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ESSAYS ON THE VALUATION OF BONDS, CALLABLE BONDS, AND FUTURES

BY

BARRY PETER LAISS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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THE UNDERSIGNED CERTIFY THEY HAVE READ, AND RECOMMEND TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH FOR ACCEPTANCE, A THESIS ENTITLED ESSAYS ON THE VALUATION OF BONDS, CALLABLE BONDS, AND FUTURES SUBMITTED BY BARRY PETER LAISS IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN FINANCE.

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ABSTRACT

I. **Multifactor Treasury Bond Pricing: The Empirical Evidence**

In this paper, a continuous time multiple state variable bond valuation model is used to estimate the prices of United States Treasury coupon bearing issues traded during 1988. State variables are constructed from the implied yields of Treasury bills as well as highly liquid longer term bond futures and are assumed to follow Ornstein-Uhlenbeck processes. The parameters of the state variables' stochastic processes are first estimated over their time series and then used as inputs in a second least squares procedure over a cross-section of bond prices to generate consistent and asymptotically normal estimates of their associated market prices of risk. Empirical results suggest that at least two state variables are necessary to explain the observed term structure.

II. **Multifactor Callable Treasury Bond Pricing**

Callable United States Treasury bonds have provisions allowing the option to be exercised at any scheduled coupon payment date during the last five years of their thirty year maturities. A European stochastic interest rate call option model is derived in a manner consistent with an underlying multiple state variable coupon bond valuation model. The value of the call feature allowing multiple discrete exercise dates is approximated using Black's pseudo-American option

approach. Estimated prices of callable bonds trading during 1988 are subsequently calculated as the theoretical price of an equivalent characteristic straight bond less the value of the call provision.

III. The Term Structure of Futures Prices

The ratio of futures and spot prices is a function of the cost of carry rate and time to maturity and converges to one at contract expiration. Therefore, this variable can be modelled in a manner similar to that of a Treasury bill. A continuous time model to explain the relationship between futures and spot prices is derived allowing the cost of carry rate to be stochastic and a closed-form solution obtained. The model is extended to a multivariate framework allowing individual components of the cost of carry rate (domestic interest rate, convenience yield, and holding costs) to each be stochastic in nature. Empirical implementation of the model is illustrated using British pound, Deutsche mark, and Japanese yen foreign currency futures. Results are then compared to those obtained using the standard non-stochastic arbitrage model of futures prices.

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CHAPTER I: Multifactor Treasury Bond Pricing: The Empirical Evidence

1. Introduction and Literature Review

The continuous time arbitrage pricing approach to the term structure of interest rates assumes that the values of default-free bonds may be expressed as functions of underlying state variables following continuous time diffusion processes. Provided that markets are perfect, standard arbitrage arguments can be employed to derive a partial differential equation governing the price movement of bonds of all maturities. Various models are distinguished by their choice and number of state variables, the way in which the associated market prices of risk are determined, and whether or not a closed form solution has been found. A summary of the relevant literature is presented in Table I-1.

Models with a single state variable, the instantaneous riskless rate of interest (r), have been suggested by Brennan and Schwartz(1977), Vasicek(1977), Dothan(1978), Cox, Ingersoll, and Ross(1985), Longstaff(1989), and Jamshidian(1989a). While these models generally allow closed form solutions, their empirical success is limited by implicitly requiring the returns on all bonds to be perfectly positively correlated. Two state variable models appearing in the literature mitigate this problem but encounter others. Preference dependent models derived by Richard(1978) and Cox, Ingersoll, and Ross(1985) incorporate the instantaneous real rate of interest and expected inflation as state variables, both of which are typically unobservable. The Cox, Ingersoll, and Ross and Longstaff models are distinct from all other work in the area by having been derived within a general equilibrium context.

Brennan and Schwartz(1979,1980,1982,1983b) have proposed a two state variable model using the instantaneous nominal interest rate and the yield on a consol bond (1). Although not necessarily observable, they can at least be adequately proxied. However, because of the choice of stochastic processes governing the state variables and the desire to eliminate estimation of the market price of consol yield risk via the properties of consol bonds, the model's solution must be obtained by numerical methods. This, in turn, resulted in the empirical necessity of assuming intertemporally constant market prices of risk. Nevertheless, the series of Brennan and Schwartz papers represent a significant contribution and include the first attempts at empirically testing arbitrage term structure models. In subsequent work, Schaefer and Schwartz(1984) obtained an approximate analytical solution of a similar model, this time based on the consol yield and the spread between the consol rate and the short rate as orthogonal state variables.

In the course of Brennan and Schwartz's early empirical work, the possible need for more than two state variables was demonstrated. Using a sample of Government of Canada bonds, Brennan and Schwartz(1980) discovered by applying factor analysis to the residuals of their two variable model that, at most, there existed the need for one state variable in addition to the short rate and the long rate. Subsequently, this result was supported using United States Treasury bond data in Brennan and Schwartz(1982). However, the appropriate number and exact specification of relevant state variables remains an open question.

A multivariate model of the term structure was developed by Langetieg(1980). It can accommodate an arbitrary number of state variables. The model assumes that the vector of state variables follows a joint elastic random walk, the instantaneous riskless rate can be represented as a linear combination of the state variables, and that the corresponding market prices of risk are, at most, time dependent. Langetieg(1980) shows that the model's closed form solution can be simplified by further assuming that the Ornstein-Uhlenbeck processes followed by the state variables have mutually uncorrelated error terms. When this last assumption is maintained, Oldfield and Rogalski(1987) find a closed form solution when the underlying state variables follow square root or arithmetic stochastic processes. As yet, no attempt has been made in the literature to test empirically the ability of the above multivariate models to describe accurately bond prices or, indeed, any term structure model with more than two state variables.

The aim of this paper is to develop and test an operational bond pricing model which is straightforward to implement and avoids several of the drawbacks encountered in previous work. The model outlined in Section 2 possesses a closed-form solution and is able to accommodate an arbitrary number of state variables. Section 3 suggests three sensible variables to describe the yield curve, which may be observed directly in highly liquid T-bill and futures markets, from which state variables can be constructed. The model estimation procedure which allows bond prices to be estimated cross-sectionally at a single point in time is discussed in Section 4. Prices of United States Treasury notes and bonds are then estimated monthly for 1988 under one, two, and three state variable scenarios. The predictive ability of the model is then examined in each of the three cases, with the result that two state variables are sufficient to describe the term structure. Concluding remarks are contained in Section 5.

2. Development of the Model

The model begins by assuming that the price at time t of a default-free discount bond paying $1 at maturity time T can be expressed as a function of a vector of n state variables $x=(x_1,...,x_n)$ and time to maturity $\tau=T-t$, denoted $B_\tau=B(t,T,x)$. This function defines the entire term structure of interest rates with particular values of $B_\tau=B(t,T,x)$ representing present value factors that can be used to price any pattern of future payments. The n state variables $x=(x_1,...,x_n)$ are assumed to follow a joint stochastic process of the form:

$$dx_i = \alpha_i(t,x)dt + \sigma_i(t,x)dZ_i, \quad i=1,...,n \tag{1}$$

where dZ_i $(i=1,...,n)$ are standard Gauss-Wiener processes with $E[dZ_i]=0$, $E[dZ_i^2]=dt$, and $E[dZ_i dZ_j]=\rho_{ij}dt$.

Provided $B(t,T,x)$ is continuous in t and x with continuous partial derivatives with respect to t and x and with continuous partial second derivatives with respect to x, Ito's lemma shows that B_τ follows the process:

$$dB_\tau = \alpha_{B\tau}dt + \Sigma_{i=1,n}\beta^i_{B\tau}dZ_i \tag{2}$$

where

$$\alpha_{B\tau} = \partial B_\tau/\partial t + \Sigma_{i=1,n}(\partial B_\tau/\partial x_i)\alpha_i + (1/2)\Sigma_{i=1,n}\Sigma_{j=1,n}(\partial^2 B_\tau/\partial x_i \partial x_j)\sigma_i\sigma_j\rho_{ij}$$

$$\beta^i_{B\tau} = (\partial B_\tau/\partial x_i)\sigma_i \ , \quad i=1,...,n.$$

Assuming frictionless markets, consider a portfolio comprised of n+2 discount bonds of varying maturities denoted by B_i in the vector $B=[B_1,...,B_{n+2}]$ with the associated vectors of parameters $\alpha=[\alpha_{B1},\ldots,\alpha_{Bn+2}]$ and $\beta^i=[\beta^i_{B1},\ldots,\beta^i_{Bn+2}]$, for $i=1,\ldots,n$. Using an arbitrage argument similar to Ross(1977), suppose the vector of weights $\omega=[\omega_1,\ldots,\omega_{n+2}]$ are chosen such that the portfolio is riskless and zero wealth is invested. To prevent arbitrage profits, the return on the portfolio must be zero. Because the vector of weights ω is orthogonal to B (zero investment), the n vectors β^i (zero risk), and α (zero return), then α can be expressed as a linear combination of B and β^i as in:

$$\alpha = rB + \phi_1\beta^1 + \ldots + \phi_n\beta^n \ , \tag{3}$$

where ϕ_i, $i=1,\ldots,n$ are scalars and r is the instantaneous nominal riskfree rate of interest. The weights ϕ_i are the market prices of risk associated with the state variables x_i, which are in general functions of x_i and t. When the $\phi_i(x_i,t)$ are not tradable, they must be empirically estimated or theoretically specified.

Substituting into equation (3) the expressions for $\alpha_{B\tau}$

and $\beta^i_{B\tau}$ from (2) yields the partial differential equation followed by any bond $B_\tau = B(t,T,x)$ given by:

$$\partial B_\tau/\partial t + \sum_{i=1,n} (\partial B_\tau/\partial x_i)\alpha_i$$
$$+ (1/2)\sum_{i=1,n}\sum_{j=1,n} (\partial^2 B_\tau/\partial x_i \partial x_j)\sigma_i(x_i)\sigma_j(x_j)\rho_{ij}$$
$$- rB - \sum_{i=1,n} (\partial B_\tau/\partial x_i)\sigma_i(x_i)\phi_i(x_i,t) = 0, \qquad (4)$$

which is subject to the terminal boundary condition $B(T,T,x)=1$. Equation (4) is the standard n state variable partial differential equation derived by Langetieg(1980) and Oldfield and Rogalski(1987). It is at this point that explicit bond pricing models are developed by specifying exact functional forms for both the stochastic processes governing the state variables and the market prices of risk.

Consequently, the general model may be specialized by invoking the following assumptions:

1) The state variables follow Ornstein-Uhlenbeck processes (mean reverting with constant variance) of the form:

$$dx_i = \kappa_i(\theta_i - x_i)dt + \gamma_i dZ_i \qquad i=1,\ldots,n \qquad (5)$$

where $\kappa_i>0$ is the speed of adjustment, θ_i is the long-run mean of state variable x_i, and γ_i is the constant standard deviation of the process;

2) the state variables are mutually orthogonal implying $\rho_{ij}=0$ for all $i \neq j$;

3) the state variables will be constructed such that they sum to the instantaneous riskfree rate interest,

$r = \sum_{i=1,n} x_i$; and

4) the market prices of risk, ϕ_i, $i=1,\ldots,n$ are constants.

Under these assumptions, the partial differential equation (4) reduces to:

$$\partial B_\tau/\partial t + \sum_{i=1,n} (\partial B_\tau/\partial x_i)\kappa_i(\theta^*_i - x_i)$$
$$+ (1/2)\sum_{i=1,n}(\partial^2 B_\tau/\partial x_i^2)\gamma_i^2 - \sum_{i=1,n} x_i B_\tau = 0, \qquad (6)$$

subject to the terminal boundary condition $B(T,T,x)=1$, and where

$$\theta^*_i = \theta_i + \phi_i\gamma_i/\kappa_i.$$

Because of the simplifying assumptions and judicious construction of state variables, the solution to (6) is the product of the solutions of n one-variable problems (see Carslaw and Jaeger(1959), Schaefer and Schwartz(1984)) as in:

$$B(t,T,x) = \prod_{i=1,n} Y_i(t,T,x_i) \qquad (7)$$

where $Y_i(t,T,x_i)$ is the solution to:

$$\partial Y_i/\partial t + (\partial Y_i/\partial x_i)\kappa_i(\theta^*{}_i - x_i) + (1/2)(\partial^2 Y_i/\partial x_i^2)\gamma_i^2 - x_i Y_i = 0, \quad (8)$$

with boundary condition $Y(T,T,x_i)=1$. Vasicek(1977) shows that the solution to (8) is given by:

$$Y(t,T,x_i) = \mathrm{EXP}[(1/\kappa_i)(1-\mathrm{EXP}(-\kappa_i(T-t)))(x_{i\infty}-x_i) - (T-t)x_{i\infty} - (\gamma_i^2/4\kappa_i^3)(1-\mathrm{EXP}(-\kappa_i(T-t)))^2] \quad (9)$$

where $x_{i\infty} = \theta^*{}_i - \gamma_i^2/2\kappa_i^2$.

After substitution of (9) into (7) and rearrangement of terms, a compact expression for the value of a discount bond with maturity date T may then be written:

$$B(t,T,x) = \mathrm{EXP}[\lambda(t,T)+\Sigma_{i=1,n} L_i(t,T)x_i]$$

where $L_i(t,T)=-(1/\kappa_i)(1-\mathrm{EXP}(-\kappa_i(T-t)))$

$$\lambda(t,T)=\Sigma_{i=1,n}[(1/\kappa_i)(1-\mathrm{EXP}(-\kappa_i(T-t)))-(T-t)][\theta_i-(\gamma_i^2/2\kappa_i^2)+(\gamma_i\phi_i/\kappa_i)] -\Sigma_{i=1,n}(\gamma_i^2/4\kappa_i^3)(1-\mathrm{EXP}(-\kappa_i(T-t)))^2. \quad (10)$$

This formulation is a special case of the Langetieg(1980) result, but was obtained via a much simpler separation of variables technique by assuming orthogonal state variables. As mentioned earlier, the function B(t,T,x) completely describes the term structure of interest rates. Consequently, the price of any coupon bearing default-free note or bond can be expressed as the sum of its cash flows at times T_j discounted by their associated present value factors $B(t,T_j,x)$. Denoting both the remaining semi-annual coupon payments and final principal payment at times T_j as cash flows C_j, allows the current invoice price $BP(t,T,x)$ of a coupon bond with maturity at time T, to be written as the quoted price $QP(t,T,x)$ plus accrued interest $AI(t)$:

$$BP(t,T,x) = QP(t,T,x)+AI(t) = \sum_{t<Tj\leq T} C_j B(t,T_j,x) \qquad (11)$$

where $AI(t)=(t+0.5-MIN(T_j))COUPON$. The function $AI(t)$ represents the interest that has accrued on the security since the last semi-annual coupon payment which is added to the quoted price the buyer agrees to pay the seller. The invoice or actual price paid for the bond equals the present value of the coupon and principal cash flows.

3. Specification and Estimation of the Model

In order to operationalize the model in Section 2, several further issues need to be considered. These include the number and choice of state variables, the estimation of the stochastic processes which they follow, and the method and time frame for estimating their associated market prices of risk.

3.1 Number and Choice of State Variables

It is necessary to select or construct state variables which are likely to embody the information contained in the term structure, be readily observable (unlike variables such as the real rate of interest or the inflation rate), and are such that the sum of the state variables will equal the instantaneous nominal risk-free rate of interest. For these reasons, state variables were constructed from an instantaneous nominal rate, r, an intermediate term yield, m, and a long term yield, l. Following Brennan and Schwartz, I took the annualized yield on the Treasury bill closest to 30 days to maturity as a proxy for the instantaneous riskless rate r. The surrogates for m and l were more difficult to choose. Treasury note and bond markets are not very liquid and a time series of yields on a constant coupon rate and constant maturity note or bond cannot be observed directly. However, futures markets for Treasury notes and bonds are highly liquid and uniformly assume delivery of a hypothetical 8% coupon security with 10 years and 20 years to maturity, respectively. Consequently, the annualized yield as implied by the futures prices of the nearest maturity treasury note and bond futures contract should serve as ideal proxies for m and l. Although data from futures markets has several desirable properties, the markets themselves have not been in operation for long. The Chicago Board of Trade (CBT) initiated trading in United States Treasury bond futures contracts in August of 1977, and United States Treasury note contracts in March, 1982. The required T-bill and futures data were collected monthly for the last trading day from the Wall Street Journal from December, 1982 through December, 1988.

Because the optimal number of state variables necessary to explain bond prices was uncertain, three cases were considered:

Case 1: One State Variable

With a single state variable, $x_1=r$, the bond pricing model reduces to the Vasicek(1977) model.

Case 2: Two State Variables

In this case, state variables $x_1=r-l$, the spread between the instantaneous rate and the long term yield, and $x_2=l$ were chosen.

Case 3: Three State Variables

With three factors the spreads $x_1=r-m$ and $x_2=m-l$, as well as the long yield, $x_3=l$, were employed.

For each of these model specifications, the stochastic processes governing the state variables will require the same estimation procedure.

3.2 Estimation Method for the Stochastic Processes

Recall that the state variables are assumed to follow Ornstein-Uhlenbeck processes of the form:

$$dx_i = \kappa_i(\theta_i - x_i)dt + \gamma_i dZ_i \qquad i=1,\ldots n$$

where θ_i is the long run mean of the series, κ_i is the speed of adjustment, and γ_i the (constant) standard deviation. A difficulty in estimating the parameters of these equations arises because the data is only sampled at discrete time intervals, whereas the Ornstein-Uhlenbeck processes are defined in continuous time. Simply substituting first differences for the differentials above leads to estimation bias as noted by Phillips(1972), Wymer(1972), and Marsh and Rosenfeld(1983). It is possible to avoid this situation by employing an exact differential equation (see Oldfield and Rogalski(1987)) of the form:

$$E[\int_{t,s} dx_i] = x_i(t)(EXP(-\kappa_i(s-t))-1) + \theta_i(1-EXP(-\kappa_i(s-t)))$$

where

$$Var[\int_{t,s} dx_i] = \gamma_i^2(1-EXP(-2\kappa_i(s-t)))/2\kappa_i.$$

The required parameters can then be obtained directly by non-linear estimation or implied by the slope and intercept coefficients of an ordinary least squares regression as in:

$$x_i(s) = a + (1+b)x_i(t) + u(s), \qquad (12)$$

where u(s) is an error term. In particular, the parameters are then acquired by setting

$$\kappa_i = -\ln(1+b)/(s-t),$$

$$\theta_i = -a/b,$$

and γ_i = SQRT[2ln(1+b)σ_u^2/((s-t)((1+b)^2-1))], where σ_u^2 is the error sample variance. The assumption of independent and identically distributed (E[u_s]=0,VAR[u_s]=σ_u^2) errors is imposed in order to make inferences about the parameter estimates. With estimates for non-linear models such as this, the resulting test statistic will not be a t-statistic but rather the square root of a Chi-square variable with one degree of freedom. The distribution of this statistic is virtually the same as a t-statistic based on more than 50 observations. Because monthly data is used, s-t will equal 1/12 when these processes are actually estimated.

3.3 Empirical Results of the Estimation of the Stochastic Processes

In each of the one, two, and three state variable cases, the appropriate stochastic processes were estimated using monthly data beginning with the period December, 1982 through January, 1988. Another month of observations was then added and the parameters reestimated. This procedure was repeated until December, 1988 was reached.

The implied parameters of the stochastic processes during each estimation period appear in Tables I-2A, I-2B, and I-2C for the one, two, and three state variable models, respectively. When only a single state variable (x_1=r) is employed, the long run mean, θ, of the short rate in each of the periods is approximately 6.3% (t>7). The speed of adjustment parameter, κ, has a typical magnitude of 1.7 (t~2) which implies that half of the adjustment occurs within five months. Estimated values of the standard deviation parameter, γ, associated with the Wiener component of the process are on average 3%. It should be noted that Ornstein-Uhlenbeck processes allow variables to take on negative values. This property will be desirable for modelling spread variables in multiple factor specifications, but not when the only variable is the short rate of interest. However, since the reported standard deviation is measured in annual terms and mean reversion occurs quickly, it is not unreasonable to assume that the short term rate is sufficiently above zero such that it has a negligible probability of becoming negative.

In the two state variable case (where $x_1=r-l$ and $x_2=l$), the long run means of each factor are significantly different from zero (t>5) in each period with the mean of the spread variable being negative and that of the long term rate proxy being positive as would be expected with an upward sloping yield curve. In addition, the spread variable adjusts very quickly to its long run mean having a large speed of adjustment coefficient of approximately 8 (t>3) on average, implying that half of the adjustment occurs within one month.

Note that the standard deviation associated with this variable has a magnitude similar to its long run mean, allowing the possibility of x_1 temporarily taking on a positive value and hence a downward sloping yield curve. Conversely, the second state variable reverts more slowly to its long run mean with a smaller standard deviation as would be expected since long term yields tend to be less volatile than the short end of the term structure. When three state variables are used (x_1=r-m, x_2=m-l, and x_3=l), similar comments apply to parameter estimates for the two spread variables and long term yield variable.

4. Estimation of the Bond Pricing Model

After the model of Section 2 has been operationalized and all the required Ornstein-Uhlenbeck parameter inputs have been estimated, the one, two, and three state variable versions of the model (equation (11)) can now be used to estimate the only remaining unknown parameters, the market prices of risk. Coupon rate and maturity characteristics as well as bid and ask prices for United States Treasury notes and bonds with maturities up to twenty-five years were obtained for the last trading day of each month during 1988 from the Wall Street Journal. On average, 159 bonds and notes were collected for each of the twelve estimation time periods. Callable bonds and "flower bonds" were omitted due to their call feature and estate tax advantages, respectively.

After adding accrued interest, the average of the dealers' bid and ask prices became the dependent variable during an iterative single equation non-linear estimation procedure which takes as its objective function the sum of squared residuals of the model given by equation (11). The assumption of independent and identically distributed errors will be imposed in order to make inferences regarding the parameter estimates. It should be noted that this approach estimates the market prices of risk ϕ_i at a single point in time over a cross-section of bonds, precluding arbitrage opportunities. In addition, the estimates of ϕ_i are consistent and asymptotically normal despite the two stage method employed. A proof demonstrating that, although estimates of the stochastic process parameters are required as inputs in the estimation of (11), the market price of risk estimates are consistent and asymptotically normal appears in Appendix 1.

Results of the non-linear estimation appear in Tables I-3A, I-3B, and I-3C in the cases of one, two, and three state variables, respectively. With only one state variable, an average root mean squared error(RMSE) of $0.77 and average mean absolute error(MAE) of $0.61 were achieved for estimates of bond prices based on a face value of $100. In addition, the market price of risk is significantly greater than zero, a result which remains consistent across time periods. The model implies that a positive market price of risk means that its associated state variable requires a positive risk premium. In the two state variable case, an average RMSE of $0.30 and MAE of $0.23 were obtained, an improvement on the single factor model. Again, the estimates of the market prices of risk associated with the state variables are consistently significantly greater than zero. With three state variables, only marginal improvement in RMSE ($0.27) and MAE ($0.19) was observed. Also note that the market prices of risk associated with the first two factors have extremely large magnitudes of opposite sign and t-statistics of equivalent size. This suggest a non-linear version of multicollinearity in the three variable model, and that little benefit is gained by including the third state variable.

One of the few available benchmarks to which to compare these results is the findings in Brennan and Schwartz(1982) which consider United States Treasury bonds and notes with maturities up to twenty years over the period 1958 through 1979. When their bond pricing model is applied in sample over the same period as the estimation of underlying stochastic processes, it resulted in a RMSE of $1.58, and $3.90 during out-of-sample pricing. Using Government of

Canada bonds, Brennan and Schwartz(1979) found a similar RMSE of $1.56, again during in-sample estimation. Consequently, the results obtained here seem quite promising.

In order to examine more closely the three models' pricing errors, actual bond prices were regressed on estimated values with the resulting regression statistics reported in Tables I-4A, I-4B, and I-4C, for each of the three cases considered. For unbiased predictions, the intercept term should be zero while the slope coefficient should be equal to one. As a result, t-ratios are calculated based on these hypotheses, rather than presenting t-statistics. These regression results should be treated cautiously since there is no guarantee that pricing errors are either independent or normally distributed. The single variable model produced relatively biased bond price estimates. During the first estimation period, for example, a slope coefficient of 0.965 (t-ratio of 7.23) was observed along with an intercept of $4.229 (t-ratio of 7.87). It is encouraging, however, to see a closer correspondence between actual and estimated bond prices in the two state variable case. During the first estimation period, for example, a slope coefficient of 0.997 (t-ratio of 2.20) was obtained along with an intercept of only $0.335 (t-ratio of 1.99). The results remain reasonably consistent over the remaining subperiods. Interestingly, the three state variable model yielded intercept and slope coefficients that were further away from hypothesized values in a majority of periods when compared to the two state variable specification. This provides added evidence that the addition of the third factor is unnecessary. Note also that these results appear to be an improvement upon the Brennan and Schwartz(1979) findings for this test which report a slope coefficient of 0.930 (t-ratio of approximately 10) and an intercept term of $7.28 (t-ratio of 10.46), during estimation of a sample of 101 Government of Canada bonds.

Further, to check whether the model's residuals were systematically related to bond characteristics such as coupon rate or time to maturity, bond price errors (actual minus estimated) were regressed on these two variables. Tables I-5A, I-5B, and I-5C present these regression results for the three cases with slope 1 being the coefficient for time to maturity in years and slope 2 being the coefficient for annual coupon rate in percent. The two variables explain on average 27% of the residual variance for the single state variable specification, but on average only 4% for either the two or three factor model. Also note that the two variable model's regression coefficients possess t-statistics which are smaller than 2.5 in a majority of periods, suggesting that this particular model version generates reasonably accurate bond price estimates without large systematic errors.

5. Concluding Remarks

In this paper a bond pricing model has been developed based on the arbitrage approach to the term structure of interest rates which assumed that the value of default-free bonds may be expressed as functions of underlying state variables following continuous time diffusion processes. A partial differential equation could then be derived in Section 3 governing the movement of bond prices of all maturities. After assuming orthogonal Ornstein-Uhlenbeck processes for the state variables, a closed form solution was obtained. Sections 3 and 4 involved the testing of the bond pricing model incorporating either one, two, or three state variables whose values are directly observable in highly liquid T-bill and futures markets. Due to the existence of a closed form solution, the model allowed market prices of risk to be exogenously estimated from a cross section of bond prices at a single point in time. The advantages of this approach resulted in an easily implementable model which generated promising bond price estimates without large systematic errors. In addition, two state variables appeared to be sufficient to explain the observed term structure. Given the empirical success of the model, further research will concentrate on the natural extension to callable bonds and options on bonds valuation.

Table I-1: Review of Continuous Time Term Structure Models

Model	Number of State Variables	State Variables	Solution Technique	Bond Pricing Tests
Brennan and Schwartz(1977)	1	r	Numerical	Simulation
Vasicek(1977)	1	r	Closed-Form	None
Dothan(1978)	1	r	Closed-Form	Simulation
CIR(1985)	1	r	Closed-Form	None
Longstaff(1989)	1	r	Closed-Form	USA bills
Jamshidian(1989a)	1	r	Closed-Form	None
Richard(1978)	2	r(real) E[inflation]	Closed-Form	None
CIR(1985)	2	r(real) E[inflation]	Closed-Form	None
Brennan and Schwartz(1979)	2	r,l	Numerical	Canada
Brennan and Schwartz(1980)	2	r,l	Numerical	Canada
Brennan and Schwartz(1982)	2	r,l	Numerical	USA
Brennan and Schwartz(1983b)	2	r,l	Numerical	None
Schaefer and Schwartz(1984)	2	r-l,l	Approximate Closed-Form	Simulation
Langetieg(1980)	N	Unspecified	Closed-Form	None
Oldfield and Rogalski(1987)	N	Unspecified	Closed-Form	None

Table I-2A: Case 1: One State Variable Implied Parameters for the Stochastic Processes
(t-statistics in parentheses)

Period	Variable	κ	θ	γ
1/88	x_1	1.67212	0.06313	0.03098
		(1.85)	(7.46)	
2/88	x_1	1.64053	0.06275	0.03070
		(1.84)	(7.40)	
3/88	x_1	1.60992	0.06239	0.03047
		(1.84)	(7.33)	
4/88	x_1	1.67100	0.06307	0.03032
		(1.92)	(7.86)	
5/88	x_1	1.67392	0.06312	0.03008
		(1.95)	(8.02)	
6/88	x_1	1.68320	0.06329	0.02986
		(1.98)	(8.23)	
7/88	x_1	1.69211	0.06351	0.02965
		(2.01)	(8.44)	
8/88	x_1	1.71571	0.06443	0.02967
		(2.03)	(8.78)	
9/88	x_1	1.72023	0.06429	0.02946
		(2.05)	(8.91)	
10/88	x_1	1.72495	0.06404	0.02926
		(2.07)	(9.02)	
11/88	x_1	1.72452	0.06385	0.02906
		(2.09)	(9.10)	
12/88	x_1	1.71740	0.06342	0.02889
		(2.09)	(9.10)	

Table I-2B: Case 2: Two State Variables Implied Parameters for the Stochastic Processes
(t-statistics in parentheses)

Period	Variable	κ	θ	γ
1/88	x_1	8.46337	-0.03303	0.03830
		(3.06)	(15.96)	
	x_2	0.39451	0.08648	0.01551
		(0.88)	(3.81)	
2/88	x_1	8.46066	-0.03301	0.03797
		(3.08)	(16.22)	
	x_2	0.38271	0.08564	0.01537
		(0.87)	(3.67)	
3/88	x_1	8.48561	-0.03317	0.03781
		(3.10)	(16.55)	
	x_2	0.44673	0.08958	0.01543
		(1.01)	(5.08)	
4/88	x_1	8.52186	-0.03312	0.03756
		(3.14)	(16.85)	
	x_2	0.46840	0.09094	0.01535
		(1.07)	(5.71)	
5/88	x_1	8.51805	-0.03316	0.03726
		(3.16)	(17.13)	
	x_2	0.48576	0.09213	0.01527
		(1.12)	(6.26)	
6/88	x_1	8.55553	-0.03302	0.03711
		(3.18)	(17.33)	
	x_2	0.46769	0.09054	0.01521
		(1.09)	(5.78)	
7/88	x_1	8.52991	-0.03298	0.03680
		(3.21)	(17.54)	
	x_2	0.48999	0.09196	0.01512
		(1.14)	(6.46)	
8/88	x_1	8.38049	-0.03269	0.03677
		(3.18)	(17.22)	
	x_2	0.49065	0.09201	0.01504
		(1.56)	(6.59)	
9/88	x_1	8.04279	-0.03248	0.03629
		(3.19)	(16.80)	
	x_2	0.47447	0.09064	0.01497
		(1.13)	(6.16)	

Table I-2B: Case 2: Two State Variables Implied Parameters for the Stochastic Processes (continued)
(t-statistics in parentheses)

10/88	x_1	7.73513	-0.03230	0.03580
		(3.20)	(16.43)	
	x_2	0.45211	0.08912	0.01490
		(1.08)	(5.63)	
11/88	x_1	7.71306	-0.03229	0.03551
		(3.26)	(16.63)	
	x_2	0.48143	0.09075	0.01487
		(1.16)	(6.46)	
12/88	x_1	7.71278	-0.03229	0.03526
		(3.30)	(16.88)	
	x_2	0.47586	0.09038	0.01476
		(1.16)	(6.41)	

Table I-2C: Case 3: Three State Variables
Implied Parameters for the Stochastic Processes
(t-statistics in parentheses)

Period	Variable	κ	θ	γ
1/88	x_1	8.82846	-0.02883	0.03822
		(3.07)	(14.55)	
	x_2	8.16759	-0.00419	0.00574
		(3.03)	(13.08)	
	x_3	0.39451	0.08648	0.01551
		(0.88)	(3.81)	
2/88	x_1	8.82837	-0.02882	0.03789
		(3.09)	(14.80)	
	x_2	8.03347	-0.00418	0.00568
		(3.05)	(13.06)	
	x_3	0.38271	0.08564	0.01537
		(0.87)	(3.67)	
3/88	x_1	8.82461	-0.02899	0.03770
		(3.11)	(15.07)	
	x_2	8.02893	-0.00418	0.00563
		(3.09)	(13.28)	
	x_3	0.44673	0.08958	0.01543
		(1.01)	(5.08)	
4/88	x_1	8.86455	-0.02894	0.03745
		(3.14)	(15.34)	
	x_2	8.01816	-0.00417	0.00558
		(3.12)	(13.48)	
	x_3	0.46840	0.09094	0.01535
		(1.07)	(5.71)	
5/88	x_1	8.85464	-0.02900	0.03715
		(3.17)	(15.60)	
	x_2	7.97471	-0.00416	0.00557
		(3.14)	(13.56)	
	x_3	0.48576	0.09213	0.01527
		(1.12)	(6.26)	
6/88	x_1	8.90429	-0.02888	0.03699
		(3.19)	(15.81)	
	x_2	7.87728	-0.00413	0.00549
		(3.15)	(13.55)	

Table I-2C: Case 3: Three State Variables Implied Parameters for the Stochastic Processes (continued)
(t-statistics in parentheses)

	x_3	0.46769	0.09054	0.01521
		(1.09)	(5.78)	
7/88	x_1	8.88337	-0.02884	0.03669
		(3.22)	(16.00)	
	x_2	7.88954	-0.00414	0.00545
		(3.19)	(13.79)	
	x_3	0.48999	0.09196	0.01512
		(1.14)	(6.46)	
8/88	x_1	8.75735	-0.02858	0.03664
		(3.20)	(15.78)	
	x_2	7.81889	-0.00411	0.00541
		(3.20)	(13.78)	
	x_3	0.49065	0.09201	0.01504
		(1.56)	(6.59)	
9/88	x_1	8.42858	-0.02839	0.03618
		(3.20)	(15.42)	
	x_2	7.77734	-0.00411	0.00537
		(3.25)	(13.82)	
	x_3	0.47447	0.09064	0.01497
		(1.13)	(6.16)	
10/88	x_1	8.15953	-0.02824	0.03571
		(3.23)	(15.17)	
	x_2	7.63523	-0.00407	0.00533
		(3.25)	(13.65)	
	x_3	0.45211	0.08912	0.01490
		(1.08)	(5.63)	
11/88	x_1	8.17885	-0.02825	0.03547
		(3.29)	(15.43)	
	x_2	7.37313	-0.00404	0.00528
		(3.23)	(13.31)	
	x_3	0.48143	0.09075	0.01487
		(1.16)	(6.46)	
12/88	x_1	8.23159	-0.02831	0.03530
		(3.33)	(15.75)	
	x_2	6.64417	-0.00395	0.00527
		(3.06)	(11.83)	
	x_3	0.47586	0.09038	0.01476
		(1.16)	(6.41)	

Table I-3A: Case 1: One State Variable
Non-Linear Estimation of Bond Prices
(t-statistics in parentheses)

Period	Obs	RMSE	MAE	ME	ϕ_1	R^2
1/88	159	0.927	0.745	0.371	1.137 (121.20)	0.9937
2/88	159	1.089	0.860	0.437	1.104 (101.84)	0.9907
3/88	159	1.018	0.798	0.384	1.355 (128.86)	0.9910
4/88	159	1.066	0.851	0.449	1.490 (127.07)	0.9896
5/88	159	0.752	0.573	0.244	1.669 (195.30)	0.9935
6/88	159	0.747	0.565	0.253	1.464 (174.39)	0.9946
7/88	159	0.673	0.496	0.170	1.637 (207.78)	0.9954
8/88	160	0.455	0.328	0.105	1.621 (299.59)	0.9977
9/88	158	0.526	0.400	0.104	1.452 (234.48)	0.9973
10/88	159	0.525	0.419	0.067	1.345 (219.11)	0.9976
11/88	160	0.470	0.386	-0.244	1.652 (288.60)	0.9975
12/88	158	0.954	0.862	-0.584	1.772 (149.33)	0.9906

Table I-3B: Case 2: Two State Variables
Non-Linear Estimation of Bond Prices
(t-statistics in parentheses)

Period	Obs	RMSE	MAE	ME	ϕ_1	ϕ_2	R^2
1/88	159	0.228	0.177	-0.032	2.728	0.645	0.9996
					(54.61)	(75.79)	
2/88	159	0.322	0.245	-0.040	2.399	0.696	0.9992
					(34.40)	(59.67)	
3/88	159	0.359	0.255	-0.039	1.649	0.906	0.9989
					(18.83)	(56.03)	
4/88	159	0.331	0.248	-0.032	1.584	0.994	0.9990
					(18.47)	(60.88)	
5/88	159	0.305	0.225	-0.029	2.624	0.885	0.9989
					(31.91)	(55.12)	
6/88	159	0.309	0.245	-0.034	3.183	0.705	0.9991
					(39.44)	(46.32)	
7/88	159	0.359	0.273	-0.038	3.451	0.729	0.9987
					(34.81)	(37.60)	
8/88	160	0.260	0.184	-0.023	4.650	0.510	0.9992
					(65.55)	(35.89)	
9/88	158	0.329	0.254	-0.038	4.548	0.427	0.9990
					(53.39)	(24.97)	
10/88	159	0.338	0.282	-0.051	4.689	0.340	0.9990
					(57.19)	(20.77)	
11/88	160	0.248	0.189	-0.037	6.187	0.143	0.9993
					(96.88)	(10.73)	
12/88	158	0.245	0.158	-0.023	8.268	0.210	0.9994
					(129.54)	(15.91)	

Table I-3C: Case 3: Three State Variables
Non-Linear Estimation of Bond Prices
(t-statistics in parentheses)

Period	Obs	RMSE	MAE	ME	ϕ_1	ϕ_2	ϕ_3	R^2
1/88	159	0.195	0.141	0.007	101.994	-614.989	0.742	0.9997
					(7.74)	(7.52)	(50.96)	
2/88	159	0.283	0.198	0.011	107.995	-646.251	0.815	0.9994
					(6.87)	(6.71)	(39.97)	
3/88	159	0.322	0.211	0.011	117.617	-713.081	1.070	0.9991
					(6.18)	(6.09)	(35.30)	
4/88	159	0.299	0.218	0.012	103.488	-624.642	1.157	0.9992
					(6.02)	(5.92)	(37.32)	
5/88	159	0.274	0.193	0.012	95.532	-564.155	1.050	0.9991
					(6.19)	(6.02)	(34.19)	
6/88	159	0.273	0.208	0.011	88.187	-512.642	0.871	0.9993
					(6.81)	(6.56)	(30.69)	
7/88	159	0.321	0.230	0.010	103.975	-608.203	0.933	0.9989
					(6.42)	(6.19)	(25.23)	
8/88	160	0.239	0.166	0.008	70.891	-404.350	0.643	0.9994
					(5.71)	(5.31)	(23.02)	
9/88	158	0.289	0.214	0.010	139.134	-843.747	0.618	0.9992
					(6.93)	(6.69)	(19.33)	
10/88	159	0.274	0.211	0.010	197.539	-1216.410	0.560	0.9994
					(9.12)	(8.89)	(20.10)	
11/88	160	0.210	0.146	0.003	89.479	-509.066	0.313	0.9995
					(8.24)	(7.64)	(12.92)	
12/88	158	0.240	0.143	-0.008	20.496	-65.370	-0.148	0.9994
					(3.53)	(2.03)	(5.29)	

Table I-4A: Case 1: One State Variable
Regression of Actual Prices on Estimated Prices
(t-ratios in parentheses)

Period	Intercept	Slope	R^2
1/88	4.229	0.965	0.9960
	(7.87)	(7.23)	
2/88	5.132	0.957	0.9942
	(8.03)	(7.39)	
3/88	4.885	0.958	0.9941
	(7.64)	(7.08)	
4/88	5.390	0.954	0.9937
	(8.23)	(7.59)	
5/88	3.969	0.965	0.9955
	(7.26)	(6.84)	
6/88	3.627	0.968	0.9963
	(7.15)	(6.69)	
7/88	2.974	0.974	0.9964
	(5.98)	(5.66)	
8/88	2.331	0.979	0.9983
	(6.78)	(6.50)	
9/88	2.379	0.979	0.9979
	(6.19)	(5.95)	
10/88	1.992	0.982	0.9980
	(5.25)	(5.10)	
11/88	-0.996	1.007	0.9982
	(2.80)	(2.12)	
12/88	-5.127	1.043	0.9958
	(8.93)	(7.95)	

Table I-4B: Case 2: Two State Variables Regression of Actual Prices on Estimated Prices
(t-ratios in parentheses)

Period	Intercept	Slope	R^2
1/88	0.335	0.997	0.9996
	(1.99)	(2.20)	
2/88	0.550	0.995	0.9992
	(2.26)	(2.44)	
3/88	0.825	0.992	0.9990
	(2.95)	(3.11)	
4/88	0.832	0.992	0.9991
	(3.17)	(3.30)	
5/88	0.924	0.991	0.9990
	(3.52)	(3.65)	
6/88	0.597	0.994	0.9991
	(2.36)	(2.51)	
7/88	0.557	0.994	0.9987
	(1.84)	(1.98)	
8/88	0.737	0.993	0.9993
	(3.32)	(3.44)	
9/88	0.724	0.993	0.9990
	(2.71)	(2.87)	
10/88	0.516	0.995	0.9991
	(1.97)	(2.17)	
11/88	0.289	0.997	0.9993
	(1.34)	(1.52)	
12/88	-0.643	1.006	0.9994
	(3.15)	(3.05)	

Table I-4C: Case 3: Three State Variables
Regression of Actual Prices on Estimated Prices
(t-ratios in parentheses)

Period	Intercept	Slope	R^2
1/88	0.498	0.996	0.9997
	(3.53)	(3.50)	
2/88	0.748	0.993	0.9994
	(3.55)	(3.52)	
3/88	0.981	0.991	0.9992
	(3.97)	(3.42)	
4/88	0.925	0.992	0.9993
	(3.94)	(3.87)	
5/88	0.936	0.991	0.9992
	(3.99)	(3.95)	
6/88	0.687	0.994	0.9993
	(3.09)	(3.06)	
7/88	0.599	0.994	0.9990
	(2.21)	(2.18)	
8/88	0.718	0.993	0.9994
	(3.51)	(3.49)	
9/88	0.793	0.993	0.9992
	(3.39)	(3.36)	
10/88	0.644	0.994	0.9994
	(3.04)	(3.01)	
11/88	0.274	0.997	0.9995
	(1.49)	(1.48)	
12/88	-0.648	1.006	0.9994
	(3.24)	(3.21)	

Table I-5A: Case 1: One State Variable
Regression of Pricing Errors on Time to Maturity(slope 1) and Coupon Rate(slope 2)
(t-statistics in parentheses)

Period	Intercept	Slope1	Slope2	R^2
1/88	1.089	-0.098	-0.029	0.286
	(4.63)	(7.20)	(1.18)	
2/88	1.288	-0.117	-0.034	0.287
	(4.70)	(7.24)	(1.19)	
3/88	1.281	-0.104	-0.045	0.274
	(4.92)	(6.75)	(1.65)	
4/88	1.319	-0.126	-0.032	0.345
	(5.17)	(8.30)	(1.22)	
5/88	0.885	-0.073	-0.032	0.235
	(4.37)	(6.07)	(1.53)	
6/88	0.826	-0.075	-0.025	0.233
	(4.10)	(6.17)	(1.20)	
7/88	0.604	-0.055	-0.020	0.147
	(3.03)	(4.60)	(0.96)	
8/88	0.479	-0.032	-0.024	0.133
	(3.49)	(3.93)	(1.67)	
9/88	0.537	-0.0031	-0.031	0.105
	(3.28)	(3.21)	(1.78)	
10/88	0.410	-0.022	-0.026	0.055
	(2.36)	(2.12)	(1.41)	
11/88	-0.393	0.064	-0.014	0.412
	(3.71)	(10.31)	(1.22)	
12/88	-1.424	0.158	0.016	0.776
	(11.45)	(21.69)	(1.22)	

Table I-5B: Case 2: Two State Variables Regression of Pricing Errors on Time to Maturity(slope 1) and Coupon Rate(slope 2)
(t-statistics in parentheses)

Period	Intercept	Slope1	Slope2	R^2
1/88	0.138	0.010	-0.022	0.072
	(1.94)	(2.50)	(3.02)	
2/88	0.150	0.012	-0.025	0.047
	(1.48)	(1.98)	(2.38)	
3/88	0.270	0.013	-0.038	0.070
	(2.42)	(1.93)	(3.25)	
4/88	0.191	0.009	-0.027	0.041
	(1.81)	(1.43)	(2.47)	
5/88	0.174	0.008	-0.025	0.042
	(1.80)	(1.46)	(2.47)	
6/88	0.113	0.008	-0.019	0.026
	(1.13)	(1.37)	(1.82)	
7/88	0.067	0.008	-0.015	0.014
	(0.57)	(1.21)	(1.20)	
8/88	0.129	0.008	-0.019	0.035
	(1.52)	(1.55)	(2.17)	
9/88	0.158	0.011	-0.026	0.041
	(1.47)	(1.80)	(2.26)	
10/88	0.083	0.014	-0.020	0.037
	(0.74)	(2.18)	(1.72)	
11/88	-0.018	0.009	-0.006	0.021
	(0.22)	(1.84)	(0.69)	
12/88	-0.243	0.001	0.022	0.047
	(2.94)	(0.23)	(2.56)	

Table I-5C: Case 3: Three State Variables Regression of Pricing Errors on Time to Maturity(slope 1) and Coupon Rate(slope 2) (t-statistics in parentheses)

Period	Intercept	Slope1	Slope2	R^2
1/88	0.172	0.002	-0.018	0.047
	(2.77)	(0.48)	(2.76)	
2/88	0.194	0.001	-0.019	0.027
	(2.14)	(0.05)	(2.01)	
3/88	0.311	0.002	-0.032	0.057
	(3.08)	(0.31)	(3.02)	
4/88	0.222	-0.001	-0.021	0.032
	(2.32)	(0.11)	(2.14)	
5/88	0.207	-0.001	-0.020	0.033
	(2.36)	(0.09)	(2.19)	
6/88	0.150	-0.002	-0.013	0.017
	(1.69)	(0.36)	(1.45)	
7/88	0.102	-0.003	-0.008	0.006
	(0.96)	(0.40)	(0.75)	
8/88	0.152	0.001	-0.015	0.022
	(1.94)	(0.13)	(1.84)	
9/88	0.200	0.001	-0.020	0.026
	(2.09)	(0.06)	(1.97)	
10/88	0.125	-0.001	-0.012	0.010
	(1.34)	(0.06)	(1.19)	
11/88	0.007	-0.001	-0.000	0.000
	(0.09)	(0.18)	(0.01)	
12/88	-0.233	-0.003	0.025	0.051
	(2.87)	(0.53)	(2.85)	

CHAPTER II: Multifactor Callable Treasury Bond Pricing

1. Introduction

Some United States Treasury bond issues have provisions which allow them to be called during a specified period, usually beginning five years prior to maturity and ending at the maturity date. This means that at any scheduled coupon payment date during this period, the Treasury has the right to force the investor to sell the bonds back to the government at par provided four months notice was given. In order to accurately price these securities, a call option model consistent with an underlying bond pricing model is required. In Chapter I, successful replication of observed yield curves via the multiple state variable bond pricing model derived by Langetieg(1980) was demonstrated. This suggests the need for a compatible multiple factor option formula which is capable of valuing the call feature on coupon bearing Treasury bonds.

The most common approach to pricing debt options derives from equilibrium theories of the term structure. Whether based on one or two state variables, early option models such as those of Courtadon(1982), Brennan and Schwartz(1983c), and Dietrich-Campbell and Schwartz(1986) relied on numerical methods in order to obtain price estimates for options on coupon paying bonds. In cases where a closed form solution has been found, the models encounter one of two problems. Jamshidian(1989b) developed an exact coupon bond option formula which is based on the Vasicek(1977) single state variable term structure model. Single state variable models possess the undesirable property that returns on bonds of all maturities are perfectly correlated. In addition, the Vasicek(1977) model performed poorly relative to multiple state variable models in empirical tests as reported in Chapter I. Other debt option models such as that of Heath, Jarrow, and Morton(1987) allow for multiple state variables but are restrictive in the sense that only options on discount bonds may be priced.

This paper proposes to value callable bonds by deriving a multiple state variable formula for a European call on a coupon bond using the Langetieg(1980) model as the underlying term structure theory. The option pricing model relies on a result by Merton(1973) who extended the Black-Scholes model to accommodate a stochastic term structure where the price of a default-free discount bond follows a diffusion process such that the bond's price is unity at maturity, the instantaneous volatility of the bond's return is at most a deterministic function of time, and that this volatility is zero at maturity. Within this framework, a closed-form solution for a European call option on a stock was obtained. However, the optioned asset could itself be another bond. Although the application of Merton's model in the case of a discount bond is relatively straightforward, the valuation of options on coupon bonds introduces added complexity.

Section 2 will briefly outline the underlying bond pricing model and related estimation issues. The multiple state variable European call option formula for coupon bonds is proposed in section 3. The fourth section discusses the specific application of the option and bond models to price callable Treasury bonds, while empirical results are presented in section 5. Concluding remarks appear in section 6.

2. The Underlying Bond Pricing Model

The model assumes that the price at time t of a default-free discount bond paying $1 at maturity time T can be expressed as a function of a vector of n state variables $x=(x_1,...,x_n)$ and time to maturity, denoted B(t,T,x). This function defines the entire term structure of interest rates with particular values of B(t,T,x) representing present value factors that can be used to price any pattern of future payments. Using standard arbitrage arguments, a partial differential equation governing the price movement of discount bonds of all maturities can be derived. In the companion paper, a solution to this differential equation was obtained under the following assumptions:

1) The state variables follow stationary

Ornstein-Uhlenbeck processes (mean reverting with constant variance) of the form:

$$dx_i = \kappa_i(\theta_i - x_i)dt + \gamma_i dZ_i \qquad i=1,\ldots,n$$

where $\kappa_i>0$ is the speed of adjustment, θ_i is the long-run mean of state variable x_i, γ_i is the constant standard deviation of the stochastic process, $E[dZ_i]=0$, $dZ_i^2=dt$, and $dZ_i dZ_j=\rho_{ij}dt$;

2) the state variables are mutually orthogonal implying $\rho_{ij}=0$ for all $i \neq j$;

3) the state variables are constructed such that they sum to the instantaneous riskfree rate of interest,

$r=\sum_{i=1,n} x_i$; and

4) the market prices of risk, ϕ_i, $i=1,\ldots,n$ are constants.

The value of a discount bond with maturity date T_j may then be written:

$$B(t,T_j,x) = \text{EXP}[\lambda(t,T_j)+\sum_{i=1,n} L_i(t,T_j)x_i] \qquad (1)$$

where $L_i(t,T_j)=-(1/\kappa_i)(1-\text{EXP}(-\kappa_i(T_j-t)))$

$$\lambda(t,T_j)=\sum_{i=1,n}(1/\kappa_i)[1-\text{EXP}(-\kappa_i(T_j-t))-(T_j-t)]$$
$$[\theta_i-(\gamma_i^2/2\kappa_i^2)+(\gamma_i\phi_i/\kappa_i)]$$

$$-\sum_{i=1,n}(\gamma_i^2/4\kappa_i^3)(1-EXP(-\kappa_i(T_j-t))^2.$$

This formulation is a special case of the Langetieg(1980) result, but was obtained via a much simpler separation of variables technique by assuming orthogonal state variables. As mentioned earlier, the function $B(t,T,x)$ completely describes the term structure of interest rates. Consequently, the price of any coupon bearing default-free note or bond can be expressed as the sum of its cash flows at times T_j discounted by their associated present value factors $B(t,T_j,x)$. Denoting both the remaining semi-annual coupon payments and final principal payment at times T_j as cash flows C_j, allows the current invoice price $BP(t,T,X)$ of a coupon bond with maturity at time T, to be written as the quoted price $QP(t,T,x)$ plus accrued interest $AI(t)$:

$$BP(t,T,x) = QP(t,T,x) + AI(t) = \sum_{t<Tj\leq T} C_j B(t,T_j,x) \quad (2)$$

where $AI(t)=(t+0.5-MIN(T_j))COUPON$. The function $AI(t)$ represents the interest that has accrued on the security since the last semi-annual coupon payment which is added to the quoted price the buyer agrees to pay the seller. The invoice or actual price paid for the bond equals the present value of the coupon and principal cash flows.

In Chapter I, successful replication of observed bond prices using a two state variable specification of the bond pricing model outlined above was demonstrated. The two state variables used were constructed from an instantaneous riskfree rate (r) and a long term yield(l), proxied by the yield on the Treasury bill with maturity closest to one month and the yield implied by the nearest maturity 8% coupon twenty year Treasury bond futures price, respectively. Both proxies are directly observable in highly liquid T-bill and T-bond futures markets. Specifically, the state variables were the spread between the instantaneous riskless rate and the long term yield (x_1=r-l) and the long term yield itself (x_2=l). Parameter estimates for the Ornstein-Uhlenbeck processes governing the state variables appear in Table I-2B of Chapter I based on a time series of observations collected monthly from December, 1982 through December, 1988 from the Wall Street Journal. Further, using data for Treasury notes and bonds of all maturities collected on the last trading day of each month during 1988, estimates of the market prices of risk associated with the two state variables were obtained and appear in Table I-3B of Chapter I. Due to the existence of a closed form solution, the model allows the market prices of risk to be estimated from a cross section of bond prices at a single point in time without the need to specify their functional forms. These parameter estimates are needed as inputs for the calculation of both the straight bond and option components of callable Treasury bonds.

3. A Multifactor Coupon Bond Option Formulation

Along with standard frictionless market assumptions, the Merton stochastic term structure option model requires a discount bond (representing the present value of the exercise price) whose instantaneous return may be described by:

$$dB/B = \alpha_B dt + \sigma_B dz_B. \qquad (3)$$

The instantaneous expected return α_B may be stochastic through dependence on the level of bond prices and different for different maturities. However, σ_B is assumed to be non-stochastic, independent of the bond price, and of a form such that it takes on a value of zero at its maturity time τ. Applying Ito's lemma to the expression for a discount bond with maturity τ (to match the call option maturity), $B(t,\tau,x)$, yields the following parameters

$$\alpha_B = \lambda_t(t,\tau) + \sum_{i=1,n} \left[L_{it}(t,\tau) x_i + L_i(t,\tau) \kappa_i (\theta_i - x_i) + L_i^2(t,\tau)\gamma_i^2 \right]$$

and

$$\sigma_B = \mathrm{SQRT}\left[\sum_{i=1,n} \left[L_i(t,\tau)\gamma_i \right]^2 \right],$$

which satisfy the requirements of the model.

Similarly, the stock or optioned asset (in this case a coupon bond) must have instantaneous return dynamics given by

$$dBP/BP = \alpha_{BP}dt + \sigma_{BP}dz_{BP} \qquad (4)$$

where α_{BP} may be a stochastic variable of quite general type including being dependent on the level of the asset's price or the returns of other assets. The standard deviation σ_{BP}, however, is restricted to be non-stochastic and, at most, a known function of time. Applying Ito's lemma to the coupon bond price after first adjusting for accrued interest,

$$BP(t,T,x) = \sum_j C_j B(t,T_j,x),$$

yields the following parameters:

$$\alpha_{BP} = \left[\sum_j C_j B(t,T_j,x)\left[\lambda_t(t,T_j) + \sum_{i=1,n}\left[L_{it}(t,T_j)x_i + L_i(t,T_j)\kappa_i(\theta_i - x_i) + L_i^2(t,T_j)\gamma_i^2\right]\right]\right]/BP(t,T,x)$$

and

$$\sigma_{BP} = SQRT\left[\sum_{i=1,n}\left[\left[\sum_j L_i(t,T_j)C_j B(t,T_j,x)/BP(t,T_j,x)\right]\gamma_i\right]^2\right].$$

Also note that the covariance between the processes dB/B and dBP/BP is:

$$\sigma_{BP,B} = \sum_{i=1,n} L_i(t,\tau)\left[\sum_j L_i(t,T_j)C_j B(t,T_j,x)/BP(t,T_j,x)\right]\gamma_i^2$$

Requirements for the drift term α_{BP} are directly satisfied in the case of a coupon bearing bond. Although expressions for the total bond price and individual cash flow prices appear in the standard deviation formula, this should not be considered a violation of Merton's assumptions. A justification can be made by considering the precise formulation of σ_{BP}. Deterministic functions of time $L_i(t,T_j)$ associated with each cash flow are weighted by the proportion of that cash flow's value to the total bond price. Because both the numerator and denominator of the weights will have highly correlated stochastic behaviour through similar dependence on the n stochastic variables x_i, the weights themselves and hence σ_{BP} should be stochastic only to a second order degree. Otherwise, the weights themselves are also functions of time. Note also that the variable σ_{BP} possesses the desirable property of decreasing to zero at the bond's maturity as well as behaving in a manner resembling duration.

Finally, direct substitution into Merton's framework results in the following specification for a European call option at time t, with strike price K, maturity time τ, and written on a coupon bond $BP(t,T,x)$, given by:

$$C(BP(t,T,x),B(t,\tau,x),t;\tau,K,v^2) = BP(t,T,x)N(h_1)-KB(t,\tau,x)N(h_2) \quad (5)$$

where $h_1 = [\ln(BP(t,T,x)/KB(t,\tau,x)) + (v^2/2)(\tau-t)]/v(\tau-t)^{1/2}$

$h_2 = h_1 - v(\tau-t)^{1/2}$

$v^2 = \sigma^2_{BP} + \sigma^2_{B} - 2\sigma_{BP,B}$

$N(\cdot)$ = the cumulative normal distribution function.

4. An Application to Callable Treasury Bonds

In general, the price of a callable bond will equal the value of a straight bond with equivalent coupon and maturity characteristics less the value of the call feature. While the price of the straight bond portion can be obtained using the bond model outlined in section 2, the specific nature of the Treasury bond call feature requires further discussion. In particular, the appropriate optioned asset under consideration and the method of handling multiple discrete call dates must be determined.

Because analysis is within a European option framework, the present value of intermediate coupon payments prior to the call maturity can be disregarded and not considered when pricing the option. This is because the optioned asset consists of only the remaining coupons and principal that may be exchanged at par on a particular exercise date. A direct analogy is the use of a stock price series adjusted by the present value of known dividends when valuing a European option on a dividend paying stock.

Recall that a Treasury bond may be called at any scheduled coupon payment date during the last five years of its thirty year maturity. Consequently, the option component involves multiple discrete call dates. The pseudo-American call option approach, first proposed by Black(1975), will serve as a first approximation for modelling the call value component in the above situation. This approach involves calculating the European option value at each possible exercise date and selecting the largest of these values as the approximation for the Treasury bond call feature. Intuitively, the pseudo-American call method is particularly appropriate for modelling callable bonds. Consider a high coupon callable bond selling at a premium. As coupons are paid over time, the bonds price will decline, approaching its face value from above. One would expect that the bond would most likely be called at the first available exercise date and that this option value will tend to dominate the value of the call feature. Alternatively, the price of a low coupon callable bond currently selling at a discount will appreciate over time, approaching the face value from below. The call feature on these bonds will typically be of small or negligible value. In fact, such a dichotomy in coupon rates can be observed in currently outstanding callable issues, providing an empirical as well as theoretical justification for the pseudo-American approximation. Consequently, with the closed form bond and European bond option models outlined above, the valuation of callable Treasury bonds becomes straightforward.

5. Empirical Results

Coupon rate and maturity characteristics as well as bid and ask prices for callable United States Treasury bonds were obtained for the last trading day of each month during 1988 from the Wall Street Journal. After o itting callable bonds with estate tax privileges, twenty-two bond observations with an average maturity of twenty years were collected in each of the twelve estimation periods. Using parameter estimates obtained in the companion paper, callable bond price estimates were calculated based on the estimate of the straight bond component (equation (2)) less the estimate of the pseudo-American call value (equation (5)). The resulting callable bond price estimates could then be compared to the average of dealers' bid and ask prices after adjusting for accrued interest. Summary statistics for this estimation procedure are reported in Table II-1 for each of the twelve estimation periods during 1988. An average root mean squared error (RMSE) of $0.54 and average mean absolute error (MAE) of $0.43 were achieved for bond prices based on a face value of $100. It is interesting to note that the Brennan and Schwartz(1982) two state variable model when applied to Treasury securities yielded a RMSE of $1.58 in sample. When the same framework was applied to just short term call options on bonds in Dietrich-Campbell and Schwartz(1986), a

RMSE of $0.67 was reported.

In order to examine more closely the pricing errors, actual callable bond prices were regressed on estimated values with the resulting regression statistics appearing in Table II-2. For unbiased predictions, the intercept term should be zero while the slope coefficient should be equal to one. As a result, t-ratios are calculated based on these hypotheses rather than presenting t-statistics. Although the actual pricing errors are small in magnitude, the callable bond price estimates typically reject these hypotheses at a 1% level of significance. During the first estimation period, for example, a slope coefficient of 0.988 (t-ratio of 3.10) was obtained along with an intercept of $1.85 (t-ratio of 3.85). Unfortunately, there are no benchmark results in the literature against which to compare the callable bond price estimates. However, to check whether the residuals were systematically related to bond characteristics such as coupon rate or time to maturity, bond price errors (actual minus estimated) were regressed on these two variables. Table II-3 presents these regression results with slope 1 being the coefficient for time to maturity in years and slope 2 being the coefficient for annual coupon rate in percent. The regressions explain, on average, 50% of the residual variance. However, only the intercept term is significantly greater than zero at a 1% level in most periods while the slope coefficients have small negative values.

Finally, to get an impression of how the callable bond pricing errors appear relative to the straight note and bond errors, these are plotted simultaneously in Figures II-1 through II-12 for each of the 12 monthly estimation periods during 1988. Callable bond pricing errors are small in magnitude and appear indistinguishable from note and bond pricing errors suggesting that the that the pseudo-American call approach gives a suitable approximation of the call feature.

6. Concluding Remarks

The pricing of callable Treasury bonds requires a unified and consistent approach to valuing the straight bond and call option components of the security. An accurate bond pricing model is particularly important since errors in this portion will tend to dominate those of the option. Given the empirical success of the multiple state variable bond pricing model of section 2, a compatible multiple state variable coupon bond option formula was proposed. A simultaneous application of the two models allowed the prices of callable Treasury bonds to be estimated with promising results. Because a joint test of this type has not yet appeared in the literature, the results obtained should be of some interest.

Table II-1: Callable Treasury Bond Pricing: Error Analysis

Period	Obs	ME	MAE	RMSE
1/88	22	0.380	0.439	0.593
2/88	22	0.199	0.407	0.572
3/88	22	-0.065	0.390	0.460
4/88	22	-0.157	0.450	0.542
5/88	22	-0.247	0.488	0.603
6/88	22	-0.129	0.595	0.743
7/88	22	-0.272	0.485	0.570
8/88	22	-0.129	0.347	0.411
9/88	22	0.003	0.513	0.623
10/88	22	0.051	0.435	0.613
11/88	22	0.008	0.249	0.320
12/88	22	0.164	0.323	0.388

Table II-2: Callable Treasury Bond Pricing Regression of Actual Prices on Estimated Prices
(t-ratios in parentheses)

Period	Intercept	Slope	R^2
1/88	1.847	0.988	0.9997
	(3.85)	(3.10)	
2/88	2.090	0.984	0.9996
	(4.00)	(3.68)	
3/88	1.704	0.985	0.9997
	(4.10)	(4.32)	
4/88	1.961	0.981	0.9997
	(4.35)	(4.77)	
5/88	2.222	0.977	0.9996
	(4.90)	(5.52)	
6/88	2.587	0.976	0.9991
	(3.53)	(3.76)	
7/88	1.626	0.983	0.9996
	(3.12)	(3.69)	
8/88	1.379	0.986	0.9998
	(3.77)	(4.19)	
9/88	2.743	0.976	0.9996
	(5.51)	(5.62)	
10/88	2.172	0.982	0.9994
	(3.57)	(3.53)	
11/88	0.629	0.994	0.9997
	(1.58)	(1.59)	
12/88	0.111	1.001	0.9997
	(0.24)	(0.12)	

Table II-3: Callable Treasury Bond Pricing
Regression of Pricing Errors on Time to Maturity(slope 1) and Coupon Rate(slope 2)
(t-statistics in parentheses)

Period	Intercept	Slope1	Slope2	R^2
1/88	1.576	-0.030	-0.058	0.3548
	(4.05)	(1.20)	(0.98)	
2/88	1.872	-0.063	-0.040	0.5614
	(4.95)	(2.57)	(0.69)	
3/88	1.481	-0.036	-0.081	0.5878
	(4.76)	(1.80)	(1.72)	
4/88	1.694	-0.033	-0.118	0.6317
	(5.06)	(1.51)	(2.34)	
5/88	1.699	-0.005	-0.182	0.6152
	(4.69)	(0.23)	(3.33)	
6/88	2.237	-0.082	-0.073	0.5950
	(4.52)	(2.55)	(0.98)	
7/88	1.313	-0.044	-0.071	0.5901
	(3.82)	(2.31)	(1.44)	
8/88	1.142	-0.019	-0.089	0.5271
	(4.01)	(1.03)	(2.06)	
9/88	2.309	-0.049	-0.133	0.7046
	(6.43)	(2.11)	(2.44)	
10/88	1.885	-0.083	-0.021	0.5946
	(4.57)	(3.09)	(0.34)	
11/88	0.438	-0.019	-0.007	0.1162
	(1.37)	(0.89)	(0.15)	
12/88	0.108	-0.036	0.074	0.1177
	(0.31)	(1.57)	(1.40)	

FIGURE II-1 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of January, 1988

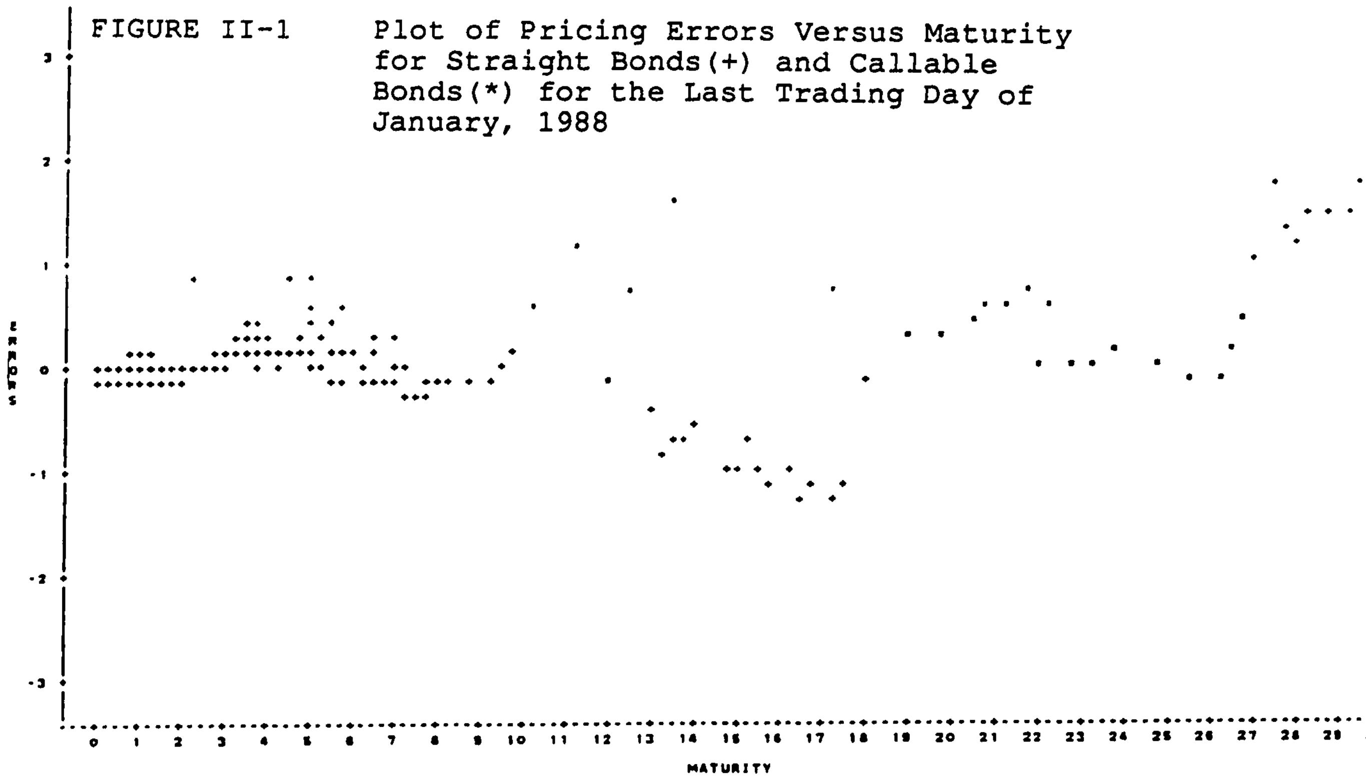


FIGURE II-2 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of February, 1988

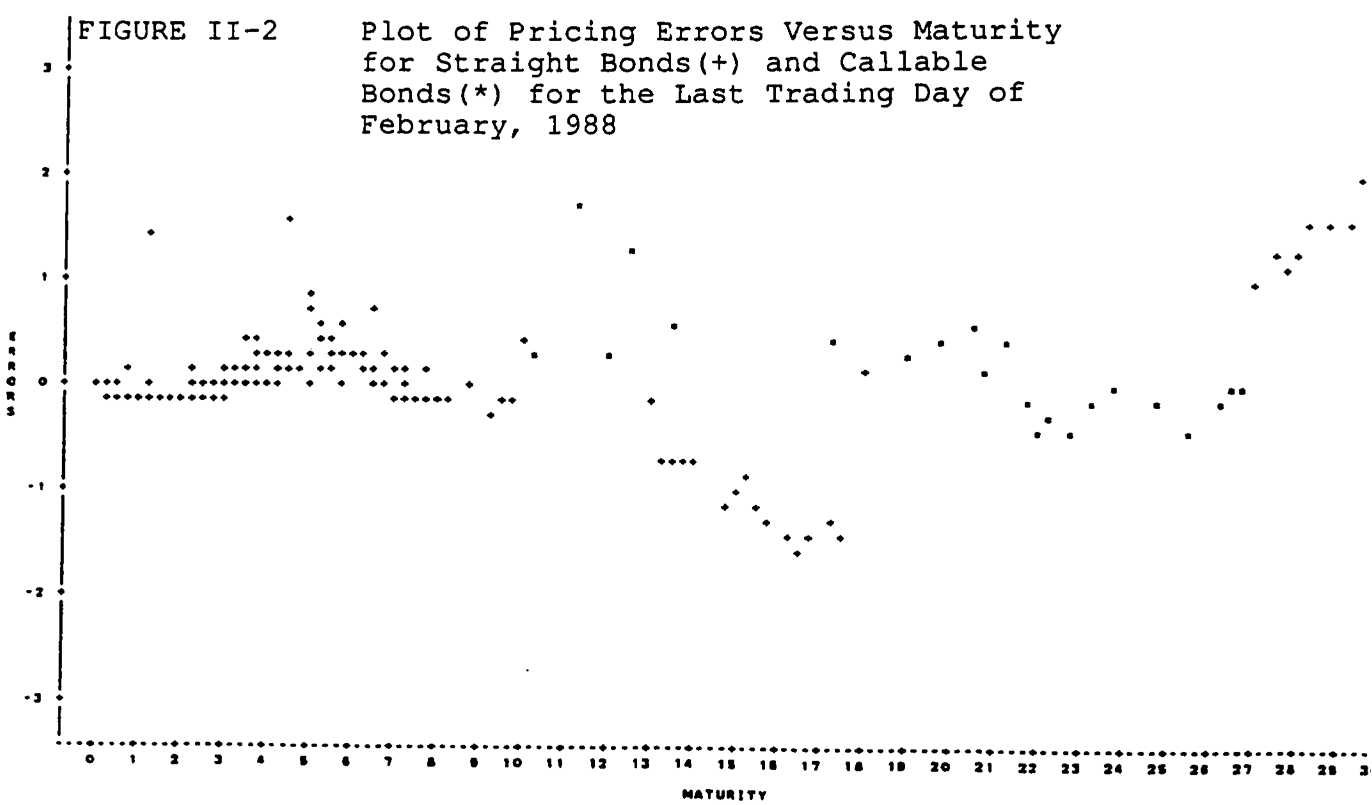


FIGURE II-3 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of March, 1988

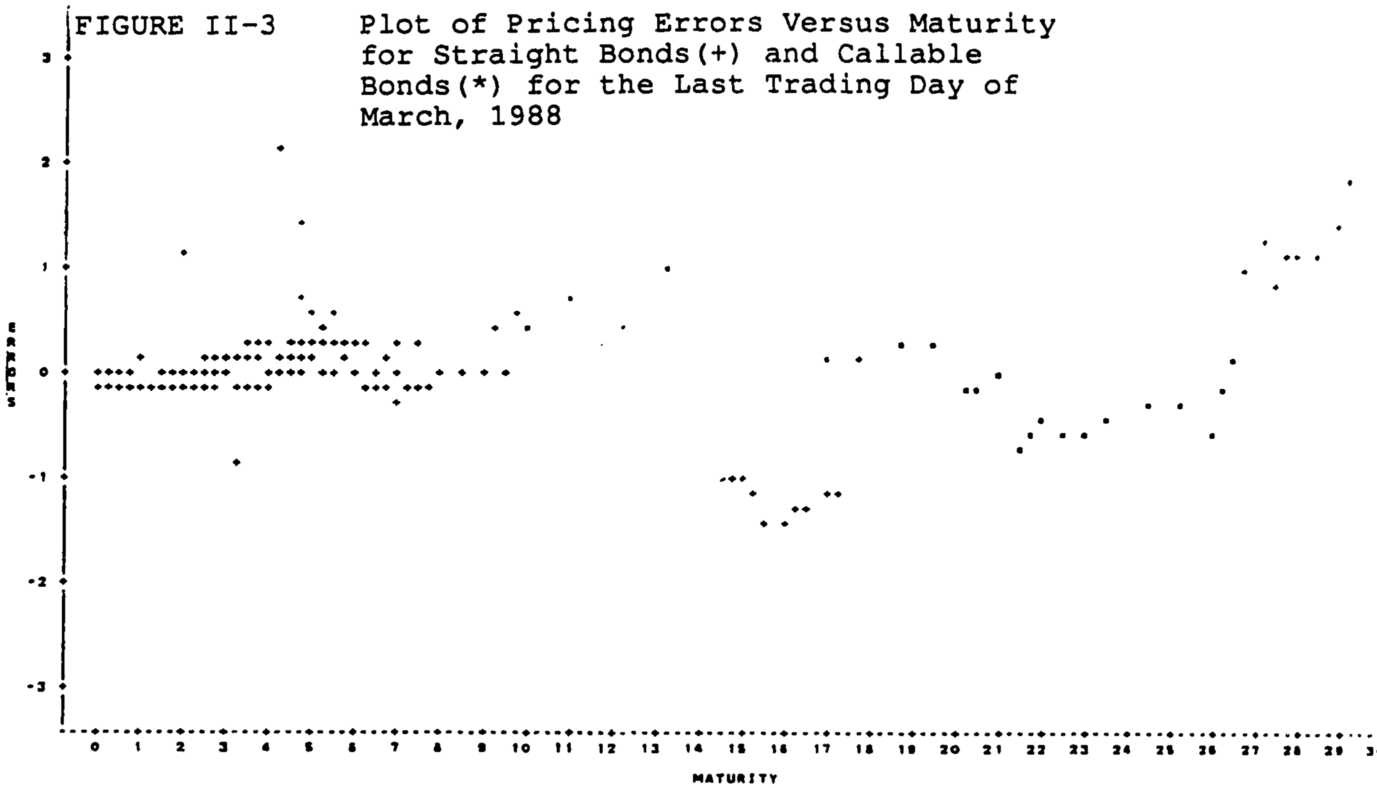


FIGURE II-4 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of April, 1988

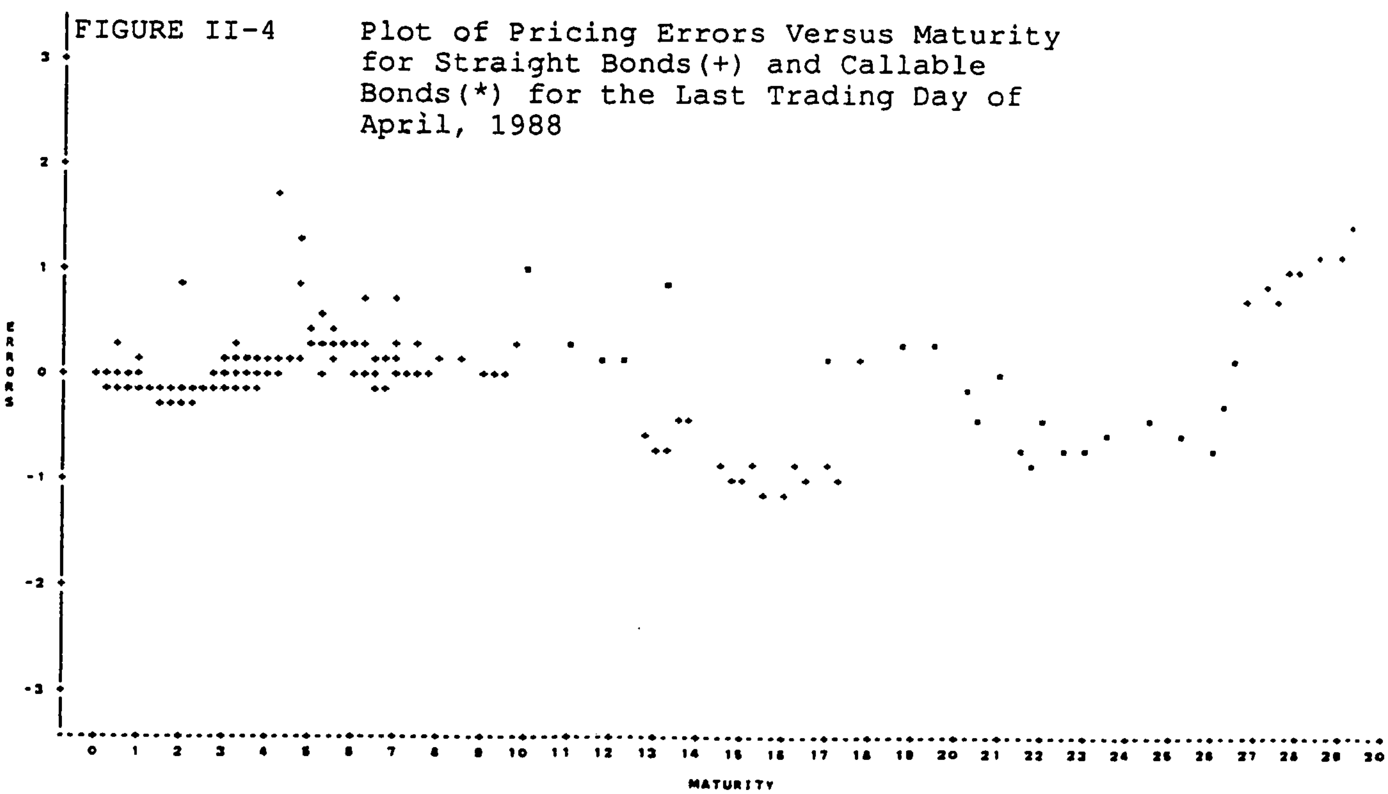


FIGURE II-5 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of May, 1988

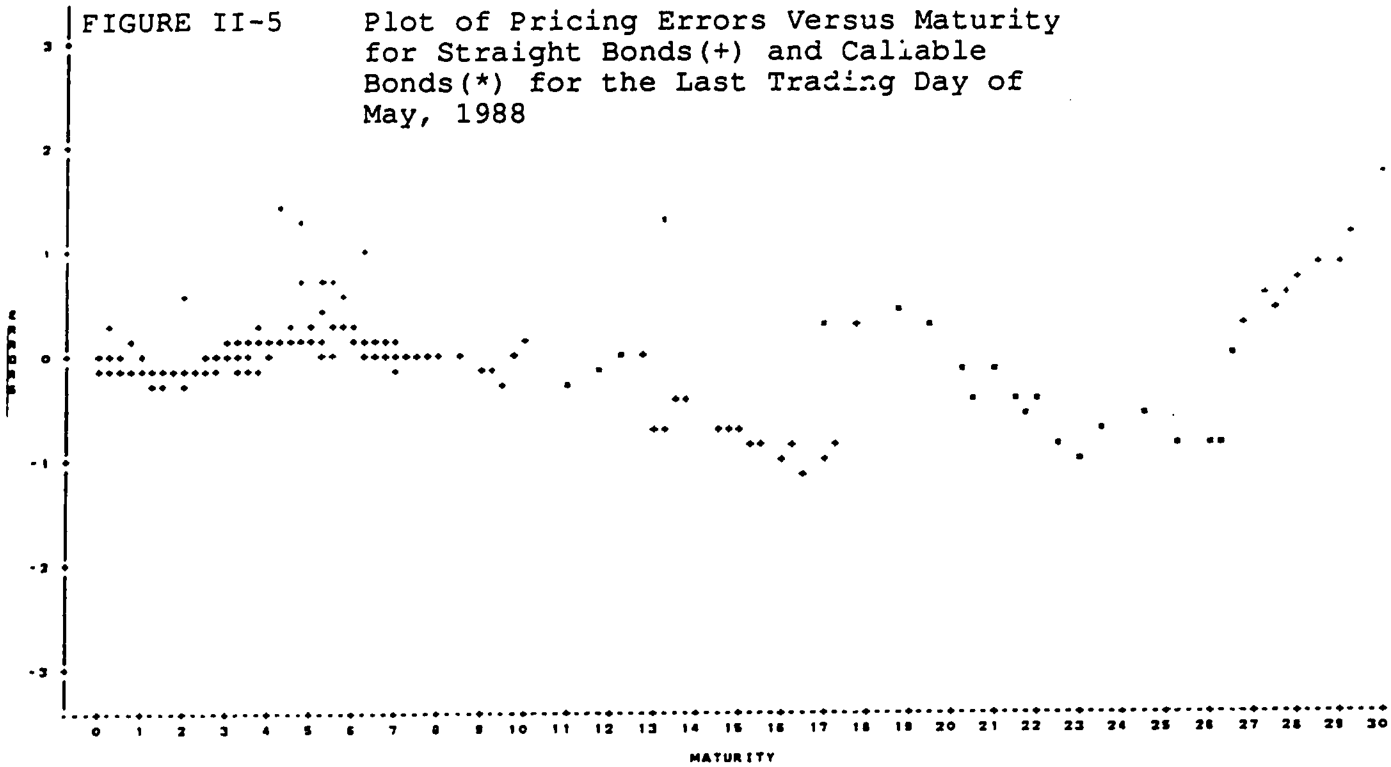


FIGURE II-6 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of June, 1988

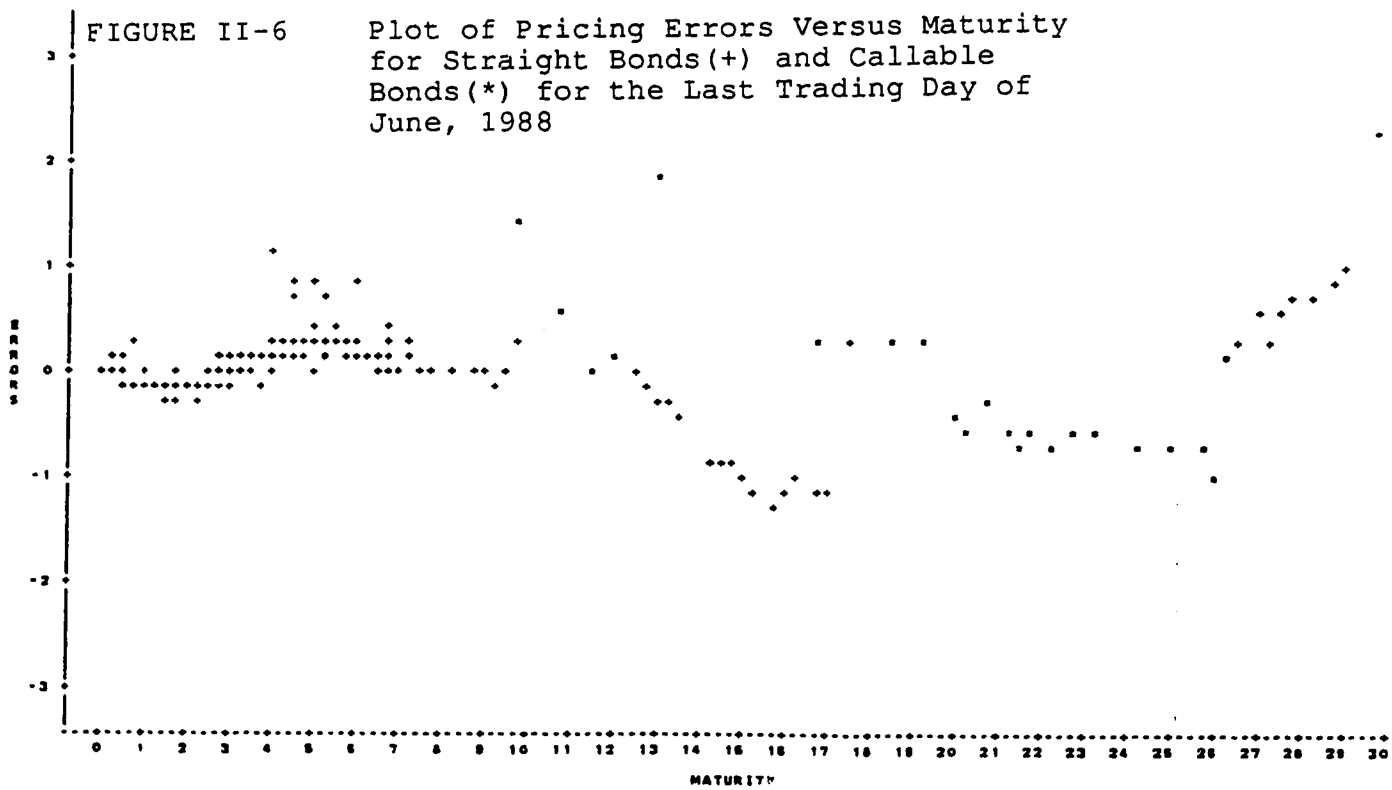


FIGURE II-7 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of July, 1988

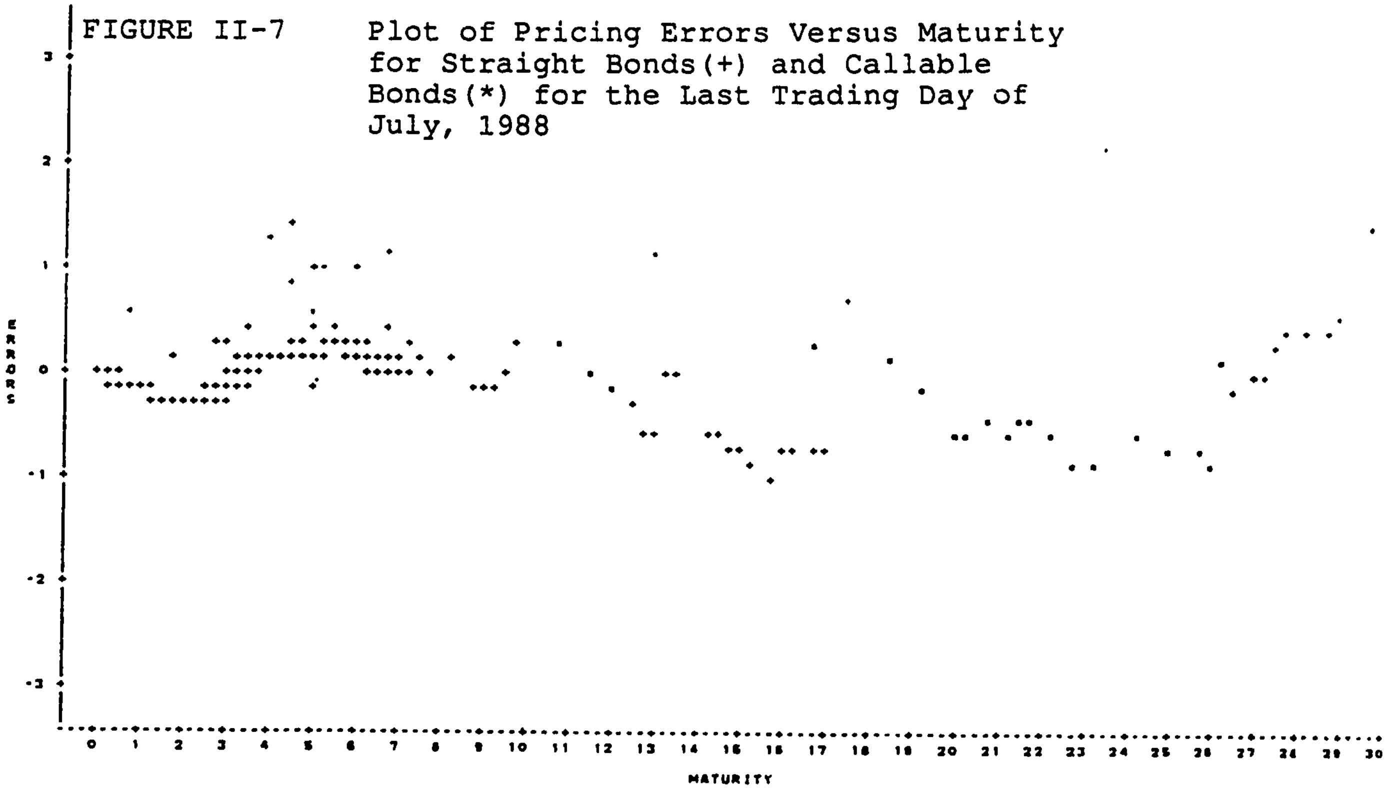


FIGURE II-8 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of August, 1988

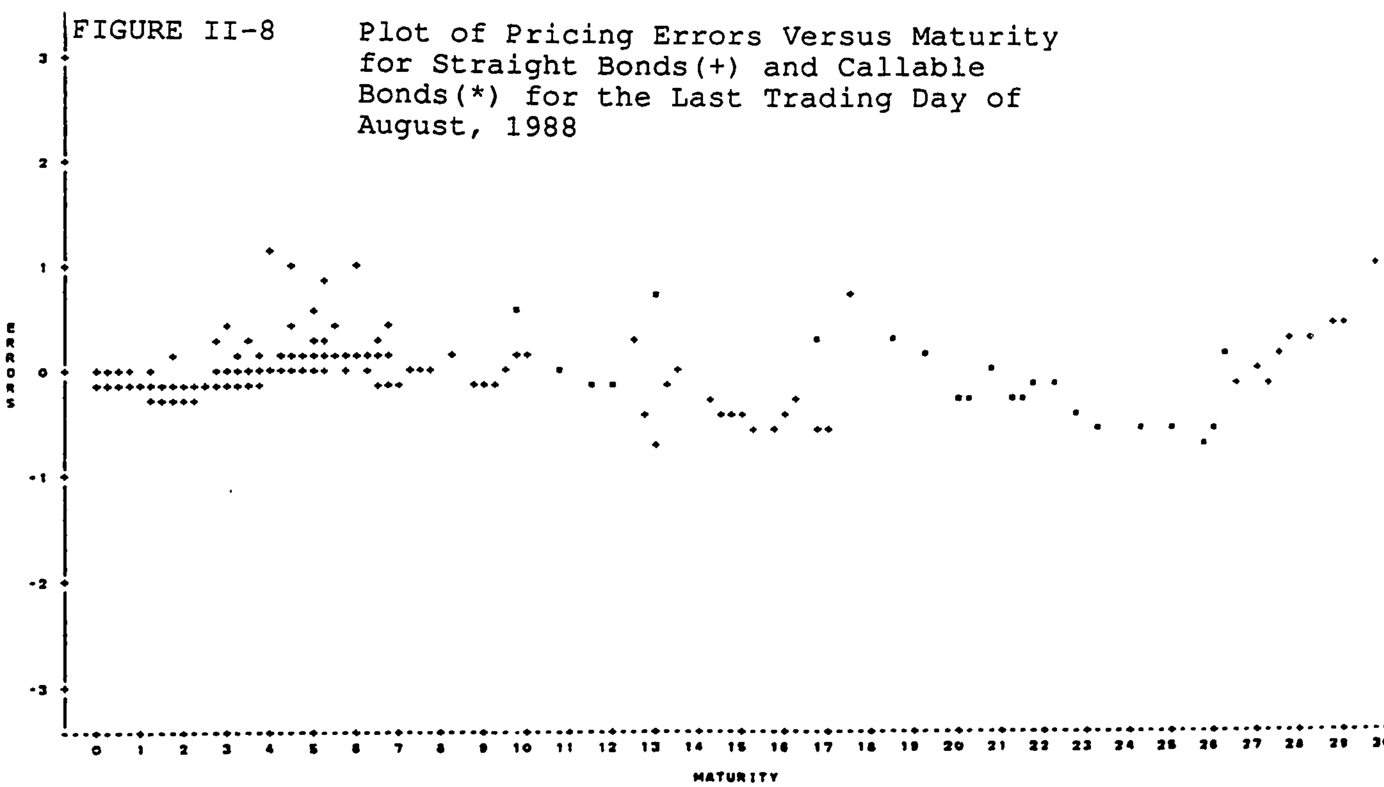


FIGURE II-9 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of September, 1988

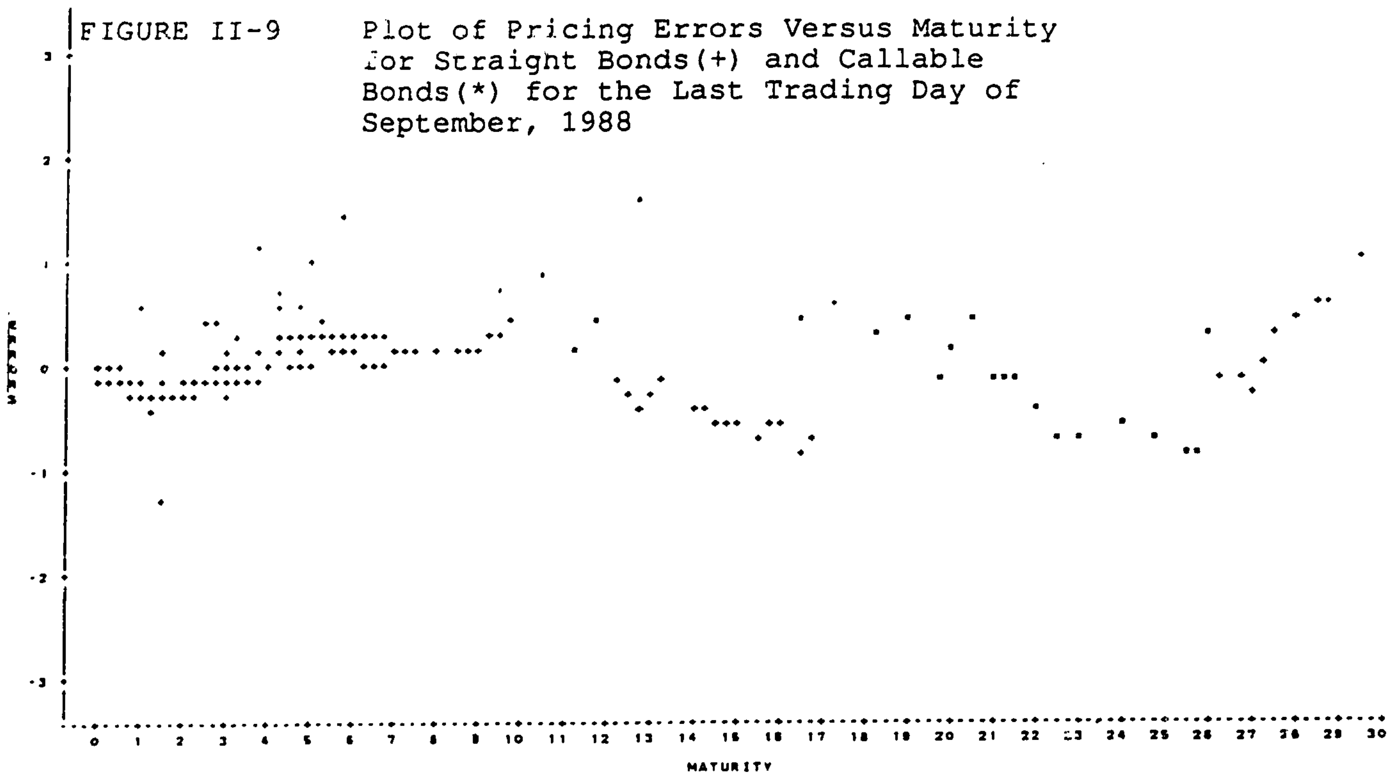


FIGURE II-10 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of October, 1988

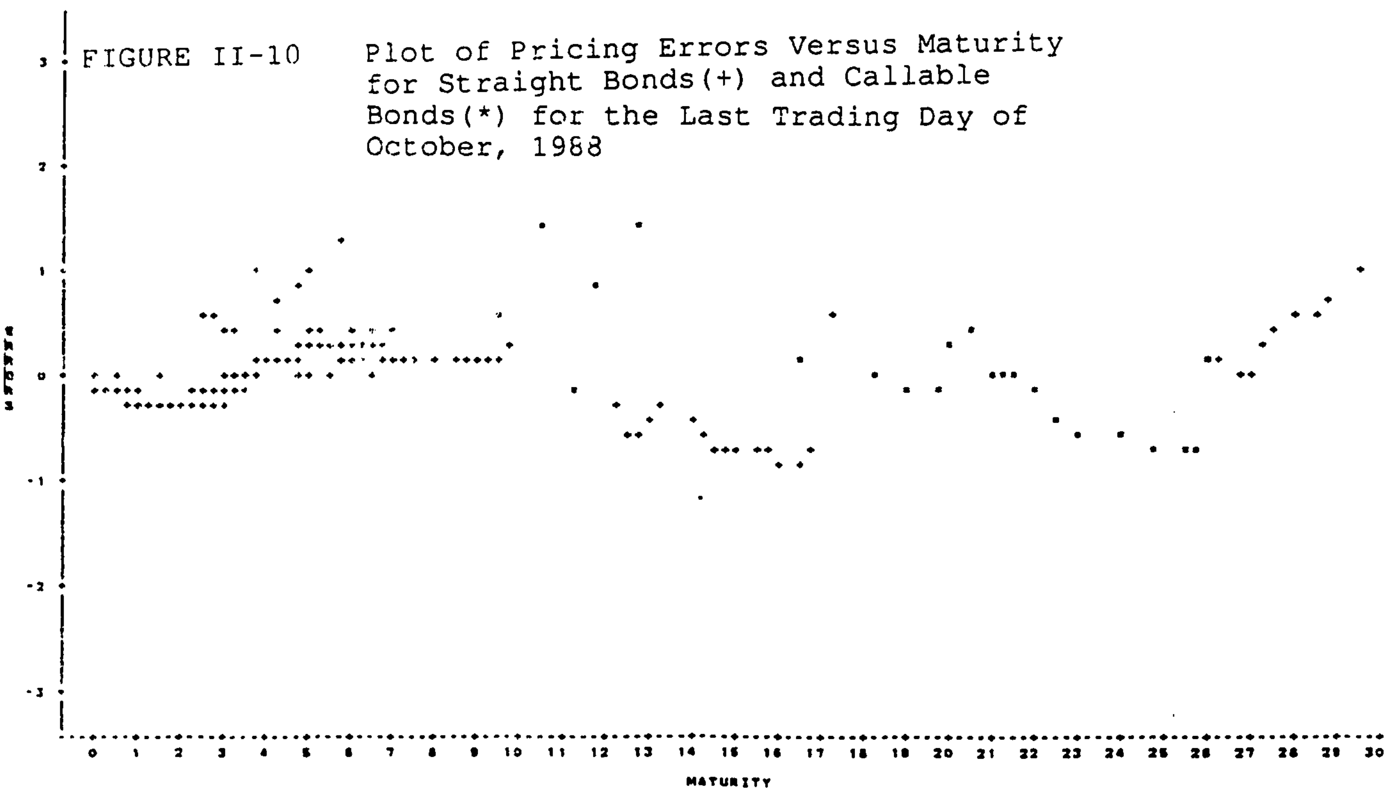


FIGURE II-11 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of November, 1988

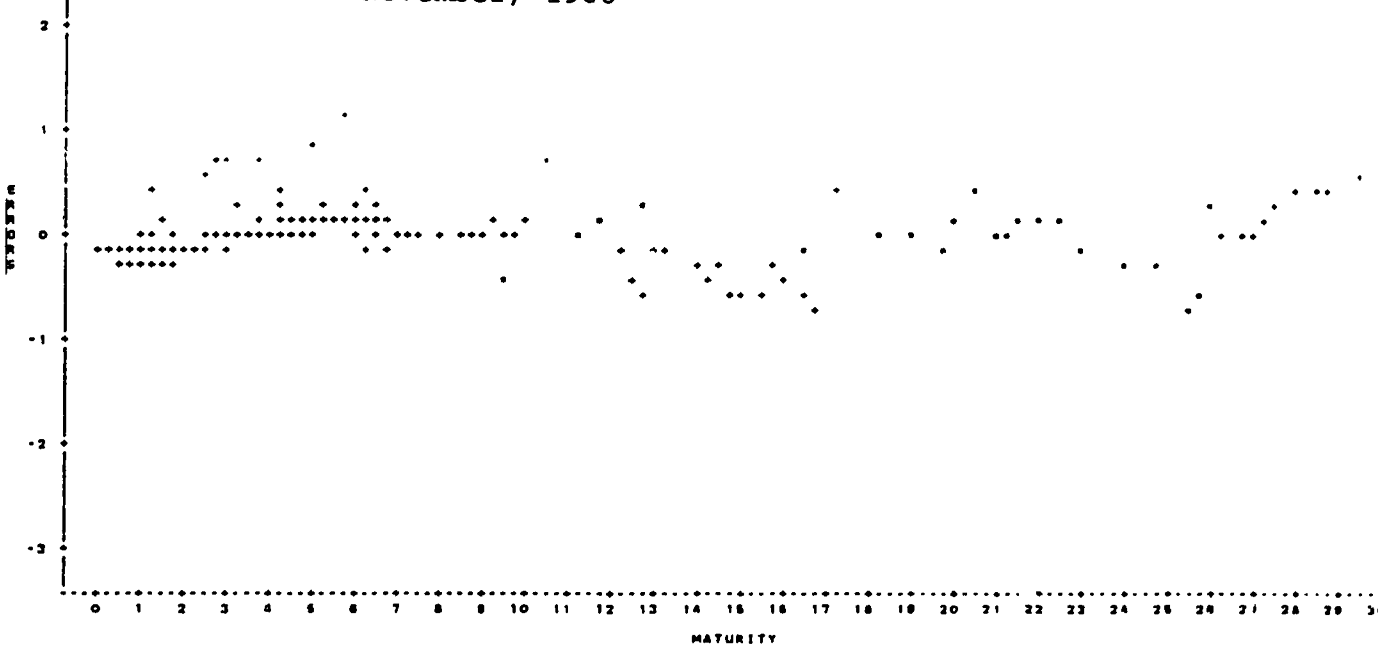
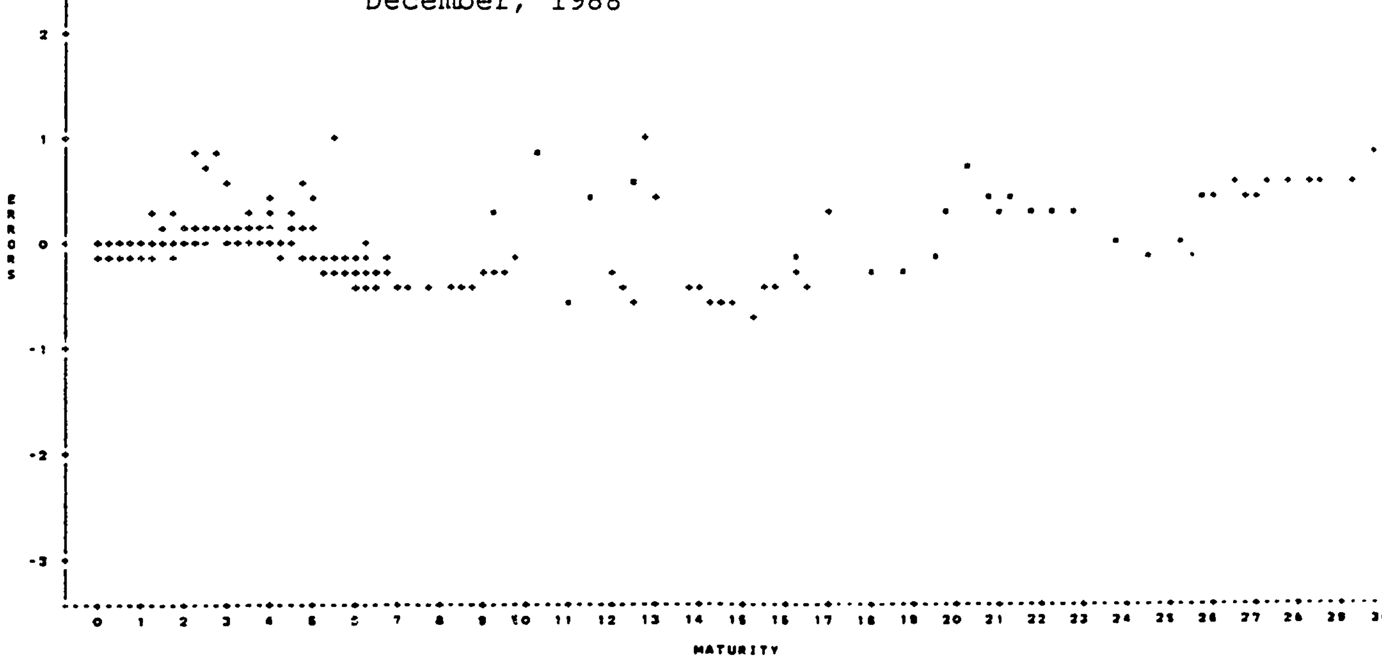


FIGURE II-12 Plot of Pricing Errors Versus Maturity for Straight Bonds(+) and Callable Bonds(*) for the Last Trading Day of December, 1988



CHAPTER III: The Term Structure of Futures Prices

1. Introduction

Futures contracts and forward contracts are similar but not identical. The fundamental difference between them is the daily resettlement or "marking-to-market" requirement of futures but not forward contracts. Jarrow and Oldfield(1981) clarified this distinction by showing that when the daily interest rate is constant, forward and futures prices are the same. Several studies such as Cox, Ingersoll, and Ross(1981a), Richard and Sundaresan(1981), and French(1983) have sought to explain the theoretical implications of the daily resettlement process on the relative prices of the two contracts by allowing stochastic interest rates in one form or another. By doing so, they were able to obtain various testable propositions.

Only Cox, Ingersoll, and Ross(1981a), however, obtained an explicit formula for futures prices. In a special case, CIR were able to value bond futures within their well known general equilibrium framework. The purpose of my paper is to model the relationship between futures and spot prices allowing not only stochastic interest rates, but more generally a stochastic cost of carry rate. In the multivariate case, each individual component of the cost of carry rate, whether the interest rate, convenience yield, or storage cost is allowed to be stochastic depending on the particular underlying asset characteristics. Section 2 illustrates how the relationship between futures and spot prices resembles that of a Treasury bill and how mathematical results from the term structure literature can be modified to derive a closed-form expression for this relationship. The model is then operationalized in section 3 and tested using British pound, Deutsche mark, and Japanese yen foreign currency futures. Results are then compared to estimates obtained using the traditional non-stochastic cost of carry model of futures and forward prices. Section 4 concludes the paper.

2. Arbitrage Models of Futures and Forward Prices

Under frictionless markets and continuous trading, simple arbitrage arguments can be invoked to value futures and forward contracts. Define $f(t,T)$ as the forward price of some commodity at time t for a contract that matures at time T and define $r(t,T)$ as the yield to maturity on a default-free discount bill that pays one dollar at time T. The current price of the bond is consequently

$$B(t,T,r) = EXP[-(T-t)r(t,T)]. \qquad (1)$$

An investor will be indifferent between $f(t,T)B(t,T.r)$ dollars today and one unit of the commodity at time T. Equivalently, by defining $S(T)$ as the unknown price of the commodity at time T, then the forward price must equal the present value of the maturity spot price multiplied by the inverse of the bond price:

$$f(t,T) = EXP[(T-t)r(t,T)]PV_{t,T}[S(T)] \tag{2}$$

where $PV_{t,T}[]$ denotes the present value at time t of a payment received at time T. If the commodity in question did not provide a convenience yield nor require storage costs, then the current commodity price, S(t), must equal the present value of the future commodity price. Alternatively, if the asset provided a continuous convenience yield, q(t,T), and incurred storage costs, c(t,T), the present value of the commodity would equal S(t)EXP[(T-t)(-q(t,T)+c(t,T))] and so the forward price may be expressed more generally as:

$$f(t,T) = S(t)EXP[(T-t)b(t,T)] \tag{3}$$

where b(t,T)=r(t,T)-q(t,T)+c(t,T) is the cost of carry rate associated with the commodity. The forward price is simply the deferred value of the current commodity price.

Cox, Ingersoll, and Ross(1981a) and French(1983) develop a similar expression for futures prices. They demonstrate that the futures price must equal the present value of the product of the maturity spot price and the gross return from rolling over one day bonds,

$$F(t,T) = PV_{t,T}\{EXP[\Sigma_{\tau=t,T-1}r(\tau,\tau+1)]S(T)\} \tag{4}$$

where F(t,T) is the futures price at time t for a contract that matures at time T and r(τ,τ+1) is the unknown continuously compounded interest rate on a one day bond from time τ to time τ+1. In the more general case where the asset provides a daily convenience yield, $q(\tau,\tau+1)$, and storage costs, $c(\tau,\tau+1)$, the futures price may be expressed

$$F(t,T) = S(t)EXP[\Sigma_{\tau=t,T-1}b(\tau,\tau+1)] \tag{5}$$

where $b(\tau,\tau+1)=r(\tau,\tau+1)-q(\tau,\tau+1)+c(\tau,\tau+1)$. Assuming continuous trading and continuous marking-to-market, (5) becomes

$$F(t,T) = S(t)EXP[\int_{t,T}b(u)du]. \tag{6}$$

When subsequent cost of carry rates are stochastic, the futures price may be expressed as the solution to

$$F(t,T) = E_t\{S(t)EXP[\int_{t,T}b(u)du]\} \tag{7}$$

subject to $F(T,T)=S(T)$, such that the futures and spot prices converge at contract expiration.

The problem can be simplified by considering, instead, the evolution over time of the variable $F(t,T)/S(t)$. Normalizing the futures price by the spot price for every time t results in a variable which no longer has a cost of carry rate of zero, unlike a futures contract itself which requires no initial investment. As a result (7) becomes

$$F(t,T)/S(t) = E_t\{EXP[\int_{t,T}b(u)du]\} \tag{8}$$

subject to $F(T,T)/S(T)=1$. By focusing on the normalized futures price instead of the futures price itself, several modified results from the theory of the term structure of interest rates can be applied. The solution to equation (8) when the cost of carry b is stochastic will have a similar form to that of a discount bond when interest rates are stochastic and allowed to take on either positive or negative values.

2.1 Univariate Case

Let $B(t,T,b)$ denote the value at time t of the normalized futures price which is analogous to an accumulation/discount factor associated with a futures contract maturing at time T with $t \leq T$. This factor will have a value of one at maturity, i.e. $B(T,T,b)=1$. It will be assumed that $b(t)$ is a continuous function of time described by the stochastic differential equation

$$db = \alpha(b,t)dt + \sigma(b,t)dZ \tag{9}$$

where $\alpha(b,t)$ is an instantaneous drift parameter, $\sigma(b.t)$ represents an instantaneous standard deviation parameter, and $Z(t)$ is a Wiener process with unit variance. Standard arbitrage arguments as in Vasicek(1977) imply (i) that the market price of risk associated with b, $\phi(b,t)$, is the same for normalized futures prices of all maturities, and (ii) that the value of $B(t,T,b)$ must satisfy the fundamental partial differential equation given by:

$$\partial B/\partial t + (\partial B/\partial b)\alpha(b,t) + (1/2)(\partial^2 B/\partial b^2)\sigma^2(b,t)$$

$$+ (\partial B/\partial b)\sigma(b,t)\phi(b,t) = -bB, \qquad (10)$$

subject to the terminal boundary condition $B(T,T,b)=1$.

It is important to recognize the difference between the right hand side of this p.d.e. and those found in the term structure literature. Whereas a riskless portfolio of discount bonds would require an instantaneous return equal to the risk free rate, a riskless portfolio of normalized futures contracts will require an instantaneous return of minus the cost of carry rate. Consider an asset which has a positive cost of carry. The normalized futures price $F(t,T)/S(t)$ will be larger than one and approach unity from above, decreasing instantaneously at a rate -b. Conversely, an asset with a negative cost of carry will have a normalized futures price approaching unity from below, increasing instantaneously at a rate -b which assumes a positive numerical value.

Techniques presented in Richard(1978) allow the solution to (10) to be given by

$$B(t,T,b) = E_t[EXP(A(T))] \qquad (11)$$

where

$$A(T) = +\int_{t,T} b(u)\,du + \int_{t,T}(1/2)\phi^2(b(u),u)\,du$$

$$+ \int_{t,T}\phi(b(u),u)\,dZ(u), \quad t\leq T.$$

The solution to $B(t,T,b)$ may be obtained once the stochastic process for b(t) and functional forms for the market price of risk $\phi(b,t)$ are specified.

It will be assumed that the cost of carry rate follows an Ornstein-Uhlenbeck process of the form:

$$db = \kappa(\theta - b)dt + \gamma dZ, \tag{12}$$

where θ is the long run mean of b, κ represents the speed of mean reversion parameter, and γ is the constant standard deviation associated with the Wiener component of the process. The Ornstein-Uhlenbeck process is particularly well suited for the model since it possesses the desirable property of allowing variables to take on either positive or negative values. When the additional assumption of a constant market price of risk ($\phi(b,t)=\phi$) is invoked, the solution to (11) is:

$$B(t,T,b) = \text{EXP}[\lambda(t,T) + L(t,T)b]$$

where $$L(t,T) = (1/\kappa)(1-\text{EXP}(-\kappa(T-t)))$$

$$\lambda(t,T) = -\,[(1/\kappa)(1-\text{EXP}(-\kappa(T-t))) - (T-t)]\,[\theta + (\gamma^2/2\kappa^2) + (\gamma\phi/\kappa)] - (\gamma^2/4\kappa^3)(1-\text{EXP}(-\kappa(T-t)))^2. \tag{13}$$

Note that if the cost of carry rate b is non-stochastic and constant, equation (13) reduces to (3) and hence forward and futures prices are equal, a conclusion which is consistent with the literature. The solution given by (13) has a similar form to the Vasicek(1977) term structure model. The differences between the two formulae are illustrated in Appendix 2, along with a proof demonstrating that (13) satisfies the fundamental partial differential equation given by (10).

2.2 The Multivariate Case

This subsection will briefly extend the previous univariate results to the multivariate case where the individual components, r, q, and c which comprise the cost of carry rate, b, each follow individual stochastic processes. The necessary mathematical restriction is that the total cost of carry rate be a linear combination of the individual variables, which, by definition, they are. Each of the three components will be expressed in vector/matrix form where $x^1=[r,q,c]$ is a vector of state variables x_i, i=1,2,3 which follow a joint stochastic process of the form

$$dx_i = \alpha_i(x,t)dt + \sigma_i(x,t)dZ_i \qquad (14)$$

where dx_i are standard Gauss-Wiener processes with $E[dZ_i]=0$, $E[dZ_i^2]=dt$, and $E[dZ_i dZ_j]=\rho_{ij}dt$. B(x,t,T) must then satisfy the partial differential equation:

$$\partial B/\partial t + \sum_{i=1,3}(\partial B/\partial x_i)\alpha_i(x,t)$$
$$+ (1/2)\sum_{i=1,3}\sum_{j=1,3}(\partial^2 B/\partial x_i \partial x_j)\sigma_i(x,t)\sigma_j(x,t)\rho_{ij}$$

$$+ \sum_{i=1,3} (\partial B/\partial x_i)\sigma_i(x,t)\phi_i(x_i,t) = -bB, \quad (15)$$

which is subject to the terminal boundary condition $B(T,T,x)=1$.

Following Richard(1978), the solution to (15) has the form:

$$B(t,T,x) = E_t[EXP(A(T))] \quad (16)$$

where

$$A(T) = +\int_{t,T} b(u)du + \int_{t,T}(1/2)\phi\sigma^T\Sigma^{-1}\phi\sigma du$$

$$+\int_{t,T}\phi\sigma^T\Sigma^{-1}\phi dZ(u), \quad t\leq T$$

and where

$$\phi\sigma^T = [\phi_1\sigma_1 \ \ \phi_2\sigma_2 \ \ \phi_3\sigma_3]$$

$$\phi dZ = [\sigma_1 dZ_1 \ \ \sigma_2 dZ_2 \ \ \sigma_3 dZ_3]$$

$$\Sigma = [\sigma_i\sigma_j\rho_{ij}] \ \ i=1,2,3 \text{ and } j=1,2,3.$$

When the market prices of risk associated with the variables are constant and the state variables follow Ornstein-Uhlenbeck processes, the solution to (16) becomes straightforward. Specifically, the joint stochastic processes is of the form:

$$dx_i = \kappa_i(\theta_i - x_i)dt + \gamma_i\, dZ_i, \quad i=1,2,3 \quad (17)$$

such that $E[dZ_i dZ_j] = 0$ for every $i\neq j$. Because the disturbances are assumed uncorrelated, a separation of variables technique employed by Langetieg(1980) and Cox, Ingersoll, and Ross(1985a) allows the multivariate solution to (16) to be expressed as the product of three single variable solutions:

$$B(t,T,x) = \Pi_{i=1,3}B(t,T,x_i) \tag{18}$$

where
$$B(t,T,x_i) = EXP[\lambda(t,T)+L_i(t,T)x_i]$$
$$L_i(t,T)= (1/\kappa_i)(1-EXP(-\kappa_i(T-t)))$$
$$\lambda(t,T)=-[(1/\kappa_i)(1-EXP(-\kappa_i(T-t)))-(T-t)]$$
$$[\theta_i+(\gamma_i^2/2\kappa_i^2)+(\gamma_i\phi_i/\kappa_i)]$$
$$-(\gamma_i^2/4\kappa_i^3)(1-EXP(-\kappa_i(T-t)))^2.$$

Note that if one or more of the variables is non-stochastic and constant, its associated single variable solution will reduce to the simple exponential form:

$$B(t,T,x_i) = EXP[(T-t)x_i] \tag{19}$$

which would be used in lieu of the more complex expression.

3. An Application to Foreign Currency Futures

The empirical implementation of the previous framework can be illustrated by modelling foreign currency futures for the British pound, Deutsche mark, and Japanese yen traded at the International Monetary Market of the Chicago Mercantile Exchange. Several issues such as the number and choice of state variables, the estimation of the stochastic processes which they follow, and the method and time frame for estimating their associated market prices of risk, need to be further examined.

3.1 Number and Choice of State Variables

It is necessary to select or construct state variables which are likely to explain the difference between futures and spot prices, are readily observable, and are such that the sum of the state variables will equal the instantaneous cost of carry rate for foreign currency. Any sensible choice of state variables for foreign currency futures will involve short term domestic and foreign interest rates. More specifically, it will be the difference between these two rates which drive futures prices, and this difference can be reasonably expected to follow a mean reverting process over short periods of time since central banks tend to align their interest rates in the absence of major policy shifts. In addition, only the Ornstein-Uhlenbeck process can be used to model the cost of carry rate in this example since it allows variables to take on negative values. Of course, cost of carry rates for premium currencies will be positive while those of discount currencies will be negative. Alternatively, a two variable version of the model could have been used where the foreign interest rate constitutes the convenience yield.

Closing Wednesday one month rates for the four countries involved were collected weekly for calendar years 1986 through 1988 from the Financial Times of London. The specific instantaneous annualized interest rate proxies were the United States and British T-bill rates, and the near equivalent Frankfurt and Tokyo money rates. The United States will be treated as the domestic economy and the United Kingdom, West Germany, and Japan each as foreign economies. In each of the three cases, the stochastic processes governing the cost of carry state variable will require the same estimation procedure.

3.2 Estimation Method for the Stochastic Processes

Recall that the state variables in the general case are assumed to follow Ornstein-Uhlenbeck processes of the form:

$$dx_i = \kappa_i(\theta_i - x_i)dt + \gamma_i \, dZ_i$$

where θ_i is the long run mean of the series, κ_i is the speed of adjustment, and γ_i the (constant) standard deviation. A difficulty in estimating the parameters of these equations arises because the data is only sampled at discrete time intervals, whereas the Ornstein-Uhlenbeck processes are defined in continuous time. Simply substituting first differences for the differentials above leads to estimation bias as noted by Phillips(1972), Wymer(1972), and Marsh and Rosenfeld(1983). It is possible to avoid this situation by employing an exact differential equation (see Oldfield and Rogalski(1987)) of the form:

$$E[\int_{t,s} dx_i] = x_i(t)(EXP(-\kappa_i(s-t))-1)$$

$$+ \theta_i(1-EXP(-\kappa_i(s-t)))$$

where

$$Var[\int_{t,s} dx_i] = \gamma_i^2(1-EXP(-2\kappa_i(s-t)))/2\kappa_i.$$

The required parameters can then be obtained directly by non-linear estimation or implied by the slope and intercept coefficients of an ordinary least squares regression as in:

$$x_i(s) = a + (1+b)x_i(t) + u(s) \qquad (20)$$

where u(s) is an error term. In particular, the parameters are then acquired by setting

$$\kappa_i = -\ln(1+b)/(s-t),$$

$$\theta_i = -a/b,$$

and $\gamma_i = SQRT[2\ln(1+b)\sigma_u^2/((s-t)((1+b)^2-1))]$, where σ_u^2 is the error sample variance. The assumption of independent and identically distributed errors ($E[u_s]=0$, $VAR[u_s]=\sigma_u^2$) is imposed in order to make inferences concerning the parameter estimates. With estimates from non-linear models of this type, the resulting test statistic will not be a t-statistic but rather the square root of a Chi-squared variable with one degree of freedom. The distribution of this statistic is virtually the same as a t-statistic based on more than 50 observations. Because weekly data is used, s-t will equal 1/52 when these processes are actually estimated.

3.3 Empirical Results of the Estimation of the Stochastic Processes

In each of the three cases, stochastic processes were estimated using weekly data beginning with the period January, 1986 through December, 1987. The two year window was then moved up by weekly intervals and the parameters were reestimated. This procedure was repeated until the last Wednesday in December of 1988 was reached. Partial results for the implied parameters at six month increments of the estimation window appear in Table III-1. The long run mean, θ, of the difference in interest rates is negative for the United Kingdom case, implying that pound sterling was a discount currency relative to the United States dollar during the estimation periods. Conversely, the Deutsche mark and Japanese yen were typically premium currencies since their long run mean costs of carry rates were positive. In each of the three cases, the parameter $\hat{\theta}$ was significantly different from zero at the 1% level. The speed of adjustment parameter, κ, took on necessary positive values in all three cases at the 1% level of significance in all but one sub-period although with relatively large changes in magnitude as the two year estimation window was moved. This parameter measures the speed of mean reversion of the process and a value of 10, for example, implies that half of the adjustment is expected to take place within approximately three and a half weeks. Estimated values of the standard deviation parameter, γ, associated with the Wiener component of the processes are on average of similar magnitude to the long run means in the case of the United Kingdom and triple the magnitudes of the long run means in the cases of West Germany and Japan.

3.4 Estimation of Normalized Futures Prices

After the model of Section 2 has been operationalized and all of the required Ornstein-Uhlenbeck parameter inputs have been estimated, the single state variable version of the model (equation (13)) can now be used to estimate the only remaining unknown parameters, the market prices of risk. For each of the three currencies, futures prices of all available contract maturities traded at the IMM of the CME were employed. Closing prices at 2:00pm Central time were collected weekly for each Wednesday in 1988 as reported in the Wall Street Journal. Spot exchange rates as quoted by Bankers Trust Co. at 3:00pm Eastern time and reported in the Wall Street Journal were also collected for each Wednesday in 1988. The resulting futures prices normalized by their associated spot prices became the dependent variable during the iterative single equation non-linear procedure to estimate (13) which takes as its objective function the sum of squared residuals from the model. The assumption of independent and identically distributed errors will be imposed in order to make inferences regarding the parameter estimates. A proof demonstrating that, although estimates of the stochastic process parameters are required as inputs in the estimation of (13), the market price of risk estimates are consistent and asymptotically normal appears in Appendix 3.

Table III-2 contains empirical results of the estimation of the market price of risk, ϕ, over the entire period and quarterly sub-periods for the normalized futures prices of the British pound, Deutsche mark, and Japanese yen. In addition to the estimates of the stochastic cost of carry model (equation (13) and denoted "S"), the traditional non-stochastic model represented by equation (3) was tested using the difference in annualized one month rates and time to contract expiration as inputs (denoted "NS"). The root mean squared error (RMSE), mean absolute error (MAE), and mean error (ME) criteria are provided for each of the estimation periods. As a further error analysis, actual normalized futures prices are regressed on estimates generated by both models, with these results appearing in Table III-3.

In the case of British pound futures, the market price of risk is significantly different from zero at a 1% level during each estimation period. Positive risk prices imply that cost of carry risk is rewarded by the market. Under either the RMSE or MAE criterion, the stochastic model yields pricing errors which are less than half those of the non-stochastic benchmark in each of the quarterly sub-periods although slightly more than one-half over the entire estimation period. Regressing actual on estimated model prices should result in an intercept term of zero and a slope of one if the estimates are unbiased. Table III-3 reports regression parameters and t-ratios based on this null hypothesis. The stochastic model resulted in rejection of the hypothesis at the 1% level of significance only in the fourth quarter sub-period, while the non-stochastic model led to rejection in every estimation period.

Results for both the Deutsche mark and Japanese yen normalized futures were more favourable. Again, highly significant positive market prices of risk were obtained. In both cases and over every estimation period, the RMSE and MAE criteria show that mispricing by the stochastic model is only approximately one quarter that of the non-stochastic model. Intercept and slope coefficients generated by the regression of actual futures prices on the stochastic model's estimates were significantly different from hypothesized values at the 1% level only during the last two quarterly estimation periods in the Deutsche mark case. By contrast, the non-stochastic model given by (3) resulted in the null hypothesis of unbiased errors being rejected at the 1% level consistently during every estimation period for both the

Deutsche mark and Japanese yen cases.

In fairness to the non-stochastic benchmark model, the use of the difference between annualized one month interest rates as an input parameter to generate futures price estimates for contract lengths up to and including nine months does not allow it to perform optimally. If it were possible, matching interest rates to contract expiration should yield better estimates. Presumably, one would expect the absolute difference in interest rates to increase, on average, for longer maturities. The use of one month rates instead of matching rates would then cause the non-stochastic model to result in underpricing of longer maturity futures for premium currencies such as the Deutsche mark and Japanese yen while overpricing the futures of discount currencies such as the British pound. However, this was not the case. The non-stochastic model consistently underpriced the normalized futures of all three currencies as shown by the large positive mean errors in Table III-2. This suggests that there may be an additional risk premium required by the market and it is this premium which the stochastic cost of carry model is designed to explicitly incorporate into futures pricing.

4. Concluding Remarks

Taking advantage of the simple observation that the ratio of futures and spot prices is a function of the cost of carry rate and time to maturity, as well as converging to one at contract expiration, allows this variable to be modelled in a manner similar to that of a Treasury bill. As a result, a closed-form solution was derived to represent explicitly the relationship between futures and spot prices when the cost of carry rate or any of its components is stochastic in nature. Implementation of the model was illustrated using British pound, Deutsche mark, and Japanese yen foreign currency futures with favourable results. However, a more precise test of the alternative non-stochastic model with interest rates more closely resembling the time to contract expiration may be needed to clearly demonstrate the empirical advantages of the proposed stochastic model.

Table III-1: Implied Parameters for the Stochastic Processes
(t-statistics in parentheses)

Panel A: United Kingdom

Period	κ	θ	γ
01/86-12/87	9.6630	-0.0431	0.0335
	(2.81)	(17.84)	
04/86-03/88	10.2424	-0.0406	0.0363
	(2.82)	(16.07)	
07/86-06/88	5.2650	-0.0381	0.0350
	(2.11)	(8.07)	
10/86-09/88	6.5893	-0.0378	0.0379
	(2.43)	(9.26)	
01/87-12/88	10.6621	-0.0372	0.0445
	(3.05)	(12.57)	

[illegible] West Germany

Period	κ	θ	γ
01/86-12/87	8.7009	0.0109	0.0310
	(2.54)	(4.38)	
04/86-03/88	15.6165	0.0109	0.0358
	(3.32)	(6.72)	
07/86-06/88	10.7241	0.0122	0.0339
	(2.90)	(5.46)	
10/86-09/88	13.2634	0.0137	0.0366
	(3.18)	(6.95)	
01/87-12/88	23.0115	0.0153	0.0417
	(4.01)	(11.84)	

Panel C: Japan

Period	κ	θ	γ
01/86-12/87	12.3192	0.0094	0.0297
	(3.55)	(5.43)	
04/86-03/88	13.6752	0.0099	0.0334
	(3.18)	(5.707)	
07/86-06/88	10.2362	0.0109	0.0314
	(2.80)	(5.03)	
10/86-09/88	10.9998	0.0136	0.0337
	(2.95)	(6.21)	
01/87-12/88	15.2900	0.0156	0.0396
	(3.43)	(8.44)	

Table III-2: Non-Linear Estimation of Normalized Futures Prices
(t-statistics in parentheses)

Period	Model	Obs	RMSE	MAE	ME	ϕ	R^2
Panel A: British Pound							
01/88-03/88	S	31	0.001629	0.001313	0.000657	7.649 (31.18)	0.871
	NS	31	0.006206	0.005400	0.005400		
04/88-06/88	S	36	0.002456	0.001851	0.000503	6.000 (25.06)	0.508
	NS	36	0.004964	0.004349	0.004349		
07/88-09/88	S	29	0.001571	0.001264	0.000488	3.099 (14.32)	0.913
	NS	29	0.004869	0.004081	0.004801		
10/88-12/88	S	30	0.001488	0.001245	0.000439	2.747 (12.86)	0.942
	NS	30	0.006603	0.005525	0.005516		
Whole Period	S	126	0.003195	0.002619	0.000461	5.040 (24.97)	0.572
	NS	126	0.005669	0.004826	0.004824		
Panel B: Deutsche Mark							
01/88-03/88	S	43	0.001327	0.000976	0.000047	10.711 (57.32)	0.979
	NS	43	0.009440	0.007627	0.007602		
04/88-06/88	S	36	0.001578	0.001119	0.000153	10.454 (39.27)	0.963
	NS	36	0.006868	0.005786	0.005786		
07/88-09/88	S	39	0.001415	0.001045	0.000375	7.363 (35.74)	0.968
	NS	39	0.007645	0.006406	0.006370		
10/88-12/88	S	37	0.001437	0.001055	0.000428	9.870 (38.01)	0.973
	NS	37	0.007922	0.006741	0.006721		
Whole Period	S	155	0.002034	0.001541	0.000284	9.527 (59.85)	0.939
	NS	155	0.008061	0.006681	0.006660		

Table III-2: Non-Linear Estimation of Normalized Futures Prices (continued)
(t-statistics in parentheses)

Period	Model	Obs	RMSE	MAE	ME	ϕ	R^2
Panel C: Japanese Yen							
01/88-03/88	S	37	0.001917	0.001462	-0.000339	8.072	0.939
						(25.64)	
	NS	37	0.007805	0.005944	0.005900		
04/88-06/88	S	49	0.001608	0.001298	-0.000087	8.528	0.971
						(52.54)	
	NS	49	0.008650	0.007094	0.007064		
07/88-09/88	S	48	0.002008	0.001516	-0.000046	7.819	0.961
						(41.56)	
	NS	48	0.008523	0.006488	0.006358		
10/88-12/88	S	50	0.002192	0.001713	0.000221	9.711	0.969
						(48.97)	
	NS	50	0.011420	0.009448	0.009393		
Whole Period	S	184	0.002232	0.001669	-0.000085	8.653	0.957
						(74.72)	
	NS	184	0.009277	0.007344	0.007279		

NS - non-stochastic "standard" model (equation (3))
S - stochastic "proposed" model (equation (13))

Table III-3: **Regression of Actual on Estimated Normalized Futures Prices**
(t-ratios in parentheses)

Period	Model	Intercept	Slope	R^2
Panel A: British Pound				
01/88-03/88	S	-0.159	1.161	0.871
		(1.92)	(1.93)	
	NS	0.432	0.569	0.900
		(12.40)	(12.23)	
04/88-06/88	S	0.234	0.765	0.508
		(1.82)	(1.82)	
	NS	0.375	0.627	0.720
		(5.64)	(5.56)	
07/88-09/88	S	-0.148	1.150	0.913
		(2.19)	(2.20)	
	NS	0.335	0.665	0.905
		(8.18)	(8.09)	
10/88-12/88	S	-0.134	1.136	0.942
		(2.55)	(2.56)	
	NS	0.[illegible]91	0.[illegible]08	0.915
		(11.[illegible])	(1[illegible]2)	
Whole Period	S	0.[illegible]	0.947	0.572
		(0.7[illegible])	(0.72)	
	NS	0.353	0.648	0.878
		(16.51)	(16.26)	
Panel B: Deutsche Mark				
01/88-03/88	S	0.008	0.992	0.972
		(0.36)	(0.35)	
	NS	-0.651	1.654	0.740
		(4.23)	(4.27)	
04/88-06/88	S	0.029	0.972	0.963
		(0.87)	(0.86)	
	NS	-0.694	1.694	0.950
		(10.38)	(10.46)	
07/88-09/88	S	0.076	0.925	0.968
		(2.71)	(2.70)	
	NS	-0.342	1.347	0.672
		(3.20)	(3.24)	
10/88-12/88	S	0.070	0.932	0.973
		(2.65)	(2.62)	
	NS	-0.673	1.676	0.839
		(5.38)	(5.44)	
Whole Period	S	0.048	0.953	0.939
		(2.29)	(2.29)	
	NS	-0.572	1.575	0.783
		(8.49)	(8.58)	

Table III-3: Regression of Actual on Estimated Normalized Futures Prices (continued)
(t-ratios in parentheses)

Period	Model	Intercept	Slope	R^2
Panel C: Japanese Yen				
01/88-03/88	S	-0.079	1.078	0.939
		(1.68)	(1.68)	
	NS	-0.758	1.761	0.633
		(3.33)	(3.36)	
04/88-06/88	S	-0.015	1.015	0.971
		(0.58)	(0.59)	
	NS	-0.810	1.810	0.902
		(9.21)	(9.29)	
07/88-09/88	S	0.001	0.999	0.961
		(0.03)	(0.03)	
	NS	-0.431	1.432	0.757
		(3.56)	(3.61)	
10/88-12/88	S	0.026	0.975	0.969
		(1.01)	(0.99)	
	NS	-0.589	1.591	0.832
		(5.64)	(5.72)	
Whole Period	S	-0.019	1.019	0.957
		(1.18)	(1.18)	
	NS	-0.558	1.561	0.813
		(9.99)	(10.12)	

NS - non-stochastic "standard" model (equation (3))
S - stochastic "proposed" model (equation (13))

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Appendix 1

(For simplicity, results presented here are expressed in terms of the one state variable discount bond model. The conclusions, however, are easily extendible to the multiple state variable coupon bond model.)

Estimates of the market price of risk, ϕ, are consistent and asymptotically normal. The single variable version of equation (10) can be rewritten as:

$$B(t,T,x)=EXP[A_1(t,T,x,\kappa,\theta,\gamma)+A_2(t,T,\kappa,\theta,\gamma)\phi]+e_i$$

where e_i are independent and identically distributed errors ($E[e_i]=0$, $VAR[e_i]=\sigma_i^2$) for observations $i=1,\ldots,N$ associated with N bonds of different maturities. True values of κ, θ, and γ are unknown, but their consistent estimates $\hat{\kappa}$, $\hat{\theta}$, and $\hat{\gamma}$ can be obtained during the first non-linear least squares estimation of the stochastic processes. Consequently, $\text{plim}\hat{\kappa}=\kappa$, $\text{plim}\hat{\theta}=\theta$, and $\text{plim}\hat{\gamma}=\gamma$, where plim is the probability limit as the number of observations, N, increases to infinity. The market price of risk, ϕ, is estimated from:

$$B(t,T,x)=EXP[A_1(t,T,x,\hat{\kappa},\hat{\theta},\hat{\gamma})+A_2(t,T,\hat{\kappa},\hat{\theta},\hat{\gamma})\phi]+v_i$$

where

$$v_i = e_i + \{EXP[A_1(t,T,x,\kappa,\theta,\gamma)+A_2(t,T,\kappa,\theta,\gamma)\phi] - EXP[A_1(t,T,x,\hat{\kappa},\hat{\theta},\hat{\gamma})+A_2(t,T,\hat{\kappa},\hat{\theta},\hat{\gamma})\phi]\}.$$

Because this equation is non-linear in ϕ, the non-linear least squares procedure will give an estimate of ϕ such that

$S(\phi)=\sum_{i=1,N} v_i^2$ is minimized. Care must be taken since v_i may be correlated with $A_1(\cdot)$ and $A_2(\cdot)$.

Suppose that $\hat{\phi}$ is the value of ϕ which minimizes $S(\phi,\hat{\kappa},\hat{\theta},\hat{\gamma})$. Consequently, $\partial S/\partial\phi=0$ where $\phi=\hat{\phi}(\hat{\kappa},\hat{\theta},\hat{\gamma})$. Then, it will be the case that:

$$\text{plim}\,\partial S(\phi,\hat{\kappa},\hat{\theta},\hat{\gamma})/\partial\phi|_{\phi=\hat{\phi}} = \text{plim}\,\partial S(\phi,\kappa,\theta,\gamma)/\partial\phi|_{\phi=\hat{\phi}} = 0.$$

This means that $\hat{\phi}$ also minimizes $S(\phi,\kappa,\theta,\gamma)$ in which the true values of κ, θ, and γ are substituted for estimated values in this expression.

Ultimately, because consistent estimates of κ, θ, and γ are used, it can also be shown that the proof given by Amemiya(1985, Theorems 4.3.1 (page 129) and 4.3.2 (page 133) can be extended to the present case assuming that the conditions of the theorems hold. The two stage method will yield market price of risk estimates such that:

(1) $\hat{\phi}$ is consistent ($\text{plim}\,\hat{\phi}=\phi$),

and (2) $\hat{\phi}$ is asymptotically normal such that:

$$SQRT(N)\{\hat{\phi}-\phi\} \longrightarrow N(0,\sigma_e^2 C^{-1})$$

where $\sigma_e^2 = N^{-1}S(\hat{\phi})$

$$C = \lim_{N\to\infty} N^{-1}\sum_{i=1,N}(\partial B(t,T,x)/\partial\phi)^2$$

$$= \lim_{N\to\infty} N^{-1}\sum_{i=1,N}(A_2[A_1+A_2\phi]EXP[A_1+A_2\phi])^2$$

Appendix 2

PART 1: Proof that the solution (13) satisfies (10)

The solution

$$B(t,T,b) = EXP[\lambda(t,T) + L(t,T)b]$$

where $L(t,T) = (1/\kappa)(1-EXP(-\kappa(T-t)))$

$$\lambda(t,T) = -[(1/\kappa)(1-EXP(-\kappa(T-t)))-(T-t)][\theta+(\gamma^2/2\kappa^2)+(\gamma\phi/\kappa)] - (\gamma^2/4\kappa^3)(1-EXP(-\kappa(T-t)))^2$$

can be seen to satisfy

$$\partial B/\partial t + (\partial B/\partial b)\kappa(\theta-b) + (1/2)(\partial^2 B/\partial b^2)\gamma^2 + (\partial B/\partial b)\gamma\phi = -bB$$

by substituting in for the partial derivatives:

i) $\partial B/\partial b = L(t,T)B(t,T,b)$

ii) $\partial^2 B/\partial b^2 = L^2(t,T)B(t,T,b)$

iii) $\partial B/\partial t = [\partial\lambda/\partial t + (\partial L/\partial t)b]B(t,T,b)$

iv) $\partial\lambda/\partial t = -\kappa\theta L(t,T) - \gamma\phi L(t,T) - (1/2)\gamma^2 L^2(t,T)$

v) $\partial L/\partial t = -EXP(-\kappa(T-t))$

PART 2: The Vasicek(1977) term structure model

In the notation of this paper if b equals the riskless rate of interest, the solution to:

$$\partial B/\partial t + (\partial B/\partial b)\kappa(\theta - b) + (1/2)(\partial^2 B/\partial b^2)\gamma^2 + (\partial B/\partial b)\gamma\phi = bB$$

is given by:

$$B(t,T,b) = EXP[\lambda(t,T) + L(t,T)b]$$

where $L(t,T) = -(1/\kappa)(1-EXP(-\kappa(T-t)))$

$$\lambda(t,T) = [(1/\kappa)(1-EXP(-\kappa(T-t))) - (T-t)][\theta - (\gamma^2/2\kappa^2) + (\gamma\phi/\kappa)] - (\gamma^2/4\kappa^3)(1-EXP(-\kappa(T-t)))^2$$

Appendix 3

Estimates of the market price of risk, ϕ, are consistent and asymptotically normal. Equation (13) can be rewritten as:

$$B(t,T,b)=EXP[A_1(t,T,b,\kappa,\theta,\gamma)+A_2(t,T,\kappa,\theta,\gamma)\phi]+e_i$$

where e_i are independent and identically distributed errors ($E[e_i]=0$, $VAR[e_i]=\sigma_i^2$) for observations $i=1,\ldots,N$ associated with N futures of different maturities. True values of κ, θ, and γ are unknown, but their consistent estimates $\hat{\kappa}$, $\hat{\theta}$, and $\hat{\gamma}$ can be obtained during the first non-linear least squares estimation of the stochastic processes. Consequently, $\text{plim}\hat{\kappa}=\kappa$, $\text{plim}\hat{\theta}=\theta$, and $\text{plim}\hat{\gamma}=\gamma$, where plim is the probability limit as the number of observations, N, increases to infinity. The market price of risk, ϕ, is estimated from:

$$B(t,T,b)=EXP[A_1(t,T,b,\hat{\kappa},\hat{\theta},\hat{\gamma})+A_2(t,T,\hat{\kappa},\hat{\theta},\hat{\gamma})\phi]+v_i$$

where

$$v_i = e_i+\{EXP[A_1(t,T,b,\kappa,\theta,\gamma)+A_2(t,T,\kappa,\theta,\gamma)\phi] -EXP[A_1(t,T,b,\hat{\kappa},\hat{\theta},\hat{\gamma})+A_2(t,T,\hat{\kappa},\hat{\theta},\hat{\gamma})\phi]\}.$$

Because this equation is non-linear in ϕ, the non-linear least squares procedure will give an estimate of ϕ such that

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$$\text{plim}\,\partial S(\phi,\hat{\kappa},\hat{\theta},\hat{\gamma})/\partial\phi|_{\phi=\hat{\phi}} = \text{plim}\,\partial S(\phi,\kappa,\theta,\gamma)/\partial\phi|_{\phi=\hat{\phi}} = 0.$$

This means that $\hat{\phi}$ also minimizes $S(\phi,\kappa,\theta,\gamma)$ in which the true values of κ, θ, and γ are substituted for estimated values in this expression.

Ultimately, because consistent estimates of κ, θ, and γ are used, it can also be shown that the proof given by Amemiya(1985, Theorems 4.3.1 (page 129) and 4.3.2 (page 133) can be extended to the present case assuming that the conditions of the theorems hold. The two stage method will yield market price of risk estimates such that:

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$$\text{SQRT}(N)\{\hat{\phi}-\phi\} \qquad N(0,\sigma_e^2 C^{-1})$$

where $\sigma_e^2 = N^{-1}S(\hat{\phi})$

$$C = \lim_{N\to\infty} N^{-1}\sum_{i=1,N}(\partial B(t,T,x)/\partial\phi)^2$$

$$= \lim_{N\to\infty} N^{-1}\sum_{i=1,N}(A_2[A_1+A_2\phi]EXP[A_1+A_2\phi])^2$$