University of Alberta

The Units of Modular Group Algebras

by

Leo Creedon ©

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ABSTRACT

Let K be a field and G a group. Let KG be the group algebra and $U(KG)$ its group of units. In this thesis we investigate the existence and explicit construction of free groups in $U(KG)$ and examine the consequences of the existence of these free groups. We place special emphasis on the modular case, where K has positive characteristic p and G contains elements of order a power of p.

As motivation, we start by giving a technique for the construction of free semigroups in group algebras with some restrictions.

We use a new method of constructing units to explicitly construct generators of free groups in $U(KG)$ and give examples in group algebras where previous techniques do not apply.

This construction relies heavily on the abundance of non-commuting pairs of elements in the finite group ring F_pG. We use combinatorial techniques to see precisely how scarce these commuting pairs of elements are.

Next we study criteria for the existence of free groups in group algebras. For a finite group G and a field K which is not algebraic over its prime subfield F_p we show that $U(KG)$ does not contain free groups $\Leftrightarrow$ G' is a p-group $\Leftrightarrow$ $U(KG)$ is soluble $\Leftrightarrow$ the torsion subset of $U(KG)$ forms a group $\Leftrightarrow$ $KG/J(KG)$ is isomorphic to a direct sum of fields $\Leftrightarrow$ the transvections of KG are contained in $1 + J(KG)$. We also

explore connections between $U(KG)$ and the finite group $U(F_pG)$. The locally finite analogues of these results are also given.

The existence or absence of free groups thus leads to an important dichotomy in the structure of the group algebra, in the spirit of the Tits Alternative.

We give similar results on $U(KG)$ where G is either an FC group or is locally nilpotent. After studying a newly defined chain of unit groups we finish by proving some results on the Jacobson radical of KG.

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Contents

Chapter 1

Introduction

1.1 Introduction

For all the work that has been done on group rings in the last fifty years the subject is in many ways still in its infancy. A counterexample to the Isomorphism Problem for integral group rings was only found in the last decade. We have little idea what the units (or zero-divisors) of an arbitrary group ring look like in terms of the group or coefficient ring.

On the other hand, we can often construct bicyclic units, Bass cyclic units and alternating units (given minor restrictions on the group algebra), and these can be used for various constructions in the group algebra.

One of the more obvious approaches to studying group rings (and more general rings) is to consider the existence of polynomial identities. The torsion and commutativity of rings can be examined in this way. It is natural to suspect that the group identities of the unit group will have some bearing on the polynomial identities of the group algebra. The existence of a free group (of rank 2) in the unit group would quickly put an end to this line of reasoning. These free groups have become the object of my affections.

In this section and the next we give some of the notation and results needed to understand the ideas presented in later chapters. The main theme of the thesis is the existence and explicit construction of free groups in group algebras, and the consequences of the existence of these free groups. As motivation, we start in Section 2.1 by giving a technique for the construction of free semigroups in group algebras with some restrictions. We then develop some of the theory needed to study the applications of a new construction of units (the GBUs). In Section 2.3 we use these GBUs to construct free groups in group algebras where previous techniques do not apply. This construction relies heavily on the abundance of non-commuting pairs of elements in the finite group ring F_pG. In Chapter 3 we use combinatorial techniques to see precisely how scarce these commuting pairs of elements are. (We also briefly mention the proportion of invertible elements of F_pG.)

In Chapter 4 we study criteria for the existence of free groups in group algebras KG (especially where G is locally finite). In Theorems 31 and 32 we see that the existence or absence of free groups in the group algebra leads to an important and surprising dichotomy in the structure of the group algebra. This is done in the spirit of the Tits Alternative and suggests that the most important question to ask of a group algebra is whether or not it contains free subgroups. In Section 4.4 we consider some results on $U(KG)$ where G is not periodic. In the following section we define a new chain of unit groups and investigate their properties. In Section 4.6 we chronicle some of the consequences of the preceding results. The Jacobson radical is used throughout the chapter, and is studied in its own right in the final section. In Chapter 5 we apply our techniques to some of the examples which motivated this work and which, as far as the author is aware, are not readily studied using existing methods. An Appendix includes some of the more tedious calculations which were performed using the software package GAP.

Notation and Conventions

Definition: Given a group G and a ring R, the **group ring** RG is the set of elements $\Sigma r_i g_i$ where $r_i \in R$ and $g_i \in G$ and the summation is over all elements of G (with only finitely many $r_i \neq 0$). So RG is a free R-module with the obvious addition, and it becomes a ring if we use the multiplication of the group to define multiplication.

N is the set of natural numbers, Z the integers, Q the rationals and C the complex numbers.

F_{p^n} is the finite field of p^n elements.

G is a group, k is a (commutative) field.

$o(x)$ is the order of an element x of G.

C_p denotes the cyclic group with p elements.

$G_p = S_p(G)$ denotes the sylow p-subgroup of the group G. $G * H$ denotes the free product of the groups G and H.

S_n is the symmetric group on n symbols.

A regular element of a ring is any element which is not a zero-divisor.

kG is the group algebra.

Define the augmentation map $\omega : kG \to k$ by $\sum a_g g \to \sum a_g$. Clearly this is a k-algebra epimorphism. ω has kernel $\Delta(G) = \langle x-1 | x \in G\rangle_{kG}$, called the augmentation ideal of kG.

$^* : kG \to kG$ is the involution $\sum \alpha_g g \to \sum \alpha_g g^{-1}$.

$tr : kG \to kG$ is the trace map defined by $tr(\sum a_g g) = a_1$.

$\mathcal{U} = \mathcal{U}(kG)$ is the (multiplicative) group of units of kG.

$V = V(KG)$ is the group of units of augmentation 1.

If $\alpha \in \mathcal{U}$ has finite order n then $\hat{\alpha}$ is the element $1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$ of kG.

A *bicyclic* unit of kG is an element of the form $1 \pm \hat{x}y(1-x^i)$ or $1 \pm (1-x^i)y\hat{x}$, where $x, y \in G$, $o(x) = n < \infty$ and $i \in \{0, 1, \ldots, n-1\}$. Note that such an element must be a unit of order p if char $k = p > 0$.

By a free group I shall mean a non-cyclic free group.

$\Lambda(G) = \{g \in G \mid [H : C_H(g)] < \infty$ for every finitely generated subgroup H of $G\}$.

$\Lambda^+(G) = \{g \in \Lambda(G) \mid g$ is a torsion element$\}$.

Note $\Lambda^+(G) \triangleleft \Lambda(G) \triangleleft G$ and $\Lambda(G)/\Lambda^+(G)$ is a torsion-free abelian group.

$J(R)$ is the Jacobson radical of the ring R.

If $\alpha = \sum \alpha_g g \in kG$ then the support of α is the set $\{g \in G \mid \alpha_g \neq 0\}$.

If H is a *subset* of G then $\pi_H : kG \to kG$ is the natural projection down to the subset of elements whose support lies in H.

If q is a prime number then π_q will denote the projection π_H, where H is the set of q-elements of G.

If $V = M_n(D)$ then the Schur index of V is the square root of the dimension of the division algebra D over its center.

1.2 Background Results

A considerable amount of progress has been made on the question of the existence of free subgroups of $\mathcal{U}(kG)$. The Tits Alternative [35] is a powerful tool here. So let us review what is known about these free groups.

Theorem 1 (Tits Theorem / Tits Alternative [35]) *Let G be a finitely generated linear group over a (commutative) field. Then either G is soluble-by-finite or G contains free groups.*

Note that the proof does not exhibit the free group explicitly or constructively.

Theorem 2 (Wedderburn-Artin Theorem) *[8, p.38-9] Every semisimple algebra is isomorphic to a direct product of matrix algebras over division algebras. Conversely, a direct product of matrix algebras over division algebras is a semisimple algebra. This decomposition is unique, up to permutation of the direct factors.*

Theorem 3 (Wedderburn-Malcev Theorem) *Let G be a finite group. Then*

$$KG \simeq \frac{KG}{J(KG)} \oplus J(KG)$$

as vector spaces.

Proof:
By [8, p.107], it suffices to show that $\frac{KG}{J(KG)}$ is separable, that is $\oplus M_{n_i}(K_i)$ is separable. Thus, by [8, p.105], we need only show that the K_i are separable algebras. By the definition of separable algebras [8, p.104], we need only show that for every extension L of the field K_i, $K_i \otimes L$ is semisimple, and this is true by [8, p.75 Corollary 4.3.6]. □

Theorem 4 *[33, p.64-5] Let G be a finite group and F a field that contains the algebraic closure of the rationals if F has characteristic 0. Then every irreducible FG-module has Schur index 1.*

From the proof of this result we see that if our field K has characteristic $p > 0$ then every finite dimensional simple image of KG is of the form $M_n(F)$, with F a commutative field containing K, where K has finite index as a subfield of F. This fact will be used in subsequent chapters.

Sehgal [32, p.200] and then Hartley and Pickel [15] proved

Theorem 5 *Let G be soluble-by-finite, and suppose $\mathcal{U}(ZG)$ does not contain a free group of rank 2. Then*

i) every finite subgroup of G is normal in G,

ii) the torsion subset T of G is a subgroup and is either abelian or the direct product of an elementary abelian 2-group and a quaternion group of order 8 (i.e. a non-abelian Hamiltonian 2-gp).

In the finite case we have the following situation:

Theorem 6 *[15, Theorem 2 p.1342] If G is finite then exactly one of the following occurs:*

i) G is abelian,

ii) G is a non-abelian Hamiltonian 2-group, and $\mathcal{U}(ZG) = \{\pm g : g \in G\}$,

iii) $\mathcal{U}(ZG)$ contains a free group.

For k a field, Gonçalves [12] showed

Theorem 7 *If G is finite and k is a field of characteristic 0 then $\mathcal{U}(kG)$ contains a free group if and only if G is non-abelian.*

and

Theorem 8 *If G is finite and k is a field of characteristic $p > 0$ then $\mathcal{U}(kG)$ does not contain a free group if and only if one of the following occurs:*

i) G is abelian,

ii) k is algebraic over its prime field F_p,

iii) $S_p(G)$, the p-Sylow subgroup of G, is normal in G, and $G/S_p(G)$ is abelian.

In [3], Bovdi gives perhaps the most comprehensive survey of the problem to date:

Theorem 9 *Let k be a field of characteristic $p \geq 0$ and suppose that $\mathcal{U}(kG)$ does not contain a free group. Then one of the following conditions holds:*

1. G is abelian;

2. G is a torsion group, $p > 0$ and k is algebraic over its prime subfield F_p;

3. $p = 0$ and

- *a. $\Lambda^+(G)$ is an abelian subgroup and each of its subgroups is normal in G;*
- *b. the centralizer $C_G(\Lambda^+(G))$ contains all elements of finite order of G;*
- *c. for every $a \in \Lambda^+(G)$, which is not central in G, k contains no root of unity of order equal to $o(a)$;*

4. $p > 0$, k is not algebraic over its prime subfield F_p and

- *a. the p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is an abelian group;*
- *b. the centralizer $C_{G/P}(A)$ contains all torsion elements of G/P;*
- *c. if A is noncentral in G/P and G/P is non-torsion, then the algebraic closure L of F_p in k is finite and for all $g \in G/P$ and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r is a multiple of $[L : F_p]$.*

5. G is not a torsion group, $p > 0$, k is algebraic over its prime subfield F_p and

- a. *the p-Sylow subgroup P of $\Lambda^+(G)$ is normal in G and $A = \Lambda^+(G)/P$ is an abelian group;*
- b. *if A is noncentral in G/P then the algebraic closure L of F_p in k is finite and for all elements g of infinite order in G/P and $a \in A$ there exists a natural number r such that $gag^{-1} = a^{p^r}$. Furthermore, each such r is a multiple of $[L : F_p]$.*

Note however that the above conditions are not sufficient for $U(KG)$ to not contain a free group; for example the possibility that G is a free group is not excluded.

Chapter 2

GBUs and free groups

We investigate the existence (and construction) of free pairs of units in the unit group of a (modular) group algebra KG. We generalise a result of Gonçalves and Passman [14] to do this, and use the programming package GAP to investigate the units of $F_2(t)D_{10}$, where F_2 is the Galois field of order 2 and t is an element transcendental over F_2.

Z. Marciniak and S. Sehgal [23, 24] have constructed free groups in arbitrary integral group rings, where the group is non-abelian. Gonçalves and Passman [14] have used a similar construction for some group rings of the form $F_p(t)G$ where t is an element transcendental over the field F_p. Here we will be extending the work of Gonçalves and Passman.

2.1 Free Semigroups

A semigroup $G = \langle S \rangle$ is freely generated by the set S if it has the property that any mapping from the set S into a semigroup H can be extended to a homomorphism of semigroups.

As motivation for the subsequent results on free groups, we give a result on the construction of free semigroups in group algebras. Comments and examples follow the proof.

Theorem 10 *Let K be a field of characteristic zero, R a subring of KG, y a unit of KG, and $X \in R$. Write $r^\sigma = y^{-1}ry$ for all $r \in R$. Assume that*

i) $Ry \subseteq yR$.

ii) The (non-negative) powers of y are right linearly independent over R.

iii) X is a regular element of R.

iv) For any positive integer n, the elements $(X^r)^\sigma X^{n-r}$, $r = 0, \ldots, n$, are linearly independent over Q.

Then X and $Y = 1 + y$ generate a free subsemigroup of KG.

Proof:

Conditions i) and ii) clearly imply that the subring $R[Y]$ generated by R and Y is isomorphic to $\oplus_{i=0}^{\infty} y^i R$ (a direct sum of R-modules), since we have the commutation rule $ry = yr^\sigma$.

Any element w of the free semigroup on $\{u, v\}$ may be written uniquely in the form $u^{i_0}vu^{i_1}v \ldots vu^{i_m}$ for integers $i_j \geq 0$. (For example, $v^3 = u^0vu^0vu^0vu^0$.) We write $I = (i_0, \ldots, i_m)$ and $w_I(u, v)$ for the element w above. The *length* $L(I)$ of $I = (i_0, \ldots, i_m)$ is defined to be $m + 1$, and we write $I' = (i_0, \ldots, i_{m-1})$. The identity element corresponds to the empty sequence.

Given an element $w_I(u, v)$ of the free semigroup, write $W(I) = w_I(X, Y) \in R[Y]$. We know that

$$W(I) = \sum_{j=0}^{\infty} y^j W_j(I),$$

where the $W_j \in R$ and the sum contains only finitely many terms (at most $L(I)$, since $W(I)$ only contains $L(I)-1$ occurrences of Y). On the other hand, we have

$$\begin{aligned} W(I) &= \sum_{j=0}^{\infty} y^j W_j(I) \\ &= \sum_{j=0}^{\infty} y^j W_j(I')(1+y)X^{i_m} \\ &= \sum_{j=0}^{\infty} y^j W_j(I')X^{i_m} + \sum_{j=0}^{\infty} y^{j+1} W_j(I')^{\sigma} X^{i_m}, \end{aligned}$$

whence $W_0(I) = W_0(I')X^{i_m}$ and $W_j(I) = (W_j(I') + W_{j-1}(I')^{\sigma})X^{i_m}$ for all $j \geq 1$. We may now use induction on the length of I to prove that

$$W_0(I) = X^n,$$

$$W_1(I) = \sum_{t=0}^{m-1} \left(X^{i_0+\ldots+i_t}\right)^{\sigma} X^{n-(i_0+\ldots+i_t)},$$

and

$$W_m(I) = (X^{i_0})^{\sigma^m}(X^{i_1})^{\sigma^{m-1}} \ldots (X^{i_{m-1}})^{\sigma}(X^{i_m}),$$

where $m = L(I)-1$ and $n = i_0 + \ldots + i_m$. Note that assumption iii) implies that $W_m(I) \neq 0$, so $W(I)$ is a polynomial in y of degree exactly $m = L(I)-1$.

In order to show that X and Y generate a free semigroup, we must prove that, if I and J are sequences for which $W(I) = W(J)$, then $I = J$. Now $W(I) = W(J)$ is equivalent to $W_j(I) = W_j(J)$ for all $j \geq 0$. Moreover, $W(I)$ and $W(J)$ have exactly the same degree in y, and hence I and J have the same length. Write $I = (i_0, \ldots, i_m)$ and $J = (j_0, \ldots, j_m)$. The W_0-terms yield $i_0 + \ldots + i_m = n = j_0 + \ldots + j_m$, and then the W_1-terms simplify to

$$\sum_{t=0}^{m-1}\left(X^{i_0+\ldots+i_t}\right)^{\sigma}X^{n-(i_0+\ldots+i_t)}=\sum_{t=0}^{m-1}\left(X^{j_0+\ldots+j_t}\right)^{\sigma}X^{n-(j_0+\ldots+j_t)}.$$

Let $\{i_0, i_0+i_1, i_0+\ldots i_{m-1}\} = \{r_1,\ldots,r_p\}$, where $r_1 < r_2 < \ldots < r_p$, and r_i occurs d_i times in the set of partial sums (remember that some of the i's may be zero, so some of the partial sums may be equal). Similarly, write $\{j_0, j_0+j_1, j_0+\ldots j_{m-1}\} = \{s_1,\ldots,s_q\}$, where $s_1 < s_2 < \ldots < s_q$, and s_i occurs e_i times in the set of partial sums. The above equation, therefore, becomes

$$\sum_{i=0}^{p} d_i\left(X^{r_i}\right)^{\sigma}X^{n-r_i}=\sum_{j=0}^{q} e_j\left(X^{s_j}\right)^{\sigma}X^{n-s_j}.$$

Assumption iv) now implies that $p = q$, $d_i = e_i$, and $r_i = s_i$ for all i. So, to begin with, we have

$$r_1 = i_0 = i_0 + i_1 = \ldots = i_0 + \ldots + i_{d_1-1} = j_0 = j_0 + j_1 = \ldots = j_0 + \ldots + j_{d_1-1},$$

whence $i_t = j_t$ for $0 \leq t \leq d_1 - 1$. Next, using r_2, we find that $i_t = j_t$ for $d_1 \leq t \leq d_2 - 1$, and so on (this can be readily set up as an inductive argument). Thus $I = J$, as required. $\square$

Example 1 *We now show how this result can be applied. The simplest choice is $y = \alpha g$, where $\alpha \in K$ is a suitable scalar, and $g \in G$. For R, we may choose QH, where H is a subgroup of G satisfying $H^g \subseteq H$. The choice of $X \in R$ would then be dictated by the above data.*

For example, suppose $x \in G$ is an element such that

$$H = \langle x, x^g, \ldots, x^{g^m}, \ldots\rangle$$

is an ordered abelian group with $gx \neq xg$. Assume, moreover, that no power of g belongs to H. Then $y = g$ and $R = QH$ satisfy i) and ii) of the theorem. Also, by *the Malcev-Neumann Theorem [29, Theorem 2.11 p.601], R is an integral domain, so iii) is satisfied for any non-zero $X \in R$. To see what condition iv) imposes on a given X, let $u = X^g/X$ in F, where F is the quotient field of R. It is then immediate that*

$$\left(X^r\right)^\sigma X^{n-r} = u^r X^n$$

for all r, so condition iv) is simply the requirement that u be transcendental over Q. Since Q is algebraically closed in F, it is sufficient to have $u \notin Q$. In other words, if X is chosen such that X^g is not a rational multiple of X, then X and $1+g$ generate a free semigroup. The simplest choice is $X = 1 + x$, where $x^g \neq x$, since $1 + x^g = q(1 + x)$ for some $q \in Q$ implies that either $x^g = x$ or $x^g = 1$, both of which are contradictions.

Note: Condition iv) is determined by the choice of Y as $1 + y$. More complicated choices for Y lead to more involved conditions of a similar nature.

2.2 Elementary Results

Let us restrict our attention to finite groups G for the remainder of the chapter. Gonçalves [12] has shown that $F_p(t)G$ contains a (non-abelian) free group inside its group of units precisely when either the Sylow-p subgroup $S_p(G)$ is not normal in G or $G/S_p(G)$ is non-abelian. His proof was non-constructive.

The Gonçalves/Passman construction [14] uses bicyclic units, that is, units of the form $1+(1-g)h\hat{g}$ or $1+\hat{g}h(1-g)$, where $g, h \in G$, g has order n and $\hat{g} = 1+g+g^2+\cdots+g^{n-1}$. This Gonçalves/Passman construction only works when you can find $g, h \in G$ such that $\langle g\rangle^h \neq \langle g\rangle$ and $\langle g, g^h\rangle$ has no elements of order p. For example, it works for most non-abelian p'-groups.

However, consider a group ring like $F_2(t)G$, where $G = D_{2m} = \langle x, y | x^m = y^2 = 1, yxy = x^{-1}\rangle$. Now the unit group will contain a free group (as $S_2(D_{2m})$ is not normal in D_{2m}), but the Gonçalves/Passman construction cannot be used since when $u, v \in D_{2m}$, and $\langle u\rangle^v \neq u$, the group $\langle u, u^v\rangle$ always contains 2-elements.

We develop a method of construction of free pairs which will apply to such group algebras and in Section 2.2 we take an example, $F_2(t)D_{10}$, and using Theorem 19 we exhibit a free group in its unit group.

Lemma 11 *Let $H \subset G$, $\alpha, \beta \in kG$, $\beta \in \mathcal{U}(kG)$. If $H^\beta, H^{\beta^{-1}} \subset H$ then $\pi_H(\beta^{-1}\alpha\beta) = \beta^{-1}\pi_H(\alpha)\beta$. (In particular this holds if $H < G$ and $H \lhd \mathcal{U}(kG)$.)*

Proof:

Let $\alpha_0 = \pi_H(\alpha)$ and $\alpha = \alpha_0 + \alpha_1$. Then $\beta^{-1}\alpha\beta = \beta^{-1}(\alpha_0+\alpha_1)\beta = \beta^{-1}\alpha_0\beta + \beta^{-1}\alpha_1\beta$. $supp(\beta^{-1}\alpha_0\beta) \subset supp(\beta^{-1}H\beta) = suppH = H$. Thus $\pi_H(\beta^{-1}\alpha_0\beta) = \beta^{-1}\alpha_0\beta$ (i.e. "all of" $\beta^{-1}\alpha_0\beta$ appears in $\pi_H(\beta^{-1}\alpha\beta)$.)

Claim: $supp(\beta^{-1}\alpha_1\beta) \cap H = \phi$ i.e. "none of" $\beta^{-1}\alpha_1\beta$ appears in $\pi_H(\beta^{-1}\alpha\beta)$. Assume not, i.e. $\alpha_1 = ag + a_1g_1 + \ldots$ with $g, g_1, g_2, \ldots \in G\backslash H$ and $supp(\beta^{-1}g\beta) \cap H \neq \phi$. Say $\beta^{-1}g\beta = a_1h_1 + \ldots + a_rh_r + b_1g_1 + \ldots + b_sg_s$, with $r \geq 1$, $k \ni a_i \neq 0$ for $i = 1, \ldots, r$, with $h_1, \ldots, h_r \in H$ and $g_1, \ldots, g_s \in G \setminus H$. Then

$$\begin{aligned} G \setminus H \ni g &= \beta(\beta^{-1}g\beta)\beta^{-1} \\ &= a_1\beta h_1\beta^{-1} + \ldots + a_r\beta h_r\beta^{-1} + b_1\beta g_1\beta^{-1} + \ldots + b_s\beta g_s\beta^{-1} \\ &= a_1h_{r+1} + \ldots + a_rh_{2r} + b_1g_1^{\beta^{-1}} + \ldots + b_sg_s^{\beta^{-1}} \end{aligned}$$

with $h_{r+1} = \beta h_1\beta^{-1}, \ldots, h_{2r} = \beta h_r\beta^{-1} \in H$.

Note $supp(g_1^{\beta^{-1}}) \cap H = ... = supp(g_s^{\beta^{-1}}) \cap H = \phi$. Indeed, if $h \in supp(g_1^{\beta^{-1}} \cap H$ then, say, $g_1^{\beta^{-1}} = c_1 h + ... \Rightarrow g_1 = c_1 h^\beta + ... = c_1 h' + ...$ with $h' \in H$ (as $H^\beta \subset H$), a contradiction as $g_1 \in G \setminus H$.

Thus $a_1 h_{r+1} + ... + a_r h_{2r} = \pi_H(g) = 0$. Recall $h_{r+i} = \beta h_i \beta^{-1}$. Assume $\beta h_i \beta^{-1} = \beta h_j \beta^{-1}$ for $i \neq j$, so $h_i = h_j$ for $i \neq j$, a contradiction. Thus $h_{r+1}, ..., h_{2r}$ are all different. Thus $a_1 = ... = a_r = 0$, contradicting our assumption. This proves the claim and hence the theorem. $\square$

Corollary 12 *Let $\alpha \in kG, \beta \in \mathcal{U}(kG)$. Then $tr(\beta^{-1}\alpha\beta) = tr(\alpha)$.*

Proof:
$1 \triangleleft \mathcal{U}(kG)$, so $\pi_1(\beta^{-1}\alpha\beta) = \beta^{-1}(\pi_1\alpha)\beta = tr(\alpha(\beta^{-1}\beta) = tr(\alpha)$. $\square$

Note the alternative definition for trace: $tr(\alpha) := \pi_1(\alpha)$. Also, $tr(\alpha) \neq 0 \Leftrightarrow 1 \in supp(\alpha)$. Thus $\pi_1(\alpha\beta) = \pi_1(\beta\alpha)$.

Corollary 13 *Let $\alpha \in kG$, $\beta \in \mathcal{U}(kG)$. Then $\pi_{Z(G)}(\beta^{-1}\alpha\beta) = \beta^{-1}\pi_{Z(G)}(\alpha)\beta$.*

Proof:
$Z(G) \triangleleft \mathcal{U}(kG)$. $\square$

Recall that ω is the augmentation map. For the main theorem of the chapter we will need the following

Lemma 14 *Let $\alpha, \beta \in kG$. If H is a subset of G with $H^x \subset H$ for all $x \in supp(\alpha)$ then $\omega(\pi_H(\alpha\beta)) = \omega(\pi_H(\beta\alpha))$. (In particular, $H \triangleleft G$ works.)*

Proof:

Let

$$\alpha = \sum_{i=1}^{n} \alpha_i x_i, \quad \beta = \sum_{j=1}^{m} \beta_j y_j.$$

Then $\alpha\beta = \Sigma \alpha_i \beta_j x_i y_j$. If $x_i y_j \in H$ then $(x_i y_j)^{x_i} = y_j x_i \in H$ by hypothesis. Thus $x_i y_j \in H \Leftrightarrow y_j x_i \in H$. So

$$\pi_H(\alpha\beta) = \sum_{i,j \in I} \alpha_i \beta_j x_i y_j \Rightarrow \pi_H(\beta\alpha) = \sum_{i,j \in I} \alpha_i \beta_j y_j x_i,$$

where I is some subset of $[1, n] \times [1, m]$. □

From the proof we see that the given hypothesis could be further weakened.

Example 2 *Note that even if* $H \lhd G$ *then we can have* $\pi_H(\alpha\beta) \neq \pi_H(\beta\alpha)$. *Let* $kG = F_2 D_{2p}$, *where* $D_{2p} = \langle x, y | x^p = y^2 = 1, x^y = x^{-1} \rangle$. *Let* $\alpha = 1 + xy$ *and* $\beta = y$. *Let* H *be the normal subgroup of* D_{2p} *containing* p *elements. Then* $\pi_H(\alpha\beta) = x$ *and* $\pi_H(\beta\alpha) = x^{-1}$.

Note also that $\pi_H(\alpha\beta) \neq \pi_H(\alpha)\pi_H(\beta)$ *[29, p.29 Exercise 5]. For example* $\pi_H(\alpha\alpha^{-1}) = 1 \neq 0$, *where* $supp(\alpha) \cap H = \phi$.

2.3 Generalised Bicyclic Units

Let us note that the proofs that Gonçalves gave for Theorems 7 and 8 were non-constructive.

Recently free groups in $\mathcal{U}(kG)$ have been constructed using bicyclic units. Marciniak and Sehgal proved

Theorem 15 *[23] If* G *is any group and* $u \in \mathcal{U}(ZG)$ *is a non-trivial bicyclic unit, then* $\langle u, u^* \rangle$ *is a non-abelian free group.* □

and

Theorem 16 *[24, Theorem 1.7] Let G be any group. If $a \in ZG$ satisfies $a^2 = 0$ and $a \neq 0$ then $\langle 1+a, 1+a^* \rangle$ is a free group.*

Motivated by Theorem 15, Gonçalves and Passman [14] used a similar idea to prove

Theorem 17 *Let k be a field of characteristic $p > 0$ containing an element t transcendental over its prime subfield. Let G be a group with two elements $x, y \in G$ such that x has order $n < \infty$, $\langle x \rangle^y \neq \langle x \rangle$ and the subgroup $\langle x, y^{-1}xy \rangle$ has no p-torsion.*

Defining

$$a = (1-x)y\hat{x}, \quad b = \hat{x}y^{-1}(1-x^\delta), \quad \hat{x} = \sum_{i=0}^{n-1} x^i,$$

where $\delta = (-1)^p$, we have

$$\mathcal{U}(kG) \supset \langle 1+ta, 1+tbab, 1+t(1-b)aba(1+b) \rangle \simeq C_p * C_p * C_p. \qquad \square$$

Corollary 18 *[14] If G is a non- abelian torsion p'-group and k is not algebraic over its prime field F_p, then $\mathcal{U}(kG)$ contains a free group.* $\square$

Note that in the general situation, if we somehow had access to (not necessarily trivial) units $\alpha, \beta \in \mathcal{U}(kG)$ with order $\alpha = n < \infty$, then letting

$$a = (1-\alpha)\beta\hat{\alpha}, \qquad b = \hat{\alpha}\beta^{-1}(1-\alpha^{-1}),$$

$1+a$ and $1+b$ are again units. Indeed, β need not even be a unit. If β is an arbitrary element of kG then defining

$$a = (1-\alpha)\beta\hat{\alpha}, \qquad b = \hat{\alpha}\beta(1-\alpha^{-1}),$$

we again get two units $1+a$ and $1+b$. Let us call such units **generalised bicyclic units (GBUs)**.

These units seem to have been unduly neglected, so we now turn our attention to them. We first determine whether there exist non-trivial GBUs.

Example 3 *F_5S_3 is semi-simple (by Maschke's Theorem) and non-commutative. Let us write $S_3 = \langle x, y \mid x^3 = y^2 = (xy)^2 = 1\rangle$. Then $\hat{S}_3$ and $\frac{1}{3}\hat{x} - \hat{S}_3 = \hat{x} - \hat{x}y$ are central idempotents of F_5S_3. Thus $F_5S_3\hat{S}_3 \simeq F_5$ and $F_5S_3(\hat{x} - \hat{x}y) \simeq F_5$ are direct summands of F_5S_3, so*

$$\begin{aligned} F_5S_3 &\simeq F_5 \oplus F_5 \oplus \textit{non-commutative semi-simple piece of order } 5^4 \\ &= F_5 \oplus F_5 \oplus M_2(F_5). \end{aligned}$$

By direct computation, F_5S_3 has 12 bicyclic units. But $M_2(F_5)$ has more than 12 GBUs (found using GAP), so there exist GBUs in F_5S_3 which are not bicyclic units.

Theorem 19 *Let G be any group, let $\alpha, \beta \in \mathcal{U}(F_pG)$, $o(\alpha) = n < \infty$, $(p, n) = 1$ and define $a = (1-\alpha)\beta\hat{\alpha}$, and $b = \hat{\alpha}\beta^{-1}(1 - \alpha^{(-1)^p})$. If $(ba)^m \neq 0$ for all integers m, then $\mathcal{U}(F_pG)$ contains a free group.*
In fact,

$$\langle 1 + ta, 1 + tbab, 1 + t(1+b)aba(1+b)\rangle \simeq C_p * C_p * C_p.$$

Proof:
$a^2 = b^2 = 0$, so the result is immediate by the lemma in [14]. □

Now we can use our GBUs to generalise Theorem 17.

Theorem 20 *Let k be a field of characteristic $p > 0$, containing a transcendental element t over its prime subfield F_p. Let G be a group with $\alpha, \beta \in \mathcal{U}(F_pG)$, $o(\alpha) = n < \infty$, $p \nmid n$, and $a = (1-\alpha)\beta\hat{\alpha}$, $b = \hat{\alpha}\beta^{-1}(1 - \alpha^{(-1)^p})$. Then if*

$$\begin{aligned} &p = 2 \text{ and } \omega\pi_2(\hat{\alpha}\beta^{-1}\alpha^2\beta) \neq \omega\pi_2(\hat{\alpha}) \text{ or} \\ &p > 2 \text{ and } \omega\pi_p(\hat{\alpha}\beta^{-1}(\alpha + \alpha^{-1})\beta) \neq 2\omega\pi_p(\hat{\alpha}) \end{aligned} \tag{2.1}$$

then $\mathcal{U}(kG)$ contains a free group.

In fact, $\langle 1+ta, 1+1bab, 1+t(1+b)aba(1+b)\rangle \simeq C_p * C_p * C_p$.

Proof:

By Theorem 19 it suffices to show that ba is not nilpotent. If ba is nilpotent then by [29, Lemma p.47], we must have $\omega\pi_p(ba) = 0$.

Case i) $p = 2$. Then

$$\begin{aligned} \omega\pi_2(ba) &= \omega\pi_2(\hat{\alpha}\beta^{-1}(1+\alpha^2)\beta\hat{\alpha}) \\ &= \omega\pi_2(\hat{\alpha}\hat{\alpha}\beta^{-1}(1+\alpha^2)\beta) \text{ (by Lemma 14)} \\ &= \omega\pi_2(\hat{\alpha}(1+\beta^{-1}\alpha^2\beta)) \text{ (as } n \text{ is odd)} \\ &= \omega\pi_2(\hat{\alpha}) + \omega\pi_2(\hat{\alpha}\beta^{-1}\alpha^2\beta) \\ &\neq 0 \text{ by assumption .} \end{aligned}$$

Case ii) $p > 2$. Then

$$\begin{aligned} \omega\pi_p(ba) &= \omega\pi_p(\hat{\alpha}\beta^{-1}(2-\alpha-\alpha^{-1})\beta\hat{\alpha}) \\ &= n\omega\pi_p(\hat{\alpha}(2-\beta^{-1}(\alpha+\alpha^{-1})\beta)) \text{ (by Lemma 14)} \\ &= n\{2\omega\pi_p(\hat{\alpha}) - \omega\pi_p(\hat{\alpha}\beta^{-1}(\alpha+\alpha^{-1})\beta)\} \\ &\neq 0 \text{ by assumption.} \end{aligned}$$

□

Note that variants of Theorems 20 and 19 will be applicable in diverse settings. The ugly equations 2.1 are used to ensure that ba is not nilpotent. These could be reformulated in terms of $\omega\pi_n$, where $n \neq p$. Indeed, if $\gamma = \sum c_g g$ is nilpotent of order r, with $r < p^m$ and fixing n, assume that $supp(\gamma)$ contains no elements of order dividing $p^m n^{m_1}$, for all $m_1 \in Z$ (except the identity element). Then

$$\begin{aligned} 0 &= \omega\pi_n(0) = \omega\pi_n(\gamma^{p^m}) \\ &= \omega\pi_n((\sum c_g g)^{p^m}) = \omega\pi_n((\sum c_g^{p^m} g^{p^m})) \\ &\quad \text{(as } \omega\pi_n \text{ annihilates } [kG, kG] \text{ by Lemma 14)} \\ &= \omega\pi_n((\sum c_g^{p^m} g)) = c_{g_1}^{p^m} + \cdots + c_{g_s}^{p^m} \text{ say,} \\ &= (c_{g_1} + \cdots + c_{g_s})^{p^m} = (\omega\pi_n(\gamma))^{p^m} \end{aligned}$$

If $p > 0$ and G is finite then $F_pG \subset kG$ and F_pG is finite, so if we choose our α, β in F_pG, we can check for nilpotency of ba directly, using GAP say.

Example 4 *Consider a group ring like $F_2(t)G$, where $G = D_{2m} = \langle x, y | x^m = y^2 = 1, yxy = x^{-1}\rangle$. Now the unit group will contain a free group (as $S_2(D_{2m})$ is not normal in D_{2m}), but the Gonçalves/Passman construction cannot be used as when $u, v \in D_{2m}$, and $\langle u\rangle^v \neq u$, then the group $\langle u, u^v\rangle$ always contains 2-elements.*

So let us take such an example, say $F_2(t)D_{10}$, and attempt to construct a free group in its unit group.

Let $G = D_{10} = \langle x, y | x^5 = y^2 = 1, yxy = x^4\rangle$ and consider the group algebra $F_2(t)G$, where t is a transcendental field element over F_2. Define
$\alpha = x + y + xy$ and $\beta = (1 + x + \hat{x}y)^2$. Direct calculation (see the Appendix for details) shows that α and β are units of order 3 and 15 respectively. Using the notation of our theorem,

$$a = (1 + \alpha)\beta\hat{\alpha} \text{ and } b = \hat{\alpha}\beta^{14}(1 + \alpha).$$

Now, direct calculation again shows that $0 \neq ba = (ba)^2$, so no power of ba is equal to zero, so Theorem 19 applies.

Thus, $\langle 1 + ta, 1 + tbab, 1 + t(1 + b)aba(1 + b)\rangle = \langle T, U, V\rangle$, say, is isomorphic to

$C_2 * C_2 * C_2$.

So clearly, TU and TV generate a free group. That is,

$$\begin{aligned}
&\langle\, (1+ta)(1+tbab), (1+ta)(1+t(1+b)aba(1+b)) \,\rangle \quad = \\
&\langle\, 1+t(x+x^4+xy+x^2y)+t^2(x^2+x^4+x^2y+x^4y), \\
&1+t(xy+x^2y+x^3y+x^4y)+t^2(x^3+x^4+x^2y+x^3y) \,\rangle
\end{aligned}$$

is a free group of rank 2. The GAP file in the Appendix shows that this is not a stable free pair. $\square$

Chapter 3

Commutativity of the group algebra

When F and G are finite, an exhaustive search can be carried out (by hand or using computer software) to check the applicability of the criterion in Theorems 20 and 19 of the previous chapter. The speed of this check is influenced by the probability that two randomly selected elements of FG commute. In this chapter we attempt to determine this probability for an arbitrary finite group algebra. If R is a finite ring, define this probability as $P(R) = \frac{1}{|R|^2} \sum_{x \in R} |C_R(x)|$. This is the total number of commuting pairs of elements divided by the total number of pairs of elements in the ring.

Desmond MacHale [21] has shown that for an arbitrary non-commutative finite ring R, $P(R) \leq 5/8$, with equality if and only if $[R : Z(R)] = 4$.

Letting J represent the Jacobson radical of R, the number $P(R/J)$ can be computed using either of the following two results:

Theorem 21 (Frobenius) *[16] If A is an $n \times n$ matrix over the field F, and if $f_1, \dots, f_k$ are its invariant factors, with $f_i | f_{i-1}$ for $i = 2, \dots, k$, then the dimension of $C_{M_n(F)}(A)$ is $\sum_{i=1}^{k}(2i-1) deg\, f_i$.* □

We need to set up some notation for the following theorem:
Let us work with $n \times n$ matrices, and let $\pi(n)$ be any partition of n. Let $b_i \geq 0$ denote the number of times i appears in the partition, so that $n = b_1 + 2b_2 + 3b_3 + \cdots$. Let $k(\pi)$ denote the total number of parts of π, that is, $k(\pi) = \sum_{i \geq 1} b_i$. Let q be a prime power. Let

$$f(n,q) = f(n) = (1 - \frac{1}{q})(1 - \frac{1}{q^2}) \cdots (1 - \frac{1}{q^n})$$

for $n \geq 1$, and $f(0) = 1$. Then:

Theorem 22 (Feit and Fine) *[10]*

$$P(M_n(F_q)) = f(n) \sum_{\pi(n)} \frac{q^{k(\pi)}}{f(b_1)f(b_2)\dots f(b_n)}.$$

□

Lemma 23 *If $R_1, \dots, R_t$ are finite rings, then*

$$P(R_1 \oplus \cdots \oplus R_t) = P(R_1) \dots P(R_t).$$

Proof:

It suffices to prove the result for $t = 2$. Here

$$P(R) = P(R_1 \oplus R_2) = \frac{1}{|R|^2} \sum_{(x_1,x_2) \in R_1 \oplus R_2} |C_R(x)|$$

$$
= \frac{1}{|R|^2} \left\{ \begin{array}{l} \sum_{(x_1,x_2)\in Z(R_1)\oplus R_2} |C_{R_2}(x_2)|.|R_1| \\ +\sum_{(x_1,x_2)\in R_1\oplus Z(R_2)} |C_{R_1}(x_1)|.|R_2| \\ +\sum_{(x_1,x_2)\in R_1\setminus Z(R_1)\oplus R_2\setminus Z(R_2)} |C_{R_1}(x_1)|.|C_{R_2}(x_2)| \\ -\sum_{(x_1,x_2)\in Z(R_1)\oplus Z(R_2)} |R_1|.|R_2| \end{array} \right\}
$$

Now let $q_1 = P(R_1)$ and $q_2 = P(R_2)$. Note that

$$
q_i = \frac{1}{|R_i|^2} \sum_{x\in R_i} |C_{R_i}(x)| = \frac{|Z(R_i)|.|R_i| + (|R_i| - |Z(R_i)|)m_i}{|R_i|^2}
$$

where m_i is the average of the sizes of the centralisers of the non-central elements of R_i. Therefore

$$
m_i = \frac{q_i|R_i|^2 - |Z(R_i)|.|R_i|}{|R_i| - |Z(R_i)|},
$$

for $i = 1, 2$ (when $q_i \neq 1$). Thus

$$
P(R) = \frac{1}{|R_1|^2|R_2|^2} \left\{ \begin{array}{l} q_2|R_2|^2|R_1||Z(R_1)| + q_1|R_1|^2|R_2|.|Z(R_2)| \\ +(|R_1| - |Z(R_1)|)(|R_2| - |Z(R_2)|)m_1m_2 \\ -|Z(R_1)|.|Z(R_2)|.|R_1|.|R_2| \end{array} \right\}
$$

$$
= \frac{1}{|R_1|.|R_2|} \left\{ \begin{array}{l} q_2|R_2|.|Z(R_1)| + q_1|R_1|.|Z(R_2)| \\ +(q_1|R_1| - |Z(R_1)|)(q_2|R_2| - |Z(R_2)|) \\ -|Z(R_1)|.|Z(R_2)| \end{array} \right\} = q_1q_2.
$$

□

Lemma 24 *Let R be a finite ring, J the Jacobson radical of R. If $J \subset Z(R)$ then $P(R) = P(R/J)$.*

Proof:

Recall Theorem 3. There is a subring $S \simeq R/J(R)$ of R such that $R = S \oplus J$ as vector spaces. Let $x = x_1 + x_2$, $y = y_1 + y_2 \in R$, with $x_1, y_1 \in S$ and $x_2, y_2 \in J$. Then

$xy = x_1y_1 + (x_1y_2 + x_2y_1 + x_2y_2)$, with $x_1y_1 \in S$ and the second part in J. Computing yx we see that $y \in C_R(x)$ if and only if $y_1 \in C_S(x_1)$ and

$$x_1y_2+x_2y_1+x_2y_2 = y_1x_2+y_2x_1+y_2x_2. \qquad (*)$$

Note that since $J \subset Z(R)$, $(*)$ always holds here. Thus $y \in C_R(x)$ if and only if $y_1 \in C_S(x_1)$. Without loss of generality, assume $q_1 = P(S) \neq 1$. Let m be the average of the sizes of the centralisers of the noncentral elements of S. Now

$$\begin{aligned} P(R) &= \tfrac{1}{|R|^2} \textstyle\sum_{x_1\in S,\, x_2\in J} |C_{S+J}(x_1 + x_2)| \\ &= \tfrac{1}{|R|^2} \left\{\textstyle\sum_{x_1\in Z(S),\, x_2\in J} |C_S(x_1)|.|J| + \sum_{x_1\in S\setminus Z(S),\, x_2\in J} |C_S(x_1)|.|J|\right\} \\ &= \tfrac{1}{|R|^2} \left\{|J|^2 \textstyle\sum_{x_1\in Z(S)} |S| + |J|^2 \sum_{x_1\in S\setminus Z(S)} |C_S(x_1)|\right\} \\ &= \tfrac{1}{|S|^2} \{|Z(S)|.|S| + (|S| - |Z(S)|)m\} \\ &= \tfrac{1}{|S|^2} \{|Z(S)|.|S| + q_1|S|^2 - |Z(S)|.|S|\} \text{ by the proof of Lemma 23} \\ &= q_1. \end{aligned}$$

□

Lemma 25 *In the notation of Lemma 24, if $S \subset Z(R)$ then $P(R) = P(J)$.*

Proof:

We use the notation of the previous lemma. Again let $x = x_1 + x_2$, $y = y_1 + y_2 \in R$, with $x_1, y_1 \in S$ and $x_2, y_2 \in J$. Recalling equation (*) from the previous proof, note that $y \in C_R(x)$ if and only if $y_2 \in C_S(x_2)$. Without loss of generality, assume $q_2 = P(J) \neq 1$. Again, let m_2 be the average size of the centraliser of a noncentral element of J. Thus

$$\begin{aligned} P(R) &= \tfrac{1}{|R|^2} \left\{\textstyle\sum_{x_1\in S,\, x_2\in Z(J)} |C_S(x_1)|.|C_J(x_2)| + \sum_{x_1\in S,\, x_2\in J\setminus Z(J)} |C_S(x_1)|.|C_J(x_2)|\right\} \\ &= \tfrac{1}{|R|^2} \left\{|S|^2|J|.|Z(J)| + |S|^2 \textstyle\sum_{x_2\in J\setminus Z(J)} |C_J(x_2)|\right\} \\ &= \tfrac{1}{|J|^2} \{|J|.|Z(J)| + (|J| - |Z(J)|)m_2\} \\ &= \tfrac{1}{|J|^2} \{|J|.|Z(J)| + q_2|J|^2 - |Z(J)|.|J|\} \\ &= q_2. \end{aligned}$$

□

In a finite group ring KG, finding $P(KG/J)$ is easy, and finding $P(J)$ is often easy, especially when G is p-seperable.

We close the chapter by quoting two more computational results of a slightly different nature. Define $\delta(R) := |U(R)|/|R|$, the probability that a randomly selected ring element is a unit. The first result may be viewed as a generalisation of Wedderburn's Theorem.

Theorem 26 *[22] Let R be a finite ring with unity 1 $(\neq 0)$. If $\delta(R) > 1-\sqrt{\frac{1}{|R|}}$ then R is a field.* □

Theorem 27 *[9] Let FG be a finite group algebra with $|F| = p^m$ and $|G| = n$. Then*

$$\delta(FG) = \delta(FG/J) = \prod_{n,q}(1-q^{-1})(1-q^{-2})\dots(1-q^{-n}),$$

where the product extends over the family of ordered pairs (n, q) corresponding to the decomposition of FG/J as the direct sum of matrix rings $M_n(F_q)$. □

We will return to the consideration of $\delta(F_pG)$ in Lemma 73 and Example 6 in the next chapter.

Chapter 4

Infinite Fields

Throughout this chapter let K denote a field of positive characteristic p which is not algebraic over F_p.

4.1 Introduction

Theorem 28 (Sehgal [32], Hartley & Pickel [15]) *Let G be a finite group. Then the following statements are equivalent:*

i) G is abelian or a Hamiltonian 2-group.

ii) $U(ZG)$ is soluble

iii) $U(ZG)$ does not contain a free subgroup.

It is a natural and interesting problem to extend this result to more general group algebras. In Section 3 we prove the following theorem:

Theorem 29 *Let G be a locally finite group, F a field whose characteristic is either 0 or $p > 0$, provided that G contains no p-elements and F is not algebraic over F_p. Then the following statements are equivalent:*

i) G is abelian

ii) $U(FG)$ is soluble

iii) $U(FG)$ does not contain a free subgroup.

The case where the coefficient field K is not algebraic over its prime subfield F_p and G contains p-elements is more complicated. Gonçalves gave the following result in 1984 [12]:

Theorem 30 *Let $|G| < \infty$, p any prime, K any field of characteristic p, not algebraic over F_p. Then the following are equivalent:*

i) $U(KG) \not\supset$ free groups

ii) G' is a $p-group$

iii) $U(KG)$ is soluble

In Section 2 we prove the following more detailed result for finite groups:

Theorem 31 *Let $|G| < \infty$, p any prime. Then the following are equivalent:*

i) $U(KG) \not\supset$ free groups

ii) G' is a $p-group$

iii) $U(KG)$ is soluble

iv) $U(F_pG)$ is soluble and F_pG is not equal to either Case ii) or iii) of Theorem 40.

v) $\dfrac{U(F_pG)}{1+J(F_pG)}$ is soluble and F_pG is not equal to either Case ii) or iii) of Theorem 40.

vi) $U\left(\dfrac{F_pG}{J(F_pG)}\right)$ is soluble and F_pG is not equal to either Case ii) or iii) of Theorem 40.

vii) $S_p(G) \lhd G$ and $G/S_p(G)$ is abelian.

viii) $G/O_p(G)$ is abelian.

ix) $\dfrac{KG}{J(KG)}$ *is isomorphic to a direct sum of fields.*

x) The unit group of $K(G/O_p(G))$ *is soluble.*

xi) $\dfrac{U(KG)}{1+J(KG)}$ *is soluble.*

xii) $\dfrac{U(KG)}{1+J(KG)}$ *is abelian.*

xiii) $\dfrac{U(KG)}{1+J(KG)}$ *does not contain a free group.*

xiv) The transvections of KG *are contained in* $1+J(KG)$.

xv) $\dfrac{U(KG)}{1+J(KG)}$ *is a* p'*−group.*

xvi) $U(KG)'$ *is a nilpotent p-group.*

xvii) The torsion subset of $U(KG)$ *forms a group.*

Note that for G finite, this gives us the following interesting result: provided that either G is not a non-abelian Hamiltonian 2-group or $p \neq 2$, we have $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$, where the statements $i) \ldots iv)$ are given below:

i) $U(KG)$ contains a free subgroup for some field of characteristic $p > 0$,

ii) $U(KG)$ contains free subgroups, where K is a field of any positive characteristic (except possibly one positive characteristic),

iii) $U(ZG)$ contains free subgroups,

iv) $U(FG)$ contains free subgroups for all fields F of characteristic 0.

In Section 3 we prove a result analogous to the previous theorem, but for locally finite groups:

Theorem 32 *Let G be a locally finite group, p any prime. Then the following are equivalent:*

i) $U(KG) \not\supset$ free groups

ii) G' is a $p-$group

vii) There exists a maximal $p-$subgroup P of G with $P \lhd G$ and G/P an abelian p'-group.

viii) $G/O_p(G)$ is abelian (and therefore a $p'-$group).

xi) $U(KG)$ is locally soluble.

xvii) The torsion subset of $U(KG)$ forms a group.

Also, if G' is finite then i) is equivalent to each of the following:

iii) $U(KG)$ is soluble.

iv) $U(F_pG)$ is soluble and the two new exceptions in [5] do not occur.

All of these results can be viewed as variations on Tits Alternative, applied to the unit group of a group ring. In Section 4 we define two new chains of unit groups, U_n and $\tilde{U}_n$, and examine their properties using the previous theorems. Section 5 lists corollaries of Theorems 2 and 3, including several results on the Jacobson radical.

We fix our notation as follows:

$O_p(G) = O_p$ denotes the group generated by all the normal p—subgroups of G.

$S_p(G)$ is a sylow-p-subgroup of G.

F_p is the field with p elements.

Let $H < G$ and K be a field. Then $I(H) := I(KH) := \omega(KH) := \Delta(H)$, the augmentation ideal of KH. Also, $\Delta(G, H) := KG\Delta(H)$.

$J(R) :=$ the Jacobson radical of the ring R.

$L(R) :=$ the sum of all the locally nilpotent ideals of R, called the Levitzki radical of R.

$N^*(R) := \{\alpha \in R \mid \alpha S$ is nilpotent for all finitely generated subrings S of $R\}$.

In a ring R, if $yx = 0$ and r is any element then the unit $1 + xry$ is called a transvection. It has inverse $1 - xry$ and has finite order n if $n = 0$ in R.

Let H be a subgroup of G. We say that H has locally finite index in G if $[V : (V \cap H)] < \infty$ for any finitely generated subgroup V of G.

Define the following chains of subgroups:

$U_0 := \tilde{U}_0 := \bar{U}_0 := G$.

$U_1 := U(F_pG)$, $U_2 := U(F_pU_1), \ldots, U_n := U(F_pU_{n-1})$

$\tilde{U}_1 := U(KG)$, $\tilde{U}_2 := U(KU_1), \ldots, \tilde{U}_n := U(KU_{n-1})$

$\bar{U}_1 := U(KG)$, $\bar{U}_2 := U(K\bar{U}_1), \ldots, \bar{U}_n := U(K\bar{U}_{n-1})$

Let $H < G$. Then define ${}_HU_0 := {}_H\tilde{U}_0 := {}_H\bar{U}_0 := H$.

Define ${}_HU_1 := U(F_pH), \ldots, {}_HU_n := U(F_{pH}U_{n-1})$.

Similarly define ${}_H\tilde{U}_1 := U(KH), \ldots, {}_H\tilde{U}_n := U(K_HU_{n-1})$.

Lastly define ${}_H\bar{U}_1 := U(KH), \ldots, {}_H\bar{U}_n := U(K_H\bar{U}_{n-1})$.

Thus $U_n = {}_GU_n$.

4.2 Finite Groups

We break up the proof of Theorem 31 into several lemmas. Parts *iv)* and *v)* of the following result were stated by Gonçalves in [12].

Lemma 33 *Let $|G| < \infty$, p any prime. Then the following are equivalent:*

i) G' is a $p-$group

ii) $G/O_p(G)$ is abelian

iii) $S_p(G) \lhd G$ and $G/S_p(G)$ is abelian

iv) $U(KG)$ does not contain free groups.

v) $U(KG)$ is soluble.

Proof:

i) $\Leftrightarrow$ *ii)*: G' is a p—group if and only if $G' \leq O_p(G)$ if and only if $G/O_p(G)$ is abelian.

ii) $\Leftrightarrow$ *iii)*: $G/O_p(G)$ abelian implies that $G/O_p(G)$ is a p'—group, so $O_p(G)$ is a Sylow-p-subgroup of G. Conversely, $S_p(G) \lhd G$ implies that $S_p(G) = O_p(G)$.

iii) $\Leftrightarrow$ *iv)*: This was shown in [12].

ii) $\Leftrightarrow$ *v)*: This follows from [19, page 106]. $\square$

Lemma 34 *Let G be finite. Then the group algebra $F_p(t)G$ contains the group ring $F_p\langle t\rangle G \simeq F_p(\langle t\rangle \times G) \simeq (F_pG)\langle t\rangle$ with $J((F_pG)\langle t\rangle) = J(F_pG)\langle t\rangle$.*

Proof:

The statement about the radicals is just [32, page 128]. Note that in general, $R(G \times H) \simeq (RG)H \simeq (RH)G$. $\square$

Lemma 35 *Let G be finite. Then the group $\dfrac{U(KG)}{1 + J(KG)}$ is either abelian or contains a free group.*

Proof:

$U(KG/J(KG)) \simeq \oplus M_{n_i}(K_i)$, where the K_i are fields containing K [33, p.64]. If all n_i equal 1 then we are in the abelian case, and if some $n_i > 1$ then $U(KG/J(KG)) \supset M_2(K)$, which is well known to contain free groups. $\square$

Lemma 36 *Let G be a finite group. Then the transvections of KG are contained in $1 + J(KG)$ if and only if G' is a p-group.*

Proof:

Define θ to be the natural map $U(KG) \to U(KG/J(KG)) \simeq \oplus M_{n_i}(K_i)$, and let θ_i be the map $U(KG) \to M_{n_i}(K_i)$.

$\Leftarrow$: Since G' is a p-group, Lemmas 33 and 35 make $U(KG)/(1 + J(KG))$ abelian, so that if $xy = 0$ then $\theta(x)\theta(y) = 0$, forcing x and y to have "disjoint projections" onto the sum of fields. Thus for any $r \in KG$ we must have $\theta(1+xry) = 1+\theta(x)\theta(r)\theta(y) = 1$, proving the implication.

$\Rightarrow$: Conversely, assume that every transvection $1 + xry$ is contained in $1 + J(KG)$. Now $xy = 0$, so $\theta(x)\theta(y) = 0 \in \oplus M_{n_i}(F_i(t))$. Therefore $\theta(x)$ and $\theta(y)$ either have "disjoint projections" or there exists an i such that $\theta_i(x), \theta_i(y) \in M_{n_i}(F_i(t))$, with $\theta_i(xry) = 0$ for all $r \in KG$. Without loss of generality, let $i = 1$.

Claim 1: $\theta_1(xry) = 0$ for all $r \in KG$ implies that one of $\theta_1(x)$ or $\theta_1(y)$ is 0.

Proof of Claim 1: let $\theta_1(x) := [x_{ij}], \theta_1(r) := [r_{ij}]$, and $\theta_1(y) := [y_{ij}]$, all $n \times n$ matrices. Thus

$$\theta_1(xr) = \begin{bmatrix} \sum_{i=1}^{n} x_{1i}r_{i1} & \sum x_{1i}r_{i2} & \dots & \sum x_{i1}r_{in} \\ \dots & \dots & \dots & \dots \\ \sum x_{ni}r_{i1} & \sum x_{ni}r_{i2} & \dots & \sum x_{ni}r_{in} \end{bmatrix}$$

Thus we may choose r such that if $\theta_1(x)$ has a non-zero entry at, say x_{ij}, we can make the i^{th} row of $\theta_1(xr)$ be anything we choose. Thus, as long as $\theta_1(y) \neq 0$, we can make the i^{th} row of $\theta_1(xry)$ be not all zeros. This proves Claim 1.

Thus, whenever $x, y \in KG$, with $xy = 0$, we must have that $\theta(x)$ and $\theta(y)$ have "disjoint projections".

Claim 2: If for every $x, y \in KG$, with $xy = 0$, we have $\theta(x)$ and $\theta(y)$ having "disjoint projections", then $\frac{KG}{J(KG)} \simeq \oplus$ fields.

Proof of Claim 2: Working by contradiction, without loss of generality assume that $\frac{KG}{J(KG)} \simeq M_2(F_1(t)) \oplus$ other factors. (The use of 2×2 matrices is for typographical convenience - note that $M_2(F_1(t))$ will be a subring of whatever noncommutative factor does occur.) Now let

$$\theta(x) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0, \ldots, 0 \right), \quad \theta(y) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, 0, \ldots, 0 \right) \in \frac{KG}{J(KG)}.$$

Note that since J is nil, we may lift $\theta(x)$ and $\theta(y)$ to orthogonal idempotents x and y of KG [29, page 49]. Thus we have $x, y \in KG$ with $xy = 0$, but with $\theta(x)$ and $\theta(y)$ not having disjoint projections. This contradicts our hypothesis and proves Claim 2. Thus $\frac{KG}{J(KG)} \simeq \oplus$ fields, so [19, pages 100-101] and Lemma 33 give us that G' is a p-group. $\square$

Lemma 37 *Let G be a finite group. Then G' is a $p-$group if and only if $\dfrac{U(KG)}{1+J(KG)}$ is a $p'-$group.*

Proof:

$\Leftarrow$: We prove the contrapositive. Let G' not be a $p-$group. Then by Lemma 36 there is a transvection (and hence a $p-$element) in $U(KG) \setminus 1 + J(KG)$, so $\dfrac{U(KG)}{1+J(KG)}$ is not a $p'-$group.

$\Rightarrow$: Let G' be a $p-$group. Then

$$\frac{U(KG)}{1+J(KG)} \simeq U\left(\frac{KG}{J(KG)}\right) \simeq U\,(\oplus \text{fields of char p})$$

by Lemmas 33 and 35, say. Now if $\theta_i(U(KG))$ contains an element x of order p then $x^p = 1$, so $(x-1)^p = 0$, so $x = 1$, a contradiction. $\square$

Lemma 38 *G' is a p-group if and only if $U(KG)'$ is a nilpotent p-group.*

Proof:

If G' is a p-group then $U(KG)/(1+J)$ is abelian, so $U' < 1 + J$, so U' is nilpotent.

If G' is not a p-group then $U(KG)$ contains free groups, and therefore so does its commutator, which cannot then be a nilpotent p-group. □

Proof of Theorem 31:

$i) \Leftrightarrow ii)$, by Lemma 33.

$i) \Leftrightarrow iii)$, by Lemma 33.

$i) \Leftrightarrow iv)$, by Theorem 28.

$iv) \Leftrightarrow v)$, as $1 + J$ is a normal nilpotent subgroup of $U(F_pG)$.

$v) \Leftrightarrow vi)$, as $U(F_pG)/(1 + J) \simeq U(F_pG/J)$.

$i) \Leftrightarrow vii)$, by Lemma 33.

$iii) \Leftrightarrow x)$, by [19, page 100].

$ii) \Leftrightarrow ix)$, by [19, pages 100-101].

$ii) \Leftrightarrow viii)$: By Lemma 33.

$iii) \Leftrightarrow xi)$: since $1 + J(KG)$ is a nilpotent normal subgroup of $U(KG)$.

$ix) \Leftrightarrow xii)$: The quotient is the direct sum of fields $\Leftrightarrow$ it is abelian.

$xiii) \Leftrightarrow xii)$: This was Lemma 35.

$ii) \Leftrightarrow xiv)$: This was Lemma 36.

$ii) \Leftrightarrow xv)$: This was Lemma 37.

$ii) \Leftrightarrow xvi)$: This was Lemma 38.

$i) \Leftrightarrow xvii)$: This will be proved later in Theorem 62. □

Example 5 *$J(F_5D_{10})$ has dimension 8 over F_5 by [18, p.459]. Thus $\frac{F_5D_{10}}{J(F_5D_{10})} \simeq F_{5^2}$ or $F_5 \oplus F_5$. In fact, by the group lattice diagram in Chapter 5 we know that $\frac{F_5D_{10}}{J(F_5D_{10})} \simeq F_5 \oplus F_5$. In particular, it is commutative. Note that $U(F_5D_{10}) > D_{10}$, which is not nilpotent. This contradicts the Lemma in [2], although the Theorem in that paper is true (see [32, p.179]). Thus $F_5(t)D_{10}$ is an example of a group ring whose unit group is soluble but not nilpotent.* □

We now give some miscellaneous results for finite groups:

Theorem 39 *Let $|G| < \infty$. Let $p \neq 2$ or 3. Then $U(F_pG)$ is soluble $\Leftrightarrow$ $U(KG) \not\supset$ free groups.*

Proof:

$\Rightarrow$: $U(F_pG)$ is soluble $\Leftrightarrow$ G' is a $p-group$ [32, page 205] $\Leftrightarrow$ $U(KG)$ is soluble $\Rightarrow U(KG) \not\supset$ free groups.

$\Leftarrow$: *Case i.)* $Syl_p(G) \not\triangleleft G$. Here (by [12]), $U(KG) \supset$ free groups, so this implication is vacuously true.

Case ii.) Let $P = Syl_p(G) \triangleleft G$. Here $U(KG) \not\supset$ free groups, so by [12] again, G/P is abelian, so $G' < P$, so G' is a $p-group$, so $U(F_pG)$ is soluble [32, page 205]. $\square$

Note: If $p = 2$ and $U(KG)$ does contain free groups then $U(F_2G)$ may or may not be soluble. Similarly for $p = 3$. The following theorem describes the situation in detail.

We need the following definitions for the next result:

$P := \langle a, b : a^8 = 1, b^2 = 1, bab^{-1} = a^3 \rangle$,

$D := \langle a, b : a^4 = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle$,

$Q := \langle a, b : a^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$.

When G is a $2-group$, define $e(G) := (|G| - (G : G'))/4$,

$r(G) := |\{N \triangleleft G : G/N \simeq P\}|$, and

$s(G) := |\{N \triangleleft G : G/N \simeq D \text{ or } Q\}|$.

Theorem 40 *[27, Theorem p.211] Let F be a field (of any characteristic) and G be a finite group. Then $U(FG)$ is soluble if and only if one of the following conditions holds:*

i.) $G \simeq G_p \rtimes A$, where A is an abelian group (i.e. G' is a $p-group$. By definition, $G_0 = 1$.)

ii.) $F = F_2$, G' is not a $2-group$ and $G^* = G/O_2(G) \simeq (C_3 \times \ldots \times C_3) \rtimes C_2$ (with C_2 acting by inversion on each element).

iii.) $F = F_3$, G' is not a $3-group$, $G \simeq G_3 \rtimes G_2$ and $e(G_2) = 2r(G_2) + s(G_2)$. □

Theorem 41 *Let G be a finite group. Then either $U(F_pG)$ is soluble or $U(KG)$ contains free groups, or both.*

Both happen precisely when either case ii.) or iii.) of Theorem 40 occurs.

Proof:

By Lemma 33, at least one happens. Both happen $\Leftrightarrow$ $U(F_pG)$ is soluble and $U(KG) \supset$ free groups $\Leftrightarrow$ $U(F_pG)$ is soluble and G' is not a p-group (by Lemma 33) $\Leftrightarrow$ Case ii.) or Case iii.) of Theorem 40 happens. □

4.3 Locally Finite Groups

Throughout this section let G be a locally finite group. Note that if F is a field of characteristic $p > 0$ which is algebraic over F_p then $U(FG)$ is a torsion group. (Indeed, letting $u = \sum u_g g \in U(FG)$, we see that $u \in U(KH)$, where K is a finite field and H is a finite group, so u is contained in a finite group, and therefore is a torsion element.)

Proof of Theorem 29:

Clearly $i) \Rightarrow ii) \Rightarrow iii)$. Let $p > 0$. Assuming $ii)$, let $x, y \in G$. Then $U(K\langle x, y\rangle)$ is soluble and $\langle x, y\rangle$ is finite, so $[x, y] = 1$ (by Theorem 31), which implies $i)$. Assuming $iii)$, we get that $U(K\langle x, y\rangle)$ does not contain free subgroups, so $[x, y] = 1$, again by Theorem 31.

In the characteristic 0 case Theorem 7 and [32, Corollary 4.14 p.205] give the result. $\square$

Lemma 42 *Let p be any fixed prime and let G be a locally finite group. Then $U(KG) \not\supset$ free groups $\Leftrightarrow G'$ is a $p-group$.*

Proof:

G' is a $p - group \Leftrightarrow H'$ is a $p - group$ for all finitely generated subgroups H of G $\Leftrightarrow U(KH) \not\supset$ free groups for all finitely generated subgroups H of G (by Theorem 31)

$\Leftrightarrow U(KG) \not\supset$ free groups (indeed, if $\langle u, v\rangle$ generate a free subgroup of $U(KG)$, then letting $H_1 := \langle supp\ u, supp\ v\rangle$ we have $|H_1| < \infty$ and $U(KH_1)$ contains a free group, a contradiction). $\square$

Lemma 43 *Let G be a locally finite group, p any prime, and let G' be finite. Then $U(KG) \not\supset$ free groups $\Leftrightarrow U(KG)$ is soluble.*

Proof:

Clearly $U(KG)$ soluble $\Rightarrow U(KG) \not\supset$ free groups. For the converse suppose first that $p \geq 3$. Then $U(KG) \not\supset$ free groups $\Rightarrow G'$ is a finite $p - group$ (by Lemma 42) $\Rightarrow U(KG)$ is soluble (by [5, Theorem 1]).

Next, let $p = 2$. Then G' a finite $2 - group \Rightarrow U(KG)$ is soluble [5, Theorem 2 with

$G' = N]$. $\square$

Lemma 44 *Let p be any fixed prime, let G be a locally finite group with G' finite. Then either $U(F_pG)$ is soluble or $U(F_p(t)G) \supset$ free groups, or both. Both happen precisely when one of the two new exceptions (in [5]) happens.*

Proof:

Assume that $U(F_p(t)G) \not\supset$ free groups. Then $U(F_p(t)G)$ is soluble (by Lemma 43), so $U(F_pG)$ is soluble.

Note that both happen $\Leftrightarrow U(F_pG)$ is soluble & G' is not a $p-group$ (by Lemma 42). In this case we can reduce to G finite to get by Theorem 41 that $p = 2$ or 3. Now this is equivalent to one of the two new exceptions happening. [5] $\square$

Proof of Theorem 32:

$i) \Leftrightarrow ii)$: Lemma 42.

$ii) \Leftrightarrow vii)$: Recall that for any group G and any subgroup H, $G' < H \Leftrightarrow H \lhd G$ & G/H is abelian. Thus, G' is a $p-group \Leftrightarrow G' < P$ a maximal $p-subgroup$ of $G \Leftrightarrow P \lhd G$ & G/P is abelian.

$ii) \Leftrightarrow viii)$: This is exactly as in the proof of Theorem 31.

$i) \Leftrightarrow xi)$: Let V be a finitely generated subgroup of $U(KG)$. Then the support of the generators of V is a finite set, and hence generates a finite subgroup H of G. Thus $V < U(KH)$. Thus $U(KG) \not\supset$ free groups $\Rightarrow U(KH) \not\supset$ free groups $\Rightarrow U(KH)$ is soluble (by Theorem 31) $\Rightarrow V$ is soluble. Thus $i) \Rightarrow xi)$, and the converse is trivial.

$i) \Leftrightarrow xvii)$: This will be proved later in Theorem 62.

Assume that G' is finite.

$i) \Leftrightarrow iii)$: This was Lemma 43.

$iii) \Leftrightarrow iv)$: $U(KG)$ is soluble $\Leftrightarrow U(KG) \not\supset$ free groups, (as $i) \Leftrightarrow iii)$) $\Leftrightarrow U(F_pG)$ is soluble and the two new exceptions don't happen (by Lemma 44). $\square$

4.4 Nilpotent and FC groups

An FC group is a group in which each element has only finitely many conjugates. Define $T(G)$ to be the *set* of torsion (periodic) elements of the group G.

Lemma 45 *i): Let G be an FC group with $P = S_p(G) \lhd G$. Then $\Delta(G,P) = J(KG)$ is nilpotent.*
ii): Let G be a locally soluble group. Then $J(KG)$ is locally nilpotent.

Proof: Note that *ii)* is just [20, Corollary 46.33 p.399].

$$J\left(\frac{KG}{\Delta(G,P)}\right) = J\left(K\frac{G}{P}\right) = 0$$

by [20, page 401 Theorem 47.1.iii]. By [29, pages 317-8], $J(KP)KG \subset N^*(KG)$. But P is a locally finite p-group, so by Lemma 75 $J(KP) = \Delta(P)$. Thus $\Delta(G,P) \subset N^*(KG) \subset J(KG)$ [29, page 323 2nd paragraph]. Hence we get that $\Delta(G,P) = J(KG)$ [20, page 18 iv]. Lastly, $J(KG)$ is nilpotent by [29, page 312] and [20, page 401 Theorem 47.1.iii]. $\square$

Lemma 46 *Let G be an FC group, T the torsion subgroup of G and $S_p(T)$ a sylow p-subgroup of T. ($S_0(T)$ is defined to equal 1). Let F be any field of characteristic $p \geq 0$. If $p > 0$ then assume that either F is not algebraic over F_p or that G contains a free abelian group of rank 2.*
If $U(FG)$ does not contain free groups then

i) $T/S_p(T)$ is abelian and every subgroup of $T/S_p(T)$ is normal in $G/S_p(T)$. Conversely, if i) holds and $T/S_p(T) < Z(G/S_p(T))$ then $U(FG)$ is locally nilpotent - by - locally nilpotent and hence does not contain free groups.

Proof:

$T \lhd G$ [6, page 201] and T is locally finite. Assume that $U(FG)$ does not contain free groups. We may assume without loss of generality that G is finitely generated. (As *i*) involves only local properties). Thus T is finite [36]. Thus $U(FT)$ does not contain free groups, so by Theorem 32, T' is a p-group, so $S_p(T) \lhd T$, $S_p(T) \lhd G$ and $T/S_p(T)$ is abelian. If necessary, quotient out $S_p(T)$ to assume that G is a p' group, it is FC, and T is a finite abelian subgroup of G. It remains to show that every subgroup of T is normal in G. Proceed as in [15, proof of Lemma 4]. Assume otherwise, so let H be a finite subgroup of G with $x \in G \setminus N_G(H)$. Define $e = |H|^{-1}\hat{H}$. Now e is an idempotent in $FH \subset FT$. Define $f = e(1 - e^x)$. Note that supp $ee^x \subset HH^x \not\subset H$, so $e \neq ee^x$, so $f \neq 0$. Thus f is a non-zero idempotent of FT with $ff^x = 0 = f^x f$. Define $e_{12} = fx, e_{21} = x^{-1}f, e_{11} = e_{12}e_{21} = f, e_{22} = e_{21}e_{12} = f^x$. Thus $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where δ_{jk} is the Kronecker delta function, $i, j, k, l \in \{1, 2\}$. Since no e_{ij} equals 0, we have that $R := \Sigma_{i,j=1}^2 Fe_{ij} \simeq M_2(F)$.

If F is not algebraic over F_p then $FG \supset R \simeq M_2(F)$ contains free groups, a contradiction.

Assume therefore that G contains a free abelian group of rank 2. Thus by [36, Theorem 1.7.ii) p.4] we may assume that G contains a central element y of infinite order such that (possibly replacing y by some power of y) $\langle y\rangle \cap \langle H, x\rangle = \{1\}$. Hence $FG \supset F\langle H, x, y\rangle \simeq F(\langle y\rangle \times \langle H, x\rangle) \simeq F\langle y\rangle \otimes_F F\langle H, x\rangle$ by [19, Corollary 1.4 p.12]. Now this contains $F\langle y\rangle \otimes_F M_2(F) \simeq M_2(F\langle y\rangle \otimes_F F)$ by [20, Proposition 16.8.i) p.97]. By [19, Corollary 1.2 p.11] this is isomorphic (as a F-algebra) to $M_2(F\langle y\rangle)$. Following the proof of [11, Theorem 2.5 p.367] note that this matrix ring contains the matrices

$$A := \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix}, \quad B := P \begin{bmatrix} y & 0 \\ 0 & y^{-1} \end{bmatrix} P^{-1},$$

where

$$P = \begin{bmatrix} 1+y & y \\ -y & 1-y \end{bmatrix}.$$

Now A and B are elements of $SL_2(F\langle y\rangle)$. By [35, Proposition 3.12], some powers of A and B generate a free group.

Conversely, assume that $i)$ holds and $T/S_p(T) < Z(G/S_p(T))$. Again we may assume that G is finitely generated. Let $\bar{G} = G/S_p(T)$. Now $\bar{G}$ is a finitely generated p' FC group, and by [36] it is a subgroup of the direct sum of a finite group and an abelian group, so $\bar{G}'$ is finite, and therefore central. Thus $[\bar{G}, \bar{G}, \bar{G}] = 1$, so $\bar{G}$ is nilpotent (of class 2) and $\bar{T} < Z(\bar{G})$, so $U(F\bar{G})$ is nilpotent [32, Theorem 3.6 p.181-2]. By the previous Lemma $1 + \Delta(G, S_p(G))$ is also nilpotent, and since $U(FG)/(1 + \Delta(G, P)) \simeq U(F\bar{G})$, we see that $U(FG)$ is nilpotent-by-nilpotent. Thus $U(FG)$ is locally nilpotent-by-nilpotent.

The characteristic 0 case is similar (and a little shorter). In the converse we get that $U(FG)$ is actually locally nilpotent. □

The hypothesis that G contains a free abelian group of rank 2 is not as restrictive as it might appear. By [36, Theorem 1.7 p.4] this is equivalent to saying that G is not isomorphic to a subgroup of $G_1 \times C_\infty$, where G_1 is a periodic FC group.

Lemma 47 *Let G be a locally nilpotent group, T the torsion subgroup of G and $S_p(T)$ a sylow p-subgroup of T. (Again $S_0(T) := 1$). Let F be a field of characteristic $p \geq 0$.*

If $p > 0$ then assume that F is not algebraic over F_p. If $U(FG)$ does not contain free groups then

i) $T/S_p(T)$ is abelian and every subgroup of $T/S_p(T)$ is normal in $G/S_p(T)$.

Conversely, if i) holds and $\frac{T}{S_p(T)} < Z\left(\frac{G}{S_p(T)}\right)$ then $U(FG)$ is locally nilpotent - by - locally nilpotent and hence does not contain free groups.

Proof:

Let $P = S_p(T)$. Then
$$\frac{U(FG)}{1 + \Delta(G, P)} \simeq U(F\frac{G}{P}).$$
Assume that $U(FG)$ does not contain free groups. Hence neither does $U(FT)$, and since T is locally finite ([6, Section 2] and use a finitely generated argument) we have that T' is a p-group. Thus $P \lhd G$ and T/P is abelian. Let $x \in T$.

Now, by Lemmas 74 and 75 we get that $J(FP) = \Delta(P)$ is a locally nilpotent ideal. A quick check (and the fact that $P \lhd G$) shows that $\Delta(G, P)$ must also be a locally nilpotent ideal. Hence $\Delta(G, P)$ is a nil ideal and we get that the preimage of a unit in $U(F\frac{G}{P})$ (under the obvious map) is a unit in $U(FG)$. Thus we have that $U(F\frac{G}{P})$ does not contain free groups and clearly $\frac{G}{P}$ contains no p-elements. If $\frac{\langle x \rangle}{P} \ntriangleleft \frac{G}{P}$ then Theorem 17 gives a contradiction. Thus every subgroup of $\frac{T}{P}$ is normal in $\frac{G}{P}$ as required.

Next we prove the converse. Again note that by Lemmas 74 and 75 we have that $\Delta(G, P)$ is a locally nilpotent ideal, so by Lemma 71, $1 + \Delta(G, P)$ is a locally nilpotent group. Hence, without loss of generality we may assume that G is a finitely generated nilpotent group without p-elements and with $T < Z(G)$. Thus by [32, p.181–2] we have that $U(FG)$ is nilpotent as required.

Again the characteristic 0 case is similar and shorter. In the converse we get that $U(FG)$ is actually locally nilpotent. □

Note that in Lemma 47 the restriction on the field is not needed for the converse.

Theorem 48 *Let G be a group which is either FC, locally nilpotent or locally finite. Let T be the torsion subgroup of G and $S_p(T)$ a sylow p-subgroup of T. Let F be a field of characteristic $p \geq 0$. If $p > 0$ then assume that F is not algebraic over F_p. If $U(FG)$ does not contain free groups then*

i) $T/S_p(T)$ is abelian and every subgroup of $T/S_p(T)$ is normal in $G/S_p(T)$.

Conversely, if i) holds and $\frac{T}{S_p(T)} < Z\left(\frac{G}{S_p(T)}\right)$ then $U(FG)$ is locally nilpotent - by - locally nilpotent and hence does not contain free groups.

Proof:

Let G be a locally finite group. If $U(FG)$ does not contain free groups then by Theorem 32 ($i) \Leftrightarrow ii)$), $i)$ above is satisfied. Conversely, assume that $i)$ holds and $\frac{T}{S_p(T)} < Z\left(\frac{G}{S_p(T)}\right)$. Then G' is a p-group. So Lemma 84 gives us that $F\frac{G}{O_p(G)} = \frac{FG}{J(FG)}$, which is commutative, so $\frac{U(FG)}{1+J(FG)}$ is abelian. Also, Lemmas 74 and 71 give us that $1 + J(FG)$ is locally nilpotent. The FC and locally nilpotent cases follows from Lemmas 46 and 47.

The characteristic 0 case is trivial. □

We finish the section with a modest generalisation of a result of Coelho and Polcino Milies [6, Theorem 2.3 p.203]:

Theorem 49 *Let G be either locally finite or nilpotent or an FC group. Let T be the set (in fact a group) of periodic elements of G anfd let F be any field of characteristic $p > 0$. Then the periodic units of $U(FG)$ form a subgroup if and only if one of the* *following conditions holds:*

i) G is abelian

ii) $G = T$ and F is algebraic over its prime field F_p

iii) The set P of p-elements in G is a subgroup, $T' \subset P$ and if T/P is non-central in G/P then Ω, the algebraic closure of F_p in F, is finite and, for all $x \in G$ and all p'-elements $a \in T$, we have that xax^{-1} is of the form $xax^{-1} = a^{p^r}y$, where $r \geq 0$ and $y \in P$. Furthermore, for every such an exponent r we have that $[\Omega : F_p] \mid r$.

Proof:

By [6, Theorem 2.3 p.203] we need only prove the case where G is locally finite. If *i)* or *iii)* occur then G' is a p-group. Either F is algebraic over F_p or it is not. If not, then Theorem 62 gives us that the periodic units of $U(FG)$ form a subgroup. If F is algebraic over F_p then the ring FG is locally finite, so its unit group is also locally finite, giving us that again the periodic units of $U(FG)$ form a subgroup.

Now assume that the periodic units of $U(FG)$ form a subgroup. We may assume that F is not algebraic over F_p, and hence by Theorem 62 we have that G' is a p-group, so that *i)* or *iii)* holds. $\square$

The characteristic 0 version of this result is a little simpler:

Theorem 50 *Let G be either locally finite or nilpotent or an FC group. Let T be the set (in fact a group) of periodic elements of G anfd let F be any field of characteristic 0. Then the periodic units of $U(FG)$ form a subgroup if and only if both of the following conditions hold:*

i) T is abelian

ii) For each $t \in T$ and each $x \in G$ there exists a positive integer i such that $xtx^{-1} = t^i$ and, for each non-central element $t \in T$, F contains no root of unity of order $o(t)$.

Proof:

The nilpotent and FC cases are just quoted from [6, Theorem 3.2 p.204]. Assume that G is locally finite. If the periodic units of $U(FG)$ form a subgroup then [6, Lemma3.1 p.203] implies that T is abelian, so i) and ii) hold. The converse is trivial since $G = T$ is abelian. □

4.5 U_n

All of the previous results relate the structure of G to that of the unit group $U(KG)$. This process can be repeated, as the definitions of U_n, $\tilde{U}_n$ and $\bar{U}_n$ suggest. This gives us a new method of constructing groups. A good deal of work has been done to investigate subgroups V of $U(KG)$ with the property that V is linearly independent over the field K [4]. In this case $KU(KG)$ contains KV, but the latter algebra can also be viewed as a subalgebra of KG. Hence we also get the chain of groups $V < U(KV) < U(KU(KV)) < \cdots < U(KG)$. Note that if V is normal in $U(KG)$ then this chain is a normal series.

Proposition 51 *Let G be finite and let n be a positive integer. Then U_n is nilpotent for some $n \geq 1$ if and only if U_n is nilpotent for all $n \geq 1$.*

Proof:

It suffices to show that U_1 nilpotent implies U_2 nilpotent. So Letting U_1 be nilpotent, it is soluble, so Theorem 31 gives us that G' is a $p-$group. Thus the nilpotency of G gives us that $G = P \times H$, where $P = O_p(G)$ and H is a p', abelian group. Thus $F_pG \simeq (F_pH)P$, where F_pH is a commutative coefficient ring. Thus, by Lemma 75,

$U(F_pG) \simeq (1+\Delta(P)) \times U(F_pH)$. Note that $1+\Delta(P)$ is a finite p-group and $U(F_pH)$ is a finite abelian p'-group, so U_1 is nilpotent and U_1' is a p-group (Theorem 31). Repeating this process we see that U_2 is nilpotent with U_2' a p-group. □

Proposition 52 *Let G be finite and let G' be a $p-$group. Then $\tilde{U}_n$ and U_n are soluble for all n, and U_n is finite. (And thus U_n' is a $p-$group for all n (by Theorem 31))*

Proof 1:

Since $U_n < \tilde{U}_n$, it suffices to show that $\tilde{U}_n$ is soluble for all n. By Theorem 31 $\tilde{U}_1$ is soluble. We proceed inductively, assuming that $\tilde{U}_n$ is soluble. So Theorem 31 applies to U_{n-1}. Now $\tilde{U}_{n+1} = U(KU_n)$ is soluble

$$\Leftrightarrow U\left(K\frac{U(F_pU_{n-1})}{1+J(F_pU_{n-1})}\right)$$

is abelian $\Leftrightarrow \frac{U(F_pU_{n-1})}{1+J(F_pU_{n-1})}$ is abelian. But this is the case by Theorem 31. Thus $\tilde{U}_{n+1}$ is soluble and our proof is completed.

Proof 2:

Let G be a finite group. We will show that the conditions of Theorem 31 apply if and only if U_1' is a $p-group$. $U(F_pG)/(1+J) = U_1/O_p(U_1)$ [1, Theorem 4] $\simeq \oplus GL(n_i, F_i)$. Now $p|\ |GL(n_i, F_i)| \Leftrightarrow n_i > 1$ (indeed, $(p, p^n-1) = 1$ and $|GL(2, F_p)| = (p^2-1)(p^2-p)$). Thus, $U_1/O_p(U_1)$ is a $p'-group \Leftrightarrow$ it is abelian $\Leftrightarrow U_1'$ is a $p-group$. □

Proposition 53 *Let G be a locally finite group with G' a $p-$group. Then for all $n \geq 0$, U_n and $\tilde{U}_n$ are locally soluble and U_n is locally finite. (Thus, by Theorem 32, U_n' is a $p-$group).*

Proof:

Note first that $U_1 = U(F_pG)$ is locally finite. (Indeed, if $u_1, \ldots, u_n \in U_1$ then $H := \langle supp\ u_1, \ldots, supp\ u_n \rangle$ is finite, so $u_1, \ldots, u_n \in U(F_pH)$, which is finite) Inductively, U_n is locally finite.

It remains to prove that $\tilde{U}_n$ is locally soluble.

Proof 1:

By Theorem 32, it suffices to prove that U_n' is a $p-$group for all n. To start our induction, note that $U_0' = G'$ is a $p-$group. Now our inductive hypothesis is that U_n' is a $p-$group, and we will show that U_{n+1}' is a $p-$group. Let $u = [u_1, v_1] \ldots [u_m, v_m] \in U_{n+1}'$, with $u, u_i, v_i \in U(F_pU_n)$. Defining

$$H := \langle supp\ u_1, \ldots supp\ u_m, supp\ v_1, \ldots supp\ v_m \rangle,$$

we see that H is a finite subgroup of U_n and $u, u_i, v_i \in U(F_pH)$ and $u \in U(F_pH)'$. Now $H < U_n$, so $H' < U_n'$, so by our inductive hypothesis H' is a $p-$group. Therefore Theorem 31 gives us that $U(KH)'$ is a $p-$group. Thus u is a $p-$element, so U_{n+1}' is a $p-$group.

Proof 2:

We will show that $\tilde{U}_n$ is locally soluble for all n. Let $u_1, \ldots, u_m \in \tilde{U}_n = U(KU_{n-1})$. Now U_{n-1} is locally finite so $H := \langle supp\ u_1, \ldots, supp\ u_m \rangle$ is finite, and $u_1, \ldots, u_m \in U(KH)$. Now $H' < U_{n-1}'$ is a $p-$group by inductive hypothesis, so $U(KH)$ is soluble by Theorem 31. Thus $u_1, \ldots, u_m$ are elements of a soluble group, so $\tilde{U}_n$ is locally soluble. $\square$

Corollary 54 *Let G be locally finite with G' a $p-$group. Then $J(K\tilde{U}_n)$ and $J(KU_n)$ are locally nilpotent.*

Proof:

Just apply [20, p.399] to $\tilde{U}_n$. $\square$

4.6 Corollaries

Many of the results of this section are generalisations to the torsion subset of $U(KG)$ of results about periodic linear groups. In the characteristic 0 case, a result of Schur [37, Corollary 9.4 p.113] states that any periodic subgroup of $GL(n_i, F_i)$ is abelian-by-finite. As we will see below, the positive characteristic case is much more interesting.

Corollary 55 *If G is a finite group then a subnormal subgroup V of $U(KG)$ is either soluble or contains a free group. If G is a locally finite group then a subnormal subgroup V of $U(KG)$ is either locally soluble or contains a free group.*

Proof:

Apply Theorem 31 iii) and viii) and [11, Theorem 2.5]. $\square$

Corollary 56 *If G is a finite group with $G' = G$ then $U(KG)/(1+J(KG))$ contains a free group.*

Proof:

Since $G' = G$, G is not nilpotent, so $G \not\leq 1 + J(KG)$. Hence $\theta(G) \neq 1$, so $\theta(G)$ is a non-abelian subgroup of $U(KG)/(1 + J(KG))$ (as G has no abelian images and by the second isomorphism theorem). Thus $U(KG)/(1+J(KG))$ is a non-abelian group, so by Theorem 31 it contains a free group.

Note that this Corollary is also a consequence of Theorem 31 parts *ii*), *xii*) and *xiii*) as $G' = G$ implies that G' is not a p-group (otherwise G is a finite p-group and hence nilpotent, a contradiction). $\square$

The following lemma is a direct consequence of [5, Theorem 1].

Lemma 57 *Let $p \neq 2$ or 3, let K be any field of characteristic p and let G be a torsion group. Then $U(KG)$ is soluble $\Leftrightarrow$ G' is a finite $p-$group.* $\square$

Theorem 58 *Let G be a locally finite group, let K be any field of characteristic $p > 0$ and let V be a finitely generated subgroup of $U(KG)$. Then V satisfies the Tits Alternative, that is, either V is soluble-by-finite or it contains a free group.*

Proof:

Since V is a finitely generated subgroup of $U(KG)$, we may consider G to be finite. Let

$$\theta : U(KG) \to \frac{U(KG)}{1+J(KG)}$$

be the natural map so that

$$\theta(V) := \frac{(1+J(KG))V}{1+J(KG)}.$$

Now $\theta(U(KG)) \simeq U(\oplus M_{n_i}(K_i))$. Define θ_i to be the projection onto the i^{th} linear group. Thus $\theta_i(U(KG)) \simeq GL_{n_i}(K_i)$, a linear group. Thus, for all i, $\theta_i(V)$ satisfies the Tits Alternative, so $\theta_i(V)$ is either a soluble-by-finite group or contains a free subgroup. Now if some $\theta_i(V)$ contains a free group then V must contain a free group (using the Second Isomorphism Theorem on $\theta(V) = \langle V, 1 + J(KG)\rangle/(1 + J(KG))$). Therefore we may assume that $\theta_i(V)$ is soluble-by-finite for all i. Thus $\theta(V) \simeq \prod$soluble-by-finite groups $\simeq$ a soluble-by-finite group. But $1 + J(KG)$ is nilpotent, so $\langle V, 1 + J(KG)\rangle$ is soluble-by-finite, so V is soluble-by-finite. $\square$

Theorem 59 *Let G be locally finite, K any field of characteristic $p > 0$. Then the General Burnside Problem has a positive answer for $U(KG)$ (i.e. a finitely generated torsion subgroup of $U(KG)$ is finite).*

Proof:

Let V be a finitely generated torsion subgroup of $U(KG)$. Then by the preceding theorem, V is either soluble-by-finite or contains free groups. Therefore V is soluble-by-finite, i.e. there exists a soluble normal subgroup S of V such that $[V : S] < \infty$. But a subgroup of finite index of a finitely generated group is finitely generated [30, page 36], so S is a finitely generated soluble torsion group, and therefore S is finite. [30, page 147]. $\square$

Lemma 60 *Let G be finite. If the set of torsion elements T of $U(KG)$ forms a group then it is soluble and locally finite.*

Proof:

By Corollary 55, T is soluble and by Theorem 59 it is locally finite. $\square$

Next we classify those locally finite groups G such that the torsion units $TU(KG)$ form a group. The characteristic 0 case has been done by S. Coelho and C. Polcino Milies:

Lemma 61 *[6, Lemma 3.1] Let G be a group such that $T(G)$ is locally finite and let F be a field of characteristic* 0. *If $TU(FG)$ is a subgroup then $T(G)$ is abelian.* $\square$

Theorem 62 *Let G be a locally finite group. If $U(KG)$ does not contain free groups then the subset T of torsion elements of $U(KG)$ forms a locally finite, soluble normal subgroup of $U(KG)$. Conversely, if T is a subgroup of $U(KG)$ then $U(KG)$ does not contain free groups.*

Proof:

Let $a_1, \ldots, a_n \in T$. Considering only the group $\langle supp(a_1), \ldots, supp(a_n) \rangle$, without loss of generality we may consider G to be finite. Assume that $o(a_1 \ldots a_n) = \infty$. Then $o(\theta(a_1 \ldots a_n)) = \infty$.

(Otherwise we get $\theta(a_1 \ldots a_n)^m = 1$ for some m, so $(a_1 \ldots a_n)^m \in 1 + J$, a $p-$group, contradicting the infinite order of $a_1 \ldots a_n$.)

Thus $o(\theta_i(a_1 \ldots a_n)) = \infty$ for some i. Now $\theta_i(KG)$ is a field, so $\theta_i(a_1 \ldots a_n)^s = \theta_i(a_1)^s \ldots \theta_i(a_n)^s = 1$ for some large s ($s = LCM\ o(a_i)$ would work). This contradiction proves that $T \lhd U(KG)$. Lemma 60 completes the assertion.

To prove the converse we use a contrapositive argument. Assume that $U(KG)$ does contain free groups, so that G' is not a $p-$group. Then choose $a_1, \ldots, a_n \in G$ such that $\langle [a_i, a_j] | i, j = 1, \ldots, n \rangle$ has order mp^a, with $m \neq 1$, $(m, p) \neq 1$. Let $H = \langle a_1, \ldots a_n \rangle$. Then H is finite and H' is not a $p-$group. Thus

$$\frac{U(KH)}{1 + J(KH)} \simeq \oplus GL_{n_i}(K_i) > GL_2(K).$$

Let t be a transcendental element of K over F_p. Define

$$d := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \quad e = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}.$$

Then $d^p = e^p = 1$, but

$$de = \begin{bmatrix} 1+t^2 & t \\ t & 1 \end{bmatrix}.$$

Thus de cannot have finite order, since t is not algebraic oven F_p. Now any preimages of d and e in $U(KH)$ will be $p-$elements (as $1 + J$ is a $p-$group). Thus $U(KH)$ contains two elements of finite order which do not generate a torsion group. Hence T is not a subgroup of $U(KG)$. $\square$

Lemma 63 *Let G be a locally finite group. Then*

$$x \in J(KG) \Leftrightarrow x \in J(KH) \textit{ for all finite subgroups } H \textit{ of } G \textit{ with supp } x \subset H.$$

Proof:

Let H be any finite subgroup of G. Then by [20, Lemma48.2.ii)], $N^*(KH) = N(KH)$. By Lemma 76 $N^*(KH) = J(KH)$ and $N^*(KG) = J(KG)$. Thus by [20, Lemma 48.6]

$$x \in J(KG) \Leftrightarrow x \in J(KH) \text{ for all finite subgroups } H \text{ of } G \text{ with supp x} \subset H. \ \square$$

Proposition 64 *Let G be a locally finite group with G' a p-group. Then $\frac{U(KG)}{1+J(KG)}$ is abelian. Consequently $U(KG)/O_p(U)$ is an abelian group without p-elements.*

Proof:

Let $x \in U(KG)'$, say $x = [a_1, b_1] \ldots [a_n, b_n]$. Let H be any finite subgroup of G containing the support of x. Define H_1 to be the finite group generated by the set $\{H, supp\, a_1, supp\, b_1, \ldots supp\, a_n, supp\, b_n\}$. Now H is a subgroup of H_1. $x \in U(KH_1)' <$

$1+J(KH_1)$ by Theorem 31, as H' is a p-group. Thus $x-1 \in J(KH_1) \cap KH \subset J(KH)$ by [20, Lemma 48.2.iii)]. Now by Lemma 63 $x-1 \in J(KG)$, so $U(KG)' < 1+J(KG)$ as required. □

It is therefore of interest to examine the sylow p-subgroups of $U(KG)$ when G is locally finite but G' is not necessarily a p-group.

Lemma 65 *Let G be locally finite and let P_1 and P_2 be sylow-p subgroups of $U(KG)$. If $H = \langle P_1, P_2 \rangle$ is a torsion group then P_1 and P_2 are conjugate in H.*

Proof:

Consider $H = \langle P_1, P_2 \rangle < U(KG)$. Now for each I, $\theta_i(H)$ is a periodic linear group and has sylow p-subgroups $\bar{Q}_1 > \theta_i(P_1)$ and $\bar{Q}_2 > \theta_i(P_2)$. Now $\bar{Q}_1$ is conjugate to $\bar{Q}_2$ [7, p.163]. Thus there exists $h_i \in H$ such that $\bar{Q}_1^{\theta_i(h_i)} = \bar{Q}_2$. Since this is true for all i, there exist Q_1, $Q_2 < U(KG)$ with $\theta_i(Q_j) > P_j$ and $h \in H$ with

$$\frac{Q_1^h}{1+J} = \frac{Q_2}{1+J}.$$

Now $\frac{Q_i}{1+J}$ is a p-group, so Q_i is a p-group. Thus $Q_i(1+J) > P_i(1+J) = P_i$, so $Q_i(1+J) = P_i$. Thus $P_1^h = P_2$. □

Note that if G' is a p-group then the p-elements of $U(KG)$ form a group, so trivially we get that the sylow-p subgroups of $U(KG)$ are conjugate. However, if G' is not a p-group then as we saw in the proof of Theorem 62, the p-elements not only do not form a group, but do not even generate a torsion group. This limits the usefulness of Lemma 65.

Lemma 66 *Let G be a finite group. Then G' is a $p-$group if and only if for all subgroups V of $U(KG)$, V' is a $p-$group.*

Proof:

If V' is a $p-$group for all subgroups V of $U(KG)$ then clearly G' is a $p-$group.

If G' is a $p-$group then $U(KG)/(1+J)$ is abelian, so $U(KG)' < 1+J$, a $p-$group, so V' is a $p-$group for all subgroups V. □

Proposition 67 *Let G be a finite group of order mp^a, with $(m,p)=1$. Then every $p-$subgroup of $U(KG)$ is locally finite, nilpotent-by-nilpotent with derived length $\leq [\sqrt{(m-1)p^a}] + mp^a - 1$, and has finite exponent dividing p^{e+e_1}, where e is the least integer such that $p^e \geq mp^a$ and e_1 is the least integer such that $p^{e_1} \geq [\sqrt{mp^a - 1}]$. In particular, the exponent divides $p^{3(m+a)/2}$. (Here $[x]$ denotes the greatest integer $\leq x$.)*

Proof:

Let P be a $p-$subgroup of $U(KG)$. So P is locally finite by Theorem 59.

We start by calculating the exponent of P. First we calculate an upper bound for the exponent of $\theta(P)$. Note that $\theta(P) < \theta(U(KG)) = \oplus GL_{n_i}(F_i)$. But $\theta(P)$ is a $p-$group, and since fields contain no $p-$elements, we need concern ourselves only with those $GL_{n_i}(F_i)$ with $n_i > 1$. Since there is always at least one $n_i = 1$ (see for example [20, equation 2 p.500]), we have that the sum of the dimensions of the $M_{n_i}(F_i)$ which are not fields is $\leq dim(KG/J) - 1 = (mp^a - dim J) - 1 \leq (mp^a - (p^a - 1)) - 1$ [20, p.501] $= (m-1)p^a$. By [37, p.27 9.1.v)], $\theta_i(P)$ has exponent dividing p^{e_1}, where $p^{e_1} \geq n_i$. Thus the exponent of $\theta(P)$ is maximised if we have one matrix ring of large dimension n_i. Note that

$$p^{\frac{m+a}{2}} \geq (p^{m+a} - 1)^{\frac{1}{2}} \geq [\sqrt{mp^a - 1}] \geq n_i.$$

So the exponent of $\theta(P)$ divides $p^{(m+a)/2}$.

It now remains to find an upper bound for the exponent of $1+J$. By [18, p.422], the nilpotency index of the ideal $J(KG)$ is $\leq dim J+1$ and this in turn is $\leq |G| = mp^a$. (Obviously $J \neq KG$, as otherwise every element of $KG = 1+J$ is a unit). Thus if $1+j \in 1+J$ and e is the least integer such that $p^e \geq mp^a$ then $(1+j)^{p^e} = 1+j^{p^e} = 1$, showing that $1+J$ has exponent dividing p^e. Note that $p^{m+a} \geq mp^a$, so that $exp(1+J) \leq p^{m+a}$. Finally

$$exp(P) \leq exp(1+J)\, exp(\theta(P)) \leq p^{e+e_1} \leq p^{\frac{3}{2}(m+a)}.$$

For nilpotency observe that $\theta_i(P)$ is nilpotent for all i [37, p.112 9.1.v)], with nilpotency class $\leq [\sqrt{(m-1)p^a}]-1$. Thus $\theta(P)$ is nilpotent with nilpotency class $\leq [\sqrt{(m-1)p^a}]-1$. Now $1+J$ is nilpotent of class $\leq |G| = mp^a$ [1, Theorem 1]. But

$$\theta(P) \simeq \frac{P(1+J)}{1+J} \simeq \frac{P}{P\cap(1+J)}.$$

Thus P is nilpotent-by-nilpotent and has derived length $\leq [\sqrt{(m-1)p^a}] - 1 + mp^a$ [31, p.39]. □

Corollary 68 *Let G be a locally finite group and F a field (of any characteristic). Then in any finitely generated subgroup of $U(FG)$ the orders of the periodic elements are jointly bounded.*

Proof:

If V is a finitely generated subgroup of $U(FG)$ then it is a subgroup of $U(FH)$ for some finite subgroup H of G. Now $1+J(FH)$ is a p-group, so by Proposition 67 it has bounded exponent. A corresponding property of finitely generated linear groups

[26, Lemma 2.1.1 p.890] completes the proof. □

We will give examples in the next chapter.

Theorem 69 *Let G be a finite group. Let V be an arbitrary subgroup of $U(KG)$. Then V is either soluble-by-locally finite or it contains free groups.*

Proof:

Just apply [37, Cor 10.17 p.145] and Lemma 72. □

4.7 The Jacobson Radical

We start by examining the structure of $J(KG)$, where G is finite. We will need the following lemmas.

For completeness we record a proof of the following result (see [1, page 73]):

Lemma 70 *Let N be a nilpotent ideal of a ring R. Then $1+N$ is a nilpotent normal subgroup of $U(R)$, of nilpotency class $\leq$ the nilpotency index of N.*

Proof:

If $x \in N$ with $x^n = 0$ then $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \pm x^{n-1} \in 1+N$, so $1+N$ is a subgroup of $U(R)$ and is normal since N is an ideal.

We will prove by induction that for $u_1, \ldots, u_n \in 1+N$, $(u_i, \ldots, u_n) - 1 \in N^n$ for all n. Here (,) denotes a group commutator and [,] denotes a Lie bracket. Write $u_i = 1 + n_i$, with $n_i \in N$ for all N. To start the induction, note that

$(u_1, u_2) = 1 + u_1^{-1}u_2^{-1}[u_1, u_2]$, so $(u_1, u_2) - 1 = u_1^{-1}u_2^{-1}\{(1+n_1)(1+n_2) - (1+n_2)(1+n_1)\} = u_1^{-1}u_2^{-1}(n_1n_2 - n_2n_1) \in N^2$. Now assume the hypothesis for $n = k$ and we will prove it for $n = k+1$. $(u_1, \dots, u_{k+1}) - 1 = ((u_1, \dots, u_k), u_{k+1}) - 1 = (u_1, \dots, u_k)^{-1}u_{k+1}^{-1}[(u_1, \dots, u_k), u_{k+1}]$
$= (u_1, \dots, u_k)^{-1}u_{k+1}^{-1}[(u_1, \dots, u_k) - 1, u_{k+1}] = (u_1, \dots, u_k)^{-1}u_{k+1}^{-1}[\alpha, u_{k+1}]$, with $\alpha \in N^k = (u_1, \dots, u_k)^{-1}u_{k+1}^{-1}(\alpha n_{k+1} - n_{k+1}\alpha) \in N^{k+1}$. This proves the induction and the Lemma. □

Lemma 71 *Let N be a nil ideal of a ring R. Then $1 + N$ is a normal subgroup of $U(R)$. If R has characteristic $p > 0$ then $1 + N$ is a p-group. If N is locally nilpotent then $1 + N$ is a locally nilpotent group.*

Proof:

The first two statements are obvious and the third is proved as follows: Let $x_1, \dots, x_n \in N$, and let N_1 be the ideal of R that they generate. Then N_1 is nilpotent so by Lemma 70, $1 + x_1, \dots, 1 + x_n \in 1 + N_1$, a nilpotent group. □

Lemma 72 *Let K be any field of characteristic p, let $|G| = p^a m < \infty$, with $(p, m) = 1$ and let G' be a p-group. Then*

i) $J(KG) = lann(\hat{P}) = \{\sum_{x \in G} x_g g \in KG | \sum_{s \in P} x_{gs} = 0 \ \forall g \in G\}$
$= \{\sum_{x \in G} x_g g \in KG | \sum_{r \in gP} x_r = 0 \ \forall g \in G\} = \Delta(G, P) = KG\, J(KP)$.

ii) $U(KG) \simeq (1 + J) \rtimes A$ *for some abelian group* A.

iii) $1 + J$ *is a normal nilpotent* $p-$ *subgroup of finite exponent in* $U(KG)$.

iv) $dim_K J(KG) = |G| - |G/P| = m(p^a - 1)$

v) When $K = F_p$ *we get* $|1 + J| = p^{m(p^a - 1)}$.

vi) $KG/J(KG)$ *is a semisimple commutative* K*-algebra of dimension* m.

vii) $J(KG)$ *is a nilpotent ideal.*

viii) $1 + J(KG) = O_p(U(KG))$ *is a nilpotent normal subgroup of* $U(KG)$.

Proof:

i) G is $p-soluble$ (in fact G is soluble) [30, p.261]. Thus $J(KG) = lann(\hat{P})$ [20, p.474] $= \{\sum_{x\in G} x_g g \in KG | \sum_{s\in P} x_{gs} = 0 \ \forall g \in G\} = \{\sum_{x\in G} x_g g \in KG | \sum_{r\in gP} x_r = 0 \ \forall g \in G\} = KG\, I(P)$ [29, p.68]. Recall that $I(P)$ is nilpotent [29, p.70].
Also, $J(KG) = KG\, J(KP)$ [20, p.461 Prop 52.11.].

ii) Recall [20, p.337] that $RG/RG\, I(N) \simeq R\frac{G}{N}$ for all $N \lhd G$. Thus $KG/J(KG) = KG/K\, I(P) \simeq K\frac{G}{P} = KA$ for some abelian group A. Thus $KG/J \simeq \oplus$fields, so $U(KG) \simeq 1 + J \rtimes A_1$, for some abelian group A_1 [25, p.402].

iii) This was done in [1, Theorem 1].

iv) $dim_K J(KG) = |G| - |G/P|$ [20, p.459].

v) Thus, if $|G| = p^a m$, with $(p, m) = 1$, and $K = F_p$, then $|1 + J| = p^{m(p^a-1)}$.

vi) Thus $KG/J(KG)$ is a semisimple K-algebra of dimension $p^a m - (p^a m - m) = m$ and is commutative by *ii)*.

vii) This is shown in [20, page 357].

viii) See [1, Theorem 4 and first line of section 4] and Lemma 70. □

Lemma 73 *Let* G *be a finite group with* $|G| = p^a m < \infty$, *with* $(p, m) = 1$ *and let* G' *be a* $p-$*group. Then* $|U(F_pG)| \in [p^{m(p^a-1)}(p-1)^m,\ p^{m(p^a-1)}(p^m-1)]$.

Proof:

By Theorem 31, $U(F_pG/J(F_pG)) \simeq U(F_pG)/(1 + J(F_pG)) \simeq U(\oplus\text{fields})$. But $F_pG/J(F_pG)$ is an F_p-algebra of cardinality $p^{p^a m}/p^{m(p^a-1)} = p^m$. So since it is a direct sum of fields, the order of its unit group is in the interval $[(p-1)^m,\ p^m-1]$. □

Example 6 $G = D_{2q}$, $p = q$. *Thus F_pG satisfies the conditions of Theorem 31.* $|G| = q^1 2 = p^a m$. *Therefore* $|U(F_q D_{2q})| \in [q^{2(q-1)}(q-1)^2,\ q^{2(q-1)}(q^2-1)]$.
In fact (see the following chapter), $|U(F_q D_{2q})| = q q^{q-1} q^{q-2}(q-1)^2 = q^{2(q-1)}(q-1)^2$.
This tells us that $F_q D_{2q}/J(F_q D_{2q}) \simeq F_q \oplus F_q$. *So* $\delta(F_q D_{2q}) = \left(\frac{q-1}{q}\right)^2$. □

We now turn our attention to the structure of $J(KG)$, where G is locally finite.

Lemma 74 *Let G be a locally finite group and K a field of characteristic $p > 0$. Then $J(KG)$ is locally nilpotent and therefore nil, so $L(KG) = J(KG)$.*

Proof:

Let H be a finitely generated subgroup of G. H is a finite group so $J(KH)$ is a nilpotent ideal of KH [20, page 357]. Hence it is locally nilpotent, so by [20, page 340] we find that $J(KG)$ is locally nilpotent. Since we always have $L(KG) \subset J(KG)$ [20, page 271], the local nilpotence of $J(KG)$ forces $L(KG) = J(KG)$. □

Lemma 75 *Let G be a locally finite $p-$group and R be a direct sum of fields. Then $J(RG) = \Delta(G)$, and $U(RG) = (1 + \Delta(G)) \times U(R)$. Note that if $R = K$ is a field then $J(KG) =$ the zero-divisors of KG and $U(KG) = KG \setminus \Delta(G)$, so KG is a local ring.*

Proof:

$\Delta(G)$ is locally nilpotent [20, Theorem 44.2 p.351] and therefore it is nil. Thus $\Delta(G) \subset J(KG)$ [20, p.21 Corollary 6.11]. Now $\frac{RG}{\Delta(G)} \simeq R = \oplus_{i=1}^{n} F_i$, where the F_i

are fields. Let π_i denote projection of RG onto the i^{th} field F_i. Now the kernel N_i of π_i is a maximal ideal containing $\Delta(G)$. Thus $J(RG) \subset \cap_{i=1}^{n} N_i = \Delta(G)$. Hence $J(KG) = \Delta(G)$, so by Lemma 71, $1 + \Delta(G)$ is a group of units. Hence, $u + \Delta(G)$ is a set of units for all $u \in U(R)$. The remark about zero-divisors is a consequence of the fact that group algebras of locally finite groups are algebraic [29, Theorem 3.11 and Lemma 3.12 p.53]. □

Lemma 76 *Let G be a locally finite group. Then $J(KG) = N^*(KG)$. Furthermore $KG/J(KG)$ is right artinian $\Leftrightarrow KG/J(KG) \simeq \oplus M_{n_i}(K_i) \Leftrightarrow [G : O_p(G)] < \infty$.*

Proof:

Notice that since G is locally finite, every subgroup must have locally finite index. Thus, by [29, page 318], $N^*(KG) = J(KG)KG = J(KG)$. The rest now follows from [29, page 409] and [19, page 4]. □

Corollary 77 *Let G be a locally finite group with $[G : O_p(G)] < \infty$. Then $O_p(U(KG)) = 1 + J(KG)$.*

Proof:

$\theta : KG \to KG/J \simeq \oplus M_{n_i}(K_i)$ is defined as in the finite case. Let $P = O_p(U(KG))$. Now $\theta_i(P) = \theta_i(O_p(U)) = O_p(\theta_i(U)) = O_p(GL_{n_i}(K_i))$. We claim that the latter group is a subgroup of $Z(GL_{n_i}(K_i)) = K_i^*$, a contradiction (unless $\theta_i(P) = 1$) since K_i contains no elements of order p. This claim will finish the proof.

By [34, p.78], either $O_p(GL_{n_i}(K_i))$ contains $SL_{n_i}(K_i)$ or $O_p(GL_{n_i}(K_i)) = GL_2(F_2)$ or $= GL_2(F_3)$. The first case does not happen as $SL_{n_i}(K_i)$ is not a p-group. Next $GL_2(F_2)$ is isomorphic to the dihedral group of six elements, so $O_2(GL_2(F_2)) = 1$ and we are done. Lastly, note that $GL_2(F_3)$ has order $2^4 3^1 = 48$. Both

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

generate sylow 3-subgroups, but they are conjugate (under $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, say) and hence $O_3(GL_2(F_3)) = 1$, completing the proof. □

Corollary 78 *If G is a locally finite group with G' a p-group then $KG/J(KG)$ is right artinian $\Leftrightarrow G/O_p(G)$ is a finite abelian p'-group.*

Proof:

Apply Lemma 76 and Theorem 32 parts *ii*) and *viii*). □

Corollary 79 *If G_1 and G_2 are locally finite groups then*

$$J(K(G_1 \times G_2)) = J(KG_1)KG_2 + KG_1J(KG_2).$$

Proof:

Lemma 20 and [29, page 329]. □

Lemma 80 *Let G be a locally finite group with $[G : O_p(G)] < \infty$. Then either $\frac{U(KG)}{1+J(KG)}$ is abelian or it contains free groups.*

Proof:

$$\frac{U(KG)}{1 + J(KG)} \simeq U(\oplus M_{n_i}(K_i)),$$

where K_i is a field containing K. If some $n_i > 1$ then $U(\oplus M_{n_i}(K_i(t))) \supset M_2(K) \supset$ a free group by the usual construction.

Hence we may assume that all $n_i = 1$, in which case the quotient is abelian. $\square$

Lemma 81 *Let G be a locally finite group with $[G : O_p(G)] < \infty$. Then the transvections of KG are contained in $1 + J(KG) \Leftrightarrow G'$ is a $p-$group.*

Proof:

Proceed as in Lemma 36.

$\Leftarrow$: Let G' be a $p-$group. By Theorem 32, $U(KG)$ does not contain free groups, so by Lemma 80, $\frac{U(KG)}{1+J(KG)}$ is abelian, so $\frac{KG}{J(KG)} \simeq \oplus$fields (Lemma 76). The rest follows as before.

$\Rightarrow$: By Lemma 76 we may use the previous proof, using the fact that $J(KG)$ is nil (Lemma 74) to lift idempotents. $\square$

These results may be assembled to give:

Theorem 82 *Let p be any prime, G be a locally finite group, with $[G : O_p(G)] < \infty$. Then the following are equivalent:*

i) $U(KG) \not\supset$ free groups

ii) G' is a $p-group$

iii) $G/O_p(G)$ is abelian

iv) $S_p(G) \lhd G$ and $G/S_p(G)$ is abelian.

v) $U(F_p(t)G)$ is locally soluble

vi) $U(KG)/(1 + J)$ is abelian

vii) The subgroup of $U(KG)$ generated by the transvections is contained in $1+J(KG)$

viii) $KG/(J(KG)) \simeq \oplus$ fields. $\square$

All of these results can be viewed as variations on Tits Alternative, applied to the unit group of a group ring.

Lemma 83 *Let K be a field of characteristic $p > 0$ and let G be an infinite locally finite group with $[G : O_p(G)] < \infty$. Then $J(KG)$ is not a nilpotent ideal.*

Proof:

By [20, page 357], $J(KG)$ is nilpotent if and only if there exist subgroups P and H of G such that P is a finite normal p—subgroup of G, H has finite index in G and $J(K\frac{H}{P}) = 0$. Now by [20, page 455], $J(K\frac{H}{P}) = 0$ implies that $O_p(H/P) = 1$, that is, $O_p(H) = P$. So choose any groups P and H satisfying these criteria. Now $O_p(G) \cap H < P$ because $P = O_p(H)$ (every p—subgroup of H which is normal in G is also normal in H). Now by the Second Isomorphism Theorem,

$$\frac{O_p(G)H}{O_p(G)} \simeq \frac{H}{O_p(G) \cap H} > \frac{P}{O_p(G) \cap H},$$

and since $[G : O_p(G)] < \infty$, we have that

$$\left|\frac{H}{P}\right| < \left|\frac{H}{O_p(G) \cap H}\right| = \left|\frac{O_p(G)H}{O_p(G)}\right| < \infty.$$

Thus, $|G| = [G : H][H : P][P : 1] < \infty$, a contradiction. □

Thus for G locally finite we see that either

i) $\frac{KG}{J}$ is semisimple, or

ii) $J(KG)$ is nilpotent,

or neither, but not both.

This makes it impossible to proceed as in the case of G finite.

Example 7 *Let F be any field of characteristic $p > 0$. Let $G = P \times (Q_1 \times Q_2 \times \cdots)$, where the P is a finite p-group, the Q_j are p'-groups with $|Q_j|$ finite and bounded and there are infinitely many Q_j's. Then $J(FG)$ is a nilpotent ideal. Indeed, G/P is an FC group, so by [20, Theorem 47.1 p.401], $J(F\frac{G}{P}) = 0$. Hence we have the nilpotency of $J(FG)$ by [29, Corollary 1.14 p.312].*

Recall that we already had that $J(KG) = \Delta(P)KG$ when G is finite.

Lemma 84 *Let G be a locally finite group with G' a $p-$group and K a field of characteristic p. Let $P = O_p(G)$. Then $J(K(G/P)) = 0$ and $J(KG) = (J(KG) \cap KP)KG = J(KP)KG = \Delta(P)KG$.*

Proof:
The fact that $J(KG/P) = 0$ is a consequence of [20, page 315 Thm38.2] (and also a consequence of [20, page 349]).
By Theorem 32, G/P is a locally finite abelian $p'-group$. Now apply [20, pages 329-330] and Lemma 75 to get that $J(KG) = (J(KG) \cap KP)KG \subset J(KP)KG = \Delta(P)KG$. Now apply [29, page 317 Lemma2.5] to get $J(KP)KG \subset N^*(KG)$. Lemma 76 now gives us the result. □

By Theorem 32 we knew that if G' is a p-group and $O_p(G)$ is finite then $U(KG)$ is soluble. We can improve this result somewhat:

Lemma 85 *Let G be a locally finite group with $O_p(G)$ finite. If G' is a p-group then $J(KG)$ is nilpotent and $U(KG)$ is a nilpotent p-group - by - abelian p'-group.*

Proof:

Now $G/O_p(G)$ is an abelian p'-group, so it is an FC-group and Lemma 84 gives us that $J(K\frac{G}{O_p(G)}) = 0$. Hence [20, Proposition 44.15 p.357] implies that $J(KG)$ is nilpotent. Lemma 84 also gives us that

$$K\frac{G}{O_p(G)} \simeq \frac{KG}{\Delta(O_p(G))KG} = \frac{KG}{J(KG)}$$

with an abelian p' unit group.

Since $1 + J(KG)$ is a nilpotent p-group we are done. $\square$

In particular, if G is locally finite with $O_p(G)$ finite then we have that the nilpotence of the unit group implies the nilpotence of the Jacobson radical.

Example 8 *In Lemma 85 we cannot remove the condition that G' is a $p-$group. If G' is not a $p-$group then $U(KG)$ contains free groups, and hence is not even soluble. However, Example 7 shows that J can still be nilpotent, even when $U(KG)$ contains free groups.* $\square$

Chapter 5

Examples

Example 9 *We study the group lattice of the unit group of certain dihedral groups. We start with $U(F_pD_{2p})$, where p is an odd prime. The results of the previous chapter will then be applied to this group and its infinite counterpart $U(KD_{2p})$.*

Write $D_{2p} = \langle x, y | x^p = y^2 = 1, x^y = x^{-1}\rangle$.

We start by calculating the radical of F_pD_{2p}. $J := J(F_pD_{2p})$ = the left annihilator of the sum of the p-elements of D_{2p} = $lann(\hat{x})$ [20, page 473, Corollary 53.13]

$$= \{\sum_{i=0}^{p-1} a_i x^i + \sum_{i=0}^{p-1} b_i x^i y | \sum_{i=0}^{p-1} a_i = 0 = \sum_{i=0}^{p-1} b_i\}.$$

Thus $|J| = p^{2(p-1)}$. Thus $|F_pD_{2p}/J| = p^{2p}/p^{2(p-1)} = p^2$, so $F_pD_{2p}/J \simeq F_{p^2}$ or $F_p \oplus F_p$. Note that F_pD_{2p}/J is generated (as an F_p-space) by $1 + J$ and $y + J$. But if $a, b \in F_p$ we have $(a + by + J)^p = a^p + b^p y + J = a + by + J$ (as $c^{p-1} = 1\ \forall c \in C_{p-1}$). Thus every element of F_pD_{2p}/J is either not a unit or has order dividing $p - 1$. Thus $F_pD_{2p}/J \neq F_{p^2}$, so $F_pD_{2p}/J = F_p \oplus F_p$.

Thus $U(F_pD_{2p}) \simeq (1+J) \rtimes (C_{p-1} \times C_{p-1})$, and we will concentrate on $V(F_pD_{2p}) \simeq (1 + J) \rtimes C_{p-1}$.

Note that $e := \frac{1}{2}(1 + y)$ and $f := \frac{1}{2}(1 - y)$ are orthogonal idempotents of F_pD_{2p}, so $F_pD_{2p}/J = F_p\frac{D_{2p}}{\langle x\rangle}$[19, p.14] $= F_p \oplus F_p = F_pe \oplus F_pf = F_p(1 + y) \oplus F_p(1 - y)$. Thus

F_pD_{2p} has augmentation ideal $\Delta(D_{2p}) = F_p(1-y)+J$. Note that not all elements of $1+\Delta(G)$ are units [19, page 42] and $J^p = 0$ [20, top of page 462], so $1+J$ has exponent p.

We will calculate $C_V(x) = \{v \in V | vx = xv\}$. Let

$$\alpha = \sum_{i=0}^{p-1} a_i x^i + \sum_{i=0}^{p-1} b_i x^i y \in C_V(x).$$

Then

$$vx - xv = \sum_{i=0}^{p-1} a_i x^{i+1} + \sum_{i=0}^{p-1} b_i x^{i-1} y - (\sum_{i=0}^{p-1} a_i x^{i+1} + \sum_{i=0}^{p-1} b_i x^{i+1} y) = 0,$$

so $\sum_{i=0}^{p-1} b_i x^{i+1} = \sum_{i=0}^{p-1} b_i x^{i-1}$. Thus $b_i = b_{i+2}$ for all i. Hence $b_0 = b_2 = b_4 = \ldots = b_{p-1} = b_1 = b_3 = \ldots = b_{p-2} = b_0$. Thus there exists $b \in F_p$ such that $b_i = b$ for all i. Thus $C_V(x) = \{\sum_{i=0}^{p-1} a_i x^i + b\sum_{i=0}^{p-1} x^i y | \sum_{i=0}^{p-1} a_i = 1\}$.
Also,

$$C_V(x) \simeq \prod_{i=1}^{p} C_p,$$

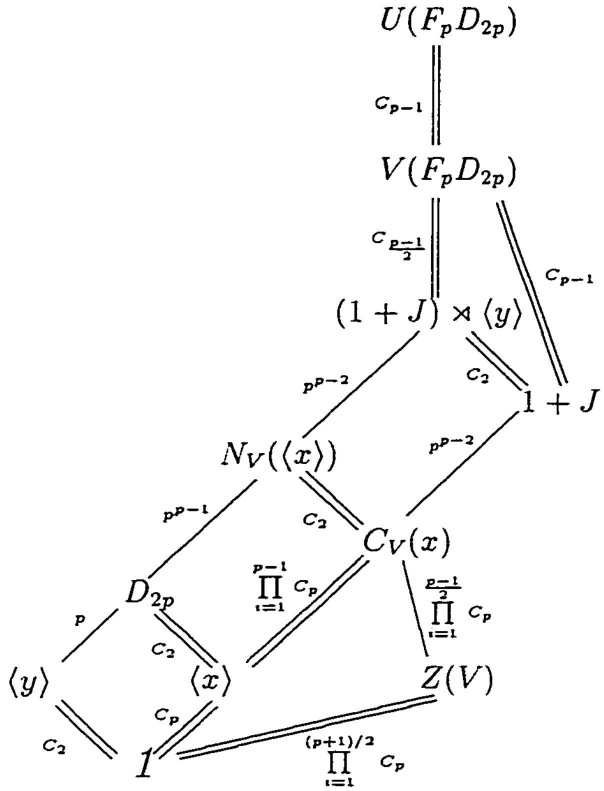
since $C_V(x)$ is abelian and being contained in $1+J$, it must have exponent p.
Note that $C_U(x) = C_{F_pD_{2p}}(x) \setminus \Delta(G)$.
Now by [19, p.44] we have

$$N_V(\langle x\rangle) = N_{D_{2p}}(\langle x\rangle)C_V(x) = D_{2p}C_V(x) = C_V(x) + yC_V(x)$$

$$= 1 + \left\{\sum_{i=0}^{p-1} a_i x^i + b\sum_{i=0}^{p-1} x^i y \middle| \sum_{i=0}^{p-1} a_i = 0\right\} + \left\{a\sum_{i=0}^{p-1} x^i + \sum_{i=0}^{p-1} b_i x^i y \middle| \sum_{i=0}^{p-1} b_i = 0\right\}.$$

Thus $|N_V(\langle x\rangle)| = 2p^p$. Obviously, $C_V(x) \lhd N_V(\langle x\rangle)$. Note that $Z(V) = C_V(x) \cap Z(F_pD_{2p}) = \{\sum_{i=0}^{p-1} a_i x^i + b\sum_{i=0}^{p-1} x^i y | \sum_{i=0}^{p-1} a_i = 1 \& a_i = a_{-i} \forall i\}$. Thus $|Z(V)| = p^{\frac{p+1}{2}}$.

Thus we get the following group lattice diagram:



Note that $1 + J(F_pD_{2p})$ *has exponent* p. *Proposition 67 gives us that* $1 + J$ *has exponent* $\leq p^{2+1+[\sqrt{1p^1}]} = p^{3+[\sqrt{p}]}$. *In* KD_6 *we have that* $1 + J$ *has exponent dividing* $3^{3+1} = 3^4$. *Alternatively, note that* $3^2 \geq 6$ *to see that* $1 + J$ *has exponent dividing* $p^{2+1} = 3^3$.

Again use Proposition 67 to see that the exponent of $1+J(F_2D_{2p})$ *is* $\leq 2^{p+1+[\sqrt{2(p-1)}]}$. *In* KD_6 *(char* $K = 2$*) we have that* $1 + J$ *has exponent dividing* $2^{4+2} = 2^6$. *Alternatively, note that* $2^3 \geq 6$ *to see that* $1 + J$ *has exponent dividing* $2^{3+2} = 2^5$.

Example 10 *Here we construct a pair of generators of a free subgroup of $U(KG)$, where G is the quaternion group of eight elements and K is a field of characteristic 3. This is of special interest bacause it is one of the few cases omitted in [13] in their construction of free pairs in group algebras. The construction in [13] relies on Theorem 17, which does not apply to hamiltonian groups because of its reliance on the non-triviality of the bicyclic units. However, we can construct non-trivial GBUs here. We write the quaternion group as $Q_8 = \langle x, y | x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle$. Consider the element $\alpha := x - x^3 - y \in F_pQ_8$. Now letting $p = 3$, we see that α is a unit of order 2. Define $a := (1-\alpha)y\hat{\alpha}$ and $b := \hat{\alpha}y^{-1}(1-\alpha)$. Now $ba = \hat{\alpha}y^{-1}(1-2\alpha+\alpha^2)y\hat{\alpha} = 2 + x + x^2 + 2x^3 + y + 2x^2y$ (see the appendix for the calculations in GAP). Thus $\omega\pi_p(ba) = tr(ba)$ (since Q_8 is a 2-group and $p \neq 2$) $= 2 \neq 0$. Thus by Theorem 20 we see that $\langle 1+ta, 1+1bab\rangle \simeq C_3 * C_3$, where t is an element of K which is transcendental over F_3. By [28, p.371] this group is also isomorphic to*

$$\left\langle R = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \right\rangle < PSL(2, Q).$$

Now

$$RS = \begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \text{ and } R^2S^2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix},$$

and these two matrices are well known to generate a free group [17, p.92]. (The fact that we are in PSL(2,Q) does not affect the freeness of the group. Indeed, considered as elements of $SL(2,Q)$, RS and R^2S^2 generate a free group of rank 2, and this group contains no normal subgroup of order 2 (in particular it does not contain $\{I, -I\}$), so mapping $\langle RS, R^2S^2\rangle$ from $SL(2,Q)$ to $PSL(2,Q)$ does not alter the structure of this subgroup. Thus $\langle (1+ta)(1+tbab), (1+ta)^2(1+tbab)^2\rangle$ is isomorphic to a free group.

The following observations may also be of use in constructing free groups in F_pQ_8. Note that for any characteristic $p \neq 5$ we have that $\alpha = x - x^3 - y \in F_pQ_8$ is a unit.

Indeed, $\alpha^2 = -2 + 3x^2 \in F_p\langle x^2\rangle = F_pC_2 \simeq F_p \oplus F_p$. *In this group algebra* α^2 *is a unit with inverse* $\frac{2}{5} + \frac{3}{5}x$. *Note that every unit of* F_pC_2 *has order dividing* $p-1$, *so* α *has order dividing* $2(p-1)$. *Again define* $a := (1-\alpha)y\hat{\alpha}$ *and* $b := \hat{\alpha}y^{-1}(1-\alpha)$. *Now* $ba = \hat{\alpha}y^{-1}(1-2\alpha+\alpha^2)y\hat{\alpha} = \hat{\alpha}(-1+2x+3x^2-2x^3+2y)\hat{\alpha} = \hat{\alpha}\beta\hat{\alpha}$, *say. Now if* ba *is nilpotent then* $\omega\pi_p(ba) = tr(ba) = 0 = tr(\hat{\alpha}^2\beta = n\ tr(\hat{\alpha}\beta)$, *where* $n = o(\alpha)$. *Thus* $tr(ba) = 0$ *implies* $tr(\hat{\alpha}\beta) = 0$, *since* p *does not divide* n. *But given the order of* α, *it is not a difficult matter to find* $tr(\hat{\alpha}\beta)$.

Claim:

$o(\alpha) = 2n$ for some n and $tr(\hat{\alpha}\beta) = \sum_{j=0}^{n-1}(-5)^{j+1}$.

Proof: First note that since the augmentation map is a group homomorphism from $U(KG)$ to K^*, the fact that $\omega(\alpha) = -1$ forces α to have even order. Now let $o(\alpha) = 2n$. Now $tr(\hat{\alpha}\beta) = tr(\beta) + tr(\alpha\beta) + \sum_{j=1}^{n-1} tr((\alpha^{2j} + \alpha^{2j+1})\beta) = -1 + -4 + \sum_{j=1}^{n-1} tr((\alpha^{2j}(\beta+\alpha\beta))$. Note that $tr((\alpha^{2j}(\beta+\alpha\beta)) = tr(\alpha^{2j}\pi_{\langle x^2\rangle}(\beta+\alpha\beta)) = tr(\alpha^{2j}(-1+3x^2+-4+2x^2)) = tr(\alpha^{2j}(-5+5x^2)) = 5tr(\alpha^{2(j-1)}\alpha^2(-1+x^2)) = 5tr(\alpha^{2(j-1)}(5-5x^2)) = (-1)^1 5^2 tr(\alpha^{2(j-1)}(-1+x^2)) = (-1)^j 5^{j+1} tr(-1+x^2) = (-5)^{j+1}$, by an easy induction. Thus $tr(\hat{\alpha}\beta) = -5 + \sum_{j=1}^{n-1}(-5)^{j+1} = \sum_{j=0}^{n-1}(-5)^{j+1}$.

□

Example 11 *Next we construct a pair of generators of a free subgroup of* $U(KG)$, *where* $G = A_5$ *is the alternating group on five elements and* K *is a field of characteristic 2. This will be of interest because the free pair generated will not be stable (fixed by the involution map), as compared with [13], where a technique based on Threoem 17 is given for the construction of stable free pairs in group algebras. Note also that Theorem 17 does not apply here since* A_5 *has no subgroup of order 15 [38, p.138].*

Consider the elements $x := (1,2,3,4,5), y := (1,2,4)\alpha := (1,2,3)\beta := 1 + \hat{y}(xyx^2 + x^3)(1+y); \in F_pA_5$. *Now letting* $p = 2$, *we see that* β *is a unit of order 2.*

Define $a := (1+\alpha)\beta\hat{\alpha}$ *and* $b := \hat{\alpha}\beta(1+\alpha)$.

Now $ba = (3,4,5)+(2,3)(4,5)+(2,4,5)+(1,2)(4,5)+(1,2,3,4,5)+(1,2,4,5,3)+(1,3,4,5,2)+(1,3)(4,5)+(1,3,2,4,5)+(1,4,5,3,2)+(1,4,5)+(1,4,5,2,3)$ *(see the appendix for the calculations in GAP). Thus* $\omega\pi_p(ba) = 3 = 1 \neq 0$. *Thus by Theorem 20 we see that*

$\langle 1+ta, 1+1bab, 1+t(1+b)(aba(1+b)\rangle \simeq C_2 * C_2 * C_2$, *where* t *is an element of* K *which is transcendental over* F_2. *By [28, p.371] this group is also isomorphic to*

$$\left\langle T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, U = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, V = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \right\rangle < PSL(2,Q).$$

Now in this notation

$$C_1 = TUV = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \text{ and } D_1 = TVU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix},$$

and these two matrices are well known to generate a free group [17, p.92]. (The fact that we are in PSL(2,Q) does not affect the freeness of the group. Indeed, considered as elements of $SL(2,Q)$, TUV *and* TVU *generate a free group of rank 2, and this group contains no normal subgroup of order 2 (in particular it does not contain* $\{I, -I\}$*), so mapping* $\langle TUV, TVU\rangle$ *from* $SL(2,Q)$ *to* $PSL(2,Q)$ *does not alter the structure of this subgroup. Thus* $\langle c,d\rangle = \langle (1+ta)(1+tbab)(1+t(1+b)aba(1+b), (1+ta)(1+t(1+b)aba(1+b))(1+tbab)\rangle$ *is isomorphic to a free group. The appendix shows that* c *and* d *are not a stable free pair.*

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Appendix A

Appendix

Here we use "LAG - Lie Algebras of Group Algebras", an extension package for GAP 3.4 developed by Richard Rossmanith (1997), to do many of the calculations mentioned in the examples. The examples are numbered in the order they appear in the thesis.

Example 4:

```
gap
Read("lag.g");
F:=GF(2);
G:=DihedralGroup(10);
FG:=GroupAlgebra(F,G);
e:=GroupAlgebraElement([()], [One(F)]);
x:=GroupAlgebraElement([(1,2,3,4,5)], [One(F)]);
xhat:= e + x + x^2 + x^3 + x^4;
y:= GroupAlgebraElement([(2,5)(3,4)], [One(F)]);
```

```
yhat:= e + y;
# The elements of D10 look like this:
# e;  Z(2)^0*()+
#x;  Z(2)^0*(1,2,3,4,5)+
#x^2;  Z(2)^0*(1,3,5,2,4)+
#x^3;  Z(2)^0*(1,4,2,5,3)+
#x^4;  Z(2)^0*(1,5,4,3,2)+
#y;  Z(2)^0*(2,5)(3,4)+
#x*y;  Z(2)^0*(1,5)(2,4)+
#x^2*y;  Z(2)^0*(1,4)(2,3)+
#x^3*y;  Z(2)^0*(1,3)(4,5)+
#x^4*y;  Z(2)^0*(1,2)(3,5)+

beta := (e + x + y + x*y + x^2*y + x^3*y + x^4*y)^2;
#beta^15;
#n := 1;
#while n <= 2^10 and beta^n <> e do
#            n := n+1;
#            od;
#n;
# gap> n;
# 15
# Thus, beta is a unit in F2D10 of order 15.

alpha:=x+y+x*y;
alpha^3;
# Z(2)^0*()+   Thus alpha is a unit of order 3.
```

```
alphahat:=e+alpha+alpha^2;


a:= (e+alpha)*beta*alphahat;

b:= alphahat*beta^14*(e+alpha);

ba:=b*a;


ba+ba^2;

# Now (ba)=(ba)^2 <> 0, so ba is not nilpotent!!


# Note that

# <(1+ta)(1+tbab), (1+ta)(1+t(1+b)aba(1+b))> = <TU,TV> = <c1,c2> say,

# is isomorphic to a free group. Now

# c1:= 1+t(a+bab)+t^2(abab),

# and d1:= 1+t(a+(1+b)aba(1+b))+t^2(a(1+b)aba(1+b)).

a+ba*b;

# = x+x^2*y+x^4+x*y

a*ba*b;

# = x^4*y+x^2+x^2*y+x^4

a+(e+b)*a*b*a*(e+b);

# = x^4*y+x^3*y+x^2*y+x*y

a*(e+b)*a*b*a*(e+b);

# = x^3*y+x^2*y+x^3+x^4


# Thus, our free group equals

# < 1+t(x+x^4+x*y+x^2*y)+t^2(x^2+x^4+x^2*y+x^4*y),

#   1+t(x*y+x^2*y+x^3*y+x^4*y)+t^2(x^3+x^4+x^2*y+x^3*y) >.

# Note that this free pair is not stable (that is, not fixed under
```

```
# the involution mapping) :

# Now c1star = 1+t(x+x^4+x*y+x^2*y)+t^2(x+x^3+x^2*y+x^4*y) and
# d1star = 1+t(x*y+x^2*y+x^3*y+x^4*y)+t^2(x+x^2+x^2*y+x^3*y).
# Thus c1*c1star = 1+t(c1start1)+
# t^2((x^2+x^4+x^2*y+x^4*y) + (x^3+x^4+x^2*y+x^3*y) +
# (x+x^4+x*y+x^2*y)*(x*y+x^2*y+x^3*y+x^4*y)
c1start2:=(x^2+x^4+x^2*y+x^4*y) + (x+x^3+x^2*y+x^4*y) +
(x+x^4+x*y+x^2*y)*(x+x^4+x*y+x^2*y);
# = 0.
# Similarly, d1*d1star = 1+t(0) +
# t^2(x+x^2+x^3+x^4+(x*y+x^2*y+x^3*y+x^4*y)*(x*y+x^2*y+x^3*y+x^4*y)
d1start2:=
x+x^2+x^3+x^4+(x*y+x^2*y+x^3*y+x^4*y)*(x*y+x^2*y+x^3*y+x^4*y);
# =0.

# Coeff of t in c1*c1star:
(x+x^4+x*y+x^2*y) + (a+b*a*b);
# gap> (x+x^4+x*y+x^2*y) + (a+b*a*b);
# Lag.Zero()
# Coeff of t^2 in c1*c1star:
(a+b*a*b)*(x+x^4+x*y+x^2*y) + (a*b*a*b) + (x+x^3+x^2*y+x^4*y);
gap> (a+b*a*b)*(x+x^4+x*y+x^2*y) + (a*b*a*b) + (x+x^3+x^2*y+x^4*y);
Lag.Zero()
# Coeff of t^3 in c1*c1star:
(x+x^4+x*y+x^2*y)*(x+x^3+x^2*y+x^4*y)
+ (x^2+x^4+x^2*y+x^4*y)*(x+x^4+x*y+x^2*y);
```

```
#gap> $+x*y+x^2*y);
#Lag.Zero()
# Coeff of t^4 in c1*c1star:
(x^2+x^4+x^2*y+x^4*y)*(x+x^3+x^2*y+x^4*y);
# gap> (x^2+x^4+x^2*y+x^4*y)*(x+x^3+x^2*y+x^4*y);
# Lag.Zero()
# Thus c1*c1star = 1, so c1^{-1} = c1star.

# Coeff of t in d1*d1star:
(x*y+x^2*y+x^3*y+x^4*y) + (x*y+x^2*y+x^3*y+x^4*y);
gap> (x*y+x^2*y+x^3*y+x^4*y) + (x*y+x^2*y+x^3*y+x^4*y);
Lag.Zero()
# Coeff of t^2 in d1*d1star:
(x*y+x^2*y+x^3*y+x^4*y)*(x*y+x^2*y+x^3*y+x^4*y)
+ (x^3+x^4+x^2*y+x^3*y) + (x+x^2+x^2*y+x^3*y);
gap> $*y) + (x+x^2+x^2*y+x^3*y);
Lag.Zero()
# Coeff of t^3 in d1*d1star:
(x*y+x^2*y+x^3*y+x^4*y)*(x+x^2+x^2*y+x^3*y)
+ (x^3+x^4+x^2*y+x^3*y)*(x*y+x^2*y+x^3*y+x^4*y);
gap> $(x*y+x^2*y+x^3*y+x^4*y);
Lag.Zero()
# Coeff of t^3 in d1*d1star:
(x^3+x^4+x^2*y+x^3*y)*(x+x^2+x^2*y+x^3*y);
gap> (x^3+x^4+x^2*y+x^3*y)*(x+x^2+x^2*y+x^3*y);
Lag.Zero()
# Thus d1*d1star = 1, so d1^{-1} = d1star.
```

Example 10:

```
gap;
Read("lag.g");
F:=GF(3);
p := 3;
x1 := [ [ Z(p)^0, Z(p)^0,  ] ,
        [ Z(p)^0, 2*Z(p)^0 ]   ] ;;
y1 := [ [ 0*Z(p), Z(p)^0,  ] ,
        [ 2*Z(p)^0, 0*Z(p) ]   ] ;;
G:=Group(x1, y1);
#GroupId(G); = Q8
FG:=GroupAlgebra(F,G);
x:=GroupAlgebraElement([x1], [One(F)]);
y:=GroupAlgebraElement([y1], [One(F)]);
e1:=x1^4;
e:=GroupAlgebraElement([e1], [One(F)]);
alpha:=x+(-1)*y+(-1)*x^3;
# Z(3)*[ [ 0*Z(3), Z(3)^0 ], [ Z(3), 0*Z(3) ] ]+
# [ [ Z(3)^0, Z(3)^0 ], [ Z(3)^0, Z(3) ] ]+Z(3)*
# [ [ Z(3), Z(3) ], [ Z(3), Z(3)^0 ] ]+
#This is a unit of order 2.
alphahat:= (e+alpha);
a:=(e-alpha)*y*(alphahat);
b:=(alphahat)*y^(-1)*(e-alpha);
ba:=b*a;
```

```
# [ [ 0*Z(3), Z(3)^0 ], [ Z(3), 0*Z(3) ] ]+Z(3)*
# [ [ 0*Z(3), Z(3) ], [ Z(3)^0, 0*Z(3) ] ]+Z(3)*
# [ [ Z(3)^0, 0*Z(3) ], [ 0*Z(3), Z(3)^0 ] ]+
# [ [ Z(3)^0, Z(3)^0 ], [ Z(3)^0, Z(3) ] ]+
# [ [ Z(3), 0*Z(3) ], [ 0*Z(3), Z(3) ] ]+
# Z(3)*[ [ Z(3), Z(3) ], [ Z(3), Z(3)^0 ] ]+

# Now this has trace = 2.
```

Example 11:

```
gap
Read("lag.g");
F:=GF(2);
G:=AlternatingGroup(5);
FG:=GroupAlgebra(F,G);
e:=GroupAlgebraElement([()], [One(F)]);
alpha:=GroupAlgebraElement([(1,2,3)], [One(F)]);
alphahat:= e + alpha + alpha^2;
x:=GroupAlgebraElement([(1,2,3,4,5)], [One(F)]);
y:=GroupAlgebraElement([(1,2,4)], [One(F)]);
yhat:= e+y+y^2;
xhat:= e+x+x^2+x^3+x^4;
# Use the idempotent trick (and xhat say) to come up with a
# new unit for beta.
beta:= e+yhat*(x*y*x^2+x^3)*(e+y);
```

```
betahat:= e + beta;
a:=(e+alpha)*beta*alphahat;
b:=alphahat*beta*(e+alpha);
ba:=b*a;
# gap> ba;
# (3,4,5)+(2,3)(4,5)+(2,4,5)+(1,2)(4,5)+(1,2,3,4,5)+(1,2,4,5,3)+
# (1,3,4,5,2)+ (1,3)(4,5)+(1,3,2,4,5)+(1,4,5,3,2)+(1,4,5)+(1,4,5,2,3)
# Note that when projected down to its 2-support, ba has
# augmentation =3 =1 neq 0 here !!!

# a;
# (3,5,4)+(2,3,5)+(2,4,5)+(2,5,4)+(1,2)(3,5)+(1,2,3,5,4)+(1,2,4,5,3)+
# (1,2,5,4,3)+(1,3,5,4,2)+(1,3,5)+(1,3,2,4,5)+(1,3,2,5,4)+(1,4,5,3,2)
# +(1,4,5)+(1,4,5,2,3)+(1,4,2,5,3)+(1,4,3,2,5)+(1,4)(2,5)+(1,5,2)+
# (1,5,3,4,2)+(1,5,3)+ (1,5)(3,4)+(1,5)(2,3)+(1,5,2,3,4)+
astar:=GroupAlgebraElement([
(3,4,5),(2,5,3),(2,5,4),(2,4,5),(1,2)(3,5),(1,4,5,3,2),(1,3,5,4,2),
(1,3,4,5,2),(1,2,4,5,3),(1,5,3),(1,5,4,2,3),(1,4,5,2,3),(1,2,3,5,4),
(1,5,4),(1,3,2,5,4),(1,3,5,2,4),(1,5,2,3,4),(1,4)(2,5),(1,2,5),
(1,2,4,3,5),(1,3,5),(1,5)(3,4),(1,5)(2,3),(1,4,3,2,5)],
 [One(F), One(F),
One(F), One(F), One(F), One(F), One(F), One(F), One(F), One(F),
One(F), One(F), One(F), One(F), One(F), One(F), One(F), One(F),
One(F), One(F), One(F), One(F), One(F), One(F)]);
# astar:=
# (3,4,5)+(2,5,3)+(2,5,4)+(2,4,5)+(1,2)(3,5)+(1,4,5,3,2)+(1,3,5,4,2)+
# (1,3,4,5,2)+(1,2,4,5,3)+(1,5,3)+(1,5,4,2,3)+(1,4,5,2,3)+(1,2,3,5,4)
```

```
# +(1,5,4)+(1,3,2,5,4)+(1,3,5,2,4)+(1,5,2,3,4)+(1,4)(2,5)+(1,2,5)+
# (1,2,4,3,5)+(1,3,5)+ (1,5)(3,4)+(1,5)(2,3)+(1,4,3,2,5);
# b;
# (3,5,4)+(2,3,5)+(2,4,5)+(2,4)(3,5)+(2,5)(3,4)+(1,2,3,4,5)+
# (1,2,3,5,4)+ (1,2,4,3,5)+(1,2,5,4,3)+(1,2,5)+(1,2,5,3,4)+
# (1,3,4,5,2)+(1,3,5)+(1,3,2,4,5)+ (1,3)(2,5)+(1,3,2,5,4)+(1,4,3,5,2)
# +(1,4,5)+(1,4)(3,5)+(1,4,5,2,3)+(1,4,2,5,3)+ (1,4,3,2,5)+
# (1,5,4,3,2)+(1,5,2)+(1,5,3,4,2)+(1,5,4)+(1,5)(3,4)+(1,5)(2,3)+
# (1,5,2,4,3)+(1,5,3,2,4)+
# bstar = b^* =
# (3,4,5)+(2,5,3)+(2,5,4)+(2,4)(3,5)+(2,5)(3,4)+
# (1,5,4,3,2)+ (1,4,5,3,2)+ (1,5,3,4,2)+(1,3,4,5,2)+
# (1,5,2)+(1,4,3,5,2)+ (1,2,5,4,3)+(1,5,3)+(1,5,4,2,3)+
# (1,3)(2,5)+(1,4,5,2,3)+(1,2,5,3,4)+(1,5,4)+(1,4)(3,5)+
# (1,3,2,5,4)+(1,3,5,2,4)+ (1,5,2,3,4)+(1,2,3,4,5)+
# (1,2,5)+(1,2,4,3,5)+(1,4,5)+(1,5)(3,4)+(1,5)(2,3)+
# (1,3,4,2,5)+(1,4,2,3,5)
bstar:=GroupAlgebraElement([(3,4,5),(2,5,3),(2,5,4),
(2,4)(3,5),(2,5)(3,4),(1,5,4,3,2),(1,4,5,3,2),
(1,5,3,4,2),(1,3,4,5,2),(1,5,2),(1,4,3,5,2),(1,2,5,4,3),
(1,5,3),(1,5,4,2,3),(1,3)(2,5),(1,4,5,2,3),(1,2,5,3,4),
(1,5,4),(1,4)(3,5),(1,3,2,5,4),(1,3,5,2,4),(1,5,2,3,4),
(1,2,3,4,5),(1,2,5),(1,2,4,3,5),(1,4,5),(1,5)(3,4),
(1,5)(2,3),(1,3,4,2,5),(1,4,2,3,5)],
[One(F),  One(F), One(F), One(F), One(F), One(F),
One(F), One(F), One(F), One(F), One(F), One(F), One(F),
One(F), One(F), One(F), One(F), One(F), One(F), One(F),
```

```
One(F), One(F), One(F), One(F), One(F), One(F), One(F),
One(F), One(F), One(F)]);

# Recall that \omega \pi_2 (ba) =1 \neq 0 here, so ba is
# not nilpotent and by Theorem 20  we get that U(KD10)
# contains free groups. In fact, it contains
# <1+ta, 1+tbab, 1+t(1+b)aba(1+b)>, which is isomorphic
# to C2*C2*C2 in the obvious way. It is also isomorphic
# to {T,U,V} iso to gamma^3 (see \cite[p.371]{Newman}.
# Here T=the 2X2 matrix[0,1,-1,0] , U=[1,1,-2,-1] and
# V=[-1,-2,1,1]. Here T,U and V are considered as
# matrices in PSL(2,Q),
# i.e. I=-I. Note: T^2=U^2=V^2=-I=I.
# Now in Newman notation: C_1=TUV=[1,3,0,1] and
# D_1=TVU=[1,0,3,1].
# Thus <TUV, TVU> is well known to be isomorphic to a
# free group (again considered as  a subgp of PSL(2,C)),
# (The fact that we are in PSL(2,C) does not affect the
# freeness of the group as the matrices in the group
# have no negative entries, so going from SL(2,C) to
# PSL(2,C) does not alter the structure of this subgp).

# Thus: <(1+ta)(1+tbab)(1+t(1+b)aba(1+b)),
#        (1+ta)(1+t(1+b)aba(1+b))(1+tbab)>
# is isomorphic to a free group, <c,d>, say.
# Now cstar=
# (1+t(1+bstar)astar*bstar*astar(1+bstar))
```

```
# *(1+t*bstar*astar*bstar)(1+t*astar)
# and we already have astar and bstar, so we compute
# cstar as follows:
# Write cstar as (1+t*f)(1+t*g)(1+t*astar), where
f:=(e+bstar)*astar*bstar*astar*(e+bstar);
g:=bstar*astar*bstar;
# So cstar = 1+t(astar+f+g)+t^2(g*astar+f*astar+f*g)+
# t^3(f*g*astar) = 1+t*h+t^2*i+t^3*j, say, where
# h:=astar+f+g;
# (3,5,4)+(2,3)(4,5)+(2,3,5)+(2,4,3)+(2,4,5)+(2,5,3)+
# (2,5,4)+(1,2)(4,5)+(1,2,3)+(1,2,3,4,5)+(1,2,3,5,4)+
# (1,2,4,3,5)+(1,2,5,4,3)+(1,2,5)+(1,2,5,3,4)+(1,3,4)+
# (1,3)(2,5)+(1,4,5,3,2)+(1,4,2)+(1,4)(3,5)+(1,4)(2,5)+
# (1,5)(3,4)+(1,5)(2,3)+ (1,5)(2,4)+

i:=g*astar + f*astar + f*g;
# ()+(2,3,4)+(2,4,5)+(2,4)(3,5)+(1,2)(3,4)+(1,2,3)+
# (1,2,3,4,5)+(1,2,4,5,3)+(1,2,5)+(1,3,2)+(1,3,4)+
# (1,3,5)+(1,3,5,2,4)+(1,3,4,2,5)+(1,4,5,3,2)+
# (1,4,3,5,2)+(1,4)(3,5)+(1,4,5,2,3)+(1,4,2,5,3)+
# (1,4)(2,5)+(1,5)(3,4)+(1,5)( 2,3)+(1,5,2,4,3)+
# (1,5,3,2,4)+

j:=f*g*astar;
# ()+(3,5,4)+(2,3)(4,5)+(2,3,4)+(2,4,5)+(2,4)(3,5)+
# (2,5,3)+(2,5)(3,4)+(1,2)(4,5)+(1,2)(3,4)+(1,2,4)+
# (1,2,4,3,5)+(1,2,5,4,3)+(1,2,5)+(1,3,2)+(1,3,5,4,2)+
```

```
# (1,3)(2,4)+(1,3,2,4,5)+(1,3,2,5,4)+(1,3,4,2,5)+
# (1,4,5,3,2)+(1,4,3,5,2)+(1,4,3)+(1,4,5)+(1,4)(2,3)+
# (1,4,2,3,5)+(1,4,2,5,3)+(1,4)(2,5)+(1,5,2)+
# (1,5,3,4,2)+(1,5,4)+(1,5)(3,4)+(1,5,4,2,3)+(1,5)(2,3)+
# (1,5,2,4,3)+(1,5,3,2,4)+

# The interesting question is: "is cstar = c or d?"
# Let us begin to answer this:
# c:=(1+ta)(1+tbab)(1+t(1+b)aba(1+b)) =
# (1+t*a)(1+t*k)(1+t*m), say, where
k:= b*a*b;
m:=(e+b)*a*b*a*(e+b);
# Then c=1+t(a+m+k)+t^2(k*m+a*m+a*k)+t^3(a*k*m).
# Now compute h+(a+m+k);
# This is not equal to zero, so cstar <> c !!!

# Now let us ask: "is cstar = d?"
# d = (1+ta)(1+t(1+b)aba(1+b))(1+tbab) =
# (1+t*a)(1+t*n)(1+t*o), say, where
n:=(e+b)*a*b*a*(e+b);
o:=b*a*b;
# Thus, d=1+t(a+n+o)+t^2(a*n+a*o+n*o)+t^3(a*n*o).
# Aside: c and d have the same coeff of t,
# (but not t^2 and t^3) as we see:
# gap> (a+n+o)+(a+m+k);
# Lag.Zero()
```

```
# Now to answer "is cstar = d?", we compute
(a+n+o)+h;
(a*n+a*o+n*o)+i;
(a*n*o)+j;
# None of these are zero, so we conclude that <c,d> is
# a non-stable free pair
# in F(t)A_5.

# Now <c,d>=F_2, so <cstar,dstar)=F_2 also.
# (Indeed, if w(cstar,dstar)=1
# then applying * we get w_*(c,d)=1.)

# Let us check to see if c*cstar = 1:
# Let us redefine c as c=1+tp+t^2q+t^3r, where:
p:= a+m+k;
q:= k*m+a*m+a*k;
r:= a*k*m;
# Then we have c*cstar = 1 + t% +t^2% +... +t^6%.
# The coeff of t is:
p+h;
# This is <> 0, so c*cstar <> 1. Thus F_2=<c,d> is
# not a subgp of the orthogonal group
# O:={x\in KG | x*xstar=1 }.
# Now <c,d> is a free group, so <cstar,dstar> is also a
# free group. (Indeed, if w(cstar,dstar)=1 is a word in
# <cstar,dstar> then applying * we get w_*(c,d)=1 in
# <c,d>.)
```