21949

National Library of Canada | Bibliothèque nationale du Canada

CANADIAN THESES ON MICROFICHE

THÈSES CANADIENNES SUR MICROFICHE

NAME OF AUTHOR/*NOM DE L'AUTEUR* PARAMJIT SINGH RANA

TITLE OF THESIS/*TITRE DE LA THÈSE* STATISTICAL INDEPENDENCE AND DISTRIBUTION OF QUADRATIC FORMS IN NORMAL RANDOM VARIABLES.

UNIVERSITY/*UNIVERSITÉ* OF ALBERTA

DEGREE FOR WHICH THESIS WAS PRESENTED/*GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* MASTER OF SCIENCE

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* Fall 1974

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* Dr. EARL NORDBROCK

DATED/*DATÉ* 2.10.74 SIGNED/*SIGNÉ* Singh Rana

PERMANENT ADDRESS/*RÉSIDENCE FIXÉ* House No: 139/20 A.
CHANDIGARH (INDIA)

NL-91 (11-73)

THE UNIVERSITY OF ALBERTA

STATISTICAL INDEPENDENCE AND DISTRIBUTION OF QUADRATIC FORMS IN NORMAL RANDOM VARIABLES

---

by

© PARAMJIT SINGH RANA

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

FALL, 1974

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled STATISTICAL INDEPENDENCE AND DISTRIBUTION OF QUADRATIC FORMS IN NORMAL RANDOM VARIABLES submitted by PARAMJIT SINGH RANA in partial fulfilment of the requirements for the degree of Master of Science.

. . . Earl Nordbrock . . .
(Supvervisor)

. . . [illegible] . . .

. . . [illegible] . . . .

Date . . . Sept. 16, 1974 . . . . .

ABSTRACT

The aim of this thesis is to review the literature relating to the distribution and independence of quadratic forms in normal random variables. In Chapter II, we have discussed Craig's theorem and its generalizations to correlated and non-central cases. Matern's result for testing the independence of non-negative quadratic forms has been discussed.

In Chapter III, several results for testing whether a given quadratic form follows a chi-square (central and non-central) distribution have been reviewed. In this direction, Cochran's theorem and Craig's theorem and their generalizations have been discussed. These discussions include the case when V (Var.-Cov. matrix) is singular. Finally, we have given some applications of the results discussed above.

ACKNOWLEGEMENT

I would like to express my sincere appreciation to my supervisor Dr. E. Nordbrock for his helpful advice and encouragement during the preparation of this thesis.

I would also like to thank Miss June Talpash, for typing the final draft of this thesis.

Finally, I wish to express my gratitude to the University of Alberta for financial assistance.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

"Quadratic forms enter into many statistics associated with normally distributed random variables. Formal analysis of variance, for example, is entirely concerned with statistics constructed from quadratic forms in random variables representing original observations (or transformations thereof)."* The frequent occurence of quadratic forms in the study of ANOVA, Regression Analysis, Econometrics and many other related fields, makes it necessary to investigate their properties. The complete justification of various results in such studies involves the independence of quadratic forms and the conditions under which a given quadratic form follows the χ^2 - distribution. The fundamental theorems in this direction are Craig's theorem (Theorem 1) and Cochran's theorem (Theorem 15).

In this thesis our studies are confined to quadratic forms in normal random variables only, and the basic aim is to review the literature pertaining to these two theorems and their subsequent developments. Basically we have followed a matrix approach in presenting the results.

In Chapter II, we state and prove Craig's theorem, which gives a criterion for testing the independence of two quadratic forms in normal random variables with mean zero. Extensions of this theorem to the correlated case and non-central case as given by

---

* Quoted from Continuous Univariate Distributions by Norman L. Johnson and Samuel Kotz. New York, Houghton Mifflin, 1970.

A.C. Aitkin [2] and O. Carpenter [8] respectively are discussed. Apart from the Craig's criterion for testing the independence of two quadratic forms (Theorem 1), B. Matern [25] has given another criterion for the independence of non-negative quadratic forms in normally correlated variables. This result along with its extension to arbitrary quadratic forms as given by Y. Kawada [17] has been discussed. The chapter is concluded by proving a criterion which deals with the independence of quadratic forms of the type $Q(X_1, X_2, \ldots, X_n)$ where the X_i's follow a multivariate normal distribution.

In Chapter III, Cochran's theorem which gives "a necessary and sufficient condition for several quadratic forms to be independently distributed as χ^2" has been proved. The extensions of this theorem to the correlated case as given in [9], [4] and [22] have been discussed. Also various results which deal with conditions under which $Q = X'AX$ follows a χ^2-distribution, for X multivariate normal with variance-covariance matrix V (possibly singular) have been discussed. In these discussions various results on idempotent matrices have been used. The knowledge of idempotent matrices and their properties has been assumed.

The last chapter is devoted to some applications of various results established in the previous two chapters.

CHAPTER II

INDEPENDENCE OF QUADRATIC FORMS

§2.1. At various places in the study of statisticas we encounter linear, bilinear and quadratic forms. The t-test, variance ratio test and certain other tests of significance are valid only on condition that the linear, quadratic and bilinear forms concerned are statistically independent. In the present chapter we shall confine our studies to the independence of quadratic forms in normal random variables only.

The central theorem of this chapter is the one due to A.T. Craig [7]. But earlier W.G. Cochran [5] obtained another result which is not so easy to apply as is Craig's result. Because of the importance of Craig's thoerem, H. Hotelling [13] and A.C. Aitkin [2] also tried to prove this elegant theorem. Later J. Ogawa [27] gave an algebraic proof of Craig's theorem after pointing out some mistakes in the original proofs of Craig and Hotelling.

Craig's theorem as stated in [7] deals with independent rv's following univariate normal distribution with mean zero. Later A.C. Aitkin [2] obtained an extension to correlated case and O. Carpenter [8] extended this theorem to the case of noncentral normal variates with equal variance.

All these results stated above deal with the quadratic forms $Q(x_1,\ldots,x_n)$ where x_i's follow univariate normal distribution; J. Ogawa [27] proved a criterion for testing the independence of two quadratic forms when the random sample is drawn from multivariate normal population.

Apart from the Craig's and Cochran's result, B. Matern [25] has given another criterion which deals with non-negative quadratic forms in normally correlated variables. Later Y. Kawada [17] extended Matern's result from non-negative case to general case.

Finally, it is worth noting that the central theorem of the present chapter, i.e., Theorem 1, has been attributed in literature to A.T. Craig; but K. Matusita [26] page 82; has claimed that he had this result in 1943 independently of A.T. Craig and thus gives another independent proof in [26].

§2.2. The central theorem for the present chapter is one due to A.T. Craig [7] in which we suppose that $x_i \sim N(0,1)$ are independently distributed rv's. If Q_1, Q_2 are two quadratic forms in $x_1, x_2, \ldots, x_n$ with associated matrices (real and symmetric) A and B respectively, then:

Theorem 1: (Craig): A necessary and sufficient condition that Q_1 and Q_2 are independent in the probability sense is that the product $AB = 0$.

Here we give Ogawa's proof (see [27]) which makes use of the following lemma for its proof (see [27] page 89).

Lemma 1: Let the non-zero characteristic roots of real symmetric matrices A, B and $C = A+B$ be $\alpha_1, \alpha_2, \ldots, \alpha_q$; $\beta_1, \beta_2, \ldots, \beta_r$ and $\gamma_1, \gamma_2, \ldots, \gamma_s$ respectively. If the relations $s = q+r$ and

$$\prod_{k=1}^{s} \gamma_k = \prod_{i=1}^{q} \alpha_i \prod_{j=1}^{r} \beta_j$$

hold, then we have the relation $AB = BA = 0$..

Proof of the Theorem:

If $M(t_1,t_2)$ denotes the joint moment generating function of Q_1 and Q_2 then it can be shown (see [15], p. 385)

$$M(t_1,t_2) = |I-2t_1A-2t_2B|^{-(1/2)} .$$

Consequently Q_1, Q_2 are independent iff

$$M(t_1,t_2) = M(t_1,0)M(0,t_2)$$

i.e., $$|I-2t_1A-2t_2B| = |I-2t_1A| \cdot |I-2t_2B| \tag{1.1}$$

(c.f., [1], page 45)

We shall establish the equivalence of (1.1) and the condition $AB = 0$.

Suppose $AB = 0$;

$$(I-2t_1A)(I-2t_2B) = (I-2t_1A-2t_2B)$$

$$\Rightarrow |I-2t_1A| \cdot |I-2t_2B| = |I-2t_1A-2t_2B| . \tag{1.2}$$

Conversely suppose (1.1) holds; since it holds for all real values of t_1 and t_2, it holds when $t_1 = t_2 = \frac{1}{2x}$ (say), substituting $2t_1 = 2t_2 = \frac{1}{x}$ in (1.2), we have

$$|xI-A| \cdot |xI-B| = x^n |xI-B-A| . \tag{1.3}$$

This relation shows that the non-zero characteristic values of A+B are identical with those of A and B as a whole. Let

$\alpha_1, \alpha_2, \ldots, \alpha_q$; $\beta_1, \beta_2, \ldots, \beta_r$ and $\gamma_1, \gamma_2, \ldots, \gamma_s$ be non-zero ch. roots of A, B and A+B respectively. Then the smallest degree term of the ch. polynomial $|xI-A|$ is

$$(-1)^q \left(\prod_{i=1}^{q} \alpha_i\right) x^{n-q} .$$

Similarly

$$(-1)^r \left(\prod_{j=1}^{r} \beta_j\right) x^{n-r} \quad \text{and} \quad (-1)^s \left(\prod_{k=1}^{s} \gamma_k\right) x^{n} \cdot x^{n-s}$$

are respectively the smallest degree terms of the ch. polynomials $|xI-B|$ and $|xI-B-A|$ respectively.

Therefore smallest degree terms on both sides of (1.3) become

$$(-1)^{q+r} \left(\prod_{i=1}^{q} \alpha_i \cdot \prod_{j=1}^{r} \beta_j\right) x^{2n-(q+r)} \quad \text{and} \quad (-1)^s \left(\prod_{k=1}^{s} \gamma_k\right) x^{2n-s}$$

respectively.

Now because of the equality in (1.3) these must be the same, consequently

$$s = q+r \quad \text{and} \quad \left(\prod_{i=1}^{q} \alpha_i\right)\left(\prod_{j=1}^{r} \beta_j\right) = \prod_{k=1}^{s} \gamma_k .$$

Therefore conditions of Lemma 1 are satisfied and thus it follows (from Lemma 1) AB = 0. /

Because of the importance of this theorem many statisticians tried to reprove this theorem differently, Hotelling [13] gave quite a rigorous proof of this theorem, but unfortunately both the original proof of Craig [7] as well as Hotelling's [13] have some mis

as pointed out later by Ogawa [27]. For the defect in Hote

proof see [27] page 95.

Theorem 1 as stated above deals with uncorrelated

~~The following is an extension due to Aitkin [2] of Theorem~~

correlated variates.

<u>Theorem 2</u>: Let $x_1, x_2, \ldots, x_n$ be normal random variables w

zero and variance covariance matrix V. If $Q_1 = X'AX$ an

$Q_2 = X'BX$ (where $X' = (x_1, \ldots, x_n)$ are two quadratic forms

Q_1 and Q_2 are independent iff $AVB = 0$. (V positive de

<u>Proof</u>: Since V is positive definite it admits a real squ

$V^{1/2}$.

Consider the transformation $Y = V^{-(1/2)}X$. Then

i.e., $Y' = (y_1, \ldots, y_n)$ are uncorrelated variates with uni

$$\therefore \quad Q_1 \text{ becomes } \quad Y'V^{1/2}AV^{1/2}Y$$

$$\text{and } Q_2 \text{ becomes } \quad Y'V^{1/2}BV^{1/2}Y \; .$$

$\therefore$ Applying Theorem 1 now we have Q_1 and Q_2 independen

$$V^{1/2}AV^{1/2}V^{1/2}BV^{1/2} = 0$$

$$\text{i.e.,} \quad V^{1/2}AVBV^{1/2} = 0 \iff AVB = 0 \; . \quad /$$

Before Craig's result (Theorem 1), Cochran [5] obta

following condition for testing the independence of two qua forms in independent $N(0,1)$ variates. However Craig's result is relatively easier to apply.

<u>Theorem 3</u>: Let $x_1, x_2, \ldots, x_n$ be normally and independently distributed with zero mean and unit variance; then the quadratic forms $Q_1 = (\frac{1}{2})X'AX$ and $Q_2 = (\frac{1}{2})X'BX$ are independent iff

$$|I-it_1A-it_2B| = |I-it_1A| \cdot |I-it_2B| .$$

<u>Proof</u>: If M_{AB}, (M_A), (M_B) denote the characteristic functions of Q_1, Q_2, (Q_1) and (Q_2) respectively, then it can be shown (cf., [30])

$$M_{AB} = |I-it_1A-it_2B|^{-(1/2)}$$

$$M_A = |I-it_1A|^{-(1/2)}$$

$$M_B = |I-it_2B|^{-(1/2)}$$

and result follows immediately on noting that Q_1 and Q_2 are independent iff $M_{AB} = M_A \cdot M_B$.

§2.3. In this section we shall discuss briefly some of the results which will be used quite implicitly in the remainder of this chapter and in the subsequent chapters. Because of their importance we shall explicitly state these results here and if necessary proofs will be outlined.

Let $Q = X'AX$ be a quadratic form in variables (not necessarily random) $(x_1, x_2, \ldots, x_n) = X'$, with associated (real) matrix A. Obviously without any loss of generality A can be assumed to be symmetric. Then there exists (cf., [14], page 255) an orthogonal transformation $X = TY$ such that

$$Q = \sum_{j=1}^{m} \lambda_j y_j^2 \tag{1.4}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are non-zero eigenvalues of A; m = rank of A. Here we recall the following:

Definition: If A is the real symmetric matrix associated with the quadratic form Q; then the DEGREES OF FREEDOM of Q is defined to be the rank of A.

Theorem 5: Let $X' = (x_1, x_2, \ldots, x_n)$. Suppose X is $N(\mu, V)$ (V non-singular), then r^{th} cumulant of $X'AX$ is

$$K_r(X'AX) = 2^{r-1}(r-1)!\,[\mathrm{tr}(AV)^r + r\mu' A(VA)^{r-1}\mu] \tag{1.5}$$

(where $\mathrm{tr}(AV)$ means trace of AV).

Proof: Let $M_Q(t)$ denote the moment generating function of quadratic form $Q = X'AX$. Then it can be shown (see for example [30] page 55) that,

$$M_Q(t) = |I-2tAV|^{-(1/2)} \exp \{-\tfrac{1}{2}\, \mu'[I-(I-2tAV)^{-1}]V^{-1}\mu\} .$$

Since cumulant generating function is the logrithm of the moment generating function, we have

$$\sum_{r=1}^{\infty} k_r t^r/r! = \log [M_Q(t)]$$

$$= -\frac{1}{2} \log|I-2tAV| - \frac{1}{2} \mu'[I-(I-2tAV)^{-1}]V^{-1}\mu . \quad (1.6)$$

Using the convention "λ_i of X" to denote the "i^{th} characteristic root of X" then for sufficiently small $|t|$ we have

$$-\frac{1}{2} \log|I-2tAV| = -\frac{1}{2} \sum_{i=1}^{n} \log [\lambda_i \text{ of } (I-2tAV)]$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \log [1 - 2t(\lambda_i \text{ of } AV)]$$

$$= \frac{1}{2} \sum_{i=1}^{n} \sum_{r=1}^{\infty} [2t(\lambda_i \text{ of } AV)]^r/r$$

$$= \sum_{r=1}^{\infty} \{(2^{r-1} t^r/r) \sum_{i=1}^{n} (\lambda_i \text{ of } AV)^r\}$$

$$= \sum_{r=1}^{\infty} (2^{r-1} t^r/r) \operatorname{tr}(AV)^r .$$

Also by direct binomial expansion, for sufficiently small $|t|$

$$I - (I-2tAV)^{-1} = -\sum_{r=1}^{\infty} 2^r t^r (AV)^r .$$

Making these substitutions in (1.6) and comparing the coefficient of t^r we get (1.5).

Theorem 6: Let X be $N(\mu, V)$, with V non-singular. Then

(i) $E(X'AX) = \operatorname{tr}(AV) + \mu'A\mu$

(ii) $\mathrm{Var}\,(X'AX) = 2\,\mathrm{tr}(AV)^2 + 4\mu' AVA\mu$

(iii) $\mathrm{Cov}\,(X'AX, X'BX) = 2\,\mathrm{tr}(AVBV) + 2\mu'(AVB+BVA)\mu$

(iv) $\mathrm{Cov}\,(X, X'AX) = 2VA\mu$.

Proof: (i) Since $X'AX$ is a scalar quantity,

$$\begin{aligned} X'AX &= \mathrm{tr}(X'AX) \\ &= \mathrm{tr}(AXX') \qquad (\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB)) \end{aligned}$$

$$\therefore\ E(X'AX) = E(\mathrm{tr}(AXX')) = \mathrm{tr}\,E(AXX') = \mathrm{tr}(AE(XX'))\ .$$

But $E(XX') = V + \mu\mu'$, therefore

$$\begin{aligned} E(X'AX) &= \mathrm{tr}\,A(V+\mu\mu') \\ &= \mathrm{tr}\,AV + \mathrm{tr}(A\mu\mu') \\ &= \mathrm{tr}\,AV + \mathrm{tr}(\mu' A\mu) \\ &= \mathrm{tr}\,AV + \mu' A\mu\ . \end{aligned}$$

(ii) It follows immediately from Theorem 5 on taking $r = 2$.

(iii) Consider the quadratic form

$$X'AX + X'BX \equiv X'(A+B)X\ .$$

$$\mathrm{Var}(X'AX+X'BX) = \mathrm{Var}(X'AX) + \mathrm{Var}(X'BX) + 2\,\mathrm{Cov}(X'AX, X'BX)$$

and

$$\begin{aligned} \mathrm{Var}(X'(A+B)X) &= 2\,\mathrm{tr}[(A+B)V]^2 + 4\mu'(A+B)V(A+B)\mu\ . \\ &\qquad\qquad \text{(from part (ii))} \\ &= 2\,\mathrm{tr}[AV]^2 + 2\,\mathrm{tr}[BV]^2 + 4\,\mathrm{tr}\,[AVBV] \\ &\quad + 4\mu' AVA\mu + 4\mu' BVB\mu + 4\mu'[AVB+BVA]\mu \end{aligned}$$

$$= \mathrm{Var}\,(X'AX) + V(X'BX) + 4\{\mathrm{tr}(AVBV) + \mu'(AVB+BVA)\mu\} .$$

Hence $\mathrm{Cov}(X'AX,X'BX) = 2\ \mathrm{tr}(AVBV) + 2\mu'(AVB+BVA)\mu$.

$$\text{(iv)}\quad \mathrm{Cov}(X,X'AX) = E\,\{(X-\mu)(X'AX-E(X'AX))\}$$

$$= E\{(X-\mu)(X'AX-\mathrm{tr}AV-\mu'A\mu)\}$$

$$= E\{(X-\mu)[(X-\mu)'A(X-\mu)+2(X-\mu)'A\mu-\mathrm{tr}AV]\}$$

$$= 2E\{(X-\mu)(X-\mu)'A\mu\} .$$

($\because$ First and third moments of $(X-\mu)$ are zero.)

i.e., $\mathrm{Cov}\,(X,X'AX) = 2VA\mu$.

The following useful results are corollaries of Theorem 6.

<u>Corollary</u>: If X is $N(0,V)$, then

(i) $E(X'AX) = \mathrm{tr}(AV)$

(ii) $\mathrm{Var}\,(X'AX) = 2\ \mathrm{tr}(AV)^2$

(iii) $\mathrm{Cov}\,(X'AX,X'BX) = 2\ \mathrm{tr}(AVBV)$ (1.7)

Now we shall state a theorem which deals with matrices and will be used frequently in Chapter III, (cf., [7]).

<u>Theorem 7</u>: Let $A_1,A_2,\ldots,A_m$ be a collection of $n \times n$ symmetric matrices where the rank of A_i is p_i, and let $A = \sum_{i=1}^{m} A_i$; where the rank of A is p. Consider the four conditions:

C_1 : Each A_i is an idempotent matrix.

C_2 : $A_i \cdot A_j = 0$ (null matrix) for all $i \neq j$.

C_3 : A is an idempotent matrix

C_4 : $p = \sum_{i=1}^{m} p_i$.

Then the following are true.

(i) C_1 and C_3 imply C_2 .

(ii) C_2 and C_3 imply C_1 .

(iii) C_1 and C_2 imply C_3 .

(iv) Any two C_1, C_2 and C_3 imply all four C_1, C_2, C_3 and C_4.

(v) C_3 and C_4 imply C_1 and C_2.

<u>Proof</u>: (i) We have

$$A = A^2 = (\sum_{i=1}^{m} A_i)^2 = \sum_{i=1}^{m} A_i^2 + \sum_{i\neq j} A_iA_j \tag{1.8}$$

$$= \sum_{i=1}^{m} A_i + \sum_{i\neq j} A_iA_j$$

$$= A + \sum_{i\neq j} A_iA_j .$$

Therefore,

$$Tr(A) = Tr(A) + Tr(\sum_{i\neq j} A_iA_j)$$

hence,

$$Tr(\sum_{i\neq j} A_iA_j) = 0$$

or equivalently

$$\sum_{i \neq j} \mathrm{Tr}(A_i A_j) = 0 \tag{1.9}$$

but now $\quad \mathrm{Tr}(A_i \cdot A_j) = \mathrm{Tr}(A_i^2 A_j^2) = \mathrm{Tr}(A_j A_i^2 A_j)$

$$= \mathrm{Tr}(A_i A_j)'(A_i A_j) \tag{*}$$

From this in view of (1.9) we conclude

$$A_i A_j = 0 \quad \text{for all } i \neq j.$$

(ii) Since $A_i A_j = 0$ for all $i \neq j$, we have

$$0 = \mathrm{Tr}(A_i A_j) = \mathrm{Tr}(A_i^2 A_j^2) = \mathrm{Tr}(A_i A_j^2) = \mathrm{Tr}(A_i^2 A_j) \tag{1.9a}$$

Also looking at (1.8) we have

$$\sum_i A_i = \sum_i A_i^2 .$$

Write

$$A_i^2 - A_i = - \sum_{j \neq i} (A_j^2 - A_j) .$$

Since A_i are symmetric, we have

$$\mathrm{Tr}(A_i^2 - A_i)'(A_i^2 - A_i) = - \mathrm{Tr}[(A_i^2 - A_i) \{ \sum_{j \neq i} (A_j^2 - A_j) \}]$$

but because of relations in (1.9a) we get

$$\mathrm{Tr}(A_i^2 - A_i)'(A_i^2 - A_i) = 0 .$$

Therefore,

$$A_i^2 - A_i = 0 \quad \text{for any } i \quad (\text{see } (*))$$

hence C_1.

(iii) We have

$$A^2 = (\sum_i A_i)^2 = \sum_i A_i^2 + \sum_{i \neq j} A_i A_j$$

$$= \sum_i A_i^2 = \sum_i A_i = A \ .$$

(iv) In order to prove this, it is sufficient (by virtue of (i), (ii) and (iii)) to prove that C_1, C_2 and C_3 imply C_4.

Since the rank of an idempotent matrix is equal to its trace,

$$\text{Rank } A = \text{Tr } A = \sum_i \text{Tr } A_i = \sum_i \text{Rank } A_i$$

equivalently

$$p = \sum_i p_i \ .$$

(v) Consider the set of equations $Ax = x$, $A_2 x = 0, \ldots, A_m x = 0$. $Ax = x$ can be written as $(A-I)x = 0$. Since A is idempotent of rank p, there exists orthogonal P such that

$$P'AP = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and thus} \quad P'(A-I)P = \begin{bmatrix} 0 & 0 \\ 0 & -I_{n-p} \end{bmatrix} .$$

Therefore rank $(A-I)$ = rank $(P'(A-I)P) = n-p$.

Hence the equations $Ax = x$, $A_2x = 0$, $A_3x = 0, \ldots, A_mx = 0$ contain at most $n-p+p_2+\ldots+p_m = n-p_1$ independent equations, and thus have at least p_1 independent solutions. Thus these equations give at least p_1 independent solutions to $A_1x = x$. By writing $A_1x = x$ as $(A_1-I)x = 0$ we see that there are exactly p_1 independent solutions x. Now therefore from the characteristic equation $A_1x = x$ we conclude all the non-zero characteristic values of A_1 are $+1$, and thus A_1 is idempotent. Similarly A_i is idempotent for $i = 1,2,\ldots,m$. Therefore C_1 follows and hence C_1 together with C_3 gives C_2 by (i) and hence the result follows.

§2.4. The criteria which we discussed in §2.2 were dealing with general quadratic forms. If we confine our attention to non-negative quadratic forms then the following is an easy criterion for testing the independence.

Theorem 8: (Matern [25]): If two non-negative quadratic forms in normally correlated variables with zero means are uncorrelated, then the two forms are independent.

Proof: Let the two forms be

$$Q_1 = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j \;, \qquad Q_2 = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ij}x_ix_j$$

where x_i's are normally correlated with mean 0. Write

$$Q_1 = \sum_{i=1}^{m} c_iy_i^2 \;, \qquad Q_2 = \sum_{i=1}^{p} d_iz_i^2 \tag{1.10}$$

where y_i's and z_i's are linear functions of x_i's (see §2.3), and all c_i and d_j are positive. Ignoring the subscripts i and j, let us suppose σ_y^2 and σ_z^2 denote respectively the variances of y and z, then we show that

$$\operatorname{cov}(y,z) = \sigma \implies \operatorname{cov}(y^2,z^2) = 2\sigma^2 .$$

For

$$\begin{aligned}
\operatorname{Cov}(y^2,z^2) &= E(y^2z^2) - \sigma_y^2\sigma_z^2 \\
&= \int\int y^2z^2 f(y,z)\,dydz - \sigma_y^2 \cdot \sigma_z^2 \\
&= \int\int y^2z^2 f(y/z) f(z)\,dydz - \sigma_y^2\sigma_z^2 \\
&= \int z^2 f(z)\left[\sigma_y^2(1-\rho^2) + \rho^2 \frac{\sigma_y^2}{\sigma_z^2} z^2\right]dz - \sigma_y^2\,\sigma_z^2 \\
&\qquad (\rho = \text{correlation } (y,z)) \\
&= \sigma_y^2(1-\rho^2)\sigma_z^2 - \sigma_y^2\sigma_z^2 + \rho^2 \frac{\sigma_y^2}{\sigma_z^2}(3\sigma_z^4) \\
&= 2\rho^2\,\sigma_y^2\sigma_z^2 \\
&= 2\sigma^2 .
\end{aligned}$$

Therefore

$$\operatorname{Cov}(Q_1,Q_2) = 2 \sum_{j=1}^{m} \sum_{i=1}^{p} c_i d_j \sigma_{ij}^2 .$$

All c_i and d_j being positive, therefore

$$\text{Cov}\ (Q_1,Q_2) = 0 \implies \sigma_{ij}^2 = 0 \ ,$$

i.e., Q_1 and Q_2 are independent. /

Also from Section 2.3 we known that if $Q_1 = X'AX$ and $Q_2 = X'BX$ where X is $N(0,V)$, then

$$\text{Cov}\ (Q_1,Q_2) = 2\ \text{tr}\ (AVBV) \ .$$

Thus in the case of independent normal variates with mean zero and unit variance, Matern's result gives

$$\text{tr}\ (AB) = 0.$$

But

$$\text{tr}\ (AB) = \text{tr}(A^{1/2}A^{1/2}B^{1/2}B^{1/2})$$

or

$$\text{tr}\ (AB) = \text{tr}(B^{1/2}A^{1/2}A^{1/2}B^{1/2}) \qquad (1.11)$$

Hence

$$AB = A^{1/2}A^{1/2}B^{1/2}\ B^{1/2} = 0 \ .$$

Thus, obviously it follows that Matern's result (Theorem 8) for the case $V = I$ is equivalent to Craig's result.

Y. Kawada [17] generalised Matern's result (Theorem 8) to the case of arbitrary quadratic forms, but only when the variance - covariance matrix $V = I$. We now give Kawada's generalization.

Theorem 9: If two quadratic forms

$$Q_1 = \sum_{i,j=1}^{n} a_{ij}x_ix_j \ , \qquad Q_2 = \sum_{i,j=1}^{n} b_{ij}x_ix_j \tag{1.12}$$

in normally correlated variables $x_1,x_2,\ldots,x_n$ with zero means and variance - covariance matrix $V = I$ satisfy the following conditions

$$F_{ij} = E(Q_1^iQ_2^j) - E(Q_1^i)E(Q_2^j) = 0 \ , \quad i,j = 1,2 \tag{1.13}$$

then the relation

$$AB = 0 \qquad (A = (a_{ij});\ B = (b_{ij}))$$

holds.

Before we outline the proof of this, we note the following simple but nonetheless important consequences.

(I) When Q_1 and Q_2 are non-negative then Theorem 8 (Matern's result) follows from Theorem 9 by taking $i = 1$, $j = 1$ in (1.13). For, then, $F_{11} = \text{Cov}\ (Q_1,Q_2)$; but

$$\text{Cov}\ (Q_1,Q_2) = 2\ \text{Tr}(AB)$$

$$\therefore \quad \text{Tr}(AB) = 0 \Longrightarrow AB = 0 \ .$$

(II) If Q_1, Q_2 in (1.12) satisfy four conditions in (1.13) then Q_1 and Q_2 are independent. This follows because $AB \doteq 0$

implies independence. (Sufficiency of Craig's Theorem).

(III) The necessity part of Craig's theorem follows from Theorem 9. i.e., If Q_1 and Q_2 are independent then $AB = 0$. It is quite clear that the independence of Q_1 and Q_2 implies (1.13) and hence $AB = 0$.

Proof of Theorem 9.

It can easily be shown that the first eight moments of x_k $(k = 1,2,\ldots,n)$ are $E(x_k^i) = 0$, $i = 1,3,5,7$. $E(x_k^2) = 1$, $E(x_k^4) = 3$, $E(x_k^6) = 15$, $E(x_k^8) = 105$. Using these values and by a straightforward calculation we have

(1) $F_{11} = 2\,\mathrm{Tr}(AB)$

(2) $F_{12} = 8\,\mathrm{Tr}(AB^2) + 4\,\mathrm{Tr}(AB)\mathrm{Tr}(B)$

(3) $F_{21} = 8\,\mathrm{Tr}(BA^2) + 4\,\mathrm{Tr}(AB)\mathrm{Tr}(A)$

(4)
$$F_{22} = 32\,\mathrm{Tr}(A^2B^2) + 16\,\mathrm{Tr}[(AB)^2] + 16\,\mathrm{Tr}(AB^2)\mathrm{Tr}(A) + 16\,\mathrm{Tr}(A^2B)\mathrm{Tr}(B) + 8\,\mathrm{Tr}(AB)\mathrm{Tr}(A)\mathrm{Tr}(B) + 8[\mathrm{Tr}(AB)]^2 .$$

(1) follows immediately from (1.7); (2) and (3) are symmetrical. For the sake of illustration, we outline here the proof of (2).

$$F_{12} = E(Q_1 \cdot Q_2^2) - E(Q_1)E(Q_2^2) .$$

On diagonalizing,

$$E(Q_1 \cdot Q_2^2) = E[(\sum_i a'_i y_i^2)(\sum_{i,j} b'_{ij} y_i y_j)^2]$$

$$= E(\sum_{i,k,\ell,m,n} a'_i b'_{k\ell} b'_{mn} y_i^2 y_k y_\ell y_m y_n)$$

$$= \sum_i 15a'_i b'^2_{ii} + \sum_{i \neq j} 3a'_i b'^2_{jj} + \sum_{i \neq j} 6a'_i b'_{ii} b'_{jj}$$

$$+ \sum_{i \neq j} 12a'_i b'^2_{ij} + \sum_{i \neq j \neq k} a'_i b'_{jj} b'_{kk} + \sum_{i \neq j \neq k} 2a'_i b$$

$$= 8 \sum_{i \neq j} a'_i b'^2_{ij} + 8 \sum_i a'_i b'^2_{ii} + 4 \sum_{i \neq j} a'_i b'_{ii} b'_{jj} + 4 \sum_i a'_i b'^2_{ii}$$

$$+ 2 \sum_{i \neq j \neq k} a'_i b'^2_{jk} + 2 \sum_{i \neq j} a'_i b'^2_{jj} + 4 \sum_{i \neq k} a'_i b'^2_{ik}$$

$$+ 2 \sum_i a'_i b'^2_{ii} + \sum_{i \neq j \neq k} a'_i b'_{jj} b'_{kk} + \sum_{i \neq j} a'_i b'^2_{jj}$$

$$+ \sum_{i \neq j} 2a'_i b'_{ii} b'_{jj} + \sum_i a'_i b'^2_{ii}$$

$$= 8\, TR(AB^2) + 4 Tr(AB) Tr(B) + 2 Tr\, A Tr B^2 + (Tr B)^2 Tr A \qquad (1$$

because

$$Tr(A) = \sum_i a'_i$$

$$Tr(B) = \sum_i b'_{ii}$$

$$Tr(B^2) = \sum_i \sum_j b'^2_{ij}$$

$$Tr(AB) = \sum_i a'_i b'_{ii}$$

and

$$Tr(AB^2) = \sum_{i,k} a'_i b'^2_{ik}$$

Now it is easy to show that

$$E(Q_1)E(Q_2^2) = 2TrA\cdot TrB^2 + (TrB)^2TrA \quad . \tag{1.15}$$

Hence (1.14) and (1.15) together give

$$F_{12} = 8Tr(AB^2) + 4Tr(AB)Tr(B) \quad .$$

Equating (1), (2), (3) and (4) to zero and simplifying we have

$$2Tr(A^2B^2) + Tr[(AB)^2] = 0 \quad . \tag{1.16}$$

Write $C = AB$, then (1.16) becomes

$$2Tr(CC') + Tr[C^2] = 0 \quad . \tag{1.17}$$

If $C = (c_{ij})$, $(i,j = 1,2,\ldots,n)$. Then (1.17) can be written in the expanded form as

$$\sum_{i,j=1}^{n} (c_{ij}^2+c_{ij}c_{ji}+c_{ji}^2) = 0 \quad . \tag{1.18}$$

In (1.18) if $c_{ij}\cdot c_{ji} > 0$, the corresponding term of the summation is positive,

if $c_{ij}c_{ji} < 0$, write $(c_{ij}^2+c_{ij}c_{ji}+c_{ji}^2)$ as $(c_{ij}+c_{ji})^2 - c_{ij}c_{ji}$ which is positive (being sum of two positive terms).

Thus in any case the left hand side of (1.18) is positive unless every $c_{ij} = 0$, $i,j = 1,2,\ldots,m$.

i.e., $C = 0$.

Hence $AB = 0$.

In (I) above we looked at the case when A, B were both non-negative. If however only A is non-negative then $F_{11} = 0$ and $F_{12} = 0$ together imply $AB = 0$.

<u>Theorem 10</u>: In the context of Theorem 9, let A be non-negative; then $F_{11} = 0$ and $F_{12} = 0$ imply $AB = 0$.

<u>Proof</u>: On solving $F_{11} = 0$ and $F_{12} = 0$ we have $Tr(AB^2) = 0$. Since A is non-negative, we can choose a real symmetric matrix A_o such that $A = A_o^2$, let $C_o = A_oB$, then $Tr(AB^2) = Tr(C_oC_o') = 0$.

But

$$2Tr(C_oC_o') = 0 \Longrightarrow \sum_{i,j=1}^{n} (c_{ij}^2 + c_{ji}^2) = 0 \quad (C_o = (c_{ij})) .$$

Therefore,

$$C_o = 0 \text{ or equivalently } A_oB = 0 .$$

Hence $AB = A_o(A_oB) = 0$ ./

Now let us suppose that $Q_1, Q_2, \ldots, Q_m$ are m real symmetric non-negative (or non-positive) quadratic forms. Write

$$Q' = \sum_{i=1}^{m} Q_i .$$

If Q is any other quadratic form (or linear form) then B.R. Bhat [3] investigates independence of Q and Q'. In the following theorem we shall give a criterion for the independence of Q and Q' for $m = 2$, this theorem can easily be extended to the case of any finite m.

Theorem 11: If Q_1 and Q_2 are two real symmetric non-negative (or non-positive) quadratic forms, then a quadratic form Q is distributed independently of Q_1+Q_2 if and only if it is distributed independently of Q_1 and of Q_2. $(X \sim N(\mu,V).)$

Proof: (Necessity): Let $Q_1 = X'A_1X$, $Q_2 = X'A_2X$ and $Q = X'BX$.

Suppose Q is distributed independently of Q_1+Q_2. Therefore

$$(A_1+A_2)VB = 0 \quad , \tag{1.19}$$

where V is variance covariance matrix of X and

$$X' = (x_1,x_2,\ldots,x_n) \quad .$$

Write VB as D, therefore

$$L_1'(A_1+A_2)D = 0 \quad , \tag{1.20}$$

where L_1 is any n-dimension column vector.

It is clear that we can assume without any loss of generality that A_2 is diagonal. Let L_1 be the first column of D. Therefore, (1.20) implies

$$L_1'(A_1+A_2)L_1 = 0 \quad .$$

$$\text{i.e.,} \quad L_1'A_1L_1 = 0 \quad \text{and} \quad L_1'A_2L_1 = 0$$

($\because$ A_1 and A_2 are both non-negative or non-positive).

If we denote L_1' by $(\ell_1,\ell_2,\ldots,\ell_n)$ and A_2 by

Diag $(d_1, d_2, \ldots, d_n)$ then $L_1' A_2 L_1 = 0$ implies

$$\sum_i d_i \ell_i^2 = 0$$

but all non-zero d_i's are positive or negative.

$$\therefore \quad \sum_i d_i \ell_i^2 = 0 \implies \sum_i d_i \ell_i = 0 \ .$$

i.e., $\quad A_2 L_1 = 0$.

If $L_1, L_2, \ldots, L_n$ denote n columns of D then (1.19) implies

$$(A_1 + A_2) L_i = 0 \ , \quad (i = 1, 2, \ldots, n)$$

$$\therefore \quad A_2 L_1 = 0 \implies A_1 L_1 = 0 \ .$$

Repeating this process by taking $L_2, L_3, \ldots, L_n$ we finally get

$$A_2 D = A_2 VB = 0$$

and

$$A_1 D = A_1 VB = 0 \ .$$

Again recalling Theorem 2, we conclude Q is distributed independently of Q_1 and of Q_2.

Sufficiency of the result is obvious as $A_1 VB = 0$ and $A_2 VB = 0$ clearly imply (1.19).

§2.5. So far we have discussed certain criteria for testing the independence of two quadratic forms. However, our discussion was invariably confined to the central normal random variables. In this section we shall extend Craig's theorem to the non-central correlated case. Finally we shall give an extension of this theorem which is due to J. Ogawa [27] when the random sample is taken from a multivariate normal distribution. First we give the following extension which is due to O. Carpenter [8].

<u>Theorem 12</u>: Let $X' = (x_1, x_2, \ldots, x_n)$ be a set of normally and independently distributed variates with equal variance σ^2 and means $\mu' = (\mu_1, \mu_2, \ldots, \mu_n)$. Let $Q_1 = (1/2)X'A_1X$ and $Q_2 = (1/2)X'A_2X$ be real symmetric quadratic forms of rank r_1 and r_2 respectively. Then a necessary and sufficient condition that Q_1 and Q_2 be statistically independent is that $A_1 \cdot A_2 = 0$.

<u>Proof</u>: We assume w.ℓ.o.g. that $\sigma^2 = 1$. Let $M(t_1,t_2)$ be the joint moment generating function of Q_1 and Q_2; then

$$M(t_1,t_2) = \exp[\frac{1}{2}\mu'(t_1A_1+t_2A_2)(I-t_1A_1-t_2A_2)^{-1}\mu]\cdot|I-t_1A_1-t_2A_2|^{-(1/2)}$$

where t_1 and t_2 are restricted to those values for which $(I-t_1A_1-t_2A_2)$ is positive definite (cf., [15], page 389).

We shall show that

$$M(t_1,t_2) = M(t_1,0)\cdot M(0,t_2)$$

if and only if

$$A_1 \cdot A_2 = 0 .$$

Assume $A_1A_2 = 0$, then clearly $(I-t_1A_1-t_2A_2) = (I-t_1A_1)(I-t_2A_2)$ and therefore,

$$|I-t_1A_1-t_2A_2| = |I-t_1A_1| \cdot |I-t_2A_2| \ , \tag{1.21}$$

$$\begin{aligned}(I-t_1A_1)(I-t_2A_2) &= (I-t_1A_1-t_2A_2)\\ &= (I-t_1A_1)+(I-t_2A_2)-I \ .\end{aligned}$$

Multiplying on the left by $(I-t_1A_1)^{-1}$ and on the right side by $(I-t_2A_2)^{-1}$ we have

$$I = (I-t_2A_2)^{-1}+(I-t_1A_1)^{-1}-(I-t_1A_1)^{-1}(I-t_2A_2)^{-1} \ .$$

$$\therefore \quad (I-t_1A_1)^{-1}(I-t_2A_2)^{-1}-I = (I-t_2A_2)^{-1}-I+(I-t_1A_1)^{-1}-I \ . \tag{1.22}$$

But $(tA)(I-tA)^{-1} = (I-tA)^{-1}-I$.

$\therefore$ Using this, (1.22) gives

$$(t_1A_1+t_2A_2)(I-t_1A_1-t_2A_2)^{-1} = t_2A_2(I-t_2A_2)^{-1}+t_1A_1(I-t_1A_1)^{-1} \tag{1.23}$$

Thus because of (1.21) and (1.23) we immediately conclude that

$$M(t_1,t_2) = M(t_1,0)M(0,t_2) \ .$$

Conversely suppose $M(t_1,t_2) = M(t_1,0)M(0,t_2)$. Then

$$e^{\frac{1}{2}\mu'(t_1A_1+t_2A_2)(I-t_1A_1-t_2A_2)^{-1}\mu} \; |I-t_1A_1-t_2A_2|^{-(1/2)}$$

$$= e^{\frac{1}{2}\mu'\{t_1A_1(I-t_1A_1)^{-1}+t_2A_2(I-t_2A_2)^{-1}\}\mu}$$

$$\cdot\ |I-t_1A_1|^{-(1/2)}|I-t_2A_2|^{-(1/2)} .$$

Since μ is arbitrary, we can expand the exponential terms as power series in μ and compare the coefficients. We conclude (from comparing the first term of the expansions

$$|I-t_1A_1-t_2A_2|^{-(1/2)} = |I-t_1A_1|^{-(1/2)}\cdot|I-t_2A_2|^{-(1/2)}$$

$$\text{i.e.,}\quad |I-t_1A_1-t_2A_2| = |I-t_1A_1|\cdot|I-t_2A_2| ,$$

but we have already shown in the proof of Theorem 1 that this implies $A_1A_2 = 0$. /

Now we shall discuss the statistical independence of two quadratic forms in the case of multivariate normal population as given by J. Ogawa [27].

Consider the k-variate normal population with means 0 and Var.-Cov. matrix V distributed according to $(2\pi)^{-k/2}|V|^{-1/2}\cdot\exp[-\frac{1}{2}(r'V^{-1}r)]dr$ where $r' = (r_1,r_2,\ldots,r_k)$; $dr = dr_1,dr_2,\ldots,dr_k$: then

<u>Theorem 13</u>: Let $X' = (X_1,X_2,\ldots,X_n)$ be a random sample of size n from a k-variate normal population. Then the quadratic forms $Q_1 = X'AX$ and $Q_2 = X'BX$ are independent in the statistical sense if and only if

$$A\hat{V}B = 0; \quad \hat{V} = V \times I \quad \text{(Kronecker product)}$$

holds for coefficient matrices A and B; where V is covariance matrix of normal population.

Outline of Proof: (for details cf., Ogawa [27], p. 99): Considering X as a vector of nk-dimension, the moment generating function of Q_1 is given by:

$$M_1(t) = (2\pi)^{-nk/2}|V|^{-(n/2)} \underbrace{\int \cdots \int}_{n} \cdot \exp\{-\frac{1}{2}|\sum_{v=1}^{n}(r_v'V^{-1}r_v)-2t(X'AX)|\}\, dr_1,\ldots,dr_n \tag{1.24}$$

where $dr_v = dx_{v1},dx_{v2},\ldots,d_{vk}$, $v = 1,2,\ldots,n.$

Choose P orthogonal such that

$$P'V^{-1}P = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_k \end{bmatrix} = D$$

and since V^{-1} is positive definite all λ_i are positive.

If we write $r_v^* = r_vP$, $v = 1,2,\ldots,n$ then

$$X = (x_{11},x_{21},\ldots,x_{12},\ldots,x_{nk})$$

is transformed to

$$X^* = (x_{11}^*,x_{21}^*,\ldots,x_{12}^*,\ldots,x_{nk}^*)$$

by the transformation matrix

$\hat{P} = P \times I$ (Kronecker product; cf. [28], p. 29). The Jacobian of this transformation is

$$\left| \frac{\partial(x)}{\partial(x^*)} \right| = |\det \hat{P}| = |\det P|^n = 1 .$$

$\therefore$ Integral (1.24) reduces to

$$M_1(t) = (2\pi)^{-nk/2}|V|^{-(1/2)} \iint \cdots \int \exp\{-\tfrac{1}{2}[(X^{*\prime}\hat{D}X^*) - 2t(X^{*\prime}\hat{P}'X^*)]\}\, dr_1^* \ldots dr_n^* , \tag{1.25}$$

where $\hat{D} = D \times I$.

If we make another transformation

$$Y = X^*\hat{Q}$$

where $\hat{Q} = Q \times I$ and

$$Q = \begin{bmatrix} \sqrt{\lambda_1} & & & 0 \\ & \sqrt{\lambda_2} & & \\ & & \ddots & \\ 0 & & & \sqrt{\lambda_k} \end{bmatrix}$$

the Jacobian of this transformation is

$$\left| \frac{\partial(x^*)}{\partial(y)} \right| = |\det \hat{Q}^{-1}| = |\det D|^{-(n/2)} = |\det V|^{n/2} .$$

Consequently (1.25) becomes

$$M_1(t) = (2\pi)^{-(nk/2)} \underbrace{\int \dots \int}_{n} \cdot$$

$$\cdot \exp \{y' \cdot y - 2t(\hat{Q}^{-1'} y' \hat{P}' A \hat{P} y \hat{Q}^{-1})\} \, dy \quad ,$$

and this gives

$$M_1(t) = |I-2tA^*|^{-(1/2)}$$

where $A^* = \hat{Q}^{-1}\hat{P}'A\hat{P}\hat{Q}^{-1}$.

Similarly the moment generating function of Q_2 is

$$M_2(t) = |I-2tB^*|^{-(1/2)}$$

where $B^* = \hat{Q}^{-1}\hat{P}'B\hat{P}\hat{Q}^{-1}$.

Then Q_1 and Q_2 are independent iff

$$A^*B^* = 0$$

i.e., $$\hat{Q}^{-1}\hat{P}'A\hat{P}\hat{Q}^{-2}\hat{P}'B\hat{P}\hat{Q}^{-1} = 0 \quad . \tag{1.26}$$

But

$$\begin{aligned} \hat{P}\hat{Q}^{-2}\hat{P}' &= (P \times I)(Q^{-2} \times I)(P' \times I) \\ &= (PD^{-1}P') \times I = V \times I = \hat{V} \quad . \end{aligned}$$

$$\therefore \quad (1.26) \iff \hat{Q}^{-1}\hat{P}'A\hat{V}B\hat{P}\hat{Q}^{-1} = 0$$

$$\iff A\hat{V}B = 0 \; . /$$

CHAPTER III

DISTRIBUTION OF QUADRATIC FORMS

§3.1. In the previous chapter, we talked about the independence of two quadratic forms in normal variates along with various other related results. Basic and indeed the most practical theorem of that chapter was the one due to Craig. In the present chapter we shall go one step further in the distribution theory of quadratic forms and shall discuss various conditions under which a given quadratic form in normal random variables follows a χ^2-distribution. Among various results to follow, our attention will mainly be focused on Cochran's theorem (Theorem 15) and various generalizations of it.

Theorem 15 is due to W.G. Cochran [5]. Earlier R.A. Fisher ([10], pages 96-98) proved another theorem of this type; and because of this similarity, Cochran's theorem is often referred to as Fisher-Cochran theorem in the literature. Apart from the original proof due to W.G. Cochran [5], J. Ogawa [27] has also given proof of this theorem which is based on a series of algebraic lemmas. G.W. Madow [23] gives the algebraic basis of Cochran's theorem and uses it to extend Cochran's theorem to the non-central case. Later G.S. James [16] pointed out by proving three theorems that we do not need all the hypothesis of Cochran's theorem for its validity.

A.G. Franklin and G. Marsaglia [9] have extended Cochran's theorem to the correlated cases and their proof has subsequently been simplified by K.S. Banerjee [4] and R.M. Loynes [22]. These results involve the notion of idempotency of matrices. A.T. Craig

[7] has given a necessary and sufficient condition for two quadratic forms to be independently distributed as χ^2; here also the idempotency of the matrix is involved.

If $X' = (x_1, x_2, \ldots, x_n)$ follows a multivariate normal distribution, what are the conditions under which the quadratic form $Q = X'AX$ follows a χ^2-distribution? This question has been discussed by various statisticians viz. B.R. Bhat [3], D.N. Shanbhag [32], [33] and I.J. Good [12]. Later G.P.H. Styan [31] specializes this question and discusses separately the conditions under which Q follows central and non-central χ^2-distribution and finally he gives a generalization of Cochran's theorem. Also in this paper he points out a mistake in I.J. Good's [12] result by giving a counter example.

Finally we remark that G.W. Madow [8] has given various generalizations of Cochran's theorem which are applicable in the Multivariate Statistical Analysis (cf., Theorems 7, 8 and 9 in [24]). The derivations of these generalizations depend upon other theorems proved in the paper.

§3.2. Let $x_1, x_2, \ldots, x_n$ be normally and independently distributed with mean zero and variance 1. Consider the quadratic form $Q = X'AX$, where $X' = (x_1, x_2, \ldots, x_n)$ and rank of $A = r$. On writing

$$Q = \sum_{j=1}^{r} \lambda_j y_j^2 \;;$$

where λ_j are the non-zero eigenvalues of A (cf., §2.3) we immediately have the following.

Theorem 14: If $x_1, x_2, \ldots, x_n$ are normally and independently distributed with mean zero and variance 1. Then the quadratic form $Q = X'AX$ is distributed as is the linear form

$$\sum_{j=1}^{r} \lambda_j z_j$$

where z_j are independent and follow χ^2-distribution with 1 d.f.; and λ_j are non-zero eigen-values of A.

Corollary 1: Under the hypothesis of Theorem 14, a necessary and sufficient condition that $Q = X'AX$ follows a χ^2-distribution is that the non-zero eigenvalues of A are all 1.

The following theorem, which is the central theorem of the present chapter, was published in 1934 by W.G. Cochran [5].

Theorem 15: (Cochran): Let $x_1, x_2, \ldots, x_n$ be normally and independently distributed with mean 0 and variance 1; let $q_1, q_2, \ldots, q_k$ be k quadratic forms in x_i's with d.f. $n_1, n_2, \ldots, n_k$ respectively and such that

$$\sum_{i=1}^{n} x_i^2 = q_1 + q_2 + \ldots + q_k \quad . \tag{3.1}$$

Then a necessary and sufficient condition that $q_1, q_2, \ldots, q_k$ are independently distributed as χ^2-distributions with d.f. $n_1, n_2, \ldots, n_k$ respectively is

$$n_1 + n_2 + \ldots + n_k = n \quad . \tag{3.2}$$

Earlier R.A. Fisher [10] obtained a similar result whi can be stated as: "If $x_1, x_2, \ldots, x_n$ have independent standard normal distributions and if $z_1, z_2, \ldots z_h$ are h $(h < n)$ ortho linear forms in the x_i's, then the quantity

$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{h} z_i^2$$

is distributed independently of $z_1, z_2, \ldots, z_h$ as χ^2 with (n degrees of freedom."

Apart from the original proof of Theorem 15, given by W.G. Cochran [5], J. Ogawa [27] has given an algebraic proof, whi depends on various algebraic lemmas. Both the proofs have been combined and modified here to prove this theorem by an applicati of Theorem 7.

Proof of Theorem 15: (Necessary Part): If $q_1, q_2, \ldots, q_k$ are independently distributed as χ^2-distributions with d.f. $n_1, n_2, \ldots$ respectively, then $q_1+q_2+\ldots+q_k$ is distributed as a χ^2-distrib with d.f. $n_1+n_2+\ldots+n_k$, by the additive property of the χ^2-distribution. But

$$q_1+q_2+\ldots+q_k = x_1^2+x_2^2+\ldots+x_n^2$$

and therefore has a χ^2-distribution with n d.f.; hence

$$n = n_1+n_2+\ldots+n_k \quad .$$

(Sufficient Part): Let $A_1, A_2, \ldots, A_k$ denote the real symmetric matrices associated with the quadratic forms $q_1, q_2, \ldots, q_k$ resp then condition (3.1) in terms of matrices can be written as

$$A_1 + A_2 + \dots + A_k = I \tag{3.3}$$

where I is the identity matrix.

Now we see that Part (v) of Theorem 7 implies

(i) $A_i^2 = A_i$, $i = 1,2,\dots,k$

(ii) $A_i \cdot A_j = 0$, $i \neq j$; $i,j = 1,2,\dots,k$.

Therefore Craig's theorem (Theorem 1) implies the independence of quadratic forms; and idempotency of the matrices implies that they follow χ^2-distribution (Corollary 1; Theorem 14)./

A simple application of Theorem 15 gives us the following:

Corollary 2: Let x_i, $i = 1,2,\dots,n$ be independent $N(0,1)$ random variables; if $\sum_{i,j} a_{ij}x_ix_j$ is distributed as χ^2 with r degrees of freedom, then $\sum_{i,j} (\delta_{ij}-a_{ij})x_ix_j$ is distributed as χ^2 with $(n-r)$ degrees of freedom and both

$$\sum_{i,j} a_{ij}x_ix_j \quad \text{and} \quad \sum_{i,j} (\delta_{ij}-a_{ij})x_ix_i$$

are independent where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Proof:

Suppose $Q = \sum_{i,j} a_{ij}x_ix_j$ is distributed as χ^2. Let

$A = (a_{ij})$, we shall show that

$$r(A) + r(I-A) = n \quad .$$

Then, the assertion of the corollary is an immediate consequence of Theorem 15.

Since Q follows χ^2 distribution, the non-zero eigenvalues of A are all 1 (Corollary 1). Therefore

$$Q = \sum_{j=1}^{r} y_j^2$$

where y_j are new variates obtained by applying an orthogonal transformation T such that

$$X = TY \quad .$$

Also,

$$\sum_{i=1}^{n} x_i^2 = X'X = Y'T'TY = Y'Y = \sum_{i=1}^{n} y_i^2 \quad .$$

Therefore,

$$\sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{r} y_i^2 = \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{r} a_{ij}x_ix_j$$

$$= \sum_{i,j} (\delta_{ij}-a_{ij})x_ix_j \quad .$$

Consequently,

$$\sum_{i=r+1}^{n} y_i^2 = \sum_{i,j} (\delta_{ij}-a_{ij})x_ix_j$$

thus $r(I-A) = n-r$ and hence $r(A)+r(I-A) = n$.

G.S. James [16] has also discussed this theorem, and he has pointed out some redundancies in the original statement of Cochran's theorem. He has claimed that we need only assume (in Theorem 15) that $q_1, q_2, \ldots, q_k$ have χ^2-distribution or that they have independent distributions; the other property and also the fact that

$$\sum_{j=1}^{n} n_j = n ,$$

then follow.

More precisely James [16] states and proves the following three theorems.

Suppose $x_1, x_2, \ldots, x_n$ have independent standard normal distributions and $q_1, q_2, \ldots, q_k$ are quadratic forms in the x_i's of ranks $n_1, n_2, \ldots, n_k$ respectively satisfying

$$\sum q_j = \sum x_i^2 .$$

Then:

<u>Theorem 16</u>: If $\sum n_j = n$, then each q_j is a χ^2 variate with n_j degrees of freedom, and the q_j are distributed independently.

<u>Theorem 17</u>: If each q_j is a χ^2-variate, then q_j are distributed independently with n_j d.f. and $\sum_j n_j = n$.

<u>Theorem 18</u>: If q_j are distributed independently, then each q_j is a χ^2-variate with n_j degrees of freedom, and $\sum n_j = n$.

We notice that Theorem 16 is the sufficiency part of Cochran's theorem, whereas Theorem 17 and Theorem 18 both state differently the necessary part of Cochran's theorem. Here we shall give the proof of these three theorems and consequently we shall have another proof of Cochran's theorem.

Proof of Theorem 16: Since q_1 has rank n_1, we can find an orthonormal transformation of the x_i to new variates ξ_i, such that

$$q_1 = \lambda_1 \xi_1^2 + \ldots + \lambda_{n_1} \xi_{n_1}^2 \qquad (\lambda_1, \ldots, \lambda_{n_1} \neq 0)$$

and

$$\sum_i x_i^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_n^2 .$$

Therefore

$$q_2+q_3+\ldots+q_k = (1-\lambda_1)\xi_1^2+\ldots+(1-\lambda_{n_1})\xi_{n_1}^2+\xi_{n_1+1}^2+\ldots+\xi_n^2 .$$

But since $q_2, q_2, \ldots, q_k$ have ranks $n_2, n_3, \ldots, n_k$, the rank of $q_2+q_3+\ldots+q_k$ cannot exceed

$$n_2 + n_3 + \ldots + n_k = n-n_1 .$$

Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_{n_1} = 1$, and we have

$$q_1 = \xi_1^2 + \ldots + \xi_{n_1}^2$$

$$q_2 + \ldots + q_k = \xi_{n_1+1}^2 + \ldots + \xi_n^2 .$$

Thus q_1 is a positive semi-definite form distributed as χ^2 with n_1 degrees of freedom, independently of $q_2+q_3+\ldots+q_k$. Similarly every q_j is a positive semi-definite form distributed as χ^2 with n_j d.f.; independently of

$$q_1 + q_2 + \ldots + q_{j-1} + q_{j+1} + \ldots + q_k \ .$$

Now, it is easy to see this implies complete independence of q_i's. /

<u>Proof of Theorem 17:</u>

Since q_1 is a quadratic form of rank n_1, we can find an orthonormal transformation to new variates ξ_i, such that,

$$q_1 = \lambda_1\xi_1^2 + \ldots + \lambda_{n_1}\xi_{n_1}^2 \qquad (\lambda_1,\lambda_2,\ldots,\lambda_{n_1} \neq 0) \ .$$

Since the ξ_i are independent standard normal variates, it follows that the moment generating function of q_1 is

$$M_{q_1} = [(1-2\lambda_1 t) \ldots (1-2\lambda_{n_1} t)]^{-(1/2)} \ .$$

But q_1 is a χ^2-variate, so that M_{q_1} is of the form $(1-2t)^{-(1/2)v_1}$, where v_1 is the number of degrees of freedom of q_1. Identifying these two experessions for M_{q_1} we have

$$v_1 = n_1 \quad \text{and} \quad \lambda_1 = \lambda_2 = \ldots = \lambda_{n_1} = 1 \ .$$

Therefore

$$q_1 = \xi_1^2 + \ldots + \xi_{n_1}^2 \ ; \qquad q_2 + \ldots + q_k = \xi_{n_1+1}^2 + \ldots + \xi_n^2 \ .$$

Hence q_1 is distributed independently of $q_2 + \ldots + q_k$ and the same is true of other q_j.

Therefore we conclude (see Proof of Theorem 16) that the q_j are mutually independent χ^2-variates with n_j degrees of freedom. Thus $\sum q_j$ is distributed as χ^2 with $\sum n_j$ degrees of freedom; but $\sum q_j = x_1^2$ is also distributed as χ^2 with n degrees of freedom. Therefore $\sum_j n_j = n$. /

Proof of Theorem 18:

Since q_1 is a quadratic form of rank n_1, there exists an orthonormal transformation to new variates ξ_i, such that

$$q_1 = \lambda_1\xi_1^2 + \ldots + \lambda_{n_1}\xi_{n_1}^2 \qquad (\lambda_1,\ldots,\lambda_n \neq 0)$$

$$q_2+\ldots+q_k = (1-\lambda_1)\xi_1^2+\ldots+(1-\lambda_{n_1})\xi_{n_1}^2+\xi_{n_1+1}^2+\ldots+\xi_n^2 \; .$$

Therefore the joint moment generating function of q_1 and $q_2+\ldots+q_k$, being the expected value of $\exp\,[q_1t+(q_2+\ldots+q_k)u]$, is

$$M_{q_1,q_2+\ldots+q_k}(t,u) = [\{1-2\lambda_1t-2(1-\lambda_1)u\}$$

$$\ldots\{1-2\lambda_{n_1}t-2(1-\lambda_{n_1})u\}\{1-2u\}^{n-n_1}]^{-(1/2)} \; . \qquad (3.9)$$

But q_1 and $q_2+\ldots+q_k$ are distributed independently; therefore

$$M_{q_1,q_2+\ldots+q_k}(t,u) = M_{q_1}(t) \cdot M_{q_2+\ldots+q_k}(u) \qquad (3.10)$$

$$= [\{1-2\lambda_1t\}\ldots\{1-2\lambda_{n_1}t\}]^{-(1/2)} \cdot$$

$$\cdot\,[\{1-2(1-\lambda_1)u\}\ldots\{1-2(1-\lambda_{n_1})u\}\{1-2u\}^{n-n_1}]^{-(1/2)}$$

Comparing (3.9) and (3.10) we have

$$\{1+\ell_1 t+m_1 u\}\ldots\{1+\ell_{n_1} t+m_{n_1} u\}$$

$$\equiv \{(1+\ell_1 t)(1+m_1 u)\}\ldots\{(1+\ell_{n_1} t)(1+m_{n_1} u)\} \qquad (3.11)$$

where

$$\ell_i = (-2\lambda_i) \quad \text{and} \quad m_i = (-2)(1-\lambda_i) \quad .$$

For fixed u, comparing the coefficients of highest powers of t on both sides of (3.11), we conclude

$$\ell_1 \cdot \ell_2 \cdot \ldots \cdot \equiv \ell_1 \ldots \ell_{n_1} (1+m_1 u) \ldots (1-m_{n_1} u)$$

identically in u.

Therefore, we get

$$m_1 = m_2 = \ldots m_{n_1} = 0$$

i.e. $$\lambda_1 = \lambda_2 = \ldots = \lambda_{n_1} = 1 \quad .$$

So that

$$q_1 = \xi_1^2 + \ldots + \xi_{n_1}^2 ; \quad q_2 + \ldots + q_k = \xi_{n_1+1}^2 + \ldots + \xi_n^2$$

and now the rest of the proof follows on the same lines as in Theorem 17. /

We shall prove the following Theorem 19 later in the section, but for present, we are just stating it to get Theorem 20.

Theorem 19: If Y is distributed as $N(\mu, I_n)$, then a necessary and sufficient condition that $Y'AY$ is distributed as $\chi'^2(k,\lambda)$ (where $\lambda = \frac{1}{2}\mu'A\mu$) is that A be an idempotent matrix of rank k.

Using Theorems 7 and 19 together with the following result (which is an extension of Craig's result) we get Theorem 20.

"If Y is distributed as $N(\mu, I_n)$, then a necessary and sufficient condition that $Y'B_1Y, Y'B_2Y, \ldots, Y'B_kY$ be jointly independent is that $B_iB_j = 0$ for all $i \neq j$."

Theorem 20: If Y is distributed as $N(\mu, I_n)$ and if

$$Y'AY = \sum_{i=1}^{k} Y'A_iY \quad ,$$

where the rank of A equals p and the rank of A_i equals p_i, then

(1) Any two of three conditions C_1, C_2, C_3 are necessary and sufficient for all the remaining conditions;

(2) Any two of the three conditions D_1, D_2, D_3 are necessary and sufficient for all the remaining conditions;

(3) Any two conditions C_i and D_j; $i \neq j$ are necessary and sufficient for all the remaining conditions;

(4) E_1 and C_3 are necessary and sufficient for all the remaining conditons;

(5) E_1 and D_3 are necessary and sufficient for all the remaining conditions.

C_1 : $Y'A_iY$ is distributed as $\chi'^2(p_i,\lambda_i)$ where
$\lambda_i = (\mu' A_i \mu)/2$ for $i = 1,2,\ldots,k$.

C_2: $Y'A_iY$ and $Y'A_jY$ are independent for all $i \neq j$.

C_3: $Y'AY$ is distributed as $\chi'^2(p,\lambda)$ where
$\lambda = (\mu' A \mu)/2$.

D_1: Each A_i is an idempotent matrix.

D_2: $A_i \cdot A_j = 0$ for all $i \neq j$.

D_3: A is an idempotent matrix.

E_1: $\sum_{i=1}^{k} p_i = p$.

In the light of the preceeding remark, proof of this theorem is immediate; moreover if in the above Theorems 19 and 20 we have Y distributed as $N(u,\sigma^2 I_n)$, then again all these results are valid except each quadratic form and each λ and λ_i must be divided by σ^2.

Now we are in a position to give the generalizations of Cochran's theorem (Theorem 15). G.W. Madow [23] has extended Cochran's theorem to non-central case as follows.

<u>Theorem 21</u>: If Y is distributed as $N(\mu,I_n)$ and if

$$Y'Y = \sum_{i=1}^{k} Y'A_iY$$

(where rank of A_i is n_i), then a necessary and sufficient condition that $Y'A_iY$ $(i = 1,2,\ldots,k)$ are independently distributed as

$\chi'^2(n_i, \ _i)$ is that $\sum_{i=1}^{k} n_i = n$.

We shall deduce this theorem from Theorem 22 which appeared in [9], however for an independent proof of a little more general version of Theorem 21, readers are referred to ([23], pages 102-103).

Theorem 22: If Y is distributed as $N(\mu,V)$ where V is $n \times n$ positive definite symmetric matrix, and if

$$Y'BY = \sum_{i=1}^{k} Y'B_iY$$

then any one of the six conditions, C_1, C_2, C_3, C_4, C_5, C_6 is necessary and sufficient for $Y'B_iY$ to be independently distributed as $\chi'^2(p_i,\lambda_i)$ where $\lambda_i = \frac{1}{2}\mu'B_i\mu$.

C_1: BV is idempotent and $\sum_{i=1}^{k} p_i = p$.

C_2: BV and each B_iV be idempotent.

C_3: BV be idempotent and $B_iVB_j = 0$ for all $i \neq j$.

C_4: $Y'BY$ be distributed as $\chi'^2(p,\lambda)$ and $p = \sum_{i=1}^{k} p_i$. $(\lambda = \frac{1}{2}\mu'B\mu)$.

C_5: $Y'BY$ be distributed as $\chi'^2(p,\lambda)$ and B_iV be idempotent, (where $\lambda = \frac{1}{2}\mu'B\mu$).

C_6: $Y'BY$ be distributed as $\chi'^2(p,\lambda)$ and $B_iVB_j = 0$ for $i \neq j$ where $\lambda = \frac{1}{2}\mu'B\mu$.

Proof: Since V is positive definite, there exists a non-singular matrix P such that $P'VP = I_n$ (see [28], page 36). Let $Z = P'Y$; then Z is distributed as $N(P'\mu, I_n)$. Also $Y'BY = Z'P^{-1}BP'^{-1}Z$, $Y'B_iY = Z'P^{-1}B_iP'^{-1}Z$, and

$$Z'(P^{-1}BP'^{-1})Z = \sum_{i=1}^{k} Z'(P^{-1}B_iP'^{-1})Z \quad . \tag{3.14}$$

If we let $A = P^{-1}BP'^{-1}$ and $A_i = P^{-1}B_iP'^{-1}$, then (3.14) can be written as

$$Z'AZ = \sum_{i=1}^{k} Z'A_iZ \quad .$$

Now our theorem follows immediately (indeed remarkably!) from Theorem 20, if we show that BV is idempotent if and only if A is idempotent, B_iV is idempotent if and only if A_i is idempotent and $B_iVB_j = 0$ for $i \neq j$ iff $A_iA_j = 0$ for $i \neq j$.

We prove this.

$$\begin{aligned} A \text{ idempotent} \quad & \text{iff} \quad A \cdot A = A \\ & \text{iff} \quad (P^{-1}BP'^{-1})(P^{-1}BP'^{-1}) = (P^{-1}BP'^{-1}) \\ & \text{iff} \quad BP'^{-1}P^{-1}B = B \\ & \text{iff} \quad BVB = B \qquad (\because \; P'^{-1}P^{-1} = V) \\ & \text{iff} \quad (BV)(BV) = (BV) \; . \end{aligned}$$

Similarly A_i is idempotent iff B_iV is idempotent.

Now $B_iVB_j = 0$ for $i \neq j$ implies $P^{-1}B_iVB_jP'^{-1} = 0$.

Therefore

$$0 = P^{-1}B_iP'^{-1}P'VPP^{-1}B_jP'^{-1} \qquad \text{for } i \neq j$$

$$= A_i' I \cdot A_j = A_i \cdot A_j \quad .$$

Also the reverse implications hold and thus the theorem is established./

Taking $B = I$ and $V = I$ in the above theorem, we see that Theorem 21 falls out right away. Taking $k = 1$ in the above theorem, we have:

Corollary: If Y is distributed as $N(\mu, V)$, where V is $n \times n$, positive definite symmetric matrix, then a necessary and sufficient condition that $Y'AY$ be distributed as $\chi'^2(p,\lambda)$ where p is rank of A and where $\lambda = \frac{1}{2}\mu' A\mu$ is that AV be idempotent.

Now we go to another result due to A.T. Craig [7] (Theorem 23) which affords a simple test as to whether the distributions are of χ^2 type. Finally we shall conclude this section by giving proof of Theorem 19, a generalization of Craig's result (Theorem 23).

Theorem 23: Let $Q_1 = X'AX$ and $Q_2 = X'BX$ be two quadratic forms in n normally and independently distributed variables with mean zero and variance one. Then Q_1 and Q_2 have independent chi-square distributions if and only if

$$AB = 0, \quad A^2 = A, \quad \text{and} \quad B^2 = B.$$

A very simple and straightforward proof of this theorem is given in ([7], pages 196-197).

We now conclude this section by proving Theorem 19.

Proof of Theorem 19: We shall first prove the sufficiency. Since

A is idempotent of rank k, there exists an orthogonal matrix P such that

$$P'AP = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} .$$

Let $Z = P'Y$. Then

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

is distributed as $N(\alpha, I_n)$ where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = P'\mu ,$$

and where Z_1 and α_1 are both $k \times 1$ vectors. Z_1 is distributed as $N(\alpha_1, I_k)$. Also

$$Y'AY = Z'P'APZ = Z_1'Z_1 .$$

Therefore it follows (see [9], Theorem F, p. 679).

$Y'AY = Z_1'Z_1$ is distributed as $\chi'^2(k,\lambda)$ where $\lambda = \frac{1}{2}\alpha_1'\alpha_1$.

Thus our result will follow if we show that $\alpha_1'\alpha_1 = \mu'A\mu$.

Write $P = (P_1, P_2)$ where P_1 is $n \times k$. Then

$$\mu'A\mu = \mu'PP'APP'\mu = \mu'(P_1,P_2)\, P'AP \begin{bmatrix} P_1' \\ P_2' \end{bmatrix} \mu$$

$$= (\mu'P_1, \mu'P_2) \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1'\mu \\ P_2'\mu \end{bmatrix}$$

$$= \mu'P_1P_1'\mu = \alpha_1'\alpha_1 .$$

(Necessity): Let us now assume that $Y'AY$ is distributed as χ

and we shall show that this implies A is idempotent of rank k

Let C be the orthogonal matrix such that $C'AC = D$, where D

a diagonal matrix and number of non-zero diagonal elements d_{ii}

equal to the rank of A. Let $Z = C'Y$ then

$$Y'AY = Z'C'ACZ = Z'DZ = \sum_{i=1}^{n} d_{ii} z_i^2 .$$

Again since Z is distributed as $N(C'\mu, I_n)$, z_i^2 is distributed as $\chi'^2(1,\lambda_i)$ where $\lambda_i = [E(z_i)]^2/2$ (see [9], Theorem F, page 679).

Since the z_i are independent, the moment generating function of $\sum_{i=1}^{n} d_{ii} z_i^2$ is

$$\prod_{i=1}^{n} (1-2td_{ii})^{-(1/2)} e^{-\lambda_i+\lambda_i(1-2d_{ii}t)^{-1}} .$$

Also, since $Y'AY$ is distributed as $\chi'^2(k,\lambda)$, the moment gene

ing function (see [21], page 49) of $Y'AY$ is

$$(1-2t)^{-k/2} e^{-\lambda+\lambda(1-2t)^{-1}} .$$

Since $Y'AY = \sum_{i=1}^{n} d_{ii} z_i^2$, these two moment generating functions a

equal and we have

$$(1-2t)^{-k/2} e^{-\lambda+\lambda(1-2t)^{-1}} = \prod_{i=1}^{n} (1-2d_{ii}t)^{-(1/2)} e^{-\lambda_i+\lambda_i(1-2d_i}$$

Both sides of above as functions of t are analytic for some neighborhood of zero. For both sides to have the same singularities we must have k of d_{ii} as 1, $n-k$ of d_{ii} as zero and $\lambda = \sum_{i=1}^{n} \lambda_i$.

Thus if $Y'AY$ is distributed as $\chi'^2(k,\lambda)$, then k of d_{ii} are equal to unity and $n-k$ of d_{ii} are equal to zero. But d_{ii} are the characteristic roots of A. Hence A must be idempotent of rank k. /

§3.4. Let us consider now the quadratic form $Q = X'AX$, where $X' = (x_1, x_2, \ldots, x_n)$ follows a multivariate normal distribution with mean vector μ, where $\mu' = (\mu_1, \mu_2, \ldots, \mu_n)$ and variance - covariance matrix V. Various results have been stated for Q to follow a chi-square distribution. B.R. Bhat [3] works with V as non-singular. I.J. Good [12], D.N. Shanbhag [32], [33] and G.P.H. Styan [31] disccuses those results even when V is singular.

To prove Theorem 24, we start with the following lemmas: This theorem has quite a few applications in the analysis of variance as we shall see in the next chapter.

Lemma 1: A real symmetric quadratic form $X'AX$ is distributed as $\chi^2(n)$ if and only if $A = V^{-1}$.

Proof: We know (see [8], page 457), $X'AX$ has χ^2-distribution if and only if

$$AVA = A \quad ; \tag{3.15}$$

Further it has n-degrees of freedom if and only if A is non-singular. Therefore (3.15) implies

$$V = A^{-1} \; . /$$

Lemma 2: A necessary and sufficient condition that $X'AX$ has a $c\chi^2$-distribution $(c \geq 0)$ is that

$$cA = A \cdot V \cdot A \; .$$

Proof: Write $X = \sqrt{c}\, Y$. Therefore $X'AX$ has a $c\chi^2$-distribution if and only if $cY'AY$ has a $c\chi^2$ or equivalently if and only if $Y'AY$ has a χ^2 distribution.

Since variance - covariance matrix of Y is $\frac{1}{c} \cdot V$, therefore, as in Lemma 1, $Y'AY$ has a chi-square distribution if and only if

$$A = A \cdot \frac{V}{c} \cdot A \quad \text{or} \quad cA = AVA \; . /$$

I.J. Good [12] has given certain necessary and sufficient conditions for a quadratic form $X'AX$ to follow chi-square distribution with k-degrees of freedom, where X is assumed to have multinormal distribution with mean 0 and variance - covariance matrix V, possibly singular. His conditions depend on the following theorem.

Theorem 25: A necessary and sufficient condition for $X'AX$ to follow a chi-square distribution with k-degrees of freedom is that AV (V may be singular) has k unit characteristic roots, the rest zero.

The proof is immediate (see [12], page 215).

Good has claimed if AV has k unit characteristic roots and the rest zero, then AV must be idempotent. Assuming the truth of this assertion he gives two results, Corollary (i) and (ii) in [12]. Unfortunately his claim is shown to be false by C.G. Khatri [19] and G.P.H. Styan [31]. Here is a counter-example from [31].

Let

$$A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad V = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Here AV has one unit characteristic root and two zero characteristic roots, and is not idempotent, for

$$AV = \begin{vmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq (AV)^2 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

In Theorem 26 below, C.G. Khatri [19] has given a necessary and sufficient condition for a quadratic form $X'AX$ to follow a χ^2 distribution, where X has a multivariate normal distribution with zero mean. Later D.N. Shanbhag [32] has shown in Theorem 27 that the condition stated in Theorem 26 is equivalent to conditions given in Theorem 27. We shall prove both of these theorems simultaneously.

<u>Theorem 26</u>: A set of necessary and sufficient conditions for $X'AX$ to follow χ^2-distribution with k degrees of freedom is

$$VAVAV = VAV \; ; \qquad r(VAV) = tr(AV) = k \quad . \tag{3.16}$$

where V is covariance matrix of X, not necessarily non-singular.

Theorem 27: Following are two sets of conditions each of which is equivalent to (3.16)

(i) $(AV)^2 = (AV)^3 \; ; \quad tr(AV) = k$

(ii) $tr[(AV)^2] = tr(AV) = k \; ; \quad r(VAV) = k$.

To prove these theorems we start with the following lemmas.

Lemma 3: If C is a non-negative or non-positive symmetric matrix and U and W are matrices such that UW is real, then

$$r(U'CUW) = r(W'U'CU) = r(CUW) = r(W'U'C) \quad .$$

Proof: To be specific, let us suppose C is non-negative. We have

$$r(U'CUW) \leq r(CUW) \quad .$$

Write $C = EE'$, E a real matrix.

$$r(U'CUW) = r(U'EE'UW) \geq r(W'U'EE'UW) = r[(E'UW)'(E'UW)]$$

$$= r(E'UW) \geq r(EE'UW) = r(CUW) \quad .$$

Therefore we get

$$r(U'CUW) = r(CUW) \quad .$$

All the remaining equalities in the lemma now follow on noting $C' = C$ and $r(B) = r(B')$, B any matrix.

Lemma 4: If C is a non-negative or non-positive symmetric matrix, and if U and W are matrices such that UW is real, then $U'CUW = 0$ if and only if $CUW = 0$.

This is an immediate consequence of Lemma 3.

Proof of Theorems 26 and 27: Let T be a real matrix such that $TT' = V$; then it can be shown (cf., [28], page 188) that $X'AX$ follows chi-square distribution with k degrees of freedom if and only if $T'AT$ is idempotent of rank k. On diagonalizing $T'AT$ we can see $X'AX$ follows chi-square distribution if and only if

$$D^2 = D \quad \text{and} \quad r(D) = k \tag{3.17}$$

where D is diagonal matrix of characteristic roots of $T'AT$. Let L be the orthogonal matrix such that

$$L'T'ATL = D .$$

If D is idempotent,

$$r(D) = tr(D) = tr(L'T'ATL) = tr(LL'T'AT)$$
$$= tr(T'AT) = tr(ATT') = tr(AV)$$

i.e., $$k = tr(T'AT) = tr(AV) ;$$

therefore (3.17) is equivalent to

$$T'AT(T'AT-I) = 0, \quad tr(T'AT) = tr(AV) = k . \tag{3.18}$$

Writing $T' = U$, $C = I$ and $W = AT(T'AT-I)$, $T'AT(T'AT-I) = 0$ becomes $CUW = 0$, therefore Lemma 4 implies that (3.18) is equivalent to

$$TT'AT(T'AT-I) = 0 \quad \text{and} \quad tr(AV) = k$$

i.e., $$(TT'A-I)TT'AT = 0 \quad \text{and} \quad tr(AV) = k \ . \tag{3.19}$$

Now $(TT'A-I)TT'AT = 0$ is of the form $W'U'C = 0$ where

$$W' = (TT'A-I)TT'A \ , \quad U = T' \quad \text{and} \quad C = I \ ,$$

therefore Lemma 4 implies (3.19) is equivalent to

$$(TT'A-I)TT'ATT' = 0 \quad \text{and} \quad tr(AV) = k$$

i.e., $$(VA-I)VAV = 0 \quad \text{and} \quad tr(AV) = k \ .$$

Hence $X'AX$ follows χ^2-distribution with k-degrees of freedom if and only if

$$VA(VAV-V) = 0 \quad \text{and} \quad tr(AV) = k \ . \tag{3.20}$$

Writing $C = V$, $U = A$, $W = (VAV-V)$ and applying Lemma 4; (3.20) is equivalent to

$$AVA(VAV-V) = 0 \quad \text{and} \quad tr(AV) = k \qquad (A = A')$$

or $(AV)^3 = (AV)^2$ and $tr(AV) = k.$

If (3.17) holds, we get

$$tr(D^2) = tr(D) \quad \text{and} \quad r(D) = k \ . \tag{3.21}$$

Writing $C = I$, $U = T'$ and $W = AT$, we see that $T'AT$ has the form CUW. Therefore Lemma 3 implies

$$r(T'AT) = r(TT'AT) .$$

Also $TT'AT$ has the form $W'U'C$ with $C = I$, $U = T'$ and $W' = TT'A$.

Therefore $r(TT'AT) = r(TT'ATT')$. Hence

$$r(D) = r(T'AT) = r(TT'ATT') = r(VAV) .$$

Therefore (3.21) becomes

$$tr(D^2) = tr(D) = k \quad \text{and} \quad r(D) = r(VAV) = k .$$

Thus we see (3.17) is equivalent to

$$tr[(AV)^2] = tr(AV) = k \quad \text{and} \quad r(VAV) = k . \tag{3.22}$$

To establish the equivalence between (ii) Theorem 27 and (3.16) we must now prove (3.21) implies (3.17) and this will complete the proof.

If d_i denotes the i^{th} diagonal element of D then (3.17) implies

$$\sum_i (d_i-1)^2 = \sum_i d_i^2 - 2\sum_i d_i + n = n-k$$

and $r(D) = k$.

i.e.,
$$\sum_i (d_i-1)^2 = n-k \quad \text{and} \quad r(D) = k . \tag{3.23}$$

Also we have $\sum_i (d_i-1)^2 \geq n-r(D)$ where the equality sign holds if and only if non-zero d_i equals unity. Hence this give (3.17) and we are done. /

Another criteria given by C.P.H. Styan [31] is as follows.

Theorem 28: A necessary and sufficient condition for $X'AX$ to follow χ^2 distribution with k degrees of freedom is

$$(AV)^2 = AV$$

if and only if $r(AV) = tr(AV) = k$ or $r(AV) = r(VAV) = k$ (where V is covariance matrix of X).

Proof of this theorem depends on the following lemma.

Lemma 5: A square matrix S not necessarily symmetric, satisfying $S^2 = S^3$, is idempotent if and only if $r(S) = tr(S)$ or $r(S) = r(S^2)$.

For proof of it see ([31], page 568).

Proof of Theorem 28:

From Theorem 27, the required necessary and sufficient condition is (3.16) which is equivalent to

$$(AV)^3 = (AV)^2 .$$

Since

$$r(VAV) = r(VAVAV) \leq r((AV)^2) \leq r(VAV) .$$

By applying Lemma 5 with $AV = S$, we get the necessary and sufficient condition as stated in Theorem 28. /

We recall, in the above discussion we assumed invariably that X has multivariate normal distribution with mean $\mu = 0$ and covariance matrix V possibly singular. C.G. Khatri [19] and Rayner and Livingstone [21] have treated the non-central case in the following theorem.

We conclude this chapter by stating this result and refer the reader to the relevant papers for its proof.

Theorem 29: A set of necessary and sufficient conditions for $X'AX$ to follow a non-central χ^2 distribution with k-degrees of freedom and non-centrality parameter λ, is

$$VAVAV = VAV \ ; \quad r(VAV) = tr(AV) = k$$

$$\mu'(AV)^2 = \mu'AV$$

$$\lambda = (1/2)\,\mu'AVA\mu = (1/2)\,\mu'A\mu \ .$$

Applying Lemma 5 to Theorem 29 we obtain

Theorem 30: A necessary and sufficient condition for $X'AX$ to follow a non-central $\chi'^2(k,\lambda)$ distribution is

$$(AV)^2 = AV \quad \lambda = (1/2)\,\mu'AVA\mu = (1/2)\,\mu'A\mu$$

if and only if

$$r(AV) = tr(AV) = k \quad \text{or} \quad r(AV) = r(VAV) = k$$

CHAPTER IV

APPLICATIONS

As pointed out before, the results discussed in the previous chapters have variety of applications in statistics. The complete justification of various results in the study of mixed models (see [34]), split plot experiments (see [6]) and random effect models (see [29]), involves simultaneous applications of results discussed in Chapters II and III. Apart from these, quadratic forms and their distribution properties have frequent use in ANOVA, Regression Analysis and Econometrics, which we illustrate by the following:

I. Let $x_1, x_2, \ldots, x_n$ be n independent normal random variables with mean zero and variance 1; then the quadratic forms

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 \quad \text{and} \quad n\bar{x}^2$$

are distributed independently as χ^2 with $n-1$ and 1 degrees of freedom respectively.

To show this, we observe $n\bar{x}^2$ is clearly χ^2 with 1 degree of freedom. Therefore, by Corollary 2, Theorem 15

$$\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2$$

is distributed as χ^2 with $(n-1)$ degrees of freedom and is independent of $n\bar{x}^2$.

II. In Chapter II, we talked about the independence of two quadratic forms. If instead, we have a quadratic form $\frac{1}{2} X'BX$ and a linear form $a'X$, where $X' = \{x_1, x_2, \ldots, x_n\}$, x_i being normally correlated variables with covariance matrix V, then these two forms are independent if and only if the quadratic form $a'(V^{-1}-\beta B)^{-1}a$ is independent of β; β arbitrary real variable. For, the joint moment generating function of forms $a'X$ and $1/2\ X'BX$ is given by

$$M(\alpha,\beta) = |I-\beta BV|^{-\frac{1}{2}} \exp\{\frac{1}{2}\alpha^2 a'(V^{-1}-\beta B)^{-1}a\} \quad (4.1)$$

(cf., [1], page 41)

The first factor of (4.1) involves β alone, and is indeed the moment generating function $M(0,\beta)$ of $\frac{1}{2} X'BX$. The second factor would be the moment generating function $M(\alpha,0)$ of $a'X$ if and only if it were independent of β. But the necessary and sufficient condition for independence of any two functions with moment generating function $M(\alpha,\beta)$ is

$$M(\alpha,\beta) = M(\alpha,0)M(0,\beta)$$

and so we get the desired criteria of independence as stated above.

We apply this criteria to estimates $\bar{x}$ and s^2 of mean and variance in a sample of n independent single values of 1-variate. Such situation occurs in the derivation of t-distribution

$$\bar{x} = \sum_{i=1}^{n} x_i/n\ ; \qquad s^2 = \sum_{i=1}^{n} (x_i-\bar{x})^2/n-1$$

To avoid unnecessary complexity we may disregard the factor $1/n$ and $1/(n-1)$ and consider the independence of

$$\sum_{i=1}^{n} x_i \quad \text{and} \quad \sum_{i=1}^{n} (x_i-\bar{x})^2 .$$

Here $a' = \{1,1,\ldots,1\}$ and

$$B = \begin{bmatrix} 1-\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & 1-\frac{1}{n} & \cdots & -\frac{1}{n} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & 1-\frac{1}{n} \end{bmatrix}$$

and V here is I the identity matrix.

Clearly $a'(I-\beta B)^{-1}a$ is in this case the sum of all the n^2 elements in $(I-\beta B)^{-1}$. ——— (*)

Write

$$(I-\beta B) = (1-\beta)I + \beta M$$

where M is the matrix having all elements equal to $1/n$. Obviously M is idempotent.

Therefore

$$(I-\beta B)^{-1} = [(1-\beta)I + \beta M]^{-1} .$$

But

$$[(1-\beta)I + \beta M]^{-1} = (1-\beta)^{-1}[I-\beta M] \tag{4.2}$$

(see the lemma below).

Also

$$I - \beta M = \begin{bmatrix} 1-\frac{\beta}{n} & -\frac{\beta}{n} & \cdots & -\frac{\beta}{n} \\ -\frac{\beta}{n} & 1-\frac{\beta}{n} & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -\frac{\beta}{n} & -\frac{\beta}{n} & \cdots & 1-\frac{\beta}{n} \end{bmatrix}$$

The sum of all elements in $(I-\beta M)$ is $n(1-\beta)$, and therefore the sum of all the elements in $(I-\beta B)^{-1}$ is equal to n, and hence by (*) $a'(I-\beta B)^{-1}a$ is equal to n and thus independent of β. Consequently it follows, $\bar{x}$ and s^2 are independent.

We conclude the proof by establishing the following lemma used in (4.2).

<u>Lemma</u>: If Q is an idempotent matrix, then

$$(\rho I-\sigma Q)^{-1} = \rho^{-1}(I + \frac{\sigma}{\rho-\sigma} Q) \qquad \rho \neq \sigma \qquad (**)$$

The lemma can easily be checked by pre- and post- multiplying the right hand side of (**) and using $Q^2 = Q$.

III. In multiple regression models, the observation vector Y is assumed to be $N(X\beta,\sigma^2 I_n)$, where X is an $(n\times p)$ $(p < n)$ matrix with known elements and of rank p, β is a $(p\times 1)$ vector of unknown parameters, and σ^2 is an unknown scalar. In these models it is often desired to test hypotheses about elements of the vector β. The technique often employed to devise test functions is the technique of analysis of variance. The procedure is to partition the total sum of squares $Y'Y$ into quadratic forms such that

$$Y'Y = \sum_{i=1}^{k} Y'A_iY$$

and use Cochran's theorem (Theorem 14) to ascertain the independence and distribution of the quantities $Y'A_iY$. Since the use of Cochran's theorem involves the knowledge of ranks of A_i, it is sometimes easy to invoke idempotency of A_i's and Craig's condition for independence of quadratic forms (Theorem 1)

Consider the multiple regression model

$$Y = X\beta + e \, , \qquad e \sim N(0,\sigma^2 I)$$

X, β and Y as above.

If we partition X and β as

$$X = (X_1, X_2), \qquad \beta = \binom{\alpha}{\gamma}$$

where X_1 is of order $n \times p_1$ and α is a $p_1 \times 1$ vector, then th above model can be written as

$$Y = X_1\alpha + X_2\gamma + e$$

To test the hypothesis $H_o : \alpha = 0$, we can form the rat

$$u = \frac{Q_1}{Q} \cdot \frac{n-p}{p_1}$$

where u is distributed as F distribution with p_1 and $n-p$ degrees of freedom. Q is minimum value of $e' \cdot e$ with respect to full model and $Q_1 = Q-Q_2$, where Q_2 is the minimum value of $e'e$ with respect to reduced model under H_0. For its justification we proceed as follows:

By minimization procedure it can be shown that

$$Q = Y'(I-XS^{-1}X')Y = Y'AY$$

and

$$Q_2 = Y'(I-X_2S_2^{-1}X_2')Y = Y'BY$$

where $S = X'X$, $S_2 = X_2'X_2$, $(I-XS^{-1}X') = A$ and $(I-X_2S_2^{-1}X_2') = B$. Now

$$A^2 = (I-XS^{-1}X')(I-XS^{-1}X')$$

$$= (I-XS^{-1}X' - XS^{-1}X' + XS^{-1}X'XS^{-1}X')$$

$$= (I-XS^{-1}X') = A.$$

Therefore A is idempotent, similarly B is idempotent. Since

$$X'(I-XS^{-1}X') = 0$$

it is clear that $X_2'(I-XS^{-1}X') = 0$ and $X_1'(I-XS^{-1}X') = 0$. These now imply that C is also idempotent, and $AC = 0$ where

$$C = B-A = (I-X_2S_2^{-1}X_2') - (I-XS^{-1}X') .$$

Since the matrices A, B and C are idempotent, we have $r(A) = n-p$, $r(B) = n-(p-p_1)$ and $r(C) = r(B)-r(A) = p_1$.

Now we are in a position to apply Theorem 20, and thus we have:

1. $Q/\sigma^2 = (Y'AY)/\sigma^2$ is distributed as $\chi'^2(n-p, \lambda_A)$

2. $Q_1/\sigma^2 = (Y'CY)/\sigma^2$ is distributed as $\chi'^2(p_1, \lambda_c)$

3. Q and Q_1 are independent.

4. $\lambda_A = (1/2\sigma^2)(\beta'X'AX\beta) = 1/2\sigma^2[\beta'X'(I-XS^{-1}X')X\beta] = 0$

therefore Q/σ^2 is distributed as $\chi^2(n-p)$.

5. $$\lambda_C = (1/2\sigma^2)[\beta'X'\{(I-X_2S_2^{-1}X_2') - (I-XS^{-1}X')\}X\beta]$$
$$= (1/2\sigma^2)[(\alpha'X_1'+\gamma'X_2')\{(I-X_2S_2^{-1}X_2')-(I-XS^{-1}X')\}(\alpha X_1+\gamma X_2)]$$
$$= (1/2\sigma^2)[\alpha'(X_1'X_1-X_1'X_2S_2^{-1}X_2X_1)\alpha] .$$

Since $X_1'X_1 - X_1'X_2S_2^{-1}X_2X_1$ is positive definite, Q_1/σ^2 has central chi-square distribution if and only if $\alpha = 0$, i.e., H_o is true.

Hence $= (Q_1/Q)\cdot[(n-p)/p_1]$ is distributed as $F'(p_1, n-p, \lambda_c)$ and reduces to central F distribution if and only if H_o is true.

IV. In III we considered the model

$$Y = X\beta + e$$

where $e \sim N(0, \sigma^2 I)$ and X was of full rank.

Now let us suppose that X is not of full rank. In such situation we proceed by finding a generalized inverse G of $X'X$. Suppose our aim is to derive a suitable test statistics for testing $H_o : X\beta = 0$. By working with the minimization procedure (least square method) it can be shown residual error sum of squares is

$$SSE = Y(I-XGX')Y \ .$$

If we denote this quadratic form by Q_1 and $Y'Y$ by Q, the regression sum of squares (S.S.R) is given by

$$S.S.R = Q-Q_1 = Y'XGX'Y \ .$$

Let us denote this quadratic form by Q_2. Now

$$Q_1/\sigma^2 = Y'(I-XGX')Y/\sigma^2 \ .$$

Since $\sigma^2 I(I-XGX')/\sigma^2 = (I-XGX')$ and $(I-XGX')$ is clearly idempotent. Therefore, by Theorem 20, we have

$$Q_1/\sigma^2 \sim \chi'^2[r(I-XGX'),\beta'X'(I-XGX')X\beta/2\sigma^2]$$

or equalently $Q_1/\sigma^2 \sim \chi^2$ with $n-r$ d.f., where $r = \text{rank of } X$.

Also because XGX' is idempotent and $(XGX')(I-XGX') = 0$, Theorem 20 together with Craig's theorem implies that Q_2/σ^2 is distributed independently of Q_1/σ^2 with

$$Q_2/\sigma^2 \sim \chi'^2[r(XGX'),\beta'X'XGX'X\beta/2\sigma^2] \ .$$

Thus

$$u = \frac{Q_2/r}{Q_1/(n-r)}$$

follows non-central F-distribution with parameters r and $n-r$ and non-centrality parameter $\beta'X'X\beta/2\sigma^2$. Hence u follows central F if and only if $X\beta = 0$, i.e., if and only if H_o is true.

BIBLIOGRAPHY

1. AITKIN, A.C. (1940); "On independence of linear and quadratic forms in samples of normally distributed variates". Proc. Royal Soc. Edinb. 60; 40-46.

2. ___________, (1950); "On the statistical independence of quadratic forms in normal variates" Biometrika 37; 93-96.

3. BHAT, B.R. (1961); "On the distribution of certain quadratic forms in normal variates". J.R.S.S. (B), 24; 148-151.

4. BANERJEE, K.S. (1964); "A note on idempotent matrices". Annals of Mathematical Statistics, 35; 880-882.

5. COCHRAN, W.G. (1934); "The distribution of quadratic forms in a normal system with applications to the analysis of covariance". Proc. of Camb. Phil. Soc., 30; 178-191.

6. COCHRAN, W.G. and COX, G. (1968); Experimental Designs, John Wiley and Sons, Inc.

7. CRAIG, A.T. (1943); "Note on the independence of certain quadratic forms". Annals of Mathematical Statistics, 15; 427-429.

8. CARPENTER, O. (1950); "Note on the extension of Craig's theorem to non-central variates". Annals of Mathematical Statistics, 21; 455-457.

9. FRANKLIN, A. GRAYBILL and MARSAGLIA, G. (1957); "Idempotent matrices and quadratic forms in the general linear hypothesis". Annals of Mathematical Statistics, 28; 678-686.

10. FISCHER, R.A. (1926); "Applications of Student's distribution". Metron, 5; 90-104.

11. GOOD, I.J. (1963); "On the independence of quadratic forms". J.R.S.S. (B), 25; [illegible]-382.

12. ___________, (1969); "Conditions for a quadratic form to have a chi-squared distribution". Biometrika, 56, 215-216.

13. HOTELLING, H. (1944); "Note on a matric theorem of A.T. Craig". Annals of Mathematical Statistics, 15; 427-429.

14. HADLEY, G. (1969); Linear Algebra. Addison-Wesley.

15. HOGG, R.V. and CRAIG, A.T.; "Introduction to Mathematical Statistics", 3rd Edition; The Macmillian Company.

16. JAMES, G.S. (1952); "Note on a theorem of Cochran". Proc. of Camb. Phil. Soc. 48, 443-446.

17. KAWADA, Y. (1950); "Independence of quadratic forms in normally correlated variables". Annals of Mathematical Statistics, 21; 614-615.

18. KHATRI, C.G. (1959); "On conditions for the forms of the type XAX' to be distributed independently or to obey Wishart distribution". Calcultta Statistical Assoc. Bull., 8; 162-168.

19. ___________, (1963); "Further contributions to Wishart and independence of second degree polynomials in normal vectors". J. Indian Statist. Assoc., 1, 61-70.

20. KENDELL, M.G.; "The advanced theory of statistics", Vol. 2, (London, 1948).

21. LIVINGSTONE, D. and RAYNER, A.A. (1965); "On the distribution of quadratic forms in singular normal variates". S. African J. Agric. Sci., 8, 357-70.

22. LOYNES, R.M. (1966); "On idempotent matrices". Annals of Mathematical Statistics, 37, 295-296.

23. MADOW, G.W., (1940); "The distribution of quadratic forms in non-central normal random variables". Annals of Mathematical Statistics.

24. ___________, (1938); "Contributions to the theory of multivariate statistical analysis". Trans. Amer. Math. Soc., 44; 454-495.

25. MATERN, B. (1949); "Independence of non-negative quadratic forms in normally correlated variables". Annals of Math. Statistics, 21; 614-615.

26. MATUSITA, K. (1948); "Note on the independence of certain statistics". Tokei sur Kenkyojo Tokyo Annals Vol. 1; 79-82.

27. OGAWA, J. (1948); "On the independence of bilinear and quadratic forms of a random sample from a normal population". Tokei suri Kenkyojo Tokyo Annals, Vol. 1; 83-108.

28. RAO, C.R., "Linear statistical inference and its applications". 2nd Edition, John Wiley and Sons.

29. ROY, S.N. and GNANDESIKAN, R. (1959); "Some contributions to ANOVA in one or more dimensions: I". Annals of Mathematical Statistics, 30; 304-317.

30. SEARLE, S.R. (1971); Linear Models. John Wiley and Sons.

31. STYAN, G.P.H. (1969); "Note on the distribution of quadratic forms of singular normal variables". Biometrika 57; 567-572.

32. SHANBHAG, D.N. (1967); "Some remarks concerning Khatri's result on quadratic forms". Biometrika, 55; 593-595.

33. __________, (1969); "On the distribution of a quadratic form". Biometrika, 57; 222-223.

34. SCHEFFE, H. (1970); The Analysis of Variance. John Wiley and Sons, Inc.