University of Alberta

(Enlarging) secondary-level mathematics teachers' mathematical knowledge: An investigation of professional development

by

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For a long time research outcomes have influenced the reality of mathematics instruction and mathematical learning on a very small scale only. Research has followed the need of school practice rather than hurrying on ahead.
– Heinrich Bauersfeld, 1977

For several decades we have been seeing increasing failure in school mathematics education, in spite of intensive efforts in many directions to improve matters. It should be very clear that we are missing something fundamental about the schooling process. But we do not even seem to be sincerely interested in this; we push for 'excellence' without regard for causes of failure or side effects of interventions; we try to cure symptoms in place of finding the underlying disease, and we focus on the passing of tests instead of meaningful goals.
– Hassler Whitney, 1985

More and better mathematics for all students.
– NCTM

To Isabelle, because.

ABSTRACT

This doctoral dissertation reports on a professional development intervention aimed at enlarging the mathematical knowledge of six secondary mathematics teachers. The program focused on offering learning opportunities to experience and explore school mathematics concepts, along different avenues from ones solely limited to procedures. A model of professional development was developed, which aimed at exploring in depth the school mathematics concepts in order for teachers to (1) learn more about the *mathematics* they teach and (2) address teaching issues emerging in relation to this mathematics.

The analysis of the sessions provides results concerning the *learning opportunities* and *impact* that this approach had on teachers. It created two types of mathematical learning experiences: a refinement of teachers' knowledge of the mathematics concepts, and the learning of "new" mathematical concepts. In addition, three different types of teaching issues were addressed: development of knowledge and anticipatory skills about students' understandings, instances of pedagogical content knowledge, and discussion of issues related to teachers' everyday practices. The teachers also demonstrated changes throughout the year concerning their mathematical understanding, what they believed and appreciated as adequate mathematical understanding, and their ways of approaching mathematical topics (different from a procedural orientation). The exploration of mathematical concepts created contexts in which teachers could appreciate and

experience school mathematics as mathematics, and these experiences gave rise to teaching issues/strategies that led teachers to reflect on potential avenues to adopt to provide their students with similar mathematical experiences. Additionally, because these six teachers were strongly oriented toward and privileged *procedures* in mathematics, the teacher educator was significant in enabling and encouraging teachers to enlarge their knowledge and perceive issues along different perspectives.

From the truism that one cannot teach what one does not know about, one important contribution of the research is that a particular focus on (school) mathematics represents a promising and fruitful point of entry for teacher education practices. It is an approach that has the potential to enlarge and enrich teachers' knowledge of mathematics, and their teaching practices, but, above all, the mathematical experiences they can offer to their students.

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I first want to thank the teachers that participated in my research project. They have brought their spontaneity, insights, expertise and knowledge to my ideas. The research has taken form because of them. For that reason, I thank them deeply. I also thank the people from the school district who initially put me in contact with them and made sure that the project could happen.

Second, I wish to thank the *Social Sciences and Humanities Research Council of Canada* (SSHRC), the *Fonds québécois de la recherche sur la société et la culture* (FQRSC), and the *University of Alberta* for awarding me important grants that supported and helped me financially throughout my doctoral studies. These were important recognitions (academic and financially) that supported and motivated my research.

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INTRODUCTION

WHAT IS THIS DISSERTATION ABOUT?

This doctoral research is about the professional development of secondary mathematics teachers. This research is not situated at the pre-service level but at the in-service[1] level, where the issues at stake are different – not least because teachers are not students anymore. They have their own classrooms, have taught for a number of years and have gained experience in teaching. Further, this research is not at the elementary level; rather it is at the secondary level. This has important implications, especially in regard to the teachers' relationship with the subject matter, that is, mathematics, where secondary teachers for the most part enjoy mathematics and also are viewed as specialists in mathematics. Mathematics is often seen as being "part of" their lives. For this reason, mathematics plays a central role in this dissertation.

Research is scarce about secondary mathematics teachers, especially in regard to their knowledge of mathematics. This research demonstrates my interest in knowing more about secondary-level mathematics teachers' mathematical knowledge, and concerns the study and impact of an intervention conducted with secondary mathematics teachers with the intention of enlarging their knowledge of the mathematical concepts that they teach.

[1] The expressions "in-service" and "professional development" will be used interchangeably throughout the dissertation.

Six mathematics teachers took part in this study, which was centred on a series of in-service activities: Carole, Claudia, Erica, Gina, Lana and Linda[2].

The emphasis of this dissertation is partly theoretical, offering a perspective for thinking about secondary mathematics teachers' mathematical knowledge and their professional development; and partly applied, showing how these theoretical thoughts can be embedded in in-service education practices. I consider the elaboration of the theoretical assertions and the theoretical frameworks constructed for this research as part of the research results, as much as the data gathered and analyzed. I invite the reader to keep this in mind throughout the reading of the work.

The dissertation is structured as follows. In Chapter 1, I situate the origin of my particular interest in secondary teachers' mathematical knowledge and their professional development, and then I specify the issues at stake and introduce the research problem and its questions that guide this study. In Chapters 2 and 3, I offer the theoretical models and constructs that I have used to ground the professional development practices, and these models will additionally be used afterwards as research analysis frames to interpret the data. In Chapter 4, I describe the methodological orientation opted for in order to address the research questions. This is complemented by a description of the research setting where the data was gathered and of the planning activities elaborated to conduct the professional development sessions. It ends with an account concerning how the data was analyzed. This leads to Chapters 5, 6 and 7 on data analysis. In each chapter, I offer a description and an interpretation, from a different but complementary angle, of the data gathered in the professional development sessions. I conclude with Chapter 8, in which I revisit the research by exploring new and significant questions that arose within the study.

This research is the illustration of my learning path as a researcher and teacher educator. Consequently, I have inserted two "research reflections" in the dissertation concerning some thoughts that I have had during this research, and where these

[2] All names used in the research are pseudonyms. To preserve anonymity among the participating teachers, all participants have been given women's names. Three other teachers, who only came once to a session and do not represent the principal participants in the study, will be mentioned on some occasions (Danielle, Holly and Nina).

reflections led me. One is about mathematics, which I have inserted as an "intermission" after Chapters 5 and 6, and the other is about teachers and professional development, which I have placed as an "afterword" following the Chapter 8. Addressed differently, these "research reflections" represent thoughts that I continuously carried with me and elaborated on as the research progressed. I felt compelled to address these issues in the dissertation, even if they are inherently speculative (and therefore reflective of my novice status as a teacher educator and as a researcher).

CHAPTER 1

THE ISSUES / “PROBLÉMATIQUE”

Introducing the Research Context

First Plans for My Doctoral Research

Mathematics teachers play a fundamental role in their students’ learning and lived experiences in mathematics. It is commonly accepted and acknowledged within the mathematics education community that professional development represents a promising and important means to improve mathematics teaching practices so as to enrich students’ learning experiences. Even (2005) claims:

> Professional development practices are the most effective ways to *improve* students’ *opportunities* to learn *mathematics*.” (p. 83, my emphasis)

To that end, a growing number of researchers have investigated means of providing rich approaches to professional development of mathematics teachers based on different, and sometimes even conflicting, theoretical underpinnings. With the intention of contribution to this growing body of literature and research, I initially had chosen to study the professional development of secondary-level mathematics teachers from the perspective of an emerging line of inquiry in education called complexity theory (Davis, 2004; Davis & Sumara, 2006). Building on complexity theory ideas of learning collectives and knowledge producing systems (e.g., Johnson, 2001), my first intentions were to create an inquiry group that would explore issues of mathematics teaching and learning and co-produce (myself included) knowledge about mathematics teaching and learning, aimed at reporting on the rich events and possibilities that this type of collective in-service structure provided for teachers' learning. However, once into the research site and attempting to install that structure, I was confronted with something that I had not anticipated. This led me to act in a very different way, and to re-orient my research focus toward issues other than the ones that I had previously aimed at (i.e., the interaction of complexity theory and professional development).

A Surprising Research Site

> I am very sequential in my way of working, because for me mathematics was more difficult, so I go step by step. Finally, with reasoning I never did it. My teacher was telling me "you do this, this, this, this, this, and you end up with the answer." Well, I was arriving at the answer, I was getting a good grade and everything went well. But, finally, I do not have the mathematician side where someone would reflect, and say "oh yes!" and solve often by deductive reasoning or something like that. That is not in my personality. (Danielle's interview, November 13th, 2005)

While attempting to create a collective inquiry group within the in-service sessions, I was confronted with an important issue. The secondary mathematics teachers whom I was working with were mathematically very competent. That is, from what I could see, they did not make mistakes or experience difficulties solving problems in mathematics; they knew how, what and when to solve. Neither did they seem to make mathematical mistakes in their teaching of concepts[3], and they enjoyed mathematics very much. However, their knowledge of mathematics was very procedural, where mathematics was understood as a set of procedures to apply and facts to know. As they explained to me and as I realized while prompting them on some issues in the first few in-service sessions, they had never been asked to explain the meaning behind and make sense of concepts in mathematics – mostly, they had learned to do things. For example, they said that they had never been asked to explain what happens when you multiply fractions together or why, when dividing fractions, you can multiply by the inverse, or to make sense of why the volume formula for a cone and a pyramid has a ratio of 1/3 in it. They normally had to

[3] Based on short observations in each teacher's classroom.

memorize these procedures and formulas and apply them to obtain answers, as Danielle's comment above points to. Moreover, they had never had to question what was "behind" these concepts. Thus, I was confronted with teachers who had a strong grasp of formulas, algorithms and symbolic manipulations – what Hiebert and Lefevre (1986) refer to as "procedural knowledge" – but the meaning behind these "procedures" and mathematical concepts appeared obscure or unfamiliar to them, as they personally acknowledged.

Early in the research project, I had organized some individual meetings to visit the teachers in their classrooms and to have individual conversations with them afterwards. This provided me with an opportunity to get to know them better and come to understand the context in which they lived and taught (Dawson, 1999). As I realized from these visits and the discussions I had with the teachers, their manner of knowing mathematics had repercussions on their teaching, in that their teaching was mostly focused on learning, memorizing and applying procedures and techniques in order to get answers. Their knowledge of mathematics seemed to prevent them, as teachers, from presenting mathematics differently. For example, I observed exponents taught as laws that had to be applied when you multiply ("understanding" exponents meant to apply these laws to find answers efficiently); logarithms taught as algebraic forms that could be transformed into exponential forms (logarithms and exponents were expressed as a series of algebraic manipulations); addition of fractions shown as an algorithm (split between adding fractions with the same denominator and with different denominators)[4]. In sum, their teaching seemed to be oriented toward knowing procedures and applying them to find answers. Thompson, Philipp, Thomspon & Boyd (1994) label such an orientation to teaching as "calculational."

> The actions of a teacher with a calculational orientation are driven by a fundamental image of mathematics as the application of calculations and procedures for deriving numerical results. (p. 86)

As they explain, this does not imply that these teachers' teaching is about applying these procedures without meaning, but simply that the focus is on procedures for "getting

[4] These represent specific examples and others could have been given (about criteria of divisibility, negative numbers, proofs and formulas, etc.) that I observed while visiting teachers or from discussions that took place in the in-service sessions and individual conversations with them.

answers." They provide a list of what they call "symptoms" of a calculational orientation, some of which I list here:

- a tendency to speak exclusively in the language of numbers and numerical operations;
- a predisposition to cast solving a problem as producing a numerical solution;
- an emphasis on identifying and performing procedures;
- a tendency to treat problem solving as flat; that is, nothing about problem solving is any more or less important than anything else, except that the answer is most important because getting the answer is the reason for solving the problem. (pp. 86-87)

As the teachers in my research frequently made clear, they had never been asked or required to "reason" mathematics[5]. It should not come as a surprise that the teachers focused strongly in their teaching on applying procedures and obtaining answers in mathematics – by contrast, for example, with what Thompson et al. call a "conceptual orientation," which is more focused "toward a rich conception of situations, ideas, and relationships among ideas" (p. 86)[6]. Ma's (1999) work, concerning elementary teachers, makes a compelling case that teachers who only possess procedural knowledge of the mathematics they teach will be unable to help their students develop "conceptual understandings" of these mathematics. In other words, teachers cannot offer students aspects of mathematics that they are not aware of or are not very familiar with themselves. Far from wanting to be negative or critical toward these teachers' practices or their knowledge of mathematics, the first few sessions with the teachers, the time in their classes, and the interviews I had with them helped me recognize the presence of and the need to address in my research this important issue about the nature of teachers' mathematical knowledge. This situation within my research site triggered an immense interest for me in regard to the professional development programme that I wanted to

[5] From the French, "On ne nous a jamais demandé de raisonner en mathématiques."

[6] I do not attempt to be critical or assert that one approach is more appropriate than the other. In fact, I attempt in this dissertation to (implicitly) demonstrate how the picture is more complex in regard to what it means to do mathematics (see, for example, Chapter 2 on mathematical activity). My dissertation is a contribution to this issue.

offer to these teachers. It brought me to question and reflect on the initial goals of my research *per se* and of my teacher education practices, which I felt had to be adapted[7].

Confronted[8] with this unanticipated element of my research site, I had to take a different approach from the one I had previously planned (using complexity theory to guide professional development practices). I felt that I could not stay put, contemplate the events and act as previously planned as if this situation did not exist. It would also have been insufficient, unethical and misplaced to simply state and close the issue by asserting that these teachers needed to improve and learn more mathematics because they were weak in their subject matter (which was false and much more complicated). Thus, I felt uncomfortable continuing on with my previous research intentions and of not directly addressing this issue; I felt this situation was a major one to deal with.

Blouin (2000), a teacher educator who works with elementary teachers and deals constantly with what she calls a growing lack of mathematical knowledge in future elementary teachers, explains that it is legitimate to start with the context or teachers' situation to provide them with an adapted education in regard to where they "are" and what they expect to receive. But, she explains that at a point we also have to realize that we, as teacher educators, have intentions and expectations (for these teachers) that we feel are important and toward which we want to act. Hence, I too felt I had to act in regard to their mathematical knowledge, I wanted to enlarge their knowledge of mathematics, I wanted them to know more mathematically than procedures and facts[9]. In addition to this, the teachers' themselves expressed an awareness of their situation (about the nature of their mathematical knowledge) and their intentions to learn and know more.

[7] Much of this interest stemmed also directly from my empathetic feeling toward these teachers, where I clearly recognized myself in them. Indeed, as I started my B.Ed. to become a mathematics teacher, I had similar procedural inclination toward mathematics, having been educated myself as they mostly did within the behavioural objectives paradigm. And, I was simply amazed by the mathematics that my mathematics teacher educators from my teacher education program at the *Université du Québec à Montréal* offered me. Mathematically speaking, these were wonderful learning years! Hence, I have to admit that an important part of the interest toward this issue of teachers' mathematical knowledge in my research comes from the direct calling it had to me.

[8] The expression "confronted" is consciously used because it represents my felt reaction and experience in regard to my research site, where I was shocked and had to reflect a lot on the phenomena I was dealing with and wanted to study.

[9] This appears to be reminiscent, on some points, of the study reported on in Jardine's (1997) article concerning the nurse who changed her previous research focus concerning hospitalized patients, and even put it aside, in favour of actions that she felt she had to do *as a nurse*. In my case, I had to set aside some research plans and focus on what I believed I had to do *as a mathematics teacher educator*.

A double-sided intention then emerged: mine and theirs[10]. For these reasons, I felt I had to investigate to know more and better understand this present issue, namely the phenomenon of procedurally-inclined or calculationally-oriented mathematics teachers.

Better Understanding the Issue of Calculationally-Oriented Teachers

One main aspect to acknowledge and understand in this situation is that these secondary mathematics teachers had strong mathematical knowledge and skill, but that their skill reflected a procedural or calculational orientation to mathematics, something that I also observed in their teaching. From my observations and discussions with them, the teachers seemed unaware of or at least unfamiliar with the deployment of "reasoning" in mathematics. As I observed, it is not that they made mistakes in their lessons nor did they demonstrate misunderstandings of the mathematics they were teaching or that they were engaged in within the in-service sessions, but simply that their knowledge seemed, as Ball (1990) hinted at, too narrow. Consequently, their knowledge should not be seen as negative or something to "erase" from their understandings, like some researchers have suggested should be done (e.g., Deblois, 2006; Nantais, 2000). Procedural knowledge of mathematics is important, even if it is limited. However important that knowledge is, it is nevertheless imperative to understand and acknowledge that this knowledge is insufficient or incomplete in regard to what "mathematics" entails. Moreover, it prevents these teachers from teaching mathematics and offering mathematical experiences to students that are conceptually rich and focused on reasoning, and not simply on facts and procedures (Ma, 1999). In short, as obvious as it may seem, teachers teach what they know, and they cannot teach aspects of mathematics that they are themselves unaware, unacquainted or unfamiliar with. These ideas point to what I refer to as the "cycle of reproduction."

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[10] This double intention appears to be important since, as studies of professional development show (e.g., Bednarz, 2000; Dawson, 1999; Krainer, 2006), it is crucial to take the context into account for the intervention to be productive and not simply imposed from above as in a top-down structure. In Chapter 3, I come back to issues of top-down structures.

The Cycle of Reproduction

> As mathematics educators continue to attempt to improve presecondary mathematics programs, principally by expanding the scope of appropriate components (problem solving, estimation, geometry, computers, etc.), we must realize that we are asking a significant percentage of teachers to teach concepts to which they themselves were never exposed as students. (Post, Harel, Behr & Lesh, 1991, p. 196)

Erlwanger's (1973) article on student Benny is considered to be an important piece of research in the mathematics education community. Erlwanger showed how a small boy, Benny, who was considered to be fairly successful in mathematics based on his grades and on his teacher's perception, most of the time did not know what he was doing mathematically[11]. Benny's situation prompts some important questions, one being: "How is it that Benny 'succeeded' in school mathematics with this sort of understanding, or of non-understanding?" The answer to this question is puzzling. Without intending to answer the question here, I have flagged it because the same set of issues arose for me on some occasions in my research. For example, one of the participating teachers in the project came to me once, discouraged, and told me the following: "I have never been asked to reason in mathematics, and I had a 95% average in mathematics!"[12] Or, when another told me that "When students ask me why [to divide a fraction you multiply by the inverse], I simply say that this is how it is!" and when I overheard the following discussion between teachers participating in the project:

> Carole: Why is it that we are not able to solve by reasoning? I don't know if you [pointing to Erica] are able to or you Claudia, but I am like you Gina...
> Gina: Thank you, at least I am not the only one!
> [...]
> Carole: You know why? It is because we have not been educated to reason in mathematics. Me, I did copy, paste, repeat, and let's go...

[11] Of course, this is our observing stance in regard to the established body of mathematics, since Benny was able to explain why and what he was doing, but from a very different base not always connected to mathematics.

[12] All sessions were held in French, the language in which all teachers taught in schools. For that matter, all excerpts from the sessions reported on will be translated from French to English by myself.

These types of utterances prompt important questions in regard to the mathematical knowledge of these teachers. They point to the apparent helplessness of these teachers when confronted with their own knowledge, as they try to get a grasp of more conceptual and reasoned ways to make sense of mathematics. This difficulty is also something toward which Russell (2000) points to:

> Many of us learned mathematics as a set of disconnected rules, facts, and procedures. As mathematics teachers, we then find it difficult to recognize the important mathematical principles and relationships underlying the mathematical work of our students. (p. 158)

To re-use the National Research Council's [NRC] (2001, p. 428) expression, this should not be seen negatively as "the teachers' fault." There is something deeper here to address. In effect, not knowing the things you were educated about and taught about is one thing, but not knowing things that you have never heard of or never knew existed is quite a different matter. Brousseau's (1988) comments given in a conference are quite eloquent in this regard:

> Je ne suis jamais critique envers l'enseignement tel qu'il se pratique. Si vous voyez 200 000 profs faire la même chose et que ça vous paraisse idiot, c'est pas parce qu'il y a 200 000 idiots. C'est parce qu'il y a un phénomène qui commande la même réaction chez tous ces gens. Et c'est ce phénomène qu'il faut comprendre. [...] On l'optimisera pas avec de l'idéologie, ni avec des leçons de morale vers les maîtres. [*I am never critical toward teaching as it is practiced. If you see 200 000 teachers doing the same thing and that it looks stupid to you, it is not because there are 200 000 stupid people. It is because there is a phenomenon that orients this same type of reaction in these people. And it is this phenomenon that we need to understand. [...] We won't improve it with an ideology, nor by moralizing to teachers.*]

This "phenomenon" hinted at is what I call the cycle of reproduction. As students, these teachers were taught mathematics in a technical way, hence when they became teachers they continued to teach the way they were themselves taught. Hiebert, Morris and Glass (2003) call this "the culture of teaching which passes along, in a relatively unexamined way, the teaching methods of the past" (p. 218). Voigt (1994) adds the following:

> Presumably, in everyday classroom processes teachers reproduce routines and background understandings which have been unintentionally developed during their schooldays. (p. 288)

This cycle has also been referred to as the "transmission of traumas" (Berdot, Blanchard-Laville & Bronner, 2001), where teachers, unable to get outside of this cycle, have their students live the same problematic situations they themselves lived as students.

> For the students of mathematics, future mathematicians or teachers of mathematics, one wonders if the teaching they receive is not like an unconscious undertaking of denial of history and its crises and also, undoubtedly, a denial of living subjects who created this history. All the objects which were problematic throughout history no longer appear as such: they are routinised and now show up at the heart of mathematics as ordinary objects like any other. They are put to work by logical construction, detemporalised and impersonal. We remain mute – or almost so – about the traumas, somewhat like children who have suffered violence about which they remain silent, hoping that "it will take care of itself." (p. 10)

Therefore, naturally, the students who will become teachers again teach the way they were taught, making the cycle go on and on. To use Whitney's (1973) words, "we are back full circle" (p. 285)[13]. A cycle of reproduction is not harmful in itself, since it can perpetuate and carry important and intended outcomes, but it becomes harmful when problematic or unintended values are conveyed and reinforced within it. It becomes a vicious cycle.

The values that are carried along this cycle are that mathematics is a set of techniques and facts (Ball, Lubiensky & Mewborn, 2001; Battista, 1999). Because this cycle acquires more and more prominence and strength as it goes on and gets reinforced over the years, the idea that mathematics is a set of facts and techniques not only becomes stronger, but *becomes* mathematics itself. After a while, mathematics becomes *that* specific set of techniques and facts. And, the same problems continues on and on, one we hope "will take care of itself" (figure 1.1).

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[13] I quote Whitney for one purpose only, that is, the fact that the problematic stance of this "cycle" has been recognized for years in the mathematics education community. Whitney's article is taken from the proceedings of the Second International Congress on Mathematical Education held in Exeter, England, in 1972. The fact that it still appears to be an important issue today demonstrates the significance of exploring and studying this issue in depth and of better understanding it.

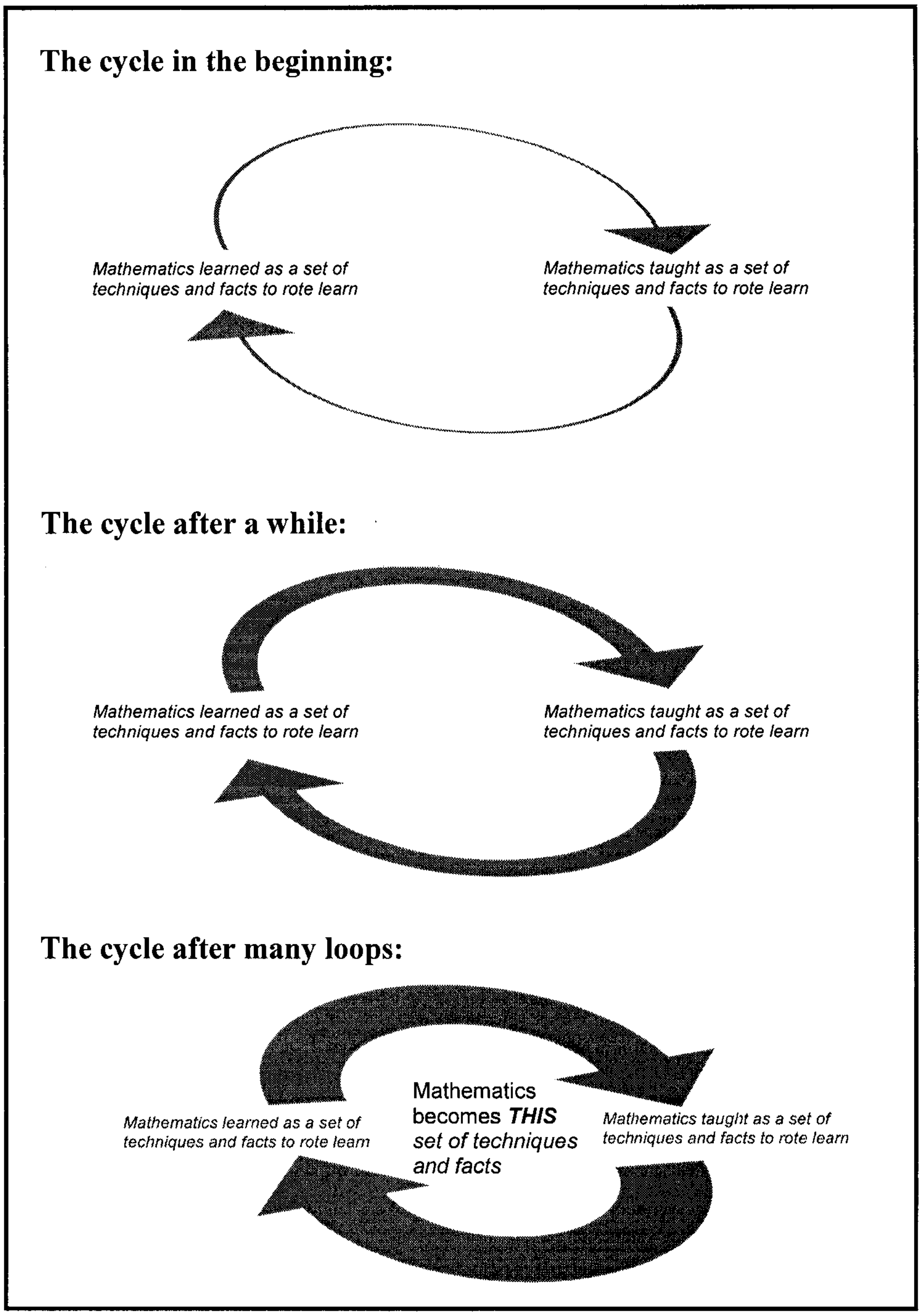

Figure 1.1. The cycle of reproduction creating mathematics as a set of techniques and facts

Further to that, this reproduction cycle has important repercussions on teachers' practices and their students' mathematical experiences.

Repercussions of Teachers' Mathematical Knowledge on their Teaching Practices and the Mathematical Experiences they Offer to Students

Teachers' own knowledge of mathematics strongly affects their teaching, which evidently affects their "reading" and understanding of the curriculum they have to teach. This is a distinction that Bauersfeld (1977) has established between the "matter meant" and the "matter taught." The former represents the mathematical structures of the discipline (curriculum's intentions) and the latter the content of the mathematics taught and shaped by the teacher in his or her practices[14].

> No doubt such fundamentally different images of the "matter meant" will influence the "matter taught," this means that they will produce different forms of mathematics teaching. (pp. 236-237)

This was also asserted by Putnam, Heaton, Prawat and Remillard (1992) in their research with elementary mathematics teachers:

> We believe that what teachers know and believe guides how they construct lessons, interpret textbooks, and interact with students. Knowledge and beliefs also provide important lenses or filters through which teachers perceive and act on various messages to change the way they teach [...]. (p. 213)

In that sense, a teacher whose knowledge is lodged in procedures and a calculational orientation will read the curriculum objectives as requests for procedures and calculations to find answers, and will directly translate such a reading of the curriculum objectives by working toward applying procedures and calculating answers in his or her teaching[15]. Thus, the nature of what is taught and talked about in the classroom is a direct consequence of the mathematical knowledge of the teacher. As Hersh (1986) has asserted, one's conception of mathematics affects one's teaching of it.

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[14] Bauersfeld also adds a third component, the "matter learned," representing the cognitive structures or the understanding that the student makes of these mathematics. For Bauersfeld, "[t]hese three forms coincide in the ideal case only" (p. 235)

[15] Cooney and Wiegel (2003) also discuss studies that show that when watching demonstrations of classroom practices oriented toward sense-making in mathematics and reasoning, teachers with strong inclinations toward procedures and calculations are not able to see the differences, and tend to focus on the technical aspects of the lesson and the material used. This shows well how teachers' procedural eye orients their interpretations and actions in regard to mathematics teaching.

> One's conception of what mathematics *is* affects one's conception of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it. [...] The issue, then, is not, What is the best way to teach? But, What is mathematics really all about? (p. 13, emphasis in the original)

The teacher then plays a leading role in shaping the learning environment in the classroom, and a teachers' knowledge is fundamental concerning what happens, almost independently of the surrounding conditions and opportunities.

> These cases suggest clearly that teaching mathematics for understanding cannot be reduced to using the right textbook, having students work in groups, using manipulatives, or using mathematics activities in real-world settings. [...] It is how teachers use these resources or tools, however, that shapes the learning environment for students. (Putnam et al., 1992, p. 214)

In addition, some studies have shown that a change in the curriculum and the textbooks is also insufficient, because the way things are realized and translated in the classroom depends in major part on the teachers' knowledge of mathematics whatever that curriculum or textbook may be, hence having little impact on what is offered mathematically in the classroom to students (Putnam et al., 1992; Ross, McDougall & Hogaboam-Gray, 2002)[16].

This shows the impact of mathematics teachers' mathematical knowledge on their practices and what is offered to their students. Thompson et al. (1994) even push these effects further by centering on the effects that a calculational orientation to teaching can have on students themselves. They explain that calculational orientations of teachers have the effect of making mathematics all about numbers and operations for students, making "[r]easoning [...] not a subject of discussion" (p. 88).

> Students who have come to view mathematics as "answers getting" not only will have difficulty focusing on their and others' reasoning but also may consider such a focus as being irrelevant to their images of what mathematics is about. (p. 88)

In addition to narrowing and biasing students' views of what mathematics is about, an approach focused on and that strongly values procedures and getting answers can have

[16] Ross et al. have reported that the only way it can have an impact is if it is conjoined with sustained in-service education for teachers.

important repercussions on students who are making sense and reason the mathematical concepts. Thompson et al. explain that it places these students in a climate dominated by procedures and calculations which can bring them to feel that they do not properly understand mathematics, and that there is something wrong with their ability to do mathematics, leading these capable students to stop trying to make sense of mathematical concepts – which is an important loss of these students' talent and capacity to understand mathematical ideas. Battista (1999) also addresses this issue, but much more strongly:

> Consequently, [teachers focusing on memorizing facts and techniques] threaten the quality of the mathematics education received not only by the general citizenry but also by future mathematicians, scientists, and engineers. Thus they endanger the entire scientific/technical infrastructure of our country [The United States]. (p. 425)

Without going into more details, this demonstrates well the direct impact that teachers' knowledge about mathematics can have on their teaching practices and on the nature of what they mathematically offer in their classrooms – the richness of the mathematical experiences offered to students. Furthermore, it demonstrates the importance of addressing this issue, and attempting at breaking this cycle of reproduction. Based on their literature review, Cooney and Wiegel (2003) warn us of the following:

> The evidence that we have examined leads to the inevitable conclusion that teachers will largely teach as they were taught in the absence of intervention. (p. 826)

Hence an intervention is required, one that will take into account the nature of these secondary mathematics teachers' knowledge and then work at expanding it to encompass more than procedures and calculations – in an attempt to break this cycle. I attempt to address this issue through professional development. Therefore, as a mathematics teacher educator and researcher, I need to understand better how I can build on teachers' (procedural) knowledge base of mathematics and what kind of professional development opportunities and experiences I should offer these teachers so that there is potential for enlarging their current knowledge of mathematics.

Attempting to Break the Cycle: What Sorts of Professional Development Experiences are Required or Suggested?

At the PME-30 (*Psychology of Mathematics Education*) meeting held in Prague (Czech Republic) in 2006, Helen Doerr raised an important point in regard to professional development when she asked the following question in a research report: "All professional development is good, so we say. It always bears results when we report on them. Hence, what is distinguishing this form of professional development from the others so that it makes this study relevant?"[17] This is an important question. Not only is it provocative, but it needs to be addressed.

There is a large and growing body of knowledge in the area of professional development of mathematics teachers. Hence, it could be tempting, since my research is within professional development, to present a literature review of the entire topic here. For example, to name a few, at the elementary level there are models such as *Cognitively Guided Instruction* (CGI), *SummerMath*, and *Teaching to the Big Ideas*. Also, although less numerous, there are promising projects at the secondary level, especially with case studies approaches (e.g., Doerr & Thomspon, 2003; Seago & Goldsmith, 2006), action research models (e.g., Raymond & Leinenbach, 2000), and the creation of communities of practices (e.g., Krainer, 2006; Lachance & Confrey, 2003), to name but a few. And, the list could be extended extensively[18].

However, as Bednarz (2000) and Krainer (2006) caution, it is essential to take the particular context into account when preparing professional development in order to address the specific issues present within that context. My intention is not to conduct professional development for the sake of conducting professional development or of simply picking one of the approaches present in the pool of possibilities available. As Doerr said, all approaches bear some result. Hence, my intention is to develop an approach to professional development that deals with the issues and questions raised within the context of my study in an attempt to understand these issues better and generate possibilities for them. In this specific case, the context calls for attention to the

[17] Obviously, this represents my own re-wording of her question.

[18] See, for instance, Zaslavsky, Chapman and Leikin's (2003) review of literature on professional development of mathematics teachers.

secondary mathematics teachers' *mathematical* knowledge. In other words, I intend to address the issue of secondary mathematics teachers' mathematical knowledge by using professional development. In that sense, professional development is here a "means to an end." So, in order to better understand the issue, and develop an effective intervention, I now turn to what prior research says about and recommends for secondary mathematics teachers' mathematical knowledge.

Addressing Secondary Mathematics Teachers' Knowledge of Mathematics: Recommendations of Research

There exist recommendations (most of them for prospective teachers) to "improve" secondary-level mathematics teachers' mathematical knowledge. However, these recommendations are often not supported by specific approaches or plans to follow, or research to demonstrate or make better sense of the possible outcomes of these recommendations. In other words, the literature in the field mostly makes theoretical recommendations for improving secondary mathematics teachers' knowledge of mathematics. This represents another stab at Doerr's question in regard to my research, namely that virtually no research is currently available that demonstrates insightful and promising results to tackle the issue of secondary-level mathematics teachers' (procedurally-inclined) mathematical knowledge. There is a need to research and understand better the issue of secondary-level procedures-oriented teachers' mathematical knowledge and explore possible and potential approaches for their professional development.

The main recommendation offered throughout the literature concerning teachers' too-narrow knowledge of mathematics – or, simply put, teachers with knowledge mostly about procedures and calculations – suggests that teachers should receive more mathematics *and* this mathematics should be at a deep and conceptual level (e.g., Bryan, 1999; Cooney & Wiegel, 2003; Even, 1993). However, as Cooney and Wiegel (2003) mention, it is not clear what type of mathematics is meant here. In the following, I attempt to understand better what seems to be a promising form of mathematical experiences to offer to procedures-inclined secondary-level mathematics teachers.

Does More Mathematics Mean More University-Level Mathematical Experiences?

When discussing the need for mathematics teachers to learn more mathematics, it is tempting to assert that teachers need to receive more mathematical content of university-level courses for future mathematicians. However, there is no consensus or there is little support that links mathematics teachers' knowledge and students' performance when looked at from the vantage point of the number of mathematical courses taken by teachers (Begle, 1979; Monk, 1994).

Discussing his review of the literature on teachers' knowledge, Begle (1979) stressed the fact that the widespread belief that the more a teacher knows about his or her subject matter (here, in terms of university mathematics courses), the more effective he or she will be as a teacher needed some "drastic modification" (p. 51). In effect, by his review, Begle was unable to demonstrate the presence of a sustained positive correlation between teachers' university mathematics credentials and students' achievements. He explained that after a certain level of mathematical study, it contributed nothing in terms of students' achievement. These results were also arrived at by Monk (1994) where he demonstrated in his study that up to a fifth course in university-level mathematics the increases in students' achievements were on a very small scale, and that after a fifth course the influence on students was basically nil. This lack of prominent influence of university-level mathematics on students' performance can also be interpreted as an interesting illustration of how teachers' previous education in mathematics as students plays a major role in their knowledge of the mathematics they have to teach and their approach to teaching this mathematics -- something I have explored elsewhere (Proulx, 2003). This is reminiscent of the idea of teachers being stuck in a cycle of reproduction, where they are caught up in a specific type and knowledge of mathematics.

Further, Begle's review points to the fact that in some cases there was the presence of negative impacts (relationship) of university-level education in mathematics on (and) students' achievements. In view of these results, Ball et al. (2001) hypothesized that teachers having received more course work in mathematics at higher levels have experienced more conventional and formal approaches to teaching mathematics, which led them to have difficulties with sound pedagogical approaches for teaching this mathematics. The strong prominence on formal and technical aspects in higher-level mathematics is conjectured to have an important effect on teachers' classroom practices, reinforcing the formal, abstract and procedural aspects of mathematics (Gattuso, 2000)[19].

These hypotheses are empirically supported, for example, by two studies. The first one is from Thompson and Thompson (1994, 1996) who studied a teacher, Bill, who had a robust understanding of the concept of rate and speed. Bill's understandings of the concepts of rate and speed was rich with plenty of connections between ideas, but was so tightly woven and hidden under calculations and operations that it made him unable to articulate clearly these understandings to one of his student in order to make the notions accessible and conceptually sound for her – even to the point that she could not make sense of his questions and felt immensely confused. Thompson and Thompson theorize that this teacher's strong knowledge base of the notion, at a formal and higher mathematical level, led him to perceive the connections and meanings as obvious, where he consistently used operations and calculations that *for him* made the connections explicit, but left them opaque for his student, creating a gap between him and his student. Concepts of rate and speed were so obvious for Bill, and his understandings of them was so tightly encapsulated and hidden within calculations and operations and formalized ways of operating, that it made them far from transparent for his student.

The second study is from Nathan and Koedinger (2000). They administered questionnaires to teachers requiring them to rate a list of algebraic problems by order of predicted difficulty for their students. The correlation between the lived difficulties of students with the problems and the teachers' predictions was very low, where teachers had overestimate the facility that students would have with formalisms and symbolism manipulations. The researchers conjectured that teachers' own facility with symbolic manipulations led them to undervalue the difficulties that these abstract forms could had on students. Discussing the study, the NRC (2001) explained that higher university-level mathematics content knowledge "by itself may be detrimental to good teaching" (p. 399).

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[19] Moreira and David (2005) arrive at similar conclusions in their study of the content offered to future teachers in university mathematics courses. They explain that the study of university-level mathematics involves and promotes forms of knowledge that are too condensed to be helpful for teachers, who need more elaborated, deconstructed and unpacked forms of knowledge. This is also in line with Adler and Davis's (2006) and Ball and Bass's (2003) ideas of unpacking mathematical ideas.

Along that line, Cooney and Wiegel (2003) conjecture that teachers with extensive pure mathematics background may be more inclined to adopt what they call a formalistic approach to teaching. This leads them to assert that the road to be taken should *not* be toward providing more formal pure mathematics training to teachers. This last assertion illustrates that the issue of teachers' knowledge of mathematics may not be situated at the level of the number of higher-level mathematics courses received.

But, the question still remains as to what are the kind of mathematical experiences to provide these teachers with? Usiskin (2001) points to an important issue in regard to mathematics teachers' knowledge of school curriculum mathematical content to teach.

> Often the more mathematics courses a teacher takes, the wider the gap between the mathematics the teacher studies and the mathematics the teacher teaches. The result of the mismatch is that teachers are often no better prepared in the content they will teach than when they were students taking that content. A beginning teacher may know little more about logarithms or factoring trinomials or congruent triangles or volumes of cones than is found in a good high school text. (p. 2)

These reports from researchers all point to the possibility that the central issue may be at the level of teachers' knowledge of the mathematics that they explicitly teach about in their classroom, the very mathematical topics that teachers were taught about when they were students.

Focusing on Teachers' Knowledge of the Mathematics they Teach

> Mathematical knowledge is a critical resource for teaching. Therefore, teacher preparation and professional development must provide significant and continuing opportunities for teachers to develop profound and useful mathematical knowledge. (NRC, 2001, p. 428)

Rather than providing mathematics teachers with more higher-level or formal mathematical experiences, the issue for professional development seems to lie directly in the mathematical education of teachers in regard to the content they have to teach. As Bryan (1999) suggests, there is a need to offer teachers opportunities "to deepen their conceptual understandings of the *content of the school mathematics curriculum*" (pp. 8-9, my emphasis). It appears then that working through the mathematical topics of the school mathematics curriculum represents a promising approach for the professional development of mathematics teachers with a calculational orientation[20]. Hence, the intervention that I need to develop for working with these teachers has to focus directly at the level of the mathematics they teach. My hypothesis is that it is not by working on more formal mathematics that teachers will enlarge their perspective and knowledge of school mathematics – and enhance their practices. It is by working on the school mathematics topics that teachers teach, precisely at a "conceptual" level, that is, deliberately working at a level different than one uniquely centred on procedures and calculations; a level aimed at deploying "reasoning" of and about the mathematics that is to be taught in teachers' classrooms. As Cooney and Wiegel (2003) assert, there is a need for teachers to study deeply the mathematics they teach, and in a far more sophisticated way than they previously did.

A question remains, however. What does it means to work at a conceptual level? What does conceptual mathematics entail? As Raman and Fernández (2005) point out, the idea of working on conceptual mathematics or on "deep understanding" seems to be an umbrella idea that is talked about a lot, but that nobody really knows what it represents:

> When Liping Ma's (1999) book *Knowing and Teaching Elementary Mathematics* burst on the scene, many people—on both side of the math wars—were quick to embrace it as exemplifying exactly the kind of knowledge that we want elementary school teachers to develop. The catch phrase, "Profound Understanding of Fundamental Mathematics" (PUFM) seemed to capture precisely what both reformers and traditionalists thought was essential mathematics for elementary school teachers. However, 5 years later we are still struggling to articulate what PUFM means and how it should play out in the training of preservice mathematics teachers. (p. 259)

In that sense, in order to know more about and define better what working on deep conceptual understandings of mathematics could mean (and on making sense and having

[20] Specifically, at the elementary level, some promising research on professional development has been attempted by Schifter and her colleagues. For example, Schifter's (1998) *Teaching to the Big Ideas* and Simon and Schifter's (1991) *SummerMath for Teachers Program* focused on having elementary teachers work on mathematical topics of the curriculum to improve their comprehension, and by the same way to initiate a reflection on their own learning processes to improve their understanding of their students' thinking. From these studies, many of these teachers changed their perspective on mathematics and their classroom teaching practices have evolved, often going from a "traditional" format to a format more focused on mathematics as a human enterprise and as inquiry (Schifter, 1998; Schifter & Fosnot, 1993).

a profound understanding of it), I intend to have a closer look at what research says concerning (procedurally-inclined) secondary-level mathematics teachers' knowledge of mathematics.

Secondary Mathematics Teachers' Knowledge of the Mathematics they Teach

What do we know about mathematics teachers' knowledge of the mathematics they teach? A large amount of research has been conducted in regard to *elementary* teachers' knowledge. Much has been written on elementary teachers' lack of understandings or absence of deep understandings of the notions they have to teach (e.g., Blouin & Gattuso, 2000; Ma, 1999); their numerous mistakes when explaining or doing mathematics (e.g., Heaton, 1992; Post et al., 1991); their negative (emotional or academic) relationship toward mathematics (e.g., Héraud, 2000); and the list could go on. However, in regard to *secondary* mathematics teachers, the literature is quite scarce (Ball et al., 2001; Cooney & Wiegel, 2003). Ball et al. (2001) suggest that this lack of research on secondary-level mathematics teachers is based on the following assumption:

> Why research has focused on elementary teachers reflects a continuing assumption that content knowledge is not a problem for secondary teachers, who, by virtue of specialized study in mathematics, know their subject. (p. 444)

As they continue to explain, this represents a biased view because some studies demonstrate the difficulties that secondary mathematics teachers experience in regard to their topics of teaching[21]. As Cooney and Wiegel (2003) caution us, secondary mathematics teachers' knowledge of mathematics should not be taken for granted.

Most of the reported studies on secondary mathematics teachers' knowledge demonstrate the fact that teachers' knowledge is insufficient or too narrow in the sense that it mostly concerns the application of mathematical procedures (e.g., Ball, 1990; Bryan, 1999) – something quite reminiscent of my own research context. For example, Ball (1990) studied the understanding of nine secondary teachers about concepts of

[21] It is important to note that a majority of the studies conducted on teachers' knowledge are with pre-service teachers, mostly because of the accessibility of participants, but also of the ethical outcomes of "testing" and "measuring" teachers (Ball et al., 2001).

division ($1\frac{3}{4} \div \frac{1}{2}$; $7 \div 0$; If $\frac{x}{0.2} = 5$, then x = ?). All secondary-level teachers succeeded in doing the correct calculations and finding the answers – except for one mistake of one teacher with the $1\frac{3}{4} \div \frac{1}{2}$ question. However, Ball highlights that most of the teachers could not provide the meaning behind these calculations. Indeed, for the first one ($1\frac{3}{4} \div \frac{1}{2}$), only five secondary-level teachers were able to generate an appropriate representation to make sense of the problems, for the second one only 4 could give a sense of the $7 \div 0$ division, and no one was able to give meaning to the "If $\frac{x}{0.2} = 5$, then x = ?" problem. These results led Ball to say that it indicates a narrow understanding of division. In that sense, the secondary teachers were able to calculate but could not make sense of these calculations by explaining the meaning behind them. These mathematical ideas were facts for them.

> One of the mathematics majors realized this and commented (on division by zero), "I just *know* that … I don't really know why … it's almost become a fact … something that's just there." (p. 142, emphasis in the original)

Bryan (1999) arrived at similar results. In his study, he interviewed nine prospective secondary-level mathematics teachers with 31 different questions on topics like exponents, division of fractions, operations on integers, slopes and lines, algebra, trigonometry, and area formulas. In total, from 279 responses, only 30 computational mistakes were made, showing how well teachers mastered the variety of different procedures and calculations. However, out of the 279 items only on 61 occasions (22% of the time) were secondary mathematics teachers able to provide an explanation and meaning for the operations or formulas inquired about. This led Bryan to say that "the competence of these future teachers to teach the very subject matter […] remains largely memorized rather than understood" (p. 9). In other words, mathematical ideas were facts for these teachers, who simply knew how to use them.

These studies highlight an important point. Secondary-level mathematics teachers, unlike many of their peers at the elementary-level, make few mistakes and errors in mathematics and for the most part are successful in mathematics, and have been successful in their mathematical careers as students (again, quite reminiscent of the teachers in my research). The issue then lies at the level of teachers' inability for whatever reason to provide meaning *behind* the procedures used and calculations made. These teachers' mathematical knowledge is often to be seen as limited and predominantly invested in the knowledge of procedures and facts, as Mewborn (2003) explains.

> By and large, teachers have a strong command of the procedural knowledge of mathematics, but they lack a conceptual understanding of the ideas that underpin the procedures. (p. 47)

In other words, secondary-level teachers often possess what Skemp (1978) calls *instrumental understanding*. This type of understanding represents a knowledge of "how" things work, and is contrasted by *relational understanding* which represents not only the knowledge of "how" things work but also of "why" they work. For example, in the case of an algorithm, a relational understanding of it represents both knowledge of "how" to use the algorithm and of the reason "why" this algorithm works; whereas an instrumental understanding would uniquely represent knowledge of "how" it works. This is one important aspect of the secondary-level mathematics teachers reported on in these studies (and the ones I worked with in my research): their knowledge of procedures is often instrumental. In that sense, a part of the work on "conceptual" mathematics needs to be placed at the level of the meaning behind the procedures and the calculations made.

But there is more. There is more to mathematics than knowing "how" and "why" procedures work, and some studies have pointed to these issues. Studies show that some secondary-level teachers also lack an understanding of the concepts themselves, aside from procedures and "getting answers," which leads researchers to explain that teachers experience difficulties with the mathematical notions under study (e.g., Even, 1993, Even & Tirosh, 1995; Hitt-Espinosa, 1998). In their respective studies, Even and Hitt-Espinosa suggest that teachers have important difficulties with the concept of functions in regard to discontinuity. As Even explains, many teachers still have an "old" definition of function as a continuous graph (drawn from one uninterrupted pencil trace) – what Hitt-Espinosa points to as an Eulerean definition. Hence, this often prevents them from recognizing and accepting alternative drawings as being representative of a function, even the discontinuous ones, because they are expecting to have "nice" graphs. In addition, Hitt-Espinosa demonstrates that this view leads teachers to draw discrete functions as continuous. This represents another part of the issue in regard to teachers' mathematical knowledge. As Even acknowledges, this is problematic since these (prospective) secondary teachers have usually encountered continuous graphs of functions in their schooling experiences as students, hence rendering them unable to make sense of other types of functions. Hitt-Espinosa also points to teachers' unfamiliarity with working with discrete situations in mathematics.

To complete his study, Hitt-Espinosa underscores how difficult it was for teachers to produce counter-examples to demonstrate specific statements about functions. In his study, he asked teachers to provide a demonstration when they considered a statement to be true, or to provide a counter-example if they believed the statement to be false (out of three cases, the first one being true, the other two being false). Whereas half of the 30 teachers could prove the adequacy of the true statement, only three teachers were able to provide a counter-example for the second one, and none were able for the third one. Hitt-Espinosa explains that this is quite telling in regard to these teachers' mathematical abilities, since producing counter-examples did not seem to be a familiar activity for them – and he noted that in Mexico, where the study was conducted, counter-examples are something rarely asked for in school textbooks.

This represents another important part of the issue concerning secondary-level mathematics teachers' mathematical knowledge. This is at the level of the mathematical concepts and notions and the interplay between them, rather than at the level of procedures and their meaning. Moreover, it does not point to difficulties or mistakes *per se*, but to a lack of awareness or unfamiliarity with working with these types of mathematical issues, something strongly linked to their previous education (again, quite reminiscent of the teachers in my research). Even (1993) explains:

> Teachers need to have learning environments that foster powerful constructions of mathematical concepts. Unfortunately, the present mathematics courses teachers typically experience do not provide such an environment. (p. 113)

Consequently, there seems to be two different types of "conceptual" knowledge or "reasoning" that the secondary mathematics teachers should work through. First, there is the development of relational understanding (Skemp, 1978) of the procedures they already know, since teachers often have not developed an understanding of the reasons "why" a procedure works or a fact is so. On that level, developing this type of knowledge would deepen their understanding of their procedural knowledge of mathematics, making it encompass the "how" but also the "why" of these procedures. It would still, though, be at the level of procedures. Therefore, there is the development of knowledge of another kind. This second kind is related to mathematical concepts and ideas, something different than procedures and more in line with the structures and relations within mathematical concepts. I elaborate more on these instances and I attempt at defining them in the next chapter. In sum, working to develop a "deep conceptual understanding" and deploy "reasoning" about the mathematics that teachers teach would seem to require, for secondary-level procedurally-inclined mathematics teachers, to take these two types of knowledge into account and provide teachers with (more) experiences in that direction, so that it impacts their current knowledge of mathematics in order to enlarge it. This leads to my research questions.

Research Questions

Teachers' knowledge of mathematical procedures represents important knowledge. Working on the school mathematics topics teachers have to teach, but at a conceptual level, requires building on teachers' current strengths in order to enlarge and enhance their knowledge of mathematics so that it encompasses more than sets of mathematical procedures and facts. Simply put, the intervention needs to aim at expanding teachers' current knowledge base of the mathematics they teach. The idea for professional development would then consist of working and exploring, with mathematics teachers, aspects of the mathematics they teach (the school mathematics of the curriculum) at a deep conceptual level. For my work, this means that I have to provide the mathematics teachers in my professional development program with rich learning opportunities about conceptual mathematics. My research intentions are to work around an approach of in-service education with secondary-level mathematics teachers in which the learning and study, in deep conceptual details, of the mathematical topics they currently teach would be the core concern of inquiry. To this end, and within these interests, I am interested in understanding better the nature of an approach along these lines. The research questions that orient and guide this study are as follows:

1- **Descriptive level - opportunities.** What type of learning opportunities does an approach focused on developing a deeper conceptual sense of the mathematical topics secondary-level mathematics teachers have to teach offer? How do teachers interact with this type of approach and what type of knowledge do they develop?

2- **Interpretative level – impact and repercussions.** How does this approach to professional development "enlarge" secondary teachers' knowledge of the mathematics they have to teach? What type of effects on teachers does this approach have? In what ways is this approach beneficial to these calculationally-oriented or procedurally-inclined secondary-level mathematics teachers?

Assumptions and Significance of the Study

The issue of teachers' knowledge of mathematics is a fundamental one in mathematics education. However, there is no sustained correlation between mathematics teachers' knowledge and their students' *achievements*. Indeed, this led Begle (1979) to assert that this link should be set aside in our studies, since we would not profit much from studying it further. However, my point of interest is different. It is not in students' achievement or success in examinations, but the instruction and mathematical experiences offered to students in the mathematics classrooms. In effect, there are strong and direct links between mathematics teachers' knowledge and what they mathematically *offer* in their classroom instruction (Fennema & Franke, 1992; Mewborn, 2003). Hence, what teachers know has a tremendous impact on what they offer to students, but more precisely on what they *can* offer in their mathematics classroom. It is a truism to say that you cannot teach what you do not know about. In that sense, teachers who possess uniquely procedural knowledge of mathematics offer their students mathematical experiences that are lodged in this same type of mathematics, which I argue represents an incomplete view of what doing mathematics entails.

In that sense, the interest in my approach to professional development is to enlarge teachers' knowledge of mathematics so that new possibilities are opened to them in regard to the mathematical concepts that they teach, where they would possibly be able to offer richer and more conceptual mathematics in their teaching. This is my working assumption in this dissertation. I do not assert or start from the assumption that enlarging teachers' mathematical knowledge will improve their students' results and scores in tests (even if that could happen). I base my study on the working assumption that by enlarging teachers' knowledge of mathematics it will possibly influence the mathematics they offer and can offer in their classroom. The idea of providing teachers with rich experiences and learning opportunities to learn "conceptual mathematics" is rooted in this assumption that if teachers know more about mathematics and enlarge their understanding of what mathematics is, then their teaching will be influenced by it and richer mathematics (not only procedures and calculations) will be offered in their classroom.

> The teachers whose mathematical knowledge appeared to be connected and conceptual were also more conceptual in their teaching, while those without this type of knowledge were more rule-based [...] when teachers have an integrated, conceptual understanding of specific subject matter, they structure their classrooms so that students are able to interact with the conceptual nature of the subject. (Fennema & Franke, 1992, p. 153)

By what teachers "offer" in their classroom, I mean teachers' ways of presenting the topics of study, their ways of understanding and assessing their students answers (e.g., Margolinas, Coulange & Bessot, 2005), the type of examples they present (e.g., Zaslavsky, Harel & Manaster, 2006), the oral explanations they give (e.g., Proulx, 2003), and so on. The richness of these teachers' "offerings"[22] has been demonstrated to be directly in line with teachers' knowledge.

> [...] the richness of the material being taught appeared to be directly related to the subject-matter knowledge of the teachers. (Fennema & Franke, 1992, p. 151)

And Sharif-Rasslan (2006) explains that the nature of these "offerings" has a profound impact on students.

> Thus it may be natural to suppose that the learner is affected by the way he learns a subject/concept, and that he is also influenced by the kind of problems that he solves. (Sharif-Rasslan, 2006)

[22] Zaslavsky and Leikin (2004) refer to these as "learning offers."

Hence, I develop an argument about what can be done to provide rich learning opportunities to teachers so that they experience conceptual mathematics, and that possibly the cycle of reproduction gets broken down (thereby allowing richer mathematics being offered and worked on in teachers' classrooms, more than only about procedures).

A Discussion about Teachers' Beliefs

> [...] teaching for understanding is complex and highly dependent on teachers' knowledge and beliefs and that facilitating meaningful change in instruction will entail helping teachers rethink and learn new mathematics content and stances toward teaching and learning. (Putnam et al., 1992, p. 225)

Because I focus and orient my practices of professional development toward providing teachers with rich mathematical experiences in regard to the mathematics they have to teach, there is a small matter I believe should be addressed: that is, the issue of teachers' beliefs. This is an important matter since, as Putnam et al. (1992) and Ross et al. (2002) have shown and explained, changes in teaching mathematics practices are directly dependent on teachers' knowledge *but* also on their beliefs. Beliefs (about mathematics and its teaching) are intertwined within teachers' mathematical knowledge and therefore influence their teaching practices (Thompson, 1992). However, as Cooney and Wiegel (2003) report in their review, teachers' beliefs about mathematics and its teaching are nested in a web of other beliefs about teaching, learning and schooling, which makes it often difficult to isolate teachers' beliefs about teaching mathematics as the sole influence impacting their practices, not always making it a fruitful site of research and intervention.

> What we see, then, is that teacher education programs can provide a basis for teachers to appreciate a broader view of mathematics and alternatives to teaching beyond telling. But we cannot expect those beliefs to transform teaching as they may be secondary to other beliefs deemed more fundamental. As noted by Skott (2001), beliefs about mathematics are often buffeted by other more centrally-held beliefs and by circumstances particular to schools and society. [...] A question then arises about the quality of teachers' mathematical background which support these new experiences [...]. (p. 802)

In their review, they also explain that research interventions that succeeded in affecting teachers' beliefs in regard to the nature of mathematics and its teaching have failed to change or impact teachers' classroom practices. For example, they report on one teacher who changed his view of mathematics to a more fallibilist perspective (in contrast to an absolutist one) where mathematics is a human endeavour for which humans made sense and constructed meaning, but who was still focusing uniquely on knowing the procedures and facts in his teaching. Again, this is linked to the web of beliefs of teachers, but also to the fact that teachers' knowledge about mathematics is one of the most influential aspects for their teaching. As the authors mention, one possibility is that the limits of his knowledge prevented him from offering something other than what he knew the topic to be, which was mainly about algorithms: "Perhaps [...] his knowledge of the subject itself was lacking which led to a superficial view of mathematics" (p. 806). Not to overemphasize the point, but this still points to the fact that "you simply cannot offer what you do not know about." This gives support to my idea of providing teachers with mathematically rich conceptual experiences of the mathematics topics and notions they have to teach – as the end of the previous quotation from Cooney and Wiegel points to – because it will offer teachers opportunities to enlarge their own mathematical background to encompass mathematical knowing larger than sets of procedures. In effect, I cannot expect teachers who mostly only have a (very good) knowledge of mathematical procedures to work toward more conceptual aspects of mathematics in their teaching – since this side is often unknown or obscure/unfamiliar to them. Ma (1999) points to that in her research:

> Limited subject matter knowledge restricts a teacher's capacity to promote conceptual learning among students. Even a strong belief of "teaching mathematics for understanding" cannot remedy or supplement a teacher's disadvantage in subject matter knowledge. A few beginning teachers in the procedurally directed group wanted to "teach for understanding." They intended to involve students in the learning process, and to promote conceptual learning that explained the rationale underlying the procedure. However, because of their own deficiency in subject matter knowledge, their conception of teaching could not be realized. (p. 36)

I am not implying that teachers' beliefs are not important and do not play a role, far from that. However, because beliefs are closely intertwined with teachers' knowledge, by addressing their mathematical knowledge so to enlarge it and make it encompass more than techniques, I believe that I will be able implicitly to attend to and impact their beliefs system about mathematics and its teaching. In that sense, without being an implicit focus of the study, but being an important issue that impacts on their teaching and knowledge, I also intend to work tacitly on and shape their belief system in regard to mathematics and its teaching. This concurs with Cooney's (2001) thoughts and experiences:

> Based on my experience with educating preservice teachers, I have come to conclude that the best entry into their beliefs systems about mathematics and the teaching of mathematics is through the study of school mathematics. It is here that reflection can become commonplace with respect to both mathematics and pedagogy. (pp. 26-27)

Significance of the Study

The issue of the mathematical knowledge of procedurally-inclined teachers raised in this first chapter has been demonstrated to be a significant dimension of teachers' knowledge, which moreover has a significant impact on the mathematical experiences that these teachers can offer to their students. This appears to be an important issue to research about and develop a better understanding of since, despite its importance, this issue has received little attention from the research community. The literature that does address these sorts of issues primarily offers recommendations for future actions, and most of them, if not all, concern prospective teachers. Hence, the development of fruitful professional development approaches to enlarge the mathematical knowledge of practicing secondary-level mathematics teachers is an avenue worth studying. My doctoral dissertation is a contribution to this issue.

The main goals of this research are: (1) to better understand the issues about and the phenomenon of secondary-level mathematics teachers' mathematical knowledge, (2) to develop an intervention (by professional development) aimed at impacting teachers' knowledge of the mathematics that they teach, and (3) to develop an understanding of what happens when the intervention is put into action with teachers *and* of what are its possible repercussions on secondary mathematics teachers. Broadly speaking, this doctoral dissertation aims at bringing new knowledge to the field of mathematics teacher education, and of mathematics education in general.

CHAPTER 2

THEORETICAL FRAMEWORKS

In this chapter, I discuss two frameworks that guide my professional development practices. In the first part, I define and clarify what doing mathematics represents and what it implies, something I call the "mathematical activity." This will ground the nature of the mathematical experiences offered (through tasks and situations) to teachers in the professional development sessions. In the second part, I develop and elaborate on the theoretical model that structures the approach that I have taken to conduct the sessions and offer these mathematical experiences to teachers. I call this model the "deep conceptual probes into the mathematics to teach."

The Mathematical Activity

In Chapter 1, I asserted that reducing mathematics to a set of procedures and calculations represents an incomplete, hence deficient view of what doing mathematics entails. To better define what I mean by this, and by what I see as a more complete picture of doing mathematics, I describe here what mathematical activity represents. In addition, since I have explained that my intentions in this professional development program were to enhance and enlarge teachers' mathematical knowledge of the mathematics they teach, theorizing mathematical activity will provide me with a frame that will be useful for the preparation of the sessions concerning the mathematical experiences that I will offer to teachers[23]. I define mathematical activity along three branches: (1) the conventions used when doing mathematics, (2) the procedures used when doing mathematics, and (3) the play with structures and relations when doing mathematics (figure 2.1).

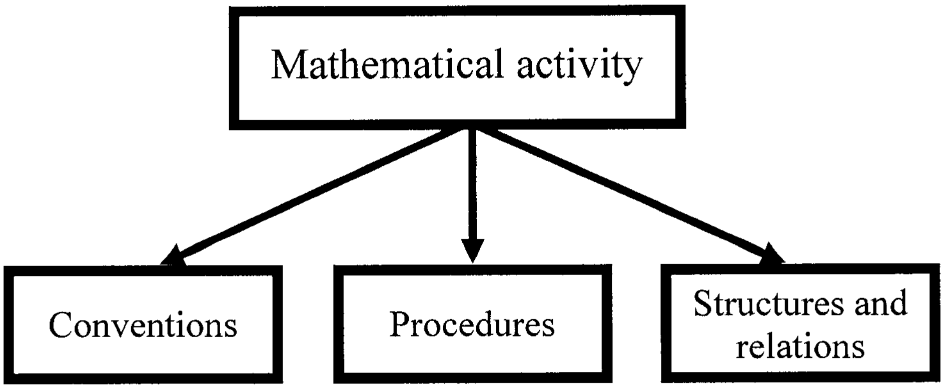

Figure 2.1. Diagram representing the three branches of the mathematical activity

Each part of the mathematical activity has specific characteristics and plays a different role, one that is important and that is complementary. However, it must be kept in mind that not everything can be usefully described as part of a single branch; some mathematics will be a combination of branches, and some will be impossible to place in one of the three branches. Nonetheless, such a distinction appears to be quite useful to better understand what doing mathematics implies.

Conventions: The Way Things Are Done in Mathematics

> The mathematics curriculum is full of conventions, which are based on choices which have been made at some time in the past. For anyone learning those conventions today, they may seem arbitrary decisions. For example, why is the x co-ordinate written first and the y co-ordinate second? This is only a convention, and there is no reason why x must be first. (Hewitt, 1999, p. 3)

When we do mathematics, we use the mathematical conventions that have been invented, promoted and used by mathematicians and other persons doing mathematics,

[23] I say more about this in the methodology chapter.

that is, the mathematics community. Conventions are standardized forms and ways agreed upon and used to communicate to others and express mathematics. Whether it is using the symbol "2" to represent the quantity "two," naming it "two," saying that a figure that has four equal sides and four right angles is called a square, or using the accepted mathematical formalisms, all of these are conventions that have to be used to communicate mathematically and that enables the mathematical doer to continue doing mathematics within a community and be able to understand what other persons doing mathematics within that community mean, and to communicate with them. These conventions, as Hewitt (1999) suggests, are arbitrary and must be "transferred" from the teacher to the students by telling: "*All* students will need to be informed of the arbitrary" (p. 4, emphasis in the original)[24]. Conventions are aspects of mathematics that are decided upon and that are accepted within the mathematics community. They are not elements to make sense of, they are elements to memorize; and one needs to know how to use them and what they represent[25]. Consider Clausen's (1991) example:

> However, I have long felt that our use of degrees to measure amounts of turn (angles) is very arbitrary. There is no way that a child (or adult) can *intuit* that there are 360 degrees in a whole turn. This is totally arbitrary, formal, true-because-Teacher-says-so knowledge. (p. 16)

Even if it could be possible to make sense of some conventions – by going back to historical roots of words or representations, by making sense of the choices made and understanding some reasons why these conventions were adopted in favour of others, or simply by following a chain of some logical thought that works for some cases – it does not change the fact that these conventions have to be learned and memorized by the mathematical doer so that he or she can use them and refer to them when doing mathematics. "If a student wishes to become part of the same [mathematical] community,

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[24] Hewitt (1999) talks about words, names, symbols, notations and conventions as the arbitrary elements. However, toward the end of his article, he tends to use more often the word "conventions" to represent *all* these ideas. I am using here the word convention as an overarching concept to represent them.

[25] This is not to say that teachers cannot work with students to help them invent new conventions to have them understand the need and relevance of creating commonly agreed on mathematical conventions, as studies of Brousseau with young pupils have insightfully shown (Salin & Brousseau, 1980). Hewitt (1999) supports this by saying: "This is perfectly possible and can be desirable at times [to have student invent conventions]: however, it does not change the fact that students will still need to be informed at some time in the future if they are to be included within a mathematics community which communicates through adopted conventions" (p. 9).

then the student needs to accept that name, rather than *question* it" (Hewitt, 1999, p. 3, emphasis in the original). Conventions need to be told and shown, because they are arbitrary.

Conventions then are the first of the three aspects inherent in the mathematical activity, that is, when we do mathematics. The next one is about the mathematical procedures[26].

Procedures: Tools to Facilitate the Mathematical Process

> The term "algorithm" sometimes provokes disdain among educators because of the oppressive ways in which traditional algorithms often are taught. In fact, algorithms are remarkable tools in mathematics and computer science. They have great practical and theoretical importance. They also are important in learning mathematics. (Bass, 2003, p. 323)

The usage of procedures in mathematics, especially in school teaching, has been a controversial subject for some decades (e.g., Bass, 2003; Battista, 1999; Schoenfeld, 2004). Many contrast, while others equate, "knowing algorithms" and "mathematical understanding." Whereas some educators argue that procedures or algorithms are an oversimplification and distortion of the mathematical ideas and do not provide a meaning to mathematics, others regard algorithms as the final goal and purpose of learning mathematics[27]. As Bass (2003) and Wu (1999) explain, these are false dichotomies, and suggest that both groups are wrong on both counts, since "both forms of knowledge are essential and are basically intertwined" (Bass, 2003, p. 326).

> Understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to

[26] A difficult aspect to classify is the usage of "mathematical definitions." They represent aspects "established" that are to be used afterwards, and in that sense possess "conventions" characteristics. However, definitions need not to be memorized – however they can be! – and can genuinely be created and even made sense of. In that regard, they are not arbitrary. Hence, this shows a possible limit of the proposed categorisation. But, to borrow from Tom Kieren (personal communication, July 2006), I classify mathematical definitions as "conventional" aspects of doing mathematics, in the sense that they are established in order to be built upon and used afterwards. It therefore figures more precisely in this branch than in any other.

[27] And in this last category of seeing algorithm as the "final goal of learning mathematics" are also people who wonder which of "mathematical understanding" or of "procedural knowledge" should come first in instruction.

> learn many mathematical concepts with understanding, and using procedures can help strengthen and develop that understanding. (NRC, 2001, p. 122)

In that sense, it is important to understand that procedures are definitely part of the activity of doing mathematics, but they *do not* represent the sole objective/activity of doing mathematics. By highlighting some important characteristics inherent to many algorithms (accuracy, generability, efficiency, ease of accurate use, and transparency[28]), Bass (2003) explains how algorithms are important in the mathematical endeavour and how remarkable they are because of their quality of generalization and their usage for a broad range of situations.

Mathematical procedures can be defined as rules and algorithms to solve mathematical tasks – what Hiebert and Lefevre (1986) call procedural knowledge. They make a distinction between two types of procedures, the ones requiring operation on mathematical symbols (e.g., an algorithm of multiplication in column), and the ones made of a list of concrete actions to follow to complete a task (e.g., to complete a square, first you do this, then you do this, etc.). For them, "they are step-by-step instructions that prescribe how to complete tasks" (p. 6). To their definition, I also add mathematical "formulas" as part of the mathematical procedures used when doing mathematics.

It is important to understand that mathematical procedures are important when doing mathematics. Therefore, it is important to know how to use them and apply them – they are efficient, their use can prevent errors, and they are simplified and simplify the operations. And so, algorithms, techniques, procedures and formulas occupy an important place in the mathematical inquiry.

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[28] In the same vein as Bass (2003), Davis and Simmt (2004) list some important aspects of the characteristics of algorithms and on what accounts they are chosen in mathematics: the retention or loss of information in the algorithm, the generalizability of the procedure to many cases, the probability of making mistakes or the error-proneness of the procedure, the facility to make corrections of errors or diagnose them if they are produced when computing, the economy of time gained or the speed of computing the answer, the facility to communicate results using the procedure, and the status of acceptance in the community.

> We must recognize the mathematical richness and usefulness of algorithms and find ways to help students develop appropriate mathematical proficiency in constructing and analyzing them. (Bass, 2003, p. 326)[29]

Mathematically, it is also important to understand "why" algorithms and procedures work, both forms of knowledge should work hand in hand (Russell, 2000)[30]. This is precisely the distinction made by Skemp (1978) between instrumental and relational understanding. Skemp's distinction appears central here, because it shows the double importance of knowing how and why, nuancing the previously stated debate about pitting procedures against understanding and only choosing one[31]. The importance of both knowing "how" and knowing "why," and not exclusively one or the other, is also highlighted by Herscovics (1980) who points out that understanding the reason for a procedure to function (the "why") does not necessarily mean that a person knows how that procedure works (the "how"). Indeed, someone could be able to explain the reasons for a procedure to work when it is in front of him or her, but not being able to use it to compute or solve a problem at a different time. This shows how important it is for both the "how" and the "why" of procedural knowledge to be mastered[32]. Because we need to make sense of the procedures that we use when doing mathematics, they are not to be memorized or rote learned but are to be made sense of – we need to understand the reasons for their functioning and how to use them. That they become automatized is clearly possible, but they should always make sense to the user if needed and not simply stated by heart.

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[29] Many research programs have shown the richness and importance of having students develop their own algorithms and make sense of procedures, and how this was central in the instruction, learning and understanding of mathematics (e.g., Carpenter, Fennema, Franke, Levi & Empson, 1999; Russell, 2000).

[30] Even and Tirosh (1995) specifically note that for teachers, only knowing "how" has important repercussions on a teachers' teaching, and that also knowing "why" things work "enables better pedagogical decisions" (p. 9).

[31] It is interesting to see that Skemp's (1978) distinction did create an important movement in which people tried to peel out different characteristics and subtleties within the relational and instrumental dyad or even generate other levels of sophistication (e.g., Buxton, 1978; Byers & Herscovics, 1977; Herscovics, 1980). I will not enter into those details here, since my intention is to elaborate on the mathematical activity, that is, *when we do mathematics*, and not on the different levels and ways of understanding a concept.

[32] Russell (2000) also highlights the case of two specific students who could make sense of the properties and operations in addition and multiplication, but did not know how the procedure worked and therefore made significant mistakes in their calculations.

Moreover to knowing "how" and "why," Russell (2000) explains that doing mathematics requires one to be computationally fluent and competent. This fluency and competency in computation is explained along three specific lines: the *efficiency* so that one does not get bogged down while solving and can operates easily, the *accuracy* so that one records well and does not make errors in the process, and *flexibility* so that one possess a diverse repertoire of approaches to be able to cope with different contexts and situations. From all this, the procedures part of mathematics is to be considered along multiple aspects: the knowing "how" procedures work, the knowing "why" they work, and the fluency and competency in using them.

Further, there is the presence of mathematical formalism in the mathematical activity, and this is also part of the procedures. Formalism can be described as the language of mathematics, the external form with which the mathematicians give shape to and vehicle their thoughts and make them accessible to others (Bourbaki, 1950). Byers and Herscovics (1977) also situate the mathematical formalism in the learning of mathematics along these lines:

> Whether we teach mathematics for intellectual growth or practical utility, the mathematics we teach is a cultural product developed by generations of mathematicians. The development of this product has been inseparable from the development of mathematical symbolism and notation. [...] Progress in mathematics demands that the student learns to cope with formal deduction; he does not have to "know logic", but he has to be able to use it. (p. 25)

Hence, mathematical formalism appears not only as a convention to adopt and reproduce, but also as a thing to work on and with/in – a "fertile research instrument" (Bourbaki, 1950, p. 231). In that sense, whereas the notations and symbolism (e.g., μ, Δ, Σ, 2^3, to name a few) themselves and what they represent lies in the previous branch of conventions, the manipulation of these symbols lies in the realm of procedures. As Byers and Herscovics explain, the symbolic character of mathematics not only saves labour or cognitive burden[33] and automatizes some processes, but makes it easier to understand and "see" new relationships in mathematics. Understanding of formal symbolisation appears as a way to understand, operate and also get to specific answers inaccessible or possibly

[33] This is also said by Bourbaki (1950) for the axiomatic method: "It should be clear from what precedes that its most striking feature is to effect a considerable economy of thought" (p. 227).

lying outside of the realm of the concrete (e.g., abstract algebra, higher degree topological dimensions, etc.). Formalism is often taken as a higher level of abstraction that leads to answers coherent within the mathematical establishment. In other words, formalism can enable mathematical thinking, and lead to higher order of thinking. An example is the formalist rules of calculus which often brings counter-intuitive results, but that are always however coherent within the foundation of mathematics. Devlin (2004) explains the same thing for geometrical proofs:

> As mathematicians from the 19^{th} century onwards ventured into ever greater heights of abstraction, the axiom-proof approach became an indispensable tool for handling concepts that were frequently counterintuitive. (p. 36)

In that sense, to do mathematics requires one to be able to work with and within the symbolisation and conventional aspects of mathematics. There are rules to follow to operate on symbolism. In addition, as was said for algorithms and rules to follow, symbolic manipulations can be interpreted in light of Skemp's (1978) distinction of instrumental and relational understanding, where it is possible to uniquely know "how" to operate, but also to know "why" it is possible or required to operate in that way (e.g., algebraic manipulations).

Again, as it is for conventions, despite their importance and usefulness, it is fundamental to understand that mathematical procedures do not represent the entire mathematical endeavour itself, or the sole mathematical enterprise[34]. Bourbaki (1950) and Brousseau (1988) express similar ideas concerning the fact that conventions and established procedures are only representing one aspect of what mathematics entails.

> Il est vrai qu'à l'occasion il faudra aussi qu'ils apprennent certaines choses qui auront été faites avant, mais c'est pas forcément l'essentiel. L'essentiel ce sera de faire fonctionner ces connaissances au fur et à mesure avec la signification qu'elles peuvent avoir. [*It is true that at some point students will have to learn some things that were produced in the past, but it does not represent the essential part. The*

[34] Procedures in mathematics (algorithms, formulas, symbolic manipulations, etc.) are what will be the more used and utilized by the students in other disciplines or events (science, professions, etc.). However, within mathematics itself, this does not appear to be sufficient.

essential will be to work with this "knowledge" in conjunction with the meaning it can have.] (Brousseau, 1988)[35]

This leads to the third aspect of the mathematical activity, the structures and relations within mathematical concepts[36].

Structures and Relations: Creation in Mathematics

> [T]he mathematician does not work like a machine, nor as the workingman on a moving belt; we can not over-emphasize the fundamental role played in his research by a special intuition, which is not the popular sense-intuition, but rather a kind of direct divination (ahead of all reasoning) of the normal behavior [...] And, for the research worker who suddenly discovers this *structure* in the phenomena which he is studying, it is like a sudden modulation which orients at one stroke in an unexpected direction the intuitive course of his thought, and which illumines with a new light the mathematical landscape in which he is moving about. [...] What all this amounts to is that mathematics has less than ever been reduced to a purely mechanical game of isolated formulas; more than ever does intuition dominate in the genesis of discoveries. (Bourbaki, 1950, pp. 227-228, my emphasis)

This quotation about intuition in mathematics – often termed Aha! experiences – brings us to the last branch of the mathematical activity. As I have mentioned before, there is more to mathematics than procedures and the understanding/reasoning of them. Mathematics is filled with concepts, notions and ideas that have structures and interrelated relationships. Hence, doing mathematics entails deducing, relating ideas, conjecturing, analyzing phenomena, judging and testing, making inferences, recognizing and describing patterns, experimenting, building models and noticing representations of phenomena, and so on. Building on Lakatos's (1976) ideas from his book *Proofs and*

[35] To take only the procedural and calculational aspects of the mathematical enterprise into consideration is to make a huge and dangerous mistake for the education in mathematics, where mathematics becomes perceived as a discipline made of facts, recipes to follow, and drill practice (Battista, 1999). Hence, procedures are part of the mathematical activity, but more is to the enterprise itself.

[36] It is worth noting that different authors that discuss issues and aspects of doing mathematics, whether they are mathematics educators or mathematicians, do not always flag the same aspects as objects of the mathematical activity. Whereas some focus on aspects of conventions (e.g., Bishop, 1977; Byers & Herscovics, 1977; Hewitt, 1999), others barely mention it (e.g., Hiebert & Lefevre, 1986; NRC, 2001; Wu, 1999). The use of techniques, which appears central in the discourse of some authors (e.g., Ashlock, 1990; Bass, 2003; NRC, 2001; Owen, 1990; Russell, 2000; Wilson, 1990; Wu, 1999), does not even appear in the discourse of others (e.g., Battista, 1999; Devlin, 2004; Hewitt, 1999; Schifter & Fosnot, 1993). In this section, I try to assemble these discourses to create a more encompassing and more coherent representation of what the mathematical activity is about – however incomplete or too roughly stated I might do it.

Refutations, Lampert (1990) explains the process of doing and producing mathematics along the lines of proving, conjecturing, creating, discovering, and so on.

> Lakatos's argument, which comes through in the person of the teacher, is that mathematics develops as a process of "conscious guessing" about relationships among quantities and shapes, with proof following a "zig-zag" path starting from conjectures and moving to the examination of premises through the use of counterexamples or "refutations." (p. 30)

This zig-zag activity[37] is placed at the heart of the production of mathematics where it is a continuous intellectual process of re-examining one's assumptions and answers, and possibly refuting them with counter-examples[38]. This zig-zag process, which could also be seen as a circular process of conjectures and refutations, is intended to embody the entire process of how mathematics is, and was historically, created.

One fundamental aspect at the heart of this is the establishment of relations and the creation of connections (Hiebert & Lefevre, 1986). By deducing, inducing and conjecturing, one is playing with and uncovering the structures and relations within mathematical concepts. An important part of doing mathematics, and what is often called working at a conceptual level, implies comparing quantities, establishing relations, creating links, establishing equivalences and differences, substituting ideas for others, assessing invariance, and so on (Hejný, Jirotková & Kratochvílová, 2006). Hence, this aspect is about uncovering and working with the structural aspects of mathematical concepts, but also about "structuring" these mathematical ideas (to give them a structure for one's own understanding). This is what the zig-zag activity is all about.

Concretely speaking, this last branch of the framework of the mathematical activity is an activity recognized as important within reform movements:

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[37] Lampert also focused on aspects of courage and modesty in the mathematical activity of proofs and refutations, based on Lakatos but also on Polya's ideas.

[38] Devlin (2004) explains well how mathematical proofs can be understood as experiments with mathematics and as an enterprise of convincing the community of mathematicians. Within this is also the idea of presenting one's solution and convincing others of its correctness and mathematical worth (Bourbaki, 1950). This, then, directly opens the space for others to enter in the process of judging and examining one's solution, what Lampert (1990) calls a "vulnerability to re-examination that allows mathematics to grow and develop" (p. 30).

> At every level of schooling, and for all students, reform documents recommend that mathematics students should be making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions, and discussing and questioning their own thinking and the thinking of others. (Lampert, 1990, pp. 32-33)

Moreover to knowing the conventions established and the set of skills and procedures useful to do mathematics – "which have been refined over the centuries to enable the solution of theoretical and practical problems" (Lampert, 1990, p. 42) – there is an intellectual activity of creation and production in/of mathematics[39]. Some researchers have even explicitly stated that this aspect of the mathematical activity is at the very heart of what doing mathematics entails (e.g., Bourbaki, 1950; Brousseau, 1988, 2006; Lakatos, 1976; Lampert, 1990). To use Hewitt's (1999) distinctions, this branch of the mathematical activity can only be cultivated and fostered by the teacher, it cannot be "given" since it lies in the realm of sense making and of creating – the role of the teacher then becomes to educate awareness.

> There are aspects of the mathematics curriculum where students do not need to be informed. These are things which students can work out for themselves and know to be correct. They are part of the mathematics curriculum which are not social conventions but rather are *properties* which can be worked out from what someone already knows. [...] So, the mathematical content which is on a curriculum can be divided up into those things which are arbitrary and those things which are necessary. (p. 4, my emphasis)[40]

The work on structures and relations is about deducing and conjecturing, hence it is why the teacher can only strive to provide situations/problems/contexts within which the student will be placed, so that he or she develops and gains awareness of the mathematical ideas, since "[m]athematics is concerned with properties – and properties

[39] Lampert explains that in her teaching she focused on each aspect, but with a different intention or agenda behind it. She talked about knowledge *of* and *about* mathematics: "This means that I needed to work on two teaching agendas simultaneously. One agenda was related to the goal of students' acquiring technical skills and knowledge in the discipline, which could be called knowledge *of* mathematics, or mathematical content. The other agenda, of course, was working toward the goal of students' acquiring the skills and dispositions necessary to participate in disciplinary discourse, which could be called knowledge *about* mathematics, or mathematical practice" (p. 44, emphasis in the original). That one finds this distinction insightful or not is not relevant here, but what is interesting is the distinction made between knowing, using and understanding techniques, *and* the activity of conceptually producing mathematics.

[40] I do not assert that Hewitt's concept of "necessary" is what I mean by "structures and relations within concepts." However, I am using his distinction about the nature of mathematics to make my point.

can be worked out or found out" (Hewitt, 1999, p. 5) by the student or the mathematics doer. This is why this branch is concerned with "producing" and "creating" mathematics.

The Importance of All the Branches of the Mathematical Activity

> Sometimes a simple skill is absolutely indispensable for the understanding of more sophisticated processes. For example, the familiar long division of one number by another provides the key ingredient to understanding why fractions are repeating decimals. (Wu, 1999, pp. 14-16)

All the three branches of the mathematical activity are essential, each in different and complementary ways; all parts complement each other, are intertwined, but also build on each other. Even if it is tempting to favour one over the other, it is important to remember that they all have a role to play and an importance in the mathematical activity. Whereas the conventions trace the ways of doing that are accepted in mathematics and how things are named and done, the procedures enable to efficiently operate and even automatize processes, and the part about structures and relations represents the site of creation and production of mathematical ideas. They are all needed to a certain, and often different, degree in the mathematical activity. In that sense, even if mathematics seems filled with symbols and vocabulary, it does not mean conventions are more important; even though it is important to automatize procedures and use them efficiently, it does not makes them more important; and, finally, even if mathematics is seen as a meaning making and human creative science, working uniquely around the structures and relations without using the other branches will not bring the mathematical doer very far. In sum, to neglect one of the branches of the mathematical activity would create a significant gap or a weakness in the mathematical process, and result in difficult instances of producing mathematics. That said, it appears that to educate someone in mathematics means to attend to the three branches of the mathematical activity.

Mathematical Activity, School Mathematics and Defining Teachers' Knowledge

> The fundamental task of a mathematician who teaches is to convey to his students not only what mathematicians know, but also what they do, and how, and why. (Moise, 1965, pp. 411-412)

Each branch has its own specific characteristics when it comes to teaching and learning mathematics. Conventions have to be presented and told by the teacher, they are not instances of making sense, they are to be *known* and used. Procedures have to be used and re-used efficiently and even automatized, so they have to be *known* to be applied concretely, but they also have to be *understood* and made sense of, so that the mathematical doer knows what he or she is doing. Finally, the structures and relations part of doing mathematics is lodged in a meaning making realm, where it is not a matter of knowing "how" and "why" procedures work or of using them, but of making sense within mathematics and of *producing/creating* meaning, of establishing links and relationships. Each part of the mathematical activity has its own particularities and characteristics and so influences how it can be dealt with in the teaching and learning of mathematics. In the following diagram (figure 2.2), I have added the above explanations to the mathematical activity framework.

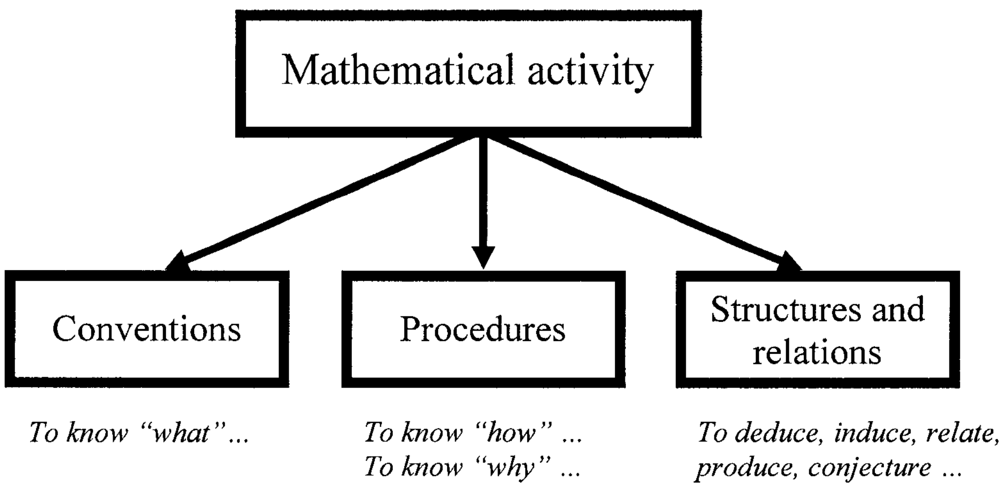

Figure 2.2. The three branches of the mathematical activity and their associated type of learning

Whereas this description of the mathematical activity represents a more genuine sketch of the activity of doing mathematics, traditional school mathematics has not always focus on all three branches. Traditional teaching mostly follows the idea that mathematics is a set of procedures and facts to memorize, an activity of mimic and drill-and-practice (Battista, 1999; Schifter & Fosnot, 1993). Unfortunately, this attitude towards mathematics has transpired teaching for years, and still represents the prominent orientation driving mathematics teaching in today's classrooms (Cooney & Wiegel, 2003; Hiebert et al., 2003). It is a situation that needs to change for it is perpetuating an incomplete image of what doing mathematics entails (Battista, 1999; Schifter & Fosnot, 1993). This situates the teaching and learning of mathematics in schools in a realm of "knowing" rather than one of "understanding," in a realm of memory rather than one of reasoning (figure 2.3).

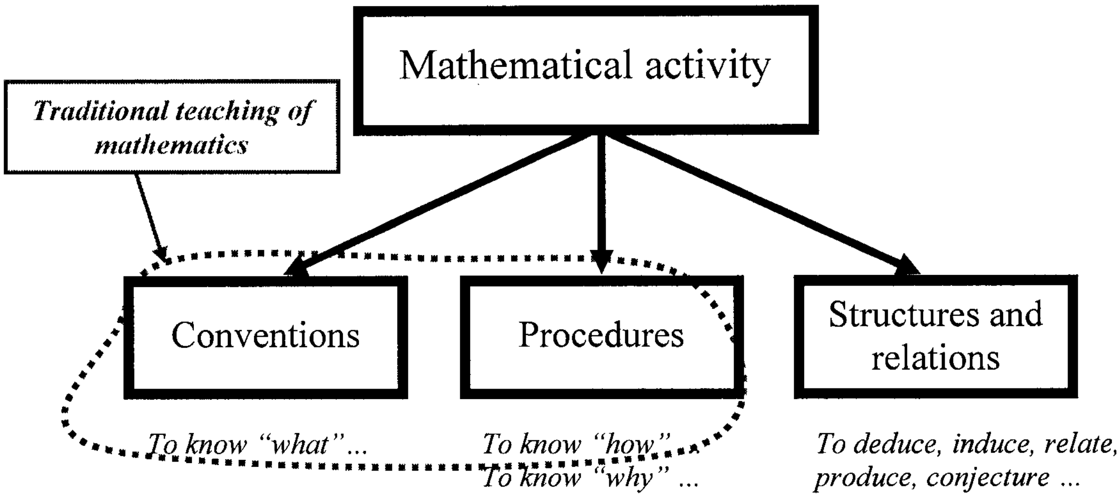

Figure 2.3. Traditional teaching and mathematical activity

This discussion brings me back to the teachers in my research. This last diagram enables me to situate teachers' actions within the traditional approach in the sense that what they know lies within this (knowledge of facts and of how to apply procedures). Most of the time, concerning topics of study, teachers know the procedures, symbolism and conventions associated with them well. They function well mathematically for any topic. However, aspects aside from this are most of the time simply unknown or unfamiliar to them. So, when I assert that I intend to work at enlarging teachers' knowledge so that it encompasses more than procedures and calculations, I mean providing teachers with opportunities to experience the "conceptual" part of mathematics, to experience more than procedures and calculations. And this "conceptual" mathematics is composed of (1) opportunities to experience and develop meaning behind and about the procedures they use (Skemp's relational understanding), and (2) opportunities to experience and interact with the structures and relations within mathematical concepts by establishing connections, patterns, properties, and so on. This is what is meant by working on "conceptual" mathematics in this dissertation.

In effect, secondary-level mathematics teachers are not familiar with, hence are in need of working on, these aspects of relational understanding and of structures and relations within mathematical concepts. This is where their knowledge is limited or too narrow[41]. The limitations in their knowledge is not only about not knowing the meaning behind procedures, it is also about the structures and relations within the mathematical concepts (of deducing, inducing, conjecturing, etc.). Hence, this is what is meant by a "conceptual" approach to mathematics, namely working along the three branches of the mathematical activity, but more precisely with a focus along developing and experiencing relational understanding of the procedures and playing/uncovering the structures and relations within mathematical concepts.

I now turn to the approach that I have used to have teachers' experience "conceptual" mathematics. I have developed a model for professional development that enables me to enact these ideas; I call this approach the "deep conceptual probes into the mathematics to teach."

A Model for Professional Development of Secondary-Level Mathematics Teachers: Deep Conceptual Probes into the Mathematics to Teach

In order to have teachers live these sorts of "conceptual" mathematics, I developed a framework that would lead and orient explorations into the mathematical contents that teachers teach. In other words, I designed a model for professional development that would build on teachers' mathematical knowledge and attempt at enlarging it by exploring "conceptual" mathematics.

I wish to repeat myself here to make a point. I recognized myself in these teachers' inclinations for procedures. This is how I personally was when I began to learn to teach mathematics in my B.Ed. But, over the course of my program, I changed. For that reason,

[41] Therefore, it is around these two aspects of relational understanding and knowledge of structures and relations that I will focus my interventions in the professional development sessions. Not that I will not intervene at the level of conventions, but the main focus will be on relational understanding and on structures and relations within mathematical concepts.

it appears relevant to look back at my own teacher education received as a B.Ed. student at the *Université du Québec à Montréal* (UQÀM) since it had such an enormous influence on my understanding of mathematics and initiated important changes in me concerning my focus on procedures (which was in effect a normal inheritance from my previous education as a student). In other words, I re-learned a lot of mathematics in those years[42].

UQÀM's Approach in Didactique of Mathematics

The UQÀM approach in *didactique* of mathematics[43] was significant in my own education. My education to become a secondary mathematics teacher happened in Quebec, Canada, at the *Université du Québec à Montréal* (UQÀM) in a 4-year baccalaureate of education (B.Ed). The UQÀM's mathematics teacher education program is distinct in that it moves away from the dominant model centered on training in the specific discipline followed by training in education, and then afterwards by some practicum in schools. The program focuses on an integration of mathematics, pedagogy, teaching mathematics, and practicum of mathematics teaching in all of its four years – in an attempt to support "[u]ne *integration constante des dimensions théoriques et pratiques* [a *constant integration of the theoretical and practical dimensions*]" (Bednarz, Gattuso & Mary, 1995, p. 19, emphasis in the original)[44].

The founding group of the UQÀM mathematics teacher educators, who were for most of them mathematicians (e.g., Claude Janvier, Nadine Bednarz, Bernadette Dufour-Janvier, to name a few), started in the 1970s with specific orientations and preoccupations toward teacher education. Having to contribute at the pre- and in-service level of

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[42] To help me to understand even better the approach used at UQÀM, I conducted an interview with one of my previous professors and founding members of the group in the 1970s, Nadine Bednarz.

[43] It is important to notice that the idea of *didactique* of mathematics that is worked on in Quebec's UQÀM is not the same as in other places in the world, for example in France or Germany, and is even different than in other Quebec universities (Concordia, University of Montreal, University Laval, etc.). The UQÀM movement in *didactique* of mathematics started with an intention to educate teachers in mathematics. In fact, the "sector" of *didactique* of mathematics in UQÀM, which is lodged in a mathematics department, was first called the "teaching of mathematics" section.

[44] I do not intend to go into all the specifics of the program here, but to point to some elements that were influential for the construction of the professional development approach. For a detailed analysis of the particularities of the program, see Bednarz (2001), Bednarz, Gattuso, and Mary (1995, 1996, 1999) and Bednarz and Proulx (2005).

education of teachers of mathematics, the group's approach emerged out of a specific intention to offer mathematics courses for already practicing teachers of mathematics – in a province wide professional development initiative for teachers of mathematics because there were many mathematics teachers who did not had a sufficient education in mathematics[45].

To better situate and contextualize the work, it is important to remember that in the 1970s, mathematics education and *didactique* of mathematics were in their infancy and not much knowledge about student's learning of mathematics was available to the field. To this end, the first courses created were aimed at teaching mathematics in schools and were structured around a comprehensive approach to the mathematical contents themselves, irrespective of students' learning of them. This was not because the group willingly set aside students' learning issues, but mostly because they did not have the knowledge of it at that time.

The group worked on building *mathematics* courses for school teachers, adapting them to school mathematics and its teaching. In addition, since they were professors in the mathematics department, they could decide on the type of mathematics courses that would be provided to prospective teachers at the pre-service level. Therefore, for both in- and pre-service level, they did not develop "traditional" mathematics courses, but created mathematics courses focused on the concepts taught in schools with the intention to develop a deep mathematical understanding of them (e.g., numeral structures, functions, algebra, geometry, probability and statistics). To create these courses, the UQÀM professors had to study the mathematical concepts deeply and develop/unearth many notions and elaborate on them.

After a number of years of working in this direction, the group started to conduct research, participate in international scientific meetings on mathematics education (and *didactique* of mathematics), and supervise future teachers in their practicum settings in secondary schools. Through these activities, the UQÀM mathematics educators became more knowledgeable about how the mathematical contents and concepts were actually learned and understood by students in schools. The UQÀM group was in a better position

[45] This project was called PERMAMA – *Perfectionnement des Maîtres en Mathématiques*.

to understand and evaluate the potential of different learning and teaching situations about specific concepts. Thus, they added this dimension to the structure of their courses. The pre-service courses were now not only focusing on deep analysis of the mathematical concepts, but also on the *learning* and *teaching* of these concepts. Mathematical topics were then analyzed and presented in this fashion, with a deep analysis of contents and also of the teaching and learning issues linked to these contents, in order to enhance the teaching of these mathematical contents for prospective teachers. In particular, the mathematics educators placed an important emphasis on the appropriation of the mathematical contents and also on the preparation for its teaching. They argue that learning to teach mathematics cannot be done in isolation of the study of the mathematical content – mathematics needs to be at the core of the activity of learning to teach mathematics (Bednarz, 2001; Bednarz et al., 1995).

Two of the main activities done in the UQÀM courses are worth noticing here. One main activity in the UQÀM program is centred on the construction of a "conceptual analysis" for mathematical content. These conceptual analyses involve deep analysis of specific mathematical content, something used afterwards as the basis for preparing and creating lesson plans. The conceptual analysis of mathematical contents (e.g., circle, trigonometry of the triangle, systems of equations, etc.) is similar to what the UQÀM mathematics educators did themselves in the 1970s to plan and prepare their mathematics courses for teachers. For the most part, it consists in finding and unearthing the key "reasonings" and concepts within the content to teach. The conceptual analyses could also be seen as the elaboration of a concept map (Skemp, 1987)[46], however it is not done in this fashion in the courses. This unpacking of key "reasonings" is important because it helps orient the teaching of the content through the delineation of the mathematical issues and ideas. This activity is primarily a lesson-planning tool, but with mathematical content at its core. In addition to this deep digging into the content and concepts, prospective teachers have to attempt at underscoring the possible aspects of students' understanding in relation to these key "reasonings." For example, prospective teachers consider possible difficulties students can experience, their frequent misconceptions, the prior knowledge

[46] Interestingly, in his book (chap. 9), Skemp also explains that concept maps represent useful tools to plan lessons.

they will come up with from the previous grades (what they would ideally need to know), and the important automatisms they need to develop in the study of these concepts. Student teachers do this by browsing into the literature on mathematics teaching and learning (professional and scientific journals, research reports, etc.), into the pedagogical resources and textbooks available, into curricular materials, and also by consulting some resource persons made available to them (teacher assistants) and even the teacher educator itself – or professors in the department – who can orient them toward some literature and ideas. Students also come up with some of these ideas on their own, inspired from their own knowledge and prior experiences in learning these contents.

Another important aspect worked on throughout the UQÀM courses is the development of an understanding and an ability to make sense of students' particular mathematical strategies, difficulties and misconceptions for diverse mathematical topics, and on reflecting on how to handle (or sometimes remediate in the case of difficulties) these issues in their classroom practices – how to intervene as teachers with the students. In other words, to develop competencies to make sense of students' possible understandings, the courses focus on future teachers' constructions of principles and knowledge to elaborate a repertoire of pedagogical strategies relevant to the teaching and learning of the mathematical concepts. Those competencies are often developed by the utilization and analysis of videotaped classroom lessons, of students' work, and of individual student interviews.

In all these activities, the mathematics educators often select tasks and place prospective teachers in situations that aim at questioning their assumptions concerning their own mathematical knowledge, and also their assumptions concerning its learning and its teaching. Again, one important idea is for prospective teachers to develop a deep sense of the mathematical concepts – because the future secondary teachers often arrive with a narrow mathematical understanding that emphasizes procedures and calculations (Bednarz, 2001; Bednarz et al., 1995). In short, the approach taken in the UQÀM teacher education program that I graduated from focuses on two specific nested elements: (1) an analysis of the school mathematical concepts to teach, with the intention to know more and understand better these very mathematical concepts, and (2) the development of an understanding of students' rapport with and learning of these mathematical concepts.

These two elements have influenced and oriented the approach to professional development that I adopt in this research.

Elaborating the Framework of Deep Conceptual Probes into the Mathematics to Teach

> Perhaps the central goal of all the teacher preparation and professional development programs is in helping teachers *understand* the mathematics they teach, how their students learn that mathematics, and how to facilitate that learning. (NRC, 2001, p. 398, emphasis in the original)

The UQÀM approach served as a basis for the model of professional development of my study. However, it has less of a direct focus on learning to teach and one more toward the mathematics content, since the primary goal of my program is the learning of the mathematics of the school curriculum along a more "conceptual" way[47].

As in UQÀM's construction of "conceptual analysis," I decided to focus on the deep exploration of concepts. However it was not to create a "conceptual analysis" with teachers or a concept map about the topic, but simply to explore deeply some mathematical ideas through the medium of tasks, problems, situations and presentations about the concepts[48]. Hence, by working on these tasks and situations, by exploring the mathematical ideas within them, it aimed at having teachers experience and learn mathematics along a "conceptual" way. In other words, it aimed at investigating deeply and "conceptually" the mathematical concepts of the curriculum, and therefore to have teachers live and experience these sorts of "conceptual" mathematics.

In addition, because the mathematics ideas worked on and explored represents (potentially) something new and unfamiliar for these teachers – being often about more than procedures and calculations – the need to inquire and make sense of what these (new) mathematical ideas can "mean" for teaching has the potential to emerge. Indeed, these explorations place teachers into a mathematical realm that they are not very familiar with, but concerns the same mathematical content that they teach, hence raising issues for

[47] The UQÀM approach could be seen, to some extent, as a warrant to the approach for professional development that I will offer here.

[48] The specific details on the type of tasks and situations used to initiate these mathematical explorations are discussed in the Chapter 4 on methodology.

them as to what it could mean for teaching. In other words, *from new mathematical ideas emerge new teaching issues*. This is a sort of "by-product" that appears to happen naturally since the location of these mathematics contents for teachers is within their teaching. In that sense, discussing teaching issues naturally unfolds from the explorations of mathematical ideas with teachers. Therefore, it should not come to a surprise that as work and explorations are undertaken on the mathematical concepts, issues of teaching and learning are brought forth. This should even be seen as a healthy aspect of a professional development program (to address teaching issues), since professional development hopefully aims at having an impact on teachers' instructional practices and students' mathematical experiences (Fennema & Franke, 1992; Zaslavsky et al., 2003). Moreover, it is to be expected that teachers taking professional development would aim for that – as did the teachers in my project.

This double endeavour of addressing mathematical issues and the teaching issues linked to them fits well and concurs with Cooney's (1994) discussion of teachers' growth in order to develop what he refers to as *mathematical* and *pedagogical power* for their future actions in the classroom[49]. It also appears similar to Russell's (2000) recommendations:

> Professional development structures and materials need to provide long-term work in which teachers immerse themselves in both examining mathematical content and learning about children's mathematical thinking through intensive institutes or regular, ongoing school-year seminars. (p. 158)

In other words, the work on the mathematics that teachers have to teach is intended to explore aspects of the mathematics of the curriculum, but with a constant eye on what

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[49] It is to be noted that Cooney (1994) mostly talks about the importance for a teacher to develop pedagogical power, making an analogy with and drawing from the construct of the development of "mathematical power" in students. I myself see both mathematical and pedagogical powers as fundamental aspects to develop in teachers (and I assume from reading his writings that Cooney does too), hence, it fits well into my argument. Therefore, because Cooney's writings (e.g., 1994, 2001, Cooney & Wiegel, 2003) are quite transparent in regard to the importance of mathematics, I associate these ideas to him. Along that line, Cooney (2001) mentions having developed material that integrates content and pedagogy about different topics, which "are designed to present different mathematical situations and engage teachers in problem solving in which they encounter such questions as 'What happens if?' and the concomitant pedagogical issues that are embedded in the mathematical considerations" (p. 27). This appears to be closely linked to the approach that I develop and offer here.

these explored mathematical concepts mean for teaching[50]. However, above all things, the approach aims at offering teachers the chance to see and explore mathematics with a different eye, to have them experience that mathematics is more than a set of procedures, facts and calculations.

I have called this frame the "deep conceptual probes into the mathematics to teach." These deep conceptual probes aim at exploring in depth the mathematical concepts and topics that teachers have to teach, in an intention (1) to learn more about the mathematics that has to be taught, and (2) to have teaching issues linked to these (newly learned) mathematical concepts emerge in the exploration (figure 2.4).

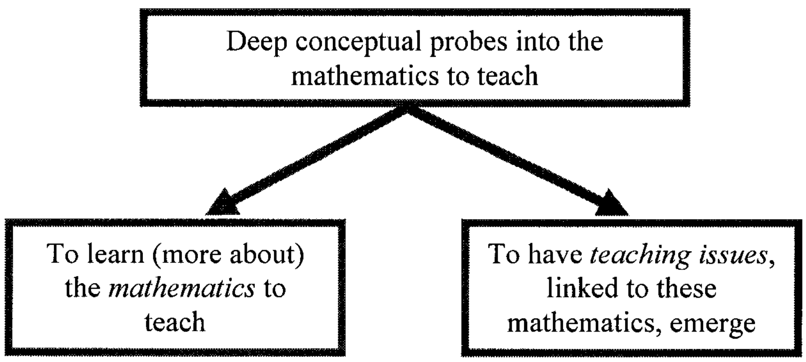

Figure 2.4. The diagram representing the model of deep conceptual probes into the mathematics to teach

Teaching Issues as Emergent Instances

This deep-conceptual-probe model is rooted in mathematics, in the sense that its starting point is in the school mathematics of the curriculum. It is by starting with an

[50] This idea of working on school mathematics and the teaching issues linked to them is an important distinction that distinguishes this type of work from one that would *only* have teachers "do" mathematics – which is something a bit too limited or not sufficient for mathematics teachers as Schifter (2001) explains. The mathematical explorations are aiming at more than simply "doing" mathematics, since the teaching/learning issues are also of major importance. The importance of working on both content (school mathematics) and pedagogy (teaching/learning issues) is also supported by the study of Saxe, Gearhart and Suad Nasir (2001) that showed that teachers taking professional development focusing on mathematics and children learning of mathematics (and motivation towards it) was more efficient concerning students' achievement on "conceptual skills" and "computational skills" than teachers taking professional development focusing uniquely on teachers reflecting on their practice. Moreover, this type of professional development also gave better students' results on "conceptual skills," and equal results on "computational skills," than teachers not taking any professional development and who focused strongly on textbooks.

exploration of diverse school mathematics topics and concepts and going deep into them that (1) new mathematics is learned, and that (2) teaching issues linked to them emerge. Whereas some teaching issues can possibly be prepared and raised in advance (some well known misconceptions on a topic, for example), most teaching issues will emerge out of the explorations on the mathematical topics themselves and will be brought up and addressed along the way within the explorations taking place. In the same way, it is believed that a lot of the mathematics that will be addressed and explored will emerge out of the current explorations themselves. Obviously, as I elaborate in the chapter on methodology, mathematical issues to work with are prepared in advance and will be directly offered to teachers, but the orientation that the discussions and explorations take (for the mathematical explorations and the teaching issues) is unpredictable and rests on a "leap of faith" – a trust commitment – from the teacher educator. In that sense, in this approach the teacher educator needs to trust that teaching issues will emerge (and further mathematical explorations too) from the mathematical concepts offered for exploration (in the form of tasks and situations). Whereas in some sense this can appear to be evident because the mathematics that will be worked on will be "new" or unfamiliar to teachers, therefore raising the need to discuss teaching issues to make sense of these approaches and these mathematics within a teaching context (what it means for teaching), in some other moments it is not always obvious what could be the kinds of possibilities and issues that are to be raised for teaching and learning concerning some topics and concepts. The same holds for the mathematics explored itself, where it is not always obvious as to where the exploration of some mathematical topics and concepts can lead to.

The contingency and emergence of teaching issues (and further mathematical explorations) are linked to the fact that the happenings of a session are neither predicted nor rigidly planned in advance and that the session follows its own path, triggered by the issues raised and the mathematical explorations undertook. This is as much for the teaching issues as it is for the mathematics itself. The teaching issues are contingent to the mathematical work being done, and the mathematical explorations are pushed further ahead on the basis of the explorations themselves. Hence, the exact events of a session are unpredictable and for this reason rest on a leap of faith from the teacher educator. As the mathematics teacher educator, I let these events flow and I try to push the explorations, even if they were not anticipated.

This unpredictable implicit character of the deep-conceptual-probes model requires a different or simply a specific mindset for the teacher educator, since the enactment of the sessions cannot be prescribed and laid out beforehand. The teacher educator has to rely on ideas of emergence. This leads to a different theoretical positioning that requires a dynamic understanding of teaching and learning, something different than what a pre-specified and linear view would offer. The next chapter provides the theoretical orientations that ground these ideas.

CHAPTER 3

EMERGENCE AND ENACTIVISM: GROUNDING THE WORK AROUND NEW THEORETICAL ORIENTATIONS

The Linear View: The Top-Down Approach

"Technicist" approaches toward professional development have dominated education for a long time (Bednarz, 2000; Zaslavsky et al., 2003). Such approaches reflect a view of teaching in which the necessary (needed) tools for teachers are decided in advance to bring ready-made solutions to predictable problems. Because the "knowledge of how to do" is already pre-decided and pre-packaged, it assumes that educating for professional development is possible through a transmission model of knowledge from leaders to other leaders or teachers in the field. This model of transmission of "information on how to teach" or "what to do," termed a top-down model, assumes that (1) there exists a predetermined set of well-polished tools for teaching and (2) that these are transferable from one person to another without any problems of comprehension – all of this leading to "the perfect application" of ideas in the classroom. This unsound model (Bednarz, 2000, 2004) is grounded in two problematic issues: a linear view of knowledge and the assumption that a pre-existing, fixed body of knowledge within which to conform exists.

A top-down model is grounded in a linear view of learning where knowledge grows at a constant and additive rate and is seen as an accumulation of facts and ideas along a continuum. This could be represented by the imagery of a linear graph, one where knowledge gets accumulated and piled up (Davis & Simmt, 2004).

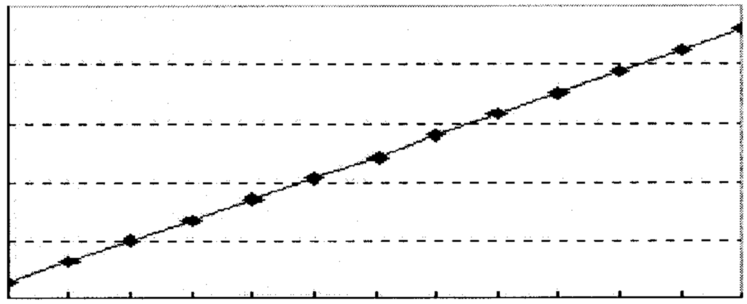

Figure 3.1. Knowledge seen as a linear accumulation of facts
(Davis & Simmt, 2004; used with permission)

With this view in mind, pre-packaged information on "how to teach" and "what to do" can be transferred to others who simply accumulate this information – the more professional development sessions you go to, the more you will know. This assumes a causal view of learning, in which the delivery of pre-packaged information is transferred to teachers who will automatically understand it and add it into their practices. Such a model does not take the teachers themselves into account in the process and how they make sense of the ideas that are brought to them. Mason (2004) criticizes this model:

> Cause and effect mechanism makes sense with machines which continue in one state until altered by fatigue or adjustment; it does not make sense when applied uncritically to organisms, and especially to human beings. Even medical doctors are beginning to realise that drugs do not have the same effect on all patients. Furthermore, the ramifications of a cause and effect mechanism for teaching and learning have been in place for many generations, and have proved not to have succeeded. (p. 1377)

Consequently, in top-down models, there are pre-determined ways and issues to know and conform to, which become the objectives of the professional development sessions. Drawing on Pimm's (1993) concept of "change merchant," Breen (1999) explains, and criticizes, that some teacher educators make a central task of convincing others of the quality of their own particular merchandise and of having people use their "magical" and infallible teaching methods. These teacher educators intend to control and strive toward creating or generating "perfect teachers" who enact assumed "good practices." Mason

(2004) condemns this tendency and explains that there is no panacea that would solve all teaching problems; there is no ultimate teaching approach. Putnam et al. (1992) also address this orientation:

> Reformers cannot simply tell teachers to teach differently. For as we have seen, there are no ready prescriptions for thoughtful teaching. (p. 226)

The problem with a top-down model is that it sets aside the teachers themselves in the learning process (Dawson, 1999). It treats teachers as if they were empty vessels to fill. From this, it follows that specific objectives are pre-determined and made to represent the goals of the sessions given to teachers. These specific, pre-decided objectives become the final goals to be obtained in the professional development sessions. All action undertaken by the teacher educator is constantly oriented toward the attainment of these predetermined objectives. In fact, the success of these professional development top-down sessions depends on having teachers acquire what was intended and on "meeting" the objectives. This suggests the imagery of a linear trajectory of taking teachers from the point *A* and bringing them to point *B* (figure 3.2).

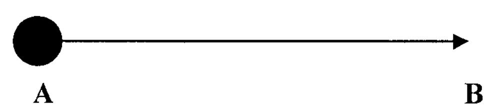

Figure 3.2. Linear imagery from A to B

Constructivist theories have been criticizing this for decades by theorizing that the learner plays a role in the learning process and that learning is not reducible to a cause and effect phenomenon (Glasersfeld, 1995). Professional development needs to build on teachers' knowledge and context and implicate teachers in the learning and knowledge-producing process, and not simply provide them with pre-packaged and digested methods to acquire and reproduce (Bednarz, 2000, 2004). Teachers' understanding cannot be "controlled" nor can it be precisely predicted. Therefore, an alternative view of learning is needed to ground different ways of acting as a teacher educator in in-service education, one that moves away from top-down approaches.

A Different View of Learning and its Implications for Acting in Professional Development

The overarching epistemology that grounds this work is known as enactivism (Maturana & Varela, 1992; Varela, 1996; Varela, Thompson & Rosch, 1991)[51]. Enactivism is a theory of cognition which views human knowledge and meaning-making as processes that are understood and theorized from a biological and evolutionary standpoint. In other words, our biology matters in the process of coming to know. Important in this theory, and specifically relevant for this work[52], is the concept of *natural drift*, a concept that has its roots in Darwin's (1867) theory of evolution and centers on two specific notions: structural coupling and structural determinism. These two concepts are in sharp contrast to causal and linear ideas of learning in which the learner "takes things in," and in that sense will enable an alternative understanding and orientation for the learning experiences and events in professional development sessions.

Natural Drift, Structural Coupling and Structural Determinism

To make sense of the process of survival of species, Darwin used the concept of "fitting." For species to survive, it must continuously adapt to its environment, to *fit* within it. If not, it would perish. In a sense, Darwin offered a pejorative or negative view of the survival of species: species that survived simply did not die – and continued to adapt. As trivial as it may seems, it created an important break from ideas of absolutism and universality which were dominating evolutionary thinking at the time[53]. The idea of "fitting" escaped notions of absolutism and of the best or fittest species. The idea was now that species were compatible and fitted within their environment; it did not represent the absolute species but simply a fitting species, one adequate for the circumstances of the moment.

The concept of fitting is not a static one in which the environment is constant and only the species evolves and continues to adapt. Darwin explained that species and

[51] Although Maturana never explicitly called himself an "enactivist," for matters of clarity and simplicity I will use this term throughout.

[52] I only report here on aspects of enactivism that are relevant for my doctoral dissertation. The foundations of the theory are detailed in Maturana and Varela (1992), and its situation within cognitive science is elaborated in Varela (1996) and Varela et al. (1991).

[53] For a concise overview of Darwin's work and times, see Howard (2000).

environment co-evolve; Maturana and Varela (1992) add that they co-adapt to each other, meaning that each influences the other in the course of evolution. In other words, the fit is an evolving one, with both parties evolving[54]. The idea of co-evolution between environment and species is key in regard to the origin of changes or adaptations of species to its environment. By co-evolving, species and environment experience a history together and influence each other in this process. This is why it sometimes seems as if some species are so compatible with their environment that they appear to be "perfectly made for it," and, inversely, that the environment seems perfectly suited for the species[55].

This co-evolution is called *structural coupling* by Maturana and Varela, because both environment and organism interact with the other and experience a mutual history of evolutionary changes and transformations. Both organism and environment undergo changes in their structure in the process of evolution and this makes them "adapted" and compatible with each other. They learn:

> Every ontogeny occurs within an environment [...] it will become clear to us that the interactions (as long as they are recurrent) between [organism] and environment will consists of reciprocal perturbations. [...] The results will be a history of mutual congruent structural changes as long as the [organism] and its containing environment do not disintegrate: there will be a *structural coupling*. (p. 75, emphasis in the original)

From this notion of structural coupling, it follows that the environment does not act as a selector nor does it predetermine or cause evolution: rather, it is a "trigger" for the species to evolve, much as the species acts as "trigger" for the environment to evolve. The authors explain that events and changes are occasioned by the environment, but they are determined by the species's structure.

> Therefore, we have used the expression "to trigger" an effect. In this way we refer to the fact that the changes that results from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system*. The same holds true for the environment: the living being is a source of perturbations and not of instructions. (p. 96, emphasis in the original)

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[54] Capra (1996) explains that this creates a shift from evolution to co-evolution.

[55] A striking example of that is reported in the article "Une incroyable association insecte-plante a été repérée" (2005), which discusses an association between ants and a plant to capture grasshoppers.

Maturana and Varela call this phenomenon *structural determinism*, meaning that it is the structure of the organism that allows for changes to occur, changes "triggered" by the interaction of the organism with its environment. They give the following example: A car that hits a tree will be destroyed, whereas the same thing would not happen to an army tank. Thus, the changes do not reside inside of the "trigger" (inside the tree), they come from the organism interacting with the "trigger." The "triggers" from the environment are essential, but they do not determine the changes. In short, changes in the organism are dependent on, but not determined by, the environment[56]. With this, structural coupling can be redefined in terms of the history of co-evolution and co-influence of species and environment, determined by each parties's structure[57]. An organism's structure allows for the changes to occur.

These notions have important implications for the professional development context. I elaborate on this in the following.

The Learners in the Teaching Situation

With the concepts of structural determinism and structural coupling, learning and change are not seen as causal events determined by external stimulus (even though they are "triggered" by that external stimulus). Rather, learning and change arise from the learner's own structure as it interacts with its environment. This demonstrates well the importance of the learners in the teaching situation. Hence, the teachers have to be taken into account in how in-service sessions (can) unfold. What is offered to teachers does not inherently possess the "power to educate" in itself, but must resonate with and be taken up by teachers in order for them to make sense of what they have been offered. In that sense, the outcomes or effects that sessions can have on teachers are determined by the teachers themselves, even if these are "triggered" by what is offered to them.

Because I intend *to build* on teachers' knowledge in order to expand it, there is no other choice than to reject a top-down linear approach. It simply does not fit here. If I

[56] Again, the same could be said for the changes in the environment in relation to the organism.

[57] Also, it is important to notice, Maturana and Varela explain that for a specific species, all other species are part of the external "environment" and do not have a specific/different status from anything else that is external to the species – even if other species can be conceptualized as having different attributes on some level other than simple elements of the environment.

accept the concept of structural determinism, then anything offered as a situation or a task for teachers to explore is at most a "trigger." The teachers' explorations will be oriented by their own understandings and meanings of these situations and tasks, and by what constitutes issues to explore for them. Varela (1996) refers to this as problem posing.

Problem Solving and Problem Posing

Varela (1996; Varela et al., 1991) explains structural determinism in terms of the difference between problem solving and problem posing. Problem solving implies already present problems situated in the world and lying "out there" waiting to be solved, independent of us as knowers. For Varela, because of our co-determination with the environment in which we live, because we have a structure and because we are coupled with that environment, problematic situations emerge for us in the sense that we specify the meaning that these situations have and how we deal with them. These problems do not lay "out there," objective and independent of our actions. We specify the problems we encounter because of our structure that enables us to act and recognize things in specific ways.

> La plus importante faculté de toute cognition vivante est précisément, dans une large mesure, de *poser* les questions pertinentes qui surgissent à chaque moment de notre vie. Elles ne sont pas prédéfinies mais *enactées*, on les *fait-émerger* sur un arrière-plan, et les critères de pertinence sont dictés par notre sens commun, d'une manière toujours contextuelle. [*The most important ability of all living cognition is precisely, to a large extent, to* pose *the relevant questions that emerge at each moment of our life. They are not predefined but* enacted, *we* bring them forth *against a background, and the relevance criteria are oriented by our common sense, always in a contextualized fashion.*] (Varela, 1996, p. 91, emphasis in the original)

In that sense, the problems we encounter and the questions we undertake are as much a part of us as they are part of the environment. We interpret events as issues to address, we see them as problems to solve. We are not acting on pre-existing situations, our co-determination and interaction with the environment creates, enables and specifies the possible situations to act towards. The problems we solve are then implicitly relevant for us and are part of our structure. Our structural determinism allows these *to be* problems for us, as the environment "triggers" them in us[58].

This is important for my research because it offers a frame that explains that the issues that will be addressed and explored in the in-service sessions will be the ones that resonate with and emerge from the teachers' structures or knowledge. Simply put, regardless of the situations and tasks that I will offer to the teachers, the issues that will be addressed or the orientation of the explorations taken cannot be pre-decided. Although these will be "triggered" by the situation or task offered, they will be explicitly determined by the teachers' knowledge (structure). Hence, the professional development sessions will go in directions in relation to the teachers' knowledge and ways of making sense of the tasks that I will offer them. In that sense, there is no linear path or trajectory that can be pre-traced and therefore, as the teacher educator, in spite of my planning and my intentions, I have no guarantee that specific issues will be dealt with. The tasks and situations offered are there to "trigger," and teachers will explore and make sense of them in the way they can. Nothing can be forced in them or directly transferred, what teachers learn is determined by who they are and what they know.

Obviously, this can lead to an understanding that the events in the session can go in any directions. In a way, this is potentially true. However, in another way, we must recognize that the work and explorations are constrained by the environment in which they are given. In that sense, the tasks and situations offered, even if they cannot prescribe the actions, still implicitly constrain the orientation of the explorations carried out. To use Davis and Simmt's (2003) expression, the tasks act as "liberating constraints." Even if it were possible, it would be surprising in the sessions to have a discussion that would be completely outside of mathematics teaching and schooling. In that sense, the interactions and explorations are constrained by the situation of professional development of mathematics teachers, even if within this "frame" it can potentially go anywhere – the problems posed and addressed will be determined by the teachers (and obviously myself as the teacher educator). Hence, and this is an important

[58] And, obviously, some "issues" of the environment that would "trigger" elements in some persons do not "trigger" the same elements in others. In that sense, the effects of the environment are not *in* the environment, they are made possible by the organism's structure in interaction with its environment.

distinction from top-down models for professional development, the explorations will address issues that are "triggered" in teachers by the tasks and situations offered. However, the way these tasks will be handled and probed into is unpredictable. They start from teachers' knowledge, builds on it, and expands it. To use Varela's (1987) expression, the path will be laid down while walking it. The explorations will take their own course. It is in that way that in the deep-conceptual-probes model of professional development (described in Chapter 2), teaching issues emerge as mathematics concepts are explored and made sense of – as much as other mathematical explorations emerge out of previous ones.

Events of professional development are not caused nor do they follow a pre-traced path. Events that happen within a professional development program are emergent and non-linear. The learning process and the events experienced are not conceived in linear terms, but rather as unfolding in relation both to the learners and to the issues explored.

Emergence and Contingence: Events Seen as Openings and Cascades

> In any of these fields, we may not know in advance where our activities are leading us, but to be deterred by that would be to accept that no solution is possible. Just as an artist like Cézanne did not know when he started out on a work whether it would have any meaning or be understood, so human beings in general simply have to follow the flow of "the spontaneous movement which binds us to others for good or ill, out of selfishness or generosity". And just as in the end Cézanne managed to extract meaning out of contingency, so humanity can create a new idea of reason if we are willing to take the risks. (Matthews, 2002, p. 15)

This quote from Matthews is grounded in Merleau-Ponty's work and draws us away from an emphasis on causality and linearity concerning the teaching or learning process. If teachers (and the teacher educator) are taken into account in the learning process of professional development, it means that where the sessions can lead or how they unfold clearly depends on the teachers' own knowledge and understanding of the events of the sessions and the tasks provided. Contrary to the linear imagery previously highlighted, which underpins top-down approaches, enactivism is framed by a completely different understanding of events, one rooted in contingency and emergence. In other words, this view is rooted in the idea that events *emerge* and *unfold* from the situations in which we are, and these cannot be predicted in advance. In using words like "emerge" and "unfold," I mean two specific aspects that I define in the next two sub-sections.

Emerging Events: Openings

When learning and working on specific concepts, the unpredictable often emerges from the interactions and explorations undertaken by the participants. Remillard and Kaye Geist (2002) call these "openings in the curriculum."

> These instances [that are] prompted most often by participants' questions, observations, challenges, or resistant stands on issues that were important to them. We have labelled these instances *openings in the curriculum* because they required facilitators to make judgments, often on-the-spot decisions, about how to guide the discourse. [...] we came to view these breaks as potentially rich spaces in the curriculum because they presented opportunities for facilitators to foster learning by capitalizing on mathematical or pedagogical issues as they arose. (p. 13, emphasis in the original)

These openings represent important instances of/for learning that emerge out of the explorations undertaken in the sessions, that represent a natural consequence if teachers are considered in the learning process and seen as bringing knowledge to the professional development situation. This is in sharp contrast to top-down prescriptive approaches where everything is predetermined and pre-packaged. It points to an emergent curriculum and not to a prescribed one (Kieren, 1995).

Remillard and Kaye Geist (2002) also explain that these openings can be lived as difficult tensions if the teacher educator is not prepared to "navigate" and take advantage of them. Openings can become points of tension resulting from feelings of uncertainty and stress concerning the need to follow the previously traced curriculum or agenda point by point. These openings, even if difficult or unsettling to manage at times, are part of what doing work with teachers entails and are natural consequences of an approach directed not to conformism or prescription but rather to the exploration of issues:

> [...] it is reasonable to suggest that all teacher educators engaging teachers in re-examining mathematics teaching and learning are likely to confront similar openings – unanticipated and at times awkward points in the conversations – through which they had to navigate. (p. 24)

To negate and set aside these emerging events is to set aside important learning opportunities for teachers. These openings emerge out of the lived moments of the sessions and demonstrate by their presence that teachers are learning and evolving: "In a sense, openings may be signals that the curriculum is working" (p. 28). And it seems to be exactly within these very openings, where the curriculum works and learning is happening, that the teacher educator should probe, since these are events emerging from the explorations and which create and stimulate an interest in the participating teachers[59] – they emerge out of the teachers' knowledge.

These events expand and deepen the explorations; they represent instances of genuine interest where issues are dug into even deeper. They cannot be predicted in advance and neither can planned pre-specified questions in line with them be. In fact, it happens to be quite the opposite. In these openings, the teacher educator probes with emerging questions that are linked to these new interests and not linked to questions that the teacher educator *has* to ask to cover some specific mandatory topics – making a difference between "planned" questions that have to be asked, and "emerging" questions that arise as interesting to ask in the moment itself. This point is aligned with Varela's idea of problem posing previously mentioned, where the problems/issues to address emerge from the teachers' knowledge itself and not inherently from the tasks or situations themselves. These openings are also closely tied with the previous explorations, leading to subsequent events that unfold from them.

Unfolding Events: Cascades

The events of the sessions, the "openings," as they emerge and take shape, and as they shape the session itself, create and constitute a momentum on their own that guides and orients the explorations in the session. These emerging "openings," as they unfold in the session, are connected to what was previously addressed and often lead to new emergent ideas or issues to look into and of which to make sense. The latter will, in its

[59] This is a good example of the importance of the teacher educator in the process of professional development, since his or her decisions to probe in one direction or another impinges strongly on the learning opportunities offered: "Our analysis highlighted the critical role teacher educators play in fostering inquiry and explorations within teachers' professional learning opportunities regardless of whether curriculum materials are involved" (Remillard & Kaye Geist, 2002, p. 29). I come back to the issue of the importance of the teacher educator later in this chapter, and also in Chapter 7.

exploration, lead to other emerging ideas, creating a series of intertwined and inter-dependent "openings." Hence, in addition to being emergent, these "openings" create and trace a path on their own, orienting and guiding the session toward more explorations, one leading to the other as in a *cascade* of events that unfold or are brought about as consequences of the previous events – like in the "snowball" effect[60]. These cascading "openings" are contingent, in the sense that they are dependent and unfold from the previously worked on "openings," taking meaning in them.

Interpreting events that happen within a session as emergent and contingent fits well with the deep-conceptual-probes model of professional development in regard to the emergence of teaching issues linked to the mathematical explorations. In effect, by deeply exploring mathematical concepts with the teachers, there will be teaching issues linked to this "newly-worked-on" mathematics that will emerge and become the subject of discussion and exploration. These un-predicted teaching and learning issues will be contingent on the events and the mathematics explored in the session, and in that sense will be said to have emerged and unfolded from the mathematical explorations – and will possibly be matters of new explorations. Again, the same can be said of the mathematical explorations themselves that can emerge and unfold from previous mathematical explorations (or from teaching issues addressed).

This entire alternate theorization of events and learning instances brings me back to where this third chapter started, namely with ideas of "objectives" for professional development sessions. I am now in a position to theorize and provide an account of my objectives for the professional development program's sessions.

Redefining Objectives: Objectives to Work on *Versus* Objectives to Attain

Objectives in education are normally seen as goals toward which we strive or elements that we want to obtain. As I have tried to explain, this leads in-service education

[60] There is an interesting link that can be traced here with the notion of *cascading failures* from network theory, where specific seemingly minor instances or events can cause an entire reorganisation of the system (network) of which it is part (for more details, see Barabasi, 2003). Also similar is the notion of *positive retroaction* defined in chaos theory (e.g., Bélair, 2004).

to become linearly conceived and aimed toward conformity. I do not imply that we cannot have specific goals in a professional development mathematics teacher education program; what I want to suggest is that it is the perception and usage of the notion of "objectives" that is maybe problematic and could be redefined.

The English word *objective* is linked to the French *objectif.* According to *Le Robert-Dictionnaire historique de la langue française*, *objectif* comes from the Latin *objectivus* which means something that constitutes an idea, a representation of the mind, but *not* an independent or predetermined reality[61]. Informed by its etymology, I am tempted to offer a redefinition of what is normally meant by *objective*. Instead of thinking of objectives as end-points to obtain, an objective could be looked at as a starting point to develop from and elaborate upon. For this, I make a distinction between *objectives to attain* and *objectives to work on*, from which I theorize that instead of fixing a goal or an objective to achieve at the end and narrowing all actions in long-term planning by focusing tightly on that objective – what Bauersfeld (1998) calls the "funnel" approach[62] – objectives and goals can be framed in terms of Davis's (2004) thesis of *expanding the space of the possible by exploring current spaces*. This changes the focus away from final products to converge onto or conform with (the objective itself), and toward an idea of an evolution from those very objectives. A shift in the underlying imagery is suggested here, from the linear trajectory (from *A* to *B*), and toward an idea of emergence and cascading, of expansion and non-directionality. Davis explains these ideas in the following excerpts:

> This [...] prompts a redescription of lessons plans as 'thought experiments' rather than 'itineraries' or 'trajectories' – as exercises in anticipation, not prespecification. So framed, a lesson plan is distinct from a lesson structure, the latter of which can only be realized in the event of teaching. (p. 182)

> Oriented by complexivist and ecological discourses, teaching and learning seem to be more about expanding the space of the possible, about creating the conditions for the emergent of the as-yet unimagined rather than about perpetuating entrenched habits of interpretation. Teaching and learning are not about convergence onto a pre-existent truth, but about divergence – about broadening what is knowable, doable, and beable. The emphasis is not on what *is*, but on what might be brought

[61] See *Le Robert-Dictionnaire historique de la langue française* for an account of how the concept evolved to its current meaning.

[62] Bauersfeld talks of a funnel pattern, "which is characterized by 'narrowing the scope of action by response expectations'" (Voigt, 1985, p. 79).

> forth. Learning thus comes to be understood as a recursively elaborative process of opening up new spaces of possibility by exploring current spaces. (p. 184, emphasis in the original)

The notion of expanding the space of the possible moves us away from ideas of conformity with and convergence toward a specific state to be or a specific way to teach. In the case of the professional development reported in this research, each session does *not* have a specific pre-determined goal to achieve, but has a theme to explore that is in itself the "objective to work on" for the session, the starting point of the session in which the tasks and situations offered will be grounded and from where the explorations will begin and emerge/unfold. For example, the objective to work on for the session 2-3 on the volume of solids was: "To have teachers experience that volume as a geometric concept is much more than only memorizing and applying formulas"[63]. It is in this sense that the exploration of the theme itself represents the very objective of the session, where there is not a list of pre-determined things to attain in the end.

The thesis of expanding the space of the possible by exploring the current spaces also fits well with the idea of building on and expanding teachers' knowledge, that is, starting from teachers' procedurally-inclined knowledge and aiming to enlarge it. The objective to work on in a session is used as a starting point upon which to expand in order to enlarge teachers' knowledge by starting from their current spaces/knowledge. Therefore, the intention is not to have teachers acquire a specific "thing," but to enlarge teachers' knowledge by expanding and deepening the space of issues addressed, and in that sense also enlarging their own spaces of action in their teaching of mathematics. The goal is to provide possibilities for exploration that have the potential to open teachers' spaces of knowledge and spaces of the possible (by exploring their current spaces of knowledge). Hence, there are no specific goals to attain in the end. Addressing the session's theme represents the objective in itself. The orientation it takes is up to the people present in the session and can go in unpredictable directions. As mentioned, the approach rests on an important "leap of faith" that events will emerge and also that teaching issues linked to the mathematical explorations will be raised and dealt with.

[63] For a list of the "objective to work on" used in each session, see Chapter 4 on methodology.

This brings into question the meaning of the "success" of a session. With top-down approaches, the success of a session is easily measured by the attainment of pre-specified objectives and a successful replication of them in the teachers' practice. In the case here it is not the same, since there are no pre-decided objectives to attain or measure. The success of a session is assessed in relation to the emergence of events, of how the session enabled the creation of possibilities; simply put, if the session "triggers" and enables issues to be brought up and explored. Zaslavsky et al. (2003) explain that teacher educators in a professional development program should aim at "*providing mathematical and pedagogical learning opportunities for teachers*" (p. 879, emphasis in the original). Its success, so to speak, lies in the production and generation of knowledge and ideas, and of learning experiences for teachers – in order for teachers to develop further their mathematical and pedagogical powers (Cooney, 1994). Professional development based on ideas of emergence aims at generating possibilities; the success of a session lies in its generativity, nothing more. Figure 3.3 gives an image of the difference between top-down approaches and what I have offered here under the heading of "objectives to work on."

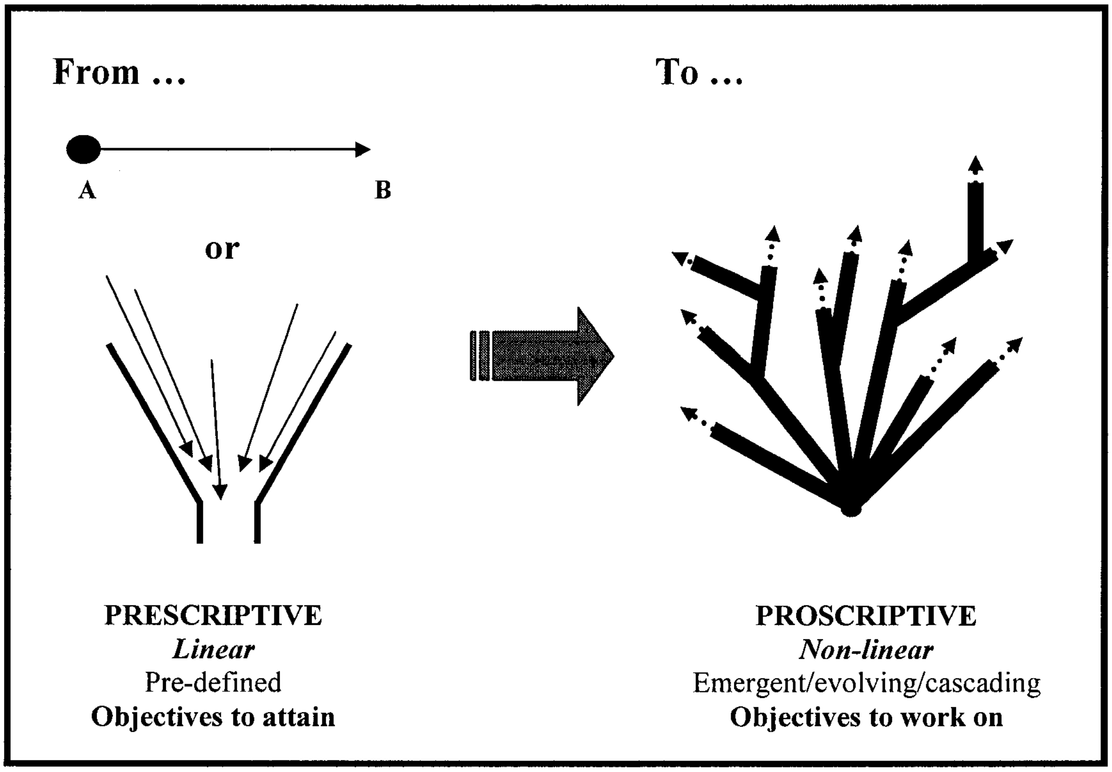

Figure 3.3. An image to contrast the "objectives to work on" thesis with a top-down or linear approach to professional development

Now that I have offered another image for in-service education – one in line with ideas of emergence, contingency and un-predictability – it prompts a redefinition of what the role of the teacher educator in an in-service education setting entails.

The Role of the Mathematics Teacher Educator

> In this way education *purposely* shapes the subjectivity of those being educated. (Osberg & Biesta, in press, p. 1, emphasis in the original)

> While the major burden is on the students to explain what they think, I actually do try to say much of what I myself *believe* on the subject of teaching and learning. I often remark on what I see in our work together and I try to say what I think about issues that students raise. (After all, I too am grateful for the occasion to learn from trying to say clearly what I think.) Yet, I have no illusions that what I say will mean the same thing to others *as it does to me*, nor that the students will, in general, give credence to what I say. But what I say does add to the assortment of things they have to think about. (Duckworth, 1987, p. 488, my emphasis)

With a new theorization of how learning happens, a new understanding of what teaching means arrives (Davis, 2004). Here, with an enactivist understanding of cognition and an idea of "triggering" and expanding the space of the possible with objectives to work on, the prominent metaphor (found in most literature on teacher education) of the facilitator of learning or "guide on the side" needs to be rethought (Kieren, 1995). Neither is the metaphor of the broadcaster of information (Cooney, 1988) in a top-down approach to teacher education, nor a representationalist view of knowledge (Kieren, 1995), very useful here. Understood through the concepts of structural determinism and structural coupling, the teacher educator becomes more than a guide and becomes a fundamental part of the teachers' learning process or, simply put, of the teaching dynamic in professional development. In the same way that for an organism everything else is part of "the environment" that "triggers," for a teacher the teacher educator is part of the "learning environment" that "triggers." Therefore, enactivism asks for more than teacher-centred or student-centred approaches. By positioning the teacher educator inside the learning environment – with which teachers structurally couple – it asks for what Kieren (1995) calls "teaching-in-the-middle," which is interaction-centred. In the following, I elaborate on what this position means for the role of the teacher educator and his or her actions.

Structural Determinism and Teaching

The concept of structural determinism implies that the individual knower has "control" over the type of effect that can be produced on him or her. The effect on teachers of what a teacher educator says or does is not pre-determined, and is determined by the teachers' structure (their knowledge). Obviously, the "inputs" of the teacher educator will provoke something, influence the process, and play a role. But the type of role and how this will be taken up is determined by the structure of the learner/teacher itself, and not by the actions and words of the teacher educator. In the same sense that the environment acted as a "trigger" on the organism or species, the teacher educator's interactions and interventions act as "triggers" in the learning process of teachers. The role of the teacher educator is then to "trigger" teachers with ideas, concepts, notions, nuances, and so on.

The teacher educator's actions are central in the teaching dynamic and are as important as the tasks or situations offered to teachers. Hence, the teacher educator plays an active role in the teaching dynamic, shapes the sphere of possibilities (Kieren, 1995) and opens/creates spaces of emergence (Davis, 2004). The teacher educator's actions act as "triggers" for teachers' learning.

Structural Coupling and Teaching

> But these actions are also such that every action influences and changes every other action and the world (in this case the classroom) in which they occur. (Kieren, 1995, p. 8)

The notion of structural coupling enables a different theorization of the teaching dynamic. Structural coupling brings the idea that both teachers and teacher educator evolve and co-adapt to each other in the learning process or the teaching dynamic. There are two major outcomes of this. The first one is that the teacher educator becomes *complicit* (Sumara & Davis, 1997) in the teachers' knowledge. The teacher educator influences what is learned by interacting and coupling with teachers. Hence, the teacher educator is "within" the teachers' knowledge and cognitive acts. By being structurally coupled with teachers, the teacher educator influences and (strongly) orients the learning that happens, hence is seen as complicit in this knowledge production. Cooney (1988) also alludes to these ideas concerning mathematics teaching:

> Both mathematicians and mathematics educators cannot escape their responsibility for shaping their students' philosophies of mathematics no matter how implicitly or subtly those philosophies may be communicated by their instructional methods, the means by which they encourage students to learn mathematics, and the means by which they assess their students' learning of mathematics. (p. 359)

Secondly, with this structural coupling, just as the teacher educator's actions act as "triggers" for the learning of teachers, the teachers' actions, comments, interactions, and so on, *reciprocally* act as "triggers" for the learning of the teacher educator.

> But this is what a good teacher does – occasions learning. To allow this to occur [the teacher] provides the possibilities or keeps open the possibilities which occasion such learning. [...] mathematics teachers can observe and learn from and with their students, helping them bring forth a world of mathematical significance and in fact, bringing it forth with them. That is, in mathematics, learning is a reciprocal activity [...] in which the students and the teacher learn from one another and the situation in which they exist. This reciprocal learning is central to bringing forth a world of mathematical significance with others. (Kieren, 1995, p. 2)

Consequently, this means that the teachers and the teacher educator develop a history together and become structurally coupled: they both learn and co-evolve in that history of relationship.

Educating Teachers

With both concepts in hand (structural coupling and structural determinism), the role of the teacher educator becomes clarified, enlarged, and maybe even rejuvenated. Literally, the teacher educator becomes someone who "educates," by acting as a "trigger" for teachers' learning. The responsibility of the teacher educator becomes to intervene in the process, to "educate" teachers, to be an accomplice in their learning and to have them learn/address issues and aspects – not just to stand passively, but to influence teachers along the way. In other words, this means that the teacher educator does not stand patiently at the side of the learner, waiting for learning to happen like a "guide on the side." A teacher educator has to get in the way, to provoke the learning process, to orient the ideas and influence what is learned, to create opportunities, to influence the teachers willingly, and so on. The teacher educator has to act as a "trigger" for the teachers constantly, so that learning and changes emerge and unfold.

Taboos of Teaching and Enactivism

It is important to note that such forms of teacher (educator) behaviour have been almost rendered taboo in the literature and might be seen as quite "surprising" assertions in this work. The understanding of these as taboos, however, most often arise from (mis-) interpretations of a constructivist theory of knowledge[64]. Because they permeate the literature and influence educational practices, I will briefly explain four of the most prominent here.

The first taboo is highlighted in Davis and Sumara (2002) by what they call "don't tell" practices. It seems that many teachers believe they should avoid, whenever they can, giving direct instructions to students. This taboo seems to be linked to the idea that teaching is not about transmitting or transferring knowledge from the teacher to the learner and is more about perturbation and construal. Teachers then stop themselves from explaining and elaborating on notions, fearing that "[...] attempts to tell are sometimes seen as violations of the constructive process, impositions on a person's sense-making rather than possible contributions to such sense-making" (p. 419).

This first taboo is linked to a second one which Bauersfeld (1994) refers to as the "method of exhaustion by repeated questioning." This practice of guiding the learner by the hand toward the answer, but without ever telling him or her exactly the words or what to do, could be seen as a consequence of the taboo "Ne jamais dire à l'élève ce qu'il peut trouver par lui-même! [*Never tell a student what he can find by himself!*]" (p. 188)

A likely cause of these two taboos is the constructivist understanding that when you tell something, it does not mean that it will be automatically and exactly understood in the way you intended. But this in no way suggests or requires that the teacher educator

[64] I have elaborated elsewhere on multiple (mis-)interpretations of constructivism (Proulx, 2006).

should not speak anymore or refrain from explaining. Indeed, the teacher educator *needs* to explain, speak and intervene to provide "triggers" in the learning situation and to create/expand that learning space of emergence. In that sense, the interventions of the teacher educator are central – they are fundamentally important to the learning process, and are at the "center" of it.

A third taboo is linked to an implicit suggestion that the learner "cannot be wrong" and that everything he or she says must be adequate since all knowledge is personal and subjective. This seems to stem from the attribution of relativist principles to constructivism or to any theory that runs against and problematizes the realistic/positivistic vision of knowledge as objective, universal, value-free, and causal. Learners are not free to create "anything" they want, any claim has to be shown compatible with and fitting the situation and lived experience to be considered viable; any claim has to be in line with the structural coupling of learners (teachers) and the teacher (teacher educator).

> It is the case that if the student can explain his/herself as fitting into or with the mathematical thinking of the community (being "not wrong") then his or her work is "good enough." (Kieren, 1995, p. 14)

In that sense, there is no problem with the teacher educator raising an opposition against a comment made by someone if the teacher educator does not consider it viable (as much as teachers can and do the same). The teacher educator raising the point that some knowledge does not fit is simply flagging the fact that it does not seem adapted to the history of structural couplings (and that his or her structure is non-adapted – in the sense of opposed – in regard to the issues just raised). In other words, not all knowledge fits and it is important to raise the possible incoherencies to continue establishing the inter-actions and the adaptations. The negotiations of meanings (Voigt, 1994) that can emerge out of disagreements are important for the learning and development of knowledge but also for the history of structural couplings between the teachers and the teacher educator. Because the teacher educator is structurally coupled and part of the teaching/learning dynamic, a second aspect concerns the fact that when the teacher educator feels something is wrong or non-adapted, it automatically also acts as a "trigger" for the teacher educator. We should not hide from the fact that the teacher educator wants teachers to learn about issues and he or she has intentions and expectations (Blouin, 2000). For that reason, the teacher educator acts in relation to what he or she believes is important for teachers to know – which raises reactions in him or her[65]. The teacher educator is not neutral (Bednarz, 2000).

In these instances, because our structure is "triggered" by questions of interest, the teacher educator cannot help but intervene since these instances are felt to be "triggers" of his or her structure. When something is interpreted and understood in a different way from one in which the teacher educator feels it should go or from what was intended at first – what Schön (1983) theorizes as the "back talk" of the context in regard to our actions – the teacher educator flags it and attempts to re-align it along *his or her* interpretation/understanding of the issue at stake: be it by a nuance, a reflection, a clarification, an opposition, and so on. In other words, because the teacher educator is part of the teaching situation, because he or she is structurally coupled to it, it is his or her role, as much as it is the participating teachers' role, to replace elements and raise concerns in regard to the non-adapted nature of comments or elements raised (from his or her own subjective perspective). In the same way that the environment shuts down non-adapted species and prevents them from reproducing themselves, this should be done for non-fitting comments/ideas. Again, it is in the sphere of interaction, in the sphere of negotiation, that the coupling takes place and that learning emerges.

Finally, since you "cannot tell" and you cannot say that someone is wrong, a fourth taboo can easily become what is called the pedagogy *du laissez-faire* – a sort of "don't need to teach" – in which the learner is seen as someone that will develop his or her knowledge by him- or herself. This renders teaching helpless and without consequences, since learners construe their own knowledge on the basis of their previous knowledge.

[65] I would be tempted to relate this to the action of the parent toward the child. What I mean by this is that the parent is eager that the child learns things (that he or she believes important), and does everything that he or she can and that is possible to do in order to act toward this goal (with all that he or she knows and within the range of possibilities available to him or her). In that sense, when something does not go in the direction that the parent thinks is an adequate direction (on the basis of his or her understanding of what is a good or bad direction), the parent *does* intervene and *does not* "let [the child] be." I would be tempted to think, not that the teacher educator is the teacher's parent, but that the teacher educator acts in that specific way toward the teacher.

This interpretation is linked to an association of a nativist point of view of knowledge to constructivism or to any theory that sees the learner as an active agent in his or her own learning process – nativism being a theory that asserts that knowledge is already inside a person at birth, and that education is a process of drawing that knowledge out. With structural determinism, enactivism explains that indeed it is the structure/knowledge of the person which allows for the changes to happen, but these are brought forth, again, by "triggers." Duckworth (1987) explains her view on this:

> Of course, when I say "working out for themselves" I do not rule out presenting people with material for them to make sense of, as I try to describe here – I structure experiences in which they learn, try to explain what they are learning, watch others learn, try to help other people explain, and hear other people's ideas. But it is the *students* who make sense of all of this. (p. 487, emphasis in the original)

The "triggers" of the teacher educator are essential for the learning of teachers to emerge, whether they are in the form of interactions, prompts, feed-back, and so on.

The Subjectivity or the Structural Dependence of the Interventions of the Teacher Educator

Enactivism points to a particular way of tackling these taboos of teaching; that is, an approach that builds heavily on concepts of structural determinism and structural coupling (and, implicitly, problem posing). Moreover, the teacher educator's interventions aim at being "triggers" for teachers, in regard to what the teacher educator believes important to be addressed, that is, in regard to his or her own views and understanding (Duckworth, 1987). In other words, the teacher educator makes choices, which are aimed at educating teachers. Again, the teacher educator is not neutral but rather orients the work from his or her own perspective (Bednarz, 2000). The relevance of the issues raised, and aimed at being addressed by the teacher educator, comes from within the structure/knowledge of the teacher educator and does not come from above, and does not represent issues that have to be tackled (as if they were universal). The teacher educator raises issues as he or she is triggered by the "back talk" of the situation and feels an interest/relevance in probing on them.

> Thus each student's learning is co-determined by the occasions each draws from the possibilities of the classroom. There is a fundamental circularity in the teaching and learning in such a classroom. The teacher is influenced by the possibilities which arise from student actions but these are influenced by the teacher. (Kieren, 1995, p. 17)

Teaching and learning become circular and mutually influenced. The teacher educator's interventions are not ultimate or absolute; they come from within the teacher educator's structure/knowledge, as much as emerging questions of teachers come from their own structure/knowledge – both are influenced by as well as influence the other.

Key Theoretical Concepts/Assumptions for the Teacher Educator's Role

By using the concepts of structural determinism and structural coupling, the dynamic of the teaching process becomes clarified. The teacher educator's interventions act as possible "triggers" in the learning process of teachers: that these will be significant in teachers' learning is determined by teachers' own structure and knowledge. This leads to the first two key theoretical assertions about the teaching dynamic:

1. *Learning is dependant on, but not determined by, teaching*;

And so unfolds that (second key assertion):

2. *If the teacher educator "tells," nothing guarantees that things will happen or be understood, but if he or she does not tell anything, nothing will happen*;

In reciprocal terms, the teachers' actions, comments and interactions will act as possible "triggers" in the learning process of the teacher educator. Moreover, teachers and teacher educator will co-influence each other, co-adapting and co-learning in the process. This leads to the third key theoretical assertions framing the teaching dynamic:

3. *The teacher educator learns in the teaching dynamic*;

The teacher educator and the teachers are both part of this structural coupling – they both learn within it – and so the teacher educator is an active element in the teaching dynamic, as much as he or she is an accomplice in teachers' learning. The teacher educator and the teachers bring forth a world of significance together (Kieren, 1995). The teacher educator does not "let learn," but actively participates and intervenes in, as much as provokes, the learning process with his or her "triggers." This leads to the fourth and fifth key theoretical assertions concerning the teaching dynamic:

4. *The teacher educator has an active role where he or she intentionally "triggers" and provokes learning issues*;

And,

5. *The teacher educator influences the knowledge and learning of the teachers (and vice-versa).*

Teachers and the teacher educator are structurally coupled in the learning process and both are driven by their own structure/knowledge in this learning process. This coupling produces a history of mutual adaptations, learning and knowledge. Obviously some knowledge is not adapted and does not "survive." The structural coupling of both parties depends on adapted understandings, and so the coupling cannot continue if one of the parties' knowledge stops adapting. Moreover, disagreements or negotiations of meaning are a fundamental part of the interactional or coupling process. This leads to the sixth key theoretical assertion framing the teaching dynamic:

6. *The teacher educator (and teachers) has (have) to flag what is believed to be inadequate knowledge, and interact/negotiate about it.*

In this section, I have tried to clarify the role of the teacher educator in a professional development setting. The teacher educator as been described here as someone who interacts, intervenes, orients and influences. In other words, the teacher educator takes an active part in the teaching dynamic and, obviously, in the learning process of teachers. The teacher educator is someone active who does not stay still, who invests and engages him or herself in the learning process of teachers.

It has been prominent in the literature to invent new metaphors to "describe" and make sense of the role of teachers and teacher educators in the learning process[66]. I am

[66] For example, in mathematics teaching, King (2001) uses the metaphor of the "jazz improvisation" to describe conceptually-oriented teachers, Brousseau (1988) uses the one of the "theatre actor" to describe how teachers are continually confronted with paradoxes, and Cooney (1988) discusses against the one of

tempted here *not* to offer any new metaphor or conceptualizations, and to read literally what the role of the teacher educator is, that is, to *educate teachers*. And for this, no one explains more concisely and clearly than Duckworth (1987) what a teacher is:

> By "teacher" I mean someone who engages learners, who seeks to involve each person wholly – mind, sense of self, sense of humor, range of interests, interactions with other people – in learning. (p. 490)

"broadcaster" of information. In education in general, for an historical account of the usage of different metaphors of teaching in relation to different theories of learning, see Davis (2004).

CHAPTER 4

METHODOLOGY

The purpose of this research is to understand better and address the issue of secondary-level mathematics teachers' knowledge of mathematics, to develop a professional development intervention aimed at enlarging teachers' mathematical knowledge, and to investigate this approach in regard to what if offers to teachers and how it impacts them.

The second and third chapters provided the theoretical underpinnings to make better sense, orient and guide the structure and functioning of the intervention, which in turn led to establish some methodological procedures. This chapter addresses the following: (1) the global methodological orientation taken in this research; (2) the research setting and data collection; (3) the planning and construction of the professional development sessions; and (4) the data analysis procedures.

Global Methodological Orientation

Qualitative Case Study

This research reports on a case study of a professional development setting that I, as the researcher and teacher educator, instigated. The decision of opting for case-study research was made on the basis that it enabled me to study the complex and dynamical aspects of an in-service program. Karsenti and Demers (2000) explain:

> Il semble que l'étude de cas soit une approche en recherche tout à fait indiquée pour l'éducation puisque plusieurs études, en particulier celles qui traitent des interactions en salle de classe ou à l'intérieur d'une école, impliquent un nombre important de variables qu'il est souvent difficile d'isoler. [*It seems that case study is a research approach well suited for education since many studies, in particular the ones which concern/deal with classroom or school interactions, involve a significant number of variables that are often difficult to isolate.*] (p. 245)

> Elle [l'étude de cas] permet de considérer et d'observer le système et les interactions d'un grand nombre de facteurs, ce qui peut aider le chercheur à mieux percevoir la complexité et la richesse de contextes ou de situations en éducation. [*It [case study] permits us to consider and observe the system and the interactions of a large number of aspects, which can help the researcher to perceive the complexity and the richness of educative contexts or situations better.*] (p. 229)

Because one of the intentions of the study was to investigate and understand better the professional development intervention in relation to its functioning and its "local" impact on teachers, it was evident that studying the very practices of a specific case using *this* approach was needed. Since the process of conducting professional development evolves[67], a flexible approach was needed. A qualitative research method, Merriam, Courtenay and Baumgartner (2003) explain, provides that flexibility:

> In addition, qualitative research is flexible. The research process continually evolves and unfolds. Because we sought to understand a dynamic, continuously evolving process, this paradigm was desirable. (p. 174)

Moreover, teachers' knowledge constitutes a dynamic and continuously evolving phenomenon, and one of the major intentions of the approach taken was to enlarge that knowledge. The qualitative paradigm could provide/support a detailed analysis of the impact of the professional development intervention on teachers' knowledge of mathematics, in order to understand how it was impacted. Therefore, I opted for a qualitative research orientation, a paradigm focused on "better understanding" phenomena and their intricacies and potentialities (rather than providing quantifiable "effects" and "results").

[67] As was made clear with elements concerning changes that had to be made in the research itself, like a modification of the first research intentions and the creation of new models to orient the practices.

> A qualitative design was chosen because qualitative research is concerned with process, and understanding the process is more important than looking for an outcome. (Merriam et al., 2003, p. 174)

This comment points to an important assumption inherent in this dissertation. In short, the intentions of the study are not to calculate the effects and to provide an already prefabricated-applicable-replicable model of professional development for dissemination. As the argument mounted in Chapter 3 against top-down approaches shows, this research is not interested in imposing itself in a top-down manner for future providers of in-service teacher education. Rather, it is intended to contribute to the field's growing conversation about secondary-level mathematics teachers' mathematical knowledge, and about the possibilities and recommendations for intervention to impact and enlarge that knowledge.

This is not a small point. In effect, the persistent intention of insisting that research provide directly applicable results is condemned by Brousseau (1988, 2006) and René de Cotret (2000) as a mistake that does not represent the fundamental purpose of doing research in mathematics education (or in *didactique* of mathematics). The fact that research *does* offer information on the issues of teaching and learning mathematics is not the point. The main intention of research is to understand better and make sense of these phenomena, and not necessarily to find direct applications[68]: "The primary purpose of research is to learn through investigation" (Wong, 1995, p. 22). The constant pressure to obtain concrete and directly applicable results – to which could be added the persistence of always wanting to measure, quantify and compare the effects of an approach in education – represents a distortion of what doing research in mathematics education is all about.

My research aims at better understanding the phenomenon of the mathematical knowledge of secondary-level mathematics teachers and ways of working and intervening with them to have them (re-)experience "conceptual" mathematics. Hence, it is not aimed at obtaining information that is directly replicable, useful and applicable, even if knowing more about an issue can inform subsequent practices and that some

[68] For Brousseau, these persistent intentions contradict the intentions of doing science, where research in science is not constantly asked about or judged by its degree of direct application (e.g., some molecules in chemistry are studied for themselves, and not with the intention of knowing right away what their direct application will be).

aspects reported on might be applicable, transferable or translatable into practice. Research in education does not aim at being consumed, as Cochran-Smith (2005) interprets it[69]. Research in mathematics education is not aimed (nor should it be) at being generalizable in order to "get the answers," but should instead aim at being *generative* (Valero & Vithal, 1998).

> Instead of evaluating a study in terms of its generalizability, which is connected to external validity, we may consider its generative capacity as an important criterion. Generativity can be taken as the extent to which a study originates new research objects for study and alternative research methodologies, as well as produces new outcomes. In other words, a generative study "unseat[s] conventional thought and thereby opens new and desirable alternatives for thought and action" (Kvale, 1996, p. 234). (p. 158)

By enabling a better understanding of phenomena, research does not aim at solving the problems and closing all the doors one by one, but rather at opening the space of possibilities. Research generates ideas and in return stimulates the generation of other ideas. The intention is not to replicate or generalize, but to generate possibilities and ideas in a continuous endeavour to move forward and make issues evolve. It is in that sense that the opposition between "knowing the measurable effects" and "better understanding and making more sense" is emphasized here. This research aimed at generating knowledge and possibilities.

Researcher's Role

> We, as teacher educators, would do well to turn to our own experiences as practitioners as the bases from which research may emerge. [...] Teaching and conducting research should be seen, not as conflicting, or even different, but, in fact, as part of the same whole. (Adler, 1993, p. 160)

My role in this research was always two-dimensional: as researcher and as mathematics teacher educator. These two roles worked hand in hand as each informed the other, but it is also important to establish and distinguish them in the methodological process. Whereas the professional development sessions could have happened without an intention to study them, I was also interested in researching these same practices. Hence,

[69] Indeed, Cochran-Smith constantly uses the metaphor of consumerism in her article to explain how researchers and practitioners should use research results and be informed by them.

I was deliberate in creating the professional development sessions for the purpose of research.

Richardson (1994) makes a distinction between these two roles as "practical inquiry" and "formal research," where both activities comprise different types of inquiry and serve different purposes. The former is aimed at improving the day-to-day practices and events of the practitioner (here, my role as teacher educator), and the latter is aimed at creating and using data to report on the events and practices (here, my role as researcher). In that sense, I was doubly involved in the dynamical process of the research – as a researcher studying the professional development program, and as the mathematics teacher educator establishing the professional development program. Consequently, there was always a dialectical (and sometimes conflicting[70]) relationship between myself as the mathematics teacher educator establishing the program (conducting the in-service sessions) and myself as the researcher studying what I had established (research on the sessions and their impact on teachers' knowledge).

This reciprocal, recursive and simultaneous nature of researcher and teacher educator in this research has been called by Cochran-Smith and Lytle (2004) "working the dialectic":

> We used this phrase because we wanted to point out that there were not distinct moments when we were only researchers or only practitioners and thus to emphasize the blurring rather than dividing of analysis and action, inquiry and experience, theorizing and doing in teacher education. (Cochran-Smith, 2005, p. 219)

Hence, it is always along, and in recognition of, this dialectic that I have acted in this research and professional development setting. The research enriched/oriented my teacher educator practices and in turn my teacher educator practices enriched/oriented my research.

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[70] I say "conflicting" because sometimes what is not working well and is difficult to manage on the level of the practice of educating teachers might happen to be interesting on the level of research. For example, tasks and events do not always work along the lines previously hoped for, and these can happen to be difficult to manage for the teacher educator. However, these can simultaneously happen to be quite interesting to analyze at the research level. At the end of Chapter 6, in the "Not seeing the mathematical activity as a panacea" section, I elaborate on this type of issue.

However, this dialectical approach is not always seen as rigorous research in academia (Cochran-Smith, 2005), or can be seen as conflicting (Wong, 1995). Most criticisms are mounted around the fact that (1) there is an absence of an objective standpoint since the researcher *is* the practitioner, (2) most data gathered are "situated," "local" and "context bound" which makes them difficult to generalize, disseminate or apply to other situations and cases, and (3) there is not always the possibility of measurable, testable and comparative effects. In sum, these are all aspects that point to the traditional positivist model of research[71] requiring objectivity, generalization and measureable effects – all requirements that have been at the core of most disagreements around the emergence of qualitative research in education, and specifically here of teacher education research (Cochran-Smith, 2005), and for which I have previously positioned myself against.

Moreover, and importantly, this position within the data and my engagement in the process as the teacher educator gives me privileged access to the practices of professional development and enriches my possibilities of making sense of them since I am intimately intertwined in them. This is in line with what Pinar implied some 30 years ago by suggesting that "to explore and understand educational experiences we must exist in them, rather than removing ourselves from them" (Adler, 1993, p. 160). Indeed, there is a growing sense concerning the worth and importance of researchers reflecting on and researching their practices as teacher educators, in order to understand these very practices better (their nature, their impact) and bring knowledge to the fore (e.g., Geddis & Wood, 1997; Korthagen, Loughran & Lunenberg, 2005; Zeichner, 1999, 2005).

Not to overemphasize the point, but whereas a positivist/rationalist would deem the situation biased, I see it as beneficial for the research since it enables me, as the researcher, to understand and analyse better the intricacies of these practices and their meaning and impact they have (had) on teachers. The in-vivo position brings strength to the interpretation of the data, in contrary to an in-vitro position (Foerster, 2003).

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[71] Adler (1993) refers to this paradigm of research as "empirical-analytical, or experimental, approaches" and continues by saying that "Alternative paradigms have opened educational research to the notion that there are multiple ways of knowing and coming to know" (p. 160).

> [...] because the activity of teaching is so imbued with human intention and inextricably embedded in the specifics of each situation. It follows, then, that teaching cannot be understood fully from the perspective of an outsider. (Wong, 1995, p. 25)

Research Setting

Participants

With the help of the superintendent and the pedagogical advisor of a francophone school district in a minority setting in a large urban area in Western Canada, secondary-level mathematics teachers were invited to take part in the professional development programme. There were ten teachers present at the first session, but only six remained in the project who constituted the principal participants of the study. The six teachers I call Carole, Claudia, Erica, Gina, Lana and Linda[72]. All participants were secondary mathematics teachers (from 8th to 12th grade) within the school district who taught and had taught at different grade levels, except for Linda who was not currently in the classroom. They were all in their thirties and early forties, and had from 5 to 15 years of experience teaching mathematics.

Structure and Calendar of the In-Service Sessions

The in-service sessions which made up the professional development program were structured around monthly 3-hour group meetings, except for the second and third meetings, which were combined sessions of 6 hours, representing respectively session 2-3 and session 4-5. Ten sessions were held in total, between September 2005 to May 2006, some 30 hours of professional development over the school year. Table 4.1 represents the calendar of the sessions with the theme worked on in each.

[72] Again, all teachers' names are female gender pseudonyms. From the four other teachers who left after the first session, two will be mentioned at times (Danielle and Nina). And, to this group can be added Holly, Gina's intern, who was present at session 4-5.

Table 4.1. Time table of the research meetings

Session	Time	Date	Topic worked on
1	3-hour	September 21st, 2005	Transition from arithmetic to algebraic thinking
2-3	6-hour	October 15th, 2005	Volume of solids
4-5	6-hour	November 10th, 2005	Area, and writing of algebraic equation from word problems
6	3-hour	December 7th, 2005	Grading students' work
7	3-hour	January 18th, 2006	Fractions and operations
8	3-hour	March 9th, 2006	Working on area conceptually
9	3-hour	April 12th, 2006	Making sense of algorithms and techniques
10	3-hour	May 17th, 2006	Analytical geometry

The importance of establishing a longitudinal and sustained professional development programme is well acknowledged in the literature (e.g., Crockett, 2002; Irwin & Britt, 1999; Jaworski & Wood, 1999). It is assumed and has been shown that professional development sustained over a long period of time enables teachers to reflect more deeply on what they are working on and also, not to neglect, they have an opportunity to experiment and attempt at trying some ideas in their very classroom (Remillard & Kaye Geist, 2002; Weinzweg, 1999), since the sessions are provided throughout the school year and teachers go back to their classroom the "next day"[73].

Data Collection

All group sessions were videotaped. A camera was installed at the back of the room to make sure that all participants were visible in its frame. The purpose of the video camera was to capture on tape the events of the sessions concerning group interactions, the discussions and the explanations given. Individual work done on paper sheets was not collected, but any material that was used to structure the sessions, provided by the teacher educator or by teachers themselves, was gathered.

[73] This is something that one of the participants (Carole) highlighted in a personal discussion, as she felt that the professional development sessions had an important impact on her practices because they happened during the school year where she was able to bring some ideas back into her practice the next day as she was often teaching similar, at times even identical, concepts to the ones explored in the sessions.

Again, following Richardson's (1994) distinction between "practical inquiry" and "formal research," the videotaping had a two-fold purpose: one for teacher education and one for research. For the former, the purpose of those videotapes was for the teacher educator to review the sessions to understand better what happened in them and possibly to adapt and modify some intentions and approaches in the following sessions. This is an idea borrowed from Steffe's (1983) "teaching experiments," where the videotapes of the previous lessons were used to plan the next ones. This idea was also used by Simon (2000) where he explains that "each subsequent intervention is guided additionally by what the researchers have learned in their previous interactions with the teachers" (p. 359). In terms of the latter purpose (that of research), the intent of videotaping was to keep a trace of the events of the sessions, to which I could return at any time. The videotapes of the sessions represent the main source of "research data" for the dissertation.

Planning and Preparing the Sessions of Professional Development

The sessions were driven by specific objectives to work on, as theorized in Chapter 3. Over the course of the entire professional development the overarching objective to work on was "To have teachers experience and explore 'conceptual' mathematics." However, each session had a particular mathematical theme with its specific tasks. In this section, I intend to give an overview of how the sessions were constructed, ranging from the choice of a theme, to the setting of an objective to work on and the choice of specific tasks to offer the teachers.

The Choice of a Mathematical Theme

Each session had its specific theme, mainly a mathematical topic that it would address. The choice of the sessions' mathematical theme was based on three sources, which were sometimes intertwined but often also represented a separate source on their own. One was my personal comfort with and interest in specific topics and the idea of conducting a session about them; a second source came from the teachers' expressed interests in specific issues, while a third source came from the sessions themselves and where it led the explorations to, when sometimes a specific topic seemed well suited to continue. Table 4.2 illustrates the main sources of influence for choosing the specific mathematical theme for each session.

Table 4.2. Principal sources for the mathematical themes of the sessions

Session	Mathematical theme	Principal source
1	Transition from arithmetical to algebraic thinking	Personal comfort/interest
2-3	Volume of solids	Personal comfort/interest
4-5	Area, and writing algebraic equations from word problems	Teachers' interests
6	Grading students' work	Work in previous session
7	Fractions and operations	Teachers' interests
8	Working on area conceptually	Personal comfort/interest
9	Making sense of algorithms and techniques	Personal comfort/interest
10	Analytical geometry	Teachers' and personal interest

But above all, and most importantly, each topic was chosen because it represented a topic for which a prominent place was usually given to procedures and their applications within school mathematics. Therefore, these represented rich topics to work on with teachers because they had a lot of potential to raise and address issues of "conceptual" mathematics – relational understanding and structure and relations – that are often neglected when they are studied in school. After a theme was decided on (at the end of a session, between sessions, or from some sessions in advance, etc.), I had personally to delve into these mathematical topics in order to prepare the objective to work on and decide on specific tasks and situations to offer to teachers in the session.

Setting an Objective to Work On for a Session

In order to construct a session, I worked at creating an objective to work on, something that would act as the overarching intention and rationale for conducting the session and that could potentially orient the explorations. In line with the overarching objective to work on of the professional development program, the ideas at the core of each session were concerned with having teachers experience "conceptual" mathematics. Hence, the objective to work on in each session had a flavour that was assumed to be uncommon or unfamiliar for teachers, because of these teachers' strong inclination toward procedures. However, as Even (1999) experienced in her project, it was assumed that some of these general intentions might provoke some reluctance or tension in teachers concerning the nature of the mathematical ideas addressed and explored, and their teaching and learning. I assumed some teachers might be sceptical about the work done in the session. This was not surprising because one main goal was to enlarge teachers' knowledge of mathematics, which will not necessarily happen smoothly. (As most attempts at introducing new curricular materials show, newness and change are not always welcome at first.) This was simply part of the intervention itself.

In order to set an objective to work on for a specific theme, I had to plunge into the mathematics and explore many aspects of it to become more knowledgeable myself. My intention was to gather insightful information and develop richer personal conceptions for each of the chosen topics in order to create sessions that would address "conceptual" mathematics. Therefore, to achieve this I explored work done on the mathematical content itself, on its teaching and on its learning by students. Four different types of sources were consulted in this endeavour:

1. Mathematics education literature (academic and professional journals, books);
2. Mathematical texts (books, journals, textbooks, reference guides and lexicons, historical texts);
3. Material of mathematics education courses taken and given (B.Ed. and graduate level);
4. Consultations with advisor, supervisory committee members and graduate students colleagues.

No one source had predominance over another and each complemented the others in the process. Hence, depending on the nature of the theme and my familiarity with it, each source was consulted and contributed to the process to different degrees. In order to render more concrete the steps taken to establish the objective to work on for a session and how things took their forms, and also to illustrate the learning path that I lived through this process and how developing a sessions was an emergent process, I offer in the following a summary of the preparation done for session 10 on analytical geometry[74].

Creating the objective to work on for session 10 on analytical geometry

To prepare for the session on analytical geometry, the notes taken as a teacher assistant for a B.Ed. course in UQÀM were consulted because one of the assignments was to create a "conceptual analysis" (see Chapter 2) about analytical geometry. From these notes, I realized the important distinction that is often created between geometry and algebra in analytical geometry and how often analytical geometry courses become reduced to algebraic manipulation focused on the algebraic notations and computations of formulas to find specific values. Two articles were consulted from these notes: Côté (1996) who focuses on giving a meaning to formulas used in analytical geometry courses (distance, mid-point, etc.), and Herscovics (1980) who reports on a teaching experiment on the concept of slope and linear relations where the students did well and made insightful sense until he brought in the formulas to use for the same ideas they already understood. These two articles strengthened my position about the futility of only using formulas in analytical geometry, or simply about their accessory presence, because all usual analytical geometry problems offered in classrooms (distance, slopes, mid-point, etc.) were solvable with the use of geometric concepts (Pythagorean theorem, intersection of lines) without a need for formulas.

To gather a sense of the emergence of this topic, I looked at Descartes' (1637/1986) appendix on geometry and at Fauvel and Gray's (1987) book on the history of mathematics (with its course booklets from Open University, 1987). This led me to understand the difference between analysis and synthesis in the history of mathematics, emerging from Descartes' work in analytical geometry. I contacted and discussed these issues with the mathematician-historian Louis Charbonneau, who referred me to an article he had written on Viète and his analytical program (Charbonneau, 2005). Based on

[74] As will be argued for the construction of tasks and situations for the sessions, the preparation and construction of the sessions could be said to represent research results on their own, ones where "conceptual" approaches were elaborated. The preparation of the sessions represented important learning opportunities for myself, as Zaslavsky et al. (2003) express: "We argue that the process of planning and implementing powerful tasks inevitably provides important learning for educators" (p. 899).

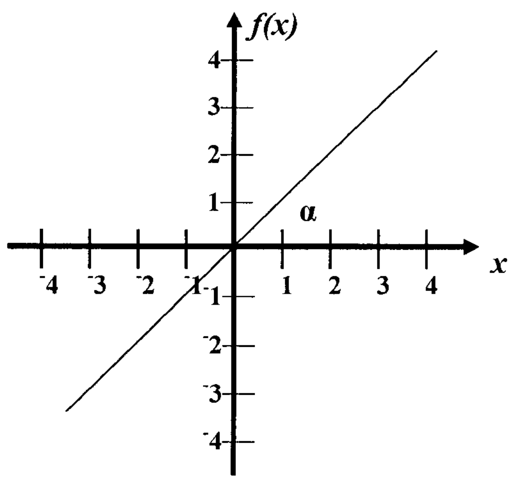
these historical sources, I came to realize that analytical geometry had first and foremost been invented to solve geometrical problems using the then novel power of algebra – making analytical geometry a bridge between geometry and algebra and a strong tool in itself to solve geometrical problems. These problems, historically, were concerned with solving and finding geometrical loci. The importance of using algebra in them was to establish algebraic equations for the locus and solve these equations to find values and properties of that locus. In other words, the usage of algebra in analytical geometry was to establish *equations* using algebra, but not algebraic *formulas* to apply in order to obtain distances and so on. This idea reinforced my disdain for the many formulas taught in school mathematics in analytical geometry, but also helped me to make sense of the intentions of bridging geometry and algebra to solve geometrical problems[75].

This background search was completed by three mathematics education articles (Kendig, 1983; Sorge & Wheatley, 1977; Zaslavsky, Hagit & Leron, 2002). Zaslavsky et al. offered a problem on rate of change illustrating the nature of analytical geometry concepts in contrast to purely geometrical ones, underscoring the bridge between algebra and geometry that analytical geometry represents. The specific task they offered (figure 4.1) makes a distinction between slope and rate of change, and introduces an enriched meaning of the concepts, the former being a geometric concept independent of the Cartesian plane and the latter being an analytical geometry concept existing uniquely in the Cartesian plane. The two sets of questions in the problem reveal a confusion about the concept where it has issues of (geometric) angles, bisectors and tangents intervening in an analytical geometry context. These are not to be taken into account in an analytical geometry context, but have to be taken into account in a purely geometric one. (For a

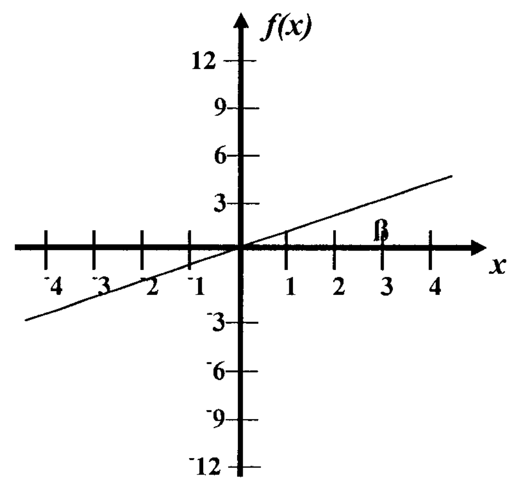
[75] It is interesting to note that Descartes himself never used the Cartesian plane – as we know it – in his annex of geometry. He was using some sort of reference or standard of two intersecting but not perpendicular lines or a single line (which we could link to our x axis) on which to establish proportions, but not to position coordinates. It seems that the origin of using axis comes from Newton's work, where he introduced a positive part and, quite a novelty for that time, a negative part, but only for the x axis. It appears that using both axes was done around mid-18th century. The use of the orthogonal plane (our current Cartesian plane) emerged from pedagogical considerations from French mathematicians at the end of the 18th and beginning 19th centuries. (I refer the reader to Boyer's 1956 book on the history of analytical geometry.) However, concerning the use of algebra in geometry, Descartes made extensive use of algebraic equations to represent and describe geometrical figures (planes, lines, curves, etc.). Hence, there is no doubt that Descartes was the pioneer at systematically using the power of algebra to solve geometric problems (following Viète's symbolism). Therefore, it *could* be said that directly linking Descartes to the invention of analytical geometry *in* the perpendicular Cartesian plane is an anachronism.

more extensive discussion on this, see Zaslavsky et al.'s paper). Therefore, analytical geometry is not uniquely geometrical, neither algebraic, but is a bridge between both.

Task 1:
The graph in figure 1 represents the function **f** such that *f*: $x \rightarrow x$



Task 2:
A student plotted the graph of the same function *f* (such that *f*: $x \rightarrow x$) in a computerized environment and got the following graph:



Answer the following questions:
1. What is the slope of the function *f*? How did you determine it?
2. Does the graph of *f* bisect the angle between the axes? How do you know?
3. Can you find the tangent of the angle between the graph and the x-axis? If you can, what is its value? How did you calculate it? If not, why not?

Figure 4.1. Problem of rate of change and of slope
(Zaslavsky et al., 2002. Used with permission)

My inquiry was completed by Avital and Barbeau's (1991) article which discusses common misconceptions of people about different mathematical topics. Some of the mathematical problems they offer and report on gave specific illustrations of the power of analytical geometry to solve geometric problems. (Indeed, some difficult geometrical problems became easily solvable by placing them into an analytical geometry context in the Cartesian plane.) This reinforced my position that analytical geometry represents a powerful tool to solve geometric problems – which was exactly Descartes' argument.

On this basis, I decided that the objective to work on for session 10 would be: "To sensitize teachers to the fact that analytical geometry is a powerful mathematical tool to solve *geometrical* problems and represents a bridge between geometry and algebra. As this bridge, analytical geometry represents much more than uniquely learning and applying formulas for distances, slopes, and so on, which first are accessory to the study and second shift the emphasis into algebraic manipulations." Table 4.3 displays the objectives to work on developed for each session.

Table 4.3. Objectives to work on for the sessions

Session	Topic	General intention
1	Transition arithmetical to algebraic thinking	To sensitize teachers to students' arithmetical strategies for solving algebraic problems, and reflect on the potentiality to take these into account in the introduction to algebra problem solving[76].
2-3	Volume of solids	To have teachers experience that volume as a geometric concept is much more than only memorizing and applying formulas.
4-5	Area and creating algebraic equations	*Area*: Identical as volume of solids. *Equations*: To sensitize teachers on the fact that writing algebraic equations is a difficult task and that there are many things that we, as mathematical solvers, take for granted and assume obvious for students.
6	Grading students' work	To have teachers reflect deeply on and question their knowledge and conceptions of mathematics concerning specific mathematical content.
7	Operations on fractions	To show teachers that it is possible to make sense and reason the operations on fractions, and that the usual algorithms are not always necessary and also can be reasoned and made sense of.
8	Working on area conceptually	To have teachers realize that there is more to area than formulas, and have them made sense of what it would mean to work within the structures and relations level.

[76] To note, even if the development of teachers' mathematical knowledge was important in session 1, it was prepared along the previous research intentions (i.e., complexity science). It is for this reason that its objective to work on mentions ideas of teaching and of sensitizing teachers to students' solutions. Because the development of teachers' knowledge of mathematics was still an important point of focus, however not the only one, I will refer to session 1 as being part of the "new" research intentions.

9	Making sense of algorithms and techniques	To sensitize teachers to the fact that there is a mathematical meaning behind the usual algorithms prominently used in mathematics, and that it is important to know "how" but also "why" they work.
10	Analytical geometry	To sensitize teachers to the fact that analytical geometry is a powerful mathematical tool to solve *geometrical* problems and represents a bridge between geometry and algebra. As this bridge, analytical geometry represents much more than uniquely learning and applying formulas for distances, slopes, and so on, which first are accessory to the study and second shift the emphasis into algebraic manipulations.

From the preparing of the objectives to work on for the sessions, interesting tasks and situations were picked up to offer to teachers in the sessions (as much as insights for adapting or developing others).

Developing Tasks and Situations to Offer Teachers

Tasks and situations were developed in line with and gravitating around the objective to work on for each session. Tasks given in professional development sessions are considered to be of fundamental value in regard to the learning experiences offered to teachers, following Zaslavsky et al. (2003). Here is how they define tasks:

> We expect powerful tasks to be open-ended, non-routine problems, in the broadest sense, that lend themselves well to collaborative work and social interactions, elicit deep mathematical and pedagogical considerations and connections, and challenge personal conceptions and beliefs about mathematics and about how one comes to understand mathematics. [...] Tasks are usually constructed or adapted by a mathematics educator in order to provide learning opportunities for others. (p. 899)

These intentions for tasks are quite bold and grand, but they do fit with the deep-conceptual-probes model, where mathematical and teaching issues are to be addressed – as much as the implicit work on beliefs and perceptions.

Specific criterion was used in order to construct and select the tasks to use in the professional development sessions. The specific form of the task was not important. It could be a specific mathematical problem to give to teachers and analyze with them, a student solution to a specific problem to analyze, a description of a teaching experiment from research on which to reflect, or a presentation of specific mathematical concepts to explore. Although the form of the task was not important, the mathematical content was central. In effect, the most important element in any task (that I designed or decided to use) was the mathematics in it, and not whether it was about teaching a concept, students' learning and difficulties, solving problems, and so on. Moreover, the idea was not to show them how to teach the notions or work on their practices of teaching, it was to offer teachers mathematically rich learning experiences. Hence, each task aimed at enlarging teachers' knowledge of the mathematics they teach, even if that task hinted at a learning or teaching situation.

The first criterion was that it had to be mathematically challenging. Mathematically challenging does not imply mathematically difficult or about higher-level mathematics, but that it engages with compelling and interesting mathematical issues about usual notions taught. For example, the Zaslavsky et al. (2002) problem on slope noted above is a good illustration of a mathematically challenging problem about a notion of the school curriculum. The second criterion, embedded in the first one, is in relation to the mathematical activity framework. In effect, the tasks offered had to be more than a simple application of a procedure and had to orient the notions to be explored into one or different realms of the mathematical activity. Hence, tasks could (1) question the usage of procedures and the meaning behind them (to develop Skemp's relational understanding), (2) enter into the structures and relations within a concept, or (3) bring forth issues of mathematical conventions[77]. In the following, I give examples of each.

Examples of tasks that questions the usage of and meaning behind procedures

One type of task was about making sense of usual algorithms used in mathematics. For example, in session 9 I gave teachers the following computation, 32.772×8.38 and $32.772 \div 8.38$, and asked them to explain how they would solve it and the meaning behind the algorithm they would use. It was expected that teachers would use the multiplication algorithm of multiplying both numbers without the decimal point and then to count the

[77] Again, because of the importance of offering opportunities to explore and learn "conceptual" mathematics, an emphasis was placed on relational understanding and structures and relations in the tasks.

number of digits after each decimal point in order to know where to place the decimal point in the answer. And, it was expected that teachers would use the division algorithm of moving the decimal point of each number the same number of times to eliminate decimals and then divide the numbers. The intention of this task was to have teachers explore and discuss the meanings and rationales behind these algorithms. For example, one way of interpreting the algorithm of multiplication is that by letting go of the decimal point when multiplying the two numbers it amounts to multiplying two numbers (in this example) that are 1000 times bigger (32772 is 1000 times bigger than 32.772) and 100 times bigger (838 is 100 times bigger than 8.38). It is as if one had multiplied the first number by 1000 and the second one by 100, making the answer 100 000 times bigger. The calculated answer then needs to be divided by 100 000, which makes the answer a 100 000 times smaller, resulting in the decimal point being moved 5 times to the left. For the division, one explanation is that the fact of moving the decimal point by three slots results in multiplying both numbers by a 1000 which amounts to keep the "balance" in the division, in the sense that the ratio from one number to the other is kept the same since a number 1000 times bigger enters the same number of times in a number that is also 1000 times bigger. Indeed, 8.38 goes into the same number of times into 32.772 as 8380 goes into 32 772 – as does 83.8 in 327.72 or 0.838 in 3.2772. So, multiplying both divisor and divided by the same amount keeps the proportion between them the same.

Another type of task offered aimed at prompting questions about the usage of algorithms and ideas taken for granted in mathematics. For example, in session 6, I gave teachers a fraction problem in which the student had unnecessarily simplified the answer (i.e., into the most simplified fraction form) (figure 4.2). It was hoped that by using this problem, discussions and reflections on the taken-for-granted assumption that fractions always need to be written in their most simplified form would emerge, in order to illustrate how this "procedure" does not have much meaning in some situations and therefore is not a compulsory transformation to operate on fractions. Indeed, $\frac{3}{6}$ is as good a fraction as $\frac{1}{2}$. This specific problem was good for prompting reflections, questions and discussions since the number of slices ordered is *not* 3 out of 4, but 6 out of 8, putting into question the reason for having simplified it to $\frac{3}{4}$.

> At Pizza Mania, the pizzas are round and we always cut them in 8 identical slices. When you arrive at the counter, you order two slices of pizza and your friend orders four. What fraction does the pizza slices ordered represent? Explain your solution.

$$\frac{2}{8}+\frac{4}{8}=\frac{6}{8}=\frac{3}{4}$$

We ordered 3 slices out of 4.

Figure 4.2. Simplification of fraction problem

Examples of tasks about structures and relations within concepts

I used two types of tasks about structures and relations, ones that showed how there is more than procedures to mathematics, and ones that worked precisely on the structures and relations within a concept. In order to show teachers what it could mean to work on more than procedures, or what it could mean to work along structures and relations, I offered teachers problems in which one needed more than knowledge of procedures to arrive at a solution. For example, in session 8 on area, I gave teachers the following problem, asking them to imagine this problem in a grade 8 class where the Pythagorean theorem has not yet been learned or is not available (inspired from Jamski, 1978).

Find the area of the following square in the geoboard.

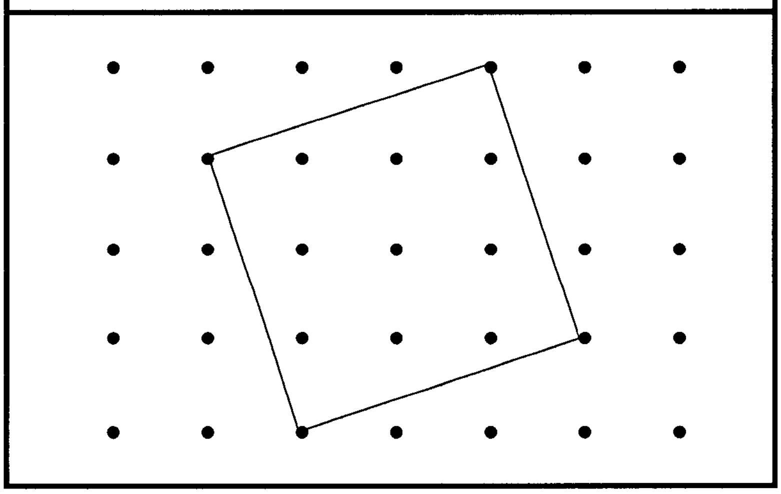


Figure 4.3. Problem of finding the area of a square in a geoboard

This problem cannot be solved directly by using procedures or area formulas – it is impossible to use the usual formula to calculate the area of the square, because the value of one side is not given in the problem. So, the usual "side times side" or "side square" is not immediately applicable. Hence, there are other skills required, skills in the realm of understanding what an area is and what it represents. What is needed here is to understand that an area is a surface and that the area of a surface is always decomposable in many smaller areas if necessary – these are pieces of knowledge about the structure of area. This reasoning is nowhere present in the knowing and understanding of formulas. But above all, and most importantly, there are deductions and meaning to draw from the problem to understand it and be able to make sense of it, links and connections that need to be made (Hejný et al., 2006). This step is more than formulaic understanding. The idea in fact is to reduce the problem into simpler problems that are easier to calculate – this step concerns the structures and relations. This is one way to solve it:

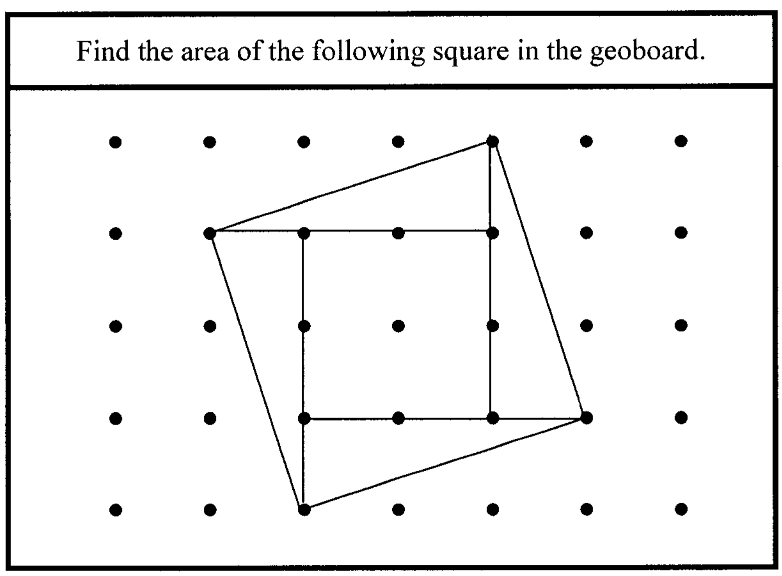


Figure 4.4. Decomposing the area of the square of the geoboard into 4 triangles and 1 smaller square

By doing this, one is now back with a situation that is easier, which is made of four identical triangles of given sides and one square with given sides. So now one can use the skills lodged in the procedures realm of the mathematical activity, that is, the formulas of area of triangle, $(base \times height) \div 2$, and square, $side \times side$, which gives 10 square-units in total. This demonstrates how other skills are needed in this problem (other than ones about procedures), in order to arrive at the answer, and also how only knowing formulas does not help and would even limit the solver here.

A second type of problem that I offered teachers in which ideas of structures and relations intervened was about structures and relations without reference to procedures. For example, again in session 8 on area, after having asked teachers to establish the entire family of rectangles with a perimeter of 20 (and whole-numbers sides), I asked teachers to consider a rectangle with 20 as area and 18 as perimeter and to establish both sets of families to which this rectangle belong (i.e., family of area 20 and family of perimeter 18). This problem, required no procedures at all but delved into the structures and relations within area and perimeter. It brought also teachers to make an important distinction between perimeter and area, not in regard to what a perimeter or an area meant, but to the relationship between both – often seen as the misconception that all rectangles with the same perimeter have the same area (Hart, 1981; Woodward & Byrd, 1983).

Examples of tasks about conventions

To offer a task that addresses conventions, any example of a procedure that utilizes notations and conventions would have been relevant, since most of the time conventions are the standardized and conventional way to communicate and represent one's work. But the intention here was to offer a task in which teachers could see how the usage of the conventions made a difference in the mathematical activity and how it is important to know them, since someone could understand a concept correctly but not be able to represent it adequately without the knowledge of the conventions – hence, showing their importance.

One task that I gave teachers in session 9 concerned the algorithm of division where I asked them to consider the following operation "$18 \div 4$" and its answer. Obviously, one answer was "4 remainder 2." But, I offered teachers to think about and explore these other answers and their adequacy: $3r6$, $2r10$, $5r^{-}2$. The usual answer when dividing requires one to ask how many times does 4 enter the most in 18, and then outline the presence of a remainder. In this case, $18 \div 4 = 4r2$. Though all four answers given are equivalent and make sense mathematically, the mathematical convention requires by definition that the remainder be bigger than or equal to zero and smaller than the divisor (i.e., $0 \leq remainder <$ divisor). Therefore, the last three answers would be ruled out because of the mathematical convention, as much as $16 \div 4 = 3r4$ would be ruled out even if it is mathematically correct. I completed the task by offering a division of integers to explore, like $^{-}18 \div 4 = ^{-}5r2$ by using the above convention, which seems counter-intuitive with regard to what had just been said for the other divisions. The mathematical convention here guides the way in which division and its algorithm are to be used. In that sense, one needs to know the adequate mathematical convention to obtain a mathematically acceptable answer, even if alternatives are conceptually meaningful.

Offering other sorts of tasks

The examples given above illustrate particular and individual tasks that were given within the sessions. However, it also happened at times that most of an entire session revolved around a specific task or a situation to explore. For example, session 1, the second part of session 2-3 and session 7 were each focused on the exploration of one larger situation. Session 1 consisted in solving an entire set of traditional algebraic problems, without using algebra, to develop arithmetical thinking (see Appendix A); the second part of session 2-3 was on exploring a teaching sequence created by Janvier (1994a, 1994b) concerned with the volume of solids – which is reported at length in Chapter 5; and session 7 was on operations on fractions where throughout the entire session the explorations were around using egg cartons, folding papers, drawing areas and delving into specific contexts to make sense of the meaning behind the usual algorithms used for the four operations on fractions (+, –, × and ÷) – see Appendix B for the events of this session.

Moreover, in the first part of session 2-3 and the second part of session 4-5, I requested teachers bring a problem that they felt was a good one concerning the mathematical topic of the session, one on volume of solids for session 2-3 and one on writing algebraic equations for session 4-5 respectively. The main intention of having teachers bring in their own set of problems was to start from where they were and what they knew in order to build on it, and also to get a better idea of what they considered important to address in these topics.

This ends the section on the preparation of the sessions of professional development. I complete this chapter by discussing how the data was analyzed, before entering directly into reporting on its analysis in the next three chapters.

Data Analysis

In her article, Crockett (2002) points to the fact that most reports on professional development uniquely describe the "structures" established to conduct professional development (e.g., inquiry groups, case studies, collaborative action research, etc.), but few of them report on the "content" or the specific activities of these sessions. In other words, most reports focus on "how" the in-service sessions were conducted and fail to address and offer insight into the "what" of the sessions. Little attention is paid to what pragmatically and concretely the specific learning opportunities of professional development sessions were. My analysis of the data from the sessions will not report on the opportunities that the established structure created and permitted, but will focus on the mathematical explorations of the sessions themselves and the learning opportunities they offered to teachers: that is, what the sessions created and how it impacted teachers' knowledge.

The actual process used to analyze the research data involved many steps. The *first step* consisted, after each session, of writing about my first impressions about specific learning moments that happened during the session in regard to the teachers. This immediate writing enabled me to start my reflections about my data and the impact of the sessions on teachers' knowledge. Having taken notes of these specific and remarkable moments, I made a short summary of the entire set of events that happened in the session. This was aimed at keeping track (of the events) for informal meetings that I would have during the following days with my advisor and/or members of my supervisory committee about the session. These meeting summaries enabled three specific things to be carried out. First, I was able to discuss the events of the session and keep a "memory" of them and, by elaborating on the session and being questioned about it, to identify other moments that I had not noted at first and to discuss them also. Second, I could interact about issues that happened to obtain direct feedback on ideas and insights about the research data and possibly be oriented toward different angles of looking at the diverse events of the session. Third, the summary accounts helped me to prepare the next session. Therefore, meeting about the data consisted of the *second step*, during which I took a lot of notes and added them to my previous ones (taken on-the-spot). This guided my next steps.

The *third step* consisted in the production of a long description of the events of the session (10 to 15 pages), during the week following each session. To do this, I watched the videotaped session in detail and took note of its events, by separating them into "moments" that were distinguishable by means of the explorations done (change of task, change of issues, specific explorations within an issue, etc.). Therefore each session, during the following week of its occurrence, was summarized, described and split into distinct moments. Also, transcriptions of conversations and events were added when they were useful to clarify the instances.

Within the third step was a *fourth step* aimed at paying specific attention to two sorts of events. The first type of event that I paid attention to was in line with my on-the-spot thoughts (and the events raised during my individual informal meetings following the sessions) concerning specific teachers' learning experiences. To these were added moments that I did not remember then, or simply that I did not "realize" when I conducted the session, which were made apparent by the video. For each of these moments, I stopped the description and first wrote a note in the middle of the description explaining the "learning experience." Second, I took a separate sheet and wrote a detailed description of the event (task, explorations made, discussions, learning, links to previous moments of other sessions, etc.) and my interpretation of it concerning the learning that had happened.

The second type of event that I paid attention to in this fourth step was my "interventions" as the teacher educator. From the videotapes, each time that I, as the teacher educator, made an intervention with an explicit intention to engage teachers, I took a note in the margins (in orange) to flag it. What I paid attention to were *active* *interventions*, and not silences or agreements with teachers which are also actions but ones that do not aim to engage teachers explicitly – hence, I did not pay attention to those. I wanted to categorize and get a sense of the types of explicit interventions that I used throughout the sessions. For this, at the end of the research year when all sessions were described in length, I defined the nature of these interventions and created a categorisation for them (somehow influenced by the process of categorisation from Glaser and Strauss's 1967 grounded theory). Therefore, the categories of action were invented along the way as I analyzed the interventions. Once one session was completed, I assembled the different types of action obtained, organized them and created an analytic grid. Then, I went back to that same session and applied the grid anew to it, in order to revise the grid if needed. After re-adapting the grid, I went to the next session and applied it, and made adaptations if needed. I did that until all sessions were analyzed. From that, I created a categorisation of the different types of interventions I used throughout the sessions as the teacher educator.

Finally, the *fifth step* consisted in coding the entire description of each session using the deep-conceptual-probes model. This was done after each session was described, and adjusted as the year progressed and new ideas and interpretations came up. To do this, I took a sheet of paper, traced a line in the middle and wrote on the left side of the line all "the mathematics explored" in the session and on the right side all "the teaching issues that emerged" during the session. Hence, each session was finely detailed in relation to what the deep-conceptual-probes model created and offered to teachers. In the end, the analysis of data consisted of each session systematically (1) summarized with remarkable moments highlighted, (2) discussed with knowledgeable researchers and detailed along important issues, (3) described in length and cut into moments, (4) with specific events of "teachers' learning" flagged and interpreted and "teacher educator's interventions" categorized, and (5) coded along the deep-conceptual-probes model.

The data is reported in Chapters 5, 6 and 7, and centers on the learning opportunities offered to teachers in order to enlarge their knowledge. In Chapter 5, I offer a description and interpretation of the events with regard to the deep-conceptual-probes model. I demonstrate how it was enacted and unfolded during the session, and what it created and offered by way of learning experiences for teachers. In Chapter 6, I provide a more detailed examination of the data by looking at the learning experiences of the teachers and tracing them in relation to their specific tasks within the diverse branches of mathematical activity. In that sense, this chapter looks at the data through the specific lens of the three branches of the mathematical activity described in Chapter 2. This enables a better understanding of the possible impact on teachers of opening the study of mathematical concepts to something other than the application of procedures. In Chapter 7, I look at the data through an enactivist lens as discussed in Chapter 3. I begin by illustrating and making sense of the phenomenon of emergence underpinning the deep-conceptual-probes model and how events continuously emerged and unfolded within the sessions. I end with an analysis and categorization of my practices as the teacher educator.

For sake of clarity, and because I wanted to address in precise detail each of these specific aspects, I have separated the analysis in three different chapters. However, this separation is mainly a heuristic fiction, since all of these aspects are enmeshed together and functioned hand in hand throughout the professional development programme, as I continuously attempted to offer teachers rich learning experiences to enlarge their mathematical knowledge. Each of the three aspects (deep-conceptual-probes model, the mathematical activity model, and enactivist principles) happened to be three different perspectives from which to look at the data. For this reason, please keep in mind each three aspects while reading the chapters. For example, when reading Chapter 5 on the deep-conceptual-probes model, the reader should continue to be sensitive to aspects of mathematical activity and the issues about enactivism (emergence and the teacher educator's actions).

CHAPTER 5

FIRST ANGLE OF ANALYSIS: DEEP CONCEPTUAL PROBES INTO THE MATHEMATICS TO TEACH

This dissertation aims to bring forth a model to address the issue of the professional development of secondary mathematics teachers who are invested in procedures. The deep-conceptual-probes theoretical and orienting model was described in Chapter 2. The purpose of this chapter is to demonstrate how this model functioned in the context of professional development sessions. In particular, I explore what it creates and provokes, in the overarching goal of enlarging secondary-level mathematics teachers' knowledge of mathematical concepts that they teach.

As Lampert (1990) does when reporting her research, I have chosen one specific session that illustrates well the type of learning spaces and opportunities that were offered and that unfolded in the professional development sessions. Or, to use Brousseau's (2006) expression, the session reported on is used as a "prototype" to represent the work done with the deep-conceptual-probes model throughout the program. The session chosen is session 2-3 on volume of solids[78]. I first present and describe at length the events that happened in that session and I insert comments (in *italics* in grey boxes) at different places within the description to underscore important elements to which I wish to draw attention. After describing the session, I discuss the events in relation to specificities that

[78] Data from other sessions are also available for consultation in the Appendixes B, C and D.

the deep-conceptual-probes model provoked and created, in order to produce a finer-grained analysis of the outcomes of this approach.

An Illustrative Example of Deep Conceptual Probes: The Volume of Solids Session

Carole: I, my teachers never did things like that when I was a student.
Gina: Wait a moment, this is not important. It is YOU who needs to do it, that is what you should tell yourself.
Erica: Me neither. My teacher never did that.

As Janvier (1994a) explains, the volume of solids is often seen as a simple concept to teach, one that only requires students to memorize a list of different formulas. Similarly, the teachers participating in the in-service sessions primarily understood volume of solids as an area of mathematics which requires the learner to know formulas and to apply them in order to calculate the numerical value of specific volumes. Therefore, working in volume was mainly an algebraic and computational task situated in the realm of substitution of values into algebraic formulas to make calculations. In other words, these teachers did not understand volume as a geometrical concept but as an algebraic–computational one; the geometrical aspects were lost in favour of the algebraic ones[79]. Because of teachers' inclination toward formulas, I aimed at going back with them to the geometry, and working on *volume as a geometrical concept* to enrich their knowledge of volume – as much for the meaning behind the formulas (Skemp's relational understanding) as for unearthing the structures and relations within the notion of volume of solids. It is important to note that the teachers' knowledge of volume of solids was not negative in itself. However, my intentions were to build on their knowledge of procedures and calculations and enlarge their knowledge so that it took in "conceptual" understandings of volume of solids as well. The objective to work on that oriented

[79] This is not surprising since algebra possess a very high social status for many people doing mathematics, which brings many to discard other aspects in favour of algebraic representations (see, for example, Schmidt & Bednarz, 1997). In Appendix D, part D.2, Lana also explains that students normally desire to know the "formula" that works and once they have it they are content, but they also almost never come back to aspects of the concepts situated in a geometric realm (except if they are asked, which often seems to create difficulties). In effect, the presence and insistence on the algebraisation of concepts is very strong in mathematics classrooms and curricular documents, reducing almost everything to algebra.

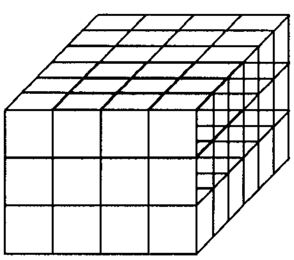
session 2-3 was “To have teachers experience that volume as a geometric concept is much more than only memorizing and applying formulas.”

The session was divided into two parts. In the first part, teachers were asked to bring in problems about the volume of solids that they thought were interesting, and which would be solved and analyzed. In the second part, I presented teachers with an innovative way to teach volume developed by Janvier's (1994a, 1994b) research programme. This teaching approach to volume is illustrated in a 33-minute videotape produced by Janvier (1994b), and taught by one of his colleagues in a grade 9 classroom[80]. This videotape was used as a starting point for the session, where it was interrupted to discuss and address specific issues *while* watching it, and oriented the discussions and work after it was finished. (These discussions continued in the beginning of the following session 4-5.) I focus on this second part in the analysis that follows.

Description of the Events

Moment 1: Volume of prisms as a piling up of layers

Volume is introduced as an accumulation of layers of the same object. This idea is discussed by the teacher in the video by piling up layers of sugar cubes, each layer being of 24 sugar cubes displaced in a 6 x 4 fashion. The teacher asks the students what would be the volume of his construction if he had 3 layers of thickness (figure 5.1).



3 X 24 cubes = 72 cubes

Figure 5.1. Representation of three layers of sugar cubes

[80] This video is a summary of a teaching sequence developed by Janvier. It is not a classroom case study for teachers to analyze. In it, there is a teacher teaching volume of solids to his students, and a narrator commenting on the actions taken and explaining the meaning behind them and their purpose.

The teacher follows this by asking students how many of these layers would there be if the volume were 108 sugar cubes, aiming at bringing the students to work with fractions of units. Moreover, this is aimed at bringing students to work little by little toward an image of continuous accumulation, in contrast to a discrete manner of accumulating layers already possessing a height[81]. Afterwards, to bring the students closer toward a possible formula, the teacher in the videotape brings his students to make a shift from volume as a "number of cubes per layer multiplied by a certain number of layers," toward volume as "area of the base of the prism times its height."

This represents the first type of mathematics introduced in the session: the idea of seeing volume as a piling up of layers. This may not have been something that was completely new to the teachers, since they were able to make sense of the approach, and may have thought about it before. However, this approach of seeing volume as a piling up is distinct from introducing volume with many formulas, one for each prism, and it focuses on a geometrical approach rather than a computational one. It represents an approach that "works for all prisms," and not exclusively for a rectangular prism or a specific prism. These are two elements that will repeatedly be addressed in the session. Specifically, the teachers will come to discuss that the piling up of layers is not a technique or a formula for finding a volume, but mainly an approach to conceive of and understand volume conceptually (even if the teacher in the videotape brings it as a formula), and also that it is applicable to all prisms. This activity is about relational understanding where the prism formulas can be made sense of. It is also about structures and relations, since it represents an understanding of what a volume is and how the volume of a prism can be interpreted and made sense of – aside from formulas and calculations, by means of piling up of layers.

<u>Moment 2: Mathematical contexts</u>

The teacher then offers his students some problems of changes of units where they have to find the volume of certain solids. Taken by this, Gina raises the issue of "contexts" in mathematics. She mentions that this type of work is *working in a context.*

Gina: You see, the other day [session 1] when we talked about problems

[81] This move toward fractions of units to get to a continuous flow from 2D to 3D is not a simple one and this issue was briefly discussed in the session. This move from piling up of discrete layers in 3D to a continuous accumulation going from 2D to 3D could indeed be mathematically criticized in this teaching sequence. Without wanting to defend the intention, the underpinning idea is mostly to instill a habit of seeing volume as an accumulation of layers, rather than making a formal infinitesimal proof.

in a context, this is what I meant.

Jérôme: Okay.

Gina: Like there, he just placed a problem in a context, it is not...

Jérôme: Like what?

Gina: Like his small centimetre, like his sugar cubes used before, all of this. For me, these are problems that are important.

Jérôme: Because they are in a context you think.

Gina: Yes, there they "see" the sugar cubes, we are talking about it, you know, there are links there, it is real. The story of the sugar[82] and all of this, they "see" it.

Jérôme: But it is not a "real-life" context, however.

Gina: No, but it is in context.

Jérôme: Ah, OK, OK.

[...]

Gina: It is not simply a problem on a sheet of paper that does not even have a drawing on it, like this [she points to her textbook].

Jérôme: Ok, now I understand what you mean by "context," I thought you meant "real-life" contexts. This is a huge discussion. Like when Lana and Erica said [in session 1] that they could not find contexts for inverse functions. These are, in fact, "contexts" that we could call "mathematical."

Gina: Yes, yes.

Jérôme: It is a mathematical context, it is not a real-life context.

Gina: In some ways, the context you create it when you do your problems. It is very different, for me, this type of problem and a problem like [opening her textbook] "Mike concluded that the signal was multiplied by ... blah blah blah," you know.

The discussion continued on with reference to contexts in curriculum documents, to real-life problems, to mathematical contexts, and so on.

This is an example of a teaching issue that emerged from the activity of the session. The type of mathematics offered in the videotape triggered Gina's reaction toward what constitutes a good mathematical context. This discussion about context represents a general teaching issue in that it is not linked to a specific piece of mathematics, but to a practice of mathematics teaching in general.

[82] At the very beginning of the video, which I have not mentioned, the teacher filled solids with sugar. This is what Gina refers to here.

Moment 3: Oblique solids and Cavalieri principle

The teacher in the videotape then goes on to consider the volume of non-conventional solids. He introduces oblique and twisted solids to the students, some made of sheets of paper and others made of plastic resembling the ones in the following (figure 5.2).

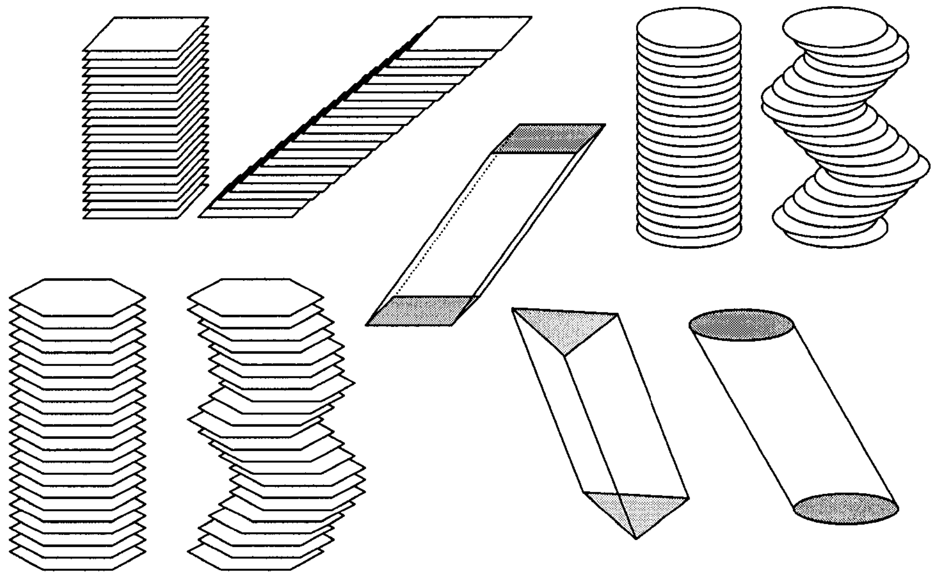


Figure 5.2. Some oblique and twisted solids

The students in the videotape explain that you can also just take the area of the base and multiply it by its height, saying that you could simply "replace" or "re-align" the sheets of paper to create straight prisms. The videotape narrator then adds: "We note that the introduction of volume by layers enables the students to imagine transformations in solids that keep the volumes invariant," which links to the introduction of Cavalieri's principle.

This introduction to oblique shapes by means of the piling up of layers consolidated the "piling-up" approach for the many types of prisms, since the teachers could see that the idea was applicable to any prism. Moreover, it opened the possibility of exploring different shapes that are not familiar nor often talked about in traditional studies of volume in classrooms. This can be seen to have enlarged their repertoire in regard to the study of volume of solids. The teachers who were absent at this session, but present at the next session 4–5, were *very intrigued and surprised as to what had been done with these "weird" oblique shapes in the previous session (I had these shapes on my desk at the beginning of session 4-5). In fact, they were even sceptical as to the worth of considering these types of solids, suggesting that maybe I was pushing a little by introducing them. For example, Linda had the following reaction.*

> *Linda:* *Jérôme, did you work on these prisms and cylinders that are not straight?*
> *Jérôme:* *Yes, yes, yes.*
> *Linda:* *It is a little bit too far, don't you think?*
> *Jérôme:* *[Talking to the other teachers that were present in session 2-3] Well, if you remember, the work on oblique prisms was quite simple with the way we approached it.*
> *Erica:* *Indeed.*

As the following will illustrate, the oblique shapes are "approachable" with this new approach to volume as piling up, and do not seem to be that much more complicated than regular or straight prisms. Therefore, it opened the way to new types of solids in the study of volume for these teachers.

The teacher in the video introduced the principle of Cavalieri which states, roughly, that if you have two solids of same height and with bases placed in the same plane, both solids will have the same volume *if* each time that a cut parallel to the base is carried out (i.e., in the same plane) it gives two surfaces that have the same area (figure 5.3).

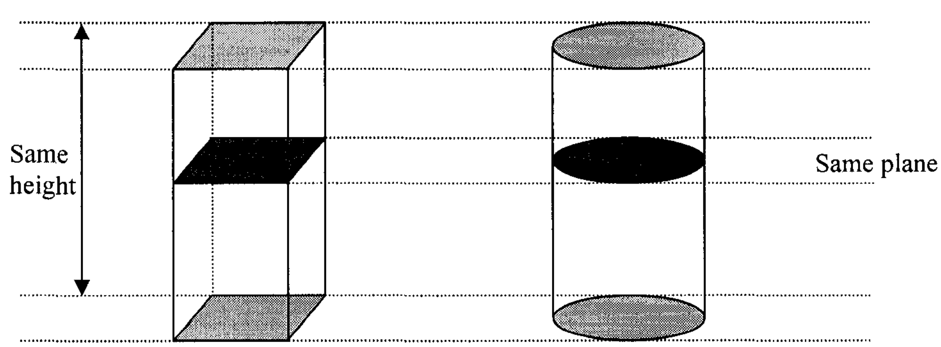


Figure 5.3. An illustration of Cavalieri's principle

This principle was new to the mathematics teachers in the session. They all concentrated deeply on what was shown on the screen and asked many questions to understand the principle and its outcomes better, which brought me to re-explain Cavalieri's principle in my own words in order to help them understand better[83].

Erica:	How can it be non-congruent but of same area? This is impossible.
Jérôme:	It is not the same 2D figure, but the area of the figure is the same.
Erica:	OK.
Claudia:	The area or volume?
Jérôme:	The area. Cavalieri's principle…
Erica:	The area of each slice?
Jérôme:	Yes. Each slice, be it a circle, an oval or anything. [Pointing to my left] Here a rectangle or a triangle, if it is the same area [I point to the right as if I had a solid on both sides] and that each time it is always the same height [I create virtual cuts in the solids].
Erica:	So, it is the same volume.
Jérôme:	In total, it is the same volume. This, is Cavalieri's principle.
Erica:	OK.
Jérôme:	But, I cannot have two solids that do not have the same height, because at a point I would cut where there would be nothing for the other.
Erica:	Humm, humm.
Jérôme:	So, the same height, if at each time that I go up I realize that it is the same area…
Claudia:	Same height, same area of the base, you will have the same area everywhere.
Erica:	To the same volume in the end.
Jérôme:	Same height and same base, and [same area] each time that I cut. Hence, at the base. Hence, in the middle. Hence, at the end.

The introduction to Cavalieri's principle represents another instance of mathematical work, but this appears to be something new to the teachers. That is, they had never heard about this principle before[84]. This principle does not focus on formulas or calculations, but mainly on aspects of comparison and relationship. In that sense, it is situated in the structure and relations domain of mathematical activity where it is not about formulas or procedures, but about relations. Here again, the explorations were pushed away from the common volume formulas, and went into a geometric realm of relations and properties. As the transcript makes quite clear, the teachers were intrigued about the meaning of

[83] The figure 5.3 above shows the general case of the Cavalieri principle, mainly that all there is needed is the same "value" of area at each layer, notwithstanding if the figures are the same shape or if they represent regular figures. As for the work done in the video and the professional development session, the work was along specific cases of the principle, that is, cases in which moreover than being of equal area, each layer was even of the same shape.

[84] This should be seen as normal since it is not present in the current curriculum. However, the teachers could have known about it, but they did not.

this geometrical principle and how it functioned, and asked many questions. All of this opened up new lenses to look at and understand the volume of solids, which potentially could enhance teachers mathematical understanding in regard to volume. Cavalieri's principle was at the center of many of the subsequent discussions.

Moment 4: Pyramids

The discussion in the video then shifted to pyramids and their volume, where the teacher in the video started by showing that a cube can be decomposed into three identical right-angled squared pyramids (figure 5.4).

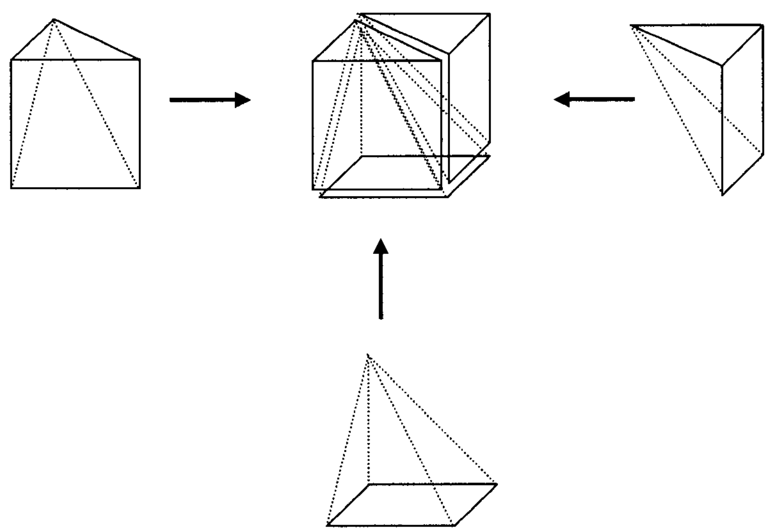


Figure 5.4. A cube decomposed in three right-angled pyramids

The teacher in the videotape followed with an inquiry into a triangular prism composed of three triangular pyramids, which were not all identical. I then stopped the videotape to get some material that I had brought for the session: pyramids, cubes, prisms, planar figures that could serve to create layers, and so on. I reproduced the previous decomposition of the cube with pyramids in front of the teachers, to show them how one pyramid is one third of (the volume of) the cube. This also brought me to introduce implicitly the terminology of "associatedness" between a pyramid and its "associated" prism (one with the same height and same base), and linking it to Cavalieri's principle.

Jérôme: This pyramid is one third of its associated prism. These are also oblique pyramids, so this is where Cavalieri's principle is very interesting. It is that this specific pyramid [pointing to the right-angled squared pyramid], if we make it straight with the apex in the middle of its base, that we will call the straight square pyramid, we can in fact prove that by cutting everywhere we always have the same area. So this one [pointing to the right-angled square pyramid] and the straight one are the same. So, if I have to measure the volume of a straight pyramid with the same squared base and a height equal to the one of the cube...

Gina: Yes.

Jérôme: It will be again a third of the cube. The interest is there.

Claudia: And the heights? Will it be the same heights?

Jérôme: Yes, you have to have the same base and same height. In fact, the theory is that same base and same height for a pyramid, and same base and same height for a cube, means that it is worth a third. So, a pyramid that has the same base and the same height that its associated cube will be a third. "Associated" in fact, when I say for a pyramid its associated prism, it means that there is the same base and same height.

[...]

Claudia: And it does not matter where is the [summit]?

Jérôme: Yes, no matter where.

Claudia: If you moved it, so if you could move it, it would always be the same height?

Jérôme: Yes.

Claudia: *That* is important.

The discussion continued on because Gina asserted that the equivalence principle was based on an average of areas and not on the fact that it was literally the same area at each layer. Consequently, the work went toward looking at what would happen if the pyramids were made of small pilings-up of paper which could then be re-aligned in many ways.

There are two sets of mathematical issues here. The first one is in relation to pyramids of different forms, that we could call oblique, and how by Cavalieri's principle under specific circumstances of height and base and equivalence of cross-sectional area they could be of identical volume (again, no measurements or computations are required, but only comparisons between them). This is then a continuation of Cavalieri's principle for pyramids, where teachers learned more about its possibilities and meaning. The second issue, that has just been introduced, and that will be worked on again afterwards, concerns the fact that a pyramid literally is a third of its "associated" prism – one which has the same *base and same height. It is not simply that "it is" or that "it happens to be" a third of a prism, but precisely that by geometric decomposition a prism can be* physically *cut up or dissected into three adequate pyramids. This will enable to give a meaning to the presence of the one third in the pyramid formula, a formula that these teachers already knew well*[85].

Erica followed up by raising the point that even if it seems obvious this equality could be quite difficult for students. Especially, because the height of the pyramid is "on" the lateral triangles for the right-angled square pyramid and therefore represents as much the height of the pyramid as the height of the triangles, whereas for a straight pyramid the height is "inside" of it and the height of the side triangles represent the "apothegm" of the pyramid (figure 5.5).

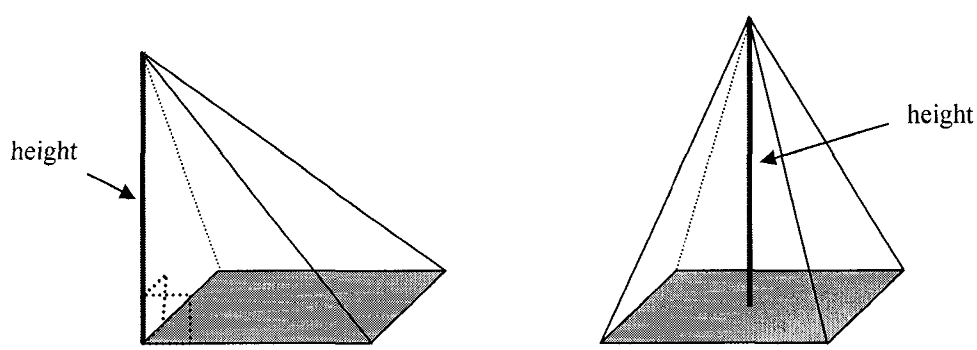


Figure 5.5. Two different positions for the height of a pyramid

This is the second teaching issue that emerged, but this time it was one explicitly linked to the mathematical issues explored. This teaching issue, linked to possible student difficulties with the different pyramids and their heights, emerged out of the mathematical work that was being done as Erica realized and pointed out to the others that it would not be an easy issue for students to understand and that it could cause difficulties in their learning.

[85] The teachers knew that a pyramid enters three times in a prism with the same base and same height, most of the time "illustrated" by filling in the pyramid with water and pouring it three times in the prism. However, here it is more than illustrating or realizing that it is entering three times, it is putting together three pyramids to create the prism – making the pyramid a third of the prism or making the prism three times a pyramid, and not uniquely accepting or realizing that "it is" three times its volume. In a sense, it is almost the difference between "showing" and "demonstrating."

I continued by re-emphasizing that Cavalieri's principle brings us to see that a straight square pyramid is also a third of its associated cube, because of its equivalence (by Cavalieri) to the right-angled square one, even if physically it cannot be re-composed into a cube (i.e., if three right-angled square pyramids are joined together, they do not create a cube). I then went back to the last ideas brought forth in the video about the three triangular pyramids that combined into a triangular prism, as in figure 5.6.

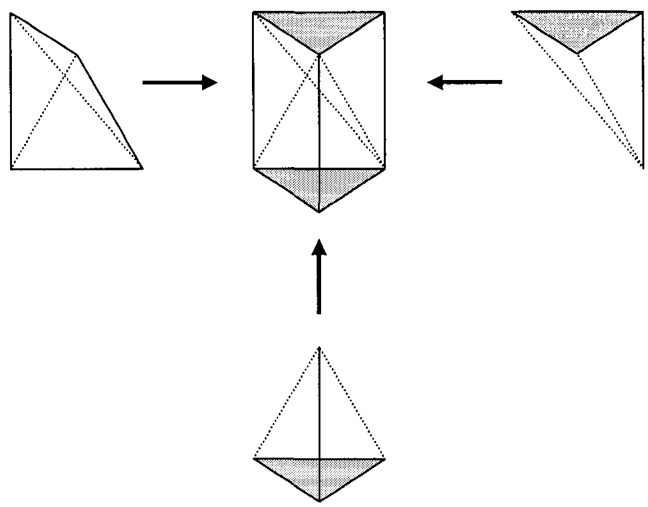


Figure 5.6. Three triangular pyramids that creates a triangular prism

In these, two are identical and the third one is different. I recalled for teachers an informal definition of a pyramid, which was that there is a base and that the lateral faces need to be all triangles. I then prompted them to consider the three pyramids as equal by *not* choosing the small triangle as the base, but one of the other triangle faces. The discussion then unfolded as to how any of these triangle sides can be chosen as a base, which helps to establish the equivalences (figure 5.7). This showed that the different pyramids had the same base and the same height, meaning that by Cavalieri's principle they were equivalent. However, this was not trivial to understand and more meaning was made out of this. For example, I had to re-explain how the third "non-conventional" pyramid was indeed a pyramid, how Cavalieri's principle could work in that case, and how it was possible to choose other triangular sides of these pyramids as the base because of the definition of a pyramid (where the lateral sides were still triangles).

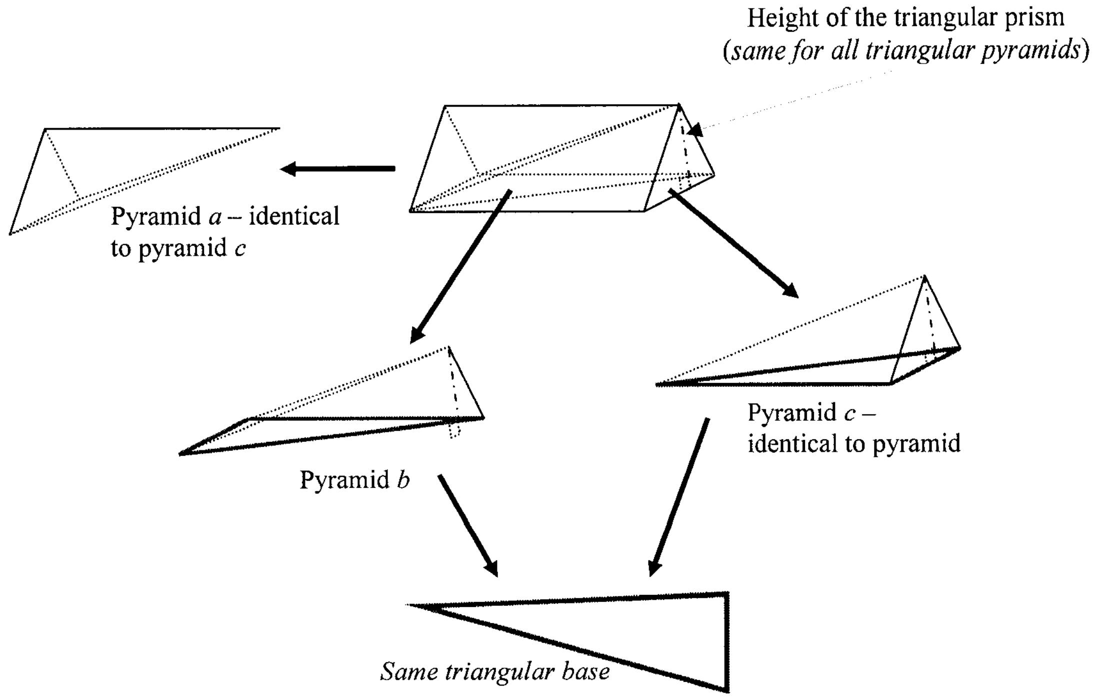


Figure 5.7. The three triangular pyramids in relation to a same base and height

To make more sense of this issue, the teacher in the videotape brought a small device in which he could move around (along the same plane) the summit of a pyramid made out of elastic bands, which showed how its volume was always kept the same however he moved the summit[86].

> *This is the continuation of the work initiated on the fact that pyramids represent a third of their associated prism. As mentioned, this gave a meaning to the presence and origin of the "⅓" in the pyramid formula. But in addition, it is for any pyramid and not exclusively for a specific type of square or triangular pyramid. This extends the applications of this relation for any type of pyramids (any pyramid has a volume which is the third of its associated prism), and creates a strong link between pyramids and prisms.*
>
> *There is also the presence of another important mathematical aspect, namely the definition of a pyramid and its possible base. (This will come back afterwards concerning the possibilities for bases of solids, in moment 7, and will be used as*

[86] This represents a general case of Cavalieri's principle, where at each layer it is the same area *but* not the same shape – since the triangles obtained by the cut are of equal area but are not congruent. However, the illustration with the device served here mainly as a "proof," and this issue was not explored deeper in the session.

an explanation at a certain point.) Indeed, stating the definition of the pyramid enabled the possibility of not necessarily placing one of the pyramids on its usual base, which in this case would normally be the smallest triangle. This appears to be counter-intuitive for two reasons. The first one is that the different side (from the other ones) is normally chosen as the base, and in this case it was the smaller triangle. But the second reason is that it is not usual to choose a base that gives different lateral faces. In the case of these triangular pyramids, choosing one of the long triangles as the base creates lateral faces that are still triangles – hence they satisfies the definition of a pyramid – but which are not identical – hence not usual or familiar. In that sense, the play with the definition enabled new meanings to emerge for the pyramids and opened up to a different sort of pyramids, not uniquely the regular ones. Irregular pyramids (with lateral triangular sides not identical) were not familiar to teachers. The exploration or analysis of the definition of a pyramid opened up some new possibilities for teachers – and these will be taken up in the subsequent explorations.

Moment 5: Formulas

The narrator in the video elaborated on the formulas that need to be memorized (or known about) in that teaching sequence. The first one being "*area of the base* **x** *height*" for the volume of prisms, and the second being "*volume of the prism* ÷ *3*" for the volume of pyramids, which also became "(*area of the base* **x** *height*) ÷ *3*".[87] This brought me to suggest the idea that maybe these were not even formulas, since these were mostly ways of seeing or "understandings."

Jérôme: I would even be ready to add that there are not really formulas that much behind this.
Gina: Because he understood, he understood its formula.
Erica: Yes, there is the area of the base and the…
Jérôme: We are able to write it, of course, but *area of the base* x *height…*
Erica: Is not a formula.
Jérôme: Is it really a formula? It is more a way to see volume. In effect we can write this thing equals this thing, but…
Erica: Humm, humm.
Gina: But once the formula is understood instead of memorized, we do not consider it as a formula.

[87] Even if the narrator and the teacher in the video talked in terms of "÷ by 3," the group and I mostly talked in terms of "a third of" to describe the relationship between pyramids and their associated prisms.

There are two aspects in this conversation. The first one is the presence of a teaching issue concerning the place of the formulas in teaching and learning volume. As Erica and Gina explained, these do not represent formulas any more and could be seen as "understandings," something that questions the place of formulas in the learning of volume of solids. In that sense, it is a reflection on the presence of formulas in learning volume and on memorizing them. This issue will come back in moment 8.

This brings me to discuss the second aspect addressed here, and it is in relation to the teachers themselves. It is possible to observe how the teachers are starting to change and "move" concerning their inclination toward formulas or the importance that they have in the study of volume, where they do not even consider them as formulas anymore. Formulas now become things that can be understood and made sense of, and not simply learned and applied. The teachers' knowledge is starting to evolve and they can now see the volume of solids in a different way, one where there is a meaning behind the formulas. Teachers are developing a relational understanding of volume formulas. Volume of solids was not reduced to the simple application of formulas anymore; these formulas could be made sense of and understood. This is what Gina meant when she said that students "understand," it is that they can know where the formulas come from.

But there is even more. By realizing that formulas can be grouped in two types, the prisms and the pyramids, it also develops the teachers' understanding in the realm of structures and relations. The acceptance of reducing all prisms to one formula has with it the idea that all prisms can be represented as "piling up of layers," hence bringing teachers to establish connections among the different prisms – and this aside from calculating their volumes. In the same sense, the reduction of all formulas of pyramids to a single one demonstrates the accepted connection between (a) the different pyramids and (b) a pyramid and its associated prism. In that sense, it illustrates the beginning of an understanding of the relations and connections among solids – an exploration of their structure. Therefore, these explorations modified and influenced what volume represented for the teachers, and how it could be made sense of. The explorations enabled teachers to develop their "conceptual" understanding both in regard to relational understanding and in regard to structures and relations (aside from the procedures and calculations).

Moment 6: Cylinders and cones

This completed Janvier's videotape on teaching volume of solids[88]. I then emphasized an aspect that had not been talked about in the video itself, that is, the volume of cylinders and cones. I explained how the cylinder could also be seen as an accumulation

[88] The videotape ends in fact with a rapid illustration of the volume of a sphere in relation to pyramids, but this issue was not explored in depth in the session.

of layers, as the prisms were, where cylinders can be defined as prisms with infinite number of sides (figure 5.8).

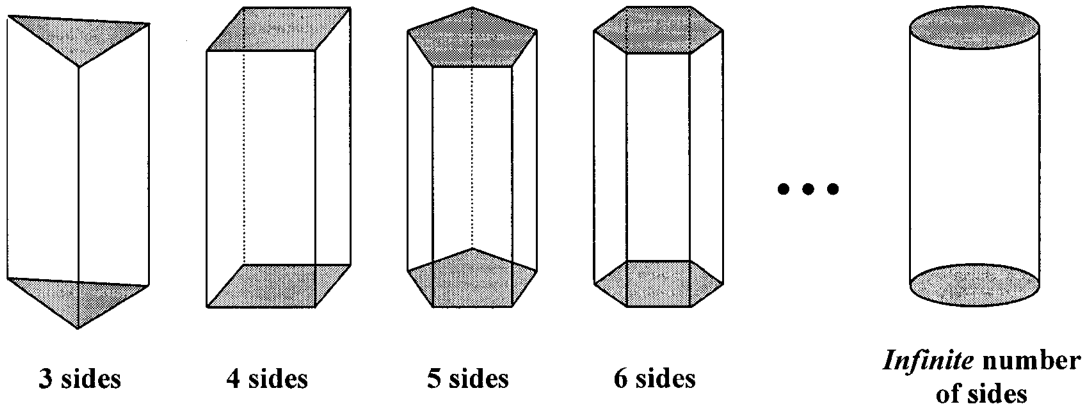


Figure 5.8. Defining the cylinder as a prism with an infinite number of sides

Consequently, the volume of a cylinder can be seen in the same way as for other prisms: that is, as the area of the base (here a circle) times its height. Along that line of thought, I showed how the cone could also be defined as a "pyramid" with an infinite number of sides, which enabled the establishment of a relationship between the cone and its associated "prism," like with any pyramid. However, in this case the prism was a cylinder. The volume of the cone then was worth a third of the volume of its associated cylinder (figure 5.9).

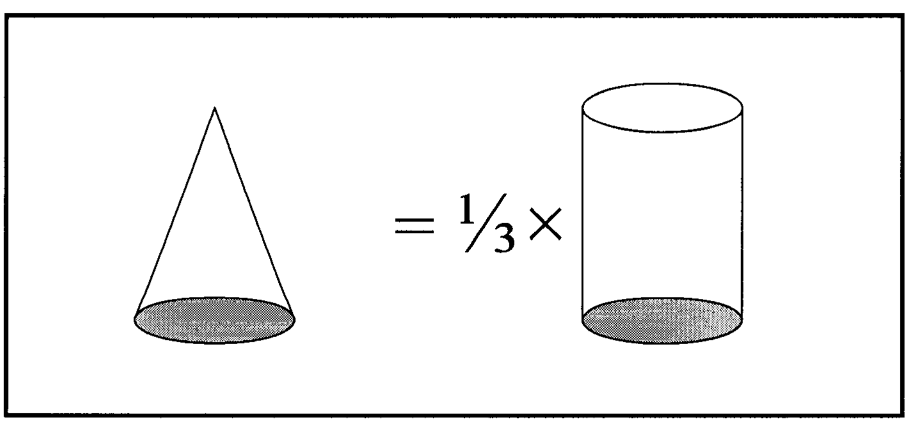


Figure 5.9. Relationship between a cone and its cylinder

The redefinition of cylinders and cones in terms of "infinite sides" is another mathematical issue addressed. This has important implications for the understanding of volume itself. First, it defined differently what a cylinder is, and linked it to prisms. In that sense, it demonstrates how the idea of "piling up of layers" could be extended to cylinders, making them a type of prism. Second, it does the same thing for the cone, by making the cone a type of pyramid possessing the same characteristics and relations that pyramids have in relation to their associated prism and also possessing the same type of formula (bringing a meaning to the presence of the "⅓"). This changes dramatically the understanding of what a cylinder and a cone represent, in regard to their formulas and the meaning behind them. But also it links the cylinders and the prisms, the cones and the pyramids and the cone and the cylinder; something aside from their formulas and in regard to the geometric solids themselves (structures and relations). Moreover, it even consolidated the previous explorations done on prisms and pyramids, adding cylinders and cones to the existing relations among solids and therefore making them part of the two formulas for prisms and pyramids (this will influence a subsequent discussion in moment 8 about formulas)[89].

Moment 7: Base of prisms

Carole then raised the point that the orientation in which the prism is placed can create difficulties for students. Or, in other words, students can experience difficulties when prisms are positioned "standing up" or "lying down" (figure 5.10).

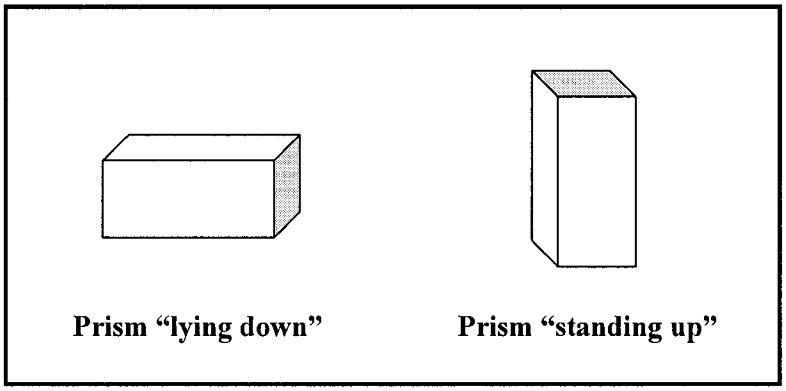


Figure 5.10. Distinction between prisms "lying down" and "standing up"

[89] This could lead to a different classification of 3D solids, where here the cylinders and cones would be in the same category than the prisms and pyramids, stepping aside from the usual "round objects" and "not round objects" categorization. This is reminiscent of de Villiers's (1994) classification of quadrilaterals, which differed from the ones usually shown in schools.

This stimulated a discussion about the fact that for a rectangular prism, it does not matter which base you choose to create the piling up, because we will always end up with the same volume. This however provoked a strong reaction of disagreement from Gina.

Gina: In my classroom, you could not call "base" the part that is at the bottom [showing one of the non-square rectangle of the rectangular prism]. If it is lying down like this [showing the rectangular prism lying down with non-square rectangle sides on the horizontal], I expect that you tell me that these are the two bases [pointing to the two squares].

Erica: Why?

Gina: Because it's a prism.

Jérôme: Gina, this is mathematically false, however.

Erica: Yeah.

Gina: Wait a moment, because the idea is that when you then work with this one [showing the hexagonal prism], you now say that it is a prism because these ones [pointing to the rectangular sides of the prism] are all rectangles and you still have your two others [showing the hexagons of the hexagonal prism].

Jérôme: Yes, but in a prism, you need the lateral faces to be rectangles. In this case [the hexagonal prism], you do not have the right. This [showing one hexagon of the hexagonal prism] is not a rectangle. In this case [showing the rectangular prism], I have the right.

Erica: Yeah.

[…]

Gina: You have the right [you can], but it confuses kids.

Erica: Yes, but then…

Jérôme: Ok yes, but then the confusion in fact, it is important that they know it however.

Erica: It is their problem [if they don't understand].

Gina: But it is important … it is their problem?!?

Erica: Of course. You cannot teach something that is false to avoid that students get it wrong, to help them understand.

Gina: But it is not false that there are two [sides in the rectangular prism] that are the same.

Jérôme: Of course, but it does not make them the bases, it is a choice you make.

Erica: It does not make them the bases, it is a choice.

I continued that discussion by paralleling it with the case of a rectangle where both sides could be called *length* or *width* and it would not matter. Gina refuses again this explanation, by saying that the small side of the rectangle (→▭) could never be called a base.

Gina: You see, I never call the small one the base.
Jérôme: I understand, but ...
Gina: You cannot do that.
Jérôme: But yes, this is exactly it, you *can* do that!
Carole: What is a base in fact? Because the vocabulary is important here.
Erica: The base is a pillar, it is what supports.
Jérôme: And you decide.
Erica: And so the base, this is the base [she puts the rectangular prism standing up], the base, this is the base [she puts the rectangular prism lying down].

That prompts Gina to say and show that in a pyramid (e.g., a hexagonal one) it is impossible, since a pyramid cannot be placed with one of its lateral triangles touching the table (lying on the table) and then suddenly calling it the base of the pyramid. Agreeing with her, I explained that of course it is not possible, but the reason being that a pyramid by definition requires that its lateral faces be triangles. I continued this idea by resorting to the example of the previous triangular pyramids that could be placed in any way with any side as the base, because all the lateral sides were still triangles. In the case of a prism, its definition requires that all lateral faces be rectangles. Gina still had problems with these ideas, mostly because, in a rectangular prism, with a pair of opposite sides that are squares and the rest being equal rectangles, the fact that the square sides are different from rectangles makes them directly and exclusively the bases of that prism. Claudia then suggested looking at a rectangular prism which has three pairs of different rectangular sides, and for that she took a videocassette box which had 3 pairs of different sides.

Claudia: [brings the videocassette box]
Gina: Yes, that's it! With this one it is not important.
Erica: Why isn't it important with this one?
Gina: Well, because you will have, they are all rectangular. You have two [of each]...
Jérôme: But the other ones also [pointing to the other rectangular prisms on the table].
Erica: They are all rectangular also [referring to the other rectangular prisms], a square is a rectangle.
Gina: Yes, yes, yes. But this one [the rectangular prism with two opposite squares on its sides] there are two identical so when you will calculate you will say, if you want to calculate the area or anything, it is easier to see the slices in that way [pointing to when the squares are taken as the bases]. [Appearing surprised, as if she just realized] *No! It is the same thing!*

Jérôme: But, that it is easier is not the same thing however.
Gina: No [agreeing that it is *not* easier]!
Erica: It is the same thing, it is the same thing. It is not easier like this [with the prism lying down] than like this [with the prism standing up].

This discussion brought Gina to understand and accept the idea. Afterwards, Carole returned to her idea about the orientation of the solids, highlighting that the idea of the choice of the base represents a really important issue because students sometimes do not see the orientation of solids in the same way their teacher does, which makes it an important issue to take into account for a teacher. Erica added to this that when she explains to her students the concept of base, she says that they need to see it as when they cut bread, and that the slices need to be identical, independent of the way they position the bread (standing up, lying down, etc.). Therefore, in the case of the hexagonal prism, there is only one possibility for obtaining identical slices. Erica added that she preferred talking in terms of bread slices, rather than of lateral sides, since it makes it more perceptible for students. Gina seemed to show interest in this approach. In the end, I explained to Gina that I understood that she wanted to avoid errors, but that understanding the definition of what a prism is was really important.

There are many issues addressed in this excerpt, and on different levels. For this reason, I only discuss here some elements concerning the teaching issues and the mathematics, but I come back later on in this chapter, and in Chapter 7, on the meaning that I am able to grasp from this small excerpt.

The main issue here was the mathematical understanding of what a base of a solid is and represents. As Gina attempted to make sense of this mathematical concept, as did the others and I by bringing more supported and clearer arguments, different mathematical aspects were brought forth and explored. Definitions of prisms and pyramids, of base, and of rectangles and planar figures were all discussed. Furthermore, counter-examples were offered to explain or refute what was offered at times. In other words, there was a lot of mathematics that was being done around the concept of the base of a solid (and of a planar figure). This exploration seemed evocative of Lampert's (1990) zig-zag activity, which brought conjectures, ideas, counter-examples, definitions, and so on, to be thrown in the exploration of the mathematical ideas. In that sense, this work was about the structures and relations, by establishing connections between the different ideas offered for the concept of a base of a solid (and a rectangle).

There was also the presence of two teaching issues. The entire conversation was triggered by Carole's comment about students' difficulties with the orientation of a prism and their different ways of perceiving it, which she returned to at the end of the session. Another teaching issue was Erica's comments about her way of teaching bases to her students, in line with the discussion that had just happened. To these teaching issues could also be added Gina's many reactions concerning how she teaches the concept of base (or mostly of what cannot be done), and also about how it was confusing for students if the bases were different from the pair of sides that are distinct from the others.

Moment 8: Back to formulas

I continued by re-initiating a discussion about the usual volume formulas and their "importance" in the teaching of volume (and the difficulties it creates for students), by contrasting them with the "new" formulas just worked on. As a comment on the formulas offered in the video, Erica mentioned that in this approach to the volume of solids all the area formulas are taken for granted because the formulas all start with "area of the base." Therefore, in her opinion, a similar approach to area needed to be developed[90].

This discussion about formulas brought the fact that students often get stuck with or only memorize/remember the well-known *LxHxW* for rectangular prisms as if it was a universal formula to calculate all volumes of solids – leading students to experience difficulties in many instances. This brought Claudia to offer the idea that the study of volume should stop being introduced by the study of rectangular prisms, and instead should be initiated by the study of another prism, for example a triangular prism. This could possibly break the strong and harmful tendency in students to always refer to *LxHxW*, given that for the triangular prism this formula or "multiplication of the three lengths together" does not function anymore. In addition, it would continue to support the idea of volume as a piling up or an accumulation of layers – which could possibly bring together the different solids and not isolating them with many different volume formulas. This seemed to please the teachers and a discussion emerged around the issue. Carole added that maybe many different prisms could be offered simultaneously to students where they would need to find a common way to establish the volume for all of them.

[90] This indeed brought me to design and begin session 4-5 with ideas about how to work with area of planar figures along the same line of thought than Janvier's approach to the volume of solids.

Claudia strongly supported this idea and even added that if students experienced problems with area formulas as they attempt to find the volumes, teachers could give the values of the area of the base in order to have students work directly on volume aspects, hence bringing students to develop and get into the "frame of mind" of creating and understanding the volume of prisms as a piling-up of layers.

This represents a teaching issue that emerged from the work on piling up of layers. Because all prisms could be seen as a piling-up of layers, it brought Claudia to think about another way to introduce the study of volume of solids, mainly with triangular prisms. In the same sense, Carole's idea of offering different prisms in order for students to find a general way to establish volumes reflects well the influence of linking the different prisms together with a common perception of piling up of layers. This represents an important effect of the "piling-up of layers" view on teachers: first, on their way to make sense and perceive of volume of prisms, and second on their reflections about possibilities for teaching this concept to students.

Carole then raised the issue that students are different, and that working on the volume geometrically is interesting for some students and will help them, but other students would prefer and would understand more with algebraic formulas. This brought Claudia to react:

> Claudia: But, does your [student who prefers formulas] really understand what volume is, or does he only understand how to use a formula?

Claudia's comment brought teachers to agree that making sense of the concept of volume was central in the study of volume of solids. Teachers started to discuss about a possible appropriate time to present the volume formulas in a teaching sequence: after or before having thought through the concept of volume. I also added that maybe teachers should never show it since, as Gina and Erica had raised before, formulas are not really needed in this type of approach to understand and work with volume.

This is another example of how the teachers' understanding of volume was evolving, as much as what it meant to know and learn it. First, Claudia's comment clearly showed how she was now seeing volume with different eyes and had been influenced by the explorations. It is worth noticing that in session 4 - 5, *Carole had the same type of comment about formulas and volume when she explained out loud to the group: "I think that only by placing numbers in a formula, you do not really understand the concept of volume."*

However, even if the discussion put into question the presence of the formulas in volume, the comments about "when" to teach the formulas – before or after the study of volumes – shows how difficult it was for the teachers to step away completely from the teaching of the usual volume formulas and from the importance of calculating in the study of volume. This is exactly in line with what Thompson et al. (1994) mean by teachers with a "calculational" orientation, who are always driven by the procedures to apply and "get answers" in the study and teaching of a concept. Even if the concept of volume changed for these teachers, formulas still seemed at the center of its study. However, the following reaction from Carole opened the way to an important change in her mathematical thinking, and her teaching.

Carole continued on by saying that she has a poster in her classroom with all the different volume formulas on it. She explained that she realized that this poster was useless because students do not need to know all of these isolated formulas. A little upset, she added that in the very textbooks she uses she also finds all these different formulas, to which I responded:

Jérôme: Textbooks are not written by God!
Carole: No, but it is the tool we use, however.
Jérôme: Oh yes, that I agree with.
Carole: I do not say I believe [the textbook] but … it's like, "My God! do we complicate the life of our students!"

Carole's reflection about the relevance of all the different volume formulas in her teaching is an important illustration of how she was changing. Her realization that the poster and the list of formulas in the textbook are not simply non-pertinent but even futile appears to be an important learning moment for her. In fact, her assertion that teachers complicate the life of their students is quite indicative of this realization in herself and the changes she was going through. Carole seemed to be changing her views about volume as a mathematical concept and about its teaching to students, as Claudia had before.

Finally, I concluded session 2-3 by saying (or even insisting on the fact) that maybe it is not about two formulas, but about one way to do something – that is, the piling-up of layers – and a relationship – that is, pyramids are one third of their associated prism. As the teachers concurred with this assertion and discussed it, Carole and Erica added that the relation from pyramids to prisms represented a ratio of 1 to 3.

This ended session 2-3 with a looking back at what the formulas offered in the videotape really meant. This discussion emerged from Carole and Claudia's assertion about the importance of formulas in teaching volume, and aimed at having the group reflect on the importance and place of formulas in the teaching of volume. It continued previous discussions about the place that volume formulas should take (and when), and also it had teachers realize, as they had started to in moment 5, that formulas do not only represent mechanical procedures but ways of understanding. Again, this is in line with the development of their relational understanding, where formulas could be made sense of, and extended into their teaching concerning possibilities for students to make sense of these formulas.

Moment 9: Families of solids

In the beginning of session 4-5, as I recounted the events of the previous session, I also added the fact that Cavalieri's principle brought in an idea of "families of solids," for example the family of rectangular prisms having the same base and same height (whether they are slanted or straight). Figure 5.11 illustrates this where there is a straight rectangular prism and an oblique one, both having the same volume – and any rectangular prism with same base and same height must also have the same volume, be it as slanted as possible, which creates an infinite family of equivalent rectangular prisms with same base and same height.

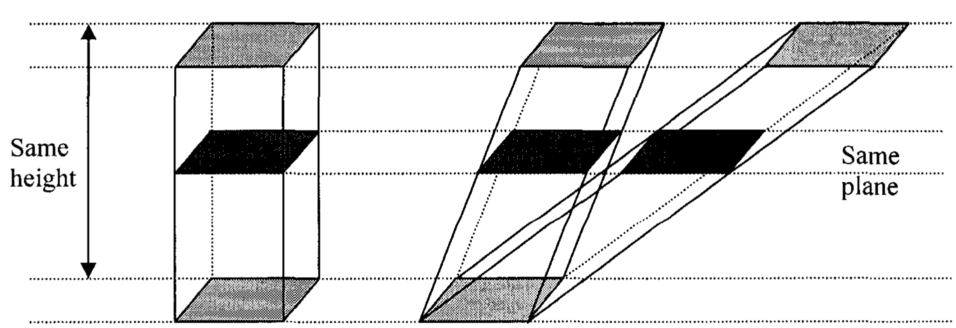


Figure 5.11. An illustration of a family of prisms

And the same can be said for pyramids, where all pyramids with the same base and same height are part of the same family, be they as slanted as possible (figure 5.12).

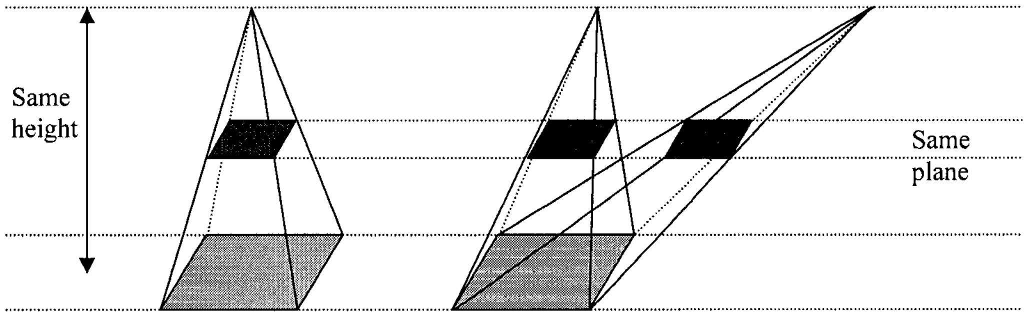


Figure 5.12. An illustration of a family of pyramids

Of course, the same could be said of cylinders, cones, hexagonal prisms, and so on[91]. This idea of "families" was re-used in the work on area of planar figures, which was offered in session 4-5 after the summary of the work on volume of solids done in the previous session 2-3[92].

The idea and representation of solids in the form of families was another mathematical issue that was worked on with the teachers, which concluded the study of volume as a geometrical concept in a way. The theory of families offered a different way of representing the links between the oblique solids and the straight-ones-usually-worked-on. Similar to the work on with the Cavalieri principle, it opened the way to the exploration of the volume of unusual solids and gave a different meaning to these unusual solids in regard to the possible relations establishable between them.

Discussion of the Deep-Conceptual-Probes Model: A Finer Analysis

Having just given a description above of the events in the session on volume of solids, I now look more closely at both the mathematics explored and the teaching issues

[91] Moreover, even if I did not talk about it in session 4-5, the families in 3D are not restricted to a 2D plane or from left to right, like in the figures 5.11 and 5.12. The family also extends from front to back and left to right, making it a 3D family extension, covering all cases of 3D solids – which is indeed needed for the previous equivalence of the three triangular pyramids that comprised the triangular prism.

[92] For a description of the families of planar figures that I created (e.g., rectangle, triangle, trapezoid, parallelogram, etc.) in line with what was offered to teachers in session 4-5, see Appendix E. The creation of the theory of families of solids and of planar figures illustrates the possibilities for the development of mathematical ideas within school mathematics, something I address in the "Intermission" after Chapter 6.

that emerged in order to illustrate in finer detail the what deep-conceptual-probes model has the potential to create and provoke.

Learning (More) about the Mathematics to Teach: Two Different Types of Learning

As illustrated in the description above, it is possible to see that there was a great deal of mathematics explored in the session – mathematics in line with the content of the school curriculum[93]. Indeed, within the deep-conceptual-probes model, mathematics occupies the central place and orients the work of the sessions. In table 5.1, I have highlighted most of the important mathematical moments that emerged from the work of the session on volume of solids[94].

Table 5.1. The mathematical explorations of the volume of solids session

Moment of the session	Mathematics worked on
Moment 1	Introduction to the volume of prisms as a piling up of layers
Moment 3	Consideration of oblique solids
Moment 3	Introduction to Cavalieri's principle
Moment 4	Comparison and equality of pyramids (by Cavalieri)
Moment 4	Decomposition of cubes into three identical pyramids (pyramid's volume as one third of its associated prism)
Moment 6	Definition of cylinders and cones as solids with infinite number of sides
Moment 7	Discussion and definition of the concept of base of a solid
Moment 9	Introduction to families of solids with Cavalieri's principle

[93] To escape polemics and confusions, in all that will follow I refrain from using or referring to the expression "Mathematics for Teaching" to describe instances related to the mathematics worked on and the emerging teaching issues addressed. There seems to be a very diverse understanding of what the term "Mathematics for Teaching" means currently in the mathematics education literature, which makes the concept difficult to define or even to get a grasp of. Indeed, it can be seen in diverse ways like the mathematical entailments of the practice of teaching mathematics (e.g., Ball & Bass, 2003; Hill & Ball, 2004; Rowland, Thwaites & Huckstep, 2005), an unpacking of mathematical ideas (Adler & Davis, 2006), an enacted-in-the-action-of-teaching knowledge focused on conceptual metaphors about mathematical concepts (e.g., Davis & Simmt, 2004, 2006), a didactical knowledge (e.g., Margolinas et al., 2005), and the list could go on. There is indeed no consensus in the community as to what "Mathematics-for-Teaching" means exactly, making it a rich notion but also a very difficult one to point to. Therefore, I will not refer to it so as to avoid being misinterpreted or misjudged in my comments or about any interpretation that I just made about my understanding of its current meanings in the literature.

[94] I will not refer to this right away, since this chapter aims more at describing the events, but many of these mathematical issues were not predicted in advance in my planning and represent emergent events that occurred in the unfolding of the session. In Chapter 7, I will discuss the issues of emergence and unfolding of events. The same applies to the teaching issues that I report on afterwards.

In these instances, a number of important mathematical ideas and concepts related to the volume of solids were explored. Within these mathematical explorations, it is possible to observe two different types of learning that mathematics teachers seemed to have experienced[95]. The first type of learning might be described as *recursive elaborations* (Davis & Simmt, 2004, 2006). The second type of learning will be referred to as "new" knowledge.

Recursive elaborations

Secondary-level mathematics teachers possess important knowledge to build on and to enlarge. In the case of volume, these teachers knew the formulas and how to apply and use them to calculate volumes of solids. It was my intention to build on this important knowledge and attempt to enlarge it by having teachers experience more "conceptual" mathematics – in the form of relational understanding and of knowledge of structures and relations within concepts. *Recursive elaborations* are an "expansion" of knowledge that teachers already "have" in the sense that these notions are not new for teachers, but are worked and re-worked on in deeper detail, thus expanding and deepening teachers' understanding of these notions and concepts[96]. Skemp (1979) refers to this as the development of the *interiority* of a concept which he relates to the quality or dimension of a concept in regard to its "wealth of interior detail" (p. 116). This is what is meant here

[95] It is difficult for me to make a clear distinction between "learning" and "experiencing" in these instances, and in fact in most thesauruses the two words are synonymous. One of my main intentions in this research was to offer and provide teachers with opportunities to *experience* the "conceptual" part of mathematics, to have them *experience* more than procedures and calculations; obviously, in order for teachers to learn. But I did not in any case "measure" that learning of teachers by tests of any sorts. Therefore, I will use "learning" and "experiencing" interchangeably since I believe that one learns within one's experiences (Glasersfeld, 1995).

[96] The idea of expansion is important, because recursive elaborations are not simple linear "iterations" or re-workings of the same concepts again and again. Different from an iteration that repeats the same process again and again on the result obtained in a loop (Briggs, 1992), a recursive elaboration changes the loops itself as it iterates. This means that as I re-work previous concepts that I knew about, my way of re-working the concepts is changed as I re-work them, which changes the concepts that I previously knew about and the new meaning that I develop. In other words, as I change my understandings of a concept, I change my way of understanding the concept, which changes the understandings that I am able to make. It is not simply a re-learning, it is a change of the entire understanding and possibilities of understanding. Another important aspect is that it is not refining and expanding to arrive at a specific point already pre-specified. The recursive elaborations are limitless, they expand in the direction that they do, without having a pre-determined path or trajectory to follow. Finally, it also needs to be understood that "expansions" and "elaborations" are not in the sense of quantitative changes, but rather of qualitative changes. Hence, the "expansions" are not metric and cannot be measured, they are qualitatively gaining refinement and depth.

by recursive elaborations and interiority, where teachers' knowledge expands in its depth and refines itself to a finer grain of understanding.

One example of this refinement is the idea of seeing the volume of prisms as a piling up of layers. The teachers knew the formula for the volume of a rectangular prism (or of any other prism), and could understand the idea of accumulating layers of area. However, they were simply "not used to" seeing volume this way and working with it in these terms; nor did they appear to have a prior understanding of the overarching effect that this understanding of layering could have on the entire concept of volume of solids, for all prisms. In some ways, it is as if they re-learned the concept of volume of solids by seeing it under the new lens of "accumulation of layers"; they recursively elaborated their description and understanding of it.

Another example concerns the volume formula for the pyramid and the "⅓" factor in it. Again, teachers knew the formula of a pyramid and could apply it to calculate a required volume. They even knew that the volume of a pyramid goes three times in a prism. However, their experience with the physical decomposition of the prisms into three pyramids, as well as the fact that there was a link between these pyramids and their associated prism (one with same height and same base as the pyramid), and finally the establishment of a link by Cavalieri between the slanted and straight pyramids, provided experience for developing different meanings for the formula of pyramid itself. The teachers' understanding went from knowing the formula and being able to show it (with pouring water), to understanding what is behind, "why" it was so, and why it worked in the way that it did (relational understanding). In that sense, the teachers' understanding of the formula of a pyramid was recursively elaborated.

More could be said about the other volume formulas. In fact, these recursive elaborations about the volume formulas, where teachers started to make more and more sense about the meaning behind the formulas, has the potential to transform the formulas from rigid procedures to apply to representations of "reasonings" or "understandings." Consider Gina and Erica's comments in *moment 5* and *8*. Formulas took on a different meaning and became "codifications of understandings." This is reminiscent of Hiebert and Lefevre's (1986) suggestion that procedures can often be seen as "observable" forms of understandings. In other words, recursive elaborations not only expand previous knowledge, but change and transform it too[97].

"New" knowledge

There are also instances where teachers learned "brand new" things, where the teachers experienced more than recursive elaborations of prior understandings. These occurrences represented instances of development of new mathematical knowledge. For example, the introduction of Cavalieri's principle was a new idea for these teachers, one they had not heard about before. This new idea enabled them to make sense of prisms and pyramids (straight or oblique) in a different way, something that brought them afterwards to see how the study of oblique solids was no more complicated than the study of straight ones (see Linda's reaction at the beginning of session 4-5). Likewise, the teachers also learned about ideas of "associatedness" and families of solids to establish relationships among prisms, pyramids, and among prisms and pyramids.

These instances represent a different type of learning experience about the learning of new ideas and concepts that are within the frame of the mathematics of the curriculum. These ideas and concepts may not be found explicitly in the program of studies (e.g., oblique solids or Cavalieri's principle), therefore do not represent aspects that teachers have explicitly to teach about, but they are however within the topic of study that they do teach about – within the volume of solids. Therefore, it is not important that teachers did not know about these mathematical ideas (it is not a failure or "gap" in their knowledge), but the learning of these "new" concepts represents a direct enlargement of the topic of study for them. These concepts happen to be interesting new mathematics that can be learned about within these school mathematics topics[98]. This "new" knowledge expand the teachers' understanding of the topics under study, in this case the volume of solids,

[97] Recursive elaborations are not only to be seen as expansions of procedural knowledge in order to develop a relational understanding of them. Recursive elaborations also concern the structures and relations, as the "piling-up of layers" example illustrates or the one about oblique solids where no formulas are taken into account.

[98] This is a point that I make in the "Intermission" about the fact that there is a lot of mathematical ideas to develop in *school* mathematics. The development of the idea of families of solids and planar figures (Appendix E) is a good example of mathematics that I have developed while preparing the sessions – mathematics that pushes farther (or deeper) the limits of school mathematics itself.

not by recursively elaborating or deepening previously known ideas, but by adding these ideas to their knowledge base of these school mathematics concepts.

Moreover, it is important to notice that the experiencing of "new" knowledge does not necessarily mean knowledge about structures and relations in mathematics. However, as mentioned in Chapter 1, secondary-level mathematics teachers are often unfamiliar with the structures and relations within mathematics concepts. Hence, it is more likely that the "new" knowledge developed will be about structures and relations within concepts. But, it can also mean "new" knowledge about procedures (or even conventions). One example of this can be seen in Appendix B (*moment 9*) about fractions, where I offered teachers a different algorithm to divide fractions (in order to make sense of it): $\frac{a}{b} \div \frac{c}{d} = \frac{a \div c}{b \div d}$. In this case, teachers developed "new" knowledge in the form of a procedure they previously did not know about.

Situating the type of mathematical work

Table 5.2 completes table 5.1 with the type of mathematical learning experiences.

Table 5.2. The mathematical explorations and the type of learning

Moment	Mathematics worked on	Type of learning
Moment 1	Introduction to the piling up of layers	Recursive elaborations
Moment 3	Consideration of oblique solids	Recursive elaborations[99]
Moment 3	Introduction to Cavalieri's principle	"New" knowledge
Moment 4	Comparison and equality of pyramids	"New" knowledge
Moment 4	Decomposition of cubes into three identical pyramids	Recursive elaborations
Moment 6	Definition of cylinders and cones as solids with infinite number of sides	Recursive elaborations
Moment 7	Discussion and definition of the concept of base of a solid	Recursive elaborations
Moment 9	Introduction to families of solids with Cavalieri's principle	"New" knowledge

[99] A point could be made that the work on oblique solids represented "new" knowledge for teachers. Indeed, even if teachers knew about their "existence," working on the volume of these objects was very unfamiliar to them – again, refer to Linda's comment in session 4-5. The work on oblique solids was at the border of recursive elaborations and "new" knowledge. An argument could also be made that some ideas could be new to some teachers and not for others, hence making the type of experience dependant on the learner. This is quite possible. However, the teachers I worked with seemed to have very similar experiences, and no instances pointed me to make this type of distinction.

It is important to note that within the in-service sessions most of the mathematical experiences were in the realm of recursive elaborations, where teachers expanded their knowledge of things they already knew about, rather than "new" knowledge. This is not surprising, since the entire professional development program had an explicit intention of building on teachers' current knowledge in order to enlarge it and to have it encompass larger experiences. In addition, because the teachers' knowledge was strongly invested in procedures, an important dimension of the program was about developing a meaning for what was behind these procedures, so that teachers potentially develop aspects of relational understanding. But there were also aspects of recursive elaborations about structures and relations (e.g., piling up of layers). In sum, since these secondary-level mathematics teachers already knew a lot, it was more about having them gain a more refined sense of their mathematical concepts and ideas than about showing them new ideas. However, on some occasions, new mathematical ideas were brought in when it was felt that it would bring important insights into the concepts studied and explored, Cavalieri's principle and families of solids (and planar figures) being examples of them. Other examples were brought in other sessions, like notions of Greek methods for subtracting squares and completing squares with area, new mathematical representations of the multiplication of negative numbers by rotating the number line (Mazur, 2003), discussions about the mathematical reasons behind Descartes' creation of analytical geometry, and so on.

In the explicit intention to enlarge the teachers' mathematical knowledge, teachers experienced two different types of learning of mathematics: recursive elaborations of what they already knew (deepening and refining their current knowledge), and learning of new notions (adding to their knowledge of mathematics). Both these experiences contributed to the enlargement of teachers' mathematical knowledge.

Emergent Teaching Issues: Three Types of Teaching Issues

While exploring the mathematics, teaching issues emerged and were addressed. These issues were related both to the teaching of the mathematical concepts explored and to students' learning of these concepts. Table 5.3 highlights the teaching issues that emerged in the course of the session on volume of solids.

The teaching issues can be categorized in three different types. The first one concerns issues of students' learning in relation to the mathematical concepts explored. The second type can be seen as instances of Shulman's (1986) *pedagogical content knowledge* and actions to be taken in class while teaching. The third type is broader and is about ideas related to teaching mathematics in general.

Table 5.3. The teaching issues of the volume of solids session

Moment of the session	Teaching issue discussed
Moment 2	Defining a mathematical context
Moment 4	Situating the height of the pyramid
Moment 5 and *8*	Discussing the place of formulas in teaching volume *and* when to teach them
Moment 7	Highlighting students' difficulties with the orientation of prisms
Moment 7	Explaining bases of prisms as slicing bread
Moment 8	Teaching volume by starting with a triangular prism, and offering different prisms to find a generative way
Moment 8	Reflecting on students who prefer formulas
Moment 8	Discussing the usefulness of the poster and the textbook's formulas

Type I: Anticipations of students' difficulties

This type of teaching issue concerns the teaching of mathematical concepts to students and their learning of them. It involves discussion of students' difficulties or of the possible difficulties students could experience with the content. Anticipation of students' difficulties appears to be an important skill that teachers need to develop, something highlighted by Bednarz (2000) in what she calls an *a priori* analysis of teaching, where teachers reflect and try to anticipate possible difficulties that students could experience when they are taught particular mathematical concepts. Such an activity represents a reflective instance that occurs in preparation for teaching and that need to be developed by teachers to plan their teaching actions.

An illustration of an anticipation of possible students difficulties happened in *moment 4* when Erica raised the fact that it could be difficult for students to understand the link between a right-angled pyramid and a straight one, especially because of the position of the height in these pyramids (where in one case it is physically "on" one of its triangular sides and in the other it is "inside" the pyramid itself). Erica's reflection was triggered by the specificity of the concepts explored concerning pyramids (straight and right-angled ones). This reflective comment illustrates her concern regarding the potential difficulties that students could experience. Other examples could be cited. Two of these examples are when Carole explained that students experience difficulties with the orientation of the prism, and also when she asserted that some students prefer being told the formula right away without knowing where it comes from. All of these represent reflective instances in which the mathematics explored brought teachers to reflect on what these newly addressed mathematics concepts could mean for teaching. The mathematical explorations fostered and triggered the development of reflections in teachers to predict and anticipate possible difficulties to pay attention to in their teaching.

The development of these reflective skills is important on two levels in relation to mathematics teaching. On one level, it represents important skills to develop for teachers' preparations and planning, where they can think and begin to anticipate and predict possible difficulties that students could experience, bringing them to adapt their teaching and what they offer in the classroom. The development of these reflective capacities are deemed fundamental for the endeavour of developing "reflective practitioners" (Schön, 1983), a concept widely accepted in the teacher education literature as central for teachers (e.g., Bednarz, 2000). In addition to developing reflective skills, on a second level this capacity to anticipate appears central in the work of teachers where they develop a growing sensitivity to students' understanding and learning. In other words, by this process teachers develop more encompassing comprehensions of students' understandings of mathematics. The knowledge of students' understandings and difficulties is underscored in the literature as a central attribute of teachers' knowledge (Fennema & Franke, 1992; Hill & Ball, 2004). However, as Margolinas et al. (2005) and Tirosh, Even and Robinson (1998) explain, it is utopian to think that teachers can be aware of all possible students' misconceptions, difficulties and understandings for all mathematics topics and concepts. This is why the development of an attitude toward and a capacity to *anticipate* possible students' errors and understandings appears central for teachers, so that they can first be more prepared in their teaching and second be more aware of students' understanding to make more meaning out of these understandings and potentially help students more effectively in return.

> What seems critical about conversations like those recounted earlier is that they allowed the teachers to practice figuring out how to handle the mathematical challenges that emerge in the act of teaching. In so doing teachers can, no doubt, work toward developing a repertoire of ideas and strategies for how to address analogous situations that they will inevitably encounter in the enactment of their everyday lessons. (Fernandez, 2005, p. 277)

The mathematical explorations conducted within the deep-conceptual-probes model brought teachers to reflect on the meaning of these new mathematical experiences for its teaching. It further developed in them a sensitivity and comprehension toward students' learning and understanding. It brought teachers to anticipate potential students' difficulties and understandings, making them develop as reflective practitioners, but also making them develop skills to understand students' understandings better, and enrich their repertoire of teaching strategies to address situations in their classrooms – something that represents fundamental teacher knowledge (Fennema & Franke, 1992).

Type II: Pedagogical content knowledge

Whereas the first type of teaching issues mentioned above concerned students' understanding, this second type of teaching issues is about teachers' possible actions to teach concepts to students. In a sense, this type of teaching issue could be said to be at the level of the preparation for teaching, and emerged from a new understanding of the mathematical notions explored. This is closely in line with Shulman's (1986) pedagogical content knowledge, which he defines as:

> [...] the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations – in a word, the ways of representing and formulating the subject that make it comprehensible to others. (p. 9)[100]

[100] I am aware that the notion of pedagogical content knowledge has been widely defined and re-defined in the community of mathematics education and along different meanings, as Zaslavsky explained in her presentation at PME-30 in 2006 (see also Zaslavsky et al., 2006). For that reason, I use Shulman's definition and elaboration of it as a foundational basis for its meaning.

In fact, the development of pedagogical content knowledge is about ways of offering mathematical notions in the most comprehensible manner, in an endeavour to make them (the most possible) accessible to students. Hence, this knowledge of "how to teach" is rooted in two aspects: in the knowledge of mathematical notions and in the knowledge of what makes a notion difficult for students (which often finds its roots in the *a priori* anticipation activity previously mentioned). Pedagogical content knowledge represents decisions based on an understanding of the concepts and of what could be good ways to make them accessible to the learner. It also represents teaching decisions and ways of presenting based on an understanding of "what makes the learning of specific topics easy or difficult" (Shulman, 1986, p. 9) for students. Knowledge of the mathematical topic brings teachers to think about ways that would make it the more accessible, and knowledge of the difficulties of students bring teachers to think of ways that would help students to overcome these difficulties or to reorganize their understandings[101]. I offer here some examples from the session.

Claudia's emerging idea that the study of volume should be started by offering a triangular prism, to stop reinforcing the *L*x*W*x*H* formula in students, represents an instance of developing pedagogical content knowledge, based on her knowledge of students' difficulties and misconceptions, as much as on a new conceptualization of volume as an accumulation of layers. Claudia was looking for a way to work and to offer the concept of volume to students that would be more comprehensible and potentially richer, and that would help students to step away from the *L*x*W*x*H* formula and the difficulties it creates for them, students often being "stuck" with it. Carole's following suggestion that many prisms should be offered to students for them to find a generative way to represent a volume is another illustration of emerging pedagogical content knowledge. Other examples of this type of teaching issue happened when Carole said that her poster of formulas and the panoply of isolated formulas in her textbook were now useless in her classroom (and her teaching), since the idea of having different isolated formulas did not made sense to her any more. Volume was now a different concept for

[101] It could be argued that Shulman could/should have separated these two forms. However, he seemed to be more interested in the teaching "actions" to take (the choice made to offer and present the subject matter) than in the source from which these decisions came from – be them from the understanding of the concepts or the knowledge of students' difficulties.

her and she would resort to other ways of offering it. This represents a teaching "choice," a specific understanding of the mathematical concepts which now affects her teaching of that notion – by intending to make it more accessible and richer to students.

These are examples of teaching "choices" that are rooted in and influenced by teachers' mathematical understandings of the concept and their reflections about, and knowledge of, students' possible difficulties: it has its roots in the mathematical explorations and in teachers' reflective skills. From these understandings, teachers were able to reflect on, discuss and find ways to make the notions more accessible to students. The events within the professional development sessions seem to have enabled and fostered the development of instances of pedagogical content knowledge in teachers.

Type III: General issues

Whereas the first two types of "teaching issues" were directly rooted in the mathematical explorations, with regard to specific mathematical concepts and issues, it is possible to observe another type of teaching issue that arose. Indeed, some of the mathematical elements worked on brought teachers to discuss broad educational issues linked to general dimensions of their teaching practice. These issues obviously concern mathematics teaching, but were not directly linked to a specific piece of mathematics. For example, motivation in mathematics, evaluation, usage of textbooks or of calculators, and so on.

One example of this happened in the volume session when Gina raised the issue of "mathematical contexts" in *moment 2*. This issue was not directly linked to a specific piece of mathematics, but rather concerned general mathematics classroom situations, in this case, the use of contexts. She explained that this is prominent in the provincial program of studies she is using. The volume session enabled her to initiate a discussion about the usage of contexts in a mathematics classroom (be it real-life, invented, mathematical, etc.) and to understand the issue better, as she was able to relate it to problems given in the video and find a language for it. It gave Gina a better understanding of the issue by making it explicit. It enabled her to support what she meant by context (indeed, she had tried in session 1 to explain to the group what she meant, but could not provide a clear sense of it). In a way, the presentation from the video could even be seen to have given Gina a warrant for her ideas, where she felt happy to see that "this was what she meant" about contexts.

In addition to this example, over the course of the professional development sessions, other general educational issues emerged and were discussed by the group. For example, subjects like the provincial examinations and their marking, the importance of establishing good communication between teachers of different grade levels to help the transition of students between the years, the difference between the Anglophone and Francophone curricula and examinations in the province, and so on. Although they were not the main issues addressed in the sessions, they emerged on many occasions and were discussed and debated, enabling teachers to develop and tackle larger issues related to their everyday teaching practices and constraints.

Situating the types of teaching issues

Table 5.4 adds to the previous table (table 5.3) for the types of teaching issues.

Table 5.4. The different types of teaching issues

Moment	Teaching issue discussed	Type of teaching issues
Moment 2	Defining a mathematical context	General issue
Moment 4	Situating the height of the pyramid	Anticipations
Moment 5 and *8*	Discussing the place of formulas in teaching volume *and* when to teach them	Pedagogical content knowledge (PCK)
Moment 7	Highlighting students' difficulties with the orientation of prisms	Anticipations
Moment 7	Explaining bases of prisms as slicing bread	PCK
Moment 8	Teaching volume by starting with a triangular prism, , and offering different prisms to find a generative way	PCK
Moment 8	Reflecting on students who prefer formulas	Anticipations
Moment 8	Discussing the usefulness of the poster and the textbook's formulas	PCK

The different mathematical explorations provoked the emergence of opportunities to address these different types of teaching issues, from students' difficulties, to possible teaching actions and to general teaching practices. More than addressing these issues, it provided a space for teachers to develop a more refined comprehension of students' understanding, reflective skills, pedagogical content knowledge, and broader understanding of their everyday practices of mathematics teaching[102]. By enlarging their understanding of the mathematical concepts that they teach, teachers also enlarged their understanding of what teaching these mathematics implied. Hence, the explorations of "conceptual" mathematics created contexts in which teachers could appreciate how their mathematical experiences gave rise to teaching issues/strategies that could afford and provide their students with similar experiences. To use Cooney's (1994) words, the approach enabled teachers to develop their pedagogical powers. The deep-conceptual-probes model for professional development helps teachers develop an enlarged sense of the mathematical content they teach, and also an enlarged sense of the teaching and learning of these concepts.

Teachers with a Calculational Orientation and Pedagogical Content Knowledge

Within the data just offered, it is possible to observe some of the implications of a teacher's calculational orientation to teaching. Consider the discussion about the base of a prism in *moment 7* with Gina. She explained what she meant when instructing her students about the concept of the base of a prism, which for her could only be one of the pair of distinct bases in the prism, and what she offered as the base of a rectangle, which could only be the longer side. For her, this conceptualization simplified the entire study of prisms (and rectangles), since it worked in the same way for all the other prisms as well. In other words, her pedagogical strategy was to look for and find the simplest way to define a base, so that it would make it easily accessible for her students. This is characteristic of a teacher with a calculational orientation: that is, to look for the simplest way, a programmable way, to express the content. Teachers with a calculational orientation are strongly inclined toward procedures, therefore this inclination brings them to create "techniques" out of the concepts to be studied in order to make the ideas

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[102] For Fernandez (2005), the development of anticipatory skills and of pedagogical content knowledge represent two sorts of critical and fundamental knowledge for teaching in a reform-minded way.

generalizable, programmable and applicable in all cases. This strategy makes the concepts "clear-cut," and without any confusion for the student who deals with these ideas – Gina said. But moreover, it makes these mathematical ideas easy to remember and memorize for the student, because they are clear and defined. This could appear to be a reasonable thing to do to help students learn, but it leads them away from complex understandings and subtleties within mathematical concepts.

What Gina offers to her students is not totally incorrect, but it is reductive. It closes opportunities down and simplifies the potential options and understandings within the concept. By simplifying the ideas in that fashion, it programs the outcomes and controls them. To interpret this situation in Hewitt's (1999) terms, it made the concept of a base of prism *not* something for students to make sense of but a "teacher-said-so" event. I am not implying that this was done intentionally to harm students. It is far from that since Gina was trying to make it simpler and more accessible to her students. The problem is that most mathematical notions are not simple, and reducing them to a simplified or a "technicized" form closes some important "reasoning" and meanings within them, meanings that may be very important. As I mentioned to Gina, the confusing or complex nature of mathematical notions need not be something for students to escape from because it is part of the concept itself. Many concepts in mathematics are complex and should be learned in that way – some notions are simply not reducible to memorized or programmable facts. For Gina, as a teacher with a calculational orientation, raising the complexity of these issues with students was too dangerous. She was aware of students' difficulties with the notion of base[103], and it is important to appreciate Gina's wisdom here, given her 20 years of teaching. As she explained: "You have the right [to use a different base], but it confuses kids"; "[...] it is easier to see slices in that way."

The question is not in regard to convincing Gina of the "positive" or "negative" side of what she does. The issue that I want to raise concerns Shulman's pedagogical content knowledge. The notion of teachers with a calculational orientation brings into question this notion of finding the "the most useful forms of representation of those ideas, the most

[103] Again, I could raise the issue brought at the end of Chapter 1 in regard to which appears more important between "students achievement and success" or "the nature of the mathematical opportunities offered to them." Raising the complexity of mathematical notions in order to make the mathematical experiences of students richer will probably not raise automatically their achievements.

powerful analogies, illustrations, examples, explanations, and demonstrations" (Shulman, 1986, p. 9). If someone read Gina's explanations, that person could say that her approach is simply wrong and that there exists a more powerful representation to give to students about the notion of base, maybe one in line with Erica's suggestion about slicing bread. But this would still be inadequate. Pedagogical content knowledge is *contingent* on the knowledge of the teacher who offers and presents the notions. There does *not* exist a most powerful or a best way of representing, or a best analogy. There are simply ways of presenting that make sense to the teacher in his or her intention to render notions accessible in order to make them learnable by students. Pedagogical content knowledge is an evolving form; it changes and develops as the teacher develops his or her understanding. As teachers' knowledge develops and becomes more complex, pedagogical content knowledge also develops and becomes more complex. This is why it is central for teachers to develop richer understandings of mathematics, that is, to work on more complexified and "conceptual" levels of mathematics. There is a need to enlarge teachers' knowledge of mathematics so that their possibilities for action are expanded and that they develop richer and more complex versions or instances of pedagogical content knowledge. Pedagogical content knowledge is rooted in individual teachers' mathematical understanding.

As complexified forms of knowledge develop for teachers, teaching these notions, however, will not become simpler but much more complex, since more options will be available to them. This is what expanding one's teaching knowledge creates. It does not simplify the possibilities of teaching, but opens them and expands them making teaching a more complex and difficult endeavour. This is reminiscent of Krygowska's explanations of what the study of *didactique* creates, where there is not one "best practice," and where it opens infinite possibilities for action:

> I knew very well how to teach fractions before starting my studies. [...] Now, I do not know anymore how to teach them and this is where the result of *didactique* of mathematics lies. I am not, in that sense, required to find the most adapted solution to my classroom from the many different possible options. Now that I know these possibilities, I feel obligated to change my conceptions within the student–teacher interaction, I have doubts, I see students' difficulties that I had never thought of before. It is the embarrassment of richness that is now the reason for my worries, for my doubts. (1973; cited in Bednarz, 2000, p. 77, my translation)

Exploring the issue of base opened possibilities for Gina (and potentially for the other teachers too). It addressed the complexity of the notion of base, and put into question its reduction to a programmable and simplified way; it opened a sensitivity to its complexities and subtleties. It does not imply, however, that Gina will immediately (or ever) change her approach to the teaching of these notions, but at least the pool from which she will be able to draw from will be more complex and she will make a choice out of this complexity. There is no certainty that these explorations will affect these teachers, but it has the potential to do it. As Carole told me many times during the year, "You completely changed my teaching, you make me reflect on my teaching continually."

Closing Comments on the Deep-Conceptual-Probes Model: Its Richness

The deep-conceptual-probes model for professional development was intended to prompt "conceptual" explorations into the mathematics that teachers teach. These explorations opened important spaces of learning for the teachers: mathematically *and* pedagogically. Concerning the mathematics explored, the model aimed at expanding teachers' mathematical knowledge of the mathematical concepts they teach, and created two types of learning experiences for the teachers. It enabled recursive elaborations of concepts that teachers already knew about, which brought them to deepen their understanding of these concepts. It also enabled the learning of new ideas and concepts that added to their understandings of and knowledge about the notions under study. The approach aimed at offering these teachers rich experiences with "conceptual" mathematics to enlarge their knowledge base, one that was oriented strongly toward procedures and calculations. In this sense, this model has the potential to enlarge and expand teachers' knowledge of mathematics.

As for the teaching issues that emerged within the mathematical explorations, the approach enabled and pushed teachers to develop important skills for teaching these topics. In effect, it brought teachers to reflect on and comprehend better students' understandings of concepts, and to develop anticipatory skills concerning students' potential difficulties with the mathematical concepts explored. It led teachers to engage in a reflective process that is deemed fundamental for teachers in their preparation for teaching. Moreover, it led teachers to develop pedagogical content knowledge where the new mathematical notions explored brought them to develop and think about diverse approaches, or simply put their approaches into question, for the teaching of these very concepts. The mathematical explorations enabled teachers to develop ideas, reflect and anticipate possibilities concerning the teaching of these mathematical notions. Finally, the approach gave teachers the opportunity and space for discussing general educational topics that occur in their everyday teaching practices. Similar instances are well documented in the in-service literature, where the idea is to create a space of interaction for teachers so they can discuss between them issues of their day-to-day practices and experiences lived in the classroom (e.g., Good & Weaver, 2003; Jalongo, 1991; Jaworski, Wood, & Dawson, 1999; Kazemi & Franke, 2004). In that sense, the mathematical explorations provided by the deep-conceptual-probes model enabled the possibility for these interactions to happen, and permitted teachers to discuss and make more sense of issues related to teaching mathematics.

This is what the deep-conceptual-probes model created and offered teachers, where it had them develop mathematical knowledge that enlarged their understanding of the subject matter they teach, and which in turn made them reflect on, and potentially inform, their teaching of these mathematics concepts. In the case of mathematics teachers, these two types of aspects (mathematics and teaching issues) cannot be separated and always go hand in hand. As teachers expand their knowledge of mathematical notions, they expand their knowledge about the teaching and learning of these notions. Even if centered on exploring the "conceptual" aspects of the mathematics that secondary mathematics teachers teach, the deep-conceptual-probes model attends to the development of both teachers' mathematical and pedagogical powers (Cooney, 1994).

This closes this chapter concerning how the deep-conceptual-probes model worked and what it created in the context of the professional development of secondary mathematics teachers. In the next chapter (Chapter 6), I analyse the impact that the work on "conceptual" mathematics had on teachers or, simply put, what attending to the three branches of mathematical activity created.

CHAPTER 6

SECOND ANGLE OF ANALYSIS:
THE MATHEMATICAL ACTIVTY

In this chapter, I discuss the data in respect of the mathematical activity and the learning experiences of the teachers. Analysis through this lens is used to give a sense of the type of mathematics that was done in regard to the three branches of the mathematical activity (conventions, procedures, and structure and relations). In other words, I explore how the three branches of the mathematical activity came into play in the sessions and what it created. With this, I illustrate how the mathematical explorations went beyond straightforward work on procedures, and how they have the potential to enlarge teachers' mathematical knowledge. This is done in an attempt to understand better the possible impact on teachers of opening the study of mathematical concepts to something other than the application of procedures (and toward the mathematical activity). I build on the data from the session on volume, and use additional data from other sessions to complete and support the various claims made[104].

At the center of the entire professional development program was mathematics. From the beginning, I have announced that my way of entering in the professional development of these procedurally-inclined mathematics teachers was through working with and providing "conceptual" learning experiences to teachers about the mathematics they teach. In effect, entering by means of any sort of task, the core intention in these tasks

[104] I remind the reader to continue considering, while reading this chapter, the aspects of the deep-conceptual-probes model treated in Chapter 5 and the ones on enactivism discussed in the next chapter.

was mathematics – as explained in the methodology chapter. It was by working and pushing on "conceptual" mathematics that I hoped to enlarge teachers' knowledge of the mathematics they teach.

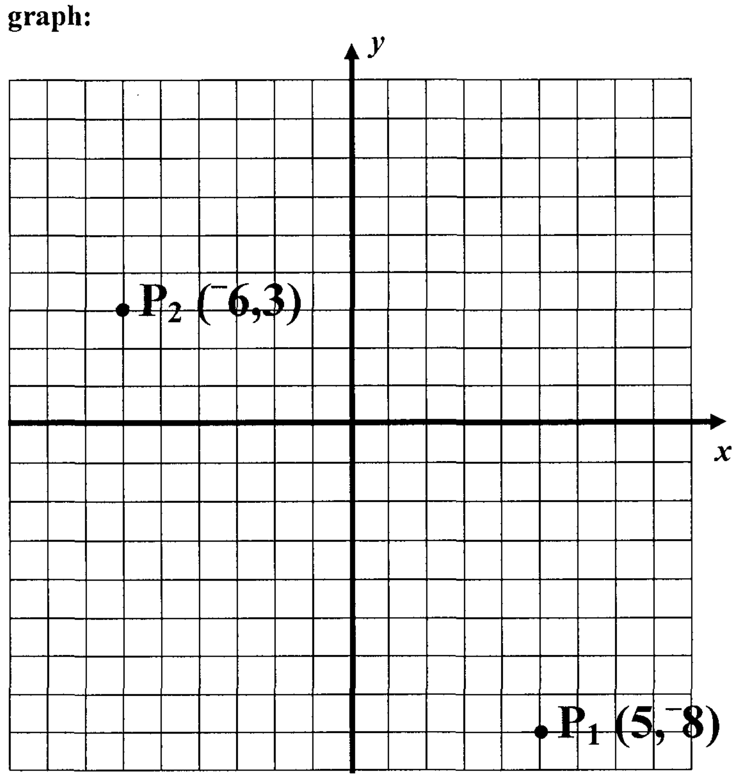
Continuously working along the line of the three branches of the mathematical activity – with a specific emphasis on relational understanding and structures and relations – was a way for me to ensure that the mathematical experiences offered to teachers were "more than just about procedures." In that sense, the mathematical work aimed at expanding teachers' knowledge in the realm of procedures (by working on relational understanding), but also in the realm of "conventions" and of "structures and relations." I describe here the data in line with the three different branches to show how each came into play on different occasions, in order to illustrate what they created and how it can potentially influence teachers' mathematical knowledge.

Conventions

In the volume session previously described, some conventional aspects of mathematics were raised during the mathematical explorations. There were many conversations about conventional aspects of mathematics as definitions were given of prisms, of pyramids, of bases, of cylinders and of cones. This had an important influence on the mathematical explorations carried out and oriented the possible conjectures (or counter-examples) made. The mathematical definitions were used as a basis or a source from which the subsequent exploration were done. In the example of the base of a prism, it opened new mathematical spaces for the teachers, and especially Gina, in order to make sense of the concept explored and refine their understanding. The established definitions influenced teachers' explanations and understanding of the concept of base, but also oriented the exploration in particular directions, for example toward finding specific cases (the videocassette box) or counter-examples (rectangle, pyramids, etc.). Definitions were part of the mathematics explored. The usage of conventional aspects in mathematics also often came into play in a subtle and implicit manner, as symbolism or names of different objects (like squares, degrees, and so on) were frequently used. In that sense, conventions were always "part" of the mathematical explorations to some degree.

In addition, there were specific instances where I intentionally brought in some tasks that were explicitly designed for the teachers to interact with and explore conventions in mathematics. One example came in session 6 when I brought in a "rate of change problem"[105]. The purpose of this session was to analyze students' responses and to give their work a grade. The following represents a task that I created to have teachers discuss conventional aspects of mathematics and the meaning behind them.

In the following graph:



Find the rate of change of the line that passes through the points P_1 and P_2. Show how you do:

$$\frac{\Delta x}{\Delta y} = \frac{5 - {}^{-}6}{{}^{-}8 - 3} = \frac{11}{{}^{-}11} = {}^{-}1$$

Figure 6.1. Inverse rate of change problem and one student's solution

[105] For the long description, see Appendix D, part D.3.

One of the first reactions was from Lana, who teaches this topic regularly. She said that this student did not understood anything and should receive a zero, because the student reversed the variations and arrived at the answer only by chance.

Lana: But in fact, this student deserves zero points.
Jérôme: Why do you say that this student deserves zero points?
Lana: He does not understand anything.
Gina: [laughing] It is only because he is a nice student.
Jérôme: What do you mean?
Lana: [laughing] He does not understand! Well, he does not understand anything because for him the rate of change he says that it is the variation in x divided by the variation in y.
Jérôme: Ok.
Lana: It is the opposite, he arrived by chance at the right answer.

This brought me to raise the point that "the order" is in fact a convention and that the student indeed only reversed both variations. This provoked a reaction of discomfort in Lana, as she agreed about the issue but looked perplexed and laughed nervously as she threw herself back on the two legs of her chair.

Jérôme: But why do you say that he does not understand a thing this one for example? Because all that this student did is to reverse x and y.
Lana: Yes.
Jérôme: But this, in fact, is only a mathematical convention.
Lana: The rate of change is always vertical on horizontal.
Jérôme: But this is a mathematical convention; it could have been horizontal over vertical.
Lana: [nervous laugh] Yeah, I agree with you [throwing herself on two legs of her chair].

This had some influence on Lana's understanding of the issue. She started to discuss the example in terms of mathematical conventions and in that sense started to change her way of speaking about it.

Lana: If the convention had been the other way around, I agree, but the convention is y over x and not x over y.

Nevertheless, Lana was still trying to find ways to convince us about the fact that it "had to be" in that order and in the course of the discussion she often tried to propose, without much success, an argument to demonstrate how it had to be Δy over Δx. For example, she discussed the meanings of positive and negative slopes, where a positive slope goes from left to right going up, which would then mean it had to be reversed if the rate of change was reversed. But still, I explained to her, that the names would simply have to be changed the other way around if the rate of change were Δx over Δy.

As Lana raised these points, it brought the group to discuss and realize the broader coherence of the body of mathematical knowledge, where aspects and notions follow each other in a coherence and build on the decisions (conventions) made. In that sense, a change in the order of the rate of change would result in many other important changes. One example highlighted was in the study of linear functions, in the equation "y=mx+b" itself. This made the issue complex because there was no reason why the order was so, making it an *arbitrary* decision to use Hewitt's (1999) term, except that the coherence of the body of mathematical knowledge was built on these decisions and many things would have to change if it was reversed[106].

The issue then became that it is possible for one to understand what a rate of change means and to be able to represent it, but also to not being able to represent it "conventionally." These discussions brought Lana to conceive of other mathematical concepts differently, as she herself was now flagging instances where there was the presence of conventions. For example, Claudia explained that if everything were reversed in order, then the "y=mx+b" would simply be changed to something like "x=my+b." This made Lana react by saying that something else would not work in regard to dependent and independent variables, but realized on the spot that it was also a convention.

> Lana: [answering to Claudia] It would be good, however, when he writes $x=2y+b$, it is good because he has it right, but he does not understands the idea of the dependent variable and independent ... that we have supposed ... This is still a convention!

[106] In the same way that if we stopped measuring with 360 degrees, but opted for 350 degrees, a lot of the coherence in the body of mathematics would be implicated, making it quite difficult to change it. Hence, the importance of conventions. But, it does not change the fact that it is still an arbitrary decision that was taken, one that did not have to be so.

This illustrates how Lana started to become more aware of the presence of conventions in mathematics, something that I do not believe she was familiar with prior to our discussion. The rate of change example made the issue of the use of conventions in mathematics more present for her in her mathematical understanding. It changed her understanding of mathematics to the point that she was able to convince others about the presence of conventions in mathematics. This happened as the discussion turned to the Cartesian plane, as Gina asserted that there were no conventions in the order of coordinates and that reversing them demonstrated a lack of understanding. This brought Lana to (once again) engage in a conversation about conventions (similar to the way that I explained the issue to her previously).

Jérôme: You do have to work with the Cartesian plane [in your teaching]?
Gina: Yes, yes.
Jérôme: So, if a student for this specific point tells you, well for this point here that would be (3,-1), tells you (-1,3), does this student receive a "0"? (see figure 6.2)

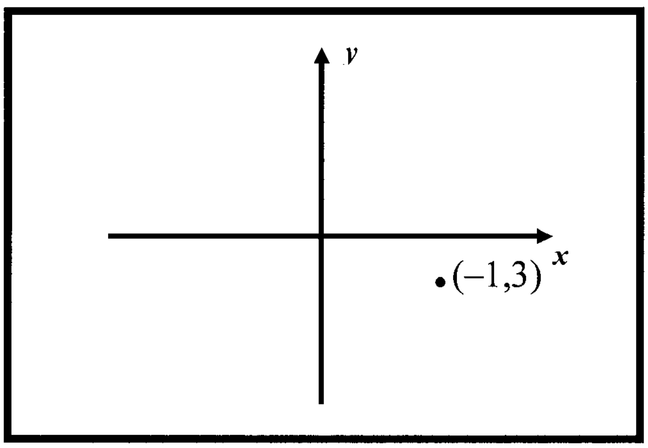


Figure 6.2. An example of inversing the coordinates in the Cartesian plane

Claudia: Yes.
Gina: Yes.
Jérôme: Why?
Gina: [hesitating] Because he is not in the right quadrant.
Lana: No, it is still a convention.
Jérôme: It is also a convention.
Lana: We again said that we would place the x first and then the y in second.

Lana demonstrated interesting instances of changes in her understanding of mathematics, and consequently in how she could now make sense of mathematical understanding (of students) in regard to the use of conventions. She was now more able to separate the proper use of conventions from the notion of understanding concepts aside form conventions. In other words, in addition to not giving zero to the student anymore, she could appreciate the presence of "understandings" in the students' answer even if that student lacked some knowledge of the conventional aspects in this answer.

Further, making this type of distinction also became apparent in the actions of other teachers. For example, in the following session, when an issue about the rate of change arose again in the discussion, it was Gina who attempted to convince Erica, who was absent in session 6, of the fact that "it was still a convention." Emphasizing the importance of placing the change in y first, Erica explained her point:

> Erica: No, it hurts much more from how high you fall than from how long you walk.
> Gina: But it is still a convention.
> Erica: Humm.

The example of the "rate of change problem" shows the potential impact that work on mathematical conventions can have on teachers' mathematical knowledge. I interpret this as another instance of recursive elaboration, where the teachers deepened their understanding of already known concepts. The change in their ways of talking about the issues (for example, the distinction between "understanding" and "using the convention properly") appears to be an important illustration of their learning. Moreover, it changed their way of appreciating someone's understanding of the concept; this distinction enabled them to see more than "this student deserves zero because he does not understand anything." In other words, the change in their mathematical understanding has the potential to influence how they might offer and work on this notion when they teach it to students in their classroom. In the end, these types of tasks and their explorations enlarged teachers' mathematical knowledge to include the presence of conventions, and made the usage of conventions an important element for teachers when one is doing mathematics. There is more than procedures to mathematics; one needs to know and make a proper usage of mathematical conventions.

Procedures

The procedures in mathematics, whether they are formulas or algorithms, were well known by the teachers. My intention when offering explorations of procedures was to start from the teachers' knowledge and expand their understanding of the very algorithms and formulas that they knew well – in most cases, to have them develop a relational understanding of these procedures. In the session on volume of solids, the work with prisms, pyramids, cylinders, cones, and so on, was explicitly aimed at deepening their understandings of the meaning behind these volume formulas.

This new understanding of the formulas had an important effect on teachers, not only in regard to their mathematical understandings, but also with respect to their reflections and understandings about teaching these same concepts. As Carole and Claudia's discussion illustrates, knowing volume was not only about formulas and playing with numbers, but was also about understanding and making sense of volume. This is why Carole began to question the relevance of the formula-poster and textbook, and why Claudia said that students who only knew formulas did not really understand volume. A similar argument could also be raised in regard to Gina and Erica's comments, where volume formulas took on a new meaning and were now seen as embodying "understandings" and not really as simple formulas to apply. The mathematical work on the formulas, on procedural aspects in the study of volume of solids, affected and changed what volume meant and how it could be known by students. It could be said that it opened new spaces and had the potential to affect their ways of teaching and addressing the topic in their classrooms.

This work on deepening teachers' knowledge of procedures affected more than their local understanding of the concept of volume of solids. These teachers began to reflect on other related topics and to question their teaching in relation to them. Gina had one of these reflections at the beginning of session 4-5. As I began to talk about area and its teaching, Gina directly commented that in area measurement, as it is for volume of solids, there could be only two "formulas" to know and learn, which would be the rectangle (B x H) and the triangle formula ((B x H)/2), and not the entire set of usual and isolated formulas found in books. To explain, she started to establish links between figures (e.g., a square is a rectangle, a parallelogram can be transformed in a rectangle, a trapezoid can be split into two triangles, etc.). This example illustrated how specific work to make meaning of mathematical notions led teachers to reflect on other notions and topics. In a way, it initiated a movement in their thoughts. Skemp (1978) had previously highlighted this concerning relational understanding:

> The connection with [motivation of people] is that if people get satisfaction from relational understanding, they may not only try to understand relationally new material which is put before them, but also actively seek out new material and explore new areas, very much like a tree extending its roots or an animal exploring new territory in search of nourishment. (p. 13)

This was an important effect of deepening teachers' understandings of procedures. It built on their current understandings and opened up new possibilities for further work in that direction. As Gina's example shows, it opened up new possibilities for action, it initiated reflections and changes in teachers in regard to other mathematical concepts apart from the exact ones explored in the sessions. This is an important implication of the work on developing relational understanding, because it extends further into other realms, establishing a sort of habit of mind.

Structures and Relations

Within the deep-conceptual-probes model, specific attention was placed on having teachers experience "conceptual" mathematics in order for them to have opportunities, following Bryan's (1999) recommendations, "to deepen their conceptual understandings of the content of the school mathematics curriculum" (pp. 8-9). As discussed, one impact that this "conceptual" work had on the mathematics teachers concerned their deepened understanding of the procedures they knew – they began to develop relational understanding. But the "conceptual" approach also aimed at working on issues of structures and relations within the mathematical concepts, something that was again different from mathematical procedures. In addition to having teachers engage with mathematical issues other than procedures, this work on structures and relations was also intended to have teachers become aware that there was more to mathematics than procedures and that some mathematical concepts could be approached and explored separately from their procedures and calculations: that is, they could be approached in regard to their structures and relations. This was a new understanding about and conception of these mathematical concepts for the teachers. As illustrated, the exploration done about Cavalieri's principle to compare and establish volumes of solids, the piling up of layers to represent the volume of prisms, the work on oblique prisms and pyramids, and the relations between families of prisms and of pyramids were all representative of explorations done in regard to the structures and relations of the mathematics concepts, where no "procedures" were required.

Enlarging teachers' knowledge of mathematics to encompass knowledge about structures and relations within concepts is important on its own, but also in regard to their teaching of these concepts. Gina's new understanding of base has the potential to open up her possibilities for teaching this concept, to complexify her approach and what she offers to her students in her teaching. For Claudia, too, there are some indications that her learning experiences with structures and relations had the potential to influence her teaching, since she was quite taken with the idea of commencing the study of volume with a triangular prism rather than a rectangular one. Her new understanding of the volume of (all) prisms as a piling up of layers brought her to see new possibilities for teaching volume and for enhancing student understanding of it. Similarly, Carole's suggestion to offer all prisms simultaneously for students to uncover a generative way of finding volumes is also representative of this influence. These are just a few of many examples that could have been given where new understanding and enlargement of teachers' knowledge about structures and relations within mathematical concepts appeared to produce/generate potential effects on their teaching.

One event in particular, which occurred in the first part of session 2-3 (where teachers brought a problem on volume of solids for the group to solve and analyze), illustrates well the potential influence that addressing issues of structure and relations can have both on teachers' mathematical knowledge and on their reflections concerning the teaching of these concepts. In session 2-3, Erica brought the following problem of optimization in which volume was the central concept (figure 6.3)[107].

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[107] See Appendix C, part C.1, for the long description of the work done.

Find the biggest rectangular box, with no lid, with a square base and of total surface area of 3600 cm^2. a) Find the constrained equation; b) Find the size to maximize and express it in relation to one variable; c) Find the dimensions of the biggest box; d) Find its volume.

Figure 6.3. Erica's optimization problem concerning volume

Whenever a teacher in the group offered a problem, she always had to explain the reasons for having chosen it as a good problem to the other teachers. Erica explained that in this problem it is important to know the difference between surface area and volume in order to establish the equations to represent the constraint. This assertion brought Gina to highlight that the question and Erica's assertion underlined important assumptions.

Gina: But this supposes that in grade 8 and 9 they have studied the relationship between volumes and surface areas.
Erica: Yes.
Gina: Well, we mostly compare area with perimeter.
Jérôme: That is interesting.
Gina: But we do not look at area and volume. Neither in grade 8 nor grade 9.

This situation prompted teachers to realize the mathematical importance of understanding the relationship between surface area and volume of solids. It became a significant relationship to know and understand in the study of volume. Based on that importance, it led teachers to reflect on the fact that it should be offered as a notion to study and work on in their teaching. Obviously teachers were aware of the existence of these notions, but having them flagged by Erica and Gina brought them to reflect on the key concepts for the study of *volume* of solids, and the relationship and distinctions between surface area and volume was now one of these key concepts for teachers. In that sense, it enlarged teachers' knowledge as these concepts became important for them. When Erica asserted that it was a very difficult issue for students to make sense of, Gina reacted again in the same way, showing how she realized that this should be fostered and worked on in their teaching.

> Gina: But there are reasons why this problem is *difficult*, it is because we do not do it [link between and passage from surface area to volume] in grade 8 or in grade 9.

This gives an illustration of how exploring mathematical issues in the realm of structure and relations and developing an awareness of their importance (even if it was not explicitly flagged in that way by them) influences and occasions teachers to reflect on their teaching of these notions.

This influence subsequently continued when Erica flagged another issue concerning the study of volume. She commented that textbooks do not focus on the reverse relationships in the study of volume and area; that is, going from the third dimension to the first dimension or from the second dimension to the first dimension. Erica elaborated by saying that students are used to going "forward" but not "backwards." She added the following concerning students' difficulties.

> Erica: Often I realize that students are weak when they need to go in a reverse mode. They are linear, and they always go from top to bottom, they never go from bottom to top, that is, starting from the result instead of the initial situation.

Again, this prompted the teachers to reflect on and realize the mathematical importance of this conceptual notion, and how they should focus on it in their teaching. As Gina explained for the relationship between surface area and volume, and Claudia did the same for the reverse relationship, the teachers knew the existence of these notions, but did not realize their level of mathematical significance. By realizing their mathematical importance, by viewing it as important in mathematics, they enlarged their knowledge of mathematics since it became "present" as an important notion for them.

Further, as they expanded their mathematical understanding of these concepts, new possibilities of acting in their classroom were opened – possibilities of offering this type of mathematics to students. This is something they could not have done or thought of doing in their previous practices, since they mentioned not being aware of the mathematical significance of the relationship between surface area and volume of solids. In that sense, it deepened and enlarged their knowledge of mathematics and, in turn, changed and affected their possibilities for teaching. These two instances of "realization," and the previous issues mentioned about volume of solids, are good illustrations of the impact that working on the structure and relations of concepts can have on (1) teachers' mathematical knowledge and (2) their teaching possibilities and actions in the classroom.

Distinguishing between "Techniques" and "Reasoning" in Mathematics: Co-Constructing a "Tool" with Teachers

As reported in Chapter 1, my research site surprised and confronted me in ways that I had not anticipated. Because of that, I had to reflect on the issue and adapt my ways of intervening with teachers in the sessions. I changed my research orientation and created new models to guide my practice. Within the models developed (offered in Chapter 2), and in order to have teachers address the issue that there is more to mathematics than procedures, I first decided to offer teachers and discuss with them a possible distinction between "techniques" and "reasoning" in mathematics. I report on this preliminary distinction and the work done with the teachers here.

While visiting the participating teachers in their classrooms, I came to realize the place of and importance given to mathematical procedures in their teaching. As I became more aware of their calculational orientation in their teaching, I realized and became more sensitized to the *importance* of knowing and mastering some techniques and automatisms in mathematics. This is something that was still vague in me at that time, and something that I felt I needed to clarify theoretically – all this led to the creation of the framework of mathematical activity presented in Chapter 2. I became more aware of the important place of procedures in mathematics, but I also felt that mathematics should not be reduced to them. This was a "discovery" that I wanted to introduce teachers to and explore with them, by grounding it in their own practices, since it appeared to be an opportunity for me to sensitize teachers to these elements in their teaching.

Introducing Teachers to the Distinction

I decided to begin the second session by offering my thoughts to teachers about a possible dialectical relation between "techniques" and "reasoning" in mathematics – that is, about their mutual importance and dependance in mathematics and how teaching has to address both and not reduce mathematics to one or the other. I explained to the teachers my reflections triggered by my presence in their classrooms (and literally having learned from them), and offered them the distinction between "techniques" and "reasoning" that I had thought about. This distinction is very close to Skemp's distinction between relational and instrumental understanding, but also hints at a part of mathematics that is not always reducible to procedures, that is, the structures and relations within concepts[108]. The main intention in offering teachers this distinction was to have them think about mathematics as more than a set of "techniques," however important they are in mathematics.

Grounding the issues in their practices, and in my reflections about the visits in their classrooms, enabled me to discuss issues concerning the nature of mathematics, their classroom practices, and my personal understanding of these matters. The discussion with them unfolded issues concerning the mutual need of both techniques and reasoning in mathematics, as Bass (2003) implies:

> Some of the public debates about education reform have pitted "basic skills," which are often characterized as knowledge of "standard" algorithms for numerical operations, against conceptual understanding. Sensible people now recognize that this is a false dichotomy. Both forms of knowledge are essential and are basically intertwined. (p. 326)

I also highlighted the problematic tendency to see mathematics as, or more simply put to reduce mathematics to, a set of techniques and facts, and to lose the "reasoning" part of the mathematical activity. I was able to talk about and refer to my own education in mathematics, rooted in the curriculum reforms of the 1970s and 1980s where behavioural objectives were dominant, and where mathematics appeared to be the solving of pages and pages of exercises in a book – or of rote learning techniques. Mason (2002) talks about this:

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[108] These ideas were close to the distinction made in figure 2.4 in Chapter 2 about traditional mathematics teaching. I did not introduce the teachers right away to the framework on mathematical activity and its three branches, since in fact that frame was still under construction and not yet explicitly elaborated for me at that time. This first distinction was a step toward it. Teachers were introduced to the mathematical activity framework later on, when it was felt relevant to do so (session 7), in order perhaps to complete the previous work on "techniques and reasoning."

> Giving people rules and mnemonics, and making them practise these to gain facility can usefully augment their understanding of what the techniques do, why they work, and to what sorts of problems they can be applied. However, it can also dominate attention by displacing the very understanding which performance on tasks is supposed to represent or indicate. (p. 17)

However, I also added the just "as problematic" tendency only to focus on "reasoning" and to set aside the importance and significance of the "technical" part of mathematics. The discussion unfolded in regard to the fact that residing solely at either pole was problematic; doing and understanding mathematics required both. This in turn implied that teaching had to focus on both.

A Tool for Facilitating Thoughts about Mathematics

I elaborate on these issues because recognizing the distinctions between "techniques" and "reasoning" opened the way to many discussions on the nature of doing mathematics. This appeared to have played a significant role in the subsequent sessions. In effect, by grounding this distinction in their practices and by discussing/negotiating the meaning of that dialectic with them, it was felt that we had *co-constructed* a "tool" that enabled us to make sense of some issues about learning and teaching mathematics. It was now part of our history of interactions, of our structural coupling, and it oriented our thoughts and discussions/interpretations. It became a common lens of analysis.

Personally, as the teacher educator, this co-constructed tool was helpful because it provided a means of intervention with these teachers. First, since the participating teachers had a strong inclination toward procedures in their teaching of mathematics, I was able to use this tool that we had all agreed on and had developed together to flag and raise some points of concern (when I aimed at making a point, when they were emphasizing "technical" aspects of mathematics, etc.), in order to have the teachers reflect on their teaching of and the relative emphases they placed within mathematics.

Second, in regard to their mathematical knowledge, it was possible for me to raise the issue of "techniques and reasoning" within the mathematical notions explored (when it was present or even when it was absent), by framing the discussion under the umbrella of "techniques and reasoning." I did that when aspects were present or absent in the discussions, in order that the teachers became more sensitized to this issue and its meaning for understanding and teaching mathematics. Finally, at the emotional level, because it was a co-constructed and agreed-upon tool, teachers did not feel badly when I used it to flag some issues about their own teaching or knowledge of mathematics. I personally did not want them to feel put down or embarrassed at these instances concerning their understandings or ways of doing (in mathematics and in teaching). As a result, this mutually built tool enabled me to feel comfortable when flagging some issues that I felt were of importance, and teachers seemed to appreciate and were receptive to my comments. In short, this tool helped me to provide teachers with opportunities to reflect on their inclination toward procedures in mathematics, and to offer opportunities to teachers to learn more about the mathematics explored and its teaching[109].

The Impact of Theorizing about "Techniques and Reasoning"

The co-constructed tool concerning "techniques" and "reasoning" had an important effect on teachers' understanding and their discourse about mathematics teaching and learning. It gave teachers some "words" to use to make sense of the activity of doing mathematics. In addition, it enlarged their knowledge of mathematics, since mathematics became not only about a mechanical application of procedures, but also about making sense and "reasoning" – they were now taking this dialectic into account when discussing and exploring mathematical issues. This did not set aside the importance of procedures in mathematics, since it was central in the discussion of "techniques and reasoning," but mostly it led teachers to understand that only knowing procedures was insufficient in mathematics and that mathematics was also an activity to make sense of. In addition, it was even aimed at by teachers themselves when they tried to make sense of some approaches and some instances of student understanding. In fact, this co-constructed tool was explicitly used by teachers themselves on a number of occasions to report on or discuss about issues. Two examples are worth looking at in detail here. The first one concerns Carole, as she was making sense of a student's solution, and the other one involves Gina, as she made sense of the skills required to write an algebraic equation from a word problem.

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[109] This section could be seen to complement the previous section on "procedures."

Carole's example

In session 2-3 on volume, when I introduced the distinction of "technique and reasoning," Carole repeatedly said that she had never been asked to reason about mathematics as a student, and said it again in session 7 while working on fractions and operations. This illustrated that it was not obvious for her to make sense of this distinction, even less in her own understanding of mathematics and her teaching of it. During the year, she began to develop a growing sensitivity to issues of "technique and reasoning" within mathematics. This sensitivity was explicitly expressed at the end of session 4-5 when I offered the following problem to make sense of (figure 6.4)[110]:

We gave the following problem to Brigitte:

"I go to the store and I buy the same number of books as discs. The books cost two dollars each, and the discs cost six dollars each. I spend 40 dollars in total."

Brigitte answered the following:

$2B + 6D = 40$, since B=D I can write
$2B + 6B = 40$
$8B = 40$
This last equation indicate that 8 books cost \$40, so one book costs \$5.

Figure 6.4. Student Brigitte's algebraic solution

In the middle of the work, when the discussion was about the meaning "Brigitte" was making and the understanding she had or did not have, Carole started to explain her own view of things in regard to the answer.

Carole: The mechanical steps are there.
Jérôme: Yes.
Carole: She wrote an adequate equation.
Jérôme: She probably arrived at B=5.
Carole: She did an adequate justification that the number of books and the

[110] The problem and its solution are taken from Bednarz (1999). Used with permission. See the Appendix D, part D.1, for the long description of the events for this task.

number of discs are equal.

Erica and me: Yes.

Carole: But she was not able to reason through her mechanical steps.

Jérôme: She was not able to go back to the problem.

Carole: No reasoning of her mechanical steps whatsoever. It is very automatic. This is demonstrating that it is a student that executes exactly what we have shown her to do at many times. It is very automatic, but there is no reasoning.

This was a clear event where Carole flagged the problematic side of solely working on the "technical" or mechanical side of mathematics, it illustrated how she saw how "limited" it was only to have a "mechanical" understanding. This was an important realization, where it was not simply flagged by myself or someone else, but came directly from her own realization and understanding of the situation. Her own understanding of the situation and her appropriation of the tool were demonstrated further when Gina's intern, Holly, disagreed with her. This brought Carole to re-explain what she meant.

Carole: She did not reason, she was able to write what was in the problem, which is different from reasoning to the solution [...] if she enacted some reasoning, she would not have made that final mistake.

This illustrated well Carole's understanding, since she was *not only* able to discern the presence and harmful effect of a unique knowledge of the "technical" side of mathematics, but could argue about it and explain it to others in order to convince them. The "techniques and reasoning" tool was an asset in Carole's development, it changed her way of looking at mathematics, and at its learning. In that sense, it enlarged her understanding of mathematics.

Gina's example

After the introduction and discussion about the "techniques and reasoning" tool in the first part of session 2-3, the teachers explained their interest in knowing more about teaching algebraic word problems and writing algebraic equations for them. As was done in session 1 on solving algebraic problems (see Appendix A), they explained that writing algebraic equations represented an aspect with which their students struggled the most (see also Bednarz & Janvier, 1996). I suggested to them that the next session (session 4-5) be used to work on this issue, and added that it represented a good illustration of the dialectic of "techniques" and "reasoning," since both parts were present in it. I raised the fact that, as it had been worked on in the first session, there was a mechanical part in solving algebra word problems as one carries out the algebraic manipulations – and that teachers themselves had explained having invented pedagogical strategies to help students proceed with these manipulations (e.g., using the metaphor of the balance where what is taken from the left must be taken from the right to keep the balance). But I also said that the writing of the equation rested uniquely on reasoning grounds which could not be reduced to a technique or a step-by-step procedure to follow. Rather, students had to make sense of the context of the problem and the relations between the data in order to write the equation. In that sense, solving algebraic word problems represented an interesting example of "techniques and reasoning," where the reasoning part is about the writing of the algebraic equation, and the technical part concerns the mechanical algebraic manipulations[111].

This brought Gina to say that she was happy that a session would be spent on this matter, because she had been desperately looking for a technique to help her students to write algebraic equations from word problems. I reacted to her comment by reminding her of the previous discussion (that had just happened) about the explorations of session 1 where it had been explained that writing the algebraic equation existed in the realm of reasoning and could not be reduced to technical aspects. In other words, there was no technique that could be invented to help the writing of algebraic equations. Looking puzzled, she agreed but said in a sceptical tone "Well, we'll see when we work on it next time."

Then, in session 4-5, we worked at making sense and better understanding what the teaching of algebraic word problems in regard to writing the implied algebraic equations. These ideas probably made their way slowly in Gina's mind, as did the work in the session 4-5, because after a while, without having raised the issue of "techniques and reasoning" yet in that session, Gina highlighted the fact that the writing of the algebraic

[111] However I did not talked of it in these terms, writing algebraic equations could be seen to lie in the realm of "structures and relations," and the algebraic manipulations to lie in "procedures."

equation was very difficult. She explained that in contrast with other parts or aspects in mathematics for which you could find an established set of "techniques" to apply (e.g., formulas in volume, algebraic manipulations, etc.), this part could not be reduced to a mechanical step-by-step procedure. And for these reasons, it placed the writing of algebraic equations in the realm of reasoning and making sense, different from "techniques." It made it non-simplifiable to a mechanical procedure, and therefore quite difficult to understand.

This is an example of how teachers started to make sense of the issue of "techniques and reasoning." Gina was now able to make a distinction that she was unable or had had a hard time doing before, and for a process (writing algebraic equations) for which she attempted to find a mechanical device to simplify it for her students for a long time[112]. In addition to the fact that the distinction between "techniques" and "reasoning" became clearer to her, it also engaged her in reflection about the mathematical issues and in that sense enlarged her understanding of mathematics and its learning by students.

On a more anecdotal level, a small event in regard to Gina's reflections and building up of meaning about these issues happened in the middle of session 4-5. Instead of calling the writing of algebraic equations a "process," I unconsciously made the mistake of calling it a "technique." Immediately and without hesitation, Gina told me, in a pedagogical tone, that writing algebraic equations was not a "technique" but "reasoning." This showed how Gina was seeing the process of writing an algebraic equation with a different eye, and how she had changed in this regard.

Final Comments on the Co-Constructed Tool of "Techniques and Reasoning"

These two examples illustrate well the utility of this co-constructed tool for teachers to make sense of in regard to the mathematics explored and its teaching. The tool had an important influence and became part of "our" history as a group as it oriented the mathematical explorations and discussions; it was part of our structural coupling, and specific patterns of interactions were beginning to be established.

[112] As explained in Chapter 5, this is another example of the attitude of a teacher with a calculational orientation who is looking for the most simplified and programmable way to express mathematical concepts in order to make them simpler for students. I come back to this issue later in this chapter.

The tool offered teachers words and expressions to reflect on and discuss the mathematics, its teaching and its learning. Moreover, it had repercussions on teachers' understanding of mathematics itself, where they were able to see that there was something more needed in mathematics than learning sets of techniques (i.e., procedures), and that only knowing procedures was not enough because one needed to enact reasoning. This was also present in their discourse where they started to highlight many instances where their students "technicized" or "automatized" mathematical notions and could not make sense of them. (See, for example, the telephone problem in Appendix C, part C.2 concerning the establishment of solution sets in graphs, where teachers explained that students have created a trick to find the appropriate regions in graphs, but were not able to deploy an adequate understanding and explanation of why it worked.) It demonstrated the development of a growing sensitivity in these teachers toward the inadequacy or incompleteness of treating and transforming mathematical notions in a technical realm (i.e., technicizing the notions). In addition, since one of my main intentions was to have teachers develop an understanding that there was more to mathematics than procedures and calculations, and in that sense attempt to enlarge their mathematical experiences and knowledge, the growing presence of these elements, the establishment of these specific patterns of interaction, in teachers' discourse appears to be very promising.

Teachers with a Calculational Orientation and Structural Determinism

In Chapter 5, I began to address the issue of teachers with a calculational orientation in relation to pedagogical content knowledge and their inclination to render programmable or simplified the mathematical concepts to their students. I now want to add to this discussion in relation to their inclination to focus on and look for procedures in mathematical concepts and ideas. Their strong inclination toward procedures leads them to look for procedures in all mathematical notions. This speaks to the influence of who they are and both what they know in mathematics and how they know it – they are influenced by their structure. Their knowledge of mathematics makes them drift toward procedures: the teachers aim at finding procedures and focus on them when they learn and explore mathematics – they are structurally determined. More than the teachers' intention to simplify mathematical concepts to render them more accessible to students as mentioned in Chapter 5, teachers with a calculational orientation often reduce mathematical activity to procedures and place the knowledge of procedures at the center of learning mathematics. I address both issues here.

Calculationally-Oriented Teachers "Technicize" the Mathematical Activity

Because of their strong inclinations toward procedures, such teachers often impose their "procedural" eye on mathematical issues. One example of the teachers' tendency to reduce to or see mathematics as a set of procedures was noticeable in Gina's interest in finding a "technique" to write algebraic equations to provide for her students. Even if she did not succeed in finding one, and the fact that it was addressed in session 1 and 2-3 as an instance of "reasoning," she still had the reaction to search for a procedure that would solve it all and make it programmable in order to reduce students' errors. It seemed as if she simply could not help it. As mentioned above, this however evolved between sessions 2-3 and 4-5 and also within session 4-5 which explicitly focused on the issue.

Nevertheless, her understanding of this issue also developed in an interesting way. She did not aim for a "technique" *per se* or a procedure to follow step-by-step but, as the other teachers did, she still wanted to find a "tool" that could help its study and prevent students' errors. As the exploration of word problems and their algebraic equations went on, many alternatives were offered as (almost) ultimate solutions to the issue. For example, Lana and Claudia suggested that teachers should try to use letters different from x and y because students experience difficulties with letters x and y, or on the contrary some other students are only capable of working with x and y, making them unable to operate if it is on different letters. As this suggestion appeared to gain relevance for them, I explained that although it could possibly make students more flexible with their usage of letters, students might still experience the same type of problems and difficulties when writing algebraic equations but with different sets of letters. I later pointed out to them that in the student Brigitte's solution (Appendix D, part D.1), where letters other than x and y were used, important problems were still experienced in regard to what the letter meant for the student.

Another solution was offered by Lana. She suggested that students write what the letters meant (e.g., x = number of cards of Steve; $x + 27$ = number of cards of Jeanne). She believed this might help students to write their equations. Again, I reminded them that if this activity was done mechanically as one of the steps, it would not help students to understand and establish the relationships and equations, because students needed to make sense of the relationships between the data and also write the equation based on an unknown and the context. In other words, however useful, Lana's solution did not represent an overarching and infallible method because students still had to understand what they were doing and to establish the many links inherent in the problem in order to write and obtain an adequate algebraic equation representing the context of the problem. But then, Gina suggested that students write all the relationships in a table and said that this would always work. But again, it was the same thing as Lana had offered, since instead of writing the relationship down, the data was simply placed in a table with each column representing one unknown value. I raised the same argument that the table was not a universal and infallible apparatus to use, since students still needed to reason.

In sum, each time that the teachers came up with "the" tool to write equations, I had to intervene and assert that despite its interest of many sorts, the suggested tool was not a guarantee of success, since students still needed to use the same reasoning. In the end, writing algebraic equations from a word problem requires reasoning. Nevertheless, these teachers still aimed at finding "the" solution to simplify their students' work and have them succeed better; despite teachers numerous failed attempts and the fact that they all agreed that writing algebraic equations existed in the realm of reasoning, which was not "technicizable." These teachers were inclined to find a universal-always-working "technique" to give to their students.

This was a constant issue with the teachers, and I frequently felt the need to nuance their attempts at finding "the" solution to simplify or program some mathematical notions. Because I intended to address and have an impact on their mathematical knowledge, it was always something that I had to take into account. Even when I believed that the session opened the way to more than procedures in mathematics, and that the teachers grew a sensitivity toward that, they were still inclined to see procedures in, and attempted at "technicizing," the mathematical content. In fact, their inclination toward seeing procedures (everywhere) in mathematics is an inherent and important part of their knowledge, of their structure. "Technicizing" mathematical concepts is an integral part of who teachers with a calculational orientation are mathematically.

Placing Procedures at the Centre of the Study of Mathematics

Another important aspect of the mathematical knowledge of teachers with a calculational orientation concerns the centrality that procedures occupy for them in the study of mathematics. For most concepts, their main goal was to know the procedures and apply them to find answers. For example, even if the explorations done on volume of solids opened new possibilities to perceive and understand volume differently from its set of formulas (e.g., relations, piling up of layers, Cavalieri's principle, families of solids), it still led Carole to raise the fact that some students prefer receiving formulas right away, demonstrating that for her the final goal was to know the formulas. The other teachers also had a similar reaction when they wondered, in face of all the possibilities offered from the exploration of the volume of solids, "When should formulas be given, before or after the study?" – prompting me to say that maybe "never" was also an alternative. Although they demonstrated a genuine interest for the explorations that had been conducted, and the meaning they made out of these explorations, at the core of their responsibilities still lay the central importance of formulas in the study and teaching of volume of solids. This again represents another important issue to take into consideration as work is attempted with teachers with a calculational orientation in order to enlarge their mathematical knowledge. Thompson et al. (1994) also flag these sets of issues.

> Teachers frequently ask us essentially this question: "After we've talked about understanding these situations, how do I introduce the standard procedures?" This question indicates to us a teacher who is grappling with a dilemma – how to reconcile an emphasis on students' reasoning with the traditional curriculum and pedagogy wherein symbols, methods, and procedures are introduced before students encounter any substantive applications. (p. 90)

I believe it is in fact this very dilemma that needs to be addressed in the professional development of teachers with a calculational orientation.

These two sets of issues – the inclination to see as and even transform mathematical ideas into procedures, and the central place afforded to procedures in the study of mathematics – represent important aspects of the knowledge of mathematics of teachers with a calculational orientation. This needs to be constantly addressed throughout the mathematical experiences offered to them, since it is a significant part of their structure.

Not Seeing the Mathematical Activity as a Panacea

I have tried to illustrate how working on each of the branches of the mathematical activity has the potential to enlarge teachers' mathematical knowledge, which in turn has the potential to trigger teachers to reflect on their teaching and to open up new possibilities concerning what they could offer in their classrooms.

However rich the potential to influence teachers' mathematical knowledge, and teachers' classroom actions, work along the three branches should not be seen as a guarantee of success because as mentioned the effect is contingent on the teachers – teachers are structure-determined beings. Indeed, there were instances when teachers, even after having experienced something new that enlarged their mathematical knowledge, expressed no intention for changes in their teaching. Two of these moments are particularly notable and can be used to demonstrate the non-causal influence on teachers' teaching of their enlargement of mathematical knowledge. These examples illustrate how it is a question of opening up and of triggering possibilities, and not of affecting a change each time. Working along the mathematical activity has the *potential* for change, but it is not an "instant change-maker" – depending on the teachers, sometimes it can have no effect at all, or so it seems. The first example involves Erica and the teaching of volume, and the second involves Lana and concerns systems of equations.

Erica: "We Don't Have Time"

In the beginning of session 4-5, I did a round table to gather details about what the teachers remembered and what they "took away" from the previous session 2-3 on volume. Erica said that she enjoyed the approach – which is quite obvious in the cassettes of session 2-3 as she often is very excited about the work being done on volume – but that as a teacher she is not always able to do these sorts of things in her classroom, because it is too much time consuming.

I was not too surprised by this comment, because it is something often said by mathematics teachers when a new approach to teaching is suggested (Even, 1999), one that obviously takes more time than in a theory-practice course, which Battista (1999) calls the traditional teaching method. My first reaction could have been to say, as is usually said, that "Spending more time here on this content will have you save more time later" or "Students will understand much more of the content, so it will be beneficial to them for understanding faster subsequent content." And it was indeed the argument that Gina used with Erica[113]. However, I felt the issue was more than just "not having the time." The session on volume of solids did not aim to say that more time should be taken to show volume concepts, but that the study of volume of solids represented something more than and different from just its formulas, and that important reasoning and conceptual issues are at stake in the learning of volume.

Therefore, as a teacher who kept on wanting only to show formulas to her students, it felt to me as if Erica did not grasp a new or influential understanding of what the study of volume of solids represented. It did not change her mathematical understanding of volume of solids to the point of having her realize that simply showing volume formulas is restrictive and is not representing much of what the study of volume consists mathematically. As she raised the fact that she did not have the time to work on volume in this way and could only take one or two lessons to teach it – in order to show the volume formulas to students – it was clear that the change in her mathematical understanding of the volume of solids, if any, did not or could not affect her teaching practices very much. In that sense, in spite of the fact that the session aimed at working on volume at a "conceptual" level in order to demonstrate how volume formulas could be made sense of (relational understanding), and that the study of volume of solids implied more than a study of its formulas, it did not automatically affect Erica's perception and conception *of its teaching*. For Erica, as she explained about having no time to do

[113] Demonstrating an important impact of the approach on Gina.

anything else other than showing formulas, volume was still at its core about its formulas, and teaching them was equivalent to teaching volume. For other teachers, like Gina for example, her understanding of the concept of volume changed to the point of having her think about doing and approaching things differently when she would teach volume subsequently. For Gina, because volume was not the same mathematical content anymore, it had to be taught differently. Even if Erica's mathematical knowledge about volume of solids had enlarged, her understanding of what should be taught was not affected to the point of having to change her ways of teaching. For Erica, volume was still centrally about its formulas, hence teaching simply its formulas made sense. This demonstrates well how a teacher's actions in the classroom are contingent and determined by the understanding and meaning that the teacher has of the content[114].

Lana: "This is How we Usually Say it"

In session 6, focused on giving a grade to students' answers, I brought students' solutions to a problem on systems of equations[115]. The first two solutions to the problem had what I would call an "incomplete character," in the sense that something was missing from the answer to make it complete (see figure 6.5 and figure 6.6 for the problem and the two students' solutions to it – what is written in bold represents the students' writing).

Solve the following system of equations:

$$\begin{array}{ll} ^1 & 2(x-3) = 1-y \\ ^2 & 2x+y=7 \end{array}$$

<u>Student solution 1:</u>

^{1} $\rightarrow \mathbf{2x - 6 = 1 - y}$
$\rightarrow \mathbf{2x + y = 7}$

Answer: ***x*** **and** ***y*** **can be any number**

Figure 6.5. First student solutions to the systems of equations problem

[114] This does not mean that the other teachers would completely change their teaching. However, they showed and demonstrated the development of reflections concerning their teaching. On the other hand, Erica made it quite clear how she would not change.

[115] Inspired from Sfard and Linchevsky (1994). The long description for this task is available in Appendix D, part D.2.

Solve the following system of equations:

$$\begin{array}{ll} 1 & 2(x-3) = 1 - y \\ 2 & 2x + y = 7 \end{array}$$

Student solution 2:

$$\begin{array}{ll} 1 & \rightarrow \mathbf{2x - 6 = 1 - y} \\ & \rightarrow \mathbf{2x + y = 7} \end{array}$$

Hence, $\begin{array}{ll} 1 & \mathbf{2x+y=7} \\ 2 & \mathbf{2x+y=7} \end{array}$

Infinity of solutions

Figure 6.6. Second student solutions to the systems of equations problem

These two solutions are incomplete in the sense that it is not only "any number" or an "infinity of solution," but it is an infinity of solutions always along the relation "*2x+y=1*," as Gina explained.

> Gina: [...] they say an infinity of solutions but it is not exactly true. [...] It is an infinity of solutions, in relation. [...] Because it is not true that x and y can be any numbers, because there is a condition between the two.

Lana agreed to this concept as she realized that the answer is maybe insufficient, but mentioned that this is how it is normally said in mathematics when two lines are superimposed – something she will raise often in the discussion – and that the meaning of "along a relation" is implicit in the answer about infinity.

> Lana: It is implicit because it is the word that we use. I understand the point of view that it is a restriction.
> Jérôme: Yes.
> Lana: But we never talk about restrictions, but I agree [that it is incomplete].
> [...]
> Lana: It is how it is talked about, everywhere, even in the diploma exams it will say "infinity of solutions" also. [...] It would not say along the curve or along the line.

For this reason, even if she realized and agreed on different occasions that the answers were incomplete, Lana would still give ten out of ten (10/10) for these two students' answers – even though the other teachers strongly disagreed with her on this point and mentioned that they themselves found the solutions mathematically incomplete, especially "student solution 1," which was simply false for them. Afterwards, Lana even explained that some students who wrote these type of answers have often automatized it, in the sense of having learned it by heart, and do not really understand what it implies. This, however, did not change her way of thinking about it. She insisted she would still accept these two answers as adequate and would not even consider highlighting the issue about the condition of "along the line" or "infinity following a relation" to her students. On the other hand, she did put into question the way it is "usually talked about," but without wanting to change her future ways of acting.

> Lana: It is the notation that we use. Is it good? Well, it depends …

Therefore, even though Lana realized the mathematical incompleteness of this type of answer, and in that sense she deepened her understanding, she did not change her view in regard to its teaching. Even though she realized that it is always discussed and written in that way in textbooks and in diploma exams, she did not see an opportunity to make this issue more adequate mathematically. And finally, even though she explained that some students who write that type of answer do it by rote and do not understand what they do, she still would accept these answers as valid and complete. She said she would not underscore the issue in her teaching of the notion, leaving it and talking about it "as it is usually said." In that sense, even though she enlarged her understanding and knowledge of this mathematical notion and saw it differently mathematically, it did not lead her to adapt her practices and affect her teaching at that moment[116].

This is reminiscent of the discussions about calculationally-oriented teachers and their pedagogical content knowledge reported in Chapter 5. Lana aimed at keeping the study of the notion simple and without any nuance to it (i.e., the nuance of infinity "along the

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[116] As for the previous example of Erica, nothing says that small things will not be changed or affected in their teaching in the long run. These issues can make their ways or be present in their thoughts when they will have to re-teach them. There are still possibilities. But as they made clear *in these sessions*, it did not pushed them to change at that time.

relation $y=mx+b$"). Even if she agreed to its more complex basis, she did not want to offer this idea to her students – something reminiscent of Gina's "base" situation.

Conclusions on the Mathematical Activity Analysis

I have tried to illustrate that the constant work along the three branches of mathematical activity impacted teachers' understandings of mathematics; it enlarged their knowledge. By recursively elaborating and acquiring "new" knowledge (Chapter 5) and also by experiencing mathematical concepts along the three branches of the mathematical activity, teachers developed what Skemp (1979) calls vari-focal knowledge, in the sense that they are now potentially able to look at (some) mathematical concepts through different lenses and levels (conventions, procedures, structures and relations, meaning behind procedures, etc.).

> 'Vari-focal' is useful for helping to think about the different ways in which a particular concept or schema can be viewed. (p. 115)[117]

Moreover, the specific work along the three branches opened up new spaces of actions for teachers concerning their teaching of these notions. Mathematical concepts were starting to be seen in a different way as teachers enlarged their mathematical knowledge concerning the presence of conventions, deepened their understandings of procedures, and tackled with issues of structures and relations in mathematics.

However, I have shown from teachers' strong inclination toward procedures that an enlargement of their mathematical knowledge did not directly guarantee automatic changes for their practices. The effect the approach can have on teachers depends on the spaces opened, but are determined by and are contingent on the teachers themselves. In the same way as for the development and enlargement of teachers' mathematical knowledge, it is the learner's structure that determines the change. Hence, changes in teachers' teaching practices are triggered by (dependent on) the learning environment offered, but determined by the teachers' structure. For that reason, work on mathematical

[117] For Skemp, at each focus, the level of quality is described in terms of its *interiority* – something I have mentioned being similar to recursive elaborations (Davis & Simmt, 2004, 2006).

activity does not represent and should not be seen as a universal remedy that will solve all the problems. It has to be framed as a trigger in terms of its potential for change.

On the other hand, without being a guarantee of influence on teachers' practices, the development of a robust understanding of the mathematics teachers have to teach is still fundamental, since without it nothing much can be expected.

> A teacher's subject matter knowledge may not automatically produce promising teaching methods or new teaching conceptions. But without solid support from subject matter knowledge, promising methods or new teaching conceptions cannot be successfully realized. (Ma, 1999, p. 38)

Consequently, everything needs to be framed in terms of potential: the tasks offered, the knowledge developed, the unfolding/emerging events and the teacher educator's actions. The last two are the topic of the next chapter. But before I turn to Chapter 7, I offer an "Intermission," to engage in a reflection about (school) mathematics and its centrality in (my approach to) the professional development of secondary mathematics teachers.

INTERMISSION

SOME THOUGHTS ON (SCHOOL) MATHEMATICS

> We as a community need to talk seriously about the implications which these results have for [mathematics] teacher reeducation programs. (Post et al., 1991, p. 198)

The Presence of Mathematics in the Professional Development Program

"Mathematics" was central in the entire professional development program that I created, from the beginning to the end, from planning to analyzing the sessions. The context of teachers' strong inclination toward procedures in mathematics led me to develop an approach to professional development intended to provide teachers with learning opportunities to experience "conceptual" mathematics. It appeared essential to work "conceptually" on school mathematics, in order to enlarge teachers' mathematical experiences and background, so that teachers could see more than procedures and calculations within the mathematics topics and concepts they have to teach. To paraphrase Bauersfeld (1977), if teachers see mathematics through a procedural eye, they will most certainly teach mathematics with that orientation. Hence, my belief was that if teachers started to see mathematics with a more encompassing eye, they would possibly begin to teach mathematics with such an orientation too.

These ideas were rooted in the working assumption that enlarging and enhancing teachers' knowledge of mathematics (to encompass more than procedures) would have an important impact on their teaching practices, enriching the mathematics they offer in their classrooms – mathematics that would not be oriented solely toward its procedural and calculational aspects. Therefore, this is one main reason why my focus was placed directly and precisely on the mathematics and not, for example, on curricular documents, classroom practices, textbook analysis, or something else.

It was the "mathematics" that always triggered the work in each session and that engaged teachers in deep conceptual probes into school mathematics. Pragmatically speaking, this appeared to be an important choice with secondary mathematics teachers, since their interest in mathematics drove much of their action and reflection. Indeed, in many instances, the mathematics teachers showed intense concentration concerning the mathematics that was offered. One of the most vibrant examples was when Lana and Erica worked on operations on fractions in session 7 (see Appendix B), where they lost track of what the rest of the group was doing and concentrated on understanding the operations and how to represent and make sense of them. In that instance, I literally felt that I had lost them in their work, as they were thoroughly engaged and were not paying attention to anything else. These types of events, where teachers were strongly invested in the mathematics offered in the session, happened throughout the sessions, sometimes resulting in teachers either switched off from the group or passionately inquiring with many questions as to what the concepts really meant. (This happened for the introduction of and work around Cavalieri's principle and oblique solids in the session on volume of solids described in Chapter 5.)

I raised the point in Chapter 1 that secondary mathematics teachers have to be seen differently from elementary teachers. The idea of placing mathematics at the center appears to be another important distinction between secondary teachers and elementary teachers. Obviously, it is not that elementary teachers cannot be strongly invested in tasks[118], far from that, but the fact is that (most) secondary teachers deeply enjoy working on mathematics, which makes it important to enter professional development practices through mathematics. Therefore, as much as it appeared important to work on mathematics to enlarge teachers' knowledge base, it also appears important to work on mathematics with secondary mathematics teachers simply because it is a perceived

[118] However, it is commonly agreed in the mathematics education community that elementary teachers' relation to mathematical topics is often problematic (see, for instance, Blouin & Gattuso, 2000).

interesting point of entry for them and it brings them to engage intensely in the ideas[119]. For that matter, the entry through school mathematics appears to be an important, relevant, promising and potentially rich approach for the professional development of secondary mathematics teachers.

The Centrality of School Mathematics in the Approach Taken

The phrasing "the mathematics teachers have to teach" is an explicit orientation in my way of speaking that I have tried to use throughout this dissertation. One expression that I have deliberately used is "the mathematics, and its teaching and learning," with some variations. I believe this is an important issue, since when working in mathematics education with the mathematics that teachers have to teach, it is almost impossible to separate the mathematics from its teaching and learning[120] – it is indeed its very context. *School mathematics* lies in a context of teaching and learning. It makes sense that a deep-conceptual-probes approach aimed at probing deeply into the mathematical concepts from which teaching issues would emerge, since they are almost impossible to separate. This is why the expression "school mathematics" has a specific sense for me and takes all its meaning, one in which there is mathematics but also its teaching and learning. It is therefore more than mathematics that is at the center of this professional development approach, it is precisely "school mathematics," the entry point being the mathematics, from which teaching and learning issues unfold.

This is the specific position that I have taken in this approach, to place school mathematics at the center, but one however that I realize could be argued against. There is currently an important confusion and even some disagreement in the mathematics education community as to what should be the object of focus in mathematics education research. Indeed, the 2006 PME-30 conference held in Prague in the Czech Republic created controversy with its main theme being "Mathematics in the centre," where many proponents argued for having "teaching of mathematics" at the center of mathematics education research, or "learning of mathematics" or "mathematics itself" or

[119] Cooney and Wiegel (2003) also point to this, and add that because of their previous success in mathematics, secondary teachers are well positioned to learn even more mathematics.

[120] Which is one argument that was made by the UQÀM group (see Bednarz et al., 1995).

"mathematical ideas" or "the place of mathematics in our technological society," and so on. In addition, the relevance of the name of the conference (Psychology of Mathematics Education) was a topic of discussion that was placed under scrutiny in regard to its emphasis on "psychology." In a way, there is currently important questioning happening in regard to the place that "mathematics" occupies, or should occupy, within "mathematics education." I will not comment further on the issue, but it seems that this "identity" problem can be easily translated to professional development practices concerning what should be at the centre of mathematics *teacher* education practices.

In the case of the education of mathematics teachers, it is possible to see many different views and approaches, each one having its own set of ideas and values and focusing on different aspects as central issues in teacher education programs (both pre- and in-service). A good example is the fact that we do not share, as a community, a common view of what represents good mathematics teaching practices (e.g., group work, inquiry-based learning, lecturing mode, "constructivist" teaching, strategic teaching, etc.). For that matter, it is difficult to mount an argument for or against or even assess any specific practice of professional development, since it can rest on opposing views about what mathematics teaching should be. In addition, as Doerr's comment at PME-30 previously cited says, all professional development brings results and is said to be of value. Hence, even if one approach to professional development bears results, it is possible to disagree with the approach to teaching offered in it. Moreover, with respect to these potential divergences on teaching, the same disagreements could and can be raised concerning curricular practices, learning theories, textbook usage, and so on.

However, from all these possible approaches to teaching mathematics, we *do* share a common thread throughout the mathematics teacher educators' community, and it is *mathematics* – its "conceptual" richness in comparison with its procedural and calculational aspects. Therefore, since mathematics is the only common thread that unites us, it should be mathematics, and more precisely school mathematics, that is to be placed at the centre of our mathematics teacher education practices and approaches, so that *rich "conceptual" mathematics is learned, and that teaching and learning issues in regard to this school mathematics be addressed.* Not surprisingly, perhaps, this is exactly the thesis of the deep-conceptual-probes model. (I refer the reader to Appendix F where I report on a discussion that I had with a colleague, Christine Suurtamm, around the centrality of school mathematics for mathematics teacher education.)

The Importance of "Conceptual" Mathematics

This move toward placing school mathematics at the center is not an obvious one and has repercussions for professional development practices, and also for the mathematics education community. Placing school mathematics at the center requires a lot in regard to mathematics. It requires the development of more conceptual approaches to these mathematical topics and notions, something that is often talked about but for which there is not much access. To repeat a quotation from Raman and Fernández (2005):

> When Liping Ma's (1999) book *Knowing and Teaching Elementary Mathematics* burst on the scene, many people—on both side of the math wars—were quick to embrace it as exemplifying exactly the kind of knowledge that we want elementary school teachers to develop. The catch phrase, "Profound Understanding of Fundamental Mathematics" (PUFM) seemed to capture precisely what both reformers and traditionalists thought was essential mathematics for elementary school teachers. However, 5 years later we are still struggling to articulate what PUFM means and how it should play out in the training of preservice mathematics teachers. (p. 259)

I believe the key to the development of conceptual approaches to mathematics resides along the lines that I have tried to develop in this doctoral research. Indeed, it is by unearthing the conceptual aspects of the mathematical notions of the school curriculum that it will become possible to offer richer mathematical content for teachers to experience and explore in mathematics teacher education practices, in such a way that it enlarges their knowledge of the mathematical notions they teach, and its teaching and learning. There is, then, a need for the development of more "conceptual" approaches of the mathematics of the curriculum, more "conceptual" content, so that these can be worked on afterwards, experienced and explored by teachers in mathematics teacher education practices. This has the potential to provide teachers, as Bryan (1999) suggests, with opportunities "to deepen their conceptual understandings of the content of the school mathematics curriculum" (pp. 8-9).

Developing and Researching the Mathematics within School Mathematics

> A practice-based approach to asking about mathematical knowledge for teaching reveals that there is much mathematics *deep inside* the school curriculum as well as *beyond it.* (Bass, 2005, p. 429, emphasis in the original)

This call for developing more "conceptual" approaches to mathematical notions of the curriculum can seem odd, since the mathematics in the curriculum is already "developed." However, as these in-service sessions have illustrated, there is a lot of important mathematics to explore and be researched within the mathematics of the curriculum. In other words, to use Bass's expression, there is a lot of mathematics "deep inside" the mathematics that is taught in schools. For this reason, to develop new comprehensive approaches means to go deep within the current mathematics taught and to develop and research it to make rich concepts and notions emerge – to develop rich comprehensive meanings for these mathematical notions.

Skemp (1987) alludes to these practices when he intends to draw out the relations between and intricacies within the mathematical concepts that are represented by the symbol system. For example, from the symbol "572" can be induced three specific numbers "5," "7" and "2," related to three specific powers of ten, and again to three operations of multiplication by these powers of ten (e.g., 5×10^2), and finally to the addition of these products (pp. 179-180). For Skemp, this represents how it is possible to unearth concepts hidden behind the "structure" of the numerals (here, "572"). This, for him, opens up to an immense realm of insightful mathematical concepts[121]:

> Once one begins this kind of analysis, it becomes evident there is a huge and almost unexplored field – enough for several doctoral theses. (p. 180)

In effect, the mathematics within school mathematics needs to be developed further, so that comprehensive approaches can be created and thought of – approaches in line with the ones that I have tried to develop in this research, approaches consonant with

[121] He also closely relates this to his idea of a concept map that lays out the connected ideas for a specific concept.

Janvier's work on volume for example[122]. This idea of developing rich comprehensive mathematics within the mathematics of the curriculum appears to open the way to a potential new field of research in mathematics education, one which focuses on developing the mathematics of the school curriculum. But this should not be misinterpreted as a call for research on mathematics as mathematicians do, namely by proving theorems and creating new conjectures in order to have the field of mathematics go forward. It is a different view of developing mathematics, one that focuses not on moving the field forward, but on digging deep into already conjectured, proved and formalized notions in order to develop conceptually rich notions and concepts. Adler and Davis (2006) also allude to this with their idea of "unpacking" the mathematical ideas[123] within a concept, saying that "[u]npacked mathematics is different from accumulated disciplinary knowledge in that it is build on ways of working within a disciplinary domain" (p. 293). In other words, whereas research in pure mathematics aims at getting the field forward, research on school mathematics aims at taking an already known pieces of mathematics and enlarging and deepening it. Figure I.1 gives an image of what is intended by "developing and researching the mathematics of school mathematics."

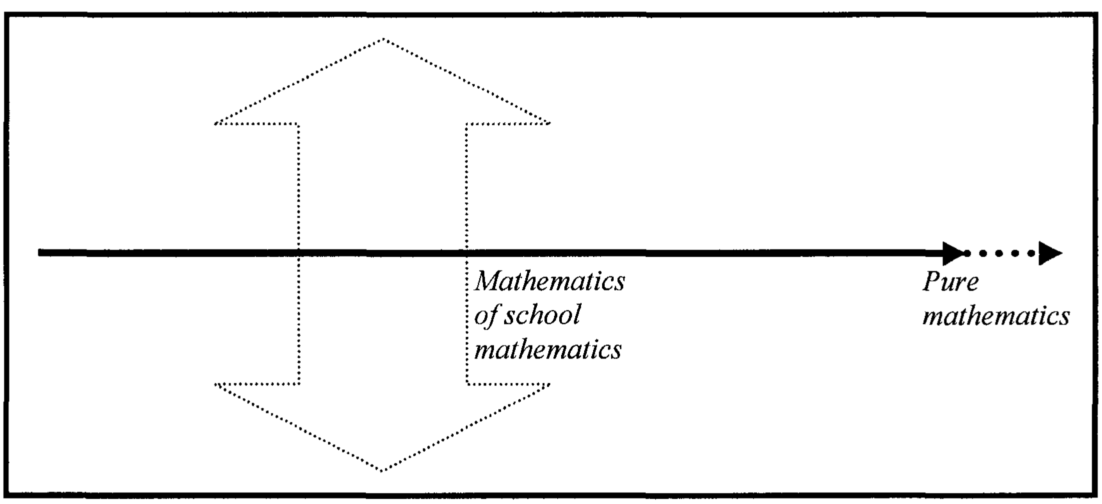

Figure I.1. An image to represent the differences between developing and researching the mathematics of school mathematics and pure mathematics research

[122] Indeed, Janvier's (1994a, 1994b) work on volume has been an important influence on my work.
[123] Based on Ball and Bass's (e.g., 2003) ideas.

There is more here, however, than Adler and Davis's idea of unpacking the mathematical concepts or Skemp's notion of drawing out the relations behind and within mathematical concepts. There is also the idea of developing new mathematical ideas and objects within these mathematics of the curriculum – the development of families of solids described in Chapter 5 and families of planar figures described in Appendix E are examples of this type of creation that I have carried out. In other words, the idea is not *only* uncovering already-known aspects within a concept or unpacking them, it is also to produce new mathematical ideas and objects within the mathematics of the curriculum.

It also distances itself from research in pure mathematics, since it draws its inspiration from the teaching and learning of these very (school mathematics) topics and notions – as mentioned, both are inseparable. For example, the development of families of planar figures (Appendix E) gained inspiration from students' difficulties with area formulas (Hart, 1981) and the prominent presence of diverse and isolated formulas in the teaching and learning of area (e.g., Hart, 1981; Janvier, 1994a)[124]. The same is true of Janvier's work on volume, where in the beginning of his book he states clearly the reasons for him to develop a richer approach to volume, based on students' difficulties with the panoply of formulas and the lack of richness in how this concept was taught in schools. The teaching and learning issues experienced with the topics render the mathematics content worth exploring because there exist some concerns or issues about its teaching and learning. The development and research of the mathematics of school mathematics would inspire itself and build on understandings of the ways these mathematics are taught in schools and are learned by students.

A Return to Mathematics: A Natural Turn?

It is interesting to note that this suggested turn of mathematics at the centre appears almost natural as a re-orientation when we look at what the mathematics education community has accomplished since its beginnings. If I allow myself to make a quick summary, I would say that mathematics education began with mathematicians interested in the learning and teaching of mathematics in schools. (Kilpatrick, 1992, 1994, explains

[124] Another influence on my work on 2D-planar figures was my own understanding of developing families of 3D-solids, which itself unfolded from the work of Janvier on volume of solids.

that it started with mathematicians *in* mathematics departments.) Naturally, being mathematicians, the importance of mathematics was very strong and it occupied an important place. One can remember that mathematics education research is often said to have seen the light of day after the events of the "new math" movement. The prominence of mathematics is quite obvious when one looks at the first issues of *Educational Studies in Mathematics* where even the assertions made in these articles are under the format of theorem proving almost in an axiomatic form (see, for instance, the articles of Hirst, 1972, and of Levi, 1971). However, mathematics education researchers quickly realized that they lacked understanding of and knowledge about how mathematics was learned by pupils, or in other words, how did students made sense of such mathematics. Again, this is reminiscent of the UQÀM group history. As it was explained to me, it appears that, among other things, this is when the PME group emerged with an intention to understand the learning of mathematics better. Since then, the mathematics education community has produced enormous amounts of insightful information about how mathematics is learned. It is then not a surprise that PME-30 conference discussed a return to the mathematics with its provocative main theme "Mathematics in the centre."

It appears natural after having produced massive amounts of information on students' (and teachers') understanding, difficulties, misconceptions, errors, thoughts, beliefs, and so on, that the loop returns back to mathematics itself. And this, I believe, is the core issue for the "development and research on the mathematics of school mathematics." As I have started to elaborate, development and research on the mathematics of school mathematics would be different from pure mathematics research where it would not be interested in pushing the field of mathematics forward, but at digging deeper into already-known concepts. But also it would be different in the sense that its inspiration for digging deeper would come from the knowledge of teaching and learning of these mathematical notions. In the return to mathematics, this proposed research endeavour would use the previously developed knowledge about learning and teaching of mathematics in order to develop itself. Figure I.2 describes the above-summarized history of mathematics education, with what I offer as a new movement of developing and researching the mathematics of school mathematics which uses the previous information on mathematics teaching and learning.

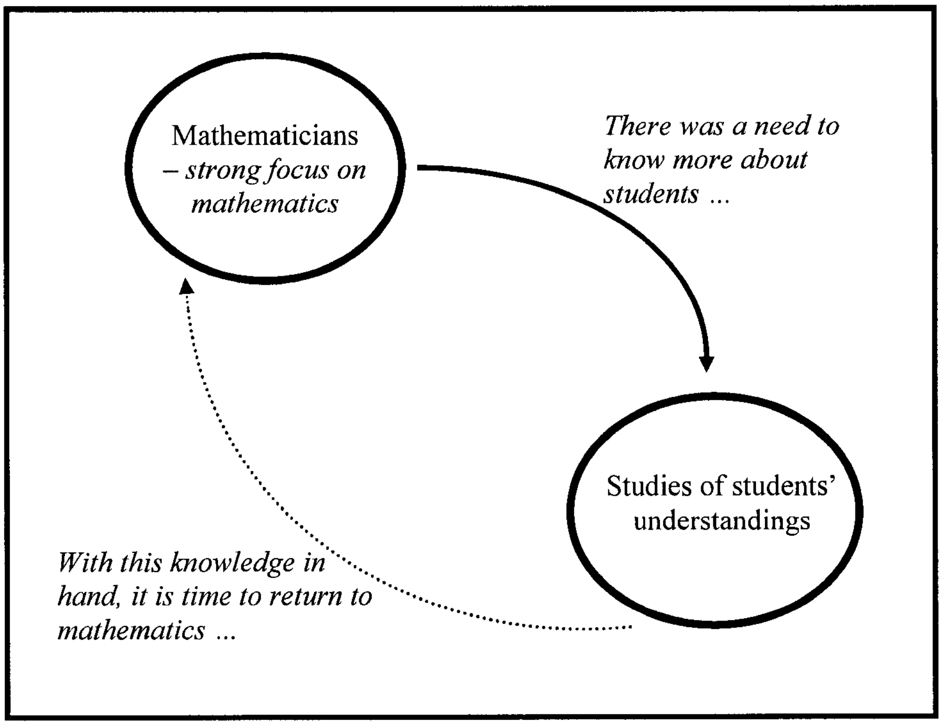

Figure I.2. The loop representing the history of mathematics education, with a new orientation toward the development and research on the mathematics of school mathematics

In addition, it is not a movement to research mathematics for the sake of researching mathematics: it also has an explicit goal of developing "conceptual" mathematics for *use* in the education of mathematics teachers. This research would have two goals: one would be to develop mathematical notions about school mathematics – developing the mathematics of school mathematics – and the other would be to re-invest these mathematical findings in the (pre- and in-service) education of mathematics teachers.

Usiskin (2001) has recently suggested than an interest should be placed on the invention and development of courses that should be given in mathematics departments to mathematics teachers in relation to the mathematics that they have to teach in high school. He calls this "teachers' mathematics"[125]. However, slightly different from what I did in my research and offer here, mostly because in Usiskin's proposal there is no focus on the teaching issues in relation to these mathematics, it still has strong interests in unearthing important concepts within the mathematics of the curriculum. In addition,

[125] This is different from and should not be connected to the research on "Mathematics for Teaching."

Usiskin has suggested that it should be considered as a coherent field of study in itself, as an instance of applied mathematics.

> This mathematics is often not known to professional mathematicians. It covers both pure and applied mathematics, algorithms and proofs, concepts and representations. [...] Teachers' mathematics is a branch of applied mathematics, applied because it emerges directly from problems in the classroom. Teachers' mathematics comes out of the teaching and learning of mathematics. (pp. 13-14)

However absent from the *content* that he offers in his university courses, it is interesting to see that Usiskin mentions its provenance from "problems" of the classroom.

Usiskin's advocacy for creating a coherent field of study – "Teachers' mathematics is not merely a bunch of mathematical topics that might be of interest to teachers but a coherent field of study [...]" (p. 3), which has potential and mathematical richness that makes it "the antithesis of a narrow research field" (p. 14) – gives warrant to my intention of developing a research agenda for the "development and research on the mathematics of school mathematics." Indeed, a field of mathematics requires that research be conducted in it, and I feel the approach that I have offered here provides one access route to this research line.

> What may be new in this work is our view of teacher' mathematics as a branch of applied mathematics, our view that this branch of mathematics is not watered-down content but more appropriate content, and our view that the body of knowledge represented in teachers' mathematics is huge and deserving of attempts by individuals and groups to structure it. (p. 14)

Concerning teacher education practices, an important addition to Usiskin's ideas is in regard to teaching issues. Within the goals of educating teachers with mathematically rich experiences, there is first an intention to have teachers learn rich mathematics, but there is also a second, and very important, intention to have issues concerning the teaching and learning of this mathematics emerge and be addressed in teacher education practices. This is the very essence of the deep-conceptual-probes model, and a fundamental issue for the education of mathematics teachers to have them develop both greater mathematical and greater pedagogical powers.

This idea of addressing teaching issues, as mentioned, takes its roots in the UQÀM approach and tradition, which is implicitly in line with what is offered here for the development of mathematical concepts within the curriculum. Therefore, this research movement that I suggest here should not be seen as a completely new orientation in mathematics education research, since, like the UQÀM group, other researchers have worked on these issues and developed deep analyses of topics in their research, although perhaps at a more implicit level. This research orientation builds on previous researchers' work as an inspiration, individuals like Freudenthal and Janvier, and more recently Hewitt and Zaslavsky to name only a few. In many research projects in mathematics education, it is not uncommon to find pieces of very insightful mathematics. The difference in the research movement that I am suggesting is that the intention to develop "conceptual" approaches to mathematics is an explicit goal in the research endeavour – it is part of the research intentions. It is deemed a research field because some mathematics is intended to be created in this digging deep into the mathematics of school mathematics, it aims to produce new knowledge about mathematics (of school mathematics).

This is obviously still at a developmental level, with directions and methodologies to develop in length. However, I have offered these ideas because this dissertation led me to develop and reflect on them, and I believe it provides a rich site for research and advancement in mathematics education.

CHAPTER 7

THIRD ANGLE OF ANALYSIS: ENACTIVISM, EMERGENCE OF EVENTS, AND TEACHER EDUCATOR PRACTICES

From the beginning, the approach taken in these professional development sessions has been oriented by the idea of building on the mathematics teachers' mathematical knowledge in order to enlarge it. In enactivist terms, the idea was to *open the space of the possible by exploring current spaces* (Davis, 2004). Within this approach, teachers are taken into account and are important concerning what happens in the sessions, since the events that can happen are dependent on them, their knowledge, their structure. To make sense of this, I contrasted this approach with top-down approaches where outcomes are pre-planned and the intention is to attain those outcomes – the knowledge to be acquired is pre-specified. The approach taken here distances itself from this. Instead of having objectives to attain, the objectives acted as starting points for the explorations to take place – these objectives were to be worked on. The events occurring in the session were dependent on the people participating in the session (myself and the teachers) as the tasks and situations offered were engaged with. By having these intentions and by enabling that setting to take place, it had repercussions on the type of learning experiences and the spaces created by/for the group.

In order to make more sense of the events that happened within the sessions, in this chapter I re-examine the data through two more lenses, one about emergence and one about the role of the teacher educator's actions. Although intertwined, each will be treated separately, with obviously some implicit connections from one to the other[126].

Enactivism and Emergence of Events

The spaces of learning created in the professional development sessions have the characteristic of being unpredictable. The explorations went into places that I, as the teacher educator, had not thought of prior to the sessions. The events that emerged and unfolded did not follow a pre-specified linear path, rather the path was laid down as it unfolded (Varela, 1987). This is what I address here, in order to illustrate the unpredictability of the learning events that happened[127], their richness, but also how an approach that aims at building on teachers' knowledge and taking their knowledge into account cannot escape from embracing the emergence of event in order for the exploration to be the most fruitful possible.

The events around the definition of the base of a solid (reported in Chapter 5, in *moment 7*) represent an interesting illustration of the emergence of unforeseen events. I use this moment as a supporting illustration here[128]. What led to this "moment" and the smaller events within the moment itself were unpredicted and emerged in relation to teachers' interests and understandings of the issues – in other words, in relation to the exploration itself. I did not and could not have predicted that these discussions and interactions would happen. It was indeed the first time that I encountered these specific issues of base since I began using Janvier's approach to volume in teacher education (with pre-service teachers and colleagues in informal presentations). These events were unpredictable, not in the sense that they were counter-intuitive or surprising, since in retrospect it makes sense that they happened, but they were unpredictable in the sense that I had not planed for them to happen. The issues were raised within the session, and I

[126] Again, I ask the reader to continue to bear in mind the aspects of the deep-conceptual-probes model and the mathematical activity, which were treated in the previous chapters.

[127] Being unpredictable does not mean being random. As explained in Chapter 3, we must recognize that the work and explorations are constrained by the environment in which they are given, which acts as "liberating constraints" (Davis & Simmt, 2003).

[128] Other ones could have been chosen since the entire deep-conceptual-probes model is based on emergence. For example, in the telephone problem (see Appendix C, part C.2), Claudia's comment about the presence of implicit constraints in word problems opened up an entire set of events concerning "constraints" and "tacit meanings" in word problems, and the way mathematics is taught in schools.

as the teacher educator simply embraced them as they unfolded and pushed the exploration further. In other words, I enabled the spaces to be opened.

Structural Determinism

My intention or "objective to work on" to have teachers experience more than formulas in volume brought me to present volume as a piling up of layers. However, I did not realize when I spoke about it, or in what the videotape offered, that volume was always referred to in terms of a solid with a vertical orientation or "standing up" (). After a while, this triggered Carole's reaction about her students' difficulties with the orientation of prisms, solids and planar figures, where she explained that some students prefer or have difficulties with one orientation or another. Without implying causal relations, if I had not only done piling ups in terms of solids "standing up," *maybe* that issue would never have come up for Carole. Her reaction was "triggered" by the implicit "standing up" situation. This represents an interesting instance of *structural determinism*, where the effect does not reside in the "trigger," that is the "standing up" of a prism, but is determined by the person's structure. It is Carole's structure and understanding of the orientation of the prism that brought her to raise this issue about the students' difficulties with orientation. The "standing up" issue itself does not possess a predetermined effect on the person who interacts with it, that is, the issue of students' difficulties with orientation does not reside in the "standing up" "trigger,"[129] but *can* "trigger" an effect on a person depending on this person's structure. This is an important point, and a fundamental one when the teachers' knowledge is taken into account, since only "triggers" can be provided and the effects of these "triggers" are unpredictable (however, on some occasions we can have an idea of where it could lead in general), and are determined by the structures of those interacting with the "triggers."

Emergence and Cascades

Carole's comment itself was the "trigger" for many other reactions and events – emergent and unfolding events that were unpredictable and that cascaded ("snowball"

[129] As all my previous experiences at the pre-service level had shown me, it never "triggered" anything like that before.

effect). Carole's comment about the orientation of prisms was followed by a discussion about the fact that "any set of bases" could work for a rectangular prism, since it did not matter which base you choose. This brought Gina to react and disagree with the ideas[130]. Gina's disagreement was not anticipated. However, I had considered that the issues on bases of solids (with the triangular base pyramids) had the potential to raise some mathematical interest in the teachers, but not to the point of having teachers react against the idea. In that sense, Gina's reaction was an emergent learning opportunity, or an "opening" to use Remillard and Kaye Geist's (2002) terminology, that was not predicted but also that unfolded from the previous comments on "any base." Hence, in addition to being emergent, Gina's reaction was also an unfolding. There is a link between the consecutive reactions: Carole' comment on orientation led to the "any base" comment, which then led to Gina's disagreement. This is what I refer to as *cascading* and *emerging* events. These reactions are emerging, in the sense that they are triggered by the events and are not predicted in advance, but they are also part of a cascade of events that unfolded from Carole's comment on orientation (which itself unfolded from the previous explorations about volume of solids). The emergence of reactions led to other emergence of reactions. Hence, the events are emergent but also interlinked, one leading to the other (when observed afterwards) as in the "snowball" effect. The session's events unfold into unpredicted directions, which then unfold into other unpredicted directions, and so on. It does not funnel, to refer to Bauersfeld's (1998) idea, but expands into many directions and possibilities.

If there were a pre-specified thread to follow, a specific objective to attain, the emergence of events into a cascade would have brought the work quite far away from that pre-decided thread, and in that sense would have threatened the success and objectives of the session. In the case here, it was the opposite that happened. The objectives of the session (to have teachers experience more than formulas in volume) were embraced to their fullest by having the explorations open doors and spaces in many directions (all linked to each other). As I will come back to later, contrary to following and attaining a

[130] Again, it shows how the reaction is determined by the structure of the person, where the other teachers did not react as Gina did.

specific pre-planned objective, the success of the session was, in fact, based on this emergence.

The Entire Set of Emerging and Cascading Events

The cascade of events did not stop there. Gina's disagreement led me to offer a mathematical definition of prisms, which led Gina to discuss the issue of students being confused by these ideas. This led Erica to interject regarding the mathematical worth of Gina's comment, and led me to offer the rectangle as an example of how the choice of the base was indeed a choice. Gina disagreed with that as well, but the entire situation around bases brought Carole to ask for a clearer definition of a base (not only of a prism), something Erica offered. This triggered a new cascade of reactions, because Gina reacted to Erica's explanation by saying the same thing but for a pyramid, in an attempt to give a counter-example to Erica's definition and the previous discussions. This led me to define the pyramid, building on the previous explorations of pyramids and triangular prisms and the previously given definition of a prism, to show Gina that her counter-example was not valid. This brought Claudia to offer Gina an explanation by giving another counter-example, which would lead Gina, it was hoped, away from her notion that "only one pair of sides" different from others could be taken as the base. This last argument that unfolded from this cascade of events succeeded in convincing Gina of the arbitrariness of the base[131]. And, it brought the discussion back to Carole's first comment that had set the cascade going in the first place. This led Erica to offer her ways of teaching this issue with students, something she offered in line with the previous discussions of piling up of layers and the definition given for prisms – talking about slicing bread into equal slices. In effect, a point could easily be made that if the previous entire discussion had been absent, there would have not been much of a point for Erica to raise this issue of how she teaches the concept of base, since it would not have been a topic of discussion. Figure 7.1 attempts to track down the emergence and cascade of events and the path created in terms of one event "leading to" the other.

[131] I have mentioned that this sequence of events is a good illustration of what the zig-zag activity (Lampert, 1990) of proving, conjecturing and deducting looks like in mathematical activity.

Discussing volume as piling up, by always placing the prisms "standing up" → Carole discusses students difficulties with orientation → "any base" comment → Gina's disagreement → definition of prism → discussion of students' difficulties/confusion → "cannot teach false ideas" → example of width and length of a rectangle → definition of "base" → counter-example of the pyramid → defining pyramids → example of the videocassette box → Gina's understanding → back to orientation of prisms → slicing bread teaching method of Erica.

Figure 7.1. Tracking down the emergence and cascading of events in terms of one "leading to" the other

At a finer grain of analysis, it is possible to look at the emergence of some events not explicitly in relation to a previous one, but to a bulk of events that "triggered" a reaction. It could be said that no specific event triggered Carole to ask for a clearer definition of a base, but rather the entire situation that was happening did. It is as if the set of moments themselves, concerning the disagreement about base from the first attempt to define a prism where the notion of base was used, made the reaction emerge. The same thing could be said for Claudia's counter-example of the videocassette box, where it is not one single event, but the entire set of events that occured in regard to Gina's difficulties to accept what was offered as a base, that "triggered" her reaction. The entire set of events could be schematized in the following way (figure 7.2).

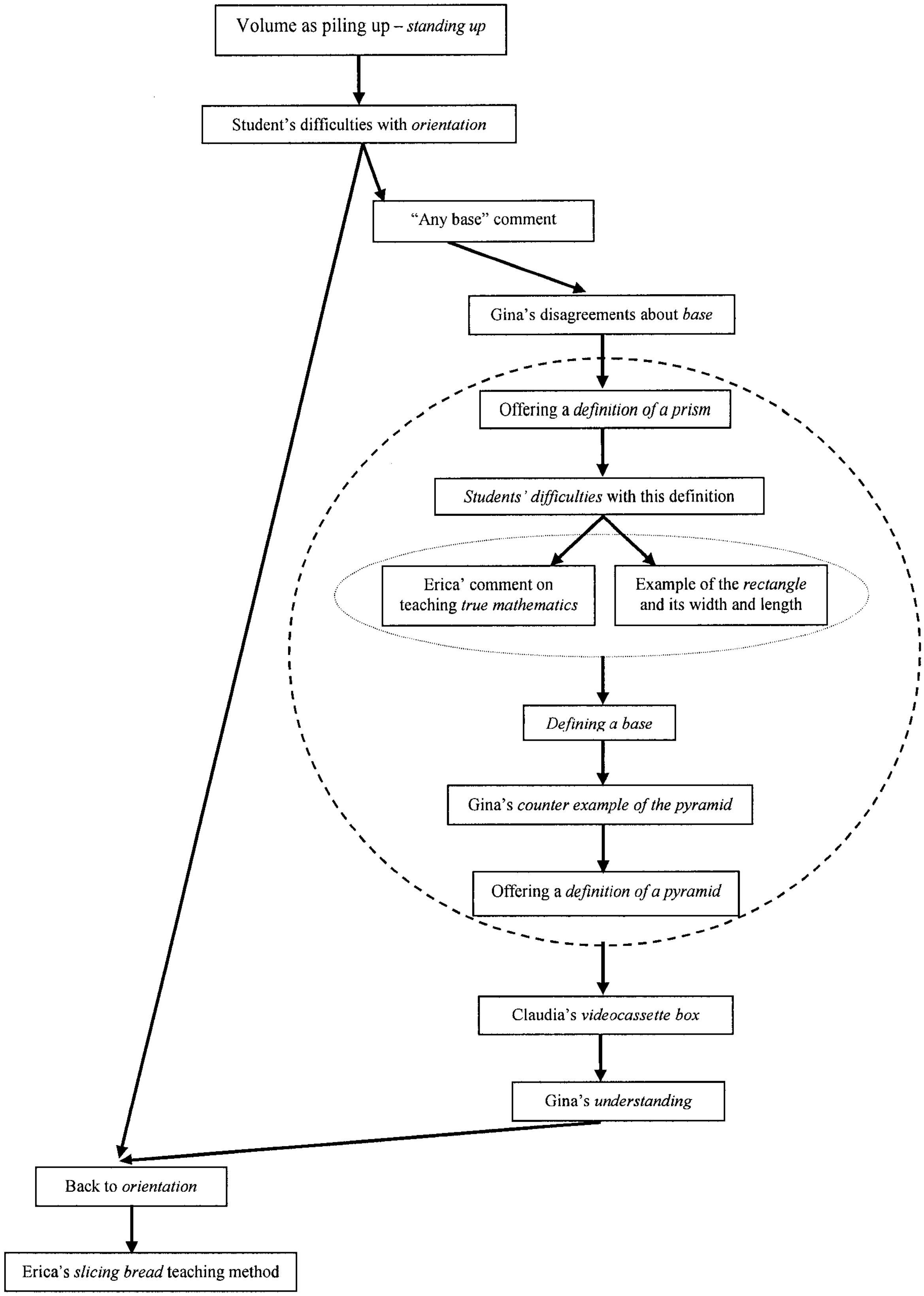

Figure 7.2. Another possible description of the emergence and cascade of events

Structural Coupling: The Events and the Viability of Assertions

In these events, the teachers and I *structurally coupled* to create meaning about the issues. In other words, we co-evolved and co-learned as these notions were addressed and explored. We evolved as we created a mutual understanding of the concepts addressed. Within this history of interaction, meaning was created concerning the concepts, but also some reactions, as I have just schematized, were based on an entire set of previous interactions. Of course, it could be argued that each reaction was embedded in the previous reactions, but the previous figure (figure 7.2) is intended to demonstrate that some reactions unfolded from the previous ones, whereas some others unfolded from *a set* of interactions. In that sense, it was the history of small previous interactions, where we structurally coupled, that triggered a reaction. It is within this mutual understanding that some reactions emerged.

Moreover, the viability of some reactions was discussed in regard to our mutually-lived history of previous interactions, where our structural coupling served as a base to judge the viability of statements. When Gina raised a counter-example to Erica's definition of a base by giving the instance of a pyramid, it was rejected on the basis of the previously-given definition of a pyramid where we had agreed that the pyramid could be placed on any base *if* the lateral sides were all triangles. The structural coupling or the history of previous interactions enabled judgements of viability in regard to what could be considered adequate or not. Along the same line, any further discussions about the base of prisms or figures would be referred to and discussed in regard to these previous interactions which were now part of the history of interactions – in the same way that Gina in session 7 commented back to Erica about the "conventional" nature of the order in the rate of change concept by drawing on the explorations of the previous session; it was now part of our history of structural coupling[132]. In that sense, two aspects were at the source of possible "disagreements" in regard to the validity of what was asserted. The first one was the participants' own structure, which could lead one of us to flag

[132] However, being in the history of interactions does not imply that it will be "remembered" at anytime. For example, in the 10th session on analytic geometry, in order to introduce elements concerning the rate of change I brought back the question used in the 6th session and, to my surprise, teachers did not seem to remember it.

problematic issues (e.g., Gina's reaction concerning the "any base" statement[133], my interjection about the fact that what Gina taught about base was "mathematically false"). The second source was the history of interactions, in which things were "established" along the way.

The Success of the Session Resting on Emergence

> In a sense, openings may be signals that the curriculum is working. (Remillard & Kaye Geist, 2002, p. 28)

As I tried to make clear when presenting the deep-conceptual-probes model, the emergence of events does not represent an interesting moment that we occasionally look at, but rather represents the *norm* within the sessions. What I mean by this is that an "objective to work on" is set and some material is brought in (e.g., Janvier's video, problems, students' solutions, etc.), but the rest depends on the emergence of events. That is the "leap of faith" the teacher educator must take. The explorations are oriented by the emergence of events and issues (mathematical and teaching/learning related).

The tasks and situations that I offered were intended to be "triggers" and starting points for the work; hence the idea was to explore them deeply so that mathematics was learned and teaching issues emerge. The success of the sessions did not rest on attaining specific notions and objectives, but on the exploration of the tasks and situations and the emergence of issues and learning experiences that would unfold. If explorations were attempted and if issues emerged from the work and possibilities were created – if "openings" happened – then the session was judged successful. The success of a session lies in its production and generation of ideas and knowledge, and not in the attainment of pre-specified products or competencies for teachers. The products are emergent and are not predictable, they are dependent on the teachers themselves. Since teachers all have different histories, the products will be necessarily different and, hence, unpredictable.

Therefore, no other session using the same material would be the same. First, the teachers in the sessions would be different and so they would bring with them different

[133] That she was right or wrong is not important here. The basis of her disagreement – be it wrong or right – still came from her own knowledge/structure.

knowledge and interests. And second, I myself would be different as the teacher educator, having this new history of previous experiences with these current teachers and the material offered. It does not mean that similar issues will not be brought forth, but rather nothing guarantees that the same outcomes will be obtained. The outcomes do not lie "in" the material, but emerge from engagement with it and its subsequent exploration. But most importantly, the intention is not even focused on having the same outcomes or products, because professional development is context dependent and the relevance of issues vary from contexts to contexts where teachers have different experiences and knowledge (Bednarz, 2000). Grounding the sessions in issues of emergence aims at producing and generating knowledge, and not at disseminating or replicating specific events of learning in teachers. Unfortunately, the materials of the session itself are not a warrant of success; it is possible that a session ends flat because of the different type of knowledge of teachers (for example, re-doing this session with teachers who would already see volume as a piling up of layers, and who would be aware of Cavalieri's principle and familiar with the volume of oblique solids probably would not have the same type of impact as it did on the teachers in my research). It is important to understand that the deep-conceptual-probes approach provides a model, one that is based on the emergence of events, and in this model some tasks and situations are offered to trigger the mathematical explorations. Nothing suggests that the "material" I have used will impact teachers in the same way.

In addition, it is important to remember that the teaching issues were all emergent – except for occasions where I intentionally prepared a specific teaching issue, for example on students' misconceptions (e.g., student Brigitte's solution in Appendix D, part D.1). All teaching issues emerged from the richness of the mathematical explorations, and provoked reactions in teachers concerning the teaching of this explored mathematics. The openness to emergence enabled important learning to happen where on-the-spot interests, interrogations and inquiries were given a chance to flourish, which brought forth important learning insights and experiences into the mathematics, and its teaching and learning. The fact that issues and learning experiences continuously emerged, and in fact formed the essence of the sessions, gives warrant to the approach taken where its intentions toward emergence got played out in the events.

As this completes the discussion about emerging events, I now turn specifically to the role that I played as the teacher educator in these sessions.

The Role of the Teacher Educator

In Chapter 3, I called for renewed practices of the teacher educator which were in line with an enactivist view of cognition. Such practices were focused in terms of "triggering" mathematical explorations and intervening to provoke learning opportunities for teachers. This is quite distinct from the notion of a "facilitator." I took a specific role as the teacher educator in the events and explorations of the session. I adopted the position of someone who actively intervenes in the process to influence and push the thinking of the mathematics teachers in order to enlarge their knowledge. For example, in the data reported for the session on volume of solids, it is quite apparent that I am prominently present in the interactions and throughout the explorations done. Quite seldom did the discussions go on without me being part of them, whether it was because I triggered the interaction, because a question was directed to me, or simply because I joined in or inserted myself into the conversations. In that sense, I was always part of the mathematical explorations as the teacher educator. This is no small issue, so I elaborate.

Fernandez (2005) has demonstrated in her study how having teachers interacting only among themselves can have shortcomings. When teachers meet and work on tasks, it does provide some potential "triggers," but it also appears to be limited at a certain point. As structure-determined beings, teachers are interpreting tasks and working on them within their realm of the possible. In the case of the teachers in my research, their strong inclination toward procedures often brought and oriented them into a relatively restricted discussion about procedures, for which I had to intervene to open potentially new horizons for them. Within their structurally-determined system, chances are that the tasks alone would not have sufficed to "trigger" teachers into realms different from ones about procedures. Fernandez comments on this concerning her lesson study experiment:

> This learning was no doubt possible because lesson study created a rich learning environment for these teachers in very much the same way that rich classroom tasks like those employed in reform classrooms set up opportunities for students to learn. However, although students learn a lot from working on such tasks, nevertheless a

> teacher who can push, solidify, and sometimes redirect their thinking is critical. Similarly, the teachers described here could have benefited from having a "teacher of teachers" help them make the most out of their lesson study work. (p. 284)

This brings her to assert that we need to know more about the strategies to engage teachers and work productively with them.

Categories of Action of the Teacher Educator

Fernandez's comments point to the importance of the teacher educator in the professional development setting. In the deep-conceptual-probes model that I offer in this dissertation, the teacher educator is needed to push and enlarge teachers' knowledge. The teacher educator's actions are acting as "triggers," in the same way that the tasks and situations offered to teachers act as "triggers." Therefore, the teacher educator's actions complete the tasks and situations offered in the sessions, in the endeavour to enlarge teachers' knowledge. It is in that sense that the teacher educator's actions are key, and it is for these reasons that I have placed a specific focus on them in the analysis. As mentioned in Chapter 4, I have categorized these actions. This categorization hopefully will enable a better understanding of potential strategies to engage and orient teachers toward avenues that they would possibly not have gone down by themselves. I give here a description of this categorisation. I am not suggesting that these are the only possible actions that can be taken, rather they simply are the ones that *I* explicitly enacted on a regular basis as the teacher educator. Here are the (8) categories of action that emerged out of the data in regard to my interventions as the teacher educator.

1. Nuancing a point, clarifying an assertion already made;
2. Bringing in research results in order to make a point;
3. Giving an interpretation of a situation;
4. Bringing forth a question, raising an issue by a question;
5. Explaining an issue or a situation;
6. Resisting a teacher's assertion;
7. Emphasizing an issue;
8. Playing Devil's advocate.

Some of the actions taken in the sessions sometimes belonged to or overlapped with different categories. For example, I could have made a nuance by using research results that I knew about. The intention was not to look for the frequency of the interventions – since some probably slipped unobserved into the background – but to give a description of the different types of interventions that I made use of. In order to understand the nature of these interventions better, I describe and illustrate them in greater detail in the following.

(1) Nuancing a point, clarifying an assertion already made

This type of intervention was made in order to bring a nuance or a subtle meaning to what had been said before. This happened often when I felt that the meaning that teachers were drawing out of an issue was simplified or that the main point had slipped into the background. It was done in order to give more substance and richness to the issues explored, and also to point to a particular non-trivial aspect that I sensed the teachers had possibly missed (or could miss).

For example, in the rate of change problem of session 6 (see Appendix D, part D.3), Gina felt uncomfortable with the fact that we were saying that a student could understand a concept without knowing the convention. For her, it felt as if we were saying that conventions are not important and could be broken without any difficulty arising. This brought me to clarify and nuance what was meant by making a distinction between the knowledge of conventions from the one about the concepts.

Gina:	The other question that we need to ask ourselves when we teach to younger students is that what they learn is always a kind of a brick for when the next teacher will have them, he will be able to place the next brick. So then, if you accept that they break the conventions because you say "yes, yes, Jo Schmo understands well" the problem is that you prevent the next brick to arrive.
Jérôme:	Ah yes, the continuity. Indeed. This is what Lana underscores [with her linear function coherence]. But at the same time, I do believe that it is ok to tell the student that it is not adequate because it is not the right mathematical convention. But here the question is mostly "does this student understands the rate of change?"
Claudia:	Right.
Jérôme:	Here, it is not the same question. I think we are obliged

> mathematically to tell the student "Stop, stop, you understand very well, but conventionally we do place the y over the x."

And later on, I gave another nuance to the importance of conventions.

> Jérôme: But I, I do not think personally, even if I say that it is only a convention, I would give 10 out of 10. The student does not know the convention, there, there are some points to loose at the mathematical level because the student does not know the convention.

Hence, the idea of making nuances was not because I disagreed with teachers or that they did not address the issue, it was mainly to flag subtle issues that I felt were very important to take into consideration.

(2) Bringing in research results in order to make a point

Depending on the explorations and conversations going on, some issues were raised about which I was aware of interesting research that had been done on the subject. On those occasions, I explained the research conducted and elaborated aspects of the issues discussed in regard to these research results.

For example, in the round-table discussion carried out at the beginning of session 2-3 to reflect on the first session, teachers commented on how they sometimes make mistakes on the board in their explanations or their calculations and that their students noted these mistakes. As some of the teachers said that they congratulated students who find them, Claudia explained that it is good that they find them because it shows how well they are paying attention and are understanding the concepts. Gina added that what was also very important for the students was to see the attitude that they, as teachers, take when they commit the mistakes, by accepting and not feeling ashamed or embarrassed by the situation since it is part of the process of doing mathematics. I agreed with Gina and underlined research done by German scholars[134] showing how teachers' ways of doing mathematics in the classroom, particularly their ways of treating errors but also of explaining mathematics, of solving problems, of negotiating meanings, and so on, give students models about how mathematics is/should be done – it implicitly demonstrates

[134] Particularly, Bauersfeld (1994, 1998), Krummheuer (1992) and Voigt (1985, 1994).

and describes to students how to do mathematics. Hence, teachers' ways of acting are very influential and impact importantly on students' own ways of engaging with/in the mathematics.

Similar to the instance noted above, I often brought in research insights which enabled me to support a point or on occasion to make one.

(3) Giving an interpretation of a situation

Frequently, I explained to the teachers my interpretation and understanding of the issues addressed. By doing this, I was offering teachers my perception and interpretation of the issue and obviously attempting to have them look at the issue from that angle. I intended for them to pay attention to specific issues that I thought were important.

For example, in the session on volume of solids, I explained to teachers that the work on formulas had to be handled with care because only focusing on applying formulas and substituting in numbers to find values was not really working on volume and was mostly about algebraic substitution, in the same way that in physics someone can plug in numbers in the formula $F=ma$ without having much understanding of force or mass. In other words, that there was more to volume than a simple application of formulas.

This represented one of my ways of bringing in issues that I wanted teachers to address, reflect on and explore. It could be seen as a very direct intervention, one known as "telling." The intention, however, behind this "telling" was not for the teachers to take up my understanding, but for me to raise issues in order to orient the explorations and conversations. I knew that it was not because I was giving my interpretation of an issue that teachers would buy it or even understand what it meant, but I knew that it had the potential to "trigger" some reactions and queries into the issue – which was exactly my intention. Sometimes I took charge of the direction of the conversation by pointing to something that they had not seemed to see, for example in the rate of change problem when no teacher had raise the fact that there was a convention; in that case, I simply oriented teachers' attention to it (see Appendix D, part D.3).

Hence, directly explaining my understanding or telling something was not a problem for me, where it enabled me to orient the explorations; not pre-deciding what the explorations of the issues would be or end up going, but raising the issues that I wanted to see addressed and explored, that I felt were important to address, or even ones that "triggered" a specific interest in me along the way. There were issues that I already knew I wanted teachers to explore and address (often being the reasons behind having chosen the tasks or situations in the first place), and other issues that arose within the sessions and for which I felt there was an interest in addressing them.

(4) Bringing forth a question, raising an issue by a question

At different moments, I attempted to "trigger" or provoke a reflection in teachers by raising a question or placing an issue under question. On these occasions, the very act of raising a question was an assertion in itself. The question's goal was to have teachers reflect on the issue, but also simply to raise the question as an issue.

An example of this happened in session 1. Teachers had to solve 10 traditional algebraic problems without using algebra (see Appendix A for the 10 problems). As the teachers realized the relevance of arithmetical ways to solve the problems, comments and concerns arose about the worth of these solutions to solve these problems, the fact that algebra is often imposed to solve problems and the fact that algebra was a better way to opt for. This brought me to raise a question that I wanted them to address, that is, "What is the goal that you want your students to achieve? That they be able to solve problems or that they be able to solve them with algebra?" I wanted them to reflect on the fact that if the goal is only to solve problems, then arithmetical solutions do the job well, but that if they wanted students to be able to solve them with algebra, then the focus was on the method of solving and not on solving the problems.

These questions were not intended to receive a direct response, but to trigger reflections and have teachers explore issues about mathematics, and its teaching and learning.

(5) Explaining an issue or a situation

At diverse times, teachers demonstrated incomprehension of an issue and I explained it to them. These situations often arose when teachers asked me directly to explain one thing or another. The explanations could concern the mathematical issues or the teaching issues that were explored and discussed.

For example, in the session on volume I explained the principle of Cavalieri to teachers because they had not heard it before and did not seem to understand what it meant. In session 6, I explained the rate of change concept to Gina because she did not know about it. Other examples could be cited. They are not all about aspects that teachers did not understand, sometimes I simply explained issues. For example, I explained how to conceive of the cylinder and the cone as solids with infinite sides in the volume session. In the session on fractions, I told teachers that the multiplication sign in a multiplication of fractions could be interpreted as an "of something" as in "½ of a ⅓" or "a ⅓ of a ½" (see Appendix B for more details). Also, at a point in the session on fractions, the teachers were not able to solve the operations on fractions using the area of rectangles and folding the sheets of paper. Instead of letting them discover by themselves how to do it, I showed them how I would solve it to give them an idea of how it was possible to make sense of the operations. This helped them afterwards both to be able to do it, but also to develop an understanding of it[135].

These cases happened to be very close to the third categorisation (giving my interpretation on an issue or "telling"), but were, however, not at the level of interpretation of a situation and were about explaining a mathematical issue. These explanations gave teachers ways of doing things that would help them make more sense of concepts. It also happened concerning teaching issues, where I would explain to a teacher who did not understand what another teacher meant. These explanations were not always linked to not knowing, where for example I explained to teachers a planning tool that I knew of to prepare lessons. Explanations about teaching, though, were less frequent than ones on mathematical aspects.

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[135] I will not attempt to go into these details, but these actions were quite reminiscent of Maturana and Varela's (1992) ideas of learning by imitation, which they explain is a genuine and very important type of learning, where a living being inspires itself from a "source" and attempts to appropriate its ways of doing for itself – imitation is not synonymous with copying. But, obviously for different reasons, it is one sort of learning that is pejoratively seen in schools, though not in sports. This can raise some thoughts ...

(6) Resisting a teacher's assertion

In my explicit intention to address teachers' prominent inclination toward procedures, I often resisted some assertions teachers made that were strongly in line with their procedural tendency. This was not aimed at being negatively provocative, far from that, but I felt important to explain to teachers my own view – and also to address directly their tendency for procedures.

This is what brought me to say to Gina that her teaching of the concept of base was "mathematically false," in order to bring into question her understanding of these ideas and make sure they were challenged. Also, I resisted Gina's idea of looking for a "technique" to write algebraic equations, because writing equations exists in the realm of reasoning, and in the session 4-5 I continuously (but mildly) objected to teachers' assertions that they had found "the" solution or "the" tool to help writing algebraic equations – by explaining how more complex the issue was and that students still had to deploy mathematical reasoning.

However, my resistance did not guaranteed an influence on teachers since, again, they are structural determined beings. This is well illustrated in Erica's comment that she did not have the time to teach volume in another way than with its formulas. My resistance to her ideas – and even Gina's resistance to them – did not seem to matter much for her.

(7) Emphasizing an issue

In order to make sure that some important issues – in my sense – were addressed and explored by teachers, I often re-emphasized and re-stated aspects that had been raised before.

For example, I not only explained what Cavalieri's principle meant, but I re-explained it afterwards in the session and in the summary held in the following session. In addition, I often referred to it in my explanations so that its importance was highlighted to teachers. In the same way, while we watched the videotape, I often pressed the pause button to highlight an issue to teachers, to re-explain it, to make sure they understood the subtleties, and so on. I did not let the video run from start to finish, but instead made sure to stop it to emphasize what *I* thought was central in it. Moreover, in these pauses, I often demonstrated to the teachers (with some material that I had brought) how the concepts worked and I also raised some other issues concerning them. Along that same line, I re-emphasized on three different occasions Claudia's idea of starting with a triangular prism to study volume, because I personally felt that this was an important insight that needed to be dealt with and reflected upon by teachers. To this could be added the many instances where I oriented teachers to reflect on the presence and meaning of formulas in the teaching of volume.

These are just some examples, but they represent a specific intention on my behalf as the teacher educator to focus on elements that I believed to be of major importance. In my sense, these issues had to be addressed and so I made sure to raise them in the clearest possible way and to emphasize them for teachers so that they address them.

(8) Playing Devil's advocate

Finally, another type of action that I used was to play Devil's advocate with teachers. I did not use this type of intervention very often though. This was done when an important point was raised and that instead of giving my interpretation of the situation or issue, which sometimes I was not sure about, I offered teachers a different view by telling them that someone else could interpret it in another way. This was done in order to have teachers reflect deeply on the issues raised.

For example, in session 6 on the problem of rate of change, at a point Lana and Gina had difficulties accepting that a student could understand rate of change if this student reversed the order of x and y. Gina explained her disagreement, and I played the Devil's advocate to counter the argument she had brought forth.

> Jérôme: This is directly where my question is, this student in fact is able to calculate the variation between the points. Conventionally in mathematics, no, because it is y on x, but at a conceptual level…
>
> Gina: I do not agree.
>
> Jérôme: What do you mean exactly?
>
> Gina: I do not agree, because if you only want to verify if he is able to subtract whole numbers, ok, yes, he should receive all his points because he understands well this part.
>
> Jérôme: You mean "$5 - {}^{-}6$".
>
> Gina: But he does not know what is the rate of change if he inversed his

formula.

Jérôme: Ok, ok, I understand what you mean.

Lana: If the convention had been the contrary, I agree, but the convention is y on x and not x on y.

Gina: It is kind like if you ask someone to calculate the mass multiplied by the acceleration to find the force of something. And then you say to this person, you verify, you give him his points because he is able to do the multiplication. I am sorry, but he does not yet know how to calculate a force.

Jérôme: Ok, I understand. But if I play the Devil's advocate, for example. This student could well understand that one needs to multiply the mass by the acceleration, but then the numbers that he or she inserts are not good.

This was a way for me to give different perspectives on the same issues to teachers, in order for them to consider these other positions and then to change, adapt or even strengthen their original views. This position was not used to demonstrate my disagreement with teachers, but mostly to have them see another perspective concerning what they were asserting, and pushing the explorations deeper. I could even sometimes switch back and forth between different advocacies to have teachers reflect on different perspectives for the same issue.

Discussing the Different Categories

These aspects are not highlighted in order to define my entire practices, but to illustrate how I was actively present and attempted to educate the teachers, and how my actions were central in the process and that the unfolding of the sessions were directly dependent on my practices – they complemented the tasks and situations offered to teachers. It is along the enactment of these actions that I have intended to push forward teachers' knowledge and orient it into "conceptual" realms about mathematics. My actions have to be seen as "triggers" that are aimed at stimulating and occasioning reflections, discussions and understandings, in other words, at stimulating the explorations.

However, my actions only possessed a potential for reaction and change or, more simply put, for learning. They aimed at influencing and at educating, but it is clearly possible that nothing at all would unfold from them. In other words, the teacher educator's actions do not cause or guarantee an influence on teachers. That teachers will seize the opportunities that I offer them is their choice alone. My focus on having them understand that mathematics is not only constituted of or reducible to a set of techniques is up to them to make sense of. In the case of Erica, it seemed that volume remained for her a set of different formulas to know and apply. As the mathematics teacher educator, I do everything I can to offer these ideas, to make them accessible and to push the explorations. I am deliberate and explicit about that. This is my responsibility as the mathematics teacher educator, I try to educate teachers. But I cannot, however, make it happen for them. It is the thesis of structural determinism; I cannot force change or influence others unless their structure allows for it. As the teacher educator, I only offer "triggers."

For that reason, it should not be seen as problematic to act as I did and to intervene as much as I did in the learning process of teachers. First, it is my role as the teacher educator to educate. Second, my actions cannot direct or cause change, they only offer potential opportunities for learning – hence, anything said or done does not have a direct "power" or "influence" over teachers. Third, and maybe most importantly, my presence appears important to complement the tasks offered and potentially orient teachers toward and into different realms of explorations that they, by their own structure, would perhaps not have been able to go or look into on their own[136].

In contrast to a top-down approach or a "change merchant" (Pimm, 1993) who aims at "giving" ways of doing and of teaching to teachers, the emphasis that I place within my actions only aims at having the issues addressed and explored. It is exploration driven. This is not a small point and represents an important distinction. My interest is in having the issues explored, which is quite far from wanting replication. The deep-conceptual-probes model for professional development centers on providing and offering learning experiences to teachers. Therefore, the outcomes of the explorations are less important than the explorations themselves.

[136] It could be argued that my interventions for teachers to explore and learn about (e.g., that mathematics is not only a set of procedures) are wrong and that I am showing them wrong stuff. However, I cannot escape from it, since I focus on what I know and on what I believe is important to be addressed.

Categories Situated at Two Main Levels of Actions

The previous section on emergence and this one about the actions taken as the teacher educator permits a possible distinction between two main levels of actions that I undertook in the sessions. The first category is the one just described, which aims at acting as "triggers" for teachers, in order to stimulate and offer teachers opportunities to learn.

At a second level of interpretation, coming from the emergence section, my actions can also be seen as *enablers*. Facing the potential insights from teachers, their demonstrated interest and the issues raised, I opened a space for these issues and ideas to be explored. I let and gave the opportunity for the spaces to be opened – I did not refrain or shut down these opportunities. That is why I call them "enablers," because the term illustrates my openness toward addressing issues that I had not planned before, and also how I embraced and promoted the exploration of these issues raised by teachers[137]. Although they are often closely linked to the current spaces being explored, these issues are on some occasions external to the core ideas worked on. But, in any case, they are potentially rich and teachers are interested in their exploration; therefore, I enable the opportunity for these issues to be addressed[138].

In sum, both attempting to "trigger" teachers' learning or "enabling" issues to be addressed aims at pushing forward the explorations and at addressing sets of issues. In that sense, my practices as a teacher educator were fundamental to the mathematical explorations being done, and were always aimed at opening the space of the possible.

I have attempted in this section to provide more information about the possible strategies and ways of acting for the teacher educator in order to open (more) spaces and offer (more) learning experiences for teachers. The categorization provided potential

[137] This theorization of actions as "enablers" could also be linked to Davis's (1997) idea of listening.

[138] For example, at the beginning of session 8, Gina brought a response sheet concerning exponents that she wanted the group to discuss in relation to the grades that had been attributed to the student. This was something aside of the main intention of the session (i.e., working on area conceptually) but which was strongly relevant to mathematics, and its teaching and learning. Also, at another time, we discussed the place of mathematics in society and how it is perceived. However not explicitly connected to our discussion in the session (which was around the three branches of the mathematical activity), this was an important discussion concerning mathematics, and one that teachers have to mingle with often in regard to the prominent students' questions about the relevance of learning mathematics. These two represent extreme cases, though, since most of the time the emergent issues were directly linked to the issues explored.

paths of action for the teacher educator in the endeavour to orient teachers toward issues that they would maybe not have thought of and to lead them to explore them. As Fernandez (2005) has explained, tasks and situations do not suffice to "trigger" teachers. The teacher educator has to intervene in the process, and that way potentially enrich teachers' learning experiences.

CHAPTER 8

REVISITING THE RESEARCH

What has been learned from this study? What more is known about secondary teachers' knowledge of the mathematics they teach? What could it mean to *enlarge* secondary teachers' knowledge of the mathematics they teach? In what ways could the approach rooted in "objectives to work on" contribute to this potential and intended enlargement? How and in what ways could this particular intervention of professional development contribute to enlarging teachers' knowledge of mathematics? And what can this study say about and to professional development approaches?

I titled this dissertation "(Enlarging) secondary-level mathematics teachers' mathematical knowledge: An investigation of professional development." Here, I want to return to the very beginning and directly address my title by exploring three specific issues: the connection between teachers' teaching and their knowledge, the meaning of "enlarging" teachers' knowledge of mathematics; and the richness of an intervention focused on mathematics.

Connection between Teachers' Knowledge and their Teaching

This research began with individual meetings with the teachers and visits to their classrooms. From those meetings and visits, as well as from the workshop sessions where discussions about teaching arose, I was able to gather a sense of their teaching. I used Thompson et al.'s (1994) theoretical construct of a "calculational" orientation to teaching to describe the teachers' ways of teaching; which were focused on procedures, applications and obtaining numerical results. However, this orientation appeared to me to be more than simply a teaching orientation.

Through the research, I become aware of a connection between the teachers' ways of teaching and their knowledge of (and way of knowing) the mathematics they teach. It seemed to me that these teachers focused on procedures because their knowledge of mathematics was focused on procedures. The teachers explained to me that their own educational experiences in mathematics revolved around memorizing procedures and applying them. Throughout the study, it was possible to see how teachers made sense and interpreted most mathematical topics along an orientation focused on procedures and their application. "Calculational" was more than a teaching orientation, it captured their orientation to mathematics in general – this is something that is not addressed in the Thompson et al. article. The teachers in my study did not focus on procedures in their teaching without reasons for doing so. They had this focus because this was what mathematics represented for them; it was what they knew about mathematics.

In addition, these orientations had a mutual influence on each other. Not only did teachers' mathematical knowledge influence their teaching, but their teaching influenced their ways of knowing mathematically. As illustrated in Chapters 5 and 6, these teachers with a "calculational" orientation tried to technicize mathematical knowledge and simplify it to make it programmable. This attitude toward teaching has important influences on their knowledge of mathematics, since this attitude also orients their ways of doing and engaging in mathematics. It leads teachers to do and understand mathematics in a certain way, one in line with what and how they teach, which leads to a technicized approach to mathematical concepts. Recall how Gina mentioned that the base could not be what we had suggested because students would have difficulties. Or remember how Lana did not intend to alter her way of talking about systems of equations, since it was counter to the way she was currently teaching it. For these teachers, the intertwining of knowledge of (school) mathematics and the teaching of that mathematics is strong and leads to a significant influence of one over the other; mathematics is both grounded and has meaning in its teaching to students.

The strong interconnection between teachers' mathematical knowledge and ways of teaching also gives credence to one of the main working assumptions stated at the beginning of this research. In attempting to enlarge teachers' knowledge of mathematics, there is the potential for impact on their teaching practices and what they offer to students in their classrooms. This interconnection shows the potential that an approach directed toward enlarging teachers' mathematical knowledge can have. This leads me into a discussion about the meaning of "enlarging" teachers' knowledge.

The Meaning of "Enlarging" Teachers' Knowledge of Mathematics

> From these meetings, that Frédéric would have preferred to happen at 3 o'clock in the morning instead of at 10, each of us came out not really better, but enriched. (di Falco & Beigbeder, 2004, p. 14, my translation)

Different ways in which teachers' knowledge was enlarged is illustrated in Chapters 5, 6, and 7. It is important for me to note that throughout these professional development encounters there was no single way in which teachers' knowledge was enlarged. The meaning of "enlargement" can be understood along different and complementary angles.

One of the ways to think about enlarging is in relation to teachers' structural determinism. The teachers' "calculational" orientation leads them to address and interpret mathematical topics in relation to that orientation. This is a feature of who they are, mathematically. As Fernandez (2005) argues, teachers left to work independently in a professional development setting can be limited. At some point, they may be unable to draw more meaning from a situation, being drawn toward the same conclusions and going around in circles. One of the roles of the teacher educator is precisely there (and Chapter 7 illustrates some of these possibilities): that is, to re-orient teachers toward new perspectives, toward other ways of seeing and interpreting, possibly away from their strong "calculational" orientation. "Enlarging," here, meant re-orienting.

A second way of making sense of the concept of enlarging (illustrated in Chapter 5) concerns the ways teachers developed their knowledge of mathematics. Teachers enlarged their knowledge in two specific ways, both grounded in their current knowledge. First, teachers gained interiority to and recursively elaborated their knowledge of mathematics. Refining what they knew and enriching it is an illustration of how they enlarged their knowledge about these mathematical concepts. Second, teachers learned new things about the topics addressed in the sessions. This represents another way that they enlarged their knowledge of specific school mathematics topics. Teachers developed knowledge about the school mathematics topics that they teach. "Enlarging," here, meant developing more knowledge about mathematics.

A third interpretation of "enlarging" concerns the nature of mathematical concepts and topics. In addition to teachers developing new mathematical knowledge about specific topics, their explorations opened new possibilities for these topics. By opening the study of some mathematical topics to more than simply procedures, the possibility that mathematics *is* about more than just procedures arose. Hence, for the teachers, the possibility that other mathematical topics be treated differently from simply as procedures was now present. This is a possibility that potentially did not exist for these teachers before the study: that is, the teachers could now imagine that mathematical topics can be worked on and studied as more than procedures. As much as the development of their knowledge in the previous point opened up to new possibilities for these specific topics, the realization that there exists more than procedures in mathematics also opened the way for new and unforeseen possibilities. "Enlarging," here, meant knowing about the presence of other possibilities for working mathematically.

A fourth way of making sense of the concept of "enlargement" concerns the new distinctions that teachers could make about mathematics. The teachers were introduced to more than a focus on different aspects from procedures, they were also initiated to two important distinctions about the activity of doing mathematics: the co-constructed tool of "techniques" *versus* "reasoning" and that about mathematical activity. This enabled teachers to make new distinctions concerning mathematics, and its learning and its teaching. Teachers now had the possibility (and, it is hoped, the capacity) to observe, recognize and distinguish some specific elements in mathematical activity. "Enlarging," here, meant being able to make new distinctions.

A fifth way of understanding "enlarging" is to pay attention to the activity itself of learning and experiencing new mathematical concepts and ideas, the processes that teachers went through. Working on challenging pieces of mathematics had significant effects on teachers. One was on the development of their capacity to probe into mathematical topics. The teachers developed (greater) aptitudes to *do* mathematics, not only to learn new things about it but to work in mathematics at a deeper level than that of procedures. Here, it was not necessarily what they learned through the activity, but the doing of the activity itself and developing a capacity to dig deeply into mathematical topics. A further effect has to do with their interest in and curiosity about undertaking this kind of activity. Unearthing some mathematical concepts raised some teachers' interest in exploring more mathematics elements[139]. Finally, another effect can be seen as the development of what might be called a habit of mind, where teachers started to think about doing similar deep analyses, but for other mathematical topics. This habit of mind became, as Skemp (1978) says, a contagion toward mathematical topics where one is oriented to do the same thing, digging deeper into other topics as well. These three effects are not at the concrete level of learning some aspects about a specific topic, but are at a meta-level as they involve teachers' attitudes toward mathematics. It is no small issue since having both the *capacity* and the *interest* to go deep into mathematical issues, and having an *orientation* toward addressing different mathematical topics along these lines is an important part of what it means to be a secondary mathematics teacher (UQÀM professor, personal communication, December 21st, 2005)[140]. These three elements represent another interpretation of an enlargement of teachers' knowledge, this time at the meta-level. "Enlarging," here, meant developing attitudes toward and aptitudes for doing mathematics and exploring topics and concepts.

A sixth and final way of interpreting "enlargement," as it emerged from this study, concerns the teaching of mathematics. As illustrated in Chapter 5, the exploration of mathematical topics and concepts raised many teaching issues. The development of anticipatory skills and instances of pedagogical content knowledge are examples of an

[139] An interesting event happened concerning Gina throughout the year. At the beginning of the project, Gina made clear to me that her reason for participating in the sessions was only to hear her colleagues' ideas and to discuss teaching with them. However, at the end of the 8th session on area, as the year came to an end, Gina asked me if we could continue to have sessions on other difficult mathematical ideas, because she wanted to continue to know more about mathematical concepts. This was quite a change from her initial intentions for participating in the professional development sessions.

[140] This came out of an interview with one of my former professors at UQÀM, who wished to remain anonymous.

enlargement of teachers' knowledge, this time about teaching. However, because these teachers' knowledge of mathematics and their ways of teaching are so intertwined, it is possible to see addressing and exploring teaching issues affecting in turn their knowledge of mathematics. An example of this happened when teachers realized the mathematical importance of students knowing the link between surface area and volume. This type of event contributed not only to teachers' knowledge about teaching (what students should know and be taught), but also to their personal knowledge of mathematics as to what is important mathematically within this topic. In that sense, the teaching issues that were raised and addressed contributed to the enlargement of teachers' knowledge of both teaching and mathematics itself. "Enlarging," here, meant knowing more about the teaching of mathematical topics.

In sum, these are all different meanings that can be attributed to the idea of "enlargement" of teachers' knowledge of mathematics: re-orienting toward perspectives, knowing more mathematically, realizing the presence of different forms of (knowing) mathematics, making new distinctions, developing aptitudes for and attitudes toward the exploration of mathematics, knowing more about teaching mathematics. These are all at the level of mathematical knowledge. At the level of mathematics *teaching*, the development of teachers' mathematical knowledge obviously has a potential impact, opening new possibilities concerning mathematics concepts and topics: there are more possibilities, more options, more ways of acting. However, how the teachers (will) take on these newly available possibilities in their teaching directly depends on them. But what is important here is that because of this professional development they do have new possibilities available to them. The pool from which they draw and play upon when they teach or learn a mathematical topic has been enriched, enlarged, unsimplified and complexified. Recall Krygowska's idea. When teachers' knowledge is enlarged, teaching a topic is complexified, since it is not about how "best" to teach but about what appears relevant to choose, out of the many numerous potential possibilities that lie in the pool of possibilities in order to teach well. Teachers' possibilities were augmented. Their pool of possibilities was enriched. This is what "enlarging" means and implies concerning teachers' teaching.

It is along that sense of opening "possibilities" that the thesis of "objectives to work on" contributes to teachers' enlargement of knowledge, since enlarging does not mean to follow a trajectory that is fixed in advance. Enlarging is opening new possibilities from the current spaces available. Further, possibilities do not mean attainment of specific good or best practices, they are simply "possibilities" for teachers' teaching. The mathematical explorations by themselves enlarged the space of possibilities, where sessions unfolded the potentialities that opened and emerged. By taking the stance of "objectives to work on," this professional development approach enabled and provided the space for explorations to happen and the opportunity to address new emerging issues. *The approach afforded the possibility of exploring* and learning about new possibilities that would complexify and enrich the teachers' pool of possibilities.

From this discussion of the possibilities for knowing and teaching mathematics, I now turn to the mathematics itself.

The Richness of Entering through Mathematics in Teacher Education

My main intention in this research was to address teachers' mathematical knowledge specifically by entering through the mathematics that they teach: that is, through the subject matter. However, the knowledge that teachers need for teaching is much more than their knowledge of the subject matter alone, teachers need to know more than their topic to teach well. Nevertheless, I realized that entering through (school) mathematics did much more than enlarge teachers' knowledge of mathematics.

In addition to them deepening and enlarging their knowledge of the mathematics that they teach, teachers developed pedagogical content knowledge, greater understanding of students' difficulties and ways of thinking, and even anticipatory skills to predict and understand possible student difficulties and understanding. As well, evidence points to the fact that teachers' beliefs about mathematics, and its teaching and learning, started to shift and be affected (e.g., vision of formulas, of conventions, of the distinction between "techniques" and "reasoning").

Fennema and Franke (1992) have elaborated a model that attempts to illustrate the different aspects of mathematics teachers' knowledge in the activity of teaching (see figure 8.1). This model includes knowledge of the mathematics to teach, knowledge of student understanding, and knowledge of general pedagogical issues about teaching. These three forms of knowledge combine together to create content-specific knowledge, something the authors explain to be in line with Shulman's notion of pedagogical content knowledge. Finally, all of this is surrounded by teachers' beliefs about mathematics, its teaching and learning (and learning and teaching in general).

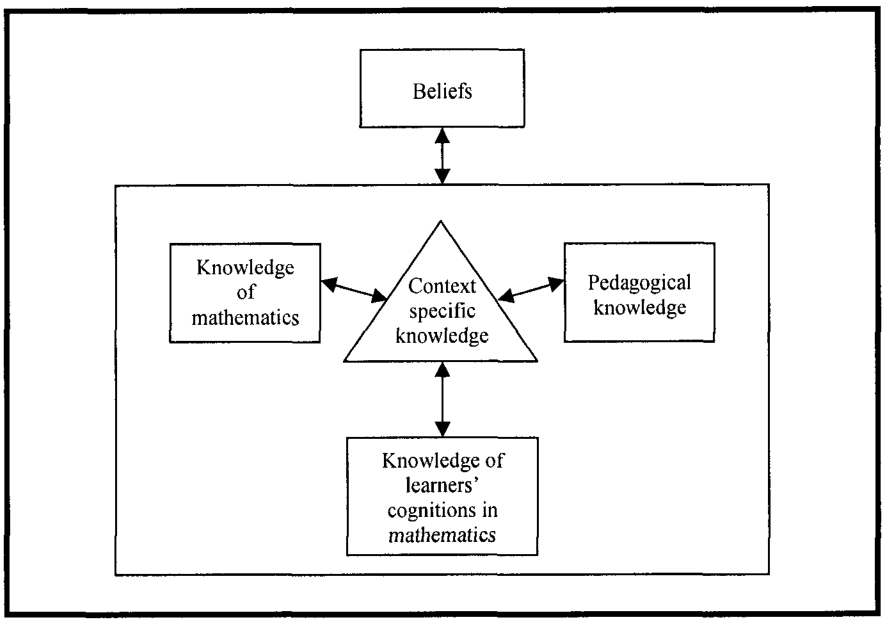

Figure 8.1. Fennema and Franke's model of teachers' knowledge

It is quite telling and interesting to realize that the approach taken in my dissertation, centred on the mathematics, enabled the development to a certain extent of all of the forms of knowledge highlighted in Fennema and Franke's model. For example, the issue of "knowledge of mathematics" was addressed by the explicit intention of the approach itself. General pedagogical issues were addressed through the emergence of the third type of teaching issues about teachers' everyday practices. Knowledge of students' understandings, was addressed by the teachers as they gathered broader comprehensions about students' understanding and developed forms of anticipatory skills to predict possible students' difficulties. Further, teachers developed forms of pedagogical content knowledge in the sessions. Finally, teachers' beliefs were seen to evolve along different lines as the year went by, confirming Cooney's (2001) assertion that, from his experience, the best entry into teachers' beliefs about mathematics and its teaching is through school mathematics.

The entry through mathematics seems to have afforded the potential for the development of fundamental forms of knowledge required to teach well. The entry through mathematics was not limited to the development of mathematical knowledge, because the mathematics addressed and explored was not stripped away from its context, namely its school/teaching context, which is exactly where it has meaning for mathematics teachers. Mathematics teachers bring with them their own experience and knowledge about the teaching of these different mathematical concepts and are themselves compelled to address these teaching issues in the course of action. Because mathematics teachers have an interest in the *teaching* of mathematical concepts, it makes the entry through mathematics a privileged entry point for the professional development of teachers.

This, I believe, makes a compelling case that entry through the content of instruction constitutes a fruitful and promising approach to the professional development of mathematics teachers[141]. Moreover, it confirmed a belief of mine that, in teacher education practices, entering through mathematics is a fruitful way of developing mathematics teachers' aptitudes for teaching mathematics, because it is both effective to have conversations and explorations going on about what it is aimed for (the mathematical ideas themselves) and efficient for developing teachers all-around knowledge, offering them learning opportunities they would in turn be able to offer to their students.

On another level, there appears to be important reasons for why school mathematics should be a privileged point of entry for the professional development of secondary mathematics teachers, and why this orientation needs to be taken into account in professional development practices. One is that it addresses teachers' knowledge of school mathematics directly, and in my sense this represents the most fundamental form

[141] I need to restate a remark made in Chapter 4 about the chosen mathematical topics. It is important to realize that the mathematical topics were not chosen randomly and were chosen because they represented topics for which the presence of procedures was important, and therefore had the potential to raise issues about procedures in mathematics. In that sense, possibly not all mathematical topics would have been efficient in creating the same outcomes as happened in this research.

of knowledge for a secondary mathematics teacher[142]. In addition, on the basis of the studies concerned with secondary mathematics teachers' knowledge (e.g., Ball, 1990; Bryan, 1999; Even, 1993; Hitt-Espinosa, 1998), teachers need to develop more "conceptual" forms and enlarge their knowledge of school mathematics because it appears to be too narrow – and this plays a major role in their teaching practices. Hence, developing a robust mathematical background in teachers offers them a pool of possibilities, a rich basis for their instruction to draw from, and this surely orients their ways of teaching. Therefore, entry through (school) mathematics ought to have a precise impact on teachers' knowledge of mathematics – something that an entry through other approaches may well not have[143]. Finally, another reason is that, for the most part, secondary teachers appreciate very much working on mathematical issues and invest themselves thoroughly in it. This is no small point, because it is part of who they are as mathematics teachers – it is part of their structure – and it will be brought forth within the sessions.

For all these reasons, and the compelling case that they make, I believe that an entry through school mathematics is a very promising approach to secondary mathematics teachers' professional development, one that has the potential to impact teachers' knowledge, their classroom practices and their students' experiences with mathematics.

Closing this Research

Taking advice from Silver, Mills, Castro, Ghousseini and Stylianides (2005), the approach to professional development of secondary-level mathematics teachers that was researched here was not intended to enter into competition with other approaches to professional development in order to say that "this one" should be the one chosen and to put down other ones. I believe that this would not make it a very formative and generative contribution to the field. Rather, my intention was to find and elaborate on a fruitful way to address, through professional development practices, an issue for which there is little

[142] Ma (1999) asserts the same thing for elementary teachers.

[143] However, some other approaches have been shown to have an impact on teachers' mathematical knowledge, to a certain extent, by entering through other means (e.g., Bednarz, 2000, uses an entry through teaching practices which shows to have *some* impact on teachers' knowledge of mathematics).

research, that is, the issue of secondary mathematics teachers' mathematical knowledge and how to enlarge it.

A fundamental feature of the approach taken was its entry point through school mathematics, that is, through the mathematics that teachers teach; and how it offered and produced learning opportunities for teachers to enlarge their knowledge. Therefore, it is not the "how" things were done, but the "what" was done – to re-use Crockett's (2002) expression – that was important. The type of tasks that I offered (problems, students' solutions, situations, videos, etc.) did not matter as much as the mathematics underpinning these tasks and situations. Therefore, this dissertation aimed at providing a fruitful point of entry from which to work (a "what"), and not a way to provide professional development (a "how"). In that sense, it does not aim to tell "how to do" professional development, but rather to raise a sensitivity toward the fundamental importance of the mathematical content underpinning any approach to the professional development or teacher education of secondary-level mathematics teachers.

AFTERWORD

SOME THOUGHTS ON PROFESSIONAL DEVELOPMENT AND ON TEACHERS[144]

Finding secondary mathematics teachers to participate in my research was difficult. From December 2004 to June 2005, I met with school principals, teachers and pedagogical advisors in order to get the project going, somewhere. Sometimes it was not possible for teachers of a school district to participate, whereas in other school districts it was only possible on a smaller scale (a couple of meetings a year). On some occasions teachers showed a lot of enthusiasm and said that "this is what professional development should be about," and on others teachers showed little or no interest at all. However, I continued to knock on doors trying to not get discouraged and, finally, at the end of June 2005, I found sufficient teachers who were interested in the project and agreed to participate. This experience raised many questions for me about the importance and status of professional development for mathematics teachers. I felt professional development had an ambiguous status for teachers. I feel compelled to address this issue.

Learning about Researching and Conducting Professional Development

Research is not a non-disruptive process (Valero & Vithal, 1998). Even with the best intentions possible, it is not a smooth endeavour. I did not know that when beginning this

[144] This afterword concerns my personal reflections as a new teacher educator, and mostly concerns my personal musings and wonderings about teachers and professional development.

study. From all that I learned and read from research reports and professional development practices, I thought that when things were right, they were right. Hence, I believed that if my professional development project was relevant and good, everything would happen smoothly and everybody would be happy. Like a fairy tale. You can imagine that the difficulties I experienced in trying to find participants were quite disequilibrating.

Some people told me that I had these "fairy tale" thoughts, because research tends to hide the disruptions and difficulties lived and to trim away the messy parts and obstacles. Because, if not, the process reported on would look too messy and not straightforward enough to make it a compelling case for research evidence. In other words, it would not look like rigorous "research" material. Jardine (1997) challenges this view.

> Don't go backwards, don't turn away from these messy secret tales that no method can outrun and make all right, as if they did not *speak* to us, as if we did not *hear* them, as if the agencies of the world were always just our own. [...] Let's reclaim the word. This *is* research. (p. 165, emphasis in the original)

It is in that sense that I write this afterword, to expose some of the reflections and thoughts that I had while trying to organize and conduct my research on professional development. They too were part of my research journey.

What is the Status of Professional Development for Teachers?

Nobody would deny that secondary-level mathematics teachers have a busy schedule. A full teaching load is heavy. From nine to five, teachers teach, plan, supervise, coach, and do the many other things that teachers do. They have little time for anything else. Ironically, however, research is clear on the fact that teachers should engage in professional development. We are at a point of tension here.

In some places in the world, governments or school boards have made it compulsory for their teachers to have a personal professional development plan, in which they have to lay out their yearly intentions and activities. Teachers are obliged to participate regularly in professional development activities. On the other hand, in other places in the world, professional development is strongly recommended but is not mandatory. In those places teachers participate in professional development activities in a sporadic manner.

Obviously, my program was not compulsory for any participant. Although the school district made space for it and the teachers agreed to participate, not all the participating teachers presented themselves to all ten sessions. At first I was emotionally affected by this situation and I questioned the value of my program and my teaching practices a lot. But, in a contradictory fashion so I felt, I was constantly receiving appreciative comments from all the participants, explaining to me how they enjoyed and learned from the sessions[145]. Hence, I had to settle my emotions and distance myself from these events so I could reflect on them at a deeper level, in order to understand better their possible underpinnings.

I first saw the teachers' absences as one of time constraints, understanding that teachers had busy schedules and were trying within them to keep both ends together – as they explained to me on some occasions[146]. But after a while, I started to sense that "time" was only one part of the issue. In other words, I felt that the issue of time brought something bigger than simply "time" itself.

I felt that the issue of time constraints also brought a question of vision – vision of what is central to someone, of one's understanding of what is important in one's profession (here, teaching mathematics), and so on. I realized that one does not invest time only in things that one likes or feels is useful. A person invests time in an "activity" if it is considered *important* for one's life. (For example, I enjoy fishing a lot and I feel it brings me inner peace, but I do not invest much time in it because fishing is not that important in my list of priorities and compulsory issues to do in my life.) In that sense, I believe it is less a question of "not having the time," rather than not feeling that spending that much time on professional development is adequate or essential – or even that professional development is not that essential in the life of a teacher. Ball et al. (2001) suggest that for many, teachers' professional development is not perceived as important

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[145] I even received echoes of their appreciation from outsiders to whom teachers had spoken.

[146] Especially the teachers who left the program after the first session (the group went from ten to six teachers). Their main reason they gave for leaving was "time constraints." I also had personal interviews with two of them (Danielle and Nina) and again "time constraints" was the main issue highlighted.

or essential because teaching is "common sense." Hence, perhaps professional development does not have a high status of importance in the "tasks" or duties of a teacher of mathematics[147].

This interpretation that the teacher does not have time could be paralleled with teachers who explain not having the time in their classroom for reform-oriented approaches[148]. Again, this rests on the teacher's understanding and interpretation of what learning mathematics implies and of the nature of specific content. This interpretation is what will determine the investment that one will put into it, or not. If mathematics is perceived by a teacher as a set of procedures to apply, it would appear irrelevant for that teacher to spend time working on something else, even if it is required by the curriculum. It seems to be a question of perception. If investing yourself in a professional development project is something that counts for you and which you believe in because you feel it is important for your evolution as a teacher, then spending regular and sustained time in professional development outside of school hours does not become a burden. On the other hand, if professional development does not make the top priority list in your understanding of the duties and tasks of a mathematics teacher, then you will perhaps be less willing to invest that much time in it.

Therefore, it is not a matter of good or poor teachers, of being motivated or un-motivated, nor of being committed or un-committed. It appears to be a question of what teachers perceive to be important, and prioritize, for themselves as professionals. Maybe spending time on professional development is "needed," "hoped for" or "intended" for teachers, but it seems that these issues will have to be dealt with in the reality of teachers' lives who perhaps do not feel and perceive these issues in the same way.

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[147] I could also add, from having taught in secondary schools, that often teachers who invest themselves in professional development are seen by others as zealous or as teachers who acknowledge having problems and are in need of resolving them – like the idea that anyone who goes to a psychiatrist is troubled or weak. Teachers participating in professional development are not seen (by all) as professionals who do what is required of them as professionals, because it does not seem to be conveyed as an essential part of what being a teacher "is" for some. Moreover, some teachers see that trying to improve yourself implies a state of failure, instead of an eagerness to continue evolving and make things better than they are.

[148] For example, Even (1999) mentions this in regard to teachers who refrain from using rich problems in their classroom because of time reasons.

What is Teachers' Status in Professional Development?

In line with issues of what is valued, another issue that can be raised concerns the status teachers have within a professional development program. Conducting in-service sessions with secondary mathematics teachers appears to be an interesting phenomenon in itself. What I mean by this is that the teacher's role in a professional development learning space is not precisely defined, as it is in their everyday role as classroom teachers. What is the status of teachers in a professional development setting? Are they teachers? Are they students? Are they colleagues? Are they a combination of all of those?

Issues of identity construction of future teachers have been raised by the studies of Blanchard-Laville and Nadot (2000), where they discuss the passage from student to teacher trainee (in someone else's classroom) and to teacher in his or her own classroom, and how future teachers struggle as they try to situate themselves and understand who they are and where they belong in these different situations. They are in a stage that is referred to as "professional adolescence" by these authors[149]. Pimm (2003) also shares these insights concerning first-year teachers in France who divide their time in their last year of training/first year of practice between their own classroom and their university teacher education center. He relates a situation when a group of "teachers" arrived late after their lunch break and then their teacher educator lectured them on their lack of professionalism. An important tension is then perceived, where they are beginning professionals but are still being supervised by a teacher educator. This creates important ambiguities for "teachers."

The same ambiguous and confusing situation can be seen to happen in professional development settings, because the status of professional development is not very well delimited in relation to its place in the teaching profession. If teachers are considered "students" in a professional development setting, they may behave as students do in a teacher-student relationship. What does it mean? It means they might arrive late, miss sessions, forget their homework, "misbehave," and so on – as students naturally do even in the "best" classrooms. But if they are colleagues, maybe these are not possible

[149] And the title of their book (Blanchard-Laville & Nadot, 2000) is very eloquent in regard to the tensions and difficulties future teachers live in these instances: "Malaise dans la formation des enseignants" [*Unease in the education of teachers*].

behaviours. In these moments, conflicts of perception and mostly of expectations can happen between teachers and the teacher educator. I feel that this difficulty is due to the non-established status of professional development in the teaching profession.

Reflecting on some Implications

What does all this mean for professional development, and on a larger scale? It is well acknowledged that teachers cannot be forced into professional development; however, it is also widely agreed upon that all teachers should take professional development because it can be beneficial for them and their students. It feels as if professional development is stuck in an impasse. Some researchers assert informally that money is the problem, some say it is the lack of support, some say it is motivation or interest, some say it is a question of time. These are all well-thought-out and legitimate claims, and maybe they are all to be considered simultaneously. My previous reflections concerning "time issues" have brought me to believe that the issue lies more in the meaning that professional development has in teachers' lives. I elaborate on the implications of this here.

As I have said previously, I conducted interviews with two of the teachers who had to leave the program. The idea was to have a small discussion with them to know more about the reasons for their departure and to understand their situations and needs. I felt this could help me to understand general professional development practices better, and the teachers' relation to it. Obviously, one main issue highlighted by them was about lack of time.

One teacher, Nina, explained to me that she really enjoyed the first session (and she was indeed very active in it), but that she did not had enough time in her schedule to allow her to participate in a year-long project. She explained that her schedule was quite full and that even if they received days off for substitute teachers to replace them[150], they still had to plan for these days, and then afterwards to work hard to recover what students obviously did not do in these "substitute-teacher day." Having a substitute teacher did not seem to be that seductive an option for these teachers. As the school schedule goes during

[150] The school district was very helpful in my project by giving teachers two days off (paid) for their participation in the project – and made sure that many meetings fell within school district days off. Also, the school district provided complete meals in the meetings (which were normally held from 4pm to 7pm).

the year, all professional development has to happen after school hours or at sporadic places in the calendar on school district days – and occasional meetings have been shown to be too limited to have an impact on teachers (Crockett, 2002). In the end, as I have said, teachers' schedules do not allow them to participate in professional development easily. And in a sense it even restricts them.

It becomes difficult to know exactly where professional development stands and fits in, and how it is to be considered. Because teachers schedules are obviously tight, adding professional development to it (whether paid or not) just makes teachers' teaching loads heavier. Therefore, adding money, hiring substitute teachers or placing sessions after school does not appear to be a good solution for teachers. In other words, it appears not simply or primarily to be a question of support. I would also say it is neither a winning formula for mathematics teacher educators, because it implicitly says that we believe that professional development is an extra-curricular and extra-scheduled activity, which relegates the importance of the role of professional development to a lower level. If professional development is to be respected and have a place in teachers' lives – as research asserts it has to – it needs to be placed and scheduled differently. Until professional development is literally inserted in teachers' teaching load, in-service work will continue to be something to fight over and will continue to be seen as an extra-curricular activity to an already burdened schedule. Issues of time and of the status of professional development will remain, and nothing will change.

When I talk about creating a place in their schedule, I am talking about inserting professional development as an activity that teachers have in the same sense and at the same level of obligation as their teaching duties. As teachers have school periods in the day to teach, they should have school periods in the day for professional development. To do that, teaching loads have to be reduced, and that remaining place changed for professional development – to give teachers fewer courses to teach and to make place in their schedules for professional development (that is planned, regular and scheduled). Professional development should *not* be added to the teaching schedule, it needs to be part of it, to be inside the teaching load (within a teacher 100% teaching load). The culture of teaching has to be impregnated with professional development *if* professional development is to have a (prioritized and respected) place in teachers' lives. Obviously, this implies time, money, support and all possible recommendations together. But most of all, it requires a change in perception from teachers, and also from the entire community interested in teaching mathematics. Until it is acknowledged that professional development is important by means of giving it a concrete place (and not simply paying lip service to it), it will continue to be a sporadic activity for teachers, one which will not have much sustained impact on teachers and on their students' experiences in mathematics.

Concluding Thoughts

> I hope that you may rightly accuse me of making more of my experience than is warranted for it is quite the opposite that typifies most conclusions of empirical research in mathematics education. They are at best safe, provide little resonance with our own experience and leave us with little desire to open our eyes and minds widely upon experiencing similar events in the future. (Brown, 1981, p. 11)

The implications for the professional development of teachers that I have just highlighted may be wishful thinking. I am probably still too much of a novice to be wiser, but it is where my thoughts are for now. An experienced teacher educator would probably know better. Anyway, researching a phenomenon does not come devoid of questions about the phenomenon itself that is being studied. And this is where these thoughts came from, from thinking about the phenomenon studied while studying it.

> We do not do what we want, but we want what we do (Prinz, 1997, p. 155; in Riegler, 2005, p. 3)

My research on professional development was not a smooth process, I lived obstacles and bumps – as I have tried to indicate in this afterword. It was tough and even discouraging at times. But in the end, I learned a lot about teachers and conducting professional development, even if I (probably) would have preferred the process to be smoother.

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APPENDIX A

LIST OF "TRADITIONAL" ALGEBRAIC PROBLEMS GIVEN IN SESSION 1

1. Two different meals were offered at the cafeteria for lunch. We serve 3 times more hamburgers than pizzas. 212 meals were served. How many hamburgers and pizzas separately were served?[151]

2. Two pupils have a stamp collection. Alex possess 27 stamps more than Josie. If together they have 181 stamps, how many stamps each have in their collection ?

3. I have 380 discs placed in three places in the house. In the living room, I have 76 discs more than in the room. In the basement, I have 114 discs more than in the living room. How many discs do I have in each place?

4. 380 students are registered in three sport activities. Basketball has 3 times more students than skating, and swimming has 2 times more students than basketball. How many students are there for each activity?

[151] All problems are taken from Bednarz (1999), my translation. Used with permission.

5. 380 students are registered in three sport activities. Basketball has 3 times more students than skating, and swimming has 114 more students than basketball. How many students are there for each activity?

6. Three chidren play with marbles. Altogether they have 201 marbles. Claude has 23 marbles more than Andrew, and Lou has 112 marbles more than Andrew. How many marbles does each pupil have?

7. Two trains carry 588 people altogether. The first one contains 12 seat cars and the second one 16 seat cars. If both trains have the same number of cars, how many people can each train carry?

8. Paula and Mary have \$154 altogether. Mary augments her amount of \$20. Now, both have the same amount of money. How much did each have in the beginning?

9. 207 persons from many regions in the world were present at the last international congress on sport doping. There were 3 times more Americans than Asians, and 16 Europeans less than Americans. The number of Africans was 7 more than the double of the number of Asians. Find how many people each country had.

10. Luc has \$3.50 less than Mike. Luc doubles his amount, whereas Mike augments his by \$1.10. Now, Luc has \$0.40 less than Mike. How much does each have?

APPENDIX B

SESSION 7: OPERATIONS WITH FRACTIONS

Lana: I never understood why in division of fractions we multiply by the inverse.
Erica: It is not trivial to explain that!

Teachers were eager to work on fractions, and to know more about them and their teaching, mostly because they explained that their students experienced difficulties with it and they wanted to help them. Hence, I prepared a session focused on offering them different tasks on operations of fractions for them to explore.

Moment 1: First manipulatives – operating with egg cartons

To begin the session, I first gave them two types of egg cartons in which to place beads in (figure B.1).

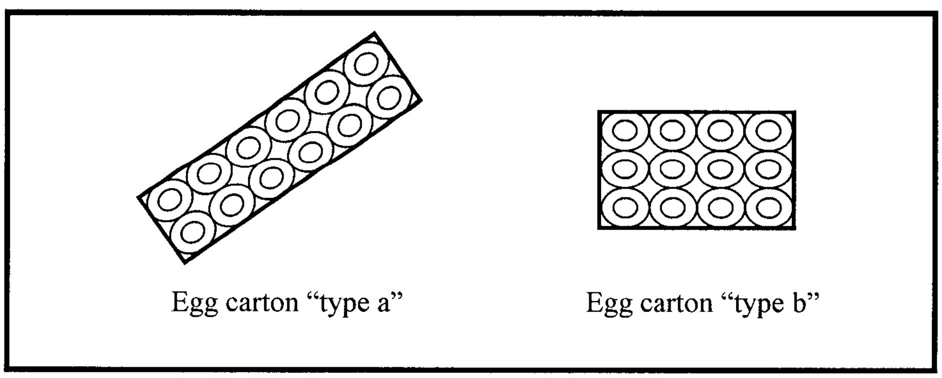


Figure B.1. The two types of egg cartons used

I began the session by asking them to represent ½ and then ¼ with the beads and the egg cartons, and then to show it in different ways. Figure B.2 shows some of these possibilities.

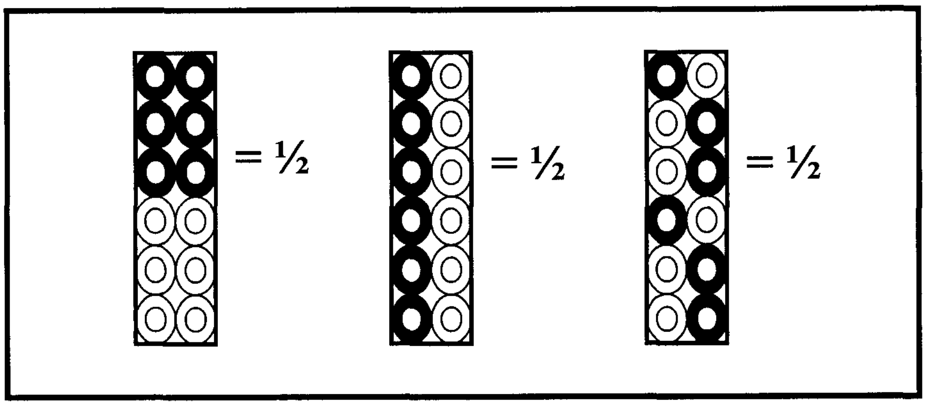


Figure B.2. Three different ways to represent ½

I followed by asking them to add fractions ¼ and ⅓ (¼+⅓). Each of them worked in their own way. This appeared to be a non-trivial task; one they were not used to doing.

Jérôme:	So, what does it gives you?
Lana:	I am not sure!
Carole:	Wait a moment [I am not ready yet]. [laughs]
Erica:	Me neither, I do not do that with this normally. I would first do it this way [showing me fraction computations with the common denominator algorithm on a piece of paper].

The teachers then explained how they arrived at $7/12$ with their egg cartons. Erica (and Carole) explained how she saw ¼ as 3 beads and ⅓ as 4 beads[152]. Lana explained that she separated the 12 slots into 3 groups of 4 and placed one bead in each of the 3 groups to create quarters. Then she separated the 12 slots into 4 groups of 3 and placed one bead in each of the 4 groups. Gina explained that it was quite easy for her because she had the second type of egg carton (type b), therefore one column was already given as

[152] To note, as they gave their answer, Erica and Carole seemed perplex as if they were doubting. Indeed, Erica laughed nervously and Carole said something that meant "Can that be it?" They afterwards explained that they were not used to work along that way and that they had to think a lot to do it.

representing a ⅓ and another one as representing a ¼. In the following figure, I display their different representations to add ⅓ + ¼ (figure B.3).

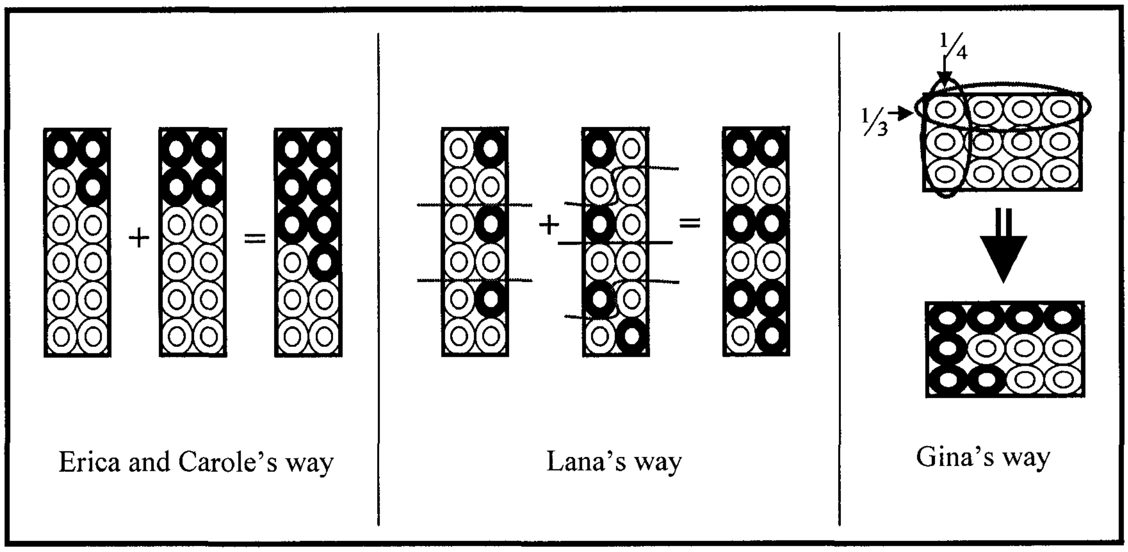


Figure B.3. Three different ways used to add ¼ + ⅓

I then flagged the fact to teachers that what I liked in this was that there is no need to find a common denominator, because it was rooted in the material, therefore it is given right away and we do not need to find or talk about it. In other words, there is no need in this material to use the common denominator algorithm to add the two fractions.

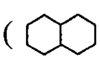
I followed by asking them to calculate $\frac{2}{6}+\frac{1}{12}$, which gave $\frac{5}{12}$ in the same type of answers as in the previous figure. After that, I asked them to compute $\frac{1}{6}+\frac{1}{3}$, which gave $\frac{6}{12}$, or a half as Lana said. With this answer, I took the opportunity to raise the fact that to obtain $\frac{6}{12}$ for the calculation of $\frac{1}{6}+\frac{1}{3}$ is quite interesting since normally the answer would be $\frac{3}{6}$ or $\frac{1}{2}$. In that sense, $\frac{6}{12}$ does not represent a "standard" answer to the question, but is still an adequate answer, hinting to the fact that there is not always a need to simplify the fraction.

Before moving to multiplication, I explained that subtraction questions would not be much more complicated and would follow the same procedure – underscoring the fact that addition and subtraction of fractions are complimentary operations and should not be isolated one from the other but mostly worked together. As I raise these ideas, Erica and

Lana talked between themselves intensively about the operations just computed and Lana explains to Erica how she succeeded in doing it. Erica and Lana seemed to be deeply involved into the mathematics they were doing with the egg cartons and fractions, and were somehow disconnected from the conversations going on among the group.

Then I asked them to do ⅓ times 2, which brought $\frac{8}{12}$. Again, that created a surprise, especially for Gina where she had to verify on a piece of paper with the standard algorithm to make sure that her answer was correct. I told her that the same thing happened to me when I first had to do it since we would almost never obtain $\frac{8}{12}$ as an answer to "⅓ x 2," where it would most of the time be $\frac{2}{3}$ – referring back to the same issue raised before for "$\frac{1}{6}+\frac{1}{3}=\frac{6}{12}$." Again, as I explained and discussed this, Lana and Erica were silent and completely disconnected as they were working through their solving. Lana then explained to the group how she did the operations by separating, as she did for the addition of "⅓ + ¼," each slots in 4 groups of 3 to create thirds, placed a bead in each third, and then doubled each of them making 8 beads in total. This brought a discussion about the plausibility of Lana's strategy, and its possible links to other ways of doing the operation.

Moment 2: A question of referent

This brings Gina to discuss how she does not like to use pattern blocks when she teaches fractions because the referent looks like two units instead of one (), which in her sense complicates the work with fractions. This difficulty with the idea of the referent brought me to talk about a previous question that had been brought in an earlier session by Linda (who was not present at this session).

How come when I add a ½ of a quantity to itself it suddenly becomes a ⅓ of it?

That question showed how the referent to which the fraction depends is very important because it changes the value of the fraction as the referent changes. As teachers recollected having worked on this question in a previous session, I offered them the following problem to push forward the conversations on the importance of the referent in fractions (adapted from Hart, 1981):

If Mary spends ½ of her amount and Johnny spends the ¼ of it, who spent the most?

As they answered that "it depends of the starting amount," all teachers raised the fact that this issue of the referent was a fundamental issue to understand fractions. Gina mentioned that she offers these types of problems to her students (with pizza contexts), and Erica explained having done it physically when she taught in grade 7 by using different sizes of cakes and distributing some parts of these cakes in relation to the fraction student chose in order to make them react (e.g., she gave a ½ of the smaller cake to a student, but ⅓ of the bigger one to another one, etc.).

Erica: And then, they understood what the "referent" meant.
Carole: It is always in relation to something.
Erica: In relation to the referent.

Moment 3: Continuing multiplication of fractions

I continued by asking them to do ⅔ x 2. Diverse answers were offered, going from $^{16}/_{24}$, $^{16}/_{12}$ to $1\,^{1}/_{3}$. The $^{16}/_{24}$ was finally dropped for $^{16}/_{12}$ and the conversation went toward the difficult reasoning that needs to take place in regard to the referent (again) and how it is not because the answer obtained is more than one that the referent changes. Each method to arrive to the answer was discussed and teachers explained how they preferred their method to another or how they had a hard time making sense of how another was explaining the operation. This brought the teachers to compare this experience with the work of students where students often opt for methods that they prefer even if they as teachers sometimes do not see these methods as the most efficient or easiest ones.

The next computation given was ½ x ⅓, for which teachers admitted to have a difficult time to make sense of. To help their understanding, I raised the issue that the "*x*" sign in the multiplication could be seen as meaning "of something" as in ½ of a ⅓ or a ⅓ of a ½. This appeared to simplified the way for some of the teachers as they were solving and attempting to make sense of it. In effect, by representing a ⅓ by four beads and taking the ½ of it, or by representing a ½ by six beads and taking the ⅓ of it, they ended up with two beads, which meant $^{2}/_{12}$.

I continued by offering them ½ x ¼, to which they strongly (but amusedly) reacted that "it does not work your problem!" because they needed to cut some of the beads in half. I however, using their amusement, pushed them by saying "we need to find an answer for it!" As they then plunged into it, these are the sorts of comments they said, demonstrating how hard they worked at it.

Erica: The ½ of a ¼. Well, it does not work because you need to take out a bead and a half. It does not work the half of a quarter.
[…]
Gina: Oh! Wait, wait, wait, I have an idea, I have an idea! I'll change, then I take half… Ha ha! I got it.
Jérôme: Ok, go on Gina.
Gina: Ok, wait a moment.
Erica: You got it! Are you sure?
Gina: Yes, but I think that I made it quite complicated to myself. I need to think about it in more details.
[…]
Carole: If you double, maybe it works?
Erica: Come on! You cannot make eights with twelfths, it is impossible!
Lana: Does it give $\frac{3}{12}$?
Erica: It does not work…!
Lana: Does it give $\frac{3}{12}$?
Erica: It gives $\frac{1}{8}$!
[laughs]
Lana: Eh boy! [discouraged]
[…]
Lana: That, it is ½. [showing 6 beads in her egg carton]
Jérôme: Yes.
Lana: I want a ¼ of this half. It means that of the four [beads], I take one.
Jérôme: Yes.
Lana: And this [pointing the two beads left], I take half of it.
Erica: This [pointing to the two beads left], you take a half of it.
Lana: It means that it is one and a half. It means that it is "1.5" over 12.
[…]
Erica: But, it is not over 12, I cannot give it "over 12."
Jérôme: Indeed, it is over 24.
Erica: It is over 24.
Jérôme: But, at the same time, you could say that it is one and a half over 12. But we know that we do not write it like that with fractions.
Erica: Indeed. It is one and a half over 12. It is the reason why I said that it was impossible with 12 since you need to take one and a half over 12.

I continued the discussion by again emphasizing that a response like $\frac{3}{24}$ is quite unusual for a ½ x ¼ question, since we would normally get $\frac{1}{8}$. This brings the discussion back to the importance of the referent which in this case changes from 12 to 24 (figure B.4).

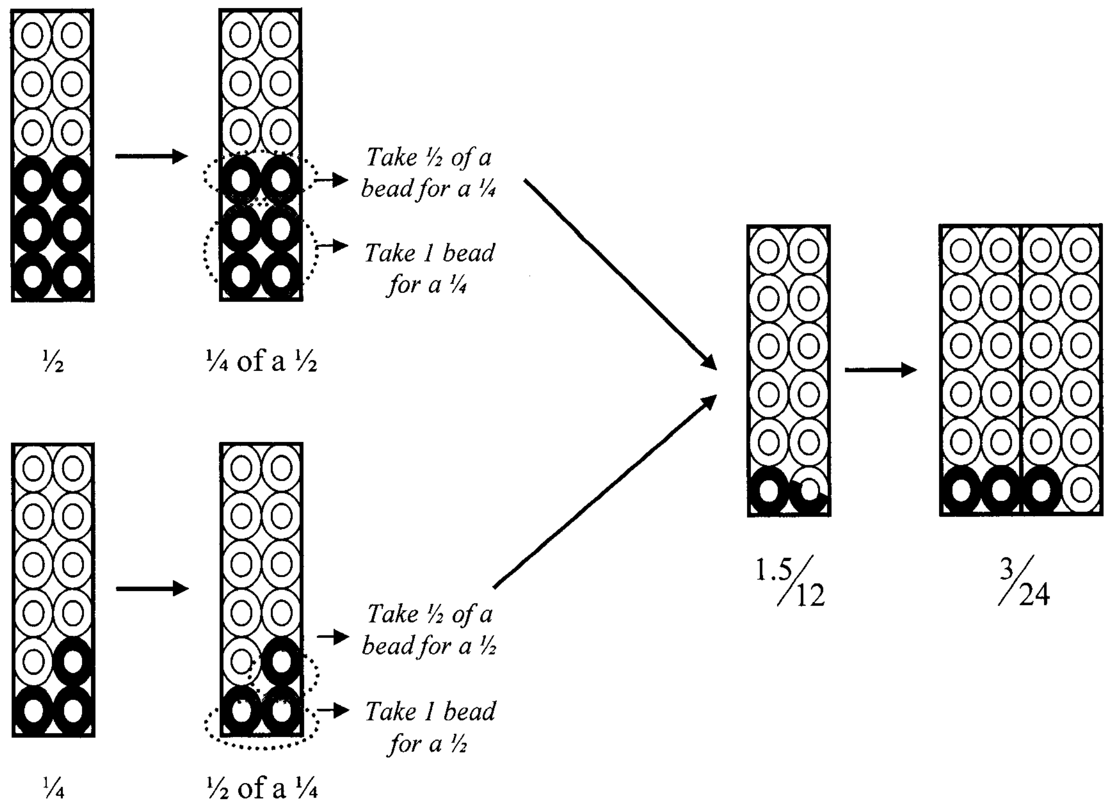


Figure B.4. Representing ½ x ¼ with egg cartons

Moment 4: Proving and understanding

This work on fractions brought Lana to say that in grade 10-11-12 courses she is able to mathematically prove everything that she teaches, but that in regard to fractions she realizes that she would not be able to prove it all, something to which Erica agrees[153]. Adding to this, Carole said that some students cannot continue to learn without understanding why a rule works, whereas some only take the rule and apply it. In line

[153] By proving, Lana meant to create a mathematical proof, with a theory or a theorem.

with Claudia's comment to Carole in the volume session[154], this brought me to raise wonder outloud if the students who only need the rule really understand what they do. My comment brought back the discussion about the difference of "techniques and reasoning," where only knowing how an algorithm works (e.g., multiplying the numerators together and the denominators together for the multiplication of fractions) is on some level mathematically insufficient. Carole agreed with this assertion, but added that she personally succeeded in school with only knowing "how" without knowing "why," but that teaching brought the need for her to understand the concepts more deeply and also that the program of studies required her to teach "with understanding."

I then added to this discussion of "techniques and reasoning" by offering them (for the first time) the diagram of the mathematical activity with its three branches (from Chapter 2). To add to Carole's comment that she succeeded in school without understanding "why" things worked, I raised the fact that traditionally mathematics teaching has concentrated on memorizing techniques and facts, and that the "understanding" parts of knowing "why" procedures worked and the making sense side of mathematics was not often the point of focus. This led Gina to say that mathematics teachers were often seen, with their multiple choice questions, as the ones with little marking and easy to mark exams because mathematics was indeed stuff to memorize and procedures to know how to apply. From her point of view, however, this important change to "meaning making" and "understanding" had also important implications for the profession/job of the mathematics teacher.

> Gina: I do ask myself the question up until what this is not a fact. Because, the reality is that if you go into "understanding" and there the students explains him or herself and it is sentences and it is paragraphs and all this, it changes enormously the marking of the mathematics teacher, the definition of what is a mathematics teacher, when you say that the answer is "1" or it is "1.5". There is also this aspect to take into account.

Supporting this change, Erica added that this is indeed where the diploma exams are going to with exams split in two parts, where one part is explicitly on reasoning and

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[154] See page 134.

explanations. She also said that this part is more difficult to mark because there is place for interpretation, to which she added:

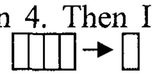
> Erica: It is a step by step process [to change the attitude toward focusing on understanding]
> Gina: It is steps toward "understanding" [pointing to the mathematical activity diagram].
> Erica: Yeah.

I ended the conversation by pointing to the fact that all the three branches of the mathematical activity were important, as teachers agreed that they all had a role to play.

<u>Moment 5: Areas and folding paper</u>

I continued by offering teachers to take sheets of paper and to do some multiplications of fractions by folding. I explained that I would be doing the representation of the folding on the white board by drawing it.

The first one that I gave them was ¼ x ⅓. Erica explained it in the following terms.

> Erica: I have folded my sheet in 4. Then I said ¼, then I kept ¼ [she shows one of the fourths: ]. Then I took my ¼ and I divided it in 3. So it gave me 1/12.
> Jérôme: [...] You did, in fact you did ⅓ of ¼, and ...
> Erica: Yes. Then I did the ¼ first, this was my ¼ and then I did the ⅓ of my ¼ [155].

The following figure describes what she did (figure B.5):

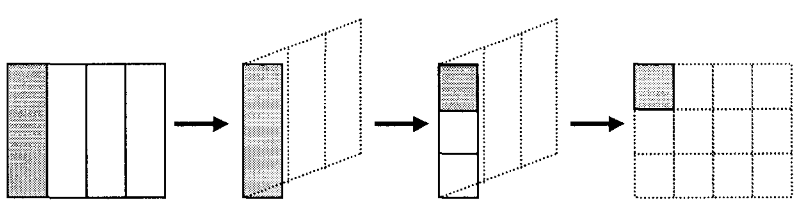


Figure B.5. An illustration of ¼ x ⅓ by folding paper

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[155] It is interesting to note Erica re-used the expression "of" to speak of how she multiplied fractions.

I then re-executed the same procedure as Erica, but inversely, that is, by doing ⅓ of ¼. Gina then offered another way to solve this, but in regard to area where she multiplied directly ¼ by ⅓ on the board, as in the following (figure B.6):

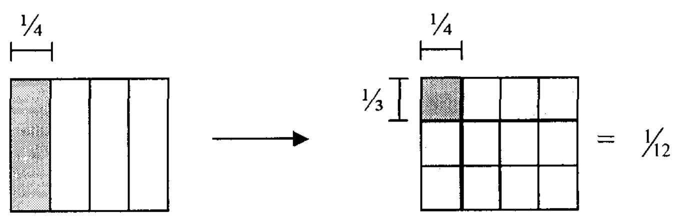


Figure B.6. Area multiplication of ¼ x ⅓

As I tried to link this explanation and way of doing to the previous one with folding papers to calculate ¼ x ⅓, Lana highlighted that it is indeed a very different way to proceed from the previous one of folding paper, since it was an area calculation and not simply a ⅓ of something else. In other words, it was really multiplying ⅓ and ¼ as trying to find the area of a rectangle from its two dimensions. This underscored that in fact it represented two approaches to multiply fractions: something "of" something, and area multiplication. In that sense, my representations on the board were more in line with the calculation of an area like Gina or Lana, than with Erica's "of" approach for folding paper[156].

I next offered them 2/5 x 3/4, which they executed in the same way (figure B.7).

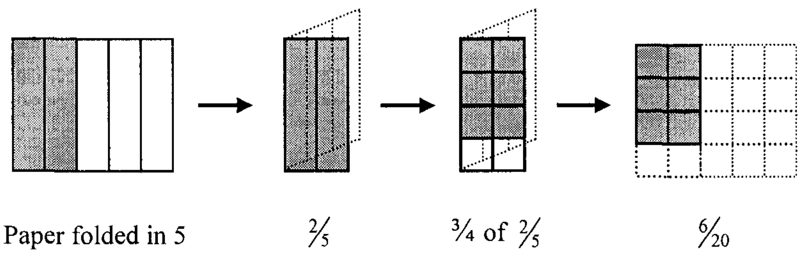


Figure B.7. An illustration of ¾ x 2/5 by folding paper

[156] This was a great learning event for me as the teacher educator. It is something that I had never thought of in these terms in the sense that these differ from one another conceptually. Indeed, Lana's comment made it clear to me how both could be linked but represented very different processes that did not used the same representations and explanations.

I also mentioned that we could do it commutatively, mainly 3/4 x 2/5 (figure B.8).

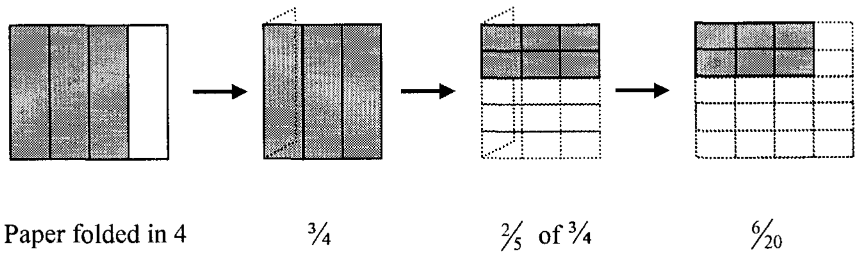


Figure B.8. An illustration of ⅖ x ¾ by folding paper

Or with the area as Lana and Gina previously did, which I re-explained to be a little different, to what Gina added "it is in the way to speak it that it is different," referring to the fact that you do not have to mention the "of" something (figure B.9).

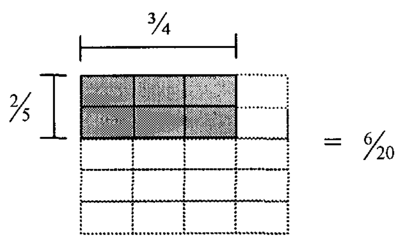


Figure B.9. An illustration of ¾ x ⅖ by area multiplication

Then Carole explained that she folded differently to get to the answer. She first folded into quarters because she wanted three of them, and then folded it in 1/5. Afterwards, she blackened 2/5 of each of the three quarters of the beginning (figure B.10).

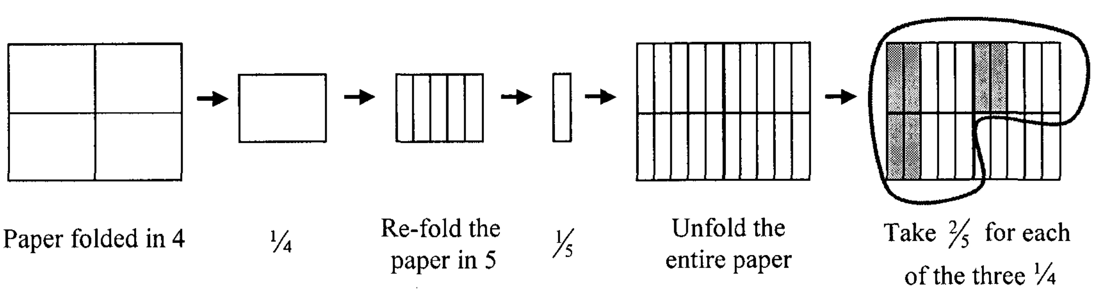


Figure B.10. Another way to fold the paper to represent ¾ x ⅖

She concluded her explanation by saying that "but we all arrived at the same answer, this is what is interesting."

Moment 6: Division by folding and area

To work on division, I gave them ⅓ divided by ¼.

Carole: Me, I am not able to do this. This, I am blocked.
[…]
Jérôme: And how do you arrive at finding the answer?
Carole: Me, I know the answer. 1 and ⅓.
Jérôme: How do you arrive at it?
Carole: Technique.
Jérôme: [as I write on the board] You do ⅓ multiplied by 4 over 1?
Carole & Erica: Yes.

Teachers were not able to explain the operation without using the algorithm. Gina attempted at explaining it by referring back to the beads and egg cartons and placing ⅓ in it (that is 4 beads), but instead she divided by 4 and not by ¼, and got $^{1}/_{12}$ [157].

Because nobody could find a way to do it, it brought me to have to show them how it could be done and explained. I however mentioned to them that I would speak of it in the "of something" way used in multiplication. In the following explanation, it is to note that all teachers were very attentive.

Jérôme: [I take a piece already folded in 3 and in 4, and I point to one column of four squares: ] This is worth ⅓.
Erica: Me, I said the third of four [squares].
Jérôme: There, you multiply.
Erica: Yes, I know. I did it by multiplying.
Jérôme: Here what I have in fact, and it is why I think it looks like how you spoke to it before [referring to the "of" in multiplication], is that I look for how many of ¼ is there inside of a ⅓.
Erica: Yes.
Jérôme: This is what a division is in fact, how many times ¼ enters in a ⅓. ⅓ is worth 4 squares. How many times ¼, which is 3 squares, enters in 4 squares? It enters once and one square. So, it is 1 and ⅓ of the quarter. Therefore, my answer is 1 ⅓.

157 This is a common mistake found in the literature on dividing fractions (see, for example, Ball, 1990).

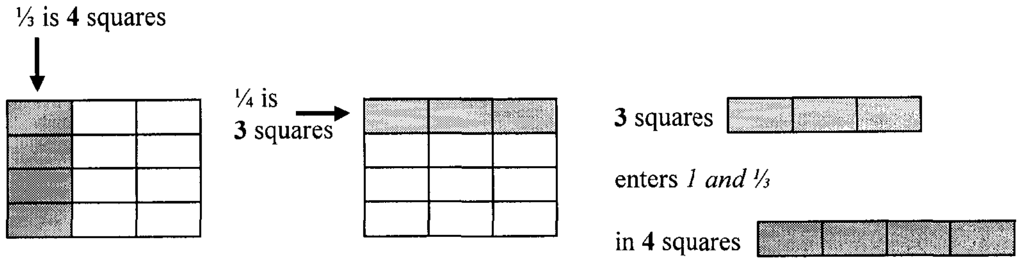


Figure B.11. Doing the division ⅓ ÷ ¼ using area

Gina: ⅓ of a quarter?
Jérôme: Yes, because how many 3 squares enters in this [pointing to a ⅓, that is, four squares]?

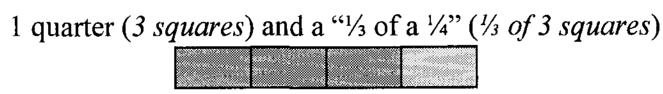


Figure B.12. A ¼ enters 1⅓ in a ⅓ *or* 3 squares enter 1⅓ in 4 squares

Jérôme: (continuing) It enters one time and a third. Each square is worth a third somehow.
Erica: Yes, that's it

This brought the discussion back to the importance of the referent.

Carole: Yes, it is because your referent is a quarter now.
Jérôme: Yes, this is it. It is "in relation to the quarter," exactly.
Erica: Yes.
Jérôme: How many times of … exactly. The referent is now a quarter.
Erica: Humm humm.

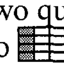
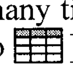
Then, Carole added:

Carole: A student that is going to understand this will really have a level of understanding of fractions that is superior.

I added to this that division of fractions represents a very difficult topic to understand and make sense of, which led Carole to nuance my assertion.

Carole: To do them not really, because it is easy with the technique. But to understand them…

As Erica said to have had many difficulties with the idea, Gina asked for a re-explanation concerning the "rest" of the division that we implied was worth ⅓, because she wondered what it was really worth where she would be more inclined to say it was worth $\frac{1}{12}$. As I tried to provide an explanation only in terms of squares by saying that three squares enters one and a third of it in four squares, Carole asked what would be the rest for the division $\frac{1}{3} \div \frac{1}{5}$. To begin this division, I started by splitting a grid into 5 and into 3, to which Lana, who had been very quiet since the beginning on division of fractions, objected strongly to my way of doing.

> Lana: Why do you say $\frac{1}{3}$? Why do you divide by 5 first? It is the $\frac{1}{3}$ that is divided by $\frac{1}{5}$.
> Jérôme: Yes.
> Lana: Why do you divide your sheet in fifths?
> Jérôme: Because I want to know what is a $\frac{1}{5}$ worth.
> Lana: No, but, when you read it is $\frac{1}{3}$ divided by $\frac{1}{5}$.
> Jérôme: Yes, yes.
> Lana: Therefore I would start … I have difficulties; well I am not able to understand what you are all doing.
> Jérôme: How would you do it?
> Lana: Well, I start with the first one. We did ⅓ ÷ ¼. I have drawn ⅓ like this [shows a part of her sheet divided in three parts].
> Jérôme: Then…
> Lana: And then I try to understand what it is.
> Jérôme: That is really fine. How many quarters will enter into this [pointing to the ⅓]?
> Lana: Yes, but what does ¼ represents?
> Jérôme: Well, this is exactly it! You have to find out how much does ¼ represents. What ⅓ is worth is this [pointing to the ⅓ on her sheet].
> Lana: Yeah.
> Jérôme: And ¼, I will try to find what it is worth. Well, if I take the same sheet and I divide it in 4 …
> Lana: Ha! Ok!
> Jérôme: One quarter is worth this [points to the three squares in the row: ]
> Lana: Ok.
> Erica: Humm humm.
> Jérôme: Therefore I am comparing two quantities in a sense. How many times *in this* [points to ] will enter my these ones [points to ]? Well, they enter once and one third.
> Lana: Ok, because, ok, ok, ok. There you have your ⅓ [looking at her sheet] and you tell yourself that you want to divide it by the ¼, but

	what does your ¼ represents …
Jérôme:	That's it, you have to find what your ¼ is worth because it is by it that you divide.
Lana:	Ok, ¼ is three squares.
Jérôme:	That's it, how many times three squares enters in four squares?
Erica:	Humm humm.
Jérôme:	One and one third[158].

The discussion continued, with teachers also explaining between themselves, and also completed the $\frac{1}{3} \div \frac{1}{5}$ operation.

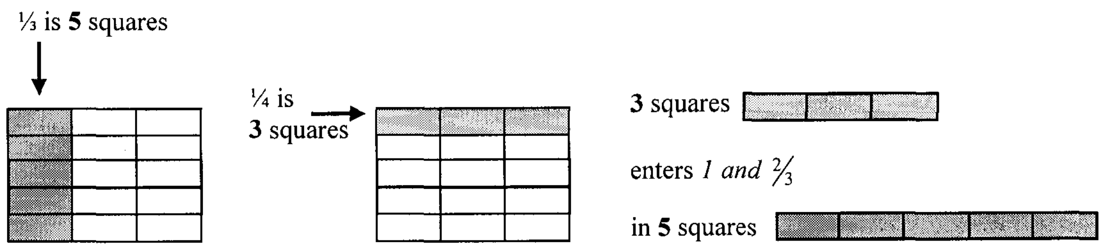


Figure B.13. Doing the division $\frac{1}{3} \div \frac{1}{5}$ using area

To step away from uniquely working with unitary fractions, I gave them the following division: ⅓ ÷ ¾. All excited, Lana shouted "$\frac{4}{9}$!" to which I agreed. Then, since teachers not being completely sure to understand it all and how to explain it, I did the operation on the board afterwards (figure B.14).

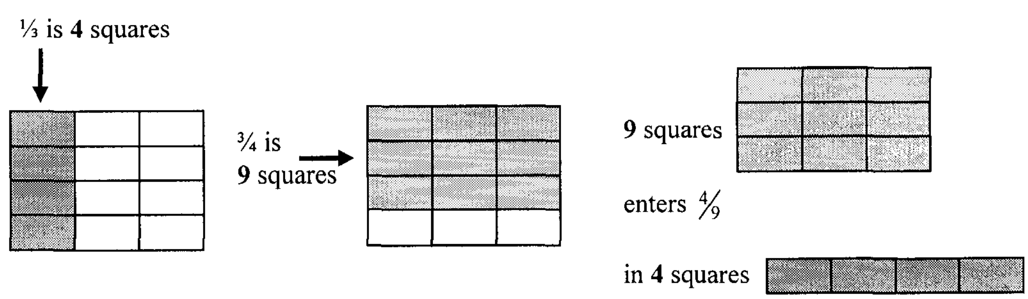


Figure B.14. Doing the division ⅓ ÷ ¾ using area

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[158] Two elements are worth noticing here. First, Lana's arising understanding of how it worked, but second my personal learning as to better explain and make explicit the reasons why I was splitting in 5 and in 3 to obtain fifths and thirds. I previously took for granted that in my explanations these reasons were transparent, but Lana made me realize the importance of placing an emphasis on, and of explaining, this.

Lana: There are 4 that fits out of the 9.
Jérôme: So, $\frac{4}{9}$ of 9 squares enter in 4 squares.
Gina: [excited] Humm!

Carole highlighted that it is easier when it is less than a whole, because it is the "rest" that causes the difficulties in the operation. She also raises the fact that students in grade 8 have enormous difficulties with "rests," where she feels that they get lost in it.

Moment 7: Word problem on division of fractions

To follow on Carole's comments and to complete the work on rests, I gave them a word problem on divisions of fractions inspired by Schifter's (1998) ribbons problem[159].

Mireille has 6 meters of material at her home. She wants to make ribbons that will measure 5/6 meters for a birthday at her school. How many ribbons can she make in total? And how much of material will she be left with? Explain how you know.

Figure B.15. The ribbons problem for division of fractions

All teachers got deeply involved in the problem, trying to solve it. After giving the answer (7 and $\frac{1}{5}$), the discussion centered around the value of the rest from the ribbons which can be as much as $\frac{1}{5}$ of a ribbon as it can be $\frac{1}{6}$ of a meter (figure B.16).

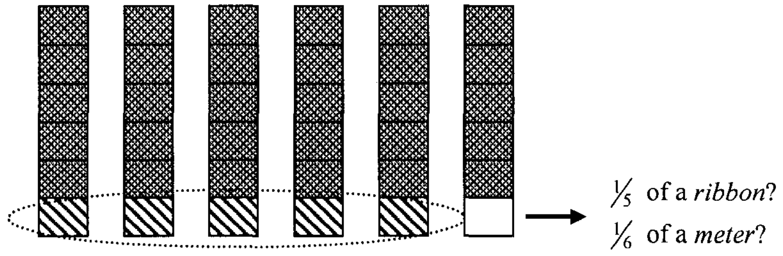

Figure B.16. Determining what is the last piece of ribbon worth?

Carole and others flagged the fact that this problem represented an excellent opportunity to discuss issues of the referent with students, because the answer was

[159] I am giving a direct translation of the French problem that I gave to the teachers, and not Schifter's.

depending on what it referred to. In that sense, it could be said to represent two values ($\frac{1}{5}$ of a ribbon and $\frac{1}{6}$ of a meter). This brought again a discussion of the importance of the referent in division of fractions (and in fractions in general). I continued by highlighting that this problem is indeed a division of fraction problem to which there is no specific need to go to the standard algorithm of "inverse and multiply."

Moment 8: Understanding and proving the "invert and multiply" algorithm

The conversation became focused on algorithms, and because Lana had previously commented that she could not always prove why it worked with fractions (in comparison to other grade 10-11-12 subjects), I offered teachers two ways that I knew of to make more sense of the algorithm of division of fractions. In other words, I offered them a way to better understand "why, when I divide ⅓ by ¼, I *can* invert and multiply?" "What enables me in mathematics to have the right to do this, to see that if I divide by ¼ it is like if I had multiplied by four?" The following line reflects the way I was speaking about how to make sense of the algorithm of division of fractions:

> Jérôme: In fact this is it. Dividing by ¼ is not multiplying by 4, but it ends up to be the same answer *as if* I had multiplied by 4.

First, I presented them the litres of water problem explanations:

If I have 6L of water to pour into glasses of 1L, it gives me 6 glasses of water:
6L ÷ 1L glass = 6 glasses.

If now the same 6L of water are poured into ½L glasses, it gives me 12 glasses, because my glasses are two times smaller so I need twice as much:
6L ÷ ½ L glass = 12 glasses.

If now I have ¼L glasses, it gives me 24 glasses, because the glasses are 2 times smaller than the previous ones of ½L or 4 times smaller than the first ones of 1L:
6L ÷ ¼ L glass = 24 glasses.

If now I have ⅛L glasses, it gives me 48 glasses:
6L ÷ ⅛ L glass = 48 glasses.

But if now I have glasses of ⅜L. These glasses are 3 times bigger than the ⅛L glasses, hence I need 3 times less glasses than with ⅛L glasses, which is 16 glasses:
6L ÷ ⅜ L glass = 16 glasses.

Figure B.17. The litres of water problem

I explained that what is realizable is that when we *divide* 6L in ½, ¼, ⅛, or ⅜, it gives the same thing *as if* we had *multiplied* by the inverse, namely by 2, 4, 8, or $8/3$. Not that "it is" multiplied by the inverse, but that it ends up giving the same answer. So, it is not the same thing or operation, but it gives the same answer. Therefore, 6 ÷ ⅜ is not 6 x $8/3$, but it gives the same answer. It is *as if* we had multiplied by $8/3$.

Completing this, Gina raised the fact that this explained the trick of "invert and multiply," but also that it made her realize that *it was* indeed a trick[160].

I then brought them another explanations to make sense of division of fractions and its algorithm and trick[161]. I explained to them that dividing by 1 is from far the simplest division that could be made. Hence, if I had to complete a complicated division, for example $5/12 \div 3/4$, I could use the concept of dividing by 1, or organize my operation so that I obtain a division by 1. If I represented the division like in the following and multiplied both numerators and denominators by 4/3, I would arrive at this:

$$\frac{\frac{5}{12}}{\frac{3}{4}} = \frac{\frac{5}{12} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{5}{12} \times \frac{4}{3}}{1} = \frac{5}{12} \times \frac{4}{3} = \frac{20}{36}$$

Jérôme: The answer that I obtain, since I have followed mathematical rules correctly ...
Lana: [to Erica] I have never thought about doing that!
Jérôme: Well then, it gives me $20/36$.
Lana: So in a sense, you can *show* that when you divide by a fraction, it comes up at doing the inverse.
Jérôme: Yes, it comes up at doing the inverse, exactly. So, dividing ends up at multiplying by the inverse.

[160] It is difficult to clearly understand what this reaction represented and exactly meant, because I did not pay close attention to it in the session, hence I did not probed deeper into what she meant. However, as I reviewed the tapes, it became clearer that Gina maybe did not see the "invert and multiply" as a trick before but mostly as a mathematical fact. I have to admit, as the teacher educator, that I had taken for granted the fact that they were all aware that it was a trick but were simply not able to explain it. In some sense, it appeared to be an illustration of how the procedures side of mathematics was engrained in Gina's way of understanding this concept, where the "invert and multiply" was not simply a trick, but the mathematics itself. My explanations for Gina had a double impact. It did not only serve to explain the trick for her, it also had her realize that *it was* indeed a trick!

[161] I thank my advisor Elaine Simmt, and also Tom Kieren, for introducing me to this idea.

Moment 8: Discussing the curriculum

This brought me to say that this small proof would not be very useful to teach beginning students on operations with fractions but that however for students of grade 10-11-12 it could be interesting.

This comment provoked a discussion about the place of fractions in the curriculum, and also on how the work on fractions differed along the different grades – where a fraction in grade 7-8 are defined in relation to a referent and are a part of a unit, and where in grades 9-10-11-12 it is more seen as an operation, like $a/b = a \div b$, which is not the case in grade 7-8. Hence, the different changes in the curriculum were discussed, in order to note where and how the conceptual change from a part/whole relationship toward an operation was done.

Moreover, because many of the teachers originally came from the province of Quebec, I brought in some differences between Quebec's and "Western Canada Protocol" curriculum, where in Quebec the fractions are worked on much earlier in the program. This brought also Lana to say that fractions should be taught at each level in the secondary grades, with questions requiring the use of fractions in them, because students are very weak in them. The session 7 ended around this discussion.

Moment 9: An interesting algorithm?

In session 8, to complete the work on fractions of the session 7, I brought an interesting procedure of a student to divide fractions[162]:

$$\frac{3}{4} \div \frac{1}{2} = \frac{3 \div 1}{4 \div 2} = \frac{3}{2}$$

Teachers were very surprised about this procedure and the fact that it worked. They wondered why this was not used all the time in schools. I explained to them that there were instances where this procedure did not bring much information on the answer. For

[162] I thank David Pimm and Mary Beisiegel for introducing me to this idea.

example, when there were no common factors between both numerators and both denominators as in the following:

$$\frac{3}{4} \div \frac{2}{5} = \frac{3 \div 2}{4 \div 5} = \frac{3/2}{4/5}$$

As is possible to see, in that case, it does not bring much insight into the answer, where it is still stuck with a division of fractions. However, the algorithm itself is always true and mimics the path taken for multiplying fractions, which is to multiply the numerators together and the denominators together. In this case, it is to divide instead of multiply. For it to give insights into the answer, common factors are needed between both numerators and between both denominators, so that the numbers simplify each other.

This brought Gina to raise the following point on the usage of common denominators and the creation of a generative way to operate on fractions.

> Gina: The other question that I ask myself is why we did not consider putting them on the common denominator? We show students to do addition and subtraction with the common denominator and then suddenly when we work with multiplication and division we take the common denominator out.
> Jérôme: It is because it does not change much [to the answer to do the common denominator].
> Gina: But it also works[163].
> Jérôme: Yes, but if there I have ¾ x ½, even if I put it on 4, how will that give me an advantage?
> Gina: It gives 2/4.
> Jérôme: It will give me 6/16. That I multiply four by four, or four by two …
> Gina: Yes, yes, it did not give much more, but …
> Jérôme: Does it change something for the division?
> Gina: No, it also works.
> Jérôme: That is, you mean, to give one way to do to everybody, to always place on common denominator to do operations?
> Gina: [laughing] Just to play the Devil's advocate here, because you were looking for another method.

[163] It is interesting again to see Gina's incline for a general way to solve and simplify the issue under a general "way of doing," as discussions in Chapter 5 and 6 led me to highlight.

However, this idea of Gina made its way and brought a realization that if the fractions are placed under the same denominator, the previous procedure would always work.

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \div \frac{2}{4} = \frac{3 \div 2}{4 \div 4} = \frac{3 \div 2}{1} = \frac{3}{2}$$

One example that did not work "before" was tried it with this new idea from Gina:

$$\frac{5}{6} \div \frac{2}{3} = \frac{5}{6} \div \frac{4}{6} = \frac{5 \div 4}{6 \div 6} = \frac{5 \div 4}{1} = \frac{5}{4}$$

This surprised me, and I even uttered out loud a "Oh!" and Gina became very excited[164].

Gina: Hey! Hey! Bingo!
Jérôme: We just found an algorithm!
Gina: [laughing happily] I bet you it will even always work, because you will always get a "1" in the bottom. This is crazy!

Other examples were checked to make sure it was right. We were all amazed by what we had developed. And indeed it worked, because placing under the common denominator leads both denominators to cancel each other out when dividing them, creating a division by "1."

The conversation ended with a small nuance concerning the overarching usage of the common denominator algorithm. As I explained them, I had some reservations to always refer to the common denominator because it became a "formula" or an algorithm that often gets applied without much understanding and in an automatized way. Agreeing, Carole added to these ideas by saying that it makes the work error prone for students because they just multiply everywhere without reflecting. Hence, even if a new mathematical idea had been developed about division of fractions, some concerns were shared about the overarching usage of the common denominator.

[164] This is another example of something that I learned as the teacher educator in this session.

APPENDIX C

EXAMPLES OF ANALYSES OF MATHEMATICAL PROBLEMS

On two occasions teachers were asked to bring good or interesting problems concerning the topic of discussion of the session: in session 2-3 on volume of solids, and in session 4-5 on writing algebraic equations from word problems. Everybody had to present their problem for the group to solve, and then explained why they had chosen it. The problems were afterwards analyzed and discussed. In this appendix, I provide two examples (summarized description) here. The first example is from session 2-3 (the optimization problem), and the second from session 4-5 (the telephone problem).

C.1: The Optimization Problem

Moment 1: The problem

Erica offered an optimization problem concerning volume that she uses in her teaching. The problem is the following:

Find the biggest rectangular box, with no lid, with a square base and of total surface area of 3600 cm^2.

a) Find the constrained equation;
b) Find the size to maximize and express it in relation to one variable;
c) Find the dimensions of the biggest box;
d) Find its volume.

Figure C.1. Erica's optimization problem concerning volume

As she offered it, everybody found it quite difficult. The discussion diverged from the solving, and rapidly turned to Erica's reasons for choosing it.

Erica explained that she wanted to take a problem linked to the reality of her teaching. She explained that it was a problem for which students needed to know well the concept of volume, a concept worked in grade 7-8-9 but not re-worked on after. Therefore, when students attempt to solve a problem like this, they experience many difficulties. In that sense, the difficulties she highlighted for her students were not necessarily concerning the concept of optimization, but of volume, area, building a box without a cover, and so on. Something she said younger students can do, whereas her students have difficulties with it. In that sense, she explained spending a lot of time re-teaching many of these concepts to her students – something she does find surprising and problematic – in order for the students to arrive at creating the equations to solve the optimization problem. She concluded by saying that she thought this problem was interesting because it regrouped many concepts from many grades, going from grade 6-7 with constructing a box without a cover, to finding maximums and minimums in grade 11, passing by area and volume.

Moment 2: Links between square units (units^2) and cubic units (units^3)

To keep a trace of all that she had mentioned, I decided to have her repeat the main ideas as I wrote them on a flip chart in front of the room. As I took notes and tried to report on all she said and had previously said, I recalled an issue she had previously flagged but not mentioned yet, so as to make her elaborate more on it.

Jérôme: There was also the idea that surface area and volume are not the same thing, no?
Erica: Indeed, they have to understand that there is a difference between surface area and volume. That square units represents a surface …
Jérôme: Yes, square units, and …
Erica: Cubic units.

Gina then highlighted that there were important suppositions concerning students' acquired knowledge for that question.

Gina: But this supposes that in grade 8 and 9 they have studied the relationship between volumes and surface areas.

Erica: Yes.
Gina: Well, we mostly compare area with perimeter.
Jérôme: That is interesting.
Gina: But we do not look at area and volume. Neither in grade 8 nor grade 9.

This prompted a discussion on the fact that the relationship between perimeter and area is worked on, but not between surface area and volume in grade 7-8-9, and that this is an important mathematical notion to work on. This led me to highlight the fact that in Erica's problem, the link between square units and cubic units is quite important to understand the problem and solve it, hence it is something important to work on in teaching.

Erica then flagged that she had not brought that problem so that other teachers felt bad about their teaching. Carole and Gina agreed on the spot and even acknowledged the importance of being aware of these issues so that their teaching can be affected.

Carole: No! I am very happy that you underline that, this is good.
Erica: Because he [Jérôme] told me to bring a *difficult* problem[165] ...
Jérôme: Indeed, indeed.
Erica: So, this is what I did.
Gina: But there are reasons why this problem is *difficult*, it is because we do not do it [link between and passage from surface area to volume] in grade 8 or in grade 9.

<u>Moment 3: Defining vocabularies: "Biggest"?</u>

Carole then asked Erica what is meant by "the biggest one" in the question, asking if it meant the larger one or the taller one[166]. Erica explained to her that it means "the box with the maximal volume," whatever its dimensions. Claudia wondered if this specification of the bigger volume should not be mentioned in the *part c* of the question,

[165] In fact, I gave the same directions to all teachers concerning the problem with the same e-mail.
[166] The expression in French was "*la plus grande*," which most of the time refers to the taller one, but in that case it was not very obvious what it meant.

so students could be aware of what is required from them. Erica replied by saying that it is implicit in the question[167]. I then supported Claudia's claim.

Jérôme:	But it is true what you say Claudia, because the "biggest one" [I make a gesture going up above my head], it is often in height.
Group:	Yeah, yeah.
Jérôme:	And it is one of the possible dimensions [for the box] in fact.
[...]	
Jérôme:	Maybe if they had written the "largest" box it would have been better[168]?
Erica:	It's, yeah, maybe the "largest," that's it.
Jérôme:	I don't know. But I think it is a very good comment Claudia.
Gina:	It is the box that has the biggest volume.
Jérôme:	Yeah, yeah, that's it.
Carole:	Well, it is just that we need to well define what "biggest" means ...
Erica:	Yes, big, really it is the biggest.
Carole:	Because "largest" could cause the same type of problems for other persons.

Moment 4: Changing the dimensions

Erica then added another issue to the idea of going from square units to cubic units (i.e., from surface area to volume). She felt textbooks do not often focus on the inverse operation, that is, going from cubic units to one dimension units (units^3 to units^1) or square units to one dimension units (units^2 to units^1) (and of course from units^3 to units^2). Erica clarified by saying that it is like if students always go forward, but never backwards. Gina agreed, but then Claudia interrupted:

Claudia:	Like if we have a volume and what is the radius?
Erica:	Yes, what is the radius.
Claudia:	That's not true, we do that.
Erica:	I know you do it, but it is not done a lot.
Claudia:	Ah, everyday [laughing]!
Erica:	Everyday! Liar [laughing].

[167] It is not clear what Erica meant here, in the sense that she could have meant that it is obvious from the context or that it is something that is not made explicit by the problem. In French, the word "*implicite*" can have both connotations.

[168] In French, I said "*plus grosse*," which seems to work better than "largest" for its sense here.

Carole disagreed a little with this and explained that it depends of the textbook, since some do it more that others. Erica agreed and added:

> Erica: Often I realize that students are weak when they need to go in an inverse mode. They are linear, and they always go from top to bottom, they never go from bottom to top, that is, starting from the result instead of the initial situation.

To resituate and clarify what Erica meant, because there seemed to be a small disagreement, I added the following explanation.

> Jérôme: It is not a critique of what is done [by teachers or textbooks], but mostly that it would be important that students be able to do it.
> Erica: That they be able to do it, that's it.
> Jérôme: Because they are not really skilled when they need to do it.

<u>Moment 5: The curriculum</u>

This brought Carole to raise the issue of communication between school years.

> Carole: But you know, what I realize is that we are making very good comments and it is something that lacks from the program revisions and in the revisions of resources. There is not enough communication between teachers. How can we know what we do not do well with students in grade 9 if we do not have any contact with grade 10?

This stimulated a discussion on the fact that in schools where all the different levels are present, teachers from different grade levels can communicate between each other. However, for most grade 7-8-9 schools, mathematics teachers are often the only mathematics teachers hence they mostly do it for themselves since they teach all grades, but without an access to grade 10 to 12.

Carole added that she did consult grade 10 programs to know more about what is done in this grade, something she asserted every teacher should do. In fact, she mentioned that she had not done it when she only taught grade 7 and 8, and that the first time she taught grade 9 she realized that she had not focused enough on fractions in grades 7 and 8, which caused many difficulties for her students because they needed to master fractions in grade 9, the program taking for granted that it was a well acquired skill from previous years – it was to be used, not studied. She thought it was taught at other grade levels, so she worked on them normally, but if she had knew that grade 8 was the last time they worked on it, she would have worked on them differently, as she now does.

Teachers then underlined the importance of that communication between the years (in regard to the curriculum topics taught and how other teachers work with them) to establish a better continuity (a bridge between topics) and be aware of the other grade level topics so to teach in a better way and place important emphasis on specific topics to prepare students for the following years.

Carole completed by re-stating her previous remark on the importance of Erica's comment about the importance of working with surface area and volume (square units and cubic units), because it is important to know more about the important topics from following grade levels to work on, something curriculum and textbooks should emphasize more in her sense. She also added that it would be important for curriculum to underline to teachers which topics are studied for the last time, where it would not be studied but utilized in the subsequent grades, possibly making teachers place a grander emphasis on them. This closed the discussion for this problem, as Gina brought hers.

C.2: The Telephone Problem

Moment 1: The problem

For session 4-5, teachers had to bring a good word problem that required the creation of an algebraic equation. Claudia offered the following problem:

> You want to make a phone call from Edmonton to Vancouver. This phone call costs \$4.36 for the first three minutes, and \$0.95 for each more minute. Create an equation permitting you to calculate the cost of your phone call if it lasts (m) minutes.

Figure C.2. Claudia's telephone problem for writing an algebraic equation

The group then tried to solve the problem, or mainly to create an equation for it.

Moment 2: The reasons for choosing it and its difficulties

To begin the discussion on the problem, I asked Claudia to explain her reasons for choosing this specific problem, to which she responded with the following.

> Claudia: I have chosen this one, not because … Well, I personally find that students have a lot of difficulties with any sort of word problems. They are difficult for them.

Claudia explained that creating algebraic equations from any word problem was a difficult task for students. In fact, she was the one who suggested that a session was spent on this topic because she felt it was very difficult one for students. She even added that because of that, she chose the problem at random in the textbook. However, by looking at the solution and discussing the problem with others, she realized that it was a difficult one. This brought Claudia to show the algebraic equation that represented the problem.

$$\boxed{\mathbf{4.36 + 0.95\,(m - 3)\ ;\ \ where\ \ m \geq 3}}$$

I then flagged the fact (surprised) that it was not \$4.36 for each of the first three minutes, but \$4.36 for the entire first three minutes, to which Linda added:

> Linda: That you speak 2 seconds or 180 seconds, it is \$4.36.
> Gina: Oh [surprised]!
> Jérôme: And then, it is \$0.95 each minute, so you do "(m – 3)."

Lana then highlighted that the difficulty for students was the fact that if the amount of minutes is between 0 and 3 then this equation does not function anymore. This brought me to say that I tried to enter the "0<m<3" condition *into* the equation but I could not find a way to achieve it, hence it needed to be expressed in a double equation bracket[169].

Erica added that it is the concept of inequality that was problematic for students, where the "condition" of the problem is an inequality and that this problem talks about both equalities and inequality – making this problem about two concepts.

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[169] The two sets of equations were not explicitly written, but mostly talked about (e.g., Linda's explanations). In fact, the set of equations is Price=4.36+0.95(m-3) if m≥3 *and* Price=4.36 if 0<m<3.

Moment 3: Clarifying the implicit/assumptions

This brought Gina to wonder about how the problem was formulated concerning these "conditions," and if more should not be said in it[170].

Gina: But in the formulation of the text of the problem, if we had add a sentence like "at the moment that you start your phone call it will cost you your $4.36 independent of the number of ...
Jérôme: Ah, you mean in the statement?
Gina: In the statement of the problem, if we added a sentence it would simplify it a lot.
Claudia: It is quite clear that things are taken for granted here.

Claudia continued by highlighting that there are things taken for granted or implicit "in" the problem, for example this issue but also that if you speak 3 minutes and a little it goes directly to 4 minutes, and if you speak 4 minutes and one second it goes to 5, and so on. Hence, it is taken for granted that it is "round up" to the next entire value. I underscored that indeed there were many implicit ideas and assumptions in word problems that were not always obvious to decode. Erica continued by explaining that students have difficulties with restrictions within a problem, like a negative amount of apples where your values (or the "x") needs to be bigger than 0. To which Lana added:

Lana: When you tell them, they understand. However, they do not realize it necessarily.
Erica: They understand it, but they do not realize it because the inequality, the condition, is not written, it is implied, as here.

The discussion continued on students' difficulties to reflect on it, *and* also about the fact that it makes it even harder because the conditions vary from one problem to another.

Moment 4: Inequalities

This led Gina to ask about when are inequalities studied and worked on in the curriculum, because she says that in grade 9 they do not really see it except very rapidly with the numeric line. Lana and I added that we had talked about it the day before (in an

[170] This appears to be reminiscent of the previous discussion on "bigger" in the optimization problem.

individual meeting) and that they had been taken out of the program. Lana added that they were seen in grade 11 applied mathematics, but only in regard to the graph.

This brought me to say that graphically there are important reasoning to develop in regard to which part of the graph satisfies the function or not. However, Lana added that most students learn to recognize by heart that "if it is >" the answer is the top part and "if it is <" the answer is the bottom part. Therefore, students do not always understand that the points in the chosen zone will always satisfy the condition, because it is automatized – something to which Erica agreed. This led me to say the following.

> Jérôme: So, even if there are opportunities to arrive at making sense of constraints, it is difficult to see them …
>
> Erica: Oh yes, they will put a line and they will say higher is ">" and lower is "<".
>
> Lana: And then, they will look for points. But do they really understand that when you blacken what is up, it means it is *all* the possible answers to your equation?

Lana continued by saying that some students understood what they did, but not all of them did. To support this entire idea, Erica gave the following example.

> Erica: I have them do a system of inequalities. I give them three lines and have them draw them and then I ask them to write possible values. And then, it's like "Well, which value you want me to find?"; "Well anyone!"; "Yes, but, how come anyone?"; "Anyone that works."; "Yes, but, where is it? It is not an intersection point."; "No, in the area"; "Well, how do I do that?"

I drew the graph of the example on the board (following her directions) (figure C.3).

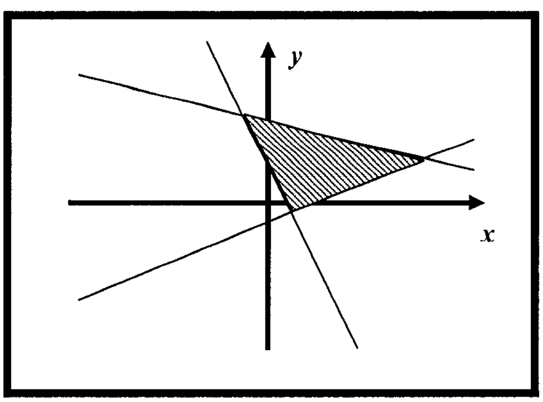


Figure C.3. Erica's graph for finding a point in a region

Linda added that students are normally ok with points along the line or at the intersection, but not with points within a region or area. Lana and Erica agreed with that fact. Gina then made a remark to the possible answer for Erica's example.

> Gina: [with a questioning tone] He could take any point that is within the region, in fact.
> Erica: Well, anyone that is in the region is a good answer.

Moment 5: Mathematics questions often have only one solution

Gina's student teacher, Holly, then added that maybe this difficulty came from the fact that it is the first time for students that, in mathematics, there was not only one possible solution, but many. Erica disagreed with this because she mentioned that in grade 9 students do see it with inequalities on the number line as Gina previously asserted. Gina and Lana replied right away that they do not work on it that much though, to which Erica said that at least it is aiming for more than one solution.

At that same time, Linda was writing explanations on the flip chart and then said that students are used to and are comfortable with, in mathematics, the idea that there is one and only one answer, and that the equality sign ("=") is not a relation for them but represents "it gives." Holly then rephrased what she meant.

> Holly: No, but, what I mean is that it has been about 10 to 15 years that they believe that mathematics are absolutes …
> Gina: And all of a sudden, for …
> Jérôme: Then there is a world of possibilities.
> Erica: Yes, I understand, but they do it anyway in grade 9, grade 10, and now they are in grade 11.

In front of possible misunderstandings in relation to the fact that "they saw it," I intervene in the discussion to explain the assertions.

> Jérôme: But just wait Erica, Gina seems to say that it is not really worked on this part.
> Gina: It is so small the amount of work we place on inequalities in grade 9.
> Erica: They see it again in grade 10 after, they do diagrams, number lines.
> Jérôme: They do it on graph, but as Lana said, it will be easy for them to

say "bigger, we look above, and smaller we look below."
Erica: But they cannot on a number line, they will say to the left or to the right.
Jérôme: Yes, yes.

This brought the conversation on the fact that even if there are many solutions possible on the number line, students have difficulties to make the links between "the left of the graph" and "any point on the left represents a possible solution." This led to the realization that maybe this is where the difficulty is, because there appears to be a conceptual difference between knowing what is bigger *and* realizing that all of them are solutions, and that there is an infinity of solutions. Many discussions about this issue continued, going from students' capacities to handle such issues like many possible solutions, to the fact that students are not often asked to do mathematics in these terms in school. I then said that I was quite surprised to learn that some students learned by heart the region solicited by the inequality signs without understanding what they did, since I never thought of it and even wondered if this trick worked all the time and if there was not a counter-example to it. Lana and Erica mentioned that there could be counter-examples with curves.

The idea of curves brought Lana to discuss another type of students' difficulties concerning the delimitation of a region. For example for the following curve (figure C.4),

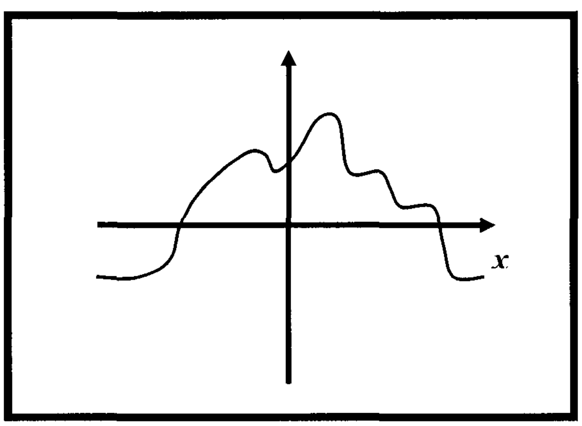


Figure C.4. Lana's example of a curve

Lana explained that some students would only blacken the upper part of the graph (for a ">" sign) that is delimited by the "x" axis and would not draw lower than it (figure C.5).

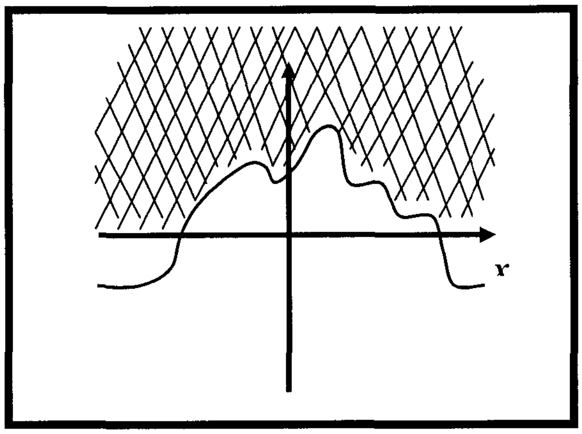


Figure C.5. Representation of students' difficulties with delimiting a region

This led me to say that it is indeed quite interesting to notice that from all the examples that had just been giving (numeric lines, graphs, Venn diagrams, regions, etc.), there are many and even plenty of opportunities offered for students in these topics to make sense of the implicit idea of constraints and diverse possible solutions. However, it is not obvious that these opportunities are taken up in teaching (i.e., that they are taken advantage from and underscored), and mainly stayed at an implicit level.

This brought Carole and Claudia to discuss these ideas in regard to sets of numbers where they are nested within one another (e.g., natural numbers are within rational numbers), leading to the fact that one number can be in more than one solution set. Carole mentioned that her students have a lot of difficulties to make these links, and that they often act on automatisms (e.g., because of the nestedness, some students think that all sets of numbers are nested, therefore placing a natural number in all other categories even in irrational numbers as if irrationals encompassed rational, naturals, etc.).

Moment 6: Issues of language and translations in mathematics

This led Linda to flag a difficulty in the west with the Francophone and Anglophone documents because in English there is no N* set which in French represents the natural numbers without zero. Hence, solutions to same examination questions are not always the same in French and English, because the same convention does not exist in both languages. This led to compare both sets of numbers and to look at the different representations and conventions used in French and in English, and what they meant.

Moreover, Carole added that it is often perceived that it is only a matter of translating from one language to the other, but however there are indeed important differences. I brought up an issue Erica and I had talked about earlier concerning the way questions in mathematics were asked in English and in French. Indeed, Anglophone phrasings did not always apply in French, especially with the often used Anglophone formulation of "if … then," which is not that common in French for mathematics questions. Linda agreed and mentioned that French questions are often longer than Anglophone ones, where attention needs to be paid to some ambiguities and subtleties that are sometimes not important in English, and vice-versa. This summed up the events for the problem of Claudia, as Lana presented hers.

APPENDIX D

EXAMPLES OF ANALYSES OF STUDENTS' SOLUTIONS

Tasks within a session sometimes consisted of analyzing students' solutions to specific problems. I provide three examples here. The first one is from session 4-5 on writing algebraic equations from word problems, and the second and third are from session 6 where the session was on evaluating students' work.

D.1: The Algebraic Word Problem: Student Brigitte Solution

Moment 1: Offering a problem and the student's solution to it

I began by offering this situation to teachers:

> We gave the following problem to Brigitte:
>
> *"I go to the store and I buy the same number of books as discs. The books cost two dollars each, and the discs cost six dollars each. I spend 40 dollars in total."*
>
> Brigitte answered the following:
>
> > 2B + 6D = 40 , since B=D I can write
> > 2B + 6B = 40
> > 8B = 40
> > *This last equation indicate that 8 books cost $40, so one book costs $5.*

Figure D.1. Student Brigitte's algebraic solution

Moment 2: Trying to make sense of the students' solution

As teachers looked at the problem and its solution, they tried to make sense of it and understand what was problematic in the student's solution.

Linda: The difficulty is that it said that there is the same number of…
Erica: The same number of books.
Linda: She bought the same number of books than of discs, but what she assumed now is that they also have the same price. This goes contrary to …
Erica: No, no, not the same price.
Linda: That's the point, it goes contrary to …
Erica: She did not assume that they had the same price.
Linda: Yes, in the end, since she said that a book costs 5 dollars.
Lana: A book costs 5 bucks.
Linda: By saying that …
Erica: Oh! Ok. Yes, I understand.
Linda: By saying that B equals to D [B=D], the 2B and the 6B, you know, you cannot say 2B+6B.
Erica: Why not? Yes.
Linda: 5 dollars.
Erica: You can do that. It is that B is equal to 5.
Linda: Ok, then…
Erica: B represents what? The *number* of books. Is that it?
Lana: Yes, that's it, because…
Erica: That is the difference.
Lana: She already has [a price for both].
Erica: It is that B is not money. B is a number of books. So, 2 times 5 equals to 10, 6 times 5 equals to 30, 30 plus 10 equals to 40. But she is right. However, the mistake she made is …
Erica & Linda: to mix up the price with the number of books.
Gina: [At the same time as Erica and Linda's previous line] With the quantity of objects.

I then took the opportunity to make a point concerning this type of student misconception about the algebraic variable, and to clarify it.

Jérôme: But in fact, this is one of students' biggest difficulties, and this is why when we write […] [remember in one of the preceding problems] I had only written "C" and you [Gina] said "no, I would personally require to write 'Chairs'," and then I did add that in fact it was not "chairs" but the *number* of chairs. So, it is the same difficulty here, that is, that the variables represent a number of, a quantity.

Erica: Humm humm

Gina: [nodding] Ok.

Jérôme: This is one of the biggest difficulties.

Erica: Ok.

From this discussion, it was understood that the B found does represent 5 dollars but the number 5, which was the unknown quantity to look for.

Moment 3: Giving a grade

Erica then asked the group how would they grade that students in regard to this answer. This brought teachers to re-analyze the students' answer for this problem.

Gina: [after reading the last sentence of Brigitte's solution] In fact, it should be that 8 objects or 8 items costs 40$.

Claudia: But the 8 is not …

Erica: No.

Jérôme: 8 is a new fictive price for …

Erica: 8x5, to justify her 5.

Jérôme: [pointing to the solution] Because the 8 here, here it is 2 dollars and here it is 6 dollars.

Claudia: The 8 are dollars.

Jérôme: This is 8 dollars, yes. A 8 dollars that never existed if you want.

This brought another way to analyze and look at the solution. Then Holly, Gina's student teacher, admitted to be mixed up and wondered about the algebraic manipulations done by Brigitte. She offered how she had solved it, and her mistakes were revealed where she mistakenly had combined $2x+6y = 40$ into $8(x+y) = 40$. As the discussion continued around Holly's solution in order to help her, Gina asked the following question.

Gina: But can we say, in the end, that it is just that the "x" is dollars?

Jérôme: That "x" is dollars?

Gina: If "x" is dollars, I am just trying to understand…

Jérôme: Ok.

Gina: If "x" is dollars, the books are 2x and the discs are, are …

Erica: You cannot say that "x" is dollars.

Jérôme: No, that's right, it is not dollars.

Claudia: You are looking for a number of objects.

Erica: It is impossible, you are looking for a number of objects.
Claudia: You do your "x" equals to ...
Erica: To find the final amount, you have to multiply a quantity times an amount of money. Then, "x" represents a quantity, a number ...
Claudia: A number of objects.
Erica: of objects.

I then rephrased the last words of Erica and Claudia.

Jérôme: That is interesting I think, because this is exactly it. "*x*" is what? It is always a number of something, it is a quantity, it is an unknown, an unknown quantity and not only an unknown. It is an unknown quantity.

This led back to Erica's previous question about grading the solution. Erica explained that she thought this was not a small mistake, but in fact a conceptual one.

Erica: For me, this is a conceptual mistake. The student who writes that does not deserve more than 50% in my sense. And when we grade these solutions, we have the tendency to mark this student as if she only had made a small mistake [...] but this student does not understands a thing here.

Then Erica tried again to re-interpret the solution given to understand it better.

Erica: In the end, the 5 represents the number of books. [...] It is just that she has 5 discs and 5 books. A total of 10 objects in all. The result that we have to say to ourselves is that 2x5 gives 10 and 6x5 gives 30. Their sum gives in the end 40. But the reason why 8B fives the same result than 2B+6B is because the quantity is identical at the beginning. 5x8 compared to 5x2 and 6x5 is like if it was commutative in the end. [...] The 5 is common to both, 2x5 + 6x5. So, in the end, the common variable is the 5, so it is (2+6) x 5. This is why it comes up to 8x5.

I then rephrased Erica's explanations.

Jérôme: Yes, this is it. When Claudia you were asking what is the 8. In fact, the 8 is a new 8 dollars as if ...
Carole: Well, each time that you buy a book plus a disc, it costs you 8 dollars.
Jérôme: That's it. 2 dollars, 6 dollars, 8 dollars.

Erica: That's it. It is that the sum of a disc and a book equals to 8 dollars. So you have two objects when she does that. In the end, this B, that is equal to 5, B is equal to two objects. B includes a disc and a book.

Moment 4: Mechanical action *versus* reasoning

Carole then made a remark about the fact that this student did all the right technical steps, but did not understood what she did.

Carole: The mechanical steps are there.
Jérôme: Yes.
Carole: She wrote an adequate equation.
Jérôme: She probably arrived at B=5.
Carole: She did an adequate justification that the number of books and the number of discs are equal.
Erica & Jérôme: Yes.
Carole: But she was not able to reason her mechanical steps.
Jérôme: She was not able to go back to the problem.
Carole: No reasoning of her mechanical steps whatsoever. It is very automatic. This is demonstrating that it is a student that executes exactly what we have show her to do at many times. It is greatly automatic, but there is no reasoning.

Holly disagreed with Carole, and Carole explained to her how she thought the student was able to write the mechanical steps of the problems, but did not made sense of these steps along the way. The discussion continued on along that line of how she did or did not understood, and about how close or far from understanding student Brigitte is, and what she does or does not understand.

Moment 5: Writing the units in algebra

Holly raised the point that maybe student Brigitte did not misunderstood, since if she had kept her units in the equation she probably would have not made that mistake. For example, writing 2$B+6$D=40$. This led me to say that in algebra units are explicitly dropped by convention, and only the numbers and the symbols are written. As Erica agreed, I mentioned that in algebra there is an explicit concatenation of symbols (e.g., $1 \cdot a$ is written a; $2 \cdot a$ is written 2a, etc.). For Holly, however, this was where the difficulty probably was for Brigitte, where she ended up without any units and attempted to make sense of what she had obtained. I agreed with Holly's idea that maybe Brigitte had lost track of this meaning along the way, and that the lost of units are maybe the cause to it, which could represent one of the pitfalls of algebra despite its power to generalize. This brought Erica to go back to one of Holly's previous comments about writing the units in algebra:

Erica:	And you say that in school you had to keep along the units everywhere in algebra?
Holly:	Sincerely, I think so. It was the first thing I had to write.
Jérôme:	I would be surprised, but however it does give a meaning [to the data].
[...]	
Jérôme:	I know teachers who would even had mark that as a mistake if you had written that.
Erica:	Yes! Me too!
[...]	
Erica:	I've never seen that!

I then flagged the fact that I believed Holly's comment to be quite relevant because if this student would have had kept her units, or at least implicitly kept it in her head, she maybe would not have made that mistake, since there would already have been some units attributed to the data.

This again brought the discussion about the fact that this represents one of algebra's difficulties, where units are explicitly dropped. And Linda raised the points that she never had thought of the fact that in algebra we do not write the units, where we write them for the (algebraic) formulas that we use (in mathematics, in physics, etc.). This opened a small discussion on the difference between (algebraic) formulas and (algebraic) equations, and this closed the discussion around this task.

D.2: The Systems of Equations Student's Solution: Infinity of Solutions

<u>Moment 1: Offering the problems and the students' solution</u>

I gave teachers the following problem, with three possible student solutions (inspired by Sfard & Linchevsky, 1994) (bold represents students' writing).

Solve the following system of equations:

$$\begin{aligned} &^1\quad 2(x-3) = 1 - y \\ &^2\quad 2x + y = 7 \end{aligned}$$

Student solution 1:

^{1} → $2x - 6 = 1 - y$
→ $2x + y = 7$

Answer: x **and** y **can be any number**

Solve the following system of equations:

$$\begin{aligned} &^1\quad 2(x-3) = 1 - y \\ &^2\quad 2x + y = 7 \end{aligned}$$

Student solution 2:

^{1} → $2x - 6 = 1 - y$
→ $2x + y = 7$

Hence, ^{1} $2x+y=7$
^{2} $2x+y=7$

Infinity of solutions

Solve the following system of equations:

$$\begin{aligned} &^1\quad 2(x-3) = 1 - y \\ &^2\quad 2x + y = 7 \end{aligned}$$

Student solution 3:

^{1} → $2x - 6 = 1 - y$
→ $2x + y = 7$
→ $y = 7 - 2x$

^{2} → $2x + y = 7$
→ $2x + (7 - 2x) = 7$
→ $7 = 7$ ⇨ $x = 0$

If $x=0$ **then** $2(0)+y=7$
$y=7$
solution: (0,7)

Figure D.2. Three possible students' solution to a system of equations problem

Moment 2: On the graph?

As teachers started to consider the solutions, Gina wondered about the fact that there were no graphical solutions provided.

Gina: No student placed it on the graph?

Jérôme: Well, some could have done it, but I only took these solutions because I thought they were interesting.

[...]

Jérôme: I would be tempted to say no, however, Gina. It is not a habit, well Lana could talk more about it since she teaches them. It is not a habit for students when they are working at the algebraic level.

Not exactly answering my question, Lana explained how she worked with the graph in her teaching and how it is important for her that students get a visual image of what the solution meant.

Lana: Well, I start with the graph, since it is more visual for them to find the intersection point. Afterwards, I go with the calculator [...] I tell them that it will always be a method to verify, and then I show them algebraically. I want them to understand the idea that it is a point of intersection [he draws two lines that cross each other on a piece of paper and shows it], and then I want them to understand, for example, that if you have two equations it will work as much for the first one as for the second one. So, visually the graph they understand it is the point of intersection. And when you do it algebraically then they understand some of the solutions, they understand that it is a point that will as much be satisfying the first than the second.

Gina spoke about what she does in her classroom, where she focuses on the relations between x and y along the graph line.

Gina: Because, me, when I do it in my classroom, obviously they only have one variable, but I insist that they do it graphically to show that x, y will equal that, but if you change [pointing to a fictive graph line] they need to see that effectively it is linear since it is all linear equations. So that they can see that "yes it makes a line," and it could be anywhere on the line.

This led me back to my first question to Lana.

Jérôme: But the students, if after a while you have shown them the algebraic method, will they be tempted to go back to the graphical method, for example to solve this specific problem [referring to the systems of equations problem just given].

Lana: No. The majority of students always want … "give me steps, what do I do first." You know, they want to have automatisms; that is what they want. But I always tell them, you can do it graphically, it is the best way to verify your answer.

Moment 3: Automatisms and infinity of solutions

This brought Gina to raise the point that the third student's solution is probably the most automatized solution, where the other two point to something else, something she disagrees a little with.

Gina: But the third one […] it is probably this that is the most automatic in terms of solving since you have to give a value to your x or to your y. […] Whereas for the other two they say an infinity of solutions but it is not exactly true.

Jérôme: Ok, why do you say that? What do you mean by this?

Gina: It is an infinity of solutions, in relation. […] You cannot have 1 and 2 [as answers].

Lana however did not grasp on Gina's last comments and went back to the third solution by explaining that for the student, because "x" disappears, it had to be equal to "0." But then, Gina re-explained what she meant.

Gina: But what I argue is that his answer in 3 is truer that the answer in 2, because it is not true that x and y can be any numbers, because there is a condition between the two.

Jérôme: Yes, indeed, they have to follow the relation that is $2x+y=7$, so it is not an infinity of solutions, neither that x and y can be any number, it is an infinity of solutions following this specific relation, the relation $2x+y=7$.

Lana: But we often write in mathematics "infinity of solutions" [uniquely].

Gina: By assuming that it is [along a relation].

This brought me to nuance the previous point just made.

Jérôme: In fact, there is an infinity of solutions.

Gina: Yes, but it is always...
Jérôme: But it is an infinity along the line.
Lana: [agreeing] the straight line.
Gina: This is why I, in fact I know well that a solution like that I would have said "no, no!" Or you show me in a graph that it is infinite with an arrow at the end of your line, or you show me in a table the relation that exist between the two [*x* and *y*] to show that, dot dot dot, indeed there is an infinity of solutions.

Lana then argued that saying "infinity of solutions" implied in itself that it is both lines that are seen as the solution to the system, in comparison with if it would only have been a simple intersection. Hence, she continued by saying that even if students would not have restricted their solution by saying "infinity of solutions along the line or relation" and uniquely said "infinity of solutions," none of them would have taken a point aside of the line to represent a possible solution to the system. In that sense, for Lana, this notion is implicit in the answer "infinity of solutions."

<u>Moment 4: Giving a grade to the solution</u>

I then went back to the third solution in regard to Gina's previous comments that it was a "truer" one, and asked them what they thought about it.

Gina: I mean, this answer is more probable in terms that there is more chances that students would give this answer than the other two.

I then asked Lana, who teaches this topic, how she would grade the third solution.

Lana: Half the points.
Jérôme: [...] Why?
Lana: This student forgot the concept that, as we said before, I agree that *x* equals "0" but *x* could be equal to "1," *x* could be equal to "2," *x* could be equal to "-1," "-¼," it does not matter.

Claudia wondered what Lana would see as a good solution, which brought the discussion back to the "infinity of solutions" issue.

Claudia: So, what would you like to see as an answer?
Lana: "Infinity of solutions." For me, this is all good [pointing to the second one].

Claudia reacted to this comment in regard to the fact that the third solution is much more elaborated than the other two[171].

Claudia: [surprised and disagreeing] So all this [pointing to the third solution] is just half of the points, and this one only writes "infinity of solutions" and it is good?
Lana: Yes.
Jérôme: You mean that this here, the third solution, it is half of the points for you.
Lana: Yes, for me.
Jérôme: And "infinity of solutions" for the second one?
Lana: All good.
Jérôme: And the first one?
Lana: I accept it.
Jérôme: Because for you, you understand that the student means "on the line."
Lana: Yes.
Jérôme: Ok, ok.

This brought Lana and Gina to discuss the fact that the idea of "infinity along a relation" is implicit in the answer and that it is how it is usually used and talked about.

Gina: It is implicit.
Lana: It is implicit because it is the word that we use. I understand the point of view that it is a restriction.
Jérôme: Yes.
Lana: But we never talk about restrictions, but I agree.

This brought me to ask Lana about how it is talked about and worked on in the textbooks they use, to which Lana explained that it is how it is always used and talked about everywhere.

Jérôme: In this manual [referring to the one she uses in her teaching], would it say "infinity of solutions" or would they say "infinity of solutions following the relation $2x+y$…"?
Lana: No, "infinity of solutions" uniquely.
Jérôme: Ah, ok, ok. So, it is in fact, you would accept it because it is how it

[171] To add to calculational teachers' discussion of Chapter 5 and 6, it is interesting to note Claudia's difficulty to discard solution 3 because in it was a long solution written with some procedures followed. Indeed, for her, it was worth more than the two other solutions. It shows how she gave a lot of credit to procedural solutions, even if false or representing important incomprehension from students, simply because there was a lot written.

is talked about in fact.

Lana: It is how it is talked about, everywhere, even in the diploma exams it will say "infinity of solutions" also.

Jérôme: Ok.

Lana: It would not say along the curve or along the line.

Gina: Or following the relation.

Claudia came back to the grading of the first student's answer, however.

Claudia: But even the first one you would give all the points? Because x and y cannot be any number.

Gina: But it means the same thing.

Lana: It means the same thing than "infinity of solutions."

I then support Claudia's claim.

Jérôme: Yes, indeed, they cannot be any ones because they need to follow the relation $2x+y=7$.

Claudia: If x is one number, y cannot be the same number.

Lana: [Agreeing] Humm humm ... but when we say "infinity of solutions" it means the same sentence as this.

Claudia: But the student does not say that here. He does not say "infinity of solutions."

Moment 5: Understanding "infinity of solutions"

Claudia's last comment led to a discussion on the differences between the first and the second solutions and if the students were demonstrating understanding.

Lana: For me, it's like [in the second one] the student learned the theory by heart. And this one [the first one] the students wrote it in his words.

Jérôme: But do you consider that this student [second one] would understand that it is really following the line or if it is only a pattern of when it is the same line I say "infinity"?

Lana: Not everybody.

Jérôme: I think this is the place where the real question is.

Lana: These are the ones that when we started with the graph, they understood what the graph really was. There are some that are "automatisms." They will say "Ah, if it is the same thing, it is infinity of solutions, since the teacher said it." They do not see it all. But I would say that the majority are really able to see that it is two lines [the answer]. [...] And it is also my role to show them in the beginning when I teach the graph, of course sometimes... because if I do not insist on the graphs and I directly go to the substitution/elimination methods, it's over.

Jérôme: Ok.

Gina: It is because there [third solution], they only find one of the points.

Lana: That's it.

This led me to say that to know if this student understood the concept, one would need to ask that student directly what he or she meant by "infinity of solutions" and if any point would work. This brought Lana to again discuss about how this concept is usually talked about.

Lana: It is the notation that we use. Is it good? It depends...

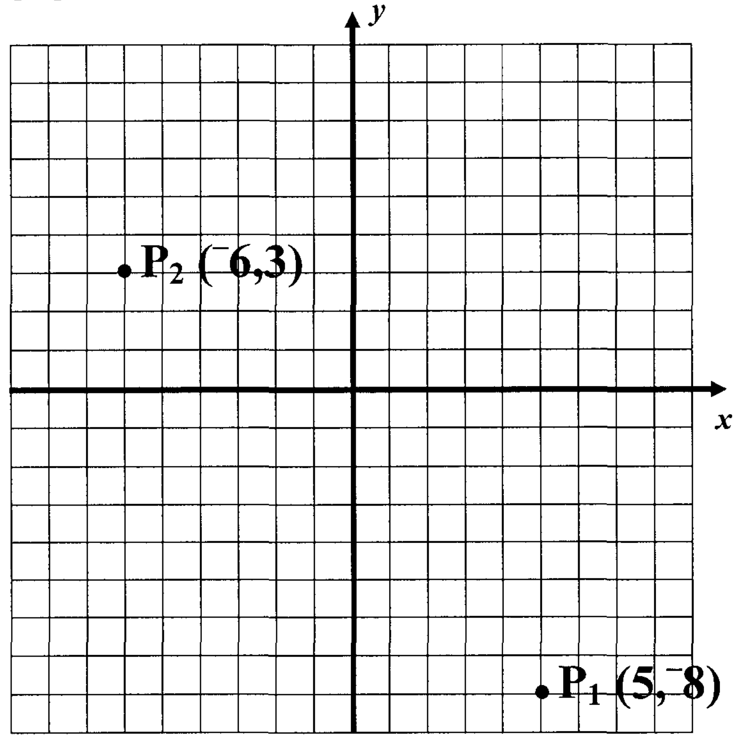
The discussion closed as Lana expressed how she felt that both first and second solutions meant the same thing, where the first one was written in the students' own words, which she implicitly supported by "this is what we use and say usually."

D.3: The Rate of Change Students' Solution: Conventions

<u>Moment 1: Offering the problem and the student's solution</u>

I offered teachers the following problem and its solution (figure D.3), and as teachers delved into it, I asked them how they would grade that solution "out of ten."

In the following graph :



Find the rate of change of the line that passes through the points P_1 and P_2. Show how you do.

$$\frac{\Delta x}{\Delta y} = \frac{5 - {}^{-}6}{{}^{-}8 - 3} = \frac{11}{{}^{-}11} = {}^{-}1$$

Figure D.3. Inverse rate of change problem and one student's solution

<u>Moment 2: Student's understanding</u>

Lana then started to discuss the steps used by the student and the worth of it in regard to understanding the concept of rate of change.

Lana: Well, there are not many steps taken.
Jérôme: What do you mean by there are not many steps taken here, can you explain.
Lana: Well, normally how I mark is that I give kind of "0.5" for the steps taken and for the answer.
Jérôme: What would you expect as steps taken here?

Lana: No, the steps taken are ok, but what I mean is that this student did not do the correct steps. I would give him half of the points, 5 out of 10 (5/10). Because he has the answer. Because a student could have hid it all, and only write "-1." I would have given him all good, since he maybe would have done it in his head.

Gina reacted surprised to this comment from Lana and flagged that it said "show me how you solve that," so the steps would have to be demonstrated to receive the entire points. For that reason, the answer Lana talked about where the student would only write "-1" would only receive one out of four (1/4). Lana agreed, but said that in a situation where it would not have been requested to write the steps, where only a numerical value would have been asked, the "-1" would have been a good answer. However, agreeing with Gina, Lana mentioned that indeed only giving the final answer would only receive "0.5" out of one (0.5/1), to which she added the following.

Lana: But in fact, this student deserves zero points.
Jérôme: Why do you say that this student deserves zero points?
Lana: He does not understand anything.
Gina: [laughing] It is only because he is a nice student.
Jérôme: What do you mean?
Lana: [laughing] He does not understand! Well, he does not understand anything because for him the rate of change he says that it is the variation in x divided by the variation in y.
Jérôme: Ok.
Lana: It is the contrary, he arrived by chance on the right answer.

Moment 3: Discussing conventions

This reaction of Lana brought me to directly raise the issue of conventions to them, in order to make sense of the solution given.

Jérôme: But why do you say that he does not understand a thing this one for example? Because all that this student do is to inverse x and y.
Lana: Yes.
Jérôme: But this, in fact, is only a mathematical convention.
Lana: The rate of change is always vertical on horizontal.
Jérôme: But this is a mathematical convention, it could have been vertical on horizontal.
Lana: [nervous laugh] Yeah, I agree with you [throwing herself on two legs of her chair].

This provoked an [intense] argumentation about this student' understanding.

Jérôme: This is directly where my question is, this student in fact is able to calculate the variation between the points. Conventionally in mathematics, no, because it is *y* on *x*, but at a conceptual level...
Gina: I do not agree.
Jérôme: What do you mean exactly?
Gina: I do not agree, because if you only want to verify if he is able to subtract whole numbers, ok, yes, he should receive all his points because he understands well this part.
Jérôme: You mean "5 – ⁻6".
Gina: But he does not know what is the rate of change if he inversed his formula.
Jérôme: Ok, ok, I understand what you mean.
Lana: If the convention had been the contrary, I agree, but the convention is *y* on *x* and not *x* on *y*.
Gina: It is kind like if you ask someone to calculate the mass multiplied by the acceleration to find the force of something. And then you say to this person, you verify, you give him his points because he is able to do the multiplication. I am sorry, but he does not yet know how to calculate a force.

The discussion continued in regard to whether this student understood the concept and of how to know more about his or her understanding. Then Gina wondered, as she explained that she did not know exactly what a rate of change was, if there was a way for the student to realize his or her mistake. This brought Lana to talk about the staircase approach.

Lana: Well, with the staircase, yes. As for myself, I accustom them by saying that the rate of change is *y* on *x*, if you have "-1" then it is "-1" on "1," it means it goes down one and you move of one [on the side].

As Gina tried to make sense of this, Claudia wondered why it was in fact *y* over *x*.

Claudia: But why is it *y* first?
Jérôme: It is a convention.
Claudia: Because us, we always do the other way, we always put *x* first.
Jérôme: That's right! In coordinates, we place *x* and *y*.
Claudia: It is always the *x* first, then the *y* when we do the coordinates.

Moment 4: Understanding the rate of change

This led Lana and I to explain how the rate of change functions in the graph (the concept itself) to Claudia and Gina. Afterwards, Gina reiterated her previous question concerning the possibility for the students to verify the answer (or realize its falseness). It led me to tell her that a "verification" would not necessarily lead one to see the mistake.

> Gina: Where I was going before with my idea is that the student could have had shown some understanding by verifying his answer.
>
> Jérôme: [...] But if he arrived with this to "-4" [a new example previously given in the discussion], he could also had verified and shown you that the movement in x is of "-4" for a movement of "1" in y.
>
> Gina: Yes, I understand what you mean.

Claudia then asked again about the order of x and y, to which I responded in relation to the same aspects as in the Cartesian plan.

> Claudia: But why have they changed it? Why having placed y and x, instead of x and y?
>
> Jérôme: [...] Well, it is a decision, in fact it is the same sort of decision as to why we placed the x before the y in the Cartesian plan.
>
> Gina: It is like the masculine wins over the feminine in the [French] grammar.
>
> Claudia: And the alphabetical order?

Lana then explained that if it is $\Delta x/\Delta y$ than what is called a positive and a negative slope visually on the graph would not work anymore because the entire orientation is changed. This led me to explain that a student can still understand and explain the variation between points even if he or she inverses them.

> Jérôme: [...] Suppose that this student understands very well, but he inverses them. This student understands well that the rate of change in fact, for him, is the difference between the x in relation with the difference between the y. Well, he will be able to explain to us how to go from one point to another anyway.

I continued by explaining how this student could explain his solution with both orders of y over x or x over y with the "-1" student solution only by switching the order. Claudia even highlighted that she herself did it with the x first. At the same time as I was explaining, Lana was constantly caught up in some calculations on her papers.

Moment 5: Going back to the students' understanding

I again raised a question concerning students' understanding, enabling me to emphasize the presence of conventions (as Lana was still invested in her calculations).

Jérôme: But I think the question is an interesting one. We ask ourselves if the student can still understand even if he does the x over the y? What is the weight of the convention, in fact, in the understanding at this level? If theoretically this student understands well all that is going on [in regard to the rate of change], if we tell him that conventionally it is the y over the x, it is maybe not a big conceptual jump to switch for the y over the x.
Gina: [agreeing] Humm humm.
Claudia: Right.

Moment 6: Mathematics as a stable and coherent body of knowledge

Finishing her calculations, Lana raised the point that with $\Delta x/\Delta y$ the linear function does not stand anymore where $y=2x+b$ becomes $y=\frac{1}{2}x+b$. Hence, you needed to have $\Delta y/\Delta x$ to keep the entire set of concepts coherently working together.

Claudia added that it would still work if you inversed x and y in the function, giving $x=2y+b$. Lana agreed and added that if it is expressed in that way, however, then something else is not understood in regard to the dependant and independent variable – which she then realized to be also a convention.

Lana: [answering to Claudia] It would be good, however, when he writes $x=2y+b$, it is good because he has it good, but he does not understands the idea of the dependant variable and independent … that we have supposed ... This is still a convention!
Jérôme: Yes, yes, yes.

This brought Gina to say that conventions are important in mathematics, because we build on them year after year in schooling.

Gina: The other question that we need to ask ourselves when we teach to younger students is that what they learn is always a kind of a brick for when the next teacher will have them, he will be able to place the next brick. So then, if you accept that they break the conventions because you say "yes, yes, Jo Schmo understands well" the problem is that you prevent the next brick to arrive.

Jérôme: Ah yes, the continuity. Indeed. This is what Lana underscores [with her linear function coherence]. But at the same time, I do believe that it is ok to tell the student that it is not adequate because it is not the right mathematical convention. But here the question is mostly "does this student understands the rate of change?"

Claudia: Right.

Jérôme: Here, it is not the same question. I think we are obliged mathematically to tell the student "Stop, stop, you understand very well, but conventionally we do place the y over the x."

Gina continued on with the importance of conventions by giving the example of grammar where a student can compose a very good essay but which would be filled with mistakes. And for that case this student should not be given all the points. I agreed to that and added that this student should however receive its points for the ideas. Gina then completed her analogy with grammar.

Gina: But the grammar would not be there at all. The conventions are a little bit like the grammar. I often tell my students that there is grammar in maths.

Moment 7: The Cartesian plan

This led me to ask Gina how she would grade the students that inversed the rate of change. Because Gina replied that she would not really know since she does not teach this topic, I gave her another example.

Jérôme: You do have to work with the Cartesian plane [in your teaching]?

Gina: Yes, yes.

Jérôme: So, if a student for this specific point tells you, well for this point here that would be (3,-1), tells you (-1,3), does this student receives a "0"? (see figure D.4)

Claudia: Yes.

Gina: Yes.

Jérôme: Why?

Gina: [hesitating] Because he is not in the right quadrant.

Lana: No, it is still a convention.

Jérôme: It is also a convention.
Lana: We again said that we would place the x first and the y in second.

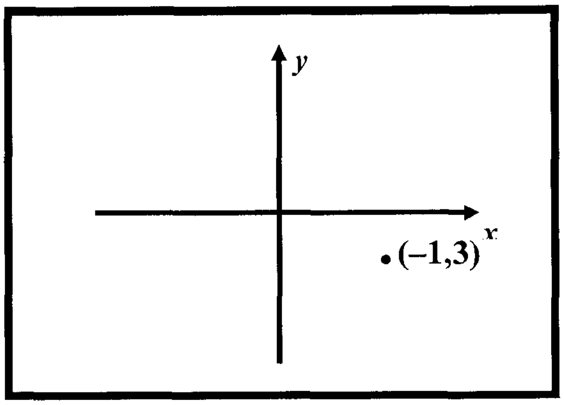


Figure D.4. An example of inversing the coordinates in the Cartesian plane

To continue on Lana's comment and bridge the current issue in line with the previous one on the rate of change and students' understanding, I added what a possible student explanation could be.

Jérôme: Because for example, if you ask this student and he answers that in fact this point is moved of "3" in x and moved of "-1" in y, there he only made a mistake concerning the convention [and not conceptually].

Gina added however that she would normally give a more difficult problem with many coordinates to place on the Cartesian plane and that if a student made one mistake then he would not end up with the same figure (e.g., a parallelogram). I then told her that if this student mixed them all up, it would not change the figure.

Jérôme: Suppose that he mixes them all up.
Gina: Ok.
Lana: He will arrive to the same answer.
Jérôme: Because he did the error of writing them all (y,x), which is quite frequent for student, it happens.
Gina: Indeed, indeed.
Jérôme: If there was not any work done on conventions, then your parallelogram [should be the same if he follows the same way of doing].
Gina: [as I am trying to display the parallelogram example on the board] I should have had brought it, but I did not know, but I had once a

student that constructed it, but the way he constructed it it inversed it completely.

Lana: Well, as Jérôme says, if you inverse them all, you are supposed to arrive at the same good answer.

[...]

Jérôme: Maybe what happened is that you gave the students the coordinates and then he placed them following the other [his] convention. Then, obviously he will come up with something inversed.

Claudia then added something concerning students' understanding.

Claudia: But if this student would do it to *all*, then at least we know that...

Jérôme: That he understands.

Claudia: That he understands how, but it is simply that he mixed them up.

Gina: He is able to place them all, there is only a need to un-mix them up.

Claudia: If he was making only one mistake, then I would not give points. But if he did it to all of them.

Gina: If he was doing it to all of them, yes I would be ready to give 50% for this one [the parallelogram], but zero for this one [the (-1,3) answer].

I added a nuance to the discussion by stressing the importance of conventions.

Jérôme: But I do not think personally, even if I say that it is only a convention, I would give 10 out of 10. The student does not know the convention, there are some points to loose at the mathematical level because the student does not know the convention.

Gina completed by saying that conventions are important and should not be thrown away because they are important for the next level where they will be *used*, and I added that they also enabled the establishment of other and future notions.

Gina: And it is important I think that the student sees [...] that the student understands that conventions have their place for the next stage.

Jérôme: Yes, in mathematics, conventions are enormously important. I mean these are decisions that need to be taken for precisely enabling us to continue on. To throw away the conventions would be to say that all the mathematical work that has been done is not good anymore.

This ended the work for this task, as the next one was on probabilities and statistics.

APPENDIX E

FAMILIES OF PLANAR FIGURES

> There is a seeming paradox here, in that it is certainly harder to learn. It is certainly easier for pupils to learn that 'area of a triangle = ½ base X height' than to learn why this is so. But they then have to learn separate rules for triangles, rectangles, parallelograms, trapeziums; whereas relational understanding consists partly in seeing all of these in relation to the area of a rectangle. It is still desirable to know the separate rules; one does not want to have to derive them afresh everytime [*sic*]. But knowing also how they are inter-related enables one to remember them as parts of a connected whole, which is easier. (Skemp, 1978, pp. 12-13)

In the beginning of session 4-5, I introduced teachers to families of prisms and pyramids in regard to the work done on volume using Cavalieri principle. I also introduced them to families of planar figures, especially rectangles, parallelograms, and triangles. Cavalieri principle works also in 2D, as Gray (1987) explains:

> The principle asserts that two plane figures have the same area if they are between the same parallels, and any line drawn parallel to the two given lines cuts off equal chords in each figure. (p. 13)

As with the volume of solids, both planar figures (straight or curved) need to be of the same height. This can be applied in a simplified way[172] to a rectangle and a parallelogram, as in the following figure (figure E.1).

[172] In the case of the session 4-5, we only worked on specific cases of the 2D Cavalieri principle, mainly between figures that were quadrilaterals and triangles – no work was done with curved figures.

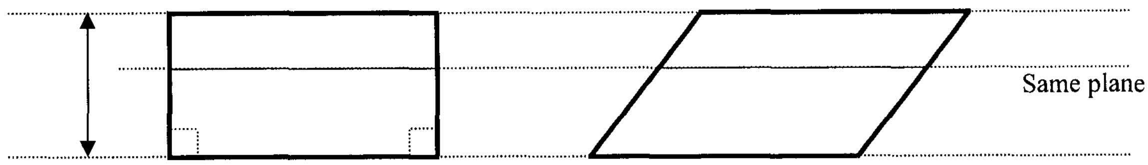


Figure E.1. Using Cavalieri principle with a rectangle and a parallelogram

The 2D Cavalieri principle brings us to be able to see "families of planar figures," for example a family of rectangle and parallelograms of same area. To see this, let the parallelogram be as slanted as you want. Hence, any parallelogram with the same height and same base would have the same area, which creates the equivalent family of rectangles and parallelograms (figure E.2).

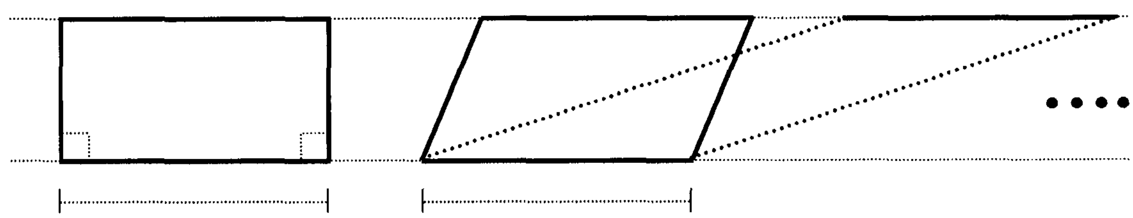


Figure E.2. Family of rectangle and parallelograms

And the same thing can be said for triangles, where all triangles with the same base and the same height are part of the same family, be they as slanted as you want. I have represented the family of triangles in a different way in the following figure (figure E.3).

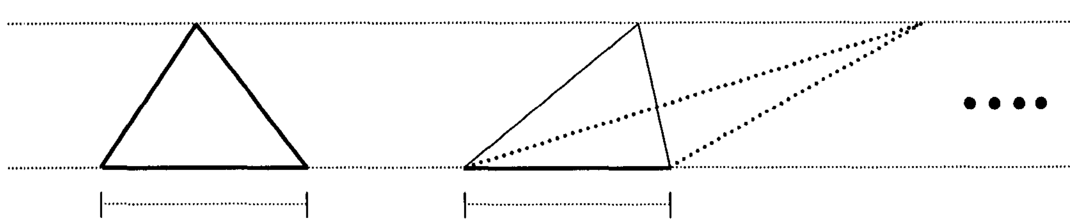


Figure E.3. A family of triangles with the same height and same base

In fact, there are no specific reasons for the triangle on the right to be this one, it could have been any triangle with the same base and same height. Indeed, the family of triangles is maybe better represented in the following way (figure E.4).

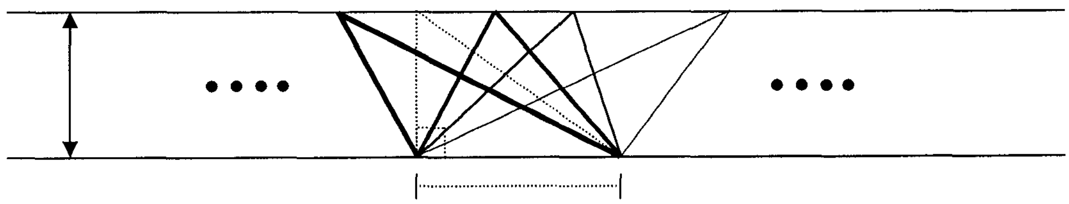


Figure E.4. Another representation of a family of triangles

It is also possible to establish a family of trapezoids, where the small base slides on the same plane. In other words, we can establish the family of trapezoids with same base and same height (figure E.5).

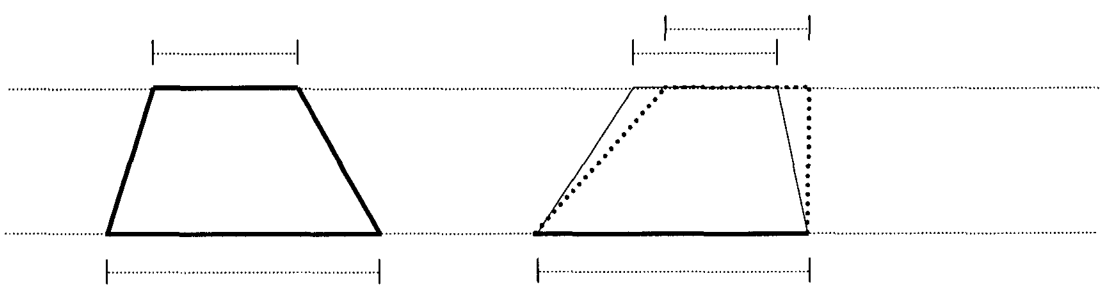


Figure E.5. A family of trapezoids with same height and same base

Moreover, as the family of trapezoids gets established, it is important to notice that a trapezoid is indeed defined as "a quadrilateral with two sides parallel" (Wolfram MathWorld) which takes into account not only standard trapezoids that we are often used to seeing (figure E.6),

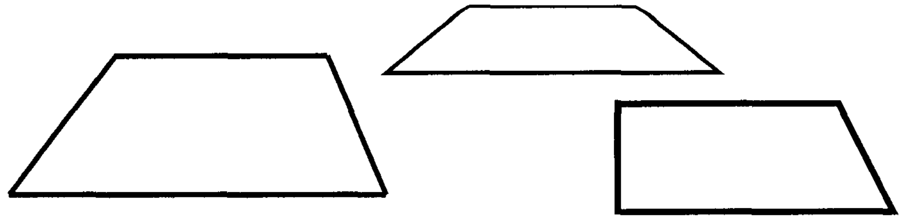


Figure E.6. Some examples of standard trapezoids but also any quadrilateral that has a pair of opposite sides that are parallel, which are indeed trapezoids, without being parallelograms (figure E.7)[173].

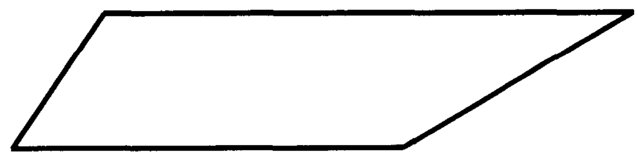


Figure E.7. Another type of trapezoid

In that sense, the family is composed of any quadrilateral that has the same height and a pair of opposite and parallel sides, making the family of trapezoids look like the following (figure E.8)[174].

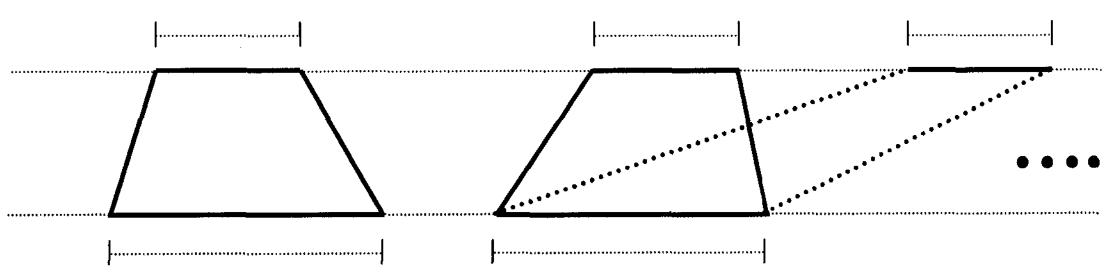


Figure E.8. A family of trapezoids with non-standard ones

[173] I say "used to seeing" mostly because from the rapid glance that I gave to textbooks and websites, only one website gave a picture different from the ones offered in figure E.6 and that was of the type presented in figure E.7 (http://id.mind.net/~zona/mmts/geometrySection/commonShapes/trapezoid/trapezoid.html). It seems indeed that this type of trapezoid is not a current form that is often studied.

[174] This work on families of planar figures represents for me an instance where I literally pushed and developed some *school* mathematics. Indeed, first, the establishment of families of planar figures represents for me the development of an interesting piece of school mathematics linking planar figures into families. Second, the development of this specific part on trapezoids brought me to a better understanding of the implications of the trapezoid's definition where it could encompass more than the "usual" trapezoids. It is the very development of the "theory" of families (applied to trapezoids) that brought me to realize the possibilities and the extent to which its definition leads. Not to overemphasize the point, there is a lot of mathematics to develop within *school* mathematics – and this came directly out of my intention to prepare in-service sessions around rich mathematical learning opportunities.

APPENDIX F

DISCUSSING SCHOOL MATHEMATICS

The main theme of the 2006 PME-30 conference was "Mathematics in the centre." This theme created much interest, discussion and controversy as to what this theme could mean and also as to know if "mathematics" really had to be at the centre of our studies and work in mathematics education. Many ideas were offered in terms of being at the centre of the work in mathematics education or, simply put, many researchers explained what was at the centre of their work, going from having "teaching mathematics" in the centre, "learning mathematics" in the centre, the discipline of "mathematics" at the centre, "mathematical ideas" in the centre, and so on. While in the last moments of the conference and within the last lecture I decided to offer my future colleague, Christine Suurtamm, what I thought should be at the centre or mainly what I thought was at the centre of my work. It appears interesting since it describes well where my thoughts are currently, and where my work heads to. I intend here to give a transcription from what we have both scribbled down on my note pad.

Jérôme: School mathematics is at the centre. School mathematics is: (1) The mathematical knowledge/notions taught in schools (that are in the curriculum) and (2) The teaching and learning of these mathematics. Hence, mathematics is at the centre, but this mathematics is not separated from its learning and teaching.

Christine: [Circling the word "taught" in my sentence – second line] May or may not be taught, but are inherent in the curriculum (e.g., proofs, notions of functions are not developed but should be – may not appear explicit [in the curriculum] …

Jérôme: For me, but I am still thinking about this…, when I work with teachers of mathematics, my way of entering is through the mathematical topics to be taught in schools *and* from them learning and teaching issues get tackled with (as mathematical notions/concepts are worked on/learned/experienced). This is why for me ~~mathematics~~ (school) mathematics is at the centre.

Christine: Yes, I agree – with some. I think exploring mathematics with teachers is a key component – it helps them think about mathematics learning with respect to themselves and their students.
I'm not sure that the important mathematical concepts are taught or explored [in schools]. My point is that the curriculum or what we see in schools does not actually focus on concepts but more on skills – so the *potential* mathematics in school could be a focus.

Jérôme: Exactly, very exactly! This is in fact the point that I raised in the job talk at Ottawa when I said that I wanted to enlarge the teachers "conceptual" understanding of volume, so that it becomes more than the formulas to apply (skills, techniques) and also [their] conceptual *geometrical* understanding of it (e.g., volume as a piling up of layers – this is a concept [not a technique]).

Christine: Yes. I knew we agreed. It's difficult to articulate what school mathematics is – it's not really what is taught in schools nor is it really the curriculum – it's what *could* be taught in schools. [Circling the word "taught" in her last sentence]. This word is problematic because it suggests to many people a transmission mode – explaining, etc.