MILNOR-WITT K-THEORY AND SYMMETRIC BILINEAR FORMS OVER FIELDS OF CHARACTERISTIC 2

by

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ABSTRACT

The central aim of this paper is to extend the main results of Morel [15] to fields of characteristic 2. In particular we will show that the n-th graded component of the Milnor-Witt K-theory, $K_n^{MW}(F)$*, is the pull-back of the following diagram:*

$$\begin{array}{ccc} & & K_n^M(F) \\ & & \downarrow \\ I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) \end{array}$$

with $K_n^M(F)$ *denoting the n-th graded component of the Milnor K-theory and* $I^n(F)$ *denoting the* $n-th$ *power of fundamental ideal in the Witt ring of symmetric bilinear forms. Our results depend on a presentation of* $I^n(F)$ *due to Arason and Baeza [1] which in turn relies on the characteristic 2 version of the Milnor conjecture proven by Kato [10].*

DEDICATION

I dedicate this to my mother.

ACKNOWLEDGEMENTS

I would like to thank my supervisor, Prof. Stefan Gille, for the endless patience, guidance and advice he has provided throughout my time as his student. I cannot overstate how influential his role has been in the development of my mathematics.

Completing this thesis has been the greatest academic challenge of my life thus far and is owed in no small degree to the continual support of my parents whom I appreciate more as each day passes.

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CHAPTER 1

Introduction

A classical result in commutative ring theory due to Serre [18] asserts that if R is a commutative Noetherian ring of Krull dimension n and P is a projective R-module of rank m then $m > n$ implies there exists a projective R-module P_0 such that

$$P \cong P_0 \oplus R^{m-n}, \text{ i.e. } P \text{ splits off a free summand.}$$

However, if $rank(P) = dim(R)$ this is not always the case. To tackle this problem, several attempts have been made to construct an obstruction class similar to the Euler class in topology. One construction by Barge and Morel [3,4] uses the Milnor K-Theory of a field F denoted by

$$K_*^M(F) = Tens_{\mathbb{Z}}(F^\times)/(u \otimes (1-u)), \quad u \in F^\times$$

and the $n-th$ power of the fundamental ideal $I(F)$ in the Witt ring of symmetric bilinear forms of F denoted by $I^n(F)$. Assuming X is a smooth integral scheme over a field F of characteristic $\neq 2$, they considered the following complexes of groups due to Kato (1.1) and Rost-Schmid (1.2):

$$C_r(X) : 0 \longrightarrow K_r^M(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} K_{r-1}^M(\kappa(x)))) \longrightarrow \bigoplus_{x \in X^{(2)}} K_{r-2}^M(\kappa(x)) \longrightarrow \cdots \quad (1.1)$$

$$D_r(X) : 0 \longrightarrow I^r(F(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} I^{r-1}(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(2)}} I^{r-2}(\kappa(x)) \longrightarrow \cdots \quad (1.2)$$

with $X^{(i)}$ denoting the set of points in X of codimension i and $\kappa(x)$ the residue field of the local ring $\mathcal{O}_{X,x}$. The complexes $C_r(X)$ and $D_r(X)$ are considered as cohomological complexes with

$$\bigoplus_{x\in X^{(i)}} K^M_{r-i}(\kappa(x))))$$

resp.

$$\bigoplus_{x\in X^{(i)}} I^{r-i}(\kappa(x))))$$

in degree i and the associated cohomology groups are denoted by $H^i(C_r(X))$ resp $H^i(D_r(X))$.

There is a natural map of complexes

$$C_k(X) \longrightarrow D_k(X)/D_{k+1}(X)$$

such that one can consider the diagram:

$$\begin{array}{ccc} & & D_k(X) \\ & & \downarrow \\ C_k(X) & \longrightarrow & D_k(X)/D_{k+1}(X) \end{array} \tag{1.3}$$

with $G_k(X)$ denoting the pull-back of (1.3). The $k-th$ Chow-Witt group, or the oriented Chow group is then defined as

$$\widetilde{CH}^k(X) := H^k(G_k(X))$$

where $H^i(G_k(X))$ is the $i-th$ cohomology group of $G_k(X)$.

Assuming X is a smooth affine variety of dimension m, Barge and Morel associated a class

$$e(P) \in \widetilde{CH}^m(X)$$

to a projective module P of rank m over X, called the Euler class of P. They showed if $m = dim(X) = rank(P) = 2$ then

$$e(P) = 0 \in \widetilde{CH}^m(X) \Longleftrightarrow P \textit{ splits off a free summand.}$$

The theory developed by Barge and Morel has been worked out as well as extended by Fasel in his thesis [7,8]. It has been successfully applied by several mathematicians to splitting problems for projective modules. A particularly impressive application of this theory, in our opinion, is the following theorem by Fasel, Rao and Swan [9]:

Theorem: *Let R be a d-dimensional normal affine algebra over an algebraically closed field k such that $char(k) = 0$. If $d = 3$, suppose moreover that R is smooth. Then every stably free R-module P of rank $d-1$ is free.*

There is another approach to Euler classes of projective modules by Morel using the $\mathbb{A}^1$-homotopy theory introduced by himself and Voevodsky [17]. In this theory, there arises a Nisnevich sheaf, $\underline{K}_n^{MW}$ called the sheaf of Milnor-Witt K-theory in weight n such that Morel was able to associate a class

$$e(P) \in H^n_{Nis}(X, \underline{K}_n^{MW})$$

to every vector bundle P of rank n over a smooth n-dimensional affine scheme. He then showed:

Theorem [16]: *Assume $n \geq 4$. If $X = Spec(A)$ is a smooth affine scheme over F of dimension $\leq n$ and ξ is an oriented algebraic vector bundle of rank n with an associated Euler class $e(\xi) \in H^n_{Nis}(X; \underline{K}_n^{MW}(F))$ then*

$$e(\xi) = 0 \in H^n_{Nis}(X; \underline{K}_n^{MW}(F)) \Longleftrightarrow \xi \text{ splits off trivial line bundle.}$$

In collaboration with Hopkins, Morel discovered a presentation of

$$K_*^{MW}(F) = \bigoplus_{n\in\mathbb{Z}} \Gamma(F, \underline{K}_n^{MW})$$

and using this he showed the following result:

Morel's Theorem: *Assume F is a field of characteristic $\neq 2$ then $K_n^{MW}(F) = \Gamma(F, \underline{K}_n^{MW})$ is the pull-back of the following diagram for every $n \in \mathbb{N}$:*

$$\begin{array}{ccc} & & K_n^M(F) \\ & & \downarrow \\ I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) \end{array}$$

As a consequence, it follows that the $i - th$ cohomology group of $G_n(X)$ is equal to $H^i_{Nis}(X; \underline{K}_n^{MW})$ whenever X is a smooth scheme over F.

The aim of this thesis is to extend *Morel's Theorem* to fields of characteristic 2. As Morels proof in characteristic $\neq 2$ relies on a presentation of $I^n(F)$ discovered by Arason and Elman in [2] we similarly rely heavily on a presentation of $I^n(F)$ discovered by Arason and Baeza in [1]. In particular, these results depend on the Milnor conjecture proven by D. Orlov, V. Vishik, and, V. Voevodsky [20] in characteristic $\neq 2$ and Kato in characteristic 2.

In Chapter 1 we outline all of the important notions in the theory of symmetric bilinear forms over fields of characteristic 2 which are necessary to the development of our main result. In particular we will follow Arason and Baeza in showing that isometry implies chain p-equivalence for anisotropic symmetric bilinear forms. We refer to the book [6] by Elman, Karpenko, Merkurjev for a standard exposition of the theory of symmetric bilinear forms in any characteristic.

In Chapter 2, following Morel, we introduce the Witt K-theory of a field and develop all the significant relations needed to provide a presentation of the Witt K-group $K_n^W(F)$ using the work of Arason and Baeza in [1]. In particular we will prove both the commutativity of symbols and Witt relation in $K_2^W(F)$ which will be used to apply a trick due to Suslin in [19] to show that

$$K_2^W(F) = I^2(F)$$

The last section of this chapter will be used to demonstrate

$$Tens_{W(F)}(I(F))/(<< u >> \otimes_{W(F)} << 1-u >>) = \oplus_{n\geq 0} K_n^W(F)$$

which has been shown by Morel in characteristic $\neq 2$ in [15].

In Chapter 3, again following Morel, we introduce both the Milnor-Witt K-theory of a field and the Milnor K-ring $K^M(F)$ due to Milnor in [13]. The main result uses the ideas established in Chapter 2 to show that

$$K_{-n}^{MW}(F) = W(F) \text{ } for \text{ } every \text{ } n \geq 1, \quad K_0^{MW}(F) = \widehat{W(F)}$$

and $K_n^{MW}(F)$ is the pull-back of the diagram:

$$\begin{array}{ccc} & & K_n^M(F) \\ & & \downarrow \\ I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) \end{array}$$

for every $n \geq 1$. The last section of this chapter will be used to demonstrate

$$Tens_{K_0^{MW}(F)}(K_1^{MW}(F))/([u]_{MW} \otimes_{K_0^{MW}(F)} [1-u]_{MW}) = \oplus_{n\geq 0} K_n^{MW}(F)$$

CHAPTER 2

Preliminaries

2.1 Symmetric Bilinear Forms

Definition 2.1.1: Let V be a finite dimensional vector space over the field F. A *symmetric bilinear form* on V is a map $b : V \times V \longrightarrow F$ satisfying the following properties for all $v_1, v_2, w_1, w_2 \in V$ and $c, d \in F$:

- $b(v, w) = b(w, v)$,
- $b(cv_1 + dv_2, w_1) = cb(v_1, w_1) + db(v_2, w_1)$.

We denote a finite dimensional vector space V equipped with a symmetric bilinear form b by (V, b) or b when appropriate.

Definition 2.1.2: A bilinear form b is called *non-degenerate* if $b(v, w) = 0$ for every $w \in V$ implies $v = 0$.

All symmetric bilinear forms will be assumed to be non-degenerate. The following proposition is a classical result in linear algebra which characterizes *Definition 2.1.2* in several different forms:

Proposition 2.1: The following are equivalent:

(1) (V, b) is non-degenerate,

(2) l: $V \longrightarrow V^{\vee}$ given by $v \longrightarrow l_v : w \longrightarrow b(v, w)$ is an isomorphism,

(3) The associated matrix $(b(e_i, e_j))$ is invertible with $e_1, \cdots, e_n$ a basis of V.

Definition 2.1.3: An *isometry* is a linear isomorphism $\phi : V \longrightarrow W$ such that

$$b(v, w) = d(\phi(v), \phi(w))$$

for all $v, w \in V$. If b and d are isometric we write $b \cong d$.

Let us consider the symmetric bilinear form defined by

$$b(x, y) = axy$$

with $a \in F^\times$. We denote b by $< a >_b$ and remark that by *Definition 2.1.3*,

$$< a >_b \cong < d >_b$$

whenever $d \in D(b)^\times$ with $D(b) = \{\ b(v, v) \mid v \in V - \{0\}\ \}$.

2.2 Hyperbolic and Metabolic Bilinear Forms

In this section we introduce two classes of symmetric bilinear forms which will play an important role in the development of the Witt ring.

To begin, we consider a finite dimensional vector space V and its associated dual space $V^\vee$ consisting of all linear functionals on V.

Definition 2.2.1: We define the *hyperbolic form* on V to be the map $b_{\mathbb{H}(\mathbb{V})}$ such that

$$b_{\mathbb{H}(V)}(v_1 + w_1^*, v_2 + w_2^*) = w_2^*(v_1) + w_1^*(v_2)$$

with $v_1, v_2 \in V$ and $w_1^*, w_2^* \in V^\vee$.

It follows by construction that $b_{\mathbb{H}(V)}$ is a symmetric bilinear form. In particular we note that $char(F) = 2$ implies

$$b_{\mathbb{H}(V)}(v_1 + w_1^*, v_1 + w_1^*) = 2w_1^*(v_1) = 0. \tag{2.1}$$

To define a *metabolic form* let us first consider (V, b) with $v \in V - \{0\}$ such that $b(v, v) = 0$. If such a vector exists we call it an *isotropic vector* and say b is *isotropic*, otherwise we say b is *anisotropic*. In this sense we define a subspace $W \subset V$ to be a *totally isotropic subspace* of V if

$$b|_W = 0.$$

It follows by *Proposition 2.1* and dimension considerations that

$$dim(W) + dim(W^{\perp}) = dim(V) \tag{2.2}$$

with $W^{\perp} = \{v \in V \mid b(v, w) = 0 \text{ for all } w \in W\}$. Therefore, a totally isotropic subspace W is contained in $W^{\perp}$ and by (2.2) we conclude that $dim(W) \leq \frac{1}{2}dim(V)$.

Definition 2.2.2: We call (V, b) a *metabolic space* equipped with a *metabolic form* b if there exists a totally isotropic subspace $W \subset V$ such that $dim(W) = \frac{1}{2}dim(V)$.

It follows immediately by (2.1) that $b_{\mathbb{H}(V)}$ is a metabolic form so a natural question to ask is whether or not *hyperbolic forms* and *metabolic forms* are equivalent since it is well-known that this is indeed the case in characteristic $\neq 2$, see [6].

Assume $dim(V) = 2$: If $v \in V$ is an *isotropic vector* then *Definition 2.1.2* implies that there exists $w \in V - F \cdot v$ such that $b(v, w) \neq 0$, which after scaling is equivalent to $b(v, w) = 1$. Therefore (V, b) is a 2-dimensional space with basis $\{v,\ w\}$ and

$$b(v, v) = 0,\ b(v, w) = 1,\ b(w, w) = x.$$

If $x \neq 0$ then we let $\{w,\ xv + w\}$ be another basis for (V, b) such that

$$b(w, w) = x,\ b(w, xv + w) = 0,\ b(xv + w, xv + w) = x.$$

We denote this b by $< x, x >_b$. It follows by (2.1) that $b \cong b_{\mathbb{H}(F)}$ implies every $w \in V$ is an *isotropic vector* in $b \cong < x, x >_b$ which is clearly a contradiction. Therefore,

$$< x, x >_b \ncong b_{\mathbb{H}(F)}.$$

If $x = 0$ then we have a linear isomorphism from $V \longrightarrow F \oplus F^{\vee}$ defined by

$$w \mapsto 1, \quad xv + w \mapsto 1^*$$

with 1^* denoting the standard basis vector in $F^{\vee}$. This implies

$$b \cong b_{\mathbb{H}(F)}. \tag{2.3}$$

Therefore, if b is a 2-dimensional metabolic form then

$$b \cong b_{\mathbb{H}(F)} \textit{ or } b \cong < x, x >_b$$

with $x \in F^{\times}$. In particular if $b(v, v) = 0$ for every $v \in V$ then b is a hyperbolic form by (2.3).

2.3 Orthogonal sum and Kronecker product

In this section we construct the orthogonal sum and tensor product of symmetric bilinear forms.

Let (V, b_1) and (W, b_2) be vector spaces with associated symmetric bilinear forms over F. We define the *orthogonal sum* of b_1 and b_2, denoted by $b_1 \perp b_2$, to be the map

$$b_1 \perp b_2 : V \oplus W \times V \oplus W \rightarrow F$$

defined by

$$(b_1 \perp b_2)((x_1, x_2), (y_1, y_2) = b_1(x_1, y_1) + b_2(x_2, y_2)$$

and $b_1 \perp b_2$ is clearly a symmetric bilinear form such that $(b_1 \perp b_2)(V, W) = 0$.

Similarily we define the *Kronecker product* or *tensor product* of b_1 and b_2, denoted by $b_1 \otimes b_2$, to be the map

$$b_1 \otimes b_2 : V \otimes W \times V \otimes W \to F$$

defined by

$$(b_1 \otimes b_2)(v_1 \otimes v_2, w_1 \otimes w_2) = b_1(v_1, w_1) \cdot b_2(v_2, w_2) \text{ for every } v_i \in V,\ w_i \in W.$$

2.4 Witt ring

In this section we introduce the Witt Cancellation Theorem and define the Witt ring of symmetric bilinear forms.

The following two results are well-known and can be found in [6]:

Theorem 2.4.1 (Bilinear Witt Decomposition Theorem): *If b is a non-degenerate symmetric bilinear form on V then there exists subspaces $U, W \subset V$ such that*

$$b = b|_U \perp b|_W$$

with $b|_U$ anisotropic and $b|_W$ metabolic. Moreover, $b|_U$ is unique up to isometry.

Theorem 2.4.2 (Witt Cancellation Theorem): *Let b_0, b_1 and b_2 be nondegenerate symmetric bilinear forms over F. If b_1 and b_2 are anisotropic then*

$$b_1 \perp b_0 \cong b_2 \perp b_0$$

implies $b_1 \cong b_2$.

To define the Witt ring of symmetric bilinear forms we first remark that the isometry classes of nondegenerate symmetric bilinear forms over F, denoted by $M(F)$, form a semiring under orthogonal sum and tensor product. The *Witt-Grothendieck group* of F, denoted by $\widehat{W(F)}$ is defined by a relation $\sim$ on $M(F) \times M(F)$ such that

$$(b_1, b_2) \sim (d_1, d_2) \tag{2.4}$$

if and only if there exists $\lambda \in M(F)$ such that

$$b_1 \perp d_2 \perp \lambda \cong d_1 \perp b_2 \perp \lambda \tag{2.5}$$

with $b_1, b_2, d_1, d_2 \in M(F)$.

To avoid confusion we denote the equivalence class of (b_1, b_2) in $\widehat{W(F)}$ by $b_1 - b_2$. It turns out $\widehat{W(F)}$ has the structure of a ring where we define addition in $\widehat{W(F)}$ by:

$$(b_1 - b_2) + (d_1 - d_2) = (b_1 \perp d_1) - (b_2 \perp d_2)$$

and multiplication in $\widehat{W(F)}$ by:

$$(b_1 - b_2)(d_1 - d_2) = ((b_1 \otimes d_1) \perp (b_2 \otimes d_2)) - ((b_1 \otimes d_2) \perp (b_2 \otimes d_1)).$$

This is clearly well-defined, associative and commutative.

It follows by (2.4) and (2.5) that

$$b_1 - b_2 = d_1 - d_2 \in \widehat{W(F)} \tag{2.6}$$

if and only if there exists a nondegenerate symmetric bilinear form λ over F such that

$$b_1 \perp d_2 \perp \lambda \cong d_1 \perp b_2 \perp \lambda. \tag{2.7}$$

To construct the *Witt ring of F* we need to quotient out the ideal $(\mathbb{H})$ consisting of all hyperbolic forms over F in $\widehat{W(F)}$.

Definition 2.4: The quotient $W(F) = \widehat{W(F)}/(\mathbb{H})$ is called the *Witt ring* of nondegenerate symmetric bilinear forms over F.

In particular, the structure of the Witt ring implies $< x, x >_b = 0$ in $W(F)$ since by definition this is equivalent to showing that

$$< x, x >_b \perp < x >_b \cong b_{\mathbb{H}(F)} \perp < x >_b . \tag{2.8}$$

Following [6] we consider the basis $\{u,\ v,\ w\}$ of $b =< x, x >_b \perp < x >_b$ such that

$$b(u, u) = x,\ \ b(v, v) = x,\ \ b(w, w) = x.$$

If we apply a change-of-basis to $\{u,\ v,\ w\}$ such that

$$\{u + w, \frac{1}{x}v + \frac{1}{x}w, u + v + w\}$$

forms a new basis we can conclude (2.8).

The following classical result will play an important role in Chapter 3 and can be found in [6]:

Theorem 2.4.3: *The Witt ring $W(F)$ is generated by nondegenerate 1-dimensional symmetric bilinear forms $< u >_b$ with $u \in F^{\times}$ subject to the following defining relations:*

(1) $< uv^2 >_b - < u >_b = 0$

(2) $2 < 1 >_b\ = 0$

(3) $< u >_b + < v >_b\ + < u + v >_b + < uv(u + v) >_b\ =$ *0 if* $u + v \neq 0$

with $u, v \in F^{\times}$.

2.5 Fundamental Ideal and $I^*(F)$

In this section we will introduce the notion of the fundamental ideal of symmetric bilinear forms and give a presentation for $I^n(F)$ *for every* $n > 0$ *due to Arason and Baeza [1].*

In *Section 2.4* we denoted every element in $\widehat{W(F)}$ by the formal expression $b_1 - b_2$ where b_1, b_2 are nondegenerate symmetric bilinear forms over F. Let us consider the map

$$dim : \widehat{W(F)} \longrightarrow \mathbb{Z}$$

defined by

$$dim(b_1 - b_2) = dim(b_1) - dim(b_2).$$

Assume there exists $d_1,\, d_2 \in M(F)$ such that

$$b_1 - b_2 = d_1 - d_2.$$

Then (2.6) and (2.7) imply there exists $b \in M(F)$ such that

$$b_1 \perp d_2 \perp b \cong d_1 \perp b_2 \perp b$$

and $dim(b_1 \perp d_2 \perp b) = dim(d_1 \perp b_2 \perp b)$ implies

$$dim(b_1 - b_2) = dim(d_1 - d_2)$$

and we conclude that dim is well-defined.

Let $\widehat{I(F)} = ker(dim : \widehat{W(F)} \longrightarrow \mathbb{Z})$. It follows that

$$dim(< u >_b - < v >_b) = dim(< u >_b) - dim(< v >_b) = 0$$

implies $< u >_b - < v >_b \in \widehat{I(F)}$. Moreover,

$$< u >_b - < v >_b = (< 1 >_b - < v >_b) - (< 1 >_b - < u >_b) \in \widehat{W(F)}$$

which implies $< 1 >_b - < u >_b$ with $u \in F^\times$ generate $\widehat{I(F)}$ as an abelian group. These results can be carried over to $W(F)$ by the following observation:

$$\widehat{I(F)} \cap (\mathbb{H}) = 0 \tag{2.9}$$

which follows immediately by definition given every element in $(\mathbb{H})$ is of the form

$$(b_1 - b_2)\cdot b_{\mathbb{H}(F)} \tag{2.10}$$

with $b_1, b_2 \in M(F)$. The multiplication operation in $\widehat{W(F)}$ implies (2.10) is equivalent to

$$(b_1 \otimes b_{\mathbb{H}(F)}) - (b_2 \otimes b_{\mathbb{H}(F)})$$

which by ([6], *Lemma 2.1*) is equal to

$$(dim(b_1)\cdot b_{\mathbb{H}(F)}) - (dim(b_2)\cdot b_{\mathbb{H}(F)})$$

with $dim(b_1)\cdot b_{\mathbb{H}(F)}$ denoting $\underbrace{b_{\mathbb{H}(F)} \perp \cdots \perp b_{\mathbb{H}(F)}}_{dim(b_1)}$. Since $(dim(b_1)\cdot b_{\mathbb{H}(F)}) - (dim(b_1)\cdot b_{\mathbb{H}(F)})$ is the additive identity we conclude

$$dim((b_1 - b_2) \otimes b_{\mathbb{H}}) = 2dim(b_1 - b_2) \neq 0$$

whenever $dim(b_1) \neq dim(b_2)$ or $(b_1 - b_2) \otimes b_{\mathbb{H}}$ is non-trivial

Definition 2.5.1: The *fundamental ideal* over F denoted by $I(F)$ is the image of $\widehat{I(F)}$ under the projection map $\widehat{W(F)} \longrightarrow W(F)$.

We then have that (2.9) implies $\widehat{I(F)} \cong I(F)$ which under the projection $\widehat{W(F)} \longrightarrow W(F)$ maps

$$< 1 >_b - < u >_b \mapsto < 1, u >_b .$$

Moreover the remarks preceeding (2.9) imply that $I(F)$ is generated by the *Pfister forms*, $< 1, u >_b := << u >>_b$ with $u \in F^{\times}$. Consider the map $\overline{d} : \widehat{W(F)} \xrightarrow{dim} \mathbb{Z} \longrightarrow \mathbb{Z}/2$ then $\overline{d}(\mathbb{H}) = 0$ implies by the universal property of quotient map that

$$\overline{dim} : W(F) \longrightarrow \mathbb{Z}/2$$

is well-defined. This allows us to formulate the following proposition which is a direct consequence of *Theorem 2.4.1*, *Definition 2.5.1* and

$$< x, x >_b \perp < x >_b \cong b_{\mathbb{H}(F)} \perp < x >_b .$$

Proposition 2.5: The commutative diagram

$$\begin{array}{ccc} \widehat{W(F)} & \xrightarrow[dim]{} & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(F) & \xrightarrow[\overline{dim}]{} & \mathbb{Z}/2 \end{array}$$

is a Cartesian square.

We define $I^n(F)$ to be the *n-th power of the fundamental ideal* $I(F)$ over F and note that $I^n(F)$ is generated by

$$<< u_1 >>_b \cdots << u_n >>_b$$

which we call the *n-fold Pfister form* and denote by

$$<< u_1, \cdots, u_n >>_b$$

with $u_i \in F^{\times}$. The main result of this section is the following:

Theorem 2.5: *For every $n \geq 1$ we define $\underline{I}^n(F)$ to be the abelian group $\mathbb{Z}[(F^{\times})^n]$ modulo the subgroup generated by the following relations:*

(1) $(u_1, \cdots, u_n)$ *where* $(u_1, \cdots, u_n) \in (F^{\times})^n$ *such that* $<< u_1, \cdots, u_n >> = 0$ *in* $W(F)$.

(2) $(a, u_2, \cdots, u_n) + (b, u_2, \cdots, u_n) - (a+b, u_2, \cdots, u_n) - (ab(a+b), u_2, \cdots, u_n)$ *with* $(a, b, u_2, \cdots, u_n) \in (F^{\times})^{n+1}$ *and* $a + b \neq 0$.

(3) $(ab, c, u_3, \cdots, u_n) + (a, b, u_3, \cdots, u_n) - (ac, b, u_3, \cdots, u_n) - (a, c, u_3, \cdots, u_n)$ *with* $(a, b, c, u_3, \cdots, u_n) \in (F^{\times})^{n+1}$ *and* $n \geq 2$.

(4) $(u_1, \cdots, u_n) - (v_1, \cdots, v_n)$ *with* $(u_1, \cdots, u_n, v_1, \cdots, v_n) \in (F^{\times})^{2n}$ *whenever* $<< u_1, \cdots, u_n >> \cong << v_1, \cdots, v_n >>$.

Then $\underline{I}^n(F) \cong I^n(F)$ *with* $(u_1, \cdots, u_n) \mapsto << u_1, \cdots, u_n >>_b$ *for every* $n \geq 1$.

Proof: The proof is due to Aarson and Baeza in [1] and uses the characteristic 2 version of the Milnor conjecture which was proven by Kato in [10].

□

Definition 2.5.2: Let $I^*(F)$ be the $\mathbb{Z}$-graded $W(F)$-algebra $\oplus_{n\in\mathbb{Z}} I^n(F)$ with $I^n(F) = W(F)$ if $n \leq 0$ and $I^n(F)$ is the *n-th power of the fundamental ideal* $I(F)$ over F whenever $n > 0$.

2.6 Chain p-equivalence

In this section we will define the notion of chain p-equivalence and following Aarson and Baeza in [1] we will provide a theorem which relates chain p-equivalence to isometry.

Definition 2.6.1: Two Pfister forms $<< u_1, \cdots, u_n >>_b$ and $<< v_1, \cdots, v_n >>_b$ with $u_i, v_i \in F^\times$ are said to be *simply p-equivalent*, denoted by

$$<< v_1, \cdots, v_n >>_b \sim << v_1, \cdots, v_n >>_b$$

if there exists $i, j \in [1, n]$ such that $<< u_i, u_j >>_b \cong << v_i, v_j >>_b$ and $u_k = v_k$ whenever $k \neq i, j$.

The definition of *chain p-equivalence* follows naturally:

Definition 2.6.2: Two Pfister forms $<< u_1, \cdots, u_n >>_b$ and $<< v_1, \cdots, v_n >>_b$ with $u_i, v_i \in F^\times$ are said to be *chain p-equivalent*, denoted by

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_n >>_b$$

if there exists $<< w_{1,i}, \cdots, w_{n,i} >>_b$ with $w_{j,i} \in F^\times$ such that

$$<< u_1, \cdots, u_n >>_b = << w_{1,1}, \cdots, w_{n,1} >>_b$$

$$<< v_1, \cdots, v_n >>_b = << w_{1,m}, \cdots, w_{n,m} >>_b$$

and $<< w_{1,i}, \cdots, w_{n,i} >>_b \sim << w_{1,i+1}, \cdots, w_{n,i+1} >>_b$ for every $i \in [1, \cdots, m-1]$.

Consider

$$<< u_1, \cdots, u_n >>_b = < 1 >_b \perp < v_1, \cdots, v_{2^n-1} >_b$$

with $v_i \in F^\times$, we say $< v_1, \cdots, v_{2^n-1} >_b$ is the *pure subform* of $<< u_1, \cdots, u_n >>_b$ and denote it by $<< u_1, \cdots, u_n >>_b^\circ$.

Lemma 2.6.1: Let $<< u_1, \cdots, u_n >>_b$ with $u_i \in F^\times$ such that

$$v \in D(<< u_1, \cdots, u_n >>_b^\circ)^\times$$

Then there exists $v_2, \cdots, v_n \in F^\times$ such that $<< u_1, \cdots, u_n >>_b \approx << v, v_2, \cdots, v_n >>_b$.

Proof: We proceed by induction on n:

Let $n = 1$ then we have $<< u_1 >>_b = < 1 >_b \perp < u_1 >_b$ and

$$v \in D(< u_1 >_b)^\times$$

implies $v = u_1 x^2$ with $x \in F^\times$. Then,

$$<< u_1 >>_b = < 1 >_b \perp < u_1 >_b \cong < 1 >_b \perp < u_1 x^2 >_b \cong < 1 >_b \perp < v >_b$$

with $< 1 >_b \perp < v_1 >_b = << v >>_b$.

Assume $n > 1$, $v \in D(<< u_1, \cdots, u_n >>_b^\circ)^\times$ is equivalent to

$$v \in D(<< u_1, \cdots, u_{n-1} >>_b^\circ \perp u_n << u_1, \cdots, u_{n-1} >>_b)^\times$$

which implies $v = x + u_n y$ with $x \in D(<< u_1, \cdots, u_{n-1} >>_b^\circ)$ and $y \in D(<< u_1, \cdots, u_{n-1} >>_b)$.

If $y = 0$ then $v = x$ and by induction we have that

$$<< u_1, \cdots, u_{n-1} >>_b \approx << v, v_2, \cdots, v_{n-1} >>_b$$

with $v_2, \cdots, v_{n-1} \in F^\times$ which implies

$$<< u_1, \cdots, u_n >>_b \approx << v, v_2, \cdots, v_{n-1}, u_n >>_b .$$

To proceed with our proof we will first need to show the following claim:

Claim: Let $<< u_1, \cdots, u_n >>_b$ with $u_i \in F^\times$ and $w \in D(<< u_1, \cdots, u_{n-1} >>_b)^\times$ then $<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << u_1, \cdots, u_{n-1}, wu_n >>_b$.

Proof of Claim:

Let $w \in D(<< u_1, \cdots, u_{n-1} >>_b)^\times = D(< 1 >_b \perp << u_1, \cdots, u_{n-1} >>_b^\circ)^\times$ which implies $w = x^2 + y$ with $x \in F$ and $y \in D(<< u_1, \cdots, u_{n-1} >>_b^\circ)$.

If $y = 0$ then $w = x^2$ and $<< u_n w >>_b \cong << u_n >>_b$.

If $y \neq 0$ then we proceed by induction assumption of *Lemma 2.6.1* which implies

$$<< u_1, \cdots, u_{n-1} >>_b \approx << y, y_2, \cdots, y_{n-1} >>_b \tag{2.11}$$

with $y_2, \cdots, y_{n-1} \in F^\times$. Therefore

$$<< u_1, \cdots, u_n >>_b \approx << y, y_2, \cdots, y_{n-1}, u_n >>_b .$$

However *Lemma A.1 (1)* implies

$$<< y, u_n >>_b \cong << y, u_n(x^2 + y) >>_b = << y, u_n w >>_b .$$

Hence,

$$<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << y, y_2, \cdots, y_{n-1}, u_n w >>_b$$

and by (2.11) this implies

$$<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << u_1, \cdots, u_{n-1}, u_n w >>_b .$$

□

Resuming where we left off, assume that $y \neq 0$. It follows by *Claim* that

$$y \in D(<< u_1, \cdots, u_{n-1} >>_b)^\times$$

implies

$$<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << u_1, \cdots, u_{n-1}, u_n y >>_b \,. \tag{2.12}$$

If $x = 0$ this gives

$$<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << v, u_1, \cdots, u_{n-1} >>_b \,.$$

Assume $x \neq 0$, then $x \in D(<< u_1, \cdots, u_{n-1} >>_b^\circ)^\times$ implies by induction that

$$<< u_1, \cdots, u_{n-1} >>_b \approx << x, x_2, \cdots, x_{n-1} >>$$

with $x_2, \cdots, x_{n-1} \in F^\times$. Therefore,

$$<< u_1, \cdots, u_{n-1}, u_n y >>_b \approx << x, x_2, \cdots, x_{n-1}, u_n y >>_b$$

which along with a result by *Lemma A.1 (2)*:

$$<< x, u_n y >>_b \cong << x + u_n y, x u_n y >>_b = << v, x u_n y >>_b$$

and (2.12) implies

$$<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << v, x_2, \cdots, x_{n-1}, x u_n y >>_b \,.$$

□

Lemma 2.6.2: Consider $<< u_1, \cdots, u_n >>_b$ with $u_i \in F^\times$. Then

$$w \in D(<< u_1, \cdots, u_{n-1} >>_b)^\times$$

implies $<< u_1, \cdots, u_{n-1}, u_n >>_b \approx << u_1, \cdots, u_{n-1}, w u_n >>_b$.

Proof: This was shown in the proof of *Lemma 2.6.1.*

□

Lemma 2.6.3: Consider $<< u_1, \cdots, u_n >>_b$, $<< v_1, \cdots, v_m >>_b$ with $u_i, v_j \in F^\times$. Then

$$w \in D(<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_m >>_b^\circ)^\times$$

implies there exists $w_2, \cdots, w_m \in F^\times$ such that

$$<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_m >>_b$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n >>_b \cdot << w, w_2, \cdots, w_m >>_b .$$

Proof: We proceed by induction on m.

If $m = 1$ then $w \in D(<< u_1, \cdots, u_n >>_b \cdot < v_1 >_b)^\times$ implies $w = xv_1$ with

$$x \in D(<< u_1, \cdots, u_n >>_b)^\times.$$

However by *Lemma 2.6.2* this implies

$$<< u_1, \cdots, u_n, v_1 >>_b \approx << u_1, \cdots, u_n, xv_1 >>_b = << u_1, \cdots, u_n, w >>_b .$$

Assume $m > 1$ then $w \in D(<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_m >>_b^\circ)^\times =$
$D\Big(<< u_1, \cdots, u_n >>_b \cdot (<< v_1, \cdots, v_{m-1} >>_b^\circ \perp v_m << v_1, \cdots, v_{m-1} >>_b)\Big)^\times$ implies
$w = e_1 + v_m d_1$ with

$$e_1 \in D\Big(<< u_1, \cdots, u_n >>_b \cdot (<< v_1, \cdots, v_{m-1} >>_b^\circ)\Big)$$

and

$$d_1 \in D\Big(<< u_1, \cdots, u_n >>_b \cdot (<< v_1, \cdots, v_{m-1} >>_b)\Big).$$

If $e_1 = 0$ then $w = v_m d_1$ which by *Lemma 2.6.2* and $d_1 \in F^\times$ implies

$$<< u_1, \cdots, u_n, v_1, \cdots, v_{m-1} >>_b \cdot << v_m >>_b$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n, v_1, \cdots, v_{m-1} >>_b \cdot << v_m d_1 >>_b$$

with $w = v_m d_1$.

Similarily, if $d_1 = 0$ then $w = e_1$ and by induction assumption

$$<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_{m-1} >>_b$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n >>_b \cdot << e_1, w_2, \cdots, w_{m-1} >>_b$$

which implies

$$<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_{m-1}, v_m >>_b$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n >>_b \cdot << e_1, w_2, \cdots, w_{m-1}, v_m >>_b .$$

If $e_1, d_1 \neq 0$ then the above considerations show

$$<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_m >>_b \tag{2.13}$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n >>_b \cdot << e_1, e_2, \cdots, e_{m-1} >>_b \cdot << v_m d_1 >>_b \tag{2.14}$$

with $e_2, \cdots, e_{m-1} \in F^\times$. It follows by *Lemma A.1 (2)* that

$$<< e_1, v_m d_1 >>_b \cong << e_1 + v_m d_1, e_1 v_m d_1 >>_b$$

which by (2.13) and (2.14) implies

$$<< u_1, \cdots, u_n >>_b \cdot << v_1, \cdots, v_m >>_b$$

is chain p-equivalent to

$$<< u_1, \cdots, u_n >>_b \cdot << w, y_2, \cdots, y_m >>_b$$

with $w = e_1 + v_m d_1$ and $y_2, \cdots, y_m \in F^\times$.

□

Theorem 2.6: *Consider* $<< u_1, \cdots, u_n >>_b$ *and* $<< v_1, \cdots, v_n >>_b$ *with* $u_i, v_i \in F^\times$:

(1) *If* $<< u_1, \cdots, u_n >>_b$ *is isotropic then*

$$<< u_1, \cdots, u_n >>_b \approx << 1, w_2, \cdots, w_n >>_b$$

with $w_i \in F^\times$.

(2) *If* $<< u_1, \cdots, u_n >>_b$ *and* $<< v_1, \cdots, v_n >>_b$ *are anisotropic then*

$$<< u_1, \cdots, u_n >>_b \cong << v_1, \cdots, v_n >>_b$$

if and only if

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_n >>_b .$$

Proof:

(1) It suffices by *Lemma 2.6.1* to show $1 \in D(<< u_1, \cdots, u_n >>_b^\circ))^\times$. We proceed by induction on n:

If $n = 1$ then $<< u_1 >>_b$ is isotropic implies $u_1 = x^2$ with $x \in F^\times$ and

$$1 \in D(< x^2 >_b)^\times.$$

Assume $n > 1$ and consider $<< u_1, \cdots, u_n >>_b$ which can be written as

$$<< u_1, \cdots, u_{n-1} >>_b \perp u_n << u_1, \cdots, u_{n-1} >>_b .$$

We consider two cases: $<< u_1, \cdots, u_{n-1} >>_b$ is isotropic and $<< u_1, \cdots, u_{n-1} >>_b$ is anisotropic.

If $<< u_1, \cdots, u_{n-1} >>_b$ is isotropic then by induction we are done.

If $<< u_1, \cdots, u_{n-1} >>_b$ is anisotropic we have that $<< u_1, \cdots, u_n >>_b$ is isotropic implies there exists $c_1, c_2 \in D(<< u_1, \cdots, u_{n-1} >>_b)$ such that

$$c_1 = u_n c_2.$$

To apply this we first observe that

$$<< u_1, \cdots, u_n >>_b^\circ$$

is equal to

$$<< u_1, \cdots, u_{n-1} >>_b^\circ \perp u_n << u_1, \cdots, u_{n-1} >>_b$$

and *Lemma A.2* implies

$$c_i << u_1, \cdots, u_{n-1} >>_b \cong << u_1, \cdots, u_{n-1} >>_b \tag{2.15}$$

with $i = 1, 2$. It follows by the above that

$$<< u_1, \cdots, u_n >>_b$$

is isometric to

$$<< u_1, \cdots, u_{n-1} >>_b^\circ \perp c_1 << u_1, \cdots, u_{n-1} >>_b .$$

However by (2.15) this is isometric to

$$<< u_1, \cdots, u_{n-1} >>_b^\circ \perp << u_1, \cdots, u_{n-1} >>_b .$$

Therefore,

$$1 \in D(<< u_1, \cdots, u_{n-1} >>_b^\circ \perp << u_1, \cdots, u_{n-1} >>_b)^\times$$

implies

$$1 \in D(<< u_1, \cdots, u_n >>_b^\circ)^\times.$$

□

(2) We begin by first showing the following claim:

Claim: Consider the anisotropic n-fold Pfister forms $<< u_1, \cdots, u_n >>_b$ and $<< v_1, \cdots, v_n >>_b$ with $u_i, v_j \in F^\times$. Assume $<< u_1, \cdots, u_n >>_b \cong << v_1, \cdots, v_n >>_b$. Let $1 \leq m \leq n$ then there exists $w_{m+1}, \cdots, w_m \in F^\times$ such that

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_m, w_{m+1}, \cdots, w_n >>_b$$

Proof of Claim: We proceed by induction on m:

If $m = 1$ then by *Lemma 2.6.1*,

$$v_1 \in D(<< u_1, \cdots, u_n >>_b^\circ)^\times = D(<< v_1, \cdots, v_n >>_b^\circ)^\times$$

implies there exists $w_2, \cdots, w_n \in F^\times$ such that

$$<< u_1, \cdots, u_n >>_b \approx << v_1, w_2 \cdots, w_n >>_b .$$

Assume $m > 1$ then by induction assumption on $m-1$, there exists $w_m, \cdots, w_n \in F^\times$ such that

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_{m-1}, w_m, \cdots, w_n >>_b . \tag{2.16}$$

It follows by our initial assumption that we can apply *Theorem 2.4.2* to

$$<< v_1, \cdots, v_n >>_b \cong << v_1, \cdots, v_{m-1}, w_m, \cdots, w_n >>_b$$

such that

$$<< v_m, \cdots, v_n >>_b \cong << w_m, \cdots, w_n >>_b .$$

Therefore,

$$<< v_1, \cdots, v_{m-1} >>_b \cdot\ << w_m, \cdots, w_n >>_b^\circ$$

is isometric to

$$<< v_1, \cdots, v_{m-1} >>_b \cdot\ << v_m, \cdots, v_n >>_b^\circ$$

which implies

$$v_m \in D(<< v_1, \cdots, v_{m-1} >>_b \cdot\ << w_m, \cdots, w_n >>_b^\circ)^\times$$

which by (2.16) and *Lemma 2.6.3* implies

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_{m-1} >>_b \cdot << v_m, b_{m+1}, \cdots, b_n >>_b$$

with $b_{m+1}, \cdots, b_n \in F^\times$.

□

Applying *Claim* to the case $m = n$ implies

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_n >>_b .$$

□

CHAPTER 3

Witt K-Theory

The aim of this chapter is to introduce the Witt K-Theory of a field F, due to Morel in [15] and establish some elementary facts and relations which will be of great importance in the next chapter.

3.1 Definition and Facts

Definition 3.1.1 The *Witt K-ring* of F is the free and graded $\mathbb{Z}$-algebra $K_*^W(F)$ generated by the symbols [u] ($u \in F^\times$) of degree 1 and one symbol η of degree -1 subject to the following relations:

$(W1)^\bullet$: For each $a \in F^\times - \{1\} : [a][1-a] = 0$,

$(W2)^\bullet$: For each $(a,b) \in (F^\times)^2 : [ab] = [a] + [b] - \eta[a][b]$,

$(W3)^\bullet$: For each $u \in F^\times : [u]\eta = \eta[u]$,

$(W4)^\bullet$: $\eta[-1] = 2$.

Theorem 3.1: *In char*$(F) = 2$ *we have that* $(W4)^\bullet$ *is equivalent to:*

$(W4)$ *:* $\eta[1] = 0 = 2$

Moreover, $K_*^W(F)$ *is a* $\mathbb{Z}/2$*-graded algebra.*

Proof: Indeed we have by $(W2)^\bullet$

$$[1] = [(-1)(-1)] = [-1] + [-1] - \eta[-1][-1]$$

which by $(W4)^\bullet$ implies

$$[1] = [-1] + [-1] - 2[-1] = 0.$$

Hence, $[-1] = [1] = 0$ and

$$2 = \eta[-1] = \eta[1] = 0.$$

Therefore, $(W4)^{\bullet} \implies (W4)$ and $(W4) \implies (W4)^{\bullet}$ follows trivially.

□

The following reformulation of *Definition 3.1.1* will be used henceforth:

Definition 3.1.2: The *Witt K-ring* of F in characteristic 2 is the free and graded $\mathbb{Z}/2$-algebra $K_*^W(F)$ generated by the symbols [u] ($u \in F^{\times}$) of degree 1 and one symbol η of degree -1 subject to the following relations:

W1: For each $a \in F^{\times} - \{1\} : [a][1+a] = 0$,

W2: For each $(a,b) \in (F^{\times})^2 : [ab] = [a] + [b] + \eta[a][b]$,

W3: For each $u \in F^{\times} : [u]\eta = \eta[u]$,

W4: $\eta[1] = 0 = 2$.

Following Morel, for any $u \in F^{\times}$ we define

$$< u >= 1 - \eta[u] = 1 + \eta[u] \tag{3.1}$$

with $< u > \in K_0^W(F)$. The following elementary relations follow as a direct consequence of *Definition 3.1.2.*

Proposition 3.1: Let $(a,b) \in (F^{\times})^2$. Then the following relations hold in $K_*^W(F)$:

(1) $[ab] = [a] + < a > [b]$,

(2) $< ab > = < a >< b >$,

(3) $< -1 > = 1$ and $[1] = 0$,

(4) $< a >$ is a unit in $K_0^W(F)$ and $< a >^{-1} = < a^{-1} >$,

(5) $[\frac{a}{b}] = [a] - < \frac{a}{b} > [b]$,

(6) $< a > [b] = [b] < a >$,

(7) $\eta[a][b] = \eta[b][a]$.

Proof:

(1) $[ab] = [a] + [b] + \eta[a][b] = [a] + (1 + \eta[a])[b] = [a]+ < a > [b]$.

(2) $< ab > = 1 + \eta[ab]$ which by (W2) implies

$$1 + \eta([a] + [b] + \eta[a][b]) = 1 + \eta[a] + \eta[b] + \eta[a]\eta[b].$$

Applying the definition $< u >= 1 + \eta[u]$ to the above:

$$1 + (1+ < a >) + (1+ < b >) + (1+ < a >)(1+ < b >) =< a >< b > .$$

(3)

$$< 1 >=< -1 >= 1 + \eta[-1] = 1$$

by $(W4)^{\bullet}$ and $[1] = 0$ follows by the proof of *Lemma 3.1*.

(4) $< a >< a^{-1} >=< (a)(a^{-1}) >=< 1 >= 1$ by (2) and (3).

(5) The following set of equalities follows directly by (1) and (3):

$$0 = [1] = [b^{-1}b] = [b^{-1}]+ < b^{-1} > [b]$$

which implies

$$[b^{-1}] =< b^{-1} > [b]$$

and we conclude

$$[\frac{a}{b}] = [ab^{-1}] = [a]+ < a > [b^{-1}] = [a]+ < ab^{-1} > [b].$$

(6) (W2) implies

$$[ab] = [a] + [b] + \eta[a][b]$$

and

$$[ba] = [b] + [a] + \eta[b][a].$$

Therefore if $[ab] = [ba]$ then $\eta[a][b] = \eta[b][a]$ and

$$< a > [b] = (1 + \eta[a])[b] = [b] + \eta[a][b]$$

which by (W3) implies

$$[b] + \eta[b][a] = [b](1 + \eta[a]) = [b] < a > .$$

(7) This follows directly by observing that

$$[ba] = [ab]$$

and using (W2).

□

We can now show following non-trivial set of relations which will be used in the next section extensively:

Corollary 3.1: Let $(a, b) \in (F^\times)^2$. Then the following relations hold in $K_*^W(F)$:

(1) $[a][-a] = [a][a] = 0$,

(2) $[a^2] = 0$, $[ab^2] = [a]$ and $< b^2 >= 1$.

Proof:

(1) Assume without loss of generality that $a \in F^\times - \{1\}$ then $-a = \frac{1-a}{1-a^{-1}}$ implies

$$[a][-a] = [a][\frac{1-a}{1-a^{-1}}]$$

which by *Proposition 3.1 (5)* implies

$$[a][\frac{1-a}{1-a^{-1}}] = [a]([1-a] + < -a > [1-a^{-1}]).$$

However by $(W1)^\bullet$ we know that $[a][1-a]=0$ hence

$$[a]([1-a]+<-a>[1-a^{-1}])=[a]<-a>[1-a^{-1}].$$

However *Proposition 3.1 (1), (3)* implies

$$0=[1]=[aa^{-1}]=[a]+<a>[a^{-1}]$$

which gives us

$$[a]=<a>[a^{-1}]. \tag{3.2}$$

Therefore,

$$\begin{aligned}
[a][-a] &= [a]<-a>[1-a^{-1}] \\
&= (<a>[a^{-1}])<-a>[1-a^{-1}] \; by \; (3.2) \\
&= <a><-a>[a^{-1}][1-a^{-1}] \; by \; Proposition \; 3.1 \; (6) \\
&= 0 \;\; by \; (W1).
\end{aligned} \tag{3.3}$$

(2) (W2) implies

$$[a^2]=[a]+[a]+\eta[a][a]$$

which by (1) implies

$$[a^2]=2[a]+\eta[a][-1]=0.$$

Similarily, (W2) and $[b^2]=0$ imply

$$[ab^2]=[a]+[b^2]+\eta[a][b^2]=[a].$$

Lastly,

$$<b^2>=1+\eta[b^2]=1$$

by $[b^2]=0$.

3.2 Witt relation and Commutativity of Symbols

In this section we will establish some deeper results in the Witt K-ring $K_*^W(F)$.

Proposition 3.2 (Witt relation): Let $(a, b) \in (F^\times)^2$ such that $a + b \neq 0$. Then

$$[a][b] = [a + b][ab(a + b)]$$

in $K_2^W(F)$.

Proof: Let us consider the right-hand side:

$$[a + b][ab(a + b)] = [a + b]([ab] + [a + b] - \eta[ab][a + b]).$$

Proposition 3.1 (7) implies

$$[a + b][ab(a + b)] = [a + b][ab] + [a + b][a + b] - \eta[ab][a + b][a + b]$$

which by *Corollary 3.1* implies

$$[a + b][ab(a + b)] = [a + b][ab].$$

If we rewrite $(a + b)$ as $a(1 + \frac{b}{a})$ and use the equality $[ab] = [ab(a^{-2})]$ by *Corollary 3.1 (3)* we have

$$[a + b][ab] = [a(1 + \frac{b}{a})][\frac{b}{a}].$$

Letting $[a(1+\frac{b}{a})] = [a]+ < a > [1+\frac{b}{a}]$ by *Proposition 3.1 (1)* allows us to conclude:

$$\begin{aligned} [a(1+\frac{b}{a})][\frac{b}{a}] &= ([a]+ < a > [1+\frac{b}{a}])[\frac{b}{a}] \\ &= [a][\frac{b}{a}]+ < a > [1+\frac{b}{a}][\frac{b}{a}] \\ &= [a][\frac{b}{a}] \ by\ (W1) \\ &= [a][ab] \ by\ Corollary\ 3.1\ (3) \\ &= [a]([a]+[b]+\eta[a][b]) \ by\ (W2) \\ &= [a][b] \ by\ Corollary\ 3.1\ (1)\ and\ (W3) \end{aligned} \tag{3.4}$$

and

$$[a+b][ab(a+b)] = [a][b].$$

□

The Witt relation will be an important tool in allowing us to determine the structure of $K_2^W(F)$ in the next section. We conclude with the following corollary which implies the commutativity of symbols in $K_*^W(F)$.

Corollary 3.2: Let $(a,b) \in (F^\times)^2$ then $[a][b] = [b][a]$ in $K_2^W(F)$.

Proof: This is a direct consequence of *Proposition 3.2*:

$$[a][b] = [a+b][ab(a+b)] = [b+a][ba(b+a)] = [b][a].$$

□

3.3 Generators and Relations in K_n^W for $n \in \mathbb{Z}$

In this section we will establish some facts regarding $K_n^W(F)$ which will be of great importance in the next chapter.

Proposition 3.3: The following hold:

(1) For $n \geq 1$, $K_n^W(F)$ is generated as an abelian group by the product of symbols

$$[u_1] \cdot \ldots \cdot [u_n]$$

with $u_i \in F^\times$.

(2) For $n \leq 0$, $K_n^W(F)$ is generated as an abelian group by

$$\eta^n < u >$$

with $u \in F^\times$.

Proof: Following the construction of $K_*^W(F)$ we have that any element in $K_n^W(F)$ is of the form $\eta^m[u_1]\cdots[u_k]$ with $k - m = n$. The result follows inductively by applying $\eta[a][b] = [a] + [b] - [ab]$ to $\eta^m[u_1]\cdots[u_k]$, reducing it to (1) if $k > m$ and (2) if $k \leq m$.

□

Corollary 3.3.1: Let $a, b \in (F^\times)$ then the following relations hold in $K_{-m}^W(F)$ for $m \geq 0$:

(1) $\eta^m < ab^2 > + \eta^m < a >= 0$,

(2) $2\eta^m < 1 >= 0$,

(3) $\eta^m < a > + \eta^m < b > + \eta^m < a + b > + \eta^m < ab(a + b) >= 0$ if $a + b \neq 0$.

Proof: It is enough to consider the case $m = 0$.

(1) This follows immediately by *Proposition 3.1 (2)* and *Corollary 3.1 (2)*.

(2) This follows by *Proposition 3.1 (3)* .

(3) Let us consider

$$\begin{aligned} < a > + < b > &= (1+\eta[a]) + (1+\eta[b]) \\ &= \eta([a]+[b]) \ \ by\ 2=0\ in\ K_*^W(F) \\ &= \eta([ab]+\eta[a][b])\ by\ (W3) \\ &= \eta([ab(a+b)^2]+\eta[a][b])\ by\ Corollary\ 3.1 \\ &= \eta([ab(a+b)^2]+\eta[a+b][ab(a+b)])\ by\ Proposition\ 3.2 \\ &= \eta([a+b]+[ab(a+b)])\ by\ (W3) \\ &= (1+\eta[a+b]) + (1+\eta[ab(a+b)])\ by\ 2=0\ in\ K_*^W(F) \\ &= < a+b > + < ab(a+b) > . \end{aligned} \tag{3.5}$$

Therefore,

$$< a > + < b > = < a+b > + < ab(a+b) >$$

and adding $< a+b > + < ab(a+b) >$ to both sides implies

$$< a > + < b > + < a+b > + < ab(a+b) > = 0.$$

□

Corollary 3.3.2: Let $a, b \in (F^{\times})$ then the following relations hold in $K_1^W(F)$:

(1) $[1] = 0$,

(2) $[ab^2] + [a] = 0$,

(3) $[a] + [b] + [a+b] + [ab(a+b)] = 0$.

Proof:

(1) This was established in *Proposition 3.1 (3)*.

(2) Similarly this follows by *Corollary 3.1 (2)*.

(3) The *Witt relation* implies

$$[a][b] = [a+b][ab(a+b)].$$

Applying (W2) to $[ab] = [ab(a+b)^2]$ (which follows by (2)) implies

$$[a] + [b] + \eta[a][b] = [a+b] + [ab(a+b)] + \eta[a+b][ab(a+b)].$$

By *Proposition 3.2* we can cancel common terms such that

$$[a] + [b] + [a+b] + [ab(a+b)] = 0.$$

□

Corollary 3.3.3: Let $a, b, c, d \in F^\times$ then the following relations hold in $K_2^W(F)$:

(1) $[a][b] = 0$ whenever $<< a, b >>_b = 0$ in $I^2(F)$,

(2) $[ab][c] + [a][b] + [ac][b] + [a][c] = 0$,

(3) $[a][b] + [c][d] = 0$ with $<< a, b >>_b \cong << c, d >>_b$.

Proof: The idea for this proof is due to Suslin [19].

Consider the map $\delta : I(F) \longrightarrow K_2^W \times (F^\times/(F^\times)^2)$ defined by

$$<< u >>_b \longmapsto (0, \overline{u})$$

where addition in $K_2^W \times (F^\times/(F^\times)^2)$ is defined by

$$(x, \overline{r}) + (y, \overline{s}) = (x + y + [r][s], \overline{rs}).$$

It is easy to see that $K_2^W \times (F^\times/(F^\times)^2)$ is an abelian group by *Corollary 3.2*. The additive identity is

$$(0, \overline{1})$$

and the additive inverse of $(x, \overline{r})$ is given by

$$(x, \overline{r}).$$

It suffices to show that δ is well-defined by checking the relations of $I(F)$ in *Theorem 2.5*:

(1) $\delta(<< 1 >>_b) = (0, \overline{1})$.

(2) $\delta(<< uv^2 >>_b + << u >>_b) = (0, \overline{uv^2}) + (0, \overline{u}) = (0, \overline{u}) + (0, \overline{u}) = (0, \overline{1})$ by *Corollary 3.1.*

(3) $\delta(<< u >>_b + << v >>_b + << u+v >>_b + << uv(u+v) >>_b)$

is equal to

$$([u][v], \overline{uv}) + ([u+v][uv(u+v)], \overline{(u+v)(uv(u+v))})$$

which by *Witt relation* and *Corollary 3.1* is equal to

$$([u][v], \overline{uv}) + ([u][v], \overline{uv}) = 0.$$

Therefore we have shown that δ is well-defined.

Claim: $\delta(I^2(F)) \subset K_2^W(F) \times \{\overline{1}\}$.

Proof of Claim: This follows from the following fact:

$$<< u, v >>_b = << u >>_b + << v >>_b + << uv >>_b .$$

Indeed,

$$\delta(<< u, v >>_b) = (0, \overline{u}) + (0, \overline{v}) + (0, \overline{uv})$$

which is precisely

$$([u][v], \overline{uv}) + (0, \overline{uv}).$$

However $([u][v], \overline{uv}) = ([u][v] + [uv][1], \overline{uv}) = (0, \overline{uv}) + ([u][v], \overline{1})$ which implies

$$([u][v], \overline{uv}) + (0, \overline{uv}) = ([u][v], \overline{1})$$

and $\delta(I^2(F)) \subset K_2^W(F) \times \{\bar{1}\}$. If we consider the projection

$$K_2^W(F) \times \{\bar{1}\} \longrightarrow K_2^W(F)$$

and apply *Theorem 2.5* we are done.

□

Corollary 3.3.4: Let $u_i, v_i \in F^\times$ then the following relations hold in $K_n^W(F)$ whenever $n \geq 3$:

(1) $[u_1] \cdots [u_n] = 0$ whenever $<< u_1, \cdots, u_n >>_b = 0$ in $I^n(F)$,

(2) $[u_1] \cdots [u_n] - [v_1] \cdots [v_n] = 0$ whenever $<< u_1, \cdots, u_n >>_b \cong << v_1, \cdots, v_n >>_b$.

Proof: We will begin by first showing the following:

Claim: If $<< u_1, \cdots, u_n >>_b$ is simply p-equivalent to $<< v_1, \cdots, v_n >>_b$ then

$$[u_1] \cdots [u_n] = [v_1] \cdots [v_n].$$

Proof of Claim: It follows by assumption that there exists $i, j \in [1, n]$ such that

$$<< u_i, u_j >>_b \cong << v_i, v_j >>_b \tag{3.6}$$

and $u_k = v_k$ whenever $k \neq i, j$. Then *Corollary 3.2* implies

$$[u_1] \cdots [u_n] - [v_1] \cdots [v_n]$$

is equal to

$$([u_i][u_j] - [v_i][v_j])[w_1] \cdots [w_{n-2}]$$

where w_k takes on the values of v_k whenever $k \neq i, j$.

It follows immediately by *Corollary 3.3.3 (3)* and (3.6) that

$$([u_i][u_j] - [v_i][v_j])[w_1] \cdots [w_{n-2}] = (0)[w_1] \cdots [w_{n-2}] = 0.$$

We can now proceed to prove the lemma:

(1): *Theorem 2.6 (1)* implies that

$$<< u_1, \cdots, u_n >>_b \approx << 1, w_2, \cdots, w_n >>_b$$

with $w_i \in F^\times$. However by induction on *Claim* this implies

$$[u_1] \cdots [u_n] = [1][w_2] \cdots [w_n] = 0$$

by *Proposition 3.1 (3)*.

(2): *Theorem 2.6 (2)* implies

$$<< u_1, \cdots, u_n >>_b \approx << v_1, \cdots, v_n >>_b$$

which by induction on *Claim* implies

$$[u_1] \cdots [u_n] = [v_1] \cdots [v_n].$$

□

3.4 $K_*^W(F)$ and $I^*(F)$

In this section we will construct an isomorphism of $W(F)-algebras$ *between* $K_*^W(F)$ *and* $I^*(F)$.

Proposition 3.4: The map $\alpha : K_*^W(F) \longrightarrow I^*(F)$ defined by

$$\eta \mapsto < 1 >_b \in W(F) = I^{-1}(F)$$

and

$$[a] \mapsto << a >>_b \in I(F)$$

with $a \in F^\times$ is well-defined.

Proof: It suffices to check relations (W1), (W2), (W3) and (W4) hold:

W1 : $\alpha([a][1+a]) =<< a >>_b \cdot << 1+a >>_b= 0$ in $I^2(F)$ since

$$<< a >>_b \cdot << 1+a >>_b$$

is isotropic implies by *Theorem 2.6 (1)* that

$$<< a >>_b \cdot << 1+a >>_b \cong << 1, u >>_b$$

with $u \in F^\times$ and

$$<< 1, u >>_b = < 1, u, 1, u >_b = 0 \in W(F).$$

W2 : $\alpha([ab] + [a] + [b] + \eta[a][b])$ is equal to

$$<< ab >>_b + << a >>_b + << b >>_b + << a, b >>_b$$

which can be rewritten as

$$< 1, ab >_b \perp < 1, a >_b \perp < 1, b >_b \perp < 1, a, b, ab >_b = 0 \in W(F).$$

W3 : $\alpha([a]\eta - \eta[a]) =< 1 >_b \cdot << a >>_b - << a >>_b \cdot < 1 >_b = 0 \in W(F).$

W4 : $\alpha(\eta[1]) =< 1 >_b \cdot << 1 >>_b = 0 \in W(F).$

□

Therefore we have that α is well-defined.

Theorem 3.4: *$\alpha : K_*^W(F) \longrightarrow I^*(F)$ is an isomorphism of $W(F)$-algebras*

Proof: α is clearly surjective following the construction of $I^*(F)$ in *Definition 2.5.3*.

Thus it suffices to show that the map $\tau_n : I^n(F) \longrightarrow K_n^W(F)$ defined by

$$<< a >>_b \mapsto [a]$$

and

$$< 1 >_b \in I^{-1}(F) \mapsto \eta$$

is well-defined. This follows immediately by *Theorem 2.5*, *Corollary 3.3.1, Corollary 3.3.2, Corollary 3.3.3* and *Corollary 3.3.4*. Therefore,

$$K_n^W(F) \longrightarrow I^n(F) \longrightarrow K_n^W(F)$$

is the identity map which is equivalent to

$$\tau_n \circ \alpha_n = id_{K_n^W(F)}$$

and we conclude that α_n is injective for every $n \in \mathbb{Z}$ which implies α is injective.

□

3.5 $I^*(F)$ and $T_*^W(I(F))$

In this section we will construct the $\mathbb{Z}/2$-graded algebra $T_*^W(I(F))$ containing the tensor algebra of $I(F)$ modulo the Steinberg relations and use the structural results in the prior sections to define an isomorphism between $T_*^W(I(F))$ and $I^*(F)$.

Definition 3.5.1: We define $T^W(I(F))$ to be the tensor algebra of the $W(F)$-modules $I(F)$ modulo the ideal generated by $<< u >>_b \otimes_{W(F)} << 1-u >>_b$ with $u \in F^\times$.

$$T^W(I(F)) = Tens_{W(F)}(I(F))/(<< u >>_b \otimes_{W(F)} << 1-u >>_b).$$

Proposition 3.5: The map $T_n^W(I(F)) \longrightarrow I^n(F)$ defined by

$$b_1 \otimes_{W(F)} \cdots \otimes_{W(F)} b_n \ \mapsto \ b_1 \cdot \ldots \cdot b_n$$

with $b_i \in I(F)$ is well-defined.

Proof: There is a canonical $W(F)$-multilinear map

$$\underbrace{I(F) \times \cdots \times I(F)}_{n \text{ times}} \longrightarrow I^n(F)$$

defined by

$$b_1 \times \cdots b_n \mapsto b_1 \cdot \cdots \cdot b_n$$

with $b_i \in I(F)$ which by the *universal property of tensor products* of $W(F)$-modules extends to a well-defined map

$$\underbrace{I(F) \otimes_{W(F)} \cdots \otimes_{W(F)} I(F)}_{n-\text{times}} \longrightarrow I^n(F). \tag{3.7}$$

To prove *Proposition 3.5* it is enough to show that

$$(<< u >>_b \otimes_{W(F)} << 1-u >>_b) \cap T_n(I(F))$$

factors through (3.7), which follows immediately by the presentation of $I^n(F)$ in *Theorem 2.5* and the fact that

$$<< u >>_b \cdot << 1-u >>_b = 0 \ \in \ I^2(F)$$

for all $u \in F^\times$

□

Let us consider $T_*^W(I(F)) = \oplus_{n\in\mathbb{Z}} T_n^W(I(F))$ with $T_n^W(I(F)) = W(F)$ for $n < 0$. We define an action of

$$< 1 >_b \in T_{-n}^W(I(F)) = W(F) \tag{3.8}$$

on $T_m^W(I(F))$ with $m, n \geq 1$ by showing the following:

Lemma 3.5: For every $n \geq 1$ there exists a unique homomorphism of $W(F)$-modules

$$\epsilon_n : T_n^W(I(F)) \longrightarrow T_{n-1}^W(I(F))$$

defined by

$$b_1 \otimes_{W(F)} b_2 \otimes_{W(F)} \cdots \otimes_{W(F)} b_n \mapsto \Big(b_1 \cdot b_2\Big) \otimes_{W(F)} \cdots \otimes_{W(F)} b_n$$

with $b_i \in I(F)$.

Proof: We begin by considering the map

$$\zeta_n : \underbrace{I(F) \times \cdots \times I(F)}_{n \text{ times}} \longrightarrow T_{n-1}^W(I(F))$$

defined by

$$b_1 \times b_2 \times \cdots \times b_n \mapsto \Big(b_1 \cdot b_2\Big) \otimes_{W(F)} \cdots \otimes_{W(F)} b_n$$

with $b_i \in I(F)$ then ζ_n is clearly $W(F)$-multilinear and the *universal propety of tensor products* implies that we have a map

$$\overline{\zeta}_n : \underbrace{I(F) \otimes_{W(F)} \cdots \otimes_{W(F)} I(F)}_{n-\text{times}} \longrightarrow T^W_{n-1}(I(F))$$

where $\overline{\zeta}_n$ is a homomorphism of $W(F)$-modules.

Therefore all that remains to show is that

$$\overline{\zeta}_n\Big((<< u >>_b \otimes_{W(F)} << 1-u >>_b) \cap T^W_n(I(F))\Big) = 0 \in T^W_{n-1}(I(F)).$$

However this is an immediate consequence of the following:

Claim: $\overline{\zeta}_n(<< u_1 >>_b \otimes_{W(F)} \cdots \otimes_{W(F)} << u_n >>_b) = 0 \in T^W_{n-1}(I(F))$ if $u_i + u_{i+1} = 1$ for some $i \geq 1$.

Proof of Claim: If $i \geq 3$ the result follows immediately since

$$\Big(<< u_1 >>_b \cdot << u_2 >>_b \Big) \otimes_{W(F)} << u_3 >>_b \otimes \cdots \otimes_{W(F)} << u_n >>_b$$

is an element of $(<< u >>_b \otimes_{W(F)} << 1-u >>_b) \cap T^W_{n-1}(I(F))$.

Assume $i = 2$, then

$$\Big(<< u_1 >>_b \cdot << u_2 >>_b \Big) \otimes_{W(F)} \cdots \otimes_{W(F)} << u_n >>_b$$

is equal to

$$<< u_1 >>_b \cdot \Big(<< u_2 >>_b \otimes_{W(F)} << u_3 >>_b \Big) \otimes \cdots \otimes_{W(F)} << u_n >>_b$$

which is an element of $(<< u >>_b \otimes_{W(F)} << 1-u >>_b) \cap T^W_{n-1}(I(F))$.

If $i = 1$, then

$$\Big(<< u_1 >>_b \cdot << u_2 >>_b \Big) \otimes_{W(F)} \cdots \otimes_{W(F)} << u_n >>_b = 0$$

since $<< u_1 >>_b \cdot << u_2 >>_b = 0 \in I(F)$.

□

It follows by the *universal property of quotient maps* that ϵ_n is well-defined and

$$\overline{\zeta}_n : \underbrace{I(F) \otimes_{W(F)} \cdots \otimes_{W(F)} I(F)}_{n-\text{times}} \longrightarrow T_n^W(I(F)) \xrightarrow{\epsilon_n} T_{n-1}^W(I(F)).$$

□

To show that $T_*^W(I(F))$ has the structure of $\mathbb{Z}/2$-graded algebra of $W(F)$-modules we define the multiplication

$$< 1 >_b \cdot << u_1 >>_b \otimes_{W(F)} \cdots \otimes_{W(F)} << u_m >>_b$$

with $< 1 >_b \in T_{-n}^W(I(F)) = W(F)$ by

$$\epsilon_{m-n+1} \circ \cdots \circ \epsilon_m \Big(<< u_1 >>_b \otimes_{W(F)} \cdots \otimes_{W(F)} << u_m >>_b \Big).$$

The right multiplication of $< 1 >_b \in T_{-n}^W(I(F)) = W(F)$ is defined mutatis mutandis. If $< u >_b \in T_{-n}^W(I(F))$ and $< v >_b \in T_{-m}^W(I(F))$ then

$$< u >_b \cdot < v >_b = < uv >_b \in T_{-(n+m)}^W(I(F)).$$

We can use the above structural results on $T_*^W(I(F))$ to define a map:

$$\theta : K_*^W(F) \longrightarrow T_*^W(I(F))$$

by

$$[u] \mapsto << u >>_b \in T_1^W(I(F)) \eta \mapsto < 1 >_b \in T_{-1}^W(I(F))$$

which is well-defined.

Theorem 3.5: *The natural map* $\beta : T_*^W(I(F)) \longrightarrow I^*(F)$ *defined by*

$$<< u >>_b \mapsto [u]$$

is an isomorphism of $W(F)$-algebras.

Proof: It is easy to see by *Proposition 3.5* that β is a well-defined surjective homomorphism of $W(F)$-algebras and the above remarks along with *Theorem 3.4* imply

$$T_*^W(I(F)) \xrightarrow{\beta} I^*(F) \xrightarrow{\cong} K_*^W(F) \xrightarrow{\theta} T^W(F)$$

is the identity map and β is injective.

□

Corollary 3.5: The map

$$Tens_{W(F)}(I(F))/(<< u >>_b \otimes_{W(F)} << 1-u >>_b) \longrightarrow \oplus_{n\geq 0} I^n(F) \tag{3.9}$$

defined by

$$<< u >>_b \mapsto << u >>_b$$

is an isomorphism of $W(F)$-algebras.

Proof: This follows immediately by *Theorem 3.4* and *Theorem 3.5*.

□

CHAPTER 4

Milnor-Witt K-Theory

In this chapter we will establish the main results of this thesis extending the work of Morel in [15] to fields of characteristic 2.

4.1 Definitions and Facts

Definition 4.1 The *Milnor-Witt K-ring* of F is the free and graded $\mathbb{Z}$-algebra $K_*^{MW}(F)$ generated by the symbols $[u]_{MW}$ $(u \in F^\times)$ of degree 1 and one symbol η_{MW} of degree -1 subject to the following relations:

MW1 : For each $a \in F^\times - \{1\}$: $[a]_{MW}[1-a]_{MW} = 0$,

MW2 : For each $(a,b) \in (F^\times)^2$: $[ab]_{MW} = [a]_{MW} + [b]_{MW} + \eta_{MW}[a][b]$,

MW3 : For each $u \in F^\times$: $[u]_{MW}\eta_{MW} = \eta_{MW}[u]_{MW}$,

MW4 : $\eta_{MW}(\eta_{MW}[-1]_{MW} + 2) = 0$.

Let us denote $h = \eta[-1]_{MW} + 2$ such that (4) can be rewritten as

$$\eta h = 0.$$

Again following Morel, for any $u \in F^\times$ we let $< u >_{MW} = 1 + \eta[u]_{MW}$.

We have the following relations in $K_*^{MW}(F)$:

Lemma 4.1.1: Let $(a,b) \in (F^\times)^2$. Then the following relations hold in $K^{MW}(F)$:

(1) $[ab]_{MW} = [a]_{MW} + < a >_{MW} [b]_{MW}$,

(2) $< ab >_{MW} = < a >_{MW} < b >_{MW}$,

(3) $< 1 >_{MW} = 1$ and $[1]_{MW} = 0$.

Proof:

(1) This follows as in *Proposition 3.1 (1)*.

(2) This follows as in *Proposition 3.1 (2)*.

(3) $\eta_{MW}(\eta_{MW}[-1]_{MW}+2)=0$ implies

$$[1]\eta_{MW}(\eta_{MW}[-1]_{MW}+2)=(<1>_{MW}-1)(<1>_{MW}+1)=0$$

which upon expanding implies

$$<1>_{MW}=1.$$

Hence $[1]_{MW}=[(1)(1)]_{MW}=[1]_{MW}+<1>_{MW}[1]_{MW}=2[1]_{MW}$ and $[1]_{MW}=0$.

□

We can reformulate $(MW4)$ in the characteristic 2 case as follows:

Lemma 4.1.2: $(MW4)$ is equivalent to

$(MW4^{\bullet}):\ 2\eta_{MW}=0=[1]$.

Proof: By (MW4) and *Lemma 4.1.1 (3)*

$$2\eta_{MW}=\eta_{MW}(\eta_{MW}[-1]_{MW}+2)=0$$

which implies $h=2$ and $\eta_{MW}h=2\eta_{MW}=0$. The reverse implication is trivial.

Lemma 4.1.3: The following hold:

(1) For $n\geq 1$, $K_m^{MW}(F)$ is generated as an abelian group by the product of symbols

$$[u_1]_{MW}\cdot\ldots\cdot[u_m]_{MW}$$

with $u_i\in F^{\times}$,

(2) For $m\leq 0$, $K_m^{MW}(F)$ is generated as an abelian group by

$$\eta_{MW}^m<u>_{MW}$$

with $u \in F^{\times}$.

Proof: The proof is identical to the one of *Proposition 3.2*.

□

Lemma 4.1.4: The map $\pi : K_*^W(F) \longrightarrow K_*^{MW}(F)/(h)$ defined by

$$[u] \mapsto [u]_{MW} + (h)$$

and

$$\eta \mapsto \eta_{MW} + (h)$$

is a well-defined morphism of $\mathbb{Z} - algebras$.

Proof: This is immediate by *Definition 3.1.2*.

□

Lemma 4.1.5: The map $\mu : K_*^{MW}(F) \longrightarrow K_*^W(F)$ sending $[u]_{MW} \mapsto [u]$ and $\eta_{MW} \mapsto \eta$ is a well-defined morphism of $\mathbb{Z} - algebras$.

Proof: This is immediate by *Definition 4.1*.

□

If we consider μ we see that $K_*^{MW}(F) \cdot h \subset Ker(\mu)$. Therefore μ factors through π and moreover

$$\overline{\mu} \circ \pi : K_*^W(F) \longrightarrow K_*^W(F)$$

is the identity map on $K_*^W(F)$ with $\overline{\mu}$ defined by

$$\mu : K_*^{MW}(F) \longrightarrow K_*^{MW}(F)/(h) \xrightarrow{\overline{\mu}} K_*^W(F).$$

We can conclude that

$$\pi_n : K_n^W(F) \longrightarrow K_n^{MW}(F)/((h) \cap K_n^{MW}(F))$$

is an isomorphism.

□

Proposition 4.1: The map $\omega : K_*^{MW}(F) \longrightarrow I^*(F)$ defined by

$$[u]_{MW} \mapsto - << u >>_b$$

and

$$\eta_{MW} \mapsto < 1 >_b \in I^{-1}(F)$$

is well-defined and surjective.

Proof: It follows by *Lemma 4.1.5* and *Theorem 3.4* that ω is well-defined and surjectivity follows by definition.

□

4.2 Milnor K-theory of a field F

In this section we will define the Milnor K-theory of a field F, originally introduced by J. Milnor in [13].

Definition 4.2: The *Milnor K-theory* of a field F is given by

$$K_*^M(F) = Tens_{\mathbb{Z}}(F^\times)/(u \otimes (1-u))$$

where $(u \otimes (1-u))$ is the ideal generated by $u \otimes (1-u)$ in $Tens_{\mathbb{Z}}(F^\times)$ with $u \in F^\times - \{1\}$.

It follows easily by *Definition 4.2* that $K_*^M(F)$ has the structure of a graded $\mathbb{Z}$-algebra.

Lemma 4.2.1: The map $\phi : K_*^{MW}(F) \longrightarrow K_*^M(F)$ defined by

$$[u] \mapsto \overline{u}, \ \eta \mapsto 0$$

is a surjective morphism of $\mathbb{Z}$-algebras.

Proof: It suffices to check that θ is well-defined. Therefore, by *Definition 4.1* we only need to check the following corresponding relations hold in $K_*^M(F)$:

MW1 : For each $a \in F^\times - \{1\}$: $[a]_{MW}[1-a]_{MW} = 0$,

MW2 : For each $(a,b) \in (F^\times)^2$: $[ab]_{MW} = [a]_{MW} + [b]_{MW} + \eta[a][b]$.

(1) follows immediately by *Definition 4.2.*

(2) follows by the group structure of $F^\times$.

□

Lemma 4.2.2: The induced homomorphism $\overline{\phi} : K_*^{MW}(F)/(\eta) \longrightarrow K_*^M(F)$ is an isomorphism.

Proof: Indeed consider the map

$$K_*^M(F) \longrightarrow K_*^{MW}(F)/(\eta)$$

defined by

$$u \mapsto [u]_{MW}.$$

To show that this map is well-defined it is enough to check:

$$u \otimes (1-u) \mapsto 0$$

with $u \in F^\times - \{1\}$ which follows immediately by *Definition 4.1.* Therefore

$$K_*^{MW}(F)/(\eta) \longrightarrow K_*^M(F) \longrightarrow K_*^{MW}(F)/(\eta)$$

is the identity on $K_*^{MW}(F)/(\eta)$ and we can conclude that $\overline{\phi}$ is an isomorphism.

Moreover this implies that

$$\overline{\phi}_n : K_n^{MW}(F)/((\eta) \cap K_n^{MW}(F)) \longrightarrow K_n^M(F)$$

is an isomorphism of groups.

□

4.3 Main Result

In this section we will combine the results of the prior sections to construct an exact sequence between $K^W_{n+1}(F)$*,* $K^{MW}_n(F)$ *and* $K^M(F)$*.*

We begin by considering the map corresponding to multiplication by η_{MW}:

$$\eta_{MW} : K^{MW}_{n+1}(F) \longrightarrow K^{MW}_n(F) \tag{4.1}$$

defined by

$$\eta^m_{MW}[u_1]_{MW} \cdots [u_k]_{MW} \mapsto \eta^{m+1}_{MW}[u_1]_{MW} \cdots [u_k]_{MW}$$

with $k - m = n + 1$. *Definition 4.1* implies that η_{MW} is well-defined in this sense. The *universal property of quotient maps* and (4.1) imply $\overline{\eta}_{MW}$ defined by

$$\eta_{MW} : K^{MW}_{n+1}(F) \longrightarrow K^{MW}_{n+1}(F)/((h) \cap K^{MW}_{n+1}(F)) \overset{\overline{\eta}_{MW}}{\longrightarrow} K^{MW}_n(F)$$

is a well-defined map.

Proposition 4.3: $K^W_{n+1}(F) \overset{\overline{\eta}_{MW} \circ \pi_{n+1}}{\longrightarrow} K^{MW}_n(F) \overset{\phi_n}{\longrightarrow} K^M_n(F) \longrightarrow 0$ is an exact sequence with

$$\overline{\eta}_{MW} \circ \pi_{n+1} : K^W_{n+1}(F) \overset{\pi_{n+1}}{\longrightarrow} K^{MW}_{n+1}(F)/(h) \overset{\overline{\eta}_{MW}}{\longrightarrow} K^{MW}_n(F)$$

and

$$\phi_n : K^{MW}_n(F) \longrightarrow K^M_n(F).$$

Proof: It is easy to see that $\phi_n(K^{MW}_n(F)) = K^M_n(F)$ by definition so it enough to check:

$$\overline{\eta}_{MW} \circ \pi_{n+1}(K^W_{n+1}(F)) = Ker(\phi_n).$$

However we know that $((\eta) \cap K^{MW}_n(F)) = \eta \cdot K^{MW}_{n+1}(F)$ and by *Lemma 4.2.2* we have that

$$Ker(\phi_n) = \eta \cdot K^{MW}_{n+1}(F).$$

It follows by definition of $\overline{\eta}_{MW}$ and (4.1),

$$\overline{\eta}_{MW} \circ \pi_{n+1}(K^W_{n+1}(F)) = \overline{\eta}_{MW}(K^{MW}_{n+1}(F)/(h)) = \eta_{MW} \cdot K^{MW}_{n+1}(F).$$

□

Theorem 4.3.1: *$K^{MW}_n(F)$ is the pull-back of the diagram:*

$$\begin{array}{ccc} & & K^M_n(F) \\ & & \downarrow \\ I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) \end{array}$$

for every $n \geq 1$.

Proof:

For every $n \geq 1$ consider the following diagram:

$$\begin{array}{ccccccccc} & & K^W_{n+1}(F) & \xrightarrow{\overline{\eta}_{MW}\circ\pi_{n+1}} & K^{MW}_n(F) & \xrightarrow{\phi_n} & K^M_n(F) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_{n+1}} & & \downarrow{\scriptstyle \omega_n} & & \downarrow{\scriptstyle \lambda_n} & & \\ 0 & \longrightarrow & I^{n+1}(F) & \longrightarrow & I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) & \longrightarrow & 0 \end{array}$$

with the map $\lambda_n : K^M_n(F) \longrightarrow I^n(F)/I^{n+1}(F)$ defined by

$$\overline{u_1 \otimes \cdots \otimes u_n} \mapsto << u_1, \cdots, u_n >>_b + I^{n+1}(F).$$

and it follows easily that λ_n is well-defined and surjective. The commutativity of the above diagram is then given by *Proposition 4.3, Theorem 3.4* and *Proposition 4.1*. The result follows immediately by *Lemma A.3*.

□

Corollary 4.3: $K^{MW}_0(F)$ is the pull-back of the canonical projection $\lambda_0 : \mathbb{Z} \longrightarrow \mathbb{Z}/2$ and $W(F) \longrightarrow \mathbb{Z}/2$.

Proof: This follow identically to *Theorem 4.3.1*.

□

Let us define $G_n(F)$, for every $n \geq 0$, to be the the pull-back of

$$\begin{array}{ccc} & & K_n^M(F) \\ & & \downarrow \\ I^n(F) & \longrightarrow & I^n(F)/I^{n+1}(F) \end{array}$$

Consider $G_*(F) = \oplus_{n\in\mathbb{Z}} G_n$ with $G_{-n} = W(F)$ whenever $n > 0$ then *Corollary 4.3* and *Proposition 2.5* imply

$$K_0^{MW}(F) \cong \widehat{W(F)} = G_0(F)$$

Moreover, the following theorem will show that $G_*(F)$ is a $\mathbb{Z}$-graded algebra isomorphic to $K_*^{MW}(F)$:

Theorem 4.3.2: *The natural homomorphism* $\Omega : K_*^{MW}(F) \longrightarrow G_*(F)$ *defined by*

$$[u]_{MW} \mapsto (<< u >>_b, u)$$

$$\eta_{MW} \mapsto < 1 >_b \in G_{-1}(F) = W(F)$$

is an isomorphism.

Proof: It follows by *Proposition 4.1* and *Lemma 4.2.1* that Ω is well-defined and surjective. Additionally, *Theorem 4.3.1* and *Corollary 4.3* imply that it is enough to show

$$K_{-n}^{MW} \cong G_{-n}(F)$$

for every $n > 0$. Assume $n > 0$ and consider the map

$$\Gamma_{-n} = \overline{\eta}_{MW} \circ \pi_{-n+1} \circ \alpha_{-n+1}^{-1} : W(F) \longrightarrow K_{-n}^{MW}(F).$$

This is well-defined and

$$K_{-n}^{MW}(F) \xrightarrow{\Omega_{-n}} W(F) \xrightarrow{\Gamma_{-n}} K_{-n}^{MW}(F)$$

is the identity map. Therefore we conclude that for every $n > 0$:

$$K_{-n}^{MW}(F) \cong W(F).$$

$\square$

4.4 $K_*^{MW}(F)$ and $T_*^W(K_1^{MW}(F))$

Definition 3.4: We define $T^W(K_1^{MW}(F))$ to be the tensor algebra of the $K_0^{MW}(F)$-modules $K_1^{MW}(F)$ modulo the ideal generated by $[u]_{MW} \otimes_{K_0^{MW}(F)} [1-u]_{MW}$ with $u \in F^\times - \{1\}$.

$$T^W(K_1^{MW}(F)) = Tens_{K_0^{MW}(F)}(K_1^{MW}(F))/([u]_{MW} \otimes_{K_0^{MW}(F)} [1-u]_{MW}).$$

Let $T_*^W(K_1^{MW}(F)) = \oplus_{n\in\mathbb{Z}} T_n^W(K_1^{MW}(F))$ with $T_{-n}^W(K_1^{MW}(F)) = K_{-1}^{MW}(F)$ for every $n \geq 1$. We define the multiplication operation on $T_*^W(K_1^{MW}(F))$,

$$\eta_{MW} < 1 >_{MW} \cdot [u_1]_{MW} \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} [u_m]_{MW}$$

with $m \geq 1$ by introducing the following lemma:

Lemma 4.4: The map $\overline{\chi}_{m+1} : T_{m+1}^W(K_1^{MW}(F)) \longrightarrow T_m^W(K_1^{MW}(F))$ defined by sending

$$y_1 \otimes_{K_0^{MW}(F)} y_2 \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} y_{m+1}$$

to

$$\Big(\eta_{MW} \cdot y_1 \cdot y_2\Big) \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} y_{m+1}$$

with $y_i \in K_1^{MW}(F)$ is well-defined.

Proof: There is a canonical $K_0^{MW}(F)$-multilinear map

$$\chi : \underbrace{K_1^{MW}(F) \times \cdots \times K_1^{MW}(F)}_{m+1 \text{ times}} \longrightarrow T_m^W(K_1^{MW}(F))$$

defined by sending

$$y_1 \times \cdots \times y_{m+1}$$

to

$$(\eta_{MW} \cdot y_1 \cdot y_2) \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} y_{m+1}$$

which by the *universal property of tensor products* of $K_0^{MW}(F)$-modules extends to a well-defined map

$$\underbrace{K_1^{MW}(F) \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} K_1^{MW}(F)}_{m+1-\text{times}} \longrightarrow T_m^W(K_1^{MW}(F)).$$

To conclude it suffices to show,

$$\chi\Big(([u]_{MW} \otimes_{K_0^{MW}(F)} [1-u]_{MW}) \cap T_m^W(K_1^{MW}(F)\Big) = 0$$

which is identical to *Lemma 3.5.* Therefore we have by the *universal property of quotient map* that $\overline{\chi}$ satisfies

$$\chi : \underbrace{K_1^{MW}(F) \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} K_1^{MW}(F)}_{m+1-\text{times}} \longrightarrow T_{m+1}^W(K_1^{MW}(F)) \xrightarrow{\overline{\chi}} T_m^W(K_1^{MW}(F))$$

and $\overline{\chi}$ is well-defined.

□

We identify the multiplication operation on $T_*^W(K_1^{MW}(F))$ by

$$\eta_{MW}^k < 1 >_{MW} \cdot [u_1]_{MW} \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} [u_m]_{MW}$$

with

$$\chi_{m-k+1} \circ \cdots \circ \chi_m\Big([u_1]_{MW} \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} [u_m]_{MW}\Big)$$

and if $< u >_{MW} \in K_0^{MW}(F)$ and $\eta_{MW}^n < v >_{MW} \in K_{-n}^{MW}(F)$ with $n \geq 1$ then

$$< u >_{MW} \cdot \eta_{MW}^n < v >_{MW} = \eta_{MW}^n < uv >_{MW} \in K_{-n}^{MW}(F)$$

and we conclude that $T_*^W(K_1^{MW}(F))$ is a $\mathbb{Z}$-graded $K_0^{MW}(F)$-module.

Proposition 4.4: The map $T_m^W(K_1^{MW}(F)) \longrightarrow K_m^{MW}(F)$ defined by

$$y_1 \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} y_m \;\mapsto\; y_1 \cdot \ldots \cdot y_m$$

is well-defined.

Proof: There is a canonical $K_0^{MW}(F)$-multilinear map

$$\underbrace{K_1^{MW}(F) \times \cdots \times K_1^{MW}(F)}_{m \text{ times}} \longrightarrow K_n^{MW}(F)$$

defined by

$$u_1 \times \ldots \times u_m \mapsto u_1 \cdot \ldots \cdot u_m$$

with $u_i \in K_1^{MW}(F)$ such that the *universal property of tensor products* of $K_0^{MW}(F)$-modules extends this to a well-defined map

$$\underbrace{K_1^{MW}(F) \otimes_{K_0^{MW}(F)} \cdots \otimes_{K_0^{MW}(F)} K_1^{MW}(F)}_{n-\text{times}} \longrightarrow K_m^{MW}(F). \tag{4.2}$$

To prove *Proposition 4.4* it is enough to show that

$$([u]_{MW} \otimes_{K_0^{MW}(F)} [1-u]_{MW}) \cap T_m^W(K_1^{MW}(F))$$

factors through (4.2), which follows immediately by *Definition 4.1*.

□

Theorem 4.4: *The map $\Delta : T_*^W(K_1^{MW}(F)) \longrightarrow K_*^{MW}(F)$ defined by*

$$[u]_{MW} \mapsto [u]_{MW}$$

and

$$\eta_{MW} < 1 >_{MW} \mapsto \eta_{MW}$$

is an isomorphism of $K_0^{MW}(F)$-algebras.

Proof: Consider the map $K_*^{MW}(F) \longrightarrow T_*^W(K_1^{MW}(F))$ defined by

$$[u]_{MW} \mapsto [u]_{MW}$$

and

$$\eta_{MW} \mapsto \eta_{MW} < 1 >_{MW} .$$

It is easy to check that the relations of *Definition 4.1* are satisfied which implies the map is well-defined. It follows by *Proposition 4.4* and considering

$$T_*^W(K_1^{MW}(F)) \xrightarrow{\Delta} K_*^{MW}(F) \longrightarrow T_*^W(K_1^{MW}(F))$$

that Δ is well-defined and injective. Therefore it is enough to show that Δ is surjective, which follows by definition.

□

CHAPTER 5

Appendix

In this section we will establish some results that are necessary to Section 2.6 and Section 4.3 but did not seem suitable to be addressed in the main text.

Lemma A.1: The following hold:

(1) $<< a, b >>_b \cong << a, bd >>_b$ with $d \in D(<< a >>_b)^\times$,

(2) $<< a, b >>_b \cong << a+b, ab >>_b$ if $a + b \neq 0$.

Proof:

(1) $<< a, b >>_b = < 1, a >_b \otimes < 1, b >_b = < 1, a, b, ab >_b$ which is equal to

$$< 1, a >_b \perp b < 1, a >_b .$$

However $bd = bx^2 + bay^2$ with $x, y \in F$ implies

$$b < 1, a >_b \cong < bd, abd >_b \cong bd < 1, a >_b$$

and we can conclude

$$< 1, a >_b \perp b < 1, a >_b \cong < 1, a >_b \perp db < 1, a >_b$$

or equivalently,

$$<< a, b >>_b \cong << a, bd >>_b .$$

(2) $<< a, b >>_b \cong << a, ab >>_b$ since by (1) we have $a = 0^2 + a(1)^2$. Clearly

$$<< a, ab >>_b \cong << ab, a >>_b$$

and applying (1) again to

$$<< ab, a >>_b$$

with $a^{-1}(a+b) = 1 + \frac{b}{a} = 1^2 + ab(a^{-2})$ implies

$$<< ab, a >>_b \cong << ab, a(a^{-1}(a+b)) >>_b \cong << ab, a+b >>_b .$$

$\square$

Lemma A.2: For every $d \in D(<< u_1, \cdots, u_n >>_b)^\times$ with $u_i \in F^\times$

$$d << u_1, \cdots, u_n >>_b \cong << u_1, \cdots, u_n >>_b$$

Proof: We proceed by induction on n.

If $n = 1$ then $d \in D(<< u_1 >>_b)^\times$ implies $d = x^2 + u_1 y^2$. Hence

$$d << u_1 >>_b = < d, u_1 d > .$$

However $d^2 = d(x^2 + u_1 y^2) = dx^2 + du_1 y^2 \in D(d << u_1 >>_b)^\times$ implies

$$< d, u_1 d > \cong < d^2, -u_1 d^2 > \cong < 1, u_1 > = << u_1 >>_b .$$

If $n = m$ then $d \in D(<< u_1, \cdots, u_m >>_b)^\times = D(< 1, u_1 >_b \otimes << u_2, \cdots, u_m >>_b)^\times$ implies

$$d = x + u_1 y$$

with $x, y \in D(<< u_2, \cdots, u_m >>_b)$.

If $x = 0$ then $d = u_1 y$ implies

$$d << u_1, \cdots, u_m >>_b = u_1 y << u_1, \cdots, u_m >>_b$$

is equal to

$$(u_1 << u_1 >>)(y << u_2, \cdots, u_m >>_b)$$

which by induction assumption and the $n = 1$ case is equal to

$$<< u_1, \cdots, u_m >>_b .$$

If $y = 0$ then $d = x$ which implies

$$d << u_1, \cdots, u_m >>_b = << u_1 >> (x << u_2, \cdots, u_m >>_b)$$

which is equal to

$$<< u_1, \cdots, u_m >>_b$$

by induction assumption.

If $x, y \neq 0$ then

$$<< u_1, \cdots, u_m >>_b = (<< u_2, \cdots, u_m >>_b \perp u_1 << u_1, \cdots, u_m >>_b)$$

which by induction assumption is equal to

$$<< u_2, \cdots, u_m >>_b \perp u_1 x^{-1} y << u_1, \cdots, u_m >>_b)$$

or

$$<< u_1 x^{-1} y >>_b << u_2, \cdots, u_m >>_b .$$

However by the base case $1 + u_1 x^{-1} y \in D(<< u_1 x^{-1} y >>_b)^{\times}$ hence

$$<< u_1, \cdots, u_m >>_b = (1 + u_1 x^{-1} y) << u_1, \cdots, u_m >>_b$$

and $x << u_2, \cdots, u_m >>_b \cong << u_2, \cdots, u_m >>_b$ implies that

$$<< u_1, \cdots, u_m >>_b \cong (1 + u_1 x^{-1} y)(x) << u_1, \cdots, u_m >>_b$$

with $(1 + u_1 x^{-1} y)(x) = x + u_1 y$.

$\square$

Lemma A.3 Let R be a commutative ring. Consider the following commutative diagram of R-modules:

$$\begin{array}{ccccccccc} & & A & \xrightarrow{f_1} & B & \xrightarrow{f_2} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \\ 0 & \longrightarrow & X & \xrightarrow[g_1]{} & Y & \xrightarrow[g_2]{} & Z & \longrightarrow & 0 \end{array}$$

If h_1 is an isomorphism and h_3 is surjective then B is a *pull-back*.

Proof:

It follows from the *universal propety of pull-back* that it is enough to show for any R-module Q such that the following diagram commutes

$$\begin{array}{ccc} Q & \xrightarrow{k_1} & C \\ \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle h_2} \\ Y & \xrightarrow[g_2]{} & Z \end{array}$$

there exists a unique $\mu : Q \longrightarrow B$ such that

$$f_2 \circ \mu = k_1 \text{ and } h_2 \circ \mu = k_2. \tag{5.1}$$

Let $q_1 \in Q$ such that

$$(h_3 \circ k_1)(q_1) = (g_2 \circ k_2)(q_1). \tag{5.2}$$

In particular we have that $k_1(q_1) \in C$ implies by exactness that

$$k_1(q_1) = f_2(b_1 + f_1(a_i))$$

with $b_1 \in B$ and $a_i \in A$. Therefore,

$$(h_3 \circ k_1)(q_1) = h_3(f_2(b_1 + f_1(a_i))).$$

However $h_3 \circ f_2 = g_2 \circ h_2$ by commutativity of the diagram implies

$$h_3(f_2(b_1 + f_1(a_i))) = (g_2 \circ h_2)(b_1 + f_1(a_i)))$$

and combining this with (5.2) implies

$$(g_2 \circ h_2)(b_1 + f_1(a_i))) = (g_2 \circ k_2)(q_1)$$

which is equivalent to

$$g_2(h_2(b_1 + f_1(a_i)) - k_2(q_1)) = 0.$$

It follows by exactness that there exists unique $x_i \in X$ such that

$$g_1(x_i) = h_2(b_1 + f_1(a_i)) - k_2(q_1) \tag{5.3}$$

and h_1 is an isomorphism implies there exists unique $a_i^* \in A$ such that

$$h_1(a_i^*) = x_i$$

which by (5.3) gives the following equation:

$$(g_1 \circ h_1)(a_i^*) = h_2(b_1 + f_1(a_i)) - k_2(q_1).$$

The commutativity of the diagram implies $(g_1 \circ h_1)(a_i^*) = (h_2 \circ f_1)(a_i^*)$ so we can conclude

$$(h_2 \circ f_1)(a_i^*) = h_2(b_1 + f_1(a_i)) - k_2(q_1)$$

which is equivalent to

$$k_2(q_1) = h_2(b_1 + f_1(a_i - a_i^*)).$$

Then by (5.1) we let

$$\mu(q_1) = b_1 + f_1(a_i - a_i^*)$$

and all that remains to show is the uniqueness of μ. Assume there exists $\mu_2 : Q \longrightarrow B$ such that

$$f_2 \circ \mu_2 = k_1 \text{ and } h_2 \circ \mu_2 = k_2$$

then by the above,

$$\mu_2(q_1) = b_1 + f_1(a_j - a_j^*)$$

and $f_1(a_i - a_i^*) = f_1(a_j - a_j^*)$ since

$$h_2(b_1 + f_1(a_i - a_i^*)) = h_2(b_1 + f_1(a_j - a_j^*))$$

implies

$$h_2(b_1) + (h_2 \circ f_1)(a_i - a_i^*)) = h_2(b_1) + (h_2 \circ f_1)(a_j - a_j^*))$$

which by $(h_2 \circ f_1) = (g_1 \circ h_1)$ and injectivity of g_1, h_1 implies

$$a_i - a_i^* = a_j - a_j^*.$$

□

CHAPTER 6

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