THE IRREDUCIBLE CHARACTERS OF 2 X 2 UNITARY MATRIX GROUPS OVER FINITE FIELDS

by

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Abstract

In this work we will construct the table of irreducible characters for the group of unitary 2 x 2 matrices over a finite field. The table and the methods for its construction will show interesting connections to the table and methods of construction of the table of irreducible characters for the general linear group.

To my wife Doris, whose patience and support made this work possible.

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List of Tables

Contents

Introduction

The irreducible characters of $GL_2(\mathbb{F}_q)$ are well known, being given for example in Fulton and Harris[FH]. Less well known are the irreducible characters of the subgroup of unitary matrices; they are stated, tersely and without much explication, in the 1963 paper “On the Characters of the Finite Unitary Groups” by Veikko Ennola[E], in Annales Academi Scientiarum Fennic Mathematica. The aim of the present work will be to construct and fully justify the character table for $U_2(\mathbb{F}_{q^2})$, hereafter denoted G. One of the chief difficulties in this task is the determination of the conjugacy classes, as in the unitary group we cannot exploit the Jordan form or rational canonical form of a matrix. Although there is no simple connection between the conjugacy classes of a group and those of its subgroups, we will see a very close resemblance between the forms of conjugacy class representatives of the general linear group and those of the unitary group. In addition, we will find that characters of the two groups have, in a loose sense, the same dimension, and that the character values of conjugacy classes with similar forms are the same. Finally, we will see that the methods used for discovering the characters for the general linear group have their almost exact counterparts for the unitary group.

Arguments concerning finite fields sometimes require special treatment where the characteristic of the field is 2. In this work the several modifications for $p = 2$ will be put in a separate chapter in order to preserve continuity of the main argument.

Representations and Characters

We will begin by recalling some facts about representations and characters. We follow Fulton and Harris here, and assume that in all cases G is a finite group.

Let V be an n dimensional complex vector space. We define a *representation* of G on V to be a homomorphism $\rho : G \to GL(V)$ from G into the group of automorphisms of V. When ρ is understood, V itself is sometimes called the representation. If W is a subspace of V such that for all $g \in G, w \in W$: $\rho(g)(w) \in W$, then we say that W is a *subrepresentation* of G. A representation V is called *irreducible* if it has no proper non-trivial subrepresentations. The concept of irreducible representation is important because it can be shown that any representation is the direct sum of irreducible representations, so that we need only seek irreducible representations of a group.

A tool that has proved useful in understanding representations is that of characters. Given a representation V of G, we define the *character* of V to be the complex function χ_V on the group given by $\chi_V(g) = \mathrm{Tr}(\rho(g))$, i.e. the trace of $\rho(g)$ on V. It is clear that the character value of g does not depend on a choice of basis, and also that character values are constant on conjugacy classes of G, i.e. it is a *class function* of G. If we denote by $\mathcal{C}$ the space of class functions on G, then we can define a Hermitian product on $\mathcal{C}$. If α and β are two class functions on G, we define $(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$; it can be shown that a representation V is irreducible if and only if its character χ_V satisfies $(\chi_V, \chi_V) = 1$. In this work, we are aiming at the complete table of irreducible characters of the unitary group; it can be shown that the number of such characters is equal to the number of conjugacy classes of G.

Finally, we recall the concept of induced representations.[1] Given a subgroup H of G, we can restrict any representation V of G to a representation of H, denoted $\mathrm{Res}_H^G V$. We would like to be able to go the other way, i.e. given a representation of H, to recover a representation of G. For the present work we are specifically interested in taking a character on H and lifting it to a character on G. To see how this might be done, suppose first that we already have a character on G, and let us

[1] This part is from section 3.3 of Fulton and Harris

see what it might mean to lift $\mathrm{Res}_H^G V$ to the original character on G. Let $W \subset V$ be a subspace of V that is invariant under the action of H. Given any $g \in G$, the subspace gW will depend only on the coset gH that g lies in, since if $g' \in gH$, then for some $h \in H$ we have $g'W = (gh)W = g(hW) = gW$. If for some $\sigma \in G/H$ we write σW for this subspace, then it may be the case that every $v \in V$ can be written uniquely as a sum of elements of such subspaces, that is $V = \bigoplus_{\sigma \in G/H} \sigma W$. If this is the case, we say that V has been *induced* by W, and we write $V = \mathrm{Ind}_H^G W$, or $\mathrm{Ind}W$. In the present work, we will have three occasions to induce a character on G from an existing one on some large subgroup. To find the character values of $\mathrm{Ind}W$, we note that any $g \in G$ maps σW to $g\sigma W$, so that the trace of g will be calculated using only those cosets σ fixed by g, i.e. if $s \in \sigma$, we want $g\sigma = \sigma$, or $s^{-1}gs \in H$ so that we get:

$$\chi_{\mathrm{Ind}W}(g) = \sum_{g\sigma=\sigma} \chi_W(s^{-1}gs), s \text{ arbitrary in } \sigma.$$

Hermitian Forms and Finite Fields

On a complex vector space V, a Hermitian form is a map $\mathcal{H} : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$, $a, b \in \mathbb{C}$.:

- $\mathcal{H}(u + v, w) = \mathcal{H}(u, w) + \mathcal{H}(v, w)$
- $\mathcal{H}(u, v + w) = \mathcal{H}(u, v) + \mathcal{H}(u, w)$
- $\mathcal{H}(au, v) = a\mathcal{H}(u, v) = \mathcal{H}(u, \overline{a}v)$
- $\mathcal{H}(v, u) = \overline{\mathcal{H}(u, v)}$

The form is called non-degenerate if for all $v \in V$, there exists a $w \in V$ such that $\mathcal{H}(v, w) \neq 0$, and a vector space having a non-degenerate Hermitian form is called a unitary space. In order to have such forms on vector spaces over a finite field, there must be something like conjugation on the field, and it is not obvious that this is always possible. Therefore in this chapter we will identify those finite fields that admit a conjugation, and we will examine two important subgroups of the group of units of the field as well as homomorphisms onto these subgroups, all of which will figure prominently in this work. In addition, we will show that all non-degenerate Hermitian forms on finite dimensional vector spaces of the same dimension are equivalent in a sense that will be defined; this is important since we shall change forms at times for computational convenience.

0.1 The Field

In what follows, we shall assume the characteristic of the field is odd. Conjugation in $\mathbb{C}$ is an automorphism of order two, thus we must identify those finite fields allowing such an automorphism. The order of a finite field $\mathbb{F}$ is necessarily p^k for some prime p and positive integer k, and the group of automorphisms of the field is cyclic of order k; therefore $\mathbb{F}$ will have an automorphism of order 2 if and only if k is even. To construct such a field, we begin with any finite field $\mathbb{F}_q$; $q = p^k$, and take a quadratic extension of it (unique up to isomorphism) to get $\mathbb{F}_{q^2}$. This is done by adjoining to $\mathbb{F}_q$ a square root of any generator of $\mathbb{F}_q^{\times}$.

Having formed the quadratic extension $\mathbb{F}_{q^2}$ of order q^2, we have the automorphism $\alpha : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ given by $\alpha(x) = x^q$. That this is an automorphism follows from the prime characteristic of the field; that it is of order two follows from the fact that the group of units of $\mathbb{F}_{q^2}$ form a multiplicative group of order $q^2 - 1$.

0.1.1 Two Important Homomorphisms on $\mathbb{F}_{q^2}^\times$

There are two important maps on $\mathbb{F}_{q^2}^\times$ that will come up frequently in this work, and we briefly describe them here:

(i) The norm map, $N : \mathbb{F}_{q^2}^\times \to \mathbb{F}_{q^2}^\times$, is given by $N(x) = x\overline{x}$, with kernel denoted $\mathcal{L} = \{x \in \mathbb{F}_{q^2}^\times | x\overline{x} = 1\}$. Recalling that $\mathbb{F}_{q^2}^\times$ is cyclic of order $q^2 - 1$, let ϵ be a generator of this group; then $\mathcal{L}$ will be generated by ϵ^{q-1}, since $x\overline{x} = 1 \Leftrightarrow x^{q+1} = 1$, and therefore $|\mathcal{L}| = q + 1$. We claim that $N(\mathbb{F}_{q^2}^\times) = \mathbb{F}_q^\times$: for any $x \in \mathbb{F}_{q^2}^\times$, $x\overline{x} \in \mathbb{F}_q^\times$, and the norm map is onto because $N(\epsilon) = \epsilon^{q+1}$, which has order $q - 1$ and so generates $\mathbb{F}_q^\times$.

(ii) Another important map is $Q : \mathbb{F}_{q^2}^\times \to \mathcal{L}$, given by $Q(x) = \dfrac{x}{\overline{x}}$. Its kernel $\mathbb{F}_q^\times$ is of order $q - 1$; this means there will be $q + 1$ cosets of the kernel, and since the order of $\mathcal{L}$ is $q + 1$, we see that the map Q is surjective. We will also find occasion to use the fact that there are $q - 1$ elements of $\mathbb{F}_{q^2}^\times$ that map to any $x \in \mathcal{L}$ under Q.

0.2 The Hermitian Form for Finite Vector Spaces Over Finite Fields

Having shown which finite fields can admit order 2 automorphisms, we define Hermitian forms on these vector spaces over such fields in the same way as for complex vector spaces, and we recall a few facts about Hermitian forms that will be important in the sequel:

- If we have chosen a basis $\beta = \{x_1, x_2, \ldots, x_n\}$ for a finite dimensional vector space V over a finite field, then each Hermitian form $\mathcal{H}$ will be associated with a Hermitian matrix M whose elements satisfy $m_{ji} = \overline{m_{ij}}$. Then for any $u, v \in V$ we will have $\mathcal{H}(u, v) = u^t M \overline{v}$.

- As is in the case of symmetric bilinear forms, if a Hermitian form $\mathcal{H}$ is non-degenerate on a sub-space W of V, (i.e. if $W \cap W^{\perp} = 0$) then $V = W \oplus W^{\perp}$.

- If we change to a new basis β', with the change of basis matrix $P : \beta' \to \beta$, then the matrix M of the Hermitian form $\mathcal{H}$ will change to $M' = P^t M \overline{P}$. This motivates the following definition of equivalence of Hermitian forms: given vector spaces V_1, V_2 with respective non-degenerate Hermitian forms $\mathcal{H}_1, \mathcal{H}_2$, we call the forms $\mathcal{H}_1$ and $\mathcal{H}_2$ *equivalent* if there exists an isomorphism $\tau : V_1 \to V_2$ such that for all $v, w \in V_1$, $\mathcal{H}_1(v, w) = \mathcal{H}_2(\tau v, \tau w)$.

- The following fact is less elementary than the previous three, and thus will need to be justified, using an argument from Grove (chapter 8) :

  > If V_1, V_2 are vector spaces of the same finite dimension, over a finite field, with respective non-degenerate Hermitian forms $\mathcal{H}_1, \mathcal{H}_2$, then the two forms are equivalent.

  To show this , we begin with the quadratic (Hermitian) form: $\mathcal{Q}(v) = \mathcal{H}(v, v)$, and show that if $\mathcal{H}$ is non-zero then for some $v \in V$, $\mathcal{Q}(v) \neq 0$. Suppose the contrary. As $\mathcal{H}$ is non-zero, we can choose $v, w \in V$ with $\mathcal{H}(v, w) = 1$. Then supposing $\mathcal{Q}(v) = 0$ for all $v \in V$, let a be arbitrary in $\mathbb{F}_{q^2}^{\times}$, so that calculation gives $0 = \mathcal{Q}(v + aw) = a + \overline{a}$. Setting $a = 1$ implies that the field characteristic must be 2, but then we have $\overline{a} = -a = a$, which is a contradiction since the conjugation map was to have order 2.

  Now we can show that a unitary space has an orthogonal basis $\{v_i\}$ with $\mathcal{Q}(v_i) = c_i \in \mathbb{F}_q^{\times}$, so that the the matrix for $\mathcal{H}$ is diagonal with $m_{ii} = c_i$. We choose v_1 with $\mathcal{Q}(v_1) = c_1 \in \mathbb{F}_q^{\times}$, and let $W_1 =< v_1 >$. It is clear that $\mathcal{H}$ is non-degenerate on W_1, so that $V = W_1 \oplus W_1^{\perp}$. Proceeding inductively on $W_1^{\perp}$ gives the result.

  Next we show that a basis exists for V such that the matrix for $\mathcal{H}$ will be the identity matrix. We have shown that the norm map is onto $\mathbb{F}_q^{\times}$, thus we take $d_i \in \mathbb{F}_{q^2}^{\times}$ and $d_i \overline{d_i} = c_i^{-1}$, so that $\mathcal{Q}(d_i v_i) = 1$. Finally, to show the equivalence of forms, suppose that we have unitary spaces V_1, V_2 with corresponding Hermitian forms $\mathcal{H}_1, \mathcal{H}_2$. We choose bases so that the matrices for both forms are identity matrices. Now if P is any matrix (of the right size of course) such that $P^t\overline{P} = I$, then P, together with the identification of the elements of V_1 and V_2 with their coordinate vectors, will provide an isomorphism from V_1 to V_2 such that for vectors v, w in V_1, $\mathcal{H}_1(v, w) = \mathcal{H}_2(Pv, Pw)$, and thus the two forms are equivalent.

All of this justifies our intended use of two forms in this work. The first given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and the second by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The notion of equivalence of forms means that switching between one and the other is only a matter of changing bases.
It is worth mentioning that, using the first form, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ will be unitary if and only if:

$$a\overline{d} + c\overline{b} = 1$$

$$a\overline{c} + \overline{a}c = 0 = b\overline{d} + \overline{b}d$$

while for the second form we require:

$$a\overline{a} + b\overline{b} = 1 = c\overline{c} + d\overline{d}$$

$$a\overline{b} + c\overline{d} = 0$$

Finally we note that the great advantage of the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is that it permits the use of the upper triangular subgroup of G (the Borel subgroup); the other form does not.

The Conjugacy Classes of $U_2(\mathbb{F}_{q^2})$

0.3 Counting the Unitary Group

Let V be a 2 dimensional vector space over $\mathbb{F}_{q^2}$ and let G be the group of unitary 2×2 matrices over the same field. We begin by finding $|G|$: if we take $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ as the matrix of our Hermitian form, then $A \in G$ will be unitary if and only if for any $u, v \in V$:

$$(Au)^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)(\overline{Av}) = u^t \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\overline{v}$$

which implies:

$$\overline{A}^t A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

So that $\overline{A}^t$ is the inverse of A. Writing A as $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the determinant of A as D, and equating the conjugate transpose with the inverse gives:

$$\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

so that $d = \overline{a}D$, and $c = -\overline{b}D$. Calculating the determinant in the usual way gives $D = ad - bc = a\overline{a}D + b\overline{b}D \Rightarrow a\overline{a} + b\overline{b} = 1$. Furthermore, since the determinant of $\overline{A}^t$ is the conjugate of the determinant of A, we see that $D\overline{D} = 1$ (and that there are therefore $q + 1$ choices for the determinants of unitary matrices in G). Thus using the standard Hermitian form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, A will unitary if and only if it is of the form:

$$\begin{pmatrix} a & b \\ -\overline{b}D & \overline{a}D \end{pmatrix}$$

where D is of norm 1, and $a\overline{a} + b\overline{b} = 1$. We count $|G|$, by taking the following cases:

- if $a = 0$, then $a\overline{a} + b\overline{b} = 1$ implies that b is of norm 1, hence there are $(q + 1)$ choices for b. Since there are $(q + 1)$ choices for the determinant, there are $(q + 1)^2$ such matrices.

- similarly, if $b = 0$ there are $(q+1)^2$ matrices.
- if a and b are both not zero then $a\overline{a} \in \mathbb{F}_q \backslash \{0, 1\}$, so there are $q - 2$ choices for $a\overline{a}$, and this determines $b\overline{b}$. Then there are $q + 1$ choices for a (the size of the cosets of the kernel of the norm map), and $q + 1$ choices for b. We still have $(q + 1)$ choices for the determinant, therefore there are $(q - 2)(q + 1)^3$ choices in this case.

Totalling the three cases, we get $|G| = 2(q+1)^2 + (q-2)(q+1)^3 = (q-1)q(q+1)^2$.

0.4 The Conjugacy Classes

Presently, we will show that the eigenvalues of any element of G lie conveniently in $\mathbb{F}_{q^2}$; for now we note that it allows us to organize the search for conjugacy class representatives by partitioning the elements of G according to their eigenvalues thus:

(i) one eigenvalue, and diagonalizable

(ii) one eigenvalue, but not diagonalizable[2]

(iii) two eigenvalues, neither of norm 1

(iv) two eigenvalues, both of norm 1

Two observations here before proceeding:

- Since the product of distinct eigenvalues equals the determinant which lies in the subgroup $\mathcal{L}$, then distinct eigenvalues must either both be norm 1, or both not norm 1.
- In order that the above partition be exhaustive of G, we need to show that all eigenvalues of unitary 2×2 matrices over $\mathbb{F}_{q^2}$ lie in $\mathbb{F}_{q^2}$. We can represent an element of G by
$$\begin{pmatrix} a & b \\ -\overline{b}D & \overline{a}D \end{pmatrix}$$
thus it suffices to show that the discriminant of the characteristic equation is a square in $\mathbb{F}_{q^2}$. It is clear that the discriminant is $(a+\overline{a}D)^2 - 4D$. Since $D \in \mathcal{L}$, it is equal[3] to $\frac{x}{\overline{x}}$ for some x in $\mathbb{F}_{q^2}$. Thus we can rewrite the discriminant as:

[2] In the general linear group, this would imply a Jordan form; in the unitary group this is not so.
[3] This is because the map Q in the previous chapter was surjective

$$\frac{(a\overline{x}+\overline{a}x)^2-4x\overline{x}}{\overline{x}^2}$$

The denominator of the fraction is a square, and since the numerator is invariant under conjugation it lies in $\mathbb{F}_q$ and is thus also a square. Therefore the discriminant is a square, and the eigenvalues of every element of G lie in $\mathbb{F}_{q^2}$.

Now we list some conjugacy class representatives, using the Hermitian form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ unless otherwise indicated. We cannot say at this point that our list is exhaustive; for example in item 2 below there could be matrices with one eigenvalue, not diagonalizable that are not conjugate in the unitary subgroup to an element of the form $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$. We will know that we have an exhaustive list only when we have accounted for all of the elements in G.

(i) $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ This requires $x\overline{x} = 1$, so that $x \in \mathcal{L}$. Therefore there are $q+1$ such class representatives. Since each of these is in the center of G, the size of each conjugacy class is 1, and we have accounted for $q+1$ elements.

(ii) $A = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$, $y \neq 0$. In order for A to be unitary we must have x of norm 1, and $\frac{y}{\overline{y}} = -\frac{x}{\overline{x}}$ (from the remark at the bottom of page 11). This last equality means that y must map to $\frac{-x}{\overline{x}}$ under the map Q mentioned in the previous chapter, and we have remarked that this gives $q-1$ choices for y . Thus naively we have $(q+1)(q-1)$ choices for this type of conjugacy class representative. We will show, however, that $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix}\right)$ if and only if y and z are both sent to the same element under the map $Q : \mathbb{F}_{q^2}^{\times} \to \mathcal{L}$, that is, they are both in the same coset of $\mathbb{F}_q^{\times}$, the kernel of the map . As a result there will be only $q+1$ such class representatives:

First let $z = ky$, $k \in \mathbb{F}_q^{\times}$. Then $k = a\overline{a}$ for some $a \in \mathbb{F}_{q^2}$ since it is in the image of the norm map. Then we will have $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix}\right)$, using conjugation by $\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{\overline{a}} \end{smallmatrix}\right)$.
Next, suppose that $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix}\right)$. Then for some $P = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$, $P^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)P = \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix}\right)$, implying that $c = 0$, and also that $d = \frac{1}{\overline{a}}$. Then $P^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)P = \left(\begin{smallmatrix} x & d\overline{d}y \\ 0 & x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & z \\ 0 & x \end{smallmatrix}\right)$, so that $z = ky,\ k \in \mathbb{F}_q^{\times}$, with $k = d\overline{d}$.

The centralizer of $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ is easily seen to be $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ where b is allowed to be zero, giving q choices for b(since either $b = 0$ or b and a map to the same element under Q), and $q+1$ choices for a, since it is in $\mathcal{L}$. Therefore the centralizer has $q(q+1)$ elements, so that each conjugacy class has $|G|/q(q+1) = (q-1)(q+1)$ elements. This type of class representative therefore accounts for $(q-1)(q+1)^2$ elements.

(iii) $A = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $y \neq x$. Since $x \neq 0$, and $x\overline{x} \neq 1$, ($y \neq x$ and $x\overline{y} = 1$ means $x\overline{x}$ cannot be equal to 1) there are $(q+1)(q-2)$ choices for x, and y is determined by x. Now since $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$ in G, using $P = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then we have $\frac{(q+1)(q-2)}{2}$ such class representatives.

The centralizer of A is the set of unitary matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ of which there are q^2-1 as $a \neq 0$, and d is determined by a. Thus the size of each conjugacy class is $|G|/(q^2-1) = q(q+1)$, so that this type of conjugacy class representative accounts for $\frac{(q+1)(q-2)}{2}q(q+1) = \frac{(q-2)q(q+1)^2}{2}$ elements.

(iv) $A = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, $y \neq 0$ (if $y = 0$ we have a scalar matrix) A has distinct eigenvalues $x \pm y$, and they are norm 1 since, from page 12 we have $x\overline{x} + y\overline{y} = 1$ and $x\overline{y} + y\overline{x} = 0$; summing these equations shows that $x + y \in \mathcal{L}$, while subtracting them shows $x - y \in \mathcal{L}$. To count these class representatives we assume first that $x = 0$; this gives $q+1$ choices for $y \in \mathcal{L}$. If x and y are not zero, we choose two elements $u_1, u_2 \in \mathcal{L}$, and let $x = \frac{u_1+u_2}{2}$, $y = \frac{u_1-u_2}{2}$ so that $x \pm y \in \mathcal{L}$. Since $x \neq 0 \neq y$, then $u_2 \neq \pm u_1$, giving $(q+1)(q-1)$ choices for this case. In all we have $q(q+1)$ representatives, but we note that $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$ using conjugation by $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$ where $a\overline{a} = -1$, and so we have $q(q+1)/2$ conjugacy class representatives of this type.

The centralizer of A is the set of elements of the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ where a or b (but not both) can be zero. Taking cases where only $a = 0$, only $b = 0$, and neither a nor b is zero, we see the order of the centralizer is $(q+1)^2$, so that the number of elements in each conjugacy class of this type is $|G|/(q+1)^2 = (q-1)q$. Therefore this type of conjugacy class accounts for $(q-1)q^2(q+1)/2$ elements.

Totalling the number of elements from our 4 types of conjugacy classes gives:

$$(q+1)+(q-1)(q+1)^2+\frac{(q-2)q(q+1)^2}{2}+\frac{(q-1)q^2(q+1)}{2} = (q-1)q(q+1)^2 = |G|$$

Since we have accounted for the number of elements in G, we can say now that our list of conjugacy class representatives was complete.

Therefore we have shown that the conjugacy classes of the unitary 2 x 2 matrices over a finite field, together with the number of elements in each class are:

representative	no. elements	no. classes	total elements
$a_x = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	1	$q+1$	$q+1$
$b_{x,y} = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$, $y \neq 0$	$(q-1)(q+1)$	$q+1$	$(q-1)(q+1)^2$
$c_{x,y} = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $y \neq x$	$q(q+1)$	$\frac{(q-2)(q+1)}{2}$	$\frac{(q-2)q(q+1)^2}{2}$
$d_{x,y} = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, $y \neq 0$	$(q-1)q$	$\frac{q(q+1)}{2}$	$\frac{(q-1)q^2(q+1)}{2}$

Table 1: Conjugacy Class Representatives of G

From the chart the total number of conjugacy classes is $(q+1)^2$, hence this is the number of irreducible characters that we must find.

The Irreducible Characters of $U_2(\mathbb{F}_{q^2})$

If $\alpha : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ is a 1 dimensional character on $\mathbb{F}_{q^2}^\times$, we can form a 1 dimensional character on G by sending any $A \in G$ to $\alpha(\det(A))$. Since the determinant of a unitary matrix is of norm 1, there will be $q+1$ such 1 dimensional characters, U_α on G:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2-y^2)$

Next we consider the permutation representation of the coset space of the Borel subgroup of G: $B = \{\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) | \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in G\}$. If $b = 0$ there are q^2-1 choices for a (it just needs to be in $\mathbb{F}_{q^2}^\times$), determining d . If $b \neq 0$, there are q^2-1 choices for a , determining d, and $q-1$ choices for b, since $b\overline{d} + \overline{b}d = 0$ implies $\dfrac{b}{\overline{b}} = -\dfrac{d}{\overline{d}}$; this means (using the map Q from page 9) that $Q(b) = -\frac{d}{\overline{d}} \in \mathcal{L}$, so there are $q-1$ choices for b. This gives $|B| = (q^2-1) + (q^2-1)(q-1) = (q-1)q(q+1)$, so that $[G : B] = q+1$. The coset representatives for B will be elements of the form $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$, together with the element $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since we are using the Hermitian form with matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we must have $t + \overline{t} = 0$. Thus $t = 0$ or $Q(t) = -1$, so there are q choices in all for t , giving the required number of coset representatives. To show that these representatives lie in distinct cosets of B, suppose first, that for coset representatives $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ r & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ r-s & 1 \end{smallmatrix}\right) \in B$. This implies that $r = s$. Next we note that $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & -t \end{smallmatrix}\right) \in B$, is impossible; therefore the previously mentioned elements form a transversal for B.

In the permutation representation of the coset space of B, the character value of $g \in G$ will be the number of cosets σ that g fixes, and $g\sigma = \sigma$ is equivalent to $s^{-1}gs \in B$ where s is arbitrary in σ. For convenience, we will use the coset representative for s. We now consider the fixed point set of each type of conjugacy class representative:

(i) $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$: This element is in B, and also in the center of G; thus for all coset representatives s,

$$s^{-1}\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \in B$$

so that this element fixes all $q+1$ cosets of B.

(ii) $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right), y \neq 0$: Since this element is in B, it certainly fixes B (with conjugation returning the original element, since we can take the identity matrix as the coset representative); to show that it fixes no other coset, we consider two cases:

- if $s = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ $t \neq 0$, then $s^{-1} = \left(\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right)$, and $s^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)s$ =
$$\begin{pmatrix} x+yt & y \\ -yt^2 & x-yt \end{pmatrix}$$
This is not in B, since both y and t are not zero.
- if $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $s^{-1} = s$, and $s^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & 0 \\ y & x \end{smallmatrix}\right)$ which is not in B since $y \neq 0$.

Therefore this type of element fixes 1 coset of B.

(iii) $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right), y \neq x$: This element is in B, and therefore fixes it, and conjugation (by say, the identity matrix) just returns $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$. But if $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ then $s^{-1}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)s = \left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$, which is in B. On the other hand if $t \neq 0$:
$$\left(\begin{smallmatrix} 1 & 0 \\ -t & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = \begin{pmatrix} x & 0 \\ yt-xt & y \end{pmatrix}$$
which is not in B since $yt - xt = 0$ implies $y = x$. Therefore this type of conjugacy class representative fixes 2 cosets of B.

(iv) $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right), y \neq 0$: This element has distinct norm 1 eigenvalues, as will $s^{-1}\left(\begin{smallmatrix} x & y \\ y & s \end{smallmatrix}\right)s$ for any coset representative s. This last expression cannot be in B then, since for $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in B$, if $a\overline{a} = 1$, then $d = \frac{1}{\overline{a}} = a$. Therefore this element fixes no cosets of B

We subtract the character of the trivial representation from that of this permutation representation to get an irreducible character V, of dimension q:

	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
V:	q	0	1	-1

Now we can tensor[4] V with U_α to get an irreducible character of dimension q: $V_\alpha = V \otimes U_\alpha$:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2 - y^2)$

There are $q+1$ such characters because there are $q+1$ of the form U_α.

The next character comes from inducing a 1 dimensional character on B. We start with 1 dimensional characters α, β on $\mathbb{F}_{q^2}^\times$. Using these, we get a 1 dimensional character ϕ, on the diagonal subgroup $\{\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) | \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \in G\}$ where $\phi[\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)] = \alpha(a)\beta(d)$. This character can be lifted to B by sending $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ to $\alpha(a)\beta(d)$; calculation shows that this is indeed a character on B, and it is this character that will be induced to G. From previous work we know :

- $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ fixes all $q+1$ cosets of B, and for any coset representative s, $s^{-1}\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ since this conjugacy class representative is in B and in the center of G, therefore the induced character value of $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ is $(q+1)\alpha(x)\beta(x)$.
- $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ fixes only B and $s^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$, so that it will have an induced character value of $\alpha(x)\beta(x)$.
- $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ fixes B (with conjugation giving the original element) and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)B$, and in the latter case, it becomes $\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$ after conjugation, so that the induced character value here is $\alpha(x)\beta(y) + \alpha(y)\beta(x)$.
- $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ fixes no cosets of B, and so has an induced character value of 0.

We will call this character $W_{\alpha,\beta}$, and its values are summarized below:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$W_{\alpha,\beta}$:	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y) + \alpha(y)\beta(x)$	0

To give conditions on α, β that make $W_{\alpha,\beta}$ irreducible, we recall that they are 1 dimensional characters on $\mathbb{F}_{q^2}^\times$. Let ϵ be the generator of $\mathbb{F}_{q^2}^\times$, and let μ be any $(q+1)$st root of unity. We claim that $W_{\alpha,\beta}$ is irreducible if and only if $\beta(\epsilon) \neq \mu\alpha(\epsilon)$.

To see this, we work out the Hermitian product of the character with itself, multiplying the value in the third column by its complex conjugate, to get $g(k) =$

[4]Here we are following Fulton and Harris, section 5.2, in their development of the character table for the general linear group.

$2+\psi(k)+\psi(k^{-1})$ where $\psi = \frac{\alpha}{\beta}$ is a 1 dimensional character on $\mathbb{F}_{q^2}^\times$, and $k = x\overline{x} \in \mathbb{F}_q^\times$. Now if $\beta(\epsilon) = \mu\alpha(\epsilon)$, then since $\mathbb{F}_q^\times$ is generated by $\epsilon^{(q+1)}$, ψ is trivial on $\mathbb{F}_q^\times$ and $g(k) = 4$ for every element in the third column. Thus the Hermitian product will be:

$$\frac{1}{|G|}\left[(q+1)^2(q+1) + (q-1)(q+1)^2 + \frac{4(q-2)q(q+1)^2}{2}\right] = 2$$

so that $W_{\alpha,\beta}$ is not irreducible; in fact inspection shows in this case that $W_{\alpha,\beta} = U_\alpha \oplus V_\alpha$.

On the other hand if $\beta(\epsilon) \neq \mu\alpha(\epsilon)$ then ψ is not trivial on $\mathbb{F}_q^\times$. We will show presently that this implies $\sum_{x\in\mathbb{F}_q^\times} \psi(x) = 0$, or $\sum_{x\in\mathbb{F}_q^\times\setminus\{1\}} \psi(x) = -1$. Supposing this to be true, we calculate the contribution of the third column to the Hermitian product indirectly: in summing $g(k)$ over the conjugacy class representatives $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, we include an extra element from each conjugacy class: $\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$. This doubles the sum, because $g(k)$ is the same for both $\left(\begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ (the k values of these elements are inverses of each other, but $g(k) = g(k^{-1})$). The reason for including these extra elements is that it allows the following grouping argument.

We are summing $g(k) = 2+\psi(k)+\psi(k^{-1})$ over all matrices of the form $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ where $k = x\overline{x}$ and $x \in \mathbb{F}_{q^2}^\times\setminus\{\mathcal{L}\}$. These elements are determined by the value of x, so there are $(q^2-1)-(q+1) = (q-2)(q+1)$ of them, and they can be put into $q+1$ sets of size $(q-2)$, in each of which k ranges over all values in $\mathbb{F}_q^\times\setminus\{1\}$. To construct such a set, for each $k \in \mathbb{F}_q^\times\setminus\{1\}$, we include one element $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ where $x\overline{x} = k$; there will be $q+1$ of these sets because that is the order of the kernel of the norm map. Summing $g(k)$ over the elements in one of these sets will give $2(q-2)-1-1 = 2(q-3)$ because $\sum_{k\in\mathbb{F}_q^\times\setminus\{1\}} \psi(k) = \sum_{k\in\mathbb{F}_q^\times\setminus\{1\}} \psi(k^{-1}) = -1$. We have $q+1$ such sets giving a total of $2(q-3)(q+1)$, but we doubled our sum by including extra elements, so the contribution of the conjugacy class representatives of the third column is $(q-3)(q+1)$. The Hermitian product is:

$$\frac{1}{|G|}\left[(q+1)^2(q+1) + (q-1)(q+1)^2 + (q-3)q(q+1)^2 + 0\right] = 1$$

and $W_{\alpha,\beta}$ is irreducible when $\beta(\epsilon) \neq \mu\alpha(\epsilon)$.

Now we justify the claim that if ψ is not trivial on $\mathbb{F}_q^\times$, then $\sum_{x\in\mathbb{F}_q^\times}\psi(x)=0$. More generally, let G be a finite group, and let ψ be a 1 dimensional non-trivial character on G. Since ψ is 1 dimensional it is a homomorphism, and since it is non-trivial, there exists $y \in G$ such that $\psi(y) \neq 0$. Therefore we can write:

$$\begin{aligned}\sum_{g\in G}\psi(g) &= \sum_{g\in G}\psi(yg)\\ &= \sum_{g\in G}\psi(y)\psi(g)\\ &= \psi(y)\sum_{g\in G}\psi(g)\end{aligned}$$

so that $(1-\psi(y))\sum_{g\in G}\psi(g)=0$, and since $\psi(y)\neq 1$, then $\sum_{g\in G}\psi(g)=0$.

To count the number of characters $W_{\alpha,\beta}$, we recall that we began with the one dimensional character on the diagonal subgroup that sent $\left(\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right)$ to $\alpha(a)\beta(d)$ where α and β were one dimensional characters on $\mathbb{F}_{q^2}^\times$. We lifted it to B, then induced to G. There are q^2-1 one dimensional characters on the diagonal subgroup, because it is isomorphic to $\mathbb{F}_{q^2}^\times$. We can form all of these characters by holding α fixed, and letting β vary over the entire group of characters. However we cannot have $\beta(\epsilon)=\mu\alpha(\epsilon)$, and this eliminates $q+1$ characters, since there are that many $(q+1)^{\text{st}}$ roots of unity. This gives $q^2-1-(q+1)=(q+1)(q-2)$, but when we induce to G, switching α and β makes no difference. Thus we get $(q+1)(q-2)/2$ irreducible characters of this form.

Thus to this point we have found

$$(q+1)+(q+1)+\frac{(q-2)(q+1)}{2}=\frac{(q+1)(q+2)}{2}\text{ irreducible characters}$$

Subtracting this from the required total of $(q+1)^2$ gives $q(q+1)/2$ remaining. To find these we begin by inducing a representation from another large subgroup of G.

Let $H\subseteq G$ be the subgroup of all matrices of the form $\left(\begin{smallmatrix} x & y\\ y & x\end{smallmatrix}\right)$. We note that in an element of this subgroup, unlike a conjugacy class representative of the same form, y can be zero. Note also that, as we saw on page 18, the eigenvalues of every element of this subgroup will have norm 1. The order of H is readily seen[5] to be $(q+1)^2$, so that $[G:H]=(q-1)q$. To find coset representatives, we define an equivalence relation on the Borel subgroup: let two elements in this subgroup be similar if and only if one is a scalar multiple of the other by a norm 1 element. Since $|B|=(q-1)q(q+1)$, and there are $q+1$ elements of norm 1, we get $(q-1)q$ equivalence classes. We claim that the set formed by taking an arbitrary element from each of these equivalence classes will be a transversal for H.

To see this, suppose first that $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \left(\begin{smallmatrix} e & f \\ 0 & h \end{smallmatrix}\right)H$. Then we have

$$\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H \Rightarrow \begin{pmatrix} \frac{a}{e} & \frac{b}{e}-\frac{df}{eh} \\ 0 & \frac{d}{h} \end{pmatrix} \in H \Rightarrow \frac{a}{e}=\frac{b}{f}=\frac{d}{h}=x, \text{ for some } \ x \in \mathbb{F}_{q^2}$$

These equalities hold since, for the matrix on the right to be in H, $\frac{a}{e}$ must equal $\frac{d}{h}$, and $\frac{b}{e}-\frac{df}{eh}$ must be equal to zero, which implies that $\frac{b}{f}=\frac{d}{h}$. Thus we have: $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)=x\left(\begin{smallmatrix} e & f \\ 0 & h \end{smallmatrix}\right)$, and $1=a\overline{d}=1(ex)(\overline{hx})=(e\overline{h})x\overline{x}=x\overline{x}$

Next, suppose that $\left(\begin{smallmatrix} e & f \\ 0 & h \end{smallmatrix}\right)=x\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $x\overline{x}=1$. Then

$$\begin{pmatrix} e & f \\ 0 & h \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}=\frac{1}{x}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}=\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{x} \end{pmatrix}$$

which is in H since $\dfrac{1}{x}$ is of norm 1.

Therefore the number of elements of B in any coset of H must be either 0 or $q+1$, but as there are $(q-1)q$ cosets of H, then there must be $q+1$ elements of B in each coset in order to account for the order of B in the partition of G into the cosets of H. Now for each conjugacy class representative g of G, we find all $\sigma \in G/H$ such that $g\sigma=\sigma$ or $s^{-1}gs \in H$, with s arbitrary in σ:

- $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$: as this element is in both H and the center of G, it fixes all cosets and $s^{-1}\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)s=\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ for any s, a coset representative of H.
- $\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$: for any coset representative $s=\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, $s^{-1}\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)s=\left(\begin{smallmatrix} x & \frac{dy}{a} \\ 0 & x \end{smallmatrix}\right)$, and this is not in H since $d, y \neq 0 \Rightarrow dy/a \neq 0$. Therefore this representative fixes no cosets.

[5]This follows from the ideas used to count the conjugacy classes with representatives of this form, while also accounting for the case where $y=0$.

- $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$: the eigenvalues of this element are both not norm 1; therefore for any coset representative s, $s^{-1}\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)s$ cannot lie in H, and this element fixes no cosets.

- $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$: for any coset representative $s = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ we have

$$s^{-1}\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)s = \begin{pmatrix} x - \dfrac{by}{d} & \dfrac{-b^2y}{ad} + \dfrac{dy}{a} \\ \dfrac{ay}{d} & x + \dfrac{by}{d} \end{pmatrix}$$

For this to be in H we require first, that $\dfrac{by}{d} = \dfrac{-by}{d} \Rightarrow b = 0$, since $y \neq 0$. This gives:

$$\begin{pmatrix} x & \dfrac{dy}{a} \\ \dfrac{ay}{d} & x \end{pmatrix}$$

.

Now we require that $\dfrac{a}{d} = \dfrac{d}{a} \Rightarrow d = \pm a$. Thus the coset representative s will be either:

$\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \in \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)H = H$, or
$\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$, with $a\overline{a} = -1$; there are $q+1$ such elements of G, all in the same coset of H.

Therefore $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ fixes two cosets of H, and we note that:

for $s = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $s^{-1}\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, whereas
for $s = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$, $s^{-1}\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)s = \left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$.

Next we take a 1 dimensional character on H as follows: if α, β are distinct 1 dimensional characters on the subgroup $\mathcal{L}$, then we have a 1 dimensional character ϕ on H by sending $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ to $\alpha(x+y)\beta(x-y)$. Inducing this to G, and writing m for $x+y$ and n for $x-y$, we get:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$\mathrm{Ind}_H^G\phi$:	$(q-1)q\alpha(x)\beta(x)$	0	0	$\alpha(m)\beta(n) + \alpha(n)\beta(m)$

To calculate the Hermitian product of this character with itself, we multiply the value in the fourth column by its complex conjugate to get: $2+\gamma(k)+\gamma(k^{-1})$ where $\gamma = \dfrac{\alpha}{\beta}$ is a non-trivial character on $\mathcal{L}$ (since α and β were distinct), and $k = \dfrac{m}{n} \in \mathcal{L}\backslash\{1\}$ since $m = n \Rightarrow y = 0$ which is not possible . Now we want to sum $g(k) = 2+\gamma(k)+\gamma(k^{-1})$ over the conjugacy class representatives $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, but in order to facilitate a convenient grouping in the sum, we include elements of the form $\left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$, which is conjugate to $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$. Since there are $q(q+1)/2$ conjugacy class representatives of the form $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, we are summing over $q(q+1)$ matrices in all. We form sets of q elements each such that in each set any $k = m/n$ will take on all values in $\mathcal{L}\backslash\{1\}$. There will be $q+1$ such sets, since for any such k, $m \in \mathcal{L}$ can be chosen freely, determining n . Summing over such a set of q matrices will give $2q-1-1 = 2q-2$, and summing over the $q+1$ such sets gives $2(q-1)(q+1)$. We divide this by 2, because $g(k)$ for $\left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$ equals $g(k)$ for $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$. The Hermitian product is:

$$\frac{1}{|G|}[(q-1)^2q^2(q+1) + (q-1)^2q(q+1)] = q-1$$

so that $\mathrm{Ind}_H^G\phi$ is not irreducible.

At this point we observe that the forms of the conjugacy class representatives for G resemble those of $G_2(\mathbb{F}_q)$, and also that the dimensions of the irreducible characters found so far for G are the same[6] as the dimensions of the first 3 irreducible characters for $G_2(\mathbb{F}_q)$: 1, q, $q+1$. Since the fourth irreducible character of the general linear group has dimension $q-1$, it seems worthwhile to try to get a character of that dimension for the unitary group as well. To this end we first tensor the characters V and $W_{\alpha,\beta}$ to get:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$V \otimes W_{\alpha,\beta}$:	$q(q+1)\alpha(x)\beta(x)$	0	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0

The point of this is that we may now imitate the method of Fulton and Harris (p 70), and consider the following virtual character $X_{\alpha,\beta}$:

$$X_{\alpha,\beta} = V \otimes W_{\alpha,\beta} - W_{\alpha,\beta} - \mathrm{Ind}_H^G\phi$$

[6] We are using "same" in a loose sense here, since q is the order of the field in one case but not the other.

with dimension: $q(q+1)-(q+1)-q(q-1)=q-1$. Its values on the conjugacy class representatives are:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$X_{\alpha,\beta}$:	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$-[\alpha(m)\beta(n)+\alpha(n)\beta(m)]$

and, recalling the contribution from the fourth column that we have already worked out, we find the Hermitian product of this character with itself to be:

$$\frac{1}{|G|}[(q-1)^2(q+1)+(q-1)(q+1)^2+(q-1)^2q(q+1)]=1$$

Therefore since the dimension is an integer greater than zero and the Hermitian product is 1, $X_{\alpha,\beta}$ is irreducible . Since α and β are distinct characters on $\mathcal{L}$, and after inducing to G, switching α and β makes no difference, there are $\frac{q(q+1)}{2}$ such characters. Summing all of the irreducible characters that we have found gives a total of :

$$(q+1)+(q+1)+\frac{(q+1)(q-2)}{2}+\frac{q(q+1)}{2}=(q+1)^2$$

which is the number of irreducible characters of G, since it is the number of conjugacy classes. We summarize the irreducible characters of $G=U_2(\mathbb{F}_{q^2})$ below, writing m for $x+y$ and n for $x-y$:

	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2-y^2)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2-y^2)$
$W_{\alpha,\beta}$:	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
$X_{\alpha,\beta}$:	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$-[\alpha(m)\beta(n)+\alpha(n)\beta(m)]$

Table 2: Irreducible Characters of G

There is another way[7] to get the final irreducible character, that does not use the somewhat unmotivated virtual character above. We have a character induced from the subgroup H:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$\mathrm{Ind}_H^G\phi$:	$(q-1)q\alpha(x)\beta(x)$	0	0	$\alpha(m)\beta(n)+\alpha(n)\beta(m)$

We will now induce another character from the subgroup $K = \{\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\}$ and combine it with the one induced from H to get $X_{\alpha,\beta}$.

We begin by finding the order of K. Since y can be zero, we have $q+1$ choices for x, and q choices for y[8]. Thus $|K| = q(q+1)$, and $[G : K] = (q-1)(q+1)$. For the conjugacy class representatives, we first consider equivalence classes of matrices of the form $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$, where two matrices are equivalent if one is a multiple of the other by an element of $\mathbb{F}_{q^2}$ of norm 1. We will take an arbitrary element from each equivalence class to form the transversal of G/K. We note that from the $q(q^2-1)$ matrices of the form $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$ we get q^2-q equivalence classes, and from the q^2-1 matrices of the form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ we get $q-1$ equivalence classes. The total number of equivalence classes is thus $(q-1)(q+1)$ as required.

Next we show that matrices from distinct equivalence classes lie in different cosets of K:

- claim: $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in \left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right)K \Leftrightarrow \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) = x\left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right),\ \ x\overline{x} = 1$. proof:

  ”$\Rightarrow$”: $\left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \in K \Rightarrow \left(\begin{smallmatrix} \frac{a}{e} & 0 \\ \frac{-ag}{eh}+\frac{c}{h} & \frac{d}{h} \end{smallmatrix}\right) \in K$
  This implies[9] that $\frac{a}{e} = \frac{d}{h} = \frac{c}{g} = x \in \mathbb{F}_{q^2}$, and $a\overline{d} = 1 \Rightarrow (ex)(\overline{hx}) = e\overline{h}x\overline{x} = x\overline{x} = 1$
  ”$\Leftarrow$”: $\left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right) = x\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right), x\overline{x} = 1 \Rightarrow \left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) = \frac{1}{x}\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{x} \end{smallmatrix}\right) \in K$

- claim: $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) \in \left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)K \Leftrightarrow \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) = x\left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)$, $x\overline{x} = 1$. proof:

  ”$\Rightarrow$”: $\left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{c}{g} & 0 \\ 0 & \frac{b}{f} \end{smallmatrix}\right) \in K \Rightarrow \frac{b}{f} = \frac{c}{g} = x \in \mathbb{F}_{q^2}$, and $1 = b\overline{c} = (fx)(\overline{gx}) = f\overline{g}x\overline{x} = x\overline{x}$.

[7] Here we are following an idea of C. Bushnell and G. Henniart pp. 47–48, in the case of GL_2
[8] For the reasons mentioned when we counted the Borel subgroup
[9] The cases with c or $g = 0$ are omitted here for simplicity; they follow in the same way.

"$\Leftarrow$": $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) = x\left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)$, $x\overline{x} = 1 \Rightarrow \left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} 0 & f \\ g & 0 \end{smallmatrix}\right)x = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \in K$

- Finally, $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \notin \left(\begin{smallmatrix} 0 & n \\ m & 0 \end{smallmatrix}\right)K$, since $\left(\begin{smallmatrix} 0 & n \\ m & 0 \end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{c}{n} & \frac{d}{n} \\ \frac{a}{m} & 0 \end{smallmatrix}\right)$, which is not in K. Therefore the transversal is as was claimed.

Next we find the cosets $\sigma \in K$ such that for a conjugacy class representative g, we get $g\sigma = \sigma$, or $s^{-1}gs \in K$ for an arbitrary s in σ:

(i) $g = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$. Here, g is in K and in the center of G, so it fixes all of the $(q-1)(q+1)$ cosets of K.

(ii) $g = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$, $y \neq 0$. First, let $\sigma = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)K$; we will show that g fixes this coset if and only if $c = 0$. Letting $s = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$, we find that

$$s^{-1}gs = \begin{pmatrix} x + \frac{cy}{a} & \frac{dy}{a} \\ \frac{-c^2y}{ad} & x - \frac{cy}{a} \end{pmatrix}$$

and this is in K if and only if $c = 0$, in which case we get

$$s^{-1}gs = \begin{pmatrix} x & d\overline{d}y \\ 0 & x \end{pmatrix}$$

where $a\overline{d} = 1 \Rightarrow \frac{d}{a} = d\overline{d}$. Therefore g fixes cosets of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)K$. There are $(q-1)$ such cosets because there are $q^2 - 1$ unitary matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$, in equivalence classes each of size $q+1$. As we range over all these cosets, $d\overline{d}$ of the coset representative $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ will take on all values in $\mathbb{F}_q^\times$; this will be important later. Next we show that g fixes no cosets of the form $g = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)K$. In this case we have

$$s^{-1}gs = \begin{pmatrix} x & 0 \\ \frac{cy}{b} & x \end{pmatrix}$$

which cannot be in K because $y \neq 0 \neq c$.

(iii) $g = \left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $y \neq x$. Here g has two eigenvalues, and every coset representative s is invertible. Thus $s^{-1}gs$ will have two eigenvalues, and so cannot be in K. Thus g fixes no cosets.

(iv) $g = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right),\ y \neq 0$. Here again, g has two eigenvalues, so that $s^{-1}gs \notin K$, implying that g fixes no cosets.

Now we create a character on K as follows: let $\psi : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$, and $\phi : \mathbb{F}_{q^2}^+ \to \mathbb{C}^\times$ be 1 dimensional characters. We also require that ϕ be nontrivial on the additive subgroup $\{t \in \mathbb{F}_{q^2}^+ \,|\, t + \bar{t} = 0\} \subseteq \mathbb{F}_{q^2}^+$. Then we define a 1 dimensional character $\gamma : K \to \mathbb{C}^\times$ by $\gamma\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) = \phi(\frac{y}{x})\psi(x)$. This seems a bit contrived, but it carries information about y, and will be just what we need. First we show that it is in fact a character:

$$\gamma\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \phi(0)\psi(1) = (1)(1) = 1.$$

$$\begin{aligned}
\gamma\left[\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} p & q \\ 0 & p \end{smallmatrix}\right)\right] &= \gamma\left(\begin{smallmatrix} px & py+qx \\ 0 & px \end{smallmatrix}\right) \\
&= \phi(\frac{py+qx}{px})\psi(px) \\
&= \phi(\frac{y}{x} + \frac{q}{p})\psi(px) \\
&= \phi(\frac{y}{x})\phi(\frac{q}{p})\psi(p)\psi(x) \\
&= \phi(\frac{y}{x})\psi(x)\phi(\frac{q}{p})\psi(p) \\
&= \gamma\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} p & q \\ 0 & p \end{smallmatrix}\right)
\end{aligned}$$

Now we give the character values for the conjugacy class representatives. For $g = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$ we have $\chi\text{Ind}_K^G g = (q-1)(q+1)\psi(x)$. For $g = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$ we get $\sum_{k \in \mathbb{F}_q^\times} \gamma\left(\begin{smallmatrix} x & ky \\ 0 & x \end{smallmatrix}\right)$(see the bottom of p 34). We can write this as $\sum_{k \in \mathbb{F}_q^\times} \phi(\frac{ky}{x})\psi(x) = \psi(x)\sum_{k \in \mathbb{F}_q^\times} \phi(\frac{ky}{x})$. To evaluate this, we write z for $\frac{y}{x}$, noting that $z + \overline{z} = 0$, so that $0, k_1z, k_2z, \ldots k_{q-1}z$, (where all k_i are distinct in $\mathbb{F}_q^\times$) is the subgroup $\{t \in \mathbb{F}_{q^2}^+ \,|\, t + \bar{t} = 0\}$ of $\mathbb{F}_{q^2}^+$ of order q, so that ϕ is nontrivial on this subgroup, which means $\phi(k_1z) + \cdots + \phi(k_{q-1}z) = -1$. Therefore $\chi\text{Ind}_K^G g = -\psi(x)$, and we have:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$\text{Ind}_K^G\gamma$:	$(q-1)(q+1)\psi(x)$	$-\psi(x)$	0	0

If we now recall the character induced from H:

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$\text{Ind}_H^G\phi$:	$(q-1)q\alpha(x)\beta(x)$	0	0	$\alpha(m)\beta(n) + \alpha(n)\beta(m)$

we might note that the difference of the dimensions is $q-1$. In fact, if we take for ψ the 1 dimensional character $\alpha\beta$ used in H, then defining $X_{\alpha\beta} = \text{Ind}_K^G\gamma - \text{Ind}_H^G\phi$ gives

representative:	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
$X_{\alpha,\beta}$:	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$-[\alpha(m)\beta(n)+\alpha(n)\beta(m)]$

and we have the fourth irreducible character. To count these representations, we note that using a different nontrivial ϕ will not make a difference; but α and β were distinct characters on $\mathcal{L}$, and inducing to G means switching α and β makes no difference, so we have $q(q+1)/2$ representations of this type, confirming our result from before.

Characteristic 2

In this chapter we will address those parts of the main argument that fail for the case $p = 2$. The central problem is the matrix $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, which serves both as a conjugacy class representative (in which case y cannot be zero), and as the form of the matrices in the subgroup denoted H (in which case y or x but not both can be zero). When it was necessary to merely count this element as a conjugacy class representative, the count was done easily using the Hermitian form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, but beginning with the listing of the first character on G, we omitted many quirks about this matrix that arise in characteristic 2, and we will address these here. However we shall begin at the beginning, modifying the problematic arguments in the same order that they appear in the main work. The assumption throughout this section is that the characteristic of the field is 2.

The Quadratic Field Extension: In characteristic 2 all elements of a finite field are squares, thus we cannot adjoin the square root of some element in $\mathbb{F}_q$ in order to get a quadratic extension. On the other hand, the map $g(x) = x^2 + x$ from $\mathbb{F}_q$ to itself is not injective since for any $a \in \mathbb{F}_q$ $g(a) = g(a+1)$. This implies (since the field is finite) that the map is not surjective, and that we can therefore find some $m \in \mathbb{F}_q$ such that $x^2 + x + m = 0$ will have no roots in the field. Thus if θ is a root of this equation (in some algebraic closure) then adjoining θ gives a quadratic extension.

Conjugation in $\mathbb{F}_{q^2}$: Each x in $\mathbb{F}_{q^2}$ will be $x = a + b\theta$ for $a, b \in \mathbb{F}_q$. Since θ is a root of $x^2 + x + m = 0$, the sum of θ and $\overline{\theta}$ is 1, so that $\overline{\theta} = \theta + 1$. It follows that $\overline{a + b\theta} = (a + b) + b\theta$.

Eigenvalues: It was easy to show that the eigenvalues of 2 x 2 unitary matrices lie in $\mathbb{F}_{q^2}$ by looking at the discriminant of the characteristic equation. In characteristic 2 however, the quadratic formula is not available, and it will take a bit more work to prove the result in this case.

The characteristic equation for $A \in G$ is $\lambda^2 + \mathrm{Tr}(A)\lambda + \det(A) = 0$. If $\mathrm{Tr}(A) = 0$, the square root of $\det(A)$ (which is in $\mathbb{F}_{q^2}$) is a root. If $\mathrm{Tr}(A) \neq 0$, then we make the substitution $y = \dfrac{\lambda}{\mathrm{Tr}(A)}$ to rewrite the characteristic equation as $y^2 + y + d = 0$, where $d = \dfrac{\det(A)}{(\mathrm{Tr}(A))^2}$. We note that $d \in \mathbb{F}_q$ since with $\det(A) = D = \frac{x}{\overline{x}}$ for some $x \in \mathbb{F}_{q^2}$

$$d = \frac{D}{(a + \overline{a}D)^2} = \frac{x\overline{x}}{(a\overline{x} + \overline{a}x)^2}$$

which is invariant under conjugation and so in $\mathbb{F}_q$. This implies that any root u, of $y^2 + y + d = 0$, lies in the unique (up to isomorphism) quadratic extension of $\mathbb{F}_q$, which is just $\mathbb{F}_{q^2}$. But now any root w of $\lambda^2 + \mathrm{Tr}(A)\lambda + \det(A) = 0$ will be equal to $u\mathrm{Tr}(A)$ and so will be in $\mathbb{F}_q^2$.

The Conjugacy Class $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right), y \neq 0$**:** This conjugacy class is meant to comprise the elements of G having distinct, norm 1 eigenvalues, but when $p = 2$, $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ has trace zero, and will therefore have 1 eigenvalue. The solution here is to use the Hermitian form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (effectively changing bases) so that the conjugacy class representative will be $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right), y \neq x$ where in this case the eigenvalues are x and y. This allows us to count the number of such conjugacy class representatives, though as we will see below, for some calculations, this conjugacy class representative must be written in a more complicated way.

Character Values: For the representations U_α and $V\alpha$ the character values for the fourth conjugacy class were given as $\alpha(x^2 - y^2)$ and $-\alpha(x^2 - y^2)$ respectively. For $p = 2$, these character values are changed to $\alpha(xy)$ and $-\alpha(xy)$, since the value was based on the determinant.

The Permutation Representation of G/B**:** The representation denoted V was derived from a permutation representation of the coset space of the Borel subgroup. A part of this argument involved showing that the conjugacy class representative $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right), y \neq 0$ fixed no cosets of B, and since $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ cannot be used when $p = 2$ this argument must be modified. If we use the other Hermitian form as we did when counting the conjugacy class representatives we face the problem that this form does not allow upper triangular matrices (apart from diagonal matrices), therefore our procedure must be a bit indirect: we begin using the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so that the class representative is $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right), y \neq x$, then we use the change of basis matrix[10] $\left(\begin{smallmatrix} 1 & \theta \\ 1 & \theta+1 \end{smallmatrix}\right)$ to change $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right), y \neq x$ into :

[10] This is the change of basis matrix that takes us from the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$

$$\begin{pmatrix} (x+y)\theta + x & (x+y)(\theta^2+\theta) \\ (x+y) & (x+y)\theta + y \end{pmatrix}$$

with $y \neq x$. This will be the conjugacy class representative in place of $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right), y \neq 0$. Now we can check all cosets of B. Denoting the above representative as A, we see:

(i) if the coset representative is $s = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$,then $s^{-1}As$ is not in B because the entries of A in the upper right and lower left positions are necessarily non-zero.

(ii) if $s = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right), t \neq 0$, then $s^{-1}As$ gives a value in the lower left element of $(x+y)(mt^2+t+1)$, where $m = \theta^2 + \theta$. This expression cannot be zero because $y \neq x$, and because we can show that $mt^2 + t + 1 = 0$ requires $t \notin \mathbb{F}_q$, whereas in characteristic 2 $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ is unitary if and only if $t \in \mathbb{F}_q$, since we will have $t + \bar{t} = 0 \Rightarrow t = \bar{t} \Rightarrow t \in \mathbb{F}_q$. To see that no $t \in \mathbb{F}_q$ is a root of $mt^2 + t + 1 = 0$, let $mt = y$, multiply the equation by m, and then write it as $y^2 + y + m = 0$. By construction any solution of this equation is not in $\mathbb{F}_q$, thus t must not be in $\mathbb{F}_q$.

Therefore A fixes no cosets of B as required.

Induction from H: The subgroup H, $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right), xy \neq 0$ that was used in this work is the subgroup consisting of elements of G that have eigenvalues of norm 1. In characteristic 2 we must write this subgroup differently; we start by using the Hermitian form whose matrix is the identity. In this way, H will be $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$ where y can be equal to x. Now we use the change of basis matrix $\left(\begin{smallmatrix} 1 & \theta \\ 1 & \theta+1 \end{smallmatrix}\right)$ (we change bases in order to be able to use matrices from the Borel subgroup) to get

$$\begin{pmatrix} (x+y)\theta + x & (x+y)(\theta^2+\theta) \\ (x+y) & (x+y)\theta + y \end{pmatrix}$$

where y can be equal to x. The character t on H that we induced used α and β, 1 dimensional characters on $\mathcal{L}$, and sent the conjugacy class representative $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ to $\alpha(m)\beta(n)$, while the induced value was $\alpha(m)\beta(n) + \alpha(n)\beta(m)$, where m, n were the eigenvalues. When we rewrite the conjugacy class representative for characteristic 2, the eigenvalues are now x, y, so that we should expect the induced value to be $\alpha(x)\beta(y) + \alpha(y)\beta(x)$.

Recall that the conjugacy class representative looks like the subgroup, with the condition that $y \neq x$. Now when we check the fix of this conjugacy class representative on cosets of H, we find that (writing S for $(x+y)$) for any coset representative $s = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$, $s^{-1}As$ is :

$$\begin{pmatrix} S\theta + x + \frac{bS}{d} & \frac{d}{a}[S(\theta^2+\theta)] \\ \frac{aS}{d} & S\theta + y + \frac{bS}{d} \end{pmatrix}$$

if this is to be in H, then from the lower left entry we have $a = d$, and from the entries on the main diagonal we have either of the following possibilities:

- $b = 0$, so that the coset representative will be (without loss of generality) $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ so that this class representative fixes H itself; to be expected , as this representative is in H.
- $b = d$, for then the upper left entry becomes $S\theta + y$, and the lower right becomes $S\theta + x$. This makes the coset representative (without loss of generality) $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ which, incidentally, is unitary only for $p = 2$. Finally, we note that the character value for this class representative upon induction to G will be $\alpha(x)\beta(y) + \alpha(y)\beta(x)$ as required.

Induction From K: In the induction from the subgroup K, we again used the conjugacy class representative $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$, $y \neq 0$. For characteristic 2 we replace it again by

$$\begin{pmatrix} (x+y)\theta + x & (x+y)(\theta^2+\theta) \\ (x+y) & (x+y)\theta + y \end{pmatrix}$$

where $y \neq x$. Now to show that this fixes no cosets of K, we note that it has distinct eigenvalues, so that no conjugate of it can lie in K.

Conclusion

We conclude with some observations about the similarities between the characters of the general linear group, and those of the unitary subgroup. For the general linear group of 2 x 2 matrices over a finite field of order q we have from Harris:

	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 1 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$
U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2-\epsilon y^2)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2-\epsilon y^2)$
$W_{\alpha,\beta}$:	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
X_ϕ:	$(q-1)\phi(x)$	$-\phi(x)$	0	$-(\phi(\zeta)+\phi(\zeta^q))$

Table 3: Irreducible Characters of $GL_2(\mathbb{F}_q)$

In the fourth row ζ is essentially one of the eigenvalues of $\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$, ζ^q is the other eigenvalue[11], and ϕ is a 1 dimensional character on $\mathbb{F}_{q^2}$. These correspond, respectively, to m, n, and $\alpha\beta$ in the fourth row of table 4.1.

Let us now compare this to our table of the characters of the unitary group:

	$\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$	$\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$
U_α:	$\alpha(x)^2$	$\alpha(x)^2$	$\alpha(x)\alpha(y)$	$\alpha(x^2-y^2)$
V_α:	$q\alpha(x)^2$	0	$\alpha(x)\alpha(y)$	$-\alpha(x^2-y^2)$
$W_{\alpha,\beta}$:	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0
$X_{\alpha,\beta}$:	$(q-1)\alpha(x)\beta(x)$	$-\alpha(x)\beta(x)$	0	$-[\alpha(m)\beta(n)+\alpha(n)\beta(m)]$

Table 4: Irreducible Characters of G

We notice first, that the dimensions in each case are $1, q, q+1, q-1$, though q is the group order only in the general linear group. Next we see that the forms of the corresponding conjugacy class representatives are almost identical. The fourth columns differ only in the presence of the epsilon in the general group, and in

[11] More precisely, $\zeta = x + y\sqrt{\epsilon}$, and $\zeta^q = x - y\sqrt{\epsilon}$.

the second column the difference arises from the fact that matrices in the unitary subgroup having one eigenvalue do not necessarily have a Jordan form. Turning to the character values themselves, we note that almost all corresponding values are the same; the exceptions arising from the epsilon, and the different form of the eigenvalues in the fourth column of the last character. We recall also, that in both tables $W_{\alpha,\beta}$ is irreducible if and only if $\alpha \neq \beta$, and also that when $\alpha = \beta$, $W_{\alpha,\beta} = U_\alpha \oplus V_\alpha$.

Next we compare the methods of construction. The first row in each table comes from mapping the determinant of each element by a 1 dimensional character. The second rows are both tensor products of the first row together with the character formed by subtracting the trivial representation from the permutation representation on the corresponding borel subgroup. The third rows are both formed by inducing a character on the Borel subgoup, and the fourth rows are found by inducing characters from subgroups having almost identical forms: $\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$ in the case of the general group, and $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ in that of the unitary group. In addition, the characters induced from each of these subgroups could be combined with others to form a virtual character that turned out to be the final one.

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