University of Alberta

N-EXTREMELY AMENABLE SEMI-TOPOLOGICAL SEMIGROUPS

by

Nicolas Bouffard ©

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ABSTRACT

A semigroup S is n-extremely left amenable if there is a set F of n multiplicative means on the space of bounded real valued functions on S which is minimal with respect to left translation. When S is a semi-topological semigroup, we replace the space of bounded real valued functions by the space of left uniformly continuous functions on S. n-extreme left amenability is related to some fixed point property.

First we characterize extreme left amenability of a semigroup in term of ultrafilters. We also give some result on the semidirect product of two extremely left amenable semigroups. Finally, we give some results related to density and to homomorphic image of semi-topological semigroups in relation to extreme left amenability.

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To my family.

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Chapter 1

Introduction

In this thesis, we are interested in studying $n-$extremely amenable semi-topological semigroups and related algebras. The subject comes from amenable semigroups, about which most of what we need for our purpose can be found in [6]. This paper proposes an interesting way to define the concept of mean, a concept which will be very important:

"A mean value, or average value, of a function is a number chosen in some reasonable fashion between the least upper bound and the greatest lower bound of the function. Here we ask that the choice be made simultaneaoustly for all functions in $m(S)$ and made in a linear way" [6].

A functional analysis definition of a mean, which is equivalent, will be given in the begining of chapter 2.

Chapter 2 contains the basic definitions and theorems that we need. We define the notion of ultrafilters and talk about the Stone-Čech compactification of a semigroup and the Bohr compactification of an abelian group. We define some interesting algebras related to a semigroup and define the notion of action of a semigroup, which we relate to the notion of extreme amenability. We also provide a characterization of extremely left amenable semigroups and extremely left amenable semi-topological semigroups.

In chapter 3, we present our main results. In particular, we are interested in seeing how we can characterize extreme left amenability using the notion of ultrafilters. We define the semidirect product of two semigroups, and give results related to the semidirect product of two extremely left amenable semigroups. We are also interested in how extreme left amenability of a group and extreme left amenability of a dense subsemigroup are related. We also present some results related to dense subgroups of an n-extremely left amenable semi-topological semigroup and the homomorphic image of an n-extremely left amenable semi-topological semigroup.

Finally, in the last chapter, we provide some examples of possible future research.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we define most of the basic notions that are needed for the next chapter.

We start this chapter with some important definitions and notations. In sections 2.4 and 2.5, we define the notion of ultrafilters and give the basic idea of the Stone-Čech compactification and the Bohr compactification which will play an important role later in characterizing extremely left amenable semigroups.

In the following two sections, 2.6 and 2.7, we present some basic characterizations of discrete extremely left amenable semigroups, and extremely left amenable semi-topological semigroups.

In section 2.8 we define some important subalgebras of $m(S)$, the space of all bounded real valued functions on a semigroup S, that will be useful later on.

In the last two sections of this chapter, we extend the notion of extreme left amenability first for semigroups acting on a Hausdorff space and then for the case where we do not necessarily have a multiplicative left invariant mean, but a left invariant mean which is the convex combination of n multiplicative means.

2.2 Definitions and notations

Definition 2.2.1. A *semigroup* is a set S with a binary operation which is associative, i.e., for any elements x, y and z in S, we have

$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

Definition 2.2.2. A *group* is a set G together with a binary operation denoted by " $\cdot$ "satisfying:

- For any $x, y, z \in G$:
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$
- There exists an element in G denoted 1, such that:
$$1 \cdot x = x \cdot 1 = x, \forall x \in G.$$
- For each element $x \in G$, there exists an element $x^{-1} \in G$ such that:
$$xx^{-1} = x^{-1}x = 1.$$

Remark 2.2.3. When the group is abelian (i.e., the binary operation is commutative) we usually denote the neutral element by 0, and use the notation $(-x)$ for the inverse of x.

Example 2.2.4. The sets $\mathbb{N}$ of natural numbers, $\mathbb{Z}$ of integers, $\mathbb{Q}$ of rational numbers, $\mathbb{R}$ of real numbers, and $\mathbb{C}$ of complex numbers are all commutative semigroups with addition and multiplication.

Example 2.2.5. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are all abelian groups under addition, and the sets $\mathbb{Q}\backslash\{0\}$, $\mathbb{R}\backslash\{0\}$, and $\mathbb{C}\backslash\{0\}$ are all abelian groups under multiplication.

Example 2.2.6. The set $M(n, \mathbb{C})$ of all $n \times n$ matrices over the complex numbers $\mathbb{C}$ under matrix multiplication is a noncommutative semigroup.

Example 2.2.7. If X is a set of cardinality greater than 1, then the set of all functions from X into X is a noncommutative semigroup under composition of functions.

Given a semigroup S with a topology, we say that the product $S \times S \rightarrow S$ is separately continuous if whenever a net $\{s_\alpha\}$ in S converges to an element $s \in S$, and t is any element of the semigroup, we have that:

$$s_\alpha t \rightarrow st$$

and

$$ts_\alpha \rightarrow ts.$$

In the same way, we say that the product is jointly continuous, if whenever $\{s_\alpha\}$ and $\{t_\beta\}$ are two nets in S such that $s_\alpha \rightarrow s$ and $t_\beta \rightarrow t$, we have:

$$s_\alpha t_\beta \rightarrow st.$$

Definition 2.2.8 (semi-topological semigroup)**.** A semi-topological semigroup is a semigroup with a Hausdorff topology such that the product $G \times G \rightarrow G$ is separately continuous.

In the same way, we define a *topological semigroup* to be a semigroup with a Hausdorff topology, such that the product is jointly continuous.

Example 2.2.9. The sets $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ with the usual topology are all topological semigroups with both addition and multiplication. They are also topological groups under addition. The usual topology on those semigroups is the topology coming from the Euclidean norm. Note that $\mathbb{Q}$ is not locally compact.

Example 2.2.10. The semigroup $M(n, \mathbb{C})$ with the usual topology is a topological semigroup. The subgroup $GL(n, \mathbb{C})$ of invertible $n \times n$ matrices over $\mathbb{C}$ is a topological group. The usual topology on $M(n, \mathbb{C})$ is the relative topology of $\mathbb{C}^{n^2}$.

Example 2.2.11. Let X be a topological space and let $C(X, X)$ be the space of continuous functions from X into X under composition of functions and the topology of pointwise convergence on X. Then $C(X, X)$ is a semitopological semigroup.

Definition 2.2.12 (The space $m(S)$)**.** Let S be a semigroup. We then define the space $m(S)$ to be the space of all bounded real-valued functions on S. Many authors use the notation $\ell^\infty(S)$ instead of $m(S)$. On the space $m(S)$ we use the topology of norm convergence, where the norm is defined by:

$$||f|| = \sup_{s \in S} |f(s)|$$

for any function $f \in m(S)$.

We denote by ℓ_s the left translation $\ell_s : m(S) \to m(S)$ defined by

$$\ell_s f(s') = f(ss')$$

for any $s, s' \in S$, and $f \in m(S)$. In the same way, we define the right translation $r_s : m(S) \to m(S)$ by

$$r_s f(s') = f(s's).$$

We denote by $CB(S)$ the space of all bounded, continuous, real-valued functions on a semi-topological semigroup S. Clearly $CB(S) \subseteq m(S)$.

Definition 2.2.13 (The space $LUC(S)$)**.** Let S be a semi-topological semigroup. The space $LUC(S)$ is the space of all left uniformly continuous functions on S. A function $f \in CB(S)$ is left uniformly continuous if the map $\theta_f : S \to CB(S)$ defined by $\theta_f(s) = \ell_s f$ is continuous when $CB(S)$ has the sup norm topology.

Note that when G is a topological group, the space $LUC(G)$ is precisely the space of bounded right uniformly continuous functions on G defined in Hewitt and Ross [13].

2.3 Amenability

Recall that a Banach space E is a normed linear space which is complete in the norm topology. If E is a Banach space, we define the dual space of E, denoted by E^*, to be the set of all continuous linear functionals ϕ on E with the following norm:

$$||\phi|| = \sup_{||f||=1} \frac{|\phi(f)|}{||f||}.$$

It is easy to see that E^* is therefore also a Banach space (see [30] and [4]).

Remark 2.3.1. For any semigroup S, the space $m(S)$ is a Banach space over the real number. Therefore, it makes sense to talk about $m(S)^*$.

We can rewrite the definition of a mean given in chapter 1 in more mathematical form in the following way: A *mean* is a linear functional $\mu \in m(S)^*$ such that

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s)$$

for all function $f \in m(S)$.

Does such a mean value exist ? If so, for which semigroup S can such a value be defined? This is the general idea from which the theory of amenable (A-mean-able) semigroups comes from. So let us start with the exact definition that will be used, which is equivalent to the previous one.

Let S be a semigroup. We define the map l_s^* on $m(S)^*$ by:

$$l_s^*\mu(f) = \mu(l_s f)$$

for all $f \in m(S), s \in S$, and $\mu \in m(S)^*$, where l_s is the left translation map already defined.

Definition 2.3.2. Let S be a semigroup and A be a norm closed, translation invariant subspace of $m(S)$ that contains constants. A *mean* on A is a linear functional $\mu : A \to \mathbb{R}$ such that $||\mu|| = 1$ and $\mu(e) = 1$, where e is the constant one function on S, and the norm is the dual norm of $m(S)^*$ coming from $m(S)$.

A mean μ on A is *left translation invariant* if $l_s^*\mu = \mu$ for all $s \in S$, where $(l_s f)(s_0) = f(ss_0)$. A mean μ on A is *multiplicative* if $\mu(fg) = \mu(f)\mu(g)$ for all $f, g \in A$.

Let S be a semigroup. We say that S is *left amenable* if there exists a left invariant mean on the space $m(S)$. See Day [6] for many properties of left amenable semigroups.

Definition 2.3.3. Let S be a semigroup. A subalgebra of $m(S)$ is extremely left amenable if it is norm closed, left translation invariant, containing constants, and has a multiplicative left invariant mean. S is extremely left amenable if $m(S)$ is extremely left amenable.

Following Namioka [26], we say that a semi-topological semigroup S is left amenable if there is a left invariant mean on the space $LUC(S)$, and similarly, we say that a semi-topological semigroup S is extremely left amenable if there exists a multiplicative left invariant mean on $LUC(S)$. If S is a discrete semigroup, then both spaces $LUC(S)$ and $m(S)$ are the same, which makes the definition of (extremely) left amenable semigroup and (extremely) left amenable semi-topological semigroup to be the same, if we use the discrete topology for any semigroup which does not have a topology.

2.4 Ultrafilters

When we talk about metric spaces, everything we need to do about convergence (or almost) can be done with sequences, but when we talk about convergence for topological spaces, we need to generalize the idea of convergence of sequences. Two main ideas have been developed: The notion of nets, and the notion of filters.

For our purpose, we need a special kind of filters: Ultrafilters. These are maximal filters.

Definition 2.4.1. A filter on a set S is a collection $\mathscr{F}$ of subsets of S such that:

1. If $F_1, F_2 \in \mathscr{F}$ then $F_1 \cap F_2 \in \mathscr{F}$.

2. If $F_1 \in \mathscr{F}$ and $F_1 \subseteq F_2$ then $F_2 \in \mathscr{F}$.

Example 2.4.2. If X is a topological space and x is any element of X, then a filter of a particular interest is the neighborhood filter U_x of all neighborhood of x in X.

Definition 2.4.3. Given a topological space X, we say that a filter $\mathscr{F}$ on X converges to some $x \in X$ if and only if $\mathscr{F}$ is finer than the neighborhood filter of x. We say that $\mathscr{F}$ clusters at x if x is an element of all the sets in the filter $\mathscr{F}$.

Definition 2.4.4. A filter $\mathscr{F}$ is an ultrafilter if it is a maximal filter, i.e., if it is not properly contained in any other ultrafilter.

Remark 2.4.5. It is easy to see that a simple application of Zorn's lemma implies that every filter is contained in some ultrafilter.

Proposition 2.4.6. *[32] Let S be a topological space. Then $\mathscr{F}$ is an ultrafilter if and only if for all subsets $E \subseteq S$ one has $E \in \mathscr{F}$ or $S - E \in \mathscr{F}$.*

Proof. ($\Leftarrow$) Suppose that $\mathscr{F}$ is a filter such that for all $E \subseteq S$ we have $E \in \mathscr{F}$ or $(S - E) \in \mathscr{F}$, and suppose that $\mathscr{F}$ is included in some ultrafilter $\mathscr{F}$. Let E_2 be an element of $\mathscr{F}$ which is not in $\mathscr{F}$. Therefore by hypothesis, $(S - E_2)$ is in $\mathscr{F}$ and by inclusion we have that $S - E_2$ is also in $\mathscr{F}$. It follows that $E_2 \cap (S - E_2) = \emptyset$ is in $\mathscr{F}$ contradicting the fact that $\mathscr{F}$ is an ultrafilter. Therefore, $\mathscr{F}$ is already an ultrafilter.

($\Rightarrow$) Now suppose that $\mathscr{F}$ is an ultrafilter, but there exists a subset $E \subset S$ such that both E and $(S - E)$ are not in $\mathscr{F}$. Then $\mathscr{F} \cup \{E\}$ generate a filter which properly contains $\mathscr{F}$. This contradicts the fact that $\mathscr{F}$ is an ultrafilter, and the proof is complete. $\square$

Proposition 2.4.7. *Let S be a set, $\mathscr{F}$ be an ultrafilter on S and S_1, S_2 be two subsets of S. Suppose $S_1 \cup S_2$ is in $\mathscr{F}$. Then at least one of S_1 or S_2 is in $\mathscr{F}$.*

Proof. Without loss of generality, we can assume S_1 is not in $\mathscr{F}$. This is clear because if S_1 is already in $\mathscr{F}$, there is nothing to be proved. Then proposition 2.4.6 implies that $S - S_1$ is in $\mathscr{F}$, and by the first property of a filter we have that

$$(S_1 \cup S_2) \cap (S - S_1) = (S_1 \cup S_2) - S_1 \in \mathscr{F}.$$

Finally since

$$(S_1 \cup S_2) - S_1 \subseteq S_2,$$

it follows that $S_2 \in \mathscr{F}$. □

It is also possible to prove the previous result using the maximality of an ultrafilter. In that case we need to prove that if A is not in $\mathscr{F}$, then there exists $C \in \mathscr{F}$ which does not intersect with A. Therefore we have:

$$C \cap (A \cup B) = (C \cap A) \cup (C \cap B) = (C \cap B) \subseteq B \in \mathscr{F}.$$

To prove that such C exists, construct the sets

$$\mathscr{F}_0 = \{C \cap A : C \in \mathscr{F}\}$$

and

$$\mathscr{F}_1 = \{D \subseteq S : \exists C \in \mathscr{F}_0, C \subseteq D\}.$$

Since $\mathscr{F}_1$ does not contain the empty set, it is an ultrafilter and since $\mathscr{F} \subseteq \mathscr{F}_1$ this contradicts the maximality of $\mathscr{F}$. Therefore, such C exists.

2.5 Stone-Čech and Bohr compactifications

In this section, we define the Stone-Čech compactification of a semigroup, and the Bohr compactification of an abelian group.

Let S be a semigroup, then we define the *first Arens product* $\odot$ on $m(S)^*$ in the following way: Let μ and ν be two functionals in $m(S)^*$ then

$$(\mu \odot \nu)(f) = \mu(\nu'(f))$$

where

$$\nu'(f(s)) = \nu(l_s f).$$

The Arens product is associative, distributive, and weak*-weak* continuous in the first variable. We also have:

$$||\mu \odot \nu|| \leq ||\mu|| \cdot ||\nu||$$

which makes $m(S)^*$ a Banach algebra. For more information on the Arens product, see [1].

Definition 2.5.1. Let S be a semigroup. We define the *Stone-Čech compactification* of S to be the set βS of all multiplicative linear functionals on $m(S)$ with the Arens product and the weak* topology. Equivalently, βS is the spectrum of the commutative Banach algebra $m(S)$.

βS is compact with this topology, and βS is also a semigroup since the Arens product is associative. However βS is not a semi-topological semigroup in general. When S is a cancellative semigroup, then βS is a semi-topological semigroup if and only if S is finite [21] or [5].

We also have the following embedding of S into βS: If $s \in S$ then we have

$$s \longrightarrow \varepsilon_s$$

defined by

$$\varepsilon_s(f) = f(s).$$

Which gives $\varepsilon_s \in \beta S$.

Proposition 2.5.2. *[28] Let βS be the Stone-Čech compactification of the semigroup S, and let $a \in \beta S$. Then the function $\mu \in m(S)^*$ defined by $\mu(f) = \tilde{f}(a)$ where $\tilde{f}$ is the unique extension of f to βS is a multiplicative mean on $m(S)$.*

For more information on the Stone-Čech compactification, see [5] and [14]. We now want to define the Bohr compactification of a locally compact abelian group which will be useful in the next section. Let $(G, \star)$ be a locally compact abelian group. A character of G is a function $\gamma : G \to \mathbb{C}$ for which $|\gamma(x)| = 1$ for all $x \in G$, and

$$\gamma(x)\gamma(y) = \gamma(x \star y).$$

Let $\hat{G}$ be the set of all continuous characters of G. Then $(\hat{G}, +)$ is an abelian group if we define the addition by:

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x)$$

for all $\gamma_1, \gamma_2 \in \hat{G}$, and $x \in G$. This group with the topology of uniform convergence on compact subset is a locally compact abelian group called the dual group of G. As well known, if G is any locally compact abelian group, then $\hat{\hat{G}} = G$. See [29].

Since G is a locally compact abelian group, there exists a unique Haar measure (up to multiplication by a positive scalar) on G, and therefore we can talk about $L^1(G)$: the Banach algebra of all integrable functions on G with the convolution product. For any function $f \in L^1(G)$, we define its Fourier transform by:

$$\hat{f}(\gamma) = \int_G f(x)\gamma(-x)dx.$$

We define the Fourier algebra $A(\hat{G})$ by:

$$A(\hat{G}) = \{\hat{f} : f \in L^1(G)\}.$$

The topology on $\hat{G}$ is precisely the weak* topology defined by $L^1(G)$.

We can prove that if G is discrete, then $\hat{G}$ is compact, and if G is compact, then $\hat{G}$ is discrete [29].

Now let $\hat{G}_d$ be the group $\hat{G}$ with the discrete topology and therefore the dual group of $\hat{G}_d$ is a compact abelian group. Also, it can be verified that G is contained in the dual group of $\hat{G}_d$. The Bohr compactification $\overline{G}$ of the abelian group G is the dual group of $\hat{G}_d$.

2.6 Discrete semigroups

Let S be a semigroup, and let X be a compact Hausdorff space. A representation of S as continuous mapping from X into X is a semigroup homomorphism

$$\Phi : S \to C(X, X),$$

where $C(X, X)$ is the set of continuous functions from X into X with composition. Recall that Φ is a semigroup homomorphism if

$$\Phi(s_1 s_2) = \Phi(s_1)\Phi(s_2)$$

for all $s_1, s_2 \in S$.

Let S be a semigroup. We say that S has the *common fixed point property on compacta* if for every compact Hausdorff space X and every representation Φ of S as continuous mapping from X into itself, there exists a common fixed point of the family Φ.

Two elements $s_1, s_2 \in S$ have a *common right zero* if there exists an element $s_3 \in S$ such that:

$$s_1 s_3 = s_2 s_3 = s_3.$$

Finally a subset $A \subseteq S$ is *left thick* in S if for every finite subset $\sigma \subseteq S$ there exists $s \in S$ such that $\sigma s \subseteq A$. Mitchell [22] has proven that if A is left thick then we can choose s to be in A without changing the definition of left thick subsets.

Theorem 2.6.1. *[22] If V is any subset of a left amenable semigroup S, then V is left thick in S if and only if there exists a left invariant mean on S whose value on the characteristic function of V is one.*

Lemma 2.6.2. *[23] Let S be a semigroup and let $\eta : S \to \Phi$ be a homomorphism of S onto Φ, where Φ is a semigroup of continuous maps from X into X. If $y \in X$ then there exists $y_0 \in X$ such that for every open neighborhood U of y_0, $U \subseteq X$, the set*

$$\{s \in S : (\eta s)y \in U\}$$

is left thick in S.

Theorem 2.6.3. *Let S be a semigroup. Then the following are equivalent (see [23], [10], and [16]):*

(a) S is extremely left amenable.

(b) Whenever $A \subset S$ is left thick in S and $A = A_1 \cup A_2$, then at least one of A_1, A_2 is left thick in S.

(c) For each finite collection of subsets $A_i \subset S$ such that $S = \cup A_i$, then at least one of the A_i is left thick in S.

(d) S has the fixed point property on compacta.

(e) Each two elements of S has a common right zero.

(f) $\beta(S)$ has a right zero.

(g) For each subset $E \subset S$, either E is left thick in S or $S - E$ is left thick in S.

Proof.
(a $\Rightarrow$ b) We want to make use of theorem 2.6.1, so we need to construct a multiplicative left invariant mean on S whose value on the characteristic function of A_1 or A_2 is 1. Let μ be a multiplicative left invariant mean on S, and let $\{B_\gamma\}_\gamma$ be the collection of all finite subsets of S directed by inclusion. Therefore, since A is left thick in S, for all A_γ there exists $s_\gamma \in S$ such that $B_\gamma s_\gamma \subseteq A$. Let $Q : S \to \beta S$ be the map defined by $Qs(f) = f(s)$ for all $f \in m(S)$. Therefore, by w^*-compactness of βS it follows that there exists a subnet $\{Qs_\delta\}_\delta$ of $\{Qs_\gamma\}_\gamma$ which is w^*-convergent to some $\mu_1 \in \beta S$. Let

$$\mu_2 = \mu \odot \mu_1.$$

Since the Arens product of two multiplicative means is a multiplicative mean and since μ is left invariant, it follows that μ_2 is a multiplicative left invariant mean on S. If we want to show that μ_2 is the multiplicative left invariant mean we need to apply theorem 2.6.1. We have that for any $s \in S$:

$$\begin{aligned}
\mu_1'(1_A)(s) &= \mu_1(\ell_s 1_A) \\
&= w^* - \lim_\delta \left(Qs_\delta(\ell_s 1_A)\right) \\
&= w^* - \lim_\delta \left(\ell_s 1_A\right)(s_\delta) \\
&= w^* - \lim_\delta \left(1_A(ss_\delta)\right) \\
&= 1.
\end{aligned}$$

This last equality follows from the fact that any $s \in S$ is eventually in some of the A_δ which implies that ss_δ is eventually in A.

$$\begin{aligned}
\mu_2(1_A) &= \mu \odot \mu_1(1_A) \\
&= \mu(\mu_1'(1_A)) \\
&= \mu(1_S) \\
&= 1.
\end{aligned}$$

Now, since μ_2 is a multiplicative mean, its value on characteristic functions is 0 or 1, and since

$$1 = \mu_2(1_A) \leq \mu_2(1_{A_1}) + \mu_2(1_{A_2}),$$

it follows by applying theorem 2.6.1 that at least one of A_1 or A_2 is left thick in S.

(b $\Rightarrow$ c) Let

$$S = \cup_{i=1}^n A_i,$$

where $A_i \subseteq S$ for all i. If $n = 2$, by (b) we know that at least one of A_1 or A_2 would be left thick in S, since clearly S is left thick in itself. Now assuming that c is true for $n = k - 1$, we want to show that this is still true for $n = k$. We have that:

$$S = A_1 \cup \left(\cup_{i=2}^{k-1} A_i \right).$$

Therefore, by (b) at least one of A_1 or $\cup_{i=2}^{k-1} A_i$ is left thick in S. In both cases, this implies that at least one of the A_i is left thick in S by the induction hypothesis.

(c $\Rightarrow$ d) Let $\eta : S \to \Phi$ be a homomorphism onto, where Φ is a semigroup of continuous maps from a compact Hausdorff space X into itself. Let y be any element of X. We denote by Γ the family of all finite subsets A_γ of S directed upwards by inclusion and denote by Δ the family of all open neighborhoods Y_δ of y_0 directed downwards by inclusion, where y_0 is coming from lemma 2.6.2. Let

$$B_\delta = \{s \in S : (\eta(s))y \in Y_\delta\}.$$

For all $\psi = (\gamma, \delta)$, there exists $s_\psi \in S$ such that

$$A_\gamma s_\psi \subseteq B_\delta.$$

Therefore, for any $s \in S$, the net ss_ψ is eventually in each B_δ, which implies that the net $\eta(ss_\psi)y$ is eventually in each Y_δ. Thus we have for all $s \in S$:

$$y_0 = \lim_\psi \eta(ss_\psi)y.$$

Now, if s_0 is a specific element of S, then for all $s \in S$ we have:

$$\begin{aligned} \eta(s)y_0 &= \eta(s) \lim_\psi (\eta(s_0 s_\psi)y) \\ &= \lim_\psi (\eta(s)\eta(s_0 s_\psi)y) \\ &= \lim_\psi (\eta(s s_0 s_\psi)y) \\ &= \lim_\psi (\eta(s_0 s_\psi)y) \\ &= y_0. \end{aligned}$$

Therefore, y_0 is a fixed point of the family Φ.

(d $\Rightarrow$ a) Let $\eta : S \to \Phi$ be the homomorphism defined by $\eta(s) = \ell_s^*$, where Φ is a semigroup of continuous maps from βS into βS. Then, by (d), there is a fixed point $\mu_0 \in \beta S$. Therefore μ_0 is clearly a multiplicative left invariant mean for the semigroup S.

(a $\Rightarrow$ e) On the semigroup S, we define the following congruence relation (i.e. an equivalence relation for which $a \sim b \Rightarrow ac \sim bc$ and $ca \sim cb$). For any $a, b \in S$:

$$a \sim b \iff \text{there exists } c \in S \text{ such that } ac = bc.$$

For any element $s \in S$, we denote its equivalence class by $\dot{s}$. Let

$$\dot{S} = \{\dot{s} : s \in S\}$$

and let

$$F : S \to \dot{S}$$

be the homomorphism defined by $F(s) = \dot{s}$ for any $s \in S$. Then $\dot{S}$ is a right cancellative extremely left amenable semigroup. By [10, corollary 2], we have that $(\dot{s})^2 = \dot{s}$ for all elements $\dot{s} \in \dot{S}$. Therefore, it follows that $\dot{b}(\dot{a})^2 = \dot{b}\dot{a}$, which implies

$$\dot{b}\dot{a} = \dot{b}$$

for all $\dot{a}, \dot{b} \in \dot{S}$. This implies that $\dot{a}\dot{S} = \dot{a}$ for all $\dot{a} \in \dot{S}$. Therefore, if $\dot{a}, \dot{b} \in \dot{S}$ such that $\dot{a} \neq \dot{b}$ we have that:

$$\dot{a}\dot{S} \cap \dot{b}\dot{S} = \emptyset,$$

which is impossible, since any two right ideals of an extremely left amenable semigroup have a non-void intersection. This implies that $\dot{S}$ contains only one element, and therefore each two elements of S have a common right zero.

(a $\Rightarrow$ f) If S is extremely left amenable, then S is in particular left amenable, so we can apply [6, Corollary 4], and it follows that if μ is a left invariant mean on S, then for any $\phi \in \beta S$:

$$\phi \odot \mu = \mu.$$

(c $\Rightarrow$ g) If E is any subset of S, then we have that:

$$S = E \cup (S - E),$$

so by applying (c), it follows that E or $S - E$ is left thick in S.

(f $\Rightarrow$ a) Suppose that βS has a right zero μ. Then for any $\phi \in \beta S$, we have that:

$$\phi \odot \mu = \mu$$

and in particular, since $1_s \in \beta S$ for any $s \in S$, we have that:

$$1_s \odot \mu = \mu.$$

But we also have that:

$$\begin{aligned} 1_s \odot \mu(f) &= 1_s(\mu'(f)) \\ &= (\mu'(f))(s) \\ &= \mu(\ell_s(f)). \end{aligned}$$

Therefore, it follows that for all $s \in S$:

$$\ell_s^* \mu = \mu,$$

which implies that μ is a multiplicative left invariant mean on S.

(e $\Rightarrow$ b) By induction, we have that for every finite subset $A \subseteq S$ there exists $a \in S$ such that $Aa = \{a\}$. Suppose S does not satisfy condition (b). Then there exists subsets $A_1, A_2 \subseteq S$ such that A_1, A_2 are not left thick in S, but the subset $A_1 \cup A_2$ is left thick in S.

Thus there exists finite subsets $A_1', A_2' \subseteq S$ such that:

$$A_1's \not\subseteq A_1$$

$$A_2's \not\subseteq A_2$$

for all $s \in S$. Let $A' = A_1' \cup A_2'$. Since A' is finite, there exists $a \in S$ such that $A'a = \{a\}$. But also we have that $A_1 \cup A_2$ is left thick, so there exists an element $b \in S$ such that $ab \in A_1 \cup A_2$, which implies that $ab \in A_1$ or $ab \in A_2$. Assume $ab \in A_1$ then:

$$A_1'(ab) \subseteq A'(ab) = \{ab\} \subseteq A_1,$$

which is a contradiction.

(g $\Rightarrow$ a) For this part of the proof, see [16]. $\square$

An interesting question when we are talking about semigroups, is to consider the special case when the semigroup is a group. Using the preceding theorem, one can easily prove that no group is extremely left amemable. In fact Mitchell proved a little more than that in theorem 2.6.4.

Theorem 2.6.4. *A two-sided cancellation semigroup S is extremely left amenable if and only if S is the trivial group.*

The following proof can be found in [23].

Proof. Suppose first that G is an abelian group which is extremely left amenable. Then G is embedded in its Bohr compactification $\overline{G}$. Then G acts on $\overline{G}$ by left multiplication. Therefore the representation Φ of G defined by:

$$\Phi(g)\overline{g} = g\overline{g}, \quad (g \in G, \overline{g} \in \overline{G})$$

has a fixed point. Therefore, there exists an element $\overline{g_0}$ in $\overline{G}$ such that $g\overline{g_0} = \overline{g_0}$ for all g in G. But since G is a group, we can simplify, which gives $g = e$ for all $g \in G$, where e is the identity in G. Therefore G is the trivial group.

Now, suppose G is an extremely amenable groupimplies that there (not necessary abelian), and let H be a subgroup of G generated by only one element of G. Therefore H is abelian, and since every closed subgroup of an extremely left amenable group is extremely left amenable, it follows that H is an extremely left amenable abelian group. Therefore, by the first part of the proof, H is the trivial group. But since H could be any subgroup of G generated by one element, it follows that G is also trivial.

Finally suppose S is a two sided cancellation semigroup which is extremely left amenable. S can be embedded in some group G' [31, corollary 3.6]. Consider G the subgroup of G' generated by S, and consider $\pi : G \to S$ to be the restriction map. Then if μ is a multiplicative left invariant mean on S then $\pi^*\mu$ is a multiplicative left invariant mean on G. Since G is an extremely left amenable group, it is trivial. It follows that S is also trivial.

□

Example 2.6.5. Let $(S, \vee, \wedge)$ be a lattice and consider the semigroup $(S, \vee)$. For any two elements $s_1, s_2 \in S$, let $s' = s_1 \vee s_2$. Then we have $s_1 \vee s' = s_2 \vee s'$, therefore by Mitchell [22, corollary 6(a)] any two elements of S have a common right zero, and by theorem 2.6.3 it follows that S is extremely left amenable [23], [10]. Recall that $(S, \vee, \wedge)$ is a lattice if for any $a, b \in S$ we have:

Commutative laws:	$a \vee b = b \vee a$	$a \wedge b = b \wedge a$
Associative laws:	$a \vee (b \vee c) = (a \vee b) \vee c$	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$
Absorption laws:	$a \vee (a \wedge b) = a$	$a \wedge (a \vee b) = a$

Example 2.6.6. [23] Let X be an infinite set, and let S be the semigroup of maps $s : X \to X$ for which the set $\{x \in X : s(x) \neq x\}$ is finite, with the composition of functions. Then for any $s_1, s_2 \in S$ define the set Y by:

$$Y = \{x \in X : s_1(x) \neq x\} \cup \{x \in X : s_2(x) \neq x\}.$$

Let $x_0 \in X - Y$, and define the map $s' \in S$ by:

$$s'(x) = x \text{ if } x \in X - Y,$$

$$s'(x) = x_0 \text{ if } x \in Y.$$

Then we have

$$s_1 s' = s' = s_2 s'.$$

Theorem 2.6.3 can now be applied to show that S is extremely left amenable.

2.7 Semi-topological semigroups

Let S be a semi-topological semigroup. Since the space $m(S)$ does not reflect the topology of S, we use a more appropriate space than $m(S)$ to reflect the topology of S.

Definition 2.7.1. Let S be a semi-topological semigroup. Then S is *extremely left amenable* if the space $LUC(S)$ has a multiplicative left invariant mean.

Lemma 2.7.2. *[26] Let S be a semi-topological semigroup. Then $LUC(S)$ is a translation-invariant closed sub-algebra of $m(S)$ that contains the constant functions.*

Theorem 2.7.3. *If G is a locally compact group, then G is extremely left amenable if and only if G is the trivial group.*

The proof of this result is due to Granirer and Lau [11] and will not be given here. However, many topological groups are extremely left amenable.

Example 2.7.4. [12] The unitary group of an infinite dimensional Hilbert space with the strong operator topology is extremely left amenable. This is one of the first examples to have been constructed and has been found by Gromov and Milman.

Example 2.7.5. [9] The group $L^0(\mathbb{I}, U(1))$ of measurable maps from the standard Lebesgue space to the circle rotation group $U(1)$, equipped with the topology of convergence in measure, is extremely left amenable.

Example 2.7.6. All generalized Levy groups are extremely left amenable. For the definition of a generalized Levy group and for the proof, see [27] and [8].

Now we want to characterize which semigroups S are extremely left amenable. The following theorem is due to Mitchell [25], and it shows that like for discrete semigroups, the extreme left amenability of a semi-topological semigroup is related to some a fixed point property. But first, we need a definition and another theorem due to Mitchell.

Definition 2.7.7. Let S be a semigroup, X be a subset of $m(S)$, and Y be a compact Hausdorff space. Let $\eta : S \to \Phi$ be a homomorphism of S onto Φ a semigroup of continuous maps from Y into Y. Then Φ is a *D-representation* of S, X on Y if the set

$$Y' = \{y \in Y : Ty(C(Y)) \subseteq X\}$$

is dense in Y, where the map Ty is defined by:

$$(Tyh)(s) = h((\eta s)y)$$

for $h \in C(Y)$ and $s \in S$. We say that S, X has the common fixed point property on compacta with respect to D-representations if, for each compact Hausdorff space Y and for each D-representation of S, X on Y, there exists in Y a common fixed point of the family Φ.

Theorem 2.7.8. *[24] Let S be a semi-topological semigroup and X a translation invariant closed subalgebra of $m(S)$ that contains the constant functions. Then the following are equivalent:*

(a) X has a multiplicative left invariant mean.

(b) S, X has the common fixed point property on compacta with respect to D-representations.

Theorem 2.7.9. *[25] Let S be a semi-topological semigroup. Then S is extremely left amenable if and only if, whenever S acts on a compact Hausdorff space Y, where the action is jointly continuous, then there is a fixed point for S in X.*

Proof.

($\Rightarrow$) Let S be an extremely left amenable semi-topological semigroup, and suppose S acts on a compact Hausdorff space Y where the action is jointly continuous. Then we want to show that this action is a D-representation, which by theorem 2.7.9 implies that there is a fixed point. For any $y \in Y$, define the map:

$$Ty : C(Y) \to m(S)$$

by:

$$(Tyh)(s) = h(sy).$$

So we need to show that the set:

$$Y' = \Big\{ y \in Y : Ty(C(Y)) \subseteq LUC(S) \Big\}$$

is dense in Y. For any $h \in C(Y)$ let $f = Tyh$. We want to show that $f \in LUC(S)$. Suppose not, then there exists a net $\{s(\gamma)\}$ in S which converges to $s \in S$ such that $\ell_{s(\gamma)} f$ does not converge uniformly to $\ell_s f$. Then there exists a real number $a > 0$, a subnet $\{s(\delta)\}$ of $\{s(\gamma)\}$, and a net $\{t(\delta)\}$ in S such that:

$$\Big| h(s(\delta)(t(\delta)y)) - h(s(t(\delta)y)) \Big| \geq a.$$

We denote $t(\delta)y$ by $y(\delta)$. By compactness of Y, the net $y(\delta)$ has a convergent subnet $y(\eta)$ which converges to $y_0 \in Y$. Therefore we have:

$$\begin{aligned} a &\leq \lim_{\eta} \Big| h(s(\eta)y(\eta)) - h(sy(\eta)) \Big| \\ &= \Big| h(sy_0) - h(sy_0) \Big| \\ &= 0, \end{aligned}$$

which is a contradiction. It follows that $f \in LUC(S)$, and the action is a D-representation, therefore there is a fixed point.

($\Leftarrow$) Let Y be the sets of multiplicative left invariant means on $LUC(S)$ and define the action $S \times Y \to Y$ by:

$$s \cdot \mu = \ell_s^* \mu$$

for any $s \in S$ and $\mu \in Y$. This is clearly an action, so we only need to prove that it is jointly continuous. Suppose for now that it is jointly continuous, then there exists a fixed point $\mu_0 \in Y$. μ_0 would then be a multiplicative left invariant mean on $LUC(S)$ which shows that if the action if jointly continuous, then S is extremely left amenable.

Now to prove that the action is jointly continuous, let $\{s(\gamma)\}$ and $\{\mu(\delta)\}$ be a net respectively in S and Y, such that $s(\gamma) \to s$ and $\mu(\delta) \to \mu$. Then for any $f \in LUC(S)$ we have:

$$\begin{aligned}
0 &\leq \lim_{\gamma,\delta} \left|(s(\gamma) \cdot \mu(\delta))f - (s \cdot \mu)f\right| \\
&= \lim_{\gamma,\delta} \left|(\mu(\delta)(\ell_{s(\gamma)}f - \ell_s f)) + ((\mu(\delta) - \mu)\ell_s f)\right| \\
&\leq \lim_{\gamma} ||\ell_{s(\gamma)}f - \ell_s f|| + \lim_{\delta} |\mu(\delta) - \mu)\ell_s f| \\
&= 0.
\end{aligned}$$

Therefore:

$$s(\gamma) \cdot \mu(\delta) \to s \cdot \mu,$$

which proves that the action is jointly continuous. □

2.8 Subalgebras of $m(S)$

When S is a semi-topological semigroup, we have already defined the space $LUC(S)$ of left uniformly continuous functions on S.

In the same way in which we defined the space $LUC(S)$, we can define the space $RUC(S)$ of right uniformly continuous functions on a semigroup S. We do that in the obvious way by changing the left shift by the right shift in the definition of the left uniformly continuous functions. Using these two spaces, we can now define the space $UC(S)$ of uniformly continuous functions by taking the intersection between these two spaces.

Definition 2.8.1. The space of uniformly continuous functions $UC(S)$ is the intersection between both spaces $LUC(S)$ and $RUC(S)$:

$$UC(S) = LUC(S) \cap RUC(S).$$

By changing the topology in the definition of the space of left uniformly continuous functions on S, we can also define the space of weak left uniformly continuous functions $WLUC(S)$:

Definition 2.8.2. Let S be a semi-topological semigroup, and let f be a function in $m(S)$. Then f is weakly left uniformly continuous (i.e. $f \in WLUC(S)$) if the map $\theta_s : f \to l_s f$ is continuous in the weak topology of $m(S)$.

From these definitions, it is easily noticeable that we have the following inclusion between $m(S)$, $LUC(S)$, WLUC(S) and $UC(S)$:

$$UC(S) \subseteq LUC(S) \subseteq WLUC(S) \subseteq m(S).$$

In particular, if S is a discrete semigroup we have that all these spaces are actually identical:

$$UC(S) = LUC(S) = RUC(S) = WLUC(S) = m(S).$$

2.9 Action of semigroup

A left action of a semigroup S on a topological space X is a mapping $S \times X \to X$ denoted by $(s, x) \to s \cdot x,\ s \in S, x \in X$ such that:

- $f_s : X \to X$ defined by $f_s(x) = s \cdot x$ is continuous for each $s \in S$.
- $(st) \cdot x = s(t \cdot x)$ for all $s, t \in S$ and all $x \in X$.

Similarly, we can define the right action $X \times S = X$ of a semigroup. When we have an action of a semigroup S on a topological space X, we can define the left and right action of a function $f \in CB(X)$ by an element $s \in S$ in the following way:

- ${}_sf(x) = f(s \cdot x), \ x \in X$, for the left action.
- ${}^sf(x) = f(x \cdot s), \ x \in X$, for the right action.

Let S be a semigroup that acts on a Hausdorff space X. A closed subalgebra A of $m(X)$ is said to be S-translation invariant if for each $f \in A$ and $s \in S$ we have ${}_sf \in A$.

Using the definition of left action, we can define the space of left uniformly continuous functions $LUC(S, X)$ (or in the same way, the space of right uniformly functions) in the following way:

Definition 2.9.1. Let S be a semigroup which acts on X a topological space. Let f be a function in the space $CB(X)$ of bounded continuous real valued functions on S. Then f is left uniformly continuous if the map:

$$s \to {}_sf$$

is continuous in the norm topology of $CB(X)$.

Notice that $LUC(S, X)$ is S-translation subalgebra of $m(X)$ for any semigroup S which acts on a Hausdorff space X.

Definition 2.9.2. Let S be a semigroup which acts on a topological space X, let A be a S-translation invariant closed subalgebra of $m(X)$, and let $\mu \in A^*$ be a mean on A. Then we say that μ is S-translation invariant if $\mu({}_sf) = \mu(f)$ for all functions $f \in LUC(S, X)$.

If S is a semi-topological semigroup which acts on itself by left action, then both spaces $LUC(S)$ and $LUC(S, S)$ coincide. As a reference for this section, we suggest the book [2] which inspired the work in this section.

2.10 n-extreme left amenability

A way to extend the notion of extremely left amenable semigroup is to consider n-extreme left amenability. Instead of considering the existence of a left invariant mean, we will consider the existence of a set of n means which is invariant by left translation. See [18] and [17].

Definition 2.10.1. A semigroup S is n-extremely left amenable if there exists a subset F of $\beta(S)$ such that $|F| = n$ and F is minimal with respect to the property $\ell_a^*(F) = F$ for all $a \in S$.

Remark 2.10.2. Suppose there exist two subsets F_1 and F_2 of $\beta(S)$ which are minimal with respect to the property $\ell_a^*(F_i) = F_i$ for $i = 1, 2$ and for all $a \in S$. Then $|F_1| = |F_2|$.

Proposition 2.10.3. *A semigroup S is k-extremely left amenable for some $k \leq n$ if and only if there exists a left invariant mean on $m(S)$ which is the convex combination of n multiplicative means.*

Let us now give some examples of semigroups which are n-extremely left amenable:

Example 2.10.4. [18] Let S be a finite group of order n, then S is n-extremely left amenable.

Example 2.10.5. [18] If S is a left amenable finite semigroup, then S is n-extremely left amenable for some integer n.

Example 2.10.6. [18] If S is any extremely left amenable semigroup, and G a group of order n, then $S \times G$ is n-extremely left amenable.

It is easy to see that all groups of order n are n-extremely left amenable. A result much harder and surprising is that we can prove that if G is a locally compact group, then the only n-extremely left amenable groups are in fact any group of order n [11].

The last example we just presented is particularly interesting, since it allow us to construct a wide variety of n-extremely left amenable semigroups.

Let S be a semi-topological semigroup, and let $F \subseteq \beta(S)$ such that $\ell_a^* F = F$ for all $a \in S$. We define an equivalence relation R by aRb if and only if $\ell_a^* \phi = \ell_b^* \phi$ for all $\phi \in F$. We denote S/F the factor semigroup defined by this relation.

Let us complete this chapter by an interesting theorem on the structure of n-extremely left amenable semigroup:

Theorem 2.10.7. *[18] If S is a semi-topological semigroup such that $LUC(S)$ is n-extremely left amenable, then there exists a collection F of n disjoint open and closed subsets of S, with union S, such that $1_A \in LUC(S)$ for all $A \in F$ and F is the decomposition of S by cosets of S/H for any finite subsets $H \subseteq \beta S$ satisfying $L_a H = H$ for all $a \in S$.*

Chapter 3

Main Results

3.1 Introduction

In this chapter, we will present mostly new results on extreme left amenability that we found during our research.

We start this chapter by giving a characterization of extremely left amenable semigroups using the notion of ultrafilter, a notion that we defined earlier in chapter 2.

In section 3.3, we define the semi-direct product of two semigroups and present results relating the semi-direct product and extreme left amenability of semigroups.

In section 3.4 and 3.5, we explore results related to density. In section 3.2 we first find analogue results from Lau [19] related to dense subsemigroup of a topological group. We prove that these results are also valid for extreme left amenability. In section 3.4, we prove some results relating dense subsemigroup and n-extreme left amenability.

We already know that the homomorphic image of an extreme left amenable semigroup is extremely left amenable. At the end of the last section of this chapter, we prove a similar result, but for n-extremely left amenable subalgebras of $CB(S)$.

3.2 Left thick subset and ultrafilters

The goal of this section is to characterize discrete semigroups which are extremely left amenable by the way of their left thick subsets by using the notion of ultrafilters.

Theorem 3.2.1. *Consider a discrete semigroup S, and let $\mathscr{F}$ be the collection of all left thick subsets of S. Assume that we can write:*

$$\mathscr{F} = \cup_{i=1}^{n} \mathscr{F}_i,$$

where all $\mathscr{F}_i$ are ultrafilters. Then S is extremely left amenable.

Proof. Suppose $S_1 \cup S_2$ is a left thick subset of S. Then $S_1 \cup S_2$ is contained in an ultrafilter $\mathscr{F}_i$ which contains only left thick subsets of S. By proposition 2.4.7, since $\mathscr{F}_i$ is an ultrafilter, we know that at least one of S_1 or S_2 is in $\mathscr{F}_i$. Then at least one of S_1 or S_2 is left thick. From theorem 2.6.3 $(b) \Rightarrow (a)$ it follows that the semigroup S is extremely left amenable. □

Example 3.2.2. Consider the semigoup S containing only two elements: $S = \{a,\ b\}$ with the following multiplication table:

↱	a	b
a	a	b
b	a	b

Then it is easy to see that there are only two ultrafilters on S, namely:

$$V_a = \{\{a\}, \{a, b\}\},$$

$$V_b = \{\{b\}, \{a, b\}\}.$$

By the definition of the semigroup S, it is also easy to see that the only left thick subsets of S are:

$$\{a\}, \{b\} \text{ and } \{a, b\}.$$

Therefore, we can write the set of all left thick subsets of S as the union of ultrafilter on S:

$$\{\{a\},\{b\},\{a,b\}\} = V_a \cup V_b.$$

Therefore, by the theorem above, the semigroup S is extremely left amenable.

Now, we want to investigate the possibility that the converse or at least a partial converse would be true. This part of the work is inspired mostly by [28]. In particular, we correct a gap in a proof in [28].

Lemma 3.2.3. *Consider βS as the Stone-Čech compactification of a semigroup S, and let $a \in S$. Then the set*

$$V_a = \{V \cap S : V \textit{ is a neighborhood of } a \textit{ in } \beta S\}$$

is an ultrafilter on S.

Proof. For a proof of this lemma, see [3] □

Lemma 3.2.4. *Let S be a semigroup, and βS its Stone-Čech compactification. For any element $s \in S$ we define $\varepsilon_s \in \beta S$ by: $\varepsilon_s(f) = \tilde{f}(s)$ where $\tilde{f}$ is the unique extension of $f \in m(S)$ to βS. Then the map $\tilde{\ell}_s : \beta S \to \beta S$ defined by $\tilde{\ell}_s\mu = \varepsilon_s \odot \mu$ is the only continuous extension of ℓ_s to βS.*

Theorem 3.2.5. *[28] If S is an extremely amenable semigroup, then there exists at least one ultrafilter on S such that all elements of this ultrafilter are left thick in S.*

Proof. Since S is extremely left amenable, by theorem 2.6.3, we know that βS has a right zero. Assume a is a right zero for βS. Then we want to show that the ultrafilter

$$V_a = \{V \cap S : V \text{ is a neighborhood of } a \text{ in } \beta S\}$$

contains only left thick subsets of S.

Let $\sigma_1 \cap S$ be an element of V_a, and let σ_2 be any finite subset of S. Then we want to find an element $v_2 \in S$ such that $\sigma_2 v_2 \subseteq \sigma_1$. Since σ_2 is finite, we can write $\sigma_2 = \{s_1, s_2, s_3, ..., s_n\}$.

Let W be a neighborhood of a in βS. Then since $\tilde{\ell}_{s_i}$ is continuous for all $i \in \{1, 2, 3, ..., n\}$, we can find neighborhoods U_i of a such that $\tilde{\ell}_{s_i}(U_i) \subseteq W$ for all $i \in \{1, 2, ..., n\}$. Let v_2 be any element in $S \cap U_1 \cap U_2 \cap ... \cap U_n$, then $\sigma_2 v_2 \subseteq (\sigma_1 \cap S)$. □

Remark 3.2.6. In the previous proof, the set $S \cap U_1 \cap U_2 \cap ... \cap U_n$ was non-empty because all U_i are neighborhoods of a, therefore contain a, and a finite intersection of open sets is open. Thus $U_1 \cap U_2 \cap ... \cap U_n$ is an open set in βS that contains a and therefore non-empty. Now we know that a might not be in S, but all open sets in βS need to intersect with some element of S. Therefore $S \cap U_1 \cap U_2 \cap ... \cap U_n$ is non-empty.

Remark 3.2.7. Note that there is a gap in the above proof when presented in [28]. There, they chose v_2 to be an element of $W \cap U_1 \cap U_2 \cap, ..., U_n$, but in that case, there does not seem to be any reason why v_2 need to be in S.

Theorem 3.2.8. *[28] Let S be an extremely left amenable semigroup, and let $E \subset S$ be left thick in S. Then E is contained in some ultrafilter U which contains only left thick subsets in S.*

Proof. Suppose S is extremely left amenable. Then there exists a right zero $a \in \beta S$ of βS.

Now we want to construct the net x_σ in the following way: for each finite subset $\sigma \subseteq S$, let x_σ be an element of E such that $yx_\sigma \in E$ for all elements y in σ. The index set of the net $\{x_\sigma : \sigma \in I\}$ is I, the set of finite subsets of S ordered by upward inclusion.

Then $\{x_\sigma : \sigma \in I\}$ as at least one cluster point. Let b be one of these cluster points and define $\tilde{r}_b(a)$ to be the only continuous extension of the right shift map r_b of the set βS. Let z be defined by

$$z = \tilde{r}_b(a).$$

Then the set

$$V_z = \{V \cap S : V \text{ is a neighborhood of } z \text{ in } \beta S\}$$

is an ultrafilter which contains only left thick subsets of S. □

3.3 Semidirect product and extreme left amenability

We begin this section by defining the notion of a semidirect product of two semigroups. We will give some results relating this product to extreme left amenability.

Let U and T be two semigroups. Then the direct product of U and T is defined to be the set:

$$S = U \times T = \{(u,t) : u \in U, t \in T\}$$

with the product:

$$(u_1,t_1)(u_2,t_2) = (u_1u_2, t_1t_2),$$

where $u_1, u_2 \in U$ and $t_1, t_2 \in T$.

The semidirect product of two semigroups is a generalization of this concept. Let $End(U)$ denote the set of all endomorphisms of U, let $Sur(U)$ denotes the set of all surjective endomorphisms of U, and let ρ be an homomorphism $\rho : T \longrightarrow End(U)$.

The semidirect product $U \rtimes_\rho T$ is the direct product, with multiplication defined by:

$$(u_1,t_1)(u_2,t_2) = (u_1\rho_{t_1}(u_2), t_1t_2)$$

for any (u_1,t_1) and (u_2,t_2) in S. It is easy to show that the set $U \times T$ with this product is a semigroup.

Now, we want to see how we can relate the semidirect product and extreme left amenability. First we need to introduce some notation. Let U and T be two semigroups, and let ρ be a homomorphism $\rho : T \longrightarrow End(U)$. We will usually write ρ_a instead of $\rho(a)$, and we define for all $a \in T$ the linear operator P_a on $m(U)$ by:

$$P_a g(u) = g(\rho_a u).$$

We define P_a^* to be the adjoint operator, i.e.,

$$P_a^* \mu(g) = \mu(P_a g).$$

In Klawe [15] it has been proved that:

Theorem 3.3.1. *If U and T are two left amenable semigroups with a homomorphism $\rho : T \to Sur(U)$, then the semigroup $S = U \rtimes_\rho T$ is also left amenable.*

The proof of this theorem is based on the following lemma, which can also be found in [15]:

Lemma 3.3.2. *If U and T are left amenable semigroups with a homomorphism $\rho : T \to Sur(U)$, then there exists a left invariant mean ϕ on $m(U)$ such that $P_a^*\phi = \phi$ for each $a \in T$.*

Now we want to relate this result to extremely left amenable semigroups. If U and T are two extremely left amenable semigroups, then from the previous lemma we know that there exists a left invariant mean ϕ on $m(U)$ such that $P_a^*\phi = \phi$ for all $a \in T$. Suppose that we can choose this mean to be also multiplicative, then we have the following result:

Theorem 3.3.3. *Let U and T be two extremely left amenable semigroups, and let ρ be a homomorphism $\rho : T \to Sur(U)$. Assume that there exists a multiplicative left invariant mean ϕ on $m(U)$ such that $P_a^*\phi = \phi$ for all $a \in T$. Then the semigroup $S = U \rtimes_a T$ is extremely left amenable.*

Proof. Let ν be a multiplicative left invariant mean on $m(T)$ and define for all $f \in m(S)$ the function $\overline{f}$ on $m(T)$ by $\overline{f}(a) = \phi(f_a)$, where $f_a(u) = f(u, a)$. Then we want to show that $\mu \in m(S)^*$ defined by $\mu(f) = \nu(\overline{f})$ for all $f \in m(S)$ is a multiplicative left invariant mean on $m(S)$, which will show that S is also extremely left amenable.

First, we want to show that μ is a mean. If $f \in m(S)$, then we have:

$$||f|| = \sup_{(u,a)\in S} |f(u,a)| \geq \sup_{u\in U} |f_a(u)| = ||f_a||$$

and clearly we have:

$$||f|| = \sup_{a\in T} ||f_a||.$$

From that equality, it follows that for $f \in m(S)$ we have:

$$\begin{aligned} ||\overline{f}|| &= \sup_{a\in T} |\overline{f}(a)| \\ &= \sup_{a\in T} |\phi(f_a)| \\ &\leq \sup_{a\in T} ||\phi|| \cdot ||f_a|| \\ &= \sup_{a\in T} ||f_a|| \\ &= ||f||. \end{aligned}$$

Finally, this implies that:

$$\begin{aligned} ||\mu|| &= \sup_{||f||=1} |\mu(f)| \\ &= \sup_{||f||=1} |\nu(\overline{f})| \\ &= \sup_{||\overline{f}||\leq 1} |\nu(\overline{f})| \\ &= ||\nu|| \\ &= 1, \end{aligned}$$

which proves that μ is a mean, since clearly $\mu(e) = 1$.

We now want to show that μ is multiplicative. Let f, g be two functions in $m(S)$, and let $a \in T, u \in U$ then we have:

$$(fg)_a(u) = (fg)(u, a) = f(u, a)g(u, a) = f_a(u)g_a(u),$$

which implies that we also have:

$$\overline{fg}(a) = \phi((fg)_a) = \phi(f_a g_a) = \phi(f_a)\phi(g_a) = \overline{f}(a)\overline{g}(a).$$

It follows that:

$$\mu(fg) = \nu(\overline{fg}) = \nu(\overline{f}\overline{g}) = \nu(\overline{f})\nu(\overline{g}) = \mu(f)\mu(g)$$

which shows that μ is multiplicative. We now need to show that μ is also left invariant, which will prove that S is extremely left amenable. We notice that:

$$\begin{aligned}(\ell_{(v,b)})f_a(u) &= \ell_{(v,b)}f(u,a)\\ &= f(v\rho_b(u), ba)\\ &= f_{ba}(v\rho_b(u))\\ &= P_b\ell_v f_{ba}(u).\end{aligned}$$

Therefore, we also have that:

$$\overline{\ell_{(v,b)}f}(a) = \phi(P_b\ell_v f_{ba}) = \phi(f_{ba}) = \ell_b\overline{f}(a),$$

which finally implies that:

$$\mu(\ell_{(v,b)}f) = \nu(\overline{\ell_{(v,b)}f}) = \nu(\ell_b\overline{f}) = \nu(\overline{f})\mu(f),$$

which proves that S is extremely left amenable. ☐

Theorem 3.3.4. *Let U and T be semigroups, and let $\rho : T \to End(U)$ be a homomorphism. Then, if $S = U \rtimes T$ is extremely left amenable, both T and U are extremely left amenable.*

Proof. For T, this follows directly from the fact that the map $\sigma_1 : S \to T$ defined by $\sigma_1(u,a) = a$ is a surjective homomorphism.

Now to show that U is extremely left amenable, we can suppose without loss of generality that T has a two-sided identity (see [15, Remark 3.7]). For each $f \in m(U)$, we define $\tilde{f} \in m(S)$ by:

$$\tilde{f}(u,a) = f(u)$$

for $u \in U$ and $a \in T$. Then, it is easy to see that we have:

$$(\ell_v f)^\sim = \ell_{(v,1)}\tilde{f}.$$

Let ν be a multiplicative left invariant mean on $m(S)$, then we define $\mu \in m(U)^*$ by:

$$\mu(f) = \nu(\tilde{f})$$

and this is easy to see that μ is a multiplicative left invariant mean on $m(U)$ since:

$$\mu(\ell_v f) = \nu((\ell_v f)\check{}) = \nu(\ell_{(v,1)}\tilde{f}) = \nu(\tilde{f}) = \mu(f),$$
$$\mu(fg) = \nu((fg)\check{}) = \nu(\tilde{f}\tilde{g}) = \nu(\tilde{f})\nu(\tilde{g}) = \mu(f)\mu(g).$$

□

3.4 Dense subsemigroup of a topological group

Let S be a topological semigroup which is a dense semigroup of a topological group G. It is easy to see that if $UC(S)$ has a multiplicative left invariant mean, then $UC(G)$ has one too. Just notice that if f is a function in $UC(G)$, then the restriction of f to S is a function in $UC(S)$. Therefore, we can define a multiplicative left invariant mean $\tilde{\mu}$ on $UC(G)$ by:

$$\tilde{\mu}(f) = \mu(f|_S),$$

where μ is a multiplicative left invariant mean on $UC(S)$ and f is a function in $UC(G)$. What we want to do in this section is go further by proving a partial converse to this result. This has been done by Lau in [19] for the case when $UC(G)$ has a left invariant mean. First we need to recall the definition of the finite intersection property for left (right) ideals and give a lemma, from [19].

Definition 3.4.1 (Finite intersection property for right ideals)**.** Let S be a semigroup. We say that S has the finite intersection property for right ideals if any finite collection of right ideals of S has a non-empty intersection. We can define in the same way the finite intersection property for left ideals.

Lemma 3.4.2. *If S is a dense subsemigroup of a topological group G then:*

1. *For each $f \in UC(S)$, there exists $F \in CB(G)$ such that $F|_S = f$.*
2. *If S has the finite intersection property for right ideals, then for each $f \in UC(S)$, there exists $F \in LUC(G)$ such that $F|_S = f$.*

3. *If S has the finite intersection property for right ideals and has the finite intersection property for left ideals, then for each $f \in UC(S)$ there exists $F \in UC(G)$ such that $F|_S = f$.*

Theorem 3.4.3. *Let G be a topological group, and S be a dense subsemigroup of G. If $CB(G)$ has a multiplicative left invariant mean, then $UC(S)$ has a multiplicative left invariant mean as well.*

Proof. From lemma 3.4.2 we know that if $f \in UC(S)$ there exists a unique function $\tilde{f} \in UC(G)$ such that $\tilde{f}|_S = f$. Let φ be a multiplicative left invariant mean on $CB(G)$ and define $\tilde{\varphi} \in UC(S)^*$ by $\tilde{\varphi}(f) = \varphi(\tilde{f})$. We want to prove that $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. First, because of the way we define $\tilde{\varphi}$, it is easy to see that

$$||\tilde{\varphi}|| = ||\varphi|| = 1$$

and

$$\tilde{\varphi}(1_G) = \varphi(1_S) = 1.$$

Therefore, $\tilde{\varphi}$ is a mean on $UC(S)$. Now we want to show that $\tilde{\varphi}$ is left invariant: We first notice that if $f \in C(S)$ and $s \in S$ then we have:

$$l_s\tilde{f} = l_s f = \tilde{l_s} f.$$

Therefore, we have:

$$\tilde{\varphi}(l_s f) = \varphi(l_s f)\tilde{} = \varphi(l_s \tilde{f}) = \varphi(\tilde{f}) = \tilde{\varphi}(f).$$

It follows that $\tilde{\varphi}$ is left invariant. So the only thing we need to prove is that $\tilde{\varphi}$ is also multiplicative. If f and g are two functions in $UC(S)$ then $\tilde{f}, \tilde{g}$ and $\tilde{f}g$ are all functions in $UC(G)$. Moreover $\tilde{f}\tilde{g} = (fg)\tilde{}$. This implies that

$$\tilde{\varphi}(fg) = \tilde{\varphi}(f)\tilde{\varphi}(g).$$

Therefore $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. □

It is important to notice that theorem 3.4.3 is false for a general semi-topological semigroup G even when G is a compact. As an example, consider the free semigroup on two generators with the discrete topology, and $G = S \cup \{z\}$ be the one point compactification of S, where $tz = zt = z$ for all $t \in G$. Then $CB(G)$ has a multiplicative left invariant mean, but $UC(S)$ does not even have a left invariant mean [25].

Theorem 3.4.4. *Let G be a topological group, and S be a dense subsemigroup of G. If $LUC(G)$ has a multiplicative left invariant mean, and S has finite intersection property for right ideals, then $UC(S)$ has a multiplicative left invariant mean too.*

Proof. From lemma 3.4.2, we know that if $f \in UC(S)$ and S has the finite intersection property for right ideals, there exists a unique function $\tilde{f} \in LUC(G)$ such that $\tilde{f}|_S = f$. Let φ be a multiplicative left invariant mean on $LUC(G)$ and define $\tilde{\varphi} \in UC(S)^*$ by $\tilde{\varphi}(f) = \varphi(\tilde{f})$. We want to prove that $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. First, because of the way we define $\tilde{\varphi}$, it is easy to see that

$$||\tilde{\varphi}|| = ||\varphi|| = 1$$

and

$$\tilde{\varphi}(1_G) = \varphi(1_S) = 1.$$

Therefore, $\tilde{\varphi}$ is a mean on $UC(S)$. Now we want to show that $\tilde{\varphi}$ is left invariant: We first notice that if $f \in LUC(S)$ and $s \in S$ then we have

$$l_s\tilde{f} = l_s f = \tilde{l_s f}.$$

Therefore, we have:

$$\tilde{\varphi}(l_s f) = \varphi(l_s f)\tilde{} = \varphi(l_s\tilde{f}) = \varphi(\tilde{f}) = \tilde{\varphi}(f).$$

It follows that $\tilde{\varphi}$ is left invariant. The only thing that we need to prove is that $\tilde{\varphi}$ is also multiplicative. If f and g are two functions in $UC(S)$ then we have

$\tilde{f}, \tilde{g}$ and $\tilde{fg}$ are all functions in $UC(G)$ and moreover we have $\tilde{f}\tilde{g} = (fg)\tilde{}$. This implies that

$$\tilde{\varphi}(fg) = \tilde{\varphi}(f)\tilde{\varphi}(g).$$

Therefore $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. $\square$

Theorem 3.4.5. *Let G be a topological group, and S be a dense subsemigroup of G. If $UC(G)$ has a multiplicative left invariant mean, and S has the finite intersection property for right ideals and finite intersection property for left ideals, then $UC(S)$ has a multiplicative left invariant mean.*

Proof. From lemma 3.4.2, we know that if $f \in UC(S)$ and S has the finite intersection property for right and left ideals, then there exists a unique function $\tilde{f} \in UC(G)$ such that $\tilde{f}|_S = f$. Let φ be a multiplicative left invariant mean on $UC(G)$ and define $\tilde{\varphi} \in UC(S)^*$ by $\tilde{\varphi}(f) = \varphi(\tilde{f})$. We want to prove that $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. First, because the way we define $\tilde{\varphi}$, it is easy to see that

$$||\tilde{\varphi}|| = ||\varphi|| = 1$$

and

$$\tilde{\varphi}(1_G) = \varphi(1_S) = 1.$$

Therefore, $\tilde{\varphi}$ is a mean on $UC(S)$. Now we want to show that $\tilde{\varphi}$ is left invariant: We first notice that if $f \in UC(S)$ and $s \in S$ then we have:

$$l_s\tilde{f} = l_s f = \tilde{l_s f}.$$

Therefore, we have:

$$\tilde{\varphi}(l_s f) = \varphi(l_s f)\tilde{} = \varphi(l_s\tilde{f}) = \varphi(\tilde{f}) = \tilde{\varphi}(f).$$

It follows that $\tilde{\varphi}$ is left invariant. The only thing that we need to prove is that $\tilde{\varphi}$ is also multiplicative. If f and g are two functions in $UC(S)$ then we have $\tilde{f}, \tilde{g}$ and $\tilde{fg}$ are all functions in $UC(G)$ and moreover we have $\tilde{f}\tilde{g} = (fg)\tilde{}$. This implies that:

$$\tilde{\varphi}(fg) = \tilde{\varphi}(f)\tilde{\varphi}(g).$$

Therefore $\tilde{\varphi}$ is a multiplicative left invariant mean on $UC(S)$. $\square$

In a more general way, we notice that the only thing we really need to have such theorems is some way to extend any functions from a subalgebra of $CB(S)$ to some subalgebra of $CB(G)$. Therefore, we have the following theorem:

Theorem 3.4.6. *Let G be a topological group, and S be a dense subsemigroup of G. Let A be a subalgebra of $CB(S)$, and B be a subalgebra of $CB(G)$ such that any function of A can be extended to a function in B. We also assume that A and B are norm closed, left invariant, and contain constants. Then if A is extremely left amenable so is B.*

3.5 Dense subsemigroup and homomorphic image of semi-topological semigroup

This section is mostly inspired by the article [20]. In this section, we first want to define the property $Q(n)$ for some pair (S, A), where S is a semi-topological semigroup and A is a norm closed, left translation invariant algebra of $m(S)$ that contains constants. Property $Q(n)$ is a fixed point property that we want to link with the existence of a left invariant mean on A, which is the average of n multiplicative mean.

Definition 3.5.1. Let X and Y be semigroups and A,B be norm closed subspaces of $m(X), m(Y)$ respectively containing constants. Then $K[A, B]$ is the set of all linear transformation T from A into B such that $T(1) = 1$ and $T(f) \geq 0$ if $f \geq 0$.

Throughout this section, S will always denote a semi-topological semigroup, and A will always denote a subalgebra of $m(S)$ that is norm closed, left translation invariant, and contains constants.

Definition 3.5.2. We say that the pair (S, A) (where S and A are defined above) has property $Q(n)$ for some $n \in \mathbb{N}$, $n \geq 1$ if whenever

$$\Phi = \{\eta(s) : s \in S\}$$

is a homomorphic representation of S as continuous mapping from a compact space Y into Y for which there exists a multiplicative function $T \in K[C(Y), A]$ such that ${}_sT(h) = T({}_{\eta(s)}h)$ for all $a \in S$ and $h \in C(Y)$, then there exists a non-empty finite subset $F \subseteq Y, |F| \leq n$ such that $\eta(s)F = F$ for all $s \in S$.

We say that mean μ is the average of k-multiplicative means for some $k \in \mathbb{N}^*$ if there exists k multiplicative means $\mu_1, \mu_2, ..., \mu_k$ such that

$$\mu = \frac{1}{k}\sum_{i=1}^{k} \mu_i.$$

Before giving the main result of this section, theorem 3.5.4, we need the following theorem, which can be found in [20].

Theorem 3.5.3. *Let S be a semigroup of transformations from a set X into X, A be a norm closed S-translation invariant subalgebra of $m(X)$ containing constants and $n \in \mathbb{N}^*$. Then A has an S-invariant mean which is the average of k-multiplicative means for some $k \in \mathbb{N}^*$, $k \leq n$ if and only if (S, A) has property $Q(n)$*

Now we want to use the previous theorem to see what happen when we consider a dense subsemigroup of a semi-topological semigroup which has property $Q(n)$ for some algebra. This is the main theorem of this section.

Theorem 3.5.4. *Let S be a semi-topological semigroup which acts on a set X, and S_0 be a dense subsemigroup of S. Let B be a norm closed S_0-translation invariant subalgebra of $m(X)$ containing constants, and A be a norm closed S-translation invariant subalgebra of B. If B has a S_0-invariant mean which is the average of k-multiplicative means then A has a S-invariant mean which is the average of k-multiplicative means.*

Proof. Since B has a S_0-invariant mean which is the average of k-multiplicative means, then (B, S_0) has property $Q(k)$. Let $\Phi = \{\eta(s) : s \in S\}$ be a homomorphic representation of S as continuous mapping from a compact space Y into Y and let $T \in K[C(Y), A]$ such that ${}_sT(h) = T({}_{\eta(s)}h)$ for all $a \in S$ and $h \in C(Y)$. Then the restriction of Φ to S_0 is a homomorphic representation of S_0 as continuous mapping from a compact space Y into Y, and T can be seen as a multiplicative function in $K[C(Y), B]$. Therefore by property $Q(k)$ we have a non empty finite subset $F \subseteq Y, |F| \leq k$ such that $\eta(s_0)F = F$ for all $s_0 \in S_0$. Now we want to show that this equality is true for all $s \in S$, which proves the theorem. Since the set $\{s \in S : \eta(s)F = F\}$ is closed in S and contains S_0, it is all S. $\square$

From this theorem, we can now deduce the following corollaries:

Corollary 3.5.5. *Let S be a semi-topological semigroup, and S_0 be a dense subsemigroup of S. If A is a norm closed S-translation invariant subalgebra of $m(S)$ containing constants, and if A has a S_0-translation invariant left invariant mean which is the average of k multiplicative means, then we can extend this mean to a S-translation invariant mean which is the average of k multiplicative means.*

Proof. Take both algebras A and B to be the same in theorem 3.5.4. The result follows directly. $\square$

Corollary 3.5.6. *Let S be a semi-topological semigroup which acts on a set X. Let A be a norm closed, S-translation invariant subalgebra of $m(X)$ which contains constants, and let B be a norm closed, S-translation invariant subalgebra of A which also contains constants. If A has a S-translation invariant mean which is the average of k-multiplicative means, then so has B.*

Proof. Since any semi-topological semigroup is dense in itself, we can take $S = S_0$ in theorem 3.5.4. The result follows directly. $\square$

In particular, we can easily deduce from the above corollary that if S is a semi-topological semigroup which is n-extremely left amenable, then any norm closed subalgebra of $LUC(S)$ which contains constants is also k-extremely left amenable for some $k \leq n$. If S is discrete, then this will be true for all norm closed, left translation invariant subalgebra of $m(S)$ contain constants, since $m(S) = LUC(S)$ when S is discrete.

Now, to conclude this section, we want to give an application of theorem 3.5.4 using a result of Pestov and Giordano, which states that any generalized Levy group is extremely left amenable. See [8] for the definition and examples of generalized Levy group.

Corollary 3.5.7. *Suppose S is a semi-topological semigroup that contains a dense subgroup which is a generalized Levy group. Then S is extremely left amenable.*

Proof. We know that every generalized Levy group is extremely left amenable [8]. Therefore by theorem 3.5.4, the result follows. □

Our next theorem is inspired by [20]:

Theorem 3.5.8. *Suppose S_1 is a semigroup of transformations from a set X into X, S_2 is a semigroup of transformations from a set Y into Y. Let A_1 be a norm closed S_1-translation invariant subalgebra of $m(X)$ containing the constants, and A_2 be a norm closed S_2-translation invariant subalgebra of $m(Y)$ containing the constants. Suppose there exists a homomorphism onto $\nu : S_1 \to S_2$ and a multiplicative $\alpha \in K[A_1, A_2]$. Then if (S_1, A_1) has property $Q(n)$, so has (S_2, A_2)*

Proof. Suppose (S_1, A_1) has property $Q(n)$. Let $\Phi_2 = \{\nu_2(s_2) : s_2 \in S_2\}$ be a homomorphic representation of S_2 as continuous mapping from a compact space Y into Y. Suppose there exists a multiplicative $T_2 \in K[C(Y), A_2]$ such that ${}_{s_2}T_2(h) = T_2({}_{\eta_2(s_2)}h)$ for all $s_2 \in S_2$ and $h \in C(Y)$. Then $\Phi_1 = \{\eta_1(s_1) : s_1 \in S_1\}$, where $\eta_1(s_1) = \eta_2(\nu(s_1))$ is a homomorphic representation of S_1 as continuous mapping from the compact space Y into Y, and $T_1 = T_2 \circ \alpha$ is a multiplicative function in $K[C(Y), A_1]$ such that ${}_{s_1}T_1(h) = T_1({}_{\eta_1(s_1)}h)$ for all $s_1 \in S_1$ and $h \in C(Y)$. Therefore using property $Q(n)$ for (S_1, A_1), there exists a subset $F \subseteq Y$, $|F| \leq n$ such that $\eta_1(s_1)F = F$ for all $s_1 \in S_1$. It follows that $\eta_2(s_2)F = F$ for all $s_2 \in S_2$. Therefore, (S_2, A_2) has property $Q(n)$. □

Chapter 4

Conclusion

4.1 Summary

The theory of n-extremely left amenable semi-topological semigroups was a subject of much research in the 60's and 70's, when most of the results on this theory were develloped. However, no locally compact groups can be n-extremely left amenable except for some trivial cases.

Recently, many important examples of extremely left amenable non-locally compact topological groups have been found. In particular, the paper [8] shows a method for finding many such groups and also shows links between non-locally compact topological groups and some related subjects of mathematics. Also, many questions are still open.

4.2 Future work

As future work, it would be interesting to investigate futher the method used in [8] and see if it could be possible to apply it in a more general way. In particular, it would be interesting to see if it could be possible to construct something similar to Levy groups or generalized Levy groups for n-extremely left amenable topological groups. In particular it would be interesting to have an example of n-extremely left amenable groups which are not the direct product of an extremely left amenable group and a group of order n.

Finally, it would be interesting to investigate futher the results of James C. S. Wong in relation to a generalization of left thick subsets and lumpy sets of M. M. Day [7] for locally semi-topological semigroups and its relation to extreme amenability. See for examples [33], [34] and [35].

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