Amenability and fixed point properties of semi-topological semigroups of non-expansive mappings in Banach spaces

by

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Abstract

In this thesis we are interested in fixed point properties of representations of semi-topological semigroups of non-expansive mappings on weak and weak* compact convex sets in Banach or dual spaces. More particularly, we study the following problems :

Problem 1 : Let $\mathcal{F}$ be any commuting family of non-expansive mappings on a non-empty weakly compact convex subset of a Banach space such that for each $f \in \mathcal{F}$ there is an x whose f-orbit has a cluster point (in the norm topology). Does $\mathcal{F}$ possess a common fixed point ?

Problem 2 : What amenability properties of a semi-topological semigroup do ensure the existence of a common fixed point for any jointly weakly continuous non-expansive representation on a non-empty weakly compact convex subset of a Banach space ?

Problem 3 : Does any left amenable semi-topological semigroup S possess the following fixed point property :

(F^*) : Whenever S defines a weak* jointly continuous non-expansive representation on a non-void weak* compact convex set in the dual of a Banach space E, there is a common fixed point for S ?

Problem 4 : Is there a fixed point proof of the existence of a left Haar measure for locally compact groups ?

Our approach is essentially based on the use of the axiom of choice through Zorn's lemma, amenability techniques and the concept of an

asymptotic center in geometry of Banach spaces. Some positive answers are obtained for problem 1; however, problem 2 is settled affirmatively for three classes of semi-topological semigroups. These classes of semigroups together with left amenable and all left reversible semi-topological semigroups possess a fixed point property which is a weak version of (F^*). We show that n-extremely left amenable discrete semigroups satisfy a fixed point property much more stronger than (F^*); whereas, n-extremely left amenable semi-topological semigroups possess the fixed point property (F^*).
Among other things, results in Browder [10], Belluce and Kirk [3,4], and Kirk [37] are generalized to non-commutative families. A result of Hsu [32, theorem 4] is extended to semi-topological semigroups. Furthermore, some results related to the work of Lim (cf. [47],[49]) are obtained. A positive answer to question 4 is established for amenable locally compact groups.

Dedicated to **Khadime Rassoul s.a.w.**

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Contents

CHAPTER 1

Introduction and Preliminaries

1.1 Introduction

In non-linear functional analysis, the study of existence of a fixed point for non-expansive mappings (i.e., constant one Lipschitzian mappings) have been investigated since the 60's; and appears as an extension of the Banach fixed point theorem in the setting of Banach spaces. But in this new situation, even on bounded closed convex sets, existence of a fixed point is not guaranteed (of course in infinite dimension). According to the Schauder fixed point theorem [58], any non-expansive mapping on non-void norm compact convex set has a non-empty fixed point set; therefore this theory has a real interest only when we consider weak topologies in infinite dimension. A natural question is therefore the following : Do non-empty weakly compact convex sets in Banach spaces have the fixed point property for non-expansive mappings ? In other words, does every non-expansive mapping on a non-void weakly compact convex set into itself possess at least one fixed point ?

In the middle of the 60's, two surprising results in this direction appeared independently; in fact, Kirk [37] showed that in a reflexive Banach space, weakly compact convex sets with normal structure have the fixed point property for non-expansive mappings; and in the same period, Browder [10] proved that the same conclusion holds for non-void bounded closed convex sets in uniformly convex spaces. Whether Kirk's condition could be removed had been a quite long-standing open question which was answered negatively in 1981 by Alspach [1]. Kirk and Browder's results had a significant impact in the development of this theory; indeed, most of the results which appeared later are more or less their generalization. Kirk in a joint work with Belluce [3], improved [37] by considering finite commuting families of non-expansive mappings; and again under the same collaboration, in [4], they proved that their previous result still holds even for arbitrary commutative families if we replace "normal structure" by a stronger condition called "complete normal structure". Lim [48], showed that [4] holds even under a normal structure setting and proved later in [47] that both complete and normal structure are actually equivalent on weak compact convex sets; which is quite a surprising result. On the other hand, Bruck [8] gave a generalization of the above results by proving that on a closed convex subset C of a Banach space E with the fixed point property and together with the conditional fixed point property for non-expansive mappings, if C is either weakly compact or bounded and separable, then any commuting family of non-expansive mappings of C possess a common fixed point.

We know that a commuting family generates an amenable semigroup, and it is clear that a point is fixed by such a family if and only if, the semigroup generated by it does. On the other hand, amenability of certain function spaces of a semigroup can be characterized using fixed point properties. These observations show that amenability and fixed point theory are related. Day characterized left amenable semigroups in terms of a linear fixed point property in his famous result "Day's fixed point theorem". This result was extended to semi-topological semigroups by Mitchell in [53]. A non-expansive version of the Markov-Kakutani fixed point theorem was proved by DeMarr [17] and it was extended by Takahashi [59] to left amenable discrete semigroups. The topological version was done by Lau [42], where he characterized left amenability of a certain function space of semi-topological semigroups in terms of a fixed point property of non-expansive mappings in locally convex spaces.

In this thesis, we are interested in the study of fixed point properties of semi-topological semigroups of non-expansive mappings on compact convex subsets of Banach spaces in weak topologies. This work is basically motivated by the work of Belluce and Kirk, an interesting hard question raised by Lau [45],[46], a result of Izzo [34], and Lim [47],[49].

In Chapter 2, we are interested in the study of the first two problems. We point out that problem 1 was posted by Belluce and Kirk in [3], where they provided an affirmative answer within the framework of strictly convex spaces. We adopt a more general setting by looking at non-expansive representations of left amenable semigroups on weakly compact convex subsets of Banach spaces. We prove that left reversible discrete semigroups satisfy problem 1 if the underlying space is uniformly convex; the geometrical property "uniform convexity" being removable if we require instead a normal structure condition, generalizing [3,theorem 3],[4,theorem 2.1],[10,theorem 1 and 2] and [37,main theorem]. For Problem 2, some class of semi-topological semigroups which answer the question are introduced.

In chapter 3, we study the fixed point property (F^*), which is a very long-standing open question raised by A. T. -M. Lau (1976 in a Halifax conference). Until now, in our best knowledge, only a few special cases have been answered positively, e.g., Lau and Takahashi in [46], proved that the answer is positive for weak* compact convex sets which are separable in the norm topology; in [6] the question was settled affirmatively for commutative semigroups. We introduce two classes of semi-topological semigroups that satisfy a weak version of (F^*) obtained by requiring a relative compactness of some orbits with respect to some suitable locally convex Hausdorff topology; and also left amenable semi-topological semigroups possess a weak version of (F^*) using a result in ℓ^1. As a consequence, a version of Lim's fixed point theorem [47, theorem 3], in the Banach space setting, is obtained for weak* compact sets in dual spaces without requiring a normal structure condition. A result of Hsu [32, theorem 4] is extended to semi-topological semigroups. Furthermore, we show that discrete semigroups with a left invariant mean which can be written as a convex combination of multiplicative means, have a fixed point property much more stronger than (F^*), whereas semi-topological semigroups with a left invariant mean of this type possess (F^*).

Chapter 4 is motivated by a nice proof of Izzo on the existence of a left Haar measure for abelian groups, see [34]. Based on it, we are able to extend his result to a class of topological groups which contains abelian groups and compact groups.

Finally, in chapter 5, we make some remarks on the results obtained, and we derive some natural questions related to them.

1.2 Preliminaries

Let S be a semi-topological semigroup; i.e., a semigroup together with a Hausdorff topology such that for all $a \in S$, the mappings $s \mapsto a.s$ and $s \mapsto s.a$ are continuous from S into itself. When $(a, b) \mapsto ab : S \times S \to S$ is jointly continuous, then S is said to be a topological semigroup.

Let $C_b(S)$ denote the Banach space of all continuous bounded real-valued functions on S with the sup norm topology. Given $s \in S$ we consider the left translation operator $\ell_s : C_b(S) \to C_b(S)$ defined by $\ell_s f(t) := f(st)$.

1.2.1 Amenability

If $\Phi \subset C_b(S)$ is a closed subspace containing the constant function e (where $e(t) = 1$ for all $t \in S$), we say that Φ is *left translation invariant* if, $\ell_s(\Phi) \subset \Phi$, for all $s \in S$. If we fix such a subspace Φ, an element m of the dual Φ^* is called a *mean* on Φ if

$$m(e) = 1 \text{ and for all } f \in \Phi \text{ we have, } f \geq 0 \Rightarrow m(f) \geq 0.$$

A mean m is called *left invariant* if $m(\ell_s f) = m(f)$ for all $s \in S$ and f in Φ. The subspace Φ is said to be *left amenable* if it has a left invariant mean. For short, we shall write sometimes "Φ has a LIM" to stand for the statement "Φ has a left invariant mean".

If in addition Φ is an algebra, then a mean m on Φ is called a *multiplicative mean* if it satisfies

$$m(f.g) = m(f).m(g), \text{ for all } f, g \in \Phi.$$

If Φ has a left invariant mean which is a convex combination of multiplicative means on Φ, then we say that Φ is *n-extremely left amenable* and then we write "Φ is n-ELA" for short. The class of n-ELA semigroups was introduced and studied in [40] and [41]. It contains the class of extremely left amenable semigroups studied first by Mitchell in [51], and after by Granirer in [27] and [28].

When S is discrete (i.e., its topology is discrete) then $C_b(S) = \ell^\infty(S)$. In this case if $\Phi = \ell^\infty(S)$ has a left invariant mean, then S is said to be a *left amenable semigroup*.

Example 1.2.1.1 (left invariant mean) Let $S = \mathbb{Z}$ the group of integers. It is well-known, see [56], if U is a free ultra-filter on S then, $f \mapsto m(f) := \lim_U \frac{1}{2n+1} \sum_{i=-n}^{n} f(i)$ is an invariant mean on $\ell^\infty(S)$.

Example 1.2.1.2 (Amenable semigroups) The class of amenable semigroups contains commutative semigroups [14], finite groups, the bicyclic semigroup $S_1 = \langle e, a, b \rangle$ generated by a unit e and two elements a, b subject to the condition $ab = e$, see [44].

Note that not all semigroups are amenable; in fact, as known, any group containing a free subgroup of two generators is not amenable, see [14] and [15]; the partially bicyclic semigroups, $S_2 := \langle e, a, b, c \rangle$ generated by a unit e and three elements a, b, c such that $ab = ac = e$ and $S_{1,1} := \langle e, a, b, c, d \rangle$ generated by a unit e and four elements a, b, c, d subject to the conditions $ac = bd = e$ are also examples of non-amenable semigroups, see [44] for more details.

A semi-topological semigroup S is said to be *left reversible* if :

$$\overline{a.S} \cap \overline{b.S} \neq \emptyset, \text{ for all } a, b \in S.$$

That is any two closed right ideals in S have a non-void intersection. The collection of all left reversible semi-topological semigroups includes all topological groups, commutative semi-topological semigroups, discrete left amenable semigroups, normal semi-topological semigroups S for which $C_b(S)$ has a left invariant mean, see [31].

Remark 1.2.1.3 Note that a semi-topological semigroup S need not be left reversible even when $C_b(S)$ has a left invariant mean; indeed, Hewitt [30] has constructed a regular Hausdorff topological space S such that the only continuous real-valued function on it are constant functions; in [26], Granirer defined a semi-topological semigroup structure on S by letting $a.b = a$ for all $a, b \in S$; moreover for all $a \in S,\ f \mapsto \delta_a(f) = f(a)$ defines a left invariant mean on $C_b(S)$. However, S is not left reversible.

1.2.2 Function spaces

• The space of *left uniformly continuous* functions on S denoted by LUC(S) is the subspace of $C_b(S)$ of those mappings f such that the mapping $s \mapsto \ell_s f : S \to C_b(S)$ is continuous, i.e., $\lim_{t\to s} \|\ell_t f - \ell_s f\| = 0$. This space was introduced jointly by Mitchell and Itzkowitz, see [33]. As known, see [53], LUC(S) is a translation invariant closed sub-algebra of $C_b(S)$ containing e and therefore the constant functions on S. We shall say that a semi-topological semigroup S is *left amenable*, if LUC(S) has a LIM.

• The space of *left multiplicatively continuous uniformly continuous* functions on S, introduced by Mitchell [53], is the subspace of $C_b(S)$ of those f with the property that the mapping $s \mapsto \ell_s f : S \to C_b(S)$ is continuous when $C_b(S)$ is given the weak topology induced by the set of multiplicative means on $C_b(S)$; i.e., $f \in$ LMC(S) if for all multiplicative mean m on $C_b(S)$ and all $s \in S$, we have $\lim_{t\to s} |m(\ell_t f) - m(\ell_s f)| = 0$.

As LUC(S), see [53], the space LMC(S) is a translation invariant closed sub-algebra of $C_b(S)$ with the constant functions in it.

If S is a semi-topological semigroup, then the following hold, cf. [53].

- LUC(S)$\subset$ LMC(S) (immediate from the definition);
- LMC(S) $= C_b(S)$ if S is compact;
- LUC(S) $=$ LMC(S) $= C_b(S)$ if S is compact and topological;
- LUC(S) $=$ LMC(S) if S is discrete or a locally compact group.

Remark 1.2.2.1. It may happen for the above inclusion to be strict. In fact, we have the following example :

Example 1.2.2.2. In [50], P. Milnes and J. S. Pym, have constructed a semi-topological semigroup S such that LMC(S) $= C_b(S)$ and a function $f \in C_b(S) \backslash$LUC(S). Therefore the first inclusion is strict for S.

Let S be a discrete semigroup. A *finite mean* on $\ell^\infty(S)$ is defined as any element of the convex hull of all point measures of S; i.e., the convex hull of all mappings δ_s, $s \in S$ with $\delta_s(f) = f(s)$ for all $f \in \ell^\infty(S)$. A net $(m_\alpha)_\alpha$ of finite means is said to be *left strongly regular*, if it satisfies the following property :

$$\text{For all } s \in S, \text{ we have } \lim_\alpha \|\ell_s^* m_\alpha - m_\alpha\| = 0.$$

The limit is taken with respect to the norm of the dual $\ell^\infty(S)^*$; where the mappings $\ell_s^* m_\alpha$ are defined by $\ell_s^* m_\alpha(f) := m_\alpha(\ell_s f)$, for all $f \in \ell^\infty(S)$. For more details about this concept please refer to [14].

1.2.3 Representations of semi-topological semigroups

Let S be a semi-topological semigroup and (Y, τ) be a Hausdorff topological space. A *representation* of S on Y is a family $\mathfrak{S} = \{\hat{s} \; ; s \in S\}$ of mappings from Y into itself subject to the following condition :

$$\text{For all } s, t \in S, \; \hat{s.t} = \hat{s} \circ \hat{t}.$$

Such a representation is said to be *separately continuous* if for all $s_o \in S$ and $y_o \in Y$, the mappings : $s \mapsto \hat{s}(y_o) : S \to K, \; y \mapsto \hat{s_o}(y) : Y \to Y$ are both continuous. The representation is *jointly continuous* if the mapping $(s, y) \mapsto \hat{s}(y) : S \times Y \to Y$ is continuous when $S \times Y$ is endowed with the product topology. Given $y \in Y$, the *orbit* of y is denoted and defined by $\mathcal{O}_y := \{\hat{s}(y) \; ; \; s \in S\}$. A point $y \in Y$ is called a *common fixed point* for S (or a *fixed point* for S), if it satisfies $\hat{s}(y) = y$ for all $s \in S$. We shall denote by $F(S)$ the set of all such y and call it the *fixed point set* of S in Y. A subset M of Y is said to be S*-invariant* if for all $s \in S$, we have $\hat{s}(M) \subset M$. We shall adopt the following notations : Given $y \in Y$ then the symbol "$s.y$" stands for $\hat{s}(y)$ and "$s.M$" for $\hat{s}(M)$.

- If Y is a compact Hausdorff space, then given a left translation invariant subspace Φ of $C_b(S)$, we say that $\mathfrak{S}$ is an A*-representation* of S, Φ on Y if for all $f \in C(Y)$ and $y \in Y$, the mapping $s \mapsto f(s.y) : S \to \mathbb{R}$ belongs to Φ, cf. [52].

- If Y is a Banach space and $M \subset Y$ is a non-empty subset, then a mapping $T : M \to M$ is said to be *non-expansive* if it satisfies the following inequality :

$$\|T(x) - T(y)\| \leq \|x - y\|, \text{ for all } x, y \in M.$$

In this case, a representation of S on M is said to be a *non-expansive representation*, if for all $s \in S$ the mapping $\hat{s} : M \to M$ is non-expansive.

Example 1.2.3.1 (Non-expansive mappings) (cf. [38])

• Let B be the unit ball of c_0, then the mapping $T : B \to B$ given by $T(x) = (x_1, 1 - |x_1|, x_2, x_3, \cdots)$ is non-expansive with fixed point set $F(T) = \{-(1, 0, 0, \cdots), (1, 0, 0, \cdots)\}$.

• If $K := \{x \in \ell^1 \,; x_i \geq 0, \ \|x\| = 1\}$ then the map $T : K \to K$ defined by $T(x) = (0, x_1, x_2, \cdots)$, is non-expansive and fixed point free.

Remark 1.2.3.2. In infinite dimension, a non-expansive mapping on a bounded closed convex set may or may not possess a fixed point. The second example shows that in the Schauder fixed point theorem (see [58]) the compactness condition is crucial and cannot be dropped in general. On the other hand, the first example shows that uniqueness of a fixed point is not guaranteed.

Example 1.2.3.3 (Non-expansive representation) Let S be a semi-topological semigroup, let $\mathrm{M}(S)$ denote the set of all means on $\mathrm{LUC}(S)$ with the weak* topology; then $\mathfrak{S} = \{\ell_s^* \,; s \in S\}$ is a jointly continuous non-expansive representation of S on $\mathrm{M}(S)$.

1.2.4 Some geometric properties of Banach spaces

• *Normal structure.* Let K be a non-void bounded closed convex subset of a Banach space E. We say that K has normal structure if for all closed convex subset M of K whose diameter $\delta(M)$ is positive, then there is $x \in M$ such that :

$$\sup_{y\in M} \|x - y\| < \delta(M)$$

This notion was introduced in 1948 by Brodskii and Milman in [9]. In general, a normal structure condition is necessary in order to ensure the existence of a fixed point for a non-expansive mapping, by virtue of Alspach's remarkable counter-example in [1] answering an open problem for 15 years. An application of this concept yields the following result due to Kirk and which can be reformulated as follows : Bounded closed convex subsets with normal structure of reflexive spaces have the fixed point property for non-expansive mappings; and actually, this theorem still holds even for non-reflexive Banach spaces, see [38].

Example 1.2.4.1 Norm compact convex subsets of Banach spaces always possess normal structure, this fact was proved by DeMarr in [17].

Note that this fails in infinite dimension when dealing with weak topologies. In fact, Alspach [1] constructed a fixed point free isometry on the following non-empty weakly compact convex set

$$K = \{f \in L^1[0,1]\ ; 0 \leq f \leq 2 \text{ a.e and } \int_{[0,1]} f d\lambda = 1\}.$$

Note that K cannot have a normal structure according to Kirk's result.

Let us fix K be a non-empty bounded subset of a Banach space E.

• *Asymptotic center.* Given a decreasing net $(W_\gamma)_\gamma$ of non-void subsets of K, its asymptotic center in K is denoted and defined by :

$$\mathrm{AC}((W_\gamma)_\gamma, K) = \{x \in K \; ; \inf_\gamma \sup_{y \in W_\gamma} \|x - y\| = r((W_\gamma)_\gamma, K)\},$$

where $r((W_\gamma)_\gamma, K) := \inf_{y\in K} \inf_\gamma \sup_{z\in W_\gamma} \|y - z\|$ is the asymptotic radius of $(W_\gamma)_\gamma$. If K is weakly compact and convex, then its asymptotic center is non-void with the same properties as K (cf. [47]).

The concept of an asymptotic center was introduced by Edelstein [21] in 1972 for bounded sequences in uniformly convex spaces allowing him to derive a fixed point theorem generalizing a result of Browder cf. [10]. Lim [47] proposed the above definition as an extension of Edelstein's. The asymptotic center of a bounded sequence $(x_n)_n$ in K denoted $\mathrm{AC}((x_n)_n, K)$, was defined as the set :

$$\{x \in K \; ; \limsup_n \|x - x_n\| = \inf_{y\in K} \limsup_n \|y - x_n\|\}.$$

Note that if we let $W_n := \{x_i \; ; i \geq n\}$ for all $n \in \mathbb{N}$, then both definitions coincide. For related results about this concept cf. [18] and [19].

• *Asymptotic normal structure.* A convex set K is said to have asymptotic normal structure, if the asymptotic center of any decreasing net $(W_\gamma)_\gamma$ of non-empty subsets of any bounded closed convex subset W of K with $\delta(W) > 0$ is a <u>proper</u> subset of W.

By definition, asymptotic normal structure $\Rightarrow$ normal structure (just by fixing any direct set Γ, and define $W_\gamma = W$ for all $\gamma \in \Gamma$). Actually, the converse holds and it was proved by Lim in [47]. This surprising result was the key for proving that any left reversible semi-topological semigroup S possesses the following fixed point property :
Any separately continuous non-expansive representation of a left reversible semi-topological semigroup on a non-void weakly compact convex subset with normal structure of a Hausdorff locally convex space possesses a common fixed point. This result generalizes a fixed point theorem of Mitchell [54] for left reversible (discrete) semigroups acting non-expansively on a compact convex subset of Banach space.

1.2.5 Some Notations

Throughout this thesis, given a non-void subset A of Banach space E or its dual, or a subset of a topological space (E, τ), we shall adopt the following notations depending on the context :

$|A|$: cardinality of A;

$\delta(A)$: diameter of A;

$\overline{A}$: closure of A with respect to the norm topology;

$\overline{A}^{wk}$: closure of A in the weak topology $\sigma(E, E^*)$ on E;

$\overline{A}^{wk*}$: closure of A in the weak* topology $\sigma(E^*, E)$;

$\overline{A}^{\tau}$: closure of A with respect to a topology τ;

$co(A)$: convex hull of A;

$\overline{co}(A)$: normed closed convex hull of A.

CHAPTER 2

Amenable semigroups and non-linear fixed point properties on weakly compact convex subsets of Banach spaces

2.1 A characterization of left amenability

Given a semigroup S, from the definition of a mean, the set $\mathrm{M}(S)$ of all means on $\ell^\infty(S)$ sits inside the unit sphere of the dual space $\ell^\infty(S)^*$. On the other hand, using the Hahn-Banach separation theorem (locally convex spaces version), one can show that the convex hull of all finite means is dense in $\mathrm{M}(S)$ in the weak* topology, and therefore every mean in $\mathrm{M}(S)$ is a weak* limit of net of finite means. Since in infinite dimension the weak* topology is not first countable, a mean need not be the limit of a sequence of finite means. However, we have the following :

Theorem 2.1.1. Let S be a countable left amenable discrete semigroup. Then, S is left amenable if and only if, there is a sequence of finite means converging weak* to a left invariant mean m on $\ell^\infty(S)$.

Proof. Clearly, one direction is obvious. For the converse, let us fix $S := \{s_1, s_2, \cdots, s_n, \cdots\}$ be a countable left amenable discrete semigroup. From [43, theorem 5.1], there is a sequence $(m_n)_n$ of finite means that is left strongly regular (cf. [35] for a proof). Since the unit ball of $\ell^\infty(S)^*$ is weak* compact, $(m_n)_n$ possesses a subnet $(m_{n_\alpha})_{\alpha\in(A,\mathcal{R})}$ that is weak* convergent. Let us set m be the weak*-limit of $(m_{n_\alpha})_\alpha$. Using the fact that $(m_n)_n$ is left strongly regular then by induction, we have :

• For $n = 1$, there is $n_1 \geq 1$ such that $n \geq n_1 \Rightarrow \|\ell^*_{s_1} m_n - m_n\| \leq \frac{1}{2}$. Then we choose an index $\alpha_1 \in A$ such that $\alpha_1 \mathcal{R} \alpha \Rightarrow n_\alpha \geq n_1$.

• For $n = 2$ then, we fix $n_2 \geq \max(2, n_1 + 1)$ such that : for $n \geq n_2$, we have $\|\ell^*_{s_i} m_n - m_n\| \leq \frac{1}{2^2}$ for all $i \in \{1, 2\}$; and an index $\alpha_2 \in A$ subject to the properties :

$$\alpha_1 \mathcal{R} \alpha_2 \text{ and } \alpha_2 \mathcal{R} \alpha \Rightarrow n_\alpha \geq n_2.$$

• By induction, if $n = p$, let us fix $n_p \geq \max(p, n_{p-1} + 1)$ with the following property : $n \geq n_p \Rightarrow \|\ell^*_{s_i} m_n - m_n\| \leq \frac{1}{2^p}$ for all $i \in \{1, \cdots, p\}$. Next we pick $\alpha_p \in A$ with the following properties :

$$\alpha_{p-1} \mathcal{R} \alpha_p \quad \text{and for all } \alpha \in A,\ \alpha_p \mathcal{R} \alpha \Rightarrow n_\alpha \geq n_p.$$

Then by induction we have constructed three sequences : $(\alpha_k)_k, (n_k)_k$ and $(m_{n_{\alpha_k}})_k$ such that, given $k \in \mathbb{N}$ the following facts hold :
$\alpha_k \mathcal{R} \alpha_{k+1},\ n_k \geq k,\ n_{k+1} > n_k,\ \alpha_k \mathcal{R} \alpha \Rightarrow n_\alpha \geq n_k$ and moreover

$$\|\ell^*_{s_i} m_n - m_n\| \leq \tfrac{1}{2^k} \text{ for all } n \geq n_k, \text{ for all } i \in \{1, \cdots, k\}.$$

Note that $(m_{n_{\alpha_k}})_k$ is a subnet of $(m_{n_\alpha})_{\alpha\in A)}$. Indeed, given $\tilde{\alpha}$ in A, since $n_k \to +\infty$, we can choose $\tilde{k} \in \mathbb{N}$ such that $n_{\tilde{k}} > n_{\tilde{\alpha}}$. Given $k \geq \tilde{k}$, if we have $\alpha_k \mathcal{R} \tilde{\alpha}$ then we would have $n_{\tilde{\alpha}} \geq n_k \geq n_{\tilde{k}} \Rightarrow n_{\tilde{\alpha}} > n_{\tilde{\alpha}}$ which is not possible. So we have necessarily $\tilde{\alpha}\mathcal{R}\alpha_k$. Therefore it follows that for all $k \geq \tilde{k}$, $\tilde{\alpha}\mathcal{R}\alpha_k$, which shows that our assertion is true. On the other hand, since m is a pointwise limit of $(m_{n_\alpha})_\alpha$, then a fortiori it is for the subnet $(m_{n_{\alpha_k}})_k$. Now we show that m is left invariant. Let $f \in \ell^\infty(S)$ and $s = s_j \in S$ fixed. Given $\epsilon > 0$, let us choose $k_{s,\epsilon} \in \mathbb{N}$ such that $k_{s,\epsilon} \geq j$ and $\frac{1}{2^{k_{s,\epsilon}}} \leq \frac{\epsilon}{\|f\|_\infty + 1}$. Then for all $k \geq k_{s,\epsilon}$, we have :

$$\begin{aligned}
|\ell_s^* m(f) - m(f)| &\leq |\ell_s^* m_{n_{\alpha_k}}(f) - \ell_s^* m(f)| + |\ell_s^* m_{n_{\alpha_k}}(f) - m_{n_{\alpha_k}}(f)| \\
&\qquad + |m_{n_{\alpha_k}}(f) - m(f)| \\
&\leq |m_{n_{\alpha_k}}(\ell_s f) - m(\ell_s f)| + \|\ell_s^* m_{n_{\alpha_k}} - m_{n_{\alpha_k}}\| \|f\|_\infty \\
&\qquad + |m_{n_{\alpha_k}}(f) - m(f)| \\
&\leq \frac{\|f\|_\infty}{2^{k_{s,\epsilon}}} + |m_{n_{\alpha_k}}(\ell_s f) - m(\ell_s f)| + |m_{n_{\alpha_k}}(f) - m(f)| \\
&\leq \epsilon + |m_{n_{\alpha_k}}(\ell_s f) - m(\ell_s f)| + |m_{n_{\alpha_k}}(f) - m(f)|
\end{aligned}$$

Using the fact that $m = \text{weak*-}\lim_k m_{n_{\alpha_k}}$, we get for k large enough

$$|\ell_s^* m(f) - m(f)| \leq 2\epsilon.$$

As ϵ is arbitrary, then we have

$$\ell_s^* m(f) = m(\ell_s f) = m(f), \text{ for all } f \in \ell^\infty(S) \text{ and for all } s \in S$$

. This means that m is a left invariant mean and to finish the proof, we let $m_k := m_{n_{\alpha_k}}$ for all $k \in \mathbb{N}$. $\square$

Example 2.1.2. Semigroups with this property includes all metrizable compact left amenable semi-topological semigroups. In fact, if S is such a semigroup, then $\mathrm{LUC}(S) = C(S)$ is (norm) separable and therefore, the unit ball of $\mathrm{LUC}(S)^*$ is a compact metric space in the weak* topology (e.g. the unit sphere $\mathbb{S}^1$). Such semigroups includes also all compact topological groups (e.g. finite groups, the unit circle).

Now we look fixed point properties. In [3], Belluce and Kirk proved that commutative families of non-expansive mappings are solutions to problem 1 if the underlying Banach space is strictly convex. This result was proved earlier for a single map by Edelstein (cf. [20]), and later extended by induction by Belluce and Kirk. We point out that in Edelstein's proof, the geometric property "strict convexity" has played an essential role. We are able to show that for uniformly convex spaces, Belluce and Kirk's result holds for left amenable semigroups of non-expansive mappings. If the underlying space is no longer uniformly convex, then if we require that the compact space on which the semigroup is acting has normal structure, then our conclusion still holds in this setting. We shall also prove that there is a class of semi-topological semigroups (including left amenable semigroups) for which one can remove both conditions "uniform convexity" and "normal structure" if we assume that representations are weakly continuous.

2.2 A non-linear common fixed point theorem for bounded closed convex sets in uniformly convex spaces

Theorem 2.2.1. Let S be a discrete semigroup. If S is left reversible, then it possesses the following fixed point property :

(F_{uc}) : Whenever $\mathfrak{S} = \{\hat{s}; s \in S\}$ is a non-expansive representation of S on a non-void bounded closed convex subset K of a uniformly convex Banach space E, then K contains a common fixed point for S.

Proof. Let us fix $x_o \in K^\tau$. Since S is left reversible, one can make S into a filtered set by letting :

$$a \leq b \; \Leftrightarrow b.S \subset a.S \;\text{ for all } a, b \in S.$$

Then $(s.x_o)_{s\in S}$ becomes a net of elements of K. Since the underlying space is uniformly convex, as known (see [2]), the asymptotic center

$$AC((s.x_o)_{s\in S}, K^\tau) := \{x \in K^\tau \; ; \; \limsup_s \|x - s.x_o\| = \inf_{y\in K^\tau} \limsup_s \|y - s.x_o\|\}$$

is a singleton. Let $AC((s.x_o)_{s\in S}, K^\tau) = \{\bar{x}\}$. Now we show that the asymptotic center is S-invariant.

- Step 1 : We assume that S is unital. Let $x \in AC((s.x_o)_{s\in S}, K^\tau)$ and $s \in S$ fixed. Given $\epsilon > 0$, there is $s_\epsilon \in S$ such that

$$\sup_{s\geq s_\epsilon} \|x - s.x_o\| \leq \inf_{y\in K^\tau} \limsup_s \|x - s.x_o\| + \epsilon \quad (1).$$

Let us pick $t_\epsilon^s \in S$ such that

$$t_\epsilon^s \geq s \text{ and } t_\epsilon^s \geq ss_\epsilon.$$

If $t \geq t_\epsilon^s$, then $t = ss_\epsilon u$ for some $u \in S$. Then using the non-expansiveness of the representation, we get :

$$\|s.x - t.x_o\| = \|s.x - ss_\epsilon u.x_o\| \leq \|x - s_\epsilon u.x_o\| \leq \sup_{s \geq s_\epsilon} \|x - s.x_o\|$$

because $s_\epsilon u \geq s_\epsilon$. Since $t \geq t_\epsilon^s$ is arbitrary, then together with (1) we obtain

$$\limsup_t \|s.x - t.x_o\| \leq \sup_{s \geq s_\epsilon} \|x - s.x_o\| \leq \inf_{y \in K^\tau} \limsup_s \|x - s.x_o\| + \epsilon$$

It follows that $s.x \in AC((s.x_o)_{s \in S}, K^\tau)$. Hence, $s.\bar{x} = \bar{x}$ for all $s \in S$.

• Step 2 : S is an arbitrary left reversible semigroup. Consider an unitization S^e of S obtained by adjoining a unit (i.e. we extend the product of S by letting $se = es := s$ and $e.e := e$). Then we have

$$sS^e \cap eS^e = (sS \cup \{s\}) \cap S^e = sS \cup \{s\} \neq \emptyset \text{ for any } s \in S.$$

Next, we extend the representation of S on K to S^e by letting $\hat{e}(x) := x$ for all $x \in K$. Then by Step 1, S^e possesses a common fixed point in K which is a fortiori a common fixed point for S. □

Corollary 2.2.2. Theorem 2.2.1 extends Browder [10, theorem 1,2].

Proof. A commuting family generates an abelian semigroup which is clearly left reversible. □

Let S be a semigroup and $(S_\alpha)_{\alpha \in J}$ be a collection of sub-semigroups of it. Then S is said to be a *direct union* of the S_α's if :

1. $S = \bigcup_\alpha S_\alpha$;
2. For all $\alpha, \beta \in J$, there is $\gamma \in J$ such that $S_\alpha \cup S_\beta \subset S_\gamma$.

Example 2.2.3. Let S be left amenable discrete semigroup. Then S is the direct union of a net of countable left amenable sub-semigroups. In fact, we know that each countable sub-semigroup of S is contained in some countable left amenable one, cf. [25]. Define

$$\mathcal{S} := \{Z \subset S \ ; \ Z \text{ is a left amenable countable sub-semigroup}\}.$$

Note that $\mathcal{S}$ is non-void because if we fix $s \in S$, the commutative semigroup $\langle s \rangle$ is countable and left amenable. Let us order $\mathcal{S}$ by letting $Z \leq Z' \Leftrightarrow Z \subset Z'$. Then it is clear that $S = \bigcup_{Z \in \mathcal{S}} Z$ and given $Z, Z' \in \mathcal{S}$, there is $Z" \in \mathcal{S}$ such that $Z \cup Z' \subset Z"$. Because Z and Z' being countable, it follows that $\langle Z \cup Z' \rangle$ is countable too, and we choose $Z" \in \mathcal{S}$ such that $Z" \supset \langle Z \cup Z' \rangle$ using [25, theorem E1].

Corollary 2.2.4. Let $S = \bigcup_{\alpha \in J} S_\alpha$ be a direct union. If each S_α is left amenable, then S possesses the fixed point property (F_{ns}).

Proof. Indeed, any semigroup which is a direct union of a sub-collection of its left amenable sub-semigroups is left amenable (see [14]). $\square$

Remark 2.2.5. We point out that the asymptotic center of a sequence is extendable to nets, (cf. [48]). On the other hand, it is known that uniformly convex spaces possess normal structure (combine [4],[47]), so it is natural to look at whether theorem 2.2.1 is extendable to weakly compact convex sets with normal structure in general Banach spaces.

2.3 Some non-linear fixed point theorems on weakly compact convex subsets of Banach spaces

In this section, we first give a sufficient condition such that the fixed point theorem 2.2.1 is partially extendable to general Banach spaces. Afterwards, we give a positive answer to problem 2 for a class of semi-topological semigroups including left amenable discrete semigroups and amenable locally compact groups. An extension will be established in chapter 3.

Definition 2.3.1. Given a representation $\mathfrak{S} = \{\hat{s}\ ; s \in S\}$ of a semi-topological semigroup S on a compact Hausdorff space (K, τ), we say that $\mathfrak{S}$ is $C(K)^+$-*continuous*, if for all $f \in C(K)$ we have :

$$f \geq 0 \Rightarrow f \circ \hat{s} : K \to \mathbb{R} \text{ is continuous.}$$

Note that any separately continuous representation of a semigroup S on a separated compact space K is automatically $C(K)^+$-continuous.

Example 2.3.2. Let $K = [0,1]$ with the usual topology, $f : [0,1] \to [0,1]$, $x \mapsto \sin(x)$, and $S = (\mathbb{N}, +)$. Then the representation $\hat{n}(x) := f^n(x)$ (where f^n denotes the n^{th} iterate of f) is $C(K)^+$-continuous.

Theorem 2.3.3. Let S be a countable semi-topological semigroup. If LMC(S) is left amenable, then S has the following fixed point property :

(F_{ns}) : Whenever K is a non-void weakly compact convex set with normal structure in a Banach space E, and $\mathfrak{S} = \{\hat{s}\ ; s \in S\}$ is a non-expansive representation of S on K such that for all $x \in K$, the mapping $s \mapsto s.x$ is continuous when K is given the weak topology, then there is in K a common fixed point for S.

The following lemmas will be needed for proving this result.

Lemma 2.3.4. Let S be a semi-topological semigroup and $\mathfrak{S}$ be a representation of S on a non-void compact Hausdorff space M. If for all $x \in M$, $s \mapsto s.x$ is continuous, then $\mathfrak{S}$ is an A-representation of S, LMC(S) on M.

Proof. Let $x \in K$ and $f \in C(M)$ fixed. Consider $\theta_x^f : S \to \mathbb{R}$ given by $\theta_x^f(t) = f(t.x)$. Then $\sup_{s \in S} |\theta_x^f(s)| \leq \sup_{y \in M} |f(y)| = \|f\|_\infty < \infty$. Therefore $\theta_x^f \in C_b(S)$. Now let $s_\alpha \to s$ be a convergent net in S and pick $m \in \beta S$ (where βS denotes the spectrum of $C_b(S)$). Consider

$$\begin{aligned} \theta_x : C(M) &\rightarrow \mathbb{R} \\ h &\mapsto m(\theta_x^h) \end{aligned}$$

Then θ_x defines a non-zero (because $m \circ \theta_x(1) = m(e) = 1$), multiplicative linear functional on $C(M)$. Therefore there is a unique $z_x \in M$ such that $m \circ \theta_x = \delta_{z_x}$, (the evaluation map at z_x). We have $\ell_z \theta_x^f = \theta_x^{f \circ \hat{z}}$ for all $z \in S$. It follows :

$$\begin{aligned} |m(\ell_{s_\alpha} \theta_x^f) - m(\ell_s \theta_x^f)| &= |m(\theta_x^{f \circ \hat{s_\alpha}}) - m(\theta_x^{f \circ \hat{s}})| \\ &= |m \circ \theta_x(f \circ \hat{s_\alpha}) - m \circ \theta_x(f \circ \hat{s})| \\ &= |\delta_{z_x}(f \circ \hat{s_\alpha}) - \delta_{z_x}(f \circ \hat{s})| \\ &= |f(s_\alpha . z_x) - f(s.z_x)| \rightarrow 0 \end{aligned}$$

Therefore, $\ell_{t_\alpha} \theta_x(f) \rightarrow \ell_t \theta_x(f)$ with respect to $\sigma(C_b(S), \beta S)$ which shows that $\theta_x^f \in$LMC(S). $\square$

For the proof of this result, we shall need the following general lemma :

Lemma 2.3.5. Let S be a semi-topological semigroup and Φ be a left translation invariant closed subspace of $C_b(S)$ containing constant functions. If Φ has a left invariant mean, then for any A-representation $\mathfrak{S} = \{\hat{s} \, ; s \in S\}$ of S, Φ on a compact Hausdorff topological space (K, τ), the following facts hold :

1. If $\mathfrak{S}$ is $C(K)^+$-continuous, any non-empty S-invariant closed subset

Ω of K contains a non-void closed set ω_τ such that :

$$\omega_\tau \subset \overline{s.\omega_\tau}^\tau, \text{ for all } s \in S.$$

2. If in addition, the mapping $\hat{s}$ is continuous for all $s \in S$, then ω_τ satisfies the following property

$$s.\omega_\tau = \omega_\tau \text{ for all } s \in S.$$

3. If there is a completely regular second countable topology $\tilde{\tau}$ on K such that $\tilde{\tau} \supset \tau$ and for all $s \in S$ the mapping $\hat{s}$ is $\tilde{\tau}$-continuous, then in part 1 the $C(K)^+$-continuity condition can be dropped.

4. If S is countable, K is a subset of a Banach space E, $\tau = \sigma(E, E^*)$, then in part 3 one can remove the second countability condition if we assume that each mapping $\hat{s}$ is norm continuous.

Proof. Let m be a left invariant mean on Φ, and let $(m_\gamma)_{\gamma \in J}$ be a net of finite means converging weak* to m. Let

$$m_\gamma := \sum_{i=1}^{n_\gamma} t_i^\gamma \delta_{s_i^\gamma} \text{ for all index } \gamma \in J.$$

Where for all γ fixed, $\sum_i t_i^\gamma = 1$, $t_i^\gamma \geq 0$. Let us pick $x_o \in \Omega^\tau$. Fix a non-void, S-invariant closed subset Ω of K. Then the mapping

$$\begin{aligned} \Theta_{x_o} : \; C(\Omega) &\to \mathbb{R} \\ f &\mapsto m(\theta^f_{x_o}) \end{aligned}$$

where $\theta^f_{x_o} : S \to \mathbb{R}$ is given by $\theta^f_{x_o}(s) := f(s.x_o)$ for all $f \in \ell^\infty(\Omega)$. Then Θ_{x_o} is a nonzero non-negative linear functional on $C(\Omega)$, and therefore, by the Riesz representation theorem, there is a regular Borel measure μ on the Borel σ-algebra of Ω (which is here a probability measure since $\mu(\Omega) = m(1)$=1) such that $\Theta_{x_o}(f) = \int_\Omega f \, d\mu$, for all $f \in C(\Omega)$. Let ω_τ denote the support of μ. Recall that the support is defined by

$$\omega_\tau := \bigcap \{F \subset \Omega^\tau \, ; F \text{ is } \tau\text{-closed and } \mu(F) = 1\}.$$

From the outer regularity of μ, for all $n \geq 1$, there is an open set O_n in Ω such that

$$\omega_\tau \subset O_n \;\text{ and }\; \mu(O_n \setminus \omega_\tau) \leq \tfrac{1}{n} \;\; (1).$$

Given an open set O of Ω containing ω_τ, then by compactness we have

$$\Omega = \bigcup_{i=1}^n (\Omega \setminus F_i) \cup O$$

where each F_i is closed and $\mu(F_i)$=1. Then it follows that $\mu(O)$=1 (2). Now let $G := \bigcap_n O_n$ and $\tilde{O}_n = O_1 \cap \cdots \cap O_n$, for all n. Then by using (1) and (2) it follows that

$$\begin{aligned} \mu(\omega_\tau) &= \mu(G) \\ &= \lim_n \mu(\tilde{O}_n) \\ &= 1. \end{aligned}$$

Now let $s \in S$. Given $\epsilon > 0$, by outer regularity let $O_\epsilon \subset \Omega$ be a τ-open set with the properties :

$$\overline{s.\omega_\tau}^\tau \subset O_\epsilon \;\text{ and }\; \mu(O_\epsilon \setminus \overline{s.\omega_\tau}^\tau) \leq \epsilon.$$

By using Urysohn's lemma, let us pick $f \in C(\Omega)$ such that $0 \leq f \leq 1$, supp$(f) \subset O_\epsilon$ and $f \equiv 1$ on $\overline{s.\omega_\tau}^\tau$. Then we have :

$$\begin{aligned}
1 = \int_\Omega d\mu &= \int_{\omega_\tau} d\mu \\
&\leq \int_\Omega f \circ \hat{s} \ d\mu \\
&= m(\theta_{x_o}^{f\circ\hat{s}}) = m(\ell_s \theta_{x_o}^f) = m(\theta_{x_o}^f) \\
&= \int_{O_\epsilon \setminus \overline{s.\omega_\tau}^\tau} f \ d\mu + \int_{\overline{s.\omega_\tau}^\tau} f \ d\mu \\
&\leq \epsilon + \mu(\overline{s.\omega_\tau}^\tau)
\end{aligned}$$

ϵ being arbitrary, it follows that $\mu(\overline{s.\omega_\tau}^\tau)$=1, and therefore from the definition of ω_τ, it follows that $\omega_\tau \subset \overline{s.\omega_\tau}^\tau$ which proves the first part. For the second part, we first note that if each mapping $\hat{s}$ is continuous, then by the forgoing, we have :

$$\omega_\tau \subset s.\omega_\tau \ \text{ for all } s \in S$$

since each . So it remains to show the reverse inclusion. Given $s \in S$, since $\hat{s}^{-1}(\omega_\tau)$ is closed, then it is enough to show that it contains ω_τ. As before given $\epsilon > 0$, from the outer regularity, there is $O_\epsilon \subset \tilde{\Omega}$ a τ-open subset such that :

$$\omega_\tau \subset O_\epsilon \ \text{ and } \ \mu(O_\epsilon \setminus \omega_\tau) \leq \epsilon.$$

Let us choose $f \in C(\Omega)$ with the properties : $0 \leq f \leq 1$, supp$(f) \subset O_\epsilon$ and $f \equiv 1$ on ω_τ. Then we have :

$$
\begin{aligned}
1 &= \int_{\Omega} d\mu = \int_{\Omega} f \ d\mu \\
&= m(\theta_{x_o}^{f}) = m(\ell_s \theta_{x_o}^{f}) \\
&= \int_{\omega_\tau} f(s.x) \ d\mu(x) \\
&= \int_{\hat{s}^{-1}(\omega_\tau)} d\mu(x) + \int_{O_\epsilon \setminus \hat{s}^{-1}(\omega_\tau)} f(s.x) \ d\mu(x) \\
&\leq \mu(\hat{s}^{-1}(\omega_\tau)) + \int_{O_\epsilon \setminus \hat{s}^{-1}(\omega_\tau)} d\mu(x) \\
&\leq \mu(\hat{s}^{-1}(\omega_\tau)) + \epsilon
\end{aligned}
$$

Therefore, as ϵ is arbitrary, it follows that $\omega_\tau \subset \hat{s}^{-1}(\omega_\tau)$; which implies $s.\omega_\tau \subset \omega_\tau$. Hence, $s.\omega_\tau = \omega_\tau$ for all $s \in S$. Finally for the last part, let $\tilde{\tau}$ be a completely regular topology on K finer than τ. Let $C_b(\Omega)$ be the Banach algebra of all bounded $\tilde{\tau}$-continuous real-valued functions on Ω, and $\beta\Omega$ be the Stone-Cĕch compactification of Ω. Fix $h : \Omega \to \beta\Omega$ be an homeomorphism from Ω onto a dense subspace of $\beta\Omega$, and define

$$
\begin{aligned}
\psi : C_b(\beta\Omega) &\to \mathbb{R} \\
f &\mapsto m(\theta_{x_o}^{f \circ h})
\end{aligned}
$$

Note that using the Hahn-Banach extension theorem if necessary, we may assume that m is a mean on $\ell^\infty(S)$ such that $\ell_s^* m = m$ on Φ. Then ψ is a well-defined non-zero non-negative linear functional, and $\beta\Omega$ being a compact Hausdorff space, there is a regular Borel measure $\tilde{\mu}$ on the Borel σ-algebra $B(\beta\Omega)$ of $\beta\Omega$ such that : $\psi(f) = \int_{\beta\Omega} f \ d\tilde{\mu}$, for all $f \in C_b(\beta\Omega)$. Given $f \in C_b(\Omega)$ we shall denote by f^β its unique continuous extension to $\beta\Omega$. We claim that for all τ-open set O in Ω containing ω_τ we have $\tilde{\mu}(h(O)) = 1$. Indeed, since (Ω, τ) is normal, pick a τ-open set U with $O \supset \overline{U}^\tau \supset U \supset \omega_\tau$. Let $f \in C(\Omega)$ such that :

$$f \equiv 1 \text{ on } \omega_\tau, \;\; 0 \leq f \leq 1, \text{supp}(f) \subset U.$$

Note that $C(\Omega, \tau) =: C(\Omega) \subset C_b(\Omega) := C_b(\Omega, \tilde{\tau})$ (since $\tau \subset \tilde{\tau}$). Then we have

$$\begin{aligned}
\tilde{\mu}(h(O)) &= \int_{h(O)} d\tilde{\mu} \\
&\geq \int_{\beta\Omega} f^\beta \, d\tilde{\mu} \\
&= m(\theta^f_{x_o}) = \int_\Omega f \, d\mu \\
&= \int_O f \, d\mu \geq \mu(\omega_\tau) = 1
\end{aligned}$$

Hence, $\tilde{\mu}(h(O)) = 1$ for all such an open set O (3). Now we show that $\tilde{\mu}(h(\omega_\tau)) = 1$. Since Ω is τ-normal and $\overline{\tau}$ is second countable, then

$$\omega_\tau = \bigcap_{i=1}^\infty \{\overline{O_i}^\tau \; ; \; O_i \; \tau\text{-open}, \, O_i \supset \omega_\tau\}.$$

Let $\tilde{O}_n := \bigcap_{i=1}^n \overline{O_i}^\tau$ for all $n \in \mathbb{N}$. Then using relation (3) it follows that

$$\begin{aligned}
\tilde{\mu}(h(\omega_\tau)) &= \tilde{\mu}(h(\bigcap_{i=1}^\infty \overline{O_i}^\tau)) \\
&= \lim_n \tilde{\mu}(\bigcap_{i=1}^\infty h(\overline{O_i}^\tau)) \\
&= \lim_n \tilde{\mu}(h(\bigcap_{i=1}^n \overline{O_i}^\tau)) = 1
\end{aligned}$$

Now let $s \in S$ fixed. Given $\epsilon > 0$, we choose an open set O_ϵ containing $\overline{s.\omega_\tau}^\tau$ such that $\mu(O_\epsilon \setminus \overline{s.\omega_\tau}^\tau) \leq \epsilon$. Fix a mapping $f \in C(\Omega)$ such that $\text{supp}(f) \subset O_\epsilon,\ 0 \leq f \leq 1$ and $f \equiv 1$ on $\overline{s.\omega_\tau}^\tau$. Then we have

$$\begin{aligned}
1 &= \tilde{\mu}(h(\omega_\tau)) = \int_{h(\omega_\tau)} d\tilde{\mu} \\
&\leq \int_{\beta\Omega} (f \circ \hat{s})^\beta d\tilde{\mu} \\
&= m(\theta_{x_o}^{f\circ\hat{s}}) = m(\ell_s \theta_{x_o}^f) = m(\theta_{x_o}^f) \\
&= \int_\Omega f\ d\mu \\
&= \int_{O_\epsilon \setminus \overline{s.\omega_\tau}^\tau} f\ d\mu + \int_{\overline{s.\omega_\tau}^\tau} f\ d\mu \\
&\leq \epsilon + \mu(\overline{s.\omega_\tau}^\tau)
\end{aligned}$$

Thus, using the arbitrariness of ϵ, it follows that $\mu(\overline{s.\omega_\tau}^\tau)$ =1. Hence, we deduce the desired result $\omega_\tau \subset \overline{s.\omega_\tau}^\tau$, for all $s \in S$. Finally, for the last part we first construct a norm separable space containing ω_τ. Let $S := \{s_i\ ;\ i \in \mathbb{N}\}$. From the definition of ω_τ, it is easy to see that it is characterized by the property :

$x \in \omega_\tau$ iff $\mu(V \cap \omega_\tau) > 0$ whenever V is a τ-neighborhood of x.

Consider the weakly separable (therefore norm separable) set given by $\overline{\{s_i.x_o\ ; i \in \mathbb{N}\}}^\tau$. If x is a point outside this set, then by Urysohn's lemma, there is $f \in C(\Omega),\ f \geq 0$ such that $f(s_i.x_o) = 0$ for all $i \in \mathbb{N}$. Then by continuity, $V := \{f > 0\}$ is a τ-neighborhood of x and we have :

$$\begin{aligned}\mu(V\cap\omega_\tau) = \int_{V\cap\omega_\tau} d\mu \;&\le\; \int_\Omega f\; d\mu \\ &= m(\theta^f_{x_o}) = \lim_\gamma m_\gamma(\theta^f_{x_o}) \\ &= \lim_\gamma \sum_{i=1}^{n_\gamma} t_i^\gamma \delta_{s_i^\gamma}(\theta^f_{x_o}) = \lim_\gamma \sum_{i=1}^{n_\gamma} t_i^\gamma f(s_i^\gamma.x_o) \\ &= 0\end{aligned}$$

Thus $\mu(V\cap\omega_\tau)=0$ which implies that x cannot be in ω_τ and therefore, $\omega_\tau \subset \overline{\{s_i.x_o\,;i\in\mathbb{N}\}}^\tau$. Hence ω_τ is norm separable. Fix $D\subset\omega_\tau$ be a countable (norm) dense subset and define

$$X := \overline{\{s_i.d\,;i\in\mathbb{N},d\in D\}\cup\{s_i.x_o\,;i\in\mathbb{N}\}}^\tau.$$

X is a norm separable (therefore second countable) subspace of Ω containing ω_τ. Now let $C_b(X)$ be the Banach space of all bounded norm continuous real-valued functions on X and $\delta : X\to\beta X$ be an homeomorphism onto a dense subset of βX. As in the previous part, there is a Borel measure $\tilde\mu$ on βX such $m(\theta^{f\circ\delta}_{x_o}) = \int_{\beta X} f\; d\tilde\mu$, for all $f\in C_b(\beta X)$. Then a similar argument as in the proof of part 3 shows that $\omega_\tau\subset\overline{s.\omega_\tau}^\tau$ for all $s\in S$. $\square$

Proof of Theorem 2.3.3. Define $\Xi := \{C\subset K\;;\,C\neq\emptyset$, τ-compact, convex and S-invariant$\}$. Then Ξ is non-void as it contains at least K. We order Ξ backwards by inclusion,

$$\text{i.e. for all } C,C'\in\Xi, \text{ we let } C\le C' \text{ if and only if } C\supset C'.$$

Then it is easy to see that $(\Xi, \leq)$ is inductive; and by Zorn's lemma, it must contain a maximal element (with respect to "$\leq$") or equivalently a minimal element K^{wk} (for "$\subset$"). By combining the lemmas 2.3.4 and 2.3.5, there is a non-empty weakly compact subset Ω_{wk} of K^{wk} such that

$$\Omega_{wk} \subset \overline{s.\Omega_{wk}}^{wk} \text{ for all } s \in S \quad (1).$$

If K^{wk} is a singleton then we are done; otherwise, $\mathrm{AC}(\Omega_{wk}; K^{wk})$ (of the constant net Ω_{wk}) would be a non-empty weakly compact convex and proper subset of K^{wk}. Furthermore, it is easy to see that it can be written as $\bigcap_{j=1}^{\infty} K_j$, where each K_j is defined by

$$K_j := \{x \in K^{wk} \ ; \ \sup_{y \in \Omega_{wk}} \|x - y\| \leq r(\Omega_{wk}; \ K^{wk}) + \tfrac{1}{j}\}.$$

In order to prove that it is S-invariant, it is enough to show that each each K_j has this property. Let $j \in \mathbb{N}$, $x \in K_j$ and $s \in S$. We know that $\Omega_{wk} \subset \overline{s.\Omega_{wk}}^{wk}$; so given $y \in \Omega_{wk}$, let $y = \lim_\alpha s.y_\alpha$; (weak limit) for some net $(y_\alpha)_\alpha$ in Ω_{wk}. From the non-expansiveness we get :

$$\begin{aligned}
\|s.x - y\| &\leq \liminf_\alpha \|s.x - s.y_\alpha\| \\
&\leq \liminf_\alpha \|x - y^\alpha\| \\
&\leq \sup_{z \in \Omega_{wk}} \|x - z\| \\
&\leq r(\Omega_{wk}; \ K^{wk}) + \frac{1}{j}
\end{aligned}$$

Since y and ϵ are arbitrary, then it follows that $s.x \in K_j$. Hence, K_j is S-invariant for all j and so is $\mathrm{AC}(\Omega_{wk}; K^{wk})$. Thus by minimality, $\mathrm{AC}(\Omega_{wk}; K^{wk}) = K^{wk}$ which leads to a contradiction. $\square$

Remark 2.3.6. The above theorem extends Belluce and Kirk [3, theorem 3], and Kirk [37, main result] because commuting families generate amenable semigroups and on the other hand, the continuity of $s \mapsto s.x$ is automatically satisfied for discrete semigroups.

Remark 2.3.7. If S is discrete, then in theorem 2.3.3 one can remove the countability condition. Indeed, when S is discrete and left amenable, S is a direct union of countable left amenable sub-semigroups (see example 2.2.3). So let $S = \bigcup_{\gamma \in J} S_\gamma$. Let K^{wk} be as in the proof of theorem 2.3.3. From the same theorem, each sub-semigroup S_γ possesses a common fixed point $x_\gamma \in K^{wk}$. Define an order on J by letting

$$\gamma \leq \gamma' \text{ if and only if } S_\gamma \subset S_{\gamma'}.$$

Then given $\gamma_o \in J$, we let

$$W_{\gamma_o} := \{x_\gamma \ ; \ \gamma \geq \gamma_o\}.$$

Then $(W_\gamma)_\gamma$ is a decreasing net of non-void subsets of K^{wk}. Assume that K^{wk} has a positive diameter then the asymptotic center $\mathrm{AC}((W_\gamma)_\gamma, K^{wk})$ is a non-empty weakly compact convex and proper subset of K^{wk}. On the other hand, we know that it coincides with $\bigcap_{j=1}^{\infty} K_j$ where

$$K_j := \{x \in K^{wk} \ ; \ \limsup_z \|x - x_\gamma\| \leq r((W_\gamma)_\gamma, K^{wk}) + \tfrac{1}{j}\}.$$

In order to prove that $\mathrm{AC}((W_\gamma)_\gamma, K^{wk})$ is S-invariant, it is enough to show that that each K_j is. Let $j \in \mathbb{N}$, $x \in K_j$ and $s \in S$. Fix $\gamma_s \in J$

such that $s \in S_{\gamma_s}$ and pick $\gamma_{s,\epsilon}$ with the properties

$$\gamma_{s,\epsilon} \geq \gamma_s \text{ and } \gamma_{s,\epsilon} \geq \gamma_\epsilon.$$

Then for all $\gamma \geq \gamma_{s,\epsilon}$, we have $s.x_\gamma = x_\gamma$, and moreover, using the non-expansiveness we get :

$$\begin{aligned}
\sup_{\gamma \geq \gamma_{s,\epsilon}} \|s.x - x_\gamma\| &= \sup_{\gamma \geq \gamma_{s,\epsilon}} \|s.x - s.x_\gamma\| \\
&\leq \sup_{\gamma \geq \gamma_{s,\epsilon}} \|x - x_\gamma\| \\
&\leq r((W_\gamma)_\gamma, K^{wk}) + \frac{1}{j}
\end{aligned}$$

Therefore $s.x \in K_j$ and it follows that the asymptotic center is S-invariant. Then by minimality we have $\mathrm{AC}((W_\gamma)_\gamma; K^{wk}) = K^{wk}$ which is absurd. □

Remark. 2.3.8. The above result in remark 2.3.7 follows from Lim [47] if we make use of the fact that discrete left amenable semigroups are left reversible. On the other hand, remark 2.3.7 extends Belluce and Kirk [3, theorem 3],[4, theorem 2.1] and Kirk [37, main result].

Remark 2.3.9. Our next result shows that a normal structure condition can be avoided if we put more "regularity" on representations. Before, we need to introduce a new class of semi-topological semigroups.

Definition 2.3.10. A semi-topological semigroup is *σ-left amenable*, if it is a direct union of separable left amenable sub-semigroups.

Remark 2.3.11. If a semi-topological semigroup S is σ-left amenable, then for short we shall write simply "S is σ-LA". On the other hand, the class of all such semi-topological semigroups includes left amenable discrete semigroups (see example 2.2.3), all separable semi-topological semigroups, and all amenable locally compact topological groups (because of the fact that each closed subgroup is also amenable).

Theorem 2.3.12. Let S be a semi-topological semigroup. If S is σ-LA, then it possesses the following fixed point property :

(F_{wk}) : Whenever $\mathfrak{S} = \{\hat{s} \ ; \ s \in S\}$ is a jointly weakly continuous non-expansive representation on a non-void weakly compact convex subset K of a Banach space E, then there is in K a common fixed point for S.

In order to prove this result, we shall need the following lemmas :

Lemma 2.3.13. Let S be a semi-topological semigroup. Then any jointly continuous representation of S on a non-empty compact Hausdorff space M defines an A-representation of S, LUC(S) on M.

Proof. See [41, lemma 2.1] (for direct proof); or [53, theorem 1]. □

Lemma 2.3.14. Let S^* be a sub-semigroup of S. Let Ω^{wk} be a non-void, S^*-invariant and weakly compact subset of K. If Ω^{wk} is minimal (i.e., any of its subsets with the same properties coincides with Ω^{wk})

then, the following facts hold :

1. For all $x \in \Omega^{wk}$, $\mathcal{O}_x := \{s.x \; ; \; s \in S^*\}$ is weakly dense in Ω^{wk}.

2. If S^* is left amenable, then $s.\Omega^{wk} = \Omega^{wk}$ for all $s \in S^*$. If in addition S^* is separable, then Ω^{wk} is compact in the norm topology.

Proof. For part 1, it is clear that S^*-orbits are S^*-invariant and their weak closures too by the weak continuity. Therefore, the minimality yields the desired result. For part 2, the first property follows by combining lemma 2.3.5 and lemma 2.3.13. For the second part, we first need to justify that the separability of S^* can be transferred to Ω^{wk} in the weak topology. In fact, let $D \subset S^*$ be a dense subset. If we fix $x \in \Omega^{wk}$, then by continuity of $s \mapsto s.x : S^* \to \Omega^{wk}$, it follows that

$$\Omega^{wk} = \overline{\mathcal{O}_x}^{wk} \subset \overline{\{s.x \; s \in D\}}^{wk} \subset \overline{\mathcal{O}_x}^{wk} = \Omega^{wk}.$$

In order to prove the norm compactness of Ω^{wk}, it is enough to show that it is totally bounded in the norm topology since its already norm closed. For that we adapt an argument used in [45] in locally convex spaces or in [32] to this context. So let us fix $\epsilon > 0$. From the separability, let $\Omega^{wk} = \overline{\varsigma}$ (countable dense subset). Then $\Omega^{wk} \subset \bigcup_{\sigma \in \varsigma} B[\sigma, \frac{\epsilon}{2}]$. Since each $B[\sigma, \frac{\epsilon}{2}]$ is weakly closed, $\{B[\sigma, \frac{\epsilon}{2}] \cap \Omega^{wk}; \sigma \in \varsigma\}$ is a countable weakly closed covering of Ω^{wk} which is weakly compact; and therefore a Baire space. So there is $\tilde{\sigma} \in \varsigma$ such that $B[\tilde{\sigma}, \frac{\epsilon}{2}] \cap \Omega^{wk}$ has a non-void interior in the relative weak topology. Let $x_\epsilon \in \Omega^{wk}$ and V_ϵ be a neighborhood of the origin (in the weak topology) such that $(x_\epsilon + V_\epsilon) \cap \Omega^{wk} \subset B[\tilde{\sigma}, \frac{\epsilon}{2}] \cap \Omega^{wk}$. Then $(x_\epsilon + V_\epsilon) \cap \Omega^{wk} \subset B[x_\epsilon, \epsilon]$. Indeed, if $z \in (x_\epsilon + V_\epsilon) \cap \Omega^{wk}$ then we have $\|x_\epsilon - z\| \leq \|x_\epsilon - \tilde{\sigma}\| + \|\tilde{\sigma} - z\| \leq \epsilon$. Now we pick a weak neighborhood V'_ϵ of the origin such that $V'_\epsilon + V'_\epsilon \subset V_\epsilon$. Then let $\delta_\epsilon > 0$ such that $B[0, \delta_\epsilon] \subset V'_\epsilon$ (this can be done because the norm topology is finer than the weak topology). We have $\Omega^{wk} \subset \bigcup_{\sigma \in \varsigma} B[\sigma, \delta_\epsilon]$. The subset $\varsigma \subset \Omega^{wk}$ being countable, let $\varsigma := \{\sigma_j \; ; j = 1, 2, \cdots\}$. Then by induction we have :

• For n=1, since $x_\epsilon \in \overline{\mathcal{O}_{\sigma_1}}^{wk}$ then, there is an element $s_1 \in S^*$ such that $\widehat{s_1}(\sigma_1) - x_\epsilon \in V'_\epsilon$.

• For n=2, since $x_\epsilon \in \overline{\mathcal{O}_{s_1.\sigma_1}}^{wk}$ then, there is an element $s_2 \in S^*$ with $\widehat{s_1}(\sigma_2) - x_\epsilon \in V'_\epsilon$.

• For n=3, since $x_\epsilon \in \overline{\mathcal{O}_{s_2 s_1.\sigma_3}}^{wk}$ then, there exists $s_3 \in S^*$ such that $\widehat{s_3 s_2 s_1}(\sigma_3) - x_\epsilon \in V'_\epsilon$.

• If n=p, since $x_\epsilon \in \overline{\mathcal{O}_{s_{p-1} \cdots s_1.\sigma_p}}^{wk}$, we pick $s_p \in S^*$ subject to the property $\widehat{s_p \cdots s_1}(\sigma_p) - x_\epsilon \in V'_\epsilon$.

Given $n \in \mathbb{N}$, if $x \in \widehat{s_n \cdots s_1}(B[\sigma_n, \delta_\epsilon] \cap \Omega)$, let $x := \widehat{s_n \cdots s_1}(\sigma_n + z_x)$ for some $z_x \in B[0, \delta_\epsilon]$. Then using the non-expansiveness, we obtain :

$$\begin{aligned} \|\widehat{s_n \cdots s_1}(\sigma_n) - x\| &= \|\widehat{s_n \cdots s_1}(\sigma_n) - \widehat{s_n \cdots s_1}(\sigma_n + z_x)\| \\ &\leq \|\sigma_n - (\sigma_n + z_x)\| = \|z_x\| \\ &\leq \delta_\epsilon. \end{aligned}$$

The above inequality yields the following inclusions :

$$\begin{aligned}\widehat{s_n\cdots s_1}(B[\sigma_n,\delta_\epsilon]\cap\Omega^{wk}) &\subset B[\widehat{s_n\cdots s_1}(\sigma_n),\delta_\epsilon]\cap\Omega^{wk}\\ &\subset x_\epsilon+V'_\epsilon+V'_\epsilon\\ &\subset x_\epsilon+V_\epsilon.\end{aligned}$$

Hence for all $n\in\mathbb{N}$, we have $\widehat{s_n\cdots s_1}(B[\sigma_n,\delta_\epsilon]\cap\Omega^{wk})\subset x_\epsilon+V_\epsilon$. Thus

$$\Omega^{wk}=\bigcup_{j=1}^{\infty}B[\sigma_j,\delta_\epsilon]\cap\Omega^{wk}\subset\bigcup_{j=1}^{\infty}\widehat{s_j\cdots s_1}^{-1}(x_\epsilon+V_\epsilon).$$

Then $\{\widehat{s_j\cdots s_1}^{-1}(x_\epsilon+V_\epsilon)\cap\Omega^{wk}\ ;j=1,2,\cdots\}$ is a weak open covering of the weakly compact set Ω^{wk}. Therefore there is $m\in\mathbb{N}$ such that $\Omega^{wk}=\bigcup_{i=1}^{m}\widehat{s_i\cdots s_1}^{-1}(x_\epsilon+V_\epsilon)\cap\Omega^{wk}$. Since $\widehat{s_m\cdots s_1}(\Omega^{wk})=\Omega^{wk}$ then

$$\begin{aligned}\Omega^{wk}=\bigcup_{j=1}^{m}\widehat{s_j\cdots s_1}^{-1}(x_\epsilon+V_\epsilon)\cap\Omega^{wk} &\subset \bigcup_{j=1}^{m}s_m\widehat{\cdots s_{j+1}}((x_\epsilon+V_\epsilon)\cap\Omega^{wk})\\ &\subset \bigcup_{j=1}^{m}s_m\widehat{\cdots s_{j+1}}(B[x_\epsilon,\epsilon]\cap\Omega^{wk})\end{aligned}$$

By non-expansiveness, we have $\|s_m\widehat{\cdots s_{i+1}}(y)-s_m\widehat{\cdots s_{i+1}}(x_\epsilon)\|\leq\epsilon$ for all $y\in B[x_\epsilon,\epsilon]$ and this shows that

$$\Omega^{wk}\subset\bigcup_{i=1}^{m}B[s_m\widehat{\cdots s_{i+1}}(x_\epsilon),\epsilon].$$

Which shows the norm total boundedness of Ω^{wk} and therefore, its compactness in the norm topology. □

We are now ready to prove the theorem.

Proof. Let $S = \bigcup_\gamma S_\gamma$ where each sub-semigroup S_γ is left amenable.

• Step 1 : Each S_γ has a non-empty fixed point set. In fact, let us fix γ. By a Zorn's lemma argument, let K^{wk} be a minimal non-empty weakly compact convex and S_γ-invariant subset of K. If we consider the restriction of the representation of S on K to $S_\gamma \times K^{wk}$, then the left amenability of S_γ implies, via lemma 2.3.14, the existence of non-void norm compact subset Ω^{wk} of K^{wk} such that $s.\Omega^{wk} = \Omega^{wk}$ for all $s \in S_\gamma$. On the other hand, if Ω^{wk} is not a singleton then from [17], there is $x_o \in \overline{co}(\Omega^{wk})$ such that :

$$\sup_{x \in \Omega^{wk}} \|x - x_o\| < \delta(\Omega^{wk}).$$

Then it is easy to see that $\bigcap_{x \in \Omega^{wk}} B[x, \sup_{z \in \Omega^{wk}} \|x_o - z\|] \cap K^{wk}$ is a proper, non-void, weakly compact, convex and S_γ-invariant subset of K^{wk}; contradicting the minimality of K^{wk}. So, necessarily Ω^{wk} is a singleton; the desired point we are looking for.

• Step 2 : Now we assume that S is an arbitrary σ-LA semigroup.

$$\text{Let us set } \gamma_i \leq \gamma_j \text{ if and only if } S_{\gamma_i} \subset S_{\gamma_j}.$$

For all γ, let $x_\gamma \in K$ be a "partial" common fixed point for S_γ, i.e., $s.x_\gamma = x_\gamma$, for all $s \in S_\gamma$. By weak compactness of K, there is a weakly convergent subnet $(S_{\gamma_t})_{t \in (T,\mathcal{R})}$ of $(S_\gamma)_\gamma$. Let $\bar{x}$ be its limit. Then $\bar{x}$ is a common fixed point for S. In fact, let $s \in S$ fixed. We choose an index γ_s such that $s \in S_{\gamma_s}$ and $t_s \in T$ with the property :

$$t_s \mathcal{R} t \Rightarrow S_{\gamma_s} \subset S_{\gamma_t}.$$

Then if $t \in T$ and $t_s \mathcal{R} t$, we have $s.x_{\gamma_t} = x_{\gamma_t}$. Therefore by passing to the limit with the continuity it follows that $s.\bar{x} = \bar{x}$. Hence $\bar{x} \in F(S)$. $\square$

Corollary 2.3.15. Let $\mathcal{F}$ be a commuting family of weakly continuous non-expansive mappings on a non-void weakly compact convex subset K of a Banach space E. Then there is a common fixed point for $\mathcal{F}$ in K.

Proof. Let $S := \langle \mathcal{F} \rangle$ be the semigroup generated by $\mathcal{F}$. $\mathcal{F}$ is amenable. If $\mathfrak{S} = \{\hat{f}\ ; f \in S\}$ with $\hat{f} \equiv f$. Then we have $F(\mathcal{F}) = F(S) \neq \emptyset$. $\square$

Remark 2.3.16. From [32, theorem 4], the above corollary remains true even if we replace $\mathcal{F}$ by a left amenable discrete semigroup of weakly continuous non-expansive mappings.

We know already that left amenable semigroups possess the fixed point property (F_{wk}). The next result shows that if we consider the discrete case, then (F_{wk}) is extendable to a more general class of semigroups. We need before to recall some well-known facts and fix some notations. Given a semigroup S, we define a new semigroup S_e obtained by adjoining a new element e to it and extend its product by letting $s.e := s =: e.s$ for all $s \in S$. This new structure is called an *unitization* of S; and S is said to be a *unital* semigroup if it has an identity element. It is known that S is left amenable if and only if S_e is, this fact follows from the proof [45, lemma 6.3] if we replace $\mathrm{AP}(S)$ by $\ell^\infty(S)$ throughout.

Let $(S_\alpha)_{\alpha \in J}$ be a family of unital semigroups with respective identities e_α . Their *direct product* $S := \prod_\alpha S_\alpha$ is a unital semigroup with identity $e = (e_\alpha)_\alpha$ and operation defined coordinatewise, i.e., given $s, t \in S$, their product is defined by $s.t := (s_\alpha . t_\alpha)_\alpha$. Next we endow S with the product topology of the discrete topologies and put

$$\varsigma := \{\sigma \subset J \,; \sigma \text{ is non-void and finite}\}$$

Then for all $\sigma \in \varsigma$, define a sub-semigroup $S_\sigma := \prod_{\gamma \in \sigma} S^*_\gamma$ of S, with :

$$S^*_\gamma := \begin{cases} S_\gamma, & if\ \gamma \in \sigma \\ \{e_\gamma\}, & if\ \gamma \notin \sigma \end{cases}$$

Theorem 2.3.17. Let $S = \prod_\alpha S_\alpha$ be a direct product. If each S_α is left amenable, then S possesses the following fixed point property :

(F'_{wk}) : Whenever $\mathfrak{S} = \{\hat{s} \ ; \ s \in S\}$ is a jointly weakly continuous nonexpansive representation of S on a non-empty weakly compact convex subset K of Banach space E then, there is a common fixed point for S.

Lemma 2.3.18. For all $\sigma \in \varsigma$, the sub-semigroup S_σ is left amenable.

Proof. We shall proceed by induction on the cardinality $|\sigma|$ of σ.

- If $|\sigma| = 1$ then nothing to prove since each S_α is left amenable.
- If $|\sigma| = 2$, let $\sigma := \{\alpha_1, \alpha_2\}$. Then $S_\sigma \cong S^*_{\alpha_1} \times S^*_{\alpha_2}$ and we know that

$S^*_{\alpha_1} \times S^*_{\alpha_2}$ is left amenable, see [39, proposition 3.4]. Therefore, S_σ as an homomorphic image of $S^*_{\alpha_1} \times S^*_{\alpha_2}$, is also left amenable (cf. [14]).

• Assume that the statement holds for $|\sigma| = n$. Let $\sigma := \{\alpha_1, \cdots, \alpha_{n+1}\}$ be in ς. $S_\sigma \cong S_{\widetilde{\sigma}} \times S_{\alpha_{n+1}}$, with $\widetilde{\sigma} := \{\alpha_1, \cdots, \alpha_n\}$ through the isomorphism $s = (s_\alpha)_\alpha \mapsto (\pi_{\alpha_1,\cdots,\alpha_n}(s), s_{\alpha_{n+1}}) : S_\sigma \to S_{\widetilde{\sigma}} \times S^*_{\alpha_{n+1}}$, where $\pi_{\alpha_1,\cdots,\alpha_n}$ is the projection of S onto $S_{\widetilde{\sigma}}$. Then from the induction and the case $|\sigma| = 2$, it follows that $S_{\widetilde{\sigma}} \times S^*_{\alpha_{n+1}}$ is left amenable; and therefore S_σ. Therefore the statement holds also for $|\sigma| = n + 1$. Hence for all $\sigma \in \varsigma$, which completes the proof. □

We are now ready to prove the theorem.

Proof. For $\alpha \in \varsigma$ fixed, since S_σ is left amenable by the previous lemma, then theorem 2.3.11 ensures the existence of a common fixed point x_σ in K for the sub-representation $\mathfrak{S}_\sigma := \{\hat{s} \ ; \ s \in S_\sigma\}$; i.e., $s.x_\sigma = x_\sigma$ for all $s \in S_\sigma$. Now we define a pre-order on ς upwards by inclusion, or more precisely,

$$\forall\, \sigma, \sigma^{'} \in \varsigma,\ \sigma \leq \sigma^{'} \ \Leftrightarrow\ \sigma \subset \sigma^{'}.$$

Then the family $(x_\sigma)_{\sigma\in\varsigma}$ becomes a net and from the weak compactness on K, it follows that it has a convergent subnet $(x_{\sigma_t})_{t\in(T,\mathcal{R})}$. Then we define $\bar{x} := \lim_t x_{\sigma_t} \in K$. Then we show next that $\bar{x}$ is the element of K we are looking for. For that, let $s = (s_\alpha)_\alpha \in S$ fixed. For every $t \in T$, we define an element s_t of S by letting :

$$s_{t,\alpha} := \begin{cases} s_\alpha, & if\ \alpha \in \sigma_t \\ \{e_\alpha\}, & otherwise \end{cases}$$

Then the net $(s_t)_{t \in T}$ converges to s. In fact, let $U = \prod_{\alpha \in J} U_\alpha$ be an elementary open neighborhood of s. From the definition of the product topology, their is $\sigma_s \in \varsigma$ such that $U_\alpha \equiv S_\alpha$ for all $\alpha \in J \setminus \sigma_s$. Now we pick $t_s \in T$ with the property : $t_s \mathcal{R} t$ implies $\sigma_s \leq \sigma_t$ (note that a such index does exist because $(x_{\sigma_t})_t$ is a subnet of $(x_\sigma)_\sigma$). Then it follows that for all $t \in T$, $t_s \mathcal{R} t \Rightarrow s_t \in U$; which proves our assertion, $s = \lim_t s_t$. On the other hand given $t \in T$, we have :

$$s_t \in S_{\sigma_t} \text{ and } s_t.x_{\sigma_t} = x_{\sigma_t} \qquad (1).$$

So from relation (1) and the joint continuity of the representation of S we obtain : $s.\bar{x} = \lim_t s_t.x_{\sigma_t} = \lim_t x_{\sigma_t} = \bar{x}$. Hence, $s.\bar{x} = \bar{x}$, $\forall\ s \in S$. $\square$

Remark 2.3.19. Note that a direct product of a family of amenable semigroups need not be amenable in general; even for groups, cf. [14].

Remark 2.3.20. Fixed point property (F'_{wk}) is a generalization of (F_{wk}) (discrete case); because on the one hand, a discrete semigroup S is left amenable if and only if its unitization S_e is (cf. [45]), and on the other hand, any representation $\mathfrak{S} = \{\hat{s}\ ; s \in S\}$ of S on a compact space K can be extended to a representation $\mathfrak{S}_e = \{\hat{s}\ ; s \in S_e\}$ of S_e by letting $\hat{e} := id_K$ (the identity map of K).

1

[1]Some results of this chapter are contained in a paper accepted for publication in the Journal of Fixed Point Theory and Applications, see [57].

CHAPTER 3

Semi-topological semigroups and non-linear common fixed point properties on weak* compact convex sets

In this chapter, we shall be interested on non-expansive representations of semi-topological semigroups. Our attention is mainly concerned with the study of problem 3, i.e., whether or not left amenable semi-topological semigroups possess the fixed point property (F^*).

We prove that, with a compactness condition of some orbits with respect to a suitable locally convex topology on the dual, then a weak version of (F^*) holds respectively for : left amenable, σ-left amenable, sequentially left amenable, and all strongly left reversible semi-topological semigroups. Whereas, n-ELA semi-topological semigroups possess the fixed point property (F^*), and discrete n-ELA semigroups have a fixed point property much stronger than (F^*). We show that the classes of semi-topological semigroups mentioned above possess (F_{wk}) and that property (F_{ns}) is extendable to locally convex spaces. Some results related to [47, theorem 3] and [49, theorem 4] are also established.

3.1 Semi-topological semigroups and weak* fixed point properties in dual spaces

Given a Banach space E, let $B_{E^{**}}$ denote the unit closed ball of the second dual E^{**}; and let $\mathrm{Ext}(B_{E^{**}})$ be the set of all extreme points of $B_{E^{**}}$ (which is of course non-void by virtue of the Krein-Milman theorem). Consider on the dual the locally convex topology τ defined by the family of semi-norms $Q := \{p_e \; ; e \in \mathrm{Ext}(B_{E^{**}})\}$ where, $p_e(f) = |e(f)|$. Then using the Krein-Milman theorem, it is easy to see that τ is separated. On the other hand, by construction $\tau \subset \sigma(E^*, E^{**})$ on E^*. If E^{**} is uniformly convex or more generally, if E is reflexive then τ is coarser than the weak* topology.

Theorem 3.1.1. Let S be a semi-topological semigroup. If S is σ-LA, then it possesses the following non-linear fixed point property :

(F^*_τ) : Whenever $\mathfrak{S} = \{\hat{s} \; ; \; s \in S\}$ is a weak* jointly continuous non-expansive representation on a non-empty weak* compact convex subset K of the dual E^* of a Banach space E, such that for all non-void weak* closed and S-invariant set $B \subset K$ with the property $s.B = B$ for all $s \in S$, there is $x \in B$ whose orbit $\mathcal{O}_x$ is relatively countably τ-compact, then there is in K a common fixed point for S.

In order to prove this result, the following lemmas are essential :

Lemma 3.1.2. Let S^* be a separable sub-semigroup of S. If K' is a non-empty weak* compact and S^*-invariant subset of K, then there is non-void weak* compact S^*-invariant subset Ω^* of K' which is minimal (for having all these properties) such that $s.\Omega^* = \Omega^*$ for all $s \in S^*$.

Proof. It follows by combining lemma 2.3.5 and lemma 2.3.13, by just taking $\tau := wk^*$ (i.e., $\sigma(E^*, E)$) and Φ=LUC(S). $\square$

Lemma 3.1.3. Let Ω^* be as in the previous lemma. Then the following facts hold :

1. For all $x \in \Omega^*$, the orbit $\mathcal{O}_x$ of x, is weak* dense in Ω^*;
2. Ω^* is $\sigma(E^*, E^{**})$-compact.

For proving this lemma the following characterization is needed.

Lemma 3.1.4. Let B be a Banach space and C be a bounded subset of B. C is relatively weakly compact if and only if, for all sequence $(x_n)_n$ in C there is a weakly convergent sequence $(y_n)_n \in E$ such that $y_n \in co(x_i\ ; i \geq n)$ for all n.

Proof of Lemma 3.1.3. For part 1, clearly orbits are S-invariant; and since for all s in S, the mapping $x \mapsto s.x$ is weak*-weak* continuous (due to the continuity of the action) then weak* closures of orbits are also S-invariant. Hence, by minimality it follows that part 1 holds. For part 2, from lemma 3.1.2, we know that Ω^* is non-void with $s.\Omega^* = \Omega^*$ for all $s \in S$. Let us fix $x \in \Omega^*$ with a relatively τ-compact orbit; and let $(z_n)_n$ be a sequence in $\mathcal{O}_x$. Since the orbit $\mathcal{O}_x$ is bounded (as a subset of K), then a fortiori so is $(z_n)_n$. Therefore by [55, corollary 0.2], there is a subsequence $(z_{n_k})_k$ of $(z_n)_n$ such that :

$$\bigcap_{k=1}^{\infty} \overline{co}^{\tau}(z_{n_i} \,; i \geq k) \subset \bigcap_{n=1}^{\infty} \overline{co}(z_i \,; i \geq n).$$

Define $F_n := \overline{\{z_{n_i} \,; i \geq n\}}^{\tau}$ for all n. Then $\{F_n \,; n \in \mathbb{N}\}$ is a decreasing sequence of τ-closed non-empty subsets of the countably compact space $\overline{\mathcal{O}_x}^{\tau}$. Therefore, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$; and this latter property allows us to be able to find a point $\xi \in \bigcap_{n=1}^{\infty} \overline{co}(z_i \,; i \geq n)$ and therefore, a sequence $(\xi_n)_n \in E^*$ such that $\xi_n \in co(z_i \,; i \geq n)$ for all n, and $\|\xi_n - \xi\| \to 0$. Then by applying the lemma 3.1.4 it follows that $\overline{\mathcal{O}_x}^{wk}$ is weakly compact; and a fortiori closed in the weak* topology. Hence, together with the first part, it follows that $\Omega^* = \overline{\mathcal{O}_x}^{wk}$ is $\sigma(E^*, E^{**})$ compact. $\square$

Lemma 3.1.5. Ω^* is compact in the norm topology.

Proof. Since Ω^* is weakly compact, see lemma 3.1.3, then weak and weak* topologies coincide on Ω^*. Therefore it follows that the sub-representation $\mathfrak{S}^* := \{\hat{s} \,; s \in S^*\}$ is jointly weakly continuous. Hence, the separability of S^* together with a similar argument as in the proof of lemma 2.3.13 yields the desired result. $\square$

Now we are ready to proceed to the proof of theorem 3.1.1.

Proof. Let $S = \bigcup_{\gamma\in J} S^*_\gamma$ such that each space LUC(S^*_γ) has a LIM. If we fix $\gamma \in J$, let K_γ be a minimal non-empty weak* compact convex and S^*_γ-invariant subset of K. From lemma 3.1.2, there is $\Omega^*_\gamma \subset K_\gamma$ norm compact subset such that $s.\Omega^*_\gamma = \Omega^*_\gamma$ for all $s \in S^*_\gamma$. Then $\mathfrak{S}^*_\gamma := \{\hat{s}\ ; s \in S^*_\gamma\}$ being a jointly weakly continuous representation, a similar argument as in Step 1 in the proof of theorem 2.3.12 shows that Ω^*_γ must be a single point $\{x_\gamma\}$, which is a common fixed point for S^*_γ. If we consider the collection $\{x_\gamma\ ; \gamma \in J\}$ and pre-order the index set J by letting $\gamma \le \gamma'$ if and only if, $S^*_\gamma \subset S^*_{\gamma'}$, then an argument similar to that used in Step 2 in the proof of theorem 2.3.12 yields the existence of a common fixed point for S. $\square$

From the proof of the previous theorem, we deduce the following:

Corollary 3.1.6. Let S be a semigroup satisfying the conditions of theorem 3.1.1. Then it has the following fixed point property :

(F^*_{wk}) : Whenever $\mathfrak{S} = \{\hat{s}\ ;\ s \in S\}$ is a weak* jointly continuous non-expansive representation on a non-empty weak* compact convex subset K of the dual E^* of a Banach space E such that for all non-void weak* closed S-invariant subset B of K with $s.B = B$ for all $s \in S$, we can find $x \in B$ whose orbit $\mathcal{O}_x$ is relatively weakly compact then, there is in K a common fixed point for S.

Proof. Indeed, $\tau \subset \sigma(E^*, E^{**})$. $\square$

Theorem 3.1.7. Let S be a semi-topological semigroup. If S is left reversible as a discrete semigroup, or separable and left reversible, then it possesses the fixed point property (F^*_τ).

Proof. From [45, corollary 3.7], given a non-void, S-invariant, weak* compact and convex subset K^* of K, there is a non-empty weak* closed set $\Omega^* \subset K^*$ such that $s.\Omega^* = \Omega^*$ for all $s \in S$. If S is separable, then by lemma 3.1.5 it follows that Ω^* is norm compact. Therefore, a similar argument as in the proof of the previous theorem shows that the assertion is true. If S is discrete and left reversible, then it can be written as a direct union of countable left reversible sub-semigroups, cf. [32, lemma 1], then in this case if $S = \bigcup_\gamma S_\gamma$ where each S_γ is left reversible and countable (a fortiori, separable), the forgoing shows that each S_γ possesses a common fixed point in K. Hence, an argument as in Step 2 in the proof of theorem 2.3.12 yields the desired result. $\square$

Remark 3.1.8. The fixed point theorem 3.1.7 shows that for separable left reversible semi-topological semigroups and (discrete) left reversible semigroups, Lim's result, cf. [47, theorem 3], in Banach spaces is extendable to weak* compact convex sets, without requiring a normal structure condition if the representation is jointly continuous in the weak* topology.

3.2 Fixed point properties of n-ELA semigroups of non-linear mappings

Let K be a non-empty convex subset of a vector space. A mapping $f : K \to K$ is said to be *affine* if :

$f(t.x + (1-t).y) = tf(x) + (1-t)f(y)$, for all $x, y \in K$ and $t \in$[0,1].

$\mathfrak{S}$ is said to be an *affine representation*, if each mapping in $\mathfrak{S}$ is affine.

Theorem 3.2.1. If S is a n-ELA discrete semigroup, then it possesses the following fixed point property :

(F_{affine}) : Whenever $\mathfrak{S} = \{\hat{s}\ ;\ s \in S\}$ is a $C(K)^+$-continuous affine representation of S on a non-empty weakly compact convex subset K of a Hausdorff locally convex topological vector space (E, Q), then S possesses a common fixed point in K.

Proof. Let $m \in co(\beta S)$ be a LIM. Then $m = \sum_{j=1}^{n} t_j m_j$. Pick $x_o \in K$ and consider as in the proof of lemma 2.3.5

$$\begin{array}{rcl} \Theta_{x_o} : C(K) & \to & \mathbb{R} \\ f & \mapsto & m(\theta^f_{x_o}) \end{array}$$

From the proof of lemma 2.3.5, $\Theta_{x_o}(f) = \int_K f\ d\mu$ for some regular Borel measure on K and $\omega_{wk} :=$ support(μ) satisfies :

$$\omega_{wk} \subset \overline{s.\omega_{wk}}^{wk} \text{ for all } s \in S \quad (1)$$

if we let $\tau = wk$. Then ω_{wk} is necessarily finite; indeed, from the definition of the support, ω_{wk} is characterized by :

$$x \in \omega_{wk} \;\Leftrightarrow\; \forall\, V \in \mathscr{V}_{wk}(x),\; \mu(V \cap \omega_{wk}) > 0.$$

Where $\mathscr{V}_{wk}(x)$ denotes the collection of all neighborhoods of x in E. On the other hand, for all j, $f \mapsto m_j(\theta^f_{x_o}) : C(K) \to \mathbb{R}$ is a nonzero multiplicative linear functional on $C(K)$; therefore $m_j(\theta^f_{x_o}) = f(x_j)$ for some unique $x_j \in K$. If $x \notin \{x_1, \cdots, x_n\}$, then by Urysohn's lemma, there is $f \in C(K)$ such that $f \geq 0$, $f(x)$=1 and $f(x_j) = 0$ for all j. Then $V := \{f > 0\}$ is a neighborhood of x and we have :

$$\begin{aligned}
\mu(V \cap \omega_{wk}) \leq \int_V f\, d\mu \;&\leq\; \int_K f\, d\mu = m(\theta^f_{x_o}) \\
&= \sum_{j=1}^{n} t_j m_j(\theta^f_{x_o}) = \sum_{j=1}^{n} t_j f(x_j) \\
&= 0
\end{aligned}$$

It follows that $\mu(V \cap \omega_{wk}) = 0$ which shows that x does not belong to ω_{wk}. Hence, $\omega_{wk} \subset \{x_1, \cdots, x_n\}$. Thus, together with (1), we get

$$|\omega_{wk}| \leq |\overline{s.\omega_{wk}}^{wk}| \leq |\omega_{wk}| \Rightarrow s.\omega_{wk} = \omega_{wk}, \text{ for all } s \in S \quad (2).$$

Let us define $\hat{x} := \frac{1}{|\omega_{wk}|} \sum_{x \in \omega_{wk}} x$. Then using (2) and the affineness of $\mathfrak{S}$, we get :

$$s.\hat{x} = \frac{1}{|\omega_{wk}|} \sum_{x \in \omega_{wk}} s.x = \frac{1}{|\omega_{wk}|} \sum_{x \in s.\omega_{wk}} x = \hat{x}$$

for all $s \in S$. Hence, $\hat{x}$ is a common fixed point for S. $\square$

Next, we establish a non-linear version of theorem 3.2.1.

Theorem 3.2.2. If S is an n-ELA discrete semigroup, then it possesses the following fixed point property :
(F_s^*) : Whenever $\mathfrak{S} = \{\hat{s}\ ;\ s \in S\}$ is a $C(K)^+$-continuous non-expansive representation of S on a non-empty weak* compact convex subset K of the dual E^* of a Banach space E, then S possesses a common fixed point in K.

Proof. Clearly $\mathfrak{S}$ defines an A-representation of $S, \ell^\infty(S)$ on each S-invariant weak* compact subset of K. Fix a minimal non-empty S-invariant weak* compact convex subset K^* of K. In lemma 2.3.5 if we let τ = wk*, then using the n-ELA property (see the previous proof), there is a non-void finite set $\omega_{wk*} \subset K^*$ such that $s.\omega_{wk*} = \omega_{wk*}$ for all $s \in S$. Since $co(\omega_{wk*})$ possesses asymptotic normal structure (since norm compact), the conclusion follows using an argument as in the proof of theorem 2.3.12. □

Corollary 3.2.3. $(F_s^*) \Rightarrow (F^*)$, i.e., if a semi-topological semigroup possesses the fixed point property (F_s^*), then a fortiori it does for (F^*). Therefore, n-ELA semi-topological semigroups do possess (F^*).

However, for n-extremely left amenable semi-topological semigroups we have the following result :

Theorem 3.2.4. If S is an n-ELA semi-topological semigroup, then it possesses the fixed point property (F^*).

Proof. From lemma 2.3.12, $\mathfrak{S}$ defines an A-representation of S,LUC(S) on K. Therefore, if we fix a minimal non-empty weak* compact convex and S-invariant set $K^* \subset K$, it follows from lemma 2.3.5 together with the fact that S is n-ELA, the existence of a non-empty finite subset ω_{wk*} of K^* such that $s.\omega_{wk*} = \omega_{wk*}$ for all $s \in S$. Let $\omega_{wk*} = \{\omega_1, \cdots, \omega_p\}$ for some $p \in [1, n]$, and consider the Chebyshev center of ω_{wk*} in K^*

$$W_{K^*}(\omega_{wk*}) := \{x \in K^* \; ; \; \max_i \|x - \omega_i\| = \inf_{y \in K^*} \max_i \|y - \omega_i\|\}.$$

Due to the weak* compactness and convexity of K^*, $W_{K^*}(\omega_{wk*})$ is a non-empty weak* compact and convex set. On the other hand, a similar argument as in the proof of theorem 2.2.1 together with the weak* lower semi-continuity of the dual norm yield

$$s.W_{K^*}(\omega_{wk*}) \subset W_{K^*}(\omega_{wk*}) \text{ for all } s \in S.$$

Therefore, by minimality it follows that $W_{K^*}(\omega_{wk*}) = K^*$. Now assume that ω_{wk*} has at least two elements. By [17, lemma 1], there is x^* in $co(\omega_{wk*})$ such that $\max_{1 \le i \le p} \|x^* - \omega_i\| < \max_{i \neq j} \|\omega_i - \omega_j\|$ (1). Let $i \neq j \in \{1, \cdots, p\}$. Since $\omega_i \in W_{K^*}(\omega_{wk*})$, then we have

$$\max_k \|\omega_i - \omega_k\| = \inf_{y \in K^*} \max_k \|y - \omega_k\|.$$

It follows using (1)

$$
\begin{aligned}
\|\omega_i - \omega_j\| &\leq \max_k \|\omega_i - \omega_k\| \\
&= \inf_{y \in K^*} \max_k \|y - \omega_k\| \\
&\leq \max_k \|x^* - \omega_k\| \\
&< \max_{p \neq q} \|\omega_p - \omega_q\|
\end{aligned}
$$

Thus, it follows that $\|\omega_i - \omega_j\| < \max_{p \neq q} \|\omega_p - \omega_q\|$ for all $i \neq j$, which is not possible. Hence ω_{wk*} must be a singleton. □

3.3 A weak* fixed point property in conjugate Banach spaces via ℓ^1

In [49], Lim proved that any decreasing net of non-void bounded subsets of ℓ^1 (considered as the dual space of c_0) has a non-empty norm compact convex asymptotic center with respect to any weak* closed convex set containing them. Using this result, we establish a non-linear fixed point theorem for semi-topological semigroups.

Theorem 3.3.1. Let S be a semi-topological semigroup. If LUC(S) has a LIM, then S has the following fixed point property :

(F^*_{isom}): Whenever $S \times K \to K$ is a jointly weak* continuous non-expansive action of S on a non-empty weak* compact convex subset K of a dual Banach space E^* such that there is a weak* closed isometry from K into ℓ^1, then there is in K a common fixed point for S.

Remark 3.3.2. In the above theorem, we consider ℓ^1 as the dual of c_0 and point out that the isometry in the above theorem need not be linear, but only closed in the weak* topology (i.e., the direct image of a weak* closed subset of K is weak* closed in ℓ^1) or weak*-weak* continuous.

Proof. Consider a non-void weak* compact subset ω_{wk*} of K with the property :

$$s.\omega_{wk*} = \omega_{wk*} \text{ for all } s \in S. \qquad (*).$$

Let $\phi : K \to \ell^1$ be the isometry whose existence is guaranteed by assumption. Then by [49], the set

$$C := \{x \in \phi(K) \; ; \sup_{y \in \phi(\omega_{wk*})} \|x - y\| = r\}$$

where $r := \inf_{y \in \phi(K)} \sup_{z \in \phi(\omega_{wk*})} \|y - z\|$, is a non-void and norm compact set. Therefore its preimage $\phi^{-1}(C)$ is a non-empty norm compact subset of E^*. Now define analogously as C the set

$$\hat{K} := \{x \in K \; ; \sup_{y \in \omega_{wk*}} \|x - y\| = \rho\}.$$

where $\rho := \inf_{y \in K} \sup_{z \in \omega_{wk*}} \|y - z\|$. The set $\hat{K}$ is non-void because one the hand, it can be written as

$$\bigcap_j \{x \in K \; ; \sup_{y \in \omega_{wk*}} \|x - y\| \leq \rho + \frac{1}{j}\}$$

and on the other hand, each set in the intersection is non-empty and weak* closed due to the weak* lower-semi-continuity of the dual norm on E^*. Hence, the weak* compactness of K forces $\hat{K}$ to be non-void. Next we show that $\hat{K}$ is norm compact. For that, it is enough to prove that it is a subset of $\phi^{-1}(C)$. Fortunately, it does. In fact, given $x \in \hat{K}$ we have $\phi(x) \in \phi(K)$ and

$$\begin{aligned}
\sup_{y \in \phi(\omega_{wk*})} \|\phi(x) - y\| &= \sup_{y \in \omega_{wk*}} \|\phi(x) - \phi(y)\| \\
&= \sup_{y \in \omega_{wk*}} \|x - y\| = \rho \\
&= \inf_{y \in K} \sup_{z \in \omega_{wk*}} \|y - z\| \\
&= \inf_{y \in K} \sup_{z \in \omega_{wk*}} \|\phi(y) - \phi(z)\| \\
&= \inf_{y \in \phi(K)} \sup_{z \in \phi(\omega_{wk*})} \|y - z\| = r
\end{aligned}$$

Therefore the inclusions holds and it follows that $\hat{K}$ is a non-empty norm compact convex subset of K. Moreover, a similar argument as in the proof a theorem 2.3.3 shows that $\hat{K}$ is also S-invariant. The restriction $S \times \hat{K} \to \hat{K}$ of the S-action on K becomes then a jointly norm continuous non-expansive action (since weak* and norm topologies agree). Using a Zorn's lemma argument if necessary, we may assume that $\hat{K}$ is minimal (i.e., there is no proper subset of $\hat{K}$ with the same properties). Then $\hat{K}$ must be a singleton because otherwise, being norm compact and convex, it has normal structure (see [17]) and therefore leads to a contradiction. □

When S is left reversible as a semi-topological semigroup, then (see [31] or [47]) S becomes a directed set if we let :

$$a \leq b \text{ iff } \{b\} \cup \overline{b.S} \subset \{a\} \cup \overline{a.S}$$

Then if fix $x \in K$ (i.e., whenever S defines an action as in theorem 3.3.1) we define $\Omega_s := \overline{s.S.x}$ for all $s \in S$, ten we obtain a decreasing net of subsets of K. Hence, it follows :

Corollary 3.3.3. All left reversible semi-topological semigroups possess the fixed point property (F^*_{isom}).

3.4 Some topological extensions

Given a semi-topological semigroup S, let $\mathrm{M}(S)$ denote the collection of all means on $\mathrm{LUC}(S)$ and let βS denote the subset of $\mathrm{M}(S)$ of all multiplicative means on $\mathrm{LUC}(S)$ (i.e., the spectrum of $\mathrm{LUC}(S)$). Motivated by theorem 2.1.1, we introduce the following subset of $\mathrm{M}(S)$:

$$\overline{co}^{seq}(\beta S) := \{m \in \mathrm{M}(S) \; ; \; m = \text{wk*-lim}_n \, m_n, \; m_n \in co(\beta S)\}$$

We shall say that S is *sequentially left amenable*, or "*seq*-LA" for short, if the Banach algebra $\mathrm{LUC}(S)$ possesses a left invariant mean in $\overline{co}^{seq}(\beta S)$.

Example 3.4.1. By virtue of theorem 2.1.1, the class of all sequentially left amenable semigroups includes all countable left amenable discrete semigroups (e.g., $\mathbb{Z}$); it includes also the collection of all compact metrizable left amenable semi-topological semigroups (e.g., the unit circle $\mathbb{S}^1$).

With this definition, theorem 3.1.1 is extendable as follows :

Theorem 3.4.2. Let S be a semi-topological semigroup. If S is σ-LA or *seq*-LA, then it possesses the fixed point property (F^*_τ).

Proof. Each S-invariant, non-void weak* compact subset of K contains a non-empty weakly compact (using lemma 3.1.3) subset ω_{wk*} such that $s.\omega_{wk*} = \omega_{wk*}$, for all $s \in S$. Using the an argument as in the proof of theorem 3.2.1 together with the sequential left amenability property and lemma 2.3.5, ω_{wk*} can be chosen to be separable in the weak topology. Hence, we conclude by an argument as in the proof of theorem 3.1.1. $\square$

Using the fixed point theorem 3.1.1, we derive the following dual version :

Theorem 3.4.4. Let S be a semi-topological semigroup. If S is either σ-LA or seq-LA, then it possesses the fixed point property (F_{wk}).

Proof.

- Step 1 : We first assume that S is separable or seq-LA. We embed E in its second dual E^{**} through the canonical injection $j : E \to E^{**}$ which is an isomorphism from (E,wk) onto $(j(E),\text{wk*})$. Then $\hat{K} := j(K)$ is a non-void weak* compact convex subset of E^{**}. We carry the S-action on K to $\hat{K}$ by letting $s * j(x) := j(s.x)$, for all $s \in S$ and $x \in K$. As readily checked, the action $S \times \hat{K} \to \hat{K}$ is jointly weak* continuous and norm non-expansive. Let $\hat{\tau}$ be the locally convex topology on E^{**} induced by the extreme points of $B_{E^{***}}[0,1]$ (the unit closed ball of the dual of E^{**}). If $\hat{B} \subset \hat{K}$ is a non-void weak* compact subset such that $s * \hat{B} = \hat{B}$ for all $s \in S$. Using a Zorn's lemma argument if necessary together with lemma 2.3.5 if S is left amenable, or [45, corollary 3.7] if S is left reversible, we may assume that $\tilde{B}$ is minimal (in the sense that, if $\tilde{B}$ is a non-void weak* compact S-invariant set contained in $\hat{B}$, then $\tilde{B} = \hat{B}$). Then $B := j^{-1}(\hat{B})$ is a minimal non-empty weakly compact S-invariant and separable subset of K with the property that $s.B = B$ for all $s \in S$. Therefore using lemma 2.3.14, it follows that B is norm compact and therefore its image $\hat{B}$ too. Thus, for all $j(x) \in \hat{K}$, the orbit $\mathcal{O}_{j(x)}$ is relatively $\hat{\tau}$-compact (since norm and $\hat{\tau}$ topologies agree on the norm closed orbit). Hence by theorem 1, there is $\hat{x} \in K$ such that $s * j(\hat{x}) = j(\hat{x})$ for all $s \in S$. Hence, $\hat{x}$ is a common fixed point for S due to the injectivity of j.

• Step 2 : Now we assume that S is an arbitrary semi-topological semigroup with either one of the properties in the theorem. From Step 1, a similar argument as in the proof of theorem 3.1.1, shows that if we consider the action of S carried on E^{**}, then

$$F(S) := \{x \in K; s * j(x) = j(x) \text{ for all } s \in S\} \neq \emptyset.$$

Hence, any $x \in K$ with $j(x) \in F(S)$ is a common fixed point for S. $\square$

Remark 3.4.5. Theorem 3.4.4 shows that beyond the class of σ-left amenable semi-topological semigroups, sequentially left amenable semi-topological semigroups do have the fixed point property (F_{wk}).
Whether or not left reversible semi-topological semigroups possess the fixed point property (F_{wk}), we have the following result in this direction:

We shall use the following concept introduced in [44]. A semi-topological semigroup S is *strongly left reversible*, if it can written as a direct union of countable left reversible sub-semigroups.

Example 3.4.6. The class of strongly left reversible semi-topological semigroups includes discrete left reversible semigroups [32, lemma 1], separable left reversible semi-topological semigroups, metrizable left reversible semi-topological semigroups [44, lemma 5.2].

Theorem 3.4.7. Strongly left reversible semi-topological semigroups possess the fixed point properties (F_{wk}) and (F_{τ}^{*}).

Proof. Indeed, if $S = \bigcup_{\alpha} S_{\alpha}$, then using [45, corollary 3.7] and a similar argument as in the proof of theorem 3.1.1, yield $F(S_{\alpha}) \neq \emptyset$ for all α. We conclude using part 2 in the proof of theorem 2.3.11. $\square$

Corollary 3.4.8. [32, theorem 4].

Proof. In fact, a discrete left reversible semigroup is strongly left reversible, cf. [32, lemma 1]. □

Remark 3.4.9. From theorem 3.4.6, one can say that in Banach spaces, normal structure condition is unnecessary if we consider jointly continuous non-expansive representations of separable semi-topological semigroups in the weak topology.

Finally, we establish an extension of (F_{ns}) (see theorem 2.3.3) to representations in Hausdorff locally convex spaces setting.

Remark 3.4.10. Even if locally convex spaces are not in general normable, using semi-norms, one can define a notion of non-expansive mapping which coincides with what we know in a normed space.

We fix a locally convex Hausdorff space E with family Q of semi-norms which induces its topology. Given a non-void set $K \subset E$, a mapping $T : K \to K$ is said to be *Q-non-expansive* or simply non-expansive, if it satisfies the following property : for all semi-norm $q \in Q$ and x, y in K we have :

$$q(T(x) - T(y)) \leq q(x - y).$$

Example 3.4.11. (Q-non-expansive map). Let S be a semi-topological semigroup. Let $\mathrm{AP}(S)$ be the translation invariant subspace of $C_b(S)$ consisting of those functions whose left orbits are relatively compact in the norm topology; more precisely, a function f belongs to $\mathrm{AP}(S)$ if, $\{\ell_s f\ ; s \in S\}$ has a compact closure in $C_b(S)$ in the supremum norm topology. Let $E := \mathrm{AP}(S)^*$ and $K := \mathrm{M}(\mathrm{AP}(S))$ (the set of all means on $\mathrm{AP}(S)$). Given f in $\mathrm{AP}(S)$, we define a semi-norm q_f on E by letting $q_f(m) = \sup_{s\in S} |m(\ell_s f)|$. Consider the locally convex topology on E induced by $Q = \{q_f\ ; f \in \mathrm{AP}(S)\}$. Then if we fix $s_o \in S$, the mapping

$$\begin{aligned} T_{s_o} : K &\to K \\ m &\mapsto \ell_{s_o}^* m \end{aligned}$$

where, $\ell_{s_o}^* m(f) := m \circ \ell_{s_o}(f) = m(\ell_{s_o} f)$, is Q-non-expansive.

From now on, we will assume E to be Hausdorff in order to ensure (via the Hahn-Banach separation theorem) that the topological dual is "huge" enough to separate points. In fact, the topological dual of a locally convex space which is not separated may be trivial (cf. [13]).

Remark 3.4.12. When E is a Banach space, then Q reduces to one semi-norm given by $q = \|.\|$; and in this case, a Q-non-expansive mapping is just the usual notion of a non-expansive mapping in normed spaces.

Theorem 3.4.13. Let S be a countable semi-topological semigroup. If LMC(S) has a LIM, then S has fixed the following fixed point property:

$(F_{ns,loc})$: Whenever $\mathfrak{S} = \{\hat{s}\,;s \in S\}$ is a Q-non-expansive representation of S on a weakly compact convex subset K with normal structure in a Frechet space E, such that for all $x \in K$, the mapping $s \mapsto s.x : S \to K$ is continuous in the relative weak topology, then K contains a common fixed point for S.

Before the proof, let us point out that the notion of an asymptotic center we used so far, remains valid in locally convex spaces; see [47].

Proof of Theorem 3.4.13. Let K^{wk} be a minimal, non-empty, weakly compact, convex and S-invariant subset of K. By virtue of lemma 2.3.4, $\mathfrak{S}$ defines an A-representation of S,LMC(S) on K^{wk}. If $\tau :=$ wk and $\tilde{\tau} = \tau_Q$, then from lemma 2.3.5, there is a non-empty subset $\omega_{wk} \subset K^{wk}$ such that $\omega_{wk} \subset \overline{s.\omega_{wk}}^{wk}$ for all $s \in S$ (1). We assert that ω_{wk} is a singleton. Indeed, otherwise there would be a semi-norm $q \in Q$ such that $\sup_{x,y \in \omega_{wk}} q(x-y) > 0$ (as E is Hausdorff). On the one hand, since normal structure and asymptotic normal structure are equivalent on K^{wk}, then the asymptotic center is properly contained in K^{wk}. Moreover, we know already that it is non-void, weakly compact and convex; and it is easy to see that it can be written as

$$\bigcap_{j=1}^{\infty} K_j := \bigcap_{j=1}^{\infty} \{x \in K^{wk} \; ; \; \sup_{y\in\omega_{wk}} q(x-y) \leq r(\omega_{wk}, K^{wk}) + \frac{1}{j}\}.$$

Now we show that it is invariant under the semigroup S. It is enough to show that each K_j is S-invariant. Let $j \in \mathbb{N}$ and $s \in S$ fixed. Let $x \in K_j$ fixed. Given $y \in \omega_{wk}$, using (1), let $y = \lim_\alpha s.y_\alpha$ with $y_\alpha \in \omega_{wk}$. From the Q-non-expansiveness, the weak lower semi-continuity of the semi-norm q, we get the following inequalities :

$$\begin{aligned} q(s.x-y) &\leq \liminf_\alpha q(s.x - s.y_\alpha) \\ &\leq \liminf_\alpha q(x-y_\alpha) \\ &\leq \sup_{z\in\omega_{wk}} q(x-z) \\ &\leq r(\omega_{wk}, K^{wk}) + \frac{1}{j} \end{aligned}$$

Since y is arbitrarily, then it follows that

$$\sup_{z\in\omega_{wk}} q(s.x-z) \leq r(\omega_{wk}, K^{wk}) + \tfrac{1}{j}$$

which shows that $s.x \in K_j$. Therefore, the asymptotic center is S-invariant. However, this is not possible from the minimality of K^{wk}. Hence, this forces K^{wk} to be a single point. □

Remark 3.4.14. $(F_{ns,loc}) \Rightarrow (F_{ns})$. This implication means that, if a semi-topological semigroup S possesses the fixed point property $(F_{ns,loc})$ then it does possess (F_{ns}).

Remark 3.4.15. The fact that discrete left amenable semigroups possess the fixed point property $(F_{ns,loc})$ follows from [49]; because for a discrete semigroup S, its left amenability implies its left reversibility and $\mathrm{LMC}(S) = \ell^\infty(S)$. However, when S is semi-topological, the left amenability of $\mathrm{LMC}(S)$ does not imply the left reversibility of S, even if $C_b(S)$ has a left invariant mean, see [31] for details.

CHAPTER 4

A fixed point proof of the existence of a left Haar measure for amenable locally compact topological groups

Existence of a left Haar measure for locally compact groups has been proven first by A. Haar in 1933 for second countable locally compact groups, see [7]; later, in 1940, A. Weil, proved the existence and uniqueness (up to a multiplicative positive constant) for general locally compact groups. Since then, many methods have been developed for constructing a left Haar measure; e.g., by using compact sets, compactly supported functions, fixed point techniques. In the latter approach, an application of the Ryll-Nardzewski fixed point theorem ensures the existence of a left Haar measure for compact topological groups, see [36]. For the class of abelian groups, a beautiful proof was established by A. Izzo [34] in 1992, using the Markov-Kakutani fixed point theorem. The existence of a left Haar measure for amenable hypergroups satisfying a positivity property for translations was proved by Wilson by a fixed point theorem [60].

In this chapter, we are able to extend Izzo's proof to a wider class of locally compact groups that includes abelian and compact groups; the so-called amenable locally compact groups.

Definition 4.1. A *topological group*, is a group G together with a Hausdorff topology τ such that the mappings : $(g, h) \mapsto g.h : G \times G \to G$ and $h \mapsto h^{-1} : G \to G$ are both continuous. We say that G is a *locally compact topological group* (or simply a locally compact group) if there is a base for the neighborhoods of the identity consisting of compact sets.

Example 4.2. Any group considered as a discrete space, is a locally compact group; the unit circle $\mathbb{S}^1$ in the usual topology is a locally compact abelian group.

Remark 4.3. Since topological groups form a sub-class of the semi-topological semigroups, one can talk about amenability. Except for a locally compact group G, its amenability is equivalent to that of $C_b(G)$.

From now on, G is an amenable locally compact group. On the normed space $C_c(G)$ we define a topology τ as follows :

Given a net $(f_\gamma)_{\gamma\in J}$ of elements of $C_c(G)$ and $f \in C_c(G)$, we say that $f_\gamma \to f$ if there is a compact subset K of G, and $\gamma_o \in J$ such that :

1. $\text{supp}(f) \subset K$ and $\bigcup_{\gamma\geq\gamma_o}\text{supp}(f_\gamma) \subset K$.
2. $\sup_{g\in K} |f_\gamma(g) - f(g)| \to 0$. i.e., $f_\gamma \to f$ uniformly on K.

Note that this is a well-defined Hausdorff topology. Because, once we know the convergent nets, then closed sets are known, and therefore the topology of the space is completely determined. On the other hand, τ is Hausdorff since it is finer that the locally convex Hausdorff topology generated by the semi-norms $p_K(f) := \sup_{g\in K} |f(g)|$, where K runs over all the compact subsets of G.

As readily checked, $(C_c(G), \tau)$ is a topological vector space. We denote by $(C_c(G), \tau)^*$ its topological dual (which is non-trivial as it contains at least evaluation mappings on $C_c(G)$). On $(C_c(G), \tau)^*$ we shall use the weak* topology.

Lemma 4.4. (A. Izzo [34]) Let G be a topological group and N be a symmetric neighborhood of e. Then there is a subset S of G such that for all $g \in G$, $(g.N.N) \cap S \neq \emptyset$ and $|g.N \cap S| \leq 1$.

Lemma 4.5. Let us fix N be a relatively compact symmetric open neighborhood of e, and let K be the collection of all non-negative functionals $\mathrm{L} \in (C_c(G), \tau)^*$ with the following properties :

1. $0 \leq \mathrm{L}(f) \leq 1$ if $0 \leq f \leq 1$ and $\mathrm{supp}(f) \subset g.N$ for some $g \in G$.
2. $\mathrm{L}(f) \geq 1$ if $f \geq 0$ and $f \equiv 1$ on $g.N.N$ for some $g \in G$.

Then K is a non-void weak*-compact convex subset of the topological dual $(C_c(G), \tau)^*$.

Before the proof, let us point out that if the non-emptiness of K is proved, then a functional in K is necessarily nonzero. Indeed, $\overline{N.N}$ is compact because $\overline{N.N} \subset \overline{N}.\overline{N}$ and the product $(g,h) \mapsto g.h$ is jointly continuous; by Urysohn's lemma, there is $f \in C_c(G)$ such that $f \equiv 1$ on $\overline{N.N}$ (a fortiori on $N.N$) and $0 \leq f \leq 1$; therefore, any element of K do not vanish at f by property 2. On the other hand, the convexity of K is immediate.

Proof. (lemma 4.5) We shall prove this lemma into two steps.

• Non-emptiness of K. Let S be as in lemma 4.4, and consider the map $\mathrm{L}(f) := \sum_{s \in S} f(s)$. L is a well-defined bounded linear functional on $C_c(G)$. For the linearity, it is enough to show that $\mathrm{L}(f)$ is a scalar for all $f \in C_c(G)$. So let $f \in C_c(G)$ fixed; then $\mathrm{supp}(f) \subset \bigcup_{i=1}^{n}(g_i.N)$ for some g_i's in G (because $\{g.N; g \in G\}$ is an open covering of $\mathrm{supp}(f)$ which is compact). On the other hand by lemma 1, we have $|(g_i.N) \cap S| \leq 1$ for each i; so $|\mathrm{supp}(f) \cap S| \leq n$ since $\mathrm{supp}(f) \cap S \subset \bigcup_{i=1}^{n}(g_i.N \cap S)$. Therefore, $\mathrm{supp}(f) \cap S = \{s_1, \cdots, s_q\}$ and $\mathrm{L}(f) = \sum_{i=1}^{q} f(s_i) \in \mathbb{R}$; moreover, from its definition, L is non-negative. Next, we shall prove its continuity. Let $f_\gamma \to f$ be a convergent net in $C_cG)$. Let C be a fixed compact subset of G containing $\mathrm{supp}(f)$ and such that, for some $\gamma_o \in J$, $\bigcup_{\gamma \geq \gamma_o} \mathrm{supp}(f_\gamma) \subset C$ and $\sup_{g \in C} |f_\gamma(g) - f(g| \to 0$. By lemma 4.4, we have $C \cap S = \{s_1, \cdots, s_p\}$ for some $p \in \mathbb{N}$. Therefore it follows $|\,\mathrm{L}(f_\gamma - f)| \leq \sum_{j=1}^{p} |f_\gamma(s_j) - f(s_j)| \leq p. \sup_C |f_\gamma(x) - f(x)| \to 0$. Hence, $\mathrm{L} \in (C_c(G), \tau)^*$. Finally we shall show that L lies in K.

i. Verification of property 1. Let $f \in C_c(G)$ with $0 \leq f \leq 1$ and $\text{supp}(f) \subset g.N$ for some g. By lemma 4.4, $S \cap g.N$ is either empty or reduces to $\{s\}$ for some $s \in S$; so L(f) lies in [0,1].

ii. Verification of property 2. Let $f \in C_c(G)$ such that : $f \geq 0$ and $f \equiv 1$ on $g.N.N$ for some g. By lemma 1, $S \cap g.N.N \neq \emptyset$. Let us pick $s \in S \cap g.N.N$. Then L$(f) \geq f(s) = 1$. Therefore L$(f) \geq 1$. Hence L belongs to K.

• Proof of the weak* compactness of K. Let $f \in C_c(g)$, $f \neq 0$. Then $\text{supp}(f) \subset \bigcup_{i=1}^{n}(g_i.N)$ for some $g_i's \in G$. Using a partition of unity subordinated to the $g_i.N's$, there are some $f_i's \in C_c(G)$ such that :

a. $\text{supp}(f_i) \subset g_i.N$, for $i = 1, \cdots, n$;

b. $0 \leq f_i \leq 1$, for all $i = 1, \cdots, n$;

c. and the sum $\sum_j f_i \equiv 1$ on $\text{supp}(f)$.

Since $f = f^+ - f^-$, $|f| = f^+ + f^-$ and the fact that $0 \leq \frac{|f|}{\|f\|} \leq 1$, where the mappings f^+ and f^- are defined by : $f^+ := \max(f, 0)$ and $f^- := \min(-f, 0)$; then $0 \leq \frac{f^+}{\|f\|} \leq 1$ and $0 \leq \frac{f^-}{\|f\|} \leq 1$. On the other hand, we have $f^+ = \sum_j f^+.f_i$ with $\text{supp}(f^+.f_i)$ subset of $g_i.N$ for all i. The same equality holds with f^- instead of f^+. Thus, for all L$\in K$,

$$\{ \text{L}(f^+),\text{L}(f^-)\} \subset [0, n.\|f\|] \Rightarrow |\text{L}(f)| \leq n.\|f\| \qquad (1).$$

Therefore we conclude that $\delta_f := \sup_{L \in K} |L(f)| < \infty$, for all f in $C_c(G)$. On the other hand, we introduce the following Hausdorff space :

$$\mathcal{K} := \prod_{f \in C_c(G)} [-\delta_f, \delta_f]$$

which is compact in the product topology by Tychonoff's theorem. Now we define a mapping

$$\Psi : K \to \mathcal{K},\ \varphi \mapsto \Psi(\varphi) := (\varphi(f))_{f\in C_c(G)}.$$

Given f in $C_c(G)$, let $\pi_f : \mathcal{K} \to \mathbb{R}$ denote the f^{th} projection. Then $\pi_f \circ \Psi$ (f^{th} component of Ψ) is continuous; because, if $\varphi_\alpha \to \varphi$ weakly*, then $\pi_f \circ \psi(\varphi_\alpha) = \varphi_\alpha(f) \to \varphi(f) = \pi_f \circ \Psi(\varphi)$; hence the continuity of Ψ follows. On the other hand, Ψ is one-to-one by construction; which implies that $\Psi : \mathcal{K} \to \Psi(\mathcal{K})$ is a bijective continuous map. The continuity of its inverse $\Psi^{-1} : \Psi(K) \to \mathcal{K}$ is straightforward because, $\Psi(\varphi_\alpha) \to \Psi(\varphi)$ iff $\pi_f \circ \Psi(\varphi_\alpha) \to \pi_f \circ \Psi(\varphi)$ for all $f \in C_c(G)$; and this is equivalent to $\varphi_\alpha \to \varphi$ weak*. Therefore, Ψ is a homeomorphism. Finally, it remains to prove that the range $\Psi(K)$ is closed in $\mathcal{K}$. Let $(\Psi(\varphi_\alpha))_\alpha$ be a net in $\Psi(K)$ converging to $(\varphi_f)_{f\in C_c(G)}$ in $\mathcal{K}$; and put $\varphi(f) := \varphi_f$ for all $f \in C_c(G)$. Then φ is linear and $\varphi_\alpha \to \varphi$ weak*; moreover, the fact that $\varphi_\alpha \to \varphi$ implies φ satisfies all the properties 1 and 2 of K and therefore, it remains to show that φ is continuous. For the continuity, let $f \in C_c(G)$ and $(f_\gamma)_{\gamma\in\Gamma}$ be a net in $C_c(G)$ converging to f. Let C be the corresponding compact subset of G in the definition of convergence of nets in $C_c(G)$; and let N be a relatively compact and symmetric neighborhood of e. Then $C \subset \bigcup_{i=1}^{n} g_i.N$ for some $g_i's \in G$ (by compactness of C). Now we fix $\gamma_o \in \Gamma$ such that $\mathrm{supp}(f) \subset C$ and

$$\gamma \geq \gamma_o \Rightarrow \mathrm{supp}(f_\gamma) \subset C.$$

Remember that $\varphi_\alpha \to \varphi$ weakly* on $C_c(G)$, and each φ_α lies in K. Thus we have for all α and γ fixed

$$|\varphi_\alpha(f_\gamma - f)| \leq \varphi_\alpha(|f_\gamma - f|) \leq n. \sup_{g\in C} |f_\gamma(g) - f(g)|, \ \forall\ \gamma \geq \gamma_o$$

The first inequality holds because each φ_α is a positive linear functional; and the second one follows from relation (1). By passing to the limit in α, we obtain then :

$$|\varphi(f_\gamma - f)| \leq \varphi(|f_\gamma - f|) \leq n. \sup_{g\in C} |f_\gamma(g) - f(g)|, \ \forall\ \gamma \geq \gamma_o$$

Therefore $\varphi(f_\gamma) \to \varphi(f)$; and this shows that $\varphi \in K$ and we have $\Psi(\varphi) = (\varphi_f)_{f\in C_c(G)}$. Hence, the limit $(\varphi_f)_{f\in C_c(G)}$ lies in the range $\Psi(\mathcal{K})$ which shows that the image of Ψ is a closed subset of the compact space $\mathcal{K}$. Hence, K is compact since homeomorphic to a compact space. $\square$

Remark 4.6. By considering the algebraic dual of $C_c(G)$ with its weak* topology, lemma 4.5 was proved in [34].

For all $g \in G$, let us define $Ł_g : (C_c(G), \tau)^* \to (C_c(G), \tau)^*$ by the equation $Ł_g(\varphi)(f) := \varphi(\ell_g f)$ for all $g \in G$. As readily checked, each $Ł_g$ is a well-defined weak*-weak* continuous mapping. Consider K the weak* compact convex set defined in lemma 4.5; then K is invariant under each $Ł_g$ (i.e., $Ł_g(K) \subset K$ for all g); in fact, let $\varphi \in K$ and $h \in G$ fixed. For condition 1, let $f \in C_c(G)$ such that $f(G) \subset$[0,1]

and for some $g \in G$, $\mathrm{supp}(f) \subset g.N$; then $\ell_h f \in C_c(G), \ell_h f(G) \in$[0,1] and $\mathrm{supp}(\ell_h f) \subset h^{-1}g.N$; therefore $Ł_h(\varphi)(f) = \varphi(\ell_h f) \in$[0,1]. For condition (2), let $f \in C_c(G)$ such that $f \geq 0$ and $f \equiv 1$ on $g.N.N$ for some $g \in G$; then $\ell_h f \in C_c(G), \ell_h f \geq 0$ and $\ell_h f \equiv 1$ on $h^{-1}g.N.N$; thus, $Ł_h(\varphi)(f) = \varphi(\ell_h f) \geq 1$. On the other hand, since non-negativity property is preserved by translations, it follows that $Ł_h(\varphi) \geq 0$. Hence, $Ł_h(\varphi) \in K$. Finally for the weak* continuity, let $g \in G$ and $(\varphi_\gamma)_\gamma \subset K$ be a weak* convergent net with limit φ, then for all $f \in C_c(G)$ we have:

$$|Ł_h(\varphi_\gamma)(f) - Ł_h(\varphi)(f)| = |\varphi_\gamma(\ell_g f) - \varphi(\ell_g f)| \to 0.$$

Therefore $Ł_h(\varphi_\gamma) \to Ł_h(\varphi)$ weakly* and the weak* continuity follows. Now let us fix $\varphi \in K$; for all $f \in C_c(G)$ let $Ł_\varphi^f : G \to \mathbb{R}$ defined by $Ł_\varphi^f(g) = Ł_g(\varphi)(f) := \varphi(\ell_g f)$. Then one can define a mapping $\xi : G \times K \to K$ by letting $\xi(g, \varphi) := g.\varphi$, with $g.\varphi(f) := Ł_\varphi^f(g)$. Then ξ is a well-defined action of G on K; indeed

$$\begin{aligned}
\xi(gh, \varphi)(f) &= (gh).\varphi(f) = \varphi(\ell_{gh} f) \\
&= \varphi(\ell_h \circ \ell_g(f)) = Ł_\varphi^{\ell_g f}(h) \\
&= h.\varphi(\ell_g f) = g.(h.\varphi)(f) \\
&= \xi(g, \xi(h, \varphi))(f)
\end{aligned}$$

Next step, we show that $Ł_\varphi^f$ lies in $C_b(G)$ for all f in $C_c(G)$ and $\varphi \in K$. Let us fix $f \in C_c(G)$ and $\varphi \in K$.

- Continuity of $Ł_\varphi^f$. If $g_\gamma \to g$ is a convergent net in G, by continuity of

φ, it is enough to show that $\ell_{g_\gamma} f \to \ell_g f$ in $(C_c(G), \tau)$. Let N_o be a fix compact neighborhood of e and let γ_o such that $\gamma \geq \gamma_o \Rightarrow g_\gamma^{-1} \in g^{-1}.N_o$. Next, we define $K_f := g^{-1}.N_o.\text{supp}(f)$ (note that K_f is a compact subset of G by the joint continuity of $(h, h') \mapsto h.h' : G\times \to G$). We have

$$[\bigcup_{\gamma \geq \gamma_o} \text{supp}(\ell_{g_\gamma} f)] \cup \text{supp}(\ell_g f) \subset K_f.$$

Therefore, $\sup_{h \in K_f} |\ell_{g_\gamma} f(h) - \ell_g f(h)| \leq \|\ell_{g_\gamma} f - \ell_g f\|$. On the other hand, as $C_c(G) \subset \text{LUC}(G)$ (cf. [23]), it follows that $\|\ell_{g_\gamma} f - \ell_g f\| \to 0$; so a fortiori $\ell_{g_\gamma} f \to \ell_g f$ uniformly on K_f. Thus, $\ell_{g_\gamma} f \to \ell_g f$ in $(C_c(G), \tau)$.

• Boundedness. $Ł_\varphi^f$ is bounded because $\text{supp}(f) \subset \bigcup_{i=1}^p g_i.N_o$ for some $g_i's$ in G; and together with relation (1) (see the proof of lemma 4.5) we get $\sup_{g \in G} \left|Ł_\varphi^f(g)\right| \leq p. \|f\| < \infty$. Hence, $Ł_\varphi^f \in C_b(G)$.

In summary, we have shown that $\xi : G \times K \to K, (g, \varphi) \mapsto g.\varphi$ induces a weak* separately continuous representation of G on K such that for all $\varphi \in K$ and $f \in C_c(G)$ the mapping $Ł_\varphi^f$ lies in $C_b(G)$.

Lemma 4.7. The representation $\mathfrak{S} = \{\xi(g, .)\ g \in G\}$ of G on K possesses a common fixed point in K. That is, there exists $\widetilde{\varphi} \in K$ such that $g.\widetilde{\varphi} = \widetilde{\varphi}$ for all $g \in G$ or equivalently, $\widetilde{\varphi}(\ell_g f) = \widetilde{\varphi}(f)$ for all $f \in C_c(G)$ and for all $g \in G$.

Proof. Let m be an invariant mean on $C_b(G)$ (G is amenable). Let $(m_\gamma)_{\gamma \in J}$ be a net of finite means on $C_b(G)$ converging pointwise to m;

with $m_\gamma = \sum_{i=1}^{n_\gamma} t_i^\gamma \delta_{g_i^\gamma}$; where $t_i^\gamma \in$[0,1], $\sum_{i=1}^{n_\gamma} t_i^\gamma = 1$. Pick $\varphi \in K$ and define $\varphi_\gamma := \sum_{i=1}^{n_\gamma} t_i^\gamma Ł_{g_i^\gamma}(\varphi)$, for all $\gamma \in J$. Then we define a net $(\varphi_\gamma)_{\gamma \in J}$ of elements of K due to its convexity. By compactness, (φ_γ) has a convergent subnet $(\varphi_{\gamma_t})_{t \in T}$. So let $\widetilde{\varphi} \in K$ be its weak*-limit. Given $f \in C_c(G)$ and $g \in G$, from a simple calculation, we get $\ell_g(Ł^f_{\widetilde{\varphi}}) = Ł^{\ell_g f}_{\widetilde{\varphi}}$. Therefore

$$\begin{aligned} g.\widetilde{\varphi}(f) &= \lim_t \varphi_{\gamma_t}(\ell_g f) = \lim_t m_{\gamma_t}(Ł^{\ell_g f}_{\varphi}) \\ &= m(Ł^{\ell_g f}_{\varphi}) = m(\ell_g Ł^f_{\varphi}) \\ &= m(Ł^f_{\varphi}) = \lim_t m_{\gamma_t}(Ł^f_{\varphi}) \\ &= \lim_t \varphi_{\gamma_t}(f) = \widetilde{\varphi}(f) \end{aligned}$$

Hence, $\widetilde{\varphi} \in K$ and satisfies $g.\widetilde{\varphi} = \widetilde{\varphi}$ for all $g \in G$. □

Definition 4.8. A Haar measure on G is a non-negative nonzero Borel measure μ on the Borel sets of G with the following properties :

1. μ is left invariant, i.e., $\mu(g.E) = \mu(E)$, for all g and any Borel set E.
2. $\mu(K) < \infty$ for all compact set $K \subset G$.
3. μ is regular, i.e., both

inner regular : for all open set $O \subset G$,

$$\mu(O) = \sup\{\mu(K)\ ; K \subset O, K \text{ compact}\}.$$

outer regular : for all Borel set $E \subset G$,

$$\mu(E) = \inf\{\mu(O)\ ; O \text{ open and } E \subset O\}.$$

Example 4.9. On the additive group $\mathbb{R}^n$, a Haar measure is given by the Lebesgue measure; if G is any countable group, then a Haar measure is given by the counting measure.

Now we are ready to prove the existence of a left Haar measure for amenable locally compact groups.

Proof. Let $\widetilde{\varphi} \in K$ be as in lemma 4.7. From the properties of K, we know that $\widetilde{\varphi}$ is a nonnegative and nonzero linear functional on $C_c(G)$; so by the Riesz representation theorem, there is a unique regular nonzero radon measure μ on the Borel sets of G such that : $\widetilde{\varphi}(f) = \int_G f \, d\mu$ for all $f \in C_c(G)$ and for all open set O in G, $\mu(O) = \sup\{\widetilde{\varphi}(f) \, ; f \in C_c(G), \operatorname{supp}(f) \subset O\}$. To finish, we show into two steps that μ is left invariant.

- For open sets. Let O be open and $g \in G$. We fix $f \in C_c(G)$ such that $\operatorname{supp}(f) \subset g.O$. Then $\operatorname{supp}(\ell_g f) \subset O$ and $\widetilde{\varphi}(f) = \widetilde{\varphi}(\ell_g f) \leq \mu(O)$. Therefore it follows $\mu(g.O) \leq \mu(O)$ by taking the supremum over all such f. Thus, we have $\mu(hg.O) \leq \mu(g.O)$ for all $h \in G$ and this implies $\mu(O) \leq \mu(g.O)$. Hence $\mu(g.O) = \mu(O)$ for all open set O and $g \in G$.
- For Borel sets. Let E be a Borel set and $g \in G$. From the outer regularity, if O is an open set containing $g.E$ then $\mu(E) \leq \mu(g^{-1}O) = \mu(O)$ (because $E \subset g^{-1}.O$ which is open). Thus, $\mu(E) \leq \mu(g.E)$ by taking the infimum over all such O. As before, by interchanging E with $g.E$ and g with g^{-1}, we get $\mu(g.E) \leq \mu(g^{-1}(g.E)) = \mu(E) \Rightarrow \mu(g.E) = \mu(E)$. $\square$

Remark 4.10. For the proof of the uniqueness of the Haar measure on G refer to [22]. We note that, the word "uniqueness" has to be understood in the following way : if there are two Haar measures μ and ν on G then, there is a multiplicative positive constant $\alpha \in (0, \infty)$ such that $\mu = \alpha.\nu$.

Remark 4.11. Our proof includes both families of topological groups namely, compact groups and commutative groups. In fact, it is well-known that commutative semigroups are amenable [5], so a fortiori abelian groups; for a compact topological group G, it is known that $C_b(G) = \mathrm{AP}(G)$ the Banach subspace of $C_b(G)$ of all almost periodic functions on G; and an application of the Ryll-Nardzewski fixed point theorem, see [11], ensures the existence of a invariant mean on $\mathrm{AP}(G)$ and so on $C_b(G)$.

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CHAPTER 5

Remarks and Open Problems

In this last chapter, we make some remarks on our work and also raise some natural questions related to some results obtained in this thesis.

5.1 Remarks on chapter 2 and related problems

• Although uniformly convex spaces are strictly convex, the proof of theorem 2.2.1 does not require this geometric property; but only the fact that the characteristic of convexity of E is zero. Recall that the characteristic of convexity is defined by $\epsilon(E) = \sup\{\epsilon \geq 0 \; ; \delta(\epsilon) = 0\}$, where $\delta :$ [0,2] $\to$ [0,1] given by

$$\delta(\epsilon) := \inf\{1 - \|\tfrac{x+y}{2}\| \; ; \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}$$

is the modulus of convexity of E. A natural question to raise is the following :

Question 1 : Is theorem 2.2.1 still true if the underlying Banach space is strictly convex ?

• According to Alspach's counter example, one cannot delete the normal structure condition in theorem 2.3.3. However, we have the following :

Question 2 : Is theorem 2.3.3 still true (even for a single map) if we replace normal structure by $C(K)^+$-continuity ?

5.2 Remarks on chapter 3 and related problems

• We proved that theorem 3.1.1 holds for σ-LA, seq-LA and separable left reversible semi-topological semigroups. So naturally we may ask :

Question 3 : Does theorem 3.1.1 still hold without a separability assumption ?

• Theorem 3.1.1 is a contribution to a conjecture of Lau whether left amenability of semi-topological semigroups can be characterized by the fixed point property (F^*). So far, the following partial answers have been obtained in the literature :

- Commutative semigroups possess the property (F^*), see [6].
- A weak version of (F^*) obtained by assuming K to be norm separable, is satisfied by left amenable semi-topological semigroups and discrete left subamenable semigroups, cf. [46].

- Left amenable semi-topological semigroups if K possesses normal structure, see [45].
- Left reversible semi-topological semigroups if the predual is an M-embedded Banach space, see [45].

Question 4 : Is the topology τ in theorem 3.1.1 minimal ? In other words, is there a locally convex topology $\tau' \subset \tau$ with the same property as τ ?

5.3 Remarks on chapter 4 with an open question

• The approach we used for proving the existence of a left Haar measure for amenable groups does not work for general locally compact groups unfortunately; because, given a non-amenable group G, it is a well-known fact that WAP(G) (the Banach subspace of $C_b(G)$ consisting of those functions whose left orbits are relatively compact in $C_b(G)$ in the weak topology) has a unique invariant mean m which vanishes at any function in $C_c(G)$, cf. [11]. For other details about almost periodicity see [5] and [16]. With this observation, the following question is natural:

Question 5 : Is there a fixed point proof of a left Haar measure for any locally compact groups ?

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