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Time Machines In the Space-Time Of Two Parallel Moving Cosmic Strings

By

Bahman K. Darian ©

A dissertation

presented to the Faculty of Graduate Studies and Research

in partial fulfilment of the requirements for the degree

of

Master of Science

in

Theoretical Physics

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Bahman Darian

Bahman K. Darian
Department of Physics
University of Alberta
Edmonton, Alberta
T6G 2J1

Date: 23 Sept 1992

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Time Machines In the Space-Time Of Two Parallel Moving Cosmic Strings " submitted by Bahman K. Darian in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics

Don N. Page
Professor D. Page, Supervisor

B.A. Campbell
Professor B. Campbell

M. Razavy
Professor M. Razavy

H.P. Künzle
Professor H. Künzle

Date: 23 Sept 1992

to

my Mother

and

my Father.

Abstract

It has long been known that many solutions of the Einstein field equations possess causal anomalies in the form of closed time-like curves (CTCs). However the matter fields in these space-times are such that time travel is extremely difficult if not impossible. In the 1980s K. Thorne gave a boost to this idea by suggesting that since certain quantum mechanical phenomena allow the violation of weak energy condition (WEC), an advanced civilization might be able to construct worm holes suitable for time travel. In 1991 R. Gott showed that CTCs are created in the space-time of two parallel moving cosmic strings (Gott space-time) which does not require the violation of the weak energy condition.

However, the asymptotic behavior of Gott space-time is completely different from the other causality violating solutions of Einstein field equations. In this thesis in the course of review of some of the work done on this space-time during the last year we see why despite causality violation Gott space-time is singularity free. A theorem related to asymptotic flatness due to F. Tipler and a theorem on topology change in causality violating space-times due to R. Geroch are mentioned in §3. The result agrees with the previous findings on the asymptotic behavior of Gott space-time; namely, Gott space-time is not asymptotically flat.

In §4 and §5 we see how straight cosmic strings represented by point particles correspond to the conjugacy classes of Poincaré group elements. It turns out that strings in Gott space-time are represented by those elements of the Poincaré group belonging to the conjugacy classes of pure boosts. What is done is essentially a more accurate formulation of what S. Deser *et al.* have done. We also review how these coordinate identifications can be represented by the double cover of $SO(2,1)$, namely $SU(1,1)$, based on the work done by S. Carroll *et al.* We reach the conclusion that $SU(1,1)$ elements representing coordinate identifications in Gott space-time

correspond to those points in (2+1)D anti-de Sitter space-time not reached by any geodesic from the origin.

Acknowledgements

I wish to thank Dr. Werner Israel for his moral support and Curt Cutler, Sean Carroll, Patrick Brady and Dr. Alex Lyons for helpful discussions. I specially want to thank Dr. Don Page, my advisor, for relentless help, extreme patience and inspiring kindness whose startling resourcefulness amazes the expert and non-expert alike.

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μ .... the line mass density of a string,

α .... the deficit angle of a straight cosmic string,

v .... velocity of string with respect to the lab frame,

u .... the velocity of string (1) with respect to the frame of string (2),

r .... generalized affine parameter

k .... extrinsic curvature,

GUT .... Grand Unified Theories,

(M, g) .... space-time M with metric g,

∂M .... boundary of space-time (M, g),

s.t. .... such that,

K^a .... a null vector,

$I^+(p)$ .... chronological future of p,

$I^-(p)$ .... chronological past of p,

$J^+(p)$ .... causal future of p,

$J^-(p)$ .... causal past of p,

$\mathcal{I}^+$ .... future null infinity,

$\mathcal{I}^-$ .... past null infinity,

$\mathcal{I}$ .... the same as ∂M,

S^n .... topological n-sphere,

S .... an achronal set S,

inf .... infimum or greatest lower bound,

$\lambda : R \rightarrow M$ .... an at least C^0 curve in M,

$U(p), U'(p), o(p)$ .... an open neighborhood of a point p,

$D^+(S)$ .... future Cauchy development of a surface S,

$H^+(S)$ .... future Cauchy horizon of S,

η, γ, λ .... a null generator of $H^+(S)$,

$R_{\alpha\beta\gamma\delta}$ .... Riemann tensor,

H^2 .... the half 2-D plane,

A .... the causality violating region,

$\dot{A}$ .... the boundary of A,

$Tr(\Omega)$ .... trace of a matrix Ω,

$\rightarrow$ .... asymptotic limit,

$\sim$ .... equivalence relation,

$\subset$ .... subset of a set,

$\cup$ .... union of two sets,

$\cap$ .... intersection of two sets.

Chapter 1

Introduction

In 1905 special relativity invalidated the Newtonian concept of absolute time and replaced the Galilean transformation laws with the Lorentzian ones. Force at a distance was replaced by interaction through a field. The causal structure of space-time was changed and the causal relations between every two points in space-time was determined by their respective null cones. Two points are causally related if one of them lies inside or on the future null cone of the other point. But despite time dilation and Lorentz contraction, Lorentz transformations respect the causal order of events (as long as tachyonic speed is ruled out and super luminal sound waves are not allowed). A glance at the Minkowski diagram reveals that Lorentz transformations are constructed in a way to preserve the causal order of events, even though the temporal order of events which are not causally related is frame dependent. This is a fortunate event because any change in the temporal order of causally related events which is formulated in terms of the creation of closed time-like curves (CTCs) would make scientific prediction impossible and immediately give rise to questions of self consistency of physical phenomena. For example what would happen if one travels back in time and kills one's parents? Questions of this kind need not cause panic as long as special relativity is concerned.

In 1915, very soon after the introduction of general relativity, it became apparent that causal order is no longer a sacred law and massive rotating bodies might deform the geometry of space-time in a way that causality violation is permitted (Kerr geometry, Gödel universe, the Van-Stockum infinite rotating cylinder [1], S.

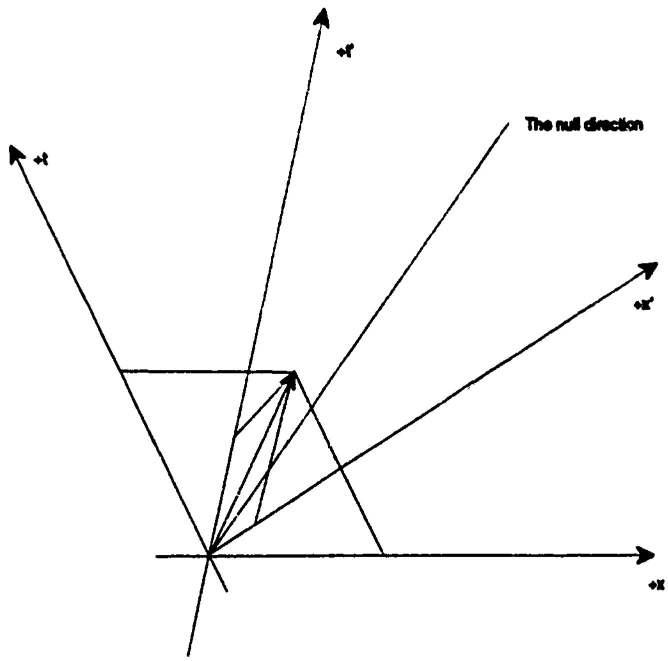

Figure 1.0.1: All observers related by a Lorentz transformation agree on the temporal order of events related by a time-like vector.

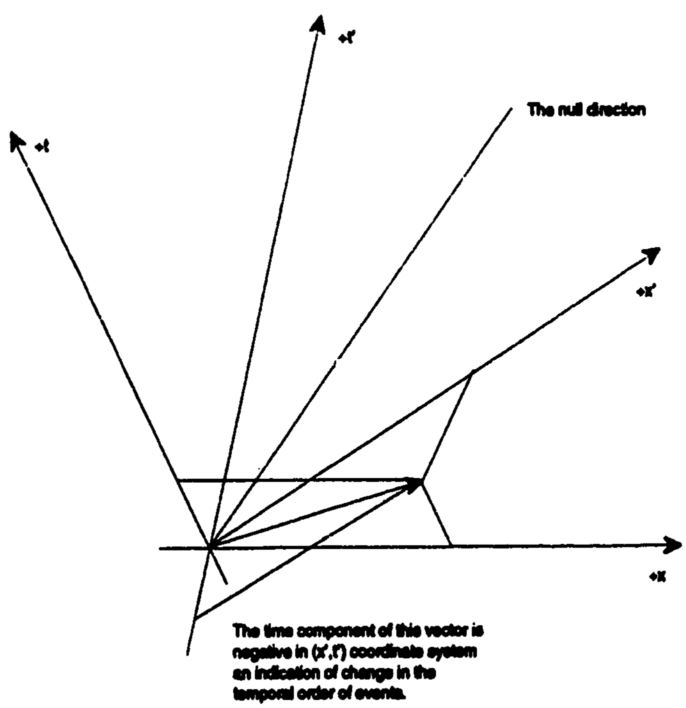

Figure 1.0.2: The temporal order of events related by a space-like vector is observer dependent.

Deser *et. als* spinning cosmons[11]). What actually happens is that the rotation of large masses causes the light cones to tip over and the closed orbits of Killing vectors which in large distances are space-like become time-like and CTCs appear. However all these space-times have features which make the time travel impossible or physically unrealistic for any observer moving along CTCs. For example Kerr space-time is not singularity free and an observer traveling on CTCs lie behind the event horizon. Causality violation in Gödel's universe is total and every point lies on a CTC. In a simply connected space-time, this implies the absence of any achronal hypersurfaces without boundary and the absence of a cosmic time function which increases along any future directed time-like curve[2]. Thorne wormhole space-time requires topology change but since it is singularity free, the weak energy condition (WEC) is necessarily violated [3]. The WEC violation also contributes to the classical stability of the Cauchy horizon by giving rise to an optical divergence effect.

In 1991, after many years working on the gravitational properties of cosmic strings, R. Gott was inspired to investigate the causality violation in the space-time of moving cosmic strings [4]. He suggested that since straight cosmic strings distort the geometry of space-time in such a way that even at large distances space-time is not Minkowskian, a particular movement or rotation of cosmic strings should result in creation of CTCs.

He achieved this goal by combining what gives rise to CTC creation in Thorne wormhole space-time, namely, the high relative velocity of two wormhole mouths, with gravitational lensing effects of straight cosmic strings. Gravitational lensing causes an observer to see a double image of a quasar behind a cosmic string. One of the rays is possibly time delayed with respect to the other one (time delay occurs when the string is not exactly between the observer and the quasar). When leaving a point A moving at nearly the speed of light, a rocket ship can reach the other side of the string before a light ray leaving A simultaneously with the rocket ship. The addition of a cosmic string parallel to the first one moving with high velocity, enables the rocket ship to travel a large distance in a very short time and if viewed from an appropriate coordinate system actually go back in time.

The discovery of CTCs in the space-time of two moving parallel cosmic strings (parallely moving parallel cosmic strings) immediately received the attention of experts on cosmology and global structure. This interest was due to several facts:

1- Unlike the case with the CTC containing Thorne's wormhole space-time, there is no violation of any of the energy conditions in Gott space-time. Therefore we do not have to rely on any phenomena based on the intrinsic quantum mechanical nature of fields, like the Casimir effect, to justify WEC violation.

2- There are apparently no event horizons or singularities which would make space-like and future null infinities causally well behaved, thereby preventing any self-inconsistent solutions of the local physical laws. Thus the question arises how singular Gott space-time is?

3- Since the metric nearby finite strings can be approximated by the metric in the space-time of straight cosmic strings, the question arose whether segments of finite strings or loops passing each other with high velocities can give rise to CTCs or not.

In this thesis we exclusively deal with CTCs in the space-time of two moving parallel cosmic strings and see how creation of CTCs in Gott space-time is solely due to the global properties of conical space. In §2 we give a precise but concise review of the definitions and concepts vital to a good understanding of those features of conical geometry and causal structure which allow the creation of CTCs. We also state and give an intuitive proof of F. Tipler's singularity theorem [1] which has been used (rather carelessly) by some authors to explain whether and why Gott space-time is singularity free. In §3.1 after a discussion on how R. Gott proves that there are in fact CTCs in the space-time of two moving parallel cosmic strings, we examine why F. Tipler's theorem does not require Gott space-time to hold singularities. In doing so extensive use of discoveries made by C. Cutler [5] on the structure of the boundary of the causality violating region are used. We also address the possibility of the existence of blue shift singularities in Gott space-time.

In §4 we make use of a methodology developed by S. Deser *et al.* [6] to see that cosmic strings in fact correspond to the conjugacy classes of Poincaré group elements or $INSO(2,1)$ (inhomogeneous Lorentz group on a plane). We examine exactly which invariant quantities of conjugacy classes of this group correspond to what physical properties of Gott space-time. It is seen that the configuration of sufficiently fast moving point particles in (2+1)D, called Gott particles, (straight cosmic string in (3+1)D) is not the only particle configuration giving rise to CTCs. S. Deser *et al.* claim (in a very concise way incomprehensible to the reader not used to their methodology) that Gott strings are physically unrealistic. In this chapter this concept is made clear.

In §5 in parallel to §4, first we give a review of the work done by S. Carroll *et al.* [7] in which they use $SU(1,1)$, the pseudo-unitary representation of Lorentz group in (2+1)D, to represent coordinate identifications in Gott space-time. In doing so first $SU(1,1)$ elements are used to investigate the decay of a stationary particle into two. Then the study of Gott space-time shows $SU(1,1)$ elements representing Gott space-time are not obtained by exponentiating Lie algebra elements. It is later seen that this is actually the same property of anti-de Sitter space-time in which there are points not reached by any geodesic from the origin. These points characterize Gott space-time. We finally try to answer the question whether this property of $SU(1,1)$ is of any physical significance or merely a property of that group irrelevant to Gott space-time.

Chapter 2

A Review of Conical Space, Causal Structure

2.1 Conical Space, Gauss-Bonnet Theorem

Before launching into rather slightly intricate features of CTCs in the space-time of two moving parallel cosmic strings, it is essential to have a good understanding of conical singularities and those generic features of conical space-time which give rise to the existence of causality violating regions.

2.1.1 Definitions

A conical singularity is a non-removable quasi-regular singular point (non-removable in the sense that space-time is not extendible beyond the singular point) in the space-time of a straight cosmic string. Such singularities arise due to a pathology in the topological structure of globally inextendible space-times. Excising these singularities leave a manifold with $S^1 \times R^3$ $(R^4 - R^2)$ topology. As it turns out (see §2.2) Gott space-time can actually be constructed from such excised space-times.

Conical singularities are a solution to $G_{\mu\nu} = 8\pi T_{\mu\nu}$ with

$$T^{\nu}_{\mu} = \mu \,\mathrm{diag}(1,0,0,1)\delta^2(r), \tag{2.1}$$

in which μ is in units of unit Plank mass per unit Plank length $(1.35 \times 10^{28}\mathrm{gcm}^{-1})$ is both mass density and tension along the string[1]. The distance is defined as,

$$ds^2 = dr^2 + dz^2 + (1-4\mu)^2 r^2 d\phi^2 - dt^2, \tag{2.2}$$

in a particular gauge. One might ask how the manifestly flat metric (2.2) gives the energy-momentum tensor (2.1), which is manifestly non-zero. The answer is that (2.1) was derived using a metric satisfying Einstein's equation with the above $T_{\mu\nu}$ by a transformation not valid at $r = 0$. With the transformation $\phi' = (1-4\mu)\phi$, (2.2) can be written in the form,

$$ds^2 = dr^2 + dz^2 + r^2 d\phi'^2 - dt^2, \tag{2.3}$$

with $0 < \phi' \leq 2\pi(1-4\mu)$. From (2.1) it is obvious that the geometry is invariant under the action of any element of inhomogeneous Lorentz group in z direction (parallel to the string).

The embedding of a t =const, z =const surface in R^4 is a cone (Fig.2.1.1). Due to the mentioned symmetry this cone is characteristic of all features of straight cosmic strings in (3+1)D. For this reason from now on space-time of a straight cosmic string is shown by a point particle in (2+1)D. Since as $r \to 0$ the ratio of the circumference of a circle centered at the origin to r is less than 2π, $r = 0$ is a non-removable singular point in this geometry. In other words at $r = 0$ the cone is not locally diffeomorphic to R^2.

It is a quasi-regular or non-p-p singularity (non-parallel propagated) since the components of Riemann tensor remain bounded (in fact identically zero) in the parallel propagated orthonormal frame of an observer approaching $r = 0$ [8][2].

---

[1] $T_0^0 = T_z^z$ means that the work done against this tension in stretching the string a unit length equals energy per unit length. The string is "conformally stretched"[8].

[2] It should be added that the quantum mechanical description of such a singularity might give a completely different result. The vacuum stress energy tensor might diverge at a quasi-regular singularity suggesting that in a self consistent calculation including quantum effect, these features would be replaced by a curvature singularity[10]

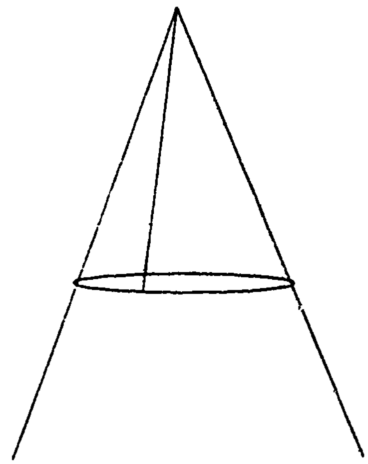

Figure 2.1.1: The embedding of t =const in R^3 is a cone. The ratio of the circumference of a circle to its radius is less than 2π as $r \to 0$.

A conical geometry can be shown in R^2 by cutting and identifying the edges of a wedge with angle $8\pi\mu$ (Fig.2.1.2). For this reason $8\pi\mu$ is called the deficit angle of the conical geometry. These jumps which later appear many times in our treatment of CTCs are not an indication of the pathological behavior of the space-time at the wedge, but they are due to the bad choice of coordinate system. In other words space-time is perfectly well behaved across the wedge and where the wedge is located is completely arbitrary as long as it does not interfere with the boundary conditions while patching different coordinate systems.

The most striking features of gravity in (2+1)D stem from the identity,

$$R_{\mu\nu\alpha\beta} \equiv -\epsilon_{\mu\nu\rho}\epsilon_{\alpha\beta\gamma}G^{\rho\gamma}, \tag{2.4}$$

equivalent to $C_{\alpha\beta\gamma\rho} \equiv 0$ (Weyl tensor vanishes), valid only in (2+1)D. This identity is an indication of the absence of curvature outside matter and shows there are no

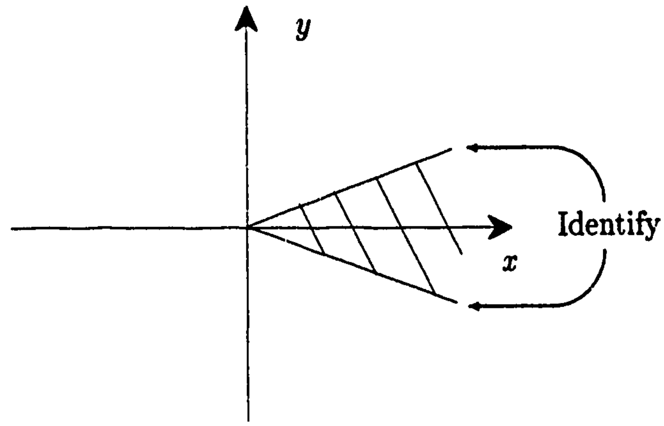

Figure 2.1.2: Conical singularity is shown by identifying the cut edges of a flat plane

dynamical degrees of freedom where $T_{\alpha\beta} = 0$, which is also obvious from the flat metric[3] (2.3). All effects of localized sources in (2+1)D are on the global geometry, which is fixed by the singularities of the world lines of the particles, arbitrary straight lines. This means in particular that the conserved quantities, total energy, momentum and angular momentum are related to the topological invariants. This property can be expressed more elegantly by the Gauss-Bonnet theorem (a generalized form of Stokes' theorem) which says $\sqrt{g}G^0_0$, in which g is the 2-metric, is the Euler invariant density, a total 2-divergence[11].

[3]For this reason (2+1)D gravity is sometimes called "gravity without curvature".

2.2 Gauss-Bonnet Theorem, Conical Geometry

A good understanding of the nature of topological invariants mentioned in the previous section is of vital importance to the justification of some of the steps taken in §4. Let D be a 2-manifold, locally homeomorphic to H^2, with intrinsic curvature R, boundary $\dot{D}$ (Fig.2.1.2)) and let $k = \frac{\delta n_a}{\delta x} e^a$ be the extrinsic curvature of $\dot{D}$. For a 2-manifold, $\dot{D}$ is a line and k is a scalar. Therefore $\dot{D}$ has no intrinsic curvature [4]. Then a corollary to the Gauss-Bonnet theorem says,

$$2\int_{\dot{D}} kdl + \int_{D} \sqrt{g} R ds = 4\pi\chi \tag{2.5}$$

where the $\chi = 1$ is the Euler number of this 2-surface. From (2.2) it is easily seen that for a cone with smooth apex the first integral on the right hand side is just $4\pi(1-4\mu)$ and is independent of the exact form of $\sqrt{g}R$ and depends only on the integral of this quantity over the 2-surface (Fig.2.2.1). But from Einstein's equations in (2+1)D,

$$16\pi\sqrt{g}T^0_0 = 2\sqrt{g}G^0_0 = -\sqrt{g}R, \tag{2.6}$$

therefore as long as $T^0_0 = 0$ for $r > \rho$, in which ρ is a certain radial distance on the cone, the metric (2.2) truly represents the geometry for $r > R$. The energy of particles (their total mass) determines the topological and intrinsic geometrical properties of a 2-surface in (2+1)D gravity, whereas the momentum and angular momentum related to the lapse function and shift vectors of these particles determine how this 2-surface evolves in (2+1)D space-time . For $\mu = 1/4$ the integration on $\dot{D}$ vanishes. In this case the space outside the source is cylindrical. For $\mu > 1/4$ the 2-surface is closed. For a closed surface with S^2 topology $\chi = 2\pi$. Thus (2.6) requires $\mu = 1/2$, in which case space-time is finite (its space-like cross sections are compact). Therefore we can have either $\mu \leq 1/4$ or $\mu = 1/2$. If $1/4 < \mu < 1/2$ there exist other mass distributions so that the total mass adds up to 1/2 [8]. One might think that causality violation

[4] The name intrinsic here follows from the nomenclature for higher dimensions.

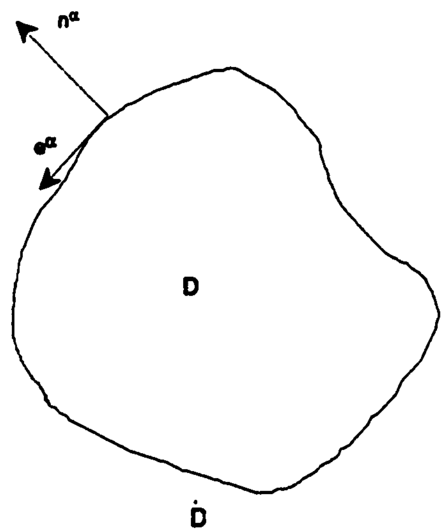

Figure 2.2.1: The integration of the intrinsic curvature on D is related to the line integral of the extrinsic curvature on $\dot{D}$.

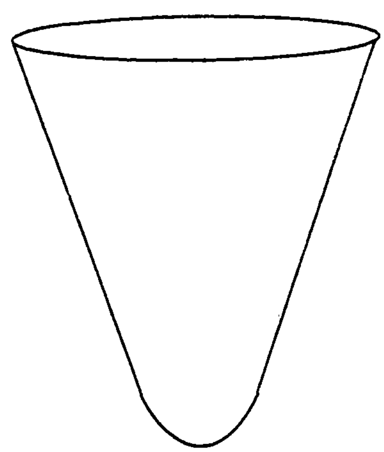

Figure 2.2.2: The apex can be smoothed out in a way that the boundary conditions are satisfied.

in Gott space-time is due to the quasi-regular singularity at $r = 0$ and since cosmic strings predicted by the GUT theories are of finite cross section, CTCs in Gott space are unphysical and this way the chronology is protected. But as it turns out there is an open neighborhood around each string in which there are no CTCs. In this case the apex can be smoothed out by replacing the point particle with a particle of finite radius satisfying boundary conditions (Fig.2.2.2). Therefore CTCs in Gott space are the generic properties of invariants described in the previous paragraph and not the singularity at $r = 0$.

2.3 Causal Structure and CTCs

It is essential for the treatment of CTCs in Gott space-time to have a basic understanding of the causal structure and for the sake of completeness a lemma from singularity theorems will be given.

2.3.1 Time Orientability

A space-time manifold M with metric g, denoted by (M, g), is time-orientable when a continuous choice of future light cone can be made for every point on this manifold [12], or, equivalently, when there is a continuous time-like vector field on the manifold [13](p.189). Two following examples clarify the meaning of this concept[5].

Example 1: The cylindrical Minkowski space [1] is an orientable space-time constructed from Minkowski space (R^4 topology) by identifying the hypersurfaces $t = 0$ and $t = 1$. The resulting space-time is flat but the topology is now $S^1 \times R^3$. The

[5]Here there is a fine distinction between the light cone and the null cone. Sometimes some authors use the term *light cone* to designate the subset of M generated by null geodesics from $p \in M$. They use the term *null cone* to designate what we call light cone, namely, the set of null vectors in the tangent space at p.

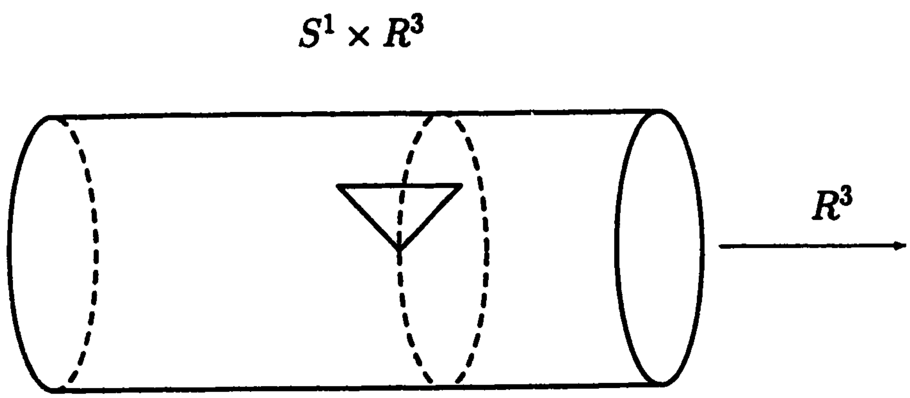

Figure 2.3.1: The cylindrical Minkowski space. The metric in $S^1 \times R^3$ defined in a way that the time coordinate is periodic. The future light cone of a point $p \in M$ is M.

circles (x, y, z) =const are closed time-like geodesics (Fig.2.3.1). Causality violation is total. The future and past of every light cone is identified with M.

Example 2: Klein-Minkowski space (Fig.2.3.2) is a non-orientable space-time formed by placing a Minkowski metric on $R^2 \times K$ (where K is the Klein non-orientable surface). It is not possible to make continuous choice of future light cones everywhere. An observer moving into his local future light cone along the CTC $pqrp$ will leave p moving forward in time but return to p going backward in time as measured by his initial light cone.

Since almost all interesting examples of causality violation occur in time orientable space-time and non-orientability makes a clear description of the causal structure impossible, from now on we assume (M, g) to be time orientable, unless stated otherwise (moreover a non-time orientable space-time becomes time orientable by going to the covering space). Given this condition the notion of future and past is determined arbitrarily at the beginning once and for all $p \in M$.

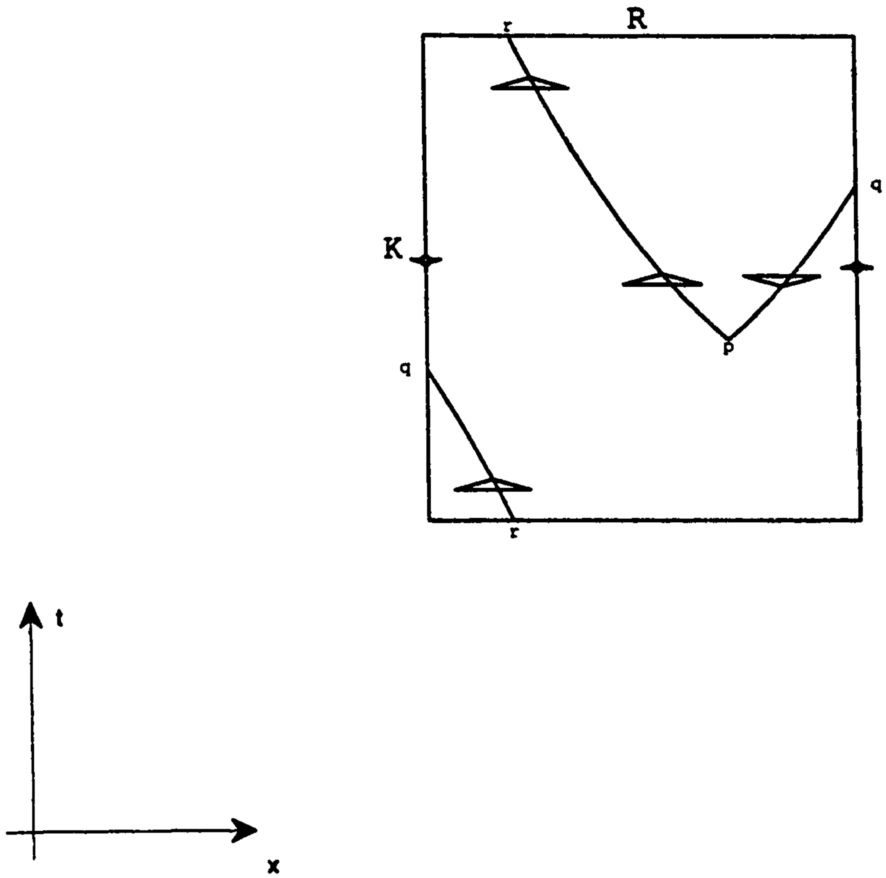

Figure 2.3.2: Klein-Minkowski space. The top and bottom of the above rectangle are identified to get a cylinder, then the other two sides are identified with a twist. The metric on the resulting Klein bottle K is induced from the covering Minkowski space, as shown. A closed time-like curve $pqrp$ is pictured together with its future light cones. The future light cone points up when $pqrp$ leaves p, but down where $pqrp$ returns tc p; the space-time is not time orientable.

2.3.2 Causal Structure

The chronological future of a point p, $I^+(p)$ is defined as the set of events that can be reached by a future directed time-like curve starting from p (future directed with respect to the future light cone).

$$I^{+(-)}(p) = \left\{ q \in M \;\middle|\; \begin{array}{l} \text{there exists a future (past) directed} \\ \text{time-like curve } \lambda(t) \text{ with } \lambda(0) = p(q) \\ \text{and } \lambda(1) = q(p) \end{array} \right\} \tag{2.7}$$

Likewise the causal future (past) of a point p, $J^{+(-)}(p)$ is the same as above, with the time-like curves replaced with causal curves. $I^{+(-)}(p)$ is open and in Minkowski space it is the interior of the future light cone of p. Likewise, $J^{+(-)}(p)$ is the future (past) light cone of p, and its interior.

An achronal set is a set no two points of which are chronologically related or

$$\nexists\, p, q \in S \text{ s.t. } p \in I^{\pm}(q), \tag{2.8}$$

For a closed and achronal set S, *edge* of S is defined as the set of points $p \in S$ such that every open neighborhood o of p contains a point $q \in I^+(p)$, a point $r \in I^-(p)$ and a time-like curve λ from r to q which does not intersect S.

For a time-like or null curve $\lambda(t)$ we say that $p \in M$ is a *future end point* of λ if for every neighborhood o of p there exists a t_0 such that $\lambda(t) \in o(p)$ for all $t > t_0$ (for M with Hausdorff property there can only be one future and one past end point). *An inextendible curve* $\lambda(t)$ in M is a curve which has no end point in M.

Future (past) Cauchy development or domain of dependence of an achronal set S;

$$D^{+(-)}(S) = \forall p \in M \left\{ \begin{array}{l} \text{every past (future) directed inextendible} \\ \text{non-space-like curve from } p \text{ intersects } S, \end{array} \right\}, \tag{2.9}$$

The definition of $\tilde{D}^{+(-)}$ is the same as above with non-space-like replaced with time-like.

The boundary of future (past) development of S is the boundary of total future (past) predictability, called the future (past) Cauchy horizon (Fig.2.3.3) and defined as

$$H^{+(-)}(S) = \left\{\forall p \in \tilde{D}^{+(-)}(S) \middle| I^{+(-)}(p) \cap \tilde{D}^{+(-)}(S) = \emptyset\right\} \qquad (2.10)$$

Lemma 1 *If S is closed, $H^+(S)$ is an achronal set generated by null geodesics which have no past end point or their past end point is on the edge of S.*

If edge of $(S) = \emptyset$, S is called a *partial Cauchy surface* (Fig.2.3.4)and if $H^{+(-)}(S) = \emptyset$ or in other words $D^+(S) \cup D^+(S) = M$, S is a *Cauchy surface.* In this case M is called *globally hyperbolic* and a striking feature of M is that knowing the induced metric, its normal derivative and the matter fields on a Cauchy surface uniquely determines M. M is fully deterministic.

A *bad partial Cauchy surface* is a partial Cauchy surface which is asymptotically null. The Penrose diagram of Minkowski space-time with a bad partial Cauchy surface is depicted in (Fig-2.3.5). These partial Cauchy surfaces are bad in a sense that a unique complete determination of a matter field on these surface does not uniquely determine that field throughout the space-time even though such space-time is free of any singularities, *i.e.* extra information can come from or sink into space-like infinity. They give no information on the causal behavior nearby space-like infinity.

A space-time $(\tilde{M}, \tilde{g})$ is called *asymptotically simple* if there is a conformal isometric imbedding of $f : (\tilde{M}, \tilde{g}) \to (\tilde{M}', \tilde{g}')$ and an at least C^3 function $\Omega > 0$ such that

$$\begin{aligned} \tilde{g}' &= \Omega^2 \tilde{g} \quad , \\ \Omega &= 0 \quad \textit{and} \;\; d\Omega \neq 0 \;\; \text{on} \;\; \partial f(\tilde{M}) \equiv \mathcal{I} \subset \tilde{M}'. \end{aligned} \tag{2.11}$$

$(\tilde{M}, \tilde{g})$ should also be void of any singularities or event horizons, therefore all null geodesics end on $\partial f(M)$ both in the past and future directions. *Asymptotic emptiness*, $R_{ab} = 0$ on $\mathcal{I}$ and Ω being C^3, guarantees $\partial f(\tilde{M})$ to be composed of two null surfaces $\mathcal{I}^-$ and $\mathcal{I}^+$, past and future null infinities respectively. A space-time (M, g) is called *asymptotically flat* if it contains an open neighborhood isometric to an open neighborhood of $\mathcal{I}^+$ and $\mathcal{I}^-$ of an asymptotically simple and empty space-time $(\tilde{M}, \tilde{g})$ [13](p.222). Roughly speaking, in Gott space-time $(\tilde{M}, \tilde{g})$ is Minkowski space-time and $(\tilde{M}', \tilde{g}')$ is the Einstein static universe[6].

2.3.3 Closed Time-Like Curves (CTCs)

A closed time-like curve is any C^0 time-like $\lambda : [0,1] \to (M, g)$ s.t. $\lambda(0) = \lambda(1)$. In the language of causality definitions, if

$$\exists (pq) \in M \text{ s.t. } p \in I^+(q) \text{ and } q \in I^+(p), \tag{2.12}$$

then p and q are connected by a CTC. The *causality (chronology) condition* is satisfied in (M, g) if there are no closed non-space-like (time-like) curves in (M, g). A space-time (M, g) is *strongly causal* if $\forall p \in (M, g)$, $\exists o(p)$ such that every causal curve through p, intersects $o(p)$ only once. We show the regions containing CTCs by A and its boundary by $\dot{A}$ respectively.

[6]The exact definition of asymptotic flatness has slightly changed over the years. The definition used here is basically the definition first used by R. Penrose [14](p.184) and later adopted by S. Hawking [13](p.310). F. Tipler defines asymptotic flatness as being exactly equal to *weakly asymptotically simple and empty* [1](p.49) even though asymptotic flatness used by R. Penrose has a slightly broader meaning. Also see [12](p.276).

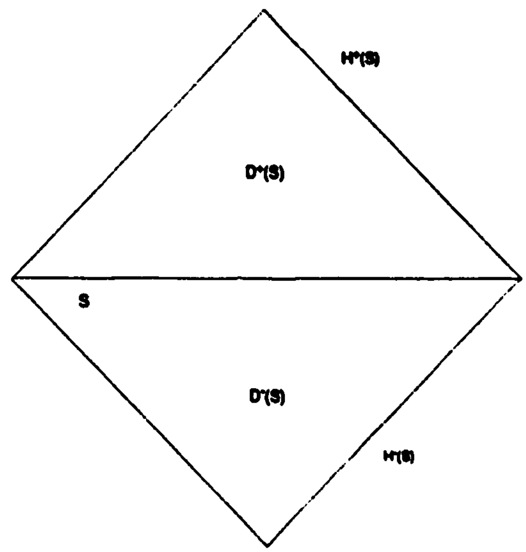

Figure 2.3.3: The schematic diagram of an achronal set and its past and future developments. 45 degree lines are light rays

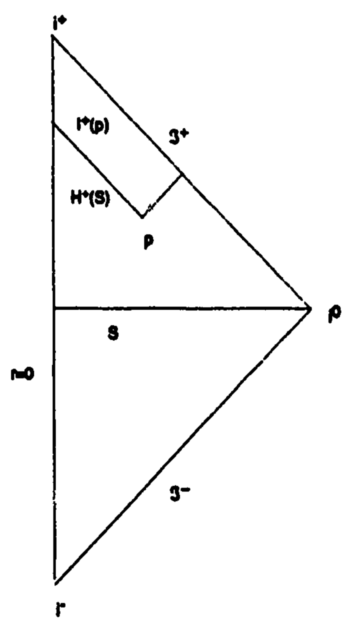

Figure 2.3.4: Penrose diagram of Minkowski space with one point removed showing an edgeless achronal set going to space-like infinity. The causal future of p could not lie in $D^+(S)$.

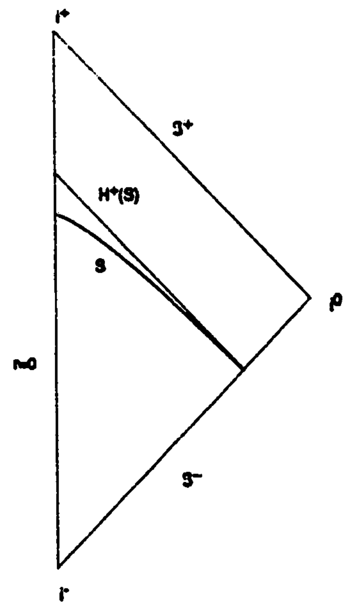

Figure 2.3.5: A bad partial Cauchy surface which is asymptotically null. This partial Cauchy surface does not give any information about the causal behavior nearby i^0.

2.3.4 Geodesic Completeness, Tipler's Singularity Theorem

Defining singularities in inextendible space-times in terms of time-like and null geodesic completeness seems to be the most natural way to define singularities. It is natural because it is unreasonable for the world line of a freely falling observer to come to an end in finite affine time (affine parameter for a null geodesic). A space-time (M, g) is *b-incomplete* if the generalized affine parameter can take only finite values. Other wise it is called b-complete. In this definition, $r(p, \mathbf{E}_i)$, the *generalized affine parameter* is defined in the terms of the tangent vector V^i in the parallely propagated basis $\mathbf{E}_i(p)$ along a geodesic parametrized by t by $r(p, \mathbf{E}_i) = \int_p (\sum_i V^i V^i)^{\frac{1}{2}} dt$. Clearly a b-complete space-time is *geodesically complete* since the definition of generalized affine parameter reduces to the ordinary affine parameter on a geodesic. [7]

[7]It is called b or *bundle* complete because completeness is actually defined in terms of the metric completeness of the manifold of the bundle of orthonormal frames on (M, g)[13](p.259). It is

P-P singularity refers to the divergence of at least one component of $R_{\alpha\beta\gamma\rho}$ in a parallel propagated orthonormal frame. One might think that b-completeness necessarily implies p-p singularity[1](p.31), but the violations of b-completeness in Taub-NUT space and the example given in [15] show that this is not the case.

Tipler's theorem says that an asymptotically flat space-time (M, g) can not be null geodesically complete if

i $R_{ab}K^a K^b \geq 0$ everywhere for all null vectors K^a. This condition is satisfied if the weak energy condition holds on (M, g),

ii the generic condition holds on (M, g) , namely $K^a K^b K_{[c} R_{d]ab[e} K_{f]} \neq 0$ for at least one point for all null geodesics in (M, g) . This condition demands that roughly speaking every geodesic encounter some effective curvature somewhere along its path. F. Tipler needs this property to make use of the following fact: given, in a space-time which is generic and satisfies the generic condition and has a complete time-like or null geodesic, then some nearby geodesic meets that one more than once [1](p.73).

iii (M, g) is partially asymptotically predictable from a partial Cauchy surface S or in other words,

$$\forall \lambda, \gamma \ \overline{D^{+}}(S) \cap \lambda \neq \emptyset, \text{ and } \overline{D^{-}}(S) \cap \gamma \neq \emptyset, \qquad (2.13)$$

in which λ is a null generator of $\mathcal{I}^{+}$, γ is a null generator of $\mathcal{I}^{-}$ and $\overline{D^{+(-)}}(S)$ is the closure of $D^{+(-)}(S)$ such that $\tilde{D}^{+(-)} = \overline{D^{+(-)}}(S)$. The importance of the space-times in which this condition holds is that in these space-times a causality violation which is visible from infinity is allowed. These spaces are causal only in a very weak sense; ***regular initial data*** exist (there is a partial Cauchy surface on which the components of the induced metric and its normal derivative are

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generalized in the sense of being defined for all C^1 curves.

well defined). The Cauchy problem is well posed for at least a non-compact subset of M, and the region near i^0, space-like infinity is causal.

iv The chronology condition is violated in $J^-(\mathcal{I}^+) \cap J^+(S)$, namely $J^-(\mathcal{I}^+) \cap J^+(S)$ contains CTCs. This condition is used to show that for space-times in which this condition holds $H^+(S) \cap \mathcal{I}^+ \neq \emptyset$.

The proof is based on a simple idea. If (M, g) is partially asymptotically predictable $\exists \eta$ s.t. η is a generator of $H^+(S)$ and $\eta \cap \mathcal{I}^+ \neq \emptyset$. Therefore η is future complete. According to lemma (1) η is also past complete, consequently the conditions (i) and (ii) require η to have a pair of conjugate points, but this contradicts the achronality of $H^+(S)$ since any null geodesic which has two conjugate points (points in which the Jacobi field vanishes), points further apart along the geodesic can be joined by a time-like curve [12](p.237)[8]. So η can not be future and past complete, a contradiction.

[8]What usually happens for example in Thorne wormhole space-time is that at a conjugate point one or more future directed null generators intersect and leave $H^+(S)$. This point is called a *caustic*[12](p.220).

Chapter 3

CTCs Produced by Moving Cosmic Strings

3.1 What Is Gott Space and How Is it Constructed?

In Gott space [4] the appearance of CTC containing region, A, is a result of the combination of the exotic behavior of conical spaces. Assume a coordinate system in which the string with deficit angle $2\alpha < \pi/2$ is always anchored to be at rest and is located at $(t_1, x_1, y_1) = (t_1, 0, d)$ parallel to z axis and the points $(t_1, (y_1 - d)\tan\alpha, y_1)$ and $(t_1, -(y_1 - d)\tan\alpha, y_1)$ are identified (Fig.3.1.1). Simple trigonometry shows that for a particle leaving x_1 at E_{ix}, moving with velocity v_p, crossing the wedge and returning to the x_1 axis, at E_{fx}, w_0 is minimized when $w_0 = w_0'$ and $\hat{E}_1 = \pi/2$. Imagine we set $t = 0$ as the particle with velocity v_p hits the identification plane at E_1 which is identified with E_2. The four important events on the particles world line are listed as below,

$$\begin{aligned}
E_i &= (-w_0/v_p, x_0, 0), \\
E_1 &= [0, (w_0 \sin\alpha - d)\tan\alpha, w_0 \sin\alpha], \\
E_2 &= [0, -(w_0 \sin\alpha - d)\tan\alpha, w_0 \sin\alpha], \\
E_f &= (w_0/v_p, -x_0, 0).
\end{aligned} \tag{3.1}$$

Now if $(t_1 x_1, y_1)$ frame is boosted (Lorentz transformed) in $+x$ direction with respect to (t_L, x_L, y_L), the lab or center of mass frame whose coordinates coincide with coordinates of (t_1, x_1, y_1) frame at $t_L = t_1 = 0$, with velocity v_s such that

$$\gamma_s^2 = \frac{x_0^2}{x_0^2 - w_0^2 v_p^2}, \tag{3.2}$$

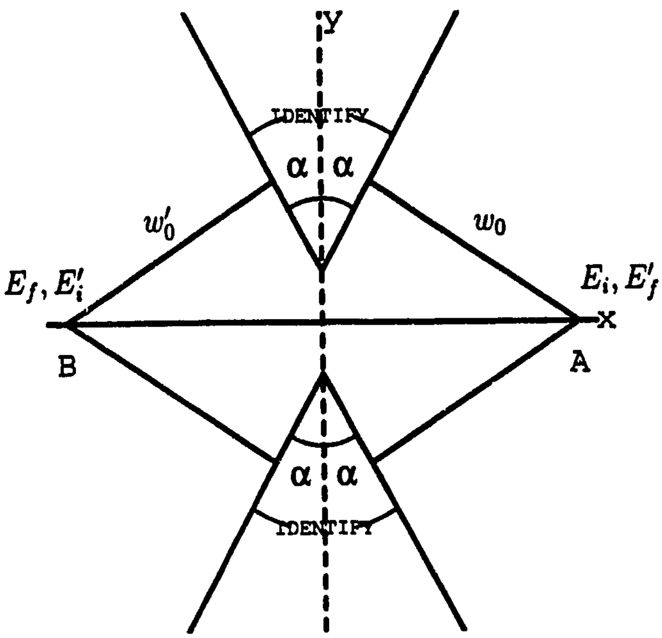

Figure 3.1.1: Two parallel string static solution; A particle leaving A at event E_i, crossing the wedge of string (1) and returning to B at event E_f. The particle then changes its direction and crosses the string (2) and returns to A at event E_f'. If the proper conditions are met, this particle returns to A before a light ray which left A at E_i, passed through the origin, reversed its direction at B and returned to A.

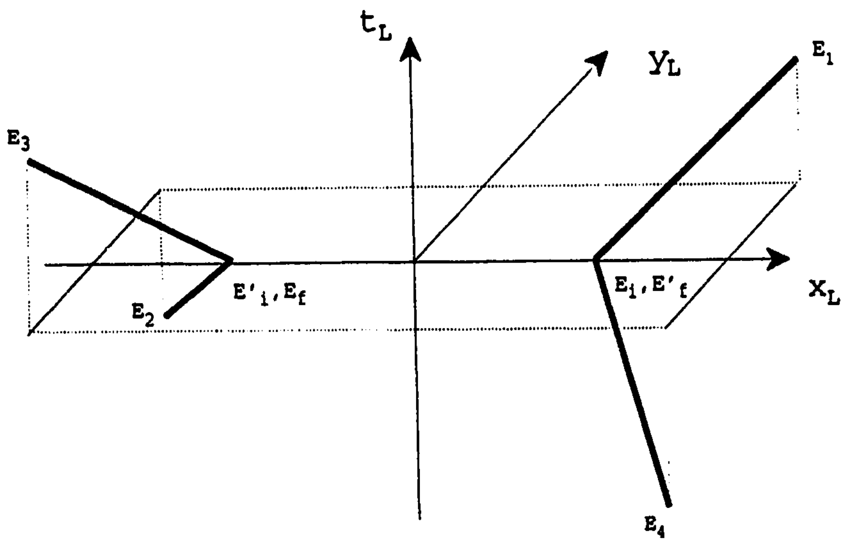

Figure 3.1.2: The space-time diagram of the same strings shown in the previous figure which have equal velocities moving in opposite directions with respect to the L frame. Identifications E_1 and E_2 are now on different t =const. surface and as the particle crosses the string towards which it is moving, it goes back in time.

the above events are simply Lorentz transformed as

$$\begin{aligned}
E_{i_L} &= (0, \gamma_s^{-1} x_0, 0), \\
E_{1_L} &= [v_s \gamma_s (w_0 \sin\alpha - d)\tan\alpha, \gamma_s (w_0 \sin\alpha - d)\tan\alpha, w_0 \sin\alpha], \\
E_{2_L} &= [-\gamma_s v_s (w_0 \sin\alpha - d)\tan\alpha, -\gamma_s (w_0 \sin\alpha - d)\tan\alpha, w_0 \sin\alpha], \\
E_{f_L} &= (0, -\gamma_s^{-1} s_0, 0).
\end{aligned} \tag{3.3}$$

Now we have the means to construct a CTC. Events E_{i_L} and E_{f_L} are on the $t_L = 0$ surface, they are simultaneous. If we add a string parallel to the first one located at $(0, 0, y_L = -d)$ at $t_L = 0$ and boost it in $-x_L$ direction the following discrete symmetries hold in the (t_L, x_L, y_L) frame;

$$\begin{aligned}
D_1 &: (t_L, x_L, y_L, z_L) = (-t_L, -x_L, y_L, z_L), \\
D_2 &: (t_L, x_L, y_L, z_L) = (t_L, -x_L, -y_L, z_L), \\
D_3 &: (t_L, x_L, y_L, z_L) = (-t_L, x_L, -y_L, z_L),
\end{aligned} \tag{3.4}$$

and because the continuity of the metric and the derivatives of the metric across the boundary $y_L = 0$ is trivially satisfied [16], these coordinate systems can be smoothly patched across this surface. The mentioned symmetries guarantee the existence of an analogous path in $y_L < 0$ region of the coordinate system Fig.(3.1.2). On this path E'_i begins on $-x$ axis and the event E'_f ends on $+x$ axis. So if E'_i and E_f are identified, $E_i = E'_f$ and the particle returns to x_0 at the same time that it leaves x_0, the departure of the particle from $x = x_0$ and its arrival to x_0 are simultaneous events. Therefore the particle follows a CTC.

From (3.2) as $(v_p \to 1^-)$ it can easily be shown that

$$\gamma_s^2 = \frac{1}{1 - w_0^2/x_0^2} > [\sin\alpha]^{-1}, \tag{3.5}$$

which is equivalent to

$$v_s > \cos\alpha \tag{3.6}$$

given

$$\lim_{x_0 \to \infty} w_0/x_0 = [\cos\alpha]. \tag{3.7}$$

This equation gives a lower bound for the string velocity.

We see the underlying reason behind this CTC creation. The coordinate identifications

$$[t_1, (y_1 - d)\tan\alpha, y_1] = [t_1, -(y_1 - d)\tan\alpha, y_1], \tag{3.8}$$

which in the rest frame of the strings truly represent the conical character of spacetime, in the lab frame are identifications along different t_L =const. surfaces. Therefore if we divert attention to Fig.3.1.2 which is a better representation of what is happening form the point of view of the observer at rest in the lab frame, the particle goes back in time when it hits one of the identification surfaces of the string towards which it is moving. As it will be shown later, boosting has caused the identification wedges in $y_L > 0$ and $y_L < 0$ to rotate in the direction of movement of the string around the y_L axis. The (1) frame is rotated clockwise and the (2) frame counter-clockwise respectively. Not surprisingly, it can be shown that if the strings make an angle ϕ with the z axis, (3.5) reduces to $\gamma_s > [\sin 4\pi\mu]/\cos\phi$, thus only the transverse components of the string velocities contribute to the creation of CTCs.

The question might arise whether cosmic strings are physically relevant and how they fit into cosmological models. It is suggested[17] that vacuum cosmic strings with $\mu \geq 10^{-6}$ produced in the early universe could provide the fluctuations necessary to cause inhomogeneities which later resulted in the formation of galaxies.

On the other hand observations put an upper bound on the mass density per length of strings. Due to the wake effect (also called the sling shot effect), as will be seen later, the transverse velocity of cosmic strings causes the light rays passing on two sides of strings to red shift or blue shift with respect to each other. Assuming relativistic velocities for these strings, the observed fluctuation in the isotropy of microwave background temperature T requires $\mu \sim \frac{\delta T}{T} \leq 10^{-4}$[8]. Then (3.5) gives $\gamma_s > 10^3$. Can strings reach such high velocities? Strings are under enormous tensions. In fact the tension is so high that straight strings barely satisfy the weak energy condition. String theories predict that collapsing loops and kinks achieve very high velocities and open ends of strings do reach the velocity of light [18].

These discoveries raise the question that since the gravitational field (rather "topological field") at small distances from loops and strings with finite length (which perhaps can be constructed in a laboratory from a substance whose tension counts for all its mass) can be approximated by (2.2), does the Gott time machine necessarily require strings with infinite length?

The other not so trivial question to answer is that when the Gott time machine is constructed or if somehow there are two infinite moving parallel cosmic strings somewhere in the universe, can they be used for time travel? The answer to the first question is postponed to the next chapter but we try to answer the second by first rewording the question. How do singularity theorems which were designed to explain causal pathologies in curved space hold in Gott space-time which has a considerably more simple structure? More precisely we should investigate the possibility of Gott time machine being marred by singularities which prevent any observer on a partial Cauchy surface to approach $H^+(S)$. One kind of these singularities are the singularities discussed in §2.3.4.

The other kind of singularities are due to infinite blue shift of a small high frequency wave packet going around the Cauchy horizon to the future. The idea behind investigating blue shift singularities is that we will finally have to show the possibility of using Gott time machine by calculating the effect of the existence of an observer on the back ground space-time in a perturbative way. For an observer approaching $H^+(S)$, the perturbation in δg of the flat metric g is of the order of the perturbation in energy-momentum density tensor. If $\delta g \ll g$ is not satisfied, using perturbation theory to study the effects of the presence of an observer on the flat space-time is no longer valid. We try to investigate the former type of singularity in §3.2 and the latter is discussed in §3.3.

3.2 Gott Space-Time and Singularity Theorems

Singularity theorems which use the causality condition are mainly divided into three categories.

1- Hawking-Penrose-Geroch singularity theorems.

2- Theorems which make extensive use of partial asymptotic predictability of causality violating space-time. This condition ensures that the region near space-like infinity i^0 is causally well behaved.

3- Theorems dealing with topology change in $M \cap S(\tau)$ in which $S(\tau)$ is the time evolution of a partial Cauchy surface S such that $S(0) = S$, *i.e.* topology change in Thorne wormhole space-time which gives rise to CTCs[1].

We briefly discuss the importance and motivation behind the construction of these theorems.

3.2.1 Causality Violation and Hawking-Penrose-Geroch Singularity Theorems

This category refers to the gravitational collapse of small objects (stars) within the universe and besides causality violation, makes extensive use of the notion of closed

[1]Almost all these theorems require the weak energy condition to hold which is not the case for Thorne wormholes.

trapped surfaces, namely, compact, edgeless 2-surface, on both sides of which the convergence of outgoing orthogonal null geodesics is positive. The existence of such surfaces means that the gravitational collapse has proceeded beyond a certain point [13](p.265)[2].

Before investigating how the theorems of this class might predict the possibility of occurrence of any singularities in the space-time of two parallel moving cosmic strings, it is useful to outline two definitions from F. Tipler [1](p.112-113).

Definition 1 *A space-time (M, g) is said to be asymptotically deterministic if;*

1- (M, g) contains a partial Cauchy surface S such that:

2- Either $H(S) \equiv H^+(S) \cup H^-(S)$ is empty, or, if not, then $\lim_{s \to a}[\inf T_{ab}K^aK^b] > 0$ on at least one of the null geodesic generators γ of $H(S)$ tangent to K^a, where a is the past (future) limit point of the affine parameter along $\gamma \in H^+(S)(H^-(S))$.

Definition 2 *The matter tensor will be said to be past stochastic along a causal geodesic segment γ with tangent vector K^a if there exists $a > 0$, $b > 0$ and $c \in N$ such that c is the number of disjoint affine parameter intervals $(s_1, s_2) \cdots (s_i, s_{i+1})$ along γ, each interval satisfying $s_{i+1} > s_i$ and $|s_i - s_{i+1}| > b$ with $T_{ab}K^aK^b \geq a$ at every point in every interval. Furthermore, c is finite if γ has a past end point or is past incomplete and infinite if γ is complete. Future stochastic matter tensors are defined similarly.*

[2]It should be noted that it is possible to construct non-compact 2-surfaces in Minkowski space-time which locally possess the characteristics of a trapped surface; the set of outgoing orthogonal null geodesics converge at each point on this surface [19](p.103). Therefore one cannot dismiss the possibility of the existence of such surfaces in Gott space-time, merely based on the absence of curvature.

F. Tipler originally constructed these definitions based on the assumption that since $H(S)$ in causality violating space-times is non-empty, where the causality violation begins it contains matter or (M, g) is at least non-flat.

Theorems of the first category are mainly constituted of five theorems [13](p.263-293);

Theorem 1 *Space-time (M, g) is not null geodesically complete if:*

1- $R_{ab}K^aK^b \geq 0$ for every non-space-like vector K^a;

2- there is a non-compact Cauchy surface H in M;

3- there is a closed trapped surface in M.

Theorem 2 *Space-time (M, g) is not null geodesically complete if:*

1- $R_{ab}K^aK^b \geq 0$ for every non-space-like vector K^a;

2- the generic condition is satisfied;

3- the chronology condition holds on M;

4- there exists at least one of the following;

i- a compact achronal set without edge;

ii- a closed trapped surface;

iii- a point p such that on every past (future) null geodesics from p, the divergence of null geodesics from p becomes negative.

Theorem 3 *There is a past incomplete non-space-like geodesic through $p \in (M, g)$ if:*

1- $R_{ab}K^aK^b \geq 0$ for every non-space-like vector K^a;

2- the strong causality condition holds on (M, g);

3- there is some past directed time-like vector W at p and a positive constant b such that if V is the unit tangent vector to the past-directed time-like geodesics through p, then on each such geodesic, the expansion $\theta \equiv V^a_{;a}$ of these geodesics becomes less than $-3c/b$ within a distance b/c from p, where $c \equiv -W^aV_a$.

Theorem 4 *Space-time is not time-like geodesically complete if:*

1- $R_{ab}K^aK^b \geq 0$ for every non-space-like vector K^a;

2- there is a compact space-like 3-surface (without edge);

3- the unit normals to this space-like 3-surface are everywhere converging (or everywhere diverging) on the surface.

Theorem 5 *Space-time is not b-bounded[3] if conditions (1)-(3) of Theorem 4 hold, and*

4- the energy-momentum tensor is non-zero somewhere on the space-like 3-surface;

5- if K^a is a non-space-like vector, then $T^{ab}K_a$ is zero or non-space-like and $T_{ab}K^aK^b \geq 0$, equality holding only if $T^{ab}K_b = 0$.

As can be seen Theorems 1-3 require (M, g) to be causal which is certainly not the case with Gott space-time. However critics might argue that any one of

[3] The exact definition of a b-bounded space-time is rather cumbersome and not necessary to our discussion at this point. Roughly speaking, this definition is an attempt to generalize the definition of geodesic completeness in terms of $\exp : T_p(M) \to M$ and affine parameter [13](p.33) to all C^1 curves in terms of the generalized affine parameter (see §2.3.3). For a more accurate account of this definition, see [13](p.51,292).

the causality conditions is too strong a condition in the above theorems. Indeed as F. Tipler has pointed out, the condition in Theorem 1 requiring the existence of a non-compact Cauchy surface and the condition in Theorem 2 requiring the chronology condition to hold in (M, g) , can be replaced with a condition with necessitates (M, g) to have a non-compact partial Cauchy surface S and either to be asymptotically deterministic or for at least one of the generators of $H^+(S)(H^-(S))$, γ, the energy-momentum tensor, T_{ab}, to be past (future) stochastic as γ approaches its past (future) affine parameter limit [1](p.111-117). But since the generators of the past (future) boundary of CTC containing region in Gott space-time (part of $H^+(S)(H^-(S))$) are past (future) complete (see §3.2.4) and $T_{ab} \equiv 0$ (specially far away from the origin so the generators of the CTC boundary do not run the risk of going into the strings with non-vanishing T_{ab}), neither is Gott space-time asymptotically deterministic nor is the energy-momentum tensor past (future) stochastic along any of the generators of $H^+(S)(H^-(S))$.

At this moment we have no concrete proof why conditions 2 and/or 3 of theorems 4,5 are not satisfied in Gott space-time, but since as S. W. Hawking *et.al* say "condition 2 may be interpreted as saying that the universe is spatially closed", which is certainly not the case with Gott space-time, it appears highly plausible that condition 2 is not satisfied [13](p.273) (also see §3.2.3).

3.2.2 Gott Space-Time, Tipler's Theorem And Topology Change

From the second category, Tipler's theorem regarding asymptotic flatness (see §2.3.4) has been mentioned in relation with Gott space-time [4][5]. Since it does not directly require compactness or finiteness and because of its generality it seems to be the most relevant theorem of the second class to this particular example.

The theorem says that if a space-time is

a- asymptotically flat,

b- the generic condition holds,

c- weak energy condition is satisfied,

d- partially asymptotically predictable from a partial Cauchy surface S,

e- the chronology condition is violated in $J^{+}(S) \cap J^{-}(\mathcal{I}^{+})$, or in other words the causality violating set is at least partly naked,

then this space-time is singular in the sense of being null geodesically incomplete. But it should be noted that time-like and null geodesic completeness are minimum conditions for a space-time to be singularity free. There are by no means sufficient [13](p.258).

Gott space-time is locally flat[4], therefore there are obviously many geodesics which encounter no effective curvature anywhere along their path. But one can not dismiss the application of the above theorem to Gott space-time hastily since one is after those properties of this space-time which are stable under small perturbations. Therefore it is regarded as physically reasonable to demand the above theorem to hold in Gott space-time because even if a given space-time is not generic, a small perturbation in the metric would result in a space-time which is generic[16].

Asymptotic flatness of Gott space-time is discussed in §3.2.4. It is shown that Gott space-time is not asymptotically flat. But as it is seen, there might be conditions under which the requirement of asymptotic flatness in Tipler's theorem can be eased so that a modified version of this theorem can be used in Gott space-time.

[4]In this context, a locally flat space-time is a space-time in which $R_{\alpha\beta\gamma\delta} \equiv 0$.

R. Geroch was the pioneer of the theorems of the third category [20]. We use a different version of one of his theorems due to F. Tipler which unlike that of Geroch does not require the space-time to be compact. The only constraint is that any possible topology change is localized in a compact region [1](p.109).

To investigate further how these theorems comply with Gott space-time we need to have a better understanding of the structure of causality violating region.

3.2.3 Partial Asymptotic Predictability of Gott Space-Time

We first prove there are sets with measure non-zero with space-like boundary[5], without CTCs [5]. To see this we tilt the wedges of (3.1) according to Fig.3.2.1 and boost the (1) frame and (2) frames (rest frames of strings (1) and (2)) in $+x_L$ and $-x_L$ of the frame of the stationary observer respectively[6]. The two following sets in the L frame are defined for $\epsilon > 0$;

$$I \equiv \{x_L \geq 0, t_1 \geq \epsilon\}\,,\; II \equiv \{x_L \leq 0, t_2 \geq \epsilon\}\,, \tag{3.9}$$

or in terms of the L coordinates (Fig.3.2.2),

$$I \equiv \{x_L \geq 0, t_L \geq vx_L + \epsilon/\gamma\}\,,\; II \equiv \{x_L \leq 0, t_L \geq -vx_L + \epsilon/\gamma\}\,, \tag{3.10}$$

now since discontinuities in t_1 and t_2 are due to "jumps" across wedges at rest in (2) and (1) frames respectively and these discontinuities are limited to $x_L \leq 0$ for (1) frame and $x_L \geq 0$ for (2) frame, t_1 is an increasing monotonic function along every future directed time-like curve in the set I and t_2 is an increasing monotonic function along every future directed time-like curve in II. So any CTC must cross the boundary between sets I and II, namely $x_L = 0$, an even number of times. But at

[5]This part is mainly based on Curt Cutler's work.

[6]Here we adopt a coordinate dependent definition of the stationary observer, namely, a stationary observer is a static observer in the Lab frame.

$x_L = 0$, $t_L = t_1 = t_2$ and there can be no closed time-like curves intersecting $x_L = 0$. It is seen $I \cup II$ can be foliated by surfaces (Fig.3.2.2)

$$\{x_L \geq 0, t_2 = k\} \cup \{x_L \geq 0, t_1 = k\}, \tag{3.11}$$

which can be smoothed out at $x_L = 0$ (so it becomes C^∞) while preserving the space-like character of these surfaces. Due to symmetry, there are similar surfaces for $t_L \leq 0$ which are the reflected images of these edgeless achronal sets. Three features of these surfaces are very important:

1- the space-like character of these surfaces is not changed under the action of any element of Lorentz group[7]. So although the existence of these surfaces was proved by placing the wedge identifications in a particular way, the geometrical properties of these surfaces are obviously coordinate independent.

2- As $\epsilon \to 0$ it is seen that there is always an arbitrary small neighborhood of the origin devoid of CTCs and at $T_L > (<) 0$ the world line of the strings enter $I \cup II$ (Fig.3.2.2). Therefore "smoothing out" the apex of the cone does not affect the structure of the CTC containing region.

3- These surfaces are not bad partial Cauchy surfaces. In other words they are not asymptotically null of the form given in §2.1.1.

The third feature gives important information on the causal behavior nearby the space-like infinity i^0. The point i^0 is in an open neighborhood which is causally well behaved. This means $D^+(S) \cap \lambda \neq \emptyset$ for at least some of the null generators λ of $\mathcal{I}^+$ and since $D^-(S) \cap \gamma \neq \emptyset$ for all generators γ of $\mathcal{I}^-$, according to the definitions given in 2.3.4. Gott space-time is *most probably* partially asymptotically predictable.

[7]Or under any diffeomorphisms in general.

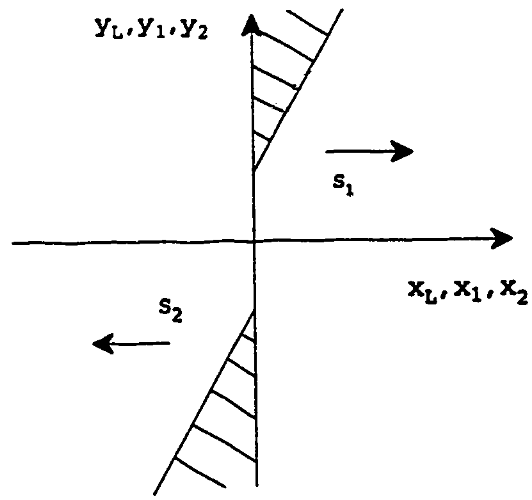

Figure 3.2.1: The identification wedges are tilted in such a way that t_1 is continuous in $x_L \geq 0$ and $t_L \geq 0$, whereas t_2 is continuous in $x_L \leq 0$ and $t_L \geq 0$.

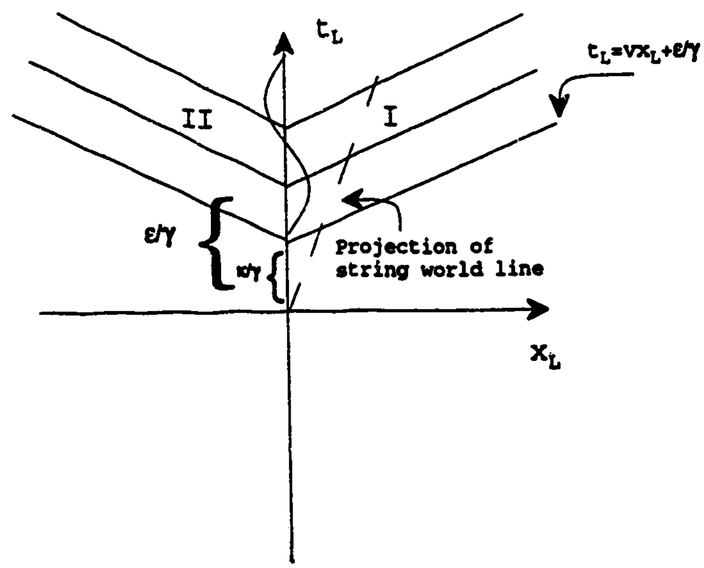

Figure 3.2.2: Space-time diagram illustrating the proof that there are regions containing no CTCs. The projections of the world lines of the strings on (x_L, t_L) plane are shown with dashed lines. Region $I \cup II$ is devoid of CTCs.

It should be added that this problem is treated lightly and if I may say carelessly by both Cutler [5] and S. Carroll *et al.* [7] Cutler asserts that the mere existence of a partial Cauchy surface guarantees the partial asymptotic predictability of this space-time. But after a full understanding of the boundary of causality violating region ($\dot{A}$) it turns out this problem merits a more careful treatment.

3.2.4 The Boundary of the Causality Violating Region

There is one property of the boundary of any causality violating subset of (M, g) which is rather generic of the boundary of the CTC containing regions. It is a null surface because it can simply neither be a space-like nor a time-like surface. It can not be space-like since any time-like curve passing through a point p infinitesimally close to a space-like surface has to cross the surface[8]. Therefore any CTC through p would have to enter the region in which there are no CTCs. So points infinitesimally close to a space-like surface can not possibly lie on a CTC. To prove that $\dot{A}$ (the boundary of the CTC containing region) is not time-like, Cutler implicitly makes the not so restrictive following assumptions that if $\dot{A}$ were time-like and

$$\begin{array}{lll} \exists (p,q) \in \dot{A} & \text{s.t.} & p \in I^{+}(q), \\ \forall (U \text{ and } U') & \text{s.t.} & p \in U \text{ and } q \in U', \\ \exists (p' \in U \text{ and } q' \in U') & \text{s.t.} & \begin{cases} p' \in I^{+}(p) \text{ and } q \in I^{+}(q') \\ q' \in I^{+}(p') \text{ and } p' \in I^{\pm}(q'), \end{cases} \end{array} \tag{3.12}$$

then $\dot{A}$ can not be time-like. The above assumption says that there are at least two points on $\dot{A}$ lying on a time-like curve such that in every open normal neighborhood of them there are two points connected by a CTC. After making this assumption the proof is only one step away. Since $I^{+}(q)$ and $I^{+}(p)$ are open sets, either the future

[8] Here we use the fact that in a space-time with a Lorentz metric and a time-like direction field, a positive definite metric can be constructed. The proximity to the null surface is defined in terms of this metric[13](p.39).

directed time-like curve λ from p to q or from q to p can be deformed in such a way that:

$$\exists r \in \lambda \text{ s.t. } r \notin A \cup \dot{A}, \text{ but } q \in I^+(r),\ r \in I^+(p) \text{ and } p \in I^+(q), \tag{3.13}$$

so r lies on a CTC, a contradiction. Therefore $\dot{A}$ is a null surface.

To gain further insight into the structure of $\dot{A}$ we accept the"supposition" that points on $\dot{A}$ lie on null geodesics that spiral around the strings an infinite number of times, they are $\lim_{n\to\infty}$ of "nth polarized hypersurfaces N"[9]which are defined as

$$\forall p \in N \Rightarrow \begin{cases} p \text{ is the past and future end point of a self-intersecting} \\ \text{null geodesic which spirals around the strings } n \text{ times.} \end{cases} \tag{3.14}$$

In this context a self-intersecting null geodesic is a null geodesic with different tangent vectors at p, distinguished from a closed null geodesic (CNG) with equal tangent vectors at p which go around the strings an infinite number of times but are either past or future geodesically incomplete [15][10].

To see which points lie on CNG we first note that a CNG must loop around both strings on a z =const. surface, since there are no identifications in the z direction, there can be no CNGs with tangent components along the z direction. We assume the geodesics loop around the strings n number of times and return to the $+x$ axis having the same direction with respect to the $-x$ axis (see Appendix-A). Therefore we define the function

$$\phi' = g(\phi) = \cot^{-1}\left\{(1-u^2)^{-1/2}[-\cot(2\alpha-\phi) + u\csc(2\alpha-\phi)]\right\}, \tag{3.15}$$

[9]This argument originally belongs to Kim and Thorne[21] which they apply to show that A in Thorne wormhole space-time is foliated by hypersurfaces on which the vacuum polarization component of energy-momentum density tensor is divergent. There is not a concrete proof that Gott space-time should obey the same rules, but it rather looks intuitively obvious that it is probably the case. Since A has no holes and discontinuities, as we approach $\dot{A}$ it becomes more and more difficult to go back in time far enough to have a CTC. There are two ways CTCs can cope with this problem. First to go back in time more effectively which means becoming null geodesics, second to loop the strings a larger number of times. As points in A approach $\dot{A}$, we expect these two effects to combine.

[10]This is a generic property of causality violating space-times which have Cauchy horizons with compactly generated null generators.

in which $u = \frac{2v}{1+v^2}$ is the velocity of frame (1) with respect to frame (2), which maps $\phi = \tan^{-1}\frac{-dy}{dx}$ in the rest frame of string (1) to ϕ' in the rest frame of string (2) (Fig.3.2.3). A *necessary* condition for the existence of a CNG is that

$$\phi = g^N(\phi), \ N \text{ even.} \tag{3.16}$$

But Fig.3.2.4 shows $g(\phi)$ has the following important properties which characterize the behavior of null rays in Gott space-time,

$$\begin{aligned} g(\zeta) &= \zeta \\ g(2\alpha-\zeta) &= 2\alpha-\zeta \\ g(\phi) &\to \zeta & \text{for} \quad & 0<\phi<\zeta, \\ g(\phi) &\to \zeta & \text{for} \quad & \zeta<\phi<2\alpha-\zeta, \\ g(\phi) &\to \pi & \text{for} \quad & 2\alpha-\zeta<\phi<\pi. \end{aligned} \tag{3.17}$$

Solving (3.16) for $N = 1$ gives

$$\cot\zeta + \cot(2\alpha-\zeta) = v[\csc\zeta + \csc(2\alpha-\zeta)]. \tag{3.18}$$

This equation uniquely determines ζ as;

$$\sin\zeta = \frac{\cos\alpha}{v}[\sin\alpha - \sqrt{v^2-\cos\alpha^2}], \tag{3.19}$$

which shows $v \geq \cos\alpha$ in compliance with (3.6). Now since for $g(\phi) \geq \pi$ and $g(\phi) < 0$ the null ray never intersects the identification wedges, the behavior of $g(\phi)$ in $0 < \phi < 2\alpha$ is all we need. Fig.3.2.4 also shows that for a CNG we can have either $\phi = \zeta$ (stable fixed point angle) or $\phi = 2\alpha - \zeta$ (unstable fixed point angle). Equation (3.17) explains the choice of nomenclature. If ϕ is slightly higher or lower than the fixed point angle, it is asymptotically driven to the fixed point angle. The behavior of CNGs with slightly higher or lower angles than the unstable fixed point angle is the opposite.

The sufficient condition for the above ray to be a CNG is that $x^\beta(\lambda_n) = x^\beta(\lambda_{n+1})$ in which λ is the curve parameter (not necessarily affine). If d is half of the

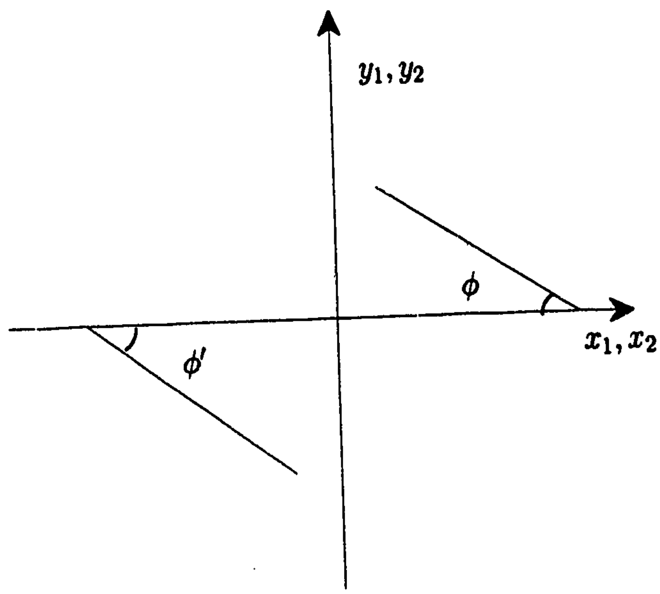

Figure 3.2.3: The function $g(\phi) = \phi'$ maps the angle measured in frame (1) with respect to the $-x$ axis to the angle ϕ in frame (2) measured with respect to the $+x$ axis. $g[g(\phi)]$ is the angle which the null ray has with respect to the $-x$ axis in frame (1) after scattering form both strings.

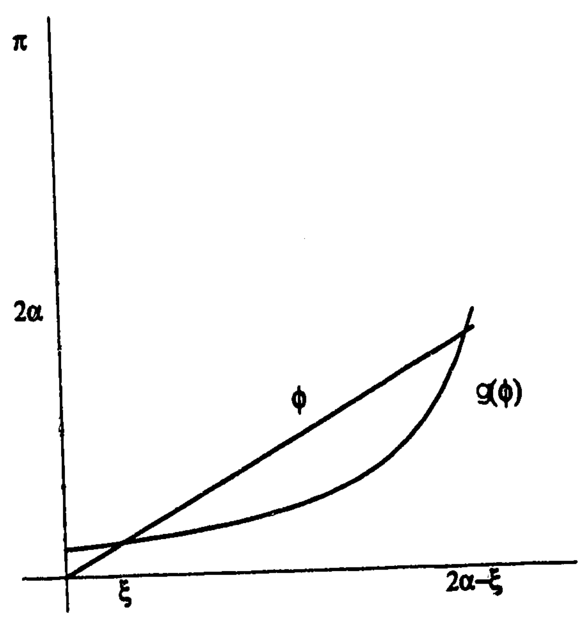

Figure 3.2.4: The behavior of $g(\phi)$ vs. ϕ; $g(\phi) < (>)\phi$ for $\phi > (<)\zeta$ and $g(\phi) > (<)\phi$ for $\phi > (<)2\alpha - \zeta$.

impact parameter, half of the distance between two strings at $t_L = 0$, then in the $d = 0$ limit we have

$$\begin{aligned} \Delta t_1 &= h[\cot\zeta + \cot(2\alpha - \zeta)], \\ \Delta x_1 &= -h[\csc\zeta + \csc(2\alpha - \zeta)] \end{aligned} \tag{3.20}$$

in which $h = \chi \sin\zeta$ and Δt_1 and Δx_1 are the distance and time required for a CNG to leave $y_1 = 0$, cross the string and return to $y_1 = 0$. Then from (3.18),

$$\Delta t_L = \frac{\Delta t_1 + v\Delta x_1}{\sqrt{1 - v^2}} = 0, \tag{3.21}$$

and therefore due to symmetry there is a CNG for ζ and one for $(2\alpha - \zeta)$ only when $d = 0$. But $d = 0$ is only the limiting case and in general $d \neq 0$ and $\Delta t_L > 0$. *There are no CNGs in a generic Gott space-time.*

Points on $\dot{A}$ lie on null geodesics which loop around the strings an infinite number of times and the absence of CNGs may look disappointing. But we should remember that the null generators of $\dot{A}$ also spiral around the strings an infinite number of times. Because if they ever stop spiralling the strings, they have to enter $I \cup II$ given in Fig.3.2.2. They never leave $\dot{A}$ simply because any tangent to a null cone is along a null ray which eventually intersects any space-like surface unless it hits a surface of coordinate discontinuity and goes back in time into somewhere before intersecting S. Then for sufficiently large n the nth polarized hypersurface would follow the null generators into $I \cup II$, a contradiction.

By a simple Lorentz transformation the null vectors whose direction of space-like components are given by the stable and unstable fixed point angles, which from now on we shall call stable and unstable null vectors, are in the lab frame;

$$\begin{aligned} \mathbf{j} &\propto \begin{cases} t_L - \cos\zeta_L \mathrm{x}_L + \sin\zeta_L \mathrm{y}_L \ , & \frac{dy_L}{dt_L} > 0 \\ t_L + \cos\zeta_L \mathrm{x}_L - \sin\zeta_L \mathrm{y}_L \ , & \frac{dy_L}{dt_L} < 0 \end{cases} \quad \text{stable null vector,} \\ \mathbf{k} &\propto \begin{cases} t_L + \cos\zeta_L \mathrm{x}_L + \sin\zeta_L \mathrm{y}_L \ , & \frac{dy_L}{dt_L} > 0 \\ t_L - \cos\zeta_L \mathrm{x}_L - \sin\zeta_L \mathrm{y}_L \ , & \frac{dy_L}{dt_L} < 0 \end{cases} \quad \text{unstable null vector,} \end{aligned} \tag{3.22}$$

in which ζ_L and $(\pi - \zeta_L)$ are the stable and unstable angles respectively seen by the observer stationary in the lab frame given by

$$\sin\zeta_L = \frac{\sqrt{1-v^2}}{v}\cot\alpha. \tag{3.23}$$

The ambiguity remains which one, **j** or **k**, are along the null generators of the boundary. To answer this question we note that in Lorentzian geometry unlike Euclidean geometry, a null surface is uniquely specified by knowing the null vector along the generators of the surface which in our case are **j** and **k**[11]. Thus $\dot{A}$ is

$$\begin{cases} -\mathbf{t}_L - \mathbf{x}_L\cos\zeta_L + \mathbf{y}_L\sin\zeta_L = -q\cos\zeta_L & \text{where } \frac{dy_L}{dt_L} > 0 \\ -\mathbf{t}_L + \mathbf{x}_L\cos\zeta_L - \mathbf{y}_L\sin\zeta_L = -q\cos\zeta_L & \text{where } \frac{dy_L}{dt_L} < 0 \\ \text{generator: stable null vector} & \end{cases}$$
$$\begin{cases} -\mathbf{t}_L + \mathbf{x}_L\cos\zeta_L + \mathbf{y}_L\sin\zeta_L = q\cos\zeta_L & \text{where } \frac{dy_L}{dt_L} > 0 \\ -\mathbf{t}_L - \mathbf{x}_L\cos\zeta_L - \mathbf{y}_L\sin\zeta_L = q\cos\zeta_L & \text{where } \frac{dy_L}{dt_L} < 0 \\ \text{generator: unstable null vector} & \end{cases} \tag{3.24}$$

We focus attention on the cross section of that part in which $\frac{dy_L}{dt_L} > 0$, in which the generators always go towards $+y$. Due to symmetry the structure of $\dot{A}$ in the $\frac{dy_L}{dt_L} < 0$ region is the reflected image of the part in which $\frac{dy_L}{dt_L} > 0$. Remember, $\zeta_L < \pi/2$ is measured with respect to the $-x$ axis, it is seen that if **j** were the null vector tangent to the generators of that part of $\dot{A}$ which is adjacent to future of CTC containing region (called future boundary), $\dot{A}$ would have to cross into $I \cup II$, a contradiction (this is obvious by looking at Fig.3.2.2 and keeping the directions of $\mathbf{j} < \pi/2$ and $\mathbf{k} > \pi/2$ in mind). Therefore **k** is along the generators of the future and **j** is along the generator of the past boundary. The future and past boundary intersect each other at the cusp

$$\begin{cases} x_L = q, \quad -t_L + y_L\sin\zeta_L = 0 & \text{for } \frac{dy_L}{dt_L} > 0 \\ x_L = -q, \quad t_L + y_L\sin\zeta_L = 0 & \text{for } \frac{dy_L}{dt_L} < 0. \end{cases} \tag{3.25}$$

[11]In Euclidean geometry we simply do not have null hypersurfaces.

Knowing there is a neighborhood of the origin not containing CTCs, we immediately conclude $q > 0$. A complete structure of A is given in Fig.3.2.5.

Identification wedges for $y_L > 0$ in L coordinate are

$$\begin{aligned} x_L - vt_L - \gamma^{-1} y_L \tan\alpha &= -\gamma^{-1} d \tan\alpha, \quad \tfrac{dy_L}{dt_L} > 0 \\ &\text{identified with} \\ x_L - vt_L + \gamma^{-1} y_L \tan\alpha &= +\gamma^{-1} d \tan\alpha, \quad \tfrac{dy_L}{dt_L} < 0. \end{aligned} \tag{3.26}$$

This equation shows as $v \to 1$, $\gamma^{-1} \to 0$ and the identification wedges close. So the boost has caused the wedges to rotate and close (Fig.3.2.7). These planes are similar to butterfly wings which are very close rather than open, *i.e.* the maximum angle between the covariant vectors orthogonal to the identification wedges, β, when $\alpha = \pi/4$ and $v = \cos\alpha$, with respect to the R^4 Euclidean metric $g = \mathrm{diag}[1,1,1,1]$ is $\beta = \cos^{-1} \frac{2-\csc\pi/4}{2+\csc\pi/4}$ and $\lim_{v\to 1} \beta = 0$.

To determine the value of q we note that because of the linearity of Lorentz transformations, D_1 in (3.4) in terms of coordinate (1) can be written as;

$$D_1 \; : \; (x_1, y_1, t_1) \; \to \; (-x_1, y_1, -t_1). \tag{3.27}$$

But the other symmetry in coordinate (1) is

$$(x_1, y_1, t_1) \; \to \; (-x_1, y_1, t_1), \tag{3.28}$$

therefore,

$$(x_1, y_1, t_1) \; \to \; (x_1, y_1, -t_1), \tag{3.29}$$

should also be a symmetry in coordinate (1). The parametric equations of the identification wedge for y_L in terms of coordinate (1) are (Fig.3.1.1)

$$\begin{aligned} x_L &= \gamma[(y_1 - d)\tan\alpha + vt_1] = \gamma[-(y_1 - d)\tan\alpha + vt_1], \\ y_1 &= y_L, \\ t_L &= \gamma[t_1 + v(y_1 - d)\tan\alpha] = \gamma[t_1 + v(y_1 - d)\tan\alpha], \end{aligned} \tag{3.30}$$

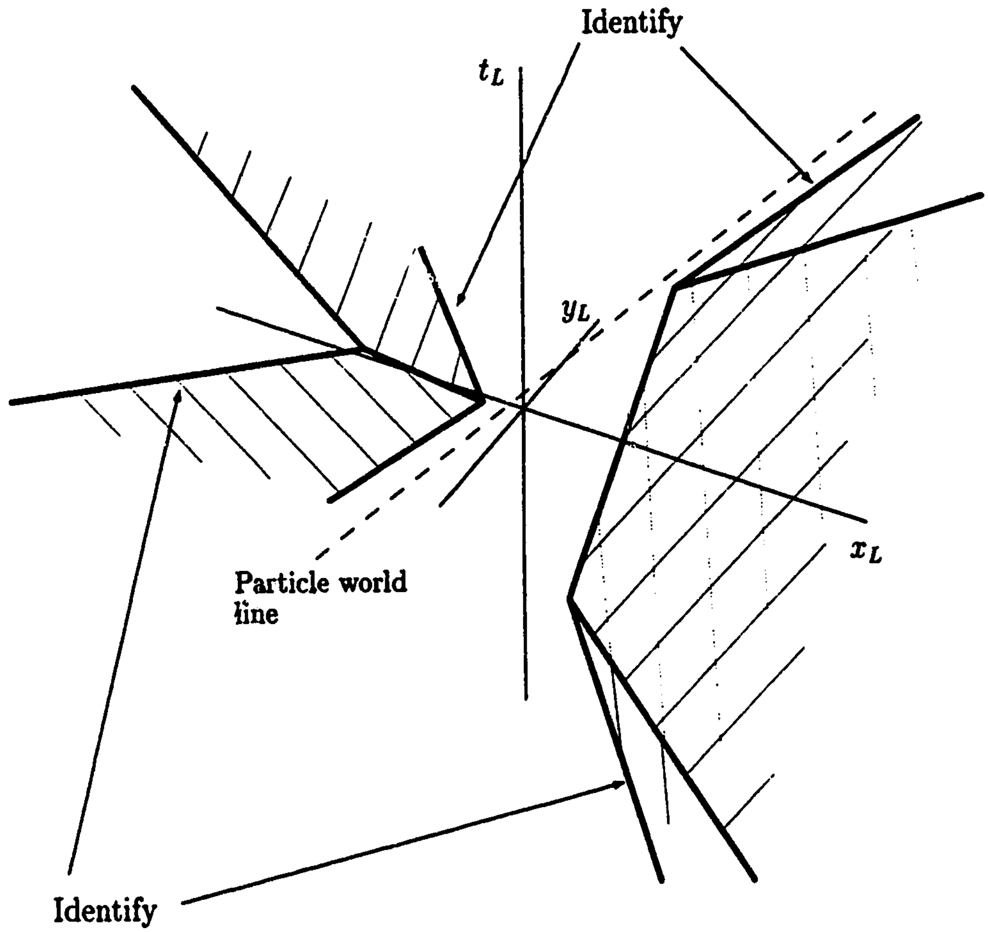

Figure 3.2.5: A sketch showing the general features of the boundary of the region containing CTCs. The past and future boundaries of this region are null planes which meet at a space-like cusp. CTCs are restricted to the region between these planes. The "identification" edges of the planes represent the intersection of CTC boundary with the string wedges. The seemingly disconnected A is actually connected and the connection is along the identification wedges which have a cusp along the particle world line. Therefore A has topology $S^1 \times R^2$. The null generators **j** and **k** of the past and future CTC boundaries, respectively, are shown inscribed. Only the world line of the particle moving at $y_L > 0$ is shown. The above diagram is the CTC boundary for $v = 0.90$, $\alpha = \pi/4 - 0.00001$ and $d = 0.1$.

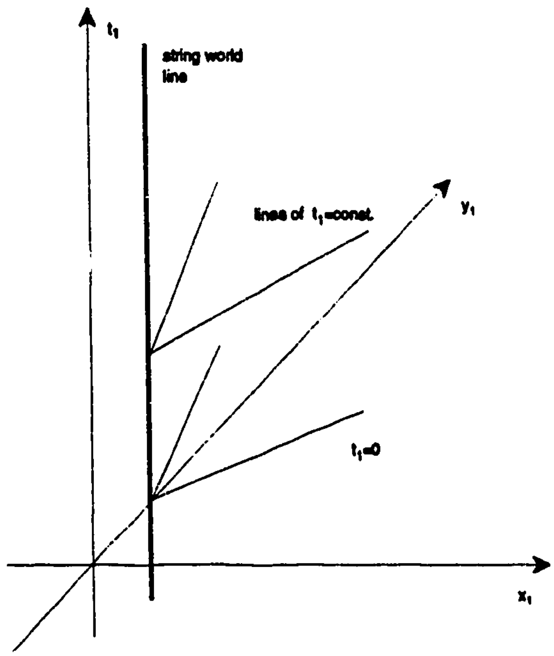

Figure 3.2.6: String world line and identification lines in the rest frame of the string at rest in coordinate system (1).

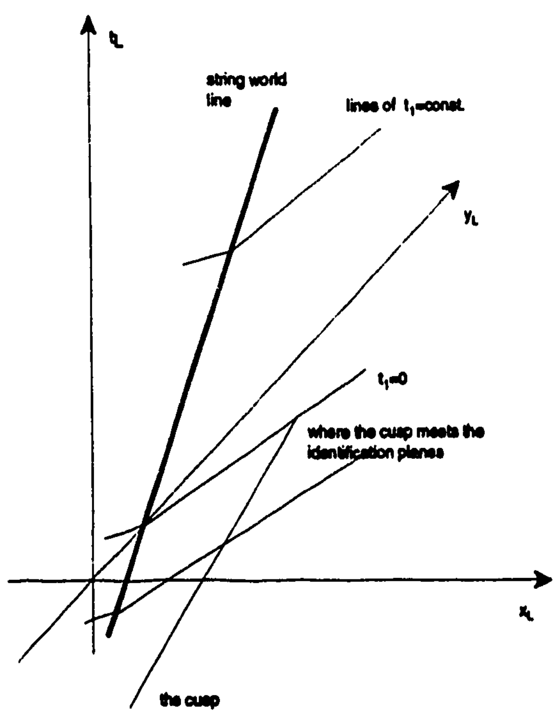

Figure 3.2.7: The boost has caused the identification wedges of the strings to rotate with respect to the L frame. The cusps meet these identification planes on $t_1, t_2 = 0$ lines.

and the only way that the intersection of the cusp with identification wedges is mapped to itself under (3.29) is that the cusp intersects the plane at r. Inserting $t_1 = 0$ on the right hand side of (3.30), and (3.26) and using (3.25) gives

$$y_1 \sin \zeta_L = vq. \tag{3.31}$$

This equation, with

$$q\sqrt{1-v^2} = x_1 := (y_1 - d) \tan \alpha, \tag{3.32}$$

gives

$$q = \frac{d \sin \alpha \cos \alpha \sqrt{1-v^2}}{v^2 - \cos \alpha^2}. \tag{3.33}$$

In the limit as $v \to \cos \alpha$, $\zeta_L \to \pi/2$ and $q \to \infty$. So the null boundary opens up and q goes to $+\infty$. An x_L, t_L cross section of Fig.3.2.5 shows how a self-intersecting null geodesic behaves (Fig.3.2.8). A self-intersecting null ray starts off with a tangent vector very close to $\mathbf{k}$. But as it develops through time, its trajectory distances itself from the boundary far enough to go back in time where it approaches $\mathbf{j}$ and the null ray begins to go forward in time again.

Now we are in a position to see how Geroch's topology change and the remaining conditions of Tipler's theorem apply in Gott space. For the CTC containing region A we have $A \cap \mathcal{I}^+ \neq \emptyset$, $A \cap \mathcal{I}^- \neq \emptyset$, and as $S(\tau)$ develops ($S(0)$ corresponds to the partial Cauchy surface of §3.2.4) through this space-time for some τ at which $S(\tau)$ is no longer a Cauchy surface, $S(\tau) \cap A \neq \emptyset$ and the intersection is not compact. Therefore if there is a topology change, such a topology change does not occur in a finite region [1](p.109). It is also obvious from Fig.3.2.5 and the definition of the past boundary that the past boundary is a subset of $H^+(S(0))$ since the domain of future development of $S(0)$ ends at $\dot{A}$.

We now investigate how the remaining conditions required by Tipler's theorem, namely $J^-(\mathcal{I}^+) \cap A \neq \emptyset$ and asymptotic flatness, comply with Gott space-time. As trivial the first one is ($\mathcal{I}^+ \cap A \neq \emptyset$ is obvious), asymptotic flatness is not so trivial. Asymptotic flatness requires an open neighborhood of $\mathcal{I}^+$ and $\mathcal{I}^-$ (which are $\dot{M}$, the boundary of M in the conformal imbedding of Minkowski space in Einstein static universe) to be isometric to an open neighborhood in Gott space which is naturally an open neighborhood of $\mathcal{I}^+$ and $\mathcal{I}^-$. But because of the conical structure of Gott space-time it has been suggested [22] that since parallel transporting a vector on a closed path around the origin in this open neighborhood changes the direction of the vector whereas parallel transporting such a vector in Minkowski space gives an identical vector, open neighborhoods of $\mathcal{I}^+$ and $\mathcal{I}^-$ in Gott space and Minkowski space are not isometric.

It should be noted that the rotation of a vector mentioned in the above paragraph, due to parallel transportation is also true for a static string (static cone) and is not due to causality violation in this space-time.

Critics might refute the asymptotic flatness of Gott space-time based on the fact that strings extend to i^0 in $\pm z$ directions, so $R_{ab} \neq 0$ on $\dot{M} = \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0$ [12](p.276). But there is some confusion about the definition of $\dot{M}$ since Hawking defines $\dot{M}$ to be $(\mathcal{I}^+ \cup \mathcal{I}^-)$ not including i^0[13]. This argument of critics does not seem to be a serious blow to the asymptotic flatness of Gott space-time as long as,

$$\exists o(i^0) \text{ s.t. } o(i^0) \cap H^+ = \emptyset \tag{3.34}$$

in which $o(i^0)$ is an open neighborhood of i^0. We tend to accept the definition given in [13](p.225) since this book gives a more careful treatment of causality violating space-times.

Another way to view the problem stated in the previous paragraph is that one major weakness of theorems of this kind is that their proof depends on the existence and structure of asymptotic infinity. The condition of asymptotic flatness in these theorems was used only to show that at least one generator of $H^+(S)$ could

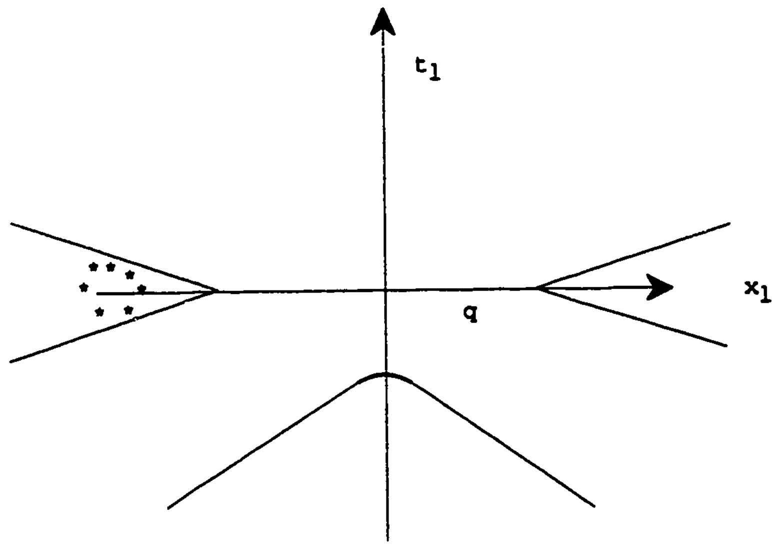

Figure 3.2.8: This figure represents the (x_L, t_L) cross section at $(y_L = 0, z_L = 0)$ of the CTC boundary depicted in Fig.3.2.5 A self-intersecting null geodesic begins with an angle very close to **k**. As it moves forward in time it distances itself away from the future boundary until it begins to go back in time so its tangent vector approaches **j** and begins to go forward in time again.

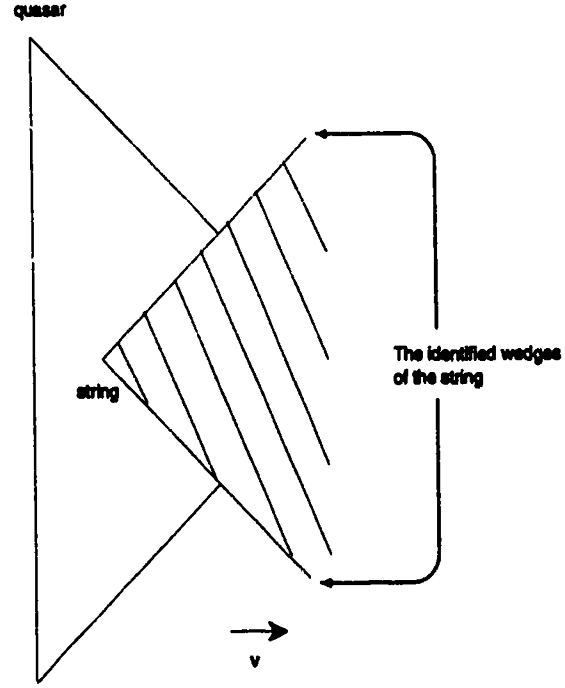

Figure 3.2.9: The string has a transverse velocity to the left where the light rays passing to the right of the string are blue shifted with respect to the light rays passing to the left.

be continued into the future for infinite affine parameter length while remaining in $\dot{H}^+(S)$. This could happen even in non-asymptotically flat space-times unless there are many causality violating regions whose Cauchy horizons intersected [1](p.86).

At the end we emphasize the point that the causal structure of Gott space-time is rather remarkable. Recall that i^0 is where space-like geodesics end. In this space-time there are those space-like geodesics which enter A and never leave A ($+x$, $-x$ axis) and there are those space-like geodesics which never even asymptotically approach A, *i.e.* space-like geodesics on space-like surfaces which foliate $I \cup II$.

3.3 Absence of Blue Shift Singularity in Gott Space-Time

Now we address the blue shift singularity which characterizes how a a small wave packet can possibly affect the background space-time. The blue shift of the light rays circulating the moving strings is basically the result of the same effect which causes anisotropy in the microwave background due to the transverse motion of infinite cosmic strings[23]. An observer at rest in the frame of the strings sees two images of the source neither of them red shifted with respect to the other one. But as soon as the string begins its transverse motion to the left according to the space through which the light ray passes, the observer has a velocity towards the source (Fig.3.2.9). Thus the observer's right image of the source is Doppler blue shifted with respect to the left image. All light rays passing to the right of the string will be blue shifted and all light rays passing to the left experience no Doppler effect[12].

We investigate a possible blue shift singularity of the null generators of $\dot{A}$. To

[12]Of course this depends on the specific choice of the deficit angle. Whether one side is moving toward the observer and the other side is receding from the observer is completely arbitrary. Only the relative blue shift has any physical meaning.

address this issue we use some of the basic ideas used by Hawking[13]. Each time a null tetrad is parallel propagated around the strings along the stable or unstable fixed point null vectors,[14] it is Lorentz boosted with a factor $\exp(h)$ with respect to the L frame of §3.2.3, and the temporal separation shrinks by $\exp(-h)$, so the power grows by $\exp(2h)$ (this is due to Lorentz contraction of a volume element).

But possible infinite blue shift is not the only factor contributing to the divergence in energy-momentum tensor. Another important geometrical quantity associated with the null ray on the Cauchy horizon is the change of cross sectional areas of a pencil of generators as the null tetrad is parallel propagated around the strings. Let

$$f = \ln \frac{S_{n+1}}{S_n}, \tag{3.35}$$

where S_n and S_{n+1} are the areas of the pencil on successive loops around the strings. The quantity f measures the amount the generators are diverging in future direction. The classical stability of $H^+(S)$ is measured by $\exp(2h - f)$, in other words $(2h - f) \leq 0$ guarantees that there is no blue shift singularity on $H^+(S)$.

For a null ray the blue shift after leaving $+x$, scattering both strings and returning to $+x$ is independent of the choice of coordinate system. Therefore we try to calculate this quantity due to the crossing from the wedge which is at rest in coordinate (1) as seen by an observer at rest in the coordinate of string (2). If ζ is the smaller fixed point angle which the null ray makes with the $-x$ axis in the rest frame of string (1) before scattering from the strings, and consequently $(2\alpha - \zeta)$ is the angle which that ray makes with $+x$ after scattering (Fig.3.2.9), since $(\omega, \mathbf{k})$ makes a 4-vector, the change in the frequency of the ray passing through wedge (1) as seen

[13] The full machinery used by Hawking to investigate this problem is rather complicated and was originally designed for non-flat metrics in four dimensions. We only need the criteria which he uses to show the stability of Cauchy horizon for the specific example given in his paper [15].

[14] It is also useful to know that the null tetrad is defined so that one of the basis vectors is tangent to the null vector along which the null tetrad is parallel propagated.

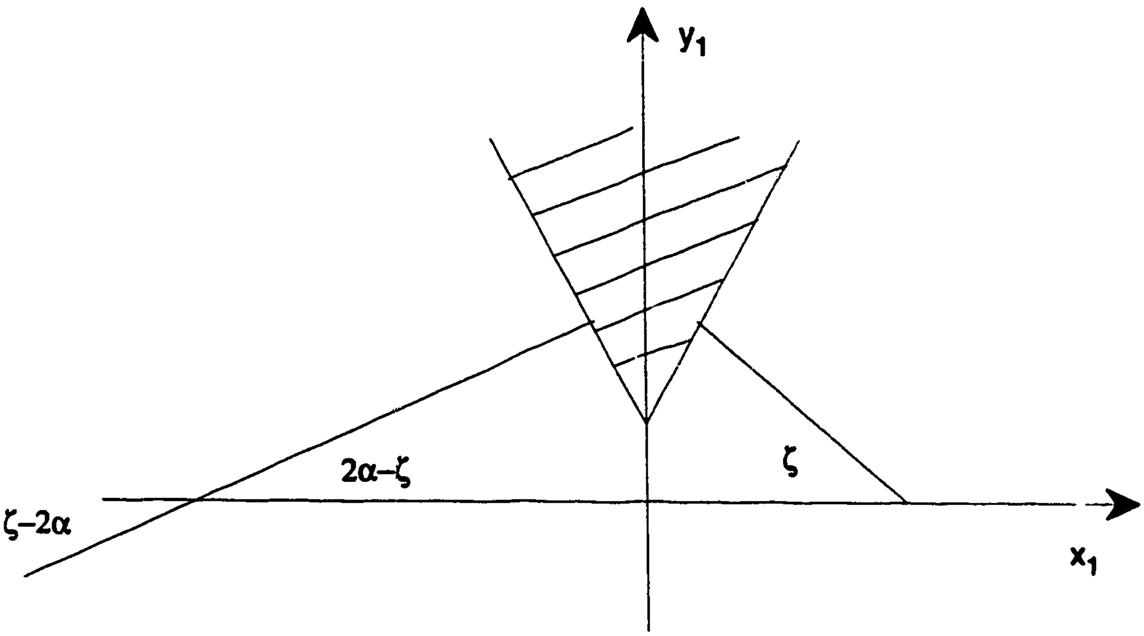

Figure 3.3.1: Scattering of an arbitrarily directed light ray from a string with angle deficit 2α as is seen by an observer at rest in frame (1).

from frame (2) is

$$\begin{aligned} \omega_{2i} &= \gamma(\omega_1 - \mathbf{u}.\mathbf{k}), \\ \omega_{2i} &= \gamma(\omega_1 - \omega_1 u \cos\zeta). \end{aligned} \tag{3.36}$$

Likewise,

$$\omega_{2f} = \gamma[\omega_1 - \omega_1 u \cos(2\alpha - \zeta)], \tag{3.37}$$

therefore from the above equations and (3.19) we conclude

$$\kappa = \frac{\omega_{2f}}{\omega_{2i}} = \frac{1 - u\cos(2\alpha - \zeta)}{1 - u\cos\zeta} > 1, \tag{3.38}$$

in which w_{2i} and w_{2f} are the frequencies of the null ray before and after crossing the wedge at rest in (1) frame. Using (3.19) it can be shown that $\exp(h) = k > (<)1$ for generators along the past (future) boundary.

To address the convergence of the generators, we see that because of the absence of curvature the null geodesic generators of $\dot{A}$ are straight lines, therefore convergence in Gott space-time can only be inversely proportional to affine parameter λ *i.e.* it can only be of the form given in Fig.3.3.2, but this contradicts the fact that there are only two discrete fixed point angles. Therefore we have $f = 0$. To investigate this point further we solved the geodesic equations of motion in the lab frame for a generator starting of at $[t_L(x_{L_0}), x_{L_0}, 0]$ on $\dot{A}$ (Fig.3.2.5). After scattering form the wedge at rest in the frame of string (1) (Fig.3.3.1, also see Appendix C) this ray returns to $y_L = 0$ at $[t'_L(x'_{L_0}), x'_{L_0}, 0]$ with

$$x'_{L_0} = a x_{L_0} - b \tag{3.39}$$

in which

$$\begin{aligned} a &= -1 \\ b &> 0. \end{aligned} \tag{3.40}$$

Therefore due to symmetry after one complete spiral around both strings

$$x_{L_0} \rightarrow x_{L_0} + 2b \tag{3.41}$$

which would not be possible if the generators of $\dot{A}$ were to have non-zero convergence since b is independent of x_{L_0}. So $f = 0$ and the generators of the null boundary are blue shifted on the past boundary and red shifted on the future boundary going towards the future with no convergence or divergence.

The result achieved so far does not strictly prove that there is no blue shift singularity, though we believe there is none. A more careful look at Fig.3.2.5 shows that even though along the past boundary blue shift causes exponential growth in the energy density, this growth is finite. The generators of the past boundary leave the boundary through the cusp and stop blue shifting. This means for a wave packet starting of at a finite time in the past, the blue shift is finite.

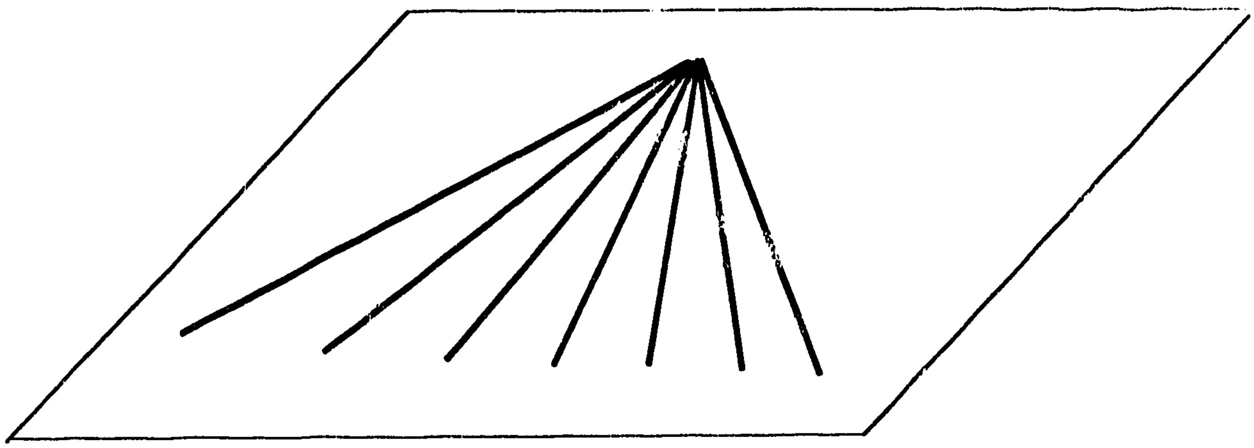

Figure 3.3.2: Due to the absence of curvature, convergence on the null boundary can only be constant.

The arguments presented in the last few sections show that Gott space-time, perturbed by a classical wave packet is most probably free of any singularities or exponential growth in the energy-momentum tensor density as the observers approach the causality violating region. Therefore if such causality violating region exists, from a classical point of view, it can be used for time travel[15].

In the next chapter we address the issue of the possibility of constructing time machines using ordinary matter, a matter which satisfies the known energy conditions. In addressing this issue we use a method employed by S. Deser *et al.* [6] using the elements of the Poincaré group.

[15]Hawking showed that for Misner space-time, the classical instability of the Cauchy horizon allows one to choose a quantum state in which the semi-classical divergence of the quantum stress tensor in the vacuum state is cancelled. However, since the Gott space-time is classically stable, this mechanism could not be used to cancel a divergence in the quantum stress tensor if such a quantum divergence occurs in this space-time.

Chapter 4

Is Gott Space-Time Physically Realistic?

As we said in §2.1.1 space-time in (2+1)D is necessarily locally flat outside matter. Several consequences follow this flatness:

i There are no gravitational waves in classical and quantum gravity in (2+1)D, which immediately follows from $C_{\alpha\beta\gamma\delta} = 0$.

ii Forces between sources are not mediated by graviton exchange, but rather interactions arise from non-trivial geometric and topological properties of the 2-surface.

iii The Einstein gravity in (2+1)D does not approach Newtonian gravity in the $G \to 0$ limit, since Newtonian gravity in (2+1)D requires the force law to have an inverse distance character whereas Einstein gravity is flat regardless of the value of G.

These characteristics of gravity in (2+1)D inspired Deser, Jackiw, 'tHooft and Teitelboim to embark on the investigation of planar gravity in 1981 [11][24][25]. They hoped this dimensional reduction of the theory would help them gain insight into formulating a way to overcome unrenormalizable infinities in (3+1)D which are absent in (2+1)D gravity.

During their study in a 1984 paper they presented solutions of the field equations in the presence of point particles having CTCs. These CTCs arise due to identifications along the t axis similar to the identifications on different t =const. planes due to the presence of moving parallel cosmic strings. They find it suggestive that the quantization of angular momentum would correspond to the quantization of jumps in time coordinate. But in the same paper they also added (without proof) that such jumps in time coordinate are not possible in a space with n moving particles where angular momentum is purely orbital, in contrast to spin angular momentum. By orbital angular momentum they mean angular momentum due to the relative spatial movement of particles about a specific point *i.e.* Gott moving parallel cosmic strings.

Following the publication of Gott's result regarding CTCs in the space-time of two moving parallel cosmic strings, they decided to use the methodology which they developed to treat spinless and spinning sources in (2+1)D, namely elements of the Poincaré group, representing the conical geometry of space-time to prove that although Gott's strings correspond to purely orbital motion of particles in (2+1)D, these particles are not physically realistic, by which they mean the particles correspond to a source with tachyonic (space-like) momentum.

In their methodology Deser *et al.* use elements of the Poincaré group (inhomogeneous Lorentz group) to construct a coordinate independent geometrical interpretation of particles with and with out spin in (2+1)D (to which Deser *et al.* refer as cosmons) and later tachyonic particles. In this section after a short review of some of the properties of the elements of the Poincaré group, we give a self-contained review of their results with emphasis on those parts which we think were treated rather lightly.

4.1 Conjugacy Classes of the Poincaré Group Elements

An element B of a group G is said to be conjugate to $A \in G$ if there exists $X \in G$ such that[26]

$$B = XAX^{-1} \text{ or } A = X^{-1}BX, \tag{4.1}$$

It can easily be proved that conjugacy is an equivalence relation and the equivalence class of mutually conjugate elements of a group is simply called a conjugacy class [1]. The importance of conjugacy classes in a group are in that the properties common to all the elements of a class represent coordinate independent physical properties of the conjugacy class. These physical properties are coded by the invariants of the conjugacy class. For example the set $(1, A, B, C, D_1, D_2)$ form a group. In this group A, B, C are reflections through the axes A, B, C of an equilateral triangle and D_n are the rotations of the triangle with angle $2\pi n/3$. (A, B, C) form a conjugacy class of this group. The Poincaré group in (2+1)D has a subgroup $\Omega(\theta)$,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{4.2}$$

rotations around the t axis in the (x, y) plane (Fig.3.3.1). Each element of this subgroup represents a conjugacy class.

Each element of pure boosts, L, in (2+1)D can be shown as,

$$\begin{aligned} L^0_0 &= \cosh u, \\ L^0_i &= L^i_0 = \hat{v}^i \sinh u, \\ L^i_j &= \delta^{ij} - \hat{v}^i\hat{v}^j(1-\cosh u), \end{aligned} \tag{4.3}$$

[1] One of the well known properties of equivalence classes are that they are disjoint.

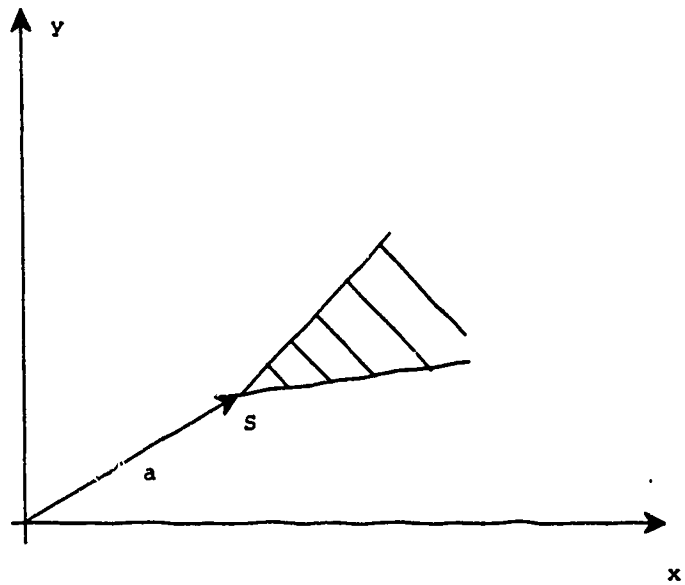

Figure 4.1.1: The rotations around t form a subgroup and each rotation represents a conjugacy class of the Poincaré group elements.

in which $\hat{v}$ is the unit vector in the direction of the boost and u the rapidity. It is easily shown (4.7) that $L(\hat{v}, u) \sim L(\hat{v}', u)$ in which $\hat{v}' \neq \hat{v}$. Therefore the parameter space of the equivalence classes of pure boosts and pure rotations is 2-dimensional.

The action of an element of the Poincaré group acting on a vector x in (2+1)D can be written as[27];

$$T(\mathbf{q}, B)(\mathbf{x}) = B(\mathbf{x}) + \mathbf{q} = \mathbf{q}B(\mathbf{x}) \tag{4.4}$$

in which B is an element of $SO(2,1)$ (homogeneous Lorentz group in (2+1)D), $\mathbf{q} = (t, x, y)$ is a translation and T is a Poincaré group element. There are two important invariants of the two mentioned conjugacy classes of T:

1- Since the action of the Poincaré group elements on vectors are represented by the action of a $SO(2,1)$ element and a translation, the traces of all the elements in a class is one of the invariants. This follows, since in this case the operation of conjugation becomes that of making a similarity transformation. To see this we investigate the conjugation of $T(\mathbf{q}, B)$ by $T(\mathbf{p}, A)$ which yields

$$\begin{aligned} T(\mathbf{p}, A)T(\mathbf{q}, B)T^{-1}(\mathbf{p}, A)(\mathbf{x}) &= \mathbf{p}A\mathbf{q}BA^{-1}\mathbf{p}^{-1}(\mathbf{x}) \\ &= ABA^{-1}(\mathbf{x}) - ABA^{-1}\mathbf{p} + A\mathbf{q} + \mathbf{p} \\ &= \Omega(\mathbf{x}) + \mathbf{r}, \end{aligned} \tag{4.5}$$

in which $\mathbf{r} = (t, x, y)$ is a translation, but

$$Tr(\Omega) = Tr(ABA^{-1}) = Tr(B), \tag{4.6}$$

therefore conjugation of the Poincaré group elements leaves the trace invariant. The importance of this quantity is that

$$\begin{aligned} Tr(B) &= 1 + 2\cos\theta \\ &\text{or} \\ Tr(B) &= 1 + 2\cosh u, \end{aligned} \tag{4.7}$$

depending on whether B belongs to the conjugacy classes of pure rotations or pure boosts respectively. As it turns out, it also corresponds to the mass of the point particles in (2+1)D.

2- The other important invariant quantity is $v^\nu q_\nu / v^2$ in which v^ν is the eigenvector of B with unit eigenvalue. The importance of this quantity lies in the fact that if B belongs to the conjugacy classes of pure rotations, in the coordinate system in which $B = \Omega(\theta)$, eigenvector of B defined by $B\mathbf{v} = \mathbf{v}$ is,

$$\mathbf{v} = (1, \vec{v}) = (1, 0, 0), \tag{4.8}$$

Therefore, this quantity represents q^0 (which is later shown to be related to the angular momentum of the particle). If B is a pure boost $v^\nu q_\nu$ represents identification along the null vector in the plane passing through axis t and $\vec{v}$ called $t - \vec{v}$ hyperplane.

These two invariants are all we need to describe the dynamics of point particles in (2+1)D and investigate the feasibility of the argument of Deser *et al.* in refuting that cosmic strings giving rise to CTCs in Gott space-time as unphysical.

4.2 Dynamics of Point particles in (2+1)D

4.2.1 Static Point Particles

To formulate the conical structure of space-time caused by straight cosmic strings as point particles in (2+1)D in terms of the elements of the Poincaré group, we show the angle deficit depicted in fig-(2.1.2) by the identification

$$x' = \Omega(2\alpha)(x) \equiv T(0,\Omega)(x). \tag{4.9}$$

The metric corresponding to this Poincaré group element is

$$ds^2 = \frac{1}{r^{8\pi\mu}}(r^2 d\phi^2 + dr^2) - dt^2, \tag{4.10}$$

which after a length rescaling ($r^{\alpha}\alpha^{-1} \to r'$, $\alpha\phi \to \phi'$, $\alpha \equiv 1 - 8\pi\mu$) gives the metric (2.3). Inserting the metric components of (4.10) in formulae for the Einstein tensor component for the general stationary metric given in Appendix-B, we derive

$$\begin{aligned} \sqrt{g}T^{00} &= \mu\delta^2(r), \\ \sqrt{g}T^{i0} &= 0, \\ \sqrt{g}T^{ij} &= 0, \end{aligned} \tag{4.11}$$

which truly represents a static point particle at the origin. The Poincaré group element identification corresponding to a static particle located at $\mathbf{a} = (a_0, \vec{a}) = (0, \vec{a})$ is

$$\mathbf{x}' = \Omega(\mathbf{x} - \mathbf{a}) + \mathbf{a} = T(\Omega, \mathbf{a})T(0, -\mathbf{a})(\mathbf{x}), \tag{4.12}$$

in which Ω is a pure rotation and therefore $\vec{a} \neq 0$ merely represents that the particle is not at the origin. This fact does not represent an invariant physical quantity. There is a representation in which $\vec{a} = 0$ with Ω being a pure rotation, but if $a_0 \neq 0$, there would be no representation rendering $a_0 = 0$ and keeping $\Omega(\theta)$ a pure rotation (or in the language of group theory, there would be no irreducible representation of $\Omega(\theta)$ in which $a_0 = 0$). Therefore a_0 is related to an invariant physical quantity.

4.2.2 Moving Particles with Angular Momentum

The identification corresponding to a particle located at a moving with velocity $\vec{v}$ is

$$\mathbf{x}' = \mathbf{a} + B(\vec{v})\Omega B^{-1}(\vec{v})(\mathbf{x} - \mathbf{a}) \equiv T(\mathbf{a}, B)T(0, \Omega)T(\mathbf{a}^{-1}, B^{-1})(\mathbf{x}) \qquad (4.13)$$

in which $B(\vec{v})\Omega B^{-1}(\vec{v})$ is a similarity transformation of Ω from the coordinate system in which the particle is at rest to the coordinate system in which the particle is moving with the velocity $\vec{v}$.

Equivalently for two particles located at (a,-a) moving along the $(+x, -x)$ directions with velocities $(\vec{v}, -\vec{v})$, the identification equation corresponding to (4.13) is written as

$$\mathbf{x}'' = \mathbf{a} + B\Omega B^{-1}\left\{-2\mathbf{a} + B^{-1}\Omega B(\mathbf{x} + \mathbf{a})\right\} = \Omega'(\mathbf{x}) + \mathbf{c}. \qquad (4.14)$$

Now for this identification to belong to the conjugacy classes of pure rotations we should have

$$Tr(B\Omega B^{-1}B^{-1}\Omega B) = Tr(\Omega') = 1 + 2\cos 2\alpha'. \qquad (4.15)$$

which requires $\left|\frac{Tr(\Omega')-1}{2}\right| < 1$.

The second invariant which signifies identifications in the time coordinate can be calculated to be

$$v^{\mu}c_{\mu} = c^0 = 2\sinh u \sin\alpha \left\{\cos\alpha(-2a_y) + \sin\alpha \cosh u(2a_x)\right\}. \qquad (4.16)$$

To interpret this equation correctly we note that in (2+1)D world, because of the above identifications, the fact that the particles have velocities $+\vec{v}$ and $-\vec{v}$ does not necessarily require $\vec{v}_{cm} = 0$ [7], moreover the mass addition formula is non-linear and follows the prescription given by (4.15). We expect as $(\alpha\vec{v}) \to 0$, $\vec{v}_{cm} \to 0$ and the mass addition formula follows that of ordinary (non-conical) space. Therefore we expect

$$\lim_{\alpha,\vec{v}\to 0} c^0 = 4\alpha\vec{a} \times \vec{v} = 8\pi J, \tag{4.17}$$

in which J is the total orbital angular momentum of the particles about the origin. The general two parameter (no charge) stationary metric describing this particle and the energy momentum tensor derived using Appendix-B can be written as (Again it is not this metric which is directly inserted in Einstein's equation. The actual metric is related to this metric by a length rescaling and is similar to (4.10)[11])

$$\begin{aligned} ds^2 &= [d(t + J\theta)]^2 - [dr^2 + r^2 d\theta^2], \quad 0 \le \theta \le 2\pi(1 - 4\mu), \\ \sqrt{g}T^{00} &= M\delta^2(\mathbf{r}), \\ \sqrt{g}T^{0i} &= (1/2)J\epsilon^{ij}\partial_j\delta(x)\delta(y), \end{aligned} \tag{4.18}$$

in which i, j refer to Cartesian coordinates, proving that this metric genuinely represents a spinning particle at the origin. Therefore we give a brief detour to study the space-time of a spinning source in (2+1)D.

4.2.3 Spinning Sources in (2+1)D

By a simple transformation $(t + J\theta) \to T$, the metric (4.18) yields the manifestly flat form

$$ds^2 = dT^2 - dr^2 - r^2 d\theta^2, \tag{4.19}$$

but because of the identifications $\mathbf{x}_{(\theta=0)} = \mathbf{x}_{[\theta=2\pi(1-4\mu)]}$, the above metric represents a CTC for $r < J$ fig-(4.2.1), namely the line r =const., $0 \le \theta \le 2\pi(1 - 4\mu)$.

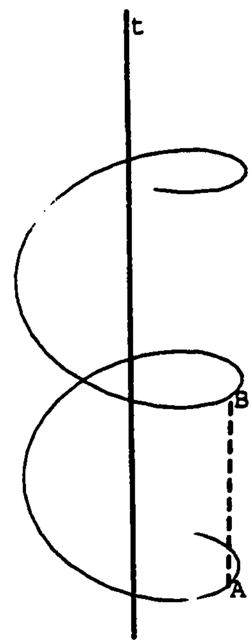

Figure 4.2.1: The line r =const., $0 \leq \phi \leq 2\pi(1-4\mu)$ is a CTC (the dashed line). The surface of constant "inertial" time T winds helically around the t axis. The event B lies in the causal future of A.

To gain further insight into an observer independent interpretation of the nature of the existence of CTCs in this (2+1)D world, we try to interpret metrics (4.18) and (4.19). For the observer in the coordinate system which is represented by the metric (4.18) traveling on a circle with radius R centered at the origin, all the physical properties of space-time can be represented by a spinning point particle with angular momentum J and mass M. As it was shown for this observer there exists CTCs for $r < J$. For the observer in the coordinate system represented by the metric (4.19) traveling on a similar circle in his/her coordinate system, the space-time is stationary and the properties of space-time can be represented by static point particles inside the circle. This observer also experiences CTCs at $r < J$ since the causal properties of space-time are coordinate independent[2].

After this brief discussion we address the possibility of the existence of CTCs

[2]More precisely the geometrical properties of space-time are diffeomorphism invariant.

in the space-time of two moving spinless particles. We note that the Gauss-Bonnet theorem §2.2 and equation (2.5) place a strong condition on the trajectory of the observers entering the causality violating region, the region containing CTCs. According to equation(2.5), the metric (4.19) truly represents the space-time of an observer at $r = l$ as long as R(the 2-curvature)$= 0$ for $r \geq l$. In other words there are no particles outside a circle of radius l centered at the origin. For an observer entering CTC violating region at a distance l from the origin this requires $R = 0$ for $l \geq J$. To see whether this condition is met if $a = l$ from(4.17) if ($\alpha = 4\pi\mu$) we have

$$2\mu v > 1 \Rightarrow \mu > 1/2, \tag{4.20}$$

which according to §2.2 necessarily requires the space-time to be finite. Pure orbital momentum, corresponding to a Poincaré group element representing a pure rotation, gives rise to CTCs only in finite space-times.

A summary of the results achieved in the last few sections is required at this point. There are at least two space-times in (2+1)D ,whose identifications belong to the conjugacy classes of pure rotations of the Poincaré group elements which permit causality violation

i space-times nearby a rotating point particle (or particles) which satisfies $R = 0$ for at least some point at $r < J$. Equation (4.17) was derived assuming angular momentum is purely orbital. Therefore for a spinning point particle, $\vec{a} = 0$ and r can be arbitrarily small so $r < J$ is satisfied. This is not a completely new result, as the underlying reason of the existence of CTCs in the stationary axially symmetric space-time of the Kerr geometry and the Van-Stockum infinite rotating cylinder [1] are manifestly similar to those of spinning point particles in (2+1)D. Like that of a spinning point particle there is a limit to how far the CTCs are spread out. This limit is proportional to the mass or mass density of the rotating object and the rate of rotation.

ii Finite space-times, space-times whose space-like cross sections are compact. Compact space-times in (3+1)D do give rise to CTCs and it is not surprising to have CTCs in (2+1)D finite space-times with moving point particles. The possibility of the existence of CTCs in these space-times was investigated by 't Hooft[28]

The question remains how Gott particles fit into this picture. Gott space-time is the space-time of two point particles moving with equal velocities in opposite directions whose space-time satisfies certain conditions. Therefore Gott space-time is not from category (i) since angular momentum in Gott space is totally orbital and due to the relative motion of point particles. It is certainly not from category (ii) either, since according to §3.2.3 this space-time possesses non-compact partial Cauchy surfaces going to i^0 (space-like infinity). To answer the question asked at the beginning of the paragraph we go back to §4.1 and equation(4.15) to investigate the conjugacy classes of the Poincaré group elements which represent the coordinate identifications in Gott space-time.

4.2.4 Point Particles with Tachyonic Center of Mass

We recall from §3.1 that a necessary condition for the Gott space-time to have CTCs is that v, the velocity of each string in the lab frame to satisfy equation(3.6) which can easily be shown to require

$$\cosh u \sin \alpha > 1, \tag{4.21}$$

in which $v = \tanh u$. On the other hand the Poincaré group element representing coordinate identifications in Gott space are certainly given by left hand side of (4.15), therefore if this element belongs to the conjugacy classes of pure rotations $\left|\frac{Tr(\Omega')-1}{2}\right| < 1$ can be shown to require (this step is not straight forward and needs some algebra)

$$\cosh u \sin \alpha < 1, \tag{4.22}$$

therefore (4.21) only if the corresponding Poincaré group element belongs to the conjugacy classes of pure boosts in which case $Tr(\Omega') = 1 + 2\cos 4\pi\mu$, requires μ to be pure imaginary. This surprising result is due to the fact that identifications in Gott space-time are boost like; coordinate identifications are restricted to the $t - \vec{v}$ hyperplane, the plane passing through the velocity vector $\vec{v}$ and t axis.

To further investigate the physical properties of these boost identified space-times we assume $\vec{v}$ is along $+x$, the corresponding metric is[6]

$$d\tau^2 = dt^2 - dx^2 - \alpha\frac{(xdt - tdx)^2}{(t^2 - x^2)} - \left[d\left(y + \beta\ln\frac{t+x}{t-x}\right)\right]^2, \qquad (4.23)$$

which after inserting into Einstein's equation for general stationary metrics given in Appendix-B yields

$$T^{yy} \propto \delta(x)\delta(t). \qquad (4.24)$$

This equation has two important implications. First it confirms our earlier statement that because of the conical geometry of space-time the velocity addition formula does not follow that of non-conical space-time, despite the fact that the particles are moving in $+x$, $-x$ directions, $T^{yy} \neq 0$. The second implication is that (4.24) is the energy-momentum tensor of matter with tachyonic velocities which is a direct consequence of the first implication.

A consequence of a space-like center of mass momentum is that CTCs do not arise from the decay of a particle with initial time-like momentum, Deser *et al.* say [6] "Gott CTCs can not be created or destroyed, but come towards the interaction region from space-like infinity. We can view this as resulting from a boundary condition at space-like infinity that one should call 'unphysical', namely the identification (4.23). The source must always be moving at its high velocity in order to ensure this CTC creation". This conclusion is generally acceptable as long as one does not try to broaden the meaning of "unphysical" beyond its limit. One should also note that tachyonic center of mass of two particles with time-like momenta is a direct consequence of non-linear mass addition formula in multiconical space-time (see §5.3). Whether one calls this phenomenon 'unphysical' seems to be more a matter of taste than a sound physical reasoning. If we assume the space-time is causally well behaved at any time, it is always causally well behaved. But there is no reason to believe we are living in a universe which is causally well behaved at any time. One should also notice that CTCs approach the origin from $\mathcal{I}^-$ and go to $\mathcal{I}^+$, not as Deser *et al.* claim from i^0, space-like infinity.

Chapter 5

Gott Space-Time and Pseudo-Unitary Representations of the Lorentz Group in (2+1)D

Following the discovery of CTCs in the space-time of two moving parallel cosmic strings, being familiar with S. Deser *et al.ś* treatment of straight cosmic strings as point particles in (2+1)D, S. Carroll *et al.* [7] decided to investigate the feasibility of time travel in Gott space-time by representing point particles in (2+1)D, unlike S. Deser *et al.*, not by the conjugacy classes of Poincaré group elements, but by non-compact pseudo-unitary group of Lorentz transformations in (2+1)D, SU(1,1) and its corresponding Lie algebra. Despite striking similarities between the representation of Gott space-time in (2+1)D by the elements of Poincaré group and the elements of SU(1,1) (SU(1,1) and SO(2,1) have the same local properties) the work done by S. Carroll *et al.* bears new results and merits the devotion of an entire chapter to their treatment of (2+1)D point particles in Gott space-time.

S. Carroll *et al.ś* approach to this problem has several advantages as well as disadvantages over the approach made by S. Deser *et al.*

i perhaps the most important advantage of the S. Carroll *et al.ś* paper is that they claim to have formulated a method which allows them to treat generic many particle systems in (2+1)D dimensional space-time[7]. In §4.2.2 we treated a system with two particles only a system with a high degree of symmetry[1]. Using

[1] One of R. Jackiw's students, Daniel Kabat has studied a highly symmetrical many particle system using Poincaré group elements. In doing so he has used the fact that to maximize the angular momentum confined to a compact region of an open (2+1)D universe, one has to lay N

this method, later they claim to have been able to prove "in an open universe with net time-like momentum, no subset of particles can possess a space-like momentum"[22].

ii S. Carroll *et al.* do not draw any conclusions on the existence of CTCs in the space-time of spinning point particles with total time-like momentum (see §4.2.2). In fact in deriving the conservation laws regarding the particle decay they implicitly assume that space-time is causal at large distances from the point particles. This is also true for spinning point particles at sufficiently large distances from the origin.

iii The geometry of Gott space-time represented by the metric (4.23) can be shown to posses space-like identifications (reminiscent of Misner spaces [13](p.173)) which when viewed from properly Lorentz transformed coordinate system bears CTCs (the CTC possessing property of a space-time is coordinate independent). But S. Carroll *et al.* do not independently show Gott space-time possesses CTCs. They merely prove that (3.5) derived by Gott can not be satisfied by a center of mass with time-like momentum.

In this chapter in the light of what was said in §3 and §4 we try to give a self-contained comprehensive review of S. Carroll *et al.*ś treatment of Gott space-time and those aspects of multiconical space-time which was not clearly revealed (*i.e.* non-linear energy addition formula). After a self-contained review of $SU(1,1)$ and how it is related to (2+1)D gravity in §5.1, in §5.2 we study the decay of a particle into

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particles rotating on a circle around the origin. Then he proves there is no way for the angular momentum to reach the limit which is required for the creation of CTCs following discussions given in §4.2.3[29]. Later on, S. Deser and R. Jackiw use the generators of $SO(2,1)$ to treat two particle systems in (2+1)D[33].

two and see how the conservation laws directly follow the causality requirements in (2+1)D. In §5.3 we finally see how Gott space-time is represented by the elements of $SU(1,1)$ and in what sense this space-time is unphysical. §5.4 is the study of a peculiar similarity between (2+1)D anti-de Sitter space-time and the matrix representations of the group of Lorentz transformations in (2+1)D, namely $SU(1,1)$.

5.1 Pseudo-Unitary representations of the Lorentz Transformations in (2+1)D

If

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \end{aligned} \tag{5.1}$$

in which σ_i are the Pauli matrices, $SU(1,1)$, the group of unimodular unitary transformations consists of those elements of $g \in SL(2,C)$ (group of unimodular 2×2 complex matrices) which satisfy the relation

$$g^\dagger \sigma_3 g = \sigma_3. \tag{5.2}$$

It can be shown that this relation requires g to be represented as[30];

$$g = \begin{pmatrix} u+iv & x-iy \\ x+iy & u-iv \end{pmatrix} = u\sigma_0 + x\sigma_1 + y\sigma_2 + iv\sigma_3, \tag{5.3}$$

where

$$u^2 + v^2 - x^2 - y^2 = 1. \tag{5.4}$$

Now if $X = x^\mu \sigma_\mu$ in which $\mu = (0,1,2)$ represents the space-time coordinates in (2+1)D, $\det[X] = x_0^2 - x_1^2 - x_2^2$ is invariant under the transformations of the type

$$X' = gXg^\dagger \tag{5.5}$$

which on its own indicates that $SU(1,1)$ is homomorphic to at least some subclass of (2+1)D Lorentz transformations [31](p.294).

To understand the Lie algebra of $SU(1,1)$ (its tangent space at identity with properly defined structure constants [12](p.169)) which is essential to a good understanding of §5.2, §5.3 and §5.4, we investigate how pure boosts and pure rotations are represented by the elements of $SU(1,1)^2$. If $J = \frac{1}{2}\sigma_3$, in complete analogy with $SU(2)$, rotations in x, y plane with angle α are easily verified to be

$$R(\alpha) = \sigma_0 \cos\alpha/2 - i\sigma_3 \sin\alpha/2 = \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = e^{i\alpha J}. \tag{5.6}$$

Boosts in (x, y) less intuitively follow the group structure of $SU(2)$ and are what makes $SU(1,1)$a non-compact Lie group. They are shown to be

$$B(\vec{\xi}) = \sigma_0 \cosh\xi/2 + \hat{\xi}.\vec{\sigma}\sinh\alpha/2 = \begin{pmatrix} \cosh\xi/2 & e^{-i\phi}\sinh\xi/2 \\ e^{i\phi}\sinh\xi/2 & \cosh\xi/2 \end{pmatrix} = e^{-i\vec{\xi}.\mathbf{k}} \tag{5.7}$$

in which $\xi = |\vec{\xi}|$, $\hat{\xi} = \vec{\xi}/\xi$, $\vec{v} = \hat{\xi}\tanh\xi$ is the velocity in (x, y) plane, $\phi = \tan^{-1}\xi_y/\xi_x$ and $k_j = \frac{i}{2}\sigma_j$ $(j = 1, 2)$. Therefore

$$J = \tfrac{1}{2}\sigma_3 \quad , \quad k_j = \tfrac{i}{2}\sigma_j \tag{5.8}$$

form a basis (generators) for the Lie algebra over $SU(1,1)$.

The representations introduced in the previous paragraph, besides showing an easy way to calculate the trace of the elements of $SU(1,1)$(Tr[g]), also suggest that $(J, \vec{K})$ form a basis for the Lie algebra of $SU(1,1)$[32](p.26). One of the known properties of non-compact Lie groups is that they may posses elements connected to the identity σ_0 but not expressible as the exponentiations of the elements of the Lie algebra [7]. In this case the connection to identity is given by multiplication of several exponentiations of the components of the Lie algebra. As a matter of fact it will be

[2]The complex extensions of the Lie algebras of $SU(1,1)$and $SU(2)$ are identical[32](p.24).

shown (see §5.3) that $SU(1,1)$transformation matrices representing Gott space-time belong to this category.

Now we are in a position to see how $SU(1,1)$helps us to understand the dynamics of point particles in (2+1)D. In analogy to §4.14 the coordinate identifications representing the embedding of the space-time of a moving particle, with rest frame deficit angle α and rapidity $\vec{\xi}$ in (2+1)D flat space-time is shown by [3]

$$T = B(\vec{\xi})R(\alpha)B^{-1}(\vec{\xi}). \tag{5.9}$$

But now the generalizations begin. In a causally well behaved space-time in which $(1,\cdots,N)$ particles interact and result in $(1,\cdots,M)$ particles the relation

$$T_{tot} \equiv T_N T_{N-1}\cdots T_1 = T'_M T'_{M-1}\cdots T'_1 \tag{5.10}$$

should hold simply because the effects of interaction move with finite velocities (this statement can not be made in space-times with causal pathologies). In §5.2 the above equation yields the conservation laws regarding the decay of one particle into two.

To derive the proper form of the conservation laws we should define the total momentum of a group of particles in a way that for one particle it reduces to the $SU(1,1)$element representing the coordinate identifications in the space-time of a single particle namely (5.6) and (5.9). The total momentum is defined as;

$$e^{-i8\pi G P^\mu J_\mu} \equiv T_{tot} \tag{5.11}$$

in which $J_0 = J$, $J_i = -\epsilon_{ij}K_j$ with $i,j = (1,2)$, $\epsilon_{12} = 1$ defined in (5.8), M the mass of the particle and $P^\mu = (\gamma M, \gamma M\vec{v})$ is the 3-momentum. Two major consequences of the above equation are as follows:

[3] S. Carroll *et al.* unlike Cutler and S. Deser *et al.* do not use the term "coordinate identification". They rather use the fact that because of the coordinate identifications, parallel transport of a vector around the origin does not map the vector into itself. More technically, this is represented by the *holonomy* group of (2+1)D space-time.

i in the rest frame of a single particle

$$e^{-i8\pi GP^\mu J_\mu} = e^{-i\alpha J} = R(\alpha), \tag{5.12}$$

so in the rest frame (5.11) truly defines the 3-momentum of the particle. That (5.11) should also define the 3-momentum in the moving frame follows from the fact that in the matrix representation of a Lie group $Be^AB^{-1} = e^{BAB^{-1}}$. Therefore if $A = R(\alpha)$, the above equation properly defines the transformation of P^μ as Lorentz vector.

ii If P^μ is the 3-momentum associated with each particle $(1, \cdots, N)$ then

$$\begin{aligned} e^{-i8\pi GP^\mu J_\mu} &= T_{tot} = T_N T_{N-1} \cdots T_1 \\ &= e^{-i8\pi GP^\mu_N J_\mu} e^{-i8\pi GP^\mu_{N-1} J_\mu} \ldots e^{-i8\pi GP^\mu_1 J_\mu} \\ &\neq e^{-i8\pi G(P^\mu_N + P^\mu_{N-1} + \cdots + P^\mu_1)J_\mu}. \end{aligned} \tag{5.13}$$

This is an indication of non-linear addition of energy and momenta of a group of particles which is characteristic of a multiconical space-time. The above inequality is also an indication of the fact that the order in which matrices are multiplied to yield T_{tot} is important since for Abelian groups the inequality at the end of (5.13) is replaced with an equality.

All the information regarding the parallel transport of a vector around a moving particle (specifying the holonomy group) is contained in (5.9). Calculating T in (5.9) explicitly yields

$$T^A(p, \alpha', \phi) = \begin{pmatrix} \sqrt{1+p^2}e^{-i\alpha'/2} & ipe^{-i\phi} \\ -ipe^{+i\phi} & \sqrt{1+p^2}e^{i\alpha'/2} \end{pmatrix} \tag{5.14}$$

in which

$$p \equiv \sinh \xi_A \sin \alpha_A/2 = \gamma_A v_A \sin \alpha/2, \tag{5.15}$$

given $0 \leq \alpha \leq \pi/2$, uniquely determines the magnitude of the special relativistic 2-momentum of the particle. The above equation should not be confused with (5.12) as the true definition of 3-momentum in multiconical space-time. In (5.14)

$$\tan \alpha'/2 \equiv \cosh \xi \tan \alpha/2 \tag{5.16}$$

is related to the total energy of the moving particle. These abbreviations greatly simplify the treatment of two particle systems.

5.2 The Decay of a Particle into Two

To have a better grasp of the physical reality of Gott space-time we first study particle decay in this space-time which is later used to analyze universes with total time-like momentum. (2+1)D particle decay is based on the a priori assumption that the space-time of a single particle is causally well behaved and any CTC creation is confined to with in a neighborhood of the origin[4]. This assumption requires the coordinate identifications at large and small distances from the origin to be the same[5]. Coordinate identification is given by combination of (5.14) and (5.6) at small and large distance respectively. In particular equating

$$T(\alpha) = T^B(p_B, \alpha'_B, \phi_B)T^A(p_A, \alpha'_A, \phi_a) \tag{5.17}$$

with (5.6) yields the conservation laws regarding the disintegration of a static particle into two, namely

$$\begin{aligned} \alpha'_A + \alpha'_B &= \alpha \quad , \\ p_A = p_B &= p \quad , \\ \phi_A - \phi_B &= \pi - \alpha/2 \quad . \end{aligned} \tag{5.18}$$

[4]Gott space-time simply does not satisfy the above criterion.

[5]The holonomy group is a discrete group. (I)-Identity which corresponds to no transformation. (II)-Integer multiples of positive parallel transport of a vector around the particle. (III)-Integer multiples of negative parallel transport of a vector around the particle.

As it is seen the very fact that Gott particles satisfy

$$\phi_A = \phi_B = \pi \tag{5.19}$$

guarantees that they can not arise from the disintegration of a particle into two. This fact corresponds to the statement in §4.2.4 that the Poincaré group elements representing coordinate identifications in Gott space-time do not belong to the conjugacy classes of pure rotations. In other words (5.18) is an indication of the fact that as the particle disintegrates, the offspring particles move back to back on the two sides of the cone and the distortion in the geometry of the space-time moves away from the center with the velocity of the faster particle. As it is seen this picture is inconsistent with particle kinematics in Gott space-time.

To give further insight what was said in the previous paragraph, a t =const. surface is drawn as the particles move away from the decaying particle. To simplify calculations the deficit angle $\alpha_{tot} = \alpha'_A + \alpha'_B$ is located in a way that $v_{y_A} = v_{y_B} = \frac{p}{\sqrt{1+p^2}}$ (Fig.5.2.1). As the particle decays, two observers on y axis in the upper and lower half plane which were stationary with respect to each other now have relative velocities $\frac{2v_y}{1+v_y^2}$. This behavior suggests that the particle decay might result in the instability of this space-time.

5.3 Gott Particles

As R. Gott showed [4] the space-time of two moving particles contains CTCs provided the particle velocities in the lab frame satisfy $\gamma \leq \frac{1}{\sin 4\pi\mu}$ (see §3.1)

$$b \equiv \cosh\xi \sin\alpha/2 > 1. \tag{5.20}$$

In the *cm* frame coordinate identifications are either described by (5.6) or (5.7). These equations also show a convenient way to determine whether coordinate identifications

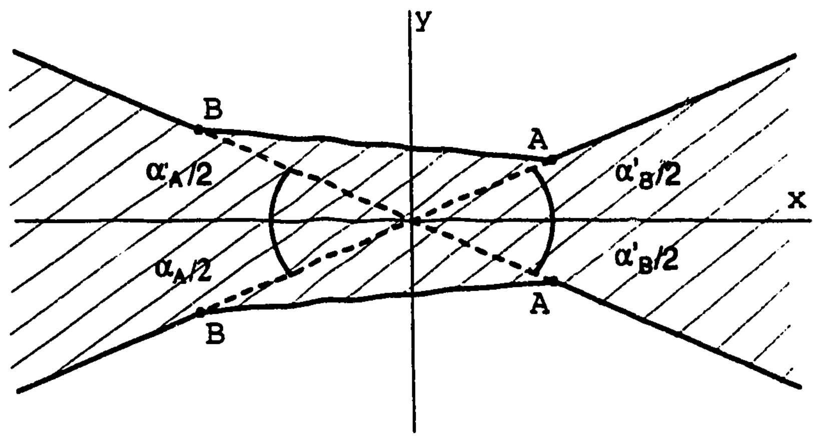

Figure 5.2.1: The decay of a particle into two.

belong to the holonomy group of pure rotations or they necessarily involve both rotations and boosts. We have

$$\frac{1}{2}\mathrm{Tr}[e^{-i8\pi GP^\mu J_\mu}] = \begin{cases} \cos\alpha/2 = \cos\sqrt{-4\pi GP^2} & P^2 < 0 \ , \\ \cosh\xi/2 = \cosh\sqrt{+4\pi GP^2} & P^2 > 0 \ . \end{cases} \tag{5.21}$$

The general form of the coordinate identification matrix for two moving particles is given in (5.17). For Gott particles

$$\begin{aligned} \alpha_A &= \alpha_B = \alpha \ , \\ \xi_A &= \xi_B = \xi \ , \\ \phi_A &= 0 \ , \\ \phi_B &= \pi \ . \end{aligned} \tag{5.22}$$

Inserting these values in (5.17) yields

$$\frac{1}{2}\mathrm{Tr}T^{BA} = 1 - 2b^2 < -1, \tag{5.23}$$

which is not covered by either of (5.21). It can be seen from (5.6) and (5.7) that no matrix of the form (5.11) can have $\mathrm{Tr}[T_{tot}] < -1$. But it can be shown that the combinations of several exponentiations

$$T_{tot} = e^{-i8\pi G P^{\mu} J_{\mu}} e^{-i8\pi G P'^{\mu} J_{\mu}} e^{-i8\pi G P''^{\mu} J_{\mu}} \tag{5.24}$$

can have negative trace. Therefore the $SU(1,1)$element representing Gott space-time can not be written as the exponentiation of an element of the Lie algebra of $SU(1,1)$. The question remains whether this property of Gott space-time has any physical significance or is merely due a group property of $SU(1,1)$. In trying to answer this question we note that $SU(1,1)$is a double cover of $SO(2,1)$. Namely

$$R(\alpha)|_{SU(1,1)} = -R(\alpha+2\pi)|_{SU(1,1)} \tag{5.25}$$

whereas they represent the same Lorentz transformation. Since Tr$[R(\alpha)]$ =-Tr$[R(\alpha+2\pi)]$, $\pm$Tr$[T^{BA}]$ represent the same Lorentz transformation and Gott space-time corresponds to the second equation in (5.21). In §5.4 we see that anti-de Sitter space-time has similar properties which are not surprisingly related to the exponential map on that space-time manifold.

To gain further insight into on how Gott space-time might be constructed from accelerating stationary particles after inserting (5.22) into the right hand side of (5.17), unlike §5.2, (5.17) is equated not to $R(\alpha)$, rather to a boosted rotation, namely (5.14). After equating the real and imaginary parts of the T_{12} component and using (5.21), we derive

$$\begin{aligned} \phi_{cm} &= \pi/2 \quad , \\ v_{cm}^2 &= \tfrac{g^2-\sin^2\alpha/2}{1-\sin^2\alpha/2} \quad . \end{aligned} \tag{5.26}$$

As it is seen, Gott space-time necessarily requires the center of mass to move with tachyonic velocities $v > c$ in the y direction. It should not escape attention that the origin of this tachyonic center of mass is due to the multiconical nature of the space-time and by no means requires violation of any conservation laws. However it shows that causality violations and tachyonic center of mass for any two particles are intertwined, therefore the existence of one necessarily requires the existence of the other . A condition exists which is similar to §4.24 and which one might call "unphysical".

Following the same procedure, S. Carroll *et al.* further prove that the decay of two particles into four can not result in a tachyonic center of mass for any two of the produced particles and then generalize their result to show "in an open universe with net time-like momentum, no subset of particles can possess a space-like momentum"[22]. It has been shown that in closed (2+1)D universes, the decay of a particle into two can result in creation of CTCs satisfying Gott condition (5.20) in the vicinity of particles once they get close enough. However this space-time is unstable and collapses before the creation of CTCs[28].

5.4 Gott Particles and (2+1)D Anti-de Sitter Space-Time

The space of constant curvature with $R < 0$ is called anti-de Sitter space Anti-deSitter space in (2+1)D has topology $S^1 \times R^2$ and its geometry is the geometry induced on the hyperboloid (Fig.5.4.1)

$$u^2 + v^2 - x^2 - y^2 = 1 \qquad (5.27)$$

in 4-D space (u, v, x, y) with the line element

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2. \qquad (5.28)$$

From (5.27) and (5.4) it is easily seen that anti-de Sitter space-time and $SU(1,1)$are *isomorphic* with group operation on this space-time defined by the isomorphism. The following coordinates are introduced on the hyperboloid:

$$\begin{aligned} u &= \cosh r \cos t' \quad , \\ v &= \cosh r \sin t' \quad , \\ x &= \sinh r \cos\phi \quad , \\ y &= \sinh r \sin\phi \quad . \end{aligned} \tag{5.29}$$

in which $0 \le t' \le 2\pi$. The line element induced on the hyperboloid can be written as

$$ds^2 = -\cosh r^2 dt'^2 + dr^2 + \sinh r^2 d\phi^2. \tag{5.30}$$

This space-time manifestly contains CTCs and has at least one time-like Killing vector $\frac{\partial}{\partial t'}$ which with the first integral ($ds^2 = g_{\mu\nu}dx^\mu dx^\nu$) yields a straight forward way to derive radial geodesics. With a coordinate rescaling

$$r' \; = \; 2\tan^{-1} e^r - \pi/2 \;\; ,0 \;\le\; r' \;\le\; \pi/2, \tag{5.31}$$

the metric (5.30) can be written in the conformal form $ds^2 = \cosh r^2 d\bar{s}^2$ in which $d\bar{s}$ is the metric for a flat (2+1)D Einstein static universe [13](p.121). In this coordinate system the tangent to geodesics can be written as [6];

$$\frac{dt'}{dr'} = \frac{\pm E}{\sqrt{E^2 + \epsilon \sec^2 r'}}, \tag{5.32}$$

in which

$$\epsilon = \begin{cases} + & \text{space-like} \\ - & \text{time-like} \\ 0 & \text{light-like} \end{cases} . \tag{5.33}$$

The conformal diagram of this space-time with geodesic lines is given : Fig.5.4.2. Region (I) corresponds to those points in anti-de Sitter space-time which can not be reached from p, an arbitrarily selected point (this space-time is homogeneous), by any geodesic. In other words for any $q \in$(I), q is not in the range of the

[6]For time-like geodesics, it is much easier to write the geodesic equation in derivative form rather than closed integrated form.

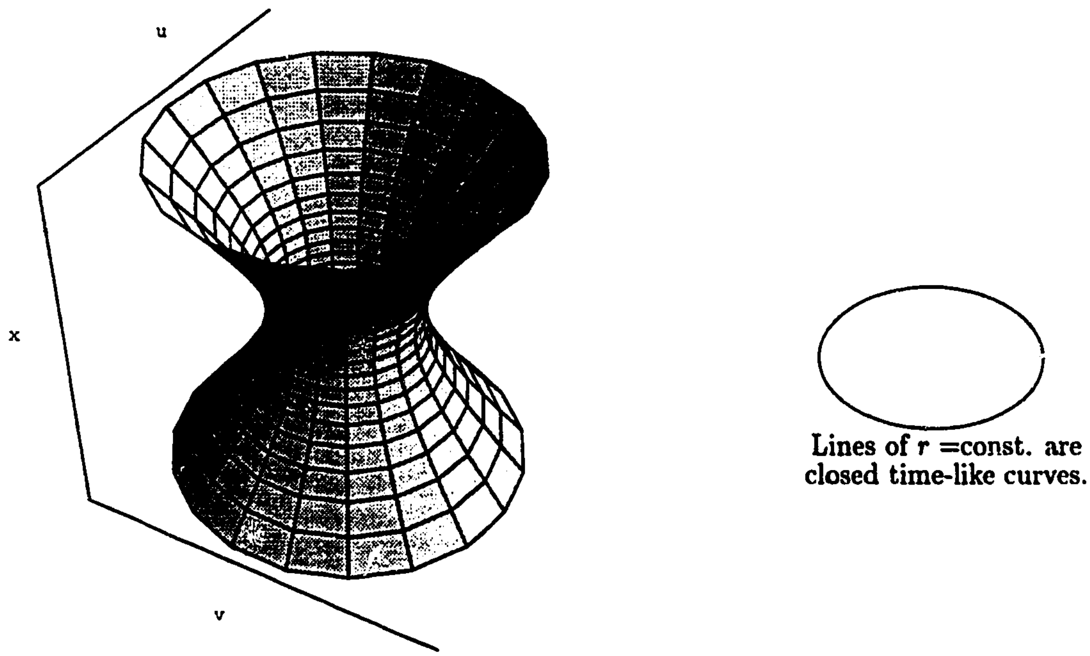

Figure 5.4.1: The geometry of anti-de Sitter space-time is the geometry induced on a hyperboloid in 4-D with the line element $ds^2 = -du^2 - dv^2 + dx^2 + dy^2$.

following map[13](p.33);

$$\exp : T_p \rightarrow \text{Ads} \tag{5.34}$$

which is the exponential map of the tangent space T_p to anti-de Sitter space-time manifold (Ads). This phenomenon can be recognized as a characteristic of $SU(1,1)$and its corresponding Lie algebra (*i.e.* tangent space at the identity). Gott particles are represented by those elements of $SU(1,1)$which are not connected to the identity by exponentiations of the elements of the Lie algebra. In other words these elements do not belong to a one parameter subgroup of $SU(1,1)$. To see this more clearly, $\frac{1}{2}\text{Tr}[g]$ from (5.4), (5.23) and (5.29) in conformal coordinates can be written as;

$$\frac{1}{2}\text{Tr}[g] = u = \frac{\cos t'}{\cos r'} < -1 \text{ for } \forall q \in (\text{I}). \tag{5.35}$$

Therefore in the light of what was said in §5.3 these elements of $SU(1,1)$cannot be written as exponentiation of Lie algebra elements.

The question remains whether the phenomenon mentioned above is of any physical significance or merely represents an aesthetic property of $SU(1,1)$. To answer this question we note that the elements of $SO(2,1)$ sufficiently represent coordinate identifications of point particles in (2+1)D space-time. But in (5.5) the Lorentz transformation of X is invariant under $g \rightarrow (-g)$, therefore homomorphism of $SU(1,1)$ onto $SO(2,1)$ elements maps g and $(-g)$ to the same $SO(2,1)$ element. In other words the points (u, v, x, y) and $(-u, -v, -x, -y)$ on the above hyperboloid (Fig.5.4.1) represent the same particle dynamics in (2+1)D and as far as this space-time is concerned, they are the same[7]. In conformal coordinates this identification is equivalent to

$$\begin{aligned} t' &= t' + \pi, \\ \phi &= \phi + \pi. \end{aligned} \tag{5.36}$$

It is easily seen that under such identifications, points in (I) on ϕ plane can be reached by space-like radial geodesics moving on $\phi + \pi$ plane.

[7]This is similar to the homomorphism of $SU(2)$ and $SO(3)$, the only difference is that the double value of $SU(1,1)$ is *not* an indication of an extra degree of freedom.

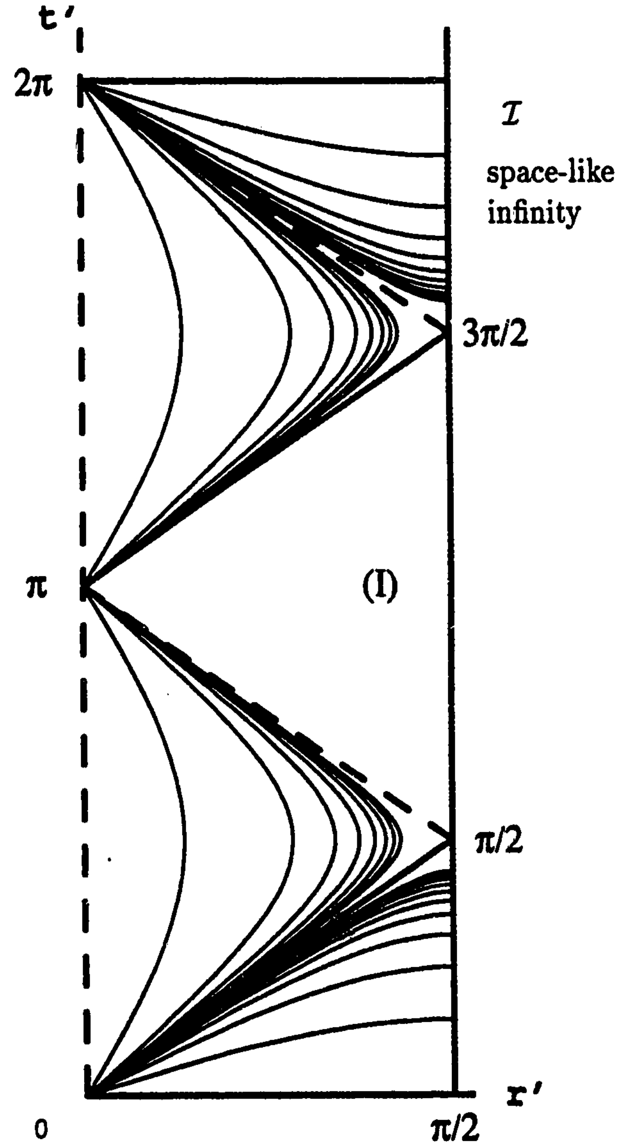

Figure 5.4.2: Conformal diagram of anti-de Sitter space-time with time-like and space-like geodesics. (I) correspond to $SU(1,1)$elements representing Gott space-time. Space-like geodesics go to space-like infinity which is a time-like surface. In $SU(1,1)$Time-like geodesics pass through $r' = 0$ at $t' = \pi$ and return to the origin at $t = 2\pi$. Under the $SU(1,1) \rightarrow SO(2,1)$ homomorphism, time-like geodesics return to the origin at $t' = \pi$.

Chapter 6

Conclusion

In conclusion, we embarked on what was treated lightly by the pioneers of investigating the structure of Gott space-time, which we tried to address more completely. If the strings are assumed to have finite thickness, Gott space-time is singularity free and not asymptotically flat, namely there is no open set isometric to an open neighborhood of $\mathcal{I}^+$ and $\mathcal{I}^-$ in Minkowski space-time, since causality violation extend to $\mathcal{I}^+$ and $\mathcal{I}^-$ in the space-time of two moving parallel cosmic strings. This fact rescues Gott space-time from having singularities, or rather rescues F. Tipler's theorem, which requires asymptotically flat space-times in which certain conditions hold to have singularities, from joining the long queue of incorrect theorems.

In trying to establish the lack of singularities in Gott space-time we found out that it has a rather interesting and perhaps bizarre causal structure. The causal behavior near i^0 depends on which space-like geodesic one approaches i^0. There are space-like geodesics in an open neighborhood never intersecting A, whereas there are space-like geodesics in an open neighborhood which intersect A and never leave that region again. However a complete understanding of the causal behavior of Gott space-time is possible if the conformal structure of this space-time is established. We tried to obtain a conformal diagram of this space-time but since it is not asymptotically flat, the problem turned out to be more difficult than what we originally thought, however, the effort was not completely fruitless. We could show that at least for some values of the parameters v, d, α and the polar angle θ, i^0 lies in a causally well behaved open neighborhood.

The absence of CNGs causes Gott space-time to be most probably free of singularities even when small classical perturbations are added. In most causality violating space-times which do have CNGs, like Thorne wormhole space-time and the example given in [15], the classical blue shift singularity is prevented by the divergence of future directed null geodesics infinitesimally close to a CNG which is caused by WEC violation (or in a less formal way, negative energy density). This is a fortunate event (from the point of view of the observers traveling in Gott space-time), because the blue-shifted future directed null generators of the past CTC boundary in Gott space-time have positive convergence and the existence of a CNG would result in a blue shift singularity (The classical stability does not necessarily guarantee that the Cauchy horizon is immune to divergence in the energy momentum density due to vacuum polarization [15].)

It seems as S. Carroll *et al.* claim, pseudo-unitary representations of Lorentz transformations in (2+1)D yield a more powerful tool to investigate many particle systems. In this method point particles are represented by $SU(1,1)$ elements. Using this method, they claim they have been able to prove that in a universe with total time-like momentum no subset of particles (each seperately assumed to have a time-like momentum) can have a net space-like momentum. After proving Gott particles cannot arise due to the decay of a particle into two, we addressed how Gott space-time corresponds to those elements of $SU(1,1)$ not connected to the identity by exponentiating Lie algebra elements. After an ansatz made by D. N. Page we see this space-time is equally indicated by those points in anti-de Sitter space-time not connected to the origin by any geodesic. In other words these points do not lie in one parameter subgroup of $SU(1,1)$. This seems to be an irrelevant group property of $SU(1,1)$ which disappears upon the homomorphic map of $SU(1,1)$ onto $SO(2,1)$.

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Appendix A

To derive the function $g(\phi_1) = \tilde{\phi}_2$ which maps the angle that a light ray makes in frame (1) with respect to the $-x_1$ axis before scattering from string (1) of a light ray to the angle $\tilde{\phi}_2$ with respect to the $+x_2$ axis in the frame of string (2) (Fig.3.2.3), we use Lorentz transformation laws as follows: First we define

$$\begin{aligned} \cos\phi_1 &= -\tfrac{dx_1}{dt} \ , \ \sin\phi_1 = \tfrac{dy_1}{dt_1} \\ \cos\phi_2 &= \tfrac{dx_2}{dt_2} \ , \ \sin\phi_2 = -\tfrac{dy_2}{dt_2} \end{aligned} \tag{A.1}$$

given Lorentz transformations,

$$x_1 = \gamma(x_2 - vt_2),\ t_1 = \gamma(t_2 - vx_2). \tag{A.2}$$

Therefore,

$$\cot\phi_2 \equiv -\frac{dx_2}{dy_2} = (1-u^2)^{-1/2}[\cot\phi_1 - u\csc\phi_1]. \tag{A.3}$$

Likewise,

$$\cot\tilde{\phi}_2 \equiv -\frac{dx_2}{dy_2} = (1-u^2)^{-1/2}[\cot\tilde{\phi}_1 - u\csc\tilde{\phi}_1], \tag{A.4}$$

in which $\tilde{\phi}_1, \tilde{\phi}_2$ are the angles seen by the observers at rest in the frame of string (1) and (2) after the scattering of the light ray from string (1) and $u = \frac{2v}{1+v^2}$ is the speed of frame (2) with respect frame (1).

To calculate $\tilde{\phi}_1$ in terms of ϕ_1 we divert attention to Fig.1.0.1 in which the deficit angle is placed below $y_1 = d$, so $\tilde{\phi} = 2\pi - (2\alpha - \phi_1)$, and after inserting in (A.4) we have

$$\cot\tilde{\phi}_2 = (1-u^2)^{-1/2}[-\cot(2\alpha - \phi_1) - u\csc(2\alpha - \phi_1)], \tag{A.5}$$

which immediately yields

$$g(\phi) = \cot^{-1}\{(1-u^2)^{-1/2}[-\cot(2\alpha - \phi_1) - u\csc(2\alpha - \phi_1)]\}. \tag{A.6}$$

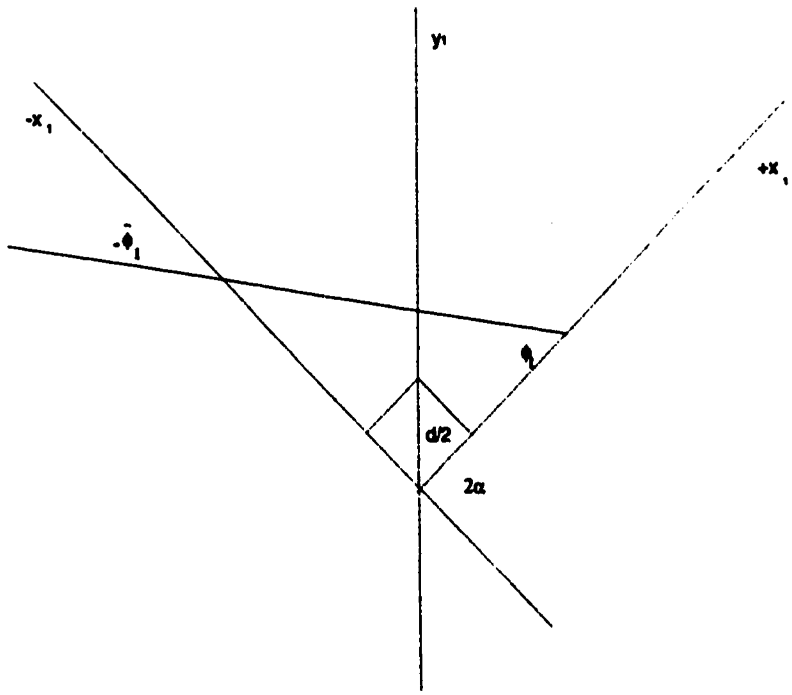

Figure 1.0.1: The angle deficit is placed below the $+x$, $-x$ axis.

Appendix B

A general stationary metric in (2+1)D can always be written as

$$ds^2 = -N^2(dt + k_i dx^i)^2 + \gamma_{ij} dx^i dx^j. \tag{B.1}$$

With this metric the Einstein equations can be written as

$$\begin{aligned}
G^{00} &\equiv \tfrac{1}{2N^2}R + \tfrac{3}{8}F^{ij}F_{ij} - k_i\nabla_j F^{ij} - 3K_i N^{-1}(\partial_j N)F^{ij} \\
&\quad -N^{-1}K_iK_j(\nabla^i\nabla^j - \gamma^{ij}\nabla^2)N + \tfrac{1}{2}N^2K_iK_jF^{ik}F^j_k - \tfrac{1}{8}N^2\gamma^{ij}K_iK_jF^{kl}F_{kl} \\
&\equiv 2\pi G T^{00}, \\
G^{0i} &\equiv -\tfrac{N^2}{2}K_jF^{jk}F^iF_k + N^{-1}k_j(\nabla^i\nabla^j - \gamma^{ij}\nabla^2)N + \tfrac{1}{2}\nabla_j F^{ij} \\
&\quad +\tfrac{3}{2}N^{-1}(\partial_j N)F^{ij} + \tfrac{1}{8}k^iN^2F_{jk} \\
&= 2\pi G T^{i0}, \\
G^{ij} &\equiv \tfrac{1}{2}N^2F^{ik}F^j_k - N^{-1}(\nabla^i\nabla^j - \gamma^{ij}\nabla^2)N - \tfrac{1}{8}\gamma^{ij}F^{kl}F_{kl} \\
&= 2\pi G T^{ij},
\end{aligned} \tag{B.2}$$

where R is the Ricci scalar constructed from the spatial metric γ_{ij}, ∇^i is the covariant derivative with respect to the spatial metric, and

$$\begin{aligned}
F_{ij} &\equiv \nabla_i K_j - \nabla_j K_i, \\
F^{ij} &\equiv \gamma^{ik}\gamma^{jl}F_{kl}.
\end{aligned} \tag{B.3}$$

Appendix C

To calculate the change in the coordinates of a null generator of the future boundary (Fig.3.2.5 and §3.3), as a null generator leaves $+x$ axis in the lab frame and scatters from the wedge at rest in the coordinate (1), its x coordinate advances in $-x$ direction *i.e.* $[t_L(x_{L_0}), x_{L_0}, 0] \rightarrow [t_L(x_{L_0} - b), x_{L_0} - b, 0]$. But this change in the x coordinate is constant for all null generators at all times *i.e.* $a = -1$. Therefore the null generators neither converge nor diverge. To calculate (a, b), we first perform the following transformation,

$$[t_L(x_{L_0}), x_{L_0}, 0] \rightarrow (t_{1_0}, x_{1_0}, 0), \tag{C.1}$$

to the coordinate system in which the string (1) is at rest. Then we perform the scattering according to Fig.1.0.1

$$(t_{1_0}, x_{1_0}, 0) \rightarrow (t'_{1_0}, x'_{1_0}, 0), \tag{C.2}$$

and perform the following transformation,

$$(t'_{1_0}, x'_{1_0}, 0) \rightarrow [t_L(x_{L_0} - b), x_{L_0} - b, 0]. \tag{C.3}$$

The result is

$$\begin{aligned} a &= -1, \\ b &= \frac{2^{3/2} d v^2 \sqrt{1-v^2} \sin\alpha \tan\alpha}{(1-v^2)\sqrt{2v^2 - 1 - \cos 2\alpha}} > 0. \end{aligned} \tag{C.4}$$