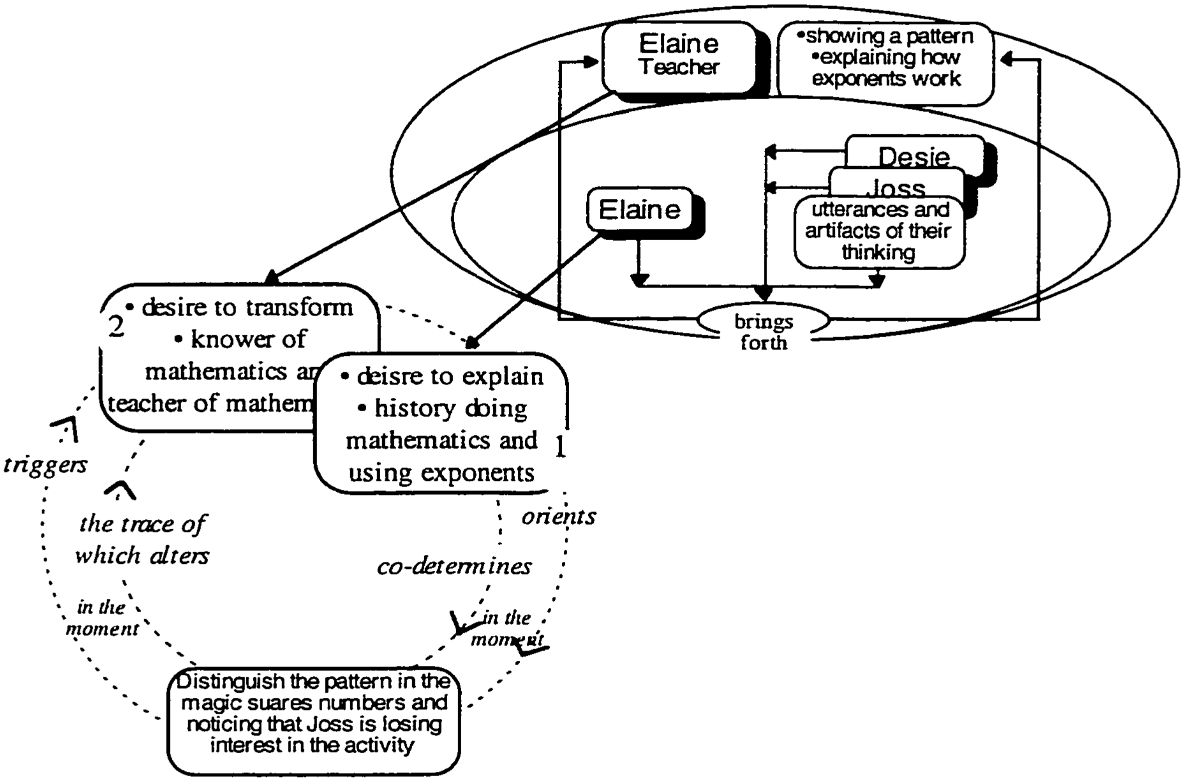

Figure 7-10. I am brought forth as "teacher" in my interactions with Desie and Joss and the artifacts of their knowing

Teaching Powers of 2

But I noticed something else in the magic numbers.

"Now I am going to show you something really neat about this. Remember the big squares that you made? This was a 2 by 2 square." I point to the square and then to my table where I had written 2^2. "This was a 3 by 3 square and look at this 3." I pointed to the exponent of 2^3. "This was a 4 by 4," I pointed to the exponent of 2^4. "This was a 5 by 5 and this was a 6 by 6."

I left but Desie carried on. "So then it would be 2 to the what power? 2 times 2 time 2 times 2 times 2 times 2. Two to the what power, would be this one?"

"6," Joss said.

"Yea. Our eighth square would be 2 to the eighth power. Look at that." Desie smiled.

"I don't get it though."

"Well it's not getting it right now," she said referring to the problem of finding the 64th square. "We just know that 256 is 2 times 2 times 2 eight times."

"I just don't understand. I think it makes no sense."

"You understand that 4 is 2 times 2?"

"Right."

"So that 8 is, 2 times 2 is 4 and 4 times 2 is 8."

"Uh um."

"So this is the same thing. So instead of writing all those twos, we end up being able to write it like this. This means two to the eighth power."

Joss picked up his empty bottle and tried to get more pop out of it. "So, I don't know what power means."

"It just means— It means that—" Desie started again. "Let's say that I wrote [4x4x4x4x4x4x4=] and this would be 4 and the little number would be?"

"6"

"Is that how many fours there are?"

"I was counting the xs," Joss explained.

"Ah, you are supposed to count the numbers. So you know what I have written there is 4 to the seventh power."

"But how much is that?"

"I would still have to go 4 times 4 is 8 and 8 times 4 is— I would still have to do it all the way to find out what 4 to the seventh power means. But maybe there is a computer that does that kind of thing."

"They have a calculator over there." Josh said as he pointed to the people at another table.

"I don't know if theirs does it or not."

Joss picked up the instructions to his mom's calculator. "The power is by solar."

"That just means the sun powers it," she told him as she tried to use the exponent key.

"Why are there so many ambulances?," asked Joss who was distracted by the sound of a siren.

"Well there's a hospital near here."

Whereas Desie and Joss seemed to have come to a halt again, I could see a relationship between the size of the square and its magic number; but I needed to use exponential notation to express the relationship. I was excited by this thought because **I realized** it was one way **they could write** the value that fit in the 64th square. My interaction with them involved my intentions as the teacher. I decided to show Joss (I assumed Desie knew) how to write the powers of two in exponential form. However, Joss did not get it, and he repeatedly said so. When his words went unheard, he said it with his actions. He tried to drink from his empty pop bottle; he pointed out that someone else in the room had a calculator; he began reading the instructions for his mother's calculator; and he was distracted by an ambulance siren. Compared to earlier in the evening when he understood what was happening, he now was easily distracted and not fully participating. Not knowing altered him—he behaved quite differently.

Back to the story

Just then Ingrid, the other researcher, called for the attention of the group. Okay, let's look back to the book."

Joss and the others listened attentively as Ingrid finished reading the story. In my view, the solution offered by the story-tellers was brilliant in the way it tamed big numbers. The king called for his royal mathematician who explained that after covering 16 squares twice 32 768 grains of rice (one full bag) would be needed for the 17th square. Then the mathematician doubles bags of rice until the 32nd square at which point a whole shipload would be needed for just the 33rd square. And the shiploads of rice are doubled until the 48th square at which point the 49th square would need all the rice that could be grown in the kingdom in a year and so on (Figure 7-11).

"You may be interested in reading the piece at the end of the book about the game of chess." said Ingrid as she closed the book and gave it to Desie.

"Do you want me to read this to you?" Desie asked Joss.

"Yea."

Desie read and Joss listened. A minute or two into her reading he put his head on the table.

"What are you doing?" she asked as she put the book down.

"I'm just tired."

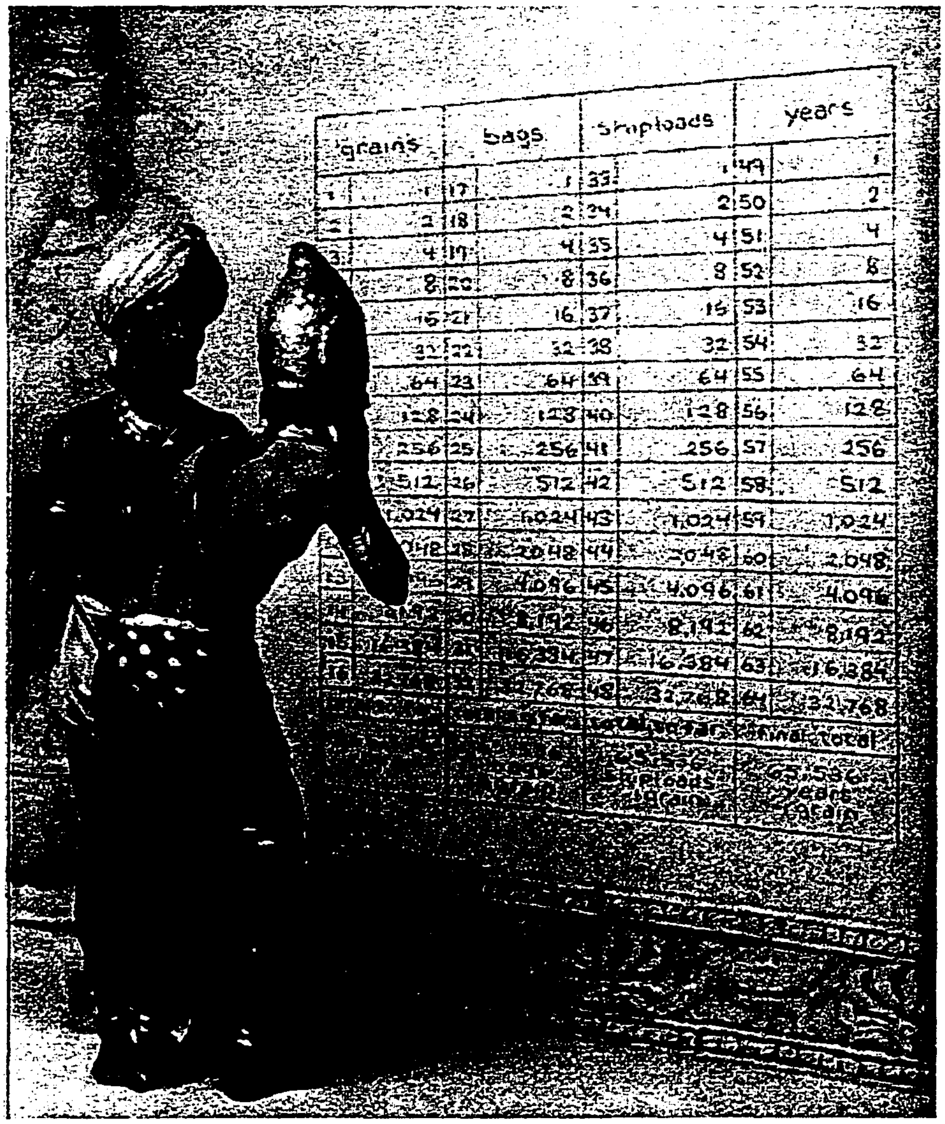

Figure 7-11. Illustration taken from the Token Gift showing the solution presented in the story (used with permission from Annick Press)

Throughout the night, Desie and Joss's mathematical thinking involved big numbers. Their activities were focused, not only, on finding the number which represented the amount of rice needed on the 64th square of the chess board but on trying to make sense of that number. The story tellers handled the big numbers beautifully. Although I found their solution most satisfying, I am not certain that Desie did, and I am quite certain that their resolution was quite unremarkable for Joss. As animated as he was throughout the night he barely responded to the story. It is hard to tell from the transcripts and video if his body is uttering his loss of engagement or intense involvement. The story itself is interesting as stories go but the mathematician's solution did not spark a visible response in Joss.

However, Joss was willing to listen to his mother read him the postscript to the story. It discusses what little is known about the history of the game of chess. Just as he has probably done hundreds of times before, Joss put his head down and closed his eyes as his mother read to him. The bedtime story is likely very familiar to Joss. I wonder if, as his mother read to him, his body embraced the familiarity of the experience—he closed his eyes and rested. Joss, the son, was brought forth in this very familiar interaction with his mother.

Reflecting on their thinking

"Seven minutes left" Desie said trying to encourage him. "Hey, we did some fancy thinking tonight. Yes we did. But we still have no idea what this number is," she pointed to the 64th square. "We know how we could get it easily, hey? Multiply this number by 2 to the 8th, this number by 2 to the 8th, this number by 2 to the 8th, this number 2 to the 8th." She pointed to the empty squares at the end of each of the last four rows of the chart. "We don't have a calculator that would do it. But if I did, at least you wouldn't have to do every single one along the way, would you?"

"Uh um." Joss had begun to doodle.

I approached their table, "Are you still learning more here?"

"He's drawn a hairy beast. I don't know why."

"What's your beast about," I asked.

"I don't know."

Desie reflected on what they had done. As she had done in all of the sessions, she made the most of her time with Joss to build his confidence. On this night, when she reviewed what they had done, Joss began to doodle. He was no longer engaged in a mathematics conversation with his mother or with me. When asked what his beast was about, he simply replied, "I don't know." This was the same response he had for the questions posed about exponents. Clearly, Joss was no longer interacting with his mother and the artifacts of their thinking with the attentiveness that he had for most of the evening.

Taming big numbers

I turned to Desie. "I think it was an interesting transition, when in the story the numbers got too big for them they changed it to a bag full of rice and then they—"

"And then the bags kept going by the same numbers, then that many shiploads, then that many years. They were using that same number— our number."

"Grains, bags, shiploads—"

"Years."

"Which is a very neat way of making the numbers doable," I asserted.

"Yea, understandable with the same number. Decreasing the bulk of it."

"Yea, when numbers get very big they get very hard to imagine."

"Well, that's with us. We just couldn't even— Once it was too big for our calculator—" she pointed to the 64th square again. "Joss said we still don't know what this number is."

"We don't," I agreed. "All we know is that—"

"32 768 years to grow." Desie had understood it from that point of view.

"Years to grow that much rice in all the rice fields of India."

"He was trying to understand what that meant, though and he couldn't understand it. I was like trying to explain, who would want to write 2 times 2 times 2... But what does 2 to the 8th mean. How would you every know what the number is? And then we looked at the calculator and it had some weird way of pushing all those strange buttons."

"Well we know what two time two is until we have 256 right? Well we know that this number right here is 2 to the 63. The number is just too big to compute." I explained.

"Well, I'm the same as him. Like me, I don't know— Like this [4^7] would be 4 to the 7th power. To me that is not a number. It doesn't mean anything."

"Yea, true. But powers do mean something. When we are measuring space, out to the stars, these begin to mean something. Because the numbers are so big that we have to make them small again to understand them."

"Understandable, okay. Well scooter time to go?" Desie gave Joss pat on the arm and stood up. She picked up the book and handed it to Ingrid.

Desie explained Joss's difficulty with the exponential notation to me. However, when I talked about why we might use exponential notation she turned away. Although she was willing to discuss her son, it appeared as though she no longer wanted to talk mathematics. By this point in the evening, even she found it simpler to suggest that she just didn't get it than to persist with trying to understand the magnitude of a number as big as 2^{63}. Although I was still trying to interact with her about mathematics, Desie no longer had the emotion which oriented her to act mathematically. Hence, she severed herself from the conversation she had been actively engaged in over the course of the evening with disinterest.

A Head For Math

"You're welcome to take the book home if you would like to borrow it." Ingrid said.

"That's okay. I'll definitely keep it in mind the next time I want to have a new story." Desie is a professional story teller. "Especially for the story telling. That would be really interesting to bring it into— What grade did you say it was for?"

"It says ages 7 and up."

"Okay, right from grade two. Wow."

"I could bring it into my class," Joss said eagerly.

"Oh, I know you could. But I could come and tell it at your class, which would be fun."

Likely, feeling confused himself, Joss grinned, "I'll confuse the teacher."

Desie turned to me, "Every week we come here and he really impresses me with the way he thinks. It's pretty weird. I never had a head for math myself."

With a connection made back to Joss's mathematics class at school, he became animated again. The thought of bringing the book to school pleased and excited Joss. As he had done with his classmate, Chris, Joss could see that the book offered a possibility for him to show off his new found mathematical skills. It excited him that he could confuse his teacher with a problem as difficult as the one posed in the story. As we saw earlier in the transcript, Joss's mathematics knowing in action brought into the moment his relationships with peers at school and with his teacher.

Mathematics: Conversation That Brings Forth Knowers

Reflecting on this mother and son's actions and interactions, provided me with an opportunity to understand mathematics knowing as a fully embodied phenomenon. In my exploration of Desie and Joss's mathematics knowing, I was struck by the prominence of the body as both a physical structure as well as an experiential one. The tone of their voices, their utterances, their giggles and laughter, the position of their bodies, and the use of their hands to point and gesture reveal the significance of their physicality. At the same time, Desie and Joss's references to others, the invitations to each other to participate, and their respect for the actions and utterances of the other suggest to me that their mathematics knowing is not separate from them as social beings. Finally, the ways in which their behaviours are co-referenced to the patterns of acting that we call mathematics indicate that, as knowers, they are brought forth as members of a culture at the same time as they bring forth that culture. Through observing their various mathematical actions and interactions, I am beginning to understand that doing mathematics is much more than simply "doing mathematics"; doing mathematics involves emotioning and languaging braided together in a conversation (Maturana, 1988). Mathematics is that conversation; it intersects in our bodyhoods in such a way that our being is permeated with the experiences of doing mathematics and reflexively that conversation is brought forth by our interacting bodyhoods.

To conclude this chapter, I summarize the ways in which Desie and Joss's mathematics knowing was brought forth from perceptually guided actions and the ways in which their knowing intersected in their bodyhoods, thereby bringing them forth as knowers. I observe that they have ideas, emotions, social relationships and culture—not in the sense of having property but in the sense of a relationship which the person bears to the term into which he or she projects him or herself (Merleau-Ponty, 1962, p. 174). From observing their actions and interactions, I distinguish their knowing as enacting personal thought, social relationships and cultural forms. Each of these dimensions of their experience, at once, constrain and make possible their knowing, at the same time, as the dimensions come together in their bodies and transform them. Further, as I discussed in the previous chapter, any change in the knowers changes the context of which they are a part; hence, potentially transforming the world they bring forth with others.

The Knower: Gesture, Utterance and Emotion

Maturana (1988) explains that emotions, not only make possible all our actions, but emotions specify the domains in which we take part (p. 48). “Emotions,” he claims are “kinds of relational behaviours in that they guide moment after moment our doings by specifying the relational domain in which we operate at any instant” (Maturana, 1998 http://www.inteco.cl/articulos/metadesign_parte1.html, p. 7). As I have tried to demonstrate, there were many instances in the session where Desie and Joss’s emotions are visible in their actions and interactions and can be seen to orient their actions. For example, when the numbers appearing on the calculator got large, Joss expressed his delight with their magnitude. As the numbers grew, he called out, “we are in the hundreds,” “we are in the thousands.” His delight was not only a response to the growing numbers but altered his bodyhood and orientated his subsequent interactions. For example, tied up in Joss’s delight was his anticipation of the rapid growth of the numbers. He did not sit back and wait for his mom to print the numbers on her sheet; instead he followed along, watching the calculator closely to catch sight of the numbers as they appeared on screen. Watching these numbers grow and anticipating their growth was an emotional experience for Joss—one that I suggest leaves a trace in his bodyhood, and, hence will be implicated in Joss’s knowing acts in the future.

Pondering on Joss’s comments and the excitement in his voice, I might ask “Which came first, his cognitive understanding of the magnitude of the numbers or the excitement he felt about what he was experiencing? Or, is it possible that his noticing was the outcome of his excitement?” In this instance, it appears that his excitement arose in the moment of distinguishing the magnitude of the numbers. His emotion is, what we might regard as, local yet fully implicated in his knowing. In the moment of knowing, Joss’s emotion readies his being and guides his perceptions so that some actions are made possible whereas others are made impossible.

I also notice Joss's emotional state is tied to his structure in another way. It depends on his history of experiences. Specifically, I am referring to the fact that Joss does not get excited about just any response; rather, he gets excited over numbers (hundreds, thousands) which in his experience, as a student of mathematics and speaker of English, are special. In Figure 7-1, Figure 7-6, Figure 7-8, and Figure 7-10 I illustrate my observation that in conversation Joss's mathematics knowing, moment by moment, triggers his emotions and leaves a trace on his structure which in turn, moment by moment, readies and determines his languaging and emotioning (conversation).

Not only is the model of interaction I proposed useful for observing mathematics knowing in action, but it can also used to explain situations when the knower's understanding is not good enough to maintain mathematical interactions. Take for example, when Joss repeatedly told both his mother and myself that he did not get it. Looking back in the transcript to where I was trying to show Joss how exponents worked for the table of magic numbers, I note that, although his and my actions were coordinated, there is no evidence to suggest that we were languaging (coordinating the coordinations of our behaviours) even though our actions were in language. There was no recursion on the interaction. In other words, at no time did it appear that Joss thought about that which we were doing as repeated multiplication, likely because he did not have an adequate understanding of multiplication with which to reflect. Thus, he did not have the opportunity to reflect on repeated multiplication as a process that leads to the big numbers in which he was interested. Neither did he reflect on the exponential notation as a way of representing the process or the product of the process. It is not surprising that he "did not get it." Without the coordination of the coordinations of our behaviours, there was nothing (no-thing) for Joss "to get" (Blumer, 1969). Furthermore, without a history of actions and interactions (structure) which included multiplication, a coordination of our coordinations was not possible.

We might also notice how Joss's emotional orientation changed in the events described here. Up until this point in the evening, Joss was quite involved with all of the activity. He watched, chatted, pondered, and questioned. However, once I began the "lesson" on exponents, his behaviour changed abruptly. He responded to my prompts with silence. Once I trivialized my questions, he obediently responded but those responses were no longer animated, and when he did speak it was little more than parroting what I said. The smiles and giggles subsided. He did not ask questions. He stopped making connections to the mathematics he already knew, to the events in his life and to the people with whom he interacted. This response was not like Joss's behaviour earlier that same evening. These two examples suggest, to me, that emotions are, at once, bodily orientations for action (knowing) and products of acts of knowing—and they co-emerge with structure determined behaviours.

The Knower: Social Relationships

Desie and Joss's mathematics knowing in action point to yet another dimension of their knowing; that is, as social beings in relationship with others. In particular, from the transcript of Desie's and Joss's activity, we note how their relationship served to help maintain their mathematical interactions and was maintained in doing mathematics together. Further to that, their mathematical knowing brought them forth as knowers in multiple dimensions throughout the evening. Joss (in particular) was brought forth as a student, classmate and son in his mathematics knowing. The social relationship, like most things we note from an enactivist perspective, is double-sided. On one hand, we might refer to the person's relationship with the other as a co-participant and the way in which the social interactions within that relationship trigger, foster and propel mathematics knowing in action. On the other hand, we might explore the way in which the person's mathematics knowing in the moment intersects in the bodyhood thus potentially impacting the person's social relationships, both inside and outside of the moment of action. I point to both with my comments, first with respect to the former and then the latter.

When two people are in relationship, they are coupled and their actions serve to maintain that coupling until such point as the coupling breaks down. Throughout the transcript of Desie and Joss' activity, there is evidence of how their actions served to maintain their relationship as well as to foster mathematics. For example, although Joss did not compute the doubles, magic numbers, or maintain the records they kept, he did follow along with his mother's actions, utter comments and offer suggestions. In doing so, he maintained a relationship with his mother as a co-participant as they did mathematics together. Of course, on the other side of this relationship is Desie who offered her thinking out loud, made public her records and repeatedly asked her son for his contributions. Joss could have lost interest in doing mathematics as his role became somewhat passive but he did not. He maintained a very active role both through his own volition and with his mother's invitations for ongoing participation. One further comment is needed here. One might suggest that Desie acted, as the teacher, and kept her son in the game (so to speak). There is some evidence that she did this, for example, at the very end of the session; but for the most part, Desie and Joss were co-participants in this activity. When she asked Joss what they might do next she was doing so not so much as mother, or teacher but as co-thinker or co-participant. She asked for suggestions because their (both his and her) mathematical thinking needed to be prompted, not simply because she was trying to keep Joss on task or to teach him what to do. Hence the interactive space created from their relationship was used to do mathematics and in doing mathematics they maintained their relationship.

Within their activity we also observe how other social relationships were implicated in their mathematics knowing and how their mathematics knowing had the potential to affect those relationships. For example, I am struck by how throughout the evening Joss's relationships with classmates and his teacher were called into the moment triggered by his mathematics knowing. Recall, how Joss commented about his interaction with Chris at school, when Joss demonstrated his cleverness by knowing 24 times 2 or recall when Joss speculated he could confuse his teacher by bringing in the book, *The Token Gift*. Both of these moments suggest, to me, that the mathematics Joss does with his mother in the parent-child mathematics program intersects with other conversations of which he is/was a part, and that have left traces. As Maturana (1988) suggests, Joss's multiple conversations intersect in his bodyhood therefore a conversation in one domain has the potential to impact a conversation in another domain.

The Knower: Cultural Forms

As human knowers interact within the constrains and possibilities of their own structures, they, at the same time, are bounded by the environment and others who have engaged in mathematics as part of their history of experience. Maturana (1998) notes:

> [T]he culture in which we live constitutes the medium in which we are realized as human beings, and we become transformed in our bodyhoods in the course of the history of our culture according to the human identity that arises and is conserved in that culture. (http://www.inteco.cl/articulos/metadesign_parte1.html)

Hence, our actions are also constrained and made possible by what we observe as culture—"the way of life and thought that we construct, negotiate, institutionalize and finally end up calling "reality" to comfort ourselves" (Bruner, 1996, p. 87).

Again, Desie and Joss's actions and interactions provide us with a number of examples of the ways in which we observe cultural forms to be implicated in their knowing and their being. It is significant, to my interpretation of mathematics cognition, that Joss commented when the numbers reached the hundreds and then again when they reached the thousands, that his mom made a remark when the numbers reached the millions, and that neither Joss nor Desie bothered to comment on the magnitude of just any numbers. As I already discussed, their actions help us understand the ways in which mathematics and the

English language are implicated in the personal knowing Joss and Desie brought forth in their activity, and the ways in which mathematics is embodied.

Recall too, how Desie had some difficulty when she tried to read out a large number. After placing the comma in the wrong position she confused thirteen 13 000 and 131 000. This triggered an interesting interaction between her and Joss—one that implicates and was implicated in culture (cultural forms) quite explicitly, and their inter-generational relationship somewhat implicitly. Joss noticed his mom's form for the numbers, and made a comment that revealed he is embedded in mathematics differently than she. Joss has learned to write large numbers with spaces instead of commas. He and his mother use different representations. Even more significantly, he sees that his knowing is different from his mother's and this is implicated in their relationship not only as collaborators but also as mother and child. He told her, "you're old fashioned mom"—the culture has changed—I am different than you.

> But at the same time, as human beings that live in conversations we are reflective beings that can become aware of the way they [sic] live and of the kind of human beings that they become (Maturana, 1998, http://www.inteco.cl/articulos/metadesign.htm).

The interaction, occasioned by Desie's form for writing a large number, offers some interesting insight into the reflexive nature of mathematics knowing. Joss's knowing is both replicative, in that he is engaging an established pattern of acting, and generative, in that it changes the culture of mathematics for those around him. When he writes numbers differently than his mother, he changes mathematics for her. This is more than teaching her mathematics (in the usual sense); he is practicing mathematics differently, and this opens different possibilities for both himself and his mother. In practicing mathematics differently, he is brought forth as a different mathematics knower than his mother.

Finally, I must speak of Joss's desire to have a calculator and to carry it everywhere. Joss's possession of a calculator permeates mathematics, in that his use of a calculator reshapes mathematics. Paper and pencils changed the nature of computing. Now computation is going through another revolution with the widespread use of electronic computing devices. One hundred years ago, there was a need in commerce and industry for people that were very good at computing. Thus, it was sensible, that in school, students would learn efficient algorithms, and practice them until they became fast and accurate. However, how do we explain a grade 3 class today that continues to operate on outdated needs and contexts? Joss's desire to carry a calculator everywhere is, not only, insightful and useful given his situation but it has a formative role in the culture of mathematics as well. Mathematics itself changes in light of Joss carrying his calculator. Mathematics is different today (for Joss), than it was at the turn of the century (for the clerk), at least, in part, due to electronic technologies. Those same technologies that contribute to a different mathematics for Joss also contribute to the changing face of professional mathematics (Borwein et al., 1996).

Understanding Differently

The night that this vignette is drawn from is only one night out of ten in which Joss and his mom did mathematics together. Pausing to make this observation is useful because we are reminded that their activity in any particular session is only a small part of their interacting together in general and interacting together with mathematics in the particular. In this case, I am reminded that Desie brought her son to the program to provide him with an opportunity to engage in mathematical thinking; but in doing this, she ended up engaging in mathematical knowing with him, and bringing forth a world of significance with her son that included mathematics.

With the examples in the previous sections, I have tried to demonstrate that we can think about Desie and Joss's mathematical cognition as problem solving, and that is useful in some ways; but, when we do this, other features of their knowing and of them as knowers go unnoticed and unexplored. We can note a "problem with big numbers" which is woven throughout their activity and interaction, over the course of the whole session.

However, I have shown that their knowing is much more than that. There are many layers of their knowing, and although not seen are present none the less. Only on the surface is their activity about finding out how many grains of rice the king will need for the token gift. Looking back over the transcript, it is surprising to note that Desie and Joss did not make any reference to rice as they worked that evening. Maybe the king's problem was not a problem for them at all. It is clear, their activity wasn't just about finding out how much rice the king needs to cover the chess board.

If we simply view mathematical cognition as problem solving, then how do we do to account for the emotions, such as desire and curiosity, that seem to be present in this situation? What about the persons' hunches and technical difficulties? How are they implicated in the mathematics knowing that co-emerged-emerges in action? What about the language Desie and Joss use and their use of mathematical notation? What about the relationship between this mother and her son, to what extent is it implicated in their mathematics knowing? I assert that it is inadequate simply to talk about Desie and Joss's activity as problem solving *per se*. If mathematics knowing is simply viewed as problem solving, then the human knower is nothing more than a problem-solver. However, if mathematics knowing is fully embodied action then the mathematics knower is brought forth in multiple dimensions of his or her living as she or he brings forth a world of significance with other people.

An enactivist interpretation would suggest that Desie and Joss were not solving problems in as much as they were acting, constrained by their own knowing (in all of its dimensions) and restrained by their environment, to bring forth a world of significance which this night included: a story of chess, big numbers, a mother and a son, a researcher, school, calculators, and classmates. If we interpret Maturana's (1988) assertion that emotioning and languaging are braided together to form our conversations, then with respect to mathematics knowing, we might suggest that mathematics is the conversation that arises from the braiding of mathematical actions (languaging) and the bodily dispositions (emotions) that propel those actions. Mathematics is not emotion free and neither is emotion independent of mathematics. As human beings, languaging and emotioning constitute our actions and interactions in our moment by moment living—our conversations—and it is through our interactions the space of the possible is expanded and the human knower transformed (Simmt and Kieren, 1999).

Chapter Eight

THE FRACTAL SPACES OF KNOWING

I sit, trying to write this final chapter after many starts and stops. I have spent days trying to understand why it is not working. Talked and talked some more, to anyone who would listen, trying to understand myself, trying to make sense of what I need to do to write the last chapter.

Today I think I shall write it. I have decided its purpose—to reflect on my research, my knowing in action. I am content with that. And so I write a new outline.

I return to my last version of this final chapter and begin to mine substance from the things that I have already written, classifying my utterances in terms of my new outline. I make a few markings, then blurt out a string of comments down the side of a page.

describes what I
mean by ordering
one's ordering
↓
do it to say "I know"
↓
thinking/
know
↓
I am thinking/
I know
↓
I am in relationship
↓
I am coupled
↓
it is part of me
↓
intimacy
↓
I know

The memory of a mother's words interrupt my stream of thought. "I now know my son in a way I never knew him before." Tears well up in my eyes.

Why does this mom's comment move me like it does? After all of my work—interpretation, diagrams, transcripts, models, explanations... Why did this particular comment emerge when I uttered, "I know"?

Tears of profound recognition.

> Thought is no internal thing and does not exist independently of the world and of words... 'Pure' thought reduces itself to a certain void of consciousness, to a momentary desire. The new sense-giving intention knows itself only by donning already available meanings, the outcome of previous acts of expression. The available meanings suddenly link up in accordance with an unknown law and once and for all a fresh cultural entity has taken on an existence. Thought and expression then are simultaneously constituted, when our cultural store [language] is put at the service of this unknown law, as our body suddenly lends itself to some new gesture in the formation of a habit. The spoken word is a genuine gesture, and it contains its meaning in the same way the gesture contains it (Merleau-Ponty 1962, p. 183).

Writing this thesis has been an act of research, not a report of research or a research report (although some could choose to read it that way). Each word uttered, each diagram drawn, and of course the many words and drawings that were changed or deleted were acts of knowing (determined by my history of interactions and oriented by my emotions) which brought me forth as a knower. Consequently, as I write this last chapter, I find I am unable simply to summarize what I have said. First of all, I am no longer the same knower that said those things. My own words are not transparent for me; they need to be enacted in my reading to be meaningful even to me. I must interpret my own utterances. Hence, in reviewing my thoughts (now the artifacts of my thoughts), I am transformed yet again, and a new world of significance co-emerges with me. Secondly, this dissertation was (in the first instance) my interaction with questions that emerged for me over the course of my study. I hesitate to offer an abstraction of my experience by ending this thesis with some concise, clear and linear summary of what I did, and with 'answers' to my research questions. I fear that this may be read without any experiences of the particular from which the abstractions emerged. This chapter, as all the others were, is my knowing in action. Hence, I will use it to continue to research by interacting with the artifacts of my own thinking as I look back on writing this thesis.

Re-membering the Research

I am stuck by my awareness that I did not set out to do research when I began the parent-child mathematics program. Rather, there was a sequence of events which, in retrospect, I distinguish as occasioning me to research, to explore mathematics knowing in action. (See chapter one).

Questions emerged, guided first by the demands of the parent-child mathematics program, then, by an interest in how I might understand the ways in which the person and the environment interact to bring forth what I observe to be mathematical activity, and, finally, by conversing about my observations of the parents' and children's mathematics knowing in action. There were questions about prompting mathematical behaviours, identifying sites of interaction and sources of perturbations, and exploring the relationships among cognition, emotion, body and mind. My interest grew to include trying to understand how personal thought, collective processes and cultural forms were implicated in mathematical knowing in action. Eventually, I formulated the question that permeated my interactions in the spaces of research: How might I characterize the mathematics knowing that is brought forth in the actions and interactions of the parents and children in this mathematics program? Further questions arose, for me, and were articulated my in writing about mathematics knowing in action. How might I understand the knowers' actions and interactions that bring forth mathematics? How might I understand the knowers that are brought forth in mathematics? Exploring those interactive spaces, the questions opened up for me, and I found complexity and complicity. There were many actions that I pointed to as mathematics knowing; but the actions (spoken or written utterance, gesture, tone, facial expression or body position) were not simply mathematics knowing. They suggested more than that to me.

Reflecting back on my role in terms of the prompts selected, the questions I posed, the participants' mathematical understanding that I validated by my comments, and what I have selected to include in this thesis, say more about my understanding about what counts as mathematics than I think we can know about the participants' understanding about mathematics. Although I have indicated this elsewhere, I will repeat it; this study is based on the premise that any of the explanations I offer make coherent the my (the observer's) experiences and observations. The explanations are not of the things in themselves; rather, they point to the relationship between the observer and the observed. Hence, as scientific explanations they must account for the observation but not the thing itself. The explanations that I have offered in this thesis, then, are offered as tools for observing and interpreting mathematics knowing in action.

Consistent with the enactive premise that our day-to-day knowing is not centred on problem solving but living, and in doing so bringing forth our lives and the world we live in, my research co-emerged with my experiences as a teacher (facilitator) of an extracurricular mathematics program for parents and children. It is not surprising, to me, that this unconventional site became a valuable site for researching mathematics knowing. Although few people are likely to conjure up an image of parents and children doing mathematics together[1] as a wonderful place to study mathematics knowing, it is in such an unfamiliar place that well-established patterns of behaviour are disrupted. The parents and children who participated in the mathematics program had impressions about what counts as mathematics, and what one does in a math class. Hence, this site was valuable in that in its novelty the parents' and children's understanding of what it means to do mathematics had the potential to be suspended, if only briefly. The setting worked to encourage rich and complex mathematical activity among the participants.

A problem, for me, doing this research was (still is) to develop ways to think about mathematics cognition all at once. An enactive interpretation, through the use of a fractal

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[1] Interestingly, biographies of many female mathematicians indicate that as children and teens those women had a significant person in their lives who encouraged their mathematical thinking outside of the traditional school (Perl, 1993).

metaphor, opened up possibilities to point to the minute and particular in the participants' complex interactions without having to reduce mathematical knowing to fundamental components of one sort or another (if one could). Fractal geometry provided me with a way of layering interpretations of the complex phenomena I was observing, based on different "scales" of interactions to produce self-similar layers of understanding. I have found it useful to think of multiple interpretations as layers that fit together (fractal-like structure) to form explanations of mathematics knowing in action. Between these layers are permeable boundaries that allow interaction among the layers; thus, each layer has the potential to affect another, which affects yet another layer and so on. (Maturana and Varela's (1992) explanation of cognition provides an interesting example of layered interpretation which shows self-similarity at the same time as growing complexity and complicity.)

I suspect that the real value of using a fractal metaphor has been that it opened up space for considering dimensions of mathematics knowing that are not often discussed in research reports about mathematics cognition (chapter three). In my work, I used the fractal metaphor to interpret mathematics cognition as bringing forth a world of significance that includes personal, social and cultural dimensions of knowing all at once. As an interpretive device, the fractal metaphor helped me understand those instances where something appeared to be one thing (personal mathematical thought, for example) and, at the same, time was observed to be of another kind (cultural or social).

In some ways, transcripts and vignettes are the best interpretation of people's mathematical activity because such forms offer context, and create a site of interaction for the reader to imagine (in our case) a parent and child doing mathematics and the many things that might involve. At the same time, it is only through a close reading of the vignettes and a discussion among educators and researchers that the many fractal layers of interpretation are created, and an understanding of mathematics cognition is brought forth.

Ordering One's Orderings

Throughout this thesis, I have been putting order to my observations of people's mathematics knowing in action by offering cases (specializing), drawing distinctions and making generalizations from patterning those distinctions to create models for further observations of the particular cases. As I discussed in the second chapter, my method of studying mathematics knowing in action involved an iterative and recursive process, which I described and displayed as a fractal research cycle. The method was iterative in that the cycle was engaged repeatedly, and it was recursive in that with each cycle, new understanding had the potential to change previous knowing hence creating different possibilities for understanding prior distinctions and understanding and potentially transforming the process of understanding itself.

Although I have described some of the ways in which my understanding was transformed by doing research, the changes in my understanding are not easily noticed in my writing. It may be that, the only way in which they appear is as contradictions or unmarked distinctions in the way I express or use a concept. For the most part, however, the many passages, sentences and words that were removed or replaced from the early chapters as I wrote new chapters and rewrote the older ones as part of my growing understanding cannot be seen; therefore, the reader can only take my word (given his or her experience) that indeed the research process was, not only iterative, but recursive; as new understandings were enacted previous ones were transformed. Writing and rewriting was (is) my knowing in action and even as I write this word I observe my own knowing changing.

A striking example of observing my own knowing change was in the process of trying to understand the relationships between the personal, social and cultural dimensions of knowing and one's bodyhood (Maturana, 1988). Recall the case of Desie and Joss who had engaged in activity around doubling. Prior to writing chapter seven, I had already worked with the transcripts and other artifacts of their activity when I wrote about Joss's knowing as triply-embodied (Simmt, 1998). Six months later, I wrote about the ways in which Joss's emotions were implicated in his knowing (Simmt, 1999). When I came to work with the transcript (and other artifacts of their activity) for the purposes of the dissertation, the case addressed me differently than it had in either of the two previous encounters; although I still understood the case in those terms. I began to understand Joss's and Desie's actions as enacting their "selves." Up until this point, I understood my study as one about mathematics knowing in action. However, with this third "reading", I began to understand my research as trying to make sense of the knower and the knowing as co-emergent or bringing each other forth in action. My understanding was transformed in a recursive act of ordering my orderings.

A similar phenomenon was observed repeatedly in the mathematical actions and interactions of the people with whom I interacted in the mathematics program. Mathematics knowing in action seems to be full of such activity. Recall how Dan and Kerri (from chapter three) put order to sets of tiles, kept a record of the arrangements, and then ordered those arrangements with a table of values. Using the numbers from the table, a generalization for anticipating the number of arrangements without actually constructing all of them was created in another act of ordering. With each ordering, new understanding was brought forth. Another example of this is illustrated by Cathy's construction of a table in response to the diagonal intruder prompt and interactions with her father and me (chapter six). She took her table (an ordered list of the number of unit squares the diagonal passes through for the $4 \times n$ rectangles) and transformed the information in it to a set of differences. She deliberately brought order to her table of differences when she preserved the counting-by-two pattern by marking the exception with "don't count." Ordering is a very deliberate act. Similarly, in chapter five, Roberta's move from the iconic representations of the rectangles she had constructed, in response to the diagonal intruder prompt, to a graph of the relationship between the variable dimension of a $3 \times n$ rectangle with the number of unit squares the diagonal passes through is another example of ordering. In this example, we observe how Roberta generalized from her ordering by being able to predict values for cases she had not drawn, and by noticing that she marked the correct ordered pair for the *3 x 12* rectangle even though she had incorrectly drew it, and recorded the wrong count on her drawing. Engaged in noticing her own thinking, rather than the artifacts of her previous thinking, Roberta did not appear to observe the mistake or the contradiction between the graph and the count from the drawing of the *3 x 12* rectangle.

The case of my research knowing and each of these examples from parent-child activity suggests to me that one of the characteristics of the knowing in action that I have been observing is ordering one's ordering.

Variable-entry Prompts

In the parent-child mathematics program, the need to provide suitable prompts was an occasion for me to listen more carefully to the participants, as I created a curriculum with them that was worth living. In reflecting on my choice of the word prompt, I realize that the choice was pragmatic. That is, I knew that the participants had quite distinct histories in mathematics: they were of different ages, they had different interests and so on. On the other hand, what I did not realize, except in reflection, was that variable-entry prompts were my means of establishing a relationship with the participants and addressing their needs (in a broad sense) as well as the opportunity for them to reformulate, moment by moment, the nature of the questions they were exploring.

The first half of this chapter suggests to me that doing the research involved a similar form of activity. I would like to suggest that the parent-child mathematics program was a variable-entry prompt for me. It was a place I could enter into research with my history of experiences but without a specific research question. As a prompt for my research, the site and my interest in the participants mathematical understanding was sufficient to generate and sustain my research program. In much the same way that the prompts were transformed by the participants in the mathematics program, the prompt for my research evolved and was transformed throughout the process of doing research as I reformulated questions, adjusted my focus, reinterpreted bits of interaction and so on.

In the examples of both mathematics knowing and research knowing, the ways in which the variable-entry prompt was taken up by participants in the former case, and the researcher in the latter instance, suggest how knowing is co-determined by the knower and his or her environment. That is, these examples of knowing in action demonstrate how in interactive spaces that were enriched by the variable-entry prompts many actions were good enough to sustain and propel the persons' knowing. Had the initial prompts been very narrow and the environmental constraints very strict, the participants would have had to engage in specific actions in order to foster mathematical thinking. Variable-entry prompts contribute to an "enactivist" environment for knowing in that they invoke a proscriptive logic in which what ever is not forbidden is allowed. Given adequate actions and interactions, the sphere of behavioural possibilities opens up and there is potential for further acts of mathematics knowing. This is quite different from the prescriptive logic that is often found in school mathematics; where the student's action must match a prescribed one in order to keep the student's sphere of behavioural possibilities from constricting very rapidly or even collapsing totally (with respect to mathematics).

Occasioning

Of particular interest, early in my research, was what "caused" these parents and children to do what they did. In part, this question was triggered by my search for appropriate prompts but, more so, it was occasioned when I studied various artifacts of the parent's and children's mathematics knowing. It seemed obvious to me that what I did, the prompts I offered, the questions I asked, and the comments I made were implicated in the actions and interactions of the parents and children in the sessions. However, it was also clear that although the participants' actions and interactions shared similarities, what the participants did with the prompts could also be quite distinctive between parent-child pairs and to some extent within each parent-child pair. Maturana's and Varela's (1992) notions of structural determinism and structural coupling were useful to me; their notions suggested, to me, that I must consider the interactions among the participants to observe how the prompt was taken up and transformed by them.

I suggest that, in part, the very distinctive behaviours (and those that share commonalties) can be attributed to the multiplicity of dimensions in which a perturbation can be taken up. What might be viewed by an observer as a unitary perturbation can be taken up in many ways by the knower. For example, when Desie took out the calculator to compute the doubles of the large numbers, Joss saw the calculator as something more than a tool to solve the problem immediately in front of him. He also was occasioned to imagine how he would like to carry it everywhere. Although Desie and Joss's use of the calculator could be viewed in the moment as electronically assisted computation, Joss's comment reveals that such activity is permeated by and permeates other dimensions of his existence. In examples like this, I find it useful to invoke a fractal image because it helps me to understand how, at once, Joss can be doing calculations with a calculator to satisfy the constraints of the prompt and his own desire for a quicker way to compute the doubles and, at the same time, be thinking about how the calculator is a neat thing to carry around everywhere. If we observe knowing as complex and having a fractal structure, then even when a prompt is offered with the intention of focusing knowing in possibly just one of these dimensions we observe that it is the knower that takes it up and integrates it into his or her experience. Therefore, the prompt can occasion knowing in a dimension that was unanticipated or in more than one dimension, all at once.

Further to this, another layer of interpretation suggests that we can understand the prompt, itself, as a occasioning multiple interactive spaces for the person who offers the prompt. That is, the one offering the prompt does so with intent. In the case of the parent-child mathematics program, I offered the prompts as a way of inviting parents and children into mathematical activity; but, immediately, that put us in relation to one another and it occasions the social interactional dynamics between us. At the same time, the prompt is an expression of my mathematics knowing and implicates the broader mathematical community in the actions and interactions in the parent-child mathematics program. The culture of mathematics is brought forth in the prompts I offer, the questions I ask, the responses I acknowledge, and the practices and explanations I validate in my interactions with the participants.

The occasioning of mathematical understanding is not unidirectional from parent to student (which is obvious in any of the illustrative cases discussed in my thesis) or from facilitator or teacher (researcher) to participant; it is reciprocal. The mathematical actions and interactions of the participants and the artifacts of their knowing occasioned my mathematics knowing. In those conversations which were occasioned and sustained by the participants' mathematics knowing in action, the culture of mathematics was brought forth.

Interaction

I have spoken at length now about the role the prompt played in the mathematical actions and interactions of the participants. I noted that, in fact, it is one of the ways in which the parents and children were occasioned to participate in collective patterns of behaviour that I understand as mathematics. However, this was only one part of the interaction that occasioned and sustained their mathematical activity. The participants selected energy-rich matter from their environments and proceeded to engage in mathematical activity as they were occasioned, moment by moment, in their interactions with the prompt, each other, me, their own thoughts, and the interactions of others. I observed the interaction between the parent and the child as significant to the mathematics knowing that emerged in action. In this section, I discuss three patterns of interaction (Kieren, Simmt, Gordon Calvert, Reid, 1996).

Parallel action

In the case of parallel action there are few opportunities for occasioning between people because if there is any interaction between participants, it involves little mathematics. There were very few instances in the parent-child mathematics program where the adult and child worked strictly in parallel, not even referencing each other's work. The parent-child mathematics program was not a situation conducive to strictly individual activity.

Co-referencing

I use the term co-referencing (Simmt, Kieren, Gordon Calvert, 1996) to refer to those situations in which the participants do not work together on the task but do interact, particularly to reference the other's work or to ask a question that satisfies a need of one of them. This is illustrated in the case of Roberta and Kristina (chapter five). Although they did talk to each other, for the most part each of them simply checked to see if her records matched with the other's. As a result, there was little evidence that their interaction occasioned each other's thinking in any remarkable way; for example, their interactions did not serve to sustain their mathematical activity. One might also interpret a good portion of Cathy's and Jake's (chapter six) interaction or Cathy's and Rebecca's (chapter three) interaction as co-referencing. In the case of Cathy's interaction with her father, we notice that they work quite independently of each other, likely because of different understandings of the situation, the different skills that each of them bring to the task, and the ways in which they prefer to work. However, it is interesting to note that Cathy's actions bear a resemblance to her father's; hence we might conclude that in fact she was occasioned by interacting with the artifacts of his interactions. In contrast, Cathy and her mother also were observed to co-reference but Cathy and her mother's ways of working are quite similar. This may be why Cathy and Rebecca are seen to be occasioned by each other even though they do not work together. In this case, we might say they co-occasioned each other's knowing.

Co-occasioning

When two people interact in such a way that their mathematical knowing is both propelled and sustained by interacting with the other, we call this co-occasioning (Simmt, Kieren, Gordon Calvert, 1996). Rebecca and Cathy demonstrate this at times in their interaction but not to the extent that Desie and Joss do. Throughout the session, in which Desie and Joss consider the doubling problem (chapter seven), we observe the back and forth nature of their conversation, and the many times when Desie's comments or questions (for example) occasioned Joss's knowing which in turn occasioned Desie's knowing and *vice versa*. In my opinion, one of the remarkable features of co-occasioning is that we witness reciprocal learning—both Desie and Joss learn from their interactions. When two (or more) people interact so intensely, there are many opportunities to enlarge their spheres of behavioural possibilities; hence, they have more energy-rich matter from which to select to transform their mathematical understanding. Dan and Kerri are another parent-child pair with whom we note a great deal of co-occasioning (see chapter three and five). In their case, there were times when their interactions were so tightly braided that it is difficult for me to ascribe the mathematics knowing that emerged to one of them or the other.

Fully Embodied Knowing

What do I mean then when I suggest that mathematics knowing is fully embodied? In the first place, I am suggesting that mathematics knowing is the utterance or gesture that co-emerges with the knower (Merleau-Ponty, 1962). It is that which arises in the actions and interactions of people in language. What ever else mathematics is, I work from the perspective that mathematics is human activity and as such exists in the phenomenal domain that arises for humans in languaging. Mathematics knowing is fully embodied in that it is an act of the bodyhood, the braiding of languaging and emotioning. At the same time, mathematics knowing in action embodies personal thought, social relationships and cultural forms and practices. In other words, an act of mathematics knowing is at once personal, social, and cultural (Davis, 1996: Simmt, 1998) and it involves interactions among the interactions of these dimensions.

Social relationships

Much is being said about the social dimensions of mathematics knowing (see for example, Lerman, 1996; Confrey, 1995a; Cobb, 1995). Interestingly, however, most of those discussions are focused on the role of social interaction in learning mathematics. When I say that mathematics knowing is social, I am certainly supporting such claims. However, I am also suggesting that mathematics knowing involves social or interpersonal relationships which co-emerge with mathematics knowing. I pointed to this view of the social domain of mathematics knowing in chapter seven in my discussion of Desie and Joss's mathematics knowing in action. Specifically, I discussed this in terms of Joss's relationship with his mother, his peers at school and his teacher as they co-emerged with his mathematics knowing in action. Here, I will elaborate with a different example.

Dan and Kerri (chapters one, three and five) participated in more sessions (26) than did any of the other pairs. When they first came to the program, Dan was very likely to take a "teacher-like" role in their interactions, and he asked Kerri very pointed questions which served to controlled her actions. However, Dan and Kerri's interaction patterns changed quite remarkably over the course their participation. Rather, than controlling her actions, Dan encouraged Kerri to take the lead in responding to the prompts, and then he supported her actions in a number of ways: he often kept records, he asked questions that he thought would help them better understand the problem, he asked her to explain her thinking, and he explained his thinking to her. There are likely a number of reasons for this change. One reason might be that many of the prompts were novel to him; as Kerri had to, he also had to work through the prompts in order to understand them. With the changes in the way they interacted, their relationship as co-participants developed and replaced the teacher/student relationship that was first enacted in their interactions. By the second and third set of sessions, Dan and Kerri's patterns of interaction had become quite stable; they interacted intensely, each playing a role in the mathematics knowing that emerged. Dan and Kerri provide a wonderful example of how a relationship contributed to joint mathematical activity and how doing mathematics played a part in forming patterns of behaviour and, indeed, the relationship between two people.

Cultural forms and practices

With an example taken from Joss and Desie's interaction, we can note the ways in which their interactions are interactions not only in the local activity but how they fit with community practices. Recall, Joss commenting to his mother about her use of a comma to write a large number and then how he teased her. In that very brief exchange, many dimensions of Joss' knowing were triggered: his relationship with his mother, his relationship with the culture; and his relationship with his teacher. Further, that which is observed as the culture of mathematics was implicated in his act of knowing. Even as a child, his practices potentially change the culture of mathematics by changing the world in which he lives and, therefore, potentially changing the worlds of others.

Another example of the multiple dimensions of knowing and the way in which these play out as interactions among interactions involves Cathy and Jake. Recall how Jake insisted Cathy proceed in a step-by-step manner when she drew rectangles for the diagonal intruder prompt. At first, I saw this interaction as social, in that it facilitated their joint action (and kept them in relationship as co-participants in mathematical activity). I also interpreted Jake's directions as being about saving paper. On one hand, his concern may have been economical or possibly relational (in terms of father teaching and directing daughter). However, maybe his comment was more about mathematics than I first thought it was. That is, Jake anticipated the need to order their results from the first stage of their inquiry, and doing the task in a step-by-step fashion would order the results, in the first place. I might observe his actions as a feature of his personal knowing; however, I conjecture his actions are more than a local act in that they have the potential to propagate a pattern of behaviour noted among mathematicians. Such systematicity is a common practice in mathematics, and when Jake insists that his daughter act this way he is ensuring she act within the practice of mathematics (as he understands it).

Personal thought

It is common in constructivist research to ask questions that are focused on the personal knowing of the individual, particularly as it is understood in relation to the schemata the child is constructing (see Steffe and Gale, 1995). In my work, I considered personal thought from a somewhat different perspective and found myself asking a different question than the constructivists have been asking: In what ways is the knower brought forth (transformed) by knowing mathematics? In this exploration, I observed for their utterances and gestures, and I explicitly considered how their knowing was oriented by their emotions, at the same time, as their knowing triggered those emotions. For example, when Kerri, after generating a number of tiling patterns with dominoes, realized that she had refuted her conjecture she uttered, “Oh shoot! That doesn’t follow my theory.” She was disappointed (but not frustrated) and persisted, looking for another relationship. On the other hand, when Joss was unable to make sense of my explanation of the number pattern I saw in his magic numbers, he became frustrated and lost interest in both mathematics and maintaining our relationship, and eventually removed himself from the room. The personal knowing that was brought forth in action and interaction was observed in terms of the knower’s utterances, gestures, and emotions.

All at once

Because I have separated the personal, social and cultural dimensions of human mathematics knowing in the previous passages, I need to observe that my research suggests that each of these is fully complicity with the others. In other words, just as I speak about certain actions or interactions that suggest mathematics knowing is personal (for example), I am able to see how it is also cultural. Desie and Joss's excitement at the doublings hitting the hundreds, thousands and millions marks is an example of how something is at once personal and cultural. The fractal image has been most useful to me in this regard (Figure 8-1). It reminds me that whenever I observe a particular action or interaction as personal, social or cultural, I am likely to see the other dimensions of that knowing act by making distinctions at a differently level or by attending to different features of the interaction. This suggests, to me, that this is why the radical constructivists, social constructivists and the socio-cultural theorists all are able to observe the same interactions and make different distinctions. Knowing is at once personal, social and cultural.

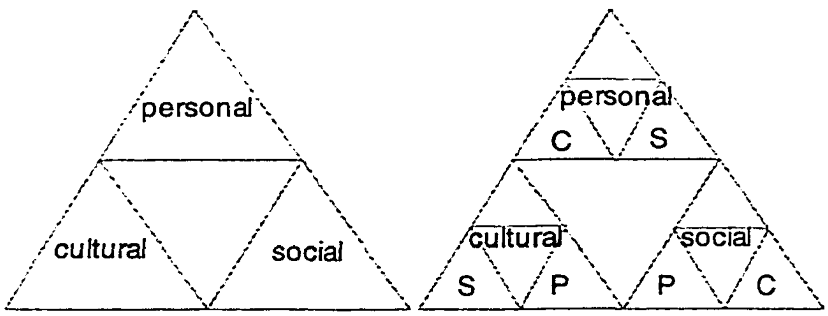

Figure 8-1. Knowing is observed as personal, social and cultural all at once

Bounding Observations and Interpretations

In this study, I considered mathematics knowing in a very local context and discussed why I view it as fully embodied. However, the enactive perspective and a fractal analysis requires that we think about self-similarity, and ask questions that involve making distinctions at different levels. I have demonstrated one way of doing that with the fractal interpretations. There are other possibilities, as well. Rather than focus on the individual, what if we focus on a group—like a mathematics class? In what ways can we say that the mathematics knowing of the class is brought forth through perceptually guided action? How might we characterise the interactions of the group? Kieren (1999) has provided leadership in this area as he studies mathematics knowing from an enactivist perspective with students in mathematics classes. Bauersfeld (see Cobb and Bauersfeld, 1995a) and his colleagues also have been studying knowing in the context of the classroom as a microculture. This, too, has been a very generative area of study, and is very closely related to the work done from an enactivist perspective. I expect it would be very useful to compare the enactivist and interactionist theories of mathematics knowing.

Further to the suggestions above, I think there would be great value in developing methods of enactivist inquiry or at least studying the methodological implications of an enactivist perspective. Currently, this is a weakness in the field. Although there are a number of people who are using (what I observe to be) enactivist notions of cognition to inform their work (Kieren, 1999; Davis, 1996; Reid, 1995; Gordon Calvert, 1999; Confrey, 1999; Bauersfeld, 1995b; Mason, 1999), there has been little published (Davis, 1996; Kieren, 1999) in terms of theoretical discussions that develop the implications of enactivism as a mode of inquiry in mathematics educational research. Without those discussions, it is difficult to situate one's work and to further develop enactivism.

The enactivist perspective occasioned me to attend to the personal, social and cultural layers of mathematics knowing in action but I was not exhaustive in my exploration of the fractal filaments of any one of these layers. For example, I did not study the parent-child relationship in any depth. Nor did I study the participants' histories with mathematics, even though structure determinism points to the significance of their histories. Finally, I did not study the political or economic dimensions of mathematics knowing in action. All of these would be well worth considering in future studies as they would help us better understand the multiple dimensions of mathematics knowing. Given my work did not address these areas but instead focused on the mathematics knowing of some parents and children in a very particular context, I will restrict the discussion of the implications of my study to two points: making space for the other and living the implications of our knowing.

Implications of Fully Embodied Knowing

In my study, I used an enactivist perspective of cognition which takes into account the complex, contextual and co-emergent mathematics knowing that is demonstrated when, in this case, a parent and child interacted with each other in mathematical activity bringing forth mathematics knowing in the moment. This enactivist perspective views mathematics knowing as highly contingent (Kieren, 1999); it makes a difference who is doing the thinking, where and with whom the thinking occurs, when it happens, and why it is happening.

Making Space for the Other

If we understand mathematics knowing as emerging from a person's actions and interactions, then we reason that those actions and interactions are implicated in one's mathematics knowing by co-determining it. This is what I wrote about in chapters three through seven. Reflecting on my interpretations expressed in those chapters, I come to understand that by asserting that mathematics knowers co-emerge with mathematics knowing, I am suggesting mathematics is not inside or outside of us but about us (Davis, 1996, p. 235). It is at once our thoughts, our relationships, and our cultural forms.

Why do mathematics? Because mathematics is an opportunity for self-realization in communion with others. Not in the sense of getting to know one's self or uncovering one's potential (so to speak), but "to realize" in the sense of making real, by bring forth a world of significance, which includes mathematics, with other people. Mathematics should be part of growing up (Kieren, in press); it is a feature of our humanity.

What does it take to know mathematics? My interpretations of mathematics knowing suggest that it co-emerges with us (knowers) in our conversations (languaging and emotioning) with others. In general terms, in order to know mathematics we must live in community with others because knowing mathematics involves interaction in the personal, social and cultural dimensions of our humanity. That interaction requires love (in Maturana's and Varela's sense (1992)); that is, making space beside ourselves for others.

I do not think it is a coincidence that within the parameters of the mathematics program for parents and children we were able to engage in and observe mathematics knowing that was different from many of the participants' experiences with school mathematics. The parents and children came to the program with the intention of being together and were willing to do mathematics together. This is not to say that everyone who came found the experience satisfying or educational. For example, in spite of the fact that Roberta was able to engage in mathematical thinking and work through some good mathematical "problems," the program did not satisfy her needs. The community with which she was hoping to interact required she could quickly respond to "word problems". That was not the form of mathematical activity that I encouraged with the math program; I was not making a space beside myself for Roberta, and neither did she make space for me. On the other hand, there were many parents and children who managed to make space beside themselves for the other, and, in doing so, they engaged in mathematics knowing together. Dan and Kerri, Rebecca and Cathy, Jake and Cathy, Calvin and Jocelyn, Greg and his mother (and many more I have not mentioned) all made space beside themselves for each other and for me. With my prompts, questions and comments I invited them to exist beside me, in a space that included mathematics. They responded to my invitation by engaging in mathematics. They reciprocated my invitation by asking me questions and offering their explanations. They shared in their humanity through mathematics.

Living the Implications of Knowing

When the parents, children and I engaged in mathematical activity, our activity intersected with the personal, social and cultural domains of our lives. In action, we brought forth worlds of significance with others, and in doing so we brought forth ourselves. In each act of bringing forth a world of significance and our "selves," we anticipate the future as our spheres of behavioural possibilities expand, making possible our next utterance, movement, action, and thought, and this happens in interaction with others. What we do, what we say, what we know makes a difference not only for ourselves but for others. As Davis (1996) suggests:

> Knowledge, rather than being understood in objective or subjective terms—whereby persons and their understandings are regarded as essentially isolated and autonomous—is recast as those patterns of acting that allow our structures to be coupled, thus entangling us in one another's existence and implicating us in one another's knowing.

This was certainly witnessed in the very local actions of the children and parents in the program. I speculate that in mathematics classes where the teacher encourages interaction among students by having them investigate together, pose questions to each other, and offer their explanations in the community, the teacher facilitates an environment in which the students' mathematics knowing has consequences for others, not just themselves. Not only does this form of teaching allow for growth in mathematical understanding (as has been well demonstrated in the work of Bauersfeld (1995a), Yackel and Cobb (1996), and Pirie and Kieren (1992)—just to name a few), but with this kind of teaching our children and youth will have opportunities for understanding how one's knowing is not without consequence for others (Kieren, 1999).

I return now to two events (involving a parent and a child) that played a significant role in my understanding of mathematics knowing in action, by triggering emotions which oriented my knowing in a difficult moment while doing this research. The first event is recounted in the preface. It involved finding the sheet with my son's name on it in his teacher's handwriting. Annoyed with the exercise she had him do, I wondered how she could have determined with nothing more than circles around typewritten words that it was my son's worksheet. Then I recognized myself. I thought about my own assessment practices, which included the use of multiple choice tests; and I began to understand what such a form of testing was (or was not) about and pangs of guilt struck me. The second event I want to return to is the one in which the mother told me she was glad she had come to the math club because she got to know her son in a way she had not known him before. On one hand, her comment pleased me but, on the other, it aroused my curiosity; I began to think about what it means to know someone in the context of mathematics. In what way does the teacher know my son by the response he circles on a test? In what way does the mother know her son differently by doing mathematics with him? What does it mean to say, "I know?"

One thing my research has taught me is that when I say I know, it is much more than knowing some "thing." It is the profound understanding that we live with the implications of our knowing, and because we live with others, our knowing has implications for them. That it is why, when I hear a mother say I know my son better, I am moved to tears. I hear her utterance as her recognition that she understands better how to make space for him to exist beside her. It is why, when I reflected on my choice to use a multiple choice test for the purpose of getting to know what my students knew, that I feel guilt. I now understand the implications of my knowing in action.

The pedagogical implications of my research are simply stated but not so simple. Mathematics knowing is a human activity. Hence, it requires we make space for others to exist beside ourselves. How do we do this? The parents and the children in my study are good examples. We must make space for the other in two ways: the first is to listen for the domain in which the other is acting, and the second is to invite him or her to listen for the domain in which you are acting. The listening I am speaking of is not evaluative listening but hermeneutic listening (Davis, 1996).

> [Hermeneutic listening] is an action that locates her [the teacher] in a complex web of existence—caught in intertwining and evolving lines of text from which one cannot extricate oneself. The teacher is not guiding a sight-seeing tour through a thoroughly mapped-out region but is dwelling in, with, and through the complexity and ambiguity of emergent knowings A full participant in the learning that is occurring, the teacher is part of the simultaneous transformation of knower and known, culture and mathematics (p. 264).

My research has transformed me. I know that I must live the implications of my knowing.

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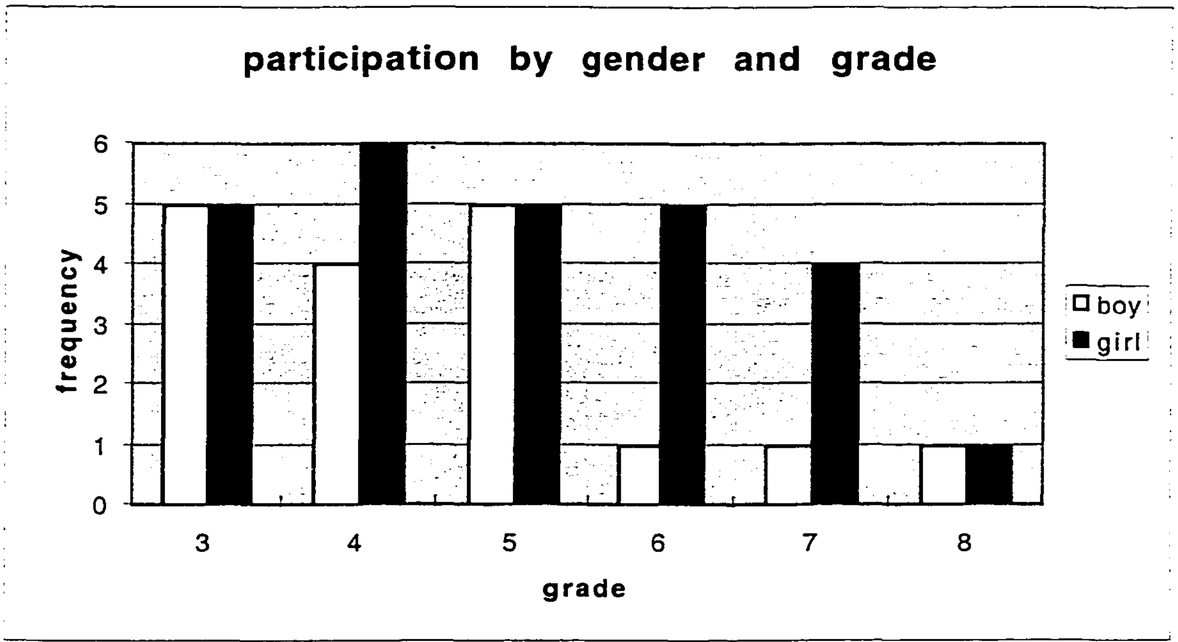
Appendix A
Summary of Sessions

Term	Location	Participants			Forms of Data Collected				
Colla-borators	# of sess-ions	# of child-ren	Boy to Girl ratio	Dad to Mom ratio	Field notes	Participant working papers	Audio tapes	Video tapes	Pre-session information sheet
Fall 1994[1] (Reid)	School (10)	8	2:6	5:3	yes	no	no	no	yes
Winter 1995 (Gordon Calvert)	School (10)	5	1:4	3:2	dialogue journal	selected	selected	no	yes
Fall 1995 (Kieren)	School (10)	8	3:5	2:6	yes	selected	selected	no	yes
Fall 1995 (Kieren, Reid, Gordon Calvert)	Uni-versity (1)	2	0:2	1:1	yes	yes	yes	yes	yes
Winter 1996	Uni-versity (5)	3	2:1	1:2	yes	yes	yes	yes	yes
Winter 1997 (Johnston)	Uni-versity (5)	7	6:1	0:5	yes	yes	yes	yes	yes
Fall 1997	School (10)	10	3:7	1:8	yes	selected	no	selected	yes
Total		43	17:26	13:27					

[1] Simmt participated in all sessions.

Appendix B
Participant Information



Appendix C

Number of Sessions in which Parent-child Pairs Participated

Pair	Number of Sessions
Dan and Kerri	26
Rebecca (Jake) and Cathy	11
Desie and Joss	5
Roberta and Kristina	5
Jolene and Calvin	10

Appendix D

List of prompts used in the sessions

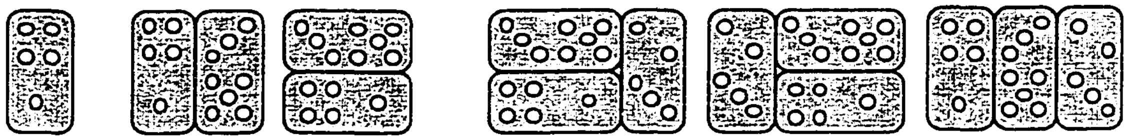
Although I have used all of these prompts in parent-child sessions, I have not used all of them each term. Some of the prompt have been used frequently (I have marked these with a *) whereas others have been used only once or twice.

Handshakes*

How many handshakes would there be if twenty people were in this room and each person shook hand with each of the others just once?

Tiling Paths:*

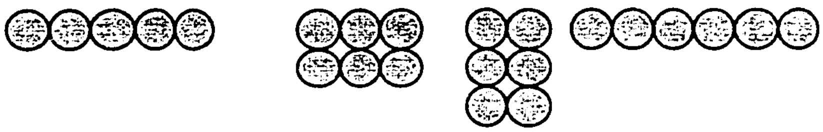
How many paths can you tile with a given number of dominoes (2x1 tiles) if the path must be two units wide. There is one path for one tile, two paths for two tiles and three paths for three tiles.



1 tile	2 tiles	3 tiles
1 way	2 ways	3 ways

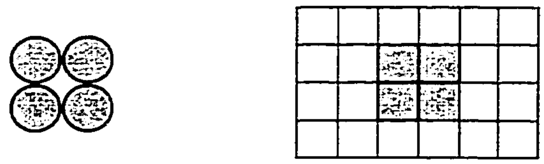
Rectangular Numbers*

Using bingo chips find the numbers which form rectangles. For example, 5 tiles can only form a 1x5 line whereas 6 bingo chips can be arranged in a 2x3 or 3x2 rectangle as well as the 1x6 line. We will call those numbers of chips for which we can form a rectangle—other than the 1x case—rectangular numbers.



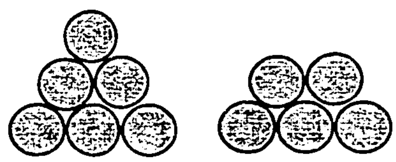
Square numbers

This is a variation of rectangular numbers and is done either with bingo chips or on graph paper. For which areas (numbers) can you produce a square on graph paper. For example, 4 chips can be arranged in a square.



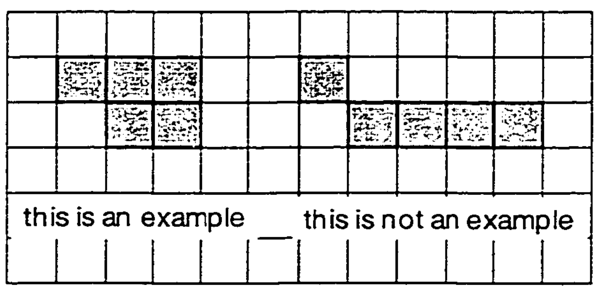
Triangular numbers

For which quantities of bingo chips can you form a triangle? For example, six chips can be arranged into a triangle but five cannot.



Pentominoes

Using graph paper, you are to make as many shapes as you can using five squares. The squares must be touching another square on at least one edge.



Common Letters

Which do you think is the most common letter used in the English language? Using a book, newspaper or magazine, try to determine the most common letters.

Halloween Statistics

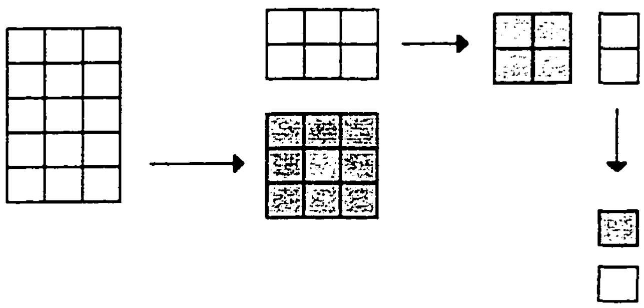
Without showing each other your candy bag, find a way to show the rest of us how much candy you collected on Halloween and the various kinds of candy you collected.

Mobius Bands

Take a strip of adding machine tape and tape the ends together. Now trace the path an ant would take walking along that path. How many sides does the band have? Cut the band along the ant's path. How many bands do you have now? Now do the same thing but put a twist in the band before you trace and cut the ant's path. Can you predict what will happen? What if the number of twists increases? What happens then?

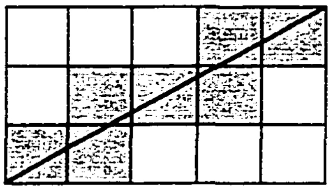
Square Take-away (Mason, Burton and Stacey, 1982):

Cut a rectangle (not a square) from a sheet of graph paper. What is the largest rectangle that can be cut from your rectangle? How many rectangles can you cut before you are left with a square? Try this for a number of different rectangles. What do you notice?



Diagonal Intruder (Stevenson, 1992)

Mark off a rectangle on a piece of graph paper. Draw in one of the diagonals. How many squares does the diagonal pass through?



Fractal Cards (Simmt and Davis, 1997)

The students construct a fractal card and then investigate the growth patterns. See reference for instructions.

Fraction Kits (Kieren, Davis, and Mason, 1996)

Using a fraction kit the students investigate equivalency, and addition and subtraction of fractions. See reference for description of activity.

Hexaflexagons (Dubiel, 1994)

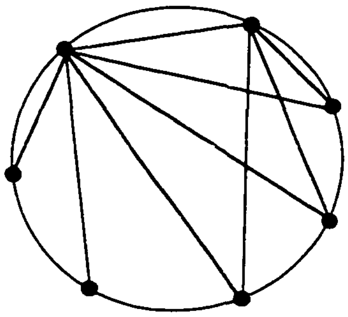
Students are given a pattern from which they construct a hexagon which folds into itself. This activity is related to Mobius Bands.

Straw constructions

Students are given scissors, tape and straws and instructed to construct a three dimensional polyhedron.

Rosettes*

Mark a set of points on a circle. How many lines will it take to connect all of the points to each of the others? Do this for different numbers of points on the circle.

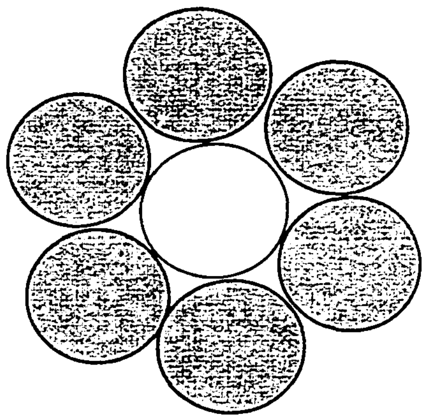


Lego© towers*

Take five pieces each of a different colour of Lego. How many different towers can you build from those five pieces? For example, a tower from the ground up might be ordered red, blue, yellow green, and white. We will call that different from a tower that is ordered red, white, blue, green, and yellow. How many different towers could you build if you had more colours?

Rings of pennies

Begin with a penny and place a row (or circle) of touching pennies around it. How many pennies are there in this row, now build another row around the row you just built. How many pennies did that take? How many pennies would the 10th row take? By the 10th row how many pennies would it take in all?



Rice bowl

A greedy land owner was approached by a peasant who asked for the opportunity to work. The greedy land owner offered the peasant a single bowl of rice for a day's work. The peasant declined the bowl of rice and instead asked for just a single grain after his first day of work and if the land owner was happy with the peasants work then the peasant requested that he be given double that on his second day, just two grains of rice, and on each day thereafter twice as much as the day before. The greedy landowner rubbed his hands in delight and agreed to the wage.

If you were the peasant which would you take? You might begin by thinking about how much rice do you think there is in a bowl and how much rice the peasant have eaten by the end of one week, two weeks and so on.

Number of squares on a chess board

How many squares are there on a chess board (8x8) ? (I say there are more than 64.)

Walk the Plank

Students pretend they are walking the plank on a pirate ship. They use two dice to instruct their movements, one to indicate direction and the other to indicate magnitude. The purpose of the activity is to find the optimal plank length in order to play the game for a suitable amount of time.

Pattern tiles

Students are offered a bucket of pattern tiles which include: regular hexagons, equilateral triangles, trapezoids and squares. They are instructed to make patterns with the tiles. The tiles are such that 2 trapezoids or 6 triangles form the hexagon. The length of the side of the square is congruent to the length of the side of the triangle.

Halloween candy count

Students bring their bag of Halloween treats to class. Their task is to report to the group about the kinds and number of treats they collected without showing us every treat in their bag.

Stars

Participants were asked which stars could be drawn without lifting their pencil from the paper or going over the same line twice.

3-D drawings

Participants were given building blocks and isometric dot paper and instructed to produce three dimensional drawings of objects they built with the blocks. Once they had a drawing they switched with each other and tried to reproduce the objects from each other's drawings.

Prompts taken from children's picture books

McKibbon, H and S. Cameron (1996). *The Token Gift*. Buffalo, NY: Annick Press.

Friedman, A and K. Howard. (1994). *A Cloak for the Dreamer.* New York: Scholastic Inc.

Anno, Masaichiro and Anno, Mitsumasa (1983). *Anno 's mysterious multiplying jar.* New York: Philomel Books.

Anno, Mitsumasa. (1992). *Anno's magic seeds.* New York: Philomel Books.

Appendix E: Parent and Child Responses to Questions Concerning Mathematics (completed in first session)

(Note participants' spelling and punctuation has been preserved.)

Anne	Greg
Do you like doing mathematics? Why or why not?	
Yes. Most of it makes sense.	No. It's hard and boring.
What is your favorite mathematical activity?	
not sure	Mony. I like to spend it
What is your least favorite activity?	
Pay bills. Its hard to make ends meet.	Dovison It is hard
Is mathematics easy or hard? Why do you think that?	
Easy. In most parts I understand it.	Both somethings are hard so are not.
What do you hope this class can do for you?	
Help me to help my son and others to understand and enjoy math and see the importance of it.	let me sped up.

Roberta	**Kristina**
Do you like doing mathematics? Why or why not?	
No. I'm very poor at math so it's always been a fear of mine.	No, because I am horrible at it and I don't always understand it (catch on quickly).
What is your favorite mathematical activity?	
Adding, multiplication, fractions and percentages.	Geometry because it is easy and fun.
What is your least favorite activity?	
Math problems that have two trains travelling in different directions at different speeds and knowing how far they will both get in 2 1/2 hrs.	Everything but geometry. (eg. measurement, fractions)
Is mathematics easy or hard? Why do you think that?	
My mind doesn't work that way—but hey, give me a computer & I'll soar.	Some are easy and some are hard. hard=measurement easy=geometry
What do you hope this class can do for you?	
I'd like to eliminate my fear of mathematical calculations and problems.	Not to be bad at math and learn and understand it.

Dan	**Kerri**
Do you like doing mathematics? Why or why not?	
Yes. I like the challenge. I like the feeling of "Oh Yea" when you think of something in a new way or gain a fresh insight.	Yes, because I like figuring things out.
What is your favorite mathematical activity?	
Puzzles. I enjoy finding solutions.	Fractions because they are fun to do and figure out.
What is your least favorite activity?	
Doing processes such as long division, addition, multiplication—it's good to know in an emergency but why bother if you have a calculator?	+, –, x, ÷ because they are boring.
Is mathematics easy or hard? Why do you think that?	
Easier than I think most people do. But there is always room to find a challenge.	It depends on the type of math, but they are in the middle.
What do you hope this class can do for you?	
Look at some basic concepts differently, but most importantly see the "oh yea" surprise or insight on my daughter's face.	To learn new things that I won't learn in school until a higher grade.

Rebecca	**Cathy**
Do you like doing mathematics? Why or why not?	
Not particularly/ Not enough grey areas.	Yes, because you can learn lots.
What is your favorite mathematical activity?	
Spending money.	Art. I need to think.
What is your least favorite activity?	
Grocery shopping/ Too may brand names and ingredients to establish best bang for buck.	I don't have one.
Is mathematics easy or hard? Why do you think that?	
Somewhat difficult if too many variables.	Hard because you have to think very hard.
What do you hope this class can do for you?	
Quality time with Cathy.	My math skills.

Jolene	Calvin
Do you like doing mathematics? Why or why not?	
Not really, because I always found it difficult.	I like it because I like working with numbers.
What is your favorite mathematical activity?	
No response.	My favorite is decimals because I like rounding and adding them.
What is your least favorite activity?	
I guess geometry, algebra.	My least favorite is fractions because I find it hard to understand.
Is mathematics easy or hard? Why do you think that?	
Very hard. I always found math difficult in school and I have been out of school for many years.	I find some easy and some hard. Somethings I can do easily and others are hard to do.
What do you hope this class can do for you?	
New up to date strategies so I can help my children more at home.	I hope to learn more about math.

Desie	Joss
Do you like doing mathematics? Why or why not?	
No. I don't feel very good at it so I fear failure & embarassment.	because I might get very high answers
What is your favorite mathematical activity?	
Playing games that include some math. The game aspect makes it fun.	addition becaus my mom shows me lost of tricks
What is your least favorite activity?	
I think any math is hateful. Because I am not confident.	subtracti~~on my~~ becaus ~~my~~ I don't work that much on it
Is mathematics easy or hard? Why do you think that?	
Hard. I developed what I think of as a block when I was very young so I've always found it hard.	easy
What do you hope this class can do for you?	
I hope to stop this same block from developing in my son so that he can feel confident and do well.	a good agecation

Appendix F: Research and professional papers based on parent child mathematics program

Term on which paper is based	Title of paper
Fall 1994	Simmt, E. (1995). The Math Club. Unpublished paper, University of Alberta.
Fall 1995	Simmt, E. (1996a). Parents and children doing mathematics: bringing forth a world of significance. Paper presented at the International Council of Psychologists Annual Convention, Banff, AB.
Fall 1995	Simmt, E. (1996b). Parents and Children and Mathematics: Occasioning in Mathematical Cognition. Paper presented at the American Educational Research Association Annual Conference, New York.
Fall 1995	Simmt, E. (1996c). $\{\text{Children}\}\cap\{\text{Parents}\}\cap\{\text{Mathematics}\}$: Researching the Intersection. In Y. Pothier (Ed.) Proceedings of the Canadian Mathematics Education Study Group Annual Conference, Halifax, NS.
Fall 1995	Simmt, E., Luce Kapler, R., Johnston, I. and Gordon Calvert, L. (1997). The Texture of Research. Paper presented at the Annual Conference of the American Educational Research Association, Chicago, IL.
Winter 1996	Simmt, E. (1997). Math Connections: A Program for Parents and Children. Delta-K, xxx.
Winter 1997	Simmt, E. (1998). The fractal nature of a mother and son's mathematical activity. Paper presentated at the Annual Conference of the American Educational Research Association, San Deigo, CA.
Winter 1997	Simmt, E. (1998). The fractal nature of a mother and son's mathematical activity. Journal of Curriculum Theorizing, xxx.
Winter 1997	Simmt, E. (1999). The complicit nature of mathematics knowing and emotion. Paper presented at the American Educational Research Association Annual Conference. April 20, 1999.
Winter 1997	Simmt E. (1999). Fractal spaces of mathematics knowing: Collecting personal thought, collective process and cultural form as events of cognition. Paper presented at the American Educational Research Association Annual Conference. April 20, 1999.

Appendix G

Fractals: The Geometry of Complexity and Chaos

Mathematics is an interpretive framework that for centuries has helped us understand various phenomena in our world. As is the case with any interpretative framework, our understanding is restrained by the tools and metaphors offered by it. A recent contribution from mathematics is the notion of fractal. A fractal is a geometric structure that is not easily defined. The term was coined by Benoit Mandelbrot (1982) to name geometrical objects that demonstrate some form of self-similarity and fractional dimension. In his early writings, Mandelbrot provided many examples of fractals but did not explicitly define the term fractal—other than to say that fractals have fractional dimension. As one writer after another used the term it took on more explicit meaning; but, a rigorous definition for 'fractal' continued to be elusive. Falconer (1990) like others defined the notion by the properties and features of fractals: fractals have a fine structure; i.e., detail on arbitrary small scales; fractals are too irregular to describe in traditional geometric language; fractals have some form of self-similarity; fractals have fractional dimension; fractals are usually defined recursively. Finally, fractal structures are often found in assoication with dynamical systems and have been called the geomtry of chaos (Glieck, 1987).

Consider a couple of simple examples of fractals. Take a line segment and remove the middle third, leaving the endpoints. Now do this again for the remaining line segments *ad infinitum*. In the structure remaining (Figure G-1), notice the self-similarity at various levels of the recursion.

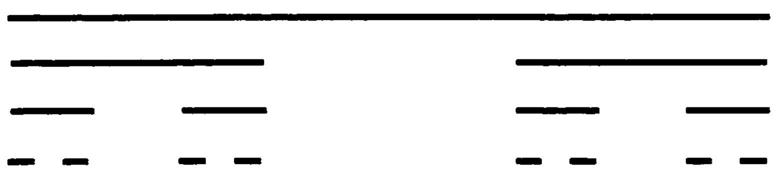


Figure G-1. Cantor Set after two recursions

Recall the process by which the object was defined—a simple rule that was repeated on the object itself numerous times. In nature, many examples of fractals can be found: clouds, mountains, and waves, to name a few. Consider a head of broccoli. Break off a piece, and the piece looks like the head. Break off a piece from the piece, and it still retains the same features of the head of broccoli except that it is smaller. The broccoli has fine detail and is self-similar at many levels of scale.

The Cantor Set and broccoli provide two illustrations of fractals and the properties of fractals that make them an interesting metaphor for understanding mathematics knowing itself. Fractals are irregular yet, at the same time they are self-similar; they have fine detail on arbitrary small scales; and they are usually defined by recursive processes. In the same way that one might study the Cantor Set or broccoli by taking just a piece of either, I hope to show that we can take a "piece" of a person's mathematics knowing and analyze it to inform our understanding of the person's mathematics knowing.

The fractal metaphor is useful for a few reasons. In the research I do, interpreting and explaining mathematics knowing-in-action, there is a need to be able to "handle" complex data. This is not a desire to "reduce" to fundamental elements in an attempt to understand how the whole works. Rather, I am suggesting that by taking a small chunk that is in itself as complex as the whole, we have a better opportunity to attend to the features of the phenomenon without being overwhelmed by the "data." The fractal metaphor provides us with a way of thinking about a phenomenon that has at any one level other levels within it. The fractal metaphor is particularly useful in an enactivist interpretation of cognition because enactivism assumes complexity and requires a non-reductive approach.