University of Alberta

On the Number of Conjugacy Classes of Non-normal Subgroups of a Group

by

Roberta L. La Haye ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Doctor of Philosophy

in

Mathematics.

Department of Mathematical Sciences

Edmonton, Alberta

Spring 1998

0-612-29058-1

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On the Number of Conjugacy Classes of Non-normal Subgroups of a Group** submitted by Roberta L. La Haye in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics.

. . . .

Dr. A. H. Rhemtulla (Supervisor)

. . . .

Dr. G. H. Cliff (Chair)

. . . .

Dr. B. N. Allison

. . . .

Dr. M. Shirvani

. . . .

Dr. L. K. Stewart

. . . .

Dr. Howard Smith (External)

Date: Jan. 22, 1998

Abstract

Let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups of a group G. This thesis is concerned with what knowledge of $\nu(G)$ can tell us about the structure of a group G. We first consider finite groups.

If $|G| = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ then $\ell(G) = n_1 + n_2 + \cdots + n_k$ is defined to be the prime length of G. We prove that if G is a finite group and $\nu(G) \neq 0$, then there is a cyclic subgroup C of prime power order contained in the centre of G such that the prime length of G/C is at most $\nu(G) + 1$. Using this result we also show that for any finite group G, either $\ell(G)$ is bounded above by $3\nu(G) + 1$ or G is the semidirect product of a p'-group A with a p-group B and G has further restrictions on its structure. A classification of all p-groups with $\nu(G) \leq p$ is also provided.

The first result mentioned above extends to infinite groups, with the subgroup C being an infinite Prüfer p-group, but only when G has finitely many non-normal subgroups. It is shown that an infinite, non-Dedekind group G with only finitely many non-normal subgroups is the direct limit of a sequence of finite, nilpotent groups each having the same conjugacy classes of non-normal subgroups.

An infinite group G with $\nu(G)$ finite and with an infinite number of non-normal subgroups does not have so nice a structure. This is to be expected because of the existence of monsters of the type constructed by S.V. Ivanov and A. Yu. Ol'shanskii. The structure of this type of group is also studied. It is shown, among other things, that such groups have only a finite number of normal subgroups and have finite FC - centres. It is also true that for a group G of this type, $\ell(G/R)$ is bounded above by $\nu(G) + 1$, where R is the finite residual of G.

Acknowledgement

I would like to express my warmest and sincerest thanks to my supervisor, Dr. Akbar Rhemtulla. His guidance and encouragement were invaluable, and his love of mathematics was inspiring. I feel very fortunate to have been his student.

Contents

List of Symbols

$G, H, \ldots$	groups and subgroups
$g, h, k, x, y, \ldots$	elements of groups
$H \leq G$	H is a subgroup of G
$H < G$	H is a proper subgroup of G
x^y	$y^{-1}xy$
H^y	$y^{-1}Hy$
$\nu(G)$	the number of conjugacy classes of non-normal subgroups of a group G
$\ell(G)$	the prime length of a group G
$\|G\|$	the order of G
G/N	the quotient group of N in G
$A \times B$	the direct product of the groups A and B
$Z(G)$	the centre of the group G
$\langle x \rangle$	the cyclic group generated by the element x
$\langle X \rangle$	the group generated by the set X
$A \rtimes B$	the semidirect product of A with B
$A \cap B$	the intersection of the groups A and B
$[g, h]$	$g^{-1}h^{-1}gh$, the commutator of elements g and h
$[G, H]$	$\langle [g, h] : g \in G, h \in H \rangle$, the commutator of G and H
$[g, h]^{x+y}$	$[g, h]^x [g, h]^y$
$G' = [G, G]$	the commutator subgroup of G
$N \lhd G$	N is a normal subgroup of G
$N \ntriangleleft G$	N is not a normal subgroup of G
$C_G(h)$	the centralizer in G of h

$C_G(H)$	the centralizer in G of the group H
$[\{g\}]$	the conjugacy class of the element g
$[\{H\}]$	the conjugacy class of the subgroup H
$[G:N]$	the index of the subgroup N in the group G
$N_G(H)$	the normalizer in G of the group H
FIZ group	a group with centre of finite index
FC group	a group such that each element has only a finite number of conjugates
BFC group	an FC group whose conjugacy classes are of boundedly finite size
$FC(G)$	the FC - centre of G
lub	least upper bound
AB	set product of groups A and B
$\nu_k(G)$	the number of conjugacy classes of subgroups of G that are not subnormal of defect $\leq k$
$\langle X : R\rangle$	group presented by generators X and relators R
$Q(8), Q_8$	the quaternion group of order 8
C_n	a cyclic group of order n
(m,n)	the greatest common divisor of integers m and n
mod	modulo
$\mu(G)$	the number of normal subgroups of G
$D(2^n)$	the dihedral group of order 2^n
$Q(2^n)$	the generalized quaternion group of order 2^n
$S(2^n)$	the quasidihedral group of order 2^n
$G \cong H$	group G is isomorphic to group H
$exp(G)$	the exponent of G
$Hom(G,H)$	the group of homomorphisms from G to H
C_{p^∞}	a Prüfer p-group

1 Introduction

1.1 Summary of Main Results

Suppose we know the number of conjugacy classes of non-normal subgroups of a group G. What can we conclude about the structure of the group? This is the underlying question that motivates the results in this thesis. For convenience, let $\nu(G)$ denote the number of conjugacy classes of non-normal subgroups.

It will be assumed that the reader is familiar with fundamental concepts of group theory (especially finite nilpotent group theory). The basic concepts and terminology can be found in any contemporary, introductory text in group theory. Examples include [Rob] and [Rot]. The list of symbols located before Chapter 1 may also prove useful.

This work began when my supervisor showed me a paper he co-authored that proved that the nilpotency class of a finite group G could be bounded above by a function of $\nu(G)$. (see [PR]). He asked me to try to find a bound for the order of the commutator subgroup of a finite group G in terms of $\nu(G)$. I was successful and subsequent investigation yielded the many results in this thesis. Many of the results proved here are in either the paper *Some Explicit Bounds in Groups With a Finite Number of Non-normal Subgroups*([L]) or the paper *Groups With a Bounded Number of Conjugacy Classes of Non-normal subgroups*([LR]).

This first Chapter serves several purposes. This first section introduces the topic of the thesis and summarizes the main results. Section 1.2 presents a brief history of the examination of conjugacy classes of non-normal subgroups and points out related areas of development. Section 1.3 outlines basic properties of $\nu(G)$ (considered as an operator on a group) and records results dealing with non-normal subgroups. The results in this section are used frequently in later Chapters. The only new result appearing in Section 1.3 is that a group G with $0 < \nu(G) < \infty$ must be torsion.

Chapters 2 and 3 concentrate on what knowledge of $\nu(G)$ can tell us about a finite group G. It is convenient to examine nilpotent groups first. This is done in Chapter 2. In Chapter 3 most results are generalized to all finite groups. Define the prime length of a finite group G, $\ell(G)$, to be the number of primes involved in the order of G, (counting multiplicity). Thus, if the order of G, $|G| = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ then $\ell(G) = n_1 + n_2 + \cdots + n_k$. The main results on finite groups are as follows:

Result 1: *If G is a finite group and $\nu(G) \neq 0$ then there exists a cyclic, central subgroup C of prime power order in G such that $\ell(G/C) \leq \nu(G) + 1$.*

Result 2: *Let G be a finite group with $\nu(G) = \nu > 0$. If G is nilpotent and $\ell(G) > 2\nu - 1$ then for some prime p, $G = A \times B$ where A is a Dedekind p'-subgroup, B is a p-subgroup and there is an element $z \in Z(B)$ such that B' is a subgroup of $\langle z \rangle$, and $\ell(G/\langle z \rangle) \leq \nu(G) + 1$. If G is not nilpotent but $\ell(G) > 3\nu - 1$ then for some prime p, $G = A \rtimes B$ where A is a Dedekind p'-subgroup and B is a p-subgroup. There exists an element $z \in B \cap Z(G)$ such that $\ell(G/\langle z \rangle) \leq \nu(G) - 1$ and $B' \leq \langle z \rangle$, $B = U\langle t \rangle$, where $U = C_B(A)$ and $B' \leq \langle t \rangle$. If H is a subgroup of G and $H \ntrianglelefteq AH$, then $B' \leq H$.*

Chapter 2 also makes some initial steps toward classifying p-groups in terms of their number of conjugacy classes of non-normal subgroups. It is shown that if G is a finite p-group then $\nu(G) = 0$, 1, or is at least p. All p-groups with $\nu(G) = p$ are given (up to isomorphism), in terms of generators and relations.

Chapters 4 and 5 concentrate on infinite groups. In Chapter 4 infinite groups with only a finite, positive number of non-normal subgroups are discussed. There we see that if G is an infinite group with only a finite, positive number of non-normal subgroups then G is the direct limit of a sequence of finite, nilpotent subgroups each having exactly the same conjugacy classes of non-normal subgroups. (Direct limits are discussed in Chapter 4). This allows us to apply the results of Chapter 2 to conclude, among other things that:

Result 3: *If G is an infinite group with a finite, positive number of non-normal subgroups then there exists a prime p and a central, Prüfer p-subgroup C of G such that $\ell(G/C) \leq \nu(G) + 1$.*

Chapter 5 deals with groups G with an infinite number of non-normal subgroups and $\nu(G) < \infty$. The structure of such groups is vastly different from the previously discussed groups. For example, such groups satisfy both the maximal and minimal condition on subgroups and if the finite residual of such a group is R, then $\ell(G/R) \leq \nu(G) + 1$ and R is a perfect group containing an infinite, simple quotient R/Q such that $0 < \nu(R/Q) < \infty$. Theorem 5.9 summarizes the properties of such groups and examples are given in Section 5.2.

1.2 Overview

The aim of this section is to put the current investigation into a historical perspective. First recall that a group G acts on a set S if there is a function

$$\alpha : S \times G \to S$$

denoted by $\alpha(s, g) = s \odot g$ such that

$i) s \odot 1 = s$ for all $s \in S$.

$ii) s \odot (gh) = (s \odot g) \odot h$ for all $s \in S$ and elements $g, h \in G$.

The action of interest to us is conjugation. Letting $S = G$ we have an action on G given by $g \odot x = g^x = x^{-1}gx$, for each $x, g \in G$. The orbit of the element g. $[\{ g \}]$ is also known as the conjugacy class of g and

$$[\{g\}] = \{g^x : x \in G\}.$$

We also say that h is conjugate to g if $g^x = h$ for some $x \in G$. It is well known that the cardinality of $[\{g\}]$ is $[G : C_G(g)]$ where $C_G(g) = \{x \in G : g^x = g\}$ is the centralizer of g in G. Recall $C_G(g) \leq G$. The conjugacy classes of elements form a partition of G.

Similarly, if S is the family of all subgroups of a group G then we have the action on S given by $H \ni x = H^x = x^{-1}Hx$. The conjugacy class of H, is

$$[\{H\}] = \{K \leq G : H^x = K \text{ for some } x \in G\}$$

and if $H^x = K$ for some $x \in G$ then H is conjugate to K. The cardinality of $[\{H\}]$ is $[G : N_G(H)]$, where $N_G(H) = \{x \in G : H^x = H\}$ is the normalizer of H in G. The subgroup $N_G(H)$ is also the largest subgroup of G in which H is normal.

Thus we have a partition of the family of all subgroups of G into conjugacy classes. The blocks of the partition containing only one element are the conjugacy classes of the normal subgroups of G and the remaining blocks of the partition are the conjugacy classes of non-normal subgroups of G.

The work of B. Neumann concerning conjugacy classes of elements and subgroups (see [N1] and [N2]) is of great importance to group theory. In [N2] he showed that the class of groups with centre of finite index (FIZ groups) is precisely the class of groups where the classes of conjugate subgroups are finite. Furthermore, he showed that the commutator subgroup G' of an arbitrary group G is finite precisely when the conjugacy classes of elements of G have boundedly finite size (ie when G is a BFC group).

The FC groups are those groups where all the conjugacy classes of elements are finite. Neumann showed in [N2] that the class of FIZ groups is properly included in the class of BFC groups which is in turn a proper subclass of the class of FC groups.

The fact that G' is finite only for a BFC group prompted Neumann to ask his Master's student J. Wiegold whether $|G'|$ can be bounded by a function of

$$n = lub\{|[\{h\}]| : h \in G\}.$$

Wiegold did bound $|G'|$ with a function of n in his Master's thesis ([W]). He and others have since improved on his initial efforts. P. Neumann and M.R. Vaughan-Lee in [NV] provide a description of the early history of this problem and give the bound

$$|G'| \leq n^{1/2(3+5\log n)}.$$

I.D. Macdonald contributed to the solution of above problem in [M]. Therein he also gave a bound for $[G : Z(G)]$ in terms of

$$m = lub\{|[\{H\}]| : H \leq G\}$$

for an FIZ group G. Namely,

$$[G : Z(G)] \leq m^{81(\log_2 m)^2}.$$

As mentioned earlier we will give a bound for the prime length of $G/Z(G)$, $\ell(G/Z(G))$, in terms of $\nu(G)$. A bound for $\ell(G')$ in terms of $\nu(G)$ will also be provided. In some sense this mimics the above mentioned results.

With respect to the further study of BFC groups and FC groups we note only that there is extensive literature on the subject, (see [T]) and that the theory of FC groups is used in dealing with an infinite group G with $\nu(G)$ finite.

In Chapter 4 we use the property that a torsion FC group is locally finite. (A group G is torsion if all of its elements have finite order and G is locally P

if all of its finitely generated subgroups are P, P a property of groups). This property was first established by Neumann in [N1] and can be found in [Rob] (see 14.5.8) for example. FC groups also make an appearance in Chapter 5 where the FC centre of a group G, $FC(G)$, which is the subgroup of G consisting of all the elements of G with a finite number of conjugates, plays an important role.

Anyone familiar with elementary group theory can appreciate the importance of normal subgroups. In 1896 Dedekind showed that any group with all of its subgroups normal must have a very special form. (See Theorem 1.1 for details). Since then many papers have dealt with some generalization of this result.

One direction of research was to generalize the concept of normal subgroups and consider groups all or most of whose subgroups were of the stated type. Examples of this phenomena include the studies of almost normal, subnormal and quasinormal subgroups noted below.

A subgroup H is an almost normal subgroup of G if H is normal in a subgroup of finite index in G. Thus the FIZ groups Neumann studied in [N2] are precisely those groups where all the subgroups are almost normal.

A subgroup H of G is subnormal in G if there is a finite series of subgroups

$$H = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_{n-1} \triangleleft H_n = G.$$

If such a series exists we say H is subnormal of defect at most n. This too has been a fruitful area of investigation (see, in particular [LS]). One notable result in this area is by Roseblade who found in [Ros] that if G is a group all of whose subgroups are subnormal of bounded defect then G is nilpotent. Subnormal subgroups will arise again in the next section and in Chapter 5.

Define a quasinormal subgroup of a group as follows. A subgroup S of G is a quasinormal subgroup of G if $SU = US$ for all subgroups U of G. Iwasawa (see [Su2]) was one author looking at describing groups with all subgroups of G being quasinormal. More generalizations of Dedekind's Theorem are noted in Chapter 6 of [LS].

Another direction of research is to restrict the number of non-normal subgroups. Hekster and Lenstra consider groups with only a finite number of non-normal subgroups in [HL] and give a very satisfying description of the structure of such groups. Their main results are stated in Chapter 4.

Since all the conjugates of a non-normal subgroup are non-normal as well, it is not surprising that someone would eventually consider imposing restrictions on the conjugacy classes of non-normal subgroups.

Some of the earlier investigation concerning the conjugacy classes of non-normal subgroups dealt with partially ordering these conjugacy classes and considering the poset.

If S_1 and S_2 are subgroups of a group G a natural partial order on the conjugacy classes of subgroups of G is defined by

$$[\{S_1\}] \leq [\{S_2\}]$$

if and only if at least one element of $[\{S_1\}]$ lies in an element of $[\{S_2\}]$. The Möbius width $\omega_c(G)$ is the maximum number t of subgroups $S_1, S_2, \ldots, S_t$ of G with the property that no S_i is conjugate to any subgroup of S_j for every $i \neq j$. If there is no such t then $\omega_c(G) = \infty$ and $\omega_c(1) = 0$. In [Br1], [BV1], and [BV2] for example, certain finite groups are characterized in terms of their Möbius width.

In [BV3] the same two authors, R. Brandl and L. Verardi, consider groups where the posets of conjugacy classes of non-normal subgroups are order isomorphic. Finally, in [Br2] Brandl decided to look at the number of conjugacy classes of non-normal subgroups of a group G, which he denoted $\nu(G)$ and we do likewise.

In [Br2] finite groups with $\nu(G) = 1$ were characterized and in [PR] J. Poland and A. Rhemtulla considered the nilpotency class of finite nilpotent groups in terms of $\nu(G)$. These results are reviewed in the next section.

With respect to infinite groups with a finite number of conjugacy classes of non-normal subgroups recent papers such as [BDF], [Sm2] and [Sm1] contribute to the current body of research. The first two papers actually consider an arbitrary infinite group G with $\nu_k(G) < \infty$ where $\nu_k(G)$ is the number of conjugacy classes of subgroups of G that are not subnormal of defect $\leq k$, k some positive integer. We state a few of the results from [BDF] in Section 1.3. (See Theorem 1.10 and Corollary 1.11). The main result of [Sm2] is a generalization of Theorem 1.10.

In [Sm1] Smith takes another avenue of research by restricting the number of conjugacy classes of subgroups that do not have a certain property. (He considers the properties of nilpotence and solvability in this context). In this work we are interested in restricting the total number of conjugacy classes of non-normal subgroups of a group. The results in the next section will be quite useful later on.

1.3 Basic Properties and Related Results

We now record important, previously known results and basic facts which are needed in later chapters. Interested readers can find proofs of the results in the given references. First note that the structure of groups G with $\nu(G) = 0$ or 1 is known. A group G has no non-normal subgroups precisely when $\nu(G) = 0$ and such groups are called Dedekind groups (or Hamiltonian groups if they are non-Abelian.)

Theorem 1.1 **(Dedekind, Baer)**(see 5.3.7 of [Rob]). *If G is a group then $\nu(G) = 0$ if and only if G is Abelian or G is the direct product of a quaternion group of order 8, an elementary Abelian 2-group and an Abelian group with all elements of odd order.*

Theorem 1.2 *Let G be a group with $\nu(G) = 1$.*

i) **(Brandl)** [Br2] *If G is a finite p-group then*

$$G \cong M(p^n) = \langle a, b : a^{p^{n-1}} = 1, b^p = 1, \text{ and } a^b = a^{p^{n-2}+1} \rangle,$$

where $n \geq 4$ if $p = 2$ and $n \geq 3$ otherwise.

ii) **(Brandl)** [Br2] *If G is finite but not a p-group then G is a non-Abelian split extension of a group N of prime order by a cyclic subgroup P of prime power order, i.e.,*

$$G = N \rtimes P,$$

and $[N, \Phi(P)] = 1$, where $\Phi(P)$ is the Frattini subgroup of P.

iii) **(Brandl, De Giovanni and Franciosi)** [BDF] *If G is an infinite group then $G/Z(G)$ is a Tarski p-group for some prime p and $Z(G)$ is a cyclic p-group. Moreover, $Z(G) = \langle g^p \rangle$ for each element g of $G \setminus Z(G)$.*

(We give the definition of a Tarski p-group in Chapter 5 where they are discussed). Now consider "ν" as an operator on a group. By this we simply mean that ν is a map from the category of all groups to the positive integers together with ∞. The following property is a vital tool for proofs in Chapters 2 and 3.

Lemma 1.3 **(Brandl)**[Br2] *Let G be a group and $N \triangleleft G$. Then $\nu(G/N) \leq \nu(G)$.*

It was pointed out by Poland and Rhemtulla in [PR] that we can think of $\nu(G/N)$ as the number of conjugacy classes of non-normal subgroups of G such that each subgroup in each of the conjugacy classes contains N. (i.e. $N \leq H$ for each conjugacy class $[\{H\}]$).

The case when $\nu(G) = \nu(G/N)$ for a finite, non-Abelian group G will often be of interest. If $\nu(G) = \nu(G/N) \neq 0$ for some non-trivial, normal subgroup of G then N is contained in the intersection of all the non-normal subgroups of G. Clearly knowledge about finite groups with the intersection of all the non-normal subgroups non-trivial would be useful. The following 2 theorems of N. Blackburn address the possibilities.

Theorem 1.4 (Blackburn) [Bl] *If G is a finite p-group and the non-normal subgroups of G have non-trivial intersection N then $p=2$ and one of the following happens:*

(i) *G is the direct product of a quaternion group $Q(8)$ of order 8, a cyclic group C_4, of order 4 and an elementary Abelian group, E;*

(ii) *G is the direct product of two quaternion groups of order 8 and an elementary Abelian group, E;*

(iii) *$G = \langle A, x \rangle$, A is Abelian, the exponent of A, $exp(A) \neq 2$, $N = \langle x^2 \rangle \leq A$, and $a^x = a^{-1}$ for all $a \in A$. Furthermore if G is not Dedekind then N is the only normal subgroup such that $\nu(G) = \nu(G/N)$.*

Theorem 1.5 (Blackburn)[Bl] *If G is a finite group that is not of prime-power order and $\nu(G) = \nu(G/N) \neq 0$ for some normal subgroup N of G then N is a p-group for some prime p and there exists a p-group P and a p'-group Q so that*

$$G = Q \rtimes P$$

where every subgroup of Q is normal in G. In fact, one of the following holds:

a) *G has a normal, Abelian subgroup A of exponent kp^n where $n \geq 1$, p is prime, and $(k, p) = 1$. G/A is cyclic of order p^r and if Au generates G/A, then u can be so chosen that u^{p^r} has order p^n. There exists an integer $v \equiv 1 \mod p^n$ such that $x^u = x^v$ for all $x \in A$.*

b) *G is the direct product of an Abelian group of odd order and one of the groups described in (i) or (ii) of the previous theorem.*

c) *G has a subgroup H of the kind described in (a) with $p = 2$ and $r = 1$. H is of index 2 and if G is generated by H and t, then t can be so chosen that $u^t = u^{-1}$, $t^2 = u^{2^n}$, and $x^t = x^\eta$ for some $\eta \equiv -1 \mod 2^n$.*

d) *G has an Abelian subgroup A of index 2. G is generated by A and t where $t^2 \in A$ and $|t^2| = 2$. If x is an element of A, then $x^t = x^\chi$ for some $\chi \equiv -1 \mod 2^n$.*

e) *G is the direct product of H, a quaternion group of order 8, and an elementary Abelian 2-group, where H is of odd order and is of the kind described in (a).*

Note that the description of G as the semidirect product of Q with P comes from the proof of Theorem 1.5 as given by Blackburn. In the discussion in Chapter 3 where this theorem is used, it is easier to work with this property than the 5 group-types mentioned.

The following result is useful when generalizing from p-groups to nilpotent groups.

Proposition 1.6 (Poland and Rhemtulla)[Pr] *If $G = A \times B$ is a finite group then*

$$\nu(G) \geq \nu(A)\nu(B) + \nu(A)\mu(B) + \mu(A)\nu(B)$$

with equality if the orders of A and B are relatively prime, where $\mu(X)$ denotes the number of normal subgroups of a group X.

A corollary to this result given in [PR] is that for each positive integer n there is a finite, nilpotent group G with $\nu(G) = n$. We will see in the examples in section 5.2 that for $n > 0$ there is an infinite, non-nilpotent group with $\nu(G) = n$ as well.

Also p-groups with cyclic subgroups of index p arise in Chapter 2 so it is convenient to include the following result from [PR] based on a well-known result (see [Su1] Theorem 4.1).

Proposition 1.7 (Poland and Rhemtulla) [PR] *Let G be a finite, non-Abelian p-group having a cyclic subgroup of index p. Then one of the following occurs:*

(i) $G \cong D(2^n) = \langle a, b : |a| = 2^{n-1}, b^2 = 1, \text{ and } a^b = a^{-1} \rangle$, *the dihedral group of class $n-1$ and $\nu(G) = 2n-4$, where $n \geq 3$.*

(ii) $G \cong S(2^n) = \langle a, b : |a| = 2^{n-1}, b^2 = 1, \text{ and } a^b = a^{-1}a^{2^{n-2}} \rangle$, *the quasi-dihedral group of class $n-1$ and $\nu(G) = 2n-5$, where $n \geq 4$.*

(iii) $G \cong Q(2^n) = \langle a, b : |a| = 2^{n-1}, b^2 = a^{2^{n-2}}, \text{ and } a^b = a^{-1} \rangle$, *the generalized quaternion group of class $n-1$ and $\nu(G) = 2n-6$ where $n \geq 3$.*

(iv) $G \cong M(p^n) = \langle a, b : |a| = p^{n-1}, b^p = 1, \text{ and } a^b = a^{p^{n-2}+1} \rangle$, *where p is a prime and $n \geq 4$ if $p = 2$, $n \geq 3$ otherwise, $\nu(G) = 1$ and the class is 2.*

Finally, consider an infinite group G with $\nu(G)$ finite. First note the following result from [BDF].(Their version is actually more general than the one given below (see Lemma 2.2 of [BDF])).

Lemma 1.8 (Brandl, De Giovanni and Franciosi) [BDF] *If G is an infinite group with $\nu(G)$ finite and H is a normal subgroup of finite index in G then $\nu(H)$ is finite as well.*

The following result appeared in the submitted paper [LR]. The proof below is similar to that of Lemma 1 in Howard Smith's paper [Sm2]. However the conditions in our hypothesis are slightly weaker and so is the conclusion.

Lemma 1.9 *Let G be a group such that $0 < \nu(G) < \infty$. Then G is periodic.*

Proof: Suppose $x \in G$ has infinite order. First assume that $\langle x \rangle \ntriangleleft G$. Since $(n, m) = 1$ implies that x^n and x^m together generate $\langle x \rangle$, then for all but at most one prime, $\langle x^{p_i} \rangle \ntriangleleft G$. In fact, there will be an infinite number of positive integers n such that $\langle x^n \rangle$ is a non-normal subgroup of G. By replacing x with some power of x if necessary, we may assume that there is an infinite set of primes p_j so that $\langle x \rangle$ is conjugate to $\langle x^{p_j^{i_j}} \rangle$ for some positive integer i_j.

Suppose that $(x^{p_j^{i_j}})^g = x^{\pm 1}, g \in G$. Then

$$\langle x \rangle < \langle x^g \rangle < \langle x^{g^2} \rangle < \ldots .$$

so that $K_{p_j} = \langle x^{g^n} \, n \geq 0 \rangle = \cup_{n \geq 0} \langle x^{g^n} \rangle$ is locally cyclic. Furthermore, since $x^{g^n} = (x^{g^{n-1}})^{\pm p_j^{i_j}}$, K_{p_j} is p_j-divisible and $K_{p_j} / \langle x \rangle \cong C_{p_j}$ where C_{p_j} is a Prüfer p_j-group (see [Rob] 4.1.5). Also, since there exists a prime $p_k \neq p_j$, a positive integer i_k and an element h in G so that $(x^{p_k^{i_k}})^h = x^{\pm 1}$, then $\langle x^h \rangle$ has order dividing $p_k^{i_k}$ mod $\langle x \rangle$ and hence $x^h \notin K_{p_j} \ntriangleleft G$. Clearly $K_{p_j} \ncong K_{p_k}$ unless $p_j = p_k$, so that we have produced an infinite number of conjugacy classes of non-normal subgroups.

Thus we may assume that all infinite cyclic subgroups of G are normal. Suppose that $\langle x \rangle \leq Z(G)$. If $\langle t \rangle \ntriangleleft G$ then by the argument above $|t| < \infty$. Also

$\langle t, x^m \rangle \ntriangleleft G$ and $\langle t, x^m \rangle$ is not conjugate to $\langle t, x^n \rangle$ if $n \neq m$, again contradicting the finiteness of $\nu(G)$.

So, if $x \in G$ has infinite order, it must generate a normal subgroup that does not intersect the centre of G and $[G : C_G(\langle x \rangle)] = 2$. Suppose that $G = C_G(\langle x \rangle)\langle h \rangle$. Then $x^h = x^{-1}$, $\langle x \rangle \cap \langle h \rangle = 1$ and for $m > 2$, $\langle h, x^m \rangle \ntriangleleft G$ as $x^2 \notin \langle h, x^m \rangle$. Also, if $(n, m) = 1$ then $\langle h, x^n \rangle$ is not conjugate to $\langle h, x^m \rangle$ and we again have a contradiction. Thus G is periodic as required. $\square$

We shall leave most of the remaining pertinent results unstated until they are needed in Chapters 4 and 5. In Chapter 4 we will need Hekster and Lenstra's classification of infinite groups with a finite number of non-normal subgroups and in Chapter 5 a few results of Isozev and Sesekin concerning groups with only a finite number of infinite conjugacy classes of non-normal subgroups are recalled.

As a final note, we point out the following. If G is an infinite group with $\nu(G)$ finite then G is nilpotent precisely when G has only a finite number of non-normal subgroups.

This follows from the fact that the infinite groups with a finite number of non-normal subgroups classified by Hekster and Lenstra are nilpotent and the following theorem and corollary from [BDF].

Theorem 1.10 (Brandl, De Giovanni and Franciosi) [BDF] *If G is an infinite group whose chief factors are either locally solvable or locally finite and $\nu_k(G) < \infty$ for some positive integer k then G is nilpotent.*

Corollary 1.11 (Brandl, De Giovanni and Franciosi) [BDF] *If G is a group with $\nu(G) < \infty$ and the chief factors are either locally solvable or locally finite then G contains only a finite number of non-normal subgroups.*

2 Results for Finite Nilpotent Groups

2.1 Bounding the Prime Length of a Central Quotient

We begin by showing that every finite p-group and every finite nilpotent group G has a cyclic, central subgroup $\langle z\rangle$ of prime power order such that the prime length of $G/\langle z\rangle$ is bounded above by $\nu(G)+1$. This fact will be used in subsequent sections to derive further information about the structure of finite nilpotent groups with a given number of conjugacy classes of non-normal subgroups. It will also be generalized in the discussion of finite groups in the next chapter.

Lemma 2.1 *Let G be a finite p-group with $\nu(G)\neq 0$. Then there is an element $z\neq 1$ in the centre $Z(G)$ of G such that $\ell(G/\langle z\rangle)$ is at most $\nu(G)-1$.*

Proof: The proof is by induction on the prime length $\ell(G)$. Let G be a counter example of least prime length. We shall split the proof into three cases. In each case we consider a quotient G/N of G, where N is a minimal normal (and hence central) subgroup of G.

Case 1. $\nu(G/N)=\nu(G)$ for some minimal normal subgroup N of G. Then for any x in G, $\langle x\rangle \ntrianglelefteq G$ implies $N\leq\langle x\rangle$. In particular, the intersection of all non-normal subgroups of G contains N and Theorem 1.4 applies, forcing us to conclude that G is a 2-group satisfying one of the following conditions.

(1) $G=Q\times C_4\times E$ where $Q\cong Q_8$, C_4 is cyclic of order 4 and E is elementary Abelian of order 2^r, for some $r\geq 0$. Let ρ equal the number of non-trivial subgroups of E. Then $\ell(G/C_4)=r+3$, and $\nu(G)\geq(\rho+1)\times\nu(Q\times C_4)$ by Proposition 1.6. We show below that $\nu(Q\times C_4)=3$ and clearly $\rho\geq r$ so that $r+3\leq\rho+3\leq 3(\rho+1)$ and so $\ell(G/C_4)\leq\nu(G)+1$.

Suppose $G = Q \times C_4 = \langle x, y\rangle \times \langle z\rangle$. Then the elements x^2, z^2 and x^2z^2 are the three elements of order 2 in G. Since $G' = \langle x^2\rangle$ and $G/\langle z^2\rangle \cong Q \times C_2$ is Dedekind, it follows that every non-normal subgroup of G contains exactly one element of order 2, namely x^2z^2.

Now $Q \times C_4$ has no elements of order larger than 4 and

$$(x^iy^jz^a)^2 = (x^i)^2(y^j)^2[x^i, y^j](z^a)^2 = x^2z^2$$

only if $a \equiv 1$ or $3 \bmod 4$ and $x^iy^j \in \{x, y, x^3, y^3, xy, (xy)^3\}$. Thus there are 6 cyclic subgroups of G properly containing $\langle x^2z^2\rangle$. Since $|G'| = 2$ it follows that the 6 cyclic subgroups are non-normal and fall into 3 conjugacy classes.

Now if $S \ntrianglelefteq G$ and S is not cyclic then $S \cong Q$ and $\langle x^2z^2\rangle$ is the unique subgroup of order 2. S would have to contain two elements from the set $\{xz, yz, x^3z, y^3z, xyz, (xy)^3z\}$. But the group generated by any 2 elements from this set contains x^2. Thus $\nu(G) = 3$ as claimed.

(2) $G = Q_1 \times Q_2 \times E$ where $Q_1 \cong Q_2 \cong Q_8$ and E is elementary Abelian of order say 2^r. Let $\langle z\rangle$ be in the centre of Q_1. Then $\ell(G/\langle z\rangle) = r + 5$. Note that $\nu(Q_1 \times Q_2) > 9$ (see below) and $\nu(G) > 9(\rho + 1)$ by Propostion 1.6 where ρ is the number of non-trivial subgroups of E. Now $r \leq \rho$ so that $\ell(G/\langle z\rangle) = r + 5 \leq 9(\rho + 1) + 1 \leq \nu(G) + 1$.

To see that $\nu(Q_1 \times Q_2) > 9$ we proceed as follows. Suppose $G = Q_1 \times Q_2 = \langle x, y\rangle \times \langle s, t\rangle$. Then $Q_1 \times Q_2$ has three elements of order 2, all central, namely x^2, s^2 and x^2s^2. The rest of the non-identity elements all have order 4.

Now $G/\langle x^2\rangle \cong G/\langle s^2\rangle \cong Q_8 \times C_2 \times C_2$. Thus all non-normal subgroups of G contain x^2s^2. If

$$(x^ay^bs^nt^m)^2 = x^2s^2$$

then x^ay^b and s^nt^m both have order 4. There are 36 such elements and hence 18 cyclic subgroups of G that are non-normal since they contain x^2s^2 strictly.

Finally, since $G' = \langle x^2 \rangle \times \langle s^2 \rangle$, for any $g \in G$

$$(x^a y^b s^n t^m)^g \in \{x^a y^b s^n t^m, x^a y^b s^n t^m x^2, x^a y^b s^n t^m x^2 s^2, x^a y^b s^n t^m s^2\}$$

which generates only 2 distinct cyclic groups so that each non-normal subgroup has exactly 2 conjugates. Also note that $\langle xs, yt \rangle$ generates a non-normal, non-cyclic subgroup of G so that $\nu(G) > 9$, as claimed.

(3) $G = \langle x, A \rangle$ is a 2−group, A is Abelian of exponent greater than two, $1 \neq N = \langle x^2 \rangle \leq A$, $a^x = a^{-1}$ for all $a \in A$ and $\nu(G) \neq 0$. Then every non-normal subgroup of G contains N and $\nu(G) = \nu(G/N)$. Notice that $Z(G)$ is the set of all elements of order at most 2 and $G' = A^2 \neq 1$. Also note that $x^4 = 1$ as $(x^2)^x = x^2 = (x^2)^{-1}$. Let $|G| = 2^n$ and $|G'| = 2^s$. There are two possibilities to consider.

(3a) Suppose $x^2 \in A^2$. Note that $A^2 \neq \langle x^2 \rangle$ since $\nu(G) \neq 0$. Thus $|A^2| = 2^s > 2$. We can construct a strictly increasing series of subgroups

$$\langle x^2 \rangle = B_1 < B_2 < \cdots < B_s = A^2$$

and obtain from this series, a second strictly increasing series

$$\langle x, B_1 \rangle < \langle x, B_2 \rangle < \cdots < \langle x, B_s \rangle = \langle x, A^2 \rangle.$$

Note that $\langle x, B_i \rangle \ntrianglelefteq G$ if $i < s$ since if $a^2 \in A^2 \backslash B_i$ then $x^a = xa^2 \notin \langle x, B_i \rangle$. Also note that by virtue of size, we can see that $\langle x, B_i \rangle$ is not conjugate to $\langle x, B_j \rangle$ if $i \neq j$. Thus there are at least $s - 1$ conjugacy classes of non-normal subgroups contained in $\langle x, A^2 \rangle$.

Now $|A/A^2| = 2^{n-1-s}$. Let

$$A/A^2 = \langle a_1 \rangle A^2/A^2 \times \langle a_2 \rangle A^2/A^2 \times \cdots \times \langle a_{n-s-1} \rangle A^2/A^2.$$

Then for each $i = 1, 2, \ldots, n-s-1$, $\langle xa_i \rangle \ntrianglelefteq G$. Indeed $(xa_i)^b = xb^2a_i \notin \langle xa_i \rangle = \{1, xa_i, x^2, x^3a_i\}$, for any $b \in A$ of order greater than 4 or any b of order 4 not containing $\langle x^2\rangle$. But if $exp(A) = 4$ and $A^2 = \langle x^2\rangle = N$ then $\nu(G/N) = 0$ - a contradiction.

Moreover, if

$$(xa_i)^g = xa_i[xa_i, g] = xa_j \text{ or } x^3a_j$$

for some $g \in G$ then $a_i \equiv a_j \bmod G' = A^2$ and so i=j. Thus $\langle xa_i\rangle$ is not conjugate to $\langle xa_j\rangle$ if $i \neq j$. Since $\langle x\rangle = \langle x, B_1\rangle$ is the only $\langle x, B_i\rangle$ of order 4 and $[x, g] \neq a_i$, and $[x, g] \neq x^2a_i$, $i \in \{1, 2, \ldots, n\}$, $\langle xa_i\rangle$ is not conjugate to any $\langle x, B_i\rangle$ either. Hence $\nu(G) \geq (n-s-1) + (s-1) = n-2$ and so $\ell(G/\langle z\rangle) \leq \nu(G) + 1$, for any $z \in Z(G) \setminus \{1\}$, as required.

(3b) Suppose $x^2 \notin A^2$. Then corresponding to the subgroups

$$1 = B_0 < B_1 < \cdots < B_s = A^2$$

we obtain s conjugacy classes of non-normal subgroups. These are represented by $\langle x\rangle, \langle x, B_1\rangle, \ldots, \langle x, B_{s-1}\rangle$. The proof of this fact is nearly identical to the one given in (3a), just remember $x^2 \notin A^2$.

Next consider

$$A/A^2 = \langle x^2\rangle A^2/A^2 \times \langle a_2\rangle A^2/A^2 \times \cdots \times \langle a_{n-s-1}\rangle A^2/A^2.$$

By the same reasoning as used in (3a), $\langle xa_i\rangle$ is non-normal for each $i \in \{2, \ldots, n-s-1\}$. Also $\langle xa_i\rangle$ and $\langle xa_j\rangle$ are not conjugate if $i \neq j$ since $(xa_i)^g = xa_i[xa_i, g]$ implies that $a_i \equiv a_j \bmod G'$ or $a_i \equiv x^2a_j \bmod G'$. Finally $\langle xa_i\rangle$ is not conjugate to $\langle x, B_j\rangle$ if $i \neq j$. Indeed, by virtue of size the only possibility is that $\langle x\rangle$ is conjugate to $\langle xa_i\rangle$ and $x^2 \not\equiv a_i \bmod A^2 = G'$. So again we have $\nu(G) \geq (n-s-2) + s = n-2$, and G is not a counter example. This concludes case 1.

Case 2. $\nu(G/N) \neq 0$ for some minimal normal subgroup N of G.

Having dealt with the situation $\nu(G/N) = \nu(G)$ in case 1, we may assume that $\nu(G/N) < \nu(G)$ and there exists a $zN \in Z(G/N)$ such that $\ell(G/\langle z, N\rangle) \leq \nu(G/N) + 1$. Note that $z^p \in Z(G)$ since

$$[z^p, g] = [z, g]^{z^{p-1}+z^{p-2}+\dots+z+1} = [z, g]^p = 1$$

for each $g \in G$ as $[z, g] \in N \subseteq Z(G)$. We now consider the various possibilities.

Case 2(a). If $z^p = 1$ then $\ell(G/N) \leq \nu(G/N) + 2 \leq \nu(G) + 1$, and the result holds for G.

Case 2(b). If $z \in Z(G)$ then G is not a counter example since

$$\ell(G/\langle z\rangle) = \ell(G/\langle z, N\rangle) + 1 \leq \nu(G/N) + 2 \leq \nu(G) + 1.$$

Case 2(c). If $N \leq \langle z\rangle$, then

$$\ell(G/\langle z^p\rangle) = \ell(G/\langle z, N\rangle) + 1 \leq \nu(G/N) + 2 \leq \nu(G) + 1$$

and the result holds in this case.

Case 2(d). If $\nu(G) > \nu(G/N) + 1$ then

$$\ell(G/\langle z^p\rangle) = \ell(G/\langle z, N\rangle) + 2 \leq \nu(G/N) + 3 \leq \nu(G) + 1.$$

and so this case is done.

In view of the above cases we may assume that $|z| = p^{n+1}$, $exp(Z(G)) = p^n$, $n > 0$, $\nu(G) = \nu(G/N) + 1$, and since $[z, G] \leq N \not\leq \langle z\rangle$, $\langle z\rangle \not\trianglelefteq G$. Let $M = \langle z^{p^n}\rangle$. Now, $\nu(G/M) \neq 0$ as $\langle z\rangle$ is not normal in G. If $\nu(G/M) = \nu(G)$ then we are done by Case 1 and if $\nu(G/M) \neq \nu(G)$ then by the argument above there is a $\overline{z}M \in Z(G/M)$ such that $\langle\overline{z}\rangle \not\trianglelefteq G$ and $\nu(G) = \nu(G/M) + 1$. Thus $N \leq \langle\overline{z}\rangle$ just as $M \leq \langle z\rangle$. Consider the minimal normal subgroup $L = \langle mn\rangle$, where $\langle m\rangle = M$ and $\langle n\rangle = N$. If $\nu(G/L) \neq 0$ then another repetition of the above argument will yield a contradiction by forcing $\nu(G/L) + 1 = \nu(G)$ so that $\langle z\rangle$ and $\langle \bar{z}\rangle$ would have to be conjugate–yet contain different minimal normal subgroups of G. Otherwise, since $M, N \subseteq G'$, $\nu(G/L) = 0$, $G/L \cong Q_8 \times E$ (E elementary abelian) and $G' = MN$. Now, $(z\bar{z})^2 = z^2\bar{z}^2 = mn$ and so $[z\bar{z}, G] = L$ else $\langle z\bar{z}\rangle \ntrianglelefteq G$ yet contains L. (Note $[\bar{z}z, G] \neq 1$ since for each $g \in G$, $[\bar{z}z, g] = [\bar{z}, g]^z[z, g] = 1$ only if $[z, g] = [\bar{z}, g] = 1$.) Let $x \in G$ such that $[x, z] = n$. Then $[x, \bar{z}] = m$. It follows that $\langle x\rangle \ntrianglelefteq G$. Since $\nu(G/N) + 1 = \nu(G/M) + 1 = \nu(G)$, either $M \leq \langle x\rangle$ or $N \leq \langle x\rangle$. Assume that $M \leq \langle x\rangle$. Then since $\nu(G/N) - 1 = \nu(G)$ we conclude that $\langle x\rangle$ is conjugate to $\langle z\rangle$. This is a contradiction because $x^g \in \{x, xn, xm, xnm\}$ for each $g \in G$ but $[x, z] = n$. This completes the second case.

Case 3. $\nu(G) > \nu(G/N) = 0$ for all minimal normal subgroups N of G. We split the proof into two cases depending on whether $p > 2$ or $p = 2$.

Case 3(a). $p > 2$. Then $G' = N$ is the unique minimal normal subgroup of G, from which it follows that $Z(G)$ is cyclic and $g^p \in Z(G)$ for all $g \in G$. Let $Z(G) = \langle z\rangle$ and let $|G/Z(G)| = p^{s+1}$. Let $\{x_i, i = 1, \ldots, s+1\}$ be a minimal set of generators of G mod $Z(G)$ such that a maximum number of the x_i satisfy $\langle x_i\rangle \ntrianglelefteq G$. If this number is zero then every subgroup of G is normal and G is Abelian, a contradiction. So assume $\langle x_i\rangle \ntrianglelefteq G, i = 1, \ldots, t$. Then $\langle x_{t+1}, \ldots, x_{s+1}, z\rangle$ is Abelian and every subgroup of this group is normal in G. This group must be cyclic because $Z(G)$ is cyclic and hence $t = s$. Now $\ell(G/Z(G)) = s + 1$ and $\nu(G) \geq s$ since $\langle x_i\rangle \ntrianglelefteq G, i = 1, \ldots, s$ and these non-normal subgroups lie in distinct conjugacy classes since $\langle x_i\rangle \not\equiv \langle x_j\rangle$ mod G'. Thus the result holds in this case.

Case 3(b). $p = 2$. Here G/N is either Abelian or isomorphic to $Q_8 \times E$ where E is an elementary Abelian 2–group. In either case N is the unique minimal normal subgroup of G. This may be argued as follows:

If M is another such subgroup then $\nu(G/M) = 0$. Take $x \in G$ where $\langle x \rangle \ntrianglelefteq G$. Since $\langle x \rangle N, \langle x \rangle M$ are normal in G, there exists an element $t \in G$ such that $x^t = x^i n = x^j m$ for some $1 \neq n \in N$ and some $1 \neq m \in M$. If $x^i = x^j$ then $n = m$ and $N = M$. If $x^i \neq x^j$ then $1 \neq \langle mn \rangle \leq \langle x \rangle$ and $\nu(G/\langle mn \rangle) \neq 0$, a contradiction.

From the uniqueness of the minimal normal subgroup N, it follows that $Z(G)$ is cyclic, say $Z(G) = \langle z \rangle$. (In other words, for all $x \neq 1 \in G$, $\langle x \rangle \lhd G$ if and only if $N \leq \langle x \rangle$). Indeed, $1 \neq \langle x \rangle \lhd G$ implies that $\langle x \rangle \cap Z(G) \neq 1$ which means that $N \leq \langle x \rangle$. Conversely, if $N \leq \langle x \rangle$ then $\langle x \rangle \lhd G$ as $\nu(G/N) = 0$. Thus $\langle x \rangle \ntrianglelefteq G$ implies $\langle x \rangle \cong C_2$. Indeed, if $\langle x \rangle \ntrianglelefteq G$ then $\langle x \rangle \cap N = 1$. If $N = G'$ then $x^2 \in Z(G)$ and so $|x| = 2$. Otherwise $G/N \cong Q_8 \times E$ and if $|x| > 2$ there exists a $y \in G$ such that $[x, y] = x^2$ or $x^2 n$. But then $\langle x^2 \rangle \ntrianglelefteq G$ yet $[x^2, y] = 1$ and $[x^2, E] = 1$ and this is a contradiction, since $G = \langle x, y, E, N \rangle$ and $x^2 \notin Z(G)$.

Now if $G/N \cong Q_8 \times E$ and $H = \langle x, y \rangle \leq G$ such that $\langle x, y \rangle N/N \cong Q_8$, then $|x| = |y| = 8$, $N \leq \langle x \rangle \cap \langle y \rangle$, $x^2 \equiv y^2 \bmod N$ and $|H| = 16$. Under these circumstances N is the unique subgroup of order 2 in H, also H is non-Abelian and Dedekind. By Dedekind's theorem no such group of order 16 exists. Thus we may assume that G/N is Abelian so that $G/Z(G)$ is an elementary Abelian 2−group.

Let $|G/Z(G)| = 2^{s+1}$. Let $\{x_i, i = 1, \ldots, s+1\}$ be a minimal set of generators of G mod $Z(G)$ such that a maximum number of the x_i satisfy $\langle x_i \rangle \ntrianglelefteq G$. Say $\{x_i, i = 1, \ldots, t\}$ is such a set. Then every subgroup of $\langle x_{t+1}, \ldots, x_{s+1}, z \rangle$ is normal in G. If this group is Abelian then it is cyclic as $Z(G)$ is cyclic. Hence $t = s$, $\ell(G/\langle z \rangle) = s + 1$ and $\nu(G) \geq s$ since $\langle x_i \rangle \ntrianglelefteq G, i = 1, \ldots, s$ and they lie in distinct conjugacy classes. In this case the result holds.

If this group is not Abelian then it is isomorphic to Q_8, $Z(G) = N$, $t = s-1$ and $\ell(G) = s-2$. Now $t > 0$ since G is not isomorphic to Q_8. Let $U = \langle x_i, i = 1, \ldots, t\rangle$ and $V = \langle x_{t+1}, x_{t+2}\rangle \cong Q_8$. For any $i \leq t$ and any $v \in V \setminus N$, $(x_i v)^2 = v^{x_i} v = v^2 n$ or $v^2 = 1$ or v^2. By the maximality of t, $(x_i v)^2 \neq 1$ as $\langle x_i v\rangle$ is not a subgroup of U. Hence $[x_i, v] = 1$ for each i and so U centralizes V. Thus U is non-Abelian. Say $[x_1, x_2] = n \neq 1$. Then $(x_1 x_2 x_{t-1})^2 = (x_1 x_2)^2 x_{t+1}^2 = n^2 = 1$ and this contradicts the maximality of t. Thus this case can not arise and the result is established. $\square$

This bound is sharp in the sense that there are p-groups in which the bound is attained. Indeed, all p-groups G with $\nu(G) = 1$ do have a central, cyclic subgroup $\langle z\rangle$ such that the prime length of $G/\langle z\rangle$ is $\nu(G) + 1$. However, improvement may be possible and a better bound could be used to improve the bounds in many of the subsequent results.

We next extend this result to all finite nilpotent groups.

Proposition 2.2 *Let G be a finite nilpotent group with $\nu(G) \neq 0$. Then for some prime p, there is a p-element $z \neq 1$ in the centre $Z(G)$ of G such that $\ell(G/\langle z\rangle) \leq \nu(G) + 1$.*

Proof: Since $\nu(G) \neq 0$ there is a prime p, such that $\nu(S_p) \neq 0$ for some Sylow p-subgroup S_p. Write G as the direct product $S_p \times K$. By Proposition 1.6 and the fact that every p-group P has a normal subgroup of each order $1, p, \ldots, |P|$,

$$\nu(G) = \nu(S_p)\nu(K) + \nu(S_p)\mu(K) + \mu(S_p)\nu(K) \geq \nu(S_p)(\ell(K) + 1),$$

where $\mu(K)$ is the number of normal subgroups of K. By Lemma 2.1, there exists an element $z \in Z(S_p)$ such that

$$\ell(G/\langle z\rangle) = \ell(G/S_p) + \ell(S_p/\langle z\rangle) \leq \ell(K) + \nu(S_p) + 1$$

and this is certainly less than or equal to $\nu(S_p)(\ell(K)+1)-1$ as $\nu(S_p) \geq 1$. This completes the proof. $\square$

Note that the above proof actually shows that if G is a finite nilpotent group, then for any prime p such that $\nu(S_p) \neq 0$, there exists a central p-element $z \neq 1$ such that $\ell(G/\langle z\rangle) \leq \nu(G) - 1$.

In fact, the bound given in the above proposition is a poor one if G is a nilpotent group that is not a p-group. This is clear from Proposition 1.6, since as the number of primes involved in the order of G grows, $\nu(G)$ increases much more quickly than the prime length of the centre. Information about the Sylow subgroups of G yields a better estimate of the prime length of $G/\langle z\rangle$. The following bound of $\ell(G/Z(G))$ is an example of this fact.

Corollary 2.3 *Let G be a finite, non-Dedekind, nilpotent group. If $G = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_k}$, where S_{p_i} is a non-trivial Sylow-p_i subgroup of G then*

$$\ell(G/Z(G)) \leq \begin{cases} \nu(S_{p_1}) + \nu(S_{p_2}) + \cdots + \nu(S_{p_k}) + k & \text{if } 2 \nmid |G| \\ \nu(S_{p_1}) + \nu(S_{p_2}) + \cdots + \nu(S_{p_k}) + k + 1 & \text{if } 2 \mid |G|. \end{cases}$$

Proof: Let G be as described above. Assume without loss of generality that $\nu(S_{p_i}) \neq 0$, $i \in \{1, 2, \ldots, s\}$ and $\nu(S_{p_i}) = 0$ for $i = s+1, \ldots, k$. If $2 \nmid |G|$ then S_{p_i} is central for each $i = s+1, \ldots, k$ and so by Lemma 2.1,

$$\begin{aligned} \ell(G/Z(G)) &= \ell(S_{p_1}/Z(S_{p_1})) + \cdots + \ell(S_{p_k}/Z(S_{p_k})) \\ &\leq \nu(S_{p_1}) + 1 + \cdots + \nu(S_{p_s}) + 1 \\ &= \nu(S_{p_1}) + \cdots + \nu(S_{p_s}) + s \\ &\leq \nu(S_{p_1}) + \cdots + \nu(S_{p_k}) + k. \end{aligned}$$

If $2 \mid |G|$ then let S_2 be the Sylow-2 subgroup. If $\nu(S_2) \neq 0$ or S_2 is Abelian then the argument above still holds. Otherwise, $S_2 = Q_8 \times E$ for some elementary

Abelian subgroup E, so that $\ell(S_2/Z(S_2)) = 2$. Again by Lemma 2.1,

$$\ell(G/Z(G)) \leq \nu(S_{p_1}) + \dots \nu(S_{p_k}) - k + 1.$$

(The extra 1 was needed because $\ell(S_2/Z(S_2)) = \nu(S_2) - 1 + 1$.) □

Note that if $G = M(p^n) \times Q_8$ for some $n \geq 3$ and some prime $p \neq 2$ ($M(p^n)$ as decribed in Proposition 1.7), then $\ell(G/Z(G)) = 4 = \nu(M(p^n)) + \nu(Q_8) + 2 - 1$ so that the bound must indeed be increased by 1 if $2 \mid |G|$.

The following result was conjectured by Brandl in [Br2] and first proved by Poland and Rhemtulla in [PR]. It follows as a natural corollary to Proposition 2.2.

Corollary 2.4 (Poland and Rhemtulla) [PR] *Let G be a finite nilpotent group of class c and let G have precisely $\nu(G) > 0$ conjugacy classes of non-normal subgroups. Then*

$$c \leq 1 - \nu(G).$$

Proof: First assume G is a p-group. Then by Lemma 2.1, $\ell(G/Z(G)) \leq \nu(G) - 1$ and since either G is Abelian or $\ell(G/Z_{c-1}(G)) \geq 2$, the upper central series of G has length at most $1 + \nu(G)$ as required.

Next assume that $G = S_{p_1} \times S_{p_2} \times \dots \times S_{p_k}$, where S_{p_i} is a non-trivial Sylow-p_i subgroup of G for each $i = 1, 2, \dots, k$. It is well-known that the nilpotency class of G is the maximum of the nilpotency classes of the S_{p_i}. If S_{p_k} has the maximal nilpotency class, $c(S_{p_k})$, then

$$c(G) = c(S_{p_k}) \leq \nu(S_{p_k}) + 1 \leq \nu(G) + 1,$$

as required. □

2.2 Restrictions on $\nu(G)$

The main purpose of this section is to point out that for p-groups with p a fixed prime, there are some values $\nu(G)$ can not attain. Specifically, we show that $\nu(G) \neq 2, 3, \ldots, p-1$ for any p-group. (p odd).

Lemma 2.5 *Let G be a finite p-group with $\nu(G) > 0$. Then either $\nu(G) = 1$ or $\nu(G) \geq p$.*

Proof: Assume not and that G is a counter example of minimal order p^n. For each minimal normal subgroup N of G, either $\nu(G/N) = 0$ or $\nu(G/N) = 1$. Let $N \leq Z(G)$ and $|N| = p$. Note that we may assume that $p > 2$.

First suppose that $\nu(G/N) = 0$, for some minimal normal subgroup N of G. Then $G' = N \leq Z(G)$ and for each $g \in G$, $g^p \in Z(G)$. We first show that $Z(G)$ must be cyclic. Indeed, suppose that $M \leq Z(G)$, $|M| = p$, and $M \cap N = 1$. Then $\nu(G/M) = 1$. The structure of p-groups with a unique conjugacy class of non-normal subgroups has been outlined, these are precisely the $p-$groups $M(p^n) = \langle a, b : a^{p^{n-1}} = b^p = 1, a^b = a^{p^{n-2}+1} \rangle$ where $n \geq 4$ in the case $p = 2$ and $n \geq 3$ otherwise (see Theorem 1.2). Hence G/M has a non-normal subgroup of order p and consequently G has a cyclic non-normal subgroup of order at most p^2. Let $M = \langle m \rangle$ and $N = \langle n \rangle$.

If $\langle g \rangle \ntrianglelefteq G$ has order p then none of the following p-subgroups are normal in G since they are of order p and are not central

$$\langle g \rangle, \langle gm \rangle, \langle gm^2 \rangle, \ldots, \langle gm^{p-1} \rangle.$$

In fact these p subgroups represent p distinct conjugacy classes of non-normal subgroups, since $G' = N$ implies that for each $h \in G$ and $j \in \{0, 1, \ldots, p-1\}$

$$(gm^j)^h = gm^j n^a = g^i m^{ki} = (gm^k)^i$$

only if $i = 1$, $a = 0$ and $j = k$ as $\langle g\rangle \cap Z(G) = 1$ and $M \cap N = 1$. Thus $\nu(G) \geq p + 1$, contradicting the assertion that G is a counter example. Thus G has no non-normal subgroups of order p.

Since $\nu(G/M) = 1$ there is a non-normal subgroup $\langle x\rangle$ of G such that $M \leq \langle x\rangle, |x| = p^2$. Furthermore, if $L_i = \langle mn^i\rangle, i \in \{0, 1, \ldots, p-1\}$ then by the above argument, $\nu(G/L_i) = 1$ and there is an $\langle x_i\rangle \ntrianglelefteq G$ such that $L_i \leq \langle x_i\rangle$ and $|\langle x_i\rangle| = p^2$. By the normality of the $L_i, i = 0, 1, \ldots, p-1$, we see that $\langle x_i\rangle$ can not be conjugate to $\langle x_j\rangle$ unless $i = j$. Thus $\nu(G) \geq p$.

So assume that $Z(G)$ is cyclic. Note that $g^p \in Z(G)$ for each $g \in G$ so that if $\langle g\rangle \ntrianglelefteq G$ then $|g| = p$. If $\langle x\rangle, \langle t\rangle \ntrianglelefteq G$ that are not conjugate, then

$$\langle x\rangle, \langle xt\rangle, \langle xt^2\rangle, \ldots, \langle xt^{p-1}\rangle$$

are distinct non-normal subgroups since

$$(xt^i)^p = x^p(t^i)^p([t^i, x])^{\binom{p}{2}} = 1.$$

Furthermore these subgroups are not conjugate. Indeed, since $G' = N$ the element xt^i is only conjugate to $xt^in^j, j \in \{0, 1, \ldots, p-1\}$ and $(xt^j)^s = x^st^{js}n^\gamma$, for some integer γ. So $xt^in^j = x^st^{js}n^\gamma$ implies $x^{1-s} = t^{js-i}n^{\gamma-j}$ and because $\langle x\rangle$ and $\langle t\rangle$ are not conjugate $s = 1, i = js$ and $\gamma = j$. Thus $i = j$ and $\nu(G) \geq p$. This contradicts the initial assumption. Thus G has a unique conjugacy class of cyclic, non-normal subgroups represented by $\langle x\rangle$.

Suppose $H \leq G$ and H is not cyclic. Then since all subgroups of order p have the form N or $\langle xn^j\rangle, j \in \{0, 1, \ldots, p-1\}$, any non-cyclic subgroup of G must contain N and hence $H \triangleleft G$. In this instance, $\nu(G) = 1$ and G is not a counter example.

If $\nu(G/N) = 1$ then by Theorem 1.2 $G = \langle x, y\rangle N/N$, $x^p \in N$, $y^{p^{n-2}} \in N$, and $[y, x] = y^{p^{n-3}}n^a$, for some $a \in \{0, 1, \ldots, p-1\}$. Note that $N \not\leq \langle y\rangle$ else $\langle y\rangle$ has index p in G and $G \cong M(p^n)$ or G is Abelian (by Theorem 4.1, Chapter 4 of

[Su1]). and $\nu(G) \leq 1$. Also. $|[x,y]| = p$. Now $G' = \langle [x,y] \rangle$ and hence $G/\langle [x,y] \rangle$ is Abelian. Indeed. $G = \{x^i y^j n^k : i, j, k \text{ are integers}\}$. Moreover $[x,y]$ is central so that for all integers i, j, a, b. $[x^i y^j, x^a y^b]$ is a power of $[x,y]$ using the standard commutator identities. (see [Rob] 5.1.5). Therefore $G' = \langle [x,y] \rangle$ as required.

So by appealing to the first case. G is not a counter example and we have the required result. □

In the previous section. we produced a bound for the prime length of a certain central quotient of G. It is natural to ask if we can produce a bound not only for the length but also for the order of the central quotient of G in terms of $\nu(G)$. In the paper [L] a bound for $[G : Z(G)]$ was produced in terms of $\nu(G)$ and the primes involved in the order of G. where G is any finite group. Consider finite groups with $\nu(G) = 1$. (see Theorem 1.2).

$$[G : Z(G)] = \begin{cases} p^2 & \text{if } G \cong \langle x, y : x^{p^{n-1}} = y^p = 1, [x,y] = x^{p^{n-2}} \rangle. \\ pq & \text{if } G \cong P \rtimes Q, |P| = p, |Q| = q^n \ p, q \text{ distinct primes.} \end{cases}$$

From these examples we can see that bounding $[G : Z(G)]$ by a function of $\nu(G)$ alone is not always possible. However. in the case that G is a p-group we can write $[G : Z(G)]$ as a function of $\nu(G)$. except in the instance that $\nu(G) \leq 1$. This follows from the previous Lemma.

Corollary 2.6 *If G is a finite p-group with $\nu(G) > 1$ then there exists an element $z \in Z(G)$ so that*

$$[G : \langle z \rangle] \leq \nu(G)^{\nu(G)+1}.$$

Proof: This follows immediately from Lemmas 2.1 and 2.5. □

We can, in fact, explicitly state all finite p-groups G with $\nu(G) = p$. We shall see that an infinite number of such groups exist. The crux of the proof is the fact that if N is a minimal normal subgroup of G then by Lemma 2.5 either $\nu(G/N) = 0, 1$ or $\nu(G/N) = \nu(G) = p$, and we know the structure of such groups.

Lemma 2.7 *Let G be a finite p-group. If $\nu(G) = p$ then either $p = 2$ and $G \cong D(8)$, or $G \cong Q(16)$ or p is any prime and*

$$G \cong \langle x, y : x^{p^{n-2}} = y^{p^2} = 1, [x, y] = x^{p^{n-3}} \rangle, n \geq 4.$$

Proof: We begin by verifying that each of the above groups do indeed satisfy $\nu(G) = p$. By Proposition 1.7, $\nu(D(8)) = \nu(Q(16)) = 2$. Consider

$$G \cong \langle x, y : x^{p^{n-2}} = y^{p^2} = 1, [x, y] = x^{p^{n-3}} \rangle, n \geq 4.$$

Note that $G' = \langle x^{p^{n-3}} \rangle$ has order p, $x^p \in Z(G)$ and that the subgroup $\langle x^p \rangle \times \langle y^p \rangle$ is central of index p^2 in G and hence equal to $Z(G)$. Also

$$\begin{aligned} (x^i y^j)^{p^a} &= (x^i)^{p^a} (y^j)^{p^a} ([y^j, x^i])^{\binom{p^a}{2}} \\ &= (x^i)^{p^a} (y^j)^{p^a} \end{aligned}$$

if $p \neq 2$ or if $p = 2$ and $a > 1$. If $p = 2$

$$(x^i y^j)^2 = (x^i)^2 (y^j)^2 [y^j, x^i].$$

It follows that all elements of order p lie in $Z(G)$: Either $p \neq 2$ and $(x^i y^j)^p = 1$ only when $(x^i)^p = 1$ and $(y^j)^p = 1$ or $p = 2$ and $(x^i y^j)^2 = 1$ means that $j \equiv 0$ or $2 \bmod 4$ as $\langle x \rangle \cap \langle y \rangle = 1$ and so $(x^i)^2 = (y^j)^2 = 1$. Hence if $H \leq G$ is not cyclic, then $G' \leq H \triangleleft G$ since any 2 distinct subgroups of order p generate a group containing G' and $H \cong Q_8$ implies $G' \subseteq H$. In fact, $\langle x^i y^j \rangle$ is normal if $|x^i| > p^2$ since under such circumstances G' lies inside the subgroup.

Finally, $\langle x^i y^j \rangle \ntrianglelefteq G$ if and only if $|x^i y^j| = p^2$ and $|y^j| = p^2$. Indeed, $\langle x^i y^j \rangle$ is non-normal precisely when it has order p^2 and does not contain G'. In other words, when $|x^i y^j| = p^2$ and $|y^j| = p^2$. The number of such subgroups is p^2. Since $|G'| = p$, each non-normal subgroup $\langle x^i y^j \rangle$ has p conjugates and so $\nu(G) = p$, as required.

Now, let G be a p-group with $\nu(G) = p$ and $|G| = p^n$. First we will dispense with a few special cases, mainly where 2-groups cause extra problems.

If G has a maximal subgroup that is cyclic then $G \cong D(8)$ or $G \cong Q(16)$. This follows from Proposition 1.7, where the p-groups G with a maximal cyclic subgroup are described and $\nu(G)$ is given as a function of n.

If G has a unique subgroup of order p then $G \cong Q(16)$. Indeed by Theorem 4.4 of [Su1], G is cyclic or $G \cong Q(2^n)$ for some n. Thus by Proposition 1.7 $n = 4$ is the only possibility.

If $\nu(G) = \nu(G/N) = p$ for some minimal normal subgroup N of G then $p = 2$ and $G \cong Q(16)$ or $G \cong \langle x, y : x^4 = y^4 = 1, [x, y] = x^2 \rangle$. Indeed, we noted in the proof of Lemma 2.1 that if $\nu(G) = \nu(G/N)$ then G is a 2-group with only three possible forms and lower bounds for $\nu(G)$ were also given in terms of n. For the first two types of 2-groups, $\nu(G) = \nu(G/N) \geq 3$, thus the only possibility is $G = \langle x \rangle A$, $|G| = 2^n$. Note $n \leq 4$, as $\nu(G) \geq n - 2$. Since neither G nor G/N is Abelian, $n = 4$ is the only valid possibility. Using GAP and the library of 2-groups contained therein.(see [Sc]), one can establish that there are, up to isomorphism, only 2 groups of order 16 with $\nu(G) = 2$. The 2 groups given above are non-isomorphic with exactly two conjugacy classes of non-normal subgroups each (Q(16) has only one element of order 2 while the second group has 3, so they are not isomorphic), and so they are the required groups.

If $\nu(G/N) = 0$ and $G' \neq N$ for some minimal normal subgroup N of G then $|G| = 8, 16$ or 32 and G has one of the forms stated in the lemma. Indeed by Theorem 1.1 $G/N \cong Q(8) \times E$ for some elementary Abelian group E and so $exp(Z(G)) = 2$ or 4 By Lemma 2.1 $[G : \langle z \rangle] = 4$ or 8, for some central element z and so $|G| = 8, 16$, or 32. Again by using GAP one establishes that there is exactly one group of order 8, 2 of order 16 and one of order 32, up to isomorphism, with $\nu(G) = 2$. We have provided descriptions of these groups in the hypothesis, as required.

Now let us consider the general case. Let N be a minimal normal subgroup of G. By the comments above and Lemma 2.5, we may assume that either $\nu(G/N) = 0$ and $G' = N$ or $\nu(G/N) = 1$.

Assume that $\nu(G/N) = 0$ and $G' = N$. Then we may assume that $Z(G)$ is not cyclic. This may be established as follows. Assume $Z(G)$ is cyclic. Then $N = G'$ is the only normal subgroup of order p. If $\langle g \rangle$ and $\langle h \rangle$ are two non-conjugate, non-normal subgroups of order p in G and $p \neq 2$ then by the same argument used in Lemma 2.5, the groups

$$\langle g \rangle, \langle h \rangle, \langle gh \rangle, \langle gh^2 \rangle, \ldots, \langle gh^{p-1} \rangle$$

represent $p+1$ distinct conjugacy classes of non-normal subgroups of order p, contradicting the fact that $\nu(G) = p$.

If $p = 2$ and $\langle g \rangle, \langle h \rangle$ are as defined above with order 2, then either $\langle gh \rangle \ntrianglelefteq G$- a contradiction or $|\langle gh \rangle| = 4$ and $\langle (gh)^2 \rangle = N$. Let $c \in Z(G)$ such that $|c| = 4$. (If no such c exists then $\ell(G) \leq \nu(G) + 2 \leq 4$ so that $|G| \leq 16$ and we know the possible structures.)

Now $\langle ghc \rangle \notin Z(G)$ and if $N = \langle n \rangle$,

$$(ghc)^2 = (gh)^2 c^2 = n^2 = 1.$$

Thus $\langle ghc \rangle \ntrianglelefteq G$. Since $ghc \neq gn$ and $ghc \neq hn$ we have a third conjugacy class of non-normal subgroups and a contradiction.

If G has exactly one conjugacy class of non-normal subgroups of order p then the elements of order p in G consist of the set $\langle g\rangle \times N \setminus \{1\}$. Let $H \leq G$ such that $|H| \geq p^2$. If H is cyclic, say $H = \langle h\rangle$ then $G' \subseteq H$ since $h^p \neq 1 \in Z(G)$. If H is not cyclic then either $G' \leq H$ or H has a unique subgroup of order p. This would mean that H is generalized quaternion and non-Abelian and hence contains G'. Thus $\nu(G) = 1$. Consequently, we can assume $Z(G)$ is not cyclic.

Let M be a second minimal normal subgroup of G. Then $\nu(G) \neq \nu(G/M)$ by the comments above and $\nu(G/M) \neq 0$ as $G' = N$. Thus $\nu(G/M) = 1$. Thus by Theorem 1.2, we may assume that there exists $n \geq 4$ ($n \geq 5$ if $p = 2$) and elements x and y in G so that $x^{p^{n-2}} \in M$, $y^p \in M$, $[x, y] \equiv x^{p^{n-3}} \bmod M$ and $\langle y\rangle M \ntrianglelefteq G$.

We may assume that $x^{p^{n-2}} = 1$ else G has a maximal subgroup which is cyclic – a possibility that has already been considered. Now suppose that $M \not\leq \langle y\rangle$ so that $|y| = p$ and $\langle y\rangle \ntrianglelefteq G$. Then the subgroups

$$\langle y\rangle, \langle ym\rangle, \ldots, \langle ym^{p-1}\rangle$$

are non-central subgroups of order p and hence non-normal. Furthermore they are pairwise non-conjugate since

$$(ym^i)^g = ym^i n^a = y^a m^{ja}$$

implies that $y^{1-a} \in Z(G)$ which means that $a \equiv 1 \bmod p$ and $i = j$. Since $\langle y\rangle M \ntrianglelefteq G$ as well, $\nu(G) \geq p+1$, a contradiction.

Thus we may assume that $M \leq \langle y\rangle \ntrianglelefteq G$ and $|y| = p^2$. Now, $M_i = \langle mn^i\rangle, i = 0, 1, \ldots, p-1$ are p minimal normal subgroups of G such that $M_i \cap N = 1$. By the above argument, for each i there is a subgroup $\langle y_i\rangle$ of order p^2 such that $M_i \leq \langle y_i\rangle \ntrianglelefteq G$. Since $\nu(G) = p$ we have just accounted for all the conjugacy classes of non-normal subgroups. If $n > 4$ then since $|x| > p^2, \langle x\rangle \triangleleft G$ and $[x, y] = x^{p^{n-3}}$. This completely determines G as the semi-direct product of $\langle x \rangle$ and $\langle y \rangle$ as $1 \neq [x, y] \in Z(G)$. In this case

$$G \cong \langle x, y : x^{p^{n-2}} = y^{p^2} = 1, [x, y] = x^{p^{n-3}} \rangle.$$

If $n = 4$ then $G/M \cong M(p^3), p \neq 2, G' = N$. Note that G has one element of order 1, $p^2 - 1$ elements of order p (as $Z(G) = MN$) and we may assume that G has 0 elements of order p^3 to avoid the existence of a maximal cyclic subgroup. Hence G has $p^4 - p^2$ elements of order p^2. Since $\nu(G) = p$ and all non-normal subgroups of G have p conjugates (as $|G'| = N$) we may assume that G has p^2 cyclic non-normal subgroups and p normal cyclic subgroups of order p^2. Let $\langle \overline{x} \rangle \lhd G$ such that $|\overline{x}| = p^2$. Then if $\langle y \rangle \ntriangleleft G$, $|y| = p^2$ and $\langle \overline{x} \rangle \cap \langle y \rangle = 1$ so that $G = \langle \overline{x} \rangle \langle y \rangle$. Now

$$[\overline{x}, y] = (\overline{x}^p)^a, \text{ for some } a = 1, 2, \ldots, p-1.$$

Replace y by a power of y if necessary and we can assume that, even when $n = 4$,

$$G \cong \langle x, y : x^{p^{n-2}} = y^{p^2} = 1, [x, y] = x^{p^{n-3}} \rangle,$$

as required.

Finally, assume that $\nu(G/M) \neq 0$ for each minimal normal subgroup M of G. Then we may assume that $\nu(G/M) = 1$. If $Z(G)$ is not cyclic then there are at least $p + 1$ minimal normal subgroups, each contained in a non-normal subgroup of order p^2. Hence $\nu(G) \geq p + 1$.

If $Z(G)$ is cyclic then we may assume that $\nu(G/N) = 1, n \geq 4$ and there exists $x, y \in G$ so that $x^{p^{n-2}} = 1, y^p \equiv 1 \bmod N, \langle y \rangle N \ntriangleleft G$ and $[x, y] \equiv x^{p^{n-3}} \bmod N$. Now, since $N \nleq \langle x \rangle$ and $[x^p, y] = 1$, we conclude $x^p \in Z(G)$ and obtain the contradiction that $|x| = p$. This establishes the lemma. □

We have now given two sets of examples of p-groups where $\nu(G)$ is "small", ($\nu(G) = 1$ and $\nu(G) = p$) but where $\ell(G)$ can be arbitrarily large. We note in the next section, that for finite nilpotent groups G, if $\ell(G)$ is much larger than $\nu(G)$ then something can be said about the structure of the group G.

2.3 Bounding the Prime Length of G and G'

Having bounded $\ell(G/\langle z\rangle)$ by a function of $\nu(G)$, we now investigate bounding $\ell(G)$ with a function of $\nu(G)$. To this end we note the following.

Lemma 2.8 *Let G be a finite p-group with $\nu(G) = \nu > 0$. If $\ell(G) > 2\nu - 1$ then there exists an element z in the centre $Z(G)$ of G, such that $G/\langle z\rangle$ is Abelian and $\ell(G/\langle z\rangle) \le \nu + 1$.*

Proof: Let G be a counter example of minimal prime length $\ell(G) > 2\nu(G) - 1$. By Lemma 2.1 there exists an element $z \in Z(G)$ such that $\ell(G/\langle z\rangle) \le \nu - 1$. By assumption $G' \not\le \langle z\rangle$. Let $w = z^{p^{r-1}}$ where p^r is the order of z. Observe that $r > 1$, otherwise $\ell(G) = \ell(G/\langle z\rangle) + 1 \le \nu - 2 \le 2\nu + 1$. Now consider the group $G/\langle w\rangle$. If $\nu(G/\langle w\rangle) = 0$ then since $G' \not\le \langle w\rangle$, $p = 2$, $|\langle G', w\rangle/\langle w\rangle| = 2$ and $Z(G/\langle w\rangle)$ is of exponent 2. Thus $|\langle z\rangle| \le 4$ and $\ell(G) \le \ell(G/\langle z\rangle) - 2 \le \nu + 3 \le 2\nu + 1$. (Recall from Theorem 1.2 that the only finite p-groups G with $\nu(G) = 1$ are $\langle a, b \colon a^{p^{n-1}} = b^p = 1, a^b = a^{p^{n-2}+1}\rangle$ where $n \ge 4$ in the case $p = 2$. These groups do not have a central quotient that is non-Abelian as the centre is cyclic and G' is of order p). If $\nu(G/\langle w\rangle) = \nu(G)$, then G is one of three types as described in Case 1 of the proof of Lemma 2.1. In all these cases, $\ell(G) \le 2\nu(G) + 1$.

Indeed, if $G \cong Q \times C_4 \times E$ as in Case 1(1) then $\ell(G) = \ell(E) + 5$ and $\nu(G) \ge 3(\ell(E) + 1)$ so that

$$\ell(G) = \ell(E) + 5 \le \nu(G) - 1 + 5 \le 2\nu(G) + 1.$$

If $G = Q_1 \times Q_2 \times E$ as in Case 1(2) then $\ell(G) = \ell(E) + 6$ and $\nu(G) \geq 9(\ell(E) + 1)$ so that

$$\ell(G) = \ell(E) + 6 \leq \nu(G) - 3 < 2\nu(G) + 1.$$

Finally, if $G = \langle x \rangle A$, as in Case 1(3) then $\nu(G) \geq \ell(G) - 2$ so that

$$\ell(G) \leq \nu(G) + 2 \leq 2\nu(G) + 1.$$

We may thus assume that $0 < \nu(G/\langle w \rangle) < \nu(G)$. By the minimality of $\ell(G)$, there exists an element $y \in G$ such that $[y, G] \leq \langle w \rangle$, $y^p \in Z(G)$, $\ell(G/\langle y, w \rangle) \leq \nu(G/\langle w \rangle) - 1$ and $G' \leq \langle w, y \rangle$ as $\ell(G/\langle w \rangle) > 2\nu(G/\langle w \rangle) - 1$. If $\langle y \rangle \cap \langle w \rangle = 1$, then $\langle y \rangle \cap \langle z \rangle = 1$ and since $\ell(G/\langle z \rangle) \leq \nu + 1$ it follows that $\ell(\langle y \rangle) \leq \nu$. Thus

$$\ell(G) = \ell(G/\langle y, w \rangle) + \ell(\langle y \rangle) + 1 \leq \nu(G/\langle w \rangle) + \nu + 2 \leq (\nu - 1) + \nu + 2 = 2\nu - 1$$

and we are done.

Hence assume $w \in \langle y \rangle$. Then $G' \leq \langle y \rangle$, $\ell(G/\langle y \rangle) \leq \nu$ and $\ell(G/\langle y^p \rangle) \leq \nu + 1$. Recall, $y^p \in Z(G)$ and so if $G' \leq \langle y^p \rangle$, then we are done.

If $G' = \langle y \rangle$ then by a Lemma of Schur (see 10.1.4 [Rob]) the exponent of G' is at most $p^{\nu+1}$ since $\ell(G/Z(G)) \leq \nu + 1$. As G' is cyclic, $\ell(G') \leq \nu + 1$; so $G' = \langle y \rangle$ implies $\ell(\langle y \rangle) \leq \nu + 1$ and hence

$$\ell(G) = \ell(G/\langle y \rangle) + \ell(\langle y \rangle) \leq \nu + \nu + 1 = 2\nu + 1.$$

This completes the proof. □

Next we extend this result to all finite nilpotent groups.

Lemma 2.9 *Let G be a finite nilpotent group with $\nu(G) = \nu > 0$. If $\ell(G) > 2\nu + 1$ then for some prime p, $G = A \times B$ where A is a Dedekind p'-subgroup, B is a p-subgroup and there is an element $z \in Z(B)$ such that B' is a subgroup of $\langle z \rangle$, and $\ell(G/\langle z \rangle) \leq \nu(G) + 1$.*

Proof: If G is a p-group then the result follows from Lemma 2.8. so assume $G = \prod S_q$. (where S_q is a Sylow-q subgroup of G). is the direct product of at least two non-trivial Sylow subgroups. Recall that a p-group of order p^n has normal subgroups of sizes p^k for each integer $0 \leq k \leq n$.(This can be proved by induction on n.) Thus. if $\mu(S_q)$ is the number of normal subgroups of S_q. then $\mu(S_q) \geq \ell(S_q) + 1$.

Suppose that G has at least two non-Dedekind Sylow subgroups. S_{p_1} and S_{p_2}. Then by Proposition 1.6.

$$\nu(G) \geq \nu(S_{p_1}) \prod_{q \neq p_1} (\ell(S_q) + 1) + \nu(S_{p_2}) \prod_{q \neq p_2} (\ell(S_q) - 1)$$

so that

$$\nu(G) \geq \nu(S_{p_1}) - \nu(S_{p_2}) + \sum \ell(S_q).$$

where the sum runs over all primes q. dividing $|G|$. In this case. $\ell(G) < \nu(G) < 2\nu(G) + 1$.

Next assume that $G = A \times B$. where A is a Dedekind p'-subgroup and B is a p-subgroup with $\nu(B) > 0$. If $\ell(B) \leq 2\nu(B) + 1$ then using the relation $\nu(G) \geq \nu(B) \prod_{p \neq q} (\ell(S_q) - 1)$. (which follows from Proposition 1.6). we have

$$\ell(G) = \sum \ell(S_q) \leq \sum_{q \neq p} \ell(S_q) + 2\nu(B) + 1 \leq 2\nu(G) + 1.$$

as required.

Finally. if $\ell(B) > 2\nu(B) + 1$ then by Lemma 2.8 above. $G = A \times B$ where A is a Dedekind p'-subgroup. B is a p-subgroup and there is an element $z \in Z(B)$ such that B' is a subgroup of $\langle z \rangle$. and $\ell(B/\langle z \rangle) < \nu(B)$. By Proposition 1.6. $\ell(G/\langle z \rangle) \leq \nu(G) + 1$. as required. □

A natural question is whether there is a converse to Lemma 2.9. where we bounded $\ell(G)$ above with a function of $\nu(G)$ except for groups with a specified structure. We can always bound $\nu(G)$ trivially by a function of $|G|$. (indeed.

the total number of subgroups of G is bounded by $2^{|G|}$). In the case when G is nilpotent and $\ell(G) > 2\nu(G) + 1$ we can obtain the following upper bound for $\nu(G)$ in terms of $\ell(G/\langle z\rangle)$ and the prime divisors of $|G|$, where $\langle z\rangle$ is a central element of G of prime power order of the type mentioned in Lemma 2.9.

Lemma 2.10 *Let G be a finite nilpotent group with $\nu(G) = \nu > 0$ and $\ell(G) > 2\nu + 1$. Suppose that $G = A \times B$ where A is a Dedekind p'-group and B is a p-group and C is a cyclic subgroup of $Z(B)$ such that $B' \leq C$ and $\ell(G/C) \leq \nu(G) + 1$. Then*

$$\nu(G) \leq \mu(A) \sum_{C<J<B} |J/C|/(p-1),$$

where $\mu(A)$ is the number of subgroups of A.

Proof: By Proposition 1.6, $\nu(G) = \mu(A)\nu(B)$. So we can assume that $A = 1$ and $G = B$ is a finite p-group. Note that if $H \leq B$, then since $B' \leq C$, H is normal in B if and only if $[HC, B] \subseteq H \cap C$. Thus the number of non-normal subgroups of B is precisely $\sum_{J,D} n_{J,D}$ where the sum ranges over all pairs of subgroups $C < J < B$ and $D \leq C$ such that $J' \leq D$ but $[J, B] \not\subseteq D$ and $n_{J,D}$ is the number of subgroups H such that $HC = J$ and $H \cap C = D$.

Indeed, if $H \ntrianglelefteq G$ then there exists a subgroup J of G so that $C < J < B$ and $HC = J$. ($HC \neq C$ else $H < C$ and $HC \neq B$ else $J' = B' = [J, B]$). Let $D = H \cap C$. Then $H' = J' \leq H \cap C = D$ but $[H, B] = [J, B] \not\subseteq D$. Thus H is accounted for by the term $n_{HC,H\cap C}$ of the summation. Conversely, if $H \leq G$ such that $HC = J$ and $H \cap C = D$ then $H \ntrianglelefteq G$ since $[H, B] \not\subseteq D = H \cap C$.

Fix J, $C < J < B$. Since the only restriction on D is that $J' \leq D < [J, B]$, the number of possible D is $\ell([J, B]/J')$. Now fix D. Either $n_{J,D} = 0$ or $n_{J,D} = |J/C|$. We prove this fact below. Assume $n_{J,D} \neq 0$. Work modulo D. Let $H \leq B$ such that $HC = J$ and $H \cap C = D$. Then we can say $HC = H \times C$ (since we are working modulo D). First note that, modulo D, $n_{J,D} = |Hom(H, C)|$.

Indeed, if $\phi \in Hom(H, C)$ then let $H_\phi = \{h\phi(h) : h \in H\}$. $H_\phi \leq B$ since, for each $h, k \in H$,

$$h\phi(h)k\phi(k) = hk\phi(h)\phi(k) = hk\phi(hk)$$

and

$$(h\phi(h))^{-1} = \phi(h)^{-1}h^{-1} = h^{-1}\phi(h^{-1}).$$

Furthermore if $\sigma \neq \phi \in Hom(H, C)$ then $H_\sigma \neq H_\phi$ since $h\phi(h) = k\sigma(k)$ implies that $hk^{-1} = \sigma(k)\phi(h^{-1}) \in H \cap C \equiv 1 \bmod D$. Also $H_\phi C = J$ and $H_\phi \cap C \equiv 1 \bmod D$ since $h\phi(h) \in C$ implies that $h \in H \cap C = 1$. Thus $n_{J,D} \geq |Hom(H, C)|$ modulo D.

Conversely, let $K \leq B$ such that $KC = J$ and $K \cap C = D$. Let $\phi : H \to C$ such that $h \mapsto z_h$ where $h = k_h z_h$, $k_h \in K$ and $z_h \in C$. ϕ is well defined since

$$h = k_h z_h = k_l z_l \Rightarrow k_h k_l^{-1} = z_h^{-1} z_l \in K \cap C \equiv 1 \bmod \mathrm{D}$$

which means that $k_l = k_h$ and $z_h = z_l$. ϕ is a homomorphism since, if $h = k_h z_h$ and $l = k_l z_l$ are two elements of H then

$$\phi(hl) = \phi(k_h z_h k_l z_l) = \phi(k_h k_l z_h z_l) = z_h z_l = \phi(h)\phi(l).$$

Thus if $n_{J,D} \neq 0$ then $n_{J,D} = |Hom(H, C)|$ modulo D. But, modulo D, $|Hom(H, C)| \leq |H|$. Indeed, modulo D, H is Abelian (as $H' \subseteq D$). Say

$$H = \langle h_1 \rangle \times \langle h_2 \rangle \times \cdots \times \langle h_k \rangle.$$

Any $\phi \in Hom(H, C)$ is completely determined by where it sends the h_i and the fact that $|\phi(h_i)| \leq |h_i|$. Hence,

$$n_{J,D} = |Hom(H, C)| \leq |h_1||h_2| \dots |h_k| = |H| \text{ modulo } D.$$

Hence if $n_{J,D} \neq 0$ then $n_{J,D} \leq |J/C|$ (with equality if $|C| \geq exp(H)$ mod D.)

Thus the total number of non-normal subgroups of B is at most

$$\sum_{C<J<B} \sum_{D} n_{J,D} \leq \sum_{C<J<B} |J/C| \ell([J,B]/J').$$

From this we can now obtain a crude upper bound for $\nu(G)$. We shall see that if $H \not\triangleleft G$ then H has at least $|[B,H]/H \cap C|$ conjugates in B. Indeed, the number of conjugates of H in B is $[B : N_B(H)]$. Since B' is cyclic we can choose $g \in B$ and $h \in H$ such that $[B,H] = \langle [g,h] \rangle$. Now, $g \in N_B(H)$ and since $B' \subseteq Z(B)$

$$[g^n, h] = ([g,h])^n,$$

so that $g^a \in N_B(H)$ if and only if $([g,h])^a \in H \cap C$.

Thus,

$$exp(B/N_B(H)) \geq exp([H,B]/H \cap C) = |[H,B]/H \cap C|,$$

since B' is cyclic.

Finally, as $D = H \cap C$ varies from $J' \leq D < [J,B]$, $|[H,B]/H \cap C|$ decreases from say p^r to p and so

$$\begin{aligned} \nu(B) &\leq \sum_{C<J<B} |J/C| \left(\frac{1}{p^r} + \frac{1}{p^{r-1}} + \cdots + \frac{1}{p}\right) \\ &= \sum_{C<J<B} |J/C| \left(\frac{1}{p^r}\right)\left(\frac{p^r-1}{p-1}\right) \\ &< \sum_{C<J<B} |J/C|/(p-1), \end{aligned}$$

as required. □

Note that, since $\mu(A)$ is a function of the prime divisors of A, it is generally not sufficient to know $\ell(G/C)$ to bound $\nu(G)$, we need to know all the prime divisors of $|G|$ as well.

Finally we bound $\ell(G')$ above by a function of $\nu(G)$. Note by Proposition 2.2 and the Lemma of Schur(10.1.4 [Rob]), the existence of such a bound is known. The following bound is more exact.

Lemma 2.11 *If G is a finite p-group, and G is not Hamiltonian, then*

$$\ell(G') \leq \nu(G).$$

Proof: Induct on n where $|G| = p^n$. If G is Abelian then $\nu(G) = 0 = \ell(G')$. In particular this is the case if $n \leq 2$. So assume $\nu(G) > 0$ and the result holds for all groups of order p^k. Let $|G| = p^{k+1}$. Let N be a minimal normal subgroup of G. By the induction assumption either $\nu(G/N) = 0$ or $\ell((G/N)') \leq \nu(G/N)$. First suppose that $\nu(G/N) \neq 0$. Then $\ell(G') \leq \nu(G/N) - 1$. If $\nu(G/N) \neq \nu(G)$ then we are done. If $\nu(G) = \nu(G/N)$ then $p = 2$ and by Case 1 of the proof of Lemma 2.1 either

1. $G \cong Q(8) \times Q(8) \times E$, $\nu(G) \geq 9$ and $|G'| = 4$ or
2. $G \cong Q(8) \times C_4 \times E$, $\nu(G) \geq 3$ and $|G'| = 2$ or
3. $G = \langle x \rangle A$, $\nu(G) \geq n - 2$ and $G' = A^2$ has length at most $n - 2$ since $A^2 \neq A$ and $x \notin A$.

In all three instances the assertion holds.

If $\nu(G/N) = 0$ then either $G' = N$, (and since $\nu(G) \neq 0$ we are done) or $G/N \cong Q(8) \times E$, E an elementary Abelian 2-group so that $\ell(G') \leq 2$. But if $\nu(G) = 1$ we know from Theorem 1.2 that $\ell(G') = 1$ so we can assume that $\nu(G) \geq 2$ and the result holds. □

Note that if G is Hamiltonian then $\nu(G) = 0$ and $\ell(G') = 1$.

Corollary 2.12 *Let G be a finite nilpotent group. Then*

$$|\ell(G')| \leq \begin{cases} \nu(G) + 1 & \text{if } 2 \mid |G|, \\ \nu(G) & \text{if } 2 \nmid |G|. \end{cases}$$

Proof: Suppose that $G = S_{p_1} \times S_{p_2} \times \cdots \times S_{p_k}$. Then the result follows immediately from the previous lemma and remark . the fact that $G' = S'_{p_1} \times S'_{p_2} \times \cdots \times S'_{p_k}$ and Proposition 1.6. $\square$

In the next chapter we consider finite groups. We extend many of the results from this Chapter to all finite groups.

3 Results for Finite Groups

In this Chapter we extend some of the results from Chapter 2 to all finite groups. We start by generalizing Proposition 2.2.

3.1 Bounding the Prime Length of a Central Quotient

Theorem 3.1 *Let G be a finite group with $\nu(G) \neq 0$. Then for some prime p, there is a p-element z in the centre $Z(G)$ of G such that $\ell(G/\langle z\rangle) \leq \nu(G) - 1$.*

Proof: The proof is by induction on the prime length $\ell(G)$.

Let G be a counter example of least prime length. Observe that if G is a simple group of order $p^\alpha q^\beta \dots r^\gamma$, then none of the subgroups of prime power order are normal so that $\nu(G) \geq \alpha + \beta + \cdots + \gamma = \ell(G)$ and the theorem holds in this case. Assume that G is not simple. By Proposition 2.2 we can also assume that G is not nilpotent. Now split the proof into four cases.

Case 1. $\nu(G/N) = \nu(G)$ for some minimal normal subgroup N of G.

As was noted in case 1 of the proof of Lemma 2.1, every non-normal subgroup of G will contain N and so by Theorem 1.5, N is of prime order p.

Indeed, N must be cyclic because there are cyclic, non-normal subgroups. N must be of prime power order else N lies in none of the Sylow subgroups and hence all are normal and G is nilpotent. N must be of prime order because it is a minimal normal subgroup of G.

Since N has order p all p'-subgroups of G are normal. If S is a Sylow p subgroup of G then $S \ntrianglelefteq G$ else G is nilpotent. Thus $G = D \rtimes S$, where S is Sylow p, D is of p' order and every subgroup of D is normal in G. Moreover $N \leq Z(G)$ since $[N, D] \leq N \cap D = 1$ and $[N, S] = 1$ since $N > [N, S]$ and N is a minimal normal subgroup of G.

We shall now show that $Z(G)$ is a cyclic subgroup of S. If $Z(G) \not\leq S$ then it has a subgroup $M = \langle m \rangle$ of prime order $q \neq p$. Now we may assume that $\nu(G/M) \neq 0$ else G/M is nilpotent and hence G would be nilpotent. Also $\nu(G/M) \neq \nu(G)$, since S is a non-normal p–subgroup intersecting M trivially. Thus there exists some element $x \in G$ such that $\ell(G/\langle x, M\rangle) \leq \nu(G/M) + 1$, $[x, G] \leq M$ and x is of prime power order. (It is clear that there is a prime r and an integer n such that $x^{r^n} \in M$, if necessary replace x by x^q to obtain an element with the desired properties). If $x \in Z(G)$ then $\ell(G/\langle x\rangle) = \ell(G/\langle x\rangle M) + 1 \leq \nu(G/M) + 2 \leq \nu(G) - 1$ and we are done. In particular this would be the case if x is a q'–element. Indeed, if $|x| = r^n$, then $1 = [x^{r^n}, g] = [x, g]^{r^n} = \overline{m}^{r^n}$ for each $g \in G$ and some $\overline{m} \in M$ and so $[x, g] = 1$, for each $g \in G$ as $|\overline{m}| = q$ and q is coprime to r.

Thus, $\langle x\rangle M$ is a q–subgroup and $x^q \in Z(G)$. If $M \leq \langle x\rangle$, then again $\ell(G/\langle x^q\rangle) = \ell(G/\langle x\rangle M) + 1 \leq \nu(G/M) + 2 \leq \nu(G) - 1$. But if $\langle x\rangle \geq M$ then $\langle x\rangle \trianglelefteq G$ and hence contains N resulting in a contradiction since $(|x|, |N|) = 1$. Thus $Z(G) \leq S$.

Suppose $Z(G)$ is not cyclic. Then there is another subgroup N_1 of order p in $Z(G)$. Note that $0 < \nu(G/N_1) < \nu(G)$ since $\nu(G/N_1) \neq \nu(G) = \nu(G/N)$ as G has a cyclic, non-normal subgroup and $\nu(G/N_1) \neq 0$ as G is not nilpotent. Hence there exists a prime power element $y \in G$ such that $[y, G] \leq N_1$ and $\ell(G/\langle y, N_1\rangle) \leq \nu(G/N_1) + 1$. If $y \in Z(G)$ or $\langle y\rangle \geq N_1$, then we are done, using the same argument as above with x replaced by y. Similarly $\nu(G) < \nu(G/N_1) + 2$ else $\ell(G/\langle y^p\rangle) \leq \nu(G) + 1$, and $y^p \in Z(G)$. The only remaining possibility is $\nu(G/N_1) + 1 = \nu(G)$. $\langle y\rangle \ntrianglelefteq G$ and so $\langle y\rangle \geq N$. Now let $N = \langle n\rangle$, $N_1 = \langle m\rangle$ and $L = \langle nm\rangle$. Repeat the above argument with N_1 replaced by L to obtain a $\langle t\rangle \ntrianglelefteq G$ such that $[t, G] = L$ and $N \leq \langle t\rangle$ and yet $\langle t\rangle$ is not conjugate to $\langle y\rangle$, since $N_1 \not\leq \langle t\rangle$. This contradicts the fact that $\nu(G/N_1) + 1 = \nu(G)$.

Thus we can assume that $Z(G)$ is cyclic.

Note that N is the unique subgroup of order p: for if U is another such subgroup then $U \triangleleft G$ since every non-normal subgroup contains N. Then $[U, D] = 1 = [U, S]$, and $Z(G)$ is not cyclic. Thus by Theorem 4.4 of [Su1] S is either cyclic or generalized quaternion $Q(2^n)$ for some integer $n \geq 3$. By hypothesis, there exists a p-element $\bar{x} \in G$ such that $[\bar{x}, G] \leq N$ and $\ell(G/\langle \bar{x}, N\rangle) \leq \nu(G/N)+1$. If S is cyclic, then $G' \leq D$, and $[\bar{x}, G] \leq D \cap N = 1$. Hence $\bar{x} \in Z(G)$ and $\ell(G/\langle \bar{x}\rangle) = \ell(G/\langle \bar{x}, N\rangle) \leq \nu(G/N) - 1 = \nu(G) - 1$ and we are done.

Thus assume S is generalized quaternion and $p = 2$. Then D is a Dedekind group of odd order and therefore is Abelian. Let V be a subgroup of prime order in D. Assume $\nu(G/V) = 0$. Then $G/V \cong Q_8 \times A$, $S \cong Q_8$ and $G' = VN$. If D is cyclic of prime power order, say $|D| = q^n$ then suppose $d \in D$ has order q^2. Since $G' = VN$, $\langle d^q \rangle = V \leq Z(G)$, contradicting the fact that $Z(G) \leq S$. If $|D| = q$ then $\ell(G/\langle \bar{x}\rangle) = 2 \leq \nu(G/N) - 1$. If G is a counter example then $\nu(G) = 1$. But if $\nu(G) = 1$, then $G \ncong C_q \rtimes Q_8$, see Theorem 1.2.

So there exists a second subgroup V_2 of prime order in D. Since $VN = V_2N$, $\nu(G/V_2) \neq 0$, and we may assume the existence of some subgroup V of prime order q in D so that $\nu(G/V) \neq 0$.

Now $\nu(G) > \nu(G/V) > 0$ since S is a non-normal 2−subgroup of G which does not contain V. By hypothesis, there exists some element $t \in G$, of prime power order, such that $[t, G] \leq V$ and $\ell(G/\langle t, V\rangle) \leq \nu(G/V) - 1$. Note that $\langle t\rangle \triangleleft G$. Indeed, $\langle t\rangle \ntriangleleft G$ implies $N \leq \langle t\rangle$ and hence t is a 2−element and so may be taken to lie in S. Then $[t, S] \leq S \cap V = 1$. So $t \in Z(S) = N = Z(G)$ and $\langle t\rangle \triangleleft G$ after all. Thus $[t, G] \leq V \cap \langle t\rangle$ so that either $V \leq \langle t\rangle$ or $\langle t\rangle \leq Z(G)$. If $t \in Z(G)$ then

$$\ell(G/\langle t\rangle) = \ell(G/\langle t, V\rangle) + 1 \leq \nu(G/V) + 2 \leq \nu(G) + 1$$

and the result holds. If $\langle t\rangle \geq V$ then $t \in D$ and $t^q \in Z(G)$ which implies that $t^q = 1$ and $\langle t\rangle = V$. In this case

$$\ell(G/N) = \ell(G/V) = \ell(G/\langle t, V\rangle) \leq \nu(G/V) - 1 < \nu(G) + 1.$$

Thus no minimal counter example exists with the property that $\nu(G) = \nu(G/N)$ for some minimal normal subgroup N.

Case 2. $Z(G) \neq 1$. In this case there exists a normal subgroup N of prime order p in $Z(G)$. If $\nu(G/N) = 0$, then we have the contradiction that G is nilpotent by the same reasoning as in Case 1. So assume, by Case 1, that $0 < \nu(G/N) < \nu(G)$. By hypothesis, there exists $z \in G$ of prime power order such that $[z, G] \leq N$ and $\ell(G/\langle z, N\rangle) \leq \nu(G/N) + 1$. Note that $z^p \in Z(G)$, as $[z^p, g] = [z, g]^p = 1$ for each $g \in G$. If $z^p = 1$ then $\ell(G/N) \leq \nu(G/N) - 2 \leq \nu(G) - 1$ so the result holds. If $z \in Z(G)$ then $\ell(G/\langle z\rangle) \leq \ell(G/\langle z, N\rangle) + 1 \leq \nu(G/N) - 2 \leq \nu(G) - 1$. So to avoid having $z \in Z(G)$ we may assume that z has order a power of p. If $N \leq \langle z\rangle$ then

$$\ell(G/\langle z^p\rangle) = \ell(G/\langle z, N\rangle) + 1 \leq \nu(G/N) + 2 \leq \nu(G) + 1$$

and G is not a counter example. If $\nu(G) > \nu(G/N) + 1$ then G can not be a counter example as

$$\ell(G)/\langle z^p\rangle) = \ell(G/\langle z\rangle N) - 2 \leq \nu(G/N) + 3 \leq \nu(G) + 1.$$

In view of above, we may assume that z is of order p^{n-1}, $n > 0$, $N \not\leq \langle z\rangle$, $\langle z\rangle \not\trianglelefteq G$ and $\nu(G) = \nu(G/N) + 1$. Now $\langle z^{p^n}\rangle = M \leq Z(G)$. Repeat the argument using M in place of N to conclude that there exists $y \in G$ such that $\ell(G/\langle y, M\rangle) \leq \nu(G/M) + 1 = \nu(G)$, $[y, G] = M$, $1 \neq y^p \in Z(G)$ and $M \not\leq \langle y\rangle$.

Observe $\langle y\rangle \ntrianglelefteq G$ since $[y, G] \leq M$. Since $\langle y\rangle$ and $\langle z\rangle$ are non-normal non-conjugate subgroups, $N \leq \langle y\rangle$. Repeat the above argument using $1 \neq L <$

$MN, L \neq M, L \neq N, |L| = p$. Then there exists $\langle x \rangle \not\triangleleft G$, x of p-power order. $\nu(G) = \nu(G/L)+1$ and $L \not\leq \langle x \rangle$. Since all non-normal subgroups of G contain L except $\langle x \rangle$ and its conjugates, we have the contradiction that both $\langle y \rangle$ and $\langle z \rangle$ are conjugate to $\langle x \rangle$ which is impossible since they contain different minimal normal subgroups. Thus we can assume the center of the minimal counter example is trivial.

Case 3. $Z(G) = 1$ and there exists a minimal normal subgroup N of G such that $\nu(G/N) = 0$. Note that $[N, G] = N$.

(3a) Suppose that N is a finite p-group so that G is solvable. Let S_p be a Sylow p-subgroup of G. Then $S_p \geq N$ so that $S_p \triangleleft G$. By the minimality of N, $[S_p, N] = 1$ and so $S_p \leq C_G(N)$. Since G is solvable, $G = S_p \rtimes Q$ where Q is a p'-group.

Let $h, g \in S_p$ such that $1 \neq [h, g]$. Then for any $x \in Q$, $[h, x] \in S_p$ and $\langle x \rangle N \triangleleft G$ so that $h^x \equiv h \mod N$. Similarly, $g^x \equiv g \mod N$. Also, $[h, g]^x = [h^x, g^x] = [h, g]$ since $S_p \leq C_G(N)$. Thus $S_p' \cap Z(S_p) \leq Z(G) = 1$ so that S_p is Abelian. It follows that $[S_p, G] \leq N$, as $[h, Q] \subseteq N$ and $[h, S_p] = 1$ for each $h \in S_p$.

Observe that N is the unique minimal normal subgroup of G. Indeed, if M is another then $M \cap N = 1$, $[M, G] = M$ and $G' \neq N$ so that $G/N \cong Q_8 \times E \times A$, for some elementary Abelian 2-group E and an Abelian 2'-group A. Then $|G'| = 2|N|$ so $|M| = 2$ and $M \leq Z(G) = 1$.

We shall next show that S_p is elementary Abelian. Suppose false: then since ${S_p}^p$ is characteristic in the normal group S_p, ${S_p}^p \triangleleft G$ and also $N \leq {S_p}^p$ since, by assumption, $1 \neq {S_p}^p$. Since $[S_p, G] \leq {S_p}^p$, it follows that S_p lies in the hypercentre of G. Indeed, if $h \in S_p$ and $x \in G$ then $[h, x] = \sigma^p$ for some $\sigma \in S_p$. Thus $[[h, x], y] = [\sigma, y]^p \in {S_p}^{p^2}$ for each $y \in G$. Consequently $h \in Z_{|S_p|}(G)$ and so S_p is contained in the hypercentre of G as claimed. But this contradicts the fact that $[N, G] = N$.

Thus $S_p = N \times L$ for some subgroup L which we may assume is normal in G by Maschke's theorem (see [Rob], Theorem 8.1.2). Thus, $S_p = N$ as N is the unique minimal normal subgroup of G and $\ell(G) \leq \nu(G) + 1$ as every non-trivial subgroup of Q is non-normal in G and every proper non-trivial subgroup of N is non-normal in G as well.

(3b) We are now left with the case where $N = \prod_1^k N_i$, where $N_1 \cong N_i$, for every i and N_1 is a non-Abelian simple group. Note that in this case every non-trivial p-subgroup for every prime p is non-normal in G. For if M is a $p-$subgroup of G and $M \triangleleft G$ then $[M, G, G] \leq N \cap M = 1$ as $G/N \cong Q_8 \times E \times A$ and so $M \leq Z(G) = 1$, a contradiction. Thus $\ell(G) \leq \nu(G)$ and the result holds.

Case 4. Final Case: $Z(G) = 1$ and $0 < \nu(G/N) < \nu(G)$ for every minimal normal subgroup N of G.

(4a) Suppose that every minimal normal subgroup N is non-Abelian. Then for each prime p, every p-subgroup of G is non-normal, $\ell(G) \leq \nu(G)$ and we are done.

(4b) Assume $|N| = p^r$ for some integer $r > 1$, for each minimal normal subgroup N of G. The subgroups of N of order $p, p^2, \ldots, p^{r-1}$, are all non-normal and represent at least $r - 1$ conjugacy classes. Since $0 < \nu(G/N) < \nu(G)$, there exists some element z of prime power order such that $[z, G] = N$ and $\ell(G/\langle z, N\rangle) \leq \nu(G/N) + 1$. Note $N \nleq \langle x\rangle$ as N is isomorphic to the direct product of r copies of C_p. Every non-trivial subgroup of $\langle z\rangle$ is non-normal. Indeed, $[z^n, G] = \langle [z^n, g] : g \in G\rangle \triangleleft G$ and this subgroup lies in N, so either $[z^n, G] = N$ or $[z^n, G] = 1$. Since $Z(G) = 1$ then for each $z^n \neq 1$, $[z^n, G] = N$. This produces $\ell(\langle z, N\rangle/N)$ additional conjugacy classes of subgroups, which do not contain N. Finally, since $\nu(G/N) \neq 0$, there exists an element $1 \neq x \in G \setminus \langle z, N \rangle$ generating a non-normal subgroup $\langle x \rangle$ that is not conjugate to any of the above (or there is a normal subgroup of order p). Thus $\nu(G) \geq \nu(G/N) - \ell(\langle z, N \rangle/N) - (\ell(N) - 1) + 1$ and G is not a counter example since

$$\ell(G) = \ell(G/\langle z, N \rangle) - \ell(\langle z, N \rangle) \leq \nu(G/N) + 1 - \ell(\langle z, N \rangle/N) - \ell(N) \leq \nu(G) - 1.$$

(4c) Thus we may assume that N has prime order p. Let $C = C_G(N)$ so that G/C is isomorphic to a cyclic group of order dividing $p-1$. By hypothesis there exists a prime power element $z \in G$ such that $[z, G] \leq N$ and $\ell(G/\langle z, N \rangle) \leq \nu(G/N) + 1$. First suppose z is a p-element. Then $z \in C$ since C contains all the Sylow p-subgroups of G. Hence $z^p \in Z(G) = 1$ and so $z^p = 1$. If $\langle z \rangle = N$ then $\ell(G) \leq \nu(G) - 1$ as claimed. otherwise $\langle z \rangle < G$. If $x \in G \setminus C$, then $\langle x \rangle < G$ as $[x, N] = N \leq \langle x \rangle$ and so $\nu(G) \geq \nu(G/N) - 2$ and

$$\ell(G) = \ell(G/\langle z, N \rangle) - 2 \leq \nu(G/N) - 3 \leq \nu(G) - 1.$$

So we may assume that z is a q-element for some prime $q \neq p$. Again $z \notin C$, otherwise it would be a central element as $[z^p, G] = [z, G]^p = 1 = [z, G]$ and $Z(G) = 1$. Conclude for the same reason that $\langle z \rangle \cap C = 1$. It now follows that $G = \langle z \rangle C$.

Indeed. note that all non-trivial subgroups of $\langle z \rangle$ are non-normal in G because $\langle z \rangle \cap C = 1$ and $1 \neq [z^{q^m}, N] \leq N$, if $z^{q^m} \neq 1$. Thus.

$$\nu(G) \geq \nu(G/N) + \ell(\langle z \rangle).$$

Now $\ell(G) = \ell(G/\langle z \rangle N) + \ell(\langle z \rangle) + 1 \leq \nu(G/N) + \ell(\langle z \rangle) + 2$. For G to be a counter example.

$$\nu(G) = \nu(G/N) + \ell(\langle z \rangle).$$

Let $x \in G \setminus \langle z \rangle C$. We can assume $|x|$ is of order coprime to p. Now, $1 \neq [x, N] \leq N$ so $\langle x \rangle \ntrianglelefteq G$. But this contradicts the fact that $\nu(G) = \nu(G/N) + \ell(\langle z \rangle)$. Hence $G = \langle z \rangle C$, as required.

We next show that $C = N$. Suppose that $c \in C \setminus N$. Consider $\langle zc \rangle$. For all $n \in N$,

$$[zc, n] = [z, n]^c [c, n] = [z, n].$$

Thus $1 \neq [zc, N] = [z, N] \leq N$. Since $N \leq \langle zc \rangle$, $\langle zc \rangle \not\lhd G$. The element $zc \in \langle z \rangle N \lhd G$ and so $\langle zc \rangle$ is not conjugate to $\langle z^a \rangle$ for any power of a. This contradicts the fact that $\nu(G) = \nu(G/N) + \ell(\langle z \rangle)$. Thus $G = N \rtimes \langle z \rangle$ and $\ell(G) = \ell(\langle z \rangle) + 1 \leq \nu(G) + 1$, as required. $\square$

3.2 Bounding the Prime Length of G and G'

We now shall generalize Lemma 2.9. The following lemma establishes some properties of finite, non-nilpotent groups whose prime length is much larger than the number of conjugacy classes of non-normal subgroups they contain.

Lemma 3.2 *Let G be a finite group with $\nu(G) = \nu > 0$. If $\ell(G) > 3\nu - 1$ then for some prime p, $G = A \rtimes B$ where A is a Dedekind p'-subgroup, B is a p-subgroup and B' is a cyclic subgroup in $Z(G)$.*

Proof: Let a central element z be chosen as in the statement of Theorem 3.1. Put $C = \langle z \rangle$. By Lemma 2.9 we may assume that G is not nilpotent so that G/C is not Dedekind. Then there is some element $x \in G \setminus C$ of prime power order such that $\langle x, Y \rangle \not\lhd G$ for every subgroup $Y \leq C$. If $|\langle x \rangle Y_1| \neq |\langle x \rangle Y_2|$ when $|Y_1| \neq |Y_2|$ then there are at least $\ell(C) + 1$ conjugacy classes of these subgroups and by the choice of z,

$$\ell(G) = \ell(G/C) + \ell(C) \leq \nu + 1 + \ell(C) \leq 2\nu.$$

In particular, if the order of x is coprime to p, then this would be the case. So we may assume that $G = A \rtimes B$ where A is a Dedekind p'-subgroup (in fact, all subgroups of A are normal in G), and B is a p-group containing C, $B \not\lhd G$.

Note that if $\nu(B) = 0$ then either B is Abelian and the result holds or $B \cong Q_8 \times E$, where E is an elementary Abelian 2−group. Otherwise this lemma still holds as

$$\ell(G) = \ell(G/C) + \ell(C) \leq \nu + 1 + 1 < 3\nu + 1.$$

Observe that if H, K are subgroups of B then H is conjugate to K in G only if H is conjugate to K in B. Indeed, if $a \in A$ and $b \in B$ so that $H^{ab} = K$ then $h^a \in B$ for each $h \in H$ and since $\langle a \rangle \lhd G$ and $B \cap A = 1$, $[h, a] = 1$ and $H^b = K$.

Thus $0 < \nu(B) < \nu(G)$ $(B \not\lhd G)$ and by Lemma 2.9, either $\ell(B) \leq 2\nu(B) + 1$ (and since $B \neq C$ then $\ell(G) \leq 3\nu + 1$ by Theorem 3.1) or there exists an element $z \in Z(B)$ such that $B' \leq \langle z \rangle$, and $\ell(B/\langle z \rangle) \leq \nu(B) + 1$. Let U be the centralizer of A in B. Since for all $a \in A$, $\langle a \rangle \lhd G$, $[B', A] = 1$ and so $B' \leq U$. Hence $B' \leq Z(G)$. □

Now we are ready to prove the general result concerning the structure of finite groups G with $\nu(G)$ much smaller than $\ell(G)$.

Theorem 3.3 *Let G be a finite group with $\nu(G) = \nu > 0$. If G is nilpotent and $\ell(G) > 2\nu + 1$ then for some prime p, $G = A \times B$ where A is a Dedekind p'−subgroup, B is a p−subgroup and there is an element $z \in Z(B)$ such that B' is a subgroup of $\langle z \rangle$, and $\ell(G/\langle z \rangle) \leq \nu(G) + 1$. If G is not nilpotent but $\ell(G) > 3\nu + 1$ then for some prime p, $G = A \rtimes B$ where A is a Dedekind p'−subgroup and B is a p−subgroup. There exists an element $z \in B \cap Z(G)$ such that $\ell(G/\langle z \rangle) \leq \nu(G) + 1$ and $B' \leq \langle z \rangle$, $B = U\langle t \rangle$, where $U = C_B(A)$ and $B' \leq \langle t \rangle$. If H is a subgroup of G and $H \not\lhd AH$, then $B' \leq H$.*

Proof: By Lemma 2.9 we may assume that G is not nilpotent and $\ell(G) > 3\nu(G) + 1$. We continue the analysis started in Lemma 3.2, using the same notation. Note that $B' \leq C$. Indeed, $\ell(G) > 3\nu - 1$, and so $\ell(C) > 2\nu$. By the previous Lemma $B' \leq \langle \bar{z} \rangle$ and $\ell(B/\langle \bar{z} \rangle) \leq \nu(B) - 1 \leq \nu$. Since $\ell(\langle \bar{z} \rangle C/\langle \bar{z} \rangle) = \ell(C/C \cap \langle \bar{z} \rangle) \leq \nu(B) - 1 \leq \nu$, $\ell(\langle \bar{z} \rangle \cap C) > \nu \geq \ell(B')$. Hence $B' \leq C$ as required.

Now assume that $\nu(B) \neq 0$. Observe that B/U is cyclic: if not then for some $x, y \in B \setminus U$, $\langle x, y, B' \rangle / B' = \langle xB' \rangle / B' \times \langle yB' \rangle / B'$. But $B' \leq \langle \bar{z} \rangle$ so that $\langle x \rangle \cap \langle \bar{z} \rangle \leq B'$ or $\langle y \rangle \cap \langle \bar{z} \rangle \leq B'$. Say the former holds.

Now for each $D \leq B$, $\langle x \rangle D \not\trianglelefteq G$ as $x \notin U$. Thus $\nu(G) \geq \ell(B/\langle x, \bar{z} \rangle) - \ell(\langle \bar{z} \rangle / \langle x \rangle \cap \langle \bar{z} \rangle)$. Now, $\ell(B') \leq \nu(B) + 1 \leq \nu(G)$. Hence $\ell(\langle x \rangle \cap \langle \bar{z} \rangle) \leq \nu(G)$. Note that $\ell(\langle \bar{z} \rangle) \geq \ell(C)$ since $\langle \bar{z} \rangle$ is a maximal cyclic subgroup of $Z(B)$ and $C \leq Z(B)$ is cyclic. Thus, $\ell(G) \leq \ell(G/C) + \ell(\langle \bar{z} \rangle / \langle \bar{z} \rangle \cap \langle x \rangle) + \ell(\langle \bar{z} \rangle \cap \langle x \rangle) \leq 3\nu(G) - 1$. Hence we can assume that $B = U\langle t \rangle$ for some $t \in B$, where $U = C_B(A)$.

Also observe that if $H \leq G$ and A does not normalize H, then $H \geq B'$. Indeed, by replacing H by a conjugate if necessary, we can assume that there exists an element $h \in (H \cap B) \setminus U$ so that $\langle h \rangle \not\trianglelefteq G$ and by a similar argument to the one used in the previous paragraph, $B' \subseteq \langle h \rangle$. In particular, if $H \leq B$ and $H \not\trianglelefteq HA$, then $H \trianglelefteq B$.

If $\nu(B) = 0$ yet $\ell(G) > 3\nu + 1$ then B is Abelian. If t is an element of maximal order in B then by 4.2.7 of [Rob] $B = \langle t \rangle \times S$ for some $S \leq B$. Now $C \cap \langle t \rangle \neq 1$ since $\ell(\langle t \rangle / \langle t \rangle \cap C) \leq \nu - 1$ and $\ell(\langle t \rangle) \geq \ell(C) > 2\nu$. Thus $S \cap C = 1$. Now if $s \in S \setminus C_B(A)$ then $\langle s \rangle \not\trianglelefteq G$ and $\langle s \rangle D \not\trianglelefteq G$ for each $1 \leq D \leq C$ which means that $\nu(G) > \ell(C)$, a contradiction. Thus even if $\nu(B) = 0$ we can still say that $B = \langle t \rangle U$ where $U = C_B(A)$.

□

The following Lemma will allow us to answer the following question: Suppose we have a finite, non-nilpotent group G and we know that the prime length of G is much larger than $\nu(G)$. Can we find an upper bound for $\nu(G)$?

Lemma 3.4 *Let G be a finite, non-nilpotent group with $\ell(G) > 3\nu(G) + 1$. Let $N = \cap\{H \triangleleft B : H \not\triangleleft G\}$ where $G = A \rtimes B$, A, B as described in Theorem 3.3. Then if $\mu(A)$ is the number of normal subgroups of A then*

$$\nu(G) = \nu(B)\mu(A) + \nu(G/N^p)$$

and $\ell(G/N^p) \leq 3\nu(G)$. In fact, if $\nu(B) \neq 0$, then $\ell(G/N^p) \leq 2\nu(G)$.

Proof: Let $G = A \rtimes B$, A, $C = \langle z \rangle$ and $B = C\langle t \rangle$ just as Theorem 3.3 describes. Let N be as defined above. First note that N is cyclic as $N \leq \langle t \rangle$. Furthermore, $N = N_1$ where

$$N_1 = \cap\{\langle b \rangle \not\triangleleft G : \langle b \rangle \triangleleft B\}.$$

Indeed, $N \subseteq N_1$ obviously. Suppose $N \neq N_1$. Then there exists an element $x \in N_1 \setminus N$ and for every $\langle b \rangle \triangleleft B$ such that $\langle b \rangle \not\triangleleft G$, $x \in \langle b \rangle$. On the other hand there exists $H \triangleleft B$, $H \not\triangleleft G$ and $x \notin H$. Hence for each $b \in H$ either $\langle b \rangle \triangleleft G$ or $\langle b \rangle \not\triangleleft B$. If $b \in C$ then $\langle b \rangle$ is not normalized by A. But by Theorem 3.3 this means that $B' \leq \langle b \rangle$ and so $\langle b \rangle \triangleleft B$, a contradiction. Hence $H \leq C$. But if $H \triangleleft B$ and $H \leq C$ then H is a normal subgroup of G, a contradiction. Thus $N = N_1$, as required.

Next note that N^p is a normal subgroup of G. Indeed, $N \triangleleft B$ as the intersection of normal subgroups of B and N^p is a characteristic subgroup of N. Either $N^p = N = 1$ or $N^p < N$ and by the definition of N, $N^p \triangleleft G$.

Now, consider $\ell(G/N^p)$. It is enough to show that $\ell(G/C \cap N^p) \leq 3\nu(G)$. Since C is cyclic, there exists a $\langle b \rangle \ntriangleleft G$, $\langle b \rangle \triangleleft B$ such that $\langle b^p \rangle \cap C = N^p \cap C$. Since every subgroup of B containing $\langle b \rangle$ is non-normal in G, we conclude that $\nu(G) \geq \ell(B/\langle b \rangle) + 1$. Furthermore, by Lemma 1.3 $\ell(A) \leq \nu(G)$. In fact, if $\nu(B) \neq 0$, then we have at least $\mu(A) > \ell(A)$ more non-normal subgroups so that in this case $\nu(G) > \ell(A) + \ell(B/\langle b \rangle)$. Also by Lemma 2.1 $\ell(\langle b \rangle / \langle b \rangle \cap C) =$

$\ell(\langle b\rangle C/C) \leq \nu(G)$ so that

$$\ell(G/\langle b^p\rangle \cap C) \leq \ell(A) + \ell(B/\langle b\rangle) + \ell(\langle b\rangle/\langle b\rangle \cap C) + 1$$

which is at most $3\nu(G)$ if $\nu(B) = 0$ and $2\nu(G)$ if $\nu(B) \neq 0$, as required.

Finally, let $H \ntrianglelefteq G$. Clearly $p \mid |H|$ as all p'-subgroups of G are normal. By replacing H with one of its conjugates if necessary, we may assume that H has a non-trivial Sylow$-p$ subgroup H_p in B. Since H is not normal in G neither is H_p. If $H_p \triangleleft B$ then $N^p \leq H_p \leq H$. Otherwise $\nu(B) \neq 0$, $\ell(G/N^p) \leq 2\nu(G)$ and the conjugacy class of H is counted in the term $\nu(B)\mu(A)$. Indeed, H is conjugate to H_pS for some $S \leq A$ and for all $1 \leq S \leq A$, $H_pS \ntriangleleft G$. Thus

$$\nu(G) \leq \nu(B)\mu(A) + \nu(G/N^p).$$

In fact, we must have equality. Recall from the proof of Lemma 3.2 that if $H, K \leq B$ then they are conjugate in G only if they are conjugate in B. So no two of the distinct conjugacy classes of non-normal subgroups in B combine into one in G. The only other possible way to have inequality is for there to exist a subgroup $H \triangleleft B$ such that N^p lies inside H and $\ell(G/N^p) \leq 2\nu(G)$. Since $B' \subseteq N$ by Theorem 3.3, this would mean that $B' = N$ and $\ell(N) \leq \nu(G) - 1$. Under such circumstances, $\ell(G) \leq 3\nu(G) + 1$, a contradiction. □

We can now give an estimate for an upper bound on $\nu(G)$ provided that we know:

1. The prime length of G is 'much' larger than $\nu(G)$.
2. The quotient groups G/C and G/N^p where N is the intersection of all the non-normal subgroups of B. ($G = A \rtimes B$ as above.) Note that, modulo N^p, B is an Abelian group.

Since we established in the above Lemma that $\ell(G/N^p)$ is bounded by a function of $\nu(G)$, the requirement of knowledge of G/N^p is no more unreasonable than the requirement that we know the group G/C as used in Lemma 2.10. In fact, we shall need Lemma 2.10 to prove the following.

Corollary 3.5 *Let G be a finite, non-nilpotent group with $\ell(G) > 3\nu(G) + 1$. Then*

$$\nu(G) \leq \mu(A) \sum_{C<J<B} |J/C|(p-1) + \nu(G/N^p),$$

where $\mu(A)$, C are as described in Lemma 2.10 and A, B and N are as described in Lemma 3.4.

Proof: By Lemma 3.4 we know that $\nu(G) = \mu(A)\nu(B) + \nu(G/N^p)$. Also $\ell(B) > 2\nu(B) + 1$ due to the fact that $\nu(B) \leq \nu(G) - 1$ and $\ell(G) > 3\nu(G) - 1$. Applying Lemma 2.10 to estimate $\nu(B)$ gives us the stated bound. $\square$

Finally we show that in all instances that G is finite we can bound the prime length of the commutator subgroup above by $\nu(G)$ or $\nu(G) - 1$.

Theorem 3.6 *Let G be a finite group. Then*

$$\ell(G') \leq \begin{cases} \nu(G) + 1 & \text{if } 2 \mid |G| \\ \nu(G) & \text{if } 2 \nmid |G|. \end{cases}$$

Proof: If the result is false, then there exists a finite counter example G of minimal order. By Corollary 2.12, G is not nilpotent. Note that G cannot be simple. Indeed, if G is a simple, non-Abelian group of order $p_1^{n_1}p_2^{n_2}\ldots p_k^{n_k}$ then $k > 1$ and G has proper subgroups of orders $p_i, p_i^2, \ldots, p_i^{n_i}$, $i \in \{1, \ldots, k\}$ so that $\nu(G) \geq \sum_{i=1}^{k} n_i = \ell(G')$.

Let N be a minimal normal subgroup of G of prime power order. Note that if $\nu(G/N) = 0$ then $\ell(G') \leq \ell(N)$ ($\ell(N) + 1$ if $2 \mid |G|$.) In this case then for G to be a counter example, $\nu(G) = \ell(N) - 1$ ($\nu(G) \geq \ell(N) - 1$ by the minimality of N) and all non-normal subgroups of G lie in N. But in this case G is nilpotent, a contradiction.

Otherwise, suppose that $\nu(G/N) \neq 0$. Then by the minimality of G as a counter example,

$$\ell(G') \leq \begin{cases} (\nu(G/N) + 1) + \ell(N) & \text{if } 2 \mid |G| \\ \nu(G/N) + \ell(N) & \text{if } 2 \nmid |G| . \end{cases}$$

Since N has prime power order, no proper subgroup of N is normal in G so if $|N| = p_i^a$ then $\nu(G) \geq \nu(G/N) + a - 1$. If $\nu(G) \geq \nu(G/N) + a$ then G is not a counter example. Assume $\nu(G) = \nu(G/N) + a - 1$.

Since G is not a p-group, all $p_j \neq p_i$ power subgroups are normal. Let M be a minimal normal subgroup of order p_j. Repeating the above argument with M replacing N, $\nu(G/M) = \nu(G)$ else G is not a counter example. So every p_i-subgroup is normal in G yielding the contradiction that G is both a counter example and nilpotent.

Finally, if N is not a prime power, say $|N| = p_1^{b_1} p_2^{b_2} \ldots p_k^{b_k}$, $k > 1$ then N has proper subgroups of at least $b_1 + b_2 + \cdots + b_k$ distinct orders and so $\nu(G) \geq \nu(G/N) + \sum_{i=1}^{k} b_i$, guaranteeing that G is not a counter example. $\square$

4 Infinite Nilpotent Groups G with $\nu(G)$ finite

First recall from Section 1.3 that we can use the descriptions "an infinite group G with only a finite number of non-normal subgroups" and "an infinite nilpotent group G with $\nu(G)$ finite" interchangeably. The structure of infinite groups with only a finite number of non-normal subgroups was first described by Hekster and Lenstra in *Groups with finitely many non-normal subgroups.* Their two main results are given below.

The main purpose of this Chapter is to extend or parallel results for finite nilpotent groups in Chapter 2 to the infinite nilpotent groups with only a finite number of conjugacy classes of non-normal subgroups. First however, we state the main results of Hekster and Lenstra's paper. The first is Theorem 2 of the paper.

Theorem 4.1 (Hekster and Lenstra) [HL] *The group G has only a finite number of non-normal subgroups if and only if G is finite, or G is Dedekind or there exists groups A, B and a prime p such that*

(a) $G \cong A \times B$.

(b) *A is a finite Dedekind group with order coprime to p;*

(c) *B has a normal, central subgroup C such that $C \cong C_{p^\infty}$ and B/C is a finite, Abelian p-group.*

Hekster and Lenstra also provided a formula to count the number of non-normal subgroups of an infinite group with only a finite number of non-normal subgroups. It was obviously motivation for Lemma 2.10 of Chapter 2.

Lemma 4.2 (Hekster and Lenstra) [HL] *Let the notation be as defined* *above. Then the number of non-normal subgroups of the group G equals*

$$\mu(A) \sum_{C<J<B} |J/C|(\ell([B,J]) - \ell([J,J]))$$

where $\mu(A)$ is the number of subgroups of A.

We shall continue to use the notation from Theorem 4.1 throughout this Chapter. This first Lemma points out the link between finite nilpotent groups and infinite nilpotent groups with only a finite number of non-normal subgroups.

Lemma 4.3 *Let G be an infinite group with only a finite, positive number of non-normal subgroups. Then there exists an infinite chain of finite subgroups of G.*

$$G_0 < G_1 < G_2 < \cdots < G_n < \cdots < G$$

such that for each $i = 0, 1, 2, \ldots$

i) G and G_i have the same non-normal subgroups.

ii) $\nu(G) = \nu(G_i)$

iii) $G' = G_i'$

iv) $G/C \cong G_i/C_i$, where C_i is a central, cyclic subgroup of G_i.

Proof: Let G be an infinite group with only a finite number of non-normal subgroups. Let A, B and C be as described in Theorem 4.1 and let $G/C = \cup_{i=1}^{n} x_i C$. Let

$$X_0 = \cup_{i=1}^{n} \{x_i\} \cup \{S \ntriangleleft G\} \cup \{c_0\}$$

where $c_0 \in C$ such that $|c_0| > |x_i|$, $i = 1, \ldots, n$. Then the set X_0 is finite. This follows from the fact that all the non-normal subgroups of G are finite, a fact proved by Hekster and Lenstra ([HL] Lemma 9). Since G is a torsion FC-group, it is locally finite, by 14.5.8 of [Rob]. Thus, $G_0 = \langle X_0 \rangle$ is a finite group. Note that if $g \in G$ and $S \not\trianglelefteq G$, then

$$S^g = S^{cx_i} = S^{x_i}$$

for some $c \in C$, $i = 1, 2, \ldots, n$. Thus G and G_0 do indeed have the same non-normal subgroups and $\nu(G) = \nu(G_0)$.

Note that $G = G_0C$ and so clearly $G' = G_0'$ and

$$G/C = G_0C/C \cong G_0/G_0 \cap C = G_0/C_0.$$

Thus G_0 has all the required properties. We can even use the $\{x_i\}, i = 1, 2, \ldots, n$ as the coset representatives of C_0 in G_0.

Now, assume that the finite subgroup G_i of G has been created with the stated properties, $i \geq 0$. Then let

$$G_{i+1} = \langle G_i, c_{i+1} \rangle$$

where $c_{i+1} \in C$ such that $|c_{i+1}| > |C_i|$. Then the above argument shows that G_{i+1} has the stated properties and contains G_i, as required. □

Next we point out that any infinite group G with only a finite number of non-normal subgroups is the direct limit of a sequence of finite groups. (Given a sequence of groups $G_1, G_2, \ldots$, and inclusions $\sigma_i : G_i \to G_{i+1}$, the direct limit group D is the union of the chain of subgroups

$$G_1 \leq G_2 \leq \ldots,$$

and we can think of $G_1, G_2, \ldots$ as being subgroups of D. This is a special case of a direct limit, see Section 1.4 of [Rob] for a precise definition).

Lemma 4.4 *Let*

$$G_0 < G_1 < G_2 < \dots$$

be an infinite sequence of finite groups such that for each $i = 0, 1, 2, \dots$

i) G_i *has the same non-normal subgroups as* G_{i-1}.

ii) $\nu(G_i) = \nu(G_{i+1})$.

Then G, *the direct limit of the* $\{G_i\}$, *is an infinite group with a finite number of non-normal subgroups, the same ones as* G_0.

Proof: First note that we are given an infinite family of groups $\{G_i : i = 0, 1, \dots\}$ with a family of monomorphisms, namely the strict inclusions $\alpha_i^j : G_i \to G_j, i \le j$ such that $\alpha_i^i = 1_i$ and $\alpha_i^j \alpha_j^k = \alpha_i^k$. Hence the direct limit G exists and is an infinite group. (See [Rob] Section 1.4 for details.)

Note that $G_i \triangleleft G$, $i = 0, 1, 2, \dots$. This follows from the fact that if $g \in G$ then $g \in G_{i+j}$ for some positive integer j and so $G_i^g = G_i$.

Also note that if $\langle x \rangle \ntriangleleft G$ then there exist finite subgroups G_i and G_j such that $x \in G_i$ and $x^g \notin \langle x \rangle$ for some $g \in G_j$. Hence $\langle x \rangle \ntriangleleft G_{i+j}$ and hence $\langle x \rangle \ntriangleleft G_0$. Thus all non-normal subgroups of G are finite and lie inside of G_0. Indeed, if $H \ntriangleleft G$, but $H \nsubseteq G_0$ then there is an element $h \in G_i \cap H \setminus G_0$ for some i. Let $\{h_0, h_1, \dots, h_n\}$ be the set of all cyclic non-normal subgroups in H. Then they, together with h, generate a subgroup S of G_0. Now $S \triangleleft G$ since it lies in $G_i \setminus G_0$. Thus since H is the join of this group with a possibly infinite number of cyclic normal subgroups, $H \triangleleft G$, as required. □

We now parallel some of the results from Chapter 2.

Theorem 4.5 *Let* G *be an infinite group with only a finite number of non-normal subgroups and let* $\nu(G) > 0$. *Then for some prime* p, *there is a Prüfer*

$p-$ group C in the centre $Z(G)$ of G such that

$$\ell(G/C) \leq \nu(G) - 1.$$

Proof: This follows immediately from Lemma 4.3 and Proposition 2.2. $\Box$

Corollary 4.6 *If G is an infinite group with only a finite number of non-normal subgroups then the intersection of the non-normal subgroups of G is trivial.*

Proof: Assume that $G = A \times B$. A, B and C as described in Theorem 4.1. Since A is finite and Dedekind, all non-normal subgroups of G contain a non-normal subgroup of B. Consider B. By Lemma 4.3 there exists a sequence of subgroups

$$B_0 < B_1 < B_2 < \ldots$$

such that B_i is a finite p-group and the exponent of the centre of B_i increases as i increases. Also the intersection of the non-normal subgroups of B_i is the same as the intersection of the non-normal subgroups of B. Assume this intersection is non-trivial. Then by Theorem 1.4 each B_i is a 2-group with centre of exponent at most 4. This is a contradiction. $\Box$

The following result can be viewed as a deduction from Lemma 4.2. (In [HL] it was noted that the number of non-normal subgroups of an infinite p-group G with $\nu(G) > 0$ was at least $p(p+1)$).

Corollary 4.7 *If G is an infinite p-group with only a finite, non-zero number of non-normal subgroups then $\nu(G) > p$.*

Proof: Apply Lemma 4.3. There is an increasing sequence of finite subgroups of G with $\nu(G) = \nu(G_i)$ for each G_i in the sequence. By Lemma 2.5 $\nu(G_i) = 0, 1$ or is at least p. But $\nu(G) \neq 0$ and we know from their structure that groups with $\nu(G) = 1$ can not be embedded into one another nor can groups with $\nu(G) = p$. (See Theorem 1.2 and Lemma 2.7.) Hence $\nu(G) > p$, as required.

□

Also we bound the number of conjugacy classes of non-normal subgroups of G in terms of p and the group G/C. Due to the fact that Lemma 4.2 tells us the exact number of non-normal subgroups of G this result is not profound.

Lemma 4.8 *Let G be a infinite group with only a finite number of non-normal subgroups and let $\nu(G) = \nu > 0$. Then*

$$\nu(G) \le \mu(A) \sum_{C<J<B} |J/C|(p-1),$$

where A, B and C are as defined in Theorem 4.1 and $\mu(A)$ is the number of subgroups of A.

Proof: Apply Lemma 4.3. Let G_0 be the first element of the series. Note that $A \subseteq G_0$ as $A \cap B = 1$. Thus $G_0 = A \times B_0$, for some $B_0 \le B$. Thus by Lemma 2.10

$$\nu(G) \le \mu(A) \sum_{C_0<J_0<B_0} |J_0/C_0|(p-1).$$

Since $B_0/C_0 \cong B/C$ the result follows. □

Finally, note the following:

Corollary 4.9 *Let G be a infinite nilpotent group with $\nu(G) < \infty$. Then*

$$|\ell(G')| \le \begin{cases} \nu(G) + 1 & \textit{if } 2 \mid |G|, \\ \nu(G) & \textit{if } 2 \nmid |G|. \end{cases}$$

Proof: The proof follows immediately from Lemma 4.3 and Corollary 2.12. □

We shall see in the next Chapter that the infinite, non-nilpotent groups with $\nu(G)$ finite do not behave as nicely.

5 Infinite Non-Nilpotent Groups G With $\nu(G)$ Finite

5.1 Structural Properties

We now consider a group G such that $0 < \nu(G) < \infty$ yet G possesses an infinite number of non-normal subgroups. In a similar vein A.V. Izosov and I.F. Sesekin considered all groups with only a finite number of infinite classes of conjugate subgroups. (see [IS]), but in their case $\nu(G)$ could still be infinite. Among other things they described the structure of such groups with FC–centre of finite index. We note one of the results from their paper below. The following result is the combination of Theorem 1 and Corollary 3 of their paper. Let $FC(G)$ denote the FC-centre of a group G.

Theorem 5.1 *If G is a group with a finite, positive number of infinite conjugacy classes of non-normal subgroups then for each $H \leq FC(G)$, the conjugacy class of H is finite. This is equivalent to saying that $[G : C_G(FC(G))]$ is finite.*

From this result Izosov and Sesekin point out that if G is an infinite group with a finite, positive number of infinite conjugacy classes of non-normal subgroups then G is not an FC-group. They go on to describe the groups G with $[G : FC(G)]$ finite and a finite, positive number of infinite conjugacy classes of non-normal subgroups . Other results from [IS] could be used in our present investigation, but in order to be fairly self-contained we will make do with the above theorem.

We now determine, through a series of Lemmas, the structure of a group G with $0 < \nu(G) < \infty$ and an infinite number of non-normal subgroups. First we note that $FC(G)$ is finite.

Lemma 5.2 *Let G be a group with $\nu(G) < \infty$ such that G has an infinite number of non-normal subgroups. Then $[G : FC(G)]$ is infinite and the FC-centre is finite so that $G/FC(G)$ has no non-trivial, finite normal subgroups.*

Proof: By Theorem 5.1 above, $[G : C_G(FC(G))]$ is finite. Let $G_2 = C_G(FC(G))$. G_2 is not Abelian by Corollary 1.11. Then $FC(G_2) = FC(G) \cap G_2$. Indeed, it is clear that $FC(G) \cap G_2 \subseteq FC(G_2)$ and if $x \in FC(G_2)$ then $x \in FC(G)$ since $[G : G_2] < \infty$. Hence it is sufficient to show that $|FC(G_2)| < \infty$ since

$$|FC(G)G_2/G_2| = |FC(G)/FC(G) \cap G_2| < \infty.$$

Observe that $FC(G_2) = Z(G_2)$.

Suppose that $Z(G_2)$ is infinite. By Lemma 1.9 G is torsion and by Theorem 4.3.11 of [Rob] either $Z(G_2)$ contains a Prüfer p-group C for some prime p or by repeated application of 4.3.11, it contains an infinite direct sum of cyclic groups of prime power orders $C_1 \times C_2 \times C_3 \times \cdots = C$.

If the former then for each positive integer i let $C_i \leq C$ such that $l(C_i) = i$. Let $x \in G_2 \setminus FC(G_2)$. Since the order of x is bounded, there exists an integer j such that

$$\langle x \rangle \cap C = \langle x \rangle \cap \langle C_1, C_2, \ldots, C_j \rangle.$$

Then $S_k = \langle x, C_1, C_2, \ldots, C_k \rangle \ntrianglelefteq G$ for each integer k (as the conjugacy class of $\langle x \rangle$ is infinite and S_k is finite.) If $j \leq k < l$, $|S_k| \neq |S_l|$ and so $\nu(G)$ is infinite, a contradiction. Hence $FC(G_2)$ and $FC(G)$ are finite. Finally, note that if $N/FC(G)$ is a finite, normal subgroup of $G/FC(G)$ then N is finite and normal and hence lies in $FC(G)$, as required.(See [LR], Lemma 10 for a shorter proof relying more heavily on [IS]). □

Lemma 5.3 *Let G be a group with $\nu(G) < \infty$ such that G has an infinite number of non-normal subgroups. Then $s \leq \nu(G)$ for any strictly increasing* *chain of subgroups*

$$1 = H_0 < H_1 < \cdots < H_s < H_{s+1} = G$$

with $H_i \lneq FC(G)$ if $H_i \lhd G$. Hence every subgroup of $G/FC(G)$ is finitely generated by at most $\nu(G) + 1$ elements.

Proof: First note that for $i \neq j$, H_i is not conjugate to H_j. Indeed, if $H_i^g = H_j$ for some $i < j$ and $g \in G$ then we have a strictly increasing series of non-normal subgroups

$$H_i < H_i^g < H_i^{g^2} < \dots .$$

Since $|g| < \infty$ this yields the contradiction that $H_i < H_i^{g^{|g|}} = H_i$.

We now produce non-normal subgroups $K_1, K_2, \dots, K_s$ no two of which are conjugate. If $H_i \ntriangleleft G$ then put $K_i = H_i$. If $H_i \lhd G$ then pick $x_i \in H_{i+1} \setminus (H_i \cup FC(G))$ (since $|H_i|$ is infinite and $FC(G)$ is finite, such an x_i exists.) Let $K_i = \langle x_i \rangle$. Note that K_i is not conjugate to any of the K_j, $j < i$.

Indeed, either

i) $K_j = H_j < H_i = K_i$ and the two subgroups of G are not conjugate by the initial comments, or

ii) $K_j = H_j < H_i \lhd G$ and all conjugates of K_j lie in the normal subgroup H_i while $K_i \nleq H_i$, or

iii) $H_j \lhd G$ and all conjugates of K_j lie in $H_i \lhd G$ and $K_i \nleq H_i$, or

iv) $H_j \lhd G$ and $H_i = K_i$ and since $|H_i|$ is infinite, K_i is not conjugate to K_j.

Thus $s \leq \nu(G)$, as required. The rest now follows. □

Lemma 5.4 *Let G be a group with $\nu(G) < \infty$ and such that G has an infinite number of non-normal subgroups. Then the finite residual. R of G.*

$$R = \cap\{H \leq G : [G : H] < \infty\}$$

has finite index in G. $R = R'$. $\nu(R) < \infty$ and $\ell(G/R) \leq \nu(G) - 1$.

Proof: By Lemmas 5.2 and 5.3. G has the minimal condition on subgroups. Thus. $[G : R]$ is finite. It follows that $\nu(R) < \infty$ by Lemma 1.8. Note that R/R' is finitely generated. Abelian and torsion by Lemmas 5.3 and 1.9 and so $R = R'$.

Consider G/R. Let $|G/R| = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$. with the p_i's distinct primes and the n_i's positive integers. Let H_{ij}/R be a subgroup of G/R such that $|H_{ij}/R| = p_i^j$. $0 \leq j \leq n_i$. Further suppose that

$$R = H_{i0} < H_{i1} < \cdots < H_{in_i}$$

for each prime p_i. Use these subgroups to produce non-conjugate. non-normal subgroups as follows. If $H_{ij} \ntriangleleft G$. then it is not conjugate to any other H_{lk} unless $j = k$ and $i = l$ because $|H_{ij}/R| = p_i^j$. Choose H_{ij} as our non-normal subgroup. If $H_{ij} \triangleleft G$. ($j > 0$) then choose $x \in H_{ij} \setminus (H_{ij-1} \cup FC(G))$. (Since $FC(G)$ is finite and the H_{ij} are infinite. this can be done.) Now $\langle x \rangle \ntriangleleft G$ since G is torsion and $x \notin FC(G)$. Note that $\langle x \rangle$ will not be conjugate to any non-normal subgroup chosen above. Indeed. $x \in H_{ij} \triangleleft G$ and $|x|$ is a power of p_i mod R. so the only possible conjugates would be the chosen non-normal subgroups arising from the chain

$$H_{i0} < H_{i1} < \cdots < H_{ij}.$$

However. either $H_{ij-1} \triangleleft G$ and $\langle x \rangle$ cannot be conjugate to any of the previous subgroups or $H_{ij-1} \ntriangleleft G$ and since it is infinite it can not be conjugate to $\langle x \rangle$.

The same reasoning shows that $\langle x \rangle$ is not conjugate to any of the other non-normal subgroups chosen, as required. Thus $\ell(G/R) \leq \nu(G) + 1$. $\square$

Note that Lemma 5.4 implies that for any group G such that $0 < \nu(G) < \infty$,

$$\ell(G/C_G(FC(G))) \leq \nu(G) - 1.$$

However, the fact that $[G : C_G(FC(G))]$ is finite is not sufficient to describe the groups with a finite number of conjugacy classes of non-normal subgroups. Indeed, Theorem 5.1 points out that all groups with only a finite number of infinite conjugacy classes of non-normal subgroups have $C_G(FC(G))$ of finite index.

Lemma 5.5 *Let G be a group with $\nu(G) < \infty$ and such that G has an infinite number of non-normal subgroups. Then*

i) Every finite, subnormal subgroup of G lies in $FC(G)$.

ii) G has only a finite number of normal subgroups and the number of normal subgroups of $G/FC(G)$ is bounded above by a function of $\nu(G)$.

iii) Every subnormal subgroup of G contained in the finite residual R is normal in R and the number of subnormal subgroups of G lying in $R/R \cap FC(G)$ is bounded above by a function of $\nu(G)$.

Proof: By Lemmas 5.2 and 5.3, G has the minimal condition on subnormal subgroups. By Theorem 13.3.8 of [Rob], the Wielandt subgroup of G, which is the intersection of the normalizers of all the subnormal subgroups of G, has finite index in G. Thus any subnormal subgroup of G has only a finite number of conjugates. It follows that if $H \leq G$ is a finite subnormal subgroup of G then $H \leq FC(G)$. It is also clear, since R is contained in the Wielandt subgroup of G, that the finite residual R normalizes every subnormal subgroup and so every subnormal subgroup of G lying in R is actually normal in R.

Next consider $\{M_\lambda : \lambda \in \Lambda\}$, the set of all minimal normal subgroups of $G/FC(G)$. The subgroups are characteristically simple, infinite and non-Abelian. For $\mu \neq \tau$, $[M_\mu, M_\tau] = 1$. Also $M_\mu \cap \langle M_\lambda : \lambda \in \Lambda \setminus \{\mu\}\rangle = 1$ since M_μ centralizes this centreless group. Thus we can construct the strictly increasing chain of subgroups

$$FC(G)/FC(G) < M_1 < M_1 \times M_2 < \dots.$$

By Lemma 5.3 there are at most $\nu(G)$ minimal normal subgroups in $G/FC(G)$.

Next, consider $G/M_i, i = 1, 2, \dots, s$. Repeat this argument to conclude G/M_i has at most $\nu(G/M_i)$ minimal normal subgroups. Since there exists an element $x_i \in M_i$, $\langle x_i \rangle \ntriangleleft G$, $\nu(G/M_i) < \nu(G)$. Repeat, and by Lemma 5.3, in at most $\nu(G)$ repetitions we will have accounted for all normal subgroups of $G/FC(G)$. Thus we may conclude that the number of normal subgroups of $G/FC(G)$ is bounded above crudely by

$$\nu(G) + \nu(G)(\nu(G) - 1) + \dots + \nu(G)! \leq \nu(G)(\nu(G)!).$$

Since $FC(G)$ is finite by Lemma 5.2, G has only a finite number of normal subgroups.

Next consider $R/R \cap FC(G)$. Recall that any subnormal subgroup of G lying in R is normal in R. Hence repeat the above argument on the minimal normal subgroups of $R/R \cap FC(G) = R/FC(R) = R/Z(R)$. Then there are again at most $\nu(G)$ minimal normal subgroups in $R/R \cap FC(G)$. Indeed, given a sequence

$$R \cap FC(G)/R \cap FC(G) < M_1 < M_1 \times M_2 < \dots,$$

with the M_i distinct minimal normal subgroups of $R/Z(R)$, we can find $x_i \in M_{i+1} \setminus (M_1 \times M_2 \times \dots \times M_i \cup (R \cap FC(G)))$ and so $\langle x_i \rangle \ntriangleleft G$ and we can have at most $\nu(G)$ such non-normal subgroups. The rest of the argument follows similarly. Just keep in mind that any strictly increasing chain of subgroups in $R/R \cap FC(G)$ is still bounded above by a function of $\nu(G)$. $\square$

Corollary 5.6 *The class of groups with only a finite number of conjugacy classes of subgroups (both normal and non-normal) consists of the finite groups together with all groups G such that $0 < \nu(G) < \infty$ and G has an infinite number of non-normal subgroups.*

Proof: This follows immediately from Lemma 5.5 above and Theorem 4.1.

Lemma 5.7 *Let G be a group with $\nu(G) < \infty$ such that G has an infinite number of non-normal subgroups. Then*

i) For all $x \in G \setminus FC(G)$, $\ell(\langle x\rangle FC(G)/FC(G)) \leq \nu(G)$.

ii) For all $x \in C_G(FC(G)) \setminus FC(G)$, $\ell(\langle x\rangle FC(G)/\langle x\rangle) < \nu(G)$.

iii) The number of primes involved in $G/FC(G) \leq \nu(G)$.

iv) The exponent of $G/FC(G)$ is bounded above by $\nu(G)$ (and the primes occuring as orders of elements of G) and so the exponent of G is finite.

Proof: If $x \in G \setminus FC(G)$ then $\langle x\rangle \ntrianglelefteq G$. Let $\ell(\langle x\rangle FC(G)/FC(G)) = n$. This determines n distinct conjugacy classes of non-normal subgroups. Hence $\ell(\langle x\rangle FC(G)/FC(G))$ is bounded above by $\nu(G)$.

If $x \in C_G(FC(G)) \setminus FC(G)$ then x generates a non-normal subgroup and for each subgroup M such that $\langle x\rangle \cap FC(G) \leq M \leq FC(G)$, $\langle x\rangle M$ is a non-normal subgroup as well. (This subgroup is finite and so cannot contain all the conjugates of $\langle x\rangle$.) This produces at least $\ell(FC(G)/\langle x\rangle \cap FC(G)) + 1$ distinct conjugacy classes of non-normal subgroups and so $\ell(\langle x\rangle FC(G)/\langle x\rangle)$ is bounded above by $\nu(G)$.

Finally suppose that $G/FC(G)$ has an element $gFC(G)$ of order $p_1^{n_1}p_2^{n_2}\dots p_k^{n_k}$. Then there exists elements $x_1, x_2, \dots, x_k$ such that $|x_i| = p_i^{n_i}, i = 1, 2, \dots, k($ mod $FC(G))$. Thus we produce $n_1 + n_2 + \dots + n_k$ distinct conjugacy class of non-normal subgroups, so that $\nu(G) \geq n_1 + n_2 + \dots + n_k$, as required. It is also now clear that the exponent of $G/FC(G)$ is at most $\nu(G)$ and the number of primes involved in $G/FC(G)$ is at most $\nu(G)$, as well. $\square$

Lemma 5.8 *Let G be a group with $\nu(G) < \infty$ and such that G has an infinite number of non-normal subgroups. Suppose that R is the finite residual of G and that*

$$J = \bigcap_{x \in R \setminus Z(R)} \langle x \rangle.$$

Then $J \leq Z(R)$ and

i) If $1 \neq J$ then for some fixed prime p, J is a cyclic p-group and $R/Z(R)$ is also a p-group.

ii) $\ell(Z(R)/J)$ is bounded above by a function of $\nu(G)$.

Proof: Note that since $R \subseteq C_G(g)$ for each $g \in FC(G)$,

$$R \cap FC(G) = FC(R) = Z(R).$$

Let J be as defined above. Clearly J is a finite, cyclic normal subgroup of R and hence lies in $Z(R)$. Suppose that $J \neq 1$. If J does not have prime power order then for each $x \in R \setminus Z(R)$, x does not have prime power order and so it is possible to generate $\langle x \rangle$ from its prime power order subgroups which must lie in $Z(R)$. This contradicts the fact that $x \in R \setminus Z(R)$.

So assume $|J| = p^a$ for some prime p. If $x \in R \setminus Z(R)$ such that x does not have p power order then there is a positive integer b so that $1 \neq x^{p^b}$ is of

p'−power order and since $J \not\leq \langle x^{p^b}\rangle$, $\langle x^{p^b}\rangle \leq Z(R)$, as required. The rest of $i)$ now follows.

Let $x \in R \setminus Z(R)$. Note that $\ell(FC(G)/\langle x\rangle \cap FC(G)) = \ell(FC(G)\langle x\rangle/\langle x\rangle) < \nu(G)$ by Lemma 5.7. Thus $\ell(Z(R)/\langle x\rangle \cap Z(R)) < \nu(G)$. If

$$\bigcap_{g\in G} \langle x\rangle^g \cap Z(R) < K \leq \langle x\rangle \cap Z(R)$$

then $K \ntrianglelefteq G$. Thus

$$\ell(\langle x\rangle \cap Z(R)/\bigcap_{g\in G} \langle x\rangle^g \cap Z(R)) \leq \nu(G)$$

and so

$$\ell(Z(R)/\bigcap_{g\in G} \langle x\rangle^g \cap Z(R)) < 2\nu(G).$$

Finally, since there are at most $\nu(G)$ conjugacy classes of cyclic, non-normal subgroups of G lying in R, and

$$J = \bigcap_{x\in R\setminus Z(R)} \bigcap_{g\in G} \langle x\rangle^g \cap Z(R)$$

then $\ell(Z(R)/J) \leq (2\nu(G))^{\nu(G)}$, as required. □

Now, we summarize the results found in this section.

Theorem 5.9 *Let G be a group with $0 < \nu(G) < \infty$ and such that G has an infinite number of non-normal subgroups. Then*

i) G is a torsion group and for all $x \in G$, $\ell(\langle x\rangle FC(G)/FC(G)) \leq \nu(G)$. Also, the exponent of $G/FC(G)$ is bounded above by $\nu(G)$, (and the primes occuring as orders of elements of G. Also the exponent of G is bounded.

ii) The FC−centre of G, $FC(G)$ is finite and equal to the FC-hypercentre of G.

iii) Any strictly increasing chain of subgroups of G no term of which is a finite normal subgroup of G, is of length at most $\nu(G) - 1$. Thus every subgroup of $G/FC(G)$ is generated by at most $\nu(G) - 1$ elements. Also the number of normal subgroups of G is finite and the number of normal subgroups of $G/FC(G)$ is bounded above by a function of $\nu(G)$.

iv) The finite residual R of G has the following properties:

a) $R = R'$.

b) $\ell(G/R) \leq \nu(G) - 1$.

c) all subnormal subgroups of R are normal in R.

d) $Z(R) = FC(R) = R \cap FC(G)$ and the number of normal subgroups of $R/Z(R)$ is bounded by a function of $\nu(G)$.

e) $FC(R/Z(R)) = 1$ and either $\ell(Z(R))$ is bounded above by a function of $\nu(G)$ or there exists a $1 \neq J \leq Z(R)$ such that

$$J = \bigcap_{x \in R \setminus Z(R)} \langle x \rangle$$

is a cyclic p-group, $\ell(Z(R)/J)$ is bounded by a function of $\nu(G)$ and $R/Z(R)$ is also a p-group.

Proof: See Lemmas 5.2 through 5.8.

Note that if G is a p-group with $\nu(G)$ finite but G has an infinite number of non-normal subgroups then $p \neq 2$. Indeed, by Theorem 14.4.3 of [Rob] all infinite 2-groups which satisfy the minimal condition are Černikov groups. Such groups have infinite exponent and so cannot be of the desired type. It may be true that other small primes can not be used as well.

Finally, we can expand on Lemma 5.8 and other previous work to obtain the following generalization of Blackburn's results on groups where the intersection of the non-normal subgroups is non-trivial.

Theorem 5.10 *Let G be a group with $0 < \nu(G) < \infty$. If*

$$J = \cap\{H \leq G : H \not\triangleleft G\} \neq 1$$

then either

i) *G is finite and its structure is outlined in Theorems 1.4 and 1.5 by Blackburn, or*

ii) *G has an infinite number of non-normal subgroups and, modulo a finite p'-group Q, of length bounded by a function of $\nu(G)$, $G = A \rtimes B$ where A is a finite p'-group, every subgroup of which is normal in G and (modulo Q) B is a p-group with*

$$J \leq Z(R) \leq R \leq B,$$

where R is described in Theorem 5.9. Also, $J \leq Z(G)$, $\nu(FC(B)) = 0$ and $\ell(FC(G)/J)$ is bounded above by a function of $\nu(G)$.

Proof: It is sufficient to consider G an infinite group with $0 < \nu(G) < \infty$. By Corollary 4.6 G must have an infinite number of non-normal subgroups. Note that $J \triangleleft G$ and that J is cyclic.

Now $\nu(R) \neq 0$ as R does not lie in the finite FC-centre of G. Thus,

$$1 \neq J \leq \cap_{x \in R \setminus Z(R)} \langle x \rangle,$$

and so by Theorem 5.9, J is a cyclic p-group for some prime p. Let $h \in G$ such that $(|h|, p) = 1$. Then $\langle h \rangle \triangleleft G$ and so $h \in FC(G)$. Thus all p' order elements of G lie in $FC(G)$ and generate a finite normal p'-subgroup, A.

Note that $\ell(Z(R)/J)$ is bounded by a function of $\nu(G)$ by Theorem 5.9. Let Q be the p' component of $Z(R)$. Then R/Q is a p-group . Consequently, $G/Q = A/Q \rtimes B/Q$ where A is a finite p'-group, B/Q is a p-group and $J \leq Z(R) \leq$

$R \leq B$. (Just let B/R be a Sylow-p subgroup of G/R). Since $\ell(G/R) \leq \nu(G) - 1$ we may conclude that $\ell(FC(G)/J)$ is bounded above by a function of $\nu(G)$.

Lastly we must show $J \leq Z(G)$. We know that R is not a 2-group by the comments made after Theorem 5.9. Thus by Theorem 1.4 we conclude that $\nu(FC(B)) = 0$ and $FC(B)$ is Abelian. Thus $[J, FC(B)] = 1$. Also, by the normality of A and J, $[J, A] = 1$ and $[J, b] = 1$ for all $b \in B \setminus FC(B)$ because $J \leq \langle b \rangle$ for such elements b. Thus J is central, as required. $\square$

We will now give some examples of infinite, non-nilpotent groups with $\nu(G)$ finite.

5.2 Some Examples

The results of the previous section do point out that there are many differences between infinite, non-nilpotent groups G with $\nu(G)$ finite and all other groups with only a finite number of conjugacy classes of non-normal subgroups. The following examples will further illustrate this fact.

By definition, a Tarski p-group (also known as a Tarski group or a Tarski monster) is an infinite group all of whose proper, non-trivial subgroups have prime order p. A. Yu Ol'shanskii has proven the existence of Tarski p-groups with exactly one conjugacy class of non-normal subgroups, provided p is a large enough prime, see [IO]. Take these groups to be our first examples of infinite, non-nilpotent groups with few conjugacy classes of non-normal subgroups. These groups are easily seen to be simple and generated by 2 elements.

Recall that Theorem 1.2 (or see [BDF]) shows that if G is an infinite group with $\nu(G) = 1$ then G has a Tarski monster as a central quotient. This theorem motivated results in the previous section. It can now be viewed as a corollary to Theorem 5.9 and a few other results.

Theorem 5.11 (Brandl, De Giovanni and Franciosi) [BDF] *If G is an infinite group with a unique conjugacy class of non-normal subgroups then $G/Z(G)$ is a Tarski p-group for some prime p and $Z(G)$ is a cyclic p-group. Moreover, $Z(G) = \langle g^p \rangle$ for each element g of $G \setminus Z(G)$.*

Proof: We know by Corollary 4.7 that G can not have a finite number of non-normal subgroups. By Theorem 5.9i), G is torsion and for each $g \in G$, $g^p \in FC(G)$ for some fixed prime p. By ii) of the same theorem, $FC(G)$ is finite. Note that $G/FC(G)$ is simple. Indeed, if a sequence

$$FC(G) < N \triangleleft G, \ N \neq G$$

exists then we can find $x \in N$ and $y \in G \setminus N$ generating non-normal and non-conjugate subgroups. Thus $G = R$ and $FC(G) = Z(G)$.

It now follows that all non-trivial subgroups of $G/FC(G)$ have prime order p since, if

$$FC(G) < H < G$$

then $H \ntriangleleft G$ and if $\langle h \rangle FC(G)/FC(G) < H/FC(G)$ then we have $\nu(G) \geq 2$. Thus, $G/Z(G)$ is a Tarski p-group.

Finally, by Lemma 5.7 ii), since $FC(G) = Z(G)$, $\ell(Z(G)/\langle g \rangle \cap Z(G)) < 1$ and so $Z(G) = \langle g^p \rangle$ for each $g \in G \setminus (Z(G)$, as required. □

We can generate many more examples of infinite, non-nilpotent groups G with $\nu(G)$ finite using the following theorem of Sergei V. Ivanov [I]. Here the set A will be a free amalgam of some groups G_α, $\alpha \in I$ (I a countable index set) if A is the union of the pairwise disjoint sets G_α, $\alpha \in I$ with identified identity elements.

Theorem 5.12 (Ivanov) [I] *Let A be the free amalgam of some groups G_α, $\alpha \in I$ such that A contains no involutions, A is finite or countable and one of the* *groups $G_\alpha, \alpha \in I$ contains an element of order $n >> 1$ (e.g. $n > 10^{96}$, $n = \infty$ is permissable). Then A can be embedded into a 2-generator group $G(A)$ ($A \subseteq G(A)$) with the following properties:*

1. *Every maximal subgroup of G is conjugate to one of the subgroups $G_\alpha \subseteq G(A)$. In particular, $G(A) = \bigcup_{x \in G(A)} x^{-1}Ax$.*

2. *If $x^{-1}Ax \cap A \neq 1$, where $x \in G(A)$, then $x \in G_\alpha$ for some α and $x^{-1}Ax \cap A = G_\alpha$.*

First apply this theorem to obtain a simple, 2-generated group with elements of $k > 1$ distinct prime orders so that $\nu(G) = k > 1$.

Example 5.13 Let A be the free amalgam of $C_{p_1}, C_{p_2}, \ldots, C_{p_k}$ with $k \geq 2$, the p_i's distinct odd primes, and C_{p_i} cyclic of order p_i and $p_1 > 10^{96}$. By Ivanov's Theorem, $A \subseteq G(A)$, $G(A)$ a group with the noted properties.

First note that, since $G(A) = \bigcup_{x \in G(A)} x^{-1}Ax$, any element of $G(A)$ has order $1, p_1, p_2, \ldots, p_{k-1}$ or p_k and if $1 \neq h \in G(A)$ then $\langle h \rangle$ is conjugate to C_{p_i} for some $i = 1, 2, \ldots, k$.

Note that $1 \neq \langle h \rangle$ is also a maximal subgroup of $G(A)$. Indeed, $\langle h^{x^{-1}} \rangle = C_p$ is maximal so $G = \langle h, x \rangle$. Let $S = \{K \leq G : \langle h \rangle \leq K, x \notin K\}$. Then S is not empty so that by Zorn's Lemma S will have a maximal element M. Now $M \leq G$ and if $M < N$ then N contains both h and x so that $N = G$. Thus M is a maximal subgroup of G and so $M = \langle h \rangle$ by size. Thus $\nu(G) = k$ and G is simple since all proper subgroups have prime order. Although Ivanov's theorem does not explicitly state that $G(A)$ is infinite, it is apparent it must be, as the Sylow subgroups are maximal and not normal and $|G| \neq p_1 p_2 \cdots p_k$ else we would violate the second condition of Ivanov's Theorem. □

Note that we can also find infinite groups G with an infinite number of non-normal subgroups and $\nu(G)$ finite so that G has non-normal subgroups not of prime power order.

Example 5.14 Let A be the free amalgam of $G_1 = C_{p_1 p_2 \ldots p_k}$ with $G_2 = C_{p_{k-1}}$, $k > 1$, and suppose the primes are distinct with $p_1 > 10^{96}$. By Ivanov's Theorem there is a group $G(A)$ with the noted properties. Hence, by similar reasoning to that used in the last example, all elements of $G(A)$ have order

$$1, p_1, p_2, p_1 p_2, p_3, \ldots, p_1 p_2 \ldots p_k \text{ or } p_{k-1}.$$

Similarly, if $1 \neq \langle h \rangle \leq G(A)$ then either $\langle h \rangle$ is conjugate to $C_{p_{k-1}}$ and is maximal or h is a subgroup of a maximal subgroup conjugate to G_1. It follows again by reasoning similar to the last example that $G(A)$ is simple, infinite and all subgroups of $G(A)$ have orders $1, p_1, p_2, p_1 p_2, p_3, \ldots, p_1 p_2 \ldots p_k$ or p_{k-1}. Consequently, $\nu(G) = (2^k - 1) + 1 = 2^k$. $\square$

The method of generating examples is now established. It is apparent that we can generate p-groups of the desired type, take A to be the free amalgam of C_{p_1} with $C_{p_1^n}$, $n \geq 1$, $p_1 > 10^{96}$ for example.

Bibliography

[Bl] N. Blackburn. *Finite groups in which the nonnormal subgroups have nontrivial intersection*. J. Algebra. **3** (1966). 30 - 37.

[Br1] R. Brandl. *Conjugacy classes of subgroups in p-groups*. J. Austral. Math. Soc. Ser. A. **53** (1992). 294 - 303.

[Br2] R. Brandl. *Groups with few non-normal subgroups*. Comm. Algebra. **23**(6) (1995). 2091 - 2098.

[BDF] R. Brandl. F. De Giovanni and S. Franciosi. *Groups with finitely many conjugacy classes of non-normal subgroups*. Proc. Roy. Irish Acad. Sect. A.. **95** (1995). 17 - 27.

[BV1] R. Brandl and L. Verardi. *Finite groups with few conjugacy classes of subgroups*. Rend. Sem. Mat. Univ. Padova. **87** (1992). 267 - 280.

[BV2] R. Brandl and L. Verardi. *Finite simple groups with few conjugacy classes of subgroups*. Japan J. Math..**18**(2). (1992). 347 - 359.

[BV3] R. Brandl and L. Verardi. *Metacyclic p-groups and their conjugacy classes of subgroups*. Glasgow Math. J.. **35**(3) (1993). 339 - 344.

[HL] N. S. Hekster and H. W. Lenstra. Jr.. *Groups with finitely many non-normal subgroups*. Arch. Math. (Basel). **54** (1990). 225 - 231.

[I] S. V. Ivanov. *On some finiteness conditions in semigroup and group theory*. Semigroup Forum. **48**. (1994). 28 - 36.

[IO] S.V. Ivanov and A. Yu. Ol'shanskii. *Some applications of graded diagrams in combinatorial group theory. Groups – St. Andrews 1989.* Vol.

2, 258 - 308, London Math. Soc. Lecture Notes Ser., **160** (1991), Cambridge University Press, Cambridge.

[IS] A. V. Izosov and I. F. Sesekin, *Groups with a finite number of infinite classes of conjugate subgroups*, Ukrainian Math. J., **40**(3) (1988), 263 - 267.

[L] R. La Haye, *Some explicit bounds in groups with a finite number of non-normal subgroups* , Comm. Alg., (to appear).

[LR] R. La Haye and A. Rhemtulla, *Groups with a bounded number of conjugacy classes of non-normal subgroups* , submitted.

[LS] J.C. Lennox and S.E. Stonehewer, *Subnormal subgroups of groups*, Clarendon Press, Oxford, 1987.

[M] I. D. Macdonald, *Some explicit bounds in groups with finite derived groups*, Proc. London Math. Soc., **11**(3) (1961), 23 - 56.

[N1] B. H. Neumann, *Groups with finite classes of conjugate elements*, Proc. London Math. Soc.,**1**(3) (1951), 178 - 187.

[N2] B. H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. **63** (1955), 76 - 96.

[NV] P. M. Neumann and M.R. Vaughan-Lee, *An essay on BFC groups*, Proc. London Math. Soc., **35**(3) (1977), 213 - 237.

[PR] J. Poland and A. Rhemtulla, *The number of conjugacy classes of non-normal subgroups in nilpotent groups*, Comm. Alg., **24**(10) (1996), 3237 - 3245.

[Rob] D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag, New York, 1982.

[**Ros**] J.E. Roseblade. *On groups in which every subgroup is subnormal*. J. Algebra. **2** (1965). 402 - 412.

[**Rot**] J. J. Rotman. *An introduction to the theory of groups*. Springer-Verlag. New York. 1995.

[**Sc**] Schoenert. Martin. et. al.. *GAP: Groups. Algorithms. and Processing*. Version 3. Release 3. Lehrstuhl D fer Mathematik. RWTH Aachen. 1993. (Available via anonymous ftp samson.math.rwth-aachen.de)

[**Sm1**] H. Smith. *Groups with finitely many conjugacy classes of subgroups of large derived length*. Bollettino Un. Mat. Ital. A (7). **9**(1) (1995). 167 - 175.

[**Sm2**] H. Smith. *Groups with finitely many conjugacy classes of subgroups with large subnormal defect*. Glasgow Math. J.. **37** (1995). 69 - 71.

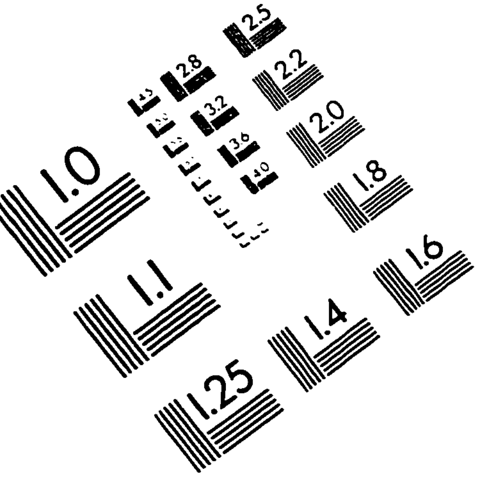
[**Su1**] M. Suzuki. *Group Theory II*. Springer-Verlag. New York. 1986.

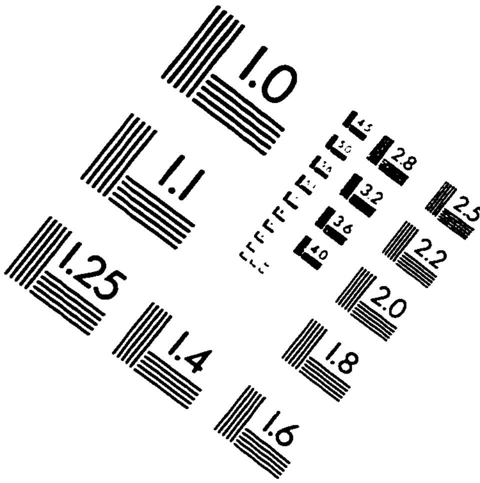
[**Su2**] M. Suzuki. *Structure of a group and the structure of its lattice of subgroups*. Springer-Verlag. Berlin. 1956.

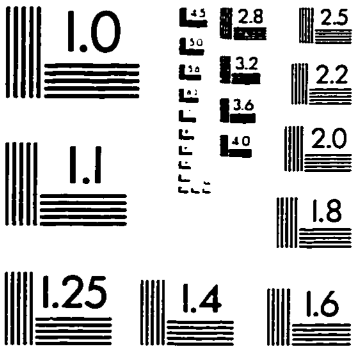
[**T**] M. J. Tomkinson. *FC-groups. Research Notes in Mathematics. 96*. Pitman (Advanced Publishing Program). Boston. 1984.

[**W**] J. Wiegold. *M. Sc. Thesis*. University of Manchester. 1955.

IMAGE EVALUATION TEST TARGET (QA-3)

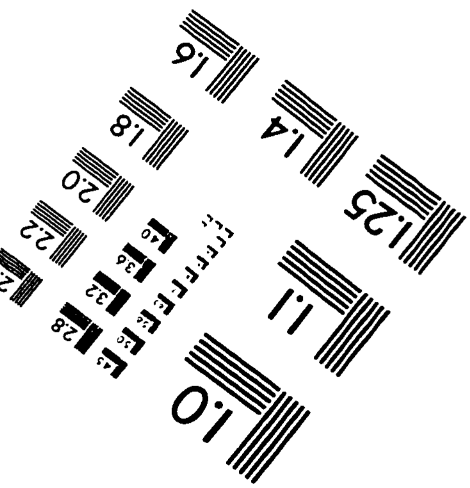






150mm

6″



APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989