University of Alberta

ESTIMATES OF THE MAXIMAL CESÁRO OPERATORS OF THE WEIGHTED ORTHOGONAL POLYNOMIAL EXPANSIONS IN SEVERAL VARIABLES

by

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Abstract

For the weight function $\prod_{j=1}^{d}|x_j|^{2\kappa_j}$ on the unit sphere $\mathbb{S}^{d-1}$, estimates of the maximal Cesàro operator of the weighted orthogonal polynomial expansions at the critical index are proved, which allow us to improve several known results in this area, including the critical index for the almost everywhere convergence of the Cesàro means, the sufficient conditions in the Marcinkiewitcz multiplier theorem, and a Fefferman-Stein type inequality for the Cesàro operators. These results on the unit sphere also enable us to establish similar results on the unit ball and on the simplex.

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Chapter 1

Introduction

Spherical Harmonic Analysis is an important branch of Harmonic analysis, a mainstream branch of mathematics that has a history of more than 200 years. It is important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, representation of gravitational fields, geoids, and the magnetic fields of planetary bodies and stars, and characterization of the cosmic microwave background radiation. In 3D computer graphics, spherical harmonics play a special role in a wide variety of topics including indirect lighting (ambient occlusion, global illumination, precomputed radiance transfer, etc.) and recognition of 3D shapes.

The purpose of this thesis is to study the pointwise convergence of Cesáro means of weighted spherical polynomial expansions on the unit sphere. For a class of product weights that are invariant under the group $\mathbb{Z}_2^d$ on the sphere, estimates of the maximal Cesàro operator of the weighted orthogonal polynomial expansions at the critical index are proved, which allow us to improve several known results in this area, including the critical index for the almost everywhere convergence of the Cesàro means, the sufficient conditions in the

Marcinkiewitcz multiplier theorem, and a Fefferman-Stein type inequality for the Cesàro operators. These results on the unit sphere also enable us to establish similar results on the unit ball and on the simplex.

This is a joint work with my supervisor Dr. Feng Dai and Dr. Sheng Wang (Guilin University of Electronic Technology, China). And our paper is now published in [1].

Below, we shall describe the main results and their background in more details with a " minimum" of definitions and notation. Necessary details and appropriate definitions will be given in the next chapter.

Let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\|= 1\}$ denote the unit sphere of $\mathbb{R}^d$ equipped with the usual rotation-invariant measure $d\sigma$, where $\|x\|$ denotes the Euclidean norm. Let

$$h_\kappa(x) := \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d, \tag{1.1}$$

where $\kappa := (\kappa_1, \cdots, \kappa_d) \in \mathbb{R}^d$ and $\kappa_{\min} := \min_{1\leqslant j\leqslant d} \kappa_j \geqslant 0$. Throughout the thesis, all functions and sets will be assumed to be Lebesgue measurable.

We denote by $L^p(h_\kappa^2; \mathbb{S}^{d-1})$, $1\leqslant p\leqslant\infty$, the L^p-space of functions defined on $\mathbb{S}^{d-1}$ with respect to the measure $h_\kappa^2(x)\, d\sigma(x)$. More precisely, $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ is the space of functions on $\mathbb{S}^{d-1}$ with finite norm

$$\|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y) \right)^{\frac{1}{p}}, \quad 1\leqslant p < \infty.$$

For $p = \infty$, $L^\infty(h_\kappa^2)$ is replaced by $C(\mathbb{S}^{d-1})$, the space of continuous functions on $\mathbb{S}^{d-1}$ with the usual uniform norm.

A spherical polynomial of degree at most n on $\mathbb{S}^{d-1}$ is the restriction to $\mathbb{S}^{d-1}$ of an algebraic polynomial in d variables of total degree n. We denote by Π_n^d the space of all spherical polynomials of degree at most n on $\mathbb{S}^{d-1}$. We denote by $\mathcal{H}_n^d(h_\kappa^2)$ the orthogonal complement of Π_{n-1}^d in Π_n^d, where it is agreed that $\Pi_{-1}^d = \{0\}$. Each element in $\mathcal{H}_n^d(h_\kappa^2)$ is then called a spherical h-harmonic polynomial of degree n on $\mathbb{S}^{d-1}$. In the case of $h_\kappa = 1$, a spherical h-harmonic is simply the ordinary spherical harmonic.

The theory of h-harmonics is developed by Dunkl (see [8, 9, 10]) for a family of weight functions invariant under a finite reflection group, of which h_κ in (1.1) is the example of the group $\mathbb{Z}_2^d$. Properties of h-harmonics are quite similar to those of ordinary spherical harmonics. For example, each $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$ has an orthogonal expansion in h-harmonics, $f = \sum_{n=0}^\infty \operatorname{proj}_n(h_\kappa^2; f)$, converging in the norm of $L^2(h_\kappa^2; \mathbb{S}^{d-1})$, where $\operatorname{proj}_n(h_\kappa^2; f)$ denotes the orthogonal projection of f onto $\mathcal{H}_n^d(h_\kappa^2)$, which can be extended to all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$.

For $\delta > -1$, the *Cesàro* (C, δ) *means* of the h-harmonic expansions are defined by

$$S_n^\delta(h_\kappa^2; f) := \sum_{j=0}^n \frac{A_{n-j}^\delta}{A_n^\delta} \operatorname{proj}_j(h_\kappa^2; f), \ \ A_{n-j}^\delta = \binom{n-j+\delta}{n-j}, \ \ n = 0, 1, \cdots,$$

whereas the *maximal Cesàro operator* of order δ is defined by

$$S_*^\delta(h_\kappa^2; f)(x) := \sup_{n \in \mathbb{N}} |S_n^\delta(h_\kappa^2; f)(x)|, \quad x \in \mathbb{S}^{d-1}.$$

One of our main goals in this thesis is to study the following weak type estimate of the maximal Cesàro operator: for $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\operatorname{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad S_*^\delta(h_\kappa^2; f)(x) > \alpha \Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0, \tag{1.2}$$

here, and in what follows, we write $\operatorname{meas}_\kappa(E) := \int_E h_\kappa^2(x) \, d\sigma(x)$ for a measurable subset $E \subset \mathbb{S}^{d-1}$. Such estimates have been playing crucial roles in spherical harmonic analysis on the sphere; for example, they can be used to establish a Marcinkiewicz type multiplier theorem for the spherical h-harmonic expansions (see [4, 6]).

The background for this problem is as follows. In the case of ordinary spherical harmonics (i.e., the case of $\kappa = 0$), it is known that (1.2) holds if and only if $\delta > \frac{d-2}{2}$. (See [4, 21]) . Indeed, in this case, since the Cesàro operators are rotation-invariant, a well-known result of Stein [18] implies that for $h_\kappa(x) \equiv 1$, (1.2) holds if and only if

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f)(x) = f(x), \quad \text{a.e. } \ x \in \mathbb{S}^{d-1}, \quad \forall f \in L^1(h_\kappa^2; \mathbb{S}^{d-1}). \tag{1.3}$$

In the case of $\kappa \neq 0$ (i.e., the weighted case), while a standard density argument shows that (1.2) implies (1.3), the result of Stein [18] is not applicable to deduce the equivalence of (1.2) and (1.3), since the measure $h_\kappa^2 d\sigma$ is no longer rotation-invariant. In fact, an estimate much weaker than (1.2) was proved and used to study (1.3) for $\delta > \lambda_\kappa := \frac{d-2}{2} + \sum_{j=1}^d \kappa_j$ in [25], whereas (1.2) itself was later proved in [6] for $\delta > \lambda_\kappa$, where the results are also applicable to the case of more general weights invariant under a reflection group. Finally, for h_κ in (1.1), it was shown in [32] that (1.3) fails for $\delta < \sigma_\kappa$ with

$$\sigma_\kappa := \lambda_\kappa - \kappa_{\min} = \frac{d-2}{2} + \sum_{j=1}^d \kappa_j - \min_{1\leqslant j\leqslant d} \kappa_j. \tag{1.4}$$

Of related interest is the fact that σ_κ is the critical index for the summablity of the Cesàro means in the space $L^1(h_\kappa^2;\mathbb{S}^{d-1})$. More precisely,

$$\lim_{N\to\infty}\|S_N^\delta(h_\kappa^2;f)-f\|_{\kappa,1}=0,\quad \forall f\in L^1(h_\kappa^2;\mathbb{S}^{d-1}) \tag{1.5}$$

if and only if $\delta>\sigma_\kappa$. (See [14, 7]).

In this thesis, we prove that if $\kappa\neq 0$, then (1.3) holds if and only if $\delta\geqslant\sigma_\kappa$, and moreover, if at most one of the κ_i is zero, then the weak estimate (1.2) holds if and only if $\delta\geqslant\sigma_\kappa$. Of special interest is the case of $\delta=\sigma_\kappa$, where our results are a little bit surprising in view of the facts that (1.5) fails at the critical index $\delta=\sigma_\kappa$, and the corresponding results in the case of $\kappa=0$ (i.e., the case of ordinary spherical harmonics) are known to be false at the critical index $\sigma_0:=\frac{d-2}{2}$.

Our results on the estimates of the maximal Cesàro operators also allow us to establish a Fefferman-Stein type inequality for the Cesàro operators and to weaken the conditions in the Marcinkiewitcz multiplier theorem that was established previously in [6]. The precise statements of our results on the sphere can be found in Theorem 3.1.1, and Corollaries 3.7.1-3.7.6 in the third chapter.

We will also establish similar results for the weighted orthogonal polynomial expansions with respect to the weight function

$$W_\kappa^B(x):=\Big(\prod_{j=1}^d|x_j|^{\kappa_j}\Big)(1-\|x\|^2)^{\kappa_{d+1}-1/2},\qquad \min_{1\leqslant i\leqslant d+1}\kappa_i\geqslant 0 \tag{1.6}$$

on the unit ball $\mathbb{B}^d=\{x\in\mathbb{R}^d:\|x\|\leqslant 1\}$, as well as for the weighted orthogonal polynomial expansions with respect to the weight function

$$W_\kappa^T(x) := \Big(\prod_{i=1}^d x_i^{\kappa_i - 1/2}\Big)(1-|x|)^{\kappa_{d+1}-1/2}, \qquad \min_{1\leqslant i\leqslant d+1} \kappa_i \geqslant 0. \tag{1.7}$$

on the simplex $\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geqslant 0, \dots, x_d \geqslant 0, 1-|x| \geqslant 0\}$, here, and in what follows, $|x| := \sum_{j=1}^d |x_j|$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The precise statements of our results on $\mathbb{B}^d$ and $\mathbb{T}^d$ can be found in Theorem 4.1.1, Corollaries 4.2.1-4.2.4, Theorem 5.1.2, and Corollaries 5.2.1-5.2.4 in the fourth and the fifth sections.

It turns out that results on the unit ball $\mathbb{B}^d$ are normally easier to be deduced directly from the corresponding results on the unit sphere $\mathbb{S}^d$, whereas in most cases, results on the simplex are not able to be deduced directly from those on the ball and on the sphere due to the differences in their orthogonal structures. (See, for instance, [6, 7, 24, 26]). In the fifth chapter of this thesis, we will develop a new technique which allows one to deduce results on the Cesàro means on the simplex directly from the corresponding results on the unit ball.

We organize this thesis as follows. Chapter two contains some preliminary materials on weighted orthogonal polynomial expansions on the unit sphere, the unit ball and the simplex. Our main results on the unit sphere are stated and proved in the third chapter. After that, in the fourth chapter, similar results are established on the unit ball. These results are deduced directly from the corresponding results on the unit sphere. Finally, in the fifth chapter we discuss how to deduce similar results on the simplex from the corresponding results on the unit ball. A new technique is developed.

Chapter 2

Preliminaries

In this chapter, we will describe some necessary materials for weighted orthogonal polynomial expansions on the sphere, the ball and the simplex. Unless otherwise stated, the main reference for the materials in this section is the book [10].

2.1 Notations

In this section, we shall introduce some necessary notations that will used frequently in the rest of the thesis. We use the notation $C_1 \sim C_2$ to mean that there exists a positive universal constant C, called the constant of equivalence, such that $C^{-1}C_1 \leqslant C_2 \leqslant CC_1$. And we note $C_1 \lesssim C_2 (C_1 \gtrsim C_2)$ if there exists a positive universal constant C such that $C_1 \leqslant CC_2 (C_1 \geqslant CC_2)$.

Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and for $x \in \mathbb{R}^d$, we write $x = (x_1, x_2, \cdots, x_d)$. The norm of x is defined by $\|x\| := \sqrt{\sum_{j=1}^d x_j^2}$. The unit sphere $\mathbb{S}^{d-1}$ and the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$ are defined by

$$\mathbb{S}^{d-1} := \{x : \|x\| = 1\}, \text{ and } \mathbb{B}^d := \{x : \|x\| \leqslant 1\}.$$

Given $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_d) \in \mathbb{Z}_2^d := \{\pm 1\}^d$, we write $\bar{x} := (|x_1|, \cdots, |x_d|)$, $|x| := \sum_{j=1}^d |x_j|$, and $x\varepsilon := (x_1\varepsilon_1, \cdots, x_d\varepsilon_d)$. We denote by $\rho(x, y)$ the geodesic distance, $\arccos x \cdot y$, of $x, y \in \mathbb{S}^{d-1}$.

The simplex $\mathbb{T}^d$ of $\mathbb{R}^d$ is defined by

$$\mathbb{T}^d = \{x \in \mathbb{R}^d : x_1 \geqslant 0, \ldots, x_d \geqslant 0, 1 - |x| \geqslant 0\}$$

2.2 Orthogonal polynomial expansions in several variables

Let Ω denote a compact domain in $\mathbb{R}^d$ endowed with the usual Lebesgue measure dx, where in the case of $\Omega = \mathbb{S}^{d-1}$, we use $d\sigma(x)$ instead of dx to denote the Lebesgue measure. Given a nonnegative product weight function W on Ω, we denote by $L^p(W; \Omega)$ the usual L^p-space defined with respect to the measure $W dx$ on Ω. For each function $f \in L^p(W; \Omega)$, we define its $\|\cdot\|_{p,W}$ norm as following

$$\|f\|_{p,W} := \left(\int_\Omega |f(x)|^p W(x) dx \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty,$$

and for $p = \infty$, we consider the space of continuous functions with the uniform norm

$$\|f\|_{\infty,W} := \sup_{x \in \Omega} |f(x)|.$$

We denote $\mathcal{V}_n^d(W)$ the space of orthogonal polynomials of degree n with respect to the weight function W on Ω. Thus, if we denote by $\Pi_n(\Omega)$ the space of all algebraic polynomials in d variables of degree at most n restricted on the domain Ω, then $\mathcal{V}_n^d(W)$ is the orthogonal complement of $\Pi_{n-1}(\Omega)$ in the space $\Pi_n(\Omega)$ with respect to the inner product of $L^2(W;\Omega)$, where it is agreed that $\Pi_{-1}(\Omega) = \{0\}$.

Since Ω is compact, each function $f \in L^2(W;\Omega)$ has a weighted orthogonal polynomial expansion on Ω, $f = \sum_{n=0}^{\infty} \operatorname{proj}_n(W;f)$, converging in the norm of $L^2(W;\Omega)$, where $\operatorname{proj}_n(W;f)$ denotes the orthogonal projection of f onto the space $\mathcal{V}_n^d(W)$. Let $P_n(W;\cdot,\cdot)$ denote the reproducing kernel of the space $\mathcal{V}_n^d(W)$; that is,

$$P_n(W;x,y) := \sum_{j=1}^{a_n^d} \varphi_{n,j}(x)\overline{\varphi_{n,j}(y)}, \quad x, y \in \Omega$$

for an orthonormal basis $\{\varphi_{n,j} : \ 1 \leqslant j \leqslant a_n^d := \dim \mathcal{V}_n^d(W)\}$ of the space $\mathcal{V}_n^d(W)$.

The projection operator $\operatorname{proj}_n(W) : L^2(W;\Omega) \mapsto \mathcal{V}_n^d(W)$ can be expressed as an integral operator

$$\operatorname{proj}_n(W;f,x) = \int_\Omega f(y) P_n(W;x,y) W(y) dy, \quad x \in \Omega, \tag{2.1}$$

which also extends the definition of $\operatorname{proj}_n(W;f)$ to all $f \in L(W;\Omega)$ since the kernel $P_n(W;x,y)$ is a polynomial in both x and y.

Let $S_n^\delta(W;f)$, $n = 0, 1, \cdots$, denote the Cesàro (C,δ) means of the weighted orthogonal polynomial expansions of $f \in L(W;\Omega)$. Each $S_n^\delta(W;f)$ can be expressed as an integral against a kernel, $K_n^\delta(W;x,y)$, called the Cesàro (C,δ)

kernel,

$$S_n^\delta(W;f,x) := \int_\Omega f(y) K_n^\delta(W;x,y) W(y) dy, \quad x \in \Omega,$$

where

$$K_n^\delta(W;x,y) := (A_n^\delta)^{-1} \sum_{j=0}^{n} A_{n-j}^\delta P_j(W;x,y), \quad x,y \in \Omega.$$

Finally, given a sequence of operators T_n, $n = 0, 1, \cdots$ on some L^p space, we denote by T_* the corresponding maximal operator defined by $T_* f(x) = \sup_n |T_n f(x)|$.

2.3 h-harmonic expansions

We restrict our discussion to h_κ in (1.1), and denote the L^p norm of $L^p(h_\kappa^2; \mathbb{S}^{d-1})$ by $\|\cdot\|_{\kappa,p}$,

$$\|f\|_{\kappa,p} := \left(\int_{\mathbb{S}^{d-1}} |f(y)|^p h_\kappa^2(y) d\sigma(y) \right)^{1/p}, \quad 1 \leqslant p < \infty$$

with the usual change when $p = \infty$. An h-harmonic on $\mathbb{R}^d$ is a homogeneous polynomial P in d variables that satisfies the equation $\Delta_h P = 0$, where $\Delta_h := \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$, and

$$\mathcal{D}_i f(x) := \partial_i + \kappa_i \frac{f(x) - f(x - 2x_i e_i)}{x_i}, \quad 1 \leqslant i \leqslant d,$$

with $e_1 = (1, 0, \cdots, 0)$, $\cdots$, $e_d = (0, \cdots, 0, 1) \in \mathbb{R}^d$. The differential-difference operators $\mathcal{D}_1, \ldots, \mathcal{D}_d$ are the Dunkl operators, which are mutually commuting. The restriction of an h-harmonic on the sphere is called a spherical h-harmonic. A spherical h-harmonic is an orthogonal polynomial with respect to the weight function $h_\kappa^2(x)$ on $\mathbb{S}^{d-1}$, and we denote by $\mathcal{H}_n^d(h_\kappa^2)$ the space of spherical h-harmonics of degree n on $\mathbb{S}^{d-1}$. Thus, using the notation of the last subsection, we have $\mathcal{H}_n^d(h_\kappa^2) \equiv \mathcal{V}_n^d(h_\kappa^2)$.

A fundamental result in the study of h-harmonic expansions is the following compact expression of the reproducing kernel (see [9, 24, 29]):

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n+\lambda_\kappa}{\lambda_\kappa} \int_{[-1,1]^d} C_n^{\lambda_\kappa}(\sum_{j=1}^d x_i y_j t_j) \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \quad (2.2)$$

where C_n^λ is the Gegenbauer polynomial of degree n, and c_κ is a normalization constant depending only on κ and d. Here, and in what follows, if some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 f(t)(1-t)^{\lambda-1} dt = \frac{f(1)+f(-1)}{2}.$$

The following pointwise estimates on the Cesàro (C, δ) kernels were proved in [7].

Theorem 2.3.1: *Let $x = (x_1, \cdots, x_d) \in \mathbb{S}^{d-1}$ and $y = (y_1, \cdots, y_d) \in \mathbb{S}^{d-1}$. Then for $\delta > -1$,*

$$|K_n^\delta(h_\kappa^2; x, y)| \leqslant c n^{d-1} \Bigg[\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(\bar{x}, \bar{y}) + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x}, \bar{y})+1)^{\delta+d/2}} + \frac{\prod_{j=1}^d (|x_j y_j| + \rho(\bar{x}, \bar{y})^2 + n^{-2})^{-\kappa_j}}{(n\rho(\bar{x}, \bar{y})+1)^{d}} \Bigg].$$

2.4 Orthogonal expansions on the unit ball

The weight function W_κ^B we consider on the unit ball $\mathbb{B}^d$ is given in (1.6) with $\kappa := (\kappa_1, \cdots, \kappa_{d+1}) \in \mathbb{R}_+^d$. It is related to the h_κ on the sphere $\mathbb{S}^d$ by

$$h_\kappa^2(x, \sqrt{1-\|x\|^2}) = W_\kappa^B(x)\sqrt{1-\|x\|^2}, \quad x \in \mathbb{B}^d, \quad (2.3)$$

in which h_κ is defined in (1.1) with $\mathbb{S}^d$ in place of $\mathbb{S}^{d-1}$. Furthermore, under the change of variables $y = \phi(x)$ with

$$\phi : x \in \mathbb{B}^d \mapsto (x, \sqrt{1-\|x\|^2}) \in \mathbb{S}^d_+ := \{y \in \mathbb{S}^d : \; y_{d+1} \geqslant 0\}, \tag{2.4}$$

we have

$$\int_{\mathbb{S}^d} g(y) d\sigma(y) = \int_{\mathbb{B}^d} \left[g(x, \sqrt{1-\|x\|^2}\,) + g(x, -\sqrt{1-\|x\|^2}\,) \right] \frac{dx}{\sqrt{1-\|x\|^2}}. \tag{2.5}$$

The orthogonal structure is preserved under the mapping (2.4) and the study of orthogonal expansions for W_κ^B on $\mathbb{B}^d$ can be essentially reduced to that of h_κ^2 on $\mathbb{S}^d$. More precisely, we have

$$\begin{aligned} P_n(W_\kappa^B; x, y) = \frac{1}{2} \big[& P_n(h_\kappa^2; (x, x_{d+1}), (y, y_{d+1})) \\ & + P_n(h_\kappa^2; (x, x_{d+1}), (y, -y_{d+1})) \big] \end{aligned} \tag{2.6}$$

where $x, y \in \mathbb{B}^d$, and $x_{d+1} = \sqrt{1-\|x\|^2}$, $y_{d+1} = \sqrt{1-\|y\|^2}$. As a consequence, the orthogonal projection, $\operatorname{proj}_n(W_\kappa^B; f)$, of $f \in L^2(W_\kappa^B; \mathbb{B}^d)$ onto $\mathcal{V}_n^d(W_\kappa^B)$ can be expressed in terms of the orthogonal projection of $F(x, x_{d+1}) := f(x)$ onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$:

$$\operatorname{proj}_n(W_\kappa^B; f, x) = \operatorname{proj}_n(h_\kappa^2; F, X), \qquad \text{with} \;\; X := (x, \sqrt{1-\|x\|^2}). \tag{2.7}$$

This relation allows us to deduce results on the convergence of orthogonal expansions with respect to W_κ^B on $\mathbb{B}^d$ from that of h-harmonic expansions on $\mathbb{S}^d$.

For $d = 1$ the weight W_κ^B in (1.6) becomes the weight function

$$w_{\kappa_2,\kappa_1}(t) = |t|^{2\kappa_1}(1-t^2)^{\kappa_2-1/2}, \qquad \kappa_i \geqslant 0, \quad t \in [-1,1], \tag{2.8}$$

whose corresponding orthogonal polynomials, $C_n^{(\kappa_2,\kappa_1)}$, are called generalized Gegenbauer polynomials, and can be expressed in terms of Jacobi polynomials,

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(t) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2-1), \end{aligned} \tag{2.9}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$, and $P_n^{(\alpha,\beta)}$ denotes the usual Jacobi polynomial of degree n and index (α,β) defined as in [23].

2.5 Orthogonal expansions on the simplex

The weight functions we consider on the simplex $\mathbb{T}^d$ are defined by (1.7), which are related to W_κ^B, hence to h_κ^2. In fact, W_κ^T is exactly the product of the weight function W_κ^B under the mapping

$$\psi : (x_1,\ldots,x_d) \in \mathbb{B}^d \mapsto (x_1^2,\ldots,x_d^2) \in \mathbb{T}^d \tag{2.10}$$

and the Jacobian of this change of variables. Furthermore, the change of variables shows

$$\int_{\mathbb{B}^d} g(x_1^2,\ldots,x_d^2)dx = \int_{\mathbb{T}^d} g(x_1,\ldots,x_d)\frac{dx}{\sqrt{x_1\cdots x_d}}. \tag{2.11}$$

The orthogonal structure is preserved under the mapping (2.10). In fact,

$R \in \mathcal{V}_n^d(W_\kappa^T)$ if and only if $R \circ \psi \in \mathcal{V}_{2n}^d(W_\kappa^B)$. The orthogonal projection, $\mathrm{proj}_n(W_\kappa^T; f)$, of $f \in L^2(W_\kappa^T; \mathbb{T}^d)$ onto $\mathcal{V}_n^d(W_\kappa^T)$ can be expressed in terms of the orthogonal projection of $f \circ \psi$ onto $\mathcal{V}_{2n}^d(W_\kappa^B)$:

$$\mathrm{proj}_n(W_\kappa^T; f, \psi(x)) = \frac{1}{2^d} \sum_{\varepsilon \in \mathbb{Z}_2^d} \mathrm{proj}_{2n}(W_\kappa^B; f \circ \psi, x\varepsilon), \quad x \in \mathbb{B}^d. \tag{2.12}$$

The fact that $\mathrm{proj}_n(W_\kappa^T)$ of degree n is related to $\mathrm{proj}_{2n}(W_\kappa^B)$ of degree $2n$ suggests that some properties of the orthogonal expansions on $\mathbb{B}^d$ cannot be transformed directly to those on $\mathbb{T}^d$.

Chapter 3

Weak estimates of the maximal Cesàro operators on the sphere

3.1 Main result

Recall that the letter κ denotes a nonzero vector $\kappa := (\kappa_1, \cdots, \kappa_d)$ in

$$\mathbb{R}_+^d := \Big\{(x_1, \cdots, x_d) \in \mathbb{R}^d : \quad x_i \geqslant 0, \quad i = 1, 2, \cdots, d\Big\},$$

and

$$\kappa_{\min} := \min_{1\leqslant j\leqslant d} \kappa_j, \quad |\kappa| = \sum_{j=1}^{d} \kappa_j, \quad \sigma_\kappa := \frac{d-2}{2} + |\kappa| - \kappa_{\min}. \tag{3.1}$$

We will keep these notations throughout this chapter. Some of our results and estimates below are not true if $\kappa = 0$.

Our main result on the unit sphere can be stated as follows:

Theorem 3.1.1: (i) *If* $\delta \geqslant \sigma_\kappa$*, then for* $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ *with* $\|f\|_{\kappa,1} = 1$*,*

$$\text{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad S_*^\delta(h_\kappa^2; f)(x) > \alpha \Big\} \leqslant C \frac{1}{\alpha}, \quad \forall \alpha > 0$$

with $\alpha^{-1}|\log \alpha|$ *in place of* α^{-1} *in the case when* $\delta = \sigma_\kappa$ *and at least two of the* κ_i *are zero.*

(ii) *If* $\delta < \sigma_\kappa$*, then there exists a function* $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ *of the form* $f(x) = f_0(|x_{j_0}|)$ *such that* $S_*^\delta(h_\kappa^2; f)(x) = \infty$ *for a.e.* $x \in \mathbb{S}^{d-1}$*, where* $1 \leqslant j_0 \leqslant d$ *and* $\kappa_{j_0} = \kappa_{\min}$*.*

3.2 Proof of Theorem 3.1.1: Part (i)

Let us first introduce several necessary notations for the proofs in the next few subsections. Recall that $\rho(x, y)$ denotes the geodesic distance $\arccos x \cdot y$ between two points $x, y \in \mathbb{S}^{d-1}$. We denote by $B(x, \theta)$ the spherical cap $\{y \in \mathbb{S}^{d-1} : \ \rho(x, y) \leqslant \theta\}$ centered at $x \in \mathbb{S}^{d-1}$ of radius $\theta \in (0, \pi]$. It is known that for any $x \in \mathbb{S}^{d-1}$ and $\theta \in (0, \pi)$

$$V_\theta(x) := \text{meas}_\kappa(B(x, \theta)) = \int_{B(x,\theta)} h_\kappa^2(y) d\sigma(y) \sim \theta^{d-1} \prod_{j=1}^{d} (x_j + \theta)^{2\kappa_j}, \tag{3.2}$$

which, in particular, implies that h_κ^2 is a doubling weight on $\mathbb{S}^{d-1}$ (see [5, 5.3]). And we denote that:

$$V(x, y) := \text{meas}_\kappa(B(x, \rho(x, y))).$$

For $f\in L^1(h_\kappa^2;\mathbb{S}^{d-1})$, we define

$$M_\kappa f(x):=\sup_{0<\theta\leqslant\pi}\frac{1}{\mathrm{meas}_\kappa(B(x,\theta))}\int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\leqslant\theta\}}|f(y)|h_\kappa^2(y)\,d\sigma(y).$$

Since the weight h_κ^2 satisfies the doubling condition and is invariant under the group $\mathbb{Z}_2^d$, the usual properties of the Hardy-Littlewood maximal functions imply that for $f\in L^1(h_\kappa^2;\mathbb{S}^{d-1})$,

$$\mathrm{meas}_\kappa\{x\in\mathbb{S}^{d-1}:\ M_\kappa f(x)>\alpha\}\leqslant C\frac{\|f\|_{\kappa,1}}{\alpha},\quad\forall\alpha>0. \tag{3.3}$$

For the proof of the first assertion in Theorem 3.1.1, we use Theorem 2.3.1 to obtain

$$|K_n^\delta(h_\kappa^2;x,y)|\leqslant CE_n^\delta(h_\kappa^2;x,y)+CR_n(h_\kappa^2;x,y), \tag{3.4}$$

where

$$E_n^\delta(h_\kappa^2;x,y):=n^{d-1}\frac{\prod_{j=1}^d(|x_jy_j|+n^{-1}\rho(\bar{x},\bar{y})+n^{-2})^{-\kappa_j}}{(n\rho(\bar{x},\bar{y})+1)^{\delta+d/2}}, \tag{3.5}$$

$$R_n(h_\kappa^2;x,y):=n^{d-1}\frac{\prod_{j=1}^d(|x_jy_j|+\rho(\bar{x},\bar{y})^2+n^{-2})^{-\kappa_j}}{(n\rho(\bar{x},\bar{y})+1)^{d}}. \tag{3.6}$$

Thus,

$$|S_n^\delta(h_\kappa^2;f,x)|\leqslant C|E_n^\delta(h_\kappa^2;f,x)|+C|T_n^\delta(h_\kappa^2;f,x)|+C|R_n(h_\kappa^2;f,x)|,$$

where

$$E_n^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}^{d-1}:\rho(\bar{x},\bar{y})\leqslant\frac{1}{2\sqrt{d}}\}} E_n^\delta(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y), \tag{3.7}$$

$$T_n^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}^{d-1}:\quad \rho(\bar{x},\bar{y})\geqslant\frac{1}{2\sqrt{d}}\}} E_n^\delta(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y), \tag{3.8}$$

$$R_n(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} R_n(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y). \tag{3.9}$$

This implies that

$$\begin{aligned}\text{meas}_\kappa\{x\in\mathbb{S}^{d-1}: S_*^\delta(h_\kappa^2; f, x) > \alpha\} \leqslant\, & \text{meas}_\kappa\{x\in\mathbb{S}^{d-1}: E_*^\delta(h_\kappa^2; f, x) > \frac{\alpha}{3C}\} \\ & + \text{meas}_\kappa\{x\in\mathbb{S}^{d-1}: T_*^\delta(h_\kappa^2; f, x) > \frac{\alpha}{3C}\} \\ & + \text{meas}_\kappa\{x\in\mathbb{S}^{d-1}: R_*(h_\kappa^2; f, x) > \frac{\alpha}{3C}\},\end{aligned}$$

where

$$\begin{aligned}& E_*^\delta(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |E_n^\delta(h_\kappa^2; f, x)|, \quad T_*^\delta(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |T_n^\delta(h_\kappa^2; f, x)| \\ & R_*(h_\kappa^2; f, x) := \sup_{n\in\mathbb{N}} |R_n(h_\kappa^2; f, x)|.\end{aligned}$$

Thus, for the proof of the stated weak estimates of $S_*^\delta(h_\kappa^2; f, x)$ in Theorem 3.1.1, it will suffice to establish the corresponding weak estimates for the maximal operators E_*^δ, T_*^δ and R_*. Namely, it suffices to prove the following three propositions:

Proposition 3.2.1: *For $\delta\geqslant\sigma_\kappa$ and each $f\in L^1(h_\kappa^2;\mathbb{S}^{d-1})$, we have that*

$$R_*(h_\kappa^2; f, x) \leqslant C M_\kappa f(x), \quad x\in\mathbb{S}^{d-1}, \tag{3.10}$$

and

$$\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \ R_*(h_\kappa^2; f, x) > \alpha\}) \leqslant C \frac{\|f\|_{1,\kappa}}{\alpha}, \quad \forall \alpha > 0. \tag{3.11}$$

Proposition 3.2.2: *For $\delta \geqslant \sigma_\kappa$ and $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$,*

$$\text{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : T_*^\delta(h_\kappa^2; f, x) > \alpha\}) \leqslant C \frac{\|f\|_{1,\kappa}}{\alpha}, \quad \forall \alpha > 0.$$

Proposition 3.2.3: *If either $\delta > \sigma_\kappa$ or $\delta = \sigma_\kappa$ and at most one of the κ_i is zero, then*

$$\text{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad E_*^\delta(h_\kappa^2; f, x) > \alpha \Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0. \tag{3.12}$$

Furthermore, if $\delta = \sigma_\kappa$ and at least two of the κ_i are zero, then

$$\text{meas}_\kappa \Big\{ x \in \mathbb{S}^{d-1} : \quad E_*^\delta(h_\kappa^2; f)(x) > \alpha \Big\} \leqslant C \frac{\|f\|_{\kappa,1}}{\alpha} \log \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

The proofs of these three propositions will be given in Sections 3.3, 3.4, 3.5 respectively.

3.3 Proof of Proposition 3.2.1

For the proof of Proposition 3.2.1, we need the following two simple lemmas.

Lemma 3.3.1: *For $x, y \in \mathbb{S}^{d-1}$,*

$$R_n(h_\kappa^2, x, y) \sim \frac{1}{1 + n\rho(\bar{x}, \bar{y})} \cdot \frac{1}{V(\bar{x}, \bar{y}) + V_{n^{-1}}(\bar{x})} \tag{3.13}$$

Proof. By (3.6), it is sufficient to show that for each $1\leqslant j\leqslant d$,

$$J_j(x,y) := (|x_jy_j|+\rho(\bar{x},\bar{y})^2+n^{-2})^{-\kappa_j} \sim (|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}. \qquad (3.14)$$

In fact, let't consider the following two cases:

Case 1. If $|x_j|\geqslant 2\rho(\bar{x},\bar{y})$, since $|x_j|\geqslant 2\rho(\bar{x},\bar{y})\geqslant 2||x_j|-|y_j||$, we have that

$$|x_j|\sim |y_j|,$$

thus

$$J_j(x,y)\sim (|x_j|^2+n^{-2}+\rho(\bar{x},\bar{y})^2)^{-\kappa_j}\sim (|x_j|+n^{-1}+\rho(\bar{x},\bar{y}))^{-2\kappa_j}.$$

Case 2. If $|x_j|\leqslant 2\rho(\bar{x},\bar{y})$, then since $|y_j|-|x_j|\leqslant\rho(\bar{x},\bar{y})$,

$$|y_j|\leqslant\rho(\bar{x},\bar{y})+|x_j|< 3\rho(\bar{x},\bar{y}),$$

thus

$$J_j(x,y)\sim (\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}\sim (|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}.$$

Hence, in either case, we have have proven (3.14).

It follows that

$$\begin{aligned}\prod_{j=1}^{d} J_j(x,y) &\sim \prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}\\ &\sim \frac{n^{-(d-1)}+\rho(\bar{x},\bar{y})^{d-1}}{V_{n^{-1}}(\bar{x})+V(\bar{x},\bar{y})}\end{aligned}$$

Then

$$R_n(h_\kappa^2, x, y) = n^{d-1} \cdot \frac{\prod_{j=1}^{d} J_j(x,y)}{(n\rho(\bar{x},\bar{y})+1)^d} \sim \frac{1}{1+n\rho(\bar{x},\bar{y})} \cdot \frac{1}{V(\bar{x},\bar{y}) + V_{n^{-1}}(\bar{x})}$$

$\square$

Lemma 3.3.2: *For $x, y \in \mathbb{S}^{d-1}$ and $\alpha \geqslant 0$, let*

$$A_n^\alpha(x,y) := \frac{n^{d-1}}{(1+n\rho(\bar{x},\bar{y}))^\alpha} \prod_{j=1}^{d} (|x_j| + \rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j}.$$

If $\alpha > d-1$ and $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$, then

$$\int_{\mathbb{S}^{d-1}} |f(y)| A_n^\alpha(x,y) h_\kappa^2(y)\, d\sigma(y) \leqslant C M_\kappa f(x),$$

where the constant C is independent of n, f and x. Furthermore, if $\alpha = d-1$ and $\varepsilon > 0$ then

$$\int_{\{y\in\mathbb{S}^{d-1}:\ \ \rho(\bar{x},\bar{y})\geqslant\varepsilon\}} |f(y)| A_n^\alpha(x,y) h_\kappa^2(y)\, d\sigma(y) \leqslant C \left|\log \frac{1}{\varepsilon}\right| M_\kappa f(x).$$

Proof. For $x \in \mathbb{S}^{d-1}$, by the last Lemma we have

$$\begin{aligned} A_n^\alpha(x,y) &= \frac{R_n^\alpha(x,y)}{(1+n\rho(\bar{x},\bar{y}))^{\alpha-d}} \\ &\sim \frac{(1+n\rho(\bar{x},\bar{y}))^{d-\alpha-1}}{V(\bar{x},\bar{y}) + V_{n^{-1}}(\bar{x})} \end{aligned}$$

Let

$$A_n^\alpha(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} A_n^\alpha(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y).$$

and

$$\tilde{A}_n^{\alpha}(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\geqslant\varepsilon\}} A_n^{\alpha}(h_\kappa^2; x, y) f(y) h_\kappa^2(y)\, d\sigma(y).$$

Then if $\alpha > d-1$,

$$\begin{aligned}
|A_n^{\alpha}(h_\kappa^2; f, x)| \leqslant & \int_{\mathbb{S}^{d-1}} \frac{|f(y)| h_\kappa^2(y)}{V(\bar{x},\bar{y}) + V_{n^{-1}}(\bar{x})} (1 + n\rho(\bar{x},\bar{y}))^{\alpha-d+1} d\sigma(y) \\
\leqslant & \int_{B(\bar{x},n^{-1})} \frac{|f(y)| h_\kappa^2(y)}{V(\bar{x},\bar{y}) + V_{n^{-1}}(\bar{x})} (1 + n\rho(\bar{x},\bar{y}))^{\alpha-d+1} d\sigma(y) \\
& + \sum_{j=0}^{\infty} \int_{\{y:\frac{2^j}{n}<\rho(\bar{x},\bar{y})\leqslant\frac{2^{j+1}}{n}\}} \frac{|f(y)| h_\kappa^2(y)}{V(\bar{x},\bar{y}) + V_{n^{-1}}(\bar{x})} (1 + n\rho(\bar{x},\bar{y}))^{\alpha-d+1} d\sigma(y) \\
\leqslant & \int_{B(\bar{x},n^{-1})} \frac{|f(y)|}{V_{n^{-1}}(\bar{x})} h_\kappa^2(y) d\sigma(y) \\
& + \sum_{j=0}^{\infty} 2^{(d-\alpha-1)j} \int_{\{y:\frac{2^j}{n}<\rho(\bar{x},\bar{y})\leqslant\frac{2^{j+1}}{n}\}} \frac{|f(y)|}{V(\bar{x},\bar{y})} h_\kappa^2(y) d\sigma(y) \\
\lesssim & M_\kappa(f)(x) + \sum_{j=0}^{\infty} \frac{2^{-j}}{\mathrm{meas}_\kappa(B(\bar{x},\frac{2^j}{n}))} \int_{B(\bar{x},\frac{2^{j+1}}{n})} |f(y)| h_\kappa^2(y) d\sigma(y) \\
\lesssim & M_\kappa(f)(x) + \sum_{j=0}^{\infty} 2^{-j} M_\kappa(f)(x) \\
\lesssim & M_\kappa(f)(x)
\end{aligned}$$

If $\alpha = d-1$, then

$$\begin{aligned}
|\tilde{A}_n^{\alpha}(h_\kappa^2; f, x)| \leqslant & \int_{\{y\in\mathbb{S}^{d-1}:\ \rho(\bar{x},\bar{y})\geqslant\varepsilon\}} \frac{|f(y)|}{V(\bar{x},\bar{y})} h_\kappa^2(y) d\sigma(y) \\
\lesssim & \sum_{j=1}^{\lceil \log_2 \frac{\pi}{\varepsilon} \rceil} \frac{1}{\mathrm{meas}_\kappa(B(\bar{x},2^j\varepsilon))} \int_{B(\bar{x},2^j\varepsilon)} |f(y)| h_\kappa^2(y) d\sigma(y) \\
\lesssim & \left|\log \frac{1}{\varepsilon}\right| M_\kappa f(x).
\end{aligned}$$

□

Proof of Proposition 3.2.1. The pointwise (3.10) follows directly from (3.9), Lemma 3.3.1 and Lemma 3.3.2, while the weak estimate (3.11) is an immediate consequence of (3.10) and (3.3). $\square$

3.4 Proof of Proposition 3.2.2

Without loss of generality, we may assume that $\|f\|_{1,\kappa}= 1$ and $\alpha > 1$. Let $\mathbb{S}_j^{d-1} := \{x \in \mathbb{S}^{d-1} : \ |x_j| \geqslant \frac{1}{2\sqrt{d}}\}$ for $1 \leqslant j \leqslant d$. Since for each $x \in \mathbb{S}^{d-1}$,

$$\max_{1\leqslant j\leqslant d} |x_j| \geqslant \frac{1}{\sqrt{d}} \|x\| = \frac{1}{\sqrt{d}},$$

it follows that $\mathbb{S}^{d-1} = \cup_{j=1}^d \mathbb{S}_j^{d-1}$. By (3.8), this implies that

$$\begin{aligned} |T_n^\delta(h_\kappa^2; f, x)| &\lesssim n^{\frac{d-2}{2}-\delta} \cdot \int_{\substack{\rho(\bar{x},\bar{y})\geqslant \frac{1}{2\sqrt{d}} \\ y\in\mathbb{S}^{d-1}}} |f(y)| \prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} h_\kappa^2(y) d\sigma(y) \\ &\leqslant \sum_{m=1}^d n^{\frac{d-2}{2}-\delta} \cdot \int_{\substack{\rho(\bar{x},\bar{y})\geqslant \frac{1}{2\sqrt{d}} \\ |y_m|\geqslant \frac{1}{\sqrt{d}}}} |f(y)| \prod_{j=1}^d (|x_j y_j| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} h_\kappa^2(y) d\sigma(y) \\ &\leqslant C \sum_{j=1}^d T_{n,j}^\delta(h_\kappa^2; f, x), \end{aligned}$$

where

$$T_{n,j}^\delta(h_\kappa^2; f, x) := \int_{\{y\in\mathbb{S}_j^{d-1}: \rho(\bar{x},\bar{y})\geqslant \frac{1}{2\sqrt{d}}\}} \frac{n^{\frac{d-2}{2}-\delta}|f(y)|}{\prod_{i=1}^d (|x_i y_i| + n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{\kappa_i}} h_\kappa^2(y) d\sigma(y).$$

Thus, it suffices to establish the weak estimates of

$$T^{\delta}_{*,j}(h^2_\kappa; f, x) := \sup_{n\in\mathbb{N}} T^{\delta}_{n,j}(h^2_\kappa; f, x)$$

for each $1\leqslant j\leqslant d$. By symmetry, we only need to consider the case of $j = 1$.

Take $\varepsilon > 0$ such that $\varepsilon^{-\kappa_1} = c\alpha$ for some absolute constant c to be specified later. Set $F_\varepsilon = \{x \in \mathbb{S}^{d-1} : |x_1|\leqslant \varepsilon\}$. A straightforward calculation then shows that

$$\begin{aligned}\text{meas}_\kappa(F_\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} |x_1|^{2\kappa_1}(1 - x_1^2)^{\frac{d-3}{2}+|\kappa|-\kappa_1} dx_1 \int_{\mathbb{S}^{d-2}} |y_2|^{2\kappa_2}\cdots |y_d|^{\kappa_d} d\sigma(y) \\ &\sim \varepsilon^{2\kappa_1+1}\leqslant C\varepsilon^{\kappa_1}\leqslant C\alpha^{-1}.\end{aligned}$$

On the other hand, if $x \in \mathbb{S}^{d-1} \setminus F_\varepsilon$, $y \in \mathbb{S}^{d-1}_1$ and $\rho(\bar{x}, \bar{y})\geqslant\frac{1}{2\sqrt{d}}$, then

$$\prod_{i=1}^{d}(|x_iy_i|+n^{-1}\rho(\bar{x}, \bar{y}) + n^{-2})^{\kappa_i}\geqslant C\varepsilon^{\kappa_1}n^{-|\kappa|+\kappa_1},$$

which implies that

$$\begin{aligned}|T^{\delta}_{n,1}(h^2_\kappa; f, x)|&\leqslant Cn^{\frac{d-2}{2}-\sigma_\kappa}\varepsilon^{-\kappa_1}n^{|\kappa|-\kappa_1}\|f\|_{1,\kappa} \\ &= Cn^{\frac{d-2}{2}+|\kappa|-\kappa_1-\sigma_\kappa}\varepsilon^{-\kappa_1} \leqslant C\varepsilon^{-\kappa_1} = Cc\alpha.\end{aligned}$$

Therefore, choosing $c > 0$ so that $Cc = \frac{1}{2}$, we deduce that

$$\text{meas}_\kappa\Big\{x \in \mathbb{S}^{d-1} : T^{\delta}_{*,1}(h^2_\kappa; f, x) > \alpha\Big\}\leqslant \text{meas}_\kappa(F_\varepsilon)\leqslant C\frac{1}{\alpha},$$

which is as desired.

3.5 Proof of Proposition 3.2.3

The proof of Proposition 3.2.3 relies on the following lemma.

Lemma 3.5.1: *Let $x, y \in \mathbb{S}^{d-1}$ be such that $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$. If i is a positive integer such that $i \leqslant d$ and $|x_i| \geqslant \frac{1}{\sqrt{d}}$, then*

$$\prod_{j=1}^{d} I_j(x, y) \leqslant C(1 + n\rho(\bar{x}, \bar{y}))^{|\kappa|-\kappa_i} \prod_{j=1}^{d} (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}, \qquad (3.15)$$

where

$$I_j(x, y) := (|x_j y_j| + n^{-1}\rho(\bar{x}, \bar{y}) + n^{-2})^{-\kappa_j}. \qquad (3.16)$$

Proof. By symmetry, we may assume that $i = 1$. Consider the following two cases:

Case 1. $\rho(\bar{x}, \bar{y}) \leqslant n^{-1}$.

In this case, note that $I_j(x, y) \sim (n^{-2} + |x_j y_j|)^{-\kappa_j}$. If $|x_j| \geqslant 2n^{-1} > 2\rho(\bar{x}, \bar{y})$, then $|x_j| \sim |y_j|$ and

$$I_j(x, y) \sim |x_j|^{-2\kappa_j} \sim (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}.$$

If $|x_j| < 2n^{-1}$, then $|y_j| < 3n^{-1}$ and

$$I_j(x, y) \sim n^{2\kappa_j} \sim (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}.$$

Thus, we have conclude that

$$\prod_{j=1}^{d} I_j(x, y) \sim \prod_{j=1}^{d} (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j},$$

which clearly implies (3.15).

Case 2. $\rho(\bar{x},\bar{y}) > n^{-1}$.

In this case, note first that if $|x_j|\geqslant 2\rho(\bar{x},\bar{y})$, then

$$I_j(x,y) \sim (|x_j|^2+n^{-1}\rho(\bar{x},\bar{y}))^{-\kappa_j} \sim |x_j|^{-2\kappa_j}\sim (|x_j|+\rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j};$$

while if $|x_j|< 2\rho(\bar{x},\bar{y})$, then

$$\begin{aligned} I_j(x,y) &\leqslant (n^{-1}\rho(\bar{x},\bar{y}) + n^{-2})^{-\kappa_j} \\ &\sim (1+n\rho(\bar{x},\bar{y}))^{\kappa_j}(\rho(\bar{x},\bar{y}) + |x_j|+n^{-1})^{-2\kappa_j} \end{aligned}$$

This means that for all $1\leqslant j\leqslant d$,

$$I_j(x,y) \leqslant C(1+n\rho(\bar{x},\bar{y}))^{\kappa_j}(\rho(\bar{x},\bar{y}) + |x_j|+n^{-1})^{-2\kappa_j}.$$

On the other hand, however, recalling that $|x_1|\geqslant \frac{1}{\sqrt{d}} \geqslant 2\rho(\bar{x},\bar{y})$, we have that $|x_1|\sim |y_1|\sim 1$, and hence

$$I_1(x,y) \sim (|x_1|+\rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_1}.$$

Therefore, putting the above together, we conclude that

$$\begin{aligned} \prod_{j=1}^{d} I_j(x,y) &= I_1(x,y)\prod_{j=2}^{d} I_j(x,y) \\ &\leqslant C(1+n\rho(\bar{x},\bar{y}))^{|\kappa|-\kappa_1}\prod_{j=1}^{d}(|x_j|+\rho(\bar{x},\bar{y}) + n^{-1})^{-2\kappa_j}, \end{aligned}$$

which is as desired. □

Now we are in a position to prove Proposition 3.2.3.

Proof of Proposition 3.2.3. Without loss of generality, we may assume that $\|f\|_{1,\kappa}= 1$ and $\alpha > 1$. As in the proof of Proposition 3.2.2, we have $\mathbb{S}^{d-1} = \bigcup_{i=1}^d \mathbb{S}_i^{d-1}$ with

$$\mathbb{S}_i^{d-1} := \{x \in \mathbb{S}^{d-1} : \ |x_i| \geqslant \frac{1}{\sqrt{d}}\}.$$

Thus, it is enough to prove that for each $1 \leqslant i \leqslant d$,

$$\mathrm{meas}_\kappa(\{x \in \mathbb{S}_i^{d-1} : \quad E_*^\delta(h_\kappa^2; f, x) > \alpha\}) \leqslant C\alpha^{-1}, \tag{3.17}$$

with $\alpha^{-1} \log \alpha^{-1}$ in place of α^{-1} in the case when $\delta = \sigma_\kappa$ and at least two of the κ_i are zero.

To prove (3.17), we consider the following cases:

Case 1. $\kappa_i > \kappa_{\min}$ or $\delta > \sigma_\kappa$

In this case, we shall prove that

$$E_*^\delta(h_\kappa^2; f, x) \leqslant CM_\kappa f(x), \quad \forall x \in \mathbb{S}_i^{d-1}, \tag{3.18}$$

from which (3.17) will follow by (3.3).

By Lemma 3.5.1, if $x \in \mathbb{S}_j^{d-1}$, $y \in \mathbb{S}^{d-1}$ and $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$, then

$$\begin{aligned} |E_n^\delta(h_\kappa^2; x, y)| &\leqslant Cn^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-\delta - \frac{d}{2}} \prod_{j=1}^d I_j(x, y) \\ &\leqslant Cn^{d-1}(1 + n\rho(\bar{x}, \bar{y}))^{-(d-1-\kappa_{\min}+\kappa_i+\delta-\sigma_\kappa)} \prod_{j=1}^d (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}. \end{aligned}$$

Since $\kappa_i - \kappa_{\min} + \delta - \sigma_\kappa > 0$ in this case, the estimate (3.18) then follows by Lemma 3.3.2.

Case 2. $\kappa_i = \kappa_{\min}$ and $\min_{j \neq i} \kappa_j > 0$.

Without loss of generality, we may assume that $i = 1$ in this case. Let $\varepsilon > 0$ be such that $\varepsilon^{d-1+2|\kappa|-2\kappa_1} = c_1^{-1}\alpha^{-1}$, where $c_1 > 0$ is an absolute constant to be specified later. Set

$$F_\varepsilon = \{x \in \mathbb{S}^{d-1} : 1 - \varepsilon^2 \leqslant |x_1| \leqslant 1\}.$$

A straightforward calculation shows that

$$\text{meas}_\kappa(F_\varepsilon) = c_\kappa \int_{1-\varepsilon^2}^{1} x_1^{2\kappa_1}(1 - x_1^2)^{\frac{d-3}{2}+|\kappa|-\kappa_1}\, dx_1 \sim \varepsilon^{d-1+2|\kappa|-2\kappa_1} \sim \alpha^{-1}.$$

Next, for $x \in \mathbb{S}_1^{d-1} \setminus F_\varepsilon$, and $y \in \mathbb{S}^{d-1}$, we set

$$J := J(x, y) = \{j : 2 \leqslant j \leqslant d,\ |x_j| < 2\rho(\bar{x}, \bar{y})\},$$
$$J' := J'(x, y) = \{2, 3, \cdots, d\} \setminus J.$$

Recall that $I_j(x, y)$ is defined in (3.16). From the proof of Lemma 3.5.1, it is easily seen that if $|x_j| \geqslant 2\rho(\bar{x}, \bar{y})$,

$$I_j(x, y) \leqslant C(|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j} \tag{3.19}$$

and that if $|x_j| < 2\rho(\bar{x}, \bar{y})$,

$$I_j(x, y) \leqslant C(1 + n\rho(\bar{x}, \bar{y}))^{\kappa_j}(\rho(\bar{x}, \bar{y}) + |x_j| + n^{-1})^{-2\kappa_j}. \tag{3.20}$$

Note also that if $x \in \mathbb{S}_1^{d-1}$ and $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$, then $|x_1| \geqslant \frac{1}{2\sqrt{d}}$ and $|y_1| \geqslant |x_1| - \rho(\bar{x}, \bar{y}) \geqslant \frac{1}{2\sqrt{d}}$. Thus, under the condition $x \in \mathbb{S}_1^{d-1}$ and $\rho(\bar{x}, \bar{y}) \leqslant \frac{1}{2\sqrt{d}}$,

$$\prod_{j=1}^{d} I_j(x, y) \leqslant C(1 + n\rho(\bar{x}, \bar{y}))^{\sum_{j\in J} \kappa_j} \prod_{j=1}^{d} (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j},$$

which, in turn, implies that

$$\begin{aligned} &|E_n^\delta(h_\kappa^2; x, y)| \\ &\leqslant C n^{d-1} (1 + n\rho(\bar{x}, \bar{y}))^{-\delta - \frac{d}{2} + \sum_{j\in J} \kappa_j} \prod_{j=1}^{d} (|x_j| + \rho(\bar{x}, \bar{y}) + n^{-1})^{-2\kappa_j}. \end{aligned} \tag{3.21}$$

If $J \subsetneqq \{2, 3, \cdots, d\}$, then $\sum\limits_{j\in J} \kappa_j \leqslant |\kappa| - \kappa_1 - \min_{2\leqslant j\leqslant d} \kappa_j$ and

$$\delta + \frac{d}{2} - \sum_{j\in J} \kappa_j \geqslant d - 1 + \min_{2\leqslant j\leqslant d} \kappa_j > d - 1.$$

On the other hand, however, if $J = \{2, 3, \cdots, d\}$, and $x \in \mathbb{S}_1^{d-1} \setminus F_\varepsilon$, then

$$\delta + \frac{d}{2} - \sum_{j\in J} \kappa_j = \delta + \frac{d}{2} - |\kappa| + \kappa_1 \geqslant d - 1,$$

and moreover,

$$\rho(\bar{x}, \bar{y}) \geqslant \frac{1}{2} \max_{2\leqslant j\leqslant d} |x_j| \geqslant \frac{\sqrt{1 - x_1^2}}{2\sqrt{d-1}} \geqslant \frac{\varepsilon}{2\sqrt{d-1}},$$

where the last step uses the fact that $1 - |x_1| > \varepsilon^2$ for $x \notin F_\varepsilon$. Thus, using (3.21) and recalling that $\varepsilon^{-(d-1+2|\kappa|-2\kappa_1)} = c_1\alpha$, we conclude that if $x \in \mathbb{S}_1^{d-1} \setminus F_\varepsilon$ and

$\rho(\bar{x},\bar{y})\leqslant\frac{1}{2\sqrt{d}}$, then

$$|E_n^\delta(h_\kappa^2;x,y)|\leqslant C\frac{n^{d-1}}{(1+n\rho(\bar{x},\bar{y}))^{d-1+\min_{2\leqslant j\leqslant d}\kappa_j}\prod_{j=1}^d(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{2\kappa_j}}$$
$$+Cc_1\alpha.$$

Since $\|f\|_{\kappa,1}=1$ and $\kappa_{\min}>0$, using Lemma 3.3.2, and choosing $c_1=(2C)^{-1}$, we deduce that for $x\in\mathbb{S}_1^{d-1}\setminus E_\varepsilon$,

$$E_*^\delta(h_\kappa^2;f,x)\leqslant CM_\kappa f(x)+\frac{1}{2}\alpha.$$

It follows that

$$\begin{aligned}&\operatorname{meas}_\kappa(\{x\in\mathbb{S}_1^{d-1}:E_*^\delta(h_\kappa^2;f,x)>\alpha\})\\ &\leqslant\operatorname{meas}_\kappa(F_\varepsilon)+\operatorname{meas}_\kappa(\{x\in\mathbb{S}_1^{d-1}\setminus F_\varepsilon:\quad E_*^\delta(h_\kappa^2;f,x)>\alpha\})\\ &\leqslant C\frac{1}{\alpha}+\operatorname{meas}_\kappa(\{x\in\mathbb{S}^{d-1}:M_\kappa f(x)\geqslant\frac{\alpha}{2C}\})\leqslant C\frac{1}{\alpha}.\end{aligned}$$

Case 3. $\kappa_i=0$, $\min_{j\neq i}\kappa_j=0$ and $\delta=\sigma_\kappa$.

Since $\kappa\neq 0$, we may assume, without loss of generality, that $i=2$ and $\kappa_1>0$. In this case, using (3.19) and (3.20), we have that for $x,y\in\mathbb{S}^{d-1}$,

$$\begin{aligned}&|E_n^\delta(h_\kappa^2;x,y)|\\ &\leqslant Cn^{d-1}\frac{\prod_{j=1}^d(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}}{(1+n\rho(\bar{x},\bar{y}))^{d-1}}\chi_{\{y\in\mathbb{S}^{d-1}:\quad |x_1|\leqslant 2\rho(\bar{x},\bar{y})\}}(y)\\ &\quad+Cn^{d-1}\frac{\prod_{j=1}^d(|x_j|+\rho(\bar{x},\bar{y})+n^{-1})^{-2\kappa_j}}{(1+n\rho(\bar{x},\bar{y}))^{d-1+\kappa_1}},\end{aligned}$$

where χ_F denotes the characteristic function of the set F. Thus, using Lemma

3.3.2, we conclude that

$$E_*^{\sigma_\kappa}(h_\kappa^2; f, x) \leqslant C\Big(\log \frac{1}{|x_1|}\Big) M_\kappa f(x).$$

Therefore, for $\|f\|_{\kappa,1} = 1$ and $\alpha > 0$,

$$\begin{aligned} &\text{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \ \ E_*^{\sigma_\kappa}(h_\kappa^2; f, x) > \alpha\} \\ \leqslant &\,\text{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \ |x_1| \leqslant \alpha^{-1}\} \\ &+ \text{meas}_\kappa\{x \in \mathbb{S}^{d-1} : \ M_\kappa f(x) > \alpha(\log \alpha)^{-1}\} \\ \leqslant &\, C\alpha^{-1}|\log \alpha|. \end{aligned}$$

3.6 Proof of Theorem 3.1.1: Part (ii)

The proof of Theorem 3.1.1 (ii) follows along the same idea as that of [16], where the Cantor-Lebesgue Theorem is combined with the Uniform Boundedness Principle to deduce a divergence result for the Cesàro means of spherical harmonic expansions. The result of [16] was later extended to the case of h-harmonic expansions in [32]. Our proof below is different from that of [32], and it leads to more information on the counterexample f, from which the corresponding results for weighted orthogonal polynomial expansions on the ball $\mathbb{B}^d$ and on the simplex $\mathbb{T}^d$ can be easily deduced.

The proof of Theorem 3.1.1 (ii) relies on several lemmas. The first lemma is a well known result on Cesàro means of general sequences (see, for instance, [33, Theorem 3.1.22, p. 78] and [33, Theorem 3.1.23, p. 78]).

Lemma 3.6.1: *Let $s_n^\delta := (A_n^\delta)^{-1} \sum_{j=0}^n A_{n-j}^\delta a_j$ denote the Cesàro (C, δ)-*

means of a sequence $\{a_j\}_{j=0}^{\infty}$ *of real numbers. Then for* $\delta \geqslant 0$

$$|a_n| \leqslant C_\delta n^\delta \max_{0\leqslant j\leqslant n} |s_j^\delta|, \quad n = 0, 1, \cdots, \tag{3.22}$$

and for $0 \leqslant \delta_1 < \delta_2$,

$$|s_n^{\delta_1}| \leqslant C_{\delta_1,\delta_2} n^{\delta_2-\delta_1} \max_{1\leqslant j\leqslant n} |s_j^{\delta_2}|, \quad n = 0, 1, \cdots. \tag{3.23}$$

The second lemma was proved in [16, Section 3.3]. It follows from the asymptotics of the Jacobi polynomials and the Riemann-Lebesgue theorem.

Lemma 3.6.2: *Let* α, $\beta \geqslant -\frac{1}{2}$, *and let* F *be a subset of* $[-1, 1]$ *with positive Lebesgue measure. Then there exists a positive integer* N *depending on the set* F *for which*

$$\sup_{t\in F} |P_n^{(\alpha,\beta)}(t)| \geqslant Cn^{-\frac{1}{2}}, \quad \forall n \geqslant N,$$

where the constant C *depends on the set* F, *but is independent of* n.

To state our next lemma, recall that the generalized Gegenbauer polynomial $C_n^{(\lambda,\mu)}$ is the weighted orthogonal polynomial of degree n with respect to the weight $|t|^{2\mu}(1-t^2)^{\lambda-\frac{1}{2}}$ on $[-1, 1]$.

Lemma 3.6.3: *Let* $f \in L(w_\kappa; [0, 1])$ *with* $w_\kappa(t) = |t|^{2\kappa_1}(1-t^2)^{\lambda_\kappa-\kappa_1-\frac{1}{2}}$. *Let* $\widetilde{f} : \mathbb{S}^{d-1} \to \mathbb{R}$ *be given by* $\widetilde{f}(x) = f(|x_1|)$. *Then* $\widetilde{f} \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ *and*

$$\mathrm{proj}_{2n}(h_\kappa^2; \widetilde{f}, x) = d_{2n}(f) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1), \quad x \in \mathbb{S}^{d-1}, \tag{3.24}$$

where

$$d_{2n}(f) := \frac{1}{\|C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}\|^2_{L^2(w_\kappa;[0,1])}} \int_0^1 f(t) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t) w_\kappa(t)\, dt. \tag{3.25}$$

Proof. We need the following formula for the reproducing kernel $P_n(h_\kappa^2;\cdot,e_1)$ of the space $\mathcal{H}_n^d(h_\kappa^2)$ (see [7, proof of Theorem 2.2 (lower bound)]):

$$P_n(h_\kappa^2; x, e_1) = \frac{n+\lambda_\kappa}{\lambda_\kappa} C_n^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1), \quad x \in \mathbb{S}^{d-1},\ n = 0, 1, \cdots, \tag{3.26}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{S}^{d-1}$.

By (2.9), it follows that $\{C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}\}_{n=0}^\infty$ is an orthogonal polynomial basis with respect to the weight $w_\kappa(t)$ on $[0,1]$. Thus, each function $f \in L(w_\kappa; [0,1])$ has a weighted orthogonal polynomial expansion $\sum_{n=0}^\infty d_{2n}(f) C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t)$ on $[0,1]$, which particularly implies that for each polynomial g of degree at most $2n$ on $[-1,1]$,

$$\int_{-1}^1 f(|t|) g(t) w_\kappa(t)\, dt = \sum_{j=0}^n d_{2j}(f) \int_{-1}^1 C_{2j}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(t) g(t) w_\kappa(t)\, dt. \tag{3.27}$$

Next, we note that (3.26) implies that the term on the right hand side of (3.24) is an h-harmonic in $\mathcal{H}_{2n}^d(h_\kappa^2)$. Thus, for the proof of (3.24), it is sufficient to verify that for each $P \in \mathcal{H}_{2n}^d(h_\kappa^2)$,

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \widetilde{f}(x) P(x) h_\kappa^2(x)\, d\sigma(x)\\ =&d_{2n}(f) \int_{\mathbb{S}^{d-1}} C_{2n}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1) P(x) h_\kappa^2(x)\, d\sigma(x).\end{aligned} \tag{3.28}$$

Indeed, for $P \in \mathcal{H}_{2n}^d(h_\kappa^2)$,

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \widetilde{f}(x)P(x)h_\kappa^2(x)\,d\sigma(x)\\ &=\int_{-1}^1 f(|x_1|)w_\kappa(x_1)\Big[\int_{\mathbb{S}^{d-2}} P(x_1,\sqrt{1-x_1^2}y)h_{\widetilde{k}}^2(y)\,d\sigma(y)\Big]\,dx_1,\end{aligned}$$

where $h_{\widetilde{k}}(y)=\prod_{j=1}^{d-1}|y_j|^{\kappa_{j+1}}$ for $y=(y_1,\cdots,y_{d-1})\in\mathbb{R}^{d-1}$. Since the weight $h_{\widetilde{k}}^2(y)$ is even in each y_j, it is easily seen that the integral over $\mathbb{S}^{d-2}$ of the last equation is an algebraic polynomial in x_1 of degree at most $2n$. Thus, it follows by (3.27) that

$$\begin{aligned}&\int_{\mathbb{S}^{d-1}} \widetilde{f}(x)P(x)h_\kappa^2(x)\,d\sigma(x)\\ &=\sum_{j=0}^n d_{2j}(f)\int_{-1}^1 C_{2j}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1)w_\kappa(x_1)\Big[\int_{\mathbb{S}^{d-2}} P(x_1,\sqrt{1-x_1^2}y)h_{\widetilde{k}}^2(y)\,d\sigma(y)\Big]\,dx_1\\ &=\sum_{j=0}^n d_{2j}(f)\int_{\mathbb{S}^{d-1}} C_{2j}^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1)P(x)h_\kappa^2(x)\,d\sigma(x).\end{aligned}$$

Since, by (3.26), $C_j^{(\lambda_\kappa-\kappa_1,\kappa_1)}(x_1)\in\mathcal{H}_j^d(h_\kappa^2)$, the desired equation (3.28) follows by the orthogonality of the spherical h-harmonics. □

Now we are in a position to prove Theorem 3.1.1 (ii).

Proof of Theorem 3.1.1(ii). Without loss of generality, we may assume that $\kappa_1=\kappa_{\min}$. Assume that the stated conclusion were not true. This would mean that $S_*^\delta(h_\kappa^2;\widetilde{f},x)$ is finite on a set $E_f\subset\mathbb{S}^{d-1}$ of positive measure for all $f\in L^1(w_\kappa;[0,1])$ and some $\delta<\sigma_\kappa$, where $\widetilde{f}(x)=f(|x_1|)$ for $x\in\mathbb{S}^{d-1}$, and

$w_\kappa(t) = |t|^{2\kappa_1}(1-t^2)^{\sigma_\kappa - \frac{1}{2}}$. By Lemma 3.6.1, this implies that

$$\sup_{n\in\mathbb{N}} n^{-\delta}|\mathrm{proj}_{2n}(h_\kappa^2; \widetilde{f}, x)| < \infty, \quad \forall x \in E_f, \forall f \in L(w_\kappa; [0,1]). \tag{3.29}$$

We will show that (3.29) is impossible unless $\delta \geqslant \sigma_\kappa$.

In fact, by(3.29),

$$E_f = \bigcup_{N=1}^{\infty} \Big\{ x \in E_f : \quad \sup_{n\in\mathbb{N}} n^{-\delta}|\mathrm{proj}_{2n}(h_\kappa^2; \widetilde{f}, x)| \leqslant N \Big\},$$

hence, there must exist a subset E_f' of E_f with positive Lebesgue measure such that

$$\sup_{x\in E_f'} \sup_{n\in\mathbb{N}} n^{-\delta}|\mathrm{proj}_{2n}(h_\kappa^2; \widetilde{f}, x)| \leqslant N_f < \infty.$$

By Lemma 3.6.3, this in turn implies that

$$\sup_{x\in E_f'} \sup_{n\in\mathbb{N}} n^{-\delta}|d_{2n}(f)||C_{2n}^{(\sigma_\kappa,\kappa_1)}(x_1)| \leqslant N_f, \tag{3.30}$$

where $d_{2n}(f)$ is defined in (3.25). Note that by (2.9),

$$C_{2n}^{(\lambda_\kappa - \kappa_1, \kappa_1)}(x_1) = \frac{\Gamma(\lambda_\kappa + n)\Gamma(\kappa_1 + \frac{1}{2})}{\Gamma(\lambda_\kappa)\Gamma(\kappa_1 + \frac{1}{2} + n)} P_n^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(2x_1^2 - 1).$$

Hence, using [23, (4.3.3)], we can rewrite (3.30) as

$$\sup_{n\in\mathbb{N}} n^{1-\delta}|\ell_n(f)| \sup_{t\in I_f} |P_n^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(t)| \leqslant N_f, \tag{3.31}$$

where $I_f := \{2x_1^2 - 1 : \quad x \in E_f'\}$, and

$$\ell_n(f) := \int_0^1 f(t) P_n^{(\sigma_\kappa - \frac{1}{2}, \kappa_1 - \frac{1}{2})}(2t^2 - 1) w_\kappa(t)\, dt. \tag{3.32}$$

Since $E'_f \subset \mathbb{S}^{d-1}$ has a positive Lebesgue measure, it is easily seen that $I_f \subset [-1,1]$ has a positive Lebesgue measure as well. Thus, (3.31) together with Lemma 3.6.3 implies that

$$\sup_{n\in\mathbb{N}} n^{\frac{1}{2}-\delta}|\ell_n(f)| < \infty, \quad \forall f \in L(w_\kappa;[0,1]). \tag{3.33}$$

Since $\{n^{\frac{1}{2}-\delta}\ell_n(f)\}_{n=0}^{\infty}$ is a sequence of bounded linear functionals on the Banach space $L(w_\kappa;[0,1])$, it follows by (3.33) and the uniform boundedness theorem that

$$\sup_{n} n^{\frac{1}{2}-\delta} \sup_{\|f\|_{L(w_\kappa;[0,1])}\leqslant 1} |\ell_n(f)| < \infty. \tag{3.34}$$

On the other hand, however, using (3.32) and [23, (7.32.2), p. 168], we have

$$\sup_{\|f\|_{L(w_\kappa;[0,1])}\leqslant 1} |\ell_n(f)| = \max_{t\in[0,1]} |P_{2n}^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(2t^2-1)| = P_{2n}^{(\sigma_\kappa-\frac{1}{2},\kappa_1-\frac{1}{2})}(1) \sim n^{\sigma_\kappa-\frac{1}{2}}.$$

Thus, (3.34) implies that

$$\sup_{n\in\mathbb{N}} n^{\frac{1}{2}-\delta} n^{\sigma_k-\frac{1}{2}} = \sup_{n\in\mathbb{N}} n^{\sigma_\kappa-\delta} < \infty,$$

which can not be true unless $\delta \geqslant \sigma_\kappa$. This completes the proof. □

3.7 Corollaries

3.7.1 The pointwise convergence

In this subsection, we devote to the investigation of almost everywhere convergence of *Cesàro* (C,δ)*-mean* S_n^δ of weighted orthogonal expansions on the unit sphere $\mathbb{S}^{d-1}$ by our weak-type estimation. What we have already known is for $\delta > \frac{d-2}{2} + |\kappa|$,

$$S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e. x \in \mathbb{S}^{d-1},$$

And for $\delta < \frac{d-2}{2} + |\kappa| - \min_{1\leqslant i\leqslant d} \kappa_i$, there exists a function $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ such that

$$\limsup_{n\to\infty} |S_n^\delta(h_\kappa^2; f, x)| = \infty, \quad a.e. x \in \mathbb{S}^d.$$

At here we proved the critical index for the a.e. convergence of *Cesàro* (C, δ)*-mean* means, that is, for $f \in L^1(h_\kappa^2; \mathbb{S}^d)$, if $\delta \geqslant \frac{d-2}{2} + |\kappa| - \kappa_{\min}$, and $\kappa_{\min} > 0$, then

$$S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e. x \in \mathbb{S}^{d-1},$$

Corollary 3.7.1: *In order that*

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{S}^{d-1}$ for all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$, it is sufficient and necessary that $\delta \geqslant \sigma_\kappa$.

Proof. For all $f \in L^1(h_\kappa^2; \mathbb{S}^{d-1})$ we can write

$$f(x) = g_m(x) + b_m(x),$$

where $g_m(x) \in \mathcal{H}_n^d$, and $\lim_{m\to\infty} \|b_m\|_{1,\kappa} = 0$. Set

$$\Lambda^\delta(f)(x) := \limsup_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - \liminf_{n\to\infty} S_n^\delta(h_\kappa^2; f, x),$$

Then by Theorem 3.1.1 (i), $\forall \varepsilon > 0$

$$\begin{aligned}\mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \Lambda^\delta(f)(x) > \varepsilon\}) &= \mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : \Lambda^\delta(b_m)(x) > \varepsilon\}) \\ &\leqslant \mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : S_*^\delta(h_\kappa^2; |b_m|, x) \gtrsim \varepsilon\}) \\ &\lesssim \frac{\|b_m\|_{1,\kappa}}{\varepsilon} \to 0, \quad \text{as } m \to \infty.\end{aligned}$$

This implies that $\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x)$ exists.

Then since $g_m(x) \in \mathcal{H}_n^d$,

$$\begin{aligned}&\mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - f(x)| > \varepsilon\}) \\ \leqslant &\mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : S_*^\delta(h_\kappa^2; |b_m|, x) > \frac{\varepsilon}{2}\}) \\ &+ \mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |b_m(x)| > \frac{\varepsilon}{2}\}) \\ \lesssim &\frac{\|b_m\|_{1,\kappa}}{\varepsilon}\end{aligned}$$

Let $m \to \infty$, we get

$$\mathrm{meas}_\kappa(\{x \in \mathbb{S}^{d-1} : |\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) - f(x)| > \varepsilon\}) = 0$$

$$\lim_{n\to\infty} S_n^\delta(h_\kappa^2; f, x) = f(x), \quad a.e.\ x \in \mathbb{S}^{d-1}.$$

Then we finish the proof of sufficiency, whereas the necessity follows directly from Theorem 3.1.1 (ii). $\square$

3.7.2 Strong estimates on L^p

Using Stein's interpolation theorem for analytic families of operators ([19]), we can deduce the following strong estimates for the maximal Cesàro operators:

Corollary 3.7.2: *If* $1 < p < \infty$ *and* $\delta > 2\sigma_\kappa|\frac{1}{2} - \frac{1}{p}|$, *then*

$$\|S_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C_p \|f\|_{\kappa,p}. \tag{3.35}$$

In particular,

$$\|S_*^{\sigma_\kappa}(h_\kappa^2; f)\|_{\kappa,p} \leqslant C_p \|f\|_{\kappa,p}, \quad 1 < p < \infty.$$

We first show S_*^δ is strong-type $(2, 2)$ for $\delta > 0$. It is sufficient to show the following lemmas. The idea of the proof is directly from the proof of Lemma 3.5 of [4].

Lemma 3.7.3: *If there exists a* $\delta_0 > 0$ *such that for all* $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$,

$$\|S_*^{\delta_0}(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim_p \|f\|_{\kappa,2}.$$

Then for all $\delta > 0$ *and for all* $f \in L^2(h_\kappa^2; \mathbb{S}^{d-1})$, *we have*

$$\|S_*^{\delta}(h_\kappa^2; f, x)\|_{\kappa,2} \lesssim_p \|f\|_{\kappa,2}.$$

Proof. Firstly, since for any $\alpha > 0$, and $\beta > \frac{1}{2}$,

$$\sum_{k=0}^{n} \left(\frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} \right)^2 \sim \sum_{k=0}^{n} \left(\frac{k^\delta (n-k)^{\beta-1}}{n^{\delta+\beta}} \right)^2 \sim n^{-1}$$

Then

$$\begin{aligned}
|S_n^{\delta+\beta}(h_\kappa^2;f,x)| &= \left|\sum_{k=0}^{n} \frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}} S_k^\delta(h_\kappa^2;f,x)\right| \\
&\leqslant \sum_{k=0}^{n} \left|\frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}}\right| \cdot |S_k^\delta(h_\kappa^2;f,x)| \\
&\leqslant \left(\sum_{k=0}^{n} \left|\frac{A_k^\delta A_{n-k}^{\beta-1}}{A_n^{\delta+\beta}}\right|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2;f,x)|^2\right)^{\frac{1}{2}} \\
&\lesssim \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2;f,x)|^2 \cdot n^{-1}\right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$S_*^{\delta+\beta}(h_\kappa^2;f,x) \leqslant \sup_n \left(\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2;f,x)|^2 \cdot n^{-1}\right)^{\frac{1}{2}}$$

Therefore, we just need to show that for all $\delta > -\frac{1}{2}$, and for all $f \in L^2(h_\kappa^2;\mathbb{S}^{d-1})$,

$$\|\sup_n (\sum_{k=0}^{n} |S_k^\delta(h_\kappa^2;f,x)|^2 \cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2} \lesssim \|f\|_{\kappa,2}$$

In fact, on one side, we know that for all $f \in L^2(h_\kappa^2;\mathbb{S}^{d-1})$,

$$\|\sup_n (\sum_{k=0}^{n} |S_k^{\delta_0}(h_\kappa^2;f,x)|^2 \cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2} \leqslant \|S_*^{\delta_0}(h_\kappa^2;f,x)\|_{\kappa,2} \lesssim \|f\|_{\kappa,2}$$

On the other side, since $(A_{n-k}^\delta)(A_n^\delta)^{-1} = \prod_{j=0}^{k}(n-j+\delta)^{-1}$ is a decreasing function of δ,

$$
\begin{aligned}
\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1} &\leqslant \sum_{k=0}^{n}\frac{1}{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2\\
&\leqslant \sum_{n=0}^{\infty}\frac{1}{n}|S_n^{\delta}(h_\kappa^2;f,x)-S_n^{\delta_0}(h_\kappa^2;f,x)|^2\\
&\sim \sum_{n=0}^{\infty}\frac{1}{n}\left|\sum_{k=0}^{n}\left(\frac{A_{n-k}^{\delta+1}}{A_n^{\delta+1}}-\frac{A_{n-k}^{\delta}}{A_n^{\delta}}\right)\mathrm{proj}_k(h_\kappa^2;f,x)\right|^2\\
&= \sum_{n=0}^{\infty}\frac{(A_n^{\delta+1})^{-2}}{n(\delta+1)^2}|\sum_{k=0}^{n}kA_{n-k}^{\delta}\,\mathrm{proj}_k(h_\kappa^2;f,x)|^2
\end{aligned}
$$

we can get

$$
\begin{aligned}
&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\\
\lesssim&\|(\delta+1)^{-1}(\sum_{n=0}^{\infty}n^{-1}(A_n^{\delta+1})^{-2}|\sum_{k=0}^{n}kA_{n-k}^{\delta}\,\mathrm{proj}_k(h_\kappa^2;f,x)|^2)^{\frac{1}{2}}\|_{\kappa,2}^2\\
=&(\delta+1)^{-2}\sum_{n=0}^{\infty}n^{-1}(A_n^{\delta+1})^{-2}\sum_{k=0}^{n}k^2(A_{n-k}^{\delta})^2\|\mathrm{proj}_k(h_\kappa^2;f,x)\|_{\kappa,2}^2\\
=&(\delta+1)^{-2}\sum_{k=0}^{\infty}\|\mathrm{proj}_k(h_\kappa^2;f,x)\|_{\kappa,2}^2\cdot k^2\sum_{n=k}^{\infty}n^{-1}(A_{n-k}^{\delta})^2(A_n^{\delta+1})^{-2}
\end{aligned}
$$

Since

$$
k^2\sum_{n=k}^{\infty}n^{-1}(A_{n-k}^{\delta})^2(A_n^{\delta+1})^{-2}\sim k^2\sum_{n=k}^{\infty}n^{-1}n^{-2(\delta+1)}(n-k)^{2\delta}\sim 1,
$$

we have

$$
\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\lesssim\|f\|_{\kappa,2}
$$

Then by using triangle inequality,

$$
\begin{aligned}
&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)|^2\cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2}\\
\leqslant&\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta_0}(h_\kappa^2;f,x)|^2\cdot n^{-1})^{\frac{1}{2}}\|_{\kappa,2}\\
&+\|\sup_n(\sum_{k=0}^{n}|S_k^{\delta}(h_\kappa^2;f,x)-S_k^{\delta_0}(h_\kappa^2;f,x)|^2k^{-1})^{\frac{1}{2}}\|_{\kappa,2}^2\\
\lesssim&\|f\|_{\kappa,2}
\end{aligned}
$$

□

By this lemma, we can get the following Lemma.

Lemma 3.7.4: *For $\delta>0$ and $f(x)\in L^2(h_\kappa^2;\mathbb{S}^{d-1})$, $\|S_*^{\delta}(h_\kappa^2;f,x)\|_{\kappa,2}\lesssim\|f\|_{\kappa,2}$.*

Proof of Theorem 3.7.2. Firstly, recalling that ([14])

$$\|S_*^{\delta}(h_\kappa^2;f)\|_\infty\leqslant C\|f\|_\infty,\quad \delta>\sigma_\kappa,$$

we deduce from Theorem 3.1.1 and the Marcinkiewitcz interpolation theorem that

$$\|S_*^{\delta}(h_\kappa^2;f)\|_{\kappa,p}\leqslant C_p\|f\|_{\kappa,p},\quad 1<p<\infty,\quad \delta>\sigma_\kappa. \tag{3.36}$$

Secondly, in Lemma 3.7.4, we have already get

$$\|S_*^{\delta}(h_\kappa^2;f)\|_{\kappa,2}\leqslant C\|f\|_{\kappa,2},\ \delta>0. \tag{3.37}$$

Thirdly, the index δ of the Cesàro (C,δ)-means can be extended analytically to $\delta\in\mathbb{C}$ with $\operatorname{Re}\delta>-1$, as can be easily seen from the definition.

Furthermore, it is well known (see [4]) that for $\delta>0$, $\varepsilon>0$ and $y\in\mathbb{R}$,

$$S_n^{\delta+\varepsilon+iy}(h_\kappa^2;f)=(A_n^{\delta+\varepsilon+iy})^{-1}\sum_{j=0}^{n}A_{n-j}^{\varepsilon-1+iy}A_j^\delta S_j^\delta(h_\kappa^2;f), \tag{3.38}$$

and

$$|A_n^{\delta+\varepsilon+iy}|^{-1}\sum_{j=0}^{n}|A_{n-j}^{\varepsilon-1+iy}|A_j^\delta\leqslant C(\varepsilon)e^{cy^2}. \tag{3.39}$$

It follows that for $\delta>0$, $\varepsilon>0$ and $y\in\mathbb{R}$,

$$S_*^{\delta+\varepsilon+iy}(h_\kappa^2;f,x)\leqslant C(\varepsilon)e^{c(\varepsilon)y^2}S_*^\delta(h_\kappa^2;f,x). \tag{3.40}$$

Finally, for each measurable function $N:\ \mathbb{S}^{d-1}\to\{0,1,\cdots\}$, define $Q_N^\alpha f(x):=S_{N(x)}^\alpha(h_\kappa^2;f,x)$ for $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$. It can be easily verified that $\{Q_N^\alpha:\ \alpha\in\mathbb{C},\ \operatorname{Re}\alpha>0\}$ is a sequence of analytic operators in the sense of [19]. On one hand, since $2|\frac{1}{p}-\frac{1}{2}|\in(0,1)$ for $p\neq 2$, it follows that for any $\delta>2\sigma_\kappa|\frac{1}{p}-\frac{1}{2}|$, we can always find $\theta\in[0,1]$ such that $2|\frac{1}{p}-\frac{1}{2}|<1-\theta<\frac{\delta}{\sigma_\kappa}$, and two numbers $\varepsilon,\varepsilon'>0$ satisfying $\delta=\theta\varepsilon+(1-\theta)(\sigma_\kappa+\varepsilon)$, and $\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{p_{\varepsilon'}}$, where $p_{\varepsilon'}=1+\varepsilon'$ if $p<2$, and $p_{\varepsilon'}=2+(\varepsilon')^{-1}$ if $p>2$. On the other hand, however, using (3.36),(3.37), (3.40), we have that for any $y\in\mathbb{R}$,

$$\|Q_N^{\varepsilon+iy}f\|_{\kappa,2}\leqslant C(\varepsilon)e^{cy^2}\|f\|_{\kappa,2},$$
$$\|Q_N^{\sigma_\kappa+\varepsilon+iy}f\|_{\kappa,p_{\varepsilon'}}\leqslant C(\varepsilon)e^{cy^2}\|f\|_{\kappa,p_{\varepsilon'}}.$$

Thus, applying Stein's interpolation theorem [19], we conclude that

$$\|Q_N^\delta f\|_{\kappa,p}\leqslant C\|f\|_{\kappa,p},\quad \delta>2\sigma_\kappa|\frac{1}{p}-\frac{1}{2}|.$$

Since the constant C in this last equation is independent of the function N, the stated estimate (3.35) follows. $\square$

3.7.3 Marcinkiewitcz multiplier theorem

We can also deduce the following vector-valued inequalities for the Cesàro operators.

Corollary 3.7.5: *For* $1 < p < \infty$, $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$ *and any sequence* $\{n_j\}$ *of positive integers,*

$$\left\| \left(\sum_{j=0}^{\infty} |S_{n_j}^{\delta}(h_\kappa^2; f_j)|^2 \right)^{1/2} \right\|_{\kappa,p} \leqslant c \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}. \tag{3.41}$$

Proof. Note first that (3.41) for $\delta > 0$ and $p = 2$ is a direct consequence of Corollary 3.7.2. Next, we prove (3.41) for $\delta > \sigma_\kappa$ and $1 < p < \infty$. Define the following positive operators:

$$\widetilde{S}_n^{\delta}(h_\kappa^2; f, x) := \int_{\mathbb{S}^{d-1}} f(y)|K_n^{\delta}(h_\kappa^2; x, y)|h_\kappa^2(y)\, d\sigma(y), \quad x \in \mathbb{S}^{d-1}, \quad n = 0, 1, \cdots.$$

It is easily seen from the proofs of Theorem 3.1.1 and Corollary 3.7.2 that

$$\|\widetilde{S}_*^{\delta}(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}, \quad 1 < p \leqslant \infty, \quad \delta > \sigma_\kappa. \tag{3.42}$$

We shall follow the approach of [20, p.104-5] that uses a generalization of the Riesz convexity theorem for sequences of functions. Let $L^p(\ell^q)$ denote the space of all sequences $\{f_k\}$ of functions for which

$$\|(f_k)\|_{L^p(\ell^q)} := \left(\int_{\mathbb{S}^{d-1}} \left(\sum_{j=0}^{\infty} |f_j(x)|^q \right)^{p/q} h_\kappa^2(x) d\sigma(x) \right)^{1/p} < \infty.$$

If T is a bounded operator on both $L^{p_0}(\ell^{q_0})$ and $L^{p_1}(\ell^{q_1})$ for some $1\leqslant p_0, q_0, p_1, q_1\leqslant\infty$, then the generalized Riesz convexity theorem (see [3]) states that T is also bounded on $L^{p_t}(\ell^{q_t})$, where

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad 0\leqslant t\leqslant 1.$$

We apply this theorem to the operator T that maps the sequence $\{f_j\}$ to the sequence $\{S_{n_j}^\delta(h_\kappa^2; f_j)\}$. By Corollary 3.7.2, T is bounded on $L^p(\ell^p)$. By (3.42), it is also bounded on $L^p(\ell^\infty)$ as

$$\left\| \sup_{j\geqslant 0} |S_{n_j}^\delta(h_\kappa^2; f_j)| \right\|_{\kappa,p} \leqslant \left\| \widetilde{S}_*^\delta\Big(h_\kappa^2; \sup_{j\geqslant 0}|f_j|\Big) \right\|_{\kappa,p} \leqslant c \left\| \sup_{j\geqslant 0}|f_j| \right\|_{\kappa,p}.$$

Thus, the Riesz convexity theorem shows that T is bounded on $L^p(\ell^q)$ if $1 < p\leqslant q\leqslant\infty$. In particular, T is bounded on $L^p(\ell^2)$ if $1 < p\leqslant 2$. The case $2 < p < \infty$ follows by the standard duality argument, since the dual space of $L^p(\ell^2)$ is $L^{p'}(\ell^2)$, where $1/p + 1/p' = 1$, under the paring

$$\langle (f_j), (g_j)\rangle := \int_{\mathbb{S}^{d-1}} \sum_j f_j(x) g_j(x) h_\kappa^2(x) d\sigma(x)$$

and T is self-adjoint under this paring.

Finally, we prove that (3.41) for the general case follows by the Stein interpolation theorem ([19]). Without loss of generality, we may assume that there are only finitely many nonzero functions f_j in (3.41). Using (3.38), (3.39), the Cauchy-Schwartz inequality, and applying the above already proven case of

(3.41), we obtain that for $\delta > 0$ and $p = 2$ or $\delta > \sigma_\kappa$ and $1 < p < \infty$,

$$\left\| \left(\sum_{j=0}^{\infty} |S_{n_j}^{\delta+\varepsilon+iy}(h_\kappa^2; f_j)|^2 \right)^{1/2} \right\|_{\kappa,p} \leqslant C(\varepsilon) e^{cy^2} \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{\kappa,p}, \tag{3.43}$$

where $y \in \mathbb{R}$ and $\varepsilon > 0$. (3.41) then follows from (3.43) via applying Stein's interpolation theorem to the family of analytic operators,

$$T^\alpha f := \sum_{j=0}^{\infty} S_{n_j}^{\alpha}(h_\kappa^2; f) g_j, \quad \operatorname{Re} \alpha > 0,$$

where (g_j) is a sequence of functions on $\mathbb{S}^{d-1}$ with $\sum_j |g_j(x)|^2 = 1$ for $x \in \mathbb{S}^{d-1}$. □

Corollary 3.7.5 allows us to weaken the condition of the Marcinkiewitcz multiplier theorem established in [6].

Corollary 3.7.6: *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \leqslant c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \leqslant c < \infty$,

where n_0 is the smallest integer $\geqslant \sigma_\kappa + 1$, $\Delta \mu_j = \mu_j - \mu_{j+1}$, and $\Delta^{\ell+1} = \Delta^\ell \Delta$.

Then $\{\mu_j\}$ defines an $L^p(h_\kappa^2; \mathbb{S}^{d-1})$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(h_\kappa^2; f) \right\|_{\kappa,p} \leqslant c \|f\|_{\kappa,p}, \qquad 1 < p < \infty,$$

where c is independent of μ_j.

In the case when the weights are invariant under a general reflection group, Corollary 3.7.6 was proved in [6] under a stronger assumption that n_0 is the smallest integer $\geqslant \sigma_\kappa + 2 + \kappa_{\min}$. The proof of Corollary 3.7.6 is based on Corollary 3.7.5 and runs along the same line as that of [4].

Chapter 4

Weak estimates on the unit ball

Analysis in weighted spaces on the unit ball $\mathbb{B}^d = \{x \in \mathbb{R}^d : \quad \|x\| \leqslant 1\}$ can often be deduced from the corresponding results on the unit sphere $\mathbb{S}^d$, due to the close connection between the weighted orthogonal polynomial expansions on $\mathbb{B}^d$ and $\mathbb{S}^d$, as described in Section 2.3, see [10, 25, 26, 28] and the reference therein. In this section, we shall develop results on $\mathbb{B}^d$ that are analogous to those on $\mathbb{S}^d$.

Throughout this section, we will use a slight abuse of notations. The letter κ denotes a fixed, nonzero vector $\kappa := (\kappa_1, \cdots, \kappa_{d+1})$ in $\mathbb{R}_+^{d+1}$ rather than in $\mathbb{R}_+^d$, and h_κ denotes the weight function $h_\kappa(x) := \prod_{j=1}^{d+1} |x_j|^{\kappa_j}$ on $\mathbb{S}^d$ rather than the weight on $\mathbb{S}^{d-1}$. Accordingly, we write

$$\kappa_{\min} := \min_{1 \leqslant j \leqslant d+1} \kappa_j, \quad |\kappa| = \sum_{j=1}^{d+1} \kappa_j, \quad \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \kappa_{\min}. \qquad (4.1)$$

For a set $E \subset \mathbb{B}^d$, we write $\mathrm{meas}_\kappa^B(E) := \int_E W_\kappa^B(x)\, dx$. Finally, recall that $S_n^\delta(W_\kappa^B; f)$ denotes the (C, δ)-means for the orthogonal polynomial expansions with respect to the weight function W_κ^B on $\mathbb{B}^d$ that is given in (1.6).

4.1 Main result

Theorem 4.1.1: (i) *If* $\delta \geqslant \sigma_\kappa := \frac{d-1}{2} + |\kappa| - \kappa_{\min}$, *then for* $f \in L(W_\kappa^B; \mathbb{B}^d)$ *with* $\|f\|_{L(W_\kappa^B;\mathbb{B}^d)} = 1$,

$$\operatorname{meas}_\kappa^B \Big\{ x \in \mathbb{B}^d : \quad S_*^\delta(W_\kappa^B; f)(x) > \alpha \Big\} \leqslant C \frac{1}{\alpha}, \quad \forall \alpha > 0,$$

with $\alpha^{-1}|\log \alpha|$ *in place of* α^{-1} *in the case when* $\delta = \sigma_\kappa$ *and at least two of the* κ_i *are zero.*

(ii) *If* $\delta < \sigma_\kappa$, *then there exists a function* $f \in L(W_\kappa^B; \mathbb{B}^d)$ *of the form* $f(x) = f_0(|x_{j_0}|)$ *such that* $S_*^\delta(W_\kappa^B; f)(x) = \infty$ *for a.e.* $x \in \mathbb{B}^d$, *where* $1 \leqslant j_0 \leqslant d+1$ *is the integer such that* $\kappa_{j_0} = \kappa_{\min}$, *and* $x_{d+1} = \sqrt{1 - \|x\|^2}$.

Proof. Given $f \in L^p(W_\kappa^B; \mathbb{B}^d)$, define $\widetilde{f} : \mathbb{S}^d \to \mathbb{R}$ by $\widetilde{f}(X) = f(x)$ for $X = (x, x_{d+1}) \in \mathbb{S}^d$. Clearly, $\widetilde{f} \circ \phi = f$, where $\phi : \mathbb{B}^d \to \mathbb{S}_+^d$ is defined in (2.4), which, using (2.5), is measure-preserving in the sense that for each $\operatorname{meas}_\kappa(E) = c_\kappa \operatorname{meas}_\kappa^B(\phi^{-1}(E))$ for each $E \subset \mathbb{S}_+^d$. Using (2.5), we also have that $\widetilde{f} \in L^p(h_\kappa^2; \mathbb{S}^d)$ and $\|\widetilde{f}\|_{L^p(h_\kappa^2;\mathbb{S}^d)} = c\|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}$. Furthermore, by (2.7),

$$S_n^\delta(h_\kappa^2; \widetilde{f}, X) = S_n^\delta(W_\kappa^B; f, x), \quad X = (x, x_{d+1}) \in \mathbb{S}^d, \quad n = 0, 1, \cdots.$$

Thus, we may identify each function $f \in L^p(W_\kappa^B; \mathbb{B}^d)$ with a function $\widetilde{f} \in L^p(h_\kappa^2; \mathbb{S}^d)$ under the measure-preserving mapping ϕ, and such an identification preserves the Cesàro means of the corresponding weighted orthogonal polynomial expansions. Consequently, the stated conclusions of Theorem 4.1.1 follow directly from the corresponding results on the sphere $\mathbb{S}^d$ that are stated in Theorem 3.1.1. □

We can also deduce the following corollaries from the corresponding results on the sphere $\mathbb{S}^d$, using a similar approach.

4.2 Corollaries

Corollary 4.2.1: *In order that*

$$\lim_{n\to\infty} S_n^\delta(W_\kappa^B; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{B}^d$ for all $f \in L(W_\kappa^B; \mathbb{B}^d)$, it is sufficient and necessary that $\delta \geqslant \sigma_\kappa$.

Corollary 4.2.2: *If $1 < p < \infty$ and $\delta > 2\sigma_\kappa|\frac{1}{2} - \frac{1}{p}|$, then*

$$\|S_*^\delta(W_\kappa^B; f)\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \leqslant C_p \|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}. \tag{4.2}$$

In particular,

$$\|S_*^{\sigma_\kappa}(W_\kappa^B; f)\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \leqslant C_p \|f\|_{L^p(W_\kappa^B;\mathbb{B}^d)}, \quad 1 < p < \infty.$$

Corollary 4.2.3: *For $1 < p < \infty$, $\delta > 2\sigma_\kappa|\frac{1}{p} - \frac{1}{2}|$ and any sequence $\{n_j\}$ of positive integers,*

$$\left\| \left(\sum_{j=0}^\infty |S_{n_j}^\delta(W_\kappa^B; f_j)|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^B;\mathbb{B}^d)} \leqslant c \left\| \left(\sum_{j=0}^\infty |f_j|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^B;\mathbb{B}^d)}. \tag{4.3}$$

Corollary 4.2.4: *Let $\{\mu_j\}_{j=0}^\infty$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \leqslant c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \leqslant c < \infty$,

where n_0 *is the smallest integer* $\geqslant \sigma_\kappa + 1$. *Then* $\{\mu_j\}$ *defines an* $L^p(W_\kappa^B; \mathbb{B}^d)$, $1 < p < \infty$, *multiplier; that is,*

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(W_\kappa^B; f) \right\|_{L^p(W_\kappa^B; \mathbb{B}^d)} \leqslant c \|f\|_{L^p(W_\kappa^B; \mathbb{B}^d)}, \qquad 1 < p < \infty,$$

where c *is independent of* μ_j.

In the case when the weights are invariant under a general reflection group, Corollary 4.2.4 was proved in [6] under a stronger assumption that n_0 is the smallest integer $\geqslant \sigma_\kappa + 2 + \kappa_{\min}$.

Chapter 5

Weak estimates on the simplex

5.1 Main result

In this section, we will show how to deduce similar results on the simplex $\mathbb{T}^d$ from those on the ball $\mathbb{B}^d$. Recall that $S_n^\delta(W_\kappa^T; f)$ denotes the (C, δ)-means of the orthogonal polynomial expansions with respect to the weight function W_κ^T on $\mathbb{T}^d$ that is given in (1.7). Our argument in this section is based on the following proposition.

Proposition 5.1.1: *Let $\psi: \mathbb{B}^d \to \mathbb{T}^d$ be the mapping defined in* (2.10)*. Then for each $f \in L(W_\kappa^T; \mathbb{T}^d)$ and $\delta \geqslant 0$,*

$$S_*^\delta(W_\kappa^B; f \circ \psi, x) \sim S_*^\delta(W_\kappa^T; f, \psi(x)), \quad x \in \mathbb{B}^d.$$

Proof. For simplicity, we set $F = f \circ \psi$. Clearly, $F \in L(W_\kappa^B; \mathbb{B}^d)$ and $F(x\varepsilon) = F(x)$ for all $\varepsilon \in \mathbb{Z}_2^d$, and $x \in \mathbb{B}^d$. In particular, this implies that

$$\operatorname{proj}_{2n+1}(W_\kappa^B; F) = 0, \quad n = 0, 1, \cdots. \tag{5.1}$$

We further claim that

$$\operatorname{proj}_n(W_\kappa^T; f, \psi(x)) = \operatorname{proj}_{2n}(W_\kappa^B; F, x). \tag{5.2}$$

Indeed, using (2.2) and (2.6), we have

$$P_n(W_\kappa^B; x\varepsilon, y\varepsilon) = P_n(W_\kappa^B; x, y), \quad x, y \in \mathbb{B}^d, \quad \varepsilon \in \mathbb{Z}_2^d, \tag{5.3}$$

and hence, for each $\varepsilon \in \mathbb{Z}_2^d$,

$$\begin{aligned}
\operatorname{proj}_{2n}(W_\kappa^B; F, x\varepsilon) &= \int_{\mathbb{B}^d} F(y) P_{2n}(W_\kappa^B; x\varepsilon, y) W_\kappa^B(y)\, dy \\
&= \int_{\mathbb{B}^d} F(y\varepsilon) P_{2n}(W_\kappa^B; x\varepsilon, y\varepsilon) W_\kappa^B(y)\, dy \\
&= \int_{\mathbb{B}^d} F(y) P_{2n}(W_\kappa^B; x, y) W_\kappa^B(y)\, dy \\
&= \operatorname{proj}_{2n}(W_\kappa^B; F, x),
\end{aligned}$$

where we used the $\mathbb{Z}_2^d$-invariance of the measure $W_\kappa^B(x)dx$ in the second step, (5.3) and the fact that $F(\cdot\varepsilon) = F(\cdot)$ in the third step. (5.2) then follows by (2.12).

Next, we prove the inequality

$$S_*^\delta(W_\kappa^T; f, \psi(x)) \leqslant C S_*^\delta(W_\kappa^B; F, x), \quad x \in \mathbb{B}^d. \tag{5.4}$$

To this end, we set

$$A_x^\delta := \frac{\Gamma(x+\delta+1)}{\Gamma(x+1)} \frac{1}{\Gamma(\delta+1)}, \quad x \geqslant 0.$$

Using asymptotic expansions for ratios of gamma functions (see [2, p.616]), we have that for $\ell = 0, 1, \cdots,$

$$\Big(\frac{d}{dx}\Big)^{\ell} A_x^{\delta} = \frac{\Gamma(\delta+\ell)}{\delta(\Gamma(\delta))^2}(x+1)^{\delta-\ell} + O\Big((x+1)^{\delta-\ell-1}\Big), \quad x \geqslant 0. \tag{5.5}$$

Define the operator

$$\tau_{2n}^{\delta}(W_\kappa^B; g, x) = \sum_{j=0}^{2n} \Phi_n(j)\,\mathrm{proj}_j(W_\kappa^B; g, x), \quad g \in L(W_\kappa^B; \mathbb{B}^d),$$

where

$$\Phi_n(x) = \begin{cases} \frac{A_{n-x/2}^{\delta}}{A_n^{\delta}} - \frac{A_{2n-x}^{\delta}}{A_{2n}^{\delta}}, & 0 \leqslant x \leqslant 2n, \\ 0, & x > 2n. \end{cases}$$

Let ℓ be an integer such that $\delta - 1 < \ell \leqslant \delta$. It is easily seen from (5.5) that for $0 < x < 2n$,

$$|\Phi_n^{(m)}(x)| \leqslant C n^{-\delta}(n - \frac{x}{2} + 1)^{\delta-m-1}, \quad m = 0, 1, \cdots, \ell+1,$$

which, in turn, implies that

$$|\triangle^{\ell+1}\Phi_n(j)| \leqslant C n^{-\delta}(n - \frac{x}{2} + 1)^{\delta-\ell-2}, \quad 0 \leqslant j \leqslant 2n-1, \tag{5.6}$$

and $\triangle^m \Phi_n(2n) = 0$ for $m = 0, 1, \cdots, \ell - 1$. Thus, using summation by parts ℓ times, we obtain

$$|\tau_{2n}^{\delta}(W_\kappa^B; g)| \leqslant C \sum_{j=0}^{2n-1} |\Delta^{\ell+1}\Phi_n(j)| j^{\ell} |S_j^{\ell}(W_\kappa^B; g)| + C|\Delta^{\ell}\Phi_n(2n)| n^{\ell} |S_{2n}^{\ell}(W_\kappa^B; g)|,$$

which, using Lemma 3.6.1, is controlled by

$$Cn^{-\delta}\sum_{j=0}^{2n}(2n-j+1)^{\delta-\ell-2}j^{\ell}j^{\delta-\ell}S_*^\delta(W_\kappa^B;g)\leqslant CS_*^\delta(W_\kappa^B;g). \tag{5.7}$$

On the other hand, however, using (5.1) and (5.2), we have

$$\begin{aligned} S_n^\delta(W_\kappa^T;f,\psi(x)) &= (A_n^\delta)^{-1}\sum_{j=0}^{n}A_{n-j}^\delta\operatorname{proj}_{2j}(W_\kappa^B;F,x) \qquad (5.8)\\ &= (A_n^\delta)^{-1}\sum_{j=0}^{2n}A_{n-j/2}^\delta\operatorname{proj}_j(W_\kappa^B;F,x)\\ &= \sum_{j=0}^{2n}\Big[\frac{A_{n-j/2}^\delta}{A_n^\delta}-\frac{A_{2n-j}^\delta}{A_{2n}^\delta}\Big]\operatorname{proj}_j(W_\kappa^B;F,x)+S_{2n}^\delta(W_\kappa^B;F,x)\\ &= \tau_{2n}^\delta(W_\kappa^B;F,x)+S_{2n}^\delta(W_\kappa^B;F,x). \qquad (5.9)\end{aligned}$$

Thus, combing (5.7) with (5.9), we deduce the estimate (5.4).

Finally, we show the converse inequality

$$S_*^\delta(W_\kappa^B;F,x)\leqslant CS_*^\delta(W_\kappa^T;f,\psi(x)),\quad x\in\mathbb{B}^d. \tag{5.10}$$

The proof is similar to that of (5.4), and we sketch it as follows.

Let m be the integer such that $2m\leqslant n<2m+1$. Then by (5.1) and (5.2),

$$\begin{aligned} S_n^\delta(W_\kappa^B;F,x) &= \sum_{j=0}^{m}\frac{A_{n-2j}^\delta}{A_n^\delta}\operatorname{proj}_{2j}(W_\kappa^B;F,x)=\sum_{j=0}^{m}\frac{A_{n-2j}^\delta}{A_n^\delta}\operatorname{proj}_j(W_\kappa^T;f,\psi(x))\\ &= \sum_{j=0}^{m}\mu_j\operatorname{proj}_j(W_\kappa^T;f,\psi(x))+S_m^\delta(W_\kappa^T;f,\psi(x)),\end{aligned}$$

where

$$\mu_j = \begin{cases} \frac{A_{n-2j}^\delta}{A_n^\delta} - \frac{A_{m-j}^\delta}{A_m^\delta}, & 0\leqslant j\leqslant m, \\ 0, & j > m. \end{cases}$$

Using (5.5) and similar to the proof of (5.7), one can easily verify that for $0\leqslant j\leqslant m$,

$$|\Delta^i \mu_j|\leqslant Cm^{-\delta}(m-j+1)^{\delta-i-1}, \quad i = 0, 1, \cdots. \tag{5.11}$$

Let ℓ be an integer such that $\delta - 1 < \ell\leqslant\delta$. Summation by parts ℓ times shows that

$$\Big|\sum_{j=0}^{m} \mu_j \operatorname{proj}_j(W_\kappa^T; f, \psi(x))\Big|\leqslant C\sum_{j=0}^{m-\ell}|\Delta^{\ell+1}\mu_j|(j+1)^\ell|S_j^\ell(W_\kappa^T; f, \psi(x))|$$
$$+ Cm^\ell \max_{0\leqslant i\leqslant\ell}|\Delta^i \mu_{m-i}||S_{m-i}^\ell(W_\kappa^T; f, \psi(x))|,$$

which, using Lemma 3.6.1, and (5.11), is controlled by $CS_*^\delta(W_\kappa^T; f, \psi(x))$. The desired inequality (5.10) then follows. □

Recall that $\kappa_{\min}$, $|\kappa|$ and σ_κ are defined in (4.1). For a set $E \subset \mathbb{T}^d$, we write $\operatorname{meas}_\kappa^T(E) := \int_E W_\kappa^T(x)\, dx$. The following result is a simple consequence of Proposition 5.1.1, Theorem 4.1.1, and (2.11).

Theorem 5.1.2: (i) *If $\delta\geqslant\sigma_\kappa := \frac{d-1}{2} + |\kappa|-\kappa_{\min}$, then for $f \in L(W_\kappa^T; \mathbb{T}^d)$ with $\|f\|_{L(W_\kappa^T;\mathbb{T}^d)}= 1$,*

$$\operatorname{meas}_\kappa^T\Big\{x \in \mathbb{T}^d : \quad S_*^\delta(W_\kappa^T; f)(x) > \alpha\Big\}\leqslant C\frac{1}{\alpha}, \quad \forall\alpha > 0,$$

with $\alpha^{-1}|\log\alpha|$ in place of α^{-1} in the case when $\delta = \sigma_\kappa$ and at least two of the κ_i are zero.

(ii) *If* $\delta < \sigma_\kappa$*, then there exists a function* $f \in L(W_\kappa^T; \mathbb{T}^d)$ *of the form* $f(x) = f_0(|x_{j_0}|)$ *such that* $S_*^\delta(W_\kappa^T; f)(x) = \infty$ *for a.e.* $x \in \mathbb{T}^d$*, where* $1 \leqslant j_0 \leqslant d+1$ *is the integer such that* $\kappa_{j_0} = \kappa_{\min}$*, and* $x_{d+1} = \sqrt{1-|x|}$.

5.2 Corollaries

As a consequence of Theorem 5.1.2, we obtain

Corollary 5.2.1: *In order that*

$$\lim_{n\to\infty} S_n^\delta(W_\kappa^T; f)(x) = f(x)$$

holds almost everywhere on $\mathbb{T}^d$ *for all* $f \in L(W_\kappa^T; \mathbb{T}^d)$*, it is sufficient and necessary that* $\delta \geqslant \sigma_\kappa$.

Corollary 5.2.2: *If* $1 < p < \infty$ *and* $\delta > 2\sigma_\kappa |\frac{1}{2} - \frac{1}{p}|$*, then*

$$\|S_*^\delta(W_\kappa^T; f)\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant C_p \|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}. \tag{5.12}$$

In particular,

$$\|S_*^{\sigma_\kappa}(W_\kappa^T; f)\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant C_p \|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}, \quad 1 < p < \infty.$$

Corollary 5.2.3: *For* $1 < p < \infty$, $\delta > 2\sigma_\kappa |\frac{1}{p} - \frac{1}{2}|$ *and any sequence* $\{n_j\}$ *of positive integers,*

$$\left\| \left(\sum_{j=0}^{\infty} |S_{n_j}^\delta(W_\kappa^T; f_j)|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant c \left\| \left(\sum_{j=0}^{\infty} |f_j|^2 \right)^{1/2} \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)}. \tag{5.13}$$

Using Corollary 4.2.3, and following the approach of [4], we have

Corollary 5.2.4: *Let $\{\mu_j\}_{j=0}^{\infty}$ be a sequence of complex numbers that satisfies*

(i) $\sup_j |\mu_j| \leqslant c < \infty$,

(ii) $\sup_j 2^{j(n_0-1)} \sum_{l=2^j}^{2^{j+1}} |\Delta^{n_0} u_l| \leqslant c < \infty$,

where n_0 is the smallest integer $\geqslant \sigma_\kappa + 1$. Then $\{\mu_j\}$ defines an $L^p(W_\kappa^T; \mathbb{T}^d)$, $1 < p < \infty$, multiplier; that is,

$$\left\| \sum_{j=0}^{\infty} \mu_j \operatorname{proj}_j(W_\kappa^T; f) \right\|_{L^p(W_\kappa^T;\mathbb{T}^d)} \leqslant c \|f\|_{L^p(W_\kappa^T;\mathbb{T}^d)}, \qquad 1 < p < \infty,$$

where c is independent of μ_j.

Corollary 5.2.4 was proved in [6] under a stronger assumption that n_0 is the smallest integer $\geqslant \sigma_\kappa + 2 + \kappa_{\min}$.

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